Matthew Foreman
Akihiro Kanamori
Editors

Handbook of Set Theory

Volume 2

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Preface

Numbers imitate space, which is of such a different nature
—Blaise Pascal

It is fair to date the study of the foundation of mathematics back to the ancient Greeks. The urge to understand and systematize the mathematics of the time led Euclid to postulate axioms in an early attempt to put geometry on a firm footing. With roots in the *Elements*, the distinctive methodology of mathematics has become *proof*. Inevitably two questions arise: *What are proofs?* and *What assumptions are proofs based on?*

The first question, traditionally an internal question of the field of *logic*, was also wrestled with in antiquity. Aristotle gave his famous syllogistic systems, and the Stoics had a nascent propositional logic. This study continued with fits and starts, through Boethius, the Arabs and the medieval logicians in Paris and London. The early germs of logic emerged in the context of philosophy and theology.

The development of analytic geometry, as exemplified by Descartes, illustrated one of the difficulties inherent in founding mathematics. It is classically phrased as the question of *how one reconciles the arithmetic with the geometric*. Are numbers one type of thing and geometric objects another? What are the relationships between these two types of objects? How can they interact? Discovery of new types of mathematical objects, such as imaginary numbers and, much later, formal objects such as free groups and formal power series make the problem of finding a common playing field for all of mathematics importunate.

Several pressures made foundational issues urgent in the 19th century. The development of alternative geometries created doubts about the view that mathematical truth is part of an absolute all-encompassing logic and caused it to evolve towards one in which mathematical propositions follow logically from assumptions that may vary with context.

Mathematical advances involving the understanding of the relationship between the completeness of the real line and the existence of solutions to equations led inevitably to anxieties about the role of infinity in mathematics.

These too had antecedents in ancient history. The Greeks were well aware of the scientific importance of the problems of the infinite which were put forth, not only in the paradoxes of Zeno, but in the work of Eudoxus, Archimedes and others. Venerable concerns about resolving infinitely divisible lines into individual points and what is now called "Archimedes' Axiom" were recapitulated in 19th century mathematics.

In response, various "constructions" of the real numbers were given, such as those using Cauchy sequences and Dedekind cuts, as a way of understanding the relationship between discrete entities, such as the integers or the rationals and the continuum. Even simple operations, such as addition of arbitrary real numbers began to be understood as infinitary operations, defined by some kind of limiting process. The notion of *function* was liberalized beyond those that can be written in closed form. Sequences and series became routine tools for solving equations.

The situation was made acute when Cantor, working on natural problems involving trigonometric series, discovered the existence of different magnitudes of infinity. The widespread use of inherently infinitary techniques, such as the use of the Baire Category Theorem to prove the existence of important objects, became deeply embedded in standard mathematics, making it impossible to simply reject infinity as part of mathematics.

In parallel 19th century developments, through the work of Boole and others, logic became once again a mathematical study. Boole's algebraization of logic made it grist for mathematical analysis and led to a clear understanding of propositional logic. Dually, logicians such as Frege viewed mathematics as a special case of logic. Indeed a very loose interpretation of the work of Frege is that it is an attempt to base mathematics on a broad notion of logic that subsumed all mathematical objects.

With Russell's paradox and the failure of Frege's program, a distinction began to be made between *logic* and *mathematics*. Logic began to be viewed as a formal epistemological mechanism for exploring mathematical truth, barren of mathematical content and in need of assumptions or axioms to make it a useful tool.

Progress in the 19th and 20th centuries led to the understanding of logics involving quantifiers as opposed to propositional logic and to distinctions such as those between first and second-order logic. With the semantics developed by Tarski and the compactness and completeness theorems of Gödel, first-order logic has become widely accepted as a well-understood, unproblematic answer to the question *What is a proof?*

The desirable properties of first-order logic include:

- Proofs and propositions are easily and uncontroversially recognizable.

- There is an appealing semantics that gives a clear understanding of the relationship between a mathematical structure and the formal propositions that hold in it.

- It gives a satisfactory model of what mathematicians actually do: the "rigorous" proofs given by humans seem to correspond exactly to the

> "formal" proofs of first-order logic. Indeed formal proofs seem to provide a normative ideal towards which controversial mathematical claims are driven as part of their verification process.

While there are pockets of resistance to first-order logic, such as constructivism and intuitionism on the one hand and other alternatives such as second-order logic on the other, these seem to have been swept aside, if simply for no other reason than their comparative lack of mathematical fruitfulness.

To summarize, a well-accepted conventional view of foundations of mathematics has evolved that can be caricatured as follows:

$$\text{Mathematical Investigation} = \text{First-Order Logic} + \text{Assumptions}$$

This formulation has the advantage that it segregates the difficulties with the foundations of mathematics into discussions about the underlying assumptions rather than into issues about the nature of reasoning.

So what are the appropriate assumptions for mathematics? It would be very desirable to find assumptions that:

1. involve a simple primitive notion that is easy to understand and can be used to "build" or develop all standard mathematical objects,

2. are *evident*,

3. are *complete* in that they settle all mathematical questions,

4. can be easily recognized as part of a recursive schema.

Unfortunately Gödel's incompleteness theorems make item 3 impossible. *Any* recursive consistent collection $\mathcal{A}$ of mathematical assumptions that are strong enough to encompass the simple arithmetic of the natural numbers will be *incomplete*; in other words there will be mathematical propositions P that cannot be settled on the basis of $\mathcal{A}$. This inherent limitation is what has made the foundations of mathematics a lively and controversial subject.

Item 2 is also difficult to satisfy. To the extent that we understand mathematics, it is a difficult and complex business. The Euclidean example of a collection of axioms that are easily stated and whose content is simple to appreciate is likely to be misleading. Instead of simple, distinctly conceived and obvious axioms, the project seems more analogous to specifying a complicated operating system in machine language. The underlying primitive notions used to develop standard mathematical objects are combined in very complicated ways. The axioms describe the operations necessary for doing this and the test of the axioms becomes how well they code higher level objects as manipulated in ordinary mathematical language so that the results agree with educated mathematicians' sense of correctness.

Having been forced to give up 3 and perhaps 2, one is apparently left with the alternatives:

2′. Find assumptions that are in accord with the intuitions of mathematicians well versed in the appropriate subject matter.

3′. Find assumptions that *describe* mathematics to as large an extent as is possible.

With regard to item 1, there are several choices that could work for the primitive notion for developing mathematics, such as *categories* or *functions*. With no *a priori* reason for choosing one over another, the standard choice of *sets* (or set membership) as the basic notion is largely pragmatic. Taking sets as the primitive, one can easily do the traditional constructions that "build" or "code" the usual mathematical entities: the empty set, the natural numbers, the integers, the rationals, the reals, $\mathbb{C}$, $\mathbb{R}^n$, manifolds, function spaces—all of the common objects of mathematical study.

In the first half of the 20th century a standard set of assumptions evolved, the axiom system called the Zermelo-Fraenkel axioms with the Axiom of Choice (ZFC). It is pragmatic in spirit; it posits sufficient mathematical strength to allow the development of standard mathematics, while explicitly rejecting the type of objects held responsible for the various paradoxes, such as Russell's.

While ZFC is adequate for most of mathematics, there are many mathematical questions that it does not settle. Most prominent among them is the first problem on Hilbert's celebrated list of problems given at the 1900 International Congress of Mathematicians, the *Continuum Hypothesis*.

In the jargon of logic, a question that cannot be settled in a theory T is said to be *independent* of T. Thus, to give a mundane example, the property of being Abelian is independent of the axioms for group theory. It is routine for normal axiomatizations that serve to synopsize an abstract concept internal to mathematics to have independent statements, but more troubling for axiom systems intended to give a definitive description of mathematics itself. However, independence phenomena are now known to arise from many directions; in essentially every area of mathematics with significant infinitary content there are natural examples of statements independent of ZFC.

This conundrum is at the center of most of the chapters in this Handbook. Its investigation has left the province of abstract philosophy or logic and has become a primarily mathematical project. The intent of the Handbook is to provide graduate students and researchers access to much of the recent progress on this project. The chapters range from expositions of relatively well-known material in its mature form to the first complete published proofs of important results. The introduction to the Handbook gives a thorough historical background to set theory and summaries of each chapter, so the comments here will be brief and general.

The chapters can be very roughly sorted into four types. The first type consists of chapters with theorems demonstrating the independence of mathematical statements. Understanding and proving theorems of this type require a thorough understanding of the mathematics surrounding the source of the problem in question, reducing the ambient mathematical constructions to combinatorial statements about sets, and finally using some method (primarily forcing) to show that the combinatorial statements are independent.

A second type of chapter involves delineating the edges of the independence phenomenon, giving proofs in ZFC of statements that on first sight would be suspected of being independent. Proofs of this kind are often extremely subtle and surprising; very similar statements are independent and it is hard to detect the underlying difference.

The last two types of chapters are motivated by the desire to *settle* these independent statements by adding assumptions to ZFC, such as large cardinal axioms. Proposers of augmentations to ZFC carry the burden of marshaling sufficient evidence to convince informed practitioners of the reasonableness, and perhaps truth, of the new assumptions as descriptions of the mathematical universe. (Proposals for axiom systems intended to *replace* ZFC carry additional heavier burdens and appear in other venues than the Handbook.)

One natural way that this burden is discharged is by determining what the supplementary axioms *say*; in other words by investigating the consequences of new axioms. This is a strictly mathematical venture. The theory is assumed and theorems are proved in the ordinary mathematical manner. Having an extensive development of the consequences of a proposed axiom allows researchers to see the overall picture it paints of the set-theoretic universe, to explore analogies and disanalogies with conventional axioms, and judge its relative coherence with our understanding of that universe. Examples of this include chapters that posit the assumption that the Axiom of Determinacy holds in a model of Zermelo-Fraenkel set theory that contains all of the real numbers and proceed to prove deep and difficult results about the structure of definable sets of reals.

Were there an obvious and compelling unique path of axioms that supplement ZFC and settle important independent problems, it is likely that the last type of chapter would be superfluous. However, historically this is not the case. Competing axioms systems have been posited, sometimes with obvious connections, sometimes appearing to have nothing to do with each other.

Thus it becomes important to compare and contrast the competing proposals. The Handbook includes expositions of some stunningly surprising results showing that one axiom system actually implies an apparently unrelated axiom system. By far the most famous example of this are the proofs of determinacy axioms from large cardinal assumptions.

Many axioms or independent propositions are not related by implication, but rather by *relative consistency* results, a crucial idea for the bulk of the Handbook. A remarkable meta-phenomenon has emerged. There appears to be a central spine of axioms to which all independent propositions are comparable in consistency strength. This spine is delineated by large cardinal axioms. There are no known counterexamples to this behavior.

Thus a project initiated to understand the relationships between disparate axiom systems seems to have resulted in an understanding of most known natural axioms as somehow variations on a common theme—at least as far as consistency strength is concerned. This type of unifying deep structure is taken as strong evidence that the axioms proposed reflect some underlying reality and is often cited as a primary reason for accepting the existence of large cardinals.

The methodology for settling the independent statements, such as the Continuum Hypothesis, by looking for evidence is far from the usual deductive paradigm for mathematics and goes against the rational grain of much philosophical discussion of mathematics. This has directed the attention of some members of the philosophical community towards set theory and has been grist for many discussions and message boards. However interpreted, the investigation itself is entirely mathematical and many of the most skilled practitioners work entirely as mathematicians, unconcerned about any philosophical anxieties their work produces.

Thus set theory finds itself at the confluence of the foundations of mathematics, internal mathematical motivations and philosophical speculation. Its explosive growth in scope and mathematical sophistication is testimony to its intellectual health and vitality.

The Handbook project has some serious defects, and does not claim to be a remotely complete survey of set theory; the work of Shelah is not covered to the appropriate extent given its importance and influence and the enormous development of classical descriptive set theory in the last fifteen years is nearly neglected. While the editors regret this, we are consoled that those two topics, at least, are well documented elsewhere. Other parts of set theory are not so lucky and we apologize.

We the editors would like to thank all of the authors for their labors. They have taken months or years out of their lives to contribute to this project. We would especially like to thank the referees, who are the unsung heroes of the story, having silently devoted untold hours to carefully reading the manuscripts simply for the benefit of the subject.

Matthew Foreman
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Introduction

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Set theory has entered its prime as an advanced and autonomous research field of mathematics with broad foundational significance, and this Handbook with its expanse and variety amply attests to the fecundity and sophistication of the subject. Indeed, in set theory's further reaches one sees tremendous progress both in its continuing development of its historical heritage, the investigation of the transfinite numbers and of definable sets of reals, as well as its analysis of strong propositions and consistency strength in terms of large cardinal hypotheses and inner models.

This introduction provides a historical and organizational frame for both modern set theory and this Handbook, the chapter summaries at the end being a final elaboration. To the purpose of drawing in the serious, mathematically experienced reader and providing context for the prospective researcher, we initially recapitulate the consequential historical developments leading to modern set theory as a field of mathematics. In the process we affirm basic concepts and terminology, chart out the motivating issues and driving initiatives, and describe the salient features of the field's internal practices. As the narrative proceeds, there will be a natural inversion: Less and less will be said about more and more as one progresses from basic concepts to elaborate structures, from seminal proofs to complex argumentation, from individual moves to collective enterprise. We try to put matters in a succinct yet illuminating manner, but be that as it may, according to one's experience or interest one can skim the all too familiar or too obscure. To the historian this account would not properly be history—it is, rather, a deliberate arrangement, in significant part to lay the ground for the coming chapters. To the seasoned set theorist there may be issues of under-emphasis or over-emphasis, of omissions or commissions. In any case, we take refuge in a wise aphorism: If it's worth doing, it's worth doing badly.

1. Beginnings

1.1. Cantor

Set theory was born on that day in December 1873 when Georg Cantor (1845–1918) established that *the continuum is not countable*—there is no one-to-one correspondence between the real numbers and the natural numbers $0, 1, 2, \ldots$. Given a (countable) sequence of reals, Cantor defined nested intervals so that any real in their intersection will not be in the sequence. In the course of his earlier investigations of trigonometric series Cantor had developed a definition of the reals and had begun to entertain infinite totalities of reals for their own sake. Now with his uncountability result Cantor embarked on a full-fledged investigation that would initiate an expansion of the very concept of number. Articulating cardinality as based on bijection (one-to-one correspondence) Cantor soon established positive results about the existence of bijections between sets of reals, subsets of the plane, and the like. By 1878 his investigations had led him to assert that there are only two infinite cardinalities embedded in the continuum: *Every infinite set of reals is either countable or in bijective correspondence with all the reals.* This was the Continuum Hypothesis (CH) in its nascent context, and the *continuum problem*, to resolve this hypothesis, would become a major motivation for Cantor's large-scale investigations of infinite numbers and sets.

In his magisterial *Grundlagen* of 1883 Cantor developed the *transfinite numbers* and the key concept of *well-ordering*, in large part to take a new, structured approach to infinite cardinality. The transfinite numbers follow the natural numbers $0, 1, 2, \ldots$ and have come to be depicted in his later notation in terms of natural extensions of arithmetical operations:

$$\begin{aligned} &\omega, \omega+1, \omega+2, \ldots \omega+\omega (= \omega{\cdot}2), \\ &\quad \ldots \omega{\cdot}3, \ldots \omega{\cdot}\omega (= \omega^2), \ldots \omega^3, \ldots \omega^\omega, \ldots \omega^{\omega^\omega}, \ldots . \end{aligned}$$

A *well-ordering* on a set is a linear ordering of it according to which every non-empty subset has a least element. Well-orderings were to carry the sense of sequential counting, and the transfinite numbers served as standards for gauging well-orderings. Cantor developed cardinality by grouping his transfinite numbers into successive number classes, two numbers being in the same class if there is a bijection between them. Cantor then propounded a basic principle: "It is always possible to bring any *well-defined* set into the form of a *well-ordered* set." Sets are to be well-ordered, and they and their cardinalities are to be gauged via the transfinite numbers of his structured conception of the infinite.

The transfinite numbers provided the framework for Cantor's two approaches to the continuum problem, one through cardinality and the other through definable sets of reals, these each to initiate vast research programs. As for the first, Cantor in the *Grundlagen* established results that reduced the continuum problem to showing that the continuum and the countable transfinite numbers have a bijection between them. However, despite several announcements Cantor could never develop a workable correlation, an emerging problem being that he could not *define* a well-ordering of the reals.

As for the approach through definable sets of reals, Cantor formulated the key concept of a *perfect* set of reals (non-empty, closed, and containing no isolated points), observed that perfect sets of reals *are* in bijective correspondence with the continuum, and showed that every closed set of reals is either countable or else have a perfect subset. Thus, Cantor showed that "CH holds for closed sets". The *perfect set property*, being either countable or else having a perfect subset, would become a focal property as more and more definable sets of reals came under purview.

Almost two decades after his initial 1873 result, Cantor in 1891 subsumed it through his celebrated *diagonal* argument. In logical terms this argument turns on the use of the validity $\neg\exists y\forall x(Pxx \longleftrightarrow \neg Pyx)$ for binary predicates P parametrizing unary predicates and became, of course, fundamental to the development of mathematical logic. Cantor stated his new, general result in terms of functions, ushering in totalities of arbitrary functions into mathematics, but his result is cast today in terms of the power set $P(x) = \{y \mid y \subseteq x\}$ of a set x: *For any set x, $P(x)$ has a larger cardinality than x.* Cantor had been extending his notion of set to a level of abstraction beyond sets of reals and the like; this new result showed for the first time that there is a set of a larger cardinality than that of the continuum.

Cantor's *Beiträge* of 1895 and 1897 presented his mature theory of the transfinite, incorporating his concepts of *ordinal number* and *cardinal number*. The former are the transfinite numbers now reconstrued as the "order-types" of well-orderings. As for the latter, Cantor defined the addition, multiplication, and exponentiation of cardinal numbers primordially in terms of set-theoretic operations and functions. Salient was the incorporation of "all" possibilities in the definition of exponentiation: If $\mathfrak{a}$ is the cardinal number of A and $\mathfrak{b}$ is the cardinal number of B then $\mathfrak{a}^{\mathfrak{b}}$ is the cardinal number of the totality, nowadays denoted ${}^{B}A$, of all functions from B into A. As befits the introduction of new numbers Cantor introduced a new notation, one using the Hebrew letter aleph, $\aleph$. $\aleph_0$ is to be the cardinal number of the natural numbers and the successive alephs

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$$

indexed by the ordinal numbers are now to be the cardinal numbers of the successive number classes from the *Grundlagen* and thus to exhaust all the infinite cardinal numbers. Cantor pointed out that the exponentiated $2^{\aleph_0}$ is the cardinal number of the continuum, so that CH could now have been stated as

$$2^{\aleph_0} = \aleph_1.$$

However, with CH unresolved Cantor did not even mention the hypothesis in the *Grundlagen*, only in correspondence. Every well-ordered set has an aleph as its cardinal number, but where is $2^{\aleph_0}$ in the aleph sequence?

Cantor's great achievement, accomplished through almost three decades of prodigious effort, was to have brought into being the new subject of set theory as bolstered by the mathematical objectification of the actual infinite and moreover to have articulated a fundamental problem, the continuum problem. Hilbert made this the very first of his famous problems for the 20th Century, and he drew out Cantor's difficulty by suggesting the desirability of "actually giving" a well-ordering of the real numbers.

1.2. Zermelo

Ernst Zermelo (1871–1953), already estimable as an applied mathematician, turned to set theory at Göttingen under the influence of Hilbert. Zermelo analyzed Cantor's well-ordering principle by reducing it to the Axiom of Choice (AC), the abstract existence assertion that every set x has a *choice function*, i.e. a function f with domain x such that for every non-empty $y \in x$,

$f(y) \in y$. Zermelo's 1904 proof of the Well-Ordering Theorem, that with AC *every set can be well-ordered*, would anticipate the argument two decades later for transfinite recursion:

With x a set to be well-ordered, let f be a choice function on the power set $P(x)$. Call $y \subseteq x$ an *f-set* if there is a well-ordering R of y such that for any $a \in y$, $a = f(\{b \in x \mid b \text{ does not } R\text{-precede } a\})$. The well-orderings of f-sets are thus determined by f, and f-sets cohere. It follows that the union of f-sets is again an f-set and must in fact be x itself.

Zermelo's argument provoked open controversy because of its appeal to AC, and the subsequent tilting toward the acceptance of AC amounted to a conceptual shift in mathematics toward arbitrary functions and abstract existence principles. Responding to his critics Zermelo in 1908 published a second proof of the Well-Ordering Theorem and then the first full-fledged axiomatization of set theory, one similar in approach to Hilbert's axiomatization of geometry and incorporating set-theoretic ideas of Richard Dedekind. This axiomatization duly avoided the emerging "paradoxes" like Russell's Paradox, which Zermelo had come to independently, and served to buttress the Well-Ordering Theorem by making explicit its underlying set-existence assumptions. Zermelo's axioms, now formalized, constitute the familiar theory Z, *Zermelo set theory*:

Extensionality (sets are equal if they contain the same members), Empty Set (there is a set having no members), Pairs (for any sets x and y there is a set $\{x, y\}$ consisting exactly of x and y), Union (for any set x there is a set $\bigcup x$ consisting exactly of those sets that are members of some member of x), Power Set (for any set x there is a set $P(x)$ consisting exactly of the subsets of x), Choice (for any set x consisting of non-empty, pairwise disjoint sets, there is a set c such that every member of x has exactly one member in c), Infinity (there is a certain, specified infinite set); and Separation (for any set x and "definite" property P, there is a set consisting exactly of those members of x having the property P).

Extensionality, Empty Set, and Pairs lay the basis for sets. Infinity and Power Set ensure sufficiently rich settings for set-theoretic constructions. Power Set legitimizes "all" for subsets of a given set, and Separation legitimizes "all" for elements of a given set satisfying a property. Finally, Union and Choice (formulated reductively in terms of the existence of a "transversal" set meeting each of a family of sets in one member) complete the encasing of the Well-Ordering Theorem.

Zermelo's axiomatization sought to clarify vague subject matter, and like strangers in a strange land, stalwarts developed a familiarity with sets guided hand-in-hand by the axiomatic framework. Zermelo's own papers, with work of Dedekind as an antecedent, pioneered the reduction of mathematical concepts and arguments to set-theoretic concepts and arguments from axioms. Zermelo's analysis moreover served to draw out what would come to be generally regarded as set-theoretic and combinatorial out of the presumptively logical, with Infinity and Power Set salient and the process being strategically advanced by the segregation of the notion of property to Separation.

Taken together, Zermelo's work in the first decade of the 20th Century initiated a major transmutation of the notion of set after Cantor. With AC Zermelo shifted the notion away from Cantor's inherently well-ordered sets, and with his axiomatization Zermelo ushered in a new abstract, prescriptive view of sets as structured solely by membership and governed and generated by axioms. Through his set-theoretic reductionism Zermelo made evident how his set theory is adequate as a basis for mathematics.

1.3. First Developments

During this period Cantor's two main legacies, the extension of number into the transfinite and the investigation of definable sets of reals, became fully incorporated into mathematics in direct initiatives. The axiomatic tradition would be complemented by another, one that would draw its life more directly from the mathematics.

The French analysts Emile Borel, René Baire, and Henri Lebesgue took on the investigation of definable sets of reals in what would be a typically "constructive" approach. Cantor had established the perfect set property for closed sets and formulated the concept of *content* for a set of reals, but he did not pursue these matters. With these as antecedents the French work would lay the basis for measure theory as well as *descriptive set theory*, the definability theory of the continuum.

Borel, already in 1898, developed a theory of *measure* for sets of reals; the formulation was axiomatic, and at this early stage, bold and imaginative. The sets measurable according to his measure are the now well-known *Borel sets.* Starting with the open intervals (a, b) of reals assigned measure $b-a$, the Borel sets result when closing off under complements and countable unions, measures assigned in a corresponding manner.

Baire in his 1899 thesis classified those real functions obtainable by starting with the continuous functions and closing off under pointwise limits—the *Baire functions*—into classes indexed by the countable ordinal numbers, providing the first transfinite hierarchy after Cantor. Baire's thesis also introduced the now basic concept of *category.* A set of reals is *nowhere dense iff* its closure under limits includes no open set, and a set of reals is *meager* (or *of first category*) *iff* it is a countable union of nowhere dense sets—otherwise, it is *of second category.* Generalizing Cantor's 1873 argument, Baire established the Baire Category Theorem: *Every non-empty open set of reals is of second category.* His work also suggested a basic property: A set of reals A has the *Baire property iff* there is an open set O such that the symmetric difference $(A-O)\cup(O-A)$ is meager. Straightforward arguments show that every Borel set has the Baire property.

Lebesgue's 1902 thesis is fundamental for modern integration theory as the source of his concept of measurability. Lebesgue's concept of measurable set subsumed the Borel sets, and his analytic definition of measurable function subsumed the Baire functions. In simple terms, any *arbitrary* subset of a Borel measure zero set is a Lebesgue measure zero, or *null*, set, and a set is *Lebesgue measurable* if it is the union of a Borel set and a null set, in which case the measure assigned is that of the Borel set. It is this "completion" of Borel measure through the introduction of arbitrary subsets which gives Lebesgue measure its complexity and applicability and draws in wider issues of constructivity. Lebesgue's subsequent 1905 paper was the seminal paper of descriptive set theory: He correlated the Borel sets with the Baire functions, thereby providing a transfinite hierarchy for the Borel sets, and then applied Cantor's diagonalization argument to show both that this hierarchy is proper (new sets appear at each level) and that there is a Lebesgue measurable set which is not Borel.

As descriptive set theory was to develop, a major concern became the extent of the *regularity properties*, those indicative of well-behaved sets of reals, of which prominent examples were Lebesgue measurability, having the Baire property, and having the perfect set property. Significantly, the context was delimited by early explicit uses of AC in the role of providing a well-ordering of the reals: In 1905 Giuseppe Vitali established that there is a non-Lebesgue measurable set, and in 1908 Felix Bernstein established that there is a set without the perfect set property. Thus, Cantor's early contention that the reals are well-orderable precluded the universality of his own perfect set property, and it would be that his new, enumerative approach to the continuum would steadily provide focal examples and counterexamples.

The other, more primal Cantorian legacy, the extension of number into the transfinite, was considerably advanced by Felix Hausdorff, whose work was first to suggest the rich possibilities for a mathematical investigation of the uncountable. A mathematician *par excellence*, he took that sort of mathematical approach to set theory and extensional, set-theoretic approach to mathematics that would come to dominate in the years to come. In a 1908 paper, Hausdorff provided an elegant analysis of scattered linear orders (those having no dense sub-ordering) in a transfinite hierarchy. He first stated the Generalized Continuum Hypothesis (GCH)

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{for every } \alpha.$$

He emphasized cofinality (the *cofinality* $\mathrm{cf}(\kappa)$ of a cardinal number κ is the least cardinal number λ such that a set of cardinality κ is a union of λ sets each of cardinality less than κ) and the distinction between *singular* ($\mathrm{cf}(\kappa) < \kappa$) and *regular* ($\mathrm{cf}(\kappa) = \kappa$) cardinals. And for the first time he broached a "large cardinal" concept, a regular limit cardinal $> \aleph_0$. Hausdorff's work around this time on sets of real functions ordered under eventual domination and having no uncountable "gaps" led to the first plausible mathematical proposition that entailed the denial of CH.

Hausdorff's 1914 text, *Grundzüge der Mengenlehre*, broke the ground for a generation of mathematicians in both set theory and topology. Early on, he defined an ordered pair of sets in terms of (unordered) pairs, formulated functions in terms of ordered pairs, and ordering relations as collections of ordered pairs. He in effect capped efforts of logicians by making these moves in mathematics, completing the set-theoretic reduction of relations and functions. He then presented Cantor's and Zermelo's work systematically, and of particular interest, he used a well-ordering of the reals to provide what is now known as Hausdorff's Paradox. The source of the later and better known Banach-Tarski Paradox, Hausdorff's Paradox provided an implausible decomposition of the sphere and was the first, and a dramatic, synthesis of classical mathematics and the new Zermelian abstract view.

A decade after Lebesgue's seminal 1905 paper, descriptive set theory came into being as a distinct discipline through the efforts of the Russian mathematician Nikolai Luzin. He had become acquainted with the work of the French analysts while in Paris as a student, and in Moscow he began a formative seminar, a major topic of which was the "descriptive theory of functions". The young Pole Wacław Sierpiński was an early participant while he was interned in Moscow in 1915, and undoubtedly this not only kindled the decade-long collaboration between Luzin and Sierpiński but also encouraged the latter's involvement in the development of a Polish school of mathematics and its interest in descriptive set theory. In an early success, Luzin's student Pavel Aleksandrov (and independently, Hausdorff) established the groundbreaking result that *the Borel sets have the perfect set property*, so that "CH holds for the Borel sets".

In the work that really began descriptive set theory, another student of Luzin's, Mikhail Suslin, investigated the *analytic sets* after finding a mistake in Lebesgue's paper. In a brief 1917 note Suslin formulated these sets in terms of an explicit operation $\mathcal{A}$ drawn from Aleksandrov's work and announced two fundamental results: *a set B of reals is Borel iff both B and its complement $\mathbb{R}-B$ are analytic;* and *there is an analytic set which is not Borel.* This was to be his sole publication, for he succumbed to typhus in a Moscow epidemic in 1919 at the age of 25. In an accompanying note Luzin announced that *every analytic set is Lebesgue measurable and has the perfect set property*, the latter result attributed to Suslin. Luzin and Sierpiński in joint papers soon provided proofs, in work that shifted the emphasis to the *co-analytic sets*, complements of analytic sets, and provided for them a basic *tree representation* based on well-foundedness (having no infinite branches) from which the main results of the period flowed.

After this first wave in descriptive set theory had crested, Luzin and Sierpiński in 1925 extended the domain of study to the *projective sets*. For $Y \subseteq \mathbb{R}^{k+1}$, the *projection of* Y is $pY = \{\langle x_1, \ldots, x_k\rangle \mid \exists y(\langle x_1, \ldots, x_k, y\rangle \in Y)\}$. Suslin had essentially noted that *a set of reals is analytic iff it is the projection of a Borel subset of* $\mathbb{R}^2$. Luzin and Sierpiński took the geometric operation of projection to be basic and defined the projective sets as those sets obtainable from the Borel sets by the iterated applications of projection and complementation. The corresponding hierarchy of projective subsets of

$\mathbb{R}^k$ is defined, in modern notation, as follows: For $A \subseteq \mathbb{R}^k$,

$$A \text{ is } \mathbf{\Sigma}^1_1 \quad \textit{iff} \quad A = pY \text{ for some Borel set } Y \subseteq \mathbb{R}^{k+1},$$

A is *analytic* as for $k = 1$, and for $n > 0$,

$$\begin{aligned} A \text{ is } \mathbf{\Pi}^1_n \quad & \textit{iff} \quad \mathbb{R}^k - A \text{ is } \mathbf{\Sigma}^1_n, \\ A \text{ is } \mathbf{\Sigma}^1_{n+1} \quad & \textit{iff} \quad A = pY \text{ for some } \mathbf{\Pi}^1_n \text{ set } Y \subseteq \mathbb{R}^{k+1}, \text{ and} \\ A \text{ is } \mathbf{\Delta}^1_n \quad & \textit{iff} \quad A \text{ is both } \mathbf{\Sigma}^1_n \text{ and } \mathbf{\Pi}^1_n. \end{aligned}$$

($\mathbf{\Sigma}^1_n$ is also written $\underset{\sim}{\Sigma}{}^1_n$; $\mathbf{\Pi}^1_n$ is also written $\underset{\sim}{\Pi}{}^1_n$; and $\mathbf{\Delta}^1_n$ is also written $\underset{\sim}{\Delta}{}^1_n$. One can formulate these concepts with continuous images instead of projections, e.g. A is $\mathbf{\Sigma}^1_{n+1}$ iff A is the continuous image of some $\mathbf{\Pi}^1_n$ set $Y \subseteq \mathbb{R}$. If the basics of continuous functions are in hand, this obviates the need to have different spaces.)

Luzin and Sierpiński recast Lebesgue's use of the Cantor diagonal argument to show that the projective hierarchy is proper, and soon its basic properties were established. However, this investigation encountered obstacles from the beginning. Whether the $\mathbf{\Pi}^1_1$ subsets of $\mathbb{R}$, the co-analytic sets at the bottom of the hierarchy, have the perfect set property and whether the $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable remained unknown. Besides the regularity properties, the properties of *separation, reduction,* and especially *uniformization* relating sets to others were studied, but there were accomplishments only at the first projective level. The one eventual success and a culminating result of the early period was the Japanese mathematician Motokiti Kondô's 1937 result, the $\mathbf{\Pi}^1_1$ Uniformization Theorem: *Every* $\mathbf{\Pi}^1_1$ *relation can be uniformized by a* $\mathbf{\Pi}^1_1$ *function.* This impasse with respect to the regularity properties would be clarified, surprisingly, by penetrating work of Gödel involving metamathematical methods.

In modern set theory, what has come to be taken for the "reals" is actually *Baire space*, the set of functions from the natural numbers into the natural numbers (with the product topology). Baire space, the "fundamental domain" of a 1930 Luzin monograph, is homeomorphic to the irrational reals and so equivalent for all purposes having to do measure, category, and perfect sets. Already by then it had become evident that a set-theoretic study of the continuum is best cast in terms of Baire space, with geometric intuitions being augmented by combinatorial ones.

During this period AC and CH were explored by the new Polish school, most notably by Sierpiński, Alfred Tarski, and Kazimierz Kuratowski, no longer as underlying axiom and primordial hypothesis but as part of ongoing mathematics. Sierpiński's own earliest publications, culminating in a 1918 survey, not only dealt with specific constructions but also showed how deeply embedded AC was in the informal development of cardinality, measure, and the Borel hierarchy. Even more than AC, Sierpiński investigated CH, and summed up his researches in a 1934 monograph. It became evident how having not only a well-ordering of the reals but one as given by CH whose initial segments are countable led to striking, often initially counter-intuitive, examples in analysis and topology.

1.4. Replacement and Foundation

In the 1920s, fresh initiatives in axiomatics structured the loose Zermelian framework with new features and corresponding axioms, the most consequential moves made by John von Neumann (1903–1957) in his doctoral work, with anticipations by Dmitry Mirimanoff in an informal setting. Von Neumann effected a Counter-Reformation of sorts that led to the incorporation of a new axiom, the Axiom of Replacement: *For any set x and property $P(v, w)$ functional on x (i.e. for any $a \in x$ there is exactly one b such that $P(a, b)$), $\{b \mid P(a, b)$ for some $a \in x\}$ is a set.* The transfinite numbers had been central for Cantor but peripheral to Zermelo; von Neumann reconstrued them as *bona fide* sets, the ordinals, and established their efficacy by formalizing transfinite recursion, the method for defining sets in terms of previously defined sets applied with transfinite indexing.

Ordinals manifest the basic idea of taking precedence in a well-ordering simply to be membership. A set x is *transitive iff* $\bigcup x \subseteq x$, so that x is "closed" under membership, and x is an *ordinal iff* x is transitive and well-ordered by $\in$. Von Neumann, as had Mirimanoff before him, established the key instrumental property of Cantor's ordinal numbers for ordinals: *Every well-ordered set is order-isomorphic to exactly one ordinal with membership.* Von Neumann took the further step of ascribing to the ordinals the role of Cantor's ordinal numbers. To establish the basic ordinal arithmetic results that affirm this role, von Neumann saw the need to establish the Transfinite Recursion Theorem, the theorem that validates definitions by transfinite recursion. The proof was anticipated by the Zermelo 1904 proof, but Replacement was necessary even for the very formulation, let alone the proof, of the theorem. Abraham Fraenkel and Thoralf Skolem had independently proposed Replacement to ensure that a specific collection resulting from a simple recursion be a set, but it was von Neumann's formal incorporation of transfinite recursion as method which brought Replacement into set theory. With the ordinals in place von Neumann completed the restoration of the Cantorian transfinite by defining the *cardinals* as the *initial ordinals*, i.e. those ordinals not in bijective correspondence with any of its predecessors. The infinite initial ordinals are now denoted

$$\omega = \omega_0, \omega_1, \omega_2, \ldots, \omega_\alpha, \ldots,$$

so that ω is to be the set of natural numbers in the ordinal construal. It would henceforth be that we take

$$\omega_\alpha = \aleph_\alpha$$

conflating extension with intension, with the left being a von Neumann ordinal and the right being the Cantorian cardinal concept. Every infinite set x, with AC, is well-orderable and hence in bijective correspondence with a unique initial ordinal ω_α, and the cardinality of x is $|x| = \aleph_\alpha$. It has become customary to use the lower case Greek letters to denote ordinals; $\alpha < \beta$ to denote $\alpha \in \beta$ construed as ordering; On to denote the ordinals; and the middle letters $\kappa, \lambda, \mu, \ldots$ to denote the initial ordinals in their role as the infinite cardinals, with κ^+ denoting the cardinal successor of κ.

Von Neumann provided a new axiomatization of set theory, one that first incorporated what we now call proper classes. A *class* is the totality of all sets that satisfy a specified property, so that membership in the class amounts to satisfying the property, and von Neumann axiomatized the ways to have these properties. Only sets can be members, and so the recourse to possibly proper classes, classes not represented by sets, avoids the contradictions arising from formalizing the known paradoxes. Actually, von Neumann took functions to be primitive in an involved framework, and Paul Bernays in 1930 re-constituted the von Neumann axiomatization with sets and classes as primitive. Classes would not remain a formalized component of modern set theory, but the informal use of classes as objectifications of properties would become increasingly liberal, particularly to convey large-scale issues in set theory.

Von Neumann (and before him Mirimanoff, Fraenkel, and Skolem) also considered the salutary effects of restricting the universe of sets to the *well-founded sets*. The well-founded sets are the sets in the class $\bigcup_\alpha V_\alpha$, where the "ranks" V_α are defined by transfinite recursion:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = P(V_\alpha); \quad \text{and} \quad V_\delta = \textstyle\bigcup_{\alpha<\delta} V_\alpha \quad \text{for limit ordinals } \delta.$$

Von Neumann entertained the Axiom of Foundation: *Every nonempty set x has an $\in$-minimal element, i.e. a $y \in x$ such that $x \cap y$ is empty.* (With AC this is equivalent to having no infinite $\in$-descending sequences.) This axiom amounts to the assertion that the *cumulative hierarchy* exhausts the universe V of sets:

$$V = \textstyle\bigcup_\alpha V_\alpha.$$

In modern terms, the ascribed well-foundedness of $\in$ leads to a ranking function $\rho : V \to \mathrm{On}$ defined recursively by $\rho(x) = \bigcup\{\rho(y) + 1 \mid y \in x\}$, so that $V_\alpha = \{x \mid \rho(x) < \alpha\}$, and one can establish results for all sets by induction on rank.

Zermelo in a 1930 paper offered his final axiomatization of set theory as well as a striking, synthetic view of a procession of models that would have a modern resonance. Proceeding in what we would now call a second-order context, Zermelo amended his 1908 axiomatization Z by adjoining both Replacement and Foundation while leaving out Infinity and AC, the latter being regarded as part of the underlying logic. The now standard axiomatization of set theory

ZFC, *Zermelo-Fraenkel with Choice*,

is recognizable if we inject Infinity and AC, the main difference being that ZFC is a first-order theory (as discussed below). "Fraenkel" acknowledges the early suggestion by Fraenkel to adjoin Replacement; and the Axiom of Choice is explicitly mentioned.

ZF, *Zermelo-Fraenkel*,

is ZFC without AC and is a base theory for the investigation of weak Choice-type propositions as well as propositions that contradict AC.

Zermelo herewith completed his transmutation of the notion of set, his abstract view stabilized by further axioms that structured the universe of sets. Replacement and Foundation focused the notion of set, with the first providing the means for transfinite recursion and induction and the second making possible the application of those means to get results about *all* sets, they appearing in the cumulative hierarchy. Foundation is the one axiom unnecessary for the recasting of mathematics in set-theoretic terms, but the axiom is also the salient feature that distinguishes investigations specific to set theory as a field of mathematics. With Replacement and Foundation in place Zermelo was able to provide natural models of his axioms, each a V_κ where κ is an *inaccessible* cardinal (regular and *strong limit*: if $\lambda < \kappa$, then $2^\lambda < \kappa$), and to establish algebraic isomorphism, initial segment, and embedding results for his models. Finally, Zermelo posited an endless procession of such models, each a set in the next, as natural extensions of their cumulative hierarchies.

Inaccessible cardinals are at the modest beginnings of the theory of *large cardinals*, now a mainstream of modern set theory devoted to the investigation of strong hypotheses and consistency strength. The journal volume containing Zermelo's paper also contained Stanisław Ulam's seminal paper on measurable cardinals, which would become focal among large cardinals. In modern terminology, a *filter over* a set Z is a family of subsets of Z closed under the taking of supersets and of intersections. (Usually excluded from consideration as trivial are $\{X \subseteq Z \mid A \subseteq X\}$ for some set $A \subseteq Z$, the *principal* filters.) An *ultrafilter U over Z* is a maximal filter over Z, i.e. for any $X \subseteq Z$, either $X \in U$ or else $Z - X \in U$. For a cardinal λ, a filter is λ*-complete* if it is closed under the taking of intersections of fewer than λ members. Finally, an uncountable cardinal κ is *measurable iff* there is a κ-complete ultrafilter over κ. In a previous, 1929 note Ulam had constructed, using a well-ordering of the reals, an ultrafilter over ω. Measurability thus generalizes a property of ω, and Ulam showed moreover that *measurable cardinals are inaccessible.* In this work, Ulam was motivated by measure-theoretic considerations, and he viewed his work as about $\{0, 1\}$-valued measures, the measure 1 sets being the sets in the ultrafilter. To this day, ultrafilters of all sorts in large cardinal theory are also called measures.

A decade later Tarski provided a systematic development of these concepts in terms of ideals. An *ideal over* a set Z is a family of subsets of Z closed under the taking of subsets and of unions. This is the "dual" notion to filters; if I is an ideal (resp. filter) over Z, then $\breve{I} = \{Z - X \mid X \in I\}$ is its dual filter (resp. ideal). An ideal is λ-*complete* if its dual filter is. A more familiar conceptualization in mathematics, Tarski investigated a general notion of ideal on a Boolean algebra in place of the power set algebra $P(Z)$. Although filters and ideals in large cardinal theory are most often said to be *on* a cardinal κ, they are more properly *on* the Boolean algebra $P(\kappa)$. Moreover, the measure-theoretic terminology has persisted: For an ideal $I \subseteq P(Z)$, the I-*measure zero* (negligible) sets are the members of I, the I-*positive measure* (non-negligible) sets are the members of $P(Z) - I$, and the I-*measure one* (all but negligible) sets are the members of the dual filter $\{Z - X \mid X \in I\}$.

Returning to the axiomatic tradition, Zermelo's 1930 paper was in part a response to Skolem's advocacy of the idea of framing Zermelo's 1908 axioms in *first-order logic*, the logic of formal languages based on the quantifiers $\forall$ and $\exists$ interpreted as ranging over the *elements* of a domain of discourse. First-order logic had emerged in 1917 lectures of Hilbert as a delimited system of logic amenable to mathematical investigation. Entering from a different, algebraic tradition, Skolem in 1920 had established a seminal result for semantic methods with the Löwenheim-Skolem Theorem, that a countable collection of first-order sentences, if satisfiable, is satisfiable in a countable domain. For this he introduced what we now call Skolem functions, functions added formally for witnessing $\exists x$ assertions. For set theory Skolem in 1923 proposed formalizing Zermelo's axioms in the first-order language with $\in$ and $=$ as binary predicate symbols. Zermelo's "definite" properties were to be those expressible in this first-order language in terms of given sets, and the Axiom of Separation was to become a *schema* of axioms, one for each first-order formula. As an argument against taking set theory as a foundation for mathematics, Skolem pointed out what has come to be called *Skolem's Paradox*: Zermelo's 1908 axioms cast in first-order logic is a countable collection of sentences, and so if they are satisfiable at all, they are satisfiable in a countable domain. Thus, we have the paradoxical existence of countable models for Zermelo's axioms although they entail the existence of uncountable sets. Zermelo found this antithetical and repugnant. However, strong currents were at work leading to a further, subtler transmutation of the notion of set as based on first-order logic and incorporating its relativism of set-theoretic concepts.

2. New Groundwork

2.1. Gödel

Kurt Gödel (1906–1978) substantially advanced the mathematization of logic by submerging metamathematical methods into mathematics. The main vehicle was the direct coding, "the arithmetization of syntax", in his celebrated 1931 Incompleteness Theorem, which worked dialectically against a program of Hilbert's for establishing the consistency of classical mathematics. But starting an undercurrent, the earlier 1930 Completeness Theorem for first-order logic clarified the distinction between the formal syntax and semantics of first-order logic and secured its key instrumental property with the Compactness Theorem.

Tarski in the early 1930s provided his systematic "definition of truth", exercising philosophers to a surprising extent ever since. Tarski simply schematized truth as a correspondence between formulas of a formal language and set-theoretic assertions about an intended structure interpreting the language and provided a recursive definition of the *satisfaction* relation, when a formula holds in the structure, in set-theoretic terms. The eventual effect of Tarski's mathematical formulation of semantics would be not only to make mathematics out of the informal notion of satisfiability, but also to enrich ongoing mathematics with a systematic method for forming mathematical analogues of several intuitive semantic notions. Tarski would only be explicit much later about satisfaction-in-a-structure for arbitrary structures, this leading to his notion of logical consequence. For coming purposes, the following affirms notation and concepts in connection with Tarski's definition.

For a first-order language, a structure N interpreting that language (i.e. a specification of a domain of discourse as well as interpretations of the function and predicate symbols), a formula $\varphi(v_1, v_2, \ldots, v_n)$ of the language with the (free) variables as displayed, and $a_1, a_2, \ldots, a_n$ in the domain of N,

$$N \models \varphi[a_1, a_2, \ldots, a_n]$$

asserts that the formula φ is satisfied in N according to Tarski's recursive definition when v_i is interpreted as a_i. A subset y of the domain of N is *first-order definable over N iff* there is a $\psi(v_1, v_2, \ldots, v_{n+1})$ and $a_1, a_2, \ldots, a_n$ in the domain of N such that

$$y = \{z \in N \mid N \models \psi[a_1, a_2, \ldots, a_n, z]\}.$$

(The first-order definability of k-ary relations is analogously formulated with v_{n+1} replaced by k variables.)

Through Tarski's recursive definition and an "arithmetization of syntax" whereby formulas are systematically coded by natural numbers, the satisfaction relation $N \models \varphi[a_1, a_2, \ldots, a_n]$ for *sets* N is definable in set theory. On the other hand, by Tarski's result on the "undefinability of truth", the satisfaction relation for V itself is not first-order definable over V.

Set theory was launched as a distinctive field of mathematics by Gödel's construction of the class L leading to the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis. In a brief 1939 account Gödel informally presented L essentially as is done today: For any set x let $\mathrm{def}(x)$ denote the collection of subsets of x first-order definable over the structure $\langle x, \in \rangle$ with domain x and the membership relation restricted to it. Then define:

$$L_0 = \emptyset; \qquad L_{\alpha+1} = \mathrm{def}(L_\alpha), \qquad L_\delta = \bigcup\{L_\alpha \mid \alpha < \delta\} \quad \text{for limit ordinals } \delta;$$

and the *constructible universe*

$$L = \bigcup_\alpha L_\alpha.$$

Gödel pointed out that L "can be defined and its theory developed in the formal systems of set theory themselves". This is actually the central feature of the construction of L. L is definable in ZF via transfinite recursion based on the formalizability of def(x), which was reaffirmed by Tarski's definition of satisfaction. With this, one can formalize the Axiom of Constructibility $V = L$, i.e. $\forall x(x \in L)$. To set a larger context, we affirm the following for a class X: for a set-theoretic formula φ, φ^X denotes φ with its quantifiers restricted to X and this extends to set-theoretic terms t (like $\bigcup x$, $P(x)$, and so forth) through their definitions to yield t^X. X is an *inner model iff* X is a transitive class containing all the ordinals such that φ^X is a theorem of ZF for every axiom φ of ZF. What Gödel did was to show in ZF that L is an inner model which satisfies AC and GCH. He thus established a relative consistency which can be formalized as an assertion: Con(ZF) implies Con(ZFC + GCH).

In the approach via def(x) it is necessary to show that def(x) remains unaltered when applied in L with quantifiers restricted to L. Gödel himself would never establish this *absoluteness of first-order definability* explicitly. In a 1940 monograph, Gödel worked in Bernays' class-set theory and used eight binary operations producing new classes from old to generate L set by set via transfinite recursion. This veritable "Gödel numbering" with ordinals eschewed def(x) and made evident certain aspects of L. Since there is a direct, definable well-ordering of L, choice functions abound in L, and AC holds there. Of the other axioms the crux is where first-order logic impinges, in Separation and Replacement. For this, "algebraic" closure under Gödel's eight operations ensured "logical" Separation for *bounded* formulas, formulas having only quantifiers expressible in terms of $\forall v \in w$, and then the full exercise of Replacement (in V) secured all of the ZF axioms in L.

Gödel's proof that L satisfies GCH consisted of two separate parts. He established the implication $V = L \rightarrow$ GCH, and, in order to apply this implication within L, that $(V = L)^L$. This latter follows from the aforementioned absoluteness of def(x), and in his monograph Gödel gave an alternate proof based on the absoluteness of his eight binary operations.

Gödel's argument for $V = L \rightarrow$ GCH rests, as he himself wrote in his 1939 note, on "a generalization of Skolem's method for constructing enumerable models". This was the first significant use of Skolem functions since Skolem's own to establish the Löwenheim-Skolem theorem, and with it, Skolem's Paradox. Ironically, though Skolem sought through his paradox to discredit set theory based on first-order logic as a foundation for mathematics, Gödel turned paradox into method, one promoting first-order logic. Gödel specifically established his "Fundamental Theorem":

> For infinite γ, every constructible subset of L_γ
> belongs to some L_β for a β of the same cardinality as γ.

For infinite α, L_α has the same cardinality as that of α. It follows from the Fundamental Theorem that in the sense of L, the power set of L_{ω_α} is included in $L_{\omega_{\alpha+1}}$, and so GCH follows in L.

The work with L led, further, to the resolution of difficulties in descriptive set theory. Gödel announced, in modern terms: *If* $V = L$, *then* (a) *there is a* $\mathbf{\Delta}^1_2$ *set of reals that is not Lebesgue measurable, and* (b) *there is a* $\mathbf{\Pi}^1_1$ *set of reals without the perfect set property.* Thus, the early descriptive set theorists were confronting an obstacle insurmountable in ZFC! When eventually confirmed and refined, the results were seen to turn on a "good" $\mathbf{\Sigma}^1_2$ well-ordering of the reals in L defined via reals coding well-founded structures and thus connected to the well-founded tree representation of $\mathbf{\Pi}^1_1$ sets. Gödel's results (a) and (b) constitute the first real synthesis of abstract and descriptive set theory, in that the axiomatic framework is incorporated into the investigation of definable sets of reals.

Gödel brought into set theory a method of construction and of argument which affirmed several features of its axiomatic presentation. Most prominently, he showed how first-order definability can be formalized and used to achieve strikingly new mathematical results. This significantly contributed to a lasting ascendancy for first-order logic which, in addition to its sufficiency as a logical framework for mathematics, was seen to have considerable operational efficacy. Moreover, Gödel's work buttressed the incorporation of Replacement and Foundation into set theory, the first immanent in the transfinite recursion and arbitrary extent of the ordinals, and the second as underlying the basic cumulative hierarchy picture that anchors L.

In later years Gödel speculated about the possibility of deciding propositions like CH with large cardinal hypotheses based on the heuristics of *reflection*, and later, *generalization*. In a 1946 address he suggested the consideration of "stronger and stronger axioms of infinity" and reflection down from V: "Any proof of a set-theoretic theorem in the next higher system above set theory (i.e. any proof involving the concept of truth, etc.) is replaceable by a proof from such an axiom of infinity". In a 1947 expository article on the continuum problem Gödel presumed that CH would be shown independent from ZF and speculated more concretely about possibilities with large cardinals. He argued that the axioms of set theory do not "form a system closed in itself" and so the "very concept of set on which they are based suggests their extension by new axioms that assert the existence of still further iterations of the operation of 'set of'". In an unpublished footnote toward a 1966 revision of the article, Gödel acknowledged "extremely strong axioms of infinity of an entirely new kind", generalizations of properties of ω "supported by strong arguments from analogy". These heuristics would surface anew in the 1960s, when the theory of large cardinals developed a self-fueling momentum of its own, stimulated by the emergence of forcing and inner models.

2.2. Infinite Combinatorics

For decades Gödel's construction of L stood as an isolated monument in the axiomatic tradition, and his methodological advances would only become fully assimilated after the infusion of model-theoretic techniques in the 1950s. In the mean time, the direct investigation of the transfinite as extension of number was advanced, gingerly at first, by the emergence of *infinite combinatorics.*

The 1934 Sierpiński monograph on CH (discussed earlier) having considerably elaborated its consequences, a new angle in the combinatorial investigation of the continuum was soon broached. Hausdorff in 1936 reactivated his early work on gaps in the orderings of functions to show that the reals can be partitioned into $\aleph_1$ Borel sets, answering an early question of Sierpiński. Hausdorff had newly cast his work in terms of functions from ω to ω, the members of Baire space or the "reals", under the ordering of *eventual dominance*: $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Work on this structure and definable sets of reals in the 1930s, and particularly of Fritz Rothberger through the 1940s, isolated what is now called the *dominating number* $\mathfrak{d}$, the least cardinality of a subset of Baire space cofinal in $\leq^*$. $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$, but absent CH $\mathfrak{d}$ assumed an independent significance as a pivotal cardinal. Rothberger established incisive results which we now cast as about the relationships to other pivotal cardinals, results which provided new understandings about the structure of the continuum but would become vacuous with the blanket assumption of CH. The investigation of $\mathfrak{d}$ and other "cardinal characteristics (or invariants) of the continuum" would blossom with the advent of forcing.

Taking up another thread, Frank Ramsey in 1930, addressing a problem of formal logic, established a generalization of the pigeonhole principle for finite sets, and in a move transcending purpose and context he also established an infinite version implicitly applying the now familiar Kőnig's Lemma for trees. In modern terms, for ordinals α, β, and δ and $n \in \omega$ the *partition relation*

$$\beta \longrightarrow (\alpha)^n_\delta$$

asserts that for any partition $f : [\beta]^n \to \delta$ of the n-element subsets of β into δ cells, there is an $H \subseteq \beta$ of order type α *homogeneous* for the partition, i.e. all the n-element subsets of H lie in the same cell. Ramsey's theorem for finite sets is: *For any $n, k, i \in \omega$ there is an $r \in \omega$ such that $r \longrightarrow (k)^n_i$.* The "Ramsey numbers", the least possible r's for various n, k, i, are unknown except in a few basic cases. The (infinite) Ramsey's Theorem is: *$\omega \longrightarrow (\omega)^n_i$ for every $n, i \in \omega$.*

A *tree* is a partially ordered set T such that the predecessors of any element are well-ordered. The *αth level* of T consists of those elements whose predecessors have order-type α, and the *height* of T is the least α such that the αth level of T is empty. A *chain* of T is a linearly ordered subset, and an *antichain* is a subset consisting of pairwise incompatible elements. A *cofinal branch* of T is a chain with elements at every non-empty level of T.

Finally, for a cardinal κ, a *κ-tree* is a tree of height κ each of whose levels has cardinality less than κ, and κ has the *tree property iff* every κ-tree has a cofinal branch. Kőnig's Lemma, of 1927, is the assertion that *ω has the tree property.*

The first systematic study of transfinite trees was carried out in Djuro Kurepa's 1935 thesis, and several properties emerging from his investigations, particularly for ω_1-trees as the first broaching context, would later become focal in the combinatorial study of the transfinite. An *Aronszajn tree* is an ω_1-tree without a cofinal branch, i.e. a counterexample to the tree property for ω_1. Kurepa acknowledged and gave Nachman Aronszajn's proof that *there is an Aronszajn tree.* A *Suslin tree* is an ω_1-tree with no uncountable chains or antichains. Kurepa reduced a hypothesis growing out of a 1920 question of Suslin about the characterizability of the ordering of the reals to a combinatorial property of ω_1, Suslin's Hypothesis (SH): *There are no Suslin trees.* Finally, a *Kurepa tree* is an ω_1-tree with at least ω_2 cofinal branches, and *Kurepa's Hypothesis* deriving from a later 1942 paper of Kurepa's is the assertion that such trees exist. Much of this would be rediscovered, and both Suslin's Hypothesis and Kurepa's Hypothesis would be resolved decades later with the advent of forcing, several of the resolutions in terms of large cardinal hypotheses. Kurepa's work also anticipated another development from a different quarter.

Paul Erdős, although an itinerant mathematician for most of his life, was the prominent figure of a strong Hungarian tradition in combinatorics, and through some seminal results he introduced major initiatives into the detailed combinatorial study of the transfinite. Erdős and his collaborators simply viewed the transfinite numbers as a combinatorially rich source of intrinsically interesting problems, the concrete questions about graphs and mappings having a natural appeal through their immediacy. One of the earliest advances was an 1943 paper of Erdős and Tarski which concluded enticingly with an intriguing list of six combinatorial problems, the positive solution to any, as it was to turn out, amounting to the existence of a large cardinal. In a footnote various implications were noted, one of them being essentially that *for inaccessible κ, the tree property for κ implies $\kappa \longrightarrow (\kappa)^2_2$*, a generalization of Ramsey's $\omega \longrightarrow (\omega)^2_2$ drawing out the Kőnig Lemma property needed.

The detailed investigation of partition relations began in earnest in the 1950s, with a 1956 paper of Erdős and Richard Rado's being representative. For a cardinal κ, set $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. What became known as *the* Erdős-Rado Theorem asserts: *For any infinite cardinal κ and $n \in \omega$,*

$$\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa.$$

This was established using the basic tree argument underlying Ramsey's results, whereby a homogeneous set is not constructed recursively, but a tree is constructed such that its branches provide homogeneous sets, and a counting argument ensures that there must be a homogeneous set of sufficient cardinality. The Erdős-Rado Theorem is the transfinite analogue of Ramsey's theorem for finite sets, with both having the form: given α, δ and n there is a β such that $\beta \longrightarrow (\alpha)^n_\delta$. However, while what the Ramsey numbers are is largely unknown, the $\beth_n(\kappa)^+$ are known to be optimal. Kurepa in effect had actually established the case $n = 1$ and shown that $\beth_1(\kappa)^+$ is the least possible, and the $\beth_n(\kappa)^+$ was also shown to be the least possible in the general case by a "negative stepping-up" lemma.

Still among the Hungarians, Géza Fodor in 1956 established a now basic fact about the uncountable that has become woven into its sense, so operationally useful and ubiquitous it has become in infinite combinatorics. For a cardinal λ and a set $C \subseteq \lambda$, C is *closed unbounded* (or "club") in λ *iff* C contains its limit (or "accumulation") points, i.e. those $0 < \alpha < \lambda$ such that $\sup(C \cap \alpha) = \alpha$, and is cofinal, i.e. $\bigcup C = \lambda$. The use of "closed" and "unbounded" are as for $\langle \lambda, < \rangle$ with the order topology. A set $S \subseteq \lambda$ is *stationary in λ iff* for any C closed unbounded in λ, $S \cap C$ is not empty. For regular $\lambda > \omega$, the intersection of fewer than λ sets closed unbounded in λ is again closed unbounded in λ, and so the closed unbounded subsets of λ generate a λ-complete filter, the *closed unbounded filter*, denoted $\mathcal{C}_\lambda$. The nonstationary subsets of λ constitute the dual *nonstationary ideal*, denoted NS_λ. Now Fodor's (or Regressive Function or "Pressing Down") Lemma: *For regular $\lambda > \omega$, if a function f is regressive on a set $S \subseteq \lambda$ stationary in λ, i.e. $f(\alpha) < \alpha$ for every $\alpha \in S$, then there is a $T \subseteq S$ stationary in λ on which f is constant.*

Fodor's Lemma is a basic fact and its proof a simple exercise now, but then it was the culmination of a progression of results beginning with a seminal 1929 observation of Aleksandrov that a regressive function on ω_1 must be constant on an uncountable set. The subsets of a regular $\lambda > \omega$ naturally separate out into the nonstationary sets, the stationary sets, and among them the closed unbounded sets as the negligible, non-negligible, and all but negligible sets according to NS_λ. Fodor's Lemma is intrinsic to stationarity, and can be cast as a substantive characterization of the concept. It would be that far-reaching generalizations of stationarity, e.g. stationary towers, would become important in modern set theory.

2.3. Definability

Descriptive set theory was to become transmuted by the turn to definability following Gödel's work. After his fundamental work on recursive function theory in the 1930s, Stephen Kleene expanded his investigations of effectiveness and developed a general theory of definability for relations on ω. In the early 1940s Kleene investigated the *arithmetical relations* on reals, those relations obtainable from the recursive relations by applications of number quantifiers. Developing canonical representations, he classified these relations into a hierarchy according to quantifier complexity and showed that the hierarchy is proper. In the mid-1950s Kleene investigated the *analytical* *relations*, those relations obtainable from the arithmetical relations by applications of function ("real") quantifiers. Again he worked out representation and hierarchy results, and moreover he established an elegant theorem that turned out to be an effective version of Suslin's characterization of the Borel sets.

Kleene was developing what amounted to the effective content of classical descriptive set theory, unaware that his work had direct antecedents in the papers of Lebesgue, Luzin, Sierpiński, and Tarski. Kleene's student John Addison then established that there is an exact correlation between the hierarchies of classical and effective descriptive set theory (as described below). The development of effective descriptive set theory considerably clarified the classical context, injected recursion-theoretic techniques into the subject, and placed definability considerations squarely at its forefront. Not only were new approaches to classical problems provided, but results and questions could now be formulated in a refined setting.

Second-order arithmetic is the two-sorted structure

$$\mathcal{A}^2 = \langle \omega, {}^{\omega}\omega, ap, +, \times, <, 0, 1 \rangle,$$

where ω and ${}^{\omega}\omega$ (Baire space or the "reals") are two separate domains connected by the binary operation $ap : {}^{\omega}\omega \times \omega \to \omega$ of *application* given by $ap(x, m) = x(m)$, and $+, \times, <, 0, 1$ impose the usual arithmetical structure on ω. The underlying language has two sorts of variables, those ranging over ω and those ranging over ${}^{\omega}\omega$, and corresponding *number quantifiers* $\forall^0$, $\exists^0$ and *function quantifiers* $\forall^1$, $\exists^1$.

For relations $A \subseteq ({}^{\omega}\omega)^k$,

$$\begin{aligned} A \text{ is } \textit{arithmetical} \quad \textit{iff} \quad & A \text{ is definable over } \mathcal{A}^2 \text{ by a formula} \\ & \text{without function quantifiers,} \\ A \text{ is } \textit{analytical} \quad \textit{iff} \quad & A \text{ is definable over } \mathcal{A}^2. \end{aligned}$$

Through the manipulation of quantifiers the analytical sets can be classified in the *analytical hierarchy*, the levels of which are the (lightface) Σ^1_n, Π^1_n, and Δ^1_n classes defined as follows: For relations $A \subseteq ({}^{\omega}\omega)^k$ and $n > 0$,

$$\begin{aligned} A \in \Sigma^1_n \quad & \textit{iff} \quad \forall \mathbf{w}(A(\mathbf{w}) \leftrightarrow \exists^1 x_1 \forall^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, \ldots, x_n)), \quad \text{and} \\ A \in \Pi^1_n \quad & \textit{iff} \quad \forall \mathbf{w}(A(\mathbf{w}) \leftrightarrow \forall^1 x_1 \exists^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, \ldots, x_n)) \end{aligned}$$

for some arithmetical $R \subseteq ({}^{\omega}\omega)^{k+n}$, where Q is $\exists^1$ if n is odd and $\forall^1$ if n is even in the first and *vice versa* in the second. Finally,

$$A \in \Delta^1_n \quad \textit{iff} \quad A \in \Sigma^1_n \cap \Pi^1_n.$$

The correlation of the effective ("lightface") and classical ("boldface") hierarchies was established by Addison in 1958 through the simple expedient of relativization to real parameters. For $a \in {}^{\omega}\omega$, *second-order arithmetic in* a is the expanded structure

$$\mathcal{A}^2(a) = \langle \omega, {}^{\omega}\omega, ap, +, \times, <, 0, 1, a \rangle$$

where a is regarded as a binary relation on ω. Replacing $\mathcal{A}^2$ by $\mathcal{A}^2(a)$ in the preceding, we get the corresponding relativized notions: *arithmetical in* a, *analytical in* a, $\Sigma^1_n(a)$, $\Pi^1_n(a)$, and $\Delta^1_n(a)$. The correlation of the hierarchies is then as follows: *Suppose that* $A \subseteq ({}^\omega\omega)^k$ *and* $n > 0$. *Then* $A \in \mathbf{\Sigma}^1_n$ *iff* $A \in \Sigma^1_n(a)$ *for some* $a \in {}^\omega\omega$, *and similarly for* $\mathbf{\Pi}^1_n$. Loosely speaking, a projective set can be analyzed with a real parameter coding the construction of the underlying Borel set, $\exists^1$ corresponding to projection, and $\forall^1$ through $\neg\exists^1\neg$ corresponding to complementation.

Joseph Shoenfield in 1961 advanced the study of projective sets into the new definability context by providing a tree representation for $\mathbf{\Sigma}^1_2$ sets based on well-foundedness as charted out to ω_1. The classical Luzin-Sierpiński tree representation of $\mathbf{\Pi}^1_1$ sets turned, in the new terms, on the f of the function quantifier $\forall f$ imputing infinite branches through a tree arithmetical in a for some $a \in {}^\omega\omega$ that must be cut off. This well-foundedness can be cast as having an order-preserving ranking function into ω_1, which Shoenfield pointed out can be recast as having an infinite branch through a tree built on the countable ordinals.

T is a *tree on* $\omega \times \kappa$ *iff* (a) T consists of pairs $\langle s, t\rangle$ where s is a finite sequence drawn from ω and t is a finite sequence drawn from κ of the same length, and (b) if $\langle s, t\rangle \in T$, s' is an initial segment of s and t' is a initial segment of t of the same length, then also $\langle s', t'\rangle \in T$. For such T, $[T]$ consists of pairs $\langle f, g\rangle$ corresponding to infinite branches, i.e. f and g are ω-sequences such that for any finite initial segment s of f and finite initial segment t of g of the same length, $\langle s, t\rangle \in T$. In modern terms, $A \subseteq {}^\omega\omega$ is κ-*Suslin iff* there is a tree on $\omega \times \kappa$ such that $A = p[T] = \{f \mid \exists g(\langle f, g\rangle \in [T])\}$. $[T]$ is a closed set in the space of $\langle f, g\rangle$'s where $f : \omega \to \omega$ and $g : \omega \to \kappa$, and so otherwise complicated sets of reals, if shown to be κ-Suslin, are newly comprehended as projections of closed sets. The analytic ($\mathbf{\Sigma}^1_1$) sets are exactly the ω-Suslin sets.

Shoenfield established that *every* $\mathbf{\Sigma}^1_2$ *set is* ω_1-*Suslin*, and his proof, emphasizing constructibility, showed that if $A \subseteq {}^\omega\omega$ is Σ^1_2, then $A = p[T]$ for a tree T on $\omega \times \omega_1$ such that $T \in L$. Shoenfield applied well-foundedness in the $\forall$ sense (no infinite descending sequences) and the $\exists$ sense (there is a ranking function) to establish that Σ^1_2 *relations are absolute (or "correct") for* L: For any $\mathbf{w} \in L$, $\mathcal{A}^2 \models \exists^1 x \forall^1 y \varphi[x, y, \mathbf{w}]$ *iff* $(\mathcal{A}^2 \models \exists^1 x \forall^1 y \varphi[x, y, \mathbf{w}])^L$ when φ has no function quantifiers.

Many substantive propositions of classical analysis as well as of metamathematical investigation are Σ^1_2 or Π^1_2, and if they can be established from $V = L$ (or just CH), then they can be established in ZF alone. It would be that in the years to come more and more projective sets of reals would be comprehended through κ-Suslin representations for larger and larger cardinals κ.

András Hajnal and Azriel Levy, in their theses of the mid-1950s, developed generalizations of L that were to become basic in a richer setting. For a set A, Hajnal formulated the *constructible closure* $L(A)$ *of* A, i.e. the smallest inner model M such that $A \in M$, and Levy formulated the *class* $L[A]$ *of sets constructible relative to* A, i.e. the smallest inner model M such that for every $x \in M$, $A \cap x \in M$. To formulate $L(A)$, define: $L_0(A) =$ the smallest transitive set $\supseteq \{A\}$ (to ensure that the resulting class is transitive); $L_{\alpha+1}(A) = \mathrm{def}(L_\alpha(A))$; $L_\delta(A) = \bigcup_{\alpha<\delta} L_\alpha(A)$ for limit $\delta > 0$; and finally $L(A) = \bigcup_\alpha L_\alpha(A)$. To formulate $L[A]$, first let $\mathrm{def}^A(x)$ denote the collection of subsets of x first-order definable over $\langle x, \in, A \cap x\rangle$, i.e. $A \cap x$ is now allowed as a predicate in the definitions. Then define: $L_0[A] = \emptyset$; $L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A])$; $L_\delta[A] = \bigcup_{\alpha<\delta} L_\alpha[A]$ for limit $\delta > 0$; and finally $L[A] = \bigcup_\alpha L_\alpha[A]$. With the "trace" $\overline{A} = A \cap L[A]$ one has $L_\alpha[\overline{A}] = L_\alpha[A]$ for every α and so $L[\overline{A}] = L[A]$.

$L(A)$ realizes the algebraic idea of building up a model starting from a set of generators, and $L[A]$ the idea of building up a model using A construed as a predicate. $L(A)$ may not satisfy AC since it may not have a well-ordering of A, yet $L[A]$ always satisfies that axiom. This distinction was only to surface later, as both Hajnal and Levy took A to be a set of ordinals, when $L(A) = L[A]$, and used these models to establish conditional independence results of the sort: if the failure of CH is consistent, then so is that failure together with $2^\lambda = \lambda^+$ for sufficiently large cardinals λ. In the coming expansion of the 1960s, both Hajnal and Levy would be otherwise engaged, with Hajnal becoming a major combinatorial set theorist and collaborator with Erdős, and Levy, a pioneer in the investigation of independence results.

2.4. Model-Theoretic Techniques

Model theory began in earnest with the appearance in 1949 of the method of diagrams in Abraham Robinson's thesis and the related method of constants in Leon Henkin's thesis, which gave a new proof of the Gödel Completeness Theorem. Tarski had set the stage with the formulation of formal languages and semantics in set-theoretic terms, and with him established at the University of California at Berkeley, a large part of the development in the 1950s and 1960s would take place there. Tarski and his students carefully laid out satisfaction-in-a-structure; *theories* (deductively closed collections of sentences) and their *models*; algebratization with *Skolem functions* and *hulls*; and *elementary substructures and embeddings*. $j : \mathcal{A} \to \mathcal{B}$ is an *elementary embedding* if for any $a_1, \ldots, a_n$ from the domain of $\mathcal{A}$, $\langle a_1, \ldots, a_n\rangle$ satisfies in $\mathcal{A}$ the same formulas that $\langle j(a_1), \ldots, j(a_n)\rangle$ does in $\mathcal{B}$; and when j is the identity $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$, denoted $\mathcal{A} \prec \mathcal{B}$. The construction of models freely used transfinite methods and soon led to new questions in set theory, but also set theory was to be decisively advanced by the infusion of model-theoretic methods.

A precursory result was a 1949 generalization by Andrzej Mostowski of the Mirimanoff-von Neumann result that every well-ordered set is order-isomorphic to exactly one ordinal with membership. A binary relation R on a set X is *extensional* if distinct members of X have distinct R-predecessors, and *well-founded* if every non-empty $Y \subseteq X$ has an R-minimal element (or, assuming AC, there is no infinite R-descending sequence). *If R is an extensional, well-founded relation on a set X, then there is a unique transitive set T and an isomorphism of $\langle X, R\rangle$ onto $\langle T, \in\rangle$, i.e. a bijection $\pi : X \to T$ such that for any $x, y \in X$, $x\,R\,y$ iff $\pi(x) \in \pi(y)$.* $\langle T, \in\rangle$ is the *transitive collapse* of X, and π the *collapsing isomorphism.* Thus, the linearity of well-orderings has been relaxed to analogues of Extensionality and Foundation, and transitive sets become canonical representatives as ordinals are for well-orderings. Well-founded relations other than membership had surfaced much earlier, most notably in the Luzin-Sierpiński tree representation of $\mathbf{\Pi}^1_1$ sets. The general transitive collapse result would come to epitomize how well-foundedness made possible a coherent theory of models of set theory.

After Richard Montague applied reflection phenomena to establish that ZF is not finitely axiomatizable, Levy also formulated reflection principles and established their broader significance. The 1960 Montague-Levy Reflection Principle for ZF asserts: *For any (first-order) formula $\varphi(v_1, \ldots, v_n)$ and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,*

$$\varphi[x_1, \ldots, x_n] \quad \textit{iff} \quad V_\alpha \models \varphi[x_1, \ldots, x_n].$$

Levy showed that this schema is equivalent to the conjunction of the Replacement schema together with Infinity in the presence of the other axioms of ZF. Moreover, he formulated reflection principles in local form that characterized the *Mahlo* cardinals, conceptually the least large cardinals after the inaccessible cardinals. Also William Hanf and Dana Scott posited analogous reflection principles for higher-order formulas, leading to what are now called the *indescribable cardinals*. The model-theoretic reflection idea thus provided a coherent scheme for viewing the bottom of an emerging hierarchy of large cardinals as a generalization of Replacement and Infinity.

In those 1946 remarks by Gödel where he broached the heuristic of reflection, Gödel also entertained the concept of ordinal definable set. A set x is *ordinal definable iff* there are ordinals $\alpha_1, \ldots, \alpha_n$ and a formula $\varphi(v_0, \ldots, v_n)$ such that $\forall y(y \in x \leftrightarrow \varphi[y, \alpha_1, \ldots, \alpha_n])$. This ostensible dependence on the satisfaction relation for V can be formally recast through a version of the Reflection Principle for ZF, so that one can define the class OD of ordinal definable sets. With $\mathrm{tc}(y)$ denoting the smallest transitive superset of y, let $\mathrm{HOD} = \{x \mid \mathrm{tc}(\{x\}) \subseteq \mathrm{OD}\}$, the class of *hereditarily ordinal definable sets.*

As noted by Gödel, HOD is an inner model in which AC, though not necessarily CH, holds. The basic results about this inner model were to be rediscovered several times. In these several ways reflection phenomena both as heuristic and as principle became incorporated into set theory, bringing to the forefront what was to become a basic feature of the study of well-foundedness.

The set-theoretic generalization of first-order logic allowing transfinitely indexed logical operations was to clarify the size of measurable cardinals. Extending familiarity by abstracting to a new domain, Tarski in 1962 formulated the *strongly compact* and *weakly compact* cardinals by ascribing natural generalizations of the key compactness property of first-order logic to the corresponding infinitary languages. These cardinals had figured in that 1943 Erdős-Tarski paper in equivalent combinatorial formulations that were later seen to imply that *a strongly compact cardinal is measurable,* and *a measurable cardinal is weakly compact.* Tarski's student Hanf then established, using the satisfaction relation for infinitary languages, that *there are many inaccessible cardinals (and Mahlo cardinals) below a weakly compact cardinal. A fortiori, the least inaccessible cardinal is not measurable.* This breakthrough was the first result about the size of measurable cardinals since Ulam's original 1930 paper and was greeted as a spectacular success for metamathematical methods. Hanf's work radically altered size intuitions about problems coming to be understood in terms of large cardinals and ushered in model-theoretic methods into the study of large cardinals beyond the Mahlo cardinals.

Weak compactness was soon seen to have a variety of characterizations, most notably *κ is weakly compact iff $\kappa \to (\kappa)^2_2$ iff $\kappa \to (\kappa)^n_\lambda$ for every $n \in \omega$ and $\lambda < \kappa$ iff κ is inaccessible and has the tree property,* and this was an early, significant articulation of the large cardinal extension of context for effecting known proof ideas and methods.

The concurrent emergence of the *ultraproduct construction* in model theory set the stage for the development of the modern theory of large cardinals. The ultraproduct construction was brought to the forefront by Tarski and his students after Jerzy Łoś's 1955 adumbration of its fundamental theorem. The new method of constructing concrete models brought set theory and model theory even closer together in a surge of results and a lasting interest in ultrafilters.

The ultraproduct construction was driven by the algebraic idea of making a structure out of a direct product of structures as modulated (or "reduced") by a filter. The particular case when all the structures are the same, the *ultrapower*, was itself seen to be substantive. To briefly describe a focal case for set theory, let N be a set, construed as a structure with $\in$, and U an ultrafilter over a set Z. On ${}^Z N$, the set of functions from Z to N, define

$$f =_U g \quad \textit{iff} \quad \{i \in Z \mid f(i) = g(i)\} \in U.$$

The filter properties of U imply that $=_U$ is an equivalence relation on ${}^Z N$, so with $(f)_U$ denoting the corresponding equivalence class of f, set ${}^Z N/U = \{(f)_U \mid f \in {}^Z N\}$. Next, the filter properties of U show that a binary relation E_U on ${}^Z N/U$ can be unambiguously defined by

$$(f)_U E_U (g)_U \quad \textit{iff} \quad \{i \in Z \mid f(i) \in g(i)\} \in U.$$

$=_U$ is thus a *congruence* relation, one that preserves the underlying structure; this sort of preservation is crucial in ultraproduct and classical, antecedent constructions with filters. (For example, in the space L^∞ in which two bounded measurable functions are equated when they agree on a set in the filter of full measure sets, the algebraic structure of + and × have many of the properties that + and × for the real numbers have. If the filter is extended to an ultrafilter, we get an ultrapower.) The *ultrapower* of N by U is then defined to be the structure $\langle {}^Z N/U, E_U\rangle$. The crux of the construction is the fundamental *Łoś's Theorem*: *For a formula* $\varphi(v_1, \ldots, v_n)$ *and* $f_1, \ldots, f_n \in {}^Z N$*,*

$$\langle {}^Z N/U, E_U\rangle \models \varphi[(f_1)_U, \ldots, (f_n)_U] \quad \textit{iff}$$
$$\{i \in Z \mid \mathcal{N} \models \varphi[f_1(i), \ldots, f_n(i)]\} \in U.$$

Satisfaction in the ultrapower is thus reduced to satisfaction on a large set of coordinates, large in the sense of U. The proof is by induction on the complexity of φ using the filter properties of U, the ultrafilter property for the negation step, and AC for the existential quantifier step.

E_U is an extensional relation, and crucially, well-founded when U is $\aleph_1$-complete. In that case by Mostowski's theorem there is a collapsing isomorphism π of the ultrapower onto its transitive collapse $\langle M, \in\rangle$. Moreover, if for $x \in N$, c_x is the constant function: $N \to \{x\}$ and $j_U : N \to M$ is defined by $j_U(x) = \pi((c_x)_U)$, then j_U is an elementary embedding, i.e. for any formula $\varphi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in N$,

$$\langle N, \in\rangle \models \varphi[a_1, \ldots, a_n] \quad \textit{iff} \quad \langle M, \in\rangle \models \varphi[j_U(a_1), \ldots, j_U(a_n)]$$

by Łoś's Theorem. When we have well-foundedness, the ultrapower is identified with its transitive collapse and denoted $\mathrm{Ult}(N, U)$.

All of the foregoing is applicable, and will be applied, with proper classes N, as long as we replace the equivalence class $(f)_U$ by sets

$$(f)^0_U = \{g \in (f)_U \mid g \text{ has minimal rank}\}$$

("Scott's trick"), and take Łoś's Theorem as a schema for formulas.

The model theorist H. Jerome Keisler established penetrating connections between combinatorial properties of ultrafilters and of their ultraproducts, and in particular took the ultrapower of a measurable cardinal κ by a κ-complete ultrafilter over κ to provide a new proof of Hanf's result that there are many large cardinals below a measurable cardinal. With Ulam's concept shown in a new light as providing well-founded ultrapowers, Dana Scott then struck on the idea of taking the ultrapower of the entire universe V by a κ-complete ultrafilter over a measurable κ, exploiting the resulting well-foundedness to get an elementary embedding $j : V \to \mathrm{Ult}(V, U)$. Importantly, κ is the *critical point*, i.e. $j(\alpha) = \alpha$ for every $\alpha < \kappa$ yet $\kappa < j(\kappa)$: Taking e.g. the identity function $\mathrm{id} : \kappa \to \kappa$, $\{\xi < \kappa \mid \alpha < \xi < \kappa\} \in U$ for every $\alpha < \kappa$, so that $\kappa \le \pi((\mathrm{id})_U) < j(\kappa)$ by Łoś's Theorem. If $V = L$, then $\mathrm{Ult}(V, U) = L$ by the definability properties of L, but this confronts $\kappa < j(\kappa)$, e.g. if κ were the least measurable cardinal. (One could also appeal to the general fact that $U \notin \mathrm{Ult}(V,U)$; that one "loses" the ultrafilter when taking the ultrapower would become an important theme in later work.) With this Scott established that *if there is a measurable cardinal, then* $V \neq L$. Large cardinal assumptions thus assumed a new significance as a means for "maximizing" possibilities away from Gödel's delimitative construction.

The ultrapower construction provided one direction of a new characterization that established a central structural role for measurable cardinals: *There is an elementary embedding* $j : V \to M$ *for some* M *with critical point* δ *iff* δ *is a measurable cardinal.* Keisler provided the converse direction: With j as hypothesized, $U_j \subseteq P(\delta)$ defined "canonically" by $X \in U_j$ *iff* $\delta \in j(X)$ is a δ-complete ultrafilter over δ. Generating ultrafilters thus via "ideal" elements would become integral to the theory of ultrafilters and large cardinals.

This characterization, when viewed with the focus on elementary embeddings, raises a point that will be even more germane, and thus will be emphasized later, in connection with strong hypotheses. That a $j : V \to M$ is elementary is not formalizable in set theory because of the appeal to the satisfaction relation for V, let alone the assertion that there is such a class j. Thus the "characterization" is really one of giving a formalization, one that provides operative sense through the ultrapower construction. Ulam's original concept was thus made intrinsic to set theory with the categorical imperative of elementary embeddings. In any event ZFC is never actually transcended in consistency results; one can always work in a sufficiently large V_α through the Reflection Principle for ZF.

In Scott's $j : V \to M = \mathrm{Ult}(V,U)$ the concreteness of the ultrapower construction delivered ${}^{\kappa}M \subseteq M$, i.e. M is closed under the taking of arbitrary (in V) κ-sequences, so that in particular $V_{\kappa+1} \cap M = V_{\kappa+1}$. Through this agreement strong reflection conclusions can be drawn. U is *normal iff* $\pi((\mathrm{id})_U) = \kappa$, the identity function is a "least non-constant" function, a property that can be easily arranged. For such U, since κ is inaccessible, it is so in M and hence by Łoś's Theorem $\{\xi < \kappa \mid \xi \text{ is inaccessible}\} \in U$—the inaccessible cardinals below κ have measure one. An analogous argument applies to any $V_{\kappa+1}$ property of κ like weak compactness, and so, as would typify large cardinal hypotheses, measurability articulates its own sense of transcendence over "smaller" large cardinals.

Normality went on to become staple to the investigation of ideals and large cardinals. Formulated for an ideal I over a cardinal λ, I is *normal iff* whenever a function f is regressive on an $S \in P(\lambda) - I$, there is a $T \in P(S) - I$ on which f is constant. Fodor's Lemma is then just the assertion that the nonstationary ideal NS_λ is normal for regular $\lambda > \omega$, and a multitude of "smallness" properties other than nonstationarity has been seen to lead to normal ideals.

Through model-theoretic methods set theory was brought to the point of entertaining elementary embeddings into well-founded models. It was soon to be transfigured by a new means for getting well-founded *extensions* of well-founded models.

3. The Advent of Forcing

3.1. Cohen

Paul Cohen (1934–2007) in April 1963 established the independence of AC from ZF and the independence of CH from ZFC. That is, Cohen established that Con(ZF) implies Con(ZF + $\neg$AC) and that Con(ZFC) implies Con(ZFC + $\neg$CH). Already prominent as an analyst, Cohen had ventured into set theory with fresh eyes and an open-mindedness about possibilities. These results solved two central problems of set theory. But beyond that, Cohen's proofs were the inaugural examples of a new technique, *forcing*, which was to become a remarkably general and flexible method for extending models of set theory. Forcing has strong intuitive underpinnings and reinforces the notion of set as given by the first-order ZF axioms with prominent uses of Replacement and Foundation. If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's method of forcing began its transformation into a modern, sophisticated one.

Cohen's approach was to start with a model M of ZF and adjoin a set G that witnesses some desired new property. This would have to be done in a minimal fashion in order that the resulting extension also model ZF, and so Cohen devised special conditions on both M and G. To be concrete, Cohen started with a countable transitive model $\langle M, \in \rangle$ of ZF. The ordinals of M would then coincide with the predecessors of some ordinal ρ, and M would be the cumulative hierarchy $M = \bigcup_{\alpha<\rho} V_\alpha \cap M$. Cohen recursively defined in M a system of terms (or "names") to denote members of the new model, working with a ramified language. In a streamlined rendition, for each $x \in M$ let $\check{x}$ be a corresponding constant; let $\dot{G}$ be a new constant; and for each $\alpha < \rho$ introduce quantifiers $\forall_\alpha$ and $\exists_\alpha$. Then define: $\dot{M}_0 = \{\dot{G}\}$, and for limit ordinals $\delta < \rho$, $\dot{M}_\delta = \bigcup_{\alpha<\delta} \dot{M}_\alpha$. At the successor stage, let $\dot{M}_{\alpha+1}$ be the collection of constants $\check{x}$ for $x \in V_\alpha \cap M$ and class terms corresponding to formulas allowing parameters from $\dot{M}_\alpha$ and quantifiers $\forall_\alpha$ and $\exists_\alpha$—a syntactical analogue of the operator def(x) for Gödel's L. Once a set G is provided from the outside, a model $M[G] = \bigcup_{\alpha<\rho} M_\alpha[G]$ would be determined by the terms.

But what properties can be imposed on G to ensure that $M[G]$ be a model of ZF? Cohen's key idea was to tie G closely to M through a partially ordered system of sets in M called *conditions* that would approximate G. While G may not be a member of M, G is to be a subset of some $Y \in M$ (with $Y = \omega$ a basic case), and these conditions would "force" some assertions about the eventual $M[G]$ e.g. by deciding some of the membership questions, whether $x \in G$ or not, for $x \in Y$. The assertions are to be just those expressible in the ramified language, and Cohen developed a corresponding *forcing relation* $p \Vdash \varphi$, "p forces φ", between conditions p and formulas φ, a relation with properties reflecting his approximation idea. For example, if $p \Vdash \varphi$ and $p \Vdash \psi$, then $p \Vdash \varphi \wedge \psi$. The conditions are ordered according to the constraints they impose on the eventual G, so that if $p \Vdash \varphi$, and q is a stronger condition, then $q \Vdash \varphi$. It was crucial to Cohen's approach that the forcing relation, like the ramified language, be definable in M.

The final ingredient which gives this whole scaffolding life is the incorporation of a certain kind of set G. Stepping out of M and making the only use of its countability, Cohen enumerated the formulas of the ramified language in a countable sequence and required that G be completely determined by a sequence of stronger and stronger conditions $p_0, p_1, p_2, \ldots$ such that for every formula φ of the ramified language exactly one of φ or $\neg\varphi$ is forced by some p_n. Such a G is called a *generic* set. The language is congenial; with the forcing conditions naturally topologized, a generic set meets every open dense set in M and is thus generic in a classical topological sense.

Cohen was able to show that the resulting $M[G]$ does indeed satisfy the axioms of ZF: Every assertion about $M[G]$ is already forced by some condition; the forcing relation is definable in M; and so the ZF axioms holding in M, most crucially Replacement and Foundation, can be applied to the ramified terms and language to derive corresponding forcing assertions about the ZF axioms holding in $M[G]$.

Cohen first described the case when $G \subseteq \omega$ and the conditions p are functions from some finite subset of ω into $\{0, 1\}$ and $p \Vdash \dot{n} \in \dot{G}$ if $p(n) = 1$ and $p \Vdash \dot{n} \notin \dot{G}$ if $p(n) = 0$. Today, a G so adjoined to M is called a *Cohen real over* M. If subsets of ω are identified with reals as traditionally construed, that G is generic can be extrinsically characterized by saying that G meets every open dense set of reals lying in M. Generally, a $G \subseteq \kappa$ analogously adjoined with conditions of cardinality less than κ is called a *Cohen subset of* κ. Cohen established the independence of CH by adjoining a set which in effect is a sequence of many Cohen reals. It was crucial that the cardinals in the ground model and generic extension coincide, and with two forcing conditions said to be *incompatible* if they have no common, stronger condition, Cohen to this end drew out and relied on the important *countable chain condition* (c.c.c.): Any antichain, i.e. collection of mutually incompatible conditions, is countable.

Cohen established the independence of AC by a version of the above scheme, where in addition to $\dot{G}$ there are also new constants $\dot{G}_i$ for $i \in \omega$, and $\dot{G}$ is interpreted by a set X of Cohen reals, each an interpretation of some $\dot{G}_i$. The point is that X is not well-orderable in the extension, since there are permutations of the forcing conditions that induce a permutation of the G_i's yet leave X fixed.

Several features of Cohen's arguments would quickly be reformulated, reorganized, and generalized, but the thrust of his approach through definability and genericity would remain. Cohen's great achievement lies in devising a concrete procedure for extending well-founded models of set theory in a minimal fashion to well-founded models of set theory with new properties but without altering the ordinals.

The extent and breadth of the expansion of set theory described henceforth dwarfs all that has been described before, both in terms of the numbers of people involved and the results established, and we are left to paint with even broader strokes. With clear intimations of a new and concrete way of building models, set theorists rushed in and, with forcing becoming method, were soon establishing a cornucopia of relative consistency results, truths in a wider sense, with some illuminating classical problems of mathematics. Just in the first weeks after Cohen's discovery, Solomon Feferman, who had been extensively consulted by Cohen as he was coming up with forcing, established further independences elaborating ¬AC and about definability; Levy soon joined in this work and pursued both directions, formulating the "Levy collapse" of an inaccessible cardinal; and Stanley Tennenbaum established the failure of Suslin's Hypothesis by generically adjoining a Suslin tree. Soon, ZFC became quite unlike Euclidean geometry and much like group theory, with a wide range of models being investigated for their own sake.

3.2. Method of Forcing

Robert Solovay above all epitomized this period of sudden expansion in set theory with his mathematical sophistication and central results about and with forcing, and in the areas of large cardinals and descriptive set theory. Following initial graduate study in differential topology, Solovay turned to set theory after hearing a May 1963 lecture by Cohen. Just weeks after, Solovay elaborated the independence of CH by characterizing the possibilities for the size of 2^κ for regular κ and made the first exploration of a range of cardinals. Building on this William Easton in late 1963 established the definitive result for powers of regular cardinals: *Suppose that* GCH *holds and F is a class function from the class of regular cardinals to cardinals such that for regular $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$ and the cofinality* $\mathrm{cf}(F(\kappa)) > \kappa$. *Then there is a (class) forcing extension preserving cofinalities in which $2^\kappa = F(\kappa)$ for every regular κ.* Thus, as Solovay had seen locally, the *only* restriction beyond monotonicity on the power function for regular cardinals is that given by a well-known constraint, the classical Zermelo-Kőnig inequality that $\mathrm{cf}(2^\kappa) > \kappa$ for any cardinal κ. Easton's result enriched the theory of forcing with the introduction of proper classes of forcing conditions, the basic idea of a product analysis, and the now familiar concept of *Easton support*. The result focused interest on the possibilities for powers of *singular* cardinals and the Singular Cardinals Hypothesis (SCH), which asserts that 2^κ for singular κ is the least possible with respect to the powers 2^μ for $\mu < \kappa$ as given by monotonicity and the Zermelo-Kőnig inequality. This requires in particular that for singular strong limit cardinals κ, $2^\kappa = \kappa^+$. With Easton's models satisfying SCH, the *singular cardinals problem*, to determine the range of possibilities for powers of singular cardinals, would become a major stimulus for the further development of set theory much as the continuum problem had been for its early development.

In the Spring of 1964 Solovay established a result remarkable for its mathematical depth and revelatory of what standard of argument was possible with forcing: *If there is an inaccessible cardinal, then in a* ZF *inner model of a forcing extension the Principle of Dependent Choices* (DC) *holds and every set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.* Solovay's inner model is precluded from having a well-ordering of the reals, but DC is a choice principle implying the regularity of ω_1 and sufficient for the formalization of the traditional theory of measure and category on the real numbers. Thus, Solovay's work vindicated the early descriptive set theorists in the sense that the regularity properties can consistently hold for all sets of reals in a *bona fide* model for the classical mathematical analysis of the reals. To prove his result Solovay applied the Levy collapse of an inaccessible cardinal to make it ω_1. For the Lebesgue measurability he introduced a new kind of forcing beyond Cohen's direct ways of adjoining new sets of ordinals or collapsing cardinals, that of adding a *random real* given by forcing with the Borel sets of positive measure as conditions and p stronger than q when $p - q$ is null. In contrast to Cohen reals, a random real meets every measure one subset of the unit interval lying in the ground model. Solovay's work not only opened the door to a wealth of different forcing arguments, but to this day his original definability arguments remain vital to descriptive set theory.

The perfect set property, central to Cantor's direct approach to the continuum problem through definability, led to the first acknowledged instance of a new phenomenon in set theory: the derivation of *equi-consistency* results between large cardinal hypotheses and combinatorial propositions about low levels of the cumulative hierarchy. Forcing showed just how relative the Cantorian concept of cardinality is, since bijective functions could be adjoined to models of set theory and powers like $2^{\aleph_0}$ can be made arbitrarily large with relatively little disturbance. For instance, large cardinals were found to satisfy substantial propositions even after they were "collapsed" to ω_1 or ω_2, i.e. a bijective function was adjoined to render the cardinal the first or second uncountable cardinals respectively. Conversely, such propositions were found to entail large cardinal propositions in an L-like inner model, mostly pointedly the very same initial large cardinal hypothesis. Thus, for some large cardinal property $\varphi(\kappa)$ and proposition ψ, there is a direction $\mathrm{Con}(\exists\kappa\varphi(\kappa)) \to \mathrm{Con}(\psi)$ established by a collapsing forcing argument, and a converse direction $\mathrm{Con}(\psi) \to \mathrm{Con}(\exists\kappa\varphi(\kappa))$ established by witnessing $\varphi(\kappa)$ in an inner model.

Solovay's result provided the forcing direction from an inaccessible cardinal to the proposition that every set of reals has the perfect set property and ω_1 is regular. But Ernst Specker in 1957 had in effect established that if this obtains, then ω_1 (of V) is inaccessible in L. Thus, Solovay's use of an inaccessible cardinal was actually necessary, and its collapse to ω_1 complemented Specker's observation. The emergence of such equi-consistency results is a subtle realization of earlier hopes of Gödel for deciding propositions via large cardinals. Forcing, however, quickly led to the conclusion that there could be no direct implication for CH itself: Levy and Solovay, also in 1964, established that measurable cardinals neither imply nor refute CH, with an argument generalizable to other inaccessible large cardinals. Rather, CH and many other propositions would be reckoned with in terms of consistency, the methods of forcing and inner models being the operative modes of argument.

Building on his Lebesgue measurability result Solovay in 1965 reactivated the classical descriptive set theory program of investigating the extent of the regularity properties (in the presence of AC) by providing characterizations in terms of forcing and definability concepts for the $\mathbf{\Sigma}^1_2$ sets, the level at which Gödel established from $V = L$ the failure of the properties. This led to the consistency relative to ZFC of the Lebesgue measurability of all $\mathbf{\Sigma}^1_2$ sets. Also, the characterizations showed that the regularity properties for $\mathbf{\Sigma}^1_2$ sets follow from existence of a measurable cardinal. Thus, although measurable cardinals do not decide CH, they do establish the perfect set property for $\mathbf{\Sigma}^1_2$ sets so that "CH holds for the $\mathbf{\Sigma}^1_2$ sets". A coda after many years: Although Solovay's use of an inaccessible cardinal for universal Lebesgue measurability seemed *ad hoc* at the time, in 1979 Saharon Shelah established in a *tour de force* that if ZF + DC and all Σ^1_3 sets of reals are Lebesgue measurable, then ω_1 is inaccessible in L.

In a separate initiative, Solovay in 1966 established the equi-consistency of the existence of a measurable cardinal and the "real-valued" measurability of $2^{\aleph_0}$, i.e. that there is a (countably additive) measure extending Lebesgue measure to all sets of reals. For the forcing direction, Solovay starting with a measurable cardinal adjoined random reals and applied the Radon-Nikodym Theorem of analysis, and for the converse direction, he starting with a real-valued measure enlisted the inner model constructed relative to the ideal of measure zero sets. This consistency result provided context for an extended investigation of the possibilities for the continuum as structured by such a measure. Through this work the concept of *saturated ideal*, first studied by Tarski, was brought to prominence as a generalization of having a measurable cardinal applicable to the low levels of the cumulative hierarchy. For an ideal over a cardinal κ, I is λ-*saturated iff* for any $\{X_\alpha \mid \alpha < \lambda\} \subseteq P(\kappa) - I$ there are $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in P(\kappa) - I$ (i.e. the corresponding Boolean algebra has no antichains of cardinality λ). The ideal of measure zero sets is $\aleph_1$-saturated, and Solovay showed that if I is any κ-complete λ-saturated ideal over κ for some $\lambda < \kappa$, then $L[I] \models$ "κ is measurable".

Solovay's work also brought to the foreground the concept of *generic ultrapower* and *generic elementary embedding*. For an ideal I over κ, forcing with the members of $P(\kappa) - I$ as conditions and p stronger than q when $p - q \in I$ engenders an ultrafilter on the ground model $P(\kappa)$. With this one can construct an ultrapower of the ground model in the generic extension and a corresponding elementary embedding. It turns out that the κ^+-saturation of the ideal ensures that this generic ultrapower is well-founded. Thus, a synthesis of forcing and ultrapowers is effected, and this raised enticing possibilities for having such large cardinal-type structure low in the cumulative hierarchy.

The development of the theory of forcing went hand in hand with this procession of central results. Solovay had first generalized forcing to arbitrary partial orders of conditions, proceeding in terms of incompatible members and dense sets and Levy's concept of *generic filter.* In his work on the Baire property for his 1964 model, Solovay came to the idea of assigning values to formulas from a complete Boolean algebra. Loosely speaking, the value would be the supremum of all the conditions forcing it. Working independently, Solovay and Scott developed the idea of recasting forcing entirely in terms of *Boolean-valued models.* This approach showed how to replace Cohen's ramified languages by a more direct induction on rank and how to avoid his dependence on a countable model. Boolean-valued functions play the role of sets, and formulas involving these functions are assigned Boolean-values by recursion respecting logical connectives and quantifiers. By establishing in ZFC that e.g. there is a complete Boolean algebra assigning the formula expressing $\neg$CH Boolean value one, a semantic construction was replaced by a syntactic one that directly secured relative consistency.

Still, the view of forcing as a way of actually extending models held the reservoir of sense and the promise of discovery, and after Shoenfield popularized an approach to the forcing relation that captured the gist of the Boolean-valued approach, forcing has been generally cast as a matter of partial orders and generic filters. Boolean algebras would nonetheless underscore and enhance the setting: partial orders are to have a maximum element 1; one is attuned to the *separativity* of partial orders, the property that ensures that they are densely embedded in their canonical Boolean completions; Boolean-values are used when illuminating; and embedding results for forcing partial orders are cast, as most algebraically informative, in terms of Boolean algebras.

By the 1970s there would be a further assimilation of both the syntactic and semantic approaches in that generic extensions would be "taken" of V. In this the current approach then, a partial order $\langle P, < \rangle$ of conditions is specified to a purpose, with $p < q$ for p being stronger than q. A class V^P of P-names defined recursively is used in forcing assertions, with a canonical name $\check{x}$ corresponding to $x \in V$. A $D \subseteq P$ is *dense* if for any $p \in P$ there is a $d \in D$ with $d \leq p$. An $F \subseteq P$ is a *filter* if (i) if $p \in F$ and $p \leq q$, then $q \in F$, and (ii) if $p_1, p_2 \in F$ then there is an $r \in F$ with $r \leq p_1$ and $r \leq p_2$. Finally, $G \subseteq P$ is a V*-generic filter* if G is a filter such that for every dense $D \subseteq P$, $G \cap D \neq \emptyset$. One posits such a G and takes a generic extension $V[G]$, its properties argued for on the basis of combinatorial properties of P. For inner or transitive set models M, one proceeds analogously to define M*-generic filters* meeting every dense set belonging to M and takes generic extensions $M[G]$.

In this one goes against the sense of V as *the* universe of all sets and Tarski's "undefinability of truth", but actually V has become *schematic* for a ground model. Generic extensions of inner models M are taken with M-generic G, and moreover, successive iterated extensions are taken, exacerbating any preoccupation with a single universe of sets. As the techniques of forcing were advanced, the methodology was itself soon to be woven into set theory as part of its postulations.

Solovay and Tennenbaum earlier in 1965 had established the consistency of Suslin's Hypothesis, that there are no Suslin trees, illuminating a classical question from 1920 with a ground-breaking use of iterated forcing to keep "killing Suslin trees" in intermediate extensions. D. Anthony Martin pointed out that the Solovay-Tennenbaum argument actually established the consistency of a closure of forcing extensions of a certain kind, an instrumental "axiom" now known as Martin's Axiom (MA): *For any* c.c.c. *partial order P and collection $\mathcal{D}$ of fewer than $2^{\aleph_0}$ dense subsets of P, there is a filter $G \subseteq P$ meeting every member of $\mathcal{D}$.* Thus method became axiom, and many consistency results could now be simply stated as direct consequences of a single umbrella proposition. CH technically implies MA, but the Solovay-Tennenbaum argument established the consistency of MA with the continuum being arbitrarily large.

While classical results with CH had worked on an $\aleph_0 \,/\, \aleph_1$ dichotomy, MA established a $<2^{\aleph_0} / \, 2^{\aleph_0}$ dichotomy. For example, Martin and Solovay established that MA implies that the union of fewer than $2^{\aleph_0}$ Lebesgue measure zero sets is again Lebesgue measure zero. Sierpiński in 1925 had established that every $\mathbf{\Sigma}^1_2$ set of reals is the union of $\aleph_1$ Borel sets. Hence, MA and $2^{\aleph_0} > \aleph_1$ implies that every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable. Many further results plied the $<2^{\aleph_0} / \, 2^{\aleph_0}$ dichotomy to show that under MA inductive arguments can be carried out in $2^{\aleph_0}$ steps that previously succeeded under CH in $\aleph_1$ steps. The continuum problem was newly illuminated as a matter of method, by showing that CH as a construction principle could be generalized to $2^{\aleph_0}$ being arbitrarily large.

Glancing across the wider landscape, forcing provided new and diverse ways of adjoin generic reals and other sets, and these led to new elucidations, for example about cardinal characteristics, or invariants, of the continuum and combinatorial structures and objects, like ultrafilters over ω. The work on Suslin's Hypothesis in hand and with the possibilities afforded by Martin's Axiom, the investigation of general topological notions gathered steam. With Mary Ellen Rudin and her students at Wisconsin breaking the ground, new questions were raised for general topological spaces about separation properties, compactness-type covering properties, separability and metrizability, and corresponding cardinal characteristics.

3.3. $0^{\#}$, $L[U]$, and $L[\mathcal{U}]$

The infusion of forcing into set theory induced a broad context extending beyond its applications and sustained by model-theoretic methods, a context which included central developments about large cardinals having their source in Scott's 1961 result that measurable cardinals contradict $V = L$. Haim Gaifman invented *iterated ultrapowers* and established seminal results about and with the technique, results which most immediately stimulated definitive work in the formative theses of Silver and Kunen.

Jack Silver in his 1966 Berkeley thesis provided a structured sense of transcendence over L in terms of the existence of a special set of natural numbers $0^{\#}$ ("zero sharp") which refined an earlier formulation of Gaifman and was quickly investigated by Solovay in terms of definability. Mostowski and Andrzej Ehrenfeucht in 1956 had developed theories whose models have *indiscernibles*, implicitly ordered members of the domain all of whose n-tuples satisfy the same formulas. They had applied Ramsey's Theorem in compactness arguments to get models generated by indiscernibles, models consequently having many automorphisms. Silver applied partition properties satisfied by measurable cardinals to produce indiscernibles *within* given structures, particularly in the initial segment $\langle L_{\omega_1}, \in\rangle$ of the constructible universe. With definability and Skolem hull arguments, Silver was able to isolate a canonical collection of sentences to be satisfied by indiscernibles, a theory whose models cohere to get L itself as generated by canonical ordinal indiscernibles—a dramatic accentuation of the original Gödel generation of L. $0^{\#}$ is that theory coded as a real, and as Solovay emphasized, $0^{\#}$ is the only possible real to satisfy a certain Π^1_2 relation, one whose complexity arises from its asserting that to every countable well-ordering there corresponds a well-founded model of the coded theory. The canonical class, closed and unbounded, of ordinal indiscernibles is often called the *Silver indiscernibles*. Having these indiscernibles substantiates $V \neq L$ in drastic ways: Each indiscernible ι has various large cardinal properties and satisfies $L_\iota \prec L$, so that by a straightforward argument the satisfaction relation for L is definable from $0^{\#}$. The theory of $0^{\#}$ was seen to relativize, and for reals $a \in {}^\omega\omega$ the analogous $a^{\#}$ for the inner model $L[a]$ would play focal roles in descriptive set theory as based on definability.

Kunen's main large cardinal results emanating from his 1968 Stanford thesis would be the definitive structure results for inner models of measurability. For U a normal κ-complete ultrafilter over a measurable cardinal κ, the inner model $L[U]$ of sets constructible relative to U is easily seen with $\overline{U} = U \cap L[U]$ to satisfy $L[U] \models$ "$\overline{U}$ is a normal κ-complete ultrafilter". With no presumption that κ is measurable (in V) and taking $U \in L[U]$ from the beginning, call $\langle L[U], \in, U\rangle$ a *κ-model iff* $\langle L[U], \in, U\rangle \models$ "U is a normal κ-complete ultrafilter over κ". Solovay observed that in a κ-model, the GCH holds above κ by a version of Gödel's argument for L and that κ is the only measurable cardinal by a version of Scott's argument. Silver then established that the full GCH holds, thereby establishing the relative consistency of GCH and measurability; Silver's proof turned on a local structure $L_\alpha[U]$ being *acceptable* in the later parlance of inner model theory.

Kunen made Gaifman's technique of iterated ultrapowers integral to the subject of inner models of measurability. For a κ-model $\langle L[U], \in, U\rangle$, the ultrapower of $L[U]$ by U with corresponding elementary embedding j provides a $j(\kappa)$-model $\langle L[j(U)], \in, j(U)\rangle$, and this process can be repeated. At limit stages, one can take the direct limit of models, which when well-founded can be identified with the transitive collapse. Indeed, by Gaifman's work these iterated ultrapowers are always well-founded, i.e. κ-models are *iterable.* Kunen showed that the λth iterate of a κ-model for any regular $\lambda > \kappa^+$ is of form $\langle L[\mathcal{C}_\lambda], \in, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]\rangle$, where $\mathcal{C}_\lambda$ again is the closed unbounded filter over λ, so that remarkably, constructing relative to a filter definable in set theory leads to an inner model of measurability. With this, there can be *comparison* of κ-models and κ'-models by iterating them up to a sufficiently large λ. This comparison possibility let to the structure results: (1) *for any κ-model and κ'-model with $\kappa < \kappa'$, the latter is an iterated ultrapower of the former,* and (2) *for any κ, there is at most one κ-model.* It then followed that if κ is measurable and U_1 and U_2 are any κ-complete ultrafilters over κ, then $L[U_1] = L[U_2]$. These various results argued forcefully for the coherence and consistency of the concept of measurability. And it would be that iterability and comparison would remain as basic features in inner model theory in its subsequent development.

Kunen's contribution to the theory of iterated ultrapowers was that iterated ultrapowers can be taken of an inner model M with respect to an ultrafilter U even if $U \notin M$, as long U is an *M-ultrafilter*, i.e. U in addition to having M related ultrafilter properties also satisfies an "amenability" condition for M. A crucial dividend was a characterization of the existence of $0^\#$ that secured its central importance in inner model theory. With $0^\#$, any increasing shift of the Silver indiscernibles provides an elementary embedding $j : L \to L$. Kunen established conversely that such an embedding generates indiscernibles, so that $0^\#$ *exists iff there is a (non-identity) elementary embedding $j : L \to L$.* Starting with such an embedding Kunen defined the corresponding ultrafilter U over the critical point and showed that U is an L-ultrafilter with which the iterated ultrapowers of L are well-founded. The successive images of the critical point were seen to be indiscernibles for L, giving $0^\#$. As inner model theory was to develop, this sharp analysis would become schematic: the "sharp" of an inner model M would encapsulate transcendence over M, and the *non-rigidity* of M, that there is a (non-identity) elementary embedding $j : M \to M$, would provide equivalent structural sense.

William Mitchell in 1972, just after completing a pioneering Berkeley thesis on Aronszajn trees, provided the first substantive extension of Kunen's inner model results and brought to prominence a new large cardinal hypothesis. For normal κ-complete ultrafilters U and U' over κ, define the *Mitchell order* $U' \lhd U$ *iff* $U' \in \mathrm{Ult}(V, U)$, i.e. there is an $f : \kappa \to V$ representing U' in the ultrapower, so that $\{\alpha < \kappa \mid f(\alpha)$ is a normal α-complete ultrafilter over $\alpha\} \in U$ and κ is already a limit or measurable cardinals. $U \lhd U$ always fails, and generally, $\lhd$ is a well-founded relation by a version of Scott's argument that measurable cardinals contradict $V = L$. Consequently, to each U can be recursively assigned a rank $o(U) = \sup\{o(U') + 1 \mid U' \lhd U\}$, and to a cardinal κ, the supremum $o(\kappa) = \sup\{o(U) + 1 \mid U$ is a normal κ-complete ultrafilter over $\kappa\}$. By a cardinality argument, if $2^\kappa = \kappa^+$ then $o(\kappa) \le \kappa^{++}$.

The hypothesis $o(\kappa) = \delta$ provided an "order" of measurability calibrated by δ, with larger δ corresponding to stronger assumptions on κ. For the investigation of these orders, Mitchell devised the concept of a *coherent* sequence of ultrafilters ("measures") and was able to establish canonicity results for inner models $L[\mathcal{U}] \models$ "$\mathcal{U}$ is a coherent sequence of ultrafilters". A coherent sequence $\mathcal{U}$ is a doubly indexed system of normal α-complete ultrafilters $\mathcal{U}(\alpha,\beta)$ over α such that $\mathcal{U}(\kappa,\beta) \lhd \mathcal{U}(\kappa,\beta')$ for $\beta < \beta'$ at the κth level, and the earlier levels contain just enough ultrafilters necessary to represent these $\lhd$ relationships in the respective ultrapowers. (Technically, if $j : V \to \mathrm{Ult}(V, \mathcal{U}(\kappa,\beta'))$, then $j(\mathcal{U})\restriction\{(\alpha,\beta) \mid \alpha \le \kappa\} = \mathcal{U}\restriction\{(\alpha,\beta) \mid \alpha < \kappa \vee (\alpha = \kappa \wedge \beta < \beta')\}$, i.e. $j(\mathcal{U})$ through κ is exactly $\mathcal{U}$ "below" (κ,β').)

Mitchell first affirmed that these $L[\mathcal{U}]$'s are *iterable* in that arbitrary iterated ultrapowers via ultrafilters in $\mathcal{U}$ and its successive images are always well-founded. He then effected a comparison: *Any $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ have respective iterated ultrapowers $L[\mathcal{W}_1]$ and $L[\mathcal{W}_2]$ such that $\mathcal{W}_1$ is an initial segment of $\mathcal{W}_2$ or vice versa.* This he achieved through a process of *coiteration* of least differences: At each stage, one finds the lexicographically least coordinate at which the current iterated ultrapowers of $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$ differ and takes the respective ultrapowers by the differing ultrafilters; the difference is eliminated as ultrafilters never occur in their ultrapowers. Note that this iteration process is external to $L[\mathcal{U}_1]$ and $L[\mathcal{U}_2]$, further drawing out the advantages of working externally to models as Kunen had first done with his M-ultrafilters. With this coiteration, Mitchell established that in $L[\mathcal{U}]$ the only normal α-complete ultrafilters over α for any α are those that occur in $\mathcal{U}$ and other propositions like GCH that showed these models to be L-like. Coiteration would henceforth be embedded in inner model theory, and with his models $L[\mathcal{U}]$ modeling $o(\kappa) = \delta$ for $\delta < \kappa^{++L[\mathcal{U}]}$, $\exists\kappa(o(\kappa) = \kappa^{++})$ would become the delimitative proposition of his analysis.

3.4. Constructibility

These various results were set against a backdrop of an increasing articulation of Gödel's original notion of constructibility. Levy in 1965 had put forward the appropriate hierarchy for the first-order formulas of set theory: A formula is Σ_0 and Π_0 if it is bounded, i.e. having only quantifiers expressible in terms of $\forall v \in w$ and $\exists v \in w$, and recursively, a formula is Σ_{n+1} if it is of the form $\exists v_1 \ldots \exists v_k \varphi$ where φ is Π_n and is Π_{n+1} if it is of the form $\forall v_1 \ldots \forall v_k \varphi$ where φ is Σ_n. Two basic points about discounting bounded quantifiers are that Σ_0 formulas are absolute for transitive structures, i.e. they hold in such structures just in case they hold in V, and that if φ is Σ_n (resp. Π_n) then $\exists v \in w\varphi$ and $\forall v \in x\varphi$ are equivalent in ZFC to Σ_n (resp. Π_n) formulas by uses of Replacement. Levy wove in Shoenfield's Σ^1_2 absoluteness result to establish the Shoenfield-Levy Absoluteness Lemma: *For any Σ_1 sentence σ,* ZF + DC $\vdash \sigma \longleftrightarrow \sigma^L$. Levy actually showed that L here can be replaced by a countable L_γ fixed for all σ, and as such the lemma can be seen as a refinement of the Reflection Principle for ZF, one that was to find wide use in the burgeoning field of admissible set theory.

Gödel's original GCH result with L was newly seen in light of the structured context for definability. For N and M construed as structures with $\in$, $j : N \to M$ is a Σ_n*-elementary embedding iff* for any Σ_n $\varphi(v_1, \ldots, v_k)$ and $x_1, \ldots, x_k \in N$, $N \models \varphi[x_1, \ldots, x_k]$ *iff* $M \models \varphi[j(x_1), \ldots, j(x_k)]$. N is a Σ_n*-elementary substructure* of M, denoted $N \prec_n M$, *iff* the identity map is Σ_n-elementary. Analysis of the satisfaction relation established that being an L_α is a Σ_1 property, and this led to the Condensation Lemma:

If α is a limit ordinal and $N \prec_1 L_\alpha$,

then the transitive collapse of N is L_β for some $\beta \leq \alpha$.

Operatively, one applies this lemma with Skolem's algebraic approach to logic by taking N to be a Σ_1 *Skolem hull* in L_α: For any Σ_0 formula $\varphi(v_1, \ldots, v_n, v_{n+1})$ and $x_1, \ldots, x_n \in L_\alpha$, if $\langle L_\alpha, \in \rangle \models \varphi[x_1, \ldots x_n, y]$ for $y \in L_\alpha$, let $f_\varphi(x_1, \ldots, x_n)$ be such a y. Then let N be the algebraic closure of some subset of L_α under these Skolem functions. The road from the Condensation Lemma to Gödel's Fundamental Theorem for the consistency of GCH is short. Generally, the lemma articulates a crucial hierarchical cohesion, and its various emanations would become fundamental to all inner model theory.

The consummate master of constructibility was to be Ronald Jensen, whose first systematic analysis transformed the subject with the introduction of the *fine structure theory* for L. Jensen's work is distinguished by the persistent pursuit of internal logical structure, the sophistication of the local apparatus developed, and a series of remarkable successes with reverberations throughout the whole expanse of set theory. After his 1964 Bonn dissertation on models of arithmetic, Jensen moved with strength into investigations with forcing and of definability, two directions that would steadily complement each other in his work. He, like Solovay, saw the great potential of forcing, and he soon derived the Easton results independently. In the direction of definability he in 1965 worked out with Carol Karp a theory of primitive recursive set functions, and with these he began his investigation of L. In his 1967 Habilitationsschrift he had definite anticipations of fine structure, although notably he had no particular application for it in mind at that time.

In 1968 Jensen made a major breakthrough by showing that $V = L$ *implies the failure of Suslin's Hypothesis*, i.e. (there is a Suslin tree)^L, applying L for the first time after Gödel to establish a relative consistency result about a classical proposition. The initial breakthrough had been when Tennenbaum had adjoined a Suslin tree with forcing and Thomas Jech had provided another forcing argument; Jensen at first pitched his construction in the guise of a forcing argument, one in fact like Jech's. This is the paradigmatic case of what would become a recurring phenomenon: A combinatorial existence assertion is first shown to be relatively consistent with ZFC using forcing, and then the assertion is shown to hold in L, the minimal inner model.

The lack of cofinal branches in Suslin trees is complemented by their abundance in Kurepa trees. Inspired by Jensen's construction the ubiquitous Solovay established: (there is a Kurepa tree)^L. Here too the relative consistency of the proposition had been established first through forcing.

Jensen isolated the combinatorial features of L that enabled these constructions and together with Kunen in 1969 worked out a larger theory. The focus was mainly on two combinatorial principles of Jensen's for a regular cardinal κ, $\diamondsuit_\kappa$ ("diamond") and a strengthening, $\diamondsuit_\kappa^+$ ("diamond plus"). Stating the first,

$\diamondsuit_\kappa$ There is a sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$ such that for any $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.

Just $\diamondsuit$ is implicitly $\diamondsuit_{\omega_1}$. $\diamondsuit_\kappa$ implies $\bigcup_{\alpha<\kappa} \kappa^{|\alpha|} = \kappa$ (so that $\diamondsuit$ implies CH) as every bounded subset of κ occurs in a $\diamondsuit_\kappa$ sequence. Indeed, a $\diamondsuit_\kappa$ sequence is an enumeration of the bounded subsets of κ that can accommodate every $X \subseteq \kappa$ in anticipatory constructions where $X \cap \alpha$ appearing in the enumeration for many α's suffices. Within a few years $\diamondsuit$ would be on par with CH as a construction principle with wide applications in topology, algebra, and analysis. (Another coda of Shelah's after many years: In 2007 he established that for successors $\lambda^+ > \omega_1$, $2^\lambda = \lambda^+$ actually implies $\diamondsuit_{\lambda^+}$, so that the two are equivalent.)

Jensen abstracted his Suslin tree result to: (1) *if $V = L$, then $\diamondsuit_\kappa$ holds from every regular $\kappa > \omega$*, and (2) *if $\diamondsuit_{\omega_1}$ holds, then there is a Suslin tree.* Solovay's result was abstracted to higher, *κ-Kurepa trees*, κ-trees with at least κ^+ cofinal branches, in terms of a new cardinal concept, *ineffability*, arrived at independently by Jensen and Kunen: *If $V = L$ and $\kappa > \omega$ is regular, then $\diamondsuit_\kappa^+$ holds iff κ is not ineffable.* Ineffable cardinals, stronger than weakly compact cardinals, would soon be seen to have a range of involvements and an elegant theory. As for "higher" Suslin trees, they would involve the use of a new combinatorial principle, one that first figured in a sophisticated forcing argument.

The crowning achievement of the investigation of Suslin's Hypothesis was its joint consistency with CH, Con(ZFC) $\to$ Con(ZFC + CH + SH), established by Jensen. In the Solovay-Tennenbaum consistency proof for SH, cofinal branches had been adjoined iteratively to Suslin trees as they arose and direct limits were taken at limit stages, a limiting process that conspired to adjoin new reals so that CH fails. Jensen, with considerable virtuosity for the time, devised a way to kill Suslin trees less directly and effected the iteration according to a curtailed tree-like indexing—so that no new reals are ever adjoined. That indexing is captured by the $\kappa = \omega_1$ case of the combinatorial principle $\square_\kappa$ ("square"):

$\square_\kappa$ There is a sequence $\langle C_\alpha \mid \alpha$ a limit ordinal $< \kappa^+\rangle$ such that for $\alpha < \kappa^+$:

(a) $C_\alpha \subseteq \alpha$ is closed unbounded in α,

(b) for β a limit point of C_α, $C_\alpha \cap \beta = C_\beta$, and

(c) for $\omega \leq \mathrm{cf}(\alpha) < \kappa$, the order-type of C_α is less than κ.

$\square_\omega$ is immediate, as witnessed by any ladder system, i.e. a sequence $\langle C_\alpha \mid \alpha$ a limit ordinal $< \omega_1\rangle$ such that C_α is of order-type ω and cofinal in α. $\square_\kappa$ for $\kappa > \omega$ brings out the tension between the desired (b) and the needed (c). As such, $\square_\kappa$ came to guide many a construction of length κ^+ based on components of cardinality $< \kappa$.

$\square_\kappa$ can be adjoined by straightforward forcing with initial approximations; Jensen established: *If $V = L$, then $\square_\kappa$ holds for every κ.* As for higher Suslin trees, a *κ-Suslin tree* is expectedly a κ-tree with no chains or antichains of cardinality κ. It was actually Jensen's work on these trees that motivated his formulation of $\square_\kappa$, and he established, generalizing his result for $\kappa = \omega_1$: (1) *for any κ, $\diamondsuit_{\kappa^+}$ and $\square_\kappa$ imply that there is a κ^+-Suslin tree,* and, for limit cardinals κ, the characterization (2) *there is a κ-Suslin tree iff κ is not weakly compact.* It is a notable happenstance that Suslin's early, 1920 speculation would have such extended ramifications in modern set theory.

Jensen's results that $\square_\kappa$ holds in L and (2) above were the initial applications of his fine structure theory. Unlike Gödel who had focused with L on relative consistency, Jensen regarded the investigation of how the constructible hierarchy grows by examining its behavior at arbitrary levels as of basic and intrinsic interest. And with his fine structure theory Jensen developed a considerable and intricate machinery for this investigation. A pivotal question became: when does an ordinal α first get "singularized", i.e. what is the least β such that there is in $L_{\beta+1}$ an unbounded subset of α of smaller order-type, and what definitional complexity does this set have? One is struck by the contrast between Jensen's attention to such local questions as this one, at the heart of his proof of $\square_\kappa$, and how his analysis could lead to major large-scale results of manifest significance.

For a uniform development of his fine structure theory, Jensen switched from the hierarchy of L_α's to a hierarchy of J_α's, the *Jensen hierarchy*, where $J_{\alpha+1}$ is the closure of $J_\alpha \cup \{J_\alpha\}$ under the "rudimentary" functions (the primitive recursive set functions without the recursion scheme). For $L[A]$, there is an analogous hierarchy of J_α^A where one also closes off under the function $x \longmapsto A \cap x$. For a set N, construed as a structure with $\in$ and possibly with some $A \cap N$ as a predicate, a relation is $\Sigma_n(N)$ *iff* it is first-order definable over N by a Σ_n formula. For every α, both $\langle J_\xi \mid \xi < \alpha\rangle$ and a well-ordering $<_L$ of L restricted to J_α are Σ_1 definable over J_α *uniformly*, in that the same formula works for all the J_α's.

In these terms, fine structure addresses the classical issue of Skolem functions through definability. For $(k+1)$-ary relations R and S,

$$\begin{aligned}&R \text{ is } \textit{uniformized} \text{ by } S \quad \textit{iff}\\ &\quad S \subseteq R \text{ and } \forall \mathbf{w}(\exists y R(\mathbf{w}, y) \longleftrightarrow \exists ! y S(\mathbf{w}, y)),\end{aligned}$$

where $\exists!$ is "there exists exactly one". This amounts to the assertion that S refines R to a function on the same $\mathbf{w}$'s and is thus a form of AC. In systematic applications of the Condensation Lemma one deduces, toward the construction of Σ_1 Skolem hulls, that $\Sigma_0(J_\alpha)$, and hence $\Sigma_1(J_\alpha)$, relations are uniformizable by $\Sigma_1(J_\alpha)$ relations that choose $<_L$-least witnesses. Weaving together all such relations into one universal one, one gets a Skolem function h Σ_1 definable over J_α uniformly, with the property that for any $X \subseteq J_\alpha$ an application of h to X yields an elementary substructure of J_α.

What about $\Sigma_2(J_\alpha)$ relations? Choosing $<_L$-least witnesses as before leads only to a $\Sigma_3(J_\alpha)$ uniformizing relation, since asserting that no predecessor in the Σ_1 definable well-ordering satisfies the Σ_2 formula adds to the quantifier complexity. Jensen saw that a palatable analysis of definability stable through various transformations would require a $\Sigma_2(J_\alpha)$ uniformizing relation. He achieved this by applying the basic elements of fine structure: As a measure of the lack of closure under definability, let the *(first) projectum* $\rho_\alpha \leq \alpha$ be the least γ for which there is a $\Sigma_1(J_\alpha)$ subset of γ which is not a member of J_α. There is then a $\Sigma_1(J_\alpha)$ map of a subset of J_{ρ_α} onto J_α, essentially a Skolem function as in the previous paragraph. The $\Sigma_1(J_\alpha)$ definitions involved here can be construed as depending on one parameter in J_α, and one can fix the $<_L$-least possibility—the *standard parameter*. With this one can consider the *projectum structure* $\langle J_{\rho_\alpha}, A_\alpha \rangle$ where A_α the *standard code*—the $<_L$-least among certain *master codes*—a predicate that codes Σ_1 satisfaction for J_α so that the part of any $\Sigma_2(J_\alpha)$ relation in J_{ρ_α} can be taken to be a $\Sigma_1(\langle J_{\rho_\alpha}, A_\alpha \rangle)$ relation. The relation can then be uniformized by a $\Sigma_1(\langle J_{\rho_\alpha}, A_\alpha \rangle)$ function, one that can subsequently be projected up to be a $\Sigma_2(J_\alpha)$ uniformizing function with the available $\Sigma_1(J_\alpha)$ mapping of a subset of J_{ρ_α} onto J_α.

The foregoing sets out the salient features of fine structure theory, and Jensen carried out this analysis in general to establish for every $n \geq 1$ the Σ_n Uniformization Theorem: *For every α, every $\Sigma_n(J_\alpha)$ relation can be uniformized by a $\Sigma_n(J_\alpha)$ relation.* In truth, as often with the thrust of method, fine structure would become autonomous in that it would be the actual fine structure workings of this lemma, rather than just its statement, which would be used. Jensen also gave expression to canonicity with what is now known as the *Downward Extension of Embeddings Lemma*, which for the foregoing situation asserts that if $e : N \to \langle J_{\rho_\alpha}, A_\alpha \rangle$ is Σ_0-elementary, then N itself is the projectum structure of a unique J_β and e can be extended uniquely to a Σ_1-elementary $\overline{e} : J_\beta \to J_\alpha$. Jensen moved forward with this fine structure theory to uncover and articulate the combinatorial structure of the constructible universe.

4. Strong Hypotheses

4.1. Large Large Cardinals

With elementary embedding having emerged as a systemic concept in set theory, Solovay and William Reinhardt at Berkeley in the late 1960s formulated inter-related large cardinal hypotheses stronger than measurability. Reinhardt conceived *extendibility*, and he and Solovay independently, *supercompactness*. A cardinal κ is *γ-supercompact iff* there is an elementary embedding $j : V \to M$ for some inner model M, with critical point κ and $\gamma < j(\kappa)$ such that ${}^{\gamma}M \subseteq M$, i.e. M is closed under the taking of arbitrary γ-sequences. κ is *supercompact iff* κ is γ-supercompact for every γ. Evidently, the heuristics of generalization and reflection were at work here, as κ is measurable *iff* κ is κ-supercompact, and stronger closure properties imposed on the target model M ensure stronger reflection properties. For example, if κ is 2^{κ}-supercompact with witnessing $j : V \to M$, then $M \models$ "κ is measurable", since ${}^{2^{\kappa}}M \subseteq M$ implies that every ultrafilter over κ is in M, and so if $U_j \subseteq P(\kappa)$ is defined canonically from j by $X \in U_j$ *iff* $\kappa \in j(X)$, then $\{\xi < \kappa \mid \xi \text{ is measurable}\} \in U_j$ by Łoś's Theorem. Supercompactness was initially viewed as an ostensible strengthening of Tarski's strong compactness in that, with the focus on elementary embedding, reflection properties were directly incorporated. Whether strong compactness is actually equivalent to supercompactness became a new "identity crisis" issue.

Reinhardt entertained a *prima facie* extension of these ideas, that there is a (non-identity) elementary embedding $j : V \to V$. With suspicions soon raised, Kunen dramatically established in 1970 that this is inconsistent with ZFC by applying an Erdős-Hajnal partition relation result, a combinatorial contingency making prominent use of the Axiom of Choice. This contingency pointed out a specific lack of closure of the target model: *For any elementary embedding $j : V \to M$ with critical point κ, let λ be the supremum of $\kappa < j(\kappa) < j^2(\kappa) < \cdots$. Then, $V_{\lambda+1} \not\subseteq M$.* This lack of closure has essentially stood as the weakest known to this day.

A net of hypotheses consistency-wise stronger than supercompactness was soon cast across the conceptual space delimited by Kunen's inconsistency. For $n \in \omega$, κ is *n-huge iff* there is an elementary embedding $j : V \to M$, for some inner model M, with critical point κ such that ${}^{j^n(\kappa)}M \subseteq M$. κ is *huge iff* κ is 1-huge. If κ is huge, then $V_\kappa \models$ "there are many supercompact cardinals". Thematically close to Kunen's inconsistency were several hypotheses articulated for further investigation, e.g. there is a (non-identity) elementary embedding $j : V_\lambda \to V_\lambda$ for some λ.

The appearance of proper classes in these various formulations raises issues about legitimacy. By Tarski's "undefinability of truth", the satisfaction relation for V is not definable in ZFC, and the elementary embedding characterization of measurability already suffers from this shortcoming. However, the γ-supercompactness of κ can be analogously formulated in terms of the existence of a "normal" ultrafilter over the set $P_\kappa\gamma = [\gamma]^{<\kappa} = \{x \subseteq \gamma \mid |x| < \kappa\}$. Similarly, n-hugeness can also be recast. As for Kunen's inconsistency, his argument can be regarded as establishing: *There is no (non-identity) elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ for any λ.*

The details on γ-supercompactness drew out new, generalizing concepts for filters (and so, for ideals). Suppose that Z is a set and F a filter over $P(Z)$ (so $F \subseteq P(P(Z))$). Then F is *fine iff* for any $a \in Z$, $\{x \in P(Z) \mid a \in x\} \in F$. F is *normal iff* whenever f is a function satisfying $\{x \in P(Z) \mid f(x) \in x\} \in F$, i.e. f is a choice function on a set in F, there is an $a \in Z$ such that $\{x \in P(Z) \mid f(x) = a\} \in F$, i.e. f is constant on a set in F. When Z is a cardinal κ and $\kappa = \{x \in P(\kappa) \mid x \in \kappa\} \in F$, then this new normality reduces to the previous concept. With an analogous reduction to filters over $P_\kappa\gamma = [\gamma]^{<\kappa} = \{x \in P(\gamma) \mid |x| < \kappa\}$, we have the formulation: κ is γ-supercompact *iff* there is a κ-complete, fine, normal ultrafilter over $P_\kappa\gamma$. This inspired a substantial combinatorial investigation of filters over sets $P_\kappa\gamma$, and a general, structural approach to filters over sets $P(Z)$.

Whether it is in these large cardinal hypotheses or the transition from V to $V[G]$ in forcing, the appeal to the satisfaction relation for V is liberal and unabashed in modern set-theoretic practice. Yet ZFC remains parsimoniously the official theory and this carries with it the necessary burden of formalization. On the other hand, it is the formalization that henceforth carries the operative sense; for example, the ultrafilter characterization of γ-supercompactness delivers through the concreteness of the ultrapower construction critical properties that become part of the concept in its use. It has become commonplace in modern set theory that informal assertions and schematic procedures often convey an incipient intentional sense, but formalization refines that sense with workable structural articulations.

Although large large cardinals were developed particularly to investigate the possibilities for elementary embeddings and were quickly seen to have a simple but elegant basic theory, what really intimated their potentialities were new forcing results in the 1970s and 1980s, especially from supercompactness, that established new relative consistencies, even of assertions low in the cumulative hierarchy. The earliest, orienting result along these lines addressed the singular cardinals problem. The "Prikry-Silver" result provided the first instance of a failure of the Singular Cardinal Hypothesis by drawing together two results of independent significance, themselves crucial as methodological advances.

Karel Prikry in his 1968 Berkeley thesis had set out a simple but elegant notion of forcing that changed the cofinality of a measurable cardinal while not collapsing any cardinals. With U a normal κ-complete ultrafilter over κ, *(basic) Prikry forcing* for U has as conditions $\langle p, A\rangle$ where p is a finite subset of κ and $A \in U$. For conditions $\langle p, A\rangle$ and $\langle q, B\rangle$, the first is stronger than the second if $p \supseteq q$ and $\alpha \in p - q$ implies $\alpha > \max(q)$, and $A \cup (p - q) \subseteq B$. A condition thus specifies a finite initial part of a new ω-cofinalizing subset of κ, and further members are to be added on top from a set large in the sense of being in U. Applying a partition property available for normal ultrafilters, Prikry established that for any condition $\langle p, A\rangle$ and forcing statement, there is a $B \subseteq A$ such that $B \in U$ and $\langle p, B\rangle$ decides the statement, i.e. extending p is unnecessary. Hence, e.g. the κ-completeness of U implies that V_κ remains unchanged in the forcing extension yet the cofinality of κ now becomes ω.

Prikry forcing may at first have seemed a curious possibility for singularization. However, that a Prikry generic sequence also generates the corresponding U in simple fashion and also results from indiscernibles made them a central feature of measurability. Prikry forcing would be generalized in various directions and for a variety of purposes. With the capabilities made available for changing cofinalities, equi-consistency connections would eventually be established between large cardinals on the one hand and formulations in connection with the singular cardinals problem on the other.

Silver in 1971 first established the relative consistency of having a measurable cardinal κ satisfying $2^\kappa > \kappa^+$, a proposition that Kunen had shown to be substantially stronger than measurability. Forcing over the model constructed by Silver with Prikry forcing yielded the first counterexample to the Singular Cardinals Hypothesis by providing a singular strong limit cardinal κ satisfying $2^\kappa > \kappa^+$.

To establish his result, Silver provided a technique for extending elementary embeddings into generic extensions and thereby preserving large cardinal properties. To get at what is at issue, suppose that $j : V \to M$ is an elementary embedding, P is a notion of forcing, and G is V-generic for P. To extend (or "lift") j to an elementary embedding for $V[G]$, the natural scheme would be to get a M-generic G' for $j(P)$ and extend j to an elementary embedding from $V[G]$ into $M[G']$. But for this to work with the forcing terms, it would be necessary to enforce

$$(*) \qquad \forall p \in G\,(j(p) \in G').$$

For getting a measurable cardinal κ satisfying $2^\kappa = \kappa^{++}$, Silver started with an elementary embedding as above with critical point κ and devised a P for adjoining κ^{++} Cohen subsets of κ. In order to establish a close connection between P and $j(P)$ toward securing $(*)$, he took P to be a uniform iteration of forcings to adjoin λ^{++} Cohen subsets of λ for every inaccessible cardinal λ up to and including κ itself. Then with the shift from κ to $j(\kappa)$, $j(P)$ can be considered a two-stage iteration of P followed by a further iteration Q. Now with G V-generic for P, G is also M-generic for P, and in $M[G]$ one should devise an H $M[G]$-generic for Q such that the combined generic $G' = G * H$ satisfies $(*)$.

But how is this to be arranged? Silver was able to control the $j(p)$'s for $p \in G$ by a single, *(strong) master condition* $q \in Q$, and build in $V[G]$ an H $M[G]$-generic over Q with $q \in H$ to satisfy $(*)$. For getting both q and H, he needed that M be closed under arbitrary κ^{++}-sequences. Thus he established: *If κ is κ^{++}-supercompact, then there is a forcing extension in which κ is measurable and $2^\kappa = \kappa^{++}$.* (To mention an elegant coda, work of Woodin and Gitik in the 1980s showed that having a measurable cardinal satisfying $2^\kappa > \kappa^+$ is equi-consistent with having a κ with $o(\kappa) = \kappa^{++}$ in the Mitchell order.) Silver's preparatory "reversed Easton" forcing with Easton support and master condition constructions of generic filters would become staple ingredients for the generic extension of elementary embeddings.

What about the use of very strong hypotheses in consistency results? A signal, 1972 result of Kunen brought into play the strongest hypothesis to that date for establish a consistency result about the low levels of the cumulative hierarchy. Earlier, Kunen had established that having a κ-complete κ^+-saturated ideal over a successor cardinal κ had consistency strength stronger than having a measurable cardinal. Kunen now showed: *If κ is huge, then there is forcing extension in which $\kappa = \omega_1$ and there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1.* With a $j : V \to M$ with critical point κ, $\lambda = j(\kappa)$, and ${}^\lambda M \subseteq M$ as given by the hugeness of κ, Kunen collapsed κ to ω_1 and followed it was a collapse of λ to ω_2 in such a way so as to be able to define a saturated ideal. Crucially, the first collapse was a "universal" collapse P iteratively constructed so that the second collapse can be absorbed into $j(P)$ in a way consistent with j applied to P, and this required ${}^\lambda M \subseteq M$. Hence, a sufficient algebraic argument was contingent on a closure property for an elementary embedding, one plucked from the emerging large cardinal hierarchy. In the years to come, Kunen's argument would be elaborated and emended to become the main technique for getting various sorts of saturated ideals over accessible cardinals. As for the proposition that there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 itself, Kunen's result set an initial, high bar for the stalking of its consistency strength, but definitive work of the 1980s would show that far less than hugeness suffices.

4.2. Determinacy

The investigation of the determinacy of infinite games is the most distinctive and intriguing development of modern set theory, and the correlations eventually achieved with large cardinals the most remarkable and synthetic. Notably, the mathematics of games first came to the attention of pioneers of set theory as an application of the emerging subject. Zermelo in a 1913 note discussed chess and worked with the concepts of *winning strategy* and *determined game*, and Kőnig in the paper that introduced his well-known tree lemma extended Zermelo's work to games with infinitely many positions. Von Neumann, lauding set-theoretic formulation, established the crucial minimax theorem, the result that really began the mathematical theory of games, and by the mid-1940s he and Oskar Morgenstern had codified the theory and its analysis of economic behavior, stimulating research for decades to come.

The investigation of infinitely long games that can be cast in a simple, abstract way would draw game-theoretic initiatives back into set theory. For a set X and $A \subseteq {}^\omega X$, let $G_X(A)$ denote the following "infinite two-person game with perfect information": There are two players, *I* and *II*. *I* initially chooses an $x(0) \in X$; then II chooses an $x(1) \in X$; then I chooses an $x(2) \in X$; then II chooses an $x(3) \in X$; and so forth:

$$\begin{array}{llllll} I: & x(0) & & x(2) & & \ldots \\ II: & & x(1) & & x(3) & \ldots \end{array}$$

Each player before making each of his moves is privy to the sequence of previous moves ("perfect information"); and the players together specify an $x \in {}^{\omega}X$. I wins $G_X(A)$ if $x \in A$, and otherwise II wins. A *strategy* is a function that tells a player what move to make given the sequence of previous moves. A *winning strategy* is a strategy such that if a player plays according to it he always wins no matter what his opponent plays. A is *determined* if either I or II has a winning strategy in $G_X(A)$.

David Gale and James Stewart in 1953 initiated the study of these games and observed that *if $A \subseteq {}^{\omega}X$ is open (in the product topology) then A is determined.* The simple argument turned on how membership is secured at a finite stage, and a basic stratagem in the further investigations of determinacy would be the reduction to such "open games". Focusing on the basic case $X = \omega$ and noting that a strategy then can itself be construed as a real, Gale and Stewart showed by diagonalizing through all strategies that *assuming* AC *there is an undetermined $A \subseteq {}^{\omega}\omega$.* Determinacy itself would come to be regarded as a regularity property, but there were basic difficulties from the beginning. Gale and Stewart asked whether all Borel sets of reals are determined, and in the decade that followed only sets very low in the Borel hierarchy were shown to be determined.

Infinitely long games involving reals had been considered as early as in the 1920s by mathematicians of the Polish school. With renewed interest in the subject in the 1950s, and with determinacy increasingly seen to be potent in its consequences, Jan Mycielski and Hugo Steinhaus in 1962 proposed the following axiom, now known as the *Axiom of Determinacy* (AD):

$$\text{Every } A \subseteq {}^{\omega}\omega \text{ is determined.}$$

With AD contradicting AC they proposed from the beginning that in the ZFC context the axiom should hold in some inner model. Solovay pointed out that the natural candidate $L(\mathbb{R})$, the constructible closure of the reals $\mathbb{R} = {}^{\omega}\omega$, observing that if AD holds then $\mathrm{AD}^{L(\mathbb{R})}$, i.e. AD holds in $L(\mathbb{R})$. Further restricted hypotheses would soon be applied to the tasks at hand: Projective Determinacy (PD) asserts that every projective $A \subseteq {}^{\omega}\omega$ is determined; $\mathbf{\Sigma}^1_n$-determinacy, that every $\mathbf{\Sigma}^1_n$ set A is determined; and so forth.

By 1964, games to specific purposes had been devised to show that for $A \subseteq {}^{\omega}\omega$ there is a closely related $B \subseteq {}^{\omega}\omega$ (a continuous preimage) so that if B is determined, then A is Lebesgue measurable, and similarly for the Baire property and the perfect set property. Moreover, AD does imply a limited choice principle, *every countable set consisting of sets of reals has a choice function.* Thus, the groundwork was laid for the reign of AD in $L(\mathbb{R})$ to enforce the regularity properties for all sets of reals there as well as a local choice principle, with unfettered uses of AC relegated to the universe at large.

In 1967 two results drew determinacy to the foreground of set theory, one about the transfinite and the other about definable sets of reals. Solovay established that AD *implies that* ω_1 *is measurable*, injecting emerging large cardinal techniques into a novel setting without AC. David Blackwell provided a new proof via the determinacy of open games of a classical result of Kuratowski that *the* $\mathbf{\Pi}^1_1$ *sets have the reduction property.* These results stimulated interest because of their immediacy and new approach to proof, that of devising a game and appealing to the existence of winning strategies to deduce a dichotomy. Martin in particular saw the potentialities at hand and soon made incisive contributions to investigations with and of determinacy. He initially made a simple but crucial observation based on the construal of strategies as reals that would have myriad applications; he showed that *under* AD *the filter over the Turing degrees generated by the cones is an ultrafilter.*

After seeing Blackwell's argument, Martin and Addison quickly and independently came to the idea of *assuming* determinacy hypotheses and pointed out that $\mathbf{\Delta}^1_2$-determinacy implies that $\mathbf{\Sigma}^1_3$ sets have the reduction property. Then Martin and Yiannis Moschovakis independently in 1968 extended the reduction property through the projective hierarchy by playing games and assuming PD, realizing a methodological goal of the classical descriptive set theorists by carrying out an inductive propagation. This was Martin's initial application of his ultrafilter on Turing cones, and the idea of ranking ordinal-valued functions via ultrafilters, so crucial in later arguments, first occurred here.

Already in 1964 Moschovakis had abstracted a property stronger and more intrinsic than reduction, the prewellordering property, from the classical analysis of $\mathbf{\Pi}^1_1$ sets. A relation $\preceq$ is a *prewellordering* if it is a well-ordering except possibly that there could be distinct x and y such that $x \preceq y$ and $y \preceq x$. While a well-ordering of a set A corresponds to a bijection of A into an ordinal, a prewellordering corresponds to a surjection onto an ordinal—a stratification of A into well-ordered layers. A class $\mathbf{\Gamma}$ of sets of reals has the *prewellordering property* if for any $A \in \mathbf{\Gamma}$ there is a prewellordering of A such that both it and its complement are in $\mathbf{\Gamma}$ in a strong sense. This property supplanted the reduction property in the Martin-Moschovakis First Periodicity Theorem, which implied that under PD the prewellordering property holds periodically for the projective classes: $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_3$, $\mathbf{\Sigma}^1_4, \ldots$.

As for Solovay's result, he in fact established that *under* AD *the closed unbounded filter* $\mathcal{C}_{\omega_1}$ *is an ultrafilter* by using a game played with countable ordinals and simulating it with reals. Martin provided an alternate proof using his ultrafilter on Turing cones, and then Solovay in 1968 used Martin's approach to establish that *under* AD ω_2 *is measurable.* With an apparent trend set, quite unexpected was the next advance. Martin in 1970 established that under AD the ω_n's for $3 \leq n < \omega$ are all singular with cofinality ω_2! This was a by-product of Martin's incisive analysis of $\mathbf{\Sigma}^1_3$ sets under AD.

Martin and Solovay had by 1969 established results about the $\mathbf{\Sigma}^1_3$ sets assuming $a^\#$ exists for every $a \in {}^\omega\omega$, and Martin went on to make explicit a "Martin-Solovay" tree representation for $\mathbf{\Sigma}^1_3$ sets. Just as Shoenfield had dualized the classical tree representation of $\mathbf{\Pi}^1_1$ sets by reconstruing well-foundedness as having order-preserving ranking functions, so too Martin was able to dualize the Shoenfield tree. For this he used the existence of sharps to order ordinal-valued functions and secure important *homogeneity* properties to establish that *if $a^\#$ exists for every $a \in {}^\omega\omega$, then every $\mathbf{\Sigma}^1_3$ set is ω_2-Suslin.* This careful analysis with indiscernibles led to the aforementioned singularity of the ω_n's for $3 \leq n < \omega$ under AD.

Martin also reactivated the earlier project of securing more and more determinacy by establishing that if there is a measurable cardinal, then $\mathbf{\Pi}^1_1$-determinacy holds, or in refined terms, *if $a^\#$ exists, then $\Pi^1_1(a)$-determinacy holds.* The proof featured a remarkably simple reduction to an open game, based on indiscernibles and homogeneity properties, of form $G_X(A)$ for a set X of ordinals. This ground-breaking proof served both to make plausible the possibility of getting PD from large cardinals as well as getting $\mathbf{\Delta}^1_1$-determinacy, Borel Determinacy, in ZFC—both directions to be met with complete success in later years.

The next advance would be by way of what would become the central structural concept in the investigation of the projective sets under determinacy. The classical issue of uniformization had been left unaddressed by the prewellordering property, and so Moschovakis in 1971 isolated a strengthening abstracted from the proof of the classical, Kondô $\mathbf{\Pi}^1_1$ Uniformization Theorem. A *scale* on a set $A \subseteq {}^\omega\omega$ is an ω-sequence of ordinal-valued functions on A satisfying certain convergence and continuity properties, and a class $\mathbf{\Gamma}$ of sets of reals has the *scale property* if for any $A \in \mathbf{\Gamma}$ there is a scale on A whose corresponding graph relations are in $\mathbf{\Gamma}$ in a strong sense. Having a scale on A corresponds to having $A = p[T]$ for a tree T in such a way that, importantly, from A is definable a member of A through a minimization process ("choosing the honest leftmost branch").

Instead of carrying out a tree dualizing procedure directly à la Shoenfield and Martin-Solovay, Moschovakis used a game argument to establish the Second Periodicity Theorem, which implied that under PD the scale property, and therefore uniformization, holds for the same projective classes as for prewellordering: $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_3$, $\mathbf{\Sigma}^1_4, \ldots$.

In the early 1970s Moschovakis, Martin, and Alexander Kechris proceeded with scales to provide a detailed analysis of the projective sets under PD in terms of Borel sets and as projections of trees, based on the *projective ordinals* $\boldsymbol{\delta}^1_n (= \underset{\sim}{\delta}^1_n)$ = the supremum of the lengths of the $\mathbf{\Delta}^1_n$ prewellorderings. For example, the $\mathbf{\Sigma}^1_{2n+2}$ sets are exactly the $\boldsymbol{\delta}^1_{2n+1}$-Suslin sets. The further analysis would be based on Moschovakis's Coding Lemma, which with determinacy provides for an arbitrary set meeting the layers of a prewellordering an appropriately definable subset meeting those same layers, and his Third Periodicity Theorem, which with determinacy asserts that when winning strategies exist there are appropriately definable such strategies. The projective ordinals themselves were subjected to considerable scrutiny, with penetrating work of Kunen particularly advancing the theory, and were found to be measurable and to satisfy strong partition properties. However, where exactly the $\boldsymbol{\delta}^1_n$ for $n \geq 5$ are in the aleph hierarchy would remain a mystery until the latter 1980s, when Steve Jackson in a *tour de force* settled the question with a deep analysis of the ultrafilters and partition properties involved. As an otherwise complete structure theory for projective sets was being worked out into the 1970s, Martin in 1974 returned to a bedrock issue for the regularity properties and established in ZFC that $\boldsymbol{\Delta}^1_1$-determinacy, Borel Determinacy, holds.

4.3. Silver's Theorem and Covering

In mid-1974 Silver established that *if κ is a singular cardinal with* $\mathrm{cf}(\kappa) > \omega$ *and $2^\lambda = \lambda^+$ for $\lambda < \kappa$, then $2^\kappa = \kappa^+$.* This was a dramatic event and would stimulate dramatic developments. There had been precious little in the way of results provable in ZFC about cardinal arithmetic, and in the early ruminations about the singular cardinals problem it was quite unforeseen that the power of a singular cardinal can be so constrained. An analogous preservation result had been observed by Scott for measurable cardinals, and telling was that Silver used large-cardinal ideas connected with generic ultrapowers.

Silver's result spurred broad-ranging investigations both into the combinatorics and avenue of proof and into larger, structural implications. The basis of his argument was a ranking of ordinal-valued functions on $\mathrm{cf}(\kappa)$. Let $\langle \gamma_\alpha \mid \alpha < \mathrm{cf}(\kappa) \rangle$ be a sequence of ordinals unbounded in κ and for $\alpha < \mathrm{cf}(\kappa)$ let $\tau_\alpha : P(\gamma_\alpha) \to 2^{\gamma_\alpha}$ be a bijection. For $X \subseteq \kappa$ let f_X on $\mathrm{cf}(\kappa)$ be defined by: $f_X(\alpha) = \tau_\alpha(X \cap \gamma_\alpha)$, noting that $X_1 \neq X_2$ implies f_{X_1} and f_{X_2} differ for sufficiently large α. Then 2^κ is mirrored through these eventually different functions, which one can work to order according to an ideal over the uncountable $\mathrm{cf}(\kappa)$. The combinatorial possibilities of such rankings led to a series of limitative results on the powers of singular cardinals of uncountable cofinality, starting with the results of Fred Galvin and Hajnal, of which the paradigmatic example is that if $\aleph_{\omega_1}$ is a strong limit cardinal, then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

In the wake of Silver's proof, Jech and Prikry defined a κ-complete ideal over κ to be *precipitous iff* the corresponding generic ultrapower à la Solovay is well-founded. They thus put the focus on a structural property of saturated ideals that Silver had simulated to such good effect. Jech and Prikry pointed out that a proof of Kunen's for saturated ideals using iterated ultrapowers can be tailored to show: *If there is a precipitous ideal over κ, then κ is measurable in an inner model.* Then Mitchell showed: *If a measurable cardinal is Levy collapsed to ω_1, then there is a precipitous ideal over ω_1.* Hence, a first equi-consistency result was achieved for measurability and ω_1.

With combinatorial characterizations of precipitousness soon in place, well-foundedness as thus modulated by forcing became a basic ingredient in a large-scale investigation of strong properties tailored to ideals and generic elementary embeddings.

The most dramatic and penetrating development from Silver's Theorem was Jensen's work on covering for L and its first extensions, the most prominent advances of the 1970s in set theory. Jensen had found Silver's result a "shocking discovery", and was stimulated to intense activity. By the end of 1974 he had made prodigious progress, solving the singular cardinals problem in the absence of $0^\#$ in three manuscripts, "Marginalia to a Theorem of Silver" and its two sequels. The culminating result featured an elegant and focal formulation of intuitive immediacy, the Covering Theorem (or "Lemma") for L: *If $0^\#$ does not exist, then for any uncountable set X of ordinals there is a $Y \in L$ with $|Y| = |X|$ such that $Y \supseteq X$.* (Without the "uncountable" there would be a counterexample using "Namba forcing".) This covering property expresses a global affinity between V and L, and its contrapositive provides a surprisingly simple condition sufficient for the existence of $0^\#$ and the ensuing indiscernible generation of L. As such, Jensen's theorem would find wide applications for implicating $0^\#$ and would provide a new initiative in inner model theory for encompassing stronger hypotheses.

The Covering Theorem gave the essence of Jensen's argument that in the absence of $0^\#$ the Singular Cardinals Hypotheses holds: Suppose that κ is singular and for reckoning with the powers of smaller cardinals consider $\lambda = \sup\{2^\mu \mid \mu < \kappa\}$. If there is a $\nu < \kappa$ such that $\lambda = 2^\nu$, then the functions f_X defined as above adapted to the present situation satisfy $f_X : \mathrm{cf}(\kappa) \to 2^\nu$, and so $\lambda \leq 2^\kappa \leq (2^\nu)^{\mathrm{cf}(\kappa)} \leq \lambda$. If on the other hand λ is the strict supremum of increasing 2^μ's, then $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$ and so the Zermelo-Kőnig inequality would dictate the least possibility for 2^κ to be λ^+. However, if for any $X \subseteq \kappa$ the range of f_X is covered by a $Y \subseteq \lambda$ with $Y \in L$ of cardinality $\mathrm{cf}(\kappa) \cdot \aleph_1$, then: there are $2^{\mathrm{cf}(\kappa)\cdot\aleph_1}$ subsets of each such Y and by the GCH in L, at most $|\lambda^{+L}|$ such Y. Hence, we would have $2^\kappa \leq 2^{\mathrm{cf}(\kappa)\cdot\aleph_1} \cdot |\lambda^{+L}| \leq \lambda^+$.

The Covering Theorem also provided another dividend that would grow in separate significance as having *weak covering property*: *Assume that $0^\#$ does not exist. If κ is singular, then $\kappa^{+L} = \kappa^+$.* If to the contrary $\kappa^{+L} < \kappa^+$, then $\mathrm{cf}(\kappa^{+L}) < \kappa$. Let $X \subseteq \kappa^{+L}$ be unbounded so that $|X| < \kappa$ and let $Y \in L$ cover X with $|Y| = |X| \cdot \aleph_1$. But then, the order-type of Y would be less than κ, contradicting the regularity of κ^{+L} in L.

A crucial consequence of weak covering is that *in the absence of $0^\#$, $\square_\kappa$ holds for singular κ*, since a $\square_\kappa$ sequence in the sense of L is then a $\square_\kappa$ sequence in V. The weak covering property would itself become pivotal in the study of inner models corresponding to stronger and stronger hypotheses, and the failure of $\square_\kappa$ for singular κ would become a delimitative proposition. Solovay had already established an upper bound on consistency by showing in the early 1970s that *if κ is λ^+-supercompact and $\lambda \geq \kappa$, then $\square_\lambda$ fails.*

Jensen's ingenious proof of the Covering Theorem for L proceeded by taking a counterexample X to covering with $\tau = \sup(X)$ and $|X|$ minimal; getting a certain Σ_1-elementary $j : J_\gamma \to J_\tau$ which contains X in its range through a Skolem hull construction so that $|\gamma| = |X|$ and, as X cannot be covered, γ is a cardinal in L; and extending j to an elementary embedding from L into L, so that $0^\#$ exists. The procedure for extending j up to some large J_δ was to consider a directed system of embeddings of structures generated by $\xi \cup p$ for some $\xi < \gamma$ and p a finite subset of J_δ, the transitized components of the system all being members of J_γ as γ is a cardinal in L, and to consider the corresponding directed system consisting of the j images. The choice of γ insured that the new directed system is also well-founded, and so isomorphic to some J_ζ. For effecting embedding extendibility, Jensen established the fine structural *Upward Extension of Embeddings Lemma*, according to which if N is the projectum structure for J_α and a Σ_1-elementary $e : N \to M$ is *strong* in that it preserves the well-foundedness of Σ_1 relations, then M itself is the projectum structure of some unique J_β and e can be extended uniquely to a Σ_1-elementary $\overline{e} : J_\alpha \to J_\beta$.

How can the proof of the Covering Theorem be adapted to establish a stronger result? The only possibility was to consider a larger inner model M and to establish that M has the *covering property*: for any uncountable set X of ordinals there is a $Y \in M$ with $|Y| = |X|$ such that $Y \supseteq X$. In groundbreaking work for inner model theory, Solovay in the early 1970s had developed a fine structure theory for inner models of measurability. Whilst a research student at Oxford University Anthony Dodd worked through this theory, and in early 1976 he and Jensen laid out the main ideas for extending the Covering Theorem to a new inner model, now known as the *Dodd-Jensen core model*, denoted K^{DJ}.

If $\langle L[U], \in, U\rangle$ is an inner model of measurability, say the κ-model, then there is a generic extension in which covering fails: If G is Prikry generic for U over $L[U]$, then G cannot be covered by any set in $L[U]$ of cardinality less than κ. Drawing back, there remains the possibility of "iterating out" the measurable cardinal: If $\langle L[U], \in, U\rangle$ is the κ-model, then $\langle L[W], \in, W\rangle$ is the λ-model for some $\lambda > \kappa$ exactly when it is an iterate of $\langle L[U], \in, U\rangle$, in which case $L[W] \subseteq L[U]$, $V_\kappa \cap L[U] = V_\kappa \cap L[W]$, and $U \notin L[W]$. Thus, if $\langle L[U_\alpha] \mid \alpha \in \mathrm{On}\rangle$ enumerates the inner models of measurability, then starting with any one of them, the process of iterating it through the ordinals converges to a proper class $\bigcap_\alpha L[U_\alpha]$ which has no inner models of measurability, with the stabilizing feature that for any γ, $V_\gamma \cap \bigcap_\alpha L[U_\alpha] = V_\gamma \cap L[U_\beta]$ for sufficiently large β. Assuming that there are inner models of measurability, K^{DJ} is in fact characterizable as this residue class. Aspiring to this, but without making any such assumption, Dodd and Jensen provided a formulation of K^{DJ} in ZFC.

K^{DJ} was the first inner model of ZFC since Gödel's L developed using distinctly new generating principles. Dodd and Jensen's approach was to take K^{DJ} as the union of L together with "mice". Loosely speaking, a *mouse* is a set $L_\alpha[U]$ such that

$$\langle L_\alpha[U], \in, U\rangle \models U \text{ is a normal ultrafilter over } \kappa$$

satisfying: (i) there is a subset of κ in $L_{\alpha+1}[U] - L_\alpha[U]$, so that U is on the verge of not being an ultrafilter; (ii) $\langle L_\alpha[U], \in, U\rangle$ is iterable in that all the iterated ultrapowers are well-founded; and (iii) fine structure conditions about a projectum below κ leading to (i). Mice can be compared by taking iterated ultrapowers, so that there is a natural prewellordering of mice, and moreover, crucial elements about L can be lifted to the new situation because there is a generalization of condensation: Σ_1-elementary substructures of mice, when transitized, are again mice. This led to $K^{\mathrm{DJ}} \models \mathrm{GCH}$, and that K^{DJ} in the sense of K^{DJ} is again K^{DJ}.

Mice generate indiscernibles through iteration, and so if $0^\#$ does not exist, then $K^{\mathrm{DJ}} = L$; if $0^\#$ exists but $0^{\#\#}$ does not, then $K^{\mathrm{DJ}} = L[0^\#]$; and this continues through the transfinite by coding sequences of sharps. On the other hand, K^{DJ} has no simple constructive analysis from below and is rather like a maximal inner model on the brink of measurability: Its own "sharp", that there is an elementary embedding $j : K \to K$, is equivalent to the existence of an inner model of measurability. Indeed, this was Dodd and Jensen's primary motivation for the formulation of K^{DJ}. They used it in place of the elementary embedding characterization of the existence of $0^\#$, together with the L-like properties of K^{DJ}, to establish the *Covering Theorem for* K^{DJ}*: If there is no inner model of measurability, then* K^{DJ} *has the covering property.* This has the attendant consequences for the singular cardinals problem. Moreover, Dodd and Jensen were able to establish a covering result for inner models of measurability that accommodates Prikry forcing. Solovay had devised a set of integers $0^\dagger$ ("zero dagger"), analogous to $0^\#$, such that $0^\dagger$ exists exactly when for some κ-model $L[U]$ there is an elementary embedding $j : L[U] \to L[U]$ with critical point above κ. Dodd and Jensen established: *If* $0^\dagger$ *does not exist yet there is an inner model of measurability, then for the* κ*-model* $L[U]$ *with* κ *least, either* (a) $L[U]$ *has the covering property, or* (b) *there is a Prikry generic* G *for* U *over* $L[U]$ *such that* $L[U][G]$ *has the covering property.* Prikry forcing provides the only counterexample to covering! Hence, the inner models thus far considered were also "core models", models on the brink so that the lack of covering leads to the next large cardinal hypothesis.

In the light of the Dodd-Jensen work, Mitchell in the later 1970s developed the core model $K[\mathcal{U}]$ for coherent sequences $\mathcal{U}$ of ultrafilters, which corresponds to his $L[\mathcal{U}]$ as K^{DJ} does to $L[U]$. The mice are now sets of form $J_\alpha[W]$ with iterability and fine structure properties, where W is an ultrafilter sequence with $\mathcal{U}$ as an initial segment. Under the assumption that there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, Mitchell established the weak covering property for $K[\mathcal{U}]$, i.e. that $(\kappa^+)^{K[\mathcal{U}]} = \kappa^+$ for singular κ. With this he showed that several propositions have at least the consistency strength of $\exists\kappa(o(\kappa) = \kappa^{++})$. One such proposition was that there is an $\aleph_1$-complete

$\aleph_2$-saturated ideal over ω_1, establishing a new lower bound in consistency strength for Kunen's consistency result from a huge cardinal. Mitchell eventually established, generalizing the Dodd-Jensen result with Prikry generic sets, a full covering theorem for $K[\mathcal{U}]$ cast in terms of coherent systems of indiscernibles and drew further conclusions about singular cardinals.

4.4. Forcing Consistency Results

Through the 1970s a wide range of forcing consistency results were established at a new level of sophistication that clarified relationships among combinatorial propositions and principles and often drew in large cardinal hypotheses and stimulated the development of method, especially in iterated forcing. A conspicuous series of results resolved questions of larger mathematics (Whitehead's Problem, Borel's Conjecture, Kaplansky's Conjecture, the Normal Moore Space Problem) in terms of relative consistency and set-theoretic principles, newly affirming the efficacy and adjudicatory character of set theory. In what follows, as we have begun to already, we pursue the larger longitudinal themes and results, necessarily saying less and less about matters of increasing complexity.

Much of the early formative work on strong large cardinal hypotheses and their integration into modern set theory through consistency results was carried out by Menachem Magidor, whose subsequent, broad-ranging initiatives have considerably advanced the entire subject. After completing his Hebrew University thesis in 1972 on supercompact cardinals, Magidor in the 1970s established a series of penetrating forcing consistency results involving strong hypotheses. In 1972–1973 he illuminated the "identity crisis" issue of whether supercompactness and strong compactness are distinct concepts by establishing: (1) *It is consistent that the least supercompact cardinal is also the least strong compact cardinal*, and (2) *It is consistent that the least strong compact cardinal is the least measurable cardinal (and so much smaller than the least supercompact cardinal).* The proofs showed how changing many cofinalities with Prikry forcing to destroy measurable cardinals can be integrated into arguments about extending elementary embeddings.

In 1974 Magidor made a basic contribution to the theory of changing cofinalities, the first after Prikry. Magidor established: *If a measurable cardinal κ is of Mitchell order $o(\kappa) \geq \lambda$ for a regular $\lambda < \kappa$, then there is a forcing extension preserving cardinals in which* $\mathrm{cf}(\kappa) = \lambda$. Generalizing Prikry forcing, Magidor's conditions consisted of a finite sequence of ordinals and a sequence of sets drawn from normal ultrafilters in the Mitchell order, the sets providing for the possible ways of filling out the sequence. Like Prikry's forcing, Magidor's may at first have seemed a curious possibility for a new singularization. However, one of the subsequent discernments of Mitchell's core model for coherent sequences of measures is that, remarkably: *If a regular cardinal κ in V satisfies $\omega < \mathrm{cf}(\kappa) < \kappa$ in a generic extension, then V has an inner model in which $o(\kappa)$ is at least that cofinality.* Thus, the capability of changing cofinalities was exactly gauged; "Prikry-Magidor" generic sets as sequences of indiscernibles would become a basic component of Mitchell's covering work.

The most salient results of Magidor's of this period were two of 1976 that provided counterweight to Jensen's covering results on the singular cardinal problem. Magidor showed: (1) *If κ is supercompact, there is a forcing extension in which κ is $\aleph_\omega$ as a strong limit cardinal yet* $2^{\aleph_\omega} > \aleph_{\omega+1}$, and (2) *If κ is a huge cardinal, then there is a forcing extension in which $\kappa = \aleph_\omega$, $2^{\aleph_n} = \aleph_{n+1}$ for $n \in \omega$, yet $2^{\aleph_\omega} > \aleph_{\omega+1}$.* Thus, forcing arguments showed that the least singular cardinal can be a counterexample to the Singular Cardinals Hypothesis; the strong elementary embedding hypotheses allowed for an elaborated Prikry forcing interspersed with Levy collapses. The Prikry-Silver and the Magidor results showed through initial incursions of Prikry forcing how to arrange high powers for singular strong limit cardinals; it would be one of the great elaborations of method that equi-consistency results would eventually be achieved with weaker hypotheses.

With respect to the Jech-Prikry-Mitchell equi-consistency of measurability and precipitousness, Magidor showed that absorptive properties of the Levy collapse of a measurable cardinal to ω_1 can be exploited by subsequently "shooting" closed unbounded subsets of ω_1 through stationary sets to get: *If there is a measurable cardinal κ, then there is a forcing extension in which $\kappa = \omega_1$ and NS_{ω_1} is precipitous.* Thus a basic, definable ideal can be precipitous, and this naturally became a principal point of departure for the investigation of ideals.

The move of Saharon Shelah into set theory in the early 1970s brought in a new and exciting sense of personal initiative that swelled into an enhanced purposiveness across the subject, both through his solutions of major outstanding problems as well as through his development of new structural frameworks. A phenomenal mathematician, Shelah from his 1969 Hebrew University thesis on has worked in model theory and eventually infused it with a transformative, abstract classification theory for models. In both model theory and set theory he has remained eminent and has produced results at a furious pace, with nearly 1000 items currently in his bibliography (his papers are currently archived at http://shelah.logic.at/).

In set theory Shelah was initially stimulated by specific problems. He typically makes a direct, frontal attack, bringing to bear extraordinary powers of concentration, a remarkable ability for sustained effort, an enormous arsenal of accumulated techniques, and a fine, quick memory. When he is successful on the larger problems, it is often as if a resilient, broad-based edifice has been erected, the traditional serial constraints loosened in favor of a wide, fluid flow of ideas, and the final result almost incidental to the larger structure. *What* has been achieved is more than a just succinctly stated theorem but rather the erection of a whole network of robust arguments.

Shelah's written accounts have acquired a certain notoriety that in large part has to do with his insistence that his edifices be regarded as autonomous conceptual constructions. Their life is to be captured in the most general forms, and this entails the introduction of many parameters. Often, the network of arguments is articulated by complicated combinatorial principles and transient hypotheses, and the forward directions of the flow are rendered as elaborate transfinite inductions carrying along many side conditions. The ostensible goal of the construction, that succinctly stated result that is to encapsulate it, is often lost in a swirl of conclusions.

Shelah's first and very conspicuous advance in set theory was his 1973, definitive results on Whitehead's Problem in abelian group theory: Is every Whitehead group, an abelian group G satisfying $\mathrm{Ext}^1(G, \mathbb{Z}) = 0$, free? Shelah established that $V = L$ implies that this is so, and that Martin's Axiom implies that there is a counterexample. Shelah thus established for the first time that a strong purely algebraic statement is undecidable in ZFC. With his L result specifically based on diamond-type principles, Shelah brought them into prominence with his further work on them, which were his first incursions into iterated forcing. As if to continue to get his combinatorial bearings, Shelah successfully attacked several problems on an Erdős-Hajnal list for partition relations, developing in particular a "canonization" theory for singular cardinals. By the late 1970s his increasing understanding of and work in iterated forcing would put a firm spine on much of the variegated forcing arguments about the continuum.

With an innovative argument pivotal for iterated forcing, Richard Laver in 1976 established the consistency of Borel's conjecture: *Every set of reals of strong measure zero is countable.* CH had provided a counterexample, and Laver established the consistency with $2^{\aleph_0} = \aleph_2$. His argument featured the adjunction of what are now called *Laver reals* in the first clearly set out *countable support iteration*, i.e. an iteration with non-trivial local conditions allowed only at countably many coordinates. The earlier Solovay-Tennenbaum argument for the consistency of MA had relied on finite support, and a Mitchell argument about Aronszajn trees, on an involved countable support with a "termspace" forcing, which would also find use. Laver's work showed that countable support iteration is both manageable and efficacious for preserving certain framing properties of the continuum to establish the consistency of propositions with $2^{\aleph_0} = \aleph_2$. Interestingly, a trade-off would develop however: while finite support iterations put all cardinals $\geq \aleph_2$ on an equal footing with respect to the continuum, countable support iterations restricted the continuum to be at most $\aleph_2$. With a range of new generic reals coming into play with the widening investigation of the continuum, James Baumgartner formulated a property common to the corresponding partial orders, *Axiom A*, which in particular ensured the preservation of ω_1. He showed that the countable support iteration of Axiom A forcings is Axiom A, thereby uniformizing the iterative adjunction of the known generic reals.

All this would retrospectively have a precursory air, as Shelah soon established a general, subsuming framework. Analyzing Jensen's consistency argument for SH + CH and coming to grips with forcing names in iterated forcing, Shelah came to the concept of *proper forcing* as a general property that preserves ω_1 and is preserved in countable support iterations. The instrumental formulation of properness is given in an appropriately broad setting:

First, for a regular cardinal λ, let $H(\lambda) = \{x \mid |\mathrm{tc}(\{x\})| < \lambda\}$, the sets hereditarily of cardinality less than λ. The $H(\lambda)$'s provide another cumulative hierarchy for V, one stratified into layers that each satisfy Replacement; whereas the V_α's for limit α satisfy every ZFC axiom except possibly Replacement, the $H(\lambda)$'s satisfy every ZFC axiom except possibly Power Set. A partial order $\langle P, < \rangle$ is *proper* if for any regular $\lambda > 2^{|P|}$ and countable $M \prec H(\lambda)$ with $P \in M$, every $p \in P \cap M$ has a $q \leq p$ such that $q \Vdash \dot{G} \cap M$ is M-generic. (Here, $\dot{G}$ a canonical name for a generic filter with respect to P, and q forcing this genericity assertion has various combinatorial equivalents.)

A general articulation of how all countable approximations are to have generic filters has been achieved, and its countable support iteration exhibited the efficacy of this remarkable move to a new plateau. Shelah soon devised variants and augmentations, and in a timely 1982 monograph *Proper Forcing* revamped forcing for combinatorics and the continuum with systemic proofs of new and old results. Proper forcing, presented in Chap. 5 of this Handbook, has become a staple part of the methods of modern set theory, with its applications wide-ranging and the development of its extended theory a fount of research.

In light of Shelah's work and Martin's Axiom, Baumgartner in the early 1980s established the consistency of a new encompassing forcing axiom, the *Proper Forcing Axiom* (PFA): *For any proper partial order P and collection $\mathcal{D}$ of $\aleph_1$ dense subsets of P, there is a filter $G \subseteq P$ meeting every member of $\mathcal{D}$.* Unlike MA, the consistency of PFA required large cardinal strength and moreover could not be achieved by iteratively taking care of the partial orders at issue, as new proper partial orders occur arbitrarily high in the cumulative hierarchy. Baumgartner established: *If there is a supercompact cardinal κ, then there is a forcing extension in which $\kappa = \omega_2$ and* PFA *holds.* In an early appeal to the full global reflection properties available at a supercompact cardinal Baumgartner iteratively took care of the emerging proper partial orders along a special diamond-like sequence that anticipates all possibilities. Laver first formulated this sequence, the "Laver diamond", toward establishing what has become a useful result for forcing theory; in a forcing extension he made a supercompact cardinal "indestructible" by any further forcing from a substantial, useful class of forcings. PFA became a widely applied forcing axiom, showcasing Shelah's concept, but beyond that, it would itself become a pivotal hypothesis in the large cardinal context.

Two points of mathematical practice should be mentioned in connection with Shelah's move into set theory. First, through his work with proper forcing it has become routine to appeal in proofs to structures $\langle H(\lambda), \in, <^*, \ldots \rangle$ for regular λ sufficiently large, with $<^*$ some well-ordering of $H(\lambda)$ and $\ldots$ including all the sets concerned. One then develops systems of elementary substructures generated uniformly by Skolem functions defined via $<^*$. This technique, in providing some of the structure available in L-like inner models, has proved highly efficacious over a wide range from combinatorics to large cardinals.

Second, several of a developing Israeli school in set theory have followed Shelah in writing "$p > q$" for p being a stronger condition than q instead of "$p < q$". The former is argued for as more natural, whereas the latter had been motivated structurally by Boolean algebras. This revisionism has no doubt led to confusion, until one realizes that it is a particular stamp of the Israeli school.

5. New Expansion

5.1. Into the 1980s

The 1980s featured a new and elaborating expansion in set theory significantly beyond the successes, already remarkable, of the previous decade. There were new methods and results of course, but more than that there were successful *maximizations* in several directions—definitive and evidently optimal results—and successful *articulations* at the interstices—new concepts and refinements that filled out the earlier explorations. A new wave of young researchers entered the fray, including the majority of the authors contributing to this Handbook, soon to become the prominent experts in their respective, newly variegated subfields. Our narrative now becomes even more episodic in increasingly inverse relation to the broad-ranging and penetrating developments, leaving accounts of details and some whole subjects to the chapter summaries at the end.

In 1977 Lon Radin toward his Berkeley thesis developed an ultimate generalization of the Prikry and Magidor forcings for changing cofinalities, a generalization that could in fact adjoin a closed unbounded subset, consisting of formerly regular cardinals, to a large cardinal κ while maintaining its regularity and further substantive properties. As graduate students at Berkeley, Hugh Woodin and Matthew Foreman saw the possibilities abounding in Radin forcing. While an undergraduate at Caltech Woodin did penetrating work on the consistency of Kaplansky's Conjecture (Is every homomorphism on the Banach algebra of continuous functions on the unit interval continuous?) and now with Radin forcing in hand would produce his first series of remarkable results. By 1979 Foreman and Woodin had the essentials for establishing: *If there is a supercompact cardinal κ, then there is forcing extension in which V_κ as a model of* ZFC *in which* GCH *fails everywhere, i.e. $2^\lambda > \lambda$ for every λ.* This conspicuously subsumed the Magidor result getting $\aleph_\omega$ a strong limit yet $2^{\aleph_\omega} > \aleph_{\omega+1}$ and put Radin forcing on the map for establishing global consistency results.

Shelah soon established two re-orienting results about powers of singular cardinals. Having come somewhat late into the game after Silver's Theorem, Shelah had nonetheless extended some of the limitative results about such powers, even to singular κ such that $\aleph_\kappa = \kappa$. Shelah subsequently established: *If there is a supercompact cardinal κ and α is a countable ordinal, then there is a forcing extension in which κ is $\aleph_\omega$ as a strong limit cardinal yet* $2^{\aleph_\omega} = \aleph_{\alpha+1}$. He thus extended Magidor's result by showing that the power of $\aleph_\omega$ can be made arbitrarily large below $\aleph_{\omega_1}$. In 1980 Shelah established the general result that *for any limit ordinal δ, $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$, so that in particular if $\aleph_\omega$ is a strong limit cardinal, then* $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$. Not only was he able to get an absolute upper bound in ZFC, but he had brought countable cofinality, the one cofinality unattended to by Silver's Theorem, into the scheme of things. Shelah's argument, based on the possible cofinalities of "reduced products" of a cofinal subset of $\aleph_\delta$, would evolve into a generally applicable method by the late 1980's, the remarkable *pcf theory.*

In 1978, Mitchell made a new breakthrough for the inner model theory of large large cardinals by developing such a model for "hypermeasurable cardinals", e.g. a measurable cardinal κ such that for some normal ultrafilter U over κ, $P(P(\kappa)) \subseteq \mathrm{Ult}(V, U)$, so that every ultrafilter over κ is in the ultrapower. This at least captured a substantial consequence of the 2^κ-supercompactness of κ, and so engendered the hope of getting L-like inner models for such strong hypotheses. Supercompactness, while increasingly relied on in relative consistency results owing to its reflection properties, was out of reach, but the Mitchell result suggested an appropriate weakening: A cardinal κ is *α-strong iff* there is an elementary embedding $j : V \to M$ for some inner model M, with critical point κ and $\alpha < j(\kappa)$ such that $V_\alpha \subseteq M$. (One can alternately require that the αth iterated power set $P^\alpha(\kappa)$ be a subset of M, which varies the definition for small α like $\alpha = 2$ but makes the definition more germane for them.) κ is *strong iff* it is α-strong for every α.

Dodd and Jensen soon simplified Mitchell's presentation in what turned out to be a basic methodological advance for the development of inner model theory. While introducing certain redundancies, they formulated a general way of analyzing an elementary embedding in terms of *extenders.* The idea, anticipated in Jensen's proof of the Covering Theorem, is that elementary embeddings between inner models can be approximated arbitrarily closely as direct limits of ultrapowers with concrete features reminiscent of iterated ultrapowers.

Suppose that N and M are inner models of ZFC, $j : N \to M$ is elementary with a critical point κ, and $\beta > \kappa$. Let $\zeta \geq \kappa$ be the least ordinal satisfying $\beta \leq j(\zeta)$; the simple ("short") case is $\zeta = \kappa$, and the general case is for the study of stronger hypotheses. For each finite subset a of β, define E_a by:

$$X \in E_a \quad \textit{iff} \quad X \in P([\zeta]^{|a|}) \cap N \ \wedge \ a \in j(X).$$

This is another version of generating ultrafilters from embeddings. E_a may not be in N, but $\langle N, \in, E_a \rangle \models$ "E_a is a κ-complete ultrafilter over $[\zeta]^{|a|}$". The (κ, β)-*extender derived from j* is $E = \langle E_a \mid a \text{ is a finite subset of } \beta \rangle$.

For each finite subset a of β, $\mathrm{Ult}(N, E_a)$ is seen to be elementarily embeddable into M, so that in particular $\mathrm{Ult}(N, E_a)$ is well-founded and hence identified with its transitive collapse, say M_a. Next, for $a \subseteq b$ both finite subsets of β, corresponding to how members of a sit in b there is a natural elementary embedding $i_{ab} : M_a \to M_b$. Finally,

$$\langle\langle M_a \mid a \text{ is a finite subset of } \beta\rangle, \langle i_{ab} \mid a \subseteq b\rangle\rangle$$

is seen to be a directed system of structures with commutative embeddings, so stipulate that $\langle M_E, \in_E\rangle$ is the direct limit, and let $j_E : N \to M_E$ be the corresponding elementary embedding. We thus have the *extender ultrapower* of N by E as a direct limit of ultrapowers. The crucial point is that the direct limit construction ensures that j_E and M_E approximate j and M "up to β", e.g. if $|V_\alpha|^M \leq \beta$, then $|V_\alpha|^M = |V_\alpha|^{M_E}$, i.e. the cumulative hierarchies of M and M_E agree up to α. Having formulated extenders derived from an embedding, a (κ, β)-*extender* is a sequence $E = \langle E_a \mid a \text{ is a finite subset of } \beta\rangle$ that satisfies various abstracted properties that enable the above construction.

In a manuscript circulated in 1980, Dodd and Jensen worked out inner models for strong cardinals. Building on the previous work of Mitchell, Dodd and Jensen formulated *coherent sequences of extenders*, built inner models relative to such, and established GCH in these models. The arguments were based on extending the established techniques of securing iterability and comparison through coiteration. The GCH result was significant as precursory for the further developments in inner model theory based on "iteration trees". Thus, with extenders the inner model theory was carried forward to encompass strong cardinals, newly arguing for the coherence and consistency of the concept. There would however be little further progress until 1985, for the aspiration to encompass stronger hypotheses had to overcome the problem of "overlapping extenders", having to carry out comparison through coiteration for local structures built on (κ_1, β_1)-extenders and (κ_2, β_2)-extenders with $\kappa_1 \leq \kappa_2 < \beta_1$. The difficulty here is one of "moving generators": if an extender ultrapower is taken with a (κ_1, β_1)-extender and then with a (κ_2, β_2)-extender, then $\kappa_2 < \beta_1$ implies that the generating features of the first extender ultrapower has been shifted by the second ultrapower and so one can no longer keep track of that ultrapower in the coiteration process. In any event, a crucial inheritance from this earlier work was the *Dodd-Jensen Lemma* about the minimality of iterations copied across embeddings, which would become crucial for all further work in inner model theory.

In the direction of combinatorics and the study of continuum, there was considerable elaboration in the 1970s and into the 1980s, particularly as these played into the burgeoning field of *set-theoretic topology.* Not only were there new elucidations and new transfinite topological examples, but large cardinals and even the Proper Forcing Axiom began to play substantial roles in new relative consistency results. The 1984 *Handbook of Set-Theoretic Topology* summed up the progress, and its many articles set the tone for further work.

In particular, Eric van Douwen's article provided an important service by standardizing notation for the cardinal characteristics, or invariants, of the continuum in terms of the lower case Fraktur letters. We have discussed the dominating number $\mathfrak{d}$, the least cardinality of a subset of Baire space cofinal under eventual dominance $<^*$. There is the *bounding number* $\mathfrak{b}$, the least cardinality of a subset of Baire space unbounded under eventual dominance $<^*$; there is the *almost disjoint number* $\mathfrak{a}$, the least cardinality of a subset of $P(\omega)$ consisting of infinite sets pairwise having finite intersection; there is a *splitting number* $\mathfrak{s}$, the least cardinality of a subset $S \subseteq P(\omega)$ such that any infinite subset of ω has infinite intersection with both a member of S and its complement; and, now, many more. The investigation of the possibilities for the cardinality characteristics and their ordering relations with each other would itself have sustained interest in the next decades, becoming a large theory to which both Chaps. 6 and 7 of this Handbook are devoted.

Conspicuous in combinatorics and topology would be the work of Stevo Todorcevic. Starting with his doctoral work with Kurepa in 1979 he carried out an incisive analysis of uncountable trees—Suslin, Aronszajn, Kurepa trees and variants—and their linearizations and isomorphism types. In 1983 he dramatically re-oriented the sense of strength for the Proper Forcing Axiom by showing that PFA *implies that* $\Box_\kappa$ *fails for every* $\kappa > \omega$. PFA had previously been shown consistent relative to the existence of a supercompact cardinal. With the failure of $\Box_\kappa$ for singular κ having been seen as having quite substantial consistency strength, PFA was itself seen for the first time as a very strong proposition. Todorcevic would go from strength to strength, making substantial contributions to the theory of partition relations, eventually establishing definitive results about ω_1 as the archetypal uncountable order-structure. His chapter in this Handbook presents that single-handedly developed combinatorial theory of sequences and walks.

Starting in 1980 Foreman made penetrating inroads into the possibilities for very strong propositions holding low in the cumulative hierarchy based on the workings of generic elementary embeddings. Extending Kunen's work and deploying Silver's master condition idea, Foreman initially used 2-huge cardinals to get model-theoretic transfer principles to hold and saturated ideals to exist among the range of $\aleph_n$'s. He would soon focus on generic elementary embeddings and corresponding ideals themselves, even making them postulational for set theory. This general area of research has become fruitful, multi-faceted, and enormous, as detailed in Foreman's chapter on this subject in this Handbook.

In a major 1984 collaboration in Jerusalem, Foreman, Magidor, and Shelah established penetrating results that led to a new understanding of strong propositions and the possibilities with forcing. The focus was on a new, maximal forcing axiom: A partial order P *preserves stationary subsets of* ω_1 *iff* stationary subsets of ω_1 remain stationary in any forcing extension by P, and with this we have *Martin's Maximum* (MM): *For any* P *preserving stationary subsets of* ω_1 *and collection* $\mathcal{D}$ *of* $\aleph_1$ *dense subsets of* P*, there is a filter* $G \subseteq P$ *meeting every member of* $\mathcal{D}$*.* This subsumes PFA and is a maximally strong forcing axiom in that there is a P which does not preserve stationary subsets of ω_1 for which the conclusion fails. Foreman, Magidor, and Shelah established: *If there is a supercompact cardinal κ, then there is a forcing extension in which $\kappa = \omega_2$ and* MM *holds.*

Shelah had considered a weakening of properness called *semiproperness*, a notion for forcing that could well render uncountable cofinalities countable. To iterate such forcings, it had to be faced that the countable cofinality of limit stages cannot be ascertained in advance, and so he developed *revised countable support iteration* (RCS) based on names for the limit stage indexing. Foreman, Magidor, and Shelah actually carried out Baumgartner's PFA consistency proof for semiproper forcings with RCS iteration to establish the consistency of the analogous *Semiproper Forcing Axiom* (SPFA). Their main advance was that, although a partial order that preserves stationary subsets of ω_1 is not necessarily semiproper, it is in this supercompact collapsing context. (Eventually, Shelah did establish that MM and SPFA are equivalent.)

Foreman, Magidor, and Shelah then established the relative consistency of several propositions by deriving them directly from MM. One such proposition was that NS_{ω_1} is $\aleph_2$-saturated. Hence, not only was the upper bound for the consistency strength of having an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 considerably reduced from Kunen's huge cardinal, but for the first time the consistency of NS_{ω_1} itself being $\aleph_2$-saturated was established relative to large cardinals. Another formative result was simply that MM implies that $2^{\aleph_0} = \aleph_2$, starting a train of thought about forcing axioms actually determining the continuum. It would be by different and elegant means that Todorcevic would show in 1990 that PFA already implies that $2^{\aleph_0} = \aleph_2$.

With their work Foreman, Magidor, and Shelah had overturned a long-held view about the scaling down of large cardinal properties. In the first flush of new hypotheses and propositions, Kunen had naturally enough collapsed a large cardinal to ω_1 in order to transmute strong properties of the cardinal into an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1, and this sort of direct connection had become the rule. The new discovery was that a collapse of a large cardinal to ω_2 instead can provide enough structure to secure such an ideal. In fact, Foreman, Magidor, and Shelah showed that even the usual Levy collapse of a supercompact cardinal to ω_2 engenders an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1. In terms of method, the central point is that the existence of sufficiently large cardinals implies the existence of substantial generic elementary embeddings with small critical points like ω_1. Woodin's later strengthenings and elaborations of these results would have far-reaching consequences.

5.2. Consistency of Determinacy

The developments of the 1980s which are the most far-reaching and presentable as sustained narrative have to do with the stalking of the consistency of determinacy. By the late 1970s a more or less complete structure theory for the projective sets was in place, a resilient edifice founded on determinacy with both strong buttresses and fine details. In 1976 the researchers had started the Cabal Seminar in the Los Angeles area, and in a few years, with John Steel and Woodin having joined the ranks, attention began to shift to sets of reals beyond the projective sets, to inner models, and to questions of overall consistency. Most of the work before the crowning achievements of the later 1980s appears in the several proceedings of the Cabal Seminar appearing in 1978, 1981, 1983, and 1988.

With the growing sophistication of methods, the inner model $L(\mathbb{R})$ increasingly became the stage for the play of determinacy, both as the domain to extend the structural consequences of AD and as the natural inner model for AD that can exhibit characterizations. Scales having held the key to the structure theory for the projective sets, Martin and Steel established a limiting case for the scale property; with the $\mathbf{\Sigma}^2_1$ sets of reals being those definable with one existential third-order quantifier, they showed that AD and $V = L(\mathbb{R})$ imply that $\mathbf{\Sigma}^2_1$ is the largest class with the scale property. Steel moreover developed a fine structure theory for $L(\mathbb{R})$, and analyzing the minimal complexity of scales there, he extended some of the structure theory under AD to sets of reals in $L(\mathbb{R})$. As for characterizations, Kechris and Woodin showed that in $L(\mathbb{R})$, AD is equivalent to the existence of many ("Suslin") cardinals that have strong partition properties. Woodin also established that in $L(\mathbb{R})$, AD is equivalent to *Turing Determinacy*, determinacy for only sets of reals closed under Turing equivalence.

The question of the overall consistency of determinacy came increasingly to the fore. Is AD consistent relative to some large cardinal hypothesis? Or, with its strong consequences, can AD subsume large cardinals in some substantial way or be somehow orthogonal? Almost a decade after his initial result that the existence of a measurable cardinal implies $\mathbf{\Pi}^1_1$-determinacy, Martin and others showed that determinacy for sets in the "difference hierarchy" built on the $\mathbf{\Pi}^1_1$ sets implies the existence of corresponding inner models with many measurable cardinals. Then in 1978 Martin, returning to the homogeneity idea of his early $\mathbf{\Pi}^1_1$ result, applied it with the Martin-Solovay tree representation for $\mathbf{\Pi}^1_2$ sets, together with algebraic properties of elementary embeddings posited close to Kunen's large cardinal inconsistency, to establish $\mathbf{\Pi}^1_2$-determinacy. A direction was set but generality only came in 1984, when Woodin showed that an even stronger large cardinal hypothesis implies $\mathrm{AD}^{L(\mathbb{R})}$. So, a mooring was secured for AD after all in the large cardinal hierarchy. With Woodin's hypothesis apparently too remote, it would now be a question of scaling it down according to the methods becoming available for proofs of determinacy, perhaps even achieving an equi-consistency result.

The rich 1984 Foreman-Magidor-Shelah work would have crucial consequences for the stalking of consistency also for determinacy. Shelah carried out a version of their collapsing argument that does not add any new reals but nonetheless gets an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1. Woodin then pointed out that with no new reals adjoined the generic elementary embedding induced by such an ideal can be used to establish that the ground model $L(\mathbb{R})$ reals are actually Lebesgue measurable. Thus Shelah and Woodin had established an outright result: *If there is a supercompact cardinal, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.* This result not only portended the possibility of getting $\mathrm{AD}^{L(\mathbb{R})}$ from a supercompact cardinal, but through the specifics of the argument stimulated the reducing of the hypothesis. While Woodin was visiting Jerusalem in June 1984, he came up with what is now known as a *Woodin cardinal.* The hypothesis was then reduced as follows: *If there are infinitely many Woodin cardinals with a measurable cardinal above them, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.* An early suggestion of optimality of hypothesis was that if the "infinitely" is replaced by "n" for some $n \in \omega$, then one can conclude that every $\mathbf{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable. The measurable cardinal hovering above would be a recurring theme, the purpose loosely speaking to maintain a stable environment with the existence of sharps.

Especially because of its subsequent centrality, it is incumbent to give an operative definition of Wooding cardinal: For a set A, κ is *α-A-strong iff* there is an elementary embedding $j : V \to M$ witnessing that κ is α-strong which moreover preserves A: $A \cap V_\alpha = j(A) \cap V_\alpha$. A cardinal δ is *Woodin iff* for any $A \subseteq V_\delta$, there is a $\kappa < \delta$ which is α-A-strong for every $\alpha < \delta$.

A Woodin cardinal, evidently a technical, consistency-wise strengthening of a strong cardinal, is an important example of concept formation through method. The initial air of contrivance gives way to seeing that Woodin cardinal seemed to encapsulate just wanted is needed to carry out the argument for Lebesgue measurability. That argument having been based on first collapsing a large cardinal to get a saturated ideal and then applying the corresponding generic elementary embedding, Woodin later in 1984 stalked the essence of method and formulated *stationary tower forcing.* An outgrowth of the Foreman-Magidor-Shelah work, this notion of forcing streamlines their forcing arguments to show that a Woodin cardinal suffices to get a generic elementary embedding $j : V \to M$ with critical point ω_1 and ${}^\omega M \subseteq M$. With a new, minimizing large cardinal concept isolated, there would now be dramatic new developments both in determinacy and inner model theory. One important scaling down result was the early 1985 result of Shelah: *If κ is Woodin, then in a forcing extension $\kappa = \omega_1$ and NS_{ω_1} is $\aleph_2$-saturated.* The large cardinal strength now seemed minimal for getting such an ideal, and there was anticipation of achieving an equi-consistency.

Steel in notes of Spring 1985 developed an inner model for a weak version of Woodin cardinal. While inner models for strong cardinals had only required linear iterations for comparison, the new possibility of overlapping extenders and moving generators had led Mitchell in 1979 to develop *iteration trees* of iterated ultrapowers for searching for possible well-founded limits of models along branches. A particularly simple example of an iteration tree is an *alternating chain*, a tree consisting of two ω-length branches with each model in the tree an extender ultrapower of the one preceding it on its branch, via an extender taken from a corresponding model in the other branch. Initially, Steel tried to avoid alternating chains, but the Foreman-Magidor-Shelah work showed that for dealing with Woodin cardinals they would be a necessary part. Their use soon led to a major breakthrough in the investigation of determinacy.

In the Fall of 1985 Martin and Steel showed that Woodin cardinals imply the existence of alternating chains in which both branches have well-founded direct limits, and used this to establish: *If there are infinitely many Woodin cardinals, then* PD *holds.* This was a culmination of method in several respects. In the earlier Martin results getting $\mathbf{\Pi}^1_1$-Determinacy and $\mathbf{\Pi}^1_2$-Determinacy, trees on $\omega \times \kappa$ for some cardinal κ had been used, to each node of which were attached ultrafilters in a coherent way that governed extensions. Kechris and Martin isolated the relevant concept of *homogeneous tree*, the point being that sets of reals which are the projections $p[T]$ of such trees T—the *homogeneously Suslin sets*—are determined. With PD, the scale property had been propagated through the projective hierarchy. Now with Woodin cardinals, having representations via homogeneous trees was propagated, getting determinacy itself. In particular, Martin and Steel established: *If $n \in \omega$ and there are n Woodin cardinals with a measurable cardinal above them, then $\mathbf{\Pi}^1_{n+1}$-determinacy holds.*

Within weeks after the Martin-Steel breakthrough, Woodin used it together with stationary towers to investigate tree representations in $L(\mathbb{R})$ to establish: *If there are infinitely Woodin cardinals with a measurable cardinal above them, then* $\mathrm{AD}^{L(\mathbb{R})}$ *holds.* With the consistency strength of AD having been gauged by this result, Woodin soon established the crowning equi-consistency result: *The existence of infinitely many Woodin cardinals is equi-consistent with the Axiom of Determinacy.* Both directions of this result, worked out with hindsight in Chaps. 22 and 23 of this Handbook, involve substantial new arguments.

This was a remarkable achievement of the concerted effort to establish the consistency strength of AD along the large cardinal hierarchy. But even this would just be a beginning for Woodin, who would go from strength to strength to establish many structural results involving AD and stronger principles, to become preeminent with Shelah in set theory.

5.3. Later Developments

We conclude our historical survey by describing here some prominent developments of the later 1980s and early 1990s, those in the broad directions of inner model theory and singular cardinal combinatorics to be elaborated in sequences of chapters of this Handbook. Other prominent developments, more individuated, are appropriately described within the chapter summaries themselves that follow at the end. Set theory would continue to expand and broaden in further directions, but we are inevitably limited in what can be covered here and in the Handbook.

In inner model theory, Martin and Steel in 1986 took the analysis of iteration trees beyond their determinacy work to develop inner models of Woodin cardinals. In order to effect comparison, they for the first time came to grips with the central iterability problem of the existence and uniqueness of iteration trees extending a given iteration tree. They were thus able to establish that "the measurable cardinal above" cannot be eliminated from their determinacy result by showing: *If $n \in \omega$ and there are n Woodin cardinals, then there is an inner model with n Woodin cardinals and a Δ^1_{n+2} well-ordering of the reals.* (The existence of such a well-ordering precludes $\mathbf{\Pi}^1_{n+1}$-determinacy.) These models were of form $L[\vec{E}]$ where $\vec{E}$ is a coherent sequence of extenders, but the comparison process used did not involve the models themselves, but rather a large model constructed from a sequence of *background extenders*, extenders in the sense of V whose restrictions to $L[\vec{E}]$ led to the sequence $\vec{E}$. With the comparison process thus external to the models, their structure remained largely veiled, and for example only CH, not GCH, could be established.

In 1987 Stewart Baldwin made a suggestion, one which Mitchell then newly forwarded, which led to a crucial methodological advance. Up to then, the extender models $L[\vec{E}\,]$ constructed relative to a coherent sequence of extenders $\vec{E}$ had each extender in the sequence "measure" all the subsets in $L[\vec{E}\,]$ of its critical point. The Baldwin-Mitchell idea was to construct only with "partial" extenders E which if indexed at γ only measures the sets in $L_\gamma[\vec{E}{\restriction}\gamma]$. This together with a previous Mitchell strategy of carrying out the comparison process using finely calibrated partial ultrapowers ("dropping to a mouse") led to a comparison process internal to $L[\vec{E}\,]$ based on the use of fine structure. The infusion of fine structure made the development of the new extender models more complex, but with this came the important dividends of a more uniform presentation, a much stronger condensation, and a more systematic comparison process. During 1987–1989, Mitchell and Steel worked out the details and showed that if there is a Woodin cardinal then there is an inner model $L[\vec{E}\,]$, L-like in satisfying GCH and so forth, in which there is a Woodin cardinal. The process involved the correlating of iteration trees for $L[\vec{E}\,]$ with iteration trees in V and applying the former Martin-Steel results. A canonical, fine structural inner model of a Woodin cardinal newly argued for the consistency of the concept, as well as provided a great deal of understanding about it as set in a finely tuned, layer-by-layer hierarchy.

What about a core model "up to" a Woodin cardinal, in analogy to K^{DJ} for $L[U]$? In 1990, Steel solved the "core model iterability problem" by showing that large cardinals in V are not necessary for showing that certain models $L[\vec{E}\,]$ have sufficient iterability properties. With this, he constructed a new core model, first building a "background certified" K^c based on extenders in V and then the "true" core model K. Steel was thus able to extend the previous work of Mitchell on the core model $K[\mathcal{U}]$ up to $\exists\kappa(o(\kappa) = \kappa^{++})$ to establish e.g.: *If there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 and* *a measurable cardinal, then there is an inner model with a Woodin cardinal.* Thus, Shelah's 1985 forcing result and Steel's, except for the artifact of "the measurable cardinal above", had calibrated an important consistency strength, and what had become a central goal of forcing and inner model theory was handily achieved.

In the early 1990s, Steel, Mitchell, and Ernest Schimmerling pushed the Jensen covering argument over the hurdles of the new fine structural Steel core model K to establish a covering lemma up to a Woodin cardinal. Schimmerling both established combinatorial principles in K as well established new consistency strengths, e.g. PFA *implies that there is an inner model with a Woodin cardinal.*

The many successes would continue in inner model theory, but we bring our narrative to a close at a fitting point. Mitchell's Chap. 18 in this Handbook is given over to the concerted study of covering over various models; Steel's Chap. 19 provides the outlines of inner model theory in general terms as well as an important application to HOD; and Schimmerling's Chap. 20 develops Steel's core model K up to a Woodin cardinal as well as provide applications across set theory.

The later 1980s featured a distinctive development that led to a new conceptual framework of applicability to singular cardinals, new incisive results in cardinal arithmetic, and a re-orienting of set theory to new possibilities for outright theorems of ZFC. Starting in late 1987 Shelah returned to the work on bounds for powers of singular cardinals and drew out an extensive underlying structure of possible cofinalities of reduced products, soon codified as *pcf theory.* With this emerged new work in *singular cardinal combinatorics*, with Shelah himself initially providing applications to model theory, partition relations, Jónsson algebras, Boolean algebras, and cardinal arithmetic. This last was epitomized by a dramatic result that exhibited how the newly seen structural constraints impose a tight bound: *If δ is a limit ordinal with $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$ then $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{+4})}$, so that in particular if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.* Quite remarkably, a ZFC result bounds $2^{\aleph_\omega}$ with a small aleph not indexed in terms of the power set operation!

Suppose that A is an infinite set of cardinals and F is a filter over A. The *product* ΠA consists of functions f with domain A such that $f(a) \in a$ for every $a \in A$. For $f, g \in \Pi A$, the relation $=_F$ defined by $f =_F g$ *iff* $\{a \in A \mid f(a) = g(a)\} \in F$ is an equivalence relation on ΠA, and the *reduced product* $\Pi A/F$ consists of the equivalence classes. We can impose order, officially on $\Pi A/F$ but still working with functions themselves, by: $f <_F g$ *iff* $\{a \in A \mid f(a) < g(a)\} \in F$.

Shelah's new theory took as central the investigation of the possible cofinalities function:

$$\mathrm{pcf}(A) = \{\mathrm{cf}(\Pi A/D) \mid D \text{ is an ultrafilter over } A\}$$

as calibrated by the ideals

$$J_{<\lambda}[A] = \{b \subseteq A \mid \mathrm{cf}(\Pi A/D) < \lambda \text{ whenever } D \text{ is an ultrafilter over } A \text{ such that } b \in D\}.$$

These concepts had appeared before in Shelah's work, notably in his 1980 result $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$, but now they became autonomous and were propelled forward by the discovery of unexpectedly rich structure.

With an eye to substantive cofinal subsets A of a singular cardinal, the abiding assumption was that A is a set of regular cardinals satisfying $|A| < \min(A)$. With this one gets that for any ultrafilter D over A, $\mathrm{cf}(\Pi A/D) < \lambda$ *iff* $D \cap J_{<\lambda}[A] \neq \emptyset$, and further, that $\mathrm{pcf}(A)$ has a maximum element. At the heart is the striking result that $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda}[A]$ together with a single set $B_\lambda \subseteq A$. Shelah in fact got "nice" generators B_λ derived from imposing the structure of elementary substructures of a sufficiently large $H(\Psi)$. This careful control on the possible cofinalities then led, when A consists of all the regular cardinals in an interval of cardinals, to $|\mathrm{pcf}(A)| \leq |A|^{+++}$, and in particular to the $\aleph_{\omega_4}$ bound mentioned above.

Shelah's work on pcf theory to 1993 appeared in his 1994 book *Cardinal Arithmetic*, and since then he has further developed the theory and provided wide-ranging applications. Through its applicability pcf theory has to a significant extent been woven into modern set theory as part of the ZFC facts of singular cardinal combinatorics. Chapter 14 of this Handbook presents a version of pcf theory and its applications to cardinal arithmetic, and the theory makes it appearance elsewhere as well, most significantly in Chap. 15.

The Singular Cardinal Hypothesis (SCH) and the train of results starting with the Prikry-Silver result of the early 1970s were to be decisively informed by results of Moti Gitik. Gitik's work exhibits a steady engagement with central and difficult issues of set theory and a masterful virtuosity in the application of sophisticated techniques over a broad range. Gitik by 1980 had established, through an iterated Prikry forcing, the conspicuous singularization result: *If there is a proper class of strongly compact cardinals, then in a* ZF *inner model of a class forcing extension every infinite cardinal has cofinality* ω. Mentioned earlier was the mid-1970s result that NS_{ω_1} being precipitous is equi-consistent with having a measurable cardinal. In 1983, Gitik established: *The precipitousness of* NS_{ω_2} *is equi-consistent with having a measurable cardinal* κ *such that* $o(\kappa) = 2$ *in the Mitchell order.* The difficult, forcing direction required considerable ingenuity because of inherent technical obstructions.

Turning to the work on SCH, in 1988 Woodin dramatically weakened the large cardinal hypothesis needed to get a measurable cardinal κ satisfying $2^\kappa > \kappa^+$, and hence the failure of SCH with the subsequent use of Prikry forcing, to a proposition technically strengthening measurability. He also showed that one can in fact get Magidor's conclusion that $\aleph_\omega$ could be the least cardinal at which GCH fails. Soon afterwards Gitik established both directions of an equi-consistency: First, he established that one can get the consistency of Woodin's proposition from just $\exists\kappa(o(\kappa) = \kappa^{++})$. Then, he applied a result from Shelah's pcf theory to Mitchell's $K[\mathcal{U}]$ analysis to establish, bettering a previous result of Mitchell, that $\exists\kappa(o(\kappa) = \kappa^{++})$ is actually necessary to get the failure of SCH. Hence, *The failure of* SCH *is equi-consistent with* $\exists\kappa(o(\kappa) = \kappa^{++})$.

Woodin's model in which GCH first fails at $\aleph_\omega$ required a delicate construction to arrange GCH below and an ingenious idea to get $2^{\aleph_\omega} = \aleph_{\omega+2}$. How about getting $2^{\aleph_\omega} > \aleph_{\omega+2}$? In a signal advance of method, Gitik and Magidor in 1989 provided a new technique to handle the general singular cardinals problem with appropriately optimal hypotheses. The Prikry-Silver two-stage approach, first making 2^κ large and then singularizing κ without adding any new bounded subsets or collapsing cardinals, had been the basic model for attacking the singular cardinals problem. Gitik and Magidor showed how to add many subsets to a large cardinal κ while *simultaneously* singularizing it without adding any new bounded subsets or collapsing cardinals. Thus, it became much easier to arrange any particular continuum function behavior below κ, like achieving GCH below, while at the same time making 2^κ arbitrarily large. Moreover, the new method smacked of naturalness and optimality.

The new Gitik-Magidor idea was to add many new Prikry ω-sequences corresponding to κ-complete ultrafilters over κ while maintaining the basic properties of Prikry forcing. There is an evident danger that if these Prikry sequences are too independent, information can be read from them that corresponds to new reals being adjoined. The solution was to start from a sufficient strong large cardinal hypothesis and develop an extender-based Prikry forcing structured on a "nice system" of ultrafilters $\langle U_\alpha \mid \alpha < \lambda\rangle$, a system such that for many $\alpha \leq \beta < \lambda$ there is a ground model function $f : \kappa \to \kappa$ such that: For all $X \subseteq \kappa$, $X \in U_\alpha$ *iff* $f^{-1}(X) \in U_\beta$. (Having such a projection function is the classical way of connecting two ultrafilters together, and one writes that $U_\alpha \leq_{\mathrm{RK}} U_\beta$ under the Rudin-Keisler partial order.) By this means one has the possibility of adding new subsets of κ, corresponding to different Prikry sequences, which are still dependent on each other so that no new bounded subsets need necessarily be added in the process. Gitik and Magidor worked out how their new approach leads to what turns out to be optimal or near optimal consistency results, and incorporating collapsing maps as in previous arguments of Magidor and Shelah, they got models in which GCH holds below $\aleph_\omega$ yet $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any prescribed countable ordinal α.

In subsequent work Gitik, together with Magidor, Mitchell, and others, have considerably advanced the investigation of powers of singular cardinals. Equi-consistency results have been achieved for large powers of singular cardinals along the Mitchell order and with α-strong cardinals, and uncountable cofinalities have been encompassed, the investigation ongoing and with dramatic successes. Much of this work is systematically presented in Gitik's Chap. 16 in this Handbook.

We now leave the overall narrative, having pursued several longitudinal themes to appropriate junctures. Stepping back to gaze at modern set theory, the thrust of mathematical research should deflate various possible metaphysical appropriations with an onrush of new models, hypotheses, and results. Shedding much of its foundational burden, set theory has become an intriguing field of mathematics where formalized versions of truth and consistency have become matters for manipulation as in algebra. As a study couched in well-foundedness ZFC together with the spectrum of large cardinals serves as a court of adjudication, in terms of relative consistency, for mathematical propositions that can be informatively contextualized in set theory by letting their variables range over the set-theoretic universe. Thus, set theory is more of an open-ended framework for mathematics rather than an elucidating foundation. It is as a field of mathematics proceeding with its own internal questions and capable of contextualizing over a broad range that set theory has become an intriguing and highly distinctive subject.

6. Summaries of the Handbook Chapters

This Handbook is divided into three volumes with the first devoted to Combinatorics, the Continuum, and Constructibility; the second devoted to Elementary Embeddings and Singular Cardinal Combinatorics; and the third devoted to Inner Models and Determinacy.

The following chapter summaries engage the larger historical contexts as they serve to introduce and summarize the contents. In many cases we build on our preceding survey as a framework and proceed to elaborate it in the directions at hand, and in some cases we introduce the topics as new offshoots and draw them in. Consequently, some summaries are shorter on account of the leads from the survey and others longer because of the new lengths to which we go.

VOLUME I

1. Stationary Sets. The veteran set theorist Thomas Jech is the author of *Set Theory* (third millennium edition, 2002), a massive and impressive text that comprehensively covers the full range of the subject up to the elaborations of this Handbook. In this first chapter, Jech surveys the work directly involving stationary sets, a subject to which he has made important contributions. In charting out the ramifications of a basic concept buttressing the uncountable, the chapter serves, appropriately, as an anticipatory guide to techniques and results detailed in subsequent chapters.

The first section provides the basic theory of stationary subsets of a regular uncountable cardinal κ. The next describes the possibilities for *stationary set reflection*: For $S \subseteq \kappa$ stationary in κ, is there an $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α? With reflection having become an important heuristic in set theory, stationary set reflection commended itself as a specific, combinatorial possibility for investigation. Focusing on the non-stationary ideal, the third section surveys the possibilities for its saturation and precipitousness.

The later sections study these various issues as adapted to notions of closed unbounded and stationary for subsets of $P_\kappa\lambda = \{x \in P(\lambda) \mid |x| < \kappa\}$, a study that the author had pioneered in the early 1970s. The wide-ranging involvements in proper forcing, Boolean algebras and stationary tower forcing are described. Of particular interest are reflection principles based on $P_{\aleph_1}\lambda$. Foreman, Magidor, and Shelah in their major 1984 work had shown that Martin's Maximum implies that a substantial reflection principle holds for stationary subsets of $P_{\aleph_1}\lambda$ for every $\lambda \geq \omega_2$. Todorcevic then showed that a stronger reflection principle SRP follows from MM, one from which substantial consequences of MM already follow, like the $\aleph_2$-saturation of NS_{ω_1}. Qi Feng and Jech subsequently formulated a streamlined principle PRS equivalent to SRP.

2. Partition Relations. In this chapter two prominent figures in the field of partition relations, András Hajnal and Jean Larson, team up to present the recent work, the first bringing to bear his expertise in relations for uncountable cardinals and the second her expertise in relations for countable ordinals. The investigation of partition relations has been a steady, rich, and concrete part of the combinatorial investigation of the transfinite, a source of intrinsically interesting problems that have stimulated the application of a variety of emerging techniques.

With the classical, 1956 Erdős-Rado Theorem $\beth_n(\kappa)^+ \longrightarrow (\kappa^+)^{n+1}_\kappa$ having established the context as the transfinite generalization of Ramsey's Theorem, extensive use of the basic tree or "ramification" method had led by the mid-1960s to an elaborately parametrized theory. This theory was eventually presented in the 1984 Erdős-Hajnal-Rado-Máté book, which is initially reflected in the first two sections of the chapter.

The next sections emphasize new methods as leading not only to new results but also providing new proofs of old results, and in this spirit they develop a 1991 method of Baumgartner, Hajnal, and Todorcevic and establish their generalizations of the Erdős-Rado Theorem. This method involves taking chains of elementary substructures of a sufficiently rich structure $\langle H(\lambda), \in, <^*, \ldots \rangle$ and associating ideals along the way. Next, the enhanced method of the recent, 1998 Foreman-Hajnal result on successors of measurable cardinals is used establish a watershed, 1972 Baumgartner-Hajnal Theorem in the special case $\omega_1 \longrightarrow (\alpha)^2_m$ for any $\alpha < \omega_1$ and $m \in \omega$. Shelah, with his considerable combinatorial prowess, has steadily made important contributions to the theory of partition relations, and several are presented, among them a recent result involving strongly compact cardinals and another invoking his pcf theory.

The investigation of partition relations for small countable ordinals was a current from the beginnings of the general theory in the late 1950s and has focused, for natural reasons, on the relation $\alpha \longrightarrow (\alpha, m)^2$ for finite m, the assertion that if the pairs from α are assigned 0 or 1, then either there is an $H \subseteq \alpha$ of order-type α all of whose pairs are assigned 0, or m elements in α

all of whose pairs are assigned 1. A formative early 1970s result was Chen-Chung Chang's that with ordinal exponentiation, $\omega^\omega \to (\omega^\omega, 3)^2$, the proof considerably simplified by Larson. Remarkably, after the passing of more than two decades Carl Darby and Rene Schipperus working independently established the first new positive and negative results, the latter by way of the same counterexamples. In the last two sections, a negative result $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 6)$ and a positive result $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)$ are established, the careful combinatorial analysis in terms of blocks of ordinals and trees illustrative of some of the most detailed work with small order-types.

3. Coherent Sequences. This chapter is a systematic account by Stevo Todorcevic of his penetrating analysis of uncountable order structures, with ω_1 being both a particular and a paradigmatic case. The chapter is a short version of his recent monograph *Walks on Ordinals and Their Characteristics* (2007), but has separate value for being more directed and closer to the historical route of discovery.

The analysis for a regular cardinal θ begins with a *C-sequence* $\langle C_\alpha \mid \alpha < \theta\rangle$ where for successors $\alpha = \beta + 1$, $C_\alpha = \{\beta\}$, and for limits α, C_α is a closed unbounded subset of α. In the case $\theta = \omega_1$, one requires that for limits α, C_α has order-type ω, so that we have a "ladder system". One can climb up, but also walk down: Given $\alpha < \beta < \theta$, let β_1 be the least member of $C_\beta - \alpha$, let β_2 the least member of $C_{\beta_1} - \alpha$, and so forth, yielding the *walk* $\beta > \beta_1 > \cdots > \beta_n = \alpha$. Through a sustained analysis Todorcevic has shown that these walks have a great deal of structure as conveyed by various "distance functions" or "characteristics" ρ on $[\theta]^2$, where $\rho(\alpha, \beta)$ packages information about the walk from β to α.

Initially, Todorcevic in 1985 used such a function to settle the main partition problem about the complexity of ω_1, by establishing the negative "square brackets partition relation" $\omega_1 \not\to [\omega_1]^2_{\omega_1}$: There is a function $f : [\omega_1]^2 \to \omega_1$ such that for any unbounded $X \subseteq \omega_1$, $f``[X]^2 = \omega_1$, i.e. for any $\zeta < \omega_1$ there are $\alpha < \beta$ both in X such that $f(\alpha, \beta) = \zeta$. Todorcevic's f was based on the property that if $S \subseteq \omega_1$ is stationary, then for any unbounded $X \subseteq \omega_1$ there are $\alpha < \beta$ both in X such that the walk from β to α has a member of S. More generally, Todorcevic introduced the *oscillation map* to effect a version of this property for regular $\theta > \omega_1$ to show that if there is a stationary $S \subseteq \theta$ which does not reflect, i.e. there is no $\alpha < \theta$ such that $S \cap \alpha$ is stationary in α, then the analogous $\theta \not\to [\theta]^2_\theta$ holds.

The first sections of the chapter develops several distance functions for the case $\theta = \omega_1$ as paradigmatic. Systematic versions of "special" Aronszajn trees and the (Shelah) result that adding a Cohen real adds a Suslin tree are presented, as well as a range of applications to Hausdorff gaps, Banach spaces, model theory, graph theory and partition relations.

The later sections encompass general θ, with initial attention given to systematic characterizations of Mahlo and weakly compact cardinals. There is soon a focus on *square* (or *coherent*) sequences, those C-sequences $\langle C_\alpha \mid \alpha < \theta\rangle$ such that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit of C_β. With these a range of applications is provided involving the principle $\square_\kappa$, higher Kurepa trees, and Jensen matrices. The oscillation map is latterly introduced, and with it the proof of the general negative square brackets partition relation as stated above. Finally, elegant characterizations of Chang's Conjecture are presented. Throughout, there is the impression that one has gotten at the immanent structure of the uncountable from which a wide range of combinatorial consequences flow.

4. Borel Equivalence Relations. Descriptive set theory as fueled by the incentive for generalization is appropriately construed as the investigation of definable sets in *Polish spaces*, i.e. separable, completely metrizable spaces. For such spaces one can define the Borel and projective sets and the corresponding hierarchies through definability. In the 1990s fresh incentives came into play that expanded the investigation into quotient spaces X/E for a Polish space X and a definable equivalence relation E on X, such quotients capturing various important structures in mathematics. New methods had to be developed, in what amounts to the investigation of definable equivalence relations on Polish spaces.

In this short chapter Greg Hjorth provides a crisp survey of Borel equivalence relations on Polish spaces as organized around the Borel reducibility ordering $\leq_B$. In an initial disclaimer, he points out how he is leaving aside several other approaches, but in any case his account provides a worthy look at a modern, burgeoning subject.

For Polish spaces X and Y, a function $f : X \to Y$ is *Borel* if the preimage of any Borel set is Borel. An equivalence relation on X is *Borel* if it is Borel as a subset of $X \times X$. If E is a Borel equivalence relation on X and F is a Borel equivalence relation on Y, then $E \leq_B F$ asserts that there is a Borel $f : X \to Y$ such that $x_1 E x_2 \leftrightarrow f(x_1) F f(x_2)$. The emphasis here is on the equivalence relations, with only the Borel sets of the underlying spaces being at issue. There is the correlative $E <_B F$, and with $\mathrm{id}(X)$ indicating the identity relation on X, an example is $\mathrm{id}(\mathbb{R}) <_B E_0$, where E_0 is the equivalence relation of eventual agreement on ${}^\omega 2$. E_0 is a reconstrual of Vitali's classical equivalence relation, with which he established that with AC there is a non-Lebesgue measurable set. The seminal Harrington-Kechris-Louveau "Glimm-Effros dichotomy" result is: *For any Borel equivalence relation E, exactly one of $E \leq_B \mathrm{id}(\mathbb{R})$ or $E_0 \leq_B E$ holds.*

Starting with this seminal result the author discusses various structure theorems, concluding with his work on *turbulence.* Next is the work on countable Borel equivalence relations, i.e. those whose equivalence classes are all countable. This topic has notable interactions across diverse fields of mathematics, and an enduring problem is how to characterize the *hyperfinite* Borel equivalence relations. The author next discusses $\leq_B$ as effective cardinality, bringing in his results with determinacy. The final topic is classification problems, problems of locating variously given Borel equivalence relations in the structure given by $\leq_B$. The range of issues here speaks to the importance and relevance of Borel equivalence relations in larger mathematics.

5. Proper Forcing. Uri Abraham provides a lucid exposition of Shelah's proper forcing. In a timely monograph *Proper Forcing* (1982) and a book *Proper and Improper Forcing* (1998), Shelah had set out his penetrating, wide-ranging work on and with proper forcing. Striking a nice balance, Abraham presents the basic theory of proper forcing and then some of the variants and their uses that illustrate its wide applicability. This chapter is commended to the reader conversant even with only the basics of forcing to assimilate what has become a staple part of the theory and practice of forcing. To be noted is that being of the Israeli school, Abraham writes "$p > q$" for p being a stronger condition than q.

In the first two sections, basic forcing notions are reviewed, and proper forcing is motivated and formulated. The basic lemma that properness is preserved in countable support iterations is carefully presented, as well as the basic fact that under CH a length $\leq \omega_2$ iteration of $\aleph_1$ size proper forcings satisfies the $\aleph_2$-chain condition and so preserves all cardinals.

A forcing partial order P is *${}^\omega\omega$-bounding iff* the ground model reals are cofinal under eventual dominance $<^*$ in the reals of any generic extension by P. The third section presents the preservation of ${}^\omega\omega$-bounding properness in countable support iterations. With this is established a finely wrought result of Shelah's, answering a question of classical model theory, that it is consistent that there are two countable elementarily equivalent structures having no isomorphic ultrapowers by any ultrafilter over ω.

A forcing partial order P is *weakly ${}^\omega\omega$-bounding iff* the ground model reals are unbounded under eventual dominance $<^*$ in the reals of any generic extension by P. The fourth section presents the preservation of weakly ${}^\omega\omega$-bounding properness, one that deftly and necessarily has to assume a stronger property at successor stages. With this is established another finely wrought result of Shelah's, answering a question in the theory of cardinal characteristics, that it is consistent with $2^{\aleph_0} = \aleph_2$ that the bounding number $\mathfrak{b}$ is less than the splitting number $\mathfrak{s}$.

The final section develops iterated proper forcing that adjoins no new reals. A relatively complex task, this has been a prominent theme in Shelah's work, and to this purpose he has come up with several workable conditions. Abraham motivates one condition, *Dee-completeness*, with his first result in set theory, and then establishes an involved preservation theorem. As pointed out, through this approach one can provide a new proof of Jensen's result that CH + SH is consistent, which for Shelah was an important stimulus in his initial development of proper forcing.

6. Combinatorial Cardinal Characteristics of the Continuum. This and the next chapters cover the recent, increasingly systematic, work across the wide swath having to do with cardinal characteristics, or invariants, of the continuum and their possible order relationships. In this chapter, the broad-ranging Andreas Blass provides a perspicuous account of combinatorial cardinal characteristics through to some of his own work. He deftly introduces characteristics in turn together with more and more techniques for their analysis, and at the end surveys the extensive forcing consistency results.

There is initially a discussion of the dominating number $\mathfrak{d}$ and the bounding number $\mathfrak{b}$, one that introduces several generalizing characteristics corresponding to an ideal $\mathcal{I}$: $\mathbf{add}(\mathcal{I})$, $\mathbf{cov}(\mathcal{I})$, $\mathbf{non}(\mathcal{I})$, $\mathbf{cof}(\mathcal{I})$. The next topic is the splitting number $\mathfrak{s}$ and related characteristics having to do with Ramsey-type partition theorems.

A systematic approach, first brought out by Peter Vojtáš, is then presented for describing many of the characteristics and the relationships among them. A triple $\mathbf{A} = \langle A_-, A_+, A\rangle$ such that $A \subseteq A_- \times A_+$ is simply a *relation*, and its *norm* $\|\mathbf{A}\|$ is the smallest cardinality of any $Y \subseteq A_+$ such that $\forall x \in A_- \exists y \in Y(\langle x, y\rangle \in A)$. The *dual* of $\mathbf{A} = \langle A_-, A_+, A\rangle$ is $\mathbf{A}^\perp = \langle A_+, A_-, \neg\breve{A}\rangle$, where $\neg\breve{A}$ is the complement of the converse $\breve{A}$ of A, i.e. $\langle x, y\rangle \in \neg\breve{A}$ *iff* $\langle y, x\rangle \notin A$. In these terms, for example, if $\mathfrak{D} = \langle {}^\omega\omega, {}^\omega\omega, <^*\rangle$, then $\|\mathfrak{D}\| = \mathfrak{d}$ and $\|\mathfrak{D}^\perp\| = \mathfrak{b}$. A *morphism* for a relation $\mathbf{A} = \langle A_-, A_+, A\rangle$ to another $\mathbf{B} = \langle B_-, B_+, B\rangle$ is a pair $\phi = (\phi_-, \phi_+)$ of functions such that $\phi_- : B_- \to A_-$; $\phi_+ : A_+ \to B_+$; and

$$\forall b \in B_- \forall a \in A_+(\langle \phi_-(b), a\rangle \in A \to \langle b, \phi_+(a)\rangle \in B).$$

It is seen that having such a morphism implies that $\|\mathbf{A}\| \geq \|\mathbf{B}\|$ and $\|\mathbf{A}^\perp\| \leq \|\mathbf{B}^\perp\|$. Through this overlay of relations and morphisms one can efficiently incorporate both categorical combinations of relations as well as conditions on morphisms, like being Borel or continuous, into the study of characteristics.

The author proceeds to discuss characteristics corresponding to the ideal $\mathcal{B}$ of meager sets and to the ideal $\mathcal{L}$ of null sets: $\mathbf{add}(\mathcal{B})$, $\mathbf{cov}(\mathcal{B})$, $\mathbf{non}(\mathcal{B})$, $\mathbf{cof}(\mathcal{B})$, $\mathbf{add}(\mathcal{L})$, $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{L})$, $\mathbf{cof}(\mathcal{L})$. The main results are established in terms of relations and morphisms, and one gets to the inequalities among these characteristics and $\mathfrak{b}$ and $\mathfrak{d}$ as given by what is known as *Cichoń's diagram*. The characteristics of measure and category are further pursued in the next chapter.

The succeeding topics have to do with cardinalities of families $F \subseteq P(\omega)$ as mediated by $\subseteq^*$, where $X \subseteq^* Y$ *iff* $X - Y$ is finite. Forcing axioms are brought into play as now particularly informative for drawing ordering conclusions. Then characteristics corresponding to maximal almost disjoint (MAD) families and independent families are investigated.

The author finally discusses characteristics related to or developed through his own work. Discussing filters and ultrafilters over ω, he gets to his principle of Near Coherence of Filters (NCF), a principle proved consistent by Shelah, and results about ultrafilters generated from filters in terms of characteristics. He then discusses his *evasion* and *prediction*, which initially had an algebraic motivation but became broadened into a combinatorial framework that provides a unifying approach to many of the characteristics.

The concluding section is largely a survey of what happens to the characteristics when one iteratively adjoins many generic reals of one kind, dealing in turn with the following reals: Cohen, random, Sacks, Hechler, Laver, Mathias, Miller. As such, this is an informative account of these various generic reals and how they mediate the continuum.

7. Invariants of Measure and Category. Tomek Bartoszynski presents the recent work on measure and category as viewed through their cardinal invariants, or characteristics. The account updates the theory presented in the substantive *Set Theory: On the Structure of the Real Line* (1995) by Bartoszynski and Haim Judah, which had stood as a standard reference for this general area for quite some time.

After putting the language of relations and morphisms (see the previous summary) in place, the author pursues an approach, one advocated by Ireneusz Recław, of emphasizing classes of sets "small" according to various criteria corresponding to the ideal invariants. One develops *Borel* morphisms that lead to inclusion relations among the classes and thence to the inequalities of Cichon's diagram. Combinatorial characterizations of membership in these classes and thus of the invariants are given, as well as a new understanding of the ideal of null sets as maximal, in terms of embedding, among analytic P-ideals.

Turning to cofinality, the author establishes Shelah's remarkable and unexpected 1999 result that it is consistent that $\mathrm{cf}(\mathbf{cov}(\mathcal{L})) = \omega$. The author then provides a systematic way of associating to each of the invariants in Cichon's diagram a generic real so that iteration with countable support increases that invariant and none of the others. Corresponding issues about the classes of small sets further draw in proper forcing techniques.

8. Constructibility and Class Forcing. In this chapter Sy Friedman describes work on the limits of possibilities for reals in terms of forcing and constructibility, the supporting technique being *Jensen coding.* In the mid-1960s Solovay, when investigating the remarkable properties of $0^\#$, raised several questions about the scope of the recently devised forcing method. For sets x, y let $x \leq_L y$ denote that x is constructible from y, i.e. $x \in L[y]$, and let $x <_L y$ be correlative. $0^\#$ cannot be adjoined to L by forcing because of its global consequences for L, but $0^\#$ was plausibly considered minimal in this respect. A (weak form of a) question of Solovay's was: If r is a real satisfying $r <_L 0^\#$, does r belong to some generic extension of L?

In 1975–1976 Jensen devised his impressive "coding the universe in a real" technique and with it established (a strong form of): *If* GCH *holds, then there is a class partial order P such that if G is P-generic, then $V[G]$ has the same cardinals and cofinalities yet for some real r there, $V[G] \models$ "$V = L[r]$".* The intricately woven P here was built using fine structure theory in L-like situations and provided a means of coding up more and more layers of the cumulative hierarchy while crucially maintaining its cardinal structure. Not only cofinalities but those properties compatible with models of form $L[r]$ all continue to hold, so that this real r veritably codes the entire universe. Jensen showed that assuming $0^\#$ exists it is consistent that there is such a real

$r <_L 0^\#$, answering Solovay's question in the negative, the intention there having been to address forcing with *set* partial orders. Not only did Jensen bring class forcing into prominence for establishing new consistency results about sets, but also for establishing outright theorems of ZFC + "$0^\#$ exists".

Starting in the mid-1980s Friedman reworked and extended the Jensen theory and established some notable results about $0^\#$ and class forcing, and this work eventually appeared in his book *Fine Structure and Class Forcing* (2000). This chapter is a short version of the book, appropriate to the task of working more directly toward several problems of Solovay and developing techniques where needed. After stating three problems of Solovay as motivation, Friedman develops the criterion of *tameness* for class partial orders for preserving ZFC and gets at the property of *relevance*, having a generic definable in $L[0^\#]$. He then provides his proof of Jensen's coding theorem assuming that $0^\#$ does not exist, this assumption allowing a comparatively simple argument free of fine structure but making appeals to the Jensen Covering Theorem. With this the Solovay problems are addressed in turn. To conclude, wide-ranging applications are given as well as a nice list of open problems.

9. Fine Structure. This and the next chapter deal with fine structure and are complementary in that they present different versions, both due initially to Jensen, as well as applications in different directions. In this chapter Ralf Schindler and Martin Zeman provide an incisive, self-contained account of Jensen's original fine structure theory for the J_α hierarchy relativized to a predicate A. Much is drawn from Zeman's book *Inner Models and Large Cardinals* (2002), but diverging from it Schindler and Zeman steer to the use of the Mitchell-Steel $r\Sigma_n$ formulas for discussing iterated projecta and embeddings. With A being a sequence of extenders this was the approach that had been taken for the use of fine structure in inner model theory. The chapter thus provides the fine structure groundwork for Chaps. 18, 19, and 20 of this Handbook.

After the preliminaries about J-structures, the chapter focuses on the *acceptable* ones, those that satisfy GCH in a strong form. The projecta of these J-structures are described, and then the Downward and Upward Extensions of Embeddings Lemmas are established. Iterated projecta are then formulated and $r\Sigma_n$ introduced for expressing preservation through embeddings using *very good* parameters. Next, standard parameters are fully analyzed and all the considerations about *soundness* and *solidity witnesses* necessary for inner model theory are given.

A later section analyzes *fine ultrapowers*, fine structure preserving ultrapowers by extenders, treating the "short" and "long" cases uniformly, and draws out the connections with the Upward Extensions of Embeddings Lemma. Finally, two illustrative applications to L are presented, with generalizable arguments: a proof, in the absence of $0^\#$, of the "countably closed" weak covering property for L and a proof of $\square_\kappa$ for $\kappa > \omega$.

10. Σ^* Fine Structure. Philip Welch considerably rounds out the discussion of fine structure by presenting the Σ^* version and the extensive work on square principles and morasses, providing commentary throughout about the interactions with inner model theory.

Σ^* fine structure is due to Jensen and detailed in Zeman's book *Inner Models and Large Cardinals* (2002). The theory is a notable advance in that it isolated the "right" classes of formulas for the articulation of fine structure results. The classes form a certain ramified version of the Levy hierarchy, the $\Sigma_k^{(n)}$ formulas for $n, k \in \omega$, which level-by-level are able to capture syntactically the semantic role of standard parameters. In particular, $\Sigma_1^{(n)}(J_\alpha)$ relations can be uniformized by $\Sigma_1^{(n)}(J_\alpha)$ relations defined uniformly for all α. And the $\Sigma_1^{(n)}$ formulas are exactly the formulas preserved by the $r\Sigma_{n+1}$ embeddings involving very good parameters.

The first section of the chapter establishes the Σ^* theory, with the treatment much as in Zeman's book. The Σ^* approach is shown to advantage in the development of the Σ^* *ultrapower*, Σ^* fine structure preserving extender ultrapowers. Then the more general pseudo-ultrapower (which corresponds to the use of "long" extenders) is developed, with a refinement toward coming applications.

The second section is devoted to square principles. Jensen had established that if $V = L$, then in addition to the principles $\square_\kappa$ a global, class version $\square$ holds. Most of the section is taken up by a Σ^* pseudo-ultrapower proof of this result, one that provides a global $\square$ sequence with uniform features.

The section concludes with an extensive and detailed description of the recent investigation of square principles in inner models. Of particular interest is the failure of $\square_\kappa$, this for singular κ precluding covering properties for inner models. Around 2000 an elucidating systemic characterization was achieved. Solovay's initial 1970s result—that if κ is λ^+-supercompact and $\lambda \geq \kappa$, then $\square_\lambda$ fails—had led to refinements, and Jensen had extracted a streamlined large cardinal concept, later dubbed *subcompactness*, still sufficient so that: *If κ is subcompact, then $\square_\kappa$ fails.* Then in a remarkable analysis, Zeman and Schimmerling established: *In "Jensen-style" extender models $L[\vec{E}]$, if $\square_\kappa$ fails, then κ is subcompact.* These results established the reach of $\square_\kappa$ well beyond current inner model theory, in that subcompact cardinals, far stronger than Woodin cardinals, are not known to have canonical inner models. By 2005 Steel established: *If $\square_\kappa$ fails for some singular strong limit cardinal κ, then* $\mathrm{AD}^{L(\mathbb{R})}$ *holds.*

The chapter is brought to an end with a survey of the extensive work on morasses. A $(\kappa, 1)$ morass is a system approximating the L_α's for $\kappa < \alpha \leq \kappa^+$ by means of L_β's for $\beta < \kappa$ and maps $f_{\beta,\beta'}$ between them as regulated by a series of conditions. Just after his development of fine structure Jensen formulated morasses and established their existence in L in order to establish model-theoretic "cardinal transfer" theorems there. A great deal of work has since been carried out on morass structures as providing approximations to large structures in terms of indexed arrays of small structures, and morasses have come to carry the weight of the extent of combinatorial structure in the constructible universe.

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11. Elementary Embeddings and Algebra. In this chapter Patrick Dehornoy describes a notable development arising out of the investigation of algebraic features of very strong elementary embeddings. After Kunen established his result that a strong large cardinal postulation is inconsistent, it was natural to investigate remaining possibilities just weaker and so still of great consistency strength. One was that there exists a (non-identity) elementary embedding $j : V_\lambda \to V_\lambda$ for some limit λ. There is a collective structure here, for letting $\mathcal{E}_\lambda$ be the set of such embeddings, $\mathcal{E}_\lambda$ is closed under functional composition $\circ$, as well as *application*: For $j, k \in \mathcal{E}_\lambda$, let $j[k] = \bigcup_{\gamma<\lambda} j(k \cap V_\gamma)$, regarding k of course as a set of ordered pairs; then $j[k]$ is in $\mathcal{E}_\lambda$ as well. Composition $\circ$ and application $[\,]$ together satisfy a handful of laws, and the latter satisfies the left distributive law $j[k[l]] = j[k][j[l]]$. Martin's 1978 result, that if there is an "iterable" elementary $j : V_\lambda \to V_\lambda$ then $\mathbf{\Pi}^1_2$-Determinacy holds, first used application $[\,]$ and these laws for j applied to itself.

Laver saw that application provided a wealth of elementary embeddings and a proliferation of critical points. With this he initiated a systematic investigation into the structure of $\mathcal{E}_\lambda$ for its own sake. In 1989 he established the freeness of the subalgebra generated by one j in $\langle \mathcal{E}_\lambda, [\,] \rangle$ subject to the left distributive law and the analogous result for $\langle \mathcal{E}, [\,], \circ \rangle$. Moreover, with his analysis Laver established that the corresponding word problem for the left distributive law is solvable, i.e. it is recursively decidable whether two given expressions in the language of one generator and $[\,]$ are equivalent according to the left distributive law. This elicited considerable interest, with a hypothesis near the limits of consistency entailing solvability in finitary mathematics. In 1992 Dehornoy eliminated the large cardinal assumption from the solvability result with an elegant argument that led to unexpected results about the Artin braid group.

Dehornoy in this chapter effectively presents the body of work on $\mathcal{E}_\lambda$ and the left distributive law. Beyond the solvability of the word problem, he also presents the Laver-Steel theorem about the set of critical points of members of $\mathcal{E}_\lambda$ having order-type ω, a result that initially applied results about the Mitchell ordering in inner model theory; Randall Dougherty's result that the growth rate of the critical points is faster than Ackermann's function; and results on the finite "Laver tables" using $\mathcal{E}_\lambda \neq \emptyset$ that thus far have not been established in ZFC alone.

12. Iterated Forcing and Elementary Embeddings. James Cummings provides a lucid exposition of that core part of the mainstream of forcing and large cardinals having to do with iterated forcing and extensions of elementary embeddings. Forcing and large cardinals are elaborated in the directions of ideals and generic elementary embeddings in the next chapter and in the direction of Prikry-type forcings in Chap. 16. Drawing on his wide-ranging knowledge, Cummings provides a well-organized account, in mainly short, crisp sections, starting from the basics and proceeding through a series of techniques, with historical progression a rough guide and conceptual complexity a steady one. This chapter is commended to the reader conversant even with only the basics of forcing and large cardinals to assimilate what have become important techniques of modern set theory.

The early sections proceed through the basics of elementary embeddings, ultrapowers and extenders, large cardinal axioms, forcing, some forcing partial orders, and iterated forcing. The first ascent is to building generic objects to extend ("lift") elementary embeddings in forcing extensions. Describing Silver's Easton support iteration and the key idea of master condition, his 1971 result is established: *If κ is κ^{++}-supercompact, then there is a forcing extension in which κ is measurable and $2^\kappa = \kappa^{++}$.* Next, Magidor's important technique of making do with an "increasingly masterful" sequence of conditions is presented. Then, the general idea of *absorption*, embedding a complex partial order into a simple one, is discussed. This is illustrated with Magidor's 1982 result (also highlighted in Chap. 15): *If there are infinitely many supercompact cardinals, then in a forcing extension in which they become the $\aleph_n$'s, every stationary subset of $\aleph_{\omega+1}$ reflects.*

Precipitousness is the subject of the two longer sections of the chapter. In the first, the Jech-Prikry-Mitchell-Magidor mid-1970s result is established, building on the previous work: *If there is a measurable cardinal κ, then there is a forcing extension in which $\kappa = \omega_1$ and NS_{ω_1} is precipitous.* This involves exploiting the absorptive properties of the initial Levy collapse with iterated "club shooting". In the second, and longest, section a proof is provided of the 1983 Gitik result: *The precipitousness of NS_{ω_2} is equi-consistent with having a measurable cardinal κ such that $o(\kappa) = 2$ in the Mitchell order.* The difficult, forcing direction exhibited Gitik's virtuosity of technique, and all the features of a "preparation forcing" before the iterated club shooting are carefully laid out: Namba forcing, RCS iteration, the S and $\mathbb{I}$ conditions.

The rest of the chapter reverts to short sections that describe a wide range of techniques and results, of which we mention the more conspicuous. Presenting Kunen's universal collapse and Silver's collapse, Kunen's focal 1972 result is established: *If κ is huge, then there is forcing extension in which $\kappa = \omega_1$ and there is an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1.* Laver's termspace forcing for introducing a universal generic object by forcing with a partial order of terms is described and applied to establish Magidor's 1973 result: *It is consistent that the least strong compact cardinal is the least measurable cardinal.* The "Laver diamond" and its original use to make supercompact cardinals "indestructible" is presented, and with this Baumgartner's 1983 consistency result is established: *If there is a supercompact cardinal κ, then there is a forcing extension in which $\kappa = \omega_2$ and* PFA *holds.* Finally, Woodin's technique of "altering generic objects" is used to establish his 1988 consistency result of getting a measurable cardinal κ satisfying $2^\kappa > \kappa^+$ from what turned out, by later work of Gitik, to be the optimal hypothesis. The incorporation of these various, historically important results in one chapter speak to how iterated forcing methods have been comprehensively assimilated in modern set theory.

13. Ideals and Generic Elementary Embeddings. In this the longest chapter of this Handbook, Matthew Foreman provides a wealth of methods and results surrounding the general theme of ideals and generic elementary embeddings. In Cummings's chapter it was shown how to extend large cardinal embeddings after forcing with a partial order P, by doing additional forcing Q. If $G \subseteq P$ is generic, then from the point of view of $V[G]$, this can be viewed as saying that forcing with Q creates a "generic" elementary embedding. Foreman's chapter takes up this theme in more generality; it is concerned with the abstract question of when such a Q exists. What is at play is the basic synthesis of forcing and ultrapowers whereby one starts with an ideal I over a cardinal κ; forces with $P(\kappa) - I$ where p is stronger than q if $p - q \in I$; produces an ultrafilter over the ground model $P(\kappa)$; and then gets a generic elementary embedding of the ground model into the corresponding ultrapower. With the possibilities of ideals occurring low in the cumulative hierarchy, so that large cardinal ideas can be applied to classical problems of set theory, an enormous subject has grown as attested to by this chapter. Indeed, in it a very wide range and variety of material have been marshalled, and this comes together with an informal and inviting engagement that provides if not proofs, sketches of proofs, and if not sketches, outlines that "show".

Not just a miscellany, the chapter has been organized in terms of overall guiding themes. At the broadest level are the "three parameters" describing the strength of a generic elementary embedding $j : V \to M$: how j moves the ordinals; how large and closed M is; and the nature of the forcing that provided j. This last is the new parameter at play beyond the "conventional" large cardinal hypotheses. Ideals through their forcing properties thus assuming a crucial role, another guiding theme is the distinction between "natural" ideals that have intrinsic definitions and ideals "induced" by elementary embeddings. As the chapter progresses, strong ideal assumptions gain an autonomy as "generic large cardinals" in their own right, and the chapter is further delineated according to consequences of generic large cardinals and consistency results about them.

Section 2 introduces the basics of generic ultrapowers and begins the study of the correspondence between combinatorial properties of ideals and structural properties of generic ultrapowers. Topics include criteria for precipitousness, the disjointing property, normality, limitations on closure, canonical functions, selectivity and the use of generic embeddings for reflection.

Section 3 provides a range of examples of natural and induced ideals. Among the natural ideals considered are the nonstationary ideals NS_λ, their important generalizations to nonstationary subsets over power sets $P(X)$, *Chang ideals*, Shelah's $I[\lambda]$ and *club guessing ideals*, *non-diamond ideals*, and *uniformization ideals*. How induced ideals arise is taken up next, with an important example being the *master condition ideals*, with their connections to proper forcing. In a general setting, *goodness* and *self-genericity* are explored for making natural ideals also induced. Self-genericity can be secured through semiproper forcing and can secure the saturation or precipitousness of natural ideals.

Section 4 takes a closer look at combinatorial properties of ideals and structural properties of generic ultrapowers. Topics include a range of saturation properties, *layered ideals*, Rudin-Keisler projections, where the ordinals go under generic elementary embeddings, and the sizes of sets in dual filters. Iterations of generic elementary embeddings are also developed as well as generic elementary embeddings arising from *towers* of ideals, i.e. sequences of ideals interrelated by projection maps.

Section 5 considers consequences of positing strong ideals, or generic large cardinals, low in the cumulative hierarchy. The wide-ranging topics include graphs and groups; Chang's Conjecture, Jónsson cardinals, and $\square_\kappa$; CH, GCH, and SCH; stationary set reflection; Suslin and Kurepa trees; partition properties; descriptive set theory; and non-regular ultrafilters. As emphasized, NS_{ω_1} being $\aleph_2$-saturated importantly has countervailing consequences.

Section 6 discusses limitative results on the possibilities for generic large cardinals. These play a role analogous to the Kunen limitation on conventional large cardinals, and indeed, argumentation for it is initially applied. A range of restrictions on ideal properties is subsequently presented, among them results that stand as remarkable successes: the Gitik-Shelah result that if κ is regular and $\delta^+ < \kappa$, then the ideal generated by NS_κ and $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \delta\}$ is not κ^+-saturated; their result that there is no $\aleph_1$-complete $\aleph_0$-dense nowhere prime ideal; the Matsubara-Shioya result that for $\omega < \kappa \leq \lambda$ with κ regular, $I_{\kappa\lambda}$ is not precipitous; and the Foreman-Magidor result that for $\omega < \kappa \leq \lambda$ with κ regular, $\mathrm{NS}_{\kappa\lambda}$ is not λ^+-saturated unless $\kappa = \lambda = \omega_1$.

Having progressed to the middle of the chapter, one sees that the chapter naturally divides into halves, the latter having to do with consistency results for strong ideal assumptions. The long Sect. 7 attends to the main consistency results for induced ideals having strong properties. After developing the basic master condition theory for extending elementary embeddings, a general theorem—the Duality Theorem—is established for characterizing the forcing necessary for constructing the elementary embedding coming from an induced ideal. With this in place, a systematic account of various forcing techniques for getting precipitous and saturated ideals is provided. Highlights are Kunen's technique for getting an $\aleph_1$-complete $\aleph_2$-saturated ideal over ω_1 from a huge cardinal; Magidor's variation for which an "almost huge" cardinal suffices; Foreman's iteration to get κ-complete κ^+-saturated ideals over κ for every regular $\kappa > \omega$; Woodin's $\aleph_1$-complete $\aleph_1$-dense ideal over ω_1 from an almost huge cardinal; and Foreman's $\aleph_1$-complete $\aleph_1$-dense uniform ideal over ω_2 from two coordinated almost huge cardinals.

Section 8 in turn attends to consistency results for natural ideals having strong properties. In the first of two main approaches, one starts with an induced ideal with strong properties and forces that ideal to be a natural ideal while retaining substantial properties. Important examples are the Magidor and Woodin arguments for getting the nonstationary ideal to be precipitous and (somewhere) saturated respectively, and the Foreman-Komjáth argument for getting the tail club guessing filter to be saturated. In the second approach, one starts with a natural ideal and manipulates its antichain structure to make the generic ultrapower have strong properties. The important example is the "antichain catching" technique of the 1984 Foreman-Magidor-Shelah work for getting the nonstationary ideal to be saturated.

Section 9 broaches the extension of the context to towers of ideals. First brought into prominence by Woodin with his stationary tower forcing, this extension allows for more flexibility in minimizing assumptions and in drawing conclusions. After considering "induced" towers, techniques based on antichain catching are presented for getting nice generic ultrapowers. The stationary towers are the "natural" towers, and examples of Woodin and Douglas Burke are described. Finally, examples of stationary tower forcing are provided.

Section 10 briefly discusses the consistency strength of ideal assumptions. How inner model theory has successfully established lower bounds complementing forcing consistency results is quickly summarized. The focus, however, is on how knowing the image of just a few sets under a generic elementary embedding suffices to show that there is a conventional large cardinal in an inner model whose embedding agrees with the generic embedding. Notably, equi-consistency results for very large cardinals like the n-huge cardinals are derived by this means.

Section 11 is a speculative discussion of the possibility of adopting generic large cardinal axioms along with their conventional cousins as additional axioms for mathematics. There is summarizing, comparisons, and prediction, and the reader could profitably read this section before surmounting all the others. Section 12 is an extensive, detailed list of open problems. These two last sections indicate the wealth of possibilities at this general confluence of the methods of forcing and ultrapowers.

14. Cardinal Arithmetic. Uri Abraham and Menachem Magidor provide a broad-based account of Shelah's pcf theory and its applications to cardinal arithmetic, an account that exhibits the gains of considerable experience.

A beginning section sets out a general theory of ordinal-valued functions modulo ideals and cofinal sequences thereof, through to the existence of *exact* upper bounds as derived from a diamond-like *club guessing* principle. Delineating consequences, Silver's Theorem and a covering result of Magidor are established forthwith.

The next sections develop the basic theory of the central pcf function as calibrated by the crucial ideals $J_{<\lambda}[A]$. The various aspects of an unexpectedly rich structure are presented, the surround of the focal result that $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda}[A]$ together with a single set $B_\lambda \subseteq A$.

The latter sections make the ascent to the applications in cardinal arithmetic. First, the general Shelah study of the cofinality of $[\mu]^\kappa = \{x \subseteq \mu \mid |x| = \kappa\}$ under $\subseteq$ is presented. One takes a sufficiently large $H_\Psi (= H(\Psi))$ and structured chains of elementary substructures to get specifically related generators B_λ. With this the 1980 Shelah result $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ is secured. Proceeding through a finer analysis leading to "transitive" generators B_λ, the now famous result, instantiated by $2^{\aleph_\omega} < \aleph_{\omega_4}$ when $\aleph_\omega$ is a strong limit, is established.

The last section is devoted to Shelah's remarkable "revised GCH" result established in the early 1990s. With his investigation of cofinalities leading to "covering" sets Shelah advocated the consideration of

$$\lambda^{[\kappa]} = \min\bigl\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \ \wedge\ \forall u \in [\lambda]^\kappa \exists x \in [\mathcal{P}]^{<\kappa}\bigl(u = \bigcup x\bigr)\bigr\}$$

as a "revised" power set operation. GCH is equivalent to the assertion that for all regular $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$. Using a variant of the pcf function, Shelah established that $\lambda^{[\kappa]} = \lambda$ for every $\lambda \geq \beth_\omega$ (where $\beth_\omega = \sup\{\beth_n \mid n \in \omega\}$ with $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$) and with $\kappa < \lambda$ sufficiently large. Thus, pcf theory provided a viable, substantive version of the GCH provable in ZFC.

15. Successors of Singular Cardinals. The investigation of combinatorial properties at successors of singular cardinals, with $\aleph_{\omega+1}$ being paradigmatic, has emerged as a distinctive subject in modern set theory. Historically, the early forcing arguments to secure substantial propositions low in the cumulative hierarchy by collapsing large cardinals to $\aleph_1$ or $\aleph_2$ did not adapt to $\aleph_{\omega+1}$. The situation became accentuated when the 1970s work on covering properties for inner models showed that the failure of $\Box_\kappa$ for singular κ would require strong large cardinal hypotheses. In the 1980s expansion, the relative consistency of strong propositions about $\aleph_{\omega+1}$ entailing the failure of $\Box_{\aleph_\omega}$ were duly achieved, and with the emergence of pcf theory a new combinatorially elaborated setting was established as well. In recent years, the conceptual space between $\Box_\kappa$-like properties and their antithetical reflection properties has become clarified through methods and principles that have particular applicability at successors of singular cardinals.

Todd Eisworth in this chapter provides a well-organized account of the modern theory for successors of singular cardinals, an account that covers the full range from consistency results to combinatorics. After a first section setting out three illustrative problems about $\aleph_{\omega+1}$ the second section takes on one, stationary set reflection, as its theme. Let $\mathrm{Refl}(\kappa)$ be the assertion that every stationary $S \subseteq \kappa$ reflects, i.e. there is an $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α. A central tension is brought to the foreground with the discussion of how $\Box_\kappa$ denies $\mathrm{Refl}(\kappa^+)$ in a strong sense and how supercompact cardinals, and even strong compact cardinals through indecomposable ultrafilters, imply versions of stationary set reflection. The rest of the section is devoted to establishing, as an entrée into the issues, Magidor's 1982 result: *If there are infinitely many supercompact cardinals, then in a forcing extension in which they become the $\aleph_n$'s,* $\mathrm{Refl}(\aleph_{\omega+1})$ *holds.*

The third section is given over to a detailed exegesis of the ideal $I[\lambda]$. Part of his deep combinatorial analysis, Shelah isolated $I[\lambda]$ after strands had appeared in his work as early as 1978, and $I[\lambda]$ has grown in importance to become a central concept. In accessible terms, $S \subseteq \lambda$ is in $I[\lambda]$ *iff* there is a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ with $a_\alpha \subseteq \alpha$ and a closed unbounded $C \subseteq \lambda$ such that every $\delta \in S \cap C$ is singular and has a cofinal $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$, each of whose initial segments appears in $\{a_\beta \mid \beta < \delta\}$. This articulates a subtle sense of fast approachability, and for singular μ, AP_μ asserts that $I[\mu^+]$ is an improper ideal, i.e. $\mu^+ \in I[\mu^+]$. $\square_\mu$ implies AP_μ, and through Shelah's incisive analysis of $I[\lambda]$, one gets to the consistency of the failure of $\mathrm{AP}_{\aleph_\omega}$ from a supercompact cardinal. The section is brought to a close with Shelah's result, a bulwark of his pcf theory, on the existence of scales: With μ singular let $A \subseteq \mu$ be a set of regular cardinals cofinal in μ of order-type $\mathrm{cf}(\mu)$ such that $\mathrm{cf}(\mu) < \min(A)$ as in pcf theory. Consider ΠA with respect to the filter $F = \{X \subseteq \mathrm{cf}(\mu) \mid |\mathrm{cf}(\mu) - X| < \mathrm{cf}(\mu)\}$ of co-bounded sets. Then Shelah showed that $\langle \Pi A, <^*_F \rangle$ has a linearly ordered, cofinal sequence of length μ^+—a *scale* for μ. (In terms of pcf theory, $\Pi A/F$ has true cofinality μ^+.)

The fourth section provides an extensive exploration of applications of scales and weak square principles. Attention soon focuses on the Foreman-Magidor Very Weak Square at μ (VWS_μ), particularly its close relationship to $I[\mu^+]$. VWS_μ is a square principle so weak that AP_μ implies it, and moreover, it is consistent to have a supercompact cardinal together with VWS_μ holding for every singular μ. The rest of the section is devoted to how scales with additional properties get us further across the divide between weak square principles and reflection properties. A family consisting of non-empty sets is *free iff* it has an injective choice function, and is *κ-free iff* every subfamily of cardinality less that κ is free. $\mathrm{NPT}(\kappa, \theta)$ is the assertion that there is a κ-free, non-free family of κ non-empty sets each of cardinality less than θ. That $\mathrm{NPT}(\kappa, \aleph_1)$ fails for any singular cardinal κ is part of Shelah's work on singular compactness. The existence of "good" scales leads to $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$, a central result of important work of Magidor and Shelah on the freeness of abelian groups. The notions of "very good" and even "better" scales provide avenues for further combinatorial elucidation.

The last section discusses square-brackets partition relations, with the focus on Jónsson algebras. The existence of such algebras was an important motivation of Shelah's development of pcf theory, and early on Shelah established that $\aleph_{\omega+1}$ carries a Jónsson algebra. The general question of whether every successor of a singular cardinal carries a Jónsson algebra remains unsolved, and the section sketches the expanse of Shelah's work here.

16. Prikry-Type Forcings. In this chapter Moti Gitik presents the full range of forcing techniques that have been developed to investigate powers of singular cardinals and the Singular Cardinal Hypothesis. With his technical virtuosity and persistence Gitik has been the main contributor to the subject, and to the organization and presentation of this chapter he brings his extensive knowledge, providing several simplifications of the previously published work. To be noted is that being of the Israeli school, Gitik writes "$p > q$" for p being a stronger condition than q.

The first half deals with the work on countable cofinality. An initial section presents the basic Prikry forcing and its variants through to a strongly compact version, all having the characteristic property of adjoining new cofinal subsets without adjoining bounded subsets or collapsing cardinals. The next several sections then present the Gitik-Magidor extender-based forcing for adjoining many Prikry sequences with optimal hypotheses. As a warm-up, the simpler case when κ is already singular, $\kappa = \sup\{\kappa_n \mid n \in \omega\}$, is presented. One posits extenders on each κ_n and uses the embeddings to develop a system of ultrafilters $U_{n\alpha}$ on κ_n for adjoining Prikry sequences t_α. The forcing itself relies on getting Cohen subsets of κ^+ to guide the construction. Then the main case of an extender-based Prikry forcing with a single extender on a regular κ is presented. This forcing elaborates the previous by singularizing κ and confronts the added difficulty that the support of a condition may have cardinality κ. Finally, the forcing that additionally brings the whole situation down to render $\kappa = \aleph_\omega$ with interwoven Levy collapses is presented.

The latter half of the chapter begins with the work on uncountable cofinality. First, the basics of Radin forcing for adjoining a closed unbounded subset to a large cardinal consisting of formerly regular cardinals is carefully presented in an extensive section. This forcing had originally been given in terms of an elementary embedding $j : V \to M$, and next, a presentation based on a coherent sequence of ultrafilters is given, this providing a treatment also encompassing Magidor forcing for changing to uncountable cofinality. Then Carmi Merimovich's extender-based Radin forcing is broached.

The last section handles iterations of general "Prikry-type forcings". Such an iteration had first occurred in Magidor's 1973 result that *it is consistent that the least strongly compact cardinal is the least measurable cardinal*, and here Magidor's proof is simplified. After discussing an interesting forcing due to Jeffrey Leaning, the section turns to Easton support iterations of Prikry-type forcings. It is observed that this provides another way of establishing the consistency of the failure of SCH from the optimal hypothesis $\exists\kappa(o(\kappa) = \kappa^{++})$. The chapter ends with five open problems about powers of singular cardinals.

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17. Beginning Inner Model Theory. In this first of several chapters on inner model theory, William Mitchell authoritatively sets out the theory from $L[U]$ and K^{DJ} through to inner models of strong cardinals, the "coarse theory" not requiring fine structure. He thus performs the service of laying out the larger features and strategies of inner model theory that will frame the later chapters. There is iteration, comparison, coherence, and coiteration, and at one end sharps and mice and the other end coherent sequences of (non-overlapping) extenders. Beyond this, he provides two illuminating discussions about the further developments that involve fine structure. One is on the advantages of the modern Baldwin-Mitchell presentation with partial extenders even for the cases that he considers. The other is about what in general the core model should be in set theory, separate from any specific large cardinal assumptions.

18. The Covering Lemma. Mitchell here draws on his experience and expertise to provide an incisive account of the covering leitmotiv for inner models, which has been central to the development of inner model theory. The Jensen argument for the Covering Lemma for L has not only stimulated the formulation of new inner models in which the argument can be applied but has proven to be robust through these models to establish various results about the global affinity between inner models and the universe.

The first two sections discuss variants of the covering lemma and their applications. What is brought out is that the basic Jensen argument as a conceptual construction can be implemented in a range of inner models, but that the conclusions that one can draw depends on the large cardinal hypotheses involved and the complexity that one wants to sustain.

The third section outlines a proof, complete except for some fine structure details, of the Jensen and Dodd-Jensen covering results for L and $L[U]$. Although proofs for these cases have been devised that do not appeal to fine structure, it is deployed here in order to maintain generalizability. In fact, the Baldwin-Mitchell approach with partial extenders is already adopted for the technical advantages of local uniformity that it provides. One significant feature of the $L[U]$ case is that a weak covering property is established first and used to study ultrapower-generated indiscernibles leading to Prikry generic sequences.

The last section is devoted largely to a proof of covering for Mitchell's core model $K[\mathcal{U}]$ for coherent sequences $\mathcal{U}$ of ultrafilters. The previous proof has now to be further elaborated on account of the possible generation of complicated systems of indiscernibles, including possibly those leading to generic sequences for the Magidor forcing for changing to uncountable cofinalities. Drawing out what is possible from the covering argument, an elaborate conclusion is articulated and established. Gitik, for one direction of his culminating equi-consistency result on the Singular Cardinals Hypothesis, had applied this covering conclusion together with elements of Shelah's pcf theory to establish that if SCH fails, then in an inner model $\exists\kappa(o(\kappa) \geq \kappa^{++})$ holds. This synthetic result is next presented as a crucial application. The section, and chapter, concludes with a discussion of how the covering proof and conclusion can be extended to a strong cardinal, and the progress made with weaker versions of covering up to a Woodin cardinal and beyond.

19. An Outline of Inner Model Theory. In this chapter John Steel provides a general theory of extender models, the canonical inner models for large cardinals, getting to his model K^c. Moreover, he provides a remarkable application, to the effect that under $\mathrm{AD}^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})}$ up to a high rank V_δ is an extender model. Since it was Steel who in the mid-1990s provided the framework and made the crucial, final advances in this inner model theory, this chapter carries the stamp of experience and authority. The next chapter provides the construction of Steel's core model K up to a Woodin cardinal, a construction based on K^c, and a range of combinatorial applications. Chapter 22 describes how iteration trees, a basic component of the K^c construction, found their first substantial use in determinacy.

After covering the basics of extenders, an early section sets out the carefully wrought definition of a *fine extender sequence* $\vec{E}$. These are coherent sequences enhanced with acceptability for $J^{\vec{E}}$ and the Baldwin-Mitchell idea of having E_α be only an extender for subsets in $J_\alpha^{\vec{E}\restriction\alpha}$. A *potential premouse* is then a structure $J_\alpha^{\vec{E}}$ where $\vec{E}$ is a fine extender sequence. With Chap. 9 preliminaries, fine structure considerations are imposed on potential premice and fine structure preserving ultrapowers are schematically described.

The next section engages the project of comparing two potential premice through coiteration. Iteration trees become central for handling overlapping extenders, and iterability for comparison is articulated in terms of games and iteration strategies for securing well-founded limits of models along branches. Fine structural considerations have become crucial to carrying out the process internally in extender models.

The succeeding section establishes the Dodd-Jensen Lemma about the minimality of iterations copied across fine structure preserving maps, as well as a weak Neeman-Steel version sufficient for present purposes. A further section deals with crucial results about solidity and condensation. These sections, elaborating the analysis starting with iterations trees, carve a fine path through a thicket of detail.

With these preparations, a culminating section provides the K^c construction and the resulting Steel *background certified core model* K^c. The model is an extender model $L[\vec{E}\,]$ for a fine extender sequence $\vec{E}$ defined according to the following stratagem: Given $\vec{E}\restriction\alpha$, an F is next adjoined if it is "certified" by a "background extender" F^*, in that F is the restriction to $J_\alpha^{\vec{E}\restriction\alpha}$ of F^*, an extender in V with sufficiently strong properties to guarantee iterability of $\vec{E}\restriction\alpha^\frown\langle F\rangle$. That such an $\vec{E}$ can be defined canonically is at the heart of the construction.

The concluding two sections bring inner model theory and determinacy together for the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$. Both sections proceed under the assumption that there are infinitely many Woodin cardinals with a measurable cardinal above them, so that in particular $\mathrm{AD}^{L(\mathbb{R})}$ holds. The main thrust of the first section is that the reals in the minimal iterable inner model M_ω satisfying "there are infinitely many Woodin cardinals" are exactly the reals in $\mathrm{OD}^{L(\mathbb{R})}$. The last section builds on this work to establish, using the (full) Dodd-Jensen Lemma, that $\mathrm{HOD}^{L(\mathbb{R})}$ is "almost" an iterate M_∞ of M_ω. Specifically, for δ the large projective ordinal $(\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta = M_\infty \cap V_\delta$. This suffices in particular to establish under $\mathrm{AD}^{L(\mathbb{R})}$ that $\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{GCH}$. It is remarkable that an inner model incipiently based on global definability can be shown to have structure as given by local definability and extender analysis.

20. A Core Model Tool Box and Guide. Building on the general theory of the previous chapter, Ernest Schimmerling develops its historical source, Steel's core model K up to a Woodin cardinal, and discusses combinatorial applications of it across set theory. Having been one of the contributors to the covering lemma theory for K and the initiating investigator of combinatorial principles there, Schimmerling provides a measured, wide-ranging account, one with careful accreditations.

The first half of the chapter is devoted to the basic theory of K. Going "up to" a Woodin cardinal, the "anti-large cardinal hypothesis" that there is no inner model with a Woodin cardinal is assumed. But moreover as Steel initially did, an additional "technical hypothesis" that there is a measurable cardinal Ω is assumed. Ω becomes regulative for the construction of K, schematically playing the role of the class On of ordinals. Regarding K^c as now a set premouse of height Ω, one works with *weasels*, other such premice, and uses the crucial simplifying property that if they have no Woodin cardinals, then their iteration trees have at most one cofinal well-founded branch. A definition of K second-order on $H(\Omega)$ is first developed, and then a first-order, recursive definition.

With K in hand, a useful "tools" section provides, without proof, a range of properties of K, from covering, forcing absoluteness and rigidity to combinatorial principles. The next section outlines a proof of the "countably closed" weak covering property for K. The proof assumes familiarity with that of analogous results as given e.g. in Chaps. 9 and 18 and very much depends on the first-order definition of K. The final section provides, without proof, applications of K and generally, core models at the level that involves iteration trees. One sees at a glance how central this inner model theory has become, with the involvements described in determinacy, trees, ideals, forcing axioms, and pcf theory.

21. Structural Consequences of AD. In this first of several chapters on determinacy, Steve Jackson surveys the structural consequences of determinacy for sets of reals. The chapter thus serves as a fitting sequel to Moschovakis's book *Descriptive Set Theory* (1980). The advances have been in two directions, the extension of the scale theory beyond the projective sets into a substantial class of sets of reals in $L(\mathbb{R})$ and the analysis of the fine combinatorial structure of cardinals provided by the computation of the projective ordinals. With both directions calibrated by the analysis of definable sets in terms of definable well-ordered stratifications, the structure theory has remarkable richness and complexity as well as overall coherence.

An early section lays the basis with a review of basic notions: scales and periodicity, the Coding Lemma, projective ordinals, Wadge reducibility—and with some topics already going beyond the scope of the Moschovakis book—

$\mathbf{\Sigma}^2_1$ sets of reals and infinite-exponent partition relations.

The next section develops the scale theory provided by Suslin cardinals under AD, the arguments mainly due to Martin. Let $S(\kappa)$ denote the class of κ-Suslin sets. A cardinal κ is *Suslin iff* $S(\kappa) - \bigcup_{\kappa'<\kappa} S(\kappa') \neq \emptyset$. That $\aleph_1$ is a Suslin cardinal is a classical result, and PD implies that the projective ordinals $\boldsymbol{\delta}^1_n$ for odd $n \in \omega$ are Suslin. The late 1970s Martin-Steel result that AD + $V = L(\mathbb{R})$ implies that $\mathbf{\Sigma}^2_1$ is the largest class with the scale property and $\mathbf{\Sigma}^2_1 = \bigcup_\kappa S(\kappa)$ provides the new, broad context. With $S(\kappa)$ taken as the analogue of the analytic sets, corresponding analogues of the projective hierarchy and projective ordinals are formulated. The scale property is then inductively propagated using Wadge reducibility and the weakly homogeneous trees available. Thus, the scale theory of the projective sets has been successfully abstracted, with the arguments applied in a suitably articulated setting.

The succeeding two sections present a schematic approach to the computation of the projective ordinals, which had been carried out by the author in a *tour de force* in the latter 1980s. $\kappa \longrightarrow (\kappa)^\lambda$ asserts that if the increasing functions from λ into κ are partitioned into two cells, then there is an $H \subseteq \kappa$ of cardinality κ such that all the increasing functions from λ into H are in one cell. The *strong partition property for* κ is the assertion $\kappa \longrightarrow (\kappa)^\kappa$ and the *weak partition property for* κ is the assertion $\forall\lambda < \kappa((\kappa \longrightarrow (\kappa)^\lambda)$. In the early 1970s Martin established under AD the strong partition property for ω_1, a striking result at the time. Kunen then carried out a detailed analysis of ultrapowers that led to the weak partition property for $\boldsymbol{\delta}^1_3$, which Martin had previously shown under AD to be $\aleph_{\omega+1}$, the third uncountable regular cardinal. In the section on "a theory of ω_1", this work is reorganized by starting with the weak partition property for ω_1 and establishing in turn the upper bound $\boldsymbol{\delta}^1_3 \leq \aleph_{\omega+1}$; the strong partition property for ω_1; the lower bound $\aleph_{\omega+1} \leq \boldsymbol{\delta}^1_3$; and the weak partition property for $\boldsymbol{\delta}^1_3$. This is done in terms of generalizable "descriptions", and the section on higher descriptions starts with the weak partition property for $\boldsymbol{\delta}^1_3$ and proceeds analogously to establish the upper bound $\boldsymbol{\delta}^1_5 \leq \aleph_{\omega^{\omega^\omega}+1}$; the strong partition property for $\boldsymbol{\delta}^1_3$; the lower bound $\aleph_{\omega^{\omega^\omega}+1} \leq \boldsymbol{\delta}^1_5$; and the weak partition property for $\boldsymbol{\delta}^1_5$. In this indicated propagation with descriptions, the author's computation of $\boldsymbol{\delta}^1_5$ and larger projective ordinals has been given a fortunate perspicuity and surveyability.

The final section explores the possibilities for extending throughout $L(\mathbb{R})$ the sort of fine analysis given by the computation of the projective ordinals. A weak square principle $\boxminus_{\kappa,\lambda}$ is established toward the goal of getting at global principles that might help propagate the inductive analysis via the Suslin cardinals.

22. Determinacy in $L(\mathbb{R})$. Woodin's culminating result that AD is equi-consistent with the existence of infinitely many Woodin cardinals figures centrally in this and the next chapters, which establish each direction of the equi-consistency in turn. In this chapter Itay Neeman develops the theme of getting determinacy from large cardinals. In getting technically optimal such results through the use of "long" games, Neeman's book *The Determinacy of Long Games* (2004) was an important contribution along these lines. In this chapter Neeman ultimately provides a complete, tailored proof of Woodin's result that if there are infinitely Woodin cardinals with a measurable cardinal above them, then $\mathrm{AD}^{L(\mathbb{R})}$. He first provides the historical and mathematical lines of approach in terms of concepts and methods of wider applicability and then proceeds with his own, well-crafted trajectory to the final conclusion.

The first several sections presents the basic, Martin-Steel theory of iteration trees. Iterability for the needed case of linear compositions of trees of length ω is articulated in terms of games and strategies and then established. The importance of Woodin cardinals is then brought out for creating complex iteration trees, the complexity discussed in terms of the author's notion of *type* for a set of formulas in place of the former Martin-Steel alternating chains.

The next sections start the ascent to the determinacy of sets in $L(\mathbb{R})$. The first vehicle is the concept of a homogeneously Suslin set of reals, a projection of a homogeneous tree and hence determined. After recasting Martin's classical $\mathbf{\Pi}^1_1$-Determinacy result, the 1985 Martin-Steel breakthrough result is presented, with its propagation of determinacy through the projective hierarchy with Woodin cardinals and iteration trees.

The last several sections make the final ascent with the author's specific approach, one based on getting determinacy by making Woodin cardinals countable with forcing rather than using stationary tower forcing as in Woodin's original proof. First, Woodin cardinals, through forcing and absoluteness, are shown to establish the determinacy of an important class of sets of reals wider than the homogeneously Suslin sets, the *universally Baire sets* of Qi Feng, Magidor, and Woodin. Second, getting at the technical heart of the matter, it is shown that given any real, models with many Woodin cardinals can be iterated to absorb the real in a further generic extension. Finally, with a least counter-example argument, AD is established in a "derived model" assuming the existence of infinitely many Woodin cardinals—getting one direction of Woodin's equi-consistency result—and assuming further the existence of a measurable cardinal above, AD is established in L($\mathbb{R}$) itself.

23. Large Cardinals from Determinacy. In this extensive, well-rounded, and sophisticated chapter Peter Koellner and Hugh Woodin set out the latter's work on getting large cardinals from determinacy hypotheses. The focal results were in place by the early 1990s, but this is the first venue where a full-fledged, systematic account is provided. With hindsight the authors are able to present a well-motivated, self-contained development organized around structural themes buttressing the extensive results.

The first two thirds of the chapter are framed as making an ascent to the Generation Theorem, an abstract theorem that provides a template for generating Woodin cardinals from refined determinacy hypotheses. In fact, the early sections add layer upon layer of complexity in an informative, well-motivated manner to get at more and more large cardinal conclusions.

Section 2 casts Solovay's seminal 1967 result that ω_1 is measurable under ZF + AD in a generalizable manner that draws out boundedness and coding techniques for getting normal ultrafilters. The generalizability is then illustrated by showing that under ZF + AD the projective ordinal $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$, "the least stable in $L(\mathbb{R})$", is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. Gearing up, Sect. 3 reviews the Moschovakis Coding Lemma and provides a strong, uniform version that will become crucial. Section 4 then establishes, as a precursor to the Generation Theorem, that under ZF + DC + AD a pivotal ordinal $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. First, reflection properties are developed that will play the role played earlier by boundedness. Then the notion of *strong normality* is used to establish that $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is λ-strong for cofinally many $\lambda < \Theta^{L(\mathbb{R})}$. Reflection properties and uniform coding are then worked to secure strong normality. Finally, with crucial appeals to AD and special properties of $\mathrm{HOD}^{L(\mathbb{R})}$, the strongness properties established for $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ are shown to relativize for $T \subseteq \Theta$ in $\mathrm{HOD}^{L(\mathbb{R})}$ to provide corresponding λ-T-strong cardinals δ_T, thus leapfrogging up to get that $\Theta^{L(\mathbb{R})}$ is Woodin in $\mathrm{HOD}^{L(\mathbb{R})}$.

The heights are reached in Sect. 5 where the work of the previous section is abstracted to establish two theorems on Woodin cardinals in a general setting. The first shows that in certain strong determinacy contexts HOD can contain many Woodin cardinals, and the second is the central Generation Theorem. The aim of this theorem is to show that the construction of Sect. 4 can be driven by lightface determinacy alone. To simulate the previous use of real parameters, the notion of *strategic determinacy* is introduced, a notion that resembles boldface determinacy but can nonetheless hold in settings with AC. Indeed, this notion is motivated by showing that it can hold in $L[S, x]$, where S is a class of ordinals and x is a real. With this in hand the Generation Theorem is finally established, and a number of instantial cases are presented.

Section 6 applies the Generation Theorem to derive the optimal amount of large cardinal strength from both lightface and boldface determinacy. The main lightface result is that ZF + DC + Δ^1_2-determinacy implies that there is a Turing cone of reals x such that $\omega_2^{L[x]}$ is a Woodin cardinal in $\mathrm{HOD}^{L[x]}$. The task here is to show that Δ^1_2-determinacy secures strategic determinacy. The main boldface result is that ZF + AD implies that in a generalized Prikry forcing extension, there are infinitely many Woodin cardinals in the corresponding HOD. The task here is to show that the Generation Theorem can be iteratively applied to generate infinitely many Woodin cardinals.

Section 7 attends to a reduction to second-order Peano Arithmetic. A first localization of the Generation Theorem shows that Δ^1_2-determinacy implies that for a Turing cone of reals x, $\omega_1^{L[x]}$ is a Woodin cardinal in $L[x]$. A second localization then shows that the proof can in fact be carried out in second-order Peano Arithmetic, to establish that if that theory plus Δ^1_2-determinacy is consistent, then so is ZFC + "On is Woodin", the latter assertion to be understood schematically.

The synthetic final Sect. 8 describes the remarkable confluences, seen in the later 1990s, of definable determinacy and inner model theory. First, actual equivalences between propositions of definable determinacy and propositions about the existence of inner models with Woodin cardinals are described. Then, the earlier HOD analysis is revisited in light of the Steel work on $\mathrm{HOD}^{L(\mathbb{R})}$, described in his Chap. 19. The full $\mathrm{HOD}^{L(\mathbb{R})}$ is not itself an extender model, but can nonetheless be comprehended as a fine-structural inner model of a new sort.

24. Forcing over Models of Determinacy. In this last chapter Paul Larson describes work of Woodin on forcing over models of determinacy. We take the opportunity here to describe that work, thereby framing the chapter. After his remarkable successes culminating in his synthetic equiconsistency results about AD and large cardinals, Woodin in the mid-1990s entered a new, middle period of his research with the investigation of $\mathbb{P}_{\max}$ forcing extensions of models of AD. Quickly becoming a far-reaching theory of maximal and canonical forcing extensions that model ZFC, the subject shed new light on the inner workings of determinacy at the level of $P(\omega_1)$ and the extent of structure in ZFC extensions, even to the possible failure of the Continuum Hypothesis.

Woodin's remarkable *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal* (1999) in nearly one-thousand pages sets out of his work into his middle period. The book's main thrust is the specification of a canonical, maximal model of ZFC in the following sense: *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and that there is a Woodin cardinal with a measurable cardinal above it. Then there is in* $L(\mathbb{R})$ *a (countably closed and homogeneous) partial order* $\mathbb{P}_{\max}$ *so that for* G $\mathbb{P}_{\max}$*-generic over* $L(\mathbb{R})$*,* $L(\mathbb{R})[G]$ *models* ZFC*, and: for any* Π_2 *(i.e.* $\forall x \exists y$*) sentence satisfied in the structure* $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$*, that sentence is already satisfied in* $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle^{L(\mathbb{R})[G]}$*, the structure relativized to the generic extension.*

With $H(\omega_2)$ suitably accommodating $P(\omega_1)$ and the intrinsic ideal NS_{ω_1} participating, Woodin argues that $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$ is the next natural extension of second-order arithmetic, which is identifiable with $\langle H(\omega_1), \in\rangle$. A pivotal, historical point about $\mathbb{P}_{\max}$ is that since $\neg$CH is equivalent to a Π_2 sentence of $\langle H(\omega_2), \in\rangle$ and there is a generic extension satisfying $\neg$CH yet preserving the hypotheses of the above result, CH actually fails in $L(\mathbb{R})[G]$. Generally, various combinatorial propositions about ω_1 are similarly consistent via "mild" forcing and are expressible as Π_2 assertions about $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$, and hence, these propositions hold in $L(\mathbb{R})[G]$. In this very substantial sense, $L(\mathbb{R})[G]$ is a canonical generic extension of $L(\mathbb{R})$.

In his book Woodin's presents an axiom that codifies his motivation for formulating $\mathbb{P}_{\max}$:

$$(*) \qquad \mathrm{AD}^{L(\mathbb{R})} \text{ and: } L(\mathcal{P}(\omega_1)) \text{ is a } \mathbb{P}_{\max}\text{-generic extension of } L(\mathbb{R}).$$

Woodin then develops a variant $\mathbb{Q}_{\max}$ of $\mathbb{P}_{\max}$ that provides extensions in which NS_{ω_1} is $\aleph_1$-dense. Woodin had famously shown that NS_{ω_1} being $\aleph_1$-dense is equivalent in ZF to AD, and with $\mathbb{Q}_{\max}$ he provides a systematic treatment of this result.

Woodin subsequently investigates $\mathbb{P}_{\max}$ extensions of AD models larger than $L(\mathbb{R})$. This enterprise is fueled by a corresponding strong form AD^+ of AD, and with it Woodin is able to starting scaling combinatorial propositions about ω_2 and even forms of Chang's Conjecture. In the final chapter of his book Woodin casts a light into the horizon with the formulation of his Ω-logic. With this new logic and AD^+, a more pristine approach can be taken to $\neg$CH, one that can subsume $\mathbb{P}_{\max}$ extensions in a more direct, albeit abstract, formulation. In work of the 21st century, Woodin will argue for the negation of the Continuum Hypothesis on the basis of his Ω-logic and a corresponding Ω Conjecture.

Larson in this final chapter of this Handbook offers a preparatory guide to Woodin's $\mathbb{P}_{\max}$, one that is to be highly appreciated for providing a patient, accessible approach. The first seven sections present a complete, self-contained analysis of the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ in an illuminating manner, proceeding incrementally by introducing hypotheses and methods as needed. After setting out the theory of iterated generic elementary embeddings fundamental to $\mathbb{P}_{\max}$, the partial order is formulated and its countable closure is established. After developing *A-iterability*, a generalized iterability property, it is applied to establish crucial structural results about the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$. Then the heralded Π_2 maximality with respect to $\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}\rangle$ is established, and assuming Woodin's axiom $(*)$, the notable minimality result that any subset of ω_1 added by a generic filter generates the entire extension.

The last several sections briefly consider $\mathbb{P}_{\max}$ extensions of larger models under AD^+; Woodin's Ω-logic and Ω Conjecture; and several variations of $\mathbb{P}_{\max}$, starting with $\mathbb{Q}_{\max}$. This sampling reflects the accomplishments with $\mathbb{P}_{\max}$ and suggests the expansive possibilities to be explored.

1. Stationary Sets

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Contents

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1. The Closed Unbounded Filter

1.1. Closed Unbounded Sets

Stationary sets play a fundamental role in modern set theory. This chapter attempts to explain this role and to describe the structure of stationary sets of ordinals and their generalizations.

The concept of stationary set first appeared in the 1950's; the definition is due to Gérard Bloch [16], and the fundamental theorem on stationary sets was proved by Géza Fodor [24]. However, the concept of a stationary set is implicit in the work of Paul Mahlo [71].

The precursor of Fodor's Theorem is the 1929 result of Pavel Alexandroff and Pavel Urysohn [2]: If $f(\alpha) < \alpha$ for all α such that $0 < \alpha < \omega_1$, then f is constant on an uncountable set.

Let us call an ordinal function f *regressive* if $f(\alpha) < \alpha$ whenever $\alpha > 0$. Fodor's Theorem (Theorem 1.5) states that every regressive function on a stationary set is constant on a stationary set. As a consequence, a set $S \subseteq \omega_1$ is stationary if and only if every regressive function on S is constant on an uncountable set.

In this section we develop the theory of closed unbounded and stationary subsets of a regular uncountable cardinal.

If X is a set of ordinals, then α is a *limit point* of X if $\alpha > 0$ and $\sup(X \cap \alpha) = \alpha$. A set $X \subseteq \kappa$ is *closed* (in the order topology on κ) if and only if X includes $\mathrm{Lim}(X)$, the set of all limit points of X less than κ.

1.1 Definition. Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is *closed unbounded* (or *club* for short) if it is closed and also an unbounded subset of κ. A set $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq \kappa$.

It is easily seen that the intersection of any number of closed sets is closed. The basic observation is that if C_1 and C_2 are both closed unbounded, then $C_1 \cap C_2$ is also closed unbounded. This leads to the following basic property.

1.2 Proposition. *The intersection of less than κ closed unbounded subsets of κ is closed unbounded.*

Consequently, the closed unbounded sets generate a κ-complete filter on κ called the *closed unbounded filter*. The dual ideal (which is κ-complete and contains all singletons) consists of all sets that are disjoint from some closed unbounded set—the nonstationary sets, and is thus called the *nonstationary* ideal, denoted NS.

If I is any nontrivial ideal on κ, then I^+ denotes the set $P(\kappa) - I$ of all *I-positive* sets. Thus, stationary subsets of κ are exactly those that are NS-positive.

1.3 Definition. Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a κ-sequence of subsets of κ. Its *diagonal intersection* is the set

$$\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \textstyle\bigcap_{\alpha<\xi} X_\alpha\};$$

its *diagonal union* is

$$\Sigma_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \textstyle\bigcup_{\alpha<\xi} X_\alpha\}.$$

The following lemma states that the closed unbounded filter is closed under diagonal intersections (or dually, that the nonstationary ideal is closed under diagonal unions):

1.4 Lemma. *If $\langle C_\alpha : \alpha < \kappa\rangle$ is a sequence of closed unbounded subsets of κ, then its diagonal intersection is closed unbounded.*

This immediately implies Fodor's Theorem:

1.5 Theorem (Fodor [24]). *If S is a stationary subset of κ and if f is a regressive function on S, then there exists some $\gamma < \kappa$ such that $f(\alpha) = \gamma$ on a stationary subset of S.*

Proof. Let us assume that for each $\gamma < \kappa$ there exists a closed unbounded set C_γ such that $f(\alpha) \neq \gamma$ for each $\alpha \in S \cap C_\gamma$. Let $C = \Delta_{\gamma<\kappa} C_\gamma$. As C is closed unbounded, there exists an $\alpha > 0$ in $S \cap C$. By the definition of C it follows that $f(\alpha) \geq \alpha$, a contradiction. ⊣

A nontrivial κ-complete ideal I on κ is called *normal* (and so is its dual filter) if I is closed under diagonal unions; equivalently, if for every $A \in I^+$, every regressive function on A is constant on some I-positive set. Thus Fodor's Theorem (or Lemma 1.4) states that the nonstationary ideal (and the club filter) is normal. In fact, the nonstationary ideal is the smallest normal κ-complete ideal on κ:

1.6 Proposition. *If F is a normal κ-complete filter on κ, then F contains all closed unbounded sets.*

Proof. If C is a club subset of κ, let $\langle a_\alpha : \alpha < \kappa\rangle$ be the increasing enumeration of C. Then

$$C \supseteq \Delta_{\alpha<\kappa}\{\xi : a_{\alpha+1} < \xi < \kappa\} \in F,$$

because F contains all final segments (being nontrivial and κ-complete). ⊣

In other words, if I is normal, then every I-positive set is stationary.

The quotient algebra $B = P(\kappa)/\mathrm{NS}$ is a κ-complete Boolean algebra, where the Boolean operations $\Sigma_{\alpha<\gamma}$ and $\Pi_{\alpha<\gamma}$ for $\gamma < \kappa$ are induced by $\bigcup_{\alpha<\gamma}$ and $\bigcap_{\alpha<\gamma}$. Fodor's Theorem implies that B is in fact κ^+-complete: If $\{X_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ, then $\Delta_{\alpha<\kappa} X_\alpha$ and $\Sigma_{\alpha<\kappa} X_\alpha$ are, respectively, the greatest lower bound and the least upper bound of the equivalence classes $X_\alpha/\mathrm{NS} \in B$. This observation also shows that if $\langle X_\alpha : \alpha < \kappa\rangle$

and $\langle Y_\alpha : \alpha < \kappa\rangle$ are two enumerations of the same collection, then $\Delta_\alpha X_\alpha$ and $\Delta_\alpha Y_\alpha$ differ only by a nonstationary set.

The following characterization of the club filter is often useful, in particular when used in its generalized form (see Sect. 6). Let $F : [\kappa]^{<\omega} \to \kappa$; an ordinal $\gamma < \kappa$ is a *closure point* of F if $F(\alpha_1, \dots, \alpha_n) < \gamma$ whenever $\alpha_1, \dots, \alpha_n < \gamma$. It is easy to see that the set Cl_F of all closure points of F is a club. Conversely, if C is a club, define $F : [\kappa]^{<\omega} \to \kappa$ by letting $F(e)$ be the least element of C greater than $\max(e)$. It is clear that $Cl_F = \mathrm{Lim}(C)$. Thus every club contains Cl_F for some F, and we have this characterization of the club filter:

1.7 Proposition. *The club filter is generated by the sets Cl_F, for all $F : [\kappa]^{<\omega} \to \kappa$. A set $S \subseteq \kappa$ is stationary if and only if for every $F : [\kappa]^{<\omega} \to \kappa$, S contains a closure point of F.*

1.2. Splitting Stationary Sets

It is not immediately obvious that the club filter is not an ultrafilter—that there exist stationary sets that are *co-stationary*, i.e. whose complement is stationary. The basic result is the following theorem of Solovay:

1.8 Theorem (Solovay [85]). *Let κ be a regular uncountable cardinal. Then every stationary subset of κ can be partitioned into κ disjoint stationary sets.*

Solovay's proof of this basic result of combinatorial set theory uses methods of forcing and large cardinals, and we shall describe it later in this section. For an elementary proof, see e.g. [49, p. 434].

To illustrate the combinatorics involved, let us prove a special case of Solovay's theorem.

1.9 Proposition. *There exist $\aleph_1$ pairwise disjoint stationary subsets of ω_1.*

Proof. For each limit $\alpha < \omega_1$, choose an increasing sequence $\langle a_n^\alpha : n \in \omega\rangle$ with limit α. We claim that there is an n such that for all $\eta < \omega_1$, there are stationary many α such that $a_n^\alpha \geq \eta$: Otherwise there exists, for each n, some η_n such that $a_n^\alpha \geq \eta_n$ for only a nonstationary set of α's. By ω_1-completeness, for all but a nonstationary set of α's the sequences $\{a_n^\alpha\}_n$ are bounded by $\sup_n \eta_n$. A contradiction.

Thus let n be such that for all η, the set $S_\eta = \{\alpha : a_n^\alpha \geq \eta\}$ is stationary. The function $f(\alpha) = a_n^\alpha$ is regressive and so by Fodor's Theorem, there is some $\gamma_\eta \geq \eta$ such that $T_\eta = \{\alpha : a_n^\alpha = \gamma_\eta\}$ is stationary. Clearly, there are $\aleph_1$ distinct values of γ_η and therefore $\aleph_1$ mutually disjoint sets T_η. ⊣

Let κ be a regular uncountable cardinal, and let $\lambda < \kappa$ be regular. Let

$$\mathbf{E}^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}.$$

For each λ, $\mathbf{E}^\kappa_\lambda$ is a stationary set. An easy modification of the proof of Proposition 1.9 above shows that for every regular $\lambda < \kappa$, every stationary subset of $\mathbf{E}^\kappa_\lambda$ can be split into κ disjoint stationary sets.

The union $\bigcup_\lambda \mathbf{E}^\kappa_\lambda$ is the set of all singular limit ordinals less than κ. Its complement is the set $\mathrm{Reg}(\kappa)$ of all regular cardinals less than κ. The set $\mathrm{Reg}(\kappa)$ is stationary just in case κ is a Mahlo cardinal.

1.3. Generic Ultrapowers

Let M be a transitive model of ZFC, and let κ be a cardinal in M. Let U be an M-*ultrafilter*, i.e. an ultrafilter on the set algebra $P(\kappa) \cap M$. Using functions $f \in M$ on κ, one can form an ultrapower $N = \mathrm{Ult}_U(M)$, which is a model of ZFC but not necessarily well-founded:

$$\begin{aligned} f =^* g &\quad\iff\quad \{\alpha : f(\alpha) = g(\alpha)\} \in U, \\ f \in^* g &\quad\iff\quad \{\alpha : f(\alpha) \in g(\alpha)\} \in U. \end{aligned}$$

The (equivalence classes of) constant functions $c_x(\alpha) = x$ provide an elementary embedding $j : (M, \in) \to (N, \in^*)$, where $j(x) = c_x$, for all $x \in M$.

An M-ultrafilter U is M-κ-*complete* if it is closed under intersections of families $\{X_\alpha : \alpha < \gamma\} \in M$, for all $\gamma < \kappa$; U is *normal* if every regressive $f \in M$ is constant on a set in U.

1.10 Proposition. *Suppose that U is a nonprincipal M-κ-complete, normal M-ultrafilter on κ. Then the ordinals of N have a well-ordered initial segment of order type at least $\kappa + 1$, $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and κ is represented in N by the diagonal function $d(\alpha) = \alpha$.*

Now let κ be a regular uncountable cardinal and consider the forcing notion $(P, <)$ where P is the collection of all stationary subsets of κ, and the ordering is by inclusion. Let B be the complete Boolean algebra $B = B(P)$, the completion of $(P, <)$. Equivalently, B is the completion of the Boolean algebra $P(\kappa)/\mathrm{NS}$. Let us consider the generic extension $V[G]$ given by a generic $G \subseteq P$. It is rather clear that G is a nonprincipal V-κ-complete normal ultrafilter on κ. Thus Proposition 1.10 applies, where $N = \mathrm{Ult}_G(V)$. The model $\mathrm{Ult}_G(V)$ is called a *generic ultrapower*.

There is more on generic ultrapowers in Foreman's chapter in this Handbook; here we use them to present the original argument of Solovay's [85]. First we prove a lemma (that will be generalized in Sect. 2):

1.11 Lemma. *Let κ be a regular uncountable cardinal, and let S be a stationary set. Then the set*

$$T = \{\alpha \in S : \text{either } \alpha \notin \mathrm{Reg}(\kappa) \text{ or } S \cap \alpha \text{ is not a stationary subset of } \alpha\}$$

is stationary.

Proof. Let C be a club and let us show that $T \cap C$ is nonempty. Let α be the least element of the nonempty set $S \cap C'$ where $C' = \mathrm{Lim}(C - \omega)$. If α is not regular, then $\alpha \in T \cap C$ and we are done, so assume that $\alpha \in \mathrm{Reg}(\kappa)$. Now $C' \cap \alpha$ is a club subset of α disjoint from $S \cap \alpha$, and so $\alpha \in T$. ⊣

We shall now outline the proof of Solovay's theorem:

Proof of Theorem 1.8. Let S be a stationary subset of κ that cannot be partitioned into κ disjoint stationary sets. By Proposition 1.9 and the remarks following its proof, we have $S \subseteq \operatorname{Reg}(\kappa)$. Let $I = \mathrm{NS}{\restriction}S$, i.e. $I = \{X \subseteq \kappa : X \cap S \in \mathrm{NS}\}$. The ideal I is *κ-saturated*, i.e. every disjoint family $W \subseteq I^+$ has size less than κ; equivalently, $B = P(\kappa)/I$ has the κ-chain condition. I is also κ-complete and normal.

Let $G \subseteq I^+$ be generic, and let $N = \mathrm{Ult}_G(V)$ be the generic ultrapower. As I is κ-saturated, N is well-founded (this is proved by showing that every name $\dot{f}$ for a function in V on κ can be replaced by an actual function on κ). Thus we have (in $V[G]$) an elementary embedding $j : V \to N$ where N is a transitive class, $j(\gamma) = \gamma$ for all $\gamma < \kappa$, and κ is represented in N by the diagonal function $d(\alpha) = \alpha$. Note that if $A \subseteq \kappa$ is any set (in V), then $A \in N$: this is because $A = j(A) \cap \kappa$; in fact A is represented by the function $f(\alpha) = A \cap \alpha$.

Now we use the fact that κ-c.c. forcing preserves stationarity (cf. Theorem 1.13 below). Thus S is stationary in $V[G]$, and because $N \subseteq V[G]$, S is a stationary set in the model N. By the ultrapower theorem we have

$$V[G] \vDash S \cap \alpha \text{ is stationary for } G\text{-almost all } \alpha.$$

This, translated into forcing, gives

$$\{\alpha \in S : S \cap \alpha \text{ is not stationary}\} \in I$$

but that contradicts Lemma 1.11. $\dashv$

Another major application of generic ultrapowers is Silver's Theorem:

1.12 Theorem (Silver [84]). *Let λ be a singular cardinal of uncountable cofinality. If $2^\alpha = \alpha^+$ for all cardinals $\alpha < \lambda$, then $2^\lambda = \lambda^+$.*

Silver's Theorem is actually stronger than this. It assumes only that $2^\alpha = \alpha^+$ for a stationary set of α's (see Sect. 2 for the definition of "stationary" when λ is not regular). The proof uses a generic ultrapower. Even though $\mathrm{Ult}_G(V)$ is not necessarily well founded, the method of generic ultrapowers enables one to conclude that $2^\lambda = \lambda^+$ when $2^\alpha = \alpha^+$ holds almost everywhere.

Silver's Theorem can be proved by purely combinatorial methods [9, 10]. In [29], Galvin and Hajnal used combinatorial properties of stationary sets to prove a substantial generalization of Silver's Theorem (superseded only by Shelah's powerful pcf theory). For further generalizations using stationary sets and generic ultrapowers, see [50] and [51].

One of the concepts introduced in [29] is the *Galvin-Hajnal norm* of an ordinal function. If f and g are ordinal functions on a regular uncountable cardinal κ, let $f < g$ if $\{\alpha < \kappa : f(\alpha) < g(\alpha)\}$ contains a club. The relation

$<$ is a well-founded partial order, and the norm $\|f\|$ is the rank of f in the relation $<$.

We remark that if $f < g$, then in the generic ultrapower (by NS), the ordinal represented by f is smaller than the ordinal represented by g.

By induction on η one can easily show that for each $\eta < \kappa^+$ there exists a *canonical* function $f_\eta : \kappa \to \kappa$ of norm η, i.e. $\|f_\eta\| = \eta$ and whenever $\|h\| = \eta$, then $\{\alpha : f_\eta(\alpha) \le h(\alpha)\}$ contains a club. (Proof: Let $f_0(\alpha) = 0$, $f_{\eta+1}(\alpha) = f_\eta(\alpha) + 1$. If $\eta < \kappa^+$ is a limit ordinal, let $\lambda = \mathrm{cf}(\eta)$ and let $\eta = \lim_{\xi\to\lambda} \eta_\xi$. If $\lambda < \kappa$, let $f_\eta(\alpha) = \sup_{\xi<\lambda} f_{\eta_\xi}(\alpha)$ and if $\lambda = \kappa$, let $f_\eta(\alpha) = \sup_{\xi<\alpha} f_{\eta_\xi}(\alpha)$.)

A canonical function of norm κ^+ may or may not exist; existence is consistent with ZFC (cf. [52]). The existence of canonical functions f_η for all η is equi-consistent with a measurable cardinal [58].

1.4. Stationary Sets in Generic Extensions

Let M and N be transitive models and let $M \subseteq N$. Let κ be a regular uncountable cardinal and let $S \in M$ be a subset of κ. Clearly, if S is stationary in the model N, then S is stationary in M; the converse is not necessarily true, and κ may even not be regular or uncountable in N. It is important to know which forcing extensions preserve stationarity and we shall return to the general case in Sect. 5. For now, we state two important special cases:

1.13 Theorem. *Let κ be a regular uncountable cardinal and let P be a notion of forcing.*

(a) *If P satisfies the κ-chain condition, then every club $C \in V[G]$ has a club subset D in the ground model. Hence every stationary S remains stationary in $V[G]$.*

(b) *If P is λ-closed for every $\lambda < \kappa$, then every stationary S remains stationary in $V[G]$.*

Proof (Outline). (a) This follows from this basic fact on forcing: if P is κ-c.c., then every unbounded $A \subseteq \kappa$ in $V[G]$ has an unbounded subset in V.

(b) Let $p \Vdash \dot{C}$ is a club; we find a $\gamma \in S$ and a $q \le p$ such that $q \Vdash \gamma \in \dot{C}$ as follows: we construct an increasing continuous ordinal sequence $\{\gamma_\alpha\}_{\alpha<\kappa}$ and a decreasing sequence $\{p_\alpha\}$ of conditions such that $p_{\alpha+1} \Vdash \gamma_{\alpha+1} \in \dot{C}$, and if α is a limit ordinal, then $\gamma_\alpha = \lim_{\xi<\alpha} \gamma_\xi$ and p_α is a lower bound of $\{p_\xi\}_{\xi<\alpha}$. There is some limit ordinal α such that $\gamma_\alpha \in S$. It follows that $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$. ⊣

We shall now describe the standard way of controlling stationary sets in generic extensions, called *shooting a club*. First we deal with the simplest case when $\kappa = \aleph_1$. Let S be a stationary subset of ω_1, and consider the following forcing P_S (cf. [12]): The forcing conditions are all bounded closed sets p of countable ordinals such that $p \subseteq S$. A condition q is stronger than p if q end-extends p, i.e. $p = q \cap \alpha$ for some α.

It is clear that this forcing produces ("shoots") a closed unbounded subset of S in the generic extension, thus the complement of S becomes nonstationary. The main point of [12] is that ω_1 is preserved and in fact $V[G]$ adds no new countable sets. Also, every stationary subset of S remains stationary.

The forcing P_S has the obvious generalization to $\kappa > \aleph_1$, but more care is required to guarantee that no new small sets of ordinals are added. For instance, this is the case when S contains the set Sing of all singular ordinals $< \kappa$. For a more detailed discussion of this problem see [1].

1.5. Some Combinatorial Principles

There has been a proliferation of combinatorial principles involving closed unbounded and stationary sets. Most can be traced back to Jensen's investigation of the fine structure of L [59] and generalize either Jensen's *diamond* ($\diamondsuit$) or *square* ($\Box$). We conclude this section by briefly mentioning diamond and *club-guessing*, and only their typical special cases.

1.14 Theorem ($\diamondsuit(\aleph_1)$, Jensen [59]). *Assume $V = L$. There exists a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ with each $a_\alpha \subseteq \alpha$, such that for every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha = a_\alpha\}$ is stationary.*

Note that every $A \subseteq \omega$ is equal to some a_α, and so $\diamondsuit(\aleph_1)$ implies $2^{\aleph_0} = \aleph_1$.

1.15 Theorem ($\diamondsuit(\mathbf{E}^{\aleph_2}_{\aleph_0})$, Gregory [40]). *Assume* GCH. *There exists a sequence $\langle a_\alpha : \alpha \in \mathbf{E}^{\aleph_2}_{\aleph_0} \rangle$ with each $a_\alpha \subseteq \alpha$, such that for every $A \subseteq \omega_2$, the set $\{\alpha < \omega_2 : A \cap \alpha = a_\alpha\}$ is stationary.*

1.16 Theorem (Club-guessing, Shelah [82]). *There exists a sequence $\langle c_\alpha : \alpha \in \mathbf{E}^{\aleph_3}_{\aleph_1} \rangle$, where each c_α is a closed unbounded subset of α, such that for every club $C \subseteq \omega_3$, the set $\{\alpha : c_\alpha \subseteq C\}$ is stationary.*

Unlike most generalizations of square and diamond, Theorem 1.16 is a theorem of ZFC but we note that the gap (between $\aleph_1$ and $\aleph_3$) is essential.

2. Reflection

2.1. Reflecting Stationary Sets

An important property of stationary sets is *reflection*. It is used in several applications, and provides a structure among stationary sets—it induces a well founded hierarchy. Natural questions about reflection and the hierarchy are closely related to large cardinal properties.

We start with a generalization of stationary sets. Let α be a limit ordinal of uncountable cofinality, say $\mathrm{cf}(\alpha) = \kappa > \aleph_0$. A set $S \subseteq \alpha$ is *stationary* if it meets every closed unbounded subset of α. The closed unbounded subsets of α generate a κ-complete filter, and Fodor's Theorem 1.5 yields this:

2.1 Lemma. *If f is a regressive function on a stationary set $S \subseteq \alpha$, then there exists a $\gamma < \alpha$ such that $f(\xi) < \gamma$ on a stationary subset of S.*

If S is a set of ordinals and α is a limit ordinal such that $\operatorname{cf}(\alpha) > \omega$, we say that S is *stationary in* α if $S \cap \alpha$ is a stationary subset of α.

2.2 Definition. Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. If $\alpha < \kappa$ and $\operatorname{cf}(\alpha) > \omega$, S *reflects at* α if S is stationary in α. S *reflects* if it reflects at some $\alpha < \kappa$.

It is implicit in the definition that $\kappa > \aleph_1$.

For our first observation, let $\alpha < \kappa$ be such that $\operatorname{cf}(\alpha) > \omega$. There is a club $C \subseteq \alpha$ of order type $\operatorname{cf}(\alpha)$ such that every element of C has cofinality $< \operatorname{cf}(\alpha)$. Thus if $S \subseteq \kappa$ is such that every $\beta \in S$ has cofinality $\geq \operatorname{cf}(\alpha)$, then S does not reflect at α. In particular, if $\kappa = \lambda^+$ where λ is regular, then the stationary set $\mathbf{E}^\kappa_\lambda$ does not reflect.

On the other hand, if $\lambda < \kappa$ is regular and $\lambda^+ < \kappa$, then $\mathbf{E}^\kappa_\lambda$ reflects at every $\alpha < \kappa$ such that $\operatorname{cf}(\alpha) > \lambda$.

To investigate reflection systematically, let us first look at the simplest case, when $\kappa = \aleph_2$. Let $E_0 = \mathbf{E}^{\aleph_2}_{\aleph_0}$ and $E_1 = \mathbf{E}^{\aleph_2}_{\aleph_1}$. The set E_1 does not reflect; can every stationary $S \subseteq E_0$ reflect?

Let us recall Jensen's Square Principle [59]: ($\square_\kappa$) There exists a sequence

$\langle C_\alpha : \alpha \in \operatorname{Lim}\,(\kappa^+)\rangle$ such that:

(i) C_α is club in α,

(ii) if $\beta \in \operatorname{Lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$, and

(iii) if $\operatorname{cf}\,\alpha < \kappa$, then $|C_\alpha| < \kappa$.

Now assume that $\square_{\omega_1}$ holds and let $\langle C_\alpha : \alpha \in \operatorname{Lim}(\omega_2)\rangle$ be a square sequence. Note that for each $\alpha \in E_1$, the order type of C_α is ω_1. It follows that there exists a countable limit ordinal η such that the set $S = \{\gamma \in E_0 : \gamma$ is the ηth element of some $C_\alpha\}$ is stationary. But for every $\alpha \in E_1$, S has at most one element in common with C_α, and so S does not reflect.

Thus $\square_{\omega_1}$ implies that there is a nonreflecting stationary subset of $\mathbf{E}^{\aleph_2}_{\aleph_0}$. Since $\square_{\omega_1}$ holds unless $\aleph_2$ is Mahlo in L, the consistency strength of "every $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects" is at least a Mahlo cardinal. This is in fact the exact strength:

2.3 Theorem (Harrington-Shelah [41]). *The following are equi-consistent:*

(i) *the existence of a Mahlo cardinal,*

(ii) *every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects.*

Theorem 2.3 improves a previous result of Baumgartner [6] who proved the consistency of (ii) from a weakly compact cardinal. Note that (ii) implies that every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects at stationary many $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.

A related result of Magidor (to which we return later in this section) gives this equi-consistency:

2.4 Theorem (Magidor [70]). *The following are equi-consistent:*

(i) *the existence of a weakly compact cardinal,*

(ii) *every stationary set $S \subseteq \mathbf{E}^{\aleph_2}_{\aleph_0}$ reflects at almost all $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.*

Here, "almost all" means all but a nonstationary set.

Let us now address the question whether it is possible that *every* stationary subset of κ reflects. We have seen that this is not the case when κ is the successor of a regular cardinal. Thus κ must be either inaccessible or $\kappa = \lambda^+$ where λ is singular.

Note that because a weakly compact cardinal is Π^1_1 indescribable, every stationary subset of it reflects. In [68], Kunen showed that it is consistent that every stationary $S \subseteq \kappa$ reflects while κ is not weakly compact. In [76] it is shown that the consistency strength of "every stationary subset of κ reflects" is strictly between greatly Mahlo and weakly compact. (For definition of greatly Mahlo, see Sect. 2.2.)

If, in addition, we require that κ be a successor cardinal, then much stronger assumptions are necessary. The argument we gave above using $\square_{\omega_1}$ works for any κ:

2.5 Proposition (Jensen). *If $\square_\lambda$ holds, then there is a nonreflecting stationary subset of $\mathbf{E}^{\lambda^+}_{\aleph_0}$.*

As the consistency strength of $\neg\square_\lambda$ for singular λ is at least a strong cardinal (as shown by Jensen), one needs at least that for the consistency of "every stationary $S \subseteq \lambda^+$ reflects". In [70], Magidor proved the consistency of "every stationary subset of $\aleph_{\omega+1}$ reflects" from the existence of infinitely many supercompact cardinals.

We mention the following applications of nonreflecting stationary sets:

2.6 Theorem (Mekler-Shelah [76]). *The following are equi-consistent:*

(i) *every stationary $S \subseteq \kappa$ reflects,*

(ii) *every κ-free abelian group is κ^+-free.*

2.7 Theorem (Tryba [90]). *If a regular cardinal κ is Jónsson, then every stationary $S \subseteq \kappa$ reflects.*

2.8 Theorem (Todorčević [88]). *If Rado's Conjecture holds, then for every regular $\kappa > \aleph_1$, every stationary $S \subseteq \mathbf{E}^{\kappa}_{\aleph_0}$ reflects.*

2.2. A Hierarchy of Stationary Sets

Consider the following operation (the *Mahlo operation*) on stationary sets. For a stationary set $S \subseteq \kappa$, the *trace* of S is the set of all α at which S reflects:

$$Tr(S) = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega \text{ and } S \cap \alpha \text{ is stationary in } \alpha\}.$$

The following basic properties of trace are easily verified.

2.9 Lemma.

(a) *If* $S \subseteq T$, *then* $Tr(S) \subseteq Tr(T)$,

(b) $Tr(S \cup T) = Tr(S) \cup Tr(T)$,

(c) $Tr(Tr(S)) \subseteq Tr(S)$,

(d) *If* $S \simeq T \bmod \mathrm{NS}$, *then* $Tr(S) \simeq Tr(T) \bmod \mathrm{NS}$.

Property (d) shows that the Mahlo operation may be considered as an operation on the Boolean algebra $P(\kappa)/\mathrm{NS}$.

If $\lambda < \kappa$ is regular, let $\mathbf{M}^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \lambda\}$, and note that $Tr(\mathbf{E}^\kappa_\lambda) = Tr(\mathbf{M}^\kappa_\lambda) = \mathbf{M}^\kappa_{\lambda^+}$.

The Mahlo operation on $P(\kappa)/\mathrm{NS}$ can be iterated α times, for $\alpha < \kappa^+$. Let

$$\begin{aligned} M_0 &= \kappa, \\ M_{\alpha+1} &= Tr(M_\alpha), \\ M_\alpha &= \Delta_{\xi<\kappa} M_{\alpha_\xi} \quad (\alpha \text{ limit, } \alpha = \sup\{\alpha_\xi : \xi < \kappa\}). \end{aligned}$$

The sets M_α are defined mod NS (the limit stages depend on the enumeration of α). The sequence $\{M_\alpha\}_{\alpha<\kappa^+}$ is decreasing mod NS, and when $\alpha < \kappa$, then $M_\alpha = \mathbf{M}^\kappa_\lambda$ where λ is the αth regular cardinal. Note that κ is (weakly) Mahlo just in case $M_\kappa = \mathrm{Reg}(\kappa)$ is stationary, and that by Lemma 1.11, $\{M_\alpha\}_\alpha$ is strictly decreasing (mod NS, as long as M_α is stationary). Following [13], κ is called *greatly Mahlo* if M_α is stationary for every $\alpha < \kappa^+$.

We shall now consider the following relation between stationary subsets of κ.

2.10 Definition (Jech [47]).

$$S < T \text{ iff } S \cap \alpha \text{ is stationary for almost all } \alpha \in T.$$

In other words, $S < T$ iff $Tr(S) \supseteq T \bmod \mathrm{NS}$. As an example, if $\lambda < \mu$ are regular, then $\mathbf{E}^\kappa_\lambda < \mathbf{E}^\kappa_\mu$. Note also that the language of generic ultrapowers gives this description of $<$:

2.11 Proposition. $S < T$ *iff* $T \Vdash S$ *is stationary in* $\mathrm{Ult}_G(V)$.

The following lemma states the basic properties of $<$.

2.12 Lemma.

(a) $A < \mathit{Tr}(A)$,

(b) *If* $A < B$ *and* $B < C$, *then* $A < C$,

(c) *If* $A \simeq A'$ *and* $B \simeq B'$ mod NS, *and if* $A < B$, *then* $A' < B'$.

By (c), $<$ can be considered a relation on $P(\kappa)/\mathrm{NS}$. By Proposition 1.11, $<$ is irreflexive and so it is a partial ordering. The next theorem shows that the partial ordering $<$ is well founded.

2.13 Theorem (Jech [47]). *The relation* $<$ *is well founded.*

Proof. Assume to the contrary that there are stationary sets such that $A_1 > A_2 > A_3 > \cdots$. Therefore there are clubs C_n such that $A_n \cap C_n \subseteq \mathit{Tr}(A_{n+1})$ for $n = 1, 2, \ldots$. For each n, let

$$B_n = A_n \cap C_n \cap \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \cap \cdots .$$

Each B_n is stationary, and for every n, $B_n \subseteq \mathit{Tr}(B_{n+1})$. Let $\alpha_n = \min(B_n)$. Since $B_{n+1} \cap \alpha_n$ is stationary, we have $\alpha_{n+1} < \alpha_n$, and therefore, a decreasing sequence $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$. A contradiction. $\dashv$

As $<$ is well founded, we can define the *order* of stationary sets $A \subseteq \kappa$, and of the cardinal κ:

$$o(A) = \sup(\{o(X) + 1 : X < A\}),$$
$$o(\kappa) = \sup(\{o(A) + 1 : A \subseteq \kappa \text{ stationary}\}).$$

We also define $o(\aleph_0) = 0$, and $o(\alpha) = o(\mathrm{cf}(\alpha))$ for every limit ordinal α.

Note that $o(\mathbf{E}^\kappa_{\aleph_0}) = 0$, and in general $o(\mathbf{E}^\kappa_\lambda) = o(\mathbf{M}^\kappa_\lambda) = \alpha$, if λ is the αth regular cardinal. Also, $o(\aleph_n) = n$, $o(\kappa) \geq \kappa + 1$ iff κ is Mahlo, and $o(\kappa) \geq \kappa^+$ iff κ is greatly Mahlo.

2.3. Canonical Stationary Sets

If λ is the αth regular cardinal, then $\mathbf{E}^\kappa_\lambda$ has order α; moreover, the set is canonical, in the sense explained below. In fact, canonical stationary sets exist for all orders $\alpha < \kappa^+$.

Let E be a stationary set of order α. If $X \subseteq E$ is stationary, then $o(X) \geq o(E)$. We call E *canonical of order* α if (i) every stationary $X \subseteq E$ has order α, and (ii) E meets every set of order α.

Clearly, a canonical set of order α is unique (mod NS), and two canonical sets of different orders are disjoint (mod NS). In the following proposition, "maximal" and $\simeq$ is meant mod NS.

2.14 Proposition (Jech [47]). *A canonical set E of order α exists iff there exists a maximal set M of order α. Then (*mod NS*)*

$$E \simeq M - Tr(M), \quad M \simeq E \cup Tr(E), \quad \textit{and} \quad Tr(E) \simeq Tr(M).$$

One can show that the sets M_α obtained by iterating the Mahlo operation are maximal (as long as they are stationary). Thus when we let $E_\alpha = M_\alpha - Tr(M_\alpha)$, we get canonical stationary sets, of all orders $\alpha < \kappa^+$ (for $\alpha < o(\kappa)$).

The canonical stationary sets E_α and the canonical function f_α of Galvin-Hajnal norm α are closely related:

2.15 Proposition (Jech [47]). *For every $\alpha < \kappa^+$, $\alpha < o(\kappa)$,*

$$E_\alpha \simeq \{\xi < \kappa : f_\alpha(\xi) = o(\xi)\}.$$

2.4. Full Reflection

Let us address the question of what is the largest possible amount of reflection, for stationary subsets of a given κ. As $A < B$ means that A reflects at almost all points of B, we would like to maximize the relation $<$. But $A < B$ implies that $o(A) < o(B)$, so we might ask whether it is possible that $A < B$ for any two stationary sets such that $o(A) < o(B)$.

By Magidor's Theorem 2.4 it is consistent that $S < \mathbf{E}^{\aleph_2}_{\aleph_1}$, and therefore $S < T$ for every S of order 0 and every T of order 1. However, this does not generalize, as the following lemma shows that when $\kappa \geq \aleph_3$, then there exist S and T with $o(S) = 0$ and $o(T) = 1$ such that $S \not< T$.

2.16 Lemma (Jech-Shelah [53]). *If $\kappa \geq \aleph_3$, then there exist stationary sets $S \subseteq \mathbf{E}^\kappa_{\aleph_0}$ and $T \subseteq \mathbf{E}^\kappa_{\aleph_1}$ such that S does not reflect at any $\alpha \in T$.*

Proof. Let S_γ, $\gamma < \omega_2$, be pairwise disjoint stationary subsets of $\mathbf{E}^\kappa_{\aleph_0}$, and let C_α, $\alpha \in \mathbf{E}^\kappa_{\aleph_1}$, be such that for every α, C_α is a club subset of α, of order type ω_1. Because at most $\aleph_1$ of the sets S_γ meet each C_α, there exists for each α some $\gamma(\alpha)$ such that $C_\alpha \cap S_{\gamma(\alpha)} = \emptyset$.

There exists some γ such that the set $T = \{\alpha : \gamma(\alpha) = \gamma\}$ is stationary; let $S = S_\gamma$. For every $\alpha \in T$, $S \cap C_\alpha = \emptyset$ and so S does not reflect at α. $\dashv$

This lemma illustrates some of the difficulties involved when dealing with reflection at singular ordinals. This problem is investigated in detail in [53], where the best possible consistency result is proved for stationary subsets of the $\aleph_n$, $n < \omega$.

Let us say that a stationary set $S \subseteq \kappa$ *reflects fully at regular cardinals* if for any stationary set T of regular cardinals $o(S) < o(T)$ implies $S < T$, and let us call *Full Reflection* the statement that every stationary subset of κ reflects fully at regular cardinals.

Full Reflection is of course nontrivial only if κ is a Mahlo cardinal. A modification of Theorem 2.4 shows that Full Reflection for a Mahlo cardinal is equi-consistent with weak compactness. The following theorem establishes the consistency strength of Full Reflection for cardinals in the Mahlo hierarchy:

2.17 Theorem (Jech-Shelah [54]). *The following are equi-consistent, for every $\alpha \le \kappa^+$:*

(i) *κ is Π^1_α-indescribable,*

(ii) *κ is α-Mahlo and Full Reflection holds.*

(A regular cardinal κ is α-Mahlo if $o(\kappa) \ge \kappa + \alpha$; κ is Π^1_1-indescribable iff it is weakly compact.)

Full Reflection is also consistent with large cardinals. The paper [56] proves the consistency of Full Reflection with the existence of a measurable cardinal. This has been improved and further generalized in [38].

Finally, the paper [91] shows that any well-founded partial order of size $\le \kappa^+$ can be realized by the reflection ordering $<$ on stationary subsets of κ, in some generic extension (using $P^2\kappa$-strong κ in the ground model).

3. Saturation

3.1. κ^+-saturation

By Solovay's Theorem 1.8 every stationary subset of κ can be split into κ disjoint stationary sets. In other words, for every stationary $S \subseteq \kappa$, the ideal $\mathrm{NS}{\restriction}S$ is not κ-saturated. A natural question is if the nonstationary ideal can be κ^+-saturated.

An ideal I on κ is κ^+-*saturated* if the Boolean algebra $P(\kappa)/I$ has the κ^+-chain condition. Thus $\mathrm{NS}{\restriction}S$ is κ^+-saturated when there do not exist κ^+ stationary subsets of S such that the intersection of any two of them is nonstationary. The existence and properties of κ^+-saturated ideals have been thoroughly studied since their introduction in [85], and involve large cardinals. The reader will find more details in Foreman's chapter in this Handbook. We shall concentrate on the special case when I is the nonstationary ideal.

The main question, whether the nonstationary ideal can be κ^+-saturated, has been answered. But a number of related questions are still open.

3.1 Theorem (Gitik-Shelah [37]). *The nonstationary ideal on κ is not κ^+-saturated, for every regular cardinal $\kappa \ge \aleph_2$.*

3.2 Theorem (Shelah). *It is consistent, relative to the existence of a Woodin cardinal, that the nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated.*

The consistency result in Theorem 3.2 was first proved in [87] using a strong determinacy assumption. That hypothesis was reduced in [92] to AD, while in [27], the assumption was the existence of a supercompact cardinal.

Shelah's result (announced in [81]) is close to optimal: by Steel [86], the saturation of NS plus the existence of a measurable cardinal imply the existence of an inner model with a Woodin cardinal.

All the models mentioned in the preceding paragraph satisfy $2^{\aleph_0} > \aleph_1$. This may not be accidental, and it has been conjectured that the saturation of NS on $\aleph_1$ implies that $2^{\aleph_0} > \aleph_1$. In fact, Woodin proved this [94] under the additional assumption that there exists a measurable cardinal. We note in passing that by [27], $2^{\aleph_0} = \aleph_1$ is consistent with NS$\restriction S$ being saturated for some stationary S.

Woodin's construction [94] yields a model (starting from AD) in which the ideal NS on $\aleph_1$ is $\aleph_1$-*dense*, i.e. the algebra $P(\omega_1)/\mathrm{NS}$ has a dense set of size $\aleph_1$. This, and Woodin's more recent work using Steel's inner model theory, gives the following equi-consistency.

3.3 Theorem (Woodin). *The following are equi-consistent:*

(i) ZF + AD*,*

(ii) *There are infinitely many Woodin cardinals,*

(iii) *The nonstationary ideal on* $\aleph_1$ *is* $\aleph_1$*-dense.*

See the Koellner-Woodin chapter of this Handbook for (i) implies (ii), and the Neeman chapter for (ii) implies (i).

As for the continuum hypothesis, Shelah proved in [80] that if NS on $\aleph_1$ is $\aleph_1$-dense, then $2^{\aleph_0} = 2^{\aleph_1}$.

We remark that the mere existence of a saturated ideal affects cardinal arithmetic, cf. [63] and [51].

Let us now return to Theorem 3.1. The general result proved in [37] is this:

3.4 Theorem (Gitik-Shelah [37]). *If* ν *is a regular cardinal and* $\nu^+ < \kappa$*, then* NS$\restriction\mathbf{E}^\kappa_\nu$ *is not* κ^+*-saturated.*

The proof of Theorem 3.4 combines an earlier result of Shelah (Theorem 3.7 below) with an application of the method of guessing clubs (as in Theorem 1.16). The earlier result uses generic ultrapowers and states that if $\kappa = \lambda^+$ and $\nu \neq \mathrm{cf}(\lambda)$ is regular, then no ideal concentrating on $\mathbf{E}^\kappa_\nu$ is κ^+-saturated.

The method of generic ultrapowers is well suited for κ^+-saturated ideals. Forcing with $P(\kappa)/I$ where I is a normal κ-complete κ^+-saturated ideal makes the generic ultrapower $N = \mathrm{Ult}_G(V)$ well founded, preserves the cardinal κ^+, and satisfies $P^N(\kappa) = P^{V[G]}(\kappa)$. It follows that all cardinals $< \kappa$ are preserved in $V[G]$, and it is obvious that if $\mathbf{E}^\kappa_\nu \in G$, then N (and therefore $V[G]$ as well) satisfies $\mathrm{cf}(\kappa) = \nu$.

Shelah's Theorem 3.7 below follows from a simple combinatorial lemma. Let λ be a cardinal and let $\alpha < \lambda^+$ be a limit ordinal. Let us call a family $\{X_\xi : \xi < \lambda^+\}$ a *strongly almost disjoint* (s.a.d.) family of subsets of α if every $X_\xi \subseteq \alpha$ is unbounded, and if for every $\vartheta < \lambda^+$ there exist ordinals $\delta_\xi < \alpha$, for $\xi < \vartheta$, such that the sets $X_\xi - \delta_\xi$, $\xi < \vartheta$, are pairwise disjoint. Note that if κ is a regular cardinal than there is a s.a.d. family $\{X_\xi : \xi < \kappa^+\}$ of subsets of κ.

3.5 Lemma. *If $\alpha < \lambda^+$ and $\mathrm{cf}(\alpha) \neq \mathrm{cf}(\lambda)$, then there exists no strongly almost disjoint family of subsets of α.*

Proof. Assume to the contrary that $\{X_\xi : \xi < \lambda^+\}$ is a s.a.d. family of subsets of α. We may assume that each X_ξ has order type $\mathrm{cf}(\alpha)$. Let f be a function mapping λ onto α. Since $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\alpha)$ there exists for each ξ some $\gamma_\xi < \lambda$ such that $X_\xi \cap f``\gamma_\xi$ is cofinal in α. There is some γ and a set $W \subseteq \lambda^+$ of size λ such that $\gamma_\xi = \gamma$ for all $\xi \in W$. Let $\vartheta > \sup(W)$. By the assumption on the X_ξ there exist ordinals $\delta_\xi < \alpha$, $\xi < \vartheta$, such that the $X_\xi - \delta_\xi$ are pairwise disjoint. Thus $f^{-1}(X_\xi - \delta_\xi)$, $\xi \in W$, are λ pairwise disjoint subsets of γ. A contradiction. $\dashv$

3.6 Corollary (Shelah [79]). *If κ is a regular cardinal and if a notion of forcing P makes $\mathrm{cf}(\kappa) \neq \mathrm{cf}(|\kappa|)$, then P collapses κ^+.*

Proof. Assume that κ^+ is not collapsed; thus in $V[G]$, $(\kappa^+)^V = \lambda^+$ where $\lambda = |\kappa|$. In V there is a s.a.d. family $\{X_\xi : \xi < (\kappa^+)^V\}$, and it remains a s.a.d. family in $V[G]$, of size λ^+. Since $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$, in $V[G]$, this contradicts Lemma 3.5. $\dashv$

3.7 Theorem (Shelah). *If $\kappa = \lambda^+$, if $\nu \neq \mathrm{cf}(\lambda)$ is regular and if I is a normal κ-complete κ^+-saturated ideal on κ, then $\mathbf{E}^\kappa_\nu \in I$.*

Proof. If not, then forcing with I-positive subsets of $\mathbf{E}^\kappa_\nu$ preserves κ^+ as well as $\mathrm{cf}(\lambda)$, and makes $\mathrm{cf}(\kappa) = \nu$; a contradiction. $\dashv$

Theorem 3.4 leaves open the following problem: If λ is a regular cardinal, can $\mathrm{NS}{\restriction}\mathbf{E}^{\lambda^+}_\lambda$ be λ^{++}-saturated? (For instance can $\mathrm{NS}{\restriction}\mathbf{E}^{\aleph_2}_{\aleph_1}$ be $\aleph_3$-saturated?) Let us also mention that for all regular ν and κ not excluded by Corollary 3.7, it is consistent that $\mathrm{NS}{\restriction}S$ is κ^+-saturated for some $S \subseteq \mathbf{E}^\kappa_\nu$ (see [33]).

If κ is a large cardinal, then $\mathrm{NS}{\restriction}\mathrm{Reg}(\kappa)$ can be κ^+-saturated, as the following theorem shows. Of course, κ cannot be too large: if κ is greatly Mahlo, then the canonical stationary sets E_α $\kappa \leq \alpha < \kappa^+$ witness nonsaturation.

3.8 Theorem (Jech-Woodin [57]). *For any $\alpha < \kappa^+$, the following are equiconsistent:*

(i) *κ is measurable of order α,*

(ii) *κ is α-Mahlo and the ideal $\mathrm{NS}{\restriction}\mathrm{Reg}(\kappa)$ on κ is κ^+-saturated.*

3.2. Precipitousness

An important property of saturated ideals is that the generic ultrapower is well-founded. It has been recognized that this property is important enough to single out and study the class of ideals that have it. The ideals for which the generic ultrapower is well founded are called *precipitous*. They are described in detail in Foreman's chapter in this Handbook; here we address the question of when the nonstationary ideal is precipitous.

Precipitous ideals were introduced by Jech and Prikry in [50]. There are several equivalent formulations of precipitousness. Let I be an ideal on some set E. An *I-partition* is a maximal family of I-positive sets such that the intersection of any two of them is in I. Let $\mathcal{G}_I$ denote the infinite game of two players who alternately pick I-positive sets S_n such that $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. The first player wins if $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

3.9 Theorem (Jech-Prikry [50, 45, 46, 30]). *Let I be an ideal on a set E. The following are equivalent:*

(i) *Forcing with $P(E)/I$ makes the generic ultrapower well-founded,*

(ii) *For every sequence $\{W_n\}_{n=1}^{\infty}$ of I-partitions there exists a sequence $\{X_n\}_{n=1}^{\infty}$ such that $X_n \in W_n$ for each n, and $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$,*

(iii) *The first player does not have a winning strategy in the game $\mathcal{G}_I$.*

The problem of whether the nonstationary ideal on κ can be precipitous involves large cardinals. For $\kappa = \aleph_1$ the exact consistency strength is the existence of a measurable cardinal:

3.10 Theorem (Jech-Magidor-Mitchell-Prikry [58]). *The following are equi-consistent:*

(i) *There exists a measurable cardinal,*

(ii) NS *on $\aleph_1$ is precipitous.*

For $\kappa \geq \aleph_2$, stronger large cardinal assumptions are involved. For $\kappa = \aleph_2$, the consistency strength is a measurable of order 2:

3.11 Theorem (Gitik [31]). *The following are equi-consistent:*

(i) *There exists a measurable cardinal of order 2,*

(ii) NS *on $\aleph_2$ is precipitous.*

See Cummings' chapter in this Handbook for proofs of both of these theorems. For the general case, the paper [47] provided lower bounds for the consistency strength of "NS is precipitous" in terms of the Mitchell order, while models with NS precipitous for $\kappa > \aleph_2$ were constructed in [33] and [27] from strong assumptions. In [35] and [36], Gitik established the exact consistency strength of "NS on κ is precipitous" when κ is the successor of a regular cardinal λ (the existence of an $(\omega, \lambda+1)$-repeat point), as well as nearly optimal lower and upper consistency bounds for κ inaccessible. Additional lower bounds for the case $\kappa = \lambda^+$ where λ is a large cardinal appear in [95].

The problem of whether the nonstationary ideal on κ can be precipitous while κ is measurable was first addressed by Kakuda in [61] who proved that many measurables are necessary. This lower bound was improved to having Mitchell order $\kappa^+ + 1$ in [47], and to a repeat point in [34]. Gitik's paper also shows that the existence of a supercompact cardinal suffices for the consistency of the nonstationary ideal on a supercompact cardinal being precipitous.

4. The Closed Unbounded Filter on $P_\kappa\lambda$

4.1. Closed Unbounded Sets in $P_\kappa A$

One of the useful tools of combinatorial set theory is a generalization of the concepts of closed unbounded set and stationary set. This generalization, introduced in [43] and [44], replaces $\langle \kappa, < \rangle$ with $\langle P_\kappa\lambda, \subset \rangle$, and is justified by the fact that the crucial Theorem 1.5 remains true under the generalization.

For κ a regular uncountable cardinal and A a set of cardinality at least κ, let $P_\kappa A$ denote the set $\{x : x \subseteq A \text{ and } |x| < \kappa\}$. Furthermore, we let $[X]^\nu = \{x \subseteq X : |x| = \nu\}$ whenever $|X| \geq \nu$ and ν an infinite cardinal.

4.1 Definition (Jech [44]). Let κ be a regular uncountable cardinal and let $|A| \geq \kappa$.

(a) A set $X \subseteq P_\kappa A$ is *unbounded* (in $P_\kappa A$) if for every $x \in P_\kappa A$ there is a $y \supseteq x$ such that $y \in X$.

(b) A set $X \subseteq P_\kappa A$ is *closed* (in $P_\kappa A$) if for any chain $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_\xi \subseteq \cdots$, $\xi < \alpha$, of sets in X, with $\alpha < \kappa$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in X.

(c) A set $C \subseteq P_\kappa A$ is *closed unbounded* if it is closed and unbounded.

(d) A set $S \subseteq P_\kappa A$ is *stationary* (in $P_\kappa A$) if $S \cap C \neq \emptyset$ for every closed unbounded $C \subseteq P_\kappa A$.

The *closed unbounded filter* on $P_\kappa A$ is the filter generated by the closed unbounded sets. We remark that when $A = \kappa$, then the set $\kappa \subseteq P_\kappa\kappa$ is closed unbounded, and the club filter on κ is the restriction to κ of the club filter on $P_\kappa\kappa$. As before, the basic observation is that the intersection of two clubs is a club, and we have again:

4.2 Proposition. *The club filter on $P_\kappa A$ is κ-complete.*

For the generalization of Theorem 1.5, let us first define the *diagonal intersection.*

$$\Delta_{a\in A} X_a = \{x \in P_\kappa A : x \in \textstyle\bigcap_{a\in x} X_a\}.$$

The generalization of Lemma 1.4 is this:

4.3 Lemma. *If $\langle C_a : a \in A\rangle$ is a sequence of closed unbounded sets in $P_\kappa A$, then its diagonal intersection is closed unbounded.*

Again, this lemma immediately implies the appropriate generalization of Theorem 1.5:

4.4 Theorem (Jech [44]). *If S is a stationary set in $P_\kappa A$ and if f is a function on S such that $f(x) \in x$ for every $x \in S - \{\emptyset\}$, then there exists some $a \in A$ such that $f(x) = a$ on a stationary subset of S.*

In Proposition 1.6 we showed that the club filter is the smallest normal filter on κ. We shall now do the same for $P_\kappa A$. A κ-complete filter F on $P_\kappa A$ is *normal* if for every $a \in A$, $\{x \in P_\kappa A : a \in x\} \in F$, and if F is closed under diagonal intersections.

The following fact (proved by induction on $|D|$) is quite useful; D is $\subseteq$-*directed* if for any $x, y \in D$ there is a $z \in D$ such that $x \cup y \subseteq z$.

4.5 Proposition. *If X is a closed set in $P_\kappa A$, then for any $\subseteq$-directed D with $|D| < \kappa$, $\bigcup D \in X$.*

Let $f : [A]^{<\omega} \to P_\kappa A$; a set $x \in P_\kappa A$ is a *closure point* of f is $f(e) \subseteq x$ whenever $e \subseteq x$. The set Cl_f of all closure points $x \in P_\kappa A$ is easily seen to be a club. More importantly, the sets Cl_f generate the club filter:

4.6 Proposition (Menas [77]). *For every closed unbounded set C in $P_\kappa A$ there is an $f : [A]^{<\omega} \to P_\kappa A$ such that $Cl_f \subseteq C$.*

Proof. By induction on $|e|$ we find an infinite set $f(e) \in C$ such that $e \subseteq f(e)$ and that $f(e') \subseteq f(e)$ whenever $e' \subseteq e$. To see that $Cl_f \subseteq C$, let x be a closure point of f. As $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$ is the union of a small $\subseteq$-directed subset of C, we have $x \in C$. $\dashv$

4.7 Corollary (Carr [18]). *If F is a normal κ-complete filter on $P_\kappa A$, then F contains all closed unbounded sets.*

Proof. Let F^+ denote the F-positive sets, those whose complement is not in F. A consequence of normality is that if $X \in F^+$ and g is a function on X such that $g(x) \in [x]^{<\omega}$ for all $x \in X$, then g is constant on a set in F^+.

Now assume that there is a club not in F. Thus there is an $f : A \to P_\kappa A$ such that the complement X of Cl_f is F-positive. For each $x \in X$ there is some $e = g(x) \in [x]^{<\omega}$ such that $f(e) \nsubseteq x$. Therefore there is some e such that $\{x : f(e) \nsubseteq x\} \in F^+$. This is a contradiction, because $\{x : f(e) \subseteq x\} \in F$. $\dashv$

As another consequence of Proposition 4.6 we consider *projections* and *liftings* of stationary sets. Let $A \subseteq B$ (and $|A| \geq \kappa$). For $X \in P_\kappa B$, the *projection* of X is the set

$$X \restriction A = \{x \cap A : x \in X\};$$

for $Y \in P_\kappa A$, the *lifting* of Y is

$$Y^B = \{x \in P_\kappa B : x \cap A \in Y\}.$$

4.8 Proposition (Menas [77]). *Let $A \subseteq B$.*

(i) *If S is stationary in $P_\kappa B$, then $S \restriction A$ is stationary in $P_\kappa A$.*

(ii) *If S is stationary in $P_\kappa A$, then S^B is stationary in $P_\kappa B$.*

Proof. (i) is easy and holds because if C is a club in $P_\kappa A$, then C^B is a club in $P_\kappa B$. For (ii), it suffices to prove that if C is a club in $P_\kappa B$, then $C \restriction A$ contains a club in $P_\kappa A$.

If $C \subseteq P_\kappa B$ is a club, by Proposition 4.6 there is an $f : [B]^{<\omega} \to P_\kappa B$ such that $Cl_f \subseteq C$. Let $g : [A]^{<\omega} \to P_\kappa A$ be as follows: let $g(E) =$ (the f-closure of $e) \cap A$. Since $Cl_f \restriction A = Cl_g$, we have $Cl_g \subseteq C \restriction A$. ⊣

When $\kappa = \aleph_1$, Proposition 4.6 can be improved by replacing f by a function with values in A, i.e. an operation on A. For $f : [A]^{<\omega} \to A$, let Cl_f denote the set $\{x : f(e) \in x$ whenever $e \subseteq x\}$. The following characterization of the club filter on $P_{\omega_1} A$ was given in [66]; this and [67] used $P_{\omega_1} A$ in the study of model theory.

4.9 Theorem (Kueker [66]). *The club filter on $P_{\omega_1} A$ is generated by the sets Cl_F where $F : [A]^{<\omega} \to A$.*

When $\kappa > \aleph_1$, then the clubs Cl_F where $F : [A]^{<\omega} \to A$, do not generate the club filter: every F has countable closure points while the set of all uncountable $x \in P_\kappa A$ is closed unbounded. However, a slight modification of Theorem 4.9 works, namely Proposition 4.10 below. Let us call the club Cl_F for $F : [A]^{<\omega} \to A$ *strongly closed unbounded.*

Let us consider $P_\kappa \lambda$ where $\lambda \geq \kappa$. We note that the set

$$\{x \in P_\kappa \lambda : x \cap \kappa \in \kappa\}$$

is closed unbounded. It turns out that the club filter is generated by adding this set to the filter generated by the strongly club sets.

4.10 Proposition (Foreman et al. [27]). *For every club C in $P_\kappa \lambda$ there exists a function $F : [\lambda]^{<\omega} \to \lambda$ such that*

$$\{x \in P_\kappa \lambda : x \cap \kappa \in \kappa \text{ and } F``[x]^{<\omega} \subseteq x\} \subseteq C.$$

Now let us consider $P_\kappa\lambda$ for $\kappa = \nu^+$ where ν is uncountable. As the set $[\lambda]^\nu$ is closed unbounded in $P_{\nu^+}\lambda$, let us consider the restriction of the club filter to $[\lambda]^\nu$. We say that a set $S \subseteq [\lambda]^\nu$ is *weakly stationary* if it meets every strongly club set. It turns out that the question whether weakly stationary sets are stationary involves large cardinals. By Proposition 4.10, this question depends on whether the set $\{x \in [\lambda]^\nu : x \cap \nu^+ \in \nu^+\}$ is in the strongly club filter. The following reformulation, implicit in [27], establishes the relation to large cardinals:

4.11 Proposition. *There exists a weakly stationary nonstationary set in $[\lambda]^\nu$ if and only if the (nonstationary) set $\{x \in [\lambda]^\nu : x \not\supseteq \nu\}$ is weakly stationary.*

4.12 Corollary. *The following are equivalent:*

(i) *The club filter on $[\omega_2]^{\aleph_1}$ is not generated by strongly club sets,*

(ii) *Chang's Conjecture.*

4.2. Splitting Stationary Sets

Let us now address the question whether stationary sets can be split into a large number of disjoint stationary sets. In particular, does Theorem 1.8 generalize to $P_\kappa A$? As only the size of A matters, and the club filter on $P_\kappa\kappa$ is basically just the club filter on κ, we shall consider subsets of $P_\kappa\lambda$ where λ is a cardinal and $\lambda > \kappa$.

We have $|P_\kappa\lambda| = \lambda^{<\kappa}$ and so the maximal possible size of a disjoint family of subsets of $P_\kappa\lambda$ is $\lambda^{<\kappa}$. While it is consistent that every stationary set splits into $\lambda^{<\kappa}$ disjoint stationary subsets (see Corollary 4.18), this is not provable in ZFC. The reason is that there may exist closed unbounded sets in $P_\kappa\lambda$ whose size is less than $\lambda^{<\kappa}$. For instance, [8] shows that there exists a club in $P_{\omega_3}\omega_4$ of size $\aleph_4^{\aleph_1}$; thus if $2^{\aleph_2} > 2^{\aleph_1} \cdot \aleph_4$, then $P_{\omega_3}\omega_4$ is not the union of $\aleph_4^{<\aleph_3}$ disjoint stationary sets. An earlier result [11] proved the consistency of a stationary set $S \subseteq P_{\omega_1}\omega_2$ such that $\mathrm{NS}{\restriction}S$ is $2^{\aleph_0}$-saturated.

A modification of Solovay's proof, using the generic ultrapower by NS, gives this:

4.13 Theorem (Gitik [32]). *Every stationary subset of $P_\kappa\lambda$ can be partitioned into κ disjoint stationary sets.*

The question of splitting stationary sets has been more or less completely solved for splitting into λ sets. Let us first observe that the nonexistence of λ disjoint stationary sets is equivalent to λ-saturation:

4.14 Lemma. *If X_α, $\alpha < \lambda$, are stationary sets in $P_\kappa\lambda$ such that $X_\alpha \cap X_\beta \in$ NS for all $\alpha \neq \beta$, then there exist pairwise disjoint stationary sets Y_α with $Y_\alpha \subseteq X_\alpha$ for all $\alpha < \lambda$.*

Proof. Let $Y_\alpha = X_\alpha \cap \{x : \alpha \in x \text{ and } \forall\beta \in x(\beta \neq \alpha \to x \notin X_\beta)\}$. $\dashv$

A long series of results by Jech [44], Menas [77], Baumgartner, DiPrisco and Marek [19], Matsubara [72, 73] established the following:

4.15 Theorem.

(i) *$P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.*

(ii) *If κ is a successor cardinal, than every stationary subset of $P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.*

(iii) *If $0^\#$ does not exist, then every stationary subset of $P_\kappa\lambda$ can be partitioned into λ disjoint stationary sets.*

A complete proof of this theorem can be found in Kanamori's book [62]. The results (ii) and (iii) are best possible, in the following sense:

4.16 Theorem (Gitik [32]). *It is consistent, relative to a supercompact cardinal, that κ is inaccessible, $\lambda > \kappa$, and some stationary set $S \subseteq P_\kappa\lambda$ cannot be partitioned into κ^+ disjoint stationary subsets.*

For a simplification of Gitik's proof, as well as further results, see [83].

The proof of Theorem 4.15 involves the following set, which clearly is stationary:

$$S_0 = \{x \in P_\kappa\lambda : |x \cap \kappa| = |x|\}.$$

This stationary set can be partitioned into λ disjoint stationary sets, and if either $\kappa = \nu^+$ or $0^\#$ does not exist, then S_0 contains a club (cf. [62]).

Clearly, in Gitik's model the set S_0 does not contain a club. Thus the statement that for some κ and λ,

$$\{x \in P_\kappa\lambda : |x \cap \kappa| < |x|\}$$

is stationary, is a consistent large cardinal statement. Its exact consistency strength (between $0^\#$ and Ramsey) is pinned down in [5] and [20].

As for splitting into $\lambda^{<\kappa}$ sets, the following result of Matsubara together with Theorem 4.15 proves the consistency result mentioned earlier:

4.17 Proposition (Matsubara [74]). *Assume* GCH. *If* $\mathrm{cf}(\lambda) < \kappa$, *then every stationary subset of $P_\kappa\lambda$ can be partitioned into λ^+ disjoint stationary sets.*

4.18 Corollary. *Assume* GCH *and that $0^\#$ does not exist. Then every stationary subset of $P_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

4.3. Saturation

By Theorem 4.15, the nonstationary ideal on $P_\kappa\lambda$ is not λ-saturated (even though $\mathrm{NS}{\restriction}S$ can be κ^+-saturated for some S). The next question is whether it can be λ^+-saturated, and the answer is again no.

4.19 Theorem (Foreman-Magidor [26]). *For every regular uncountable cardinal κ and every cardinal $\lambda > \kappa$, the nonstationary ideal on $P_\kappa\lambda$ is not λ^+-saturated.*

See Foreman's chapter in this Handbook for this result. We note that special cases of this theorem had been proved earlier, cf. [4, 73, 60, 17].

When dealing with λ^+-saturation, we naturally employ generic ultrapowers and use the fact that the ultrapower is well founded; a normal λ^+-saturated ideal on $P_\kappa\lambda$ is precipitous. The nonstationary ideal on $P_\kappa\lambda$ (for regular λ) can be precipitous. The consistency, a result of Goldring [39], is relative to a Woodin cardinal, and strengthens an earlier result in [27]. On the other hand, the paper [75] gives instances of κ and λ for which NS on $P_\kappa\lambda$ cannot be precipitous.

5. Proper Forcing and Other Applications

5.1. Proper Forcing

One of the most fruitful applications of the club filter on $P_{\omega_1}A$ is Shelah's concept of *proper forcing*. As proper forcing is discussed in detail in Abraham's chapter in this Handbook, I shall only give a brief account in this section. The rest of Sect. 5 deals with applications of the club filter on $P_{\omega_1}A$ in the theory of Boolean algebras.

When dealing with closed unbounded sets in $P_{\omega_1}A$ we may as well restrict ourselves to infinite sets, and thus consider the space $[A]^{\aleph_0}$ (where A is an uncountable set). By Kueker's Theorem 4.9, a set $X \subseteq [A]^{\aleph_0}$ is in the club filter just in case it contains the set Cl_F of all closure points of some operation on A. Equivalently, X contains all elementary submodels of some model with universe A. A useful modification of this is the following consequence of Proposition 4.8.

5.1 Proposition. *A set $X \subseteq [A]^{\aleph_0}$ is in the club filter if for some sufficiently large λ, X contains $M \cap A$, for all countable $M \prec \mathcal{H}_\lambda$ such that $A \in M$.*

Here $\mathcal{H}_\lambda = \langle H_\lambda, \in \rangle$ where H_λ is the set of all sets hereditarily of power $< \lambda$; "sufficiently large" means $2^{|\mathrm{TC}(A)|} < \lambda$.

Let us now turn to proper forcing. First we remark that the preservation Theorem 1.13 generalizes to $[A]^{\aleph_0}$:

5.2 Theorem.

(a) *If P satisfies the countable chain condition, then every club $C \in V[G]$ in $[A]^{\aleph_0}$ has a club subset in the ground model. Hence every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.*

(b) *If P is countably closed, then every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.*

For a proof, we refer the reader to [7, Theorem 2.3] or [48, p. 87]. This leads to the important definition, cf. [79]:

5.3 Definition. A notion of forcing P is *proper* if for every uncountable set A, every stationary subset of $[A]^{\aleph_0}$ remains stationary in $V[G]$.

There are several equivalent definitions of properness, most using the club filter on $[A]^{\aleph_0}$. Let me state one of them (see [79, p. 77], [48, p. 97]):

5.4 Proposition. *A complete Boolean algebra B is proper if and only if for every nonzero $a \in B$ for every uncountable λ and every collection $\{a_{\alpha\beta} : \alpha, \beta < \lambda\}$ such that $\Sigma_{\beta<\lambda} a_{\alpha\beta} = a$ for every $\alpha < \lambda$, there exists a club $C \subseteq [\lambda]^{\aleph_0}$ such that $\Pi_{\alpha\in x}\Sigma_{\beta\in x} a_{\alpha\beta} \neq 0$ for all $x \in C$.*

5.2. Projective and Cohen Boolean Algebras

The club filter on $[A]^{\aleph_0}$ turns out to be a useful tool in the study of Boolean algebras. Here we present a uniform approach to the investigation of two related concepts, projective and Cohen Boolean algebras. For simplicity, we consider only atomless Boolean algebras of uniform density.

5.5 Definition.

(a) A Boolean algebra B is *projective* if for some Boolean algebra C, the free product $B \oplus C$ is a free Boolean algebra.

(b) A Boolean algebra B is a *Cohen algebra* if its completion is isomorphic to the completion of a free Boolean algebra.

Projective algebras have several other (equivalent) definitions and are projective in the sense of universal algebra; we refer to [64] for details. Forcing with Cohen algebras adds Cohen reals. In the present context, it is the following equivalences that make these two classes interesting: Let A be a subalgebra of a Boolean algebra B. A is a *relatively complete* subalgebra of B, $A \leq_{\mathrm{rc}} B$, if for each $b \in B$ there is a smallest element $a \in A$ such that $b \leq a$. A is a *regular subalgebra* of B, $A \leq_{\mathrm{reg}} B$, if every maximal antichain in A is maximal in B. Let $\langle A_1 \cup A_2\rangle$ denote the subalgebra generated by $A_1 \cup A_2$.

5.6 Theorem.

(a) (Ščepin [78]) *A Boolean algebra B is projective if and only if the set $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club C with the property that for all $A_1, A_2 \in C$, $\langle A_1 \cup A_2\rangle \in C$.*

(b) (Koppelberg [65], Balcar-Jech-Zapletal [3]) *A Boolean algebra B is Cohen if and only if the set $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{reg}} B\}$ contains a club with the property that for all $A_1, A_2 \in C$, $\langle A_1 \cup A_2\rangle \in C$.*

This leads naturally to the following concepts:

5.7 Definition.

(a) (Ščepin) A Boolean algebra B is *openly generated* if the set $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club.

(b) (Fuchino-Jech) A Boolean algebra B is *semi-Cohen* if the set $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{reg}} B\}$ contains a club.

Openly generated (also called rc-filtered) and semi-Cohen Boolean algebras are investigated systematically in [42] and [3], respectively.

Our first observation is that every projective algebra is openly generated and every Cohen algebra is semi-Cohen; and if $|B| = \aleph_1$ and B is openly generated (or semi-Cohen), then B is projective (or Cohen). Because a σ-closed forcing preserves stationary sets in $[B]^{\aleph_0}$ (by Theorem 5.2), we have:

5.8 Corollary. *B is openly generated (resp. semi-Cohen) iff $V^P \vDash B$ is projective (resp. Cohen), where P is the σ-closed collapse of $|B|$ onto $\aleph_1$.*

An immediate consequence is that the completion of a semi-Cohen algebra is semi-Cohen.

Using some simple algebra and Proposition 4.8, one can show that every rc-subalgebra of an openly generated algebra is openly generated and every regular subalgebra of a semi-Cohen algebra is semi-Cohen. Consequently, we have:

5.9 Corollary.

(a) (Ščepin) *If B is projective and $A \leq_{\mathrm{rc}} B$ has size $\aleph_1$, then A is projective.*

(b) (Koppelberg) *If B is a Cohen algebra and $A \leq_{\mathrm{reg}} B$ has size $\aleph_1$, then A is Cohen.*

Finally, the use of the club filter yields a simple proof of the following theorem:

5.10 Theorem.

(a) (Ščepin [78], Fuchino [28]) *The union of any continuous $\leq_{\mathrm{rc}}$-chain of openly generated algebras is openly generated.*

(b) (Balcar-Jech-Zapletal [3]) *The union of any continuous $\leq_{\mathrm{reg}}$-chain of semi-Cohen algebras is semi-Cohen.*

Proof. Let B be the union and let λ be sufficiently large; by Proposition 5.1 it suffices to show that for every countable $M \prec \mathcal{H}_\lambda$ such that $B \in M$, $B \cap M$ is a relatively complete (resp. regular) subalgebra of B. It is not very difficult to prove this. ⊣

6. Reflection

In Sect. 2 we introduced the important concept of reflection. One can expect that its generalization to $P_\kappa\lambda$ will be equally important. This is indeed the case, and in particular, reflection of stationary sets in $[\lambda]^{\aleph_0}$ at sets of cardinality $\aleph_1$ plays a significant role in applications of Martin's Maximum.

Let us begin with a generalization of reflection of which very little is known (see [55] for a consistency result): Let κ be inaccessible, and let $\lambda > \kappa$. For each $x \in P_\kappa\lambda$, let $\kappa_x = x \cap \kappa$; note that for almost all x, κ_x is a cardinal. When κ_a is regular uncountable, we say that a stationary S reflects at a if $S \cap P_{\kappa_a} a$ is a stationary subset of $P_{\kappa_a} a$.

The following argument shows that there are limitations to reflection: Let $S \subseteq \mathbf{E}^\lambda_{\aleph_0}$ and $T \subseteq \mathbf{E}^\lambda_{\aleph_1}$ be such that S does not reflect at any $\alpha \in T$ (see Lemma 2.16). Let $\widehat{S} = \{x \in P_\kappa\lambda : \sup(x) \in S\}$ and $\widehat{T} = \{a \in P_\kappa\lambda : \sup(a) \in T\}$. Then $\widehat{S}$ does not reflect at any $a \in \widehat{T}$.

A similar generalization leads to significant results in the large cardinal theory and we shall now investigate this generalization.

6.1. Reflection Principles

In [27], Foreman, Magidor and Shelah introduced Martin's Maximum and proved a number of consequences. Let us recall that *Martin's Maximum* (MM) states that whenever P is a notion of forcing that preserves stationary subsets of $\aleph_1$, and D is a family of $\aleph_1$ dense subsets of P, then there exists a D-generic filter on P. By [27] Martin's Maximum is consistent relative to a supercompact cardinal.

Among the consequences of MM proved in [27] are the following:

- The nonstationary ideal on $\aleph_1$ is $\aleph_2$-saturated.
- For every regular $\kappa \geq \aleph_2$, every stationary set $S \subseteq \mathbf{E}^\kappa_{\aleph_0}$ contains a closed set of order type ω_1.
- $2^{\aleph_0} = \aleph_2$.
- For every regular $\kappa \geq \aleph_2$, $\kappa^{\aleph_0} = \kappa$.

The authors of [27] introduced the following *Reflection Principle* and proved that it follows from MM.

If S is a stationary subset of $[\lambda]^{\aleph_0}$ and $X \in [\lambda]^{\aleph_1}$ we say that S *reflects at* X if $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.

6.1 Definition (Reflection Principle, Foreman-Magidor-Shelah [27]). For every regular $\lambda \geq \aleph_2$, every stationary set $S \subseteq [\lambda]^{\aleph_0}$ reflects at some $X \in [\lambda]^{\aleph_1}$ such that $X \supseteq \omega_1$.

For a given regular λ, let us call the property in Definition 6.1 *Reflection Principle at* λ. As for the extra condition $X \supseteq \omega_1$, this is not just an ad hoc requirement. Its role is clarified in the following two propositions. (Compare this with the remark following Theorem 2.3.)

6.2 Proposition (Feng-Jech [21]). *Let $\lambda \geq \aleph_2$ be a regular cardinal.*

(a) *Reflection Principle at λ holds iff for every stationary set $S \subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S$ reflects at $X\}$ is stationary in $[\lambda]^{\aleph_1}$.*

(b) *Every stationary $S \subseteq [\lambda]^{\aleph_0}$ reflects at some $X \in [\lambda]^{\aleph_1}$ iff for every stationary set $\subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S$ reflects at $X\}$ is weakly stationary in $[\lambda]^{\aleph_1}$.*

For $\lambda = \aleph_2$ the assumption $X \supseteq \omega_1$ can be dropped; it is unknown if the same is true in general:

6.3 Proposition (Feng-Jech [21]). *Reflection Principle at $\aleph_2$ holds if and only if every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at some $X \in [\omega_2]^{\aleph_1}$.*

The significance of this and related reflection principles is illustrated by the fact that they imply the major consequences of MM. Firstly, Reflection Principle at $\aleph_2$ implies that the continuum is at most $\aleph_2$:

6.4 Theorem (Shelah [80], Todorčević [89]). *If every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at some $X \in [\omega_2]^{\aleph_1}$, then $2^{\aleph_0} \leq \aleph_2$.*

Proof. For each uncountable $\alpha < \omega_2$, let $C_\alpha \subseteq [\alpha]^{\aleph_0}$ be a club of cardinality $\aleph_1$, and let $D = \bigcup_{\omega_1 \leq \alpha < \omega_2} C_\alpha$. By Propositions 6.3 and 6.2(a), the set D contains a club, and we have $|D| = \aleph_2$. However, it is proved in [11] that every club in $[\omega_2]^{\aleph_0}$ has cardinality $\aleph_2^{\aleph_0}$; hence $2^{\aleph_0} \leq \aleph_2$. ⊣

Reflection Principle at $\aleph_2$ is not particularly strong; it is equi-consistent with the existence of a weakly compact cardinal. A modification of Magidor's construction [70] gives a model in which every stationary $S \subseteq [\omega_2]^{\aleph_0}$ reflects at $[\alpha]^{\aleph_0}$ for almost all $\alpha \in \mathbf{E}^{\aleph_2}_{\aleph_1}$.

The general Reflection Principle, for all regular $\lambda \geq \aleph_2$, is a stronger large cardinal property. A modification of the proof of Theorem 25 in [27] shows that the Reflection Principle implies that the nonstationary ideal on ω_1 is presaturated (i.e. precipitous, and forcing with $P(\omega_1)/\mathrm{NS}$ preserves ω_2). This has strong large cardinal consequences.

The Reflection Principle follows from MM and in fact from a weaker forcing axiom MA^+ (σ-closed). (This latter axiom is known to be strictly weaker than MM.) In fact, MA^+ (σ-closed) implies (cf. [14]) for every regular $\lambda \geq \aleph_2$, for every stationary set $\cdot S \subseteq [\lambda]^{\aleph_0}$, the set $\{X \in [\lambda]^{\aleph_1} : S$ reflects at $X\}$ meets every ω_1-closed unbounded set C in $[\lambda]^{\aleph_1}$. This reflection principle was introduced in [23].

Todorčević formulated a strengthening of the Reflection Principle and proved that his *Strong Reflection Principle* (SRP) implies that the nonstationary ideal on ω_1 is $\aleph_2$-saturated, that every stationary subset of $\mathbf{E}^{\kappa}_{\aleph_0}$ contains a closed copy of ω_1 and that for every regular $\kappa \geq \aleph_2$, $\kappa^{\aleph_0} = \kappa$ (cf. [15]).

In [22], another reflection principle is introduced, called *Projective Stationary Reflection*, and proved to be equivalent to the Strong Reflection Principle (SRP = PSR).

6.5 Definition (Feng-Jech [22]). A stationary set $S \subseteq [A]^{\aleph_0}$ where $A \supseteq \omega_1$, is *projective stationary* if for every club $C \subseteq [A]^{\aleph_0}$, the projection $(S \cap C) \restriction \omega_1 = \{x \cap \omega_1 : x \in S \cap C\}$ to ω_1 contains a club.

6.6 Definition (Projective Stationary Reflection (PSR), Feng-Jech [22]). For every regular $\lambda \geq \aleph_2$, every projective stationary set $S \subseteq [H_\lambda]^{\aleph_0}$ contains an increasing continuous $\in$-chain $\{N_\alpha : \alpha < \omega_1\}$ of elementary submodels of H_λ.

If $S \subseteq [H_\lambda]^{\aleph_0}$ is stationary, let P_S be the forcing notion consisting of countable increasing continuous $\in$-chains $\{N_\alpha : \alpha < \gamma\} \subseteq S$ of elementary submodels of H_λ. The set S is projective stationary just in case P_S preserves stationary subsets of ω_1. Thus PSR follows from Martin's Maximum. It is also proved in [22] that PSR implies the Reflection Principle.

6.7 Theorem (Feng-Jech [22]). *Assume* PSR.

(a) *For every regular $\kappa \geq \aleph_2$, every stationary set $S \subseteq \mathbf{E}^\kappa_{\aleph_0}$ contains a closed set of order type ω_1.*

(b) *The nonstationary ideal on ω_1 is $\aleph_2$-saturated.*

Proof. (a) This is proved by applying PSR to the projective stationary set

$$\{N \in [H_\kappa]^{\aleph_0} : S \in N \prec H_\kappa \text{ and } \sup(N \cap \kappa) \in S\},$$

where S is a given stationary subset of $\mathbf{E}^\kappa_{\aleph_0}$.

(b) Let A be a maximal antichain of stationary subsets of ω_1. Then the set

$$X = \{N \in [H_{\omega_2}]^{\aleph_0} : A \in N \prec H_{\omega_2} \text{ and } N \cap \omega_1 \in S \text{ for some } S \in A \cap N\}$$

is projective stationary. By PSR, there exists an $\in$-chain $\{N_\alpha : \alpha < \omega_1\} \subseteq X$, and we let $N = \bigcup_{\alpha<\omega_1} N_\alpha$. One can verify that $A \subseteq N$, and therefore $|A| \leq \aleph_1$. ⊣

Finally, recent work of Woodin shows that Strong Reflection implies that $2^{\aleph_0} = \aleph_2$, in fact $\boldsymbol{\delta}^1_2 = \omega_2$:

6.8 Theorem (Woodin [94]). *Assume* SRP. *Then the set $\{N \in [H_{\omega_3}]^{\aleph_1} : N \prec H_{\omega_3}$ and the order type of $N \cap \omega_3$ is $\omega_1\}$ is weakly stationary. This together with the saturation of the nonstationary ideal, implies that $\boldsymbol{\delta}^1_2 = \omega_2$.*

6.2. Nonreflecting Stationary Sets

The results about reflecting stationary sets in $[\lambda]^{\aleph_0}$ at sets of size $\aleph_1$ do not generalize to $[\lambda]^\kappa$ for $\kappa \geq \aleph_1$. For instance, the analog of the Reflection Principle is false:

6.9 Proposition. *If λ is sufficiently large, then it is not the case that every stationary $S \subseteq [\lambda]^{\aleph_1}$ reflects at some $X \in [\lambda]^{\aleph_2}$ such that $X \supseteq \omega_2$.*

In Sect. 6.1 we mentioned that the Reflection Principle implies that NS on ω_1 is presaturated. To prove Proposition 6.9, one first shows that the generalization of the Reflection Principle would yield presaturation of NS on ω_2, thus (as in Shelah's Corollary 3.7) a forcing notion that changes the cofinality of ω_2 to ω while preserving $\aleph_1$ and $\aleph_3$. But that is impossible.

Specific examples of nonreflecting stationary subsets of $[\lambda]^{\aleph_1}$ are given in [25]. That paper also explains why the consistency proof of Reflection Principle does not generalize. A model of MA^+ (σ-closed) is obtained by Lévy collapsing (to $\aleph_1$) cardinals below a supercompact. A crucial fact is that the collapse preserves stationary sets in $[\lambda]^{\aleph_0}$ (Theorem 5.2(a)). Unfortunately, the analog of this is false in general, as $<\kappa$-closed forcing can destroy stationary sets in $P_\kappa\lambda$.

Following [27], a model $N \prec H_\lambda$ is *internally approachable* (IA) if there exists a chain $\langle N_\alpha : \alpha < \gamma\rangle$ whose initial segments belong to N, with $N = \bigcup_{\alpha<\gamma} N_\alpha$. Let κ be a regular uncountable cardinal and let $\lambda \geq \kappa$ be regular. The set $P_\kappa H_\lambda \cap \mathrm{IA}$ is stationary and its projection to κ contains a club. Moreover, every countable N is internally approachable and so $[H_\lambda]^{\aleph_0} \cap \mathrm{IA}$ contains a club.

It is proved in [25] that every $<\kappa$-closed forcing preserves stationary subsets of $P_\kappa H_\lambda \cap \mathrm{IA}$, and that the $<\kappa$-closed collapse of H_λ shoots a club through $P_\kappa H_\lambda \cap \mathrm{IA}$. As for a generalization of Reflection Principle, they prove that if cardinals between κ and a supercompact are collapsed to κ, then in the resulting model, every stationary set $S \subseteq P_\kappa H_\lambda \cap \mathrm{IA}$ reflects at a set of size κ.

7. Stationary Tower Forcing

In this last section we give a brief description of stationary tower forcing, introduced by Woodin in [93]. See also [69] and Foreman's chapter in this Handbook.

Let δ be an inaccessible cardinal. Let Q and P be the following notions of forcing (*stationary tower forcing*):

A forcing condition in Q is a pair (A, S) where $A \in V_\delta$ and S is a stationary subset of $[A]^{\aleph_0}$; $(A, S) < (B, T)$ if $A \supseteq B$ and $S{\restriction}B \subseteq T$.

A forcing condition in P is a pair (A, S) where $A \in V_\delta$ and S is a weakly stationary subset of $P_{|A|}A$; $(A, S) < (B, T)$ if $A \supseteq B$ and $S{\restriction}B \subseteq T$.

In fact, stationary tower forcing is somewhat more general than these two examples, and uses the following generalization of stationary sets (considered e.g. in [20]). A set S is *stationary* in $P(A)$ if $S \subseteq P(A)$ and if for every $F : [A]^{<\omega} \to A$, S contains a closure point of F, i.e. a set $X \subseteq A$ such that $F(e) \in X$ for all $e \in [X]^{<\omega}$. As in Proposition 4.8, projections and liftings of stationary sets are stationary. Also, the analog of Theorem 4.4 holds. Note that the sets $S{\restriction}P_\kappa A$ are exactly the weakly stationary sets in $P_\kappa A$, and $S{\restriction}\{X \in P_\kappa\lambda : X \cap \kappa \in \kappa\}$ are the stationary sets in $P_\kappa\lambda$.

The general version of stationary tower forcing uses conditions (A, S) where S is stationary in $P(A)$.

If G is a generic filter on Q, then for each $A \in V_\delta$, the set $G_A = \{S : (A, S) \in G\}$ is a V-ultrafilter on $([A]^{\aleph_0})^V$; similarly for P. Moreover, if $A \subseteq B$, then G_B projects to G_A. In $V[G]$ we form a limit ultrapower $M = \mathrm{Ult}_G(V)$ by the G_A, $A \in V_\delta$. The elements of M are represented by functions (in V) whose domain is some $A \in V_\delta$. Let $j : V \to M$ be the *generic embedding*, i.e. the elementary embedding from V into the limit ultrapower.

The ultrapower has a well founded initial segment up to at least δ: each ordinal $\alpha \le \delta$ is represented by the function $f_\alpha(x) = x \cap \alpha$. The identity function $\mathrm{id}(x) = x$ represents the set $j``V_\delta$. Woodin's main tool is the following:

7.1 Theorem (Woodin [93]). *Suppose δ is a Woodin cardinal. If G is a generic on either Q or P, then the generic ultrapower $\mathrm{Ult}_G(V)$ is well founded, and the model M is closed under sequences of length $< \delta$.*

When forcing with Q, one has $\mathrm{crit}(j) = \omega_1$ and $j(\omega_1) = \delta$. For applications, see [93].

Forcing with P gives more flexibility and yields various strong forcing results. We conclude this section with a typical application. Assume that $\aleph_\omega$ is strong limit. Let

$$S = \{X \in [V_{\aleph_{\omega+1}}]^{\aleph_\omega} : X \cap \aleph_{\omega+1} \in \aleph_{\omega+1} \text{ and } \mathrm{cf}(X \cap \aleph_{\omega+1}) = \aleph_3\}$$

and let G be a generic on P such that $S \in G$. Then $\mathrm{crit}(j) = \aleph_{\omega+1}$ and $\mathrm{cf}^M(\aleph_{\omega+1}) = \aleph_3$. As $P^{V[G]}(\omega_n) = P^M(\omega_n) = P^V(\omega_n)$ for all n, we conclude that forcing with P (below $(V_{\aleph_{\omega+1}}, S)$) changes the cofinality of $\aleph_{\omega+1}$ to $\aleph_3$ while preserving $\aleph_\omega$.

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2. Partition Relations

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1. Introduction

The study of partition relations dates back to 1930, when Frank P. Ramsey [49] proved his oft-cited theorem.

1.1 Theorem (Ramsey's Theorem). *Assume* $1 \leq r$, $k < \omega$ *and* $f : [\omega]^r \to k$ *is a partition of the* r *element subsets of* ω *to* k *pieces. Then there is an infinite subset* $X \subseteq \omega$ *homogeneous with respect to this partition. That is, for some* $i < k$, $f``[X]^r = \{i\}$.

In 1941, Ben Dushnik and Edwin Miller [9] looked at partitions of the set of all pairs of elements of an uncountable set, involving Paul Erdős in solving one of their more difficult problems (see Theorem 7.4). In 1942, Erdős [10] proved some basic generalizations of Ramsey's Theorem, including among others the theorem generally called the Erdős-Rado Theorem for pairs. In the early fifties, Erdős and Richard Rado [15, 17] initiated a systematic investigation of quantitative generalizations of this result. They called it the partition calculus. There are cases in mathematical history when a well-chosen notation can enormously enhance the development of a branch of mathematics and a case in point is the ordinary partition symbol (see Definition 1.3)

$$\alpha \to (\beta_\xi)^r_{\xi<\gamma}$$

invented by Rado [16], reducing Ramsey's Theorem to $\omega \to (\omega)^r_\gamma$ for $1 \leq r$, $\gamma < \omega$. It became clear that a careful analysis of the problems according to the size and nature of the parameters leads to an inexhaustible array of problems, each seemingly simple and natural. These classical investigations were completed in the 1965 paper [18] of Erdős, András Hajnal and Rado, and were extended in the book [19] written jointly with Attila Máté. We cite this compendium from time to time for proofs we omit and as a resource for some open problems we include.

In 1967, after the first post-Cohen set theory conference, held in Los Angeles, Erdős and Hajnal wrote a list of unsolved problems for the ordinary partition symbol and related topics. This paper [12] appeared in print four years later.

A great many new results were proved by the *then* young researchers. However, unlike many other classical problems, these problems have resisted full solution. The introduction of new methods and the discovery of new ideas usually has given only incremental progress, and objectively, we are as far as ever from complete answers. However, small steps requiring new methods have been continuously made, quite a few of them during the writing of this paper, and we will concentrate on them.

For easy reference, in the ordinary partition relation $\alpha \to (\beta_\xi)^r_\gamma$, we call α the *resource*, β_ξ the *goals*, and γ the *set of colors*. We will be focusing on two main subjects:

1. New ZFC theorems obtained via the elementary submodel method both for ordinary partition relations and for polarized partition relations (see Definition 1.5).

2. The new results obtained in the late nineties for partition relations with a countable resource.

Section 2 describes the classical proofs of the (balanced) form of the Erdős-Rado Theorem and the Positive Stepping Up Lemma. These are the results where the resource is regular and the goals are equal and of the form τ, or $\tau+1$ for some cardinal τ. In Sect. 2.3 we state but do not prove the Negative Stepping Up Lemma complementing these results.

In Sect. 3, we describe the elementary submodel method and in particular, the use of nonreflecting ideals first introduced in [4]. We give an alternate proof of the balanced Erdős-Rado Theorem, and give a proof of the unbalanced form of it using the new method.

In Sect. 4, especially in Sect. 4.2, we fully develop the method of elementary submodels. We give streamlined proofs of both the balanced and unbalanced forms of the Baumgartner-Hajnal-Todorcevic Theorems [4] in Sects. 4.3 and 4.4. These results generalize the Erdős-Rado Theorem to allow goals which are ordinals more complex than cardinals τ and their ordinal successors, $\tau+1$. We state a result of Matthew Foreman and Hajnal [20] for the successors of measurable cardinals. Using the methods of the Foreman-Hajnal proof, in Sect. 4.5, we give a direct proof of a special case of the Baumgartner-Hajnal Theorem [2].

In Sect. 5, we discuss the Milner-Rado Paradox and the new ordinal $\Omega(\kappa) < \kappa^+$ introduced in the Foreman-Hajnal result [20], which is related to a form of the Milner-Rado Paradox.

In Sect. 6, we discuss a new development, the first in the twenty-first century. Solving a problem of Foreman and Hajnal, Saharon Shelah [59] proved that if there is a strongly compact cardinal, then there are cardinals κ such that $\kappa^+ \to (\kappa+2)^2_\omega$.

In Sect. 7, we briefly discuss the case of singular resources. We state, but do not prove, several theorems on this subject from the 1965 Erdős, Hajnal and Rado paper [18] and the 1975 Shelah paper [56].

In Sect. 8, we describe a new variant of the elementary submodel method called *double ramification*, which was invented by Baumgartner and Hajnal to establish their Theorem 8.2.

In Sect. 8.1, we use it for the proof of

$$(*) \qquad \binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}^{1,1}_{\gamma}$$

where κ is weakly compact and $\gamma < \kappa$. Result $(*)$ was previously known only if $\gamma < \omega$ (see the discussion before Theorem 8.2). In Sect. 8.2, we use the method for the proof of Shelah's Theorem [58] stating that $(*)$ holds for κ a singular strong limit cardinal (of uncountable cofinality) which satisfies $2^\kappa > \kappa^+$ and for $\gamma < \mathrm{cf}(\kappa)$.

In Sect. 9, we discuss the spectacular progress by Carl Darby [7, 8] and Rene Schipperus [53, 51] on the cases where the resource α is a countable ordinal, listing their negative partition results in Theorem 9.9, and give a sample counterexample, $\omega^{\omega^2} \nrightarrow (\omega^{\omega^2}, 6)^2$. This example is not optimal, but was chosen to illustrate the methods of Darby without all the complicating detail.

In Sect. 10, we outline a proof of a special case of the positive results by Schipperus that $\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, 3)^2$ for $\beta \geq 2$ the sum of one or two indecomposable ordinals (Darby independently proved the result for $\beta = 2$).

We close this section with some background definitions.

1.1. Basic Definitions

1.2 Definition. Let X be a set, $r < \omega$ and β, γ be ordinals.

1. A map $f : [X]^r \to \gamma$ is called an *r-partition of X with γ colors.*
2. For $\xi < \gamma$, a subset $Y \subseteq X$ is called *homogeneous for f in color ξ* if $f``[Y]^r = \{\xi\}$.
3. The set $Y \subseteq X$ is *homogeneous for f* if it is homogeneous for f in some color $\xi < \gamma$.
4. A linearly ordered set X has *order type β*, in symbols, $\mathrm{ot}(X) = \beta$, if it is order isomorphic to β.

1.3 Definition. Let α, β_ξ for $\xi < \gamma$, and γ be ordinals and suppose $1 \leq r < \omega$. The *ordinary partition symbol*

$$\alpha \to (\beta_\xi)^r_\gamma$$

means that the following statement is true.

> For every r-partition of α with γ colors, $f : [\alpha]^r \to \gamma$, there exist $\xi < \gamma$ and $X \subseteq \alpha$ such that $\mathrm{ot}(X) = \beta_\xi$ and X is homogeneous for f in color ξ.

We write

$$\alpha \not\to (\beta_\xi)^r_\gamma$$

to indicate that the negation of this statement is true. If all β_ξ equal β, then we write

$$\alpha \to (\beta)^r_\gamma \quad (\text{or } \alpha \not\to (\beta)^r_\gamma).$$

A further more or less self explanatory abbreviation is $\alpha \to (\beta_0, (\beta)_\gamma)^2$ in case $\beta_\xi = \beta$ for $1 \le \xi < \gamma$.

1.4 Remark. Note that the notation of Definition 1.3 is so devised that if we start with a positive partition relation $\alpha \to (\beta_\xi)^r_\gamma$, then the truth of the assertion is preserved under increasing the *resource* ordinal α on the left-hand side of the arrow ($\to$) and decreasing the ordinal *goals* β_ξ, or the colors γ on the right-hand side of the arrow. And this latter statement holds, with some exceptions, for the exponent r as well (see [19]).

We stated Definition 1.3 in this generality, because it will suffice for most of what we will prove. It should be clear that further generalizations can be made. For example, a similar symbol $\Theta \to (\Theta_\xi)^\delta_\gamma$ can be defined where $\Theta, \Theta_\xi, \delta$ are order types, by starting with an arbitrary ordered set $\langle X, \prec\rangle$ for which $\mathrm{ot}(X, \prec) = \Theta$, partitioning its subsets of order type δ,

$$[X]^\delta = \{Y \subseteq X : \mathrm{ot}(Y, \prec) = \delta\},$$

into γ color classes, and as above, looking for homogeneous subsets of the prescribed color and order type. As general Ramsey theory developed in both finite and infinite combinatorics, problems were considered in which the set partitioned was a subset of $[X]^\delta$ rather than all of $[X]^\delta$, and the homogeneous sets consisted of possibly other kinds of subsets of $[X]^\delta$. Partition relations proliferated. For a review of some of them we refer to [19], since we can not try to cover all of them in the limit space of this chapter.

In [18], among other generalizations, polarized partitions were introduced. In fact, this paper is the only place in the published literature where these relations are systematically discussed.

1.5 Definition. Let α,β be ordinals and suppose that $\alpha_0, \alpha_1 \le \alpha$ and $\beta_0, \beta_1 \le \beta$. The *polarized partition relation*

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to \begin{pmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\end{pmatrix}$$

means that the following statement is true.

> For all ordered sets A and B of order type α, β respectively, and all partitions $f : A \times B \to 2$, there is an $i < 2$ and sets $A_i \subseteq A$, $B_i \subseteq B$ such that $\mathrm{ot}(A_i) = \alpha_i$, $\mathrm{ot}(B_i) = \beta_1$ and $f``A_i \times B_i = \{i\}$.

2. Basic Partition Relations

2.1. Ramsey's Theorem

2.1 Definition. Assume $\langle X, \prec\rangle$ is an ordered set and $f : [X]^r \to \gamma$ is an r-partition of length γ of X, $1 \le r < \omega$.

1. For $V \in [X]^{r-1}$, define $f_V : X - V \to \gamma$ by
$$f_V(u) = f(V \cup \{u\}).$$
2. f is *endhomogeneous on* X if for every $V \in [X]^{r-1}$, the function f_V is homogeneous on $X| \succ V = \{u \in X : V \prec u\}$.
3. Let
$$X^- = \begin{cases} X - \{m\} & \text{if } X \text{ has a maximal element } m, \\ X & \text{otherwise.} \end{cases}$$
4. Assume f is endhomogeneous on X. Define $f^- : [X^-]^{r-1} \to \gamma$ by $f^-(V) = \eta$ iff $\forall u \in X| \succ V(f_V(u) = \eta)$ for $V \in [X^-]^{r-1}$.

The next lemma follows immediately from the definitions.

2.2 Lemma. *Using the above notation, if f is endhomogeneous on X, $Y \subseteq X^-$ and f^- is homogeneous on Y then f is homogeneous on Y and on $Y \cup \{m\}$ if m is the maximal element of X.*

We first give a direct proof of the well-known Ramsey's Theorem using non-principal ultrafilters and postponing the more natural *ramification* method to the next section for two reasons. First, Erdős and Rado considered this approach part of their "combinatorics" (Erdős called the ultrafilters "measures"). Second, having given a proof here, we do not have to adapt the formulation of the ramification to cover the case when the resource is a regular limit cardinal.

2.3 Theorem (Ramsey's Theorem).
$$\omega \to (\omega)^r_k \quad \textit{for } 1 \le r, k < \omega.$$

Proof. By induction on r. For $r = 1$ the claim is obvious. Assume $r > 1$ and $f : [\omega]^r \to k$. Let U be a non-principal ultrafilter on ω and $V \in [\omega]^{r-1}$. Define $\tilde{f}(V)$ and $A(V)$ as follows: let $\tilde{f}(V) = i$ for the unique $i < k$ for which the set $A(V, i) := \{u \in \omega - V : f_V(u) = i\}$ is in U, and set $A(V) := A(V, \tilde{f}(V))$.

We can choose by induction on n an increasing sequence $\langle x_n : n < \omega\rangle$ of integers satisfying $x_n \in \bigcap\{A(V) : V \in [\{x_j : j < n\}]^r\}$ for $n < \omega$. Let $X = \{x_n : n < \omega\}$. Then $f^-{\restriction}[X]^{r-1} = \tilde{f}{\restriction}[X]^{r-1}$ and f is endhomogeneous on X. By the induction hypothesis, there is a $Y \subseteq X$ with $\mathrm{ot}(Y) = \omega$ so that Y is homogeneous for f^-. Finally, by Lemma 2.2, Y is the desired set homogeneous for f. $\dashv$

2.2. Ramification Arguments

2.4 Remark (A brief history). The first transfinite generalization of Ramsey's theorem appeared in the paper [9] of Dushnik and Miller. They proved $\kappa \to (\kappa, \omega)^2$ for regular κ and Erdős proved this for singular κ as well. His proof was included in [9]. This theorem, unique of its kind, logically belongs to Sect. 7 where we will discuss it briefly.

The basic theorems about partition relations with exponent $r = 2$ were first stated and proved in 1942 in an almost forgotten paper of Erdős [10]. There he proved $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ for $\kappa \geq \omega$; he indicated the counterexamples $2^\kappa \not\to (3)^2_\kappa$ and $2^\kappa \not\to (\kappa^+)^2_2$; and he proved $\omega_2 \to (\omega_2, \omega_1)^2$ assuming CH. The Erdős-Rado Theorem for exponent larger than 2 was proved later in [17]. (See Corollary 2.10.) Kurepa also worked on related questions quite early (see the discussion by Todorcevic in Section C of [37]).

Few theorems have been provided with as many simplified proofs as the Erdős-Rado Theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. Erdős and Rado used the so called "ramification method". We will present this method in the proof of the next theorem. After some "streamlining," it still seems to be the simplest way for obtaining *balanced* partition relations for cardinals, ones in which all the goals are the same cardinal. For the *unbalanced* case, we will present a method worked out in [4]. This method will be used in the proofs of a number of more recent results which will be presented in later sections. Given limitations of time and energy, and a desire for coherence, we decided to focus on results amenable to this method.

2.5 Theorem. *Assume* $2 \leq r < \omega$, $\kappa \geq \omega$, $\gamma < \kappa$, $\lambda = 2^{<\kappa}$ *and*

$$f : [\lambda^+]^r \to \gamma.$$

Then there exists an $X \subseteq \lambda^+$ *with* $\mathrm{ot}(X) = \kappa + 1$ *such that* f *is endhomogeneous*

Proof. For $\alpha < \lambda^+$, define an increasing sequence $\overline{\beta}^\alpha = \langle \beta^\alpha_\eta : \eta < \varphi_\alpha \rangle$ of ordinals less then α and an ordinal φ_α by transfinite recursion on η. For $\alpha = 0$, set $\varphi_0 = 0$ and let $\overline{\beta}^0$ be the empty sequence. For positive α, to start the recursion, let $\beta^\alpha_q := q$ for $q < \max(\{\alpha, r-1\})$, and for $\alpha < r - 1$, let $\varphi_\alpha = \alpha$. To continue the recursion, assume $r - 2 < \eta$ and β^α_ζ is defined for $\zeta < \eta$. Let $\hat{\beta}^\alpha_\eta = \sup(\{\beta^\alpha_\zeta + 1 : \zeta < \eta\})$, and define sets

$$\begin{aligned} B^\alpha_\eta &:= \{\beta^\alpha_\zeta : \zeta < \eta\}, \\ A^\alpha_\eta &:= \{\beta < \alpha : \hat{\beta}^\alpha_\eta \leq \beta \wedge (\forall V \in [B^\alpha_\eta]^{r-1})(f_V(\beta) = f_V(\alpha))\}. \end{aligned}$$

Let $\beta^\alpha_\eta := \min(A^\alpha_\eta)$ if $A^\alpha_\eta \neq \emptyset$. If $A^\alpha_\eta = \emptyset$, put $\varphi_\alpha = \eta$. Clearly for each $\alpha < \lambda^+$, the set $B^\alpha_{\varphi_\alpha} \cup \{\alpha\}$ is an endhomogeneous set of order type $\varphi_\alpha + 1$, and we may define f^-_α on $[B^\alpha_{\varphi_\alpha}]^{r-1}$ as in Definition 2.1. If $\beta \in B^\alpha_{\varphi_\alpha}$, then it is easy to show by induction on $\eta < \varphi_\beta$ that $\beta^\beta_\eta = \beta^\alpha_\eta$. Thus if $\beta \in B^\alpha_{\varphi_\alpha}$, then f^-_α agrees with f^-_β on $[B^\beta_{\varphi_\beta}]^{r-1}$.

Define a relation $\prec$ on λ^+ by $\beta \prec \alpha$ iff $\beta \in B^\alpha_{\varphi_\alpha}$. It is easy to verify that $T := \langle \lambda^+, \prec \rangle$ is a tree on λ^+ and $\operatorname{rank}_T(\alpha) = \varphi_\alpha$ for $\alpha < \lambda^+$. T is called the *canonical partition tree* of f on λ^+, and T_φ, as usual, denotes the $\{\alpha < \lambda^+ : \operatorname{rank}_T(\alpha) = \varphi\}$.

For $\alpha < \lambda^+$, let $C_\alpha : [\varphi_\alpha]^{r-1} \to \gamma$ be defined by $C_\alpha(U) = f^-_\alpha(V)$ where $V = \{\beta^\alpha_\zeta : \zeta \in U\}$. It follows by transfinite induction on φ that for $\alpha, \beta \in T_\varphi$, if $C_\alpha = C_\beta$, then $\alpha = \beta$. Hence $|T_\varphi| \le |\gamma|^{|\varphi|} \le \lambda$ for $\varphi < \kappa$. Then $|\bigcup_{\varphi<\kappa} T_\varphi| \le \lambda$, $T_\kappa \ne \emptyset$ and for all $\alpha \in T_\kappa$, $B^\alpha_\kappa \cup \{\alpha\}$ is a set of order type $\kappa + 1$ which is endhomogeneous for f. ⊣

2.6 Remark. Note that $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ can hold for singular κ. Indeed it is easy to see that either $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$ or $\operatorname{cf}((2^{<\kappa})^{<\kappa}) = \operatorname{cf}(\kappa)$ and $2^{<\kappa} = \sup(\{(2^\tau)^+ : \tau < \kappa\})$. The proof described above gives Theorem 2.5 under the condition $\gamma \le \lambda$ provided $\lambda^{<\kappa} = \lambda$.

2.7 Theorem (The Stepping Up Lemma). *Assume $\kappa \ge \omega$, $1 \le r < \omega$, $\gamma < \kappa$ and $\kappa \to (\alpha_\xi)^r_\gamma$. Then*

$$(2^{<\kappa})^+ \to (\alpha_\xi + 1)^{r+1}_\gamma.$$

This is an immediate consequence of Lemma 2.2 and Theorem 2.5.

2.8 Definition. Define $\exp_i(\kappa)$ by recursion on $i < \omega$:

$$\exp_0(\kappa) = \kappa,$$
$$\exp_{i+1}(\kappa) = 2^{\exp_i(\kappa)}.$$

2.9 Theorem (The Erdős-Rado Theorem). *Assume $\kappa \ge \omega$, $\gamma < \operatorname{cf}(\kappa)$. Then for all $2 \le r < \omega$,*

$$\exp_{r-2}(2^{<\kappa})^+ \to (\kappa + (r-1))^r_\gamma.$$

Proof. Starting from the trivial relation $\kappa \to (\kappa)^1_\gamma$ for $\gamma < \operatorname{cf}(\kappa)$, we get $(2^{<\kappa})^+ \to (\kappa+1)^2_\gamma$, by Theorem 2.7. This is the case $r = 2$ of the theorem. The result follows by induction on r with repeated applications of Theorem 2.7. ⊣

A better known but weaker form of the theorem is the following.

2.10 Corollary. *Assume $\kappa \ge \omega$. Then for all $1 \le r < \omega$,*

$$\exp_{r-1}(\kappa)^+ \to (\kappa^+ + (r-1))^r_\kappa.$$

Note that while Theorem 2.9 guarantees for example that $\kappa^+ \to (\kappa+1)^2_\gamma$ holds for $\gamma < \operatorname{cf}(\kappa)$ for a singular strong limit cardinal κ, Corollary 2.10 does not say anything about this case.

2.3. Negative Stepping Up Lemma

2.11 Theorem (The Negative Stepping Up Lemma). *Assume $\kappa > 0$ is a cardinal, $2 \leq r < \omega$, $1 \leq \gamma$ and $\kappa \nrightarrow (\lambda_\xi)^r_\gamma$, where each $\lambda_\xi > 0$ is a cardinal. Then $2^\kappa \nrightarrow (1+\lambda_\xi)^{r+1}_\gamma$, provided at least one of the following conditions hold:*

1. *$\gamma \geq 2$, $\kappa, \lambda_0, \lambda_1 \geq \omega$ and λ_0 is a regular cardinal;*
2. *$\gamma \geq 2$, $\kappa, \lambda_0 \geq \omega$, λ_0 is a regular cardinal, and $r \geq 4$;*
3. *$\gamma \geq 2$, $\kappa, \lambda_0, \lambda_1 \geq \omega$, and $r \geq 4$;*
4. *$\kappa \geq \omega$ and $\lambda_\xi < \omega$ for all $\xi < \gamma$.*

For a proof, we refer the reader to the compendium by Erdős, Hajnal, Máté and Rado [19], which includes additional negative stepping up results. We do quote one related open problem from that reference.

2.12 Question (Problem 25.8 in [19]). Assume GCH. Does

$$\aleph_{\omega_{\omega+1}+1} \nrightarrow (\aleph_{\omega_{\omega+1}+1}, (4)_\omega)^3?$$

The following theorem provides a context for this question.

2.13 Theorem. *Assume* GCH*. Then*

1. *$\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_\omega)^2$; and*
2. *$\aleph_{\omega_{\omega+1}} \nrightarrow (\aleph_{\omega_{\omega+1}}, (3)_\omega)^2$.*

3. Partition Relations and Submodels

For the rest of this paper we will adopt the following conventions. Whenever we write "$H(\tau)$", τ will be a regular cardinal, and "$H(\tau)$" will stand for a structure $\mathfrak{A}$ with domain the collection of sets $H(\tau)$ which are of hereditary cardinality $< \tau$. The structure $\mathfrak{A}$ will be an expansion of $\langle H(\tau), \in, \triangle\rangle$, where $\triangle$ is a fixed well-ordering of $H(\tau)$. The expansion will depend on context, and will usually include all of the relevant "data" for the proof at hand. Note that the well-ordering $\triangle$ yields well-defined Skolem hulls for all sets $X \subseteq H(\tau)$.

3.1 Definition. Assume $\kappa \geq \omega$, $2^{<\kappa} = \lambda$. Let $H := H(\lambda^{++})$. A set N is said to be *suitable for* κ if it satisfies the following conditions: $\langle N, \in\rangle \prec H$, $|N| = \lambda$, $[N]^{<\mathrm{cf}(\kappa)} \subseteq N$, $[N]^{<\kappa} \subseteq N$ if $\lambda^{<\kappa} = \lambda$, $\lambda+1 \subseteq N$, $\alpha := N \cap \lambda^+ \in \lambda^+$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)$. The ordinal $\alpha(N) = \alpha$ will be called the *critical ordinal of* N. Note that $\alpha \subseteq N$ by assumption.

We assume that the reader is familiar with the theory of stationary subsets of an ordinal. To make our terminology definite, for a limit ordinal α, a subset $B \subseteq \alpha$ is a *club* if B is cofinal (unbounded) and closed in the order topology of α. A set $S \subseteq \alpha$ is *stationary* if $B \cap S \neq \emptyset$ for every club subset of α. The notation $\mathrm{Stat}(\alpha)$ will denote the set of stationary subsets of α.

We will make use of the following facts about elementary submodels.

3.2 Facts. Let $\lambda = 2^{<\kappa}$. For every set A with $|A| \leq \lambda$ and $A \in H(\lambda^{++})$, there is an elementary chain $\langle N_0, \in\rangle \prec \cdots \prec \langle N_\alpha, \in\rangle \prec \cdots \prec H$, with $A \subseteq N_0$, indexed by $\alpha < \lambda^+$ that is continuous, and internally approachable (i.e. $N_\beta \in N_{\alpha+1}$ for all $\beta \leq \alpha$), and the set

$$S_0 = \{\alpha < \lambda^+ : \alpha(N_\alpha) = \alpha \text{ and } N_\alpha \text{ is suitable for } \kappa\}$$

the intersection of a club in λ^+ with $S_{\mathrm{cf}(\kappa),\lambda^+} = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)\}$.

3.3 Definition. A subset $S \subseteq H(\lambda^{++})$ is *amenable* for this sequence if $S \cap \alpha \in N_{\alpha+1}$ for $\alpha \in S_0$. A function g is *amenable* if $g \restriction \alpha \in N_{\alpha+1}$ for all $\alpha \in S_0$.

Note that S_0 itself may be assumed to be amenable.

In this section we will only use the existence of one N suitable for κ. The ideals defined below were introduced in [4] for regular κ. In most of the later applications we will only consider the regular case.

3.4 Definition. Let N be suitable for $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$, $\alpha(N) = \alpha$. We define a set $I = I_\alpha = I(N) \subseteq \mathcal{P}(\alpha)$ as follows. For $X \subseteq \alpha$,

$$X \in I \quad \Longleftrightarrow \quad (\exists Y)(Y \subseteq \lambda^+ \wedge Y \in N \wedge \alpha \notin Y \wedge |X - Y| < \kappa).$$

Note that for regular κ, the last clause can be replaced by $X \subseteq Y$.

3.5 Lemma. *Let N be suitable for $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$, $\alpha(N) = \alpha$. We define a set $\mathcal{F} = \mathcal{F}_\alpha$ as follows:*

$$\mathcal{F}_\alpha := \{Z \in N : Z \subseteq \lambda^+ \wedge \alpha \in Z\}.$$

Then (i) *$X \notin I = I_\alpha$ if and only if $|X \cap Z| \geq \kappa$ for all $Z \in \mathcal{F}_\alpha$; and* (ii) *the elements Z of $\mathcal{F}_\alpha$ are stationary subsets of λ^+.*

Proof. (i) follows directly from Definition 3.4. To see that (ii) holds, we verify that $\alpha \in Z \subseteq \lambda^+$, $Z \in N$ imply that Z is stationary. Otherwise $Z \cap \beta = \emptyset$ for some club $B \in N$. Then $B \cap \alpha$ is cofinal in α, by elementarity and $\alpha \in B$ since B is closed. ⊣

3.6 Lemma. *If N is suitable for κ, then $I = I(N)$ is a $\mathrm{cf}(\kappa)$-complete proper ideal on $\alpha = \alpha(N)$. Moreover, if $\lambda^{<\kappa} = \lambda$, then I is κ-complete.*

Proof. The completeness clearly follows from $[N]^{<\mathrm{cf}(\kappa)} \subseteq N$ and $[N]^{<\kappa} \subseteq N$ respectively. To see that $\alpha \notin I$, let $Z \in N$ be a subset of λ^+ with $\alpha \in Z$. It is enough to show that $|Z \cap \alpha| = \lambda$. Since $Z \in N$, also $\sup(Z) \in N$. As $\alpha \in Z$ and $N \cap \lambda^+ = \alpha$, it follows that $\sup(Z) = \lambda^+$. Then *a fortiori* there is a one-to-one function $g : \lambda \to Z$. Hence there is a $g \in N$ like this. Using $\lambda + 1 \subseteq N$, we get that $\mathrm{ran}(g) \subseteq N \cap \lambda^+ = \alpha$. ⊣

In what follows we will often suppress details like those given above.

3.7 Definition. Assume N is suitable for κ, $\lambda = 2^{<\kappa}$ and $\alpha = \alpha(N)$. For $X \subseteq \alpha$, we say X *reflects the properties of* α if $X \cap Z \neq \emptyset$ for all $Z \in \mathcal{F}_\alpha$.

3.8 Lemma. *Assume N is suitable for κ, $\lambda = 2^{<\kappa}$ and $\alpha = \alpha(N)$. If $X \subseteq \alpha$ and $X \in I^+$, then X reflects the properties of α, so we call $I = I_\alpha$ the* non-reflecting ideal on α (induced by N).

Notation. Assume $f : [X]^2 \to \gamma$ is a function, $\eta < \gamma$ and $\alpha \in X$. For simplicity, we often write $f(\alpha, \beta)$ for $f(\{\alpha, \beta\})$, specifying which of the ordinals α, β is smaller, if necessary. Denote the set $\{\beta < \alpha : f(\alpha, \beta) = \eta\}$ by $f(\alpha; \eta)$.

3.9 Lemma (Connection Lemma). *Assume $\kappa \geq \omega$ and $\lambda = 2^{<\kappa}$. Further suppose that N is suitable for κ with $\alpha(N) = \alpha$, $f \in N$ is a 2-partition of λ^+ with $\gamma < \mathrm{cf}(\kappa)$ colors, and $X \subseteq f(\alpha; \eta) \cap \alpha$ for some $\eta < \gamma$ is such that $X \notin I = I(N)$. Then there is some $Y \subseteq X$ with $\mathrm{ot}(Y) = \mathrm{cf}(\kappa)$ so that $Y \cup \{\alpha\}$ is homogeneous for f in color η.*

Proof. Let Z be a subset of $X \cup \{\alpha\}$ maximal with respect to the following properties: $\alpha \in Z$ and Z is homogeneous for f in color η. If $|Z| \geq \mathrm{cf}(\kappa)$, then we are done. Assume by way of contradiction that $|Z| < \mathrm{cf}(\kappa)$. Then $\sup(Z \cap \alpha) < \alpha$ and $Z \cap \alpha \in N$. Let $A = \bigcap\{f(u; \eta) : u \in Z \cap \alpha\}$. Then $A \in N$ and $\alpha \in A$. Hence, by the reflection property, $A \cap (X - \sup(Z \cap \alpha)) \neq \emptyset$. If $y \in A \cap (X - \sup(Z \cap \alpha))$, then $\{y\} \cup Z$ is homogeneous for f in color η, contradicting the maximality of Z. ⊣

3.10 Theorem (Erdős-Rado Theorem (unbalanced form)). *Let κ be an infinite cardinal and $\gamma < \mathrm{cf}(\kappa)$. Then*

$$(2^{<\kappa})^+ \to ((2^{<\kappa})^+, (\mathrm{cf}(\kappa) + 1)_\gamma)^2.$$

Proof. Let $\lambda = 2^{<\kappa}$, and suppose $f : [\lambda^+]^2 \to \gamma$ is a 2-partition of λ^+ into γ colors. Use Facts 3.2 to choose N suitable for κ with $f \in N$. For notational simplicity, let $\alpha = \alpha(N)$ and $I = I(N)$. If $f(\alpha; \eta) \cap \alpha \notin I$ for some $1 \leq \eta < \gamma$, then we are done by Lemma 3.9. By Lemma 3.6, we may assume that $\alpha - f(\alpha; 0) \subseteq \bigcup\{f(\alpha; \eta) \cap \alpha : 1 \leq \eta < \gamma\} \in I$. By Definition 3.4, there is a set $Z \in N$ with $Z \subseteq \lambda^+$ and $\alpha \in Z$ for which $|Z - f(\alpha; 0)| < \kappa$. Define a set W in $H(\lambda^{++})$ as follows:

$$W := \{\beta \in Z : |Z - f(\beta; 0)| < \kappa\}.$$

Then $W \in N$ and $\alpha \in W$. Then by Lemma 3.5 we infer that $W \in \mathrm{Stat}(\lambda^+)$ and for $g(\delta) := \{\beta < \delta : f(\beta, \delta) \neq 0\}$, we have $|g(\delta)| < \kappa$ for all $\delta \in W$. By Fodor's Set Mapping Theorem [19], there is a stationary subset $S \subseteq W$ free for g (i.e. $\gamma \notin g(\delta)$ for all $\delta \neq \gamma \in S$), and S is homogeneous for f in color 0. $\dashv$

Note that with some abuse of notation we have proved the following stronger result.

3.11 Theorem. *Let $\kappa \geq \omega$, $\lambda = 2^{<\kappa}$ and suppose $\gamma < \mathrm{cf}(\kappa)$. Then*

$$\lambda^+ \to (\mathrm{Stat}(\lambda^+), (\mathrm{cf}(\kappa) + 1)_\gamma)^2.$$

This theorem should be compared with the case $r = 2$ of Theorem 2.9 and it should be observed that while for regular κ, the above theorem is a strengthening of Corollary 2.10, for singular κ the results are incomparable. It should also be noted that using Theorem 2.7, the above result can be stepped up to the following.

3.12 Corollary. *Assume $\kappa \geq \omega$ and $\gamma < \mathrm{cf}(\kappa)$. Then for all $1 \leq r < \omega$,*

$$\exp_{r-2}(2^{<\kappa})^+ \to ((2^{<\kappa})^+, (\kappa + (r-1))_\gamma)^r.$$

Finally it should be remarked that we did not try to state the strongest possible forms of the Erdős-Rado theorems. Clearly the methods give similar results in cases where the resource cardinal κ is a regular limit cardinal. For a detailed discussion we refer to [19].

4. Generalizations of the Erdős-Rado Theorem

4.1. Overview

In this section we focus on the problem of what positive relations of the form

$$(2^{<\kappa})^+ \to (\alpha_\xi)^2_\gamma$$

can be proved for regular κ and $\gamma < \kappa$ in ZFC. The case for singular κ will be almost entirely omitted because of limitations of space. Many problems remain unsolved, and the simplest of these will be stated at the end of this subsection. We start by discussing limitations, the first of which comes from the next theorem.

4.1 Theorem (Hajnal [25], Todorcevic). *If $2^\kappa = \kappa^+$, then*

$$\kappa^+ \not\to (\kappa^+, \kappa + 2)^2.$$

Proof Outline. We only sketch the proof given in [25], omitting Todorcevic's proof for singular κ, which has been circulated in unpublished notes. Let $\{A_\alpha : \alpha < \kappa^+\}$ be a well-ordering of $[\alpha]^\kappa$. Define a sequence of sets $B_\alpha \in [\kappa^+]^\kappa$ for $\alpha < \kappa^+$ by transfinite recursion on α, in such a way that the following two conditions are satisfied:

1. $|B_\alpha \cap B_\beta| < \kappa$ for all $\beta < \alpha$;
2. $B_\alpha \cap A_\beta \neq \emptyset$ for all $\beta < \alpha$ for which $|A_\beta - \bigcup\{B_\gamma : \gamma \in F\}| = \kappa$ for all $F \in [\alpha]^{<\kappa}$.

To complete the proof, for $\beta < \alpha < \kappa^+$, set $f(\beta, \alpha) = 1$ if and only if $\beta \in B_\alpha$.

The constraint that $|B_\alpha \cap B_\beta| < \kappa$ for all $\beta < \alpha < \kappa^+$ implies that f has no homogeneous subsets of order type $\kappa + 2$ for color 1. The assertion that it has no homogeneous subsets of order type κ^+ for color 0 follows from the claim below.

4.2 Claim. *Assume A is a subset of size κ^+. Then there is a subset B of A of size κ which is not almost contained in the union of fewer than κ many B_β's.*

On the one hand, if fewer than κ many B_β's meet A in a set of size κ, then any subset $B \subseteq A$ of size κ in the complement of the union of these B_α's proves the claim. Otherwise, choose a sequence $B_\beta(\eta)$ indexed by $\eta < \kappa$ of κ many sets whose intersection with A has cardinality κ, and let B be the union of the intersections $A \cap B_\beta(\eta)$. $\dashv$

Henceforth we will assume that the goals, α_ξ, are all ordinals, $\alpha_\xi < \kappa^+$ for $\xi < \gamma$.

For $\kappa = \omega$, the best possible result, $\omega_1 \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and k finite was conjectured by Erdős and Rado [15] in 1952 and proved by James Baumgartner and Hajnal [2] in 1971, already in a more general form. Using a self-explanatory extension of the ordinary partition relation for linear order types, it says

$$\Theta \to (\omega)^1_\omega \quad \text{implies} \quad \Theta \to (\alpha)^2_k \quad \text{for all } \alpha < \omega_1,\ k < \omega.$$

Soon after it was generalized (also in a self-explanatory way) by Todorcevic to partial orders [63]. Schipperus [52] proved a topological version. The Baumgartner-Hajnal proof used "Martin's Axiom + absoluteness". An elementary proof not using this kind of argument was given by Fred Galvin [21] in 1975. We will treat this theorem later in Sect. 4.5, where we will also give a brief history of earlier work on this conjecture, because some of these approaches served as starting points for other investigations.

We will treat first the case $\kappa = \mathrm{cf}(\kappa) > \omega$. The reason for this strange order is really technical. The results to be presented for the case $\kappa > \omega$ were proved later and much of the method of using elementary substructures was worked out while proving them. We will give a new proof of the

Baumgartner-Hajnal Theorem which can be extended to successors of measurable cardinals and uses the methods developed for the treatment of the cases $\kappa > \omega$.

For the cases $\kappa > \omega$, there are further limitations.

4.3 Theorem. *Assume that $\kappa = \tau^+ \geq \omega_1$ and* GCH *holds. Then there are κ-complete, κ^+-c.c. forcing conditions showing the consistency of the following negative partition relations:*

$$\kappa^+ \not\to (\kappa : \tau)^2_2 \quad \textit{and} \quad \kappa^+ \not\to (\kappa : 2)^2_\tau.$$

Here the relations mean that there are no homogeneous sets of the form $[A, B] := \{\{\alpha, \beta\} : \alpha \in A \wedge \beta \in B\}$ where $A < B$, $\mathrm{ot}(A) = \kappa$, and $\mathrm{ot}(B) = \tau$ or $\mathrm{ot}(B) = 2$ respectively. The forcing results are due to Hajnal and stated in [13]. The first result, $\kappa^+ \not\to (\kappa : \tau)^2_2$, was shown by Rebholz [50] to be true in L. It is interesting to remark that while the proofs of Theorem 4.1 really give $\kappa^+ \nrightarrow (\kappa^+, (\kappa : 2))^2$ in the relevant cases, these two statements are really not equivalent. In [35], Komjáth proves it consistent with ZFC that $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ and $\omega_1 \to (\omega_1, (\omega : 2))^2$ hold.

In view of the limitations above, the following result of Baumgartner, Hajnal and Todorcevic [4], which we prove in Sect. 4.3 (see Theorem 4.12), is the best possible balanced generalization of the Erdős-Rado Theorem for finitely many colors to ordinal goals: for all regular uncountable cardinals κ and finite γ, if $\rho < \kappa$ is an ordinal with $2^{|\rho|} < \kappa$, then

$$(2^{<\kappa})^+ \to (\kappa + \rho)^2_\gamma.$$

Note that for $\gamma = 2$, this result was proved much earlier by Shelah in Sect. 6 of [55].

As a generalization of the unbalanced form, we prove in Sect. 4.4 (see Theorem 4.18) that for all regular uncountable cardinals κ and all finite m, γ,

$$(2^{<\kappa})^+ \to (\kappa^{\omega+2} + 1, (\kappa + m)_\gamma)^2.$$

In this discussion we have restricted ourselves to 2-partitions, since the situation is different for larger tuples. For example, Albin Jones [27, 31] has shown that for all finite m, n, $\omega_1 \to (\omega + m, n)^3$, complementing the result of Erdős and Rado [17] who showed $\omega_1 \nrightarrow (\omega + 2, \omega)^3$. Eric Milner and Karel Prikry [43] proved that $\omega_1 \to (\omega + \omega + 1, 4)^3$.

We conclude this subsection with some open questions.

4.4 Question. For which $\alpha < \omega_1$ and which $n < \omega$ does the partition relation $\omega_1 \to (\alpha, n)^3$ hold?

4.5 Question. Are the following statements provable in ZFC + GCH?

1. $\omega_3 \to (\omega_2 + \omega, \omega_2 + \omega_1)^2$?
2. $\omega_3 \to (\omega_2 + 2)^2_\omega$?

Though there are additional limitations for $\gamma \geq \omega$, which we will discuss in Sect. 5, both theorems may actually generalize for infinite γ with $2^{|\gamma|} < \kappa$, but nothing like this is known with the exception of the following very recent result a proof of which will be given in Sect. 6.

4.6 Theorem (Shelah [59]). *$\lambda^+ \to (\kappa+\mu)^2_\mu$ for $\mu < \kappa = \mathrm{cf}(\kappa)$ and $\lambda = 2^{<\kappa}$, under the assumption that $\mu < \sigma \leq \kappa$ for some strongly compact cardinal σ.*

4.2. More Elementary Submodels

In this subsection we prove a generalization of Connection Lemma 3.9 for regular κ. Let $\lambda = 2^{<\kappa}$ and assume that $\langle\langle N_\alpha, \in\rangle : \alpha < \kappa^+\rangle$ is a sequence of submodels of $H := H(\lambda^{++})$ satisfying the requirements outlined in Facts 3.2, with $A = \{f\}$ where $f : [\lambda^+]^2 \to \gamma$ is a given 2-partition of λ^+ with γ colors. For notational convenience, we will let

$$S_0 := \{\alpha < \lambda^+ : \alpha \cap N_\alpha = \alpha \text{ and } N_\alpha \text{ is suitable for } \kappa\}.$$

For $\alpha \in S_0$, we will write I_α for the ideal $I(N_\alpha)$ of Definition 3.4.

4.7 Lemma (Set Mapping Lemma). *Assume that $S \subseteq S_0$ is stationary and $g : S \to \mathcal{P}(\lambda^+)$ is a set mapping so that $g(\alpha) \subseteq \alpha$ and $g(\alpha) \cap S \in I_\alpha$ for all $\alpha \in S$. Then there is a stationary set $S' \subseteq S$ which is free for g. That is, $g(\alpha) \cap S' = \emptyset$ for all $\alpha \in S'$. Moreover, if S and g are amenable, then so is S'.*

Proof. Since S is a set of limit ordinals, for each $\alpha \in S$, we can choose $\beta_\alpha < \alpha$ and $Y_\alpha \subseteq \lambda^+$ so that $\alpha \notin Y_\alpha \in N_{\beta_\alpha}$ and $g(\alpha) \subseteq Y_\alpha$. By Fodor's Theorem, first β_α and then Y_α stabilize on a stationary set. That is, for some stationary $S' \subseteq S$ and some $Y \subseteq \lambda^+$, we have $\alpha \notin Y$ and $g(\alpha) \subseteq Y$ for all $\alpha \in S'$. ⊣

4.8 Corollary. *Suppose $S \subseteq S_0$. An element $\alpha \in S$ is a* reflection point *of S if $S \cap \alpha \notin I_\alpha$. Then the set $S - \tilde{S}$ is non-stationary, where $\tilde{S}$ denotes the set of reflection points of S. Moreover, if S is amenable, then so is S'.*

Proof. Assume by way of contradiction that $S' := S - \tilde{S}$ is stationary, and define $g(\alpha) := S' \cap \alpha$ for $\alpha \in S'$. By the Set Mapping Lemma 4.7, there is a stationary subset $S'' \subseteq S'$ so that S'' is free for g. On the other hand, if $\beta < \alpha$ are both in $S'' \subseteq S'$, then $\beta \in g(\alpha) := S' \cap \alpha$, contradicting the freeness of S'' for g. ⊣

4.9 Definition. For $\alpha < \lambda^+$ and $\sigma \in {}^{<\omega}\gamma$, we define ideals $I(\alpha, \sigma)$ by recursion on $|\sigma|$. To start the recursion, we set

$$I(\alpha, \emptyset) := \begin{cases} \mathcal{P}(\alpha) & \text{if } \alpha \notin S_0, \quad \text{and} \\ I_\alpha & \text{if } \alpha \in S_0. \end{cases}$$

If $\sigma = \tau^\frown\langle i\rangle$ and $I(\alpha, \tau)$ has been defined, then for all $X \subseteq \alpha$,

$$X \in I(\alpha, \sigma) \quad \Longleftrightarrow \quad \{\beta < \alpha : X \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau)\} \in I(\alpha, \emptyset).$$

4.10 Lemma. *Suppose $\alpha < \lambda^+$ and $\sigma \in {}^{<\omega}\gamma$.*

1. *$I(\alpha, \sigma)$ is a κ-complete ideal;*
2. *if $\alpha \notin S_0$, then $I(\alpha, \sigma) = \mathcal{P}(\alpha)$;*
3. *$I(\alpha, \emptyset) \subseteq I(\alpha, \sigma)$.*

Proof. In the special case of $\sigma = \emptyset$, (1) follows either from Lemma 3.6 or the triviality that $\mathcal{P}(\alpha)$ is κ-complete. Use recursion on $|\sigma|$ to complete the proof of (1), since at each successor stage, $I(\alpha, \tau^\frown\langle i\rangle)$ is gotten by averaging κ-complete ideals according to a κ-complete ideal.

Note that (2) follows immediately from the definition of $I(\alpha, \sigma)$.

(3) is also proved by induction on $|\sigma|$ simultaneously for all $\alpha < \lambda^+$. For $\alpha \notin S_0$, it follows from the second item, so assume $\alpha \in S_0$. It is trivial for $\sigma = \emptyset$, so assume it is true for $I(\alpha, \tau)$ where $\sigma = \tau^\frown\langle i\rangle$, and let $X \in I(\alpha, \emptyset) = I_\alpha = I(N_\alpha)$ be arbitrary. By definition of $I(N_\alpha)$, there is some $Y \subseteq \lambda^+$ so that $\alpha \notin Y \in N_\alpha$ and $X \subseteq Y$. Since α is limit, there is a $\beta_0 < \alpha$ with $Y \in N_{\beta_0}$. Since the sequence of submodels is continuous, $Y \in N_\beta$ for all β with $\beta_0 < \beta < \alpha$, and for $\beta \notin Y$, we either have $X \cap \beta \in I_\beta$ if $\beta \in S_0$ or have $X \cap \beta \in I(\beta, 0)$ otherwise. Hence by the induction hypothesis, $X \cap \beta \in I(\beta, \tau)$ for $\beta \notin Y$ with $\beta_0 < \beta < \alpha$. That is, if $\beta < \alpha$ and $X \cap \beta \notin I(\beta, \tau)$, then $\beta \in Y \cup (\beta_0 + 1)$. So $X \in I(\alpha, \sigma)$, since $\alpha \notin Y - (\beta_0 + 1) \in N_\alpha$. ⊣

We postpone the proof that some of these ideals are proper.

4.11 Lemma (Second Connection Lemma). *Suppose $X \subseteq \alpha$, $X \notin I(\alpha, \sigma)$ and suppose $i \in \mathrm{ran}(\sigma)$. Then there is a subset $Y \subseteq X \cup \{\alpha\}$ with $\mathrm{ot}(Y) = \kappa + 1$ homogeneous for f in color i.*

Proof. The proof is by induction on $|\sigma|$. If $\sigma = \emptyset$, then there is nothing to prove. Next suppose $\sigma = \tau^\frown\langle j\rangle$ for some $j < \gamma$. By Lemma 4.10, we know that $X \cap \beta \notin I(\beta, \tau)$ for some $\beta < \alpha$ with $\beta \in X$. Thus the induction hypothesis gives the statement for $i \in \mathrm{ran}(\tau)$. Next assume $i = j$. Then by Lemma 4.10(3), we know that $X \notin I_\alpha$ and Connection Lemma 3.9 yields the desired result. ⊣

4.3. The Balanced Generalization

In this subsection we will prove, as announced earlier, the following balanced generalization of the Erdős-Rado Theorem.

4.12 Theorem (Baumgartner, Hajnal, Todorcevic [4]). *Suppose κ is a regular uncountable cardinal, γ is finite and $\rho < \kappa$ is an ordinal with $2^{|\rho|} < \kappa$. Then*

$$(2^{<\kappa})^+ \to (\kappa + \rho)^2_\gamma.$$

For notational simplicity, we are fixing κ, $\lambda = 2^{<\kappa}$, a 2-partition $f : [\lambda^+]^2 \to \gamma$, and ρ as in the statement of the theorem throughout this subsection, and we continue the notation introduced in Sects. 4.1 and 4.2. In what follows, it will be convenient to look at the least indecomposable ordinal $\xi \geq \rho$, rather than ρ directly. In preparation for the proof, we give several preliminary facts about ideals.

4.13 Definition. For ordinals ξ, sets $x \subseteq \lambda^+$ and sequences $\sigma \in {}^{<\omega}\gamma$, define *$x$ is (ξ,σ)-canonical for f* by recursion on $|\sigma|$. To begin the recursion, we say *x is $(\xi,\emptyset)$-canonical for f* if $x = \{\alpha\}$ for some $\alpha < \lambda^+$. For $\sigma = \tau^\frown\langle i\rangle$, we say *$x$ is (ξ,σ)-canonical for f* if x is the union of a $<$-increasing sequence $\langle x_\eta : \eta < \xi\rangle$ so that each x_η is (ξ,τ)-canonical for $\eta < \xi$ and $f(u,v) = i$ for all $u \in x_\eta$ and $v \in x_\zeta$ with $\eta < \zeta < \xi$.

The following lemma is left to the reader as an exercise.

4.14 Lemma. *Assume that ξ is an indecomposable ordinal and $\sigma \in {}^{n}\gamma$ for some $n < \omega$. Then*

1. $\mathrm{ot}(x) = \xi^n$ *for all x which are (ξ,σ)-canonical for f;*
2. *if x is (ξ,σ)-canonical for f, then every $y \subseteq x$ with $\mathrm{ot}(y) = \xi^n$, is also (ξ,σ)-canonical for f and $J := \{z \subseteq y : \mathrm{ot}(z) < \xi^n\}$ is a proper ideal;*
3. *if x is (ξ,σ)-canonical for f, then for every $i \in \mathrm{ran}(\sigma)$, there is some $y \subseteq x$ with $\mathrm{ot}(y) = \xi$ which is homogeneous for f in color i.*

4.15 Lemma (Reflection Lemma). *Assume $X \notin I(\alpha,\sigma)$ for some $\alpha < \lambda^+$, $\sigma \in {}^{<\omega}\gamma$, and further suppose that $\xi < \kappa$ is indecomposable. Then there is a set $x \subseteq X$ which is (ξ,σ)-canonical for f.*

Proof. The proof is by induction on $|\sigma|$. To start, notice the lemma is vacuously true for $\sigma = \emptyset$. Next suppose $\sigma = \tau^\frown\langle i\rangle$. Construct a sequence $\langle x_\eta : \eta < \xi\rangle$ by recursion on $\eta < \xi$. Assume that $\zeta < \xi$ and that the sets $x_\eta \subseteq X \cap f(\alpha;i)$ are (ξ,τ)-canonical for f for $\eta < \zeta$. Let $Z = \{\beta < \lambda^+ : (\forall \eta < \zeta)(\forall \delta \in x_\eta)(f(\delta,\beta) = i)\}$. Since $\langle x_\eta : \eta < \xi\rangle \in N_\alpha$, we have $Z \in N_\alpha$ and $\alpha \in Z$. Since $\{\beta < \alpha : X \cap \beta \cap f(\alpha;i) \notin I(\beta,\tau)\} \notin I_\alpha$, we can choose $\beta < \alpha$ so that $\beta \in Z \in N_\beta$, $X \cap \beta \cap f(\alpha;i) \notin I(\beta,\tau)$ and $\sup(\bigcup\{x_\eta < \zeta\}) < \beta$. By the induction hypothesis, we can choose a set $x_\zeta \subseteq X \cap Z$ which is (ξ,τ)-canonical for f with $x_\eta < x_\zeta$ for all $\eta < \zeta$.

This recursion defines $\langle x_\zeta : \zeta < \xi\rangle$, and $x = \bigcup\{x_\zeta : \zeta < \xi\}$ is the required set (ξ,σ)-canonical for f. ⊣

We need one more lemma which will be used in the proof of the unbalanced version (Theorem 4.18) as well.

4.16 Lemma. *Assume $S \subseteq S_0$ is stationary and $\Gamma \subseteq \gamma$ is non-empty. Then there are $S' \subseteq S$ stationary and $\sigma \in {}^{<\omega}\Gamma$ with σ one-to-one such that*

1. *$S \cap \beta \cap f(\alpha; j) \in I(\beta, \sigma)$, for every $\beta, \alpha \in S'$ with $\beta < \alpha$ and every $j \in \Gamma - \mathrm{ran}(\sigma)$; but*

2. *$S \cap \alpha \notin I(\alpha, \sigma)$ for $\alpha \in S'$.*

Moreover, if S is amenable, then so is S'.

Proof. Let σ be of maximal length so that $\mathrm{ran}(\sigma) \subseteq \Gamma$, σ is one-to-one, and

$$S'' := \{\alpha \in S : S \cap \alpha \notin I(\alpha, \sigma)\} \quad \text{is stationary.}$$

For $j \in \Gamma - \mathrm{ran}(\sigma)$, let

$$g_j(\alpha) := \{\beta < \alpha : S \cap \beta \cap f(\alpha; j) \notin I(\beta, \sigma)\}.$$

By the maximality of σ, it follows that $g_j(\alpha) \cap S'' \in I_\alpha$ for all but non-stationarily many $\alpha \in S$. By Lemma 4.7, there is a stationary subset $S' \subseteq S''$ which is free for g_j. ⊣

Let $\mathbb{S} := \{\sigma \in {}^{<\omega}\gamma : \sigma \text{ is one-to-one}\}$.

For $\alpha < \lambda^+$ and $\sigma \in \mathbb{S}$, say (X, Y) *fits* (α, σ) if $X \subseteq \alpha$, $X \notin I(\alpha, \sigma)$ and $f(\beta; j) \cap X \in I(\alpha, \sigma)$ for all $\beta \in Y$ and $j \notin \mathrm{ran}(\sigma)$.

From Lemma 4.16 we get the following corollary by applying the lemma with $\Gamma = \gamma$.

4.17 Corollary. *For every stationary set $S \subseteq S_0$, there are $\sigma \in \mathbb{S}$, $\alpha \in S$ and a stationary subset $S' \subseteq S$ so that $(S \cap \alpha, S')$ fits (α, σ).*

With these lemmas in hand, we turn to the proof of the main theorem of this subsection.

Proof of Theorem 4.12. Using Corollary 4.17, we define $\alpha_m \in S_0$, $\sigma_m \in \mathbb{S}$, and stationary $Z_m \subseteq S_0$ by recursion on m so that the following conditions are satisfied:

1. $\alpha_0 < \cdots < \alpha_m < \cdots$; $Z_0 \supseteq \cdots \supseteq Z_m \supseteq \cdots$; and

2. $(Z_m \cap \alpha_m, Z_{m+1})$ fits (α_m, σ_m).

Since $\mathbb{S}$ is finite, $\sigma_k = \sigma_n$ for some $k < n < \omega$. We conclude that there are a sequence $\sigma \in \mathbb{S}$, ordinals $\beta_0 < \beta_1$, and sets X_0, X_1 such that the following statement is true:

> $X_0 < X_1$, $X_i \notin I(\beta_i, \sigma)$ for $i < 2$, and $f(\eta; j) \cap X_0 \in I(\beta_0, \sigma)$ for every $j \notin \mathrm{ran}(\sigma)$ and every $\eta \in X_1$.

Let ξ be the least indecomposable ordinal with $\rho \leq \xi$. By the Reflection Lemma 4.15, there is a $y \subseteq X_1$ such that y is (ξ, σ)-canonical for f.

We shrink X_0 to $X = X_0 - \bigcup\{f(\delta; j) : j \notin \mathrm{ran}(\sigma) \text{ and } \delta \in y\}$. Then $X \notin I(\beta_0, \sigma)$ since $I(\beta_0, \sigma)$ is κ-complete, $|y| < \kappa$ and $f(\delta; j) \in I(\beta_0, \sigma)$ for $j \notin \mathrm{ran}(\sigma)$, $\delta \in y \subseteq X_1$.

Let $J = \{Z \subseteq y : Z$ is not (ξ, σ)-canonical for $f\}$. By Lemma 4.14, J is a proper ideal on y.

For every $\delta \in X$, there is an $i(\delta) \in \operatorname{ran}(\sigma)$ so that $f(\delta; i) \cap y \notin J$. Thus for every $\delta \in X$, by Lemma 4.14(3), there is a $y(\delta) \subseteq y$ of order type ρ such that $\{\delta\} \cup y(\delta)$ is homogeneous for f in color $i(\delta)$.

Using the fact that $\omega^{|\rho|} = 2^{|\rho|} \cdot \omega < \kappa$, we now obtain $i_0 \in \operatorname{ran}(\sigma)$, $y' \subseteq y$ and $X' \subseteq X$ with $X' \notin I(\alpha, \sigma)$ so that $i(\delta) = i_0$ and $y(\delta) = y'$ for all $\delta \in X'$. Thus $f(\delta_0, \delta_1) = i_0$ for all $\delta_0 \in X'$ and $\delta_1 \in y'$.

By the Second Connection Lemma 4.11, we get an $X'' \subseteq X'$ of order type κ homogeneous for f in color i_0. Finally $X'' \cup y'$ is the required set of order type $\kappa + \rho$ homogeneous for f in color i_0. ⊣

4.4. The Unbalanced Generalization

4.18 Theorem (Baumgartner, Hajnal, Todorcevic [4]). *Suppose κ is a regular uncountable cardinal, and m, γ are finite. Then*

$$(2^{<\kappa})^+ \to (\kappa^{\omega+2} + 1, (\kappa + m)_\gamma)^2.$$

This subsection is devoted to the proof of this theorem, and for notational convenience we set $\lambda = 2^{<\kappa}$ throughout. Also, fix a partition $f : [\lambda^+]^2 \to 1 + \gamma$. We also continue to use the notation introduced in Sects. 4.1, 4.2 and 4.3.

The strategy of the proof is to derive Theorem 4.18 from the following auxiliary assumption:

$$Q(\kappa) \qquad 2^{<\kappa} = \kappa \quad \text{and} \quad \forall \langle f_\alpha : \alpha < \kappa^+ \rangle \subseteq {}^\kappa\kappa \ \exists g \in {}^\kappa\kappa \ (f_\alpha \prec g)$$

where $\prec$ is the relation of eventual domination on ${}^\kappa\kappa$ (i.e. $h_1 \prec h_2$ iff $h_1(\alpha) < h_2(\alpha)$ for all but less than κ many α).

Then as in the original proof of the Baumgartner-Hajnal Theorem [2], we observe that the assumption $Q(\kappa)$ is unnecessary, and therefore that Theorem 4.18 holds in ZFC.

Let us justify this observation before going on to prove the theorem from the assumption of $Q(\kappa)$.

Let P_0 be the natural κ-closed forcing for collapsing $2^{<\kappa}$ onto κ. Then in V^{P_0} we have $\lambda = \kappa$. Working in V^{P_0} and using a standard iterated forcing argument (as in [1]) we can force every sequence of functions in ${}^\kappa\kappa$ of length κ to be eventually dominated via a partial ordering P_1 that is κ-closed and has the λ^+-chain condition. Let $P = P_0 * P_1$. Then P is κ-closed and in V^P, both $\lambda = \kappa$ and $Q(\kappa)$ hold. Note that in V^P, we will have $2^\kappa > \kappa^+$, since this inequality is implied by $Q(\kappa)$.

Assuming we have proved Theorem 4.18 under the assumption of $Q(\kappa)$, we may assume it holds in V^P. Suppose that $f : [\lambda^+]^2 \to \gamma + 1$ is a 2-partition in V. Then in V^P, there is some $A \subseteq \lambda^+$ such that either (a) A is homogeneous for f in color 0 and $\operatorname{ot}(A) = \kappa^{\omega+2} + 1$, or (b) A is homogeneous for f in color $i > 0$ and $\operatorname{ot}(A) = \kappa + m$. Suppose (a) holds. Note that $\kappa^{\omega+2}+1$ is the same whether computed in V or in V^P. Let $h : \kappa \to \kappa^{\omega+2} + 1$ be a bijection with $h \in V$. In V^P, fix an order-isomorphism $j : \kappa^{\omega+2} + 1 \to A$. Now, working in V, find a decreasing sequence $\langle p_\xi : \xi < \kappa\rangle$ of elements of P and a sequence $\langle \alpha_\xi : \xi < \kappa\rangle$ of elements of λ^+ such that for all ξ, $p_\xi \Vdash j(h(\xi)) = \alpha_\xi$. This is easy to do by recursion on ξ, using the fact that P is κ-closed. But now it is clear that $\{\alpha_\xi : \xi < \kappa\} \in V$ has order type $\kappa^{\omega+2}+1$ and is homogeneous for f in color 0. Case (b) may be handled the same way.

For the rest of this subsection, assume $Q(\kappa)$ holds. We may also assume that $\kappa > \omega$ since for $\kappa = \omega$ we have the much stronger result Theorem 4.30.

First we prove a consequence of $Q(\kappa)$.

4.19 Lemma. *Assume $Q(\kappa)$. For all positive $\ell < \omega$ and every sequence $\langle X_\alpha : \alpha < \kappa^+\rangle$ of subsets of κ^ℓ of order type $< \kappa^\ell$, there is a sequence $\langle Z_\nu : \nu < \kappa\rangle$ of subsets of κ^ℓ of order type $< \kappa^\ell$ such that every X_α is a subset of some Z_ν.*

Proof. Use induction on ℓ. For $\ell = 1$, the sets $X_\alpha \subseteq \kappa^1 = \kappa$ are bounded and we may define $Z_\nu := \nu$.

For the induction step, assume $\langle X_\alpha : \alpha < \kappa^+\rangle$ is a given sequence of subsets of κ^{k+1} of order type $< \kappa^{k+1}$. Write $\kappa^{k+1} = \bigcup_{\nu<\rho} U_\rho$ as the union of an increasing sequence $U_0 < \cdots < U_\rho < \cdots$ in which $\operatorname{ot}(U_\rho) = \kappa^k$. For each $\alpha < \kappa^+$ and $\rho < \kappa$, define

$$Y_{\alpha,\rho} := \begin{cases} X_\alpha \cap U_\rho, & \text{if } \operatorname{ot}(X_\alpha \cap U_\rho) < \kappa^k,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Since each U_ρ is isomorphic to κ^k, we may apply the induction hypothesis to each sequence $\langle Y_{\alpha,\rho} : \alpha < \kappa^+\rangle$ to get $\langle W_{\mu,\rho} : \mu < \kappa\rangle$, so that every $Y_{\alpha,\rho}$ is a subset of some $W_{\mu,\rho}$ and each $W_{\mu,\rho}$ is a subset of U_ρ of order type less than κ^k.

For each $\alpha < \kappa^+$, define $g_\alpha : \kappa \to \kappa$ by $g_\alpha(\rho)$ is the least μ so that $Y_{\alpha,\rho} \subseteq W_{\mu,\rho}$. Choose an increasing $g : \kappa \to \kappa$ eventually dominating all the g_α for $\alpha < \kappa$. Define

$$Z_\nu := \bigcup\nolimits_{\mu<\nu} \cup \bigcup\{W_{\mu,\rho} : \rho \geq \nu \wedge \mu \leq g(\rho)\}.$$

Then $\langle Z_\nu : \nu < \kappa\rangle$ satisfies the requirements of the lemma for $\ell = k + 1$.

Therefore by induction, the lemma follows. $\dashv$

From this point forward in the subsection, we assume that there is no homogeneous set for color 0 of the order type required. We may also assume that the result is true for $\gamma' < \gamma$.

4.20 Lemma. *Assume $S \subseteq S_0$ is stationary. For all $\Sigma \subseteq [1, \gamma]$ with $\Sigma \neq \emptyset$, there are a stationary set $S' \subseteq S$ and a one-to-one function $\sigma \in {}^{<\omega}\Sigma$ such that the following two properties hold:*

1. *for every stationary* $S'' \subseteq S'$ *there is some* $\alpha \in S''$ *with* $S'' \cap \alpha \notin I(\alpha, \sigma)$*;*
2. *for all* $j \in \Sigma - \operatorname{ran}(\sigma)$ *and all* $\beta, \alpha \in S'$*, if* $\beta < \alpha$*, then* $f(\alpha; j) \cap \beta \cap S' \in I(\beta, \sigma)$*.*

Proof. By induction on $|\Sigma|$. For the basis case of $|\Sigma| = 1$, suppose $\Sigma = \{i\}$ for some positive $i \leq \gamma$. Then either $\operatorname{ran}(\sigma) = \{i\}$, the first property holds with $S' = S$ and the second holds vacuously, or by the Set Mapping Lemma 4.7, there is a stationary subset $S' \subseteq S$ free for color i.

For the induction step, assume the lemma is true for some non-empty proper subset $T \subseteq [1, \gamma]$ and let $i \in [1, \gamma] - T$. We must show the statement is also true for $\Sigma = T \cup \{i\}$. Let $S_T \subseteq S$ and τ witness that the lemma is true for T. Consider two cases depending on whether or not the following statement is true, where $\operatorname{Stat}(S_T) := \operatorname{Stat}(\lambda^+) \cap \mathcal{P}(S_T)$:

$$(*) \qquad \forall S^* \in \operatorname{Stat}(S_T)\ \exists \alpha \in S^*(\{\beta < \alpha : S^* \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau)\} \notin I_\alpha).$$

For the first case, assume that $(*)$ holds. Then we can choose $S_\Sigma = S_T$ and $\sigma = \tau^\frown\langle i \rangle$, since the first item holds by $(*)$ and the second remains true since no new j comes into play.

For the second case, assume that $(*)$ fails and choose a stationary $S^* \subseteq S_T$ showing the failure. Define

$$g(\alpha) := \{\beta < \alpha : S^* \cap \beta \cap f(\alpha; i) \notin I(\beta, \tau)\}.$$

Applying the Set Mapping Lemma 4.7 to g and S^*, we get a stationary $S_\Sigma \subseteq S^*$ free for g which together with $\sigma = \tau$ satisfy the required two conditions. ⊣

Our next lemma uses the fact that by $Q(\kappa)$, we have $2^{<\kappa} = \kappa$. For notational convenience, for each $\alpha \in S_0$, define

$$\mathcal{F}_\alpha := \{Z \in N_\alpha : Z \subseteq \kappa^+ \wedge \alpha \in Z\}.$$

Also, for any $0 < \ell \leq \gamma$ and any one-to-one function $\sigma \in {}^{\ell-1}[1, \gamma]$, call a set Y (α, σ)-*slim* if $Y \subseteq S_0$, $\operatorname{ot}(Y) = \kappa^\ell$, $Y \notin I(\alpha, \sigma)$, and for all $W \subseteq Y$, the equivalence $W \notin I(\alpha, \sigma)$ if and only if $\operatorname{ot}(W) = \kappa^\ell$ holds.

4.21 Lemma. *For all one-to-one functions* $\sigma \in {}^{<\omega}[1, \gamma]$*, for all* $X \subseteq S_0$ *with* $X \notin I(\alpha, \sigma)$*, if* $\ell - 1$ *is the length of* σ*, then there exists* $Y \subseteq X$ *such that* Y *is* (α, σ)*-slim.*

Proof. To start the induction, note that if $X \notin I(\alpha, \emptyset) = I_\alpha$ for some $\alpha \in S_0$, then there is some $Y \subseteq X$ with $\operatorname{ot}(Y) = \kappa$ so that $Y \notin I_\alpha$. This implication is true because $\mathcal{F}_\alpha$ has cardinality at most κ and can be diagonalized in X. Then Y is $(\alpha, \emptyset)$-slim, by the κ-completeness of I_α. The rest follows by induction on the length of σ. ⊣

The following corollary is immediate from the previous two lemmas.

4.22 Corollary. *There are a stationary set $S_1 \subseteq S_0$, a nonempty subset $\Sigma \subseteq [1,\gamma]$ and a one-to-one function $\sigma \in {}^{\ell-1}\Sigma$ such that the following two conditions hold:*

1. *for all stationary $S \subseteq S_1$, there are $\alpha \in S$ and $X \subseteq \alpha$ of order type κ^ℓ so that $X \notin I(\alpha,\sigma)$;*
2. *for all $\beta < \alpha \in S_1$ and all $j \in [1,\gamma] - \Sigma$, one has $f(\alpha; j) \cap \beta \in I(\beta,\sigma)$.*

For notational convenience, write $X = \Sigma_{\nu<\kappa} X_\nu$ to indicate that $X_0 < \cdots < X_\nu < \cdots$ and $X = \bigcup_{\nu<\kappa} X_\nu$. For the remainder of this section, let $S_1 \subseteq S_0$, σ and ℓ as in the previous corollary be fixed.

4.23 Definition. For $\alpha \in S_0$, define $\mathcal{H}(\alpha,n)$ by recursion on $n < \omega$. To start the recursion, define

$$\mathcal{H}(\alpha,0) := \{X \subseteq S_1 : X \text{ is } (\alpha,\sigma)\text{-slim}\}.$$

If $\mathcal{H}(\alpha,n)$ has been defined, then $X \in \mathcal{H}(\alpha,n+1)$ if and only if the following conditions are satisfied:

1. $X \subseteq S_1$ and there exists $\langle X_\nu \in \mathcal{H}(\alpha,n) : \nu < \kappa\rangle$ with $X = \Sigma_{\nu<\kappa} X_\nu$;
2. for all $F \in \mathcal{F}_\alpha$, there exists ν_F so that $X_\nu \subseteq F$ for all $\nu > \nu_F$;
3. for all $\nu < \nu' < \kappa$ and $x \in X_\nu$, $y \in X_{\nu'}$, one has $f(x,y) = 0$.

Note that every $X \in \mathcal{H}(\alpha,n)$ has $\mathrm{ot}(X) = \kappa^{\ell+n}$ and X contains a subset of order κ^n homogeneous for f in color 0. Furthermore, every $Y \subseteq X$ of order type $\kappa^{\ell+n}$ has a subset in $\mathcal{H}(\alpha,n)$.

We now prove the lemma containing the main idea of the proof.

4.24 Lemma (Key Lemma). *Suppose $\alpha \in S_1$, $n < \omega$ and $X \subseteq S_1$ with $X \in \mathcal{H}(\alpha,n)$. Then there are $\beta_0 \in S_1$ with $\beta_0 > \alpha$ and $\langle T_\nu \subseteq X : \nu < \kappa\rangle$ with* $\mathrm{ot}(T_\nu) = \kappa^{\ell+n}$ *so that for all $\beta \in S_1$ with $\beta > \beta_0$, there is some $\nu < \kappa$ such that* $\mathrm{ot}(T_\nu - f(\beta;0)) < \kappa^{\ell+n}$.

Proof. Let M be a maximal subset of S_1 with the property that for all $V \in [M]^{<\omega}$, $\mathrm{ot}(\bigcap\{X - f(\beta;0) : \beta \in V\}) = \kappa^{\ell+n}$. We claim that $|M| \leq \kappa$ and then we are done, by the maximality of M.

Assume for the sake of a contradiction that $|M| = \kappa^+$, and let

$$\Pi := \left\{\bigcap_{\beta\in V} X - f(\beta;0) : V \in [M]^{<\omega}\right\}.$$

Extend $\Pi \cup \{X - Y : Y \subseteq X \wedge \mathrm{ot}(X) < \kappa^{\ell+n}\}$ to an ultrafilter U on X. Then for every $\beta \in M$, there is a $j(\beta) \in \Sigma$ so that $X \cap f(\beta; j(\beta)) \in U$. Hence there is some $j \in \Sigma$ so that the set $M_j := \{\beta \in M : j(\beta) = j\}$ has cardinality κ^+. By $\kappa^+ \to (\kappa^+, n)^2$, there is a set $H \subseteq M_j$ of size n which is homogeneous for f in color j. Now $X \cap \bigcap\{f(\beta; j) : \beta \in H\}$ is in U, so it must have order type $\kappa^{\ell+n}$. By Lemma 4.11 it contains a set W of type κ homogeneous for f in color j. This is the contradiction that proves the lemma. $\dashv$

4.25 Lemma. *Assume $S \subseteq S_1$ is stationary. Then for all $n < \omega$, there are $\alpha \in S$ and $X \subseteq S$ so that $X \in \mathcal{H}(\alpha, n)$.*

Proof. Work by induction on n. For the basis case, $n = 0$, the statement follows from Corollary 4.22 and Lemma 4.21.

For the induction step, a standard ramification argument gives the result. Assume the claim is true for some n. Let $\alpha \in S$ be arbitrary. We define a sequence

$$\{X_\xi : \xi < \kappa\} \subseteq \mathcal{H}(\alpha_\xi, n)$$

by recursion on $\xi < \kappa$. Assume that $X_\eta \in \mathcal{H}(\alpha_\eta, n)$, $X_\eta \subseteq S \cap f(\alpha; 0)$ are defined for $\eta < \xi$. Let $S_\xi = \{\beta \in S : \bigcup\{X_\eta : \eta < \xi\} \subseteq f(\beta; 0)\}$. Then $\alpha \in S_\xi$ and $S_\xi \in N_\alpha$. Then S_ξ is stationary, and so by the induction hypothesis it contains a subset $X \in \mathcal{H}(\alpha_\xi, n)$ for some $\alpha_\xi \in \alpha \cap S_\xi$. By elementarity, we may assume $X \in N_\alpha$. By the Key Lemma, there are $T_\nu \subseteq X$ for $\nu < \kappa$ such that $\mathrm{ot}(T_\nu) = \kappa^{\ell+n}$ and $|S - \bigcup_{\nu<\kappa} Z_\nu| \leq \kappa$ where

$$Z_\nu = \{\beta < \kappa : \mathrm{ot}(T_\nu - f(\beta; 0)) < \kappa^{\ell+n}\}.$$

Then, by elementarity $S - \bigcup_{\nu<\kappa} Z_\nu \subseteq \alpha$, hence $\alpha \in Z_\nu$ for some $\nu < \kappa$ and $X_\nu = T_\nu \cap f(\alpha; 0)$ satisfies the requirement. $\bigcup_{\nu<\kappa} X_\nu \in \mathcal{H}(\alpha, n+1)$ and as a bonus we have that $\bigcup_{\nu<\kappa} X_\nu \subseteq f(\alpha; 0)$. ⊣

The same ramification argument gives the next lemma as well.

4.26 Lemma. *Assume $S \subseteq S_1$ is stationary. Then there exist an increasing sequence $\langle \alpha_\xi \in S : \xi < \kappa \rangle$ and a family $\langle X_{\xi,n} \subseteq S : \xi < \kappa \wedge n < \omega \rangle$ with each $X_{\xi,n} \in \mathcal{H}(\alpha_\xi, n)$ so that if either $\xi < \eta$ or $\xi = \eta$ and $k < \ell$, then $X_{\xi,k} < X_{\eta,\ell}$ and $f(x, y) = 0$ for all $x \in X_{\xi,k}$, $y \in X_{\eta,\ell}$.*

The above lemma gives the result for $\kappa^{\omega+1}$, since the set

$$X := \bigcup\{X_{\xi,n} : \xi < \kappa \wedge n < \omega\}$$

is homogeneous for f in color 0.

To finish the proof, we use yet another ramification argument.

4.27 Lemma. *Let X be a set of order type $\kappa^{\omega+1}$ as described above, and let $X_n := \bigcup\{X_{\xi,n} : \xi < \kappa\}$. Note that $\mathrm{ot}(X_n) = \kappa^{\ell+n+1}$. Let*

$$J := \{Y \subseteq X : \exists n_0 < \omega \ \forall n > n_0 \ (\mathrm{ot}(Y \cap X_n) < \kappa^{\ell+n})\}.$$

Then J is an ideal and there are $\{T_\nu \in J^+ : \nu < \kappa\}$ and $\beta_0 \in S_1$ such that for all $\beta \in S_1$ with $\beta > \beta_0$, the set $T_\nu - f(\beta; 0)$ is in J.

Let M be a maximal subset of S_1 so that $\bigcap_{\beta \in V} X - f(\beta; 0) \notin J$ for finite $V \subseteq M$.

To see that $|M| = \kappa$, we proceed just like in the proof of Lemma 4.24. We only need the fact that if $Z \subseteq X$ and $Z \notin J$, then for all $j \in \Sigma$, the set Z contains a subset of type κ homogeneous for f in color j.

Since $|M| = \kappa$, the set $\Pi := \{\bigcap_{\beta\in V} X - f(\beta;0) : V \in [M]^{<\omega}\}$ is a family of size κ such that for all $\beta \notin M$, there is some $Z \in \Pi$ so that $Z - f(\beta;0) \subseteq Y$ for some $Y \in \Pi$.

The next lemma is the final tool we need.

4.28 Lemma. *Assume $T \in J^+$. Then there is a $\overline{J} \subseteq J$ with $|\overline{J}| \leq \kappa$ such that for all $\beta \in S_1$ with $T - f(\beta;0) \in J$, there is a $Y \in \overline{J}$ so that $T - f(\beta;0) \subseteq Y$.*

Proof. Choose $J_n \subseteq [X_n]^{\ell+n+1}$ with $|J_n| \leq \kappa$ so that for all $\beta \in S_1$ with $\mathrm{ot}(X_n - f(\beta;0)) < \kappa^{\ell+n+1}$ there is a $Y_n \in J_n$ with $T - f(\beta;0) \subseteq Y_n$. Let

$$J_0 := \left\{\bigcup_{n<\omega} Y_n : \forall n < \omega\ Y_n \in J_n\right\}.$$

Note that $|J_0| = \kappa^\omega = \kappa$. Finally, set

$$\overline{J} := \left\{A \cup B : A \in J_0 \text{ and } B = \bigcup\{X_i : i \leq n\} \text{ for some } n < \omega\right\}.$$

Then $\overline{J}$ will do the job. $\dashv$

4.5. The Baumgartner-Hajnal Theorem

Here is a brief overview of the history of the Baumgartner-Hajnal Theorem and some of its generalizations. Erdős and Rado conjectured that $\omega_1 \to (\alpha)^2_k$ and $\lambda_0 \to (\alpha)^2_k$, for λ_0 the order type of the reals, and for all $k < \omega$, $\alpha < \omega_1$.

Fred Galvin figured out, for order types Θ, that $\Theta \to (\omega)^1_\omega$ would be the right necessary and sufficient condition for $\Theta \to (\alpha)^2_k$ to hold for all $\alpha < \omega_1$.

Hajnal [25] proved in 1960 that $\lambda_0 \to (\eta_0, \alpha \vee \alpha^*)^2$ where η_0 is the order type of the rationals. More significantly, Galvin proved $\lambda_0 \to (\alpha)^2_2$, for $\alpha < \omega_1$, but contrary to the first expectations, this proof provided no clues for the general case. For the resource ω_1, Galvin could only prove $\omega_1 \to (\omega^2, \alpha)^2$ for $\alpha < \omega_1$.

Another result of Prikry [48] said $\omega_1 \to (\alpha, (\omega : \omega_1))^2$. This result was later generalized by Todorcevic [64] to

$$\omega_1 \to ((\alpha)_k, (\alpha : \omega_1))^2 \quad \text{for all } \alpha < \omega_1.$$

Finally we mention a very significant consistency result of Todorcevic [63] that PFA (Proper Forcing Axiom) implies

$$\omega_1 \to (\omega_1, \alpha)^2 \quad \text{for all } \alpha < \omega_1.$$

(For context, recall that PFA implies that $\mathfrak{c} = \omega_2$.)

Before going back to the main line of discussion, we make another detour. It was already asked in the Erdős-Hajnal problem lists [12, 13] if the partition relations $\omega_2 \to (\alpha)^2_2$ were consistent for $\alpha < \omega_2$. Though there is nothing to refute such consistency, the results going in this direction are weak and rare.

The first consistency result was obtained by Richard Laver [40] in 1982, and independently discovered by Akihiro Kanamori [33], using what is now called a *Laver ideal* I on κ (a non-trivial, κ-complete ideal with the strong saturation property that given κ^+ sets not in the ideal, there are κ^+ of them so that the intersection of any $< \kappa$ of these is also not in the ideal). He proved that if there is a Laver ideal on κ, then

$$\kappa^+ \to (\kappa \cdot 2 + 1, \alpha)^2 \quad \text{holds for all } \alpha < \kappa^+.$$

Laver also proved the consistency of the hypothesis that there is a Laver ideal on ω_1 and derived as a corollary the consistency (relative to a large cardinal, of course) of

$$\omega_2 \to (\omega_1 \cdot 2 + 1, \alpha)^2 \quad \text{holds for all } \alpha < \omega_2.$$

Foreman and Hajnal [20] tried to get a stronger consistency result for ω_2 from the stronger assumption that ω_1 carries a dense ideal, and indeed, they proved that in this case

$$\omega_2 \to ({\omega_1}^2 + 1, \alpha)^2 \quad \text{holds for all } \alpha < \omega_2.$$

They however discovered that their proof gives a much stronger result for successors.

4.29 Theorem (Foreman and Hajnal [20]). *Suppose $\kappa > \omega$ is measurable and $m < \omega$. Then $\kappa^+ \to (\alpha)^2_m$ for all $\alpha < \Omega(\kappa)$.*

Here $\kappa < \Omega(\kappa) < \kappa^+$ is a rather large ordinal. We will comment about these results in detail in Sect. 5, but for lack of space and energy we will not include proofs.

4.30 Theorem (Baumgartner and Hajnal [2]). *If an order type Θ satisfies $\Theta \to (\omega)^1_\omega$, then it also satisfies $\Theta \to (\alpha)^2_k$ for all $\alpha < \omega_1$ and finite k.*

4.31 Corollary. *For all $\alpha < \omega_1$ and $m < \omega$,*

$$\omega_1 \to (\alpha)^2_m.$$

So we decided to give a proof of Corollary 4.31 using the ideas of the Foreman-Hajnal proof. This will serve two purposes. It will make the text almost complete as far as the old results are concerned, and it will communicate most of the ideas of the new Foreman-Hajnal proof.

Notation. Let $\langle\langle N_\alpha, \in\rangle : \alpha < \omega_1\rangle$ be a sequence of elementary submodels of $H(\omega_2)$ satisfying Facts 3.2 with $\lambda = \kappa = \omega$, $A = \{f\}$ where $f : [\omega_1]^2 \to m$, and

$$S_0 := \{\alpha < \omega_1 : \omega_1 \cap N_\alpha = \alpha \text{ and } N_\alpha \text{ is suitable for } \omega\}.$$

Here S_0 is a club set in ω_1. We may assume S_0 is amenable.

4.32 Definition. We define S_ρ by transfinite recursion on $\rho < \omega_1$: S_0 has already been defined; $S_{\rho+1} := \tilde{S}_\rho$, the set of reflection points of S_ρ (see Corollary 4.8); and $S_\rho := \bigcap_{\sigma<\rho} S_\sigma$ for ρ a limit.

4.33 Lemma. *For all $\rho < \omega_1$, the set S_ρ is amenable.*

Proof. Use induction on ρ and Corollary 4.8 to prove that $\langle S_\sigma : \sigma < \rho \rangle \subseteq N_{\alpha+1}$ for $\alpha \in S_\rho$. The details and the remainder of the proof are left to the reader. $\dashv$

Next we are going to define *diagonal sets*, *cross sets*, and *cross systems*.

4.34 Definition. For $\alpha \in S_0$, for the sake of brevity, we put

$$\mathcal{F}_\alpha := \{Z \in N_\alpha : Z \subseteq \omega_1 \wedge \alpha \in Z\}.$$

(Note that for $X \subseteq \alpha$, we have $X \notin I_\alpha$ if and only if $X \cap Z \neq \emptyset$ for all $Z \in \mathcal{F}_\alpha$; see the discussion of notation after Lemma 3.6.)

Call $D \subseteq \alpha$ a *diagonal set for $\alpha \in S_0$* if $\sup(D) = \alpha$ and $|D - Z| < \omega$ for all $Z \in \mathcal{F}_\alpha$.

Clearly every diagonal set D for α has order type ω, and every cofinal subset of it is also diagonal. Moreover, a diagonal set D for α is *reflecting for α* in the sense described after Lemma 3.6.

4.35 Lemma. *For all $\alpha \in S_0$ and $X \subseteq \alpha$ with $X \notin I_\alpha$, there is a diagonal set $D \subseteq X$ for α. If $X \in N_{\alpha+1}$, then D can be chosen in $N_{\alpha+1}$.*

Proof. Since $|\mathcal{F}_\alpha| = \omega$, we can diagonalize it. $\dashv$

Notation. Assume that $\langle D_n : n < \omega \rangle$ is a sequence of sets of ordinals and $\alpha \in S_0$. Then the sequence *converges to α in N_α*, in symbols, $D_n \implies \alpha$, if and only if for every $Z \in \mathcal{F}_\alpha$ there is some n_0 so that for all $n > n_0$, $D_n \subseteq Z$.

For a set D of ordinals, we denote by $\overline{D}$ its closure in the ordinal topology.

4.36 Definition. By transfinite recursion on $\rho < \omega_1$, we define, for $\alpha \in S_\rho$, the concept *D is a cross set of rank ρ for α* as follows:

1. For $\alpha \in S_0$, the set $\{\alpha\}$ is cross set of rank 0 for α.
2. For $\rho > 0$, the set D is cross set of rank ρ for α if $\alpha \in S_\rho$ and there is a witnessing sequence $\langle D_n : n < \omega \rangle$ satisfying the following conditions:
   (a) each D_n is a cross set of rank ρ_n for α_n for some $\rho_n < \rho$ and for $\alpha_n := \sup(D_n)$;
   (b) $D_0 \cup \{\alpha_0\} < \cdots < D_n \cup \{\alpha_n\} < \cdots$;
   (c) $\overline{D}_n \implies \alpha$;
   (d) if $\rho = \sigma + 1$, then $\rho_n = \sigma$ for all $n < \omega$; if ρ is a limit, then $\rho = \sup(\rho_n)$;
   (e) $D = \bigcup_{n<\omega} D_n$.

4.37 Remark. Note that a cross set D of rank 1 for α is a diagonal set for α, and if $\{\alpha_n : n < \omega\}$ is the set of $\alpha_n := \sup(D_n)$ for a witnessing sequence for D, then $\{\alpha_n : n < \omega\}$ is also a diagonal set for α.

The next lemma is proved by induction on ρ.

4.38 Lemma. *If D is a cross set for α of rank ρ, then* $\mathrm{ot}(D) = \omega^\rho$.

We now define the concept of a cross system of rank ρ for α. Informally, this is just the closure of a cross set of rank ρ for α, equipped with functions that remember the sets appearing in the definition of the cross set of rank α.

4.39 Definition. By transfinite recursion on $\rho < \omega_1$, we define, for $\alpha \in S_\rho$, the concept $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ *is a cross system of rank ρ for α* as follows:

1. For $\alpha \in S_0$, a quadruple $\mathcal{D} = \langle \overline{D}, <_D \mathrm{rank}_D, \mathrm{succ}_d \rangle$ is a *cross system of rank* 0 *for α* if and only if $D = \{\alpha\}$, $<_D = \emptyset$, $\mathrm{rank}(\alpha) = 0$, and $\mathrm{succ}(\alpha) = \emptyset$.

2. For $\rho > 0$, a quadruple $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ is a *cross system of rank ρ for α* with underlying cross set D if there is a witnessing sequence $\langle \mathcal{D}_n : n < \omega \rangle$ of cross systems so that

   (a) $\mathcal{D}_n$ is a cross system of rank ρ_n for α_n for all $n < \omega$;

   (b) $D = \bigcup\{D_n : n < \omega\}$ is a cross set with witnessing sequence $\langle D_n : n < \omega \rangle$, where D_n underlies $\mathcal{D}_n$;

   (c) $\overline{D} = \bigcup\{\overline{D}_n : n < \omega\} \cup \{\alpha\}$;

   (d) $<_D$ is defined by $\alpha <_D \beta$ for all $\beta \in \overline{D} - \{\alpha\}$, and $<_D \restriction \overline{D}_n = <_{D_n}$ for $n < \omega$.

   (e) under $<_D$, D is a (rooted) tree with root α;

   (f) $\mathrm{rank}_D : \overline{D} \to \rho + 1$ is defined by $\mathrm{rank}_D(\alpha) = \rho$, and $\mathrm{rank}_D \restriction \overline{D}_n = \mathrm{rank}_{D_n}$ for $n < \omega$.

   Finally, $\mathrm{succ}_D(\beta)$ is just a redundant notation for the set of immediate successors of β in the tree under $<_D$.

Note that for $\rho > 0$ and $n < \omega$, under the notation of Definition 4.36, $\mathrm{succ}_D(\alpha) = \{\alpha_n : n < \omega\}$ and $\mathrm{rank}_D(\alpha_n) = \rho_n$.

Note that the underlying set of a cross system is definable as the set of elements in $\overline{D}$ of rank 0.

4.40 Lemma. *Assume $\mathcal{D} = \langle \overline{D}, <_D, \mathrm{rank}_D, \mathrm{succ}_D \rangle$ is a cross system of rank ρ for α. Then for all $\beta \in \overline{D}$, $\mathrm{rank}_D(\beta) = \emptyset$ if and only if $\beta \in D$.*

The next two lemmas are proved by induction on ρ.

4.41 Lemma (Reflection Lemma). *Assume $\mathcal{D}$ is a cross system of rank ρ for α. Then for $\gamma \in \overline{D} - D$, $\mathrm{succ}_D(\gamma)$ is a diagonal set for γ.*

4.42 Definition. Assume $\mathcal{D}$ is a cross system of rank ρ for α with underlying set D. We say that C is a *full subset of* $\overline{D}$ if $\alpha \in C$ and $C \cap \operatorname{succ}_D(\beta)$ is infinite for $\beta \in C$ with $\operatorname{rank}_D(\beta) > 0$.

4.43 Lemma (Induction lemma for cross systems). *Assume $\mathcal{D}$ is a cross system of rank ρ for α with underlying set D. For every full subset C of $\overline{D}$, there is a set $B \subseteq C \cap D$ so that $\overline{B} \subseteq C$ and B is the underlying set for a cross system of rank ρ for α.*

4.44 Definition. By recursion on $\rho < \omega_1$ define, for $\alpha \in S_\rho$, the concept $\mathcal{D}$ *is an f-canonical cross system of rank ρ for α* as follows.

1. For $\alpha \in S_0$, the unique cross system of rank 0 for α is an f-canonical cross set of rank 0.
2. For $\rho > 0$, $\mathcal{D}$ is an f-canonical cross system of rank ρ for α if it is a cross system of rank ρ for α with a witnessing sequence $\langle \mathcal{D}_n : n < \omega \rangle$ for which the following additional conditions hold:
   - (g) for $n < \omega$, $\mathcal{D}_n$ is an f-canonical cross system of rank ρ_n for α_n;
   - (h) there is some i so that $f(\beta, \gamma) = i$ for all $\beta \in D_n$ and $\gamma \in D_p$ with $n < p < \omega$.

This usage is slightly different from the use of the word "canonical" in Definition 4.13. In this section we do not use the term (ξ, σ)-canonical.

The following is one of the oldest ideas in the subject.

4.45 Lemma (Homogeneity Lemma). *For all $\sigma < \omega_1$ there is some $\rho < \omega_1$ so that if $\mathcal{D}$ is an f-canonical cross system of rank ρ, then there is a set $H \subseteq D$ of order type ω^σ which is homogeneous for f.*

The proof is left to the reader. Detailed proofs can be found in both [2] and in [21] of Galvin, where the first elementary proof of Theorem 4.30 was given.

We need one more technical lemma, a strengthening of Lemma 4.43, before launching into the main proof.

4.46 Lemma (Induction lemma for canonical cross systems). *Assume $\mathcal{D}$ is an f-canonical cross system of rank ρ for α. Suppose C is a full subset of $\overline{D}$. Then there is a set $B \subseteq C \cap D$ so that $\overline{B} \subseteq C$ and B is the underlying set of an f-canonical system of rank ρ for α.*

Proof. Use induction on ρ and the fact that every cofinal subset of a diagonal set for β is diagonal for β. $\dashv$

By the Homogeneity Lemma 4.45, the following lemma will be sufficient to prove Corollary 4.31.

4.47 Lemma (Main Lemma). *For all $\rho < \omega_1$, $\alpha \in S_\rho$ and $F \in \mathcal{F}_\alpha$, there is an f-canonical system $\mathcal{D}$ of rank ρ for α with $\overline{D} \subseteq S_0 \cap F$ and $\mathcal{D} \in N_{\alpha+1}$.*

Note that it would be sufficient to prove Lemma 4.47 without the last clause, which is needed to support induction.

The rest of this section is devoted to the proof of Lemma 4.47. We need further preliminaries. In what follows, U is a fixed non-principal ultrafilter on ω with $U \in N_0$.

4.48 Definition. Define, by recursion on $\rho < \omega_1$, *deference functions* $i_{\mathcal{D}}$ where $\mathcal{D}$ is a cross system of rank ρ for α. For $\alpha \in S_0$ and a cross system $\mathcal{D}$ of rank 0 for α, define $i_{\mathcal{D}}(\xi)$ for ξ with $\alpha < \xi < \omega_1$ by $i_{\mathcal{D}}(\xi) = i$ if and only if $f(\{\alpha, \xi\}) = i$. Assume $\rho > 0$ and deference functions have been defined for cross systems of rank $\sigma < \rho$. For a cross system $\mathcal{D}$ of rank ρ for α, define $i_{\mathcal{D}}(\xi)$ for ξ with $\alpha < \xi < \omega_1$ by $i_{\mathcal{D}}(\xi) = i$ if and only if $\{n < \omega : i_{\mathcal{D}_n}(\xi) = i\} \in U$ where $\langle \mathcal{D}_n : n < \omega \rangle$ is the witnessing sequence of cross systems for $\mathcal{D}$.

Notice that if $\mathcal{D} \in N_{\alpha+1}$, then the deference function $i_{\mathcal{D}} : \omega_1 - (\alpha+1) \to m$ is also in $N_{\alpha+1}$. Note also that $i_{\mathcal{D}}(\xi)$ can be defined "inside $\mathcal{D}$" for a fixed ξ, as follows.

4.49 Definition. Assume $\mathcal{D}$ is a cross system of rank ρ for α and $\alpha < \xi < \omega_1$. Define $j_{\mathcal{D}}(\beta, \xi)$ for $\beta \in \overline{D}$ by transfinite recursion on $\mathrm{rank}_D(\beta)$ as follows. If $\mathrm{rank}_D(\beta) = 0$, then $j_{\mathcal{D}}(\beta, \xi) = f(\{\beta, \xi\})$. For $\sigma > 0$ and β with $\mathrm{rank}_D(\beta) = \sigma$, set $j_{\mathcal{D}}(\beta, \xi) = j$ for that $j < m$ so that $\{n < \omega : j_{\mathcal{D}}(\beta_n, \xi) = j\} \in U$, where β_n is the nth element of $\mathrm{succ}_D(\beta)$.

The proof that these two definitions coincide is left to the reader.

4.50 Lemma. *Assume $\mathcal{D}$ is a cross system of rank ρ for α. Then for all ξ with $\alpha < \xi < \omega_1$, $j_{\mathcal{D}}(\alpha, \xi) = i_{\mathcal{D}}(\xi)$.*

Note that $j_{\mathcal{D}}$ is an element of $N_{\alpha+1}$ if $D \in N_{\alpha+1}$.

Next we use a fixed enumeration of pairs of natural numbers to define a *standard well-ordering* for $\overline{D}$ where $\mathcal{D}$ is a cross system. For the remainder of this section, assume $\varphi : \omega \times \omega \to \omega - \{0\}$ is a fixed bijection which is monotonic in both variables, and which is in N_0.

4.51 Definition. Define, by recursion on positive $\rho < \omega_1$, for cross systems $\mathcal{D}$ of rank ρ, a *standard well-ordering of $\overline{D}$.*

1. For $\alpha \in S_1$, if $D = \{\alpha_n : n < \omega\}$ is the underlying set of a cross system $\mathcal{D}$ of rank 1, then the standard well-ordering of $\overline{D}$ has least element $d_0 = \alpha$, and for positive k, has kth element $d_k = \alpha_{k-1}$.

2. For $\rho > 1$, if $D = \bigcup\{D_n : n < \omega\}$ is the underlying set of a cross system $\mathcal{D}$ of rank ρ where D_n is the underlying set of $\mathcal{D}_n$ of the witnessing sequence of $\mathcal{D}$, then the standard well-ordering of $\overline{D}$ has least element $d_0 = \alpha$, and for positive $k = \varphi(n, j)$, has kth element $d_k = d_{n,j}$, where $d_{n,j}$ is the jth element of $\overline{D_n}$.

By some abuse of notation, we write d_n for the nth element of the standard well-ordering.

4.52 Lemma. *For all positive $\rho < \omega_1$ and all $\alpha \in S_\rho$, if $\mathcal{D}$ is a cross system of rank ρ for α and $\langle d_k : k < \omega \rangle$ is the standard well-ordering of $\overline{D}$, then for all positive $n < \omega$, there is some $m < n$ so that $d_n \in \mathrm{succ}_D(d_m)$.*

Proof. The proof is by induction on ρ over the recursive definition of standard well-orderings. ⊣

Proof of the Main Lemma 4.47. The proof is by induction on ρ. For $\rho = 0$, the lemma is trivial.

For the induction step, assume $\rho > 0$ and the lemma is true for all $\sigma < \rho$. Let $\alpha \in S_\rho$ and $F \in \mathcal{F}_\alpha$ be arbitrary. If $\rho = \sigma + 1$, then let $\rho_n = \sigma$ for all $n < \omega$. If ρ is a limit, then let $\langle \rho_n : n < \omega \rangle \in N_{\alpha+1}$ be a strictly increasing cofinal sequence with limit ρ, and assume $\rho_0 \geq 1$.

Now, for all $n < \omega$, $\alpha \in S_{\rho_n+1}$, so α is a limit of ordinals in S_{ρ_n} and $\alpha \in \tilde{S}_{\rho_n}$. Temporarily fix an enumeration of $\mathcal{F}_\alpha$ as $\{G_n : n < \omega\}$. By definition of $\tilde{S}_{\rho_n}$, $(S_{\rho_n} \cap F \cap G_0 \cap \cdots \cap G_n) \cap \alpha \notin I_\alpha$.

Define by recursion sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle \mathcal{D}_n : n < \omega \rangle$. To start, choose $\alpha_0 \in (S_{\rho_0} \cap F \cap G_0) \cap \alpha$ large enough so that F, $G_0 \in N_{\alpha_0}$. Then F, $G_0 \in \mathcal{F}_{\alpha_n}$. Use the induction hypothesis on ρ_0, α_0, $F'_0 = F \cap G_0$ to find an f-canonical cross system $\mathcal{D}_0 \in N_{\alpha_0+1}$ of rank ρ_0 for α_0 so that $\overline{D}_0 \subseteq S_0 \cap F'_0$.

Continue, taking care to make sure the sequence of α_n's increases to α. If α_n has been defined, then choose $\alpha_{n+1} \in (S_{\rho_{n+1}} \cap F \cap G_0 \cap \cdots \cap G_{n+1} - (\alpha_n{+}1)) \cap \alpha$ large enough so that $F, G_0, G_0, \ldots, G_{n+1} \in N_{\alpha_0}$. Then $F, G_0, \ldots, G_{n+1} \in \mathcal{F}_{\alpha_{n+1}}$. Use the induction hypothesis on ρ_{n+1}, α_{n+1}, $F'_{n+1} = F'_n \cap G_{n+1} \cap \omega_1 - (\alpha_{n+1} + 1)$ to find an f-canonical cross system $\mathcal{D}_{n+1} \in N_{\alpha_{n+1}+1}$ of rank ρ_{n+1} for α_{n+1} so that $\overline{D}_{n+1} \subseteq S_0 \cap F'_{n+1}$.

Also, since m is finite, there is an infinite subsequence of $\langle \alpha_n : n < \omega \rangle \in N_{\alpha+1}$ and an $i < m$ so that $i_{\mathcal{D}_n}(\alpha) = i$ for all n in the subsequence. By shrinking if necessary, we may assume, without loss of generality, that this subsequence is the entire sequence. Now $\langle D_n : n < \omega \rangle$ is a witnessing sequence for a cross set of rank ρ for α by construction. Hence $\langle \mathcal{D}_n : n < \omega \rangle$ is a witnessing sequence for a cross system of rank ρ for α.

Finally, as $N_\alpha, \alpha \in N_{\alpha+1}$, and since S_ρ is amenable by Lemma 4.33, we may assume that $\langle \mathcal{D}_n : n < \omega \rangle$ is defined in $N_{\alpha+1}$.

Claim. *There is an infinite set $T \subseteq \omega$ with $T \in N_{\alpha+1}$ and a family $\{C_n : n \in T\}$ so that C_n is a full subset of $\overline{D_n}$ for $n \in T$ and $f(\beta, \gamma) = i$ for all $\beta \in C_n$ and $\gamma \in C_p$ with $n, p \in T$ and $n < p$.*

The induction step of the Main Lemma follows from the claim by Lemma 4.46, as each C_n can be replaced by an f-canonical system $\mathcal{C}_n \in N_{\alpha_n+1}$ and $\langle \mathcal{C}_n : n \in T \rangle$ is the witnessing sequence of the desired f-canonical system of rank ρ for α.

To prove the claim, we will pick elements of $\{\alpha\} \cup \bigcup \{\overline{D}_n : n \in \omega\}$ according to a certain bookkeeping. We pick α first. Infinitely often we pick a new element n for T, larger than any element of T picked earlier. Our choice of

n means we have picked the top point α_n of $\overline{D}_n$. For each point n of T, we promise that infinitely often we will pick an element of $\overline{D}_n$ according to the standard well-ordering of $\overline{D}_n$.

For notational convenience, let $n(\beta)$ denote that value of n with $\beta \in \overline{D}_n$.

Assume we have picked a finite non-empty set $A \subseteq \{\alpha\} \cup \bigcup\{\overline{D_n} : n < \omega\}$ which satisfies the following condition:

$$*(A) \qquad \begin{array}{l} \text{For any } n < p,\ \beta \in \overline{D}_n \cap A \text{ and } \xi \in \overline{D}_p \cap A, \\ j_{\mathcal{D}_n}(\beta, \xi) = j_{\mathcal{D}_n}(\beta, \alpha) = i. \end{array}$$

We have to pick a new point γ for A so that the enlarged set still satisfies the condition $*(A \cup \{\gamma\})$.

For the first scenario, suppose we want to add a new α_p to A. That is, we want to add a new value p to T. Let

$$Z_0 = Z_0(A) = \bigcap\{\{\xi : j_{\mathcal{D}_{n(\beta)}}(\beta, \xi) = i\} : \beta \in A\}.$$

Note that Z_0 is in N_α and $\alpha \in Z_0$. As $\mathrm{succ}(\alpha)$ is reflecting, we can choose the desired $\alpha_p \in \mathrm{succ}(\alpha)$ as large as we want.

For the second scenario, assume we want to pick a β to add to A so that $\beta \in \overline{D}_p$ for some $p \in T$ where $\alpha_p \in A$ and so that $\beta \in \mathrm{succ}(\gamma)$ for some $\gamma \in A \cap \overline{D}_p$. There are three cases, $\alpha_p = \min(A \cap \mathrm{succ}_{\mathcal{D}}(\alpha))$, $\alpha_p = \max(A \cap \mathrm{succ}_{\mathcal{D}}(\alpha))$, and $\min(A \cap \mathrm{succ}_{\mathcal{D}}(\alpha)) < \alpha_p < \max(A \cap \mathrm{succ}_{\mathcal{D}}(\alpha))$. We sketch only the last, and leave the others to the reader. Let $A^- := A \cap \bigcup\{\overline{D}_n : n < p\}$, and $A^+ := A \cap \bigcup\{\overline{D}_n : n > p\}$, and define

$$Z^+ = Z^+(A) := \bigcap\{\delta \in \mathrm{succ}_{\mathcal{D}_p}(\gamma) : j_{\mathcal{D}_p}(\delta, \xi) = i \wedge \xi \in A^+\}.$$

Now Z^+ is a subset of $\mathrm{succ}_{\mathcal{D}_p}(\gamma)$ which is a reflecting subset of γ by the Reflection Lemma 4.41. Since by $*(A)$, $j_{\mathcal{D}_p}(\gamma, \xi) = i$ for $\xi \in A$, and A is finite, it follows that Z^+ is a reflecting subset of γ. Next define

$$Z^- = Z^-(A) := \bigcap\{\xi < \omega_1 : j_{\mathcal{D}_{n(\delta)}}(\delta, \xi) = i \wedge \delta \in A^-\}.$$

By Lemma 4.50, $Z^- \in N_{\max(A^-)+1}$. Since $\max(A^-) < \gamma$, it follows that $Z^- \in N_\gamma$. By $*(A)$, $\gamma \in Z^-$. Hence $Z^+ \cap Z^-$ is infinite and any element of $Z^+ \cap Z^-$ is a suitable choice for β.

Use the technique of "jumping around" and these two scenarios to intertwine the recursive definitions of T and of all the C_n's for $n \in T$. Specifically, use the standard well-ordering of α to define a sequence $\langle \eta_k : k < \omega \rangle$. At stage 0, pick $\eta_0 = \alpha$. Suppose η_ℓ has been defined for $\ell < k$. Look at d_k. If $d_k \in \mathrm{succ}_D(\alpha)$, then use the first scenario to choose $\eta_k \in \mathrm{succ}_D(\alpha)$. If $d_k \in \mathrm{succ}_D(\eta_\ell)$ for some $\ell < k$, then use the second scenario to choose $\eta_k \in \mathrm{succ}_D(\eta_\ell)$. Otherwise, set $\eta_k = \eta_{k-1}$. Finally, let $E = \{\eta_k : k < \omega\}$.

Let $T = \{p < \omega : (\exists k)(\eta_k = \alpha_p)\}$. Since the standard order lists all the successors of α, the set T is infinite and in $N_{\alpha+1}$. For $p \in T$, let $C_p = E \cap \overline{D}_p$. Temporarily fix $p \in T$. For any $\gamma \in C_p$, since $\alpha_p = d_\ell$ for some ℓ, and

$\operatorname{succ}_{D_n}(\gamma)$ forms an infinite monotonic subsequence of $\{d_k : k < \omega\}$, the set C_p has infinitely many successors of γ. Thus C_p is full. Therefore T and the sets $\{C_p : p \in T\}$ are the ones required to prove the claim.

As noted above, the claim suffices to complete the induction step of the Main Lemma, so it follows. ⊣

5. The Milner-Rado Paradox and $\Omega(\kappa)$

Erdős and Rado considered Ramsey's Theorem to be a generalization of the pigeon-hole principle (for cardinals). In 1965, Milner and Rado [44] turned around this view, noting that the pigeon-hole principle is a partition relation with exponent 1, and that a partition relation with exponent 1 and ordinal resource and goal would be a pigeon-hole principle for ordinals.

A case in point of this approach is the easily checked family of partition relations $\kappa^n \to (\kappa^n)^1_\gamma$ for $\kappa \geq \omega$, $n < \omega$, and $\gamma < \operatorname{cf}(\kappa)$. Soon Milner and Rado discovered that basically nothing stronger is true.

5.1 Theorem (Milner-Rado [44]). *For all cardinals $\kappa \geq \omega$ and all $\alpha < \kappa^+$,*

$$\alpha \not\to (\kappa^n)^1_{n<\omega}.$$

Proof. It is sufficient to prove

$$(*) \qquad \kappa^\rho \not\to (\kappa^n)^1_{n<\omega} \quad \text{for } \rho < \kappa^+.$$

Clearly we may assume $\kappa > \omega$. We prove $(*)$ by transfinite induction on ρ. We can write $\kappa^\rho = \bigcup_{\nu<\sigma} A_\nu$ with $A_0 < \cdots < A_\nu < \cdots$ and each $\operatorname{ot}(A_\nu) = \kappa^{\rho_\nu}$ for some $\rho_\nu < \rho$, where $\sigma = \operatorname{cf}(\rho)$ if $\operatorname{cf}(\rho) > 1$ and $\sigma = \kappa$ otherwise. By the induction hypothesis, each $A_\nu = \bigcup_{n<\omega} A_{\nu,n}$ where $\operatorname{ot}(A_{\nu,n}) < \kappa^n$ for $\nu < \sigma$, $n < \omega$. In the case of $\sigma = \omega$, define a witnessing partition $\kappa^\rho = \bigcup_{j<\omega} B_j$ where $B_j = A_{\nu,n}$ for $j = 2^\nu(2n+1)$. In the case of $\sigma > \omega$, let $B_0 := \emptyset$, $B_{n+1} := \bigcup\{A_{\nu,n} : \nu < \sigma\}$. Clearly $\kappa^\rho = \bigcup_{n<\omega} B_n$; and $\operatorname{ot}(B_{n+1}) \leq \sum_{\nu<\sigma} \kappa^n \leq \kappa^{n+1} < \kappa^\omega$. ⊣

We state one consequence of the above theorem giving further limitations on to positive relations (as discussed in Theorem 4.3).

5.2 Theorem. *For all cardinals $\kappa \geq \omega$, $\kappa^+ \not\to (\kappa^n)^2_{n<\omega}$.*

Proof. For $\alpha < \kappa^+$, use Theorem 5.1 to choose partitions $\alpha = \bigcup_{n<\omega} A^\alpha_n$ with $\operatorname{ot}(A^\alpha_n) < \kappa^n$ for each $n < \omega$. Define $f : [\kappa^+]^2 \to \omega$ as follows: for $\alpha < \beta < \kappa^+$, set $f(\alpha,\beta) = n+1$ if and only if $\alpha \in A^\beta_n$. ⊣

The word *paradox* was used in reference to Theorem 5.1 because this result was so contrary to expectations. It turned out that the phenomena described in Theorem 5.1 is involved in many problems concerning uncountable cardinals, and often it leads to unexpected difficulties.

In this section we are trying to turn this tide and use the paradox in our favor. For the remainder of this section, let κ be a fixed infinite cardinal.

5.3 Definition. For $\alpha < \kappa^+$, call a partition $\alpha = \bigcup_{\gamma\in\Gamma} A_\gamma$ with $\Gamma < \kappa$ an MR-*decomposition of* α if there is a sequence $\langle n_\gamma : \gamma < \Gamma\rangle \in {}^\Gamma\omega$ such that $\mathrm{ot}(A_\gamma) = \kappa^{n_\gamma}$.

From Theorem 5.1 and the fact that any $\delta < \kappa^n$ is the finite sum of ordinals of the form $\kappa^m \cdot \nu$ where $m < n$ and $\nu < \kappa$, we get the following corollary.

5.4 Corollary. *Each* $\alpha < \kappa^+$ *has an* MR-*decomposition.*

Another way to put Definition 5.3 is that α has an MR-decomposition if there are sequences $\langle n_\gamma : \gamma < \Gamma\rangle \in {}^\Gamma\omega$ and functions $\Psi_\gamma : [\kappa]^{n_\gamma} \to \alpha$ for $\gamma < \Gamma < \kappa$ such that Ψ_γ is the canonical monotone map from $[\kappa]^{n_\gamma}$ ordered lexicographically into α.

The next definition from [20] is motivated by this formulation.

5.5 Definition. Call $\alpha < \kappa^+$ *codeable* if there are $\Gamma < \kappa$ and sequences $\langle n_\gamma : \gamma < \Gamma\rangle \in {}^\Gamma\omega$ and $\langle \Psi_\gamma : \gamma < \Gamma\rangle$ so that $\Psi_\gamma : [\kappa]^{n_\gamma} \to \alpha$ for $\gamma < \Gamma$ and for every $A \in [\kappa]^\kappa$,

$$\mathrm{ot}\big(\textstyle\bigcup_{\gamma<\Gamma} \Psi_\gamma\text{“}[A]^{n_\gamma}\big) = \alpha.$$

5.6 Definition. Let $\Omega(\kappa)$ be defined as the least ordinal $\Omega \leq \kappa^+$ so that each $\alpha < \Omega$ is codeable.

Note that this definition from [20] is only interesting if κ is a large cardinal, say at least a Jónsson cardinal.

The following list of properties of $\Omega(\kappa)$ proved in [20] gives some sense of this ordinal for a measurable cardinal $\kappa > \omega$.

1. $\Omega(\kappa) < \kappa^+$;
2. $\Omega(\kappa)$ is closed under the operations of ordinal addition, multiplication, exponentiation, and taking fixed points of these operations;
3. $\Omega(\kappa)$ cannot be changed by (κ,∞)-distributive forcing;
4. if $V \subseteq W$ and both V and W are models of ZFC + “κ is measurable”, then $\Omega(\kappa)^V \leq \Omega(\kappa)^W$;
5. by using generic elementary embeddings in the situation of 4., it is possible to make $\Omega(\kappa)^V < \Omega(\kappa)^W$.

Moreover, $\Omega(\kappa)$ is big, e.g. if U is a normal ultrafilter on κ and ν is the least ordinal such that $L_\nu[U] \cap \kappa^{<\kappa} = L[U] \cap \kappa^{<\kappa}$, then $L[U] \models \Omega(\kappa) = \nu$. Since the statement $\delta < \Omega(\kappa)$ is upwards absolute, this implication shows that the value of $\Omega(\kappa)^V$ is at least as big as ν. Moreover ν is much bigger than, for example, the first $\eta > \kappa$ such that $L_\eta[U]$ is an admissible structure, but much to our regret, we must omit the proofs.

However, we have to confess that we know very little about the combinatorial properties involved in the definitions of $\Omega(\kappa)$. In fact, we do not know if $\Omega(\kappa)$ would become smaller if we stipulated that the mappings Ψ_γ be monotone.

6. Shelah's Theorem for Infinitely Many Colors

In this section we prove Shelah's Theorem 4.6, that $\lambda^+ \to (\kappa+\mu)^2_\mu$ for $\mu < \kappa = \mathrm{cf}(\kappa)$ and $\lambda = 2^{<\kappa}$, under the assumption that $\mu < \sigma \le \kappa$ for some strongly compact cardinal σ.

We say that $B \subseteq \lambda^+$ *has essential colors for* $g, \mathcal{I}$, where g is a 2-partition of λ^+ and $\mathcal{I}$ is a normal ideal on λ^+, if $B \notin \mathcal{I}$ and every $C \subseteq B$ with $C \notin \mathcal{I}$ satisfies $g``[C]^2 = g``[B]^2$.

6.1 Lemma (Reduction to essential colors). *Assume $\mu < \kappa = \mathrm{cf}(\kappa)$, and $\lambda := 2^{<\kappa}$. Further suppose that $g : [\lambda^+]^2 \to \mu$ is a 2-partition of λ^+ with μ colors, I is a normal ideal concentrating on S_{κ,λ^+}, and $A \subseteq \lambda^+$ is not in I.*

Then there are a subset $B \subseteq A$ and a normal ideal $J \supseteq I$, such that B has essential colors for g, J.

Proof. By the normality of I and Facts 3.2 we can choose $N \prec H(\lambda^{++})$ suitable for κ such that $g, I, A \in N$, $N \cap \lambda^+ = \alpha < \lambda^+$, $\alpha \in A$, and N satisfies the following condition:

$$(*) \qquad \text{for all } C \in N, \text{ if } \alpha \in C \subseteq \lambda^+, \text{ then } C \notin I.$$

To see this situation may be assumed, choose an elementary chain $N_0 \prec \cdots \prec N_\alpha \prec H(\lambda^{++})$ as in Sect. 4.4 and use normality to see that

$$\{\alpha \in S_0 : (*) \text{ fails for some } C\} \in I.$$

To prove the lemma, define a decreasing sequence $\langle A_\xi : \xi < \kappa \rangle$ of subsets of λ^+ by recursion on $\xi < \kappa$. To start the recursion, let $A_0 := A$. Assume $0 < \xi < \kappa$ and A_ζ is defined for $\zeta < \xi$ in such a way that

$$A_\zeta \in N \quad \text{and} \quad \alpha \in A_\zeta \subseteq \lambda^+, \quad \text{for } \zeta < \xi.$$

Put $A_\xi = \bigcap_{\zeta<\xi} A_\zeta$ in case ξ is a limit ordinal.

Suppose A_ζ has been defined, and set $\Gamma_\zeta = g``[A_\zeta]^2$. Let I_ζ be the normal ideal generated on A_ζ from

$$I \cap \mathcal{P}(A_\zeta) \cup \{x \subseteq A_\zeta : g``[x]^2 \subsetneqq \Gamma_\zeta\}.$$

If $A_\zeta \notin I_\zeta$, then set $A_{\zeta+1} = A_\zeta$. If $A_\zeta \in I_\zeta$, then it is a finite or diagonal union of elements of the generating set. We treat the case where there is a sequence $\mathcal{B}_\zeta = \langle B_{\zeta,\eta} : \eta < \lambda^+ \rangle$ such that $A_\zeta = \bigcup_{\eta<\lambda^+} B_{\zeta,\eta}$, and for $\eta < \lambda^+$, $B_{\zeta,\eta} \cap (\eta+1) = \emptyset$, and either $B_{\zeta,\eta} \in I$ or $g``[B_{\zeta,\eta}]^2 \subsetneqq \Gamma_\zeta$.

Then, by elementarity, there is a sequence $\mathcal{B}_\zeta \in N$ as described above. Moreover, $\alpha \in B_{\zeta,\eta}$ for some $\eta < \lambda^+$ with $\eta < \alpha$ and $\eta \in N$, and thus $B_{\zeta,\eta} \in N$ for this η. We set $A_{\zeta+1} = B_{\zeta,\eta}$ for this η. Note that in this case, $\alpha \in A_{\zeta+1} \notin I$ and $g``[A_{\zeta+1}]^2 \subsetneqq g``[A_\zeta]^2$. This defines the sequence $\langle A_\eta : \eta < \kappa\rangle$.

Since g maps pairs from λ^+ into μ, there are at most $\mu < \kappa$ many ζ with $A_\zeta \subsetneqq A_{\zeta+1}$. Let ζ be the least ordinal with $A_\zeta = A_{\zeta+1}$, and set $B := A_\zeta$. Then I_ζ is a proper ideal on B. The ideal J generated from $I \cup I_\zeta$ is normal, and $B \notin J$. So by definition of I_ζ, B has essential colors for g, J. ⊣

Given a 2-partition g, we say that y and z are *color equivalent over* x and write $y \equiv^g_x z$ if $x < y$, $x < z$, $\mathrm{ot}(y) = \mathrm{ot}(z)$, and the order isomorphism $\pi : x \cup y \to x \cup z$ has $\pi{\restriction}x = \mathrm{id}$ and is color preserving: $g(\zeta, \eta) = g(\pi(\zeta), \pi(\eta))$.

6.2 Corollary. *For any 2-partition $g : [\lambda^+]^2 \to \mu$, and any normal ideal J, if B has essential colors for g, J, then there is a set $C \subseteq B$ with $B - C \in J$ such that for all $\alpha \in C$, for all $x \in [\alpha]^{<\kappa}$, and for all $\gamma \in \Gamma := g``[B]^2$, the set $D(\alpha, x, \gamma)$ is J-positive, where*

$$D(\alpha, x, \gamma) := \{\beta \in C : \alpha < \beta \wedge \{\alpha\} \equiv^g_x \{\beta\} \wedge g(\alpha, \beta) = \gamma\}.$$

Proof. To see that the set B has the desired property, assume to the contrary that for all α in some J-positive set $X \subseteq B$, there are $x(\alpha) \in [\alpha]^{<\kappa}$ and $\gamma(\alpha) \in g``[B]^2$ such that the set $D(\alpha, x(\alpha), \gamma(\alpha)) \in J$. By normality and $\mathrm{cf}(\alpha) = \kappa$, there are $Y \subseteq X$ with $Y \notin J$ such that for some x, γ one has $x(\alpha) = x$, $\gamma(\alpha) = \gamma$ for all $\alpha \in Y$. Then for some $Z \subseteq Y$ with $Z \notin J$ the condition $\{\alpha\} \equiv^g_x \{\beta\}$ holds for all $\alpha, \beta \in Z$. If for each $\alpha \in Z$ the set $\{\beta \in Z : g(\alpha, \beta) = \gamma\} \in J$, then, because of the normality, for the set $W := \{\delta \in Z : \forall \beta \in \delta \cap Z\ (g(\beta, \delta) \neq \gamma)\}$ both $W \notin J$ and $\gamma \notin g``[W]^2$ would hold, contradicting the fact that B has essential colors for g, J. ⊣

The above lemma and corollary are to be used with different 2-partitions, and hence were stated in generality. Now fix a 2-partition $f : [\lambda^+]^2 \to \mu$ for which we seek a homogeneous set of type $\kappa + \mu$.

6.3 Lemma (Pulldown Lemma). *There is a subset $S_0 \subseteq S_{\kappa,\lambda^+}$ closed in S_{κ,λ^+} such that for all $\alpha \in S_0$, for all $x \in [\alpha]^{<\kappa}$, and for all $z \in [\lambda^+ - (\alpha + 1)]^{<\kappa}$, there is a $y \in [\alpha - \sup(x)]^{<\kappa}$ such that $y \equiv^f_x z$.*

Proof. Let S_0 be as in Facts 3.2. Then Lemma 6.3 is true by reflection. ⊣

The Pulldown Lemma 6.3 does not say anything about the colors of edges that go between the sets y and z, while Corollary 6.2 detailed a situation in which any essential color may be pre-selected.

We apply Lemma 6.1 to f and the smallest normal ideal on λ^+, the non-stationary ideal, to get $B_0 \subseteq S_0$ and J_0, so J_0 is a normal ideal extending the non-stationary ideal, and B_0 has essential colors for f, J_0. We apply Corollary 6.2 to get $A_0 \subseteq B_0$ so that $B_0 - A_0 \in J_0$ and the other conditions of the corollary hold for all $\alpha \in A_0$. Then we choose $\alpha_0 \in A_0$, and put $T := A_0 - \alpha_0$.

6.4 Lemma. *There exists a function $h : T \times T \to \mu$ such that for all $x \in [\alpha_0]^{<\kappa}$ and $z \in [T]^{<\sigma}$ there is a $y \in [\alpha_0 - \sup(x)]^{<\sigma}$ such that*

(a) $y \equiv^f_x z$ via $\pi : x \cup y \to x \cup z$; and

(b) $f(\zeta, \zeta') = h(\pi(\zeta), \zeta')$ for all $\zeta \in y$, $\zeta' \in z, \pi(\zeta) \neq \zeta'$.

Proof. As σ is strongly compact is suffices to show that for every $Z \in [T]^{<\sigma}$ there exists a function $H : Z \times Z \to \mu$ as required.

Assume for the sake of contradiction that for every $H : Z \times Z \to \mu$ there is an $x_H \in [\alpha]^{<\sigma}$ such that for all $y \subseteq \alpha - \sup(x_H)$ satisfying (a), the function given by (b) is not H.

Let $x = \bigcup\{x_H : \ H : Z \times Z \to \mu\}$. Then $|x| < \sigma$ as $|x| \le \mu^{|Z|} < \sigma$, since σ is strongly inaccessible.

By Lemma 6.3, there is a y satisfying (a). Then (b) defines a function $H : Z \times Z \to \mu$. By the definition of x, the set $x_H \subseteq x$ is a set on which the function defined by (b) for y is not H, and that is a contradiction. ⊣

Now we define $k : [\lambda^+]^2 \to \mu \times \mu$ for $u, v \in \lambda^+$ with $u < v$ by

$$k(u, v) = \langle f(u, v), h(u, v)\rangle.$$

Next apply Lemma 6.1 and Corollary 6.2 to k and the normal ideal J_0 and the set T.

6.5 Corollary. *We get a normal ideal $J_1 \supseteq J_0$, a non-empty set $\Gamma \subseteq \mu \times \mu$, and subsets $S_1 \subseteq B_1 \subseteq T$ with $B_1 \notin J_1$, $B_1 - S_1 \in J_1$ such that B_1 has essential colors for k, J_1, and for each $\alpha \in S_1$ and for each $x \in [\alpha]^{<\kappa}$ and $\langle\gamma, \delta\rangle \in \Gamma$ the set $E(\alpha, x, \langle\gamma, \delta\rangle)$ is J_1-positive, where*

$$E(\alpha, x, \langle\gamma, \delta\rangle) := \{\beta \in S_1 : \alpha < \beta \wedge \{\alpha\} \equiv^k_x \{\beta\} \wedge k(\alpha, \beta) = \langle\gamma, \delta\rangle\}.$$

6.6 Lemma. *There is a subset $a \in [S_1]^{<\sigma}$ such that for every partition of a, say $a = \bigcup\{a_\zeta : \zeta < \mu\}$, there is a $\zeta < \mu$ such that for every $\gamma < \mu$, there is a subset $b_{\zeta,\gamma}$ of a_ζ of type μ homogeneous for f in the color γ.*

Proof. Notice that $S_1 \notin J$. We claim that if $A \subseteq S_1$ does not contain a subset of order type μ homogeneous for f in color γ for some $\gamma \in \mu$, then $A \in J$. Indeed, for each $\alpha \in A$ choose a maximal subset $M_\alpha \subseteq (\alpha + 1) \cap A$ homogeneous for f in color γ with $\alpha \in M_\alpha$. If $A \notin J$ then, by the normality of J, M_α is constant on a set not in J, and that yields, using the normality of J, a set not in J not containing any edge of color γ in f, just as in the proof of the Erdős-Rado Theorem 3.10. Hence S_1 has the property that any partition of it into μ pieces has a part A which contains a homogeneous subset of type μ for every $\gamma \in f``[S_1]^2$.

By the strong compactness of σ, there must be a set $a \subseteq S_1$ of size $< \sigma$ satisfying the same statement as S_1 about f, all partitions into μ parts and the existence of homogeneous subsets of type μ for all colors $\gamma \in f``[S_1]^2$. ⊣

We now describe the construction of the required homogeneous set.

Recall that immediately following Lemma 6.3 we chose α_0. Next choose a as in Lemma 6.1 Then choose $\alpha_1 \in S_1$ satisfying Corollary 6.5.

Then $\alpha_0 < a < \alpha_1$.

Define $a_{\gamma,\delta} := \{u \in a : k(u, \alpha_1) = \langle\gamma, \delta\rangle\}$.

By Lemma 6.6 there is a $\langle\gamma_0, \delta_0\rangle \in \Gamma$ such that a_{γ_0,δ_0} contains a subset of type μ homogeneous for color γ for every γ, hence it contains a subset $b \subseteq \alpha_{\gamma_0,\delta_0}$ of type $\mathrm{ot}(b) = \mu$ homogeneous for f in color δ_0. This will be "our color" and b will be the "μ-part" of our set. We are going to construct the "λ-part" of the set by transfinite recursion on $\xi < \kappa$ as follows. Assume $\xi < k$ and we have constructed $X = X_\xi$ of order type ξ homogeneous for f in color δ_0 and so that all edges from X to $b \cup \{\alpha_1\}$ have color δ_0.

We now apply Corollary 6.2 to α_1, and $X \cup b$ and we obtain an $\alpha_2 \in S_1$, with $\alpha_1 < \alpha_2$ such that $\{\alpha_1\} \equiv^k_{X\cup b} \{\alpha_2\}$ and $k(\alpha_1, \alpha_2) = \langle\gamma_0, \delta_0\rangle$.

As a corollary of this we have $\delta_0 = f(u, \alpha_1) = f(u, \alpha_2)$ for $u \in X$, since $\delta_0 = f(u, \alpha_1)$ for $u \in X$ is assumed, and $h(v, \alpha_1) = h(v, \alpha_2) = \delta_0$ for $v \in b$, by the choice of b.

Apply Lemma 6.4 for α_0 to $X$and $b \cup \{\alpha_1, \alpha_2\} \subseteq T$. We get $b' \cup \{\alpha_1', \alpha_2'\}$. We claim that $X_{\xi+1} = X \cup \{\alpha_2'\}$ is homogeneous in color δ_0 and sends all edges to $b \cup \{\alpha_1\}$ of color δ_0.

Indeed $f(u, \alpha_2') = f(u, \alpha_2) = \delta_0$ for $u \in X$ by the equivalence over X. For $v \in b$, we have $f(\alpha_2', v) = f(v, \alpha_2') = h(v, \alpha_2) = h(v, \alpha_1) = \delta_0$. By choice of α_2, we have $k(\alpha_1, \alpha_2) = \langle g(\alpha_1, \alpha_2),\ h(\alpha_2, \alpha_1)\rangle = \langle\gamma_0, \delta_0\rangle$. Hence $f(\alpha_2', \alpha_1) = \delta_0$ also.

7. Singular Cardinal Resources

It should be clear to the attentive reader that neither the ramification method as described in Remark 2.4 nor its refinements discussed up to now can yield any specific partition results for a singular resource. To get such results the method of *canonization* was invented in [18].

7.1 Definition. Assume $f : [\kappa]^r \to \gamma$ is an r-partition of length γ of κ, and $\langle A_\nu : \nu < \mu\rangle$ is a sequence of disjoint subsets of κ. Then f is said to be *canonical on* $\langle A_\nu : \nu < \mu\rangle$ if $f(x) = f(y)$ for all $x, y \in A := \bigcup_{\nu<\mu} A_\nu$ whenever x, y are positioned the same way in the sequence, i.e. if

$$|x \cap A_\nu| = |y \cap A_\nu| \quad \text{for all } \nu < \mu.$$

The idea is that, for a singular cardinal κ, we want to find a sequence $\langle A_\nu : \nu < \mathrm{cf}(\kappa)\rangle$ with $|A_\nu| < \kappa$ for $\nu < \mathrm{cf}(\kappa)$, and $A := \bigcup\{A_\nu : \nu < \mathrm{cf}(\kappa)\}$ of power κ such that f is canonical on $\langle A_\nu : \nu < \mathrm{cf}(\kappa)\rangle$ and use it to piece together large homogeneous sets. The following is the classical canonization theorem.

7.2 Theorem (General Canonization Lemma [18]). *Suppose that $\tau \geq 2$ is a cardinal, $r \geq 1$ is an integer, $\langle \kappa_\xi : \xi < \mu \rangle$ is a strictly increasing sequence of infinite cardinals with $\kappa_0 \geq \tau^{|\mu|}$ and $\exp_{\binom{r}{2}}(\kappa_\xi) < \exp_{\binom{r}{2}}(\kappa_\eta)$ for $\xi < \eta < \mu$. For any disjoint union $A = \dot{\bigcup}\{A_\nu : \nu < \mu\}$, and any coloring $f : [A]^r \to \tau$, if $|A_\nu| \geq (\exp_{\binom{r}{2}}(\kappa_\nu))^+$ for all $\nu < \mu$, then there are sets $B_\nu \subseteq A_\nu$ for $\nu < \mu$ so that $|B_\nu| \geq {\kappa_\nu}^+$ and the sequence $\langle B_\nu : \nu < \mu \rangle$ is canonical with respect to f.*

We are omitting the proof, since any reader with some experience in combinatorics should be able to reconstruct it, and since neither this proof nor the subsequent proofs fall into the line of the methods we are describing. We include canonization results because we think that no chapter on partition relations would be complete without them.

Here is the very first application of Theorem 7.2.

7.3 Theorem (Reduction Theorem). *Assume $\kappa > \mathrm{cf}(\kappa)$ is a strong limit cardinal. Then $\kappa \to (\kappa, \kappa_\nu)^2_{1 \leq \nu < \gamma}$ if and only if $\mathrm{cf}(\kappa) \to (\mathrm{cf}(\kappa), \kappa_\nu)^2_{1 \leq \nu < \gamma}$.*

Indeed, the next theorem is the only one obtained for a singular resource using a method different from canonization. The elementary proof of the theorem is left to the reader (see [19]).

7.4 Theorem (Erdős; Dushnik and Miller [9]). *For every infinite cardinal κ, $\kappa \to (\kappa, \omega)^2$.*

See also [19] for a proof. The General Canonization Lemma implies Theorem 7.4 for singular strong limit κ and for $\mathrm{cf}(\kappa) > \omega$ it yields $\kappa \to (\kappa, \omega + 1)^2$. It has been a longstanding problem if this partition relation holds if we do not assume that κ is strong limit. Recently Shelah [54] proved this partition relation holds under the much weaker condition that $2^{\mathrm{cf}(\kappa)} < \kappa$.

Erdős, Hajnal and Rado in [18] pursued the idea of finding the right generalization of the form $\kappa \to (\kappa, \omega_1)^2$ for singular κ. The first possible case is $\kappa = \aleph_{\mathfrak{c}^+}$, where $\mathfrak{c} = 2^\omega$, and the Reduction Theorem 7.3 gives a positive answer in case κ is a strong limit. The very first question of the Erdős-Hajnal problem list [12] asks if this additional hypothesis is necessary. Shelah and Stanley in [60] and [61] proved that the partition relation $\kappa \to (\kappa, \omega_1)^2$ can be both false and true if κ is not a strong limit cardinal. A description of this deep result is beyond the scope of this section.

There is one more *canonization* result that we want to mention. It was isolated during the discussion of the ordinary partition relation in the book [19] that the following result should be true, and Shelah later proved it.

7.5 Theorem (Shelah [56]). *Assume that κ is a singular cardinal of weakly compact cofinality. If $\kappa < 2^{<\kappa}$ and $2^\rho < 2^{<\kappa}$ for $\rho < \kappa$, then*

$$2^{<\kappa} \to (\kappa)^2_2.$$

To prove this partition relation, Shelah worked out a new group of canonization results in [56]. We only state here one of the main results. Call a sequence of cardinals $\langle \kappa_\nu : \nu < \mu \rangle$ *exponentially increasing* if $\xi < \nu < \mu$ implies $2^{\kappa_\xi} < 2^{\kappa_\nu}$. A sequence of sets $\langle B_\nu : \nu < \mu \rangle$ is *weakly canonical* if $f(u) = f(v)$ whenever $u, v \in [\bigcup_{\nu<\mu} B_\nu]^r$ and $|u \cap B_\nu| = |v \cap B_\nu| \leq 1$ for every $\nu < \mu$. A set $F \subseteq \mathcal{P}(A)$ *sustains* A over κ if for every $X \subseteq A$ with $|X| = (2^\kappa)^+$, there is a $Y \in F$ so that $Y \subseteq X$ and $|Y| = \kappa^+$.

7.6 Theorem (Shelah's Canonization Lemma [56]). *Suppose $\langle \kappa_\xi : \xi < \mu \rangle$ is an exponentially increasing sequence of infinite cardinals with $\kappa_0 \geq \tau, \mu, \omega$, for a cardinal $\tau \geq 2$. Then for any disjoint union $A = \dot{\bigcup}\{A_\nu : \nu < \mu\}$, any sequence $\langle F_\nu \subseteq \mathcal{P}(A_\nu) : \nu < \mu \rangle$, and any coloring $f : [A]^2 \to \tau$, if $|A_\nu| > 2^{\kappa_\nu}$ and F_ν sustains A_ν for all $\nu < \mu$, then there is a sequence $\langle B_\nu : \nu < \mu \rangle$ weakly canonical with respect to f with $|B_\nu| = {\kappa_\nu}^+$ for all $\nu < \mu$.*

8. Polarized Partition Relations

Polarized partition relations were defined in the introduction. We do not have the space to give an orderly discussion of the problems and results on this partition relation. Rather, we will only give a few examples, where the method of elementary submodels described in the previous section can be resourcefully used. The first appearance in the literature of the use of elementary submodels for the proofs of polarized partition relations is the following theorem of Jones which generalizes a result of Erdős, Hajnal and Rado [18] from 1965:

8.1 Theorem (Jones [30]). *Let κ be an infinite cardinal and $\lambda = 2^{<\kappa}$. Then the following polarized partition relation holds:*

$$\binom{\lambda^+}{\lambda^+} \to \begin{pmatrix} \lambda^+ & & \gamma & \kappa+1 \\ & or & & \\ \gamma & & \lambda^+ & \kappa+1 \end{pmatrix}^{1,1}.$$

In the remainder of this section, we apply the method of elementary submodels using the "method of double ramification".

8.1. Successors of Weakly Compact Cardinals

The first example is chosen with an eye to a clean presentation of the method.

8.2 Theorem (Baumgartner and Hajnal [3]). *Suppose that κ is a weakly compact cardinal. Then*

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}^{1,1}_{\gamma} \quad \text{for } \gamma < \kappa.$$

Before going into the details of the proof, we give some historical remarks and state an open problem. In [26], Hajnal proved that for measurable κ, the following partition relations holds:

$$\binom{\kappa^+}{\kappa} \to \binom{\alpha}{\kappa}^{1,n}_{<\kappa} \quad \text{for } n < \omega \text{ and } \alpha < \kappa^+.$$

In an early paper of Choodnovsky [6], it was claimed that

$$\binom{\kappa^+}{\kappa} \to \binom{\alpha}{\kappa}^{1,1}_{<\kappa} \quad \text{for } \alpha < \kappa^+$$

remains valid for weakly compact κ, but no proof was given. Realizing that this claim was by no means obvious, both Kanamori [32] and Wolfsdorf [66] published proofs that the relation is true for two colors:

$$\binom{\kappa^+}{\kappa} \to \binom{\alpha}{\kappa}^{1,1}_{2} \quad \text{for } \alpha < \kappa^+.$$

Theorem 8.2 was generalized in the thesis of Jones [29, 28], who proved, using elementary submodels, that for weakly compact cardinals κ,

$$\binom{\kappa^+}{\kappa} \to \left(\binom{\alpha}{\kappa}_m, \binom{\kappa^n}{\kappa}_\gamma\right)^{1,1} \quad \text{for } m, n < \kappa,\ \gamma < \kappa,\ \alpha < \kappa^+.$$

To the best of our knowledge, the following problem remains unsolved.

8.3 Question. Does the partition relation

$$\binom{\kappa^+}{\kappa} \to \binom{\alpha}{\kappa}^{1,1}_{\omega} \quad \text{hold for all weakly compact } \kappa \ge \omega,\ \alpha \ge \kappa^\omega?$$

The rest of this subsection is devoted to the proof of Theorem 8.2 for $\kappa > \omega$. To that end, let $\kappa > \omega$ be a weakly compact cardinal, and let $f : \kappa^+ \times \kappa \to \gamma$ be a fixed partition. We outline background assumptions below, using work from earlier sections.

8.4 Definition. Let $\langle\langle N_\alpha, \in\rangle : \alpha < \kappa^+\rangle$ be a sequence of elementary submodels of $H(\kappa^{++})$ satisfying Facts 3.2 with $\lambda = \kappa^{<\kappa} = \kappa$ and $A = \{f\}$. Let $\langle I_\alpha : \alpha < \kappa^+\rangle$ be the ideals defined in Definition 3.4 and let

$$S_0 := \{\alpha < \kappa^+ : \alpha(N_\alpha) = \alpha \wedge \mathrm{cf}(\alpha) = \kappa \wedge N_\alpha \text{ is suitable}\}$$

as defined in Sect. 4.2. Note that for $\alpha \in S_0$, I_α is a κ-complete proper ideal, by Lemma 3.6.

8.5 Definition. Call $\mathcal{N} = \langle N_{\alpha,\xi} : \alpha < \kappa^+ \wedge\ \xi < \kappa\rangle$ a *double ramification system for* $\langle N_\alpha : \alpha < \kappa^+\rangle$ as in Definition 8.4 if for each $\alpha < \kappa^+$, the sequence $\langle N_{\alpha,\xi} : \xi < \kappa\rangle \in N_{\alpha+1}$ is an increasing continuous sequence of elementary submodels of N_α with $\bigcup\{N_{\alpha,\xi} : \xi < \kappa\} = N_\alpha$ such that $|N_{\alpha,\xi}| < \kappa$ for $\xi < \kappa$.

We use the name *double ramification system* since, as we explained in the proof of the Erdős-Rado Theorem, the N_α's play the role of the ramification system of Erdős and Rado.

Just like in Facts 3.2, using general facts about elementary submodels, and the uncountability and strong Mahloness of κ, we can see that there is a system satisfying the next definition.

8.6 Definition. Let $\mathcal{N} = \langle N_{\alpha,\xi} : \alpha < \kappa^+ \wedge\ \xi < \kappa\rangle$ be a double ramification system such that for each $\alpha \in S_0$ there is a $T^0_\alpha \subseteq \kappa$, with $T^0_\alpha \in \mathrm{Stat}(\kappa)$ satisfying the following conditions for all $\xi \in T^0_\alpha$:

1. $N_{\alpha,\xi} \cap \kappa = \xi > \gamma$;
2. ξ is a regular cardinal; and
3. $[N_{\alpha,\xi}]^{<\xi} \subseteq N_{\alpha,\xi}$.

Next we relativize certain important sets to the submodels of the double ramification system.

8.7 Definition. For each $\alpha \in S_0$ and $\xi \in T^0_\alpha$, define the following sets:

1. $X_{\alpha,\xi} := N_{\alpha,\xi} \cap \kappa^+$;
2. $I_{\alpha,\xi} := \{X \subseteq \xi : (\exists Y)(Y \subseteq \kappa \wedge Y \in N_{\alpha,\xi} \wedge \xi \notin Y \wedge X \subseteq Y)\}$;
3. $\hat{I}_{\alpha,\xi} := \{X \subseteq X_{\alpha,\xi} : (\exists Y)(Y \subseteq \kappa^+ \wedge Y \in N_{\alpha,\xi} \wedge \alpha \notin Y \wedge X \subseteq Y)\}$.

8.8 Lemma. *For $\alpha \in S_0$ and $\xi \in T^0_\alpha$, both $I_{\alpha,\xi}$ and $\hat{I}_{\alpha,\xi}$ are ξ-complete ideals, and $I_{\alpha,\xi}$ is proper.*

Proof. The first statement follows from the fact that $[N_{\alpha,\xi}]^{<\xi} \subseteq N_{\alpha,\xi}$. To see that $I_{\alpha,\xi}$ is proper, then just like in Lemma 3.6, assume $Z \subseteq \kappa$, $\xi \in Z$ and $Z \in N_{\alpha,\xi}$. Then $\sup(Z) \in N_{\alpha,\xi}$, hence $\sup(Z) = \kappa$ and $\sup(Z) \cap \xi = \xi$. This implies $\xi \notin I_{\alpha,\xi}$. ⊣

Note that $\hat{I}_{\alpha,\xi}$ is proper for many α and ξ as well (see Corollary 8.11 below).

Notation. For all $\nu < \gamma$, let

$$f^{\downarrow}(\alpha;\nu) := \{\xi < \kappa : f(\alpha,\xi) = \nu\} \quad \text{for } \alpha < \kappa^+,$$
$$f^{\uparrow}(\xi;\nu) := \{\alpha < \kappa^+ : f(\alpha,\xi) = \nu\} \quad \text{for } \xi < \kappa.$$

8.9 Definition. For $\alpha \in S_0$ and $\xi \in T^0_\alpha$, let

$$a_{\alpha,\xi} := \{\nu < \gamma : f^{\downarrow}(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi}\}.$$

Note that $a_{\alpha,\xi} \neq \emptyset$ by Lemma 8.8 and the fact that $\gamma < \xi$.

8.10 Lemma (Main Lemma). *There are subsets $a \subseteq \gamma$ and $S \subseteq S_0$ with $S \in \mathrm{Stat}(\kappa^+)$, and for each $\alpha \in S$, there is a subset $T_\alpha \subseteq T^0_\alpha$ with $T_\alpha \in \mathrm{Stat}(\kappa)$, so that $f(\alpha,\eta) \in a = a_{\alpha,\eta}$ for all $\alpha \in S$ and $\eta \in \bigcup\{T_\beta : \beta \in S\}$.*

Proof. First thin each T^0_α for $\alpha \in S_0$ to a stationary subset T^1_α so that for some a_α, one has $a_{\alpha,\xi} = a_\alpha$ for all $\xi \in T^1_\alpha$. Then thin S_0 to a stationary subset S_1 so that for some $a \subseteq \gamma$ and for all $\alpha \in S_1$, $a_\alpha = a$. We may assume without loss of generality that $\gamma < \xi$ for all $\xi \in T^1_\alpha$.

Notice that for all $\alpha \in S_1$ and all $\xi \in T^1_\alpha$, if $\nu \notin a$, then $f^\downarrow(\alpha;\nu) \cap \xi \in I_{\alpha,\xi}$. Hence, by the definition of $f^\downarrow$ and the ξ-completeness of $I_{\alpha,\xi}$, it follows that $\{\eta < \xi : f(\alpha,\eta) \notin a\} \in I_{\alpha,\xi}$. By Definition 8.7, for $\alpha \in S_1$ and $\xi \in T^1_\alpha$, we can choose sets $Y_{\alpha,\xi} \subseteq \kappa$ such that $\xi \notin Y_{\alpha,\xi} \in N_{\alpha,\xi}$ and $\{\eta < \xi : f(\alpha,\eta) \notin a\} \subseteq Y_{\alpha,\xi}$. Using Fodor's Theorem twice, we get $Y \subseteq \kappa$, $S \subseteq S_1$ with $S \in \mathrm{Stat}(\kappa^+)$, and $\langle T_\alpha \subseteq T^1_\alpha : \alpha \in S\rangle$ such that $T_\alpha \in \mathrm{Stat}(\kappa)$ for all $\alpha \in S$, and $Y_{\alpha,\xi} = Y$ for $\alpha \in S$ and $\xi \in T_\alpha$.

Consequently, for all $\xi \in \bigcup\{T_\beta : \beta \in S\}$, we have $\xi \notin Y$, since $\xi \notin Y_{\beta,\xi} = Y$. However, if $\alpha \in S$ and $\eta < \kappa$ are such that $f(\alpha,\eta) \notin a$, then for some $\xi \in T_\alpha$, one has $\eta \in Y_{\alpha,\xi} = Y$, so the theorem follows. ⊣

8.11 Corollary. *There is an $\alpha < \kappa^+$, so that for κ-many ξ, the following condition holds:*

$$(+) \qquad (\exists \nu < \gamma)(f^\downarrow(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi} \wedge f^\uparrow(\xi;\nu) \cap X_{\alpha,\xi} \notin \hat{I}_{\alpha,\xi}).$$

Proof. Let α be such that $S \cap \alpha \notin I_\alpha$. Such an α must exist by Corollary 4.8. A standard argument shows that if $S \cap \alpha \notin I_\alpha$, then $W = \{\xi < \kappa : S \cap \alpha \cap X_{\alpha,\xi} \in \hat{I}_{\alpha,\xi}\}$ is non-stationary in κ. By Main Lemma 8.10, $f(\beta,\xi) \in a$ for $\xi \in T_\alpha$ and $\beta \in S \cap \alpha \cap X_{\alpha,\xi}$. Hence $f^\uparrow(\xi;\nu) \cap X_{\alpha,\xi} \notin \hat{I}_{\alpha,\xi}$ for some $\nu \in a$ and for every $\xi \in T_\alpha - W$. On the other hand, $f^\downarrow(\alpha;\nu) \cap \xi \notin I_{\alpha,\xi}$ for all $\nu \in a$ and for every $\xi \in T_\alpha$. ⊣

8.12 Lemma (Compactness Lemma). *Assume that for some $\alpha < \kappa^+$ there are κ-many ξ so that for some $A_\xi \subseteq X_{\alpha,\xi}$, $B_\xi \subseteq \xi$ with $\mathrm{ot}(A_\xi) = \mathrm{ot}(B_\xi) = \xi$, the set $A_\xi \times B_\xi$ is homogeneous for f. Then there are $A \subseteq \kappa^+$, $B \subseteq \kappa$ with $\mathrm{ot}(A) = \kappa + 1$ and $\mathrm{ot}(B) = \kappa$ such that $A \times B$ is homogeneous for f.*

Proof. Use the weak compactness of κ via its Π^1_1-indescribability. ⊣

After all these preliminaries, Theorem 8.2 now follows from Corollary 8.11, the Compactness Lemma 8.12 above, and the Reflection Lemma below.

8.13 Lemma. *Assume that for α as in Corollary 8.11 and for some $\nu < \gamma$, the ordinal ξ satisfies the formula (+) of Corollary 8.11. Then there are $A \subseteq X_{\alpha,\xi}$, $B \subseteq \xi$ with $\mathrm{ot}(A_\xi) = \mathrm{ot}(B_\xi) = \xi$ so that $A \times B$ is homogeneous for f in color ν.*

Proof. Let $\overline{A} := f^{\uparrow}(\xi;\nu) \cap X_{\alpha,\xi}$ and let $\overline{B} = f^{\downarrow}(\alpha;\nu) \cap \xi$. Since (+) holds for ν and ξ, we know that $\overline{B} \notin I_{\alpha,\xi}$ and $\overline{A} \notin \hat{I}_{\alpha,\xi}$. These last two statements imply the existence of the sets A, B as required. Indeed, we can define sequences $A = \{a_\mu : \mu < \xi\} \subseteq \overline{A}$ and $B = \{b_\mu : \mu < \xi\} \subseteq \overline{B}$ by transfinite recursion on $\mu < \xi$ so that for all $\mu', \mu'' < \xi$,

$$\begin{aligned} f(a_{\mu'}, b_{\mu''}) = \nu, \\ a_{\mu'} \in f^{\uparrow}(\xi;\nu), \\ b_{\mu''} \in f^{\downarrow}(\alpha;\nu). \end{aligned}$$

At stage $\mu < \xi$, assume this has been done for $\mu', \mu'' < \mu$. First choose a_μ. Toward that end, let

$$Z^-_\mu := \{\beta < \kappa^+ : f(\beta, b_{\mu''}) = \nu \text{ for all } \mu'' < \mu\}.$$

Then $\alpha \in Z^-_\mu$ since $b_{\mu''} \in f^{\downarrow}(\alpha;\nu)$ for all $\mu'' < \mu$. Since f, $\{b_{\mu''} : \mu'' < \mu\} \in N_{\alpha,\xi}$, it follows that $Z^-_\mu \in N_{\alpha,\xi}$. So $Z^-_\mu \cap \overline{A} - \{a_{\mu'} : \mu' < \mu\}$ is not in $\hat{I}_{\alpha,\xi}$, so we can choose α_μ from it.

Then choose b_μ similarly using $f^{\uparrow}(\xi;\nu)$ in the role of $f^{\downarrow}(\alpha;\nu)$ and $I_{\alpha,\xi}$ instead of $\hat{I}_{\alpha,\xi}$ and taking care to make $f(a_{\mu'}, b_\mu) = \nu$ for $\mu' \leq \mu$. ⊣

8.2. Successors of Singular Cardinals

In this subsection we investigate the following question.

8.14 Question. Assume κ is a singular strong limit cardinal and $\gamma < \kappa$. Under what circumstances does the following partition relation hold?

$$(*) \qquad \binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}^{1,1}_{\gamma}.$$

The problem was isolated in Problem 11 of [18], where it was asked if $(*)$ holds for $\kappa = \aleph_{\omega_1}$ under GCH. In the same paper, it was proved that $(*)$ holds provided $\mathrm{cf}(\kappa) = \omega$, but we omit the proof of this fact.

After about thirty years, a shocking partial result was proved by Shelah.

8.15 Theorem (Shelah [58]). *Assume κ is a singular strong limit cardinal of uncountable cofinality. Then $(*)$ holds if $2^\kappa > \kappa^+$.*

For another proof of this result, see Kojman [34]. A little extra information is contained in an unpublished result of Foreman, which we prove here using the result of Shelah.

8.16 Theorem (Foreman unpublished). *Suppose that κ is a singular strong limit cardinal in V and $(2^\kappa)^V > (\kappa^+)^V$. Then there is a κ-complete partial order P which satisfies the $(2^\kappa)^+$-chain condition so that*

$$V^P \models 2^\kappa = \kappa^+ \quad \text{and} \quad \binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}^{1,1}_{\gamma} \quad \text{for } \gamma < \kappa.$$

Proof. We can choose for P the κ^+-complete Levy collapse of $2^{<\kappa}$ to κ^+. For every $p \in P$ and every name for a partition $\dot{f}$, we can define in V a decreasing sequence $\langle p_\alpha \mid \alpha < \kappa^+\rangle$ of conditions and a function $g : \kappa^+ \times \kappa \to \gamma$ such that $p_0 = p$ and

$$\forall \alpha < \kappa^+ \; \forall \beta \leq \alpha \; \forall \xi < \kappa p_\alpha \Vdash \dot{f}(\beta, \xi) = g(\beta, \xi).$$

By Theorem 8.15, we can choose A, B such that $A \times B$ is homogeneous for g and $|A| = |B| = \kappa$. For some $\alpha < \kappa$, we have $A, B \subseteq \alpha$ and then

$$p_\alpha \Vdash \exists A \; \exists B(|A| = |B| = \kappa \wedge A \times B \text{ is homogeneous for } \dot{f}).$$

Hence V^P satisfies the claim. $\dashv$

All other problems remain unsolved, even for $\gamma = 2$. For notational convenience, for the rest of this section let $\mu = \mathrm{cf}(\kappa)$. We may assume that $\mu > \omega$, and we will embark on a lengthy proof of a mild strengthening of the result of Shelah.

8.17 Theorem. *Suppose that κ is a singular strong limit cardinal of uncountable cofinality μ. Then $(**)$ holds if $2^\kappa > \kappa^+$:*

$$(**) \qquad \binom{\kappa^+}{\kappa} \to \binom{\kappa+1}{\kappa}_\gamma .$$

The proof we are going to describe will be a double ramification, quite similar in structure to the proof of Theorem 8.2 and different from the simplified proof of Theorem 8.15 in Kojman [34].

8.18 Definition. Choose $\vec{\kappa} = \langle \kappa_\nu : \nu < \mu\rangle$ to be an increasing continuous sequence of cardinals satisfying the following properties:

1. $\sup(\{\kappa_\nu : \nu < \mu\}) = \kappa$;
2. $\mu < \kappa_0$; and
3. $2^{\kappa_\nu} < \kappa_{\nu+1} = \mathrm{cf}(\kappa_{\nu+1})$ for $\nu < \mu$.

We use results of Shelah's pcf theory [57] (see also the Abraham-Magidor chapter in this Handbook) to guarantee the existence of the sequence delineated in the next definition.

8.19 Definition. Choose $\vec{\lambda} = \langle \lambda_\nu : \nu < \mu\rangle$ to be an increasing sequence of regular cardinals with $\kappa_\nu < \lambda_\nu < \kappa$ for $\nu < \mu$ such that the product $\Pi := \prod_{\nu<\mu} \lambda_\nu$ satisfies

$$(\forall \{\varphi_\alpha : \alpha < \kappa^+\} \subseteq \Pi)(\exists \varphi \in \Pi)(\forall \alpha < \kappa^+)(\varphi_\alpha \prec \varphi)$$

where $\prec$ is the relation of eventual domination on Π.

We now choose a sequence of models to serve as the skeleton of a double ramification.

8.20 Definition. Let $A := \mu \cup \{\mu, f, \vec{\kappa}, \vec{\lambda}\}$. Using Facts 3.2, we can choose an increasing chain $\langle\langle N_\alpha, \in\rangle : \alpha < \kappa^+\rangle$ of elementary submodels of $H(\kappa^{++})$ with $A \in N_0$ such that

$$S_0 := \{\alpha < \kappa^+ : \alpha(N_\alpha) = \alpha > \kappa \wedge \mathrm{cf}(\alpha) = \mu \wedge N_\alpha \text{ is suitable for } \kappa\}$$

is a club in S_{μ,κ^+}. As in Definition 3.4, we define

$$I_\alpha := \{X \subseteq \alpha^+ : \exists Y\ (Y \subseteq \kappa^+ \wedge Y \in N_\alpha \wedge \alpha \notin Y \wedge |X - Y| < \kappa)\},$$

and note that since κ is singular, the last condition may no longer be replaced by $X \subseteq Y$.

8.21 Facts. The following statements hold.

1. I_α is a μ-complete proper ideal for all $\alpha \in S_0$;
2. for every stationary $S \subseteq S_0$, there is some $\alpha \in S$ so that $S \cap \alpha \notin I_\alpha$;
3. for every $\alpha \in S_0$, every $X \in \mathcal{P}(\alpha) - I_\alpha$ and every $\tau < \kappa$, there is some $W \subseteq X$ with $|W| = \tau$ so that $W \in N_\alpha$.

Proof. The first item follows from Lemma 3.6, and the second from Corollary 4.8. To see that the third item holds, fix $\alpha \in S_0$, and assume $X \in \mathcal{P}(\alpha) - I_\alpha$. By the definition of I_α, we have $|X| \geq \kappa$. Let $\tau < \kappa$ be given. Since $\mathrm{cf}(\kappa) = \mu < \kappa$, there is a $\beta < \alpha$ with $|X \cap \beta| \geq \tau$. Since $N_\alpha \prec H(\kappa^{++})$ and $\beta \in N_\alpha$, there is some U in N_α with $|U| < \kappa$ and $|X \cap U| \geq \tau$. Then any $W \subseteq X \cap U$ with $|W| = \tau$ satisfies the requirement of the item since $|\mathcal{P}(U)| < \kappa$ and therefore $\mathcal{P}(U) \subseteq N_\alpha$. ⊣

For notational convenience, we use the same names for our double ramification system here as in the proof of Theorem 8.2.

8.22 Definition (Double ramification). For each $\alpha \in S_0$, we choose $<_\alpha$, a well-ordering of type κ of N_α. Choose $\mathcal{N} = \langle N_{\alpha,\nu} : \alpha < \kappa^+ \wedge \nu < \mu\rangle$ for the skeleton chosen above so that for $\alpha \in S_0$, the sequence $\langle N_{\alpha,\nu} : \nu < \mu\rangle$ is increasing, continuous and internally approachable and satisfies the following conditions:

1. $A \in N_{\alpha,0}$;
2. $\kappa_\nu \subseteq N_{\alpha,\nu}$, $|N_{\alpha,\nu}| = \kappa_\nu$, and $N_{\alpha,\nu}$ contains the νth section of $N_{\alpha,\nu}$ in the well-ordering $<_\alpha$ for each $\nu < \mu$.

Next we relativize certain important sets to the submodels of the double ramification system.

Notation. For each $\alpha \in S_0$, define the set $X_{\alpha,\nu} := N_{\alpha,\nu} \cap \lambda_\nu$ for $\nu < \mu$ and the function $\varphi_\alpha : \mu \to \kappa$ so that $\varphi_\alpha(\nu) := \sup(X_{\alpha,\nu})$.

The following facts follow from Definition 8.19 of Π and $\vec{\lambda}$.

8.23 Lemma. *For every $\alpha \in S_0$, the function φ_α is in Π, and there is a function $\varphi \in \Pi$ which eventually dominates all the φ_α for $\alpha \in S_0$. That is, for each $\alpha \in S_0$, there is some $\nu_\alpha < \mu$, so that $\varphi_\alpha(\nu) < \varphi(\nu)$ for all ν with $\nu_\alpha \leq \nu < \mu$.*

For the remainder of this section, fix a function φ which eventually dominates all the φ_α for $\alpha \in S_0$, and let ν_α as above be the point at which domination sets in.

8.24 Definition. For $\alpha \in S_0$ and ν with $\nu_\alpha \leq \nu < \mu$, define

$$I_{\alpha,\nu} := \{X \subseteq \nu : \exists Y\ (Y \subseteq \lambda \wedge Y \in N_{\alpha,\nu} \wedge \varphi(\nu) \notin Y \wedge |X - Y| < \kappa_\nu)\}.$$

8.25 Lemma. *Let $\alpha \in S_0$ and ν with $\nu_\alpha \leq \nu < \mu$ be given. Then*

1. *$I_{\alpha,\nu}$ is a proper ideal;*
2. *for each $X \subseteq X_{\alpha,\nu}$ with $X \in I^+_{\alpha,\nu}$, there is a $W \subseteq X$ with $|W| = \kappa_\nu$ so that $W \in N_{\alpha,\nu+1}$.*

Proof. For the first item, note that the set $I_{\alpha,\nu}$ is an ideal because $N_{\alpha,\nu}$ is closed with respect to finite unions. To see that $X_{\alpha,\nu} \notin I_{\alpha,\nu}$, let $Z \in N_{\alpha,\nu}$ be a subset of λ_ν with $\varphi(\nu) \in Z$. It is enough to show $|Z \cap X_{\alpha,\nu}| \geq \kappa_\nu$. Now $Z \in N_{\alpha,\nu}$ and $\sup(Z) \in N_{\alpha,\nu}$. Hence $\sup(Z) = \lambda_\nu$. Thus there is a one-to-one function $g : \kappa_\nu \to Z$. Using the fact that κ_ν and λ_ν are in $N_{\alpha,\nu}$, by elementarity, there is a function $g \in N_{\alpha,\nu}$ like this. Using the fact that $\kappa_\nu + 1 \subseteq N_{\alpha,\nu}$, we get that $\operatorname{ran}(g) \subseteq N_{\alpha,\nu} \cap \lambda_\nu = X_{\alpha,\nu}$.

For the second item, there is a subset $W \subseteq X$ with $|W| = \kappa_\nu$ by Definition 8.24. Also, by Definition 8.22, we know that $X_{\alpha,\nu} \in N_{\alpha,\nu+1}$, $2^{\kappa_\nu} < \kappa_{\nu+1}$ and $\mathcal{P}(X_{\alpha,\nu}) \subseteq N_{\alpha,\nu+1}$. Therefore $W \in N_{\alpha,\nu+1}$ as required. ⊣

Recall the notation $f^\downarrow(\alpha; i)$ introduced after Lemma 8.8:

$$f^\downarrow(\alpha; i) := \{\xi < \kappa : f(\alpha, \xi) = i\} \quad \text{for } \alpha < \kappa^+,\ i < \gamma.$$

Using the facts that $\gamma, \omega < \mu$ and $2^\mu < \kappa$, we can show directly that

$$\binom{\kappa^+}{\mu} \to \binom{\operatorname{Stat}(\kappa^+)}{\operatorname{Stat}(\mu)}^{1,1}_{\gamma}.$$

We get the next lemma by applying this partition relation to the coloring $f \circ \varphi$ of $\kappa^+ \times \mu$.

8.26 Lemma. *There are $S \subseteq S_0$, $T \subseteq \mu$, $\overline{\nu} < \mu$ and $i < \gamma$ such that $S \in \operatorname{Stat}(\kappa^+)$, $T \in \operatorname{Stat}(\mu)$, $\overline{\nu} \cap T = \emptyset$, φ“$T \subseteq f^\downarrow(\alpha; i)$ and $\nu_\alpha = \overline{\nu}$ for all $\alpha \in S$.*

We now prove our main claim.

8.27 Lemma (Main Claim). *There is an $\alpha \in S$ such that $S \cap \alpha \notin I_\alpha$ and*

$$\{\nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu}\} \in \mathrm{Stat}(\mu).$$

Proof. By Corollary 4.8, it is sufficient to see that

$$\{\alpha \in S : \{\nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu}\} \in \mathrm{Stat}(\mu)\} \in \mathrm{Stat}(\kappa^+).$$

Let $T_\alpha := \{\nu \in T : f^{\downarrow}(\alpha, i) \cap X_{\alpha,\nu} \in I_{\alpha,\nu}\}$ for $\alpha \in S$. Assume by way of contradiction that for some $S' \in \mathrm{Stat}(\kappa^+) \cap \mathcal{P}(S)$, one has $T_\alpha \in \mathrm{Stat}(\mu)$ for all $\alpha \in S'$.

For $\alpha \in S'$, $\nu \in T_\alpha$, choose $Y_{\alpha,\nu}$ satisfying the following conditions: $Y_{\alpha,\nu} \subseteq \lambda_\nu$, $Y_{\alpha,\nu} \in N_{\alpha,\nu}$, $\varphi(\nu) \notin Y_{\alpha,\nu}$, and $|f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} - Y_{\alpha,\nu}| < \kappa_\nu$. For each $\alpha \in S'$, by Fodor's Theorem, the sets $Y_{\alpha,\nu}$ stabilize on a stationary subset of T_α. That is, for each $\alpha \in S'$, there are $T'_\alpha \subseteq T_\alpha$ with $T'_\alpha \in \mathrm{Stat}(\mu)$, Y_α and $\rho_\alpha < \kappa$ such that $Y_{\alpha,\nu} = Y_\alpha$ and $|f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} - Y_{\alpha,\nu}| \leq \rho_\alpha$ for $\nu \in T'_\alpha$ and

$$Y_\alpha \cap \{\varphi(\nu) : \nu \in T'_\alpha\} = \emptyset.$$

Note that $\bigcup\{X_{\alpha,\nu} : \nu \in T'_\alpha\} = \kappa$, hence

$$|f^{\downarrow}(\alpha; i) - Y_\alpha| \leq \rho_\alpha.$$

Now, using Fodor's Theorem again, Y_α stabilizes on a stationary subset of S'. That is, there are $T' \in \mathrm{Stat}(\mu)$, Y and ρ such that for some $S'' \in \mathrm{Stat}(\kappa^+) \cap \mathcal{P}(S')$, one has $T'_\alpha = T'$, $Y_\alpha = Y$ and $\rho_\alpha = \rho$ for all $\alpha \in S''$.

Now choose two elements $\alpha', \beta' \in S''$ with $\alpha' < \beta'$, and let $\nu' \in T'$ be such that $\beta' \in N_{\alpha',\nu'}$ and $\kappa_{\nu'} > \rho$. Since $\alpha' \in S'' \subseteq S$ and $\nu' \in T' \subseteq T$, it follows that $f(\beta', \varphi(\nu')) = i$ by Lemma 8.26. In other words, $\varphi(\nu') \in f^{\downarrow}(\beta'; i)$. However, $f^{\downarrow}(\beta'; i) \in N_{\alpha',\nu'}$, hence

$$f^{\downarrow}(\beta'; i) \cap X_{\alpha',\nu'} \notin I_{\alpha',\nu'}.$$

This last fact contradicts the inequality $|f^{\downarrow}(\beta'; i) - Y| < \rho$ and the lemma follows. $\dashv$

To finish the proof of Theorem 8.17 using the Main Claim 8.27, we want to define sequences $\langle A_\xi : \xi < \mu\rangle$ with $A_\xi \subseteq \kappa$ and $\langle B_\xi : \xi < \mu\rangle$ with $B_\xi \subseteq S_0$ so that the sets are pairwise disjoint, $|A_\xi| = |B_\xi| = \kappa_\xi$, $A_\xi, B_\xi \in N_{\alpha,\nu_\xi}$ for some $\nu_\xi \in T^0$, where $T^0 := \{\nu \in T : f^{\downarrow}(\alpha; i) \cap X_{\alpha,\nu} \notin I_{\alpha,\nu}\}$ is the set defined in the Main Claim 8.27, and f is constantly i on the set

$$\textstyle\bigcup_{\xi<\mu} B_\xi \cup \{\alpha\} \times \bigcup_{\xi<\mu} A_\xi.$$

To carry out an induction of length μ to define the desired sequences, we only need the following lemma.

8.28 Lemma. *Assume $A, B \in N_{\alpha,\nu}$ for some $\nu \in T^0$, $B \subseteq S$, $\rho < \kappa$, and f is homogeneous of color i on $(B \cup \{\alpha\}) \times A$. Then the following two statements hold.*

1. *There is a $C \in [\kappa - (A \cup B)]^\rho$ with $C \subseteq \bigcap\{f^\downarrow(\beta; i) : \beta \in B \cup \{\alpha\}\}$ so that for some $\nu' \in T^0$ with $\kappa_{\nu'} > \nu$, one has $C \in N_{\alpha,\nu'}$.*

2. *There is a $D \in [S - (A \cup B)]^\rho$ with $A \subseteq \bigcap\{f^\downarrow(\beta; i) : \beta \in D\}$ so that for some $\nu' \in T^0$ with $\kappa_{\nu'} > \nu$, one has $D \in N_{\alpha,\nu'}$.*

Proof. For the first item, choose $\nu' \in T^0$ with $\nu' > \nu$ and $\kappa_{\nu'} > \rho$. By the definition of S, we know $f(\beta, \varphi(\nu')) = i$ for $\beta \in B \cup \{\alpha\}$. By the Main Claim 8.27, we know that $f^\downarrow(\alpha; i) \cap X_{\alpha,\nu'} \notin I_{\alpha,\nu'}$. Let $Z = \bigcap\{f^\downarrow(\beta; i) : \beta \in B\}$. Then $Z \in N_{\alpha,\nu'}$ and $\varphi(\nu') \in Z$. Hence $|Z \cap f^\downarrow(\alpha; i) \cap X_{\alpha,\nu'}| \geq \rho$ by Lemma 8.25, and we can choose a subset of this intersection for C.

For the second item, the set $Z := \bigcap\{f^\uparrow(\eta; i) : \eta \in A\}$ is in $N_{\alpha,\nu}$ and $\alpha \in Z$. Since $S \cap \alpha \notin I_\alpha$, we can choose a suitable D by Facts 8.21. $\dashv$

9. Countable Ordinal Resources

9.1. Some History

In this section we look at ordinal partition relations of the form $\alpha \to (\beta, m)^2$ for limit ordinals α and β of the same cardinality. The goal m will be taken to be finite, since if $\pi : \alpha \to |\alpha|$ is a one-to-one mapping, then the partition defined on pairs $x < y < \alpha$ by

$$f(x, y) = \begin{cases} 0, & \text{if } x < y \text{ and } \pi(x) < \pi(y), \\ 1, & \text{if } x < y \text{ and } \pi(x) > \pi(y) \end{cases}$$

shows that $\alpha \not\to (|\alpha| + 1, \omega)^2$.

This particular branch of the partition calculus dates back to the 1950's, in particular to the seminal paper of Erdős and Rado [17] which introduced the partition calculus for linear order types and to the paper of Ernst Specker [62], in which he proves the following theorem.

9.1 Theorem (Specker [62]). *The following partition relations hold:*

1. *$\omega^2 \to (\omega^2, m)^2$ for all $m < \omega$;*

2. *$\omega^n \not\to (\omega^n, 3)^2$ for all $3 \leq n < \omega$.*

The finite powers of ω are all *additively indecomposable* (AI), since they cannot be written as the sum of two strictly smaller ordinals. It is well-known that the additively indecomposable ordinals are exactly those of the form ω^γ (see Exercise 5 on page 43 of Kunen [36]). We will focus on additively indecomposable α and β. There are additional combinatorial complications for decomposable ordinals.

For notational convenience in discussions of $\alpha \to (\beta, m)^2$, call α the *resource*, β the 0-*goal* and m the 1-*goal*.

For a specified countable 0-goal β and finite 1-goal m, it is possible to determine an upper bound for the resource α needed to ensure that the positive partition relation holds. In particular, Erdős and Milner showed $\omega^{1+\mu m} \to (\omega^{1+\mu}, 2^m)^2$. This result dates back to 1959 and a proof appeared in Milner's thesis in 1962. See also pages 165–168 of [65] where the proof is given via the following stepping up result:

9.2 Theorem. *Suppose γ, δ are countable and k is finite. If $\omega^\gamma \to (\omega^{1+\delta}, k)^2$, then $\omega^{\gamma+\delta} \to (\omega^{1+\delta}, 2k)^2$.*

9.3 Corollary (Erdős and Milner [14]). *If $m < \omega$ and $\mu < \omega_1$, then $\omega^{1+\mu\cdot\ell} \to (\omega^{1+\mu}, 2^\ell)^2$.*

The partition calculus for finite powers of ω is largely understood via the results below of Nosal. Her work built on Corollary 9.3 and earlier work by Galvin (unpublished), Hajnal, Haddad and Sabbagh [24], Milner [42].

9.4 Theorem (Nosal [46, 47]).

1. *If $1 \le \ell < \omega$, then $\omega^{2+\ell} \to (\omega^3, 2^\ell)^2$ and $\omega^{2+\ell} \not\to (\omega^3, 2^\ell + 1)^2$.*
2. *If $1 \le \ell < \omega$ and $4 \le r < \omega$, then $\omega^{1+r\cdot\ell} \to (\omega^{1+r}, 2^\ell)^2$ and $\omega^{r+r\cdot\ell} \not\to (\omega^{1+r}, 2^\ell + 1)^2$.*

Some progress has been made for the case in which the goal is ω^4. Nosal showed in her thesis that $\omega^6 \not\to (\omega^4, 3)^2$, which is sharp, since $\omega^7 \to (\omega^4, 4)^2$ by Corollary 9.3. Darby (unpublished) has shown that $\omega^9 \not\to (\omega^4, 5)^2$.

9.2. Small Counterexamples

In this section we look at partition relations of the form $\alpha \not\to (\alpha, m)^2$ for limit ordinals α and $m < \omega$.

In the previous section, we noted that Specker proved that $\omega^n \not\to (\omega^n, 3)^2$. In the 1970's, Galvin used *pinning*, defined below, to exploit the counterexample $\omega^3 \not\to (\omega^3, 3)^2$ to the full.

9.5 Definition. Suppose α and β are ordinals. A mapping $\pi : \alpha \to \beta$ is a *pinning map of α to β* if $\mathrm{ot}(X) = \alpha$ implies $\mathrm{ot}(\pi``X) = \beta$ for all $X \subseteq \alpha$. We say *α can be pinned to β*, in symbols, $\alpha \to \beta$, if there is a pinning map of α to β.

9.6 Theorem (Galvin [22]). *For all countable ordinals $\beta \geq 3$, if β is not* AI *and $\alpha = \omega^\beta$, then $\alpha \not\to (\alpha, 3)^2$.*

The first countable ordinal not covered by the Specker and Galvin results mentioned so far is ω^ω. Chang showed that $\omega^\omega \to (\omega^\omega, 3)^2$ and Milner modified his proof to work for all $m < \omega$.

9.7 Theorem (Chang [5]; Milner; see also [38, 65]). *For all $m < \omega$,*

$$\omega^\omega \to (\omega^\omega, m)^2.$$

Chang's original manuscript was about 90 pages long, and he received \$250 from Erdős for this proof, one of the largest sums Erdős had paid to that time. Erdős continued to focus attention on partition relations of the form $\alpha \to (\alpha, m)^2$ through offering money. In 1985, he [11] offered \$1000 for a complete characterization of those countable α for which $\alpha \to (\alpha, 3)^2$.

9.8 Definition. Any ordinal α can be uniquely written as the sum of AI ordinals, $\alpha = \alpha_0 + \cdots + \alpha_k$ with $\alpha_0 \geq \cdots \geq \alpha_k$. This sum is called the *additive normal form* (ANF) of α, and in this case, we say *the* ANF *of α has $k+1$ summands.* The summand α_k is called the *final summand.* The *initial part of the* ANF *of α* is $\alpha_0 + \cdots + \alpha_{k-1}$ if $k > 0$ and, for notational convenience, is 0 if α is AI.

An AI ordinal α is *multiplicatively indecomposable* (MI) if it is cannot be written as a product $\gamma \cdot \delta$ where γ, δ are AI and $\alpha > \gamma \geq \delta$. Any AI ordinal α can be written uniquely as a product of MI ordinals $\alpha = \alpha_0 \cdot \cdots \cdot \alpha_k$ with $\alpha_0 \geq \cdots \geq \alpha_k$. This product is called the *multiplicative normal form* (MNF) of α, and in this case, we say *the* MNF *of α has $k+1$ factors.* The factor $\widehat{\alpha} := \alpha_k$ is called the *final factor.* The *initial part of the* MNF *of α* is $\overline{\alpha} := \alpha_0 + \cdots + \alpha_{k-1}$ if $k > 0$ and, for notational convenience, is $\overline{\alpha} := 1$ if α is MI.

Note that if $\alpha = \omega^\beta$, then α is MI exactly when β is AI. Thus Galvin's result (Theorem 9.6) may be rephrased to say that for all countable ordinals $\alpha > \omega^2$, if α is not MI, then $\alpha \not\to (\alpha, 3)^2$. In the 1990's, Darby [7] and Schipperus [53, 51], working independently, came up with new families of counterexamples for MI ordinals α. Larson [39] built on their work to improve one of the results obtained by both of them.

9.9 Theorem.

1. (Darby) *If $\beta = \omega^{\alpha+1}$ and $m \to (4)^3_{2^{32}}$, then $\omega^{\omega^\beta} \not\to (\omega^{\omega^\beta}, m)^2$.*
2. (Darby; Schipperus; Larson) *If $\beta \geq \gamma \geq 1$, then $\omega^{\omega^{\beta+\gamma}} \not\to (\omega^{\omega^{\beta+\gamma}}, 5)^2$.*
3. (Darby; Schipperus) *If $\beta \geq \gamma \geq \delta \geq 1$, then $\omega^{\omega^{\beta+\gamma+\delta}} \not\to (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2$.*
4. (Schipperus) *If $\beta \geq \gamma \geq \delta \geq \varepsilon \geq 1$, then $\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2$.*

We plan to sketch a proof that there is some finite k so that $\omega^{\omega^2} \not\to (\omega^{\omega^2}, k)^2$, using the basic approach developed by Darby and some of his construction lemmas. Surprisingly, the partition counterexamples developed by Darby and Schipperus were the same, even if their approaches to uniformization were at least cosmetically different.

Rather than working directly with the ordinals, we use collections of finite increasing sequences from ω under the lexicographic ordering. Since our sequences are increasing, we will identify them with the set of their elements.

We write $\mathbf{s}^\frown\mathbf{t}$ for the concatenation of the two sequences under the assumption that the last element of $\mathbf{s}$ is smaller than the first element of $\mathbf{t}$, in symbols $\mathbf{s} < \mathbf{t}$.

We extend the notion of concatenation from individual sequences to sets of sequences by setting

$$S^\frown T := \{\mathbf{s}^\frown\mathbf{t} \mid \mathbf{s} \in S \wedge \mathbf{t} \in T \wedge \mathbf{s} < \mathbf{t}\}.$$

9.10 Definition. Define sets G_α for $\alpha = \omega^\ell$ by recursion on $1 \leq \ell < \omega$.

$$G_\omega := \{\langle m\rangle^\frown\langle k_1, k_2, \dots, k_m\rangle \mid m < k_1 < k_2 < \cdots < k_m < \omega\},$$

$$G_{\omega^{k+1}} := \bigcup\Big\{\{\langle m\rangle\}^\frown\overbrace{G_{\omega^k}{}^\frown \cdots {}^\frown G_{\omega^k}}^{m \text{ copies}} \mid m < \omega\Big\}.$$

Given a collection of sequences S and a particular sequence t, write $S(t) := \{s \in S \mid t \sqsubseteq s\}$ for the set of extensions of t in S.

9.11 Lemma. *For $1 \leq \ell, m, p < \omega$, $\mathrm{ot}(G_{\omega^\ell}(\langle m\rangle)) = (\omega^{\omega^{\ell-1}})^m$, $\mathrm{ot}(G_{\omega^\ell}) = \omega^{\omega^\ell}$, and*

$$\mathrm{ot}\Big(\overbrace{G_{\omega^\ell}{}^\frown \cdots {}^\frown G_{\omega^\ell}}^{p \text{ copies}}\Big) = (\omega^{\omega^\ell})^p.$$

Proof. First observe that $\mathrm{ot}(G_\omega(\langle m\rangle)) = \omega^m$ for all $1 \leq m < \omega$ and $\mathrm{ot}(G_\omega) = \omega^\omega$. Next notice that for subsets S and $T \subseteq [\omega]^{<\omega}$ which have indecomposable order types and which have arbitrarily large first elements, the order type of the concatenation $S^\frown T$ is the product of the order types $\mathrm{ot}(T) \cdot \mathrm{ot}(S)$. Then use induction on ℓ, m, and p. ⊣

9.12 Remark. Darby [7, Definition 2.8] defines G_α for all $\alpha < \omega_1$ so that $\mathrm{ot}(G_\alpha) = \omega^\alpha$ using a nice ladder system to assign to each limit ordinal an increasing cofinal sequence of type ω. In particular, for $\alpha = \overline{\alpha} \cdot \omega$ where $\overline{\alpha}$ is an AI ordinal, the cofinal sequence is $\alpha_m = \overline{\alpha} \cdot m$.

Our main interest is in G_α for α AI. We defined G_{ω^k} for $k < \omega$ in Definition 9.10. If $\alpha = \overline{\alpha} \cdot \omega$ where $\overline{\alpha}$ is an AI ordinal, then

$$G_\alpha = \bigcup\Big\{\{\langle m\rangle\}^\frown\overbrace{G_{\overline{\alpha}}{}^\frown \cdots {}^\frown G_{\overline{\alpha}}}^{m \text{ copies}} \mid m < \omega\Big\}.$$

If $\alpha \geq \omega^\omega$ is an AI ordinal not of the form $\alpha = \overline{\alpha}\cdot\omega$, then the cofinal sequence is a strictly increasing sequence $\langle \alpha_m : m < \omega\rangle$ of AI ordinals and G_α is the union of $\{\langle m\rangle\}^\frown G_{\alpha_m}$.

Recall we write $\mathbf{s} \sqsubseteq \mathbf{t}$ to indicate that $\mathbf{s}$ is an *initial segment* of $\mathbf{t}$, and $\mathbf{s} \sqsubset \mathbf{t}$ to indicate it is a *proper initial segment.*

9.13 Definition. For any collection of increasing sequences $S \subseteq [\omega]^{<\omega}$, let S^* denote the collection of initial segments of elements of S. For any $\mathbf{s} \in S^*$, let $S(\mathbf{s}) := \{\mathbf{t} \in S \mid \mathbf{s} \sqsubseteq \mathbf{t}\}$ be the set of all extensions of $\mathbf{s}$ that are in S.

9.14 Definition (See Definition 3.1 of [7]). Suppose $\omega < \alpha = \overline{\alpha}\cdot\widehat{\alpha} < \omega_1$ is AI but not MI with initial part $\overline{\alpha}$ and final factor $\widehat{\alpha}$. Call a non-empty sequence $\mathbf{p} \in G^*_\alpha$ a *level prefix of* G_α if $\mathrm{ot}(G_\alpha(\mathbf{p})) = \omega^\gamma$ where the final summand in the ANF of γ is $\overline{\alpha}$.

The next lemma is of particular interest when $\mathbf{s}$ is a level prefix.

9.15 Lemma (See Lemma 2.9 of [7]). *Suppose $\gamma \leq \alpha < \omega_1$ where the* ANF *of γ is $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_k$ for $k > 0$. Further suppose that $\mathbf{s} \in G^*_\alpha \setminus \{\emptyset\}$. If* $\mathrm{ot}(G_\alpha(\mathbf{s})) = \omega^\gamma$*, then $G_\alpha(\mathbf{s}) = \{\mathbf{s}\}^\frown G_{\gamma_k}{}^\frown \cdots {}^\frown G_{\gamma_0}$.*

Proof. We only prove this in the special case where $\alpha = \overline{\alpha}\cdot\omega$ and $\gamma = \overline{\alpha}\cdot n$. In this case, $\mathbf{s}$ has an extension in $G_\alpha(\langle m\rangle) = \{\langle m\rangle\}^\frown G_{\overline{\alpha}}{}^\frown \cdots {}^\frown G_{\overline{\alpha}}$ for $m = \min(\mathbf{s})$ by Definition 9.10 or Remark 9.12. Let $\mathbf{t} \sqsubseteq \mathbf{s}$ be the longest initial segment of $\mathbf{s}$ for which $G_\alpha(\mathbf{t})$ is the concatenation of $\{\mathbf{t}\}$ with some finite number of copies of $G_{\overline{\alpha}}$. There must be such a $\mathbf{t}$ since $\langle m\rangle$ has this property. If $\mathbf{s} = \mathbf{t}$, then we are done. So assume by way of contradiction that $\mathbf{u} = \mathbf{s} \setminus \mathbf{t} \neq \emptyset$. By the maximality of $\mathbf{t}$, it follows that $\mathbf{u} \in G^*_{\overline{\alpha}} \setminus G_{\overline{\alpha}}$. Since $\mathbf{u} \neq \emptyset$, $G_{\overline{\alpha}}(\mathbf{u})$ has order type δ for some $\delta < \omega^{\overline{\alpha}}$ with $\delta > 1$. Let r be the number of copies of $G_{\overline{\alpha}}$ in the decomposition of $G_\alpha(\mathbf{t})$. If $r = 1$, then $G_\alpha(\mathbf{s}) = \{\mathbf{t}\}^\frown G_{\overline{\alpha}}(\mathbf{u})$ has order type $\delta < \omega^{\overline{\alpha}}$. If $r > 1$, then $G_\alpha(\mathbf{s})$ is the concatenation of $\{\mathbf{t}\}^\frown G_{\overline{\alpha}}(\mathbf{u})$ with $r-1$ copies of $G_{\overline{\alpha}}$, so has order type $\omega^{\overline{\alpha}\cdot(r-1)}\cdot\delta$, by the argument of Lemma 9.11. In both cases, since $\delta \neq 1$ and $\delta \neq \omega^{\overline{\alpha}}$, we have a contradiction to the assumption that $\mathrm{ot}(G_\alpha(\mathbf{s})) = \omega^{\overline{\alpha}\cdot n}$. $\dashv$

9.16 Definition (See Definition 3.1 of [7]). Suppose the MNF of $\alpha < \omega_1$ has at least four factors. Call $\mathbf{t} \in G^*_\alpha$ a *sublevel prefix of* G_α if there are a level prefix $\mathbf{p}$ for G_α and a level prefix $\mathbf{q}$ for $G_{\overline{\alpha}}$ so that $\mathbf{t} = \mathbf{p}^\frown\mathbf{q}$. Call $\mathbf{u} \in G^*_\alpha$ a *sub-sublevel prefix of* G_α if there are a sublevel prefix $\mathbf{t}$ for G_α and a level prefix $\mathbf{r}$ for $G_{\overline{\overline{\alpha}}}$ so that $\mathbf{u} = \mathbf{t}^\frown\mathbf{r}$.

If we look at a pair $s \leq_{\mathrm{lex}} t$ from G_α, if s and t are disjoint as sets, then they partition one another into convex segments. That is, s and t can be expressed as concatenations, $s = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{n-1}({}^\frown s_n)$ and $t = t_0{}^\frown t_1{}^\frown \cdots {}^\frown t_{n-1}$ where $s_0 < t_0 < s_1 < t_1 < \cdots < s_{n-1} < t_{n-1} (< s_n)$.

The next definition uses Definition 9.16 to identify four types of segments used in the proofs of the negative partition relations 2–4 of Theorem 9.9.

9.17 Definition. Suppose the MNF of $\alpha < \omega_1$ has at least four factors. Further suppose that $\mathbf{s} \in G_\alpha$ has been decomposed into a convex partition $\mathbf{s} = \mathbf{s}_0{}^\frown\mathbf{s}_1{}^\frown \cdots {}^\frown\mathbf{s}_n$ where $\mathbf{s}_0 < \mathbf{s}_1 < \cdots < \mathbf{s}_n$.

1. Call $\mathbf{s}_i$ a $\square$-*segment of* $\mathbf{s}$ if $i = 0$ or $i = n$ or there are a level prefix $\mathbf{t}$ of G_α and $\mathbf{a} \in G_{\overline{\alpha}}$ so that $\mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{t} \sqsubset \mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{t}^\frown\mathbf{a}$.

2. Call $\mathbf{s}_i$ a $\triangle$-*segment of* $\mathbf{s}$ if it is not a $\square$-segment of $\mathbf{s}$ and there are a sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{b} \in G_{\overline{\overline{\alpha}}}$ so that $\mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{u} \sqsubset \mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{u}^\frown\mathbf{b}$.

3. Call $\mathbf{s}_i$ a $\blacksquare$-*segment of* $\mathbf{s}$ if it is not a $\square$ or $\triangle$-segment of $\mathbf{s}$ and there are a sub-sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{c} \in G_{\overline{\overline{\overline{\alpha}}}}$ so that $\mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1} \sqsubset \mathbf{v} \sqsubset \mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubset \mathbf{v}^\frown\mathbf{c}$.

4. Call $\mathbf{s}_i$ a $\bullet$-*segment of* $\mathbf{s}$ there are a sub-sublevel prefix $\mathbf{u}$ of G_α and $\mathbf{c} \in G_{\overline{\overline{\overline{\alpha}}}}$ so that $\mathbf{v} \sqsubset \mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1}$ and $\mathbf{s}_0{}^\frown \cdots {}^\frown\mathbf{s}_{i-1}{}^\frown\mathbf{s}_i \sqsubseteq \mathbf{v}^\frown\mathbf{c}$.

For simplicity, we include an example for which only $\square$-segments are needed to illustrate the technique. We have chosen to give an example that is easy to discuss rather than an optimal one.

9.18 Proposition. *The following partition relation holds:* $\omega^{\omega^2} \not\to (\omega^{\omega^2}, 6)^2$.

The remainder of this section is devoted to the proof of Proposition 9.18. We define a graph Γ on $G = G_{\omega_2}$ below. Then in Lemma 2, we show it has no 1-homogeneous set of size 6. After considerably more work, in Lemma 9.31, we show it has no 0-homogeneous subset of order type ω^{ω^2}. These two lemmas complete the proof.

9.19 Definition. Let $G = G_{\omega^2}$. Call a coordinate x of $\mathbf{x} \in G$ a *box coordinate* if it is either the minimum or the maximum of $\mathbf{x}$ or if $x = \min(\mathbf{x} - \mathbf{p})$ for some level prefix $\mathbf{p} \sqsubseteq \mathbf{x}$. Define a graph $\Gamma : [G]^2 \to 2$ by $\Gamma(\mathbf{x}, \mathbf{y}) = 1$ if and only if there are convex partitions

$$\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4 \quad \text{and} \quad \mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3$$

with $X_0 < Y_0 < X_1 < Y_1 < X_2 < Y_2 < X_3 < Y_3 < X_4$ so that all of X_0, X_2, X_4 are $\square$-segments of $\mathbf{x}$, Y_0, Y_3 are $\square$-segments of $\mathbf{y}$, and none of X_1, X_3, Y_1, Y_2 have box coordinates of $\mathbf{x}$, $\mathbf{y}$, respectively.

For notational convenience, let $\gamma^-(\mathbf{x}, \mathbf{y}) = \max(Y_1)$, $\gamma^+(\mathbf{x}, \mathbf{y}) = \min(Y_2)$, $\delta^-(\mathbf{x}, \mathbf{y})$ be the largest box coordinate of Y_0, and $\delta^+(\mathbf{x}, \mathbf{y})$ be the smallest box coordinate of Y_3. The graphical display below shows how the two sequences are interlaced and which have box coordinates if $\Gamma(\mathbf{x}, \mathbf{y}) = 1$.

$$\begin{array}{ccccccccc} \boxed{X_0} & & X_1 & & \boxed{X_2} & & X_3 & & \boxed{X_4} \\ & \boxed{Y_0} & & Y_1 & & Y_2 & & \boxed{Y_3} & \end{array}$$

9.20 Lemma. *The graph* Γ *has no* 1*-homogeneous set of size six.*

Proof. The proof starts with a series of claims which delineate basic properties of the partition.

Claim A. Suppose $\mathbf{x} < \mathbf{y}$, $\Gamma(\mathbf{x}, \mathbf{y}) = 1$.

1. There is a box coordinate $x \in \mathbf{x}$ with $\min(\mathbf{y}) < x < \max(\mathbf{y})$.

2. For any box coordinate $x \in \mathbf{x}$ with $\min(\mathbf{y}) < x < \max(\mathbf{y})$, the inequalities $\gamma^-(\mathbf{x}, \mathbf{y}) < x < \gamma^+(\mathbf{x}, \mathbf{y})$ hold.

3. There is no sequence $x < y < x' \in \mathbf{x}$ where $\min(\mathbf{y}) < x \in \mathbf{x}$, $x' < \max(\mathbf{y})$ and y is a box coordinate of $\mathbf{y}$.

Proof. Use the diagram above to verify these basic properties. $\dashv$

Claim B. Suppose $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}_< \subseteq G$ is 1-homogeneous for Γ. If $\Box x \in \mathbf{x}$, $\Box y \in \mathbf{y}$, and $\Box z \in \mathbf{z}$ are box coordinates and $\min(\mathbf{z}) < \Box x, \Box y < \max(\mathbf{z})$, then either $\Box x, \Box y < \Box z$ or $\Box z < \Box x, \Box y$.

Proof. Suppose the hypothesis holds but the conclusion fails. Then either (a) $\Box x < \Box z < \Box y$ or (b) $\Box y < \Box z < \Box x$. Note that $\min(\mathbf{y}) < \min(\mathbf{z}) < \Box x$ and $\Box x < \max(\mathbf{z}) < \max(\mathbf{y})$, since $\mathbf{y} < \mathbf{z}$. By Claim A(2), $\gamma^-(\mathbf{x}, \mathbf{y}) < \Box x < \gamma^+(\mathbf{x}, \mathbf{y})$. Use the definition of Γ to find $x^-, x^+ \in \mathbf{x}$ such that $\delta^-(\mathbf{x}, \mathbf{y}) < x^- < \gamma^-(\mathbf{x}, \mathbf{y})$ and $\gamma^+(\mathbf{x}, \mathbf{y}) < x^+ < \delta^+(\mathbf{x}, \mathbf{y})$. If (a) holds, then either $\Box x < \Box z < x^+$ or $\gamma^+(\mathbf{x}, \mathbf{y}) < \Box z < \Box y$ is a sequence that contradicts Claim A(3). If (b) holds, then either $\Box y < \Box z < \gamma^-(\mathbf{x}, \mathbf{y})$ or $x^- < \Box z < \Box x$ is a sequence that contradicts Claim A(3). Thus the above claim follows. $\dashv$

Claim C. Suppose $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}_< \subseteq G$ is 1-homogeneous for Γ. If $\Box x \in \mathbf{x}$, $\Box y \in \mathbf{y}$ are box coordinates with $\min(z) < \Box x, \Box y < \max(z)$, then some coordinate z of $\mathbf{z}$ lies between $\Box x$ and $\Box y$.

Proof. For the first case, suppose $\Box x < \Box y$. In this case, let $z = \gamma^+(\mathbf{x}, \mathbf{z})$. Then $z \in \mathbf{z}$ and by Claim A, $\Box x < z$. By definition of Γ, there is some $x' \in \mathbf{x}$ with $z < x' < \max(\mathbf{z})$. Since $\mathbf{y} < \mathbf{z}$, it follows that $x' < \max(\mathbf{y})$, so $x' < \delta^+(\mathbf{x}, \mathbf{y}) \leq \Box y$. By transitivity, $\Box x < z < \Box y$. The second case for $\Box y < \Box x$ is left to the reader with the hint that $z = \gamma^-(\mathbf{x}, \mathbf{z})$ works. $\dashv$

Now prove the lemma from the claims. Assume by way of contradiction that $U = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}\}_< \subseteq G$ is 1-homogeneous for Γ. Use Claim A to choose box coordinates $\varepsilon_0 \in \mathbf{a}$, $\varepsilon_1 \in \mathbf{b}$, $\varepsilon_2 \in \mathbf{c}$, $\varepsilon_3 \in \mathbf{d}$, $\varepsilon_4 \in \mathbf{e}$, so that $\min(\mathbf{f}) < \varepsilon_i < \max(\mathbf{f})$. Let $ijk\ell$ be a permutation of 0123 so that $\varepsilon_i < \varepsilon_j < \varepsilon_k < \varepsilon_\ell$. Use Claim C to choose coordinates $e', e'' \in \mathbf{e}$ and $f' \in \mathbf{f}$ with $\varepsilon_i < e' < \varepsilon_j < f' < \varepsilon_k < e'' < \varepsilon_\ell$. By Claim B, either (a) $\varepsilon_4 < \varepsilon_i$ or (b) $\varepsilon_\ell < \varepsilon_4$. Choose coordinate $f'' \in \mathbf{f}$ between ε_4 and the appropriate one of ε_i and ε_ℓ.

Let $\mathbf{x}, \mathbf{y} \in U$ be such that $\varepsilon_i \in \mathbf{x}$ and $\varepsilon_\ell \in \mathbf{y}$. By Claim A, $\delta^-(\mathbf{x}, \mathbf{f}) < \gamma^-(\mathbf{x}, \mathbf{f}) < \varepsilon_i$ and $\varepsilon_\ell < \gamma^+(\mathbf{y}, \mathbf{f}) < \delta^+(\mathbf{y}, \mathbf{f})$.

Let $\mathbf{e} = E_0{}^\frown E_1{}^\frown E_2{}^\frown E_3{}^\frown E_4$, $\mathbf{f} = F_0{}^\frown F_1{}^\frown F_2{}^\frown F_3$ be the partition that witnesses $\Gamma(\mathbf{e}, \mathbf{f}) = 1$. Note that $\varepsilon_4 \in E_2$.

If (a) holds, then $\delta^+(\mathbf{y}, \mathbf{f}) \in F_3$, and

$$\varepsilon_4 < f'' < e' < f' < e'' < \delta^+(\mathbf{y}, \mathbf{f}).$$

However, this inequality contradicts the definition of g, since there are only two blocks between E_2 and F_3. If (b) holds, then $\delta^-(\mathbf{x}, \mathbf{f}) \in F_0$, and

$$\delta^-(\mathbf{x}, \mathbf{f}) < e' < f' < e'' < f'' < \varepsilon_4.$$

This inequality also contradicts the definition of g, since there are only two blocks between F_0 and E_2. In either case we have reached the contradiction required to prove the lemma. ⊣

Now we turn to the task of showing that every subset $X \subseteq G$ of order type ω^{ω^2} includes a pair $\{\mathbf{x}, \mathbf{y}\}_< \subseteq X$ so that $\Gamma(\mathbf{x}, \mathbf{y}) = 1$. The first challenge is to guarantee that when we build a segment of one of $\mathbf{x}$ and $\mathbf{y}$, we will be able to extend it starting above the segment of the other that we will have constructed in the meanwhile. To that end, we introduce *β-prefixes* and *maximal β-prefixes.*

9.21 Definition. Suppose $\alpha < \omega_1$. Call a sequence $\mathbf{s} \in G^*_\alpha$ a *β-prefix* of $W \subseteq G_\alpha$ if $\mathrm{ot}(W(\mathbf{s})) = \beta$, and a *maximal β-prefix* if no proper extension is a β-prefix.

9.22 Lemma (Galvin; see Lemma 4.5 of [7]). *Suppose $\mathbf{s} \in G^*_\alpha$ and β is* AI. *If $W \subseteq G_\alpha$ has* $\mathrm{ot}(W(\mathbf{s})) \geq \beta$, *then there is an extension $\mathbf{t} \sqsupseteq \mathbf{s}$ so that $\mathbf{t}$ is a maximal β-prefix for W.*

The proof of the above lemma depends on the fact that the sequences in G_α are well-founded under extension. We use the next lemma for sequences $\mathbf{r}$ which are either maximal ω^2-prefixes or maximal ω^3-prefixes.

9.23 Lemma. *Suppose $\delta < \beta \leq \omega^\alpha$ for AI δ and β. Further suppose $W \subseteq G_\alpha$ and $\mathbf{r}$ is a maximal β-prefix for W. Then $\mathbf{r}$ has infinitely many one point extensions $r^\frown\langle p\rangle \in W^*$ with* $\mathrm{ot}(W(r^\frown\langle p\rangle)) \geq \delta$. *Also, for any sequence $\mathbf{s}$, there is a sequence $\mathbf{t}$ so that $\mathbf{s} < \mathbf{t}$, $\mathbf{r}^\frown\mathbf{t} \in W^*$, and $\mathbf{r}^\frown\mathbf{t}$ is a maximal δ-prefix for W.*

Proof. Since $\mathbf{r}$ is a maximal β-prefix for W, $\mathrm{ot}(W(\mathbf{r}^\frown\langle p\rangle)) < \beta$ for all $p < \omega$. Consequently, since β is AI, it follows that $\sum_{q<p<\omega} \mathrm{ot}(W(\mathbf{r}^\frown\langle p\rangle)) = \beta$ for all $q < \omega$. Since $\sum_{q<p<\omega} \gamma_p \leq \delta$ if each $\gamma_p < \delta$, it follows that for infinitely many $p < \omega$, $W(\mathbf{r}^\frown\langle p\rangle)$ has order type $\geq \delta$. Thus given $\mathbf{s}$, there is a $p > \max(\mathbf{s})$ with $\mathrm{ot}(W(\mathbf{r}^\frown\langle p\rangle)) \geq \delta$. In particular, $W(\mathbf{r}^\frown\langle p\rangle) \neq \emptyset$. To complete the proof, apply Lemma 9.22 to get $\mathbf{t} \sqsupseteq \langle p\rangle$ so that $\mathbf{r}^\frown\mathbf{t}$ is a maximal δ-prefix. ⊣

In our construction of $\mathbf{x}$, $\mathbf{y}$, we must be able to iterate the process of extending to a level prefix. To that end, we introduce the notion of *levels.*

9.24 Definition (See Definition 5.2 of [7]). Suppose α is AI but not MI and $\mathbf{q}$ is a level prefix of G_α. The *level of W prefixed by* $\mathbf{q}$ is the set

$$L(W, \mathbf{q}) := \{\mathbf{a} \in G_{\overline{\alpha}} \mid W(\mathbf{q}^\frown \mathbf{a}) \neq \emptyset\}.$$

A non-empty sequence $\mathbf{s} \in G^*_\alpha - G_\alpha$ *ends in the level of W prefixed by* $\mathbf{q}$ if there is some $\mathbf{a} \in L(W, \mathbf{q})$ so that $\mathbf{q} \sqsubseteq \mathbf{s} \sqsubset \mathbf{q}^\frown \mathbf{a}$.

Next we state without proof a series of lemmas from Darby [7] that lead up to Lemma 9.29. The interested reader can fill in the proofs for the case where $\alpha = \omega^\ell < \omega^\omega$.

9.25 Lemma (See Lemmas 4.6, 4.7 of [7]). *Suppose $\delta \leq \gamma \leq \alpha < \omega_1$, where δ, γ are* AI *and $\gamma \cdot \delta \leq \alpha$. If $\mathbf{s} \in G^*_\alpha$ is a maximal $\gamma \cdot \delta$-prefix for $W \subseteq G_\alpha$, then the following set has order type δ:*

$$W(\beta, \mathbf{s}) := \{\mathbf{p} \in G^*_\alpha \mid \mathbf{s} \sqsubseteq \mathbf{p} \textit{ and } \mathbf{p} \textit{ is a maximal } \gamma\textit{-prefix for } W\}.$$

9.26 Lemma (See Lemma 5.5 of [7]). *Suppose $\alpha < \omega_1$ is* AI *but not* MI*, $\mathbf{q}$ is a level prefix of G_α and $W \subseteq G_\alpha$. If $\mathbf{s}$ ends in level $L(W, \mathbf{q})$ and* $\operatorname{ot}(W(\mathbf{s})) \geq \omega^{\overline{\alpha} \cdot n}$*, then for any $\gamma < \overline{\alpha}$, there is an $\mathbf{a} \in L(W, \mathbf{q})$ so that $\mathbf{s} \sqsubset \mathbf{q}^\frown \mathbf{a}$ and* $\operatorname{ot}(W(\mathbf{q}^\frown \mathbf{a})) \geq \omega^{\overline{\alpha}(n-1)+\gamma}$*.*

9.27 Lemma (See Lemma 5.6 of [7]). *Suppose $\alpha < \omega_1$ is* AI *but not* MI*, $W \subseteq G_\alpha$ and every level of W has order type $\leq \omega^\delta$. If $\mathbf{s} \in G^*_\alpha$ and* $\operatorname{ot}(G_\alpha(\mathbf{s})) = \omega^{\overline{\alpha} \cdot \beta}$*, then* $\operatorname{ot}(W(\mathbf{s})) \leq \omega^{\delta \cdot \beta}$*.*

9.28 Lemma (See Lemma 5.7 of [7]). *Suppose $\alpha < \omega_1$ is* AI *but not* MI*, $W \subseteq G_\alpha(\langle m \rangle)$ and* $\operatorname{ot}(W) > \omega^\gamma$*. Then for any δ so that $\delta \cdot m < \gamma$, there is a level of W of order type $> \omega^\delta$.*

The following lemma of Darby, mildly rephrased since the general definition of G_α has been omitted, is the key to constructing pairs 1-colored by any generalization of the graph Γ to a Γ_α defined for $\alpha = \overline{\alpha} \cdot \omega$, since it allows one to plan ahead: one takes a sufficiently large set, thins it to something tractable, dives into a large level to work within, knowing that on exit from the level, one will have a large enough set of extensions to continue according to plan.

9.29 Lemma (See Lemma 5.9 of [7]). *Suppose α is* AI *but not* MI*, $0 < m < \omega$ and* $\operatorname{ot}(G_\alpha(\langle m \rangle)) = \omega^{\overline{\alpha} \cdot \beta}$*. Further suppose $W \subseteq G_\alpha(\langle m \rangle)$ and* $\operatorname{ot}(W) \geq \omega^{\overline{\alpha} \cdot n + \varepsilon}$ *where $\varepsilon \leq \overline{\alpha}$ and $0 < n < \omega$, and assume δ is such that $\delta \cdot \beta < \varepsilon$. Then there is a set $U \subseteq W$ and a level prefix $\mathbf{q}$ so that $U = U(\mathbf{q})$,* $\operatorname{ot}(L(U, \mathbf{q})) > \omega^\delta$ *and* $\operatorname{ot}(U(\mathbf{q}^\frown \mathbf{a})) \geq \omega^{\overline{\alpha} \cdot (n-1)+\varepsilon}$ *for all $\mathbf{a} \in L(U, \mathbf{q})$.*

Here our focus is on ω^{ω^k} for finite k, that is, on $\alpha = \omega^k$. In this case, $G_\alpha(\langle m \rangle)$ has order type $\omega^{\omega^{k-1} \cdot m}$, so the β of the previous lemma is simply m. The following weaker version of the above lemma suffices for our purposes.

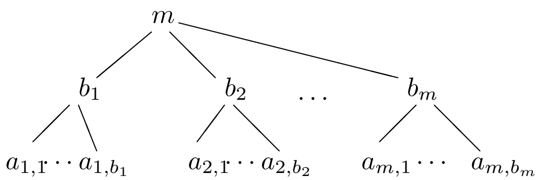
9.30 Lemma. *Suppose* $\alpha = \overline{\alpha} \cdot \omega$ *is* AI *but not* MI, $0 < n \leq m < \omega$, *and* $W \subseteq G_\alpha(\langle m \rangle)$ *has order type* $\geq \omega^{\overline{\alpha} \cdot n}$. *Further assume* δ *is such that* $\delta \cdot m < \overline{\alpha}$. *Then there is a set* $U \subseteq W$ *and a level prefix* $\mathbf{q}$ *so that* $U = U(\mathbf{q})$, $\mathrm{ot}(L(U, \mathbf{q})) > \omega^\delta$ *and* $\mathrm{ot}(U(\mathbf{q}^\frown \mathbf{a})) \geq \omega^{\overline{\alpha} \cdot (n-1)}$ *for all* $\mathbf{a} \in L(U, \mathbf{q})$.

9.31 Lemma. *Suppose* $W \subseteq G_{\omega^2}$ *has order type* ω^{ω^2}. *Then there is a pair* $\mathbf{x}$, $\mathbf{y}$ *from* W *so that* $\Gamma(\mathbf{x}, \mathbf{y}) = 1$.

Proof. We revisit the set G_{ω^2} to better understand how it is constructed by unraveling the recursive construction. A typical element σ is

$$\langle m \rangle^\frown \langle b_1 \rangle^\frown \langle a_1^1, \ldots, a_{b_1}^1 \rangle^\frown \langle b_2 \rangle^\frown \langle a_1^2, \ldots, a_{b_2}^2 \rangle \cdots {}^\frown \langle b_m \rangle^\frown \langle a_1^m, \ldots, a_{b_m}^m \rangle.$$

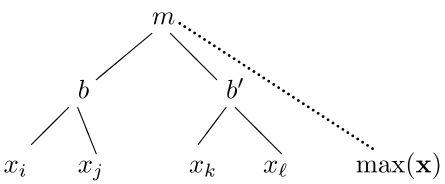
Notice that the initial element, m, tells how many levels there will be, and each level starts with a box coordinate, b_i, which determines the order type of the level, ω^{b_i}. To make the identification of the various types of elements visually immediate, we fold the sequence σ into a tree, with the initial element at the top, the box coordinates as immediate successors, and the remaining coordinates as terminal nodes. To rebuild the sequence from the tree, one walks through the tree in depth first, left-to-right order.

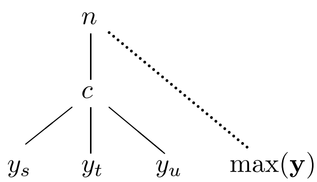


Use Lemmas 9.22, 9.23, and 9.30 to build $\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4$ and $\mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3$ one convex segment at a time so that

$$X_0 < Y_0 < X_1 < Y_1 < X_2 < Y_2 < X_3 < Y_3 < X_4.$$

For notational convenience, we plan to let $i < j < k < \ell$ be such that $\max(X_0) = x_i$, $\max(X_1) = x_j$, $\max(X_2) = x_k$, $\max(X_3) = x_\ell$. Similarly, we plan to let $s < t < u$ be such that $\max(Y_0) = y_s$, $\max(Y_1) = y_t$, $\max(Y_2) = y_u$. In addition it will be convenient to write b for the largest box coordinate of X_0, b' for the largest box coordinate of X_2, and $c = \delta^-(\mathbf{x}, \mathbf{y})$ for the largest box coordinate of Y_0. Here is a pair of subtrees of the trees we get by folding the sequences we build for $\mathbf{x}$ and $\mathbf{y}$, that include only the critical coordinates named above, together with $\max(\mathbf{x})$, $\max(\mathbf{y})$. These subtrees highlight the relationships between the critical coordinates, and allow one to see at a glance which of the segments are $\square$-segments.





Observe that since G is the union of $G(\langle 0\rangle), G(\langle 1\rangle), G(\langle 2\rangle), \ldots$, it follows that for $\beta < \omega^2$, there are infinitely many $m_\beta < \omega$ with $\mathrm{ot}(W \cap G(\langle m_\beta\rangle)) \geq \omega^\beta$. We start our construction by choosing m so that $U_0 := W \cap G(\langle m\rangle)$ has order type at least $\omega^{\omega\cdot 4}$.

Next we apply Lemma 9.30 to find a set $U_1 \subseteq U_0$ and a level prefix $\mathbf{p}$ so that $U_1 = U_1(\mathbf{p})$, $\mathrm{ot}(L(U_1, \mathbf{p})) > \omega^5$, and $\mathrm{ot}(U_1(\mathbf{p}^\frown \mathbf{a})) \geq \omega^{\omega\cdot 3}$ for all $\mathbf{a} \in L(U_1, \mathbf{p})$. Apply Lemma 9.22 to get $\mathbf{u}$, a maximal ω^4 prefix in $L(U_1, \mathbf{p})$. Then $b = \min(\mathbf{u})$ is the box coordinate of our diagram. We set $X_0 = \mathbf{p}^\frown \mathbf{u}$ and note that $\max(\mathbf{u}) = x_i$ on our diagram.

Choose $n > x_i$ so that $V_0 := W \cap G(\langle n\rangle)$ has order type at least $\omega^{\omega\cdot 4}$. Continue as in the previous step. Use Lemma 9.30 to find a set $V_1 \subseteq V_0$ and a level prefix $\mathbf{q}$ so that $V_1 = V_1(\mathbf{q})$, $\mathrm{ot}(L(V_1, \mathbf{q})) > \omega^7$, and $\mathrm{ot}(V(\mathbf{q}^\frown \mathbf{a})) \geq \omega^{\omega\cdot 3}$ for all $\mathbf{a} \in L(V_1, \mathbf{q})$. Let $\mathbf{v}$ be a maximal ω^6 prefix in $L(V_1, \mathbf{q})$. Then $c = \min(\mathbf{v})$ is the box coordinate of our diagram. We set $Y_0 = \mathbf{q}^\frown \mathbf{v}$ and note that $\max(\mathbf{v}) = y_s$ on our diagram.

By Lemma 9.23, there is a sequence X_1 with $Y_0 < X_1$ so that $\mathbf{u}^\frown X_1$ is a maximal ω^3 prefix in $L(U_1, \mathbf{p})$. Note that $X_0{}^\frown X_1$ is not a level prefix nor is any one point extension.

By Lemma 9.23, there is a sequence Y_1 with $X_1 < Y_1$ so that $\mathbf{v}^\frown Y_1$ is a maximal ω^5 prefix in $L(V_1, \mathbf{q})$.

By Lemma 9.23, the sequence $\mathbf{u}^\frown X_1$ has infinitely many one point extensions in $L(U_1, \mathbf{p})^*$. By choosing a suitable one point extension and then extending it into $L(U_1, \mathbf{p})$, we find $\mathbf{w}$ so that $Y_1 < \mathbf{w}$ and $\mathbf{u}^\frown X_1{}^\frown \mathbf{w} \in L(U_1, \mathbf{p})$. By choice of U_1 and $\mathbf{p}$, we know $\mathrm{ot}(U_1(\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w}))) \geq \omega^{\omega\cdot 3}$. Use Lemma 9.30 to find $U_2 \subseteq U_1(\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w}))$ and a level prefix $\mathbf{p}'$ so that $U_2 = U_2(\mathbf{p}')$, $\mathrm{ot}(L(U_2, \mathbf{p}')) > \omega^5$, and $\mathrm{ot}(U_2(\mathbf{p}'^\frown \mathbf{a})) \geq \omega^{\omega\cdot 3}$ for all $\mathbf{a} \in L(U_2, \mathbf{p}')$. Then $\mathbf{p}^\frown(\mathbf{u}^\frown X_1{}^\frown \mathbf{w}) \sqsubseteq \mathbf{p}'$. Apply Lemma 9.22 to get $\mathbf{u}'$, a maximal ω^4 prefix in $L(U_2, \mathbf{p}')$. Then $b' = \min(\mathbf{u}')$ is another box coordinate in our diagram. Then $\mathbf{p}'^\frown \mathbf{u}'$ is not a level prefix of U_2, nor is any one point extension of it a level prefix. We set $X_2 = \mathbf{p}' \setminus (X_0{}^\frown X_1)$, and note that $\max(X_2) = \max(\mathbf{u}') = x_k$ on our diagram.

By Lemma 9.23, there is a sequence Y_2 with $X_2 < Y_2$ so that $\mathbf{v}^\frown Y_1{}^\frown Y_2$ is a maximal ω^4 prefix in $L(V_1, \mathbf{q})$.

By Lemma 9.23, there is a sequence X_3 with $Y_2 < X_3$ so that $\mathbf{u}'^\frown X_3$ is a maximal ω^3 prefix in $L(U_2, \mathbf{p}')$.

By Lemma 9.23, the sequence $\mathbf{v}^\frown Y_1{}^\frown Y_2$ has infinitely many one point extensions in $L(V_1, \mathbf{q})^*$. Hence by first choosing a suitable one point extension and then extending it into $L(V_1, \mathbf{q})$, and finally extending it into V_1, we can find Y_3 so that $X_3 < Y_3$ and $\mathbf{y} = Y_0{}^\frown Y_1{}^\frown Y_2{}^\frown Y_3 \in V_1 \subseteq W$.

By Lemma 9.23, the sequence $\mathbf{u}'^\frown X_3$ has infinitely many one point extensions in $L(U_2, \mathbf{p}')^*$. Hence by first choosing a suitable one point extension and then extending it into $L(U_2, \mathbf{p}')$, and finally extending it into U_2, we can find X_4 so that $Y_3 < X_4$ and $\mathbf{x} = X_0{}^\frown X_1{}^\frown X_2{}^\frown X_3{}^\frown X_4 \in U_2 \subseteq W$.

By construction, X_0, X_2, X_4 and Y_0, Y_3 are all $\square$-segments, while X_1, X_3 and Y_1, Y_2 have no box coordinates. Thus $\mathbf{x}, \mathbf{y}$ witnesses the fact that W is not a 0-homogeneous set for Γ. $\dashv$

Lemmas 9.20 and 9.31 show that Γ is a witness to $\omega^{\omega^2} \nrightarrow (\omega^{\omega^2}, 6)^2$. The coloring can easily be generalized to ω^{ω^α} where α is decomposable, since it was described using only box segments and segments without box coordinates. Hence the proof of Lemma 9.20 carries through for these generalizations. In the proof of Lemma 9.31, we have taken advantage of the fact that $\alpha = 2$ is a successor ordinal, but use of lemmas from Darby's paper allow one to modify the given construction suitably.

The proof of the previous lemma gives some evidence for the following remark.

9.32 Remark. We have the following heuristic for building pairs. Suppose σ is a list of specifications of convex segments detailing which have box, triangle, bar (or dot) coordinates and which do not. If the first two and last two segments are to be box segments, then for any ordinal α of sufficient decomposability for the description to make sense, there is a disjoint pair $\mathbf{x}, \mathbf{y} \in G_{\omega^\alpha}$ so that the sequence of convex segments they create fits the description.

For the actual construction, one needs to iterate the process of taking levels and look at the approach taken carefully.

10. A Positive Countable Partition Relation

The previous section focused on countable counterexamples. Here we survey positive ordinal partition relations of the form $\alpha \to (\alpha, m)^2$ for countable limit ordinals α and sketch the proof of one of them.

Darby [7] and Schipperus [53, 51] independently extended Chang's positive result for ω^ω and $m = 3$ to larger countable limit ordinals.

10.1 Theorem (Chang for $\beta = 1$ (see Theorem 9.7); Darby for $\beta = 2$ [7]; Schipperus for $\beta \geq 2$ [53]). *If the additive normal form of $\beta < \omega_1$ has one or two summands, then $\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, 3)^2$.*

Recall that Erdős [11] offered \$1000 for a complete characterization of the countable ordinals α for which $\alpha \to (\alpha, 3)^2$. It is not difficult to show that additively decomposable ordinals fail to satisfy this partition relation. Recall that additively indecomposable ordinals are powers of ω. Specker showed that finite powers of ω greater than ω^2 fail to satisfy it. Galvin showed (see Theorem 9.6) that additively decomposable powers of ω greater than ω^2 fail to satisfy it. Thus attention has been on indecomposable powers of ω, $\alpha = \omega^{\omega^\beta}$, that is, the countable ordinals that are multiplicatively indecomposable. Schipperus (see Theorem 9.9) showed that if the additive normal form of β has at least four summands, then $\alpha \not\to (\alpha, 3)^2$. Thus to complete the characterization of which countable ordinals α satisfy this partition relation it suffices to characterize it for ordinals of the form $\alpha = \omega^{\omega^\beta}$ where the additive normal form of β has exactly three summands. We list below the first open case.

10.2 Question. Does $\omega^{\omega^3} \to (\omega^{\omega^3}, 3)^2$?

In light of Theorem 9.9, Darby and Larson have completed the characterization of the set of $m < \omega$ for which $\omega^{\omega^2} \to (\omega^{\omega^2}, m)^2$ with the following result.

10.3 Theorem (Darby and Larson [8]). $\omega^{\omega^2} \to (\omega^{\omega^2}, 4)^2$.

We complete this subsection with a sketch of the Schipperus proof that $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)^2$, using somewhat different notation than he used originally. The sketch will be divided into seven subsections:

1. representation of ω^{ω^ω} as a collection $\mathcal{T}(\omega)$ of finite trees;
2. analysis of *node labeled* trees;
3. description of a two-player game $\mathcal{G}(h, N)$ for h a 2-partition of $\mathcal{T}(\omega)$ into 2 colors and $N \subseteq \omega$ infinite;
4. uniformization of play of the game $\mathcal{G}(h, N)$ via constraint on the second player to a *conservative style* of play determined by an infinite set $H \subseteq N$ and a bounding function b;
5. construction of a three element 1-homogeneous set when the first player has a winning strategy for all games in $\mathcal{G}(h, N)$ in which the second player makes conservative moves;
6. construction of an almost 0-homogeneous set of order type ω^{ω^ω} when the first player has no such strategy;
7. completion of the proof.

10.1. Representation

Recall that, by convention, we are identifying a finite set of natural numbers with the increasing sequence of its members. The trees we have in mind for our representation are subsets of $[\omega]^{<\omega}$ which are trees under the subset relation, and the subset relation is the same as the end-extension relation when the subsets are regarded as increasing sequences.

In the proof that the coloring Γ had no independent subset of order type ω^{ω^2}, we found it convenient to fold an element

$$\mathbf{x} = \langle m, n_1, a_1^1, \ldots, a_{n_1}^1, n_2, a_1^2, \ldots, a_{n_2}^2, \ldots, n_m, a_1^m, \ldots, a_{n_m}^m \rangle$$

of G_{ω^2} into a tree with root $\langle m\rangle$, immediate successors $\langle m, n_i\rangle$ and terminal nodes $\langle m, n_i, a^i_j\rangle$. Then we could walk through the tree, node by node, so that the maximum element of each node continually increased along the walk, just as the elements of $\mathbf{x}$ increase.

We already have representations of ω^{ω^β} from the previous section as sets of increasing sequences under the lexicographic ordering. The definition of those sets is recursive, so we fold these sets up into trees recursively. Specifically, the next definition uses the representations of G_{ω^β} detailed in Definition 9.10 and Remark 9.12.

10.4 Definition. Define by recursion on $\beta \le \omega$ a sequence of *folding maps*, $F_\beta : G_{\omega^\beta} \to \mathcal{T}$:

1. For $\tau = \langle k\rangle \in G_{\omega^0} = G_1$, set $F_0(\tau) := \{\langle k\rangle\}$.
2. For $\tau = \langle m\rangle^\frown\sigma_1{}^\frown\sigma_2{}^\frown\cdots{}^\frown\sigma_m \in G_{\omega^{n+1}}$, set
$$F_{n+1}(\tau) := \{\langle m\rangle\} \cup \bigcup\{\{\langle m\rangle\}^\frown F_n(\sigma_i) : 1 \le i \le m\}.$$
3. For $\tau = \langle m\rangle^\frown\sigma \in G_{\omega^\omega}$, set $F_\omega(\tau) := \{\langle m\rangle\} \cup \{\langle m\rangle\}^\frown F_m(\sigma)$.

Let $\mathcal{T}(\beta)$ be the range of F_β.

Prove the following lemmas by induction on β.

10.5 Lemma. *For each $\beta \le \omega$, the mapping F_β is one-to-one and $\tau = \bigcup F_\beta(\tau)$. Thus, $<_{\text{lex}}$ on G_{ω^β} induces an order $<$ on $\mathcal{T}(\beta)$.*

10.6 Lemma. *For all $\beta \le \omega$ and all infinite $H \subseteq \omega$, the collection of sequences in $G_{\omega^\beta} \cap [H]^{<\omega}$ has order type ω^{ω^β}, and hence so does the collection of trees in $\mathcal{T}(\beta, H) := \mathcal{T}(\beta) \cap \mathcal{P}([H]^{<\omega})$.*

Let $\mathcal{T}$ be the collection of all finite trees $(T, \sqsubseteq)$ of increasing sequences with the property that if $s, t \in T$ and as sets, $s \subseteq t$, then as sequences, $s \sqsubseteq t$. Identify each $t \in T \in \mathcal{T}$ with the set of its elements. Then $\sqsubseteq$ and $\subseteq$ coincide, so this identification permits one to use set operations on the nodes of T.

10.7 Lemma. *For all $\beta < \omega_1$, for all $T \in \mathcal{T}(\beta)$, the following conditions are satisfied:*

1. *(transitivity) $s \sqsubset t \in T$ implies $s \in T$;*
2. *(closure under intersection) for all $s, t \in T$, $s \cap t$ is an initial segment of both s and t;*
3. *(rooted) $(T, \sqsubseteq)$ is a rooted tree with $\emptyset \notin T$;*
4. *(node ordering) for all $s \neq t$ in T, exactly one of the following holds:*

(a) $s \sqsubset t$,

(b) $t \sqsubset s$,

(c) $s \leq_{\text{lex}} t$ *and* $s < t - (s \cap t)$,

(d) $t \leq_{\text{lex}} s$ *and* $t < s - (s \cap t)$.

10.8 Definition. For all $\beta < \omega_1$, for all $T \in \mathcal{T}(\beta)$, order the nodes of T by $u < v$ if and only if $u \sqsubset v$ or $u <_{\text{lex}} v$.

10.9 Lemma. *For all $\beta < \omega_1$, for all non-empty initial segments S, T of trees in $\mathcal{T}(\beta)$, $\bigcup S \sqsubset \bigcup T$ if and only if $S \sqsubset T$.*

Proof. By Lemma 10.7, if $\emptyset \neq S \sqsubset T \sqsubseteq T' \in \mathcal{T}(\beta)$, then $\bigcup S \sqsubset \bigcup T$. For $\beta = 0$, the reverse implication is trivially true, and for $\beta > 0$, it is true by definition of the fold map and the induction hypothesis. $\dashv$

10.10 Definition. For all $\beta \leq \omega$, define $e_\beta : [\omega]^{<\omega} \to \{-1\} \cup (\beta + 2)$ by recursion:

$$e_\beta(\emptyset) = \beta + 1;$$
$$e_\beta(\sigma^\frown\langle m\rangle) = \begin{cases} -1 & \text{if } e_\beta(\sigma) \leq 0, \\ e_\beta(\sigma) - 1 & \text{if } e_\beta(\sigma) > 0 \text{ successor}, \\ \max(\sigma) & \text{if } e_\beta(\sigma) = \omega \text{ limit}. \end{cases}$$

We refer to $e_\beta(x)$ as the *ordinal of* x.

Use induction on β, the definition of F_β, and the previous lemma to prove the next lemma.

10.11 Lemma. *For all $\beta \leq \omega$, for all $T \in \mathcal{T}(\beta)$, for all $t \in T$, $e_\beta(t) \geq 0$, and if $e_\beta(t) > 0$, then t has a proper extension $u \in T$.*

The following consequence of the recursive nature of Definition 10.10 is useful in induction proofs.

10.12 Lemma. *For all $\beta \leq \omega$, for all $\langle m\rangle^\frown\tau \in [\omega]^{<\omega}$, $e_\beta(\langle m\rangle) = \beta$ and if $\tau \neq \emptyset$ and $\gamma = e_\beta(\langle m, \max(\tau)\rangle) \geq 0$, then $e_\beta(\langle m\rangle^\frown\tau) = e_\gamma(\tau)$.*

10.13 Definition. Suppose $T \in \mathcal{T}$. For all $t \in T$, let $\sharp(t, T)$ be the number of successors of t in T.

10.14 Lemma. *For all $\beta \leq \omega$, for all $T \in \mathcal{T}(\beta)$, for all $t \in T$,*

$$\sharp(t, T) = \begin{cases} 0, & \text{if } e_\beta(t) = 0, \\ 1, & \text{if } e_\beta(t) = \omega \text{ is a limit}, \\ \max(t), & \text{if } e_\beta(t) \text{ is a successor}. \end{cases}$$

10.15 Lemma. *For all $\beta \leq \omega$, for all $T \subseteq [\omega]^{<\omega}$, $T \in \mathcal{T}(\beta)$ if and only if T satisfies the four conclusions of Lemma 10.7, and for all $t \in T$, $e_\beta(t) \geq 0$ and $\sharp(t, T)$ has the value specified in Lemma 10.14.*

Proof. By Lemmas 10.7, 10.11, and 10.14, if $(T, \sqsubseteq) \in \mathcal{T}(\beta)$, then it satisfies the given list of conditions.

To prove the other direction, work by induction on β to show that if $T \subseteq [\omega]^{<\omega}$ satisfies the given conditions for β, then $\bigcup T \in G_{\omega^\beta}$ and $T = F_\beta(\bigcup T) \in \mathcal{T}(\beta)$. ⊣

10.16 Definition. For $0 < \beta \leq \omega$ and $\emptyset \neq S \sqsubset T \in \mathcal{T}(\beta)$, the *critical node of S*, in symbols $\mathrm{cri}(S)$, is the largest $s \in S$ with $\sharp(s, S)$ smaller than the value predicted in Lemma 10.14. For notational convenience, let $\mathrm{cri}(\emptyset) = \emptyset$, and set $\mathrm{cri}(T) = \emptyset$ for $T \in \mathcal{T}(\beta)$.

The next lemma shows why the name was chosen.

10.17 Lemma. *For $0 \leq \beta \leq \omega$ and $S \sqsubseteq T \in \mathcal{T}(\beta)$, if $t := \min(T - S)$, then $t = \mathrm{cri}(S)^\frown\langle\max(t)\rangle$.*

Proof. Let $m < \omega$ be such that $\langle m\rangle \in T$. Then $\langle m\rangle$ is the least element of T. If $S = \emptyset$, then $t = \langle m\rangle = \mathrm{cri}(S)^\frown\langle\max(t)\rangle$ and the lemma follows. Otherwise, $\langle m\rangle$ must be in S, and because it is the root of T, $\langle m\rangle \sqsubset t := \min(T - S)$. Let $r = t - \{\max(t)\}$. Then $\langle m\rangle \sqsubseteq r \sqsubset t$, $\sharp(r, S) < \sharp(r, T)$, so r is an element of S with $\sharp(r, S)$ smaller than the value specified in Lemma 10.14.

If $p \in T$ and $p <_{\mathrm{lex}} t$, then $p \in S$, since $S \sqsubset T$ and $T = \min(T - S)$. Moreover, if $p <_{\mathrm{lex}} t$ and $p \sqsubset q \in T$, then $q <_{\mathrm{lex}} t$. Hence if $p <_{\mathrm{lex}} t$, then $\sharp(p, S) = \sharp(p, T)$ takes on the value specified in Lemma 10.14. Thus $\mathrm{cri}(S) \sqsubset t$, so $\mathrm{cri}(S) \sqsubseteq r$. It follows that $r = \mathrm{cri}(S)$ and $t = \mathrm{cri}(S)^\frown\langle\max(t)\rangle$ as required. ⊣

10.18 Lemma. *For all $\beta \leq \omega$, the set of initial segments of trees in $\mathcal{T}(\beta)$ is well-founded under $\sqsubset$.*

Proof. The proof is by induction on β. For $\beta = 0$, the lemma is clearly true, since the longest possible sequences are those of the form $\emptyset, \langle m\rangle$ for some $m < \omega$.

Next suppose the lemma is true for $k < \omega$ and $\beta = k+1$. Let $S_0, S_1, \ldots$ be an arbitrary $\sqsubset$-increasing sequence, and without loss of generality, assume it has at least two trees in it. Then there is some $m < \omega$ so that $\langle m\rangle \in S_1$. By the definition of the fold map F_k, it follows that for $i > 1$, the tree S_i satisfies $\bigcup S_i = \langle m\rangle^\frown\sigma_{i,1}{}^\frown \cdots {}^\frown\sigma_{i,n_i}$ for some $n_i \leq m$, where $F_k(\sigma_{i,j}) \in \mathcal{T}(k)$ for $j < n_i$, and for some $\sigma' \sqsupseteq \sigma_{i,n_i}$, $F_k(\sigma') \in \mathcal{T}(k)$, so $\sigma_{i,n_i} = \bigcup T_i$ for T_i an initial segment of a tree in $\mathcal{T}(k)$. If $i < \ell$ and T_i, T_ℓ are such that $n_i = n_\ell$, then for $j < n_i$, $\sigma_{i,j} = \sigma_{\ell,j}$. Thus by the induction hypothesis, for each n with $1 \leq n \leq m$, there can be at most finitely many trees in the sequence with $n_i = n$. Hence the sequence must be finite, and the lemma is true for $\beta = k+1$.

The proof for $\beta = \omega$ is similar, since for all initial segments S of trees in $\mathcal{T}(\omega)$, either $S = \emptyset$, $S = \{\langle m\rangle\}$, or $S = \{\langle m\rangle\}^\frown S'$ for some $m < \omega$ and some S' which is an initial segment of a tree in $\mathcal{T}(m)$. The details are left to the reader.

Therefore, by induction, the lemma holds for all $\beta \leq \omega$. ⊣

10.2. Node Labeled Trees

A typical proof of a positive partition relation for a countable ordinal for pairs includes a uniformization of an arbitrary 2-partition into 2 colors, but only for those pairs for which some easily definable additional information is also uniformized. We will introduce node labellings to provide that extra information, but before we do so, we examine convex partitions of disjoint trees and the partition nodes that determine them.

10.19 Definition. For trees S^0, S^1 from $\mathcal{T}(\beta)$ with $\bigcup S^0 \cap \bigcup S^1 = \emptyset$, call $t \in S^\varepsilon$ a *partition node* if $t < \max(S^\varepsilon)$ and there is some $u \in S^{1-\varepsilon}$ with $\max(t) < \max(u) < \min(\bigcup S^\varepsilon - (1 + \max(t))$.

For notational convenience, write $T(\emptyset, t]$ for the initial segment of T consisting of all nodes $s \leq t \in T$, and, for $t < u$ in T, write $T(t, u]$ for $\{s \in T : t < s \leq u\}$. With this notation in hand, we can state the lemma below justifying the label *partition nodes*. This lemma follows from Lemmas 10.7 and 10.9.

10.20 Lemma. *Suppose S^0, S^1 are in $\mathcal{T}(\beta)$ and $\bigcup S^0 \cap \bigcup S^1 = \emptyset$. Further suppose $t^0_0, t^0_1, \ldots, t^0_{k-1} \in S^0$ and $t^1_0, t^1_1, \ldots, t^1_{\ell-1} \in S^1$ are the partition nodes of these trees if any exist. Set $t^0_{-1} = t^1_{-1} = \emptyset$, $t^0_k = \max(S^0)$, $t^0_\ell = \max(S^1)$. Then every node of S^ε is in one and only one $S^\varepsilon(t^\varepsilon_{i-1}, t^\varepsilon_i]$, and the sets $\sigma^\varepsilon_i = \bigcup S^\varepsilon(t^\varepsilon_{i-1}, t^\varepsilon_i] - t^\varepsilon_{i-1}$ satisfy*

$$\sigma^0_0 < \sigma^1_0 < \sigma^0_1 < \sigma^1_1 < \cdots < \sigma^0_{\ell-1} < \sigma^1_{\ell-1} (< \sigma^0_{k-1}).$$

Now we introduce node labellings. For simplicity, this concept is given a general form.

10.21 Definition. Suppose $\beta \leq \omega$ and $N \subseteq \omega$ is infinite. For any initial segment $S \sqsubseteq T \in \mathcal{T}(\beta)$, a function C is a *node labeling of S into N* if $C : S \to [N]^{<\omega}$ satisfies $\max(C(s)) < \max(s)$ for all $s \in S$ with $C(s) \neq \emptyset$.

We carry over from $\mathcal{T}(\beta)$ the notions of extension, complete tree and trivial tree. In particular, call (T, D) a *(proper) extension of* (S, C), in symbols, $(S, C) \sqsubset (T, D)$, if $S \sqsubset T$ and $D{\restriction}S = C$. Call (T, D) *complete (for β)* if $T \in \mathcal{T}(\beta)$; call it *trivial* if $(T, D) = (\emptyset, \emptyset)$.

Call a pair S, T from $\mathcal{T}(\beta)$ *local* if S and T have a common root; otherwise it is global. Similarly, call (S, C), (T, D) *local* if S, T is local and otherwise call it global.

10.22 Definition. A pair $((S^0, C^0), (S^1, C^1))$ is *strongly disjoint* if (a) either $S^0 = \emptyset = S^1$ or $(\bigcup S^0 \cup \mathrm{ran}(C^0)) \cap (\bigcup S^1 \cup \mathrm{ran}(C^1)) = \emptyset$ and (b) for all $s, t \in S^0 \cup S^1$, whenever $\max(s) < \max(t)$ and $C^\varepsilon(t) \neq \emptyset$, then also $\max(s) < \min(C^\varepsilon(t))$.

10.23 Definition. Call a pair $((S^0, C^0), (S^1, C^1))$ of node labeled trees *clear* if $S^0 < S^1$, $((S^0, C^0), (S^1, C^1))$ is strongly disjoint, all partition nodes $t \in S^0 \cup S^1$ are leaf nodes ($e_\beta(t) = 0$), and if for all $\varepsilon < 2$ and all $s \in S^\varepsilon$,

- $C^\varepsilon(s) = \emptyset$ if $e_\beta(s) = 0$;
- $C^\varepsilon(s) = \{\sharp(s, S^\varepsilon(\emptyset, t]) : s \sqsubset t \in S^\varepsilon \text{ is a partition node}\}$ if $e_\beta(s)$ is a successor ordinal;
- $C^\varepsilon(s) = \{e_\beta(t) : s \sqsubset t \in S^\varepsilon \,\&\, |C^\varepsilon(t)| > 1\}$ if $e_\beta(s) = \omega$ is a limit ordinal.

Call a pair S^0, S^1 of trees from $\mathcal{T}(\beta)$ *clear* if it is local or if it is global and there are node labellings C^0, C^1 with $((S^0, C^0), (S^1, C^1))$ clear.

For $\beta > \omega$, the value of the node labeling for s with $e_\beta(s)$ limit is more complicated to describe.

Notice that for $2 \leq \beta \leq \omega$, if $(S^0, C^0), (S^1, C^1)$ is a global clear pair and neither C^0 nor C^1 is constantly the empty set, then all initial segments of partition nodes are identifiable: they are the root of the tree, successor nodes whose node label is non-empty, and nodes of ordinal 0 whose immediate predecessor has non-empty node label that identifies it as a successor which is a partition node.

From the definition of *clear*, if u is a partition node of one of a pair of trees, say (S, C) then for each initial segment s whose ordinal $e_\omega(s)$ is a successor, the node label $C(s)$ must have as a member the number of immediate successors of s which are less than or equal to u in the lexicographic order. If we index the immediate successors of s in S in increasing lexicographic order starting with 1, then this value is the *index* of the immediate successor of s which is an initial segment of u. This analysis motivates the next definition.

10.24 Definition. Consider a node labeled tree (S, C) with root $\langle m \rangle$. A non-root node t of (S, C) is a *prepartition node* if for all $s \sqsubset t$ with $e_\beta(s)$ a successor ordinal, $\sharp(s, S(\emptyset, t]) \in C(s)$, and if $e_\omega(s) \in C(\langle m \rangle)$ whenever $\beta = \omega$ and $|C(s)| > 1$ The root is a prepartition node if $S \in \mathcal{T}(0)$ or $C(\langle m \rangle) \neq \emptyset$ or (S, C) has a non-root prepartition node. Call (S, C) *relaxed* if $S \notin \mathcal{T}(0)$ and $\max(S)$ is a prepartition node of ordinal 0.

Node labeled trees, clear pairs, prepartition nodes and relaxed initial segments are used in the game introduced in the next section.

10.3. Game

In this section we develop the game $\mathcal{G}(h, N)$ in which two players collaborate to build a pair of node labeled trees.

Here is a brief description of the game. Player I, the architect, plays specifications for Player II, the builder, telling him (a) which tree to extend, (b) whether to complete the tree or to build it to the next decision point, and (c) what the size of the node label of the next node to be constructed is, if it is not already determined. In turn, the builder extends the designated tree by a series of steps, adding a node and node label at each step using elements of N, until he reaches the next decision point on the given tree, if he has been so directed, or until he completes the tree. The architect wins if the pair $((S, C), (T, D))$ created at the end of the play of the game is a global clear pair with $h(S, T) = 1$; otherwise the builder wins.

Before giving a detailed description of the general game, as a warm-up exercise, consider a 2-partition h into 2 colors, an infinite set N, and the game $\mathcal{G}_0(h, N)$ in which the architect plays the strategy σ_0 directing the builder to complete the first tree and then complete the second tree. The builder can use a fold map to fold an initial segment of N into a tree S and assign the constantly $\emptyset$ node labeling C to create his first response, (S, C). Then he can fold a segment of N starting above $\bigcup S$ into a tree T and assign the constantly $\emptyset$ node labeling D to create his second response, (T, D). By construction, the pair $((S, C), (T, D))$ is clear, since there are no partition nodes, so $\{S, T\}$ is a clear global pair. If all pairs $\{X, Y\}$ of trees created using nodes from N in this game have $h(X, Y) = 1$, then playing another game, starting with (T, D) as the initial move of the builder and ending with (U, E), one builds a triple $\{S, T, U\}$ each pair of which h takes to color 1. Thus if σ_0 is a winning strategy for the architect, then the architect can arrange for a triangle to be constructed.

As a second warm-up exercise, consider a 2-partition h into 2 colors, an infinite set N with $0 \notin N$, and the game $\mathcal{G}_1(h, N)$ in which the architect plays the strategy σ_1 directing the builder to build the first tree to the next decision point starting from a root node whose node label has 0 elements, to start and complete the second tree, and then to complete the first tree.

In response to the architect's first set of specifications, the builder uses the least element n_0 of N to build the root, $\langle n_0 \rangle$ and gives it the empty set as node label. He then uses the next two elements of N, namely n_1 and n_2 by setting $\langle n_0, n_2 \rangle$ as the immediate successor of the root with node label $C_0(\langle n_0, n_2 \rangle) = \{n_1\}$. He continues with successive elements of N, extending the critical node of the tree create to that point, giving the new node an empty label unless the node to be created is the successor of a prepartition node whose index is the sole element of the node label of the prepartition node, in which case he extends and labels it as he did the successor of the root. He continues until he has created and labeled a prepartition node u whose ordinal is $e_\omega(u) = 0$, and the pair (S_0, C_0) he has built is his response.

In response to the architect's second set of specifications, the builder uses elements of N larger than any used so far to build a tree T in $\mathcal{T}(\omega)$ and gives it the constantly $\emptyset$ labeling. Then he responds to the final set of specifications of the architect by completing S_0 to S in $\mathcal{T}(\omega)$ and extending C_0 to C with all new nodes receiving empty node labels.

In the brief description of the game, the architect was allowed to direct the builder to stop at the next decision point. The decision point is either when a partition node has been created and it is time to switch to the other tree or when the next node to be created is permitted to have a node label whose size is greater than 2. Notice that if the architect switches trees after the builder has created a prepartition node with ordinal 0, then that node becomes a partition node.

10.25 Definition. A *decision node* of (S, C) is a prepartition node t with ordinal $e_\omega(t)$ such that either $e_\omega(t) = 0$ or $e_\omega = \ell + 1$ is a successor ordinal with $\ell \in C(t\restriction 1)$, t is the critical node of S and $1+\sharp(t, S)$ is an element of $C(t)$.

In the game $\mathcal{G}_0(h, N)$, the final pair of trees S, T had the property that $\min(\bigcup S) < \min(\bigcup T)$ and $\max(\bigcup S) < \max(\bigcup T)$. Call such a pair an *outside* pair. In the game $\mathcal{G}_1(h, N)$, the final pair of trees S, T had the property that $\min(\bigcup S) < \min(\bigcup T)$ and $\max(\bigcup S) > \max(\bigcup T)$. Call such a pair an *inside* pair.

10.26 Definition. Suppose $N \subseteq \omega$ is infinite and h is a 2-partition of $\mathcal{T}(\omega)$ into 2 colors. Then $\mathcal{G}(h, N)$ is a two player game played in rounds. Player I is the architect who issues specifications, and Player II is the builder whose creates or extends one of a given pair of trees in round ℓ to $((S_\ell, C_\ell), (T_\ell, D_\ell))$. Note that if the second tree has not been started in round ℓ, then $T_\ell = D_\ell = \emptyset$.

The architect's moves: In the initial round, the architect declares the type of pair to be produced, either inside or outside. In round ℓ, the architect specifies the tree to be created or extended (first or second), specifies whether the extension is to completion with all new nodes receiving empty labels or to the point at which a decision node is created and labeled (completion or decision), and specifies the size of the label for the next node to be created. In her initial move, the architect must specify the first tree be created. She may not direct the builder to extend a tree which is complete.

The builder's moves: In round ℓ, the builder creates or extends the specified tree through a series of steps in which he adds one node and its label using elements of N larger than any used to that point. If he has been directed to continue to completion, he does so while assigning the empty set node label to all new nodes. Otherwise he adds nodes one at a time, until he creates the first decision node. He adds a node after determining the size of the node label, and choosing the node label, since all elements of the node label must be smaller than the single point used to extend the critical node. The size of the label of the first node to be created is specified by the architect's move. Otherwise, the builder determines if the node will be a prepartition node with non-zero ordinal. If so, its node label has one element and otherwise its node label is empty.

Stopping condition: Play stops at in round ℓ if both trees are complete.

Payoff set: The architect wins if both S_ℓ and T_ℓ are complete, the pair is inside or outside as specified at the onset, the pair $((S_\ell, C_\ell), (T_\ell, D_\ell))$ is a global clear pair and $h(S_\ell, T_\ell) = 1$; otherwise, the builder wins.

We are particularly interested in this game when we have a fixed 2-partition, $h : [\mathcal{T}(\beta)]^2 \to 2$, but the game may be modified to work with 2-partitions into more colors. This game may also be modified to require the builder to use an initial segment of an infinite sequence from N specified by the architect in her move or be modified to start with a specified pair of node labeled trees.

10.27 Lemma. *Suppose $N \subseteq \omega$ is infinite and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Then every run of $\mathcal{G}(h, N)$ stops after finitely many steps.*

Proof. Use Lemma 10.18. ⊣

10.4. Uniformization

In this subsection, we prove the key dichotomy in which one or the other player has a winning strategy, at least up to some constraints on the play. Basically, we build a tree out of the plays of the game, show it is well-founded, and use recursion on the tree to define an infinite subset $H \subseteq \omega$ so that plays where the builder uses sufficiently large elements of H are uniform enough to allow us to prove the dichotomy.

10.28 Definition. Suppose $N \subseteq \omega$ is infinite, and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Let $\mathcal{S}(N)$ be the set of sequences of consecutive moves in the game $\mathcal{G}(h, N)$, including the empty sequence.

10.29 Lemma. *For infinite $N \subseteq \omega$, $(\mathcal{S}(N), \sqsubset)$ is a rooted, well-founded tree.*

Proof. The root is the empty sequence. End-extension clearly is a tree order on $\mathcal{S}(N)$, and $\sqsubset$ is well-founded since every game is finite. ⊣

The basic idea for the builder is to use elements from a specified set and to always start high enough.

10.30 Definition. Suppose N is an infinite set with $1 < \min(N)$ and no two consecutive integers in N. Then a function $b : \mathcal{S}(N) \to \omega$ is a bounding function if $b(\emptyset) = 0$, and if $s \sqsubseteq t$, then $b(s) \leq b(t)$.

Use a bounding function and an infinite set to delineate *conservative* moves for the builder.

10.31 Definition. Suppose $H \subseteq N \subseteq \omega$ is infinite with $1 < \min(N)$ that b is a bounding function. If $\vec{R}$ is a position in the game $\mathcal{G}(h, N)$ ending with a move by the architect, then a move $((S_\ell, C_\ell), (T_\ell, D_\ell))$ for the builder is *conservative for b and H* if all new nodes and node labels are created using elements of H greater than $b(\vec{R})$.

10.32 Lemma (Ramsey Dichotomy). *Suppose $N \subseteq \omega$ is infinite, and h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors. Then there is an infinite subset $H \subseteq N$ and a bounding function b so that $1 < \min(H)$, no two consecutive integers are in H, and the following statements hold:*

1. *for every position $\vec{R} \in \mathcal{S}(N)$ ending in a play for the architect, there is a conservative (for b and H) move for the builder; and*
2. *either the architect has a strategy σ by which she wins $\mathcal{G}(h, N)$ if the builder plays conservatively, the builder wins every run of $\mathcal{G}(h, N)$ by playing conservatively (for b and H).*

Before we tackle the proof of the dichotomy, we introduce some preliminary definitions and lemmas.

10.33 Definition. Call a set $B \subseteq [\omega]^{<\omega}$ *thin* if no u from B is a proper initial segment of any other v from B. Call B a *block for $N \subseteq \omega$* if for every infinite set $H \subseteq N$, there is exactly one $u \in B$ which is an initial segment of H. Call it a *block* if it is a block for ω.

Note that if B is a block, then it is thin. A major tool of the proof of the dichotomy is the following theorem.

10.34 Theorem (Nash-Williams Partition Theorem). *Let $N \subseteq \omega$ be infinite. For any finite partition of a thin set $c : W \to n$, there is an infinite set $M \subseteq N$ so that c is constant on $W \restriction M$.*

For a proof see [45] or [23]. The terminology *thin* comes from [23].

Here are some easy examples of blocks.

10.35 Lemma. *The families $\{\emptyset\}$, and $[\omega]^k$ for $k < \omega$ are blocks.*

10.36 Lemma. *Suppose $w \subseteq \omega$ is an increasing sequence, and $B \subseteq [\omega]^{<\omega}$ is thin. Then there is at most one initial segment u of w with $u \in B$. If B is a block, then there is exactly one such initial segment.*

10.37 Lemma. *Suppose $H \subseteq N \subseteq \omega$ is infinite, h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors, and b is bounding function. For every position $\vec{R} \in \mathcal{S}(N)$ ending in a move by the architect, there is some $k \geq b(\vec{R})$ and a block $\mathcal{B}(\vec{R})$ for $H - k$ such that for all $B \in \mathcal{B}(\vec{R})$, the builder can build his responding move using all elements of B.*

Proof. Recall the architect may not direct the builder to extend a complete tree, so if the architect has just moved, the tree she directs the builder to extend is not complete. Thus the builder's individual steps are specified up to the choice of elements of N, and his stopping point is determined by his individual steps. Hence the set of sequences of new elements used is thin. Moreover, for any infinite increasing sequence w from H above $b(\vec{R})$ and above the largest element of N used in prior moves, the builder can create a move using an initial segment of w. Therefore the set of possible moves is a block. $\dashv$

At this point we are prepared to prove the main result of this section.

Proof of Ramsey Dichotomy 10.32. Without loss of generality, assume $1 < \min(N)$ and N has no two consecutive elements, since otherwise one can shrink N to an infinite set for which these conditions hold. These conditions assure that no decision node is an immediate successor of another decision node.

Let ρ^* be the rank of $\mathcal{S}(N)$. Use recursion on $\mu \leq \rho^*$ to define a sequence $\langle M_\mu \subseteq N : \mu < \rho^* \rangle$ and a valuation $v : \mathcal{S}(N) \to 2$.

For $\mu = 0$, the sequences $\vec{R}$ of rank 0 are ones in which the last move completes the play of the game. Let $M_0 = N$, and define $v(\vec{R}) = 0$ on a sequence of rank 0 if the game ends with a win for the architect and $v(\vec{R}) = 1$ otherwise.

Next suppose that $0 < \mu < \rho^*$, and v has been defined on all nodes of rank less than μ. Enumerate all the nodes of rank μ as $\vec{R}^0_\mu, \vec{R}^1_\mu, \ldots$ and let M^{-1}_μ be $M_{\mu-1}$ if μ is a successor ordinal and let M^{-1}_μ be a diagonal intersection of a sequence M_ν for a set of ν cofinal in μ otherwise.

Extend v to the nodes of rank μ and define sets M^i_μ by recursion. For the first case, suppose $\vec{R}^i_\mu$ ends with a move for the builder, and set $M^i_\mu = M^{i-1}_\mu$. If there is some move a^i_μ with $\vec{R}^i_\mu {}^\frown \langle a^i_\mu \rangle \in \mathcal{S}(N)$ and $v(\vec{R}^i_\mu {}^\frown \langle a^i_\mu \rangle) = 1$, then set $v(\vec{R}^i_\mu) = 1$, and otherwise set $v(\vec{R}^i_\mu) = 0$.

For the second case, assume $\vec{R}^i_\mu$ ends with an move for the architect. Let $\mathcal{B}(\vec{R}^i_\mu)$ be the block of Lemma 10.37 for the set M^{i-1}_μ and the position $\vec{R}^i_\mu$. Define $c : \mathcal{B}(\vec{R}^i_\mu) \to 2$ by $c(d) = v(\vec{R}^i_\mu {}^\frown \langle P(d) \rangle)$ where $P(d)$ is the unique approved move for the builder whose new elements are created using exactly the elements of d. Apply the Nash-Williams Partition Theorem 10.34 to c to get an infinite set $M^i_\mu \subseteq M^{i-1}_\mu$ and let $v(R^i_\mu)$ be the constant value of c on $\mathcal{B}(\vec{R}^i_\mu)$ restricted to M^i_μ.

Continue by recursion as long as possible, extending v to all nodes of rank μ. If there are only finitely many of them, let M_μ be M^i_μ where $\vec{R}^i_\mu$ is the last one. If there are infinitely many, let M_μ be a diagonal intersection of the sets M^i_μ.

Since every non-empty sequence of moves in the game $\mathcal{G}(N)$ extends the empty sequence, this root of $\mathcal{S}(N)$ has the largest rank of any element of

$\mathcal{S}(N)$, namely rank $\rho^* - 1$. Let $H = M_{\rho^*-1}$. Let $v(\emptyset)$ be 1 if there is some move a by the architect so that $v(\langle a\rangle) = 1$, and set $v(\emptyset) = 0$ otherwise.

Define b on $\mathcal{S}(N)$ by recursion. Let $b(\vec{R}) = 2$ for all $\vec{R} \in \mathcal{S}(N)$ with $|\vec{R}| \leq 1$. Continue by recursion on $|\vec{R}|$. For notational convenience, let $\vec{R}^-$ be obtained from $\vec{R} \in \mathcal{S}(N) - \{\emptyset\}$ by omission of the last entry. If $b(\vec{R}^-)$ has been defined and the last move in $\vec{R}$ is $\mathcal{B}_\ell = ((S_\ell, C_\ell), (T_\ell, D_\ell))$ for the builder, then let $b(\vec{R})$ be the least b greater than $b(\vec{R}^-)$ and any element of $\bigcup(S_\ell \cup \operatorname{ran}(C_\ell) \cup T_\ell \cup \operatorname{ran}(D_\ell))$. If $b(\vec{R}^-)$ has been defined, the last move in $\vec{R}$ is a_ℓ for the architect, and $\vec{R} = \vec{R}^i_\mu$, then let $b(\vec{R})$ be the least b greater than $b(\vec{R}^-)$ so that for all d in the restriction of $\mathcal{B}(\vec{R}^i_\mu)$ to subsets of H with $\min(d) > b$, there is a conservative move for the builder for position $\vec{R}$ with new elements d. The existence of a value for $b(\vec{R})$ in this latter case follows from the fact that $H \subseteq^* M^i_\mu$ by construction, and by Lemma 10.37.

Since all $\vec{R}$ in $\mathcal{S}(N)$ are finite, this recursion extends b to all of $\mathcal{S}(N)$. This definition of H and b guarantees that the builder can always respond with conservative moves to plays of the architect.

If $v(\emptyset) = 1$, then the strategy for the architect is to keep $v(\vec{R}) = 1$. Given the definition of v, the architect will always succeed, as long as the builder moves conservatively with H and b. If $v(\emptyset) = 0$, and the builder always moves conservatively with H and b, then he will win, again by the recursive definition of v and the definition of winning the game. ⊣

10.5. Triangles

For this section we assume that $h : [\mathcal{T}(\omega)]^2 \to 0$ is fixed and that an infinite set $H \subseteq \omega$ and a bounding function b are given so that the architect has a winning strategy σ for games of $\mathcal{G}(h, H)$ in which the builder plays conservatively for b and H. The goal is to outline how one uses the strategy of the architect to construct a triangle.

10.38 Lemma. *Suppose σ is a strategy for the architect with which she wins $\mathcal{G}(h, N)$ if the builder moves conservatively for H, b. Then there is a three element 1-homogeneous set for h.*

Proof. Consider the possibilities for $\sigma(\emptyset)$. The architect must declare the pair to be built will be inside or outside, the initial move is to complete the first tree or construct it to a decision point and must declare the size d of the node label of the initial node constructed. We construct our triangles by playing multiple interconnected games in which the architect uses σ, the builder plays conservatively for H and b, and plays sufficiently large that his plays work in all the relevant games. While technically we should report a pair of node labeled trees for each play of the builder, for simplicity, we frequently only mentioned the one just created or modified.

Case 1. Using σ, the architect specifies the builder constructs a complete tree in her initial move.

Then the architect must call for an outside pair and must set $d = 0$, since otherwise the pair constructed will not be clear. The builder responds via conservative play with a complete tree (S, C) whose node labeling is constantly the empty set. The strategy σ must then specify that the builder constructs a second complete tree whose initial node has a node label of size 0. The builder responds via conservative play with a complete tree (T, D) whose node labeling is constantly $\emptyset$. Since σ is a winning strategy, $h(S, T) = 1$.

Next the architect shifts to the game where the builder has responded to the opening move with (T, D), applies the strategy σ, to which the builder responds with (U, E), a (third) complete tree whose node labeling is constantly $\emptyset$ starting sufficiently large for this response to be appropriate for the game where the builder has responded to the opening move with (S, C). Since σ is a winning strategy, $h(T, U) = 1 = h(S, U)$. Thus $\{S, T, U\}$ is the required triangle.

Case 2. Using σ, the architect declares the pair will be an inside pair, and specifies the initial node label size $d = 0$ and that the builder constructs to a decision node.

The proof in this case is similar to the last, with the architect starting one game to which the builder responds with a first tree (S_0, C_0) where the decision node is a prepartition node of ordinal zero, since no levels were coded for introducing decision nodes with successor ordinals. Thus the next play for the architect is to direct the builder to create a complete tree all of whose nodes are labeled by $\emptyset$.

The architect stops moving on the first game and, using σ, starts a new game, directing the builder to start high enough that the tree constructed could be the beginning of his response in the first game. The builder responds with a tree (T_0, D_0) where the decision node is a prepartition node of ordinal zero The architect continues this game using σ and the builder responds with a complete tree (U, E) all of whose nodes are labeled with $\emptyset$. After the architect and builder each move a final time on this game, the builder has created a complete tree (T, D) extending (T_0, D_0). Since σ is a winning strategy, $h(T, U) = 1$.

Now return to the first game: the builder plays (T, D') where D' is the constantly empty set node labeling; The architect uses σ to respond and requires the builder to construct high enough that his response works in the game where the builder plays (U, E) as well as the one where the builder plays (T, D'). Since σ is a winning strategy, $h(S, T) = h(S, U)$, Thus $\{S, T, U\}$ is the required triangle.

Case 3. Using σ, the architect declares the pair will be an outside pair, and specifies the initial node label size $d = 0$ and that the builder constructs to a decision node.

The proof in this case is similar to the last, so only the list of subtrees to be constructed is given. Start with (S_0, C_0) and (T_0, D_0) as responses to the first two moves of the architect in the first game. Next build (U_0, E_0) and

(S, C) as second and third moves in a game where (S_0, C_0) is the first move, and (U_0, E_0) is started high enough to be a response in the game starting with (T_0, D_0). Finally build (T, D) and (U, E) in the game starting with responses (T_0, D_0) and (U_0, E_0) and continuing high enough that play using (S, C) in the appropriate games is conservative.

In the remaining two cases, we use σ and conservative play for the builder to create trees S, T, U with node labellings (S, C^1) , (S, C^2), (T, D^0), (T, D^1), (U, E^0) and (U, E^1) through plays $\mathcal{G}_{0,1}$, $\mathcal{G}_{0,2}$, $\mathcal{G}_{1,2}$ of the game $\mathcal{G}(h, H)$. We pay special attention to the creation of the initial segments up to the first partition nodes for each pair and to the terminal segments, after the last partition nodes. We refer to the remainder of the run as “the mid-game”.

Case 4. Using σ, the architect declares the pair will be an inside pair, and specifies the initial node label size $d > 0$ and directs the builder to construct the first tree to a decision node.

We start by displaying a schematic overview of the construction:

$$S \quad T \quad U \quad \boxed{T \quad U} \quad \boxed{S \quad U} \quad U \quad \boxed{S \quad T} \quad T \quad S$$

Next we outline the steps to be taken.

1. Choose from H codes for d levels for S and U; choose d larger levels for S and T; start the initial segment of S with respect to T; continue it to get the initial segment of S with respect to U (the difference is in the node labellings only), and apply σ to the results to determine the sizes d', d'' of node labels for the roots of T, U in $\mathcal{G}_{0,1}$, $\mathcal{G}_{0,2}$, respectively.

2. Choose d' levels for T's interaction with U; choose d larger levels for T's interaction with S; start the initial segment of T with respect to S; continue it to get the initial segment of T with respect to U; and apply σ to determine the size d''' of the node label of the root of U for $\mathcal{G}_{1,2}$.

3. Choose d''' levels for U's interaction with T; choose d'' larger levels for U's interaction with S; start the initial segment of U with respect to S; continue it to get the initial segment of U with respect to T.

4. Play the mid-game of $\mathcal{G}_{1,2}$ to the call for the completion of U.

5. The initial segments of T and U with respect to S are complete, so update the node labellings C^0 and C^1.

6. Play the mid-game of $\mathcal{G}_{0,2}$ until the architect calls for the completion of S. In particular, play until U is complete.

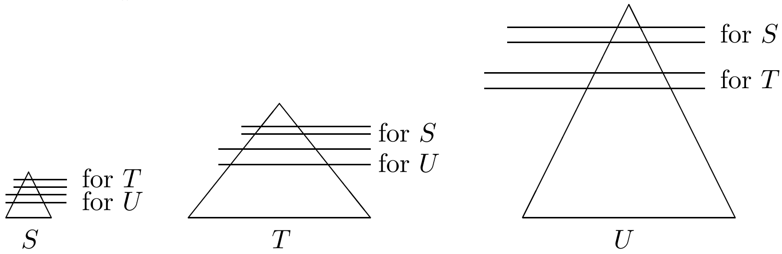
7. Update the node labeling E^1 for U by labeling all the new nodes by the empty set.

8. Complete the play of the game $\mathcal{G}_{0,1}$, starting by extending the part of S created in the play of the mid-game $\mathcal{G}_{0,2}$. Such a start is possible, since the levels of S for interaction with T are larger than those for interaction with U.

9. Update the node labellings C^2 for S and D^2 for T by labeling all the new nodes by the empty set.

Care must be taken to direct the builder to start high enough that all moves in the tree plays of $\mathcal{G}(h, H)$ are conservative. Since the construction of the initial segments calls for introducing levels, we describe the first such step in greater detail.

We know that we will need to choose *levels* for splitting of S with respect to T and U, and for splitting T with respect to U. Depending on the strategy σ, we may need to choose levels for the splitting of T with respect to S and for the splitting of U with respect to S and T. Here is a picture of the approach we plan to take on these splitting levels, in the general case where we need levels for all pairs.



To start the construction, choose $2d+1$ elements from H above $b(\langle\sigma(\emptyset)\rangle)$ ending in m^0, and use them to define $C^1(\langle m^0\rangle)$ and $C^2(\langle m^0\rangle)$ satisfying $C^2(\langle m^0\rangle) < C^1(\langle m^0\rangle)$.

Start playing a game $\mathcal{G}_{0,1}$ where the architect starts with $R_0^{0,1} = \sigma(\emptyset)$ and the builder must use the elements of $C^1(\langle m^0\rangle)$ and m^0 to start his initial move, $R_1^{0,1}$. Continue to play until the architect's last move $R_p^{0,1}$ before directing the builder to switch to the second tree. One can identify this point in the run of the game, since it is the first time the architect has stopped on a node, call it v_0, whose level is one more than $\min(C^1(\langle m^0\rangle))$. Let (S^1_{p-1}, C^1_{p-1}) be the tree paired with $(\emptyset, \emptyset)$ by the builder in his last move.

Let C^2 be the node labeling of S^1_{p-1} with the value of $C^2(\langle m^0\rangle)$ specified above, with the empty set assigned for nodes which are not initial segments of v_0, and for initial segments of v_0 longer than the root, are the singletons needed to guarantee that v_0 is a prepartition node. Then the architect directs the builder to extend this node labeled tree to a response $R_1^{0,2}$ to $\sigma(\emptyset)$ in the second game $\mathcal{G}_{0,2}$. The two players continue the game until the architect, in $R_q^{0,2}$, directs the builder to switch to the second tree to start with a node label of size d'' and to go to a decision node. Such a move is the only one that will lead to a clear pair. Let (S^2_{q-1}, C^2_{q-1}) be the tree played by the builder in his previous move.

Return to game $\mathcal{G}_{0,1}$ and require the builder to respond to $R_p^{0,1}$ with (S_{p+1}^1, C_{p+1}^1) for $S_{p+1}^1 = S_{q-1}^2$ and C_{p+1}^1 the node labeling where all new nodes that are not initial segments of the largest node are labeled with the empty set and initial segments of the largest node are labeled minimally so that it is a prepartition node. Let d' be the size of the node label for the root of the second tree determined by the architect's use of σ in response to this move of the builder.

The remaining details are left to the reader. The careful reader will note that there is one possibility in which the architect initially calls for $d = 1$, specifies a node label of size 2 at the first decision node, and after the completion of the first full segment, calls for an empty node label for the root of the second tree. The construction proceeds as above but is simpler, so these details are also left to the reader.

As in the previous cases, since σ is a winning strategy for the architect, the set $\{S, T, U\}$ we have constructed is the required triangle.

Case 5. Using σ, the architect declares the pair will be an outside pair, and specifies the initial node label size $d > 0$ and directs the builder to construct the first tree to a decision node.

This case is substantially like the previous one, so we give the schematic below to guide the reader and a few comments on how to move from one section to the next.

$$S \quad T \quad \boxed{S \quad T} \quad U \quad \boxed{S \quad U} \quad S \quad \boxed{T \quad U} \quad T \quad U$$

We start by building initial segments of S and T. We begin by choosing d small levels for the interaction of S with T and d larger levels for the interaction of S with U. We start to build the initial segment of S with respect to its convex partition by U, then extend that start to build the initial segment of S with respect to its convex partition by T. We obtain the size d' of the root node label of the second tree in $\mathcal{G}_{0,1}$ by applying σ, choose d' small levels for the interaction of T with S, and d larger levels for the interaction of T with U. We start building the initial segment of T with respect to U, then extend it to the initial segment of T with respect to S.

We play the mid-game of $\mathcal{G}_{0,1}$ until the architect calls for the completion of S. In the process we have completed the initial segments of S and T with respect to U, so we update C^2 and D^2, and apply σ to the current state of play of $\mathcal{G}_{0,2}$ to find d'' and to the current state of play of $\mathcal{G}_{1,2}$ to find d'''.

We choose d'' smaller levels for the interaction of U with respect to S and d''' larger levels for the interaction of U with respect to T. We start building the initial segment of U with respect to T, then extend it to the initial segment of U with respect to S.

We play the mid-game of $\mathcal{G}_{0,2}$ until the builder has completed the construction of S and the architect has called for the completion of U. In the process we have completed the initial segment of U with respect to T, and the final segment of S with respect to T so we update E^1 and C^1.

Then we play the mid-game of $\mathcal{G}_{1,2}$ and complete the play of that game with the final segments of T and U. Finally, we update D^0 and E^0 on the new elements of T and U which complete the games $\mathcal{G}_{0,1}$ and $\mathcal{G}_{0,2}$.

As in the previous cases, since σ is a winning strategy for the architect, the set $\{S, T, U\}$ we have constructed is the required triangle. $\dashv$

10.6. Free Sets

Our next goal is the construction of a subset of $\mathcal{T}(\omega)$ of order type ω^{ω^ω} which is 0-homogeneous for global pairs.

Recall the characterization of subsets of G_ω of order type at least ω^s that dates back to the late 1960's or early 1970's. (see [42, 41, 65]).

10.39 Definition. A non-empty set $S \subseteq \{\sigma \in G_\omega : \min(\sigma) = n\}$ is *free above coordinate* k if for every $\mathbf{x} = \langle x_0, x_1, \ldots, x_n\rangle \in S$, there is an infinite set $N \subseteq \omega$ so that for each $x' \in N$, the set of extensions of $\langle x_0, x_1, \ldots, x_k, x'\rangle$ in S is non-empty. The set S is *free in* s *coordinates* if there are s coordinates above which it is free.

10.40 Lemma (See Lemma 7.2.2 of [65]). *A set* $S \subseteq \{\sigma \in G_\omega : \min(\sigma) = n\}$ *has* $\mathrm{ot}(S) \geq \omega^s$ *if and only if there is a subset* $V \subseteq S$ *so that* V *is free in* s *coordinates.*

We would like to adapt this idea to sets of node labeled trees from $\mathcal{T}(\beta)$. By an abuse of notation, write $t \in (T, D) \in X$ to mean that $t \in T$ for some $(T, D) \in X$. The next definition facilitates our discussion. Recall that $e_\beta(s)$ is the ordinal of s.

10.41 Definition. For $\beta \leq \omega$ and any $s \in (S, C) \in \mathcal{T}^*(\beta)$, call s a *signal node* if either $|C(s)| > 1$ or $e_\beta(s)$ limit and $|C(s)| = 1$.

Recall Definition 10.24 of relaxed initial segments of trees in $\mathcal{T}(\beta)$. The first three parts of the next definition guarantee that locally Γ-free sets have nice regularity properties, and the last three guarantee (1) signal nodes are introduced whenever there is no constraint, (2) signal nodes are given large node labels, and (3) there are arbitrarily large starts for extensions of relaxed initial segments of trees in the collection. The definition of Γ-free from locally Γ-free guarantees that there are arbitrarily large new starts for trees as well.

10.42 Definition. Suppose $\beta \leq \omega$ and $0 \notin \Gamma \in [\beta + 1]^{<\omega}$. A non-empty set X of node labeled trees from $\mathcal{T}(\beta)$ is *locally* Γ*-free for* β if the following conditions are satisfied:

1. (commonality) if $\beta > 0$, then every tree in X has a proper relaxed initial segment and every local pair from X has a common proper relaxed initial segment and otherwise is disjoint;
2. (conformity) if $r \in (S, C) \in X$ and $k \in C(r) \neq \emptyset$, then there is some relaxed $(T, D) \sqsubseteq (S, C)$ so that $r \sqsubset \max(T)$ and if the ordinal of r is a successor, then $\sharp(r, T) = k$;

3. (Γ-signality) for any signal node $r \in (S, C) \in X$, either $e_\beta(r) \in \Gamma$ or for some $p \sqsubset r$ with $e_\beta(p) = \omega$, there is a $k \in C(p)$ so that $e_\beta(r) = k$;

4. (Γ-forecasting) for any relaxed $(S, C) \sqsubseteq (T, D) \in X$, if $\gamma_i \in \Gamma$, then there is some signal node $r \sqsubset \max(S)$ with $e_\beta(r) = \gamma_i$; and if $p \sqsubset \max(S)$ is a signal node, $k \in C(p)$, and $e_\beta(p) = \omega$ is a limit ordinal, then there is some signal node $r \sqsubset \max(S)$ with $e_\beta(r) = k$;

5. (signal size) for any signal node $r \in (S, C) \in X$, the inequality $|C(r)| < \max(r)$ holds, and $\max(t) < \max(r)$ implies $\max(t) < |C(r)|$ for all $t \in (T, D) \in X$;

6. (push-up) for every $k < \omega$ and every relaxed initial segment $(T, D) \sqsubset (U, E) \in X$, there is some complete extension $(V, F) \sqsupset (T, D)$ in X whose new elements start above k, i.e. $k < \min(\bigcup V \cup \operatorname{ran}(F) - \bigcup T \cup \operatorname{ran}(D))$.

We say X is Γ*-free for* β if it is locally Γ-free for β, and for all $k < \omega$, there is some $\langle m \rangle \in (S, C) \in X$ such that $k < |C(\langle m \rangle)|$ if $\beta \in \Gamma$ and $k < m$ otherwise.

By an abuse of notation, for a collection X of node labeled trees from $\mathcal{T}(\beta)$, we let $\operatorname{ot}(X) = \operatorname{ot}(\{S : \exists C (S, C) \in X\})$.

10.43 Lemma. *For all $\beta \leq \omega$, for all $0 \notin \Gamma \in [\beta + 1]^{<\omega}$, if X is Γ-free for β, then $\operatorname{ot}(X) \geq \zeta(\beta, \Gamma)$ where*

$$\zeta(\beta, \Gamma) := \begin{cases} \omega & \text{if } \beta = 0, \\ \omega^2 & \text{if } \beta > 0 \text{ and } \Gamma = \emptyset, \\ \omega^{\omega^{|\Gamma|}} & \text{if } \beta > 0 \text{ and } \omega \notin \Gamma \neq \emptyset, \text{ and} \\ \omega^{\omega^\omega} & \text{otherwise.} \end{cases}$$

Proof. Relaxed trees, especially with a specified node as an initial segment of the max, play an important role in the definition of free and locally free. Here is some notation to facilitate the discussion. For any set X of node labeled trees, define $X(t) := \{(T, D) \in X : t \in (T, D)\}$.

10.44 Claim. *If X is Γ-free for $\beta = 0$ and $0 \notin \Gamma \subseteq 1$, then $\operatorname{ot}(X) \geq \omega$.*

Proof. Since $0 \notin \Gamma \subseteq 1$, it follows that $\Gamma = \emptyset$. Since any Γ-free for $\beta = 0$ set X has arbitrarily large roots, it must have order type at least ω. $\dashv$

For $1 \leq \beta \leq \omega$, $\Gamma \subseteq \beta + 1$, Y a set of node labeled trees from $\mathcal{T}(\beta)$ and $m < \omega$, define $\rho(\beta, \Gamma, Y, m) := 0$ unless $Y(\langle m \rangle) \neq \emptyset$ is locally Γ-free for β and there is some $(S, C) \in Y$ with $\langle m \rangle \in (S, C)$, and in the latter case, set

$$\rho(\beta, \Gamma, Y, m) := \begin{cases} 1, & \text{if } \Gamma = \emptyset, \\ \omega^{\omega^\ell}, & \text{if } \Gamma \neq \emptyset \text{ and } \beta = \max(\Gamma) \text{ limit}, \\ \omega^{\omega^\mu \cdot \ell}, & \text{otherwise,} \end{cases}$$

where, for non-empty Γ, $\ell := |C(s)| - 1$ for s the least signal node of (S, C), $\mu := |\Gamma| - 1$. This function is well-defined, since if $Y(\langle m\rangle) \neq \emptyset$ is locally Γ-free for β with Γ non-empty, then all elements of $Y(\langle m\rangle)$ have a proper relaxed initial segment in common with (S, C) which must include the least signal node of (S, C).

Let $*(\beta, \Gamma)$ be the following statement.

$*(\beta,\Gamma)$ For all locally Γ-free for β sets Y, if $\langle m\rangle \in (S, C) \in Y$, then $\mathrm{ot}(Y(\langle m\rangle)) \geq \rho(\beta, \Gamma, Y, m)$.

10.45 Claim. *For all $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta + 1$, if X is Γ-free for β and $*(\beta, \Gamma)$ holds, then* $\mathrm{ot}(X) \geq \zeta(\beta, \Gamma)$.

Proof. Use induction on n to prove the claim for subsets $\Gamma \subseteq \omega$ of size n.

To start the induction, consider subsets of size 0. If X is $\emptyset$-free for $\beta \geq 1$, then by definition, $X(\langle m\rangle)$ is non-empty for infinitely many m, and by commonality and push-up, $\mathrm{ot}(X(\langle m\rangle)) \geq \omega$, so $\mathrm{ot}(X) \geq \omega^2 = \zeta(\beta, \emptyset)$.

Next assume the claim is true for subsets of size k and that $n = k + 1$. If X is Γ-free for $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta + 1$ satisfies $\omega \notin \Gamma$ and $|\Gamma| = k + 1$, then there are arbitrarily large ℓ for which there are $m \in (S, C) \in X$ with $\ell < |C(\langle m\rangle)|$ if $\beta \in \Gamma$ and with $\ell < m$ otherwise. In the latter case, by Γ-forecasting and by signal size, there are arbitrarily large ℓ for which the first signal node $s \in (S, C) \in X$ has $\ell < |C(s)|$. Since $*(\beta, \Gamma)$ holds, it follows that there are arbitrarily large $\ell < m$ with $\mathrm{ot}(X(\langle m\rangle)) \geq \omega^{\omega^k \ell}$ for $k = |\Gamma| - 1$, hence $\mathrm{ot}(X) \geq \omega^{\omega^{k+1}} = \zeta(\beta, \Gamma)$ as desired.

Therefore by induction, the claim holds for all finite subsets $\Gamma \subseteq \omega$.

To complete the proof, consider Γ with $\omega \in \Gamma$. Then $\beta = \omega$. Suppose X is Γ-free for ω and $\omega \in \Gamma$. Then the root node of every tree in X is a signal node. Also X has arbitrarily large values for $|C(\langle m\rangle)|$ by the definition of Γ-free for $\beta = \omega \in \Gamma$. Hence from $*(\omega, \Gamma)$ it follows that $\mathrm{ot}(X(\langle m\rangle)) \geq \omega^{\omega^\ell}$ for $\ell = |C(\langle m\rangle)| - 1$, so $\mathrm{ot}(X) = \omega^{\omega^\omega} = \zeta(\omega, \Gamma)$ as required. ⊣

10.46 Claim. *For all $\beta \geq 1$ and $0 \notin \Gamma \subseteq \beta+1$, the statement $*(\beta, \Gamma)$ holds.*

Proof. Suppose Y is locally $\emptyset$-free for $\beta \geq 1$ and $\langle m\rangle \in (S, C) \in Y$. Then by commonality and push-up, $\mathrm{ot}(Y(\langle m\rangle)) \geq \omega$, so $*(\beta, \emptyset)$ holds.

Use induction on β to show that for all non-empty $0 \notin \Gamma \subseteq \beta + 1$, the statement $*(\beta, \Gamma)$ holds. For the basis case, $\beta = 1$, the only case to be considered is $\Gamma = \{1\}$. Suppose Y is locally $\{1\}$-free and $\langle m\rangle \in (S, C) \in Y$. Then $\langle m\rangle$ is a signal node, and $Z := \{\bigcup T : (T, D) \in Y(\langle m\rangle)\}$ is free in $|C(\langle m\rangle)|$ coordinates in the sense of Definition 10.39 by conformity and push-up. Thus Z has order type $\omega^{|C(\langle m\rangle)|}$ by Lemma 10.40. Hence $Y(\langle m\rangle)$ has this order type as well, so $*(1, \{1\})$ holds.

For the induction step, assume $*(\beta')$ is true for all β' with $1 \leq \beta' < \beta$. Suppose Γ is non-empty with $0 \notin \Gamma \subseteq \beta + 1$, Y is locally Γ-free for β and $\langle m\rangle \in (S, C) \in Y$. It follows that $Y(\langle m\rangle)$ is also locally Γ-free for β. Let

(S^-, C^-) be the minimal proper relaxed initial segment of (S, C), required by commonality. Then (S^-, C^-) is a common initial segment of all trees in Y. Let $\langle m, m^-\rangle$ be the unique initial segment of $\max(S^-)$ of length 2.

Case 1. $\max(\Gamma) < \beta$ or $\max(\Gamma) = \beta = \omega$.

For each $(T, D) \in Y$, the *derived tree* $(\hat{T}, \hat{D})$ is defined by $\hat{t} \in \hat{T}$ if and only if $\langle m, m^-\rangle \sqsubseteq \langle m\rangle^\frown \hat{t} \in T$, and $\hat{D}(\hat{t}) = D(\langle m\rangle^\frown \hat{t})$.

Let Z be the collection of derived trees. Note that $\langle m^-\rangle$ is an element of every tree in Z. Let $\beta' = \beta - 1$ and $\Gamma' = \Gamma$ if β is finite, and let $\beta' = m$ and $\Gamma' = (\Gamma - \{\omega\}) \cup C(\langle m\rangle)$ otherwise. Then $Z = Z(\langle m^-\rangle)$ is locally Γ'-free for β'. Also, $\mathrm{ot}(Y(\langle m\rangle)) \geq \mathrm{ot}(Z(\langle m^-\rangle))$, so in this case, the desired inequality follows by the induction hypothesis.

Case 2. $\Gamma = \{\zeta + 1\}$.

Consider the set $E \subseteq \mathcal{T}(1)$ of $\langle m, k_1, k_2, \ldots, k_m\rangle$ such that there is a $(T, D) \in Y$ such that for all $1 \leq i \leq m$, $\langle m, k_i\rangle \in T$. By conformity and push-up, the set E is free in $\ell = |C(\langle m\rangle)| - 1$ many coordinates, so it has order type ω^ℓ, by Lemma 10.40. Thus $\mathrm{ot}(Y(\langle m\rangle)) \geq \mathrm{ot}(E) = \omega^\ell = \rho(\beta, \gamma, Y, m)$ as required.

Case 3. $\zeta + 1 \in \Gamma \neq \{\zeta + 1\}$.

Notice that every tree (T, D) in $Y(\langle m\rangle)$ may be thought of as a collection of m node labeled trees from $\mathcal{T}(\zeta)$ extending from the root $\langle m\rangle$.

Call an initial segment (T, D) of a tree in $Y(\langle m\rangle)$ *large* if $\max(T)$ is a prepartition node with ordinal 0 such that $\sharp(s, T) = \max(C(s))$ for all proper $s \sqsubset t$ with $|s| > 1$. Every element of $Y(\langle m\rangle)$ has exactly $|C(\langle m\rangle)|$ many large initial segments.

Let $\Gamma' = \Gamma - \{\zeta + 1\}$ and set $\mu = |\Gamma'|$. Fix attention on a large (T, D) for which $\sharp(\langle m\rangle, T) < \max(C(\langle m\rangle))$, and let k be the least element of $C(\langle m\rangle)$ greater than $\sharp(\langle m\rangle, T)$. Let $E(T, D)$ be the set of initial segments (T', D') of elements of Y extending (T, D) to a tree with root $\langle m\rangle$ extended by exactly k subtrees from $\mathcal{T}(\zeta)$. Then $E(T, D)$ has order type ω^{ω^μ}, since the collection of trees that occur for the kth slot are Γ'-free for ζ. In fact the set of maximal large initial segments of these trees also has order type ω^{ω^μ}, since each has exactly ω extensions in $E(T, D)$ and ω^{ω^μ} is multiplicatively indecomposable. From this analysis, it follows that $\mathrm{ot}(Y(\langle m\rangle)) \geq \omega^{\omega^\mu \cdot \ell}$, where $\ell = |C(\langle m\rangle)| - 1$, so $*(\beta, \Gamma)$ holds in this final case.

Therefore by induction on β, the claim follows. $\dashv$

Now the lemma follows from Claims 10.44, 10.45 and 10.46. $\dashv$

10.47 Lemma. *Suppose h is a 2-partition of $\mathcal{T}(\omega)$ with 2 colors and $N \subseteq \omega$ is infinite with $1 < \min(N)$ and no two consecutive integers are in N. Further suppose a bounding function b and $H \subseteq N$ infinite are such that the builder wins every run of $\mathcal{G}(h, N)$ by playing conservatively for b and H. Then there is a set $Y \subseteq \mathcal{T}(\omega)$ of order type ω^{ω^ω} so that $h(S, T) = 0$ for all global pairs from Y.*

Proof. We will use recursion to build a $\{\omega\}$-free for ω set X such that every global pair $((S,C),(T,D))$ from X has a coarsening $((S,C'),(T,D'))$ which is a final play in a run of $\mathcal{G}(h,N)$ in which the builder plays conservatively for b and H. (By a *coarsening*, we mean that $C'(s) \subseteq C(s)$ and $D'(t) \subseteq D(t)$ for all $s \in S$, $t \in T$.) Since the builder wins the game, $h(S,T) = 0$ for such pairs. Thus $Y = \{S : (\exists C)((S,C) \in X)\}$ is the desired set, since, by Lemma 10.43, Y has order type ω^{ω^ω}.

To start the recursion, let X_0 be the set with only $(\emptyset,\emptyset)$ in it. For positive $j < \omega$, we enumerate the node labeled trees in $\bigcup_{i<j} X_i$ which are proper initial segments, starting with $(\emptyset,\emptyset) = (S'_{j,0}, C'_{j,0})$ and ending with (S'_{j,n_j}, C'_{j,n_j}). Speaking generally, in stage j, for each $k \leq n_j$, we consider the kth initial segment, $(S'_{j,k}, C'_{j,k})$, use moves of the architect and builder in $\mathcal{G}(h,H)$ to create a relaxed or complete extension, $(S_{j,k}, C_{j,k})$, using elements of H larger than anything mentioned up to that point. Then we let X_k be the set of all $(S_{j,k}, C_{j,k})$ for $k \leq n_j$.

A simple induction shows that there are only finitely many proper initial segments to be considered in each stage and they fall into at most three types: *trivial* (i.e. $(\emptyset,\emptyset)$), *ready for completion* (i.e. a relaxed initial segment (T,D) such that for all $s \subseteq \max(T)$ whose ordinal is a successor, $\sharp(s,T) = \max(D(s))$), or relaxed but not ready for completion.

In stage j, for the trivial initial segment, one starts $\mathcal{G}(h,H)$ at the beginning. Otherwise, for the kth initial segment, one continues a game in which the first tree is $(S'_{j,k}, C'_{j,k})$ and the second tree is the relaxed initial segment constructed to extend $(\emptyset,\emptyset)$ in this stage, namely $(S_{j,0}, C_{j,0})$.

In the games played, the architect uses the following strategy. She always directs the builder to create or extend the first tree. If the architect is making her first move on the kth initial segment and it is relaxed, then she declares the next node label size to be 0 and calls for completion if $(S'_{j,k}, C'_{j,k})$ is ready for completion, and for decision otherwise. Recall that if the architect calls for completion, then the node label of new elements is the empty set. Otherwise, the architect uses the least element of H larger than any used to that point as the size of the next node label, and calls for construction to the next decision node.

The builder always responds conservatively for H, b, and always plays large enough to have the play remain conservative for any possible game that could be constructed using coarsenings of the given trees.

Play stops at the end of the first move by the builder in which he creates a tree $(S_{j,k}, C_{j,k})$ which is relaxed or complete.

In any stage, with any starting initial segment, after finitely many steps of the game, the builder has constructed the required relaxed or complete extension. Since there are only finitely many trees to extend in a given round, eventually each round is finished. Therefore, the construction stops after ω rounds with a set $\overline{X} = \bigcup X_j$ of trees. Let X be the set of complete trees in $\overline{X}$. By construction, X is $\{\omega\}$-free, so by Lemma 10.43, the set $Y := \{S : (\exists C)((S,C) \in X)\}$ has order type ω^{ω^ω}.

To check that Y is the required set, suppose that $((S^0, C^0), (S^1, C^1))$ is a global pair from the set X with $(S^0, C^0) < (S^1, C^1)$. By the construction, every partition node of $(S^\varepsilon, C^\varepsilon)$ is the maximum of some relaxed segment of $(S^\varepsilon, C^\varepsilon)$, and every splitting node r has $e_\beta(r)$ in $C(\langle m^\varepsilon\rangle)$, where a *splitting node* $r \in S^\varepsilon$ is one of the form $s \cap t$ for distinct partition nodes $s, t \in S^\varepsilon$. Hence there are coarsenings (S^0, D^0) and (S^1, D^1) so that for all $r \in S^\varepsilon$,

$$D^\varepsilon(r) = \begin{cases} \{\sharp(r, S^\varepsilon(\emptyset, s]) : r \sqsubset s \text{ partition node}\} & e_\omega(r) \text{ successor,} \\ \{e_\omega(t) : r \sqsubset t \text{ splitting node}\} & e_\omega(r) \text{ limit,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $((S^0, D^0), (S^1, D^1))$ satisfies Definition 10.23 and is a global clear pair. If $\max(\bigcup S^0) > \max(\bigcup S^1)$, then the pair is inside, and otherwise it is outside. Use this knowledge in the architect's initial move; use the values of $|D^\varepsilon(r)|$ for the sizes of the node labels in the architect's moves; and orchestrate her moves to create the pair of node labeled trees when the builder is required to use the elements of $\bigcup S^0 \cup \operatorname{ran}(D^0) \cup \bigcup S^1 \cup \operatorname{ran}(D^1)$. Since the architect has no winning strategy, and the builder's plays were large enough for any coarsening, it follows that this run of the game is a win for the builder. Thus $h(S^0, S^1) = 0$ as desired. $\dashv$

10.7. Completion of the Proof

In this subsection, we complete the proof that $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)^2$ by assembling the appropriate lemmas. We start with $h : [\mathcal{T}(\omega)]^2 \to 2$. We apply the Ramsey Dichotomy 10.32 to h and $N = \omega$ to get $H \subseteq \omega$ infinite, a bounding function b and a favored player.

If the architect has a winning strategy by which she wins $\mathcal{G}(h, N)$ when the builder plays conservatively, then there is a 1-homogeneous triangle by Lemma 10.38.

Otherwise, the builder wins every run of $\mathcal{G}(h, N)$ by playing conservatively, so by Lemma 10.47, there is a set Y of order type ω^{ω^ω} so that all global pairs get color 0. Partition Y into sets Y_n so that $Y_0 < Y_1 < \cdots$, all pairs from Y_n are local, and $\operatorname{ot}(Y_n) \geq \omega^{\omega^{1+2n}}$. Apply Corollary 9.3 to each Y_n. If for some n, the result is a 1-homogeneous triangle, we are done. Otherwise, we get 0-homogeneous sets $Z_n \subseteq Y_n$ of order type $\omega^{\omega^{1+n}}$, and $Z = \bigcup Z_n$ is the 0-homogeneous set required for completion of the proof of the theorem.

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3. Coherent Sequences

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Contents

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A transfinite sequence $C_\xi \subseteq \xi$ $(\xi < \theta)$ of sets may have a number of "coherence properties" and the purpose of this chapter is to study some of them, as well as some of their uses. Here, "coherence" usually means that the C_ξ's are chosen in some canonical way, beyond the natural requirement that C_ξ be closed and unbounded in ξ for all ξ. For example, choosing a canonical "fundamental sequence" of sets $C_\xi \subseteq \xi$ for $\xi < \varepsilon_0$ relying on the specific properties of the Cantor normal form for ordinals below the first ordinal satisfying the equation $x = x^x$ is a basis for a number of important results in proof theory. In set theory, one is interested in longer sequences as well and usually has a different perspective in applications, so one is naturally led to use some other tools beside the Cantor normal form. It turns out that the sets C_ξ can not only be used as "ladders" for climbing up in recursive constructions but also as tools for "walking" from an ordinal to a smaller one. This notion of a "walk" and the corresponding "distance functions" constitute the main body of study in this chapter. We show that the resulting "metric theory of ordinals" not only provides a unified approach to a number of classical problems in set theory but also has its own intrinsic interest. For example, from this theory one learns that the triangle inequality of an ultrametric

$$e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$$

has three versions, depending on the natural ordering between the ordinals α, β and γ, that are of a quite different character and are occurring in quite different places and constructions in set theory. The most frequent occurrence is the case $\alpha < \beta < \gamma$ when the triangle inequality becomes something that one can call the "transitivity" of e. Considerably more subtle is the case $\alpha < \gamma < \beta$ of this inequality. It is this case of the inequality that captures most of the coherence properties found in this chapter. Another thing one learns from this theory is the special role of the first uncountable ordinal in this theory. Any natural coherence requirement on the sets C_ξ $(\xi < \theta)$ that one finds in this theory is satisfiable in the case $\theta = \omega_1$. The first uncountable cardinal is the only cardinal on which the theory can be carried out without relying on additional axioms of set theory. The first uncountable cardinal is the place where the theory has its deepest applications as well as its most important open problems. This special role can perhaps be explained by the fact that many set-theoretical problems, especially those coming from other fields of mathematics, are usually concerned only about the duality between the countable and the uncountable rather than some intricate relationship between two or more uncountable cardinalities. This is of course not to say that an intricate relationship between two or more uncountable cardinalities may not be a profitable detour in the course of solving such a problem. In fact, this is one of the reasons for our attempt to develop the metric theory of ordinals without restricting ourselves only to the realm of countable ordinals.

The chapter is organized as a discussion of five basic distance functions on ordinals, $\rho, \rho_0, \rho_1, \rho_2$ and ρ_3, and the reader may choose to follow the analysis of any of these functions in various contexts. The distance functions will naturally lead us to many other derived objects, most prominent of which is the "square-bracket operation" that gives us a way to transfer the quantifier "for every unbounded set" to the quantifier "for every closed and unbounded set". This reduction of quantifiers has proven to be quite useful in constructions of various mathematical structures, some of which have been mentioned or reproduced here.

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1. The Space of Countable Ordinals

This is by far the most interesting space considered in this chapter. There are many mathematical problems whose combinatorial essence can be reformulated as a problem about ω_1, the smallest uncountable structure. What we mean by "structure" is ω_1 together with a system C_α $(\alpha < \omega_1)$ of fundamental sequences, i.e. a system with the following two properties:

(a) $C_{\alpha+1} = \{\alpha\}$,

(b) C_α is an unbounded subset of α of order-type ω, whenever α is a countable limit ordinal > 0.

Such a sequence we shall simply call a *C-sequence*, here and when we generalize to ω_1 replaced by a general regular κ and C_α is a closed unbounded subset of α for limit $\alpha < \kappa$.

Despite its simplicity, this structure can be used to derive virtually all other known structures that have been defined so far on ω_1. There is a natural recursive way of picking up the fundamental sequences C_α, a recursion that refers to the Cantor normal form which works well for, say, ordinals $< \varepsilon_0$.[1] For longer fundamental sequences one typically relies on some other principles of recursive definition and one typically works with fundamental sequences with as few extra properties as possible. We shall see that the following assumption is what is frequently needed and will therefore be implicitly assumed whenever necessary:

(c) If α is a limit ordinal, then C_α does not contain limit ordinals.

1.1 Definition. A *step* from a countable ordinal β towards a smaller ordinal α is the minimal point of C_β that is $\geq \alpha$. The cardinality of the set $C_\beta \cap \alpha$, or better to say the order-type of this set, is the *weight* of the step.

1.2 Definition. A *walk* (or a *minimal walk*) from a countable ordinal β to a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$ such that for each $i < n$, the ordinal β_{i+1} is the step from β_i towards α.

[1] One is tempted to believe that the recursion can be stretched all the way up to ω_1 and this is probably the way P.S. Alexandroff found his famous Pressing Down Lemma (see [1] and [2, appendix]).

Analysis of this notion leads to several two-place functions on ω_1 that give a rich structure with many applications. So let us describe some of these functions.

1.3 Definition. The *full code* of the walk is the function $\rho_0 : [\omega_1]^2 \longrightarrow \omega^{<\omega}$ defined recursively by

$$\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

where $\rho_0(\alpha, \alpha) = 0$,[2] and the symbol $^\frown$ refers to the sequence obtained by concatenating the one-term sequence $\langle |C_\beta \cap \alpha| \rangle$ with the already known finite sequence $\rho_0(\alpha, \min(C_\beta \setminus \alpha)$ of integers. Clearly, knowing $\rho_0(\alpha, \beta)$ and the ordinal β one can reconstruct the (*upper*) trace

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \dots, \beta_n\},$$

remembering that $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$, of the walk from β to α. The *lower trace* is defined to be

$$\mathrm{L}(\alpha, \beta) = \{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\},$$

where $\lambda_i = \max(\bigcup_{j=0}^{i} C_{\beta_j} \cap \alpha)$ for $i < n$ and so $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$.

1.4 Definition. The *full lower trace* of the minimal walk is the function $\mathrm{F} : [\omega_1]^2 \longrightarrow [\omega_1]^{<\omega}$ defined recursively by

$$\mathrm{F}(\alpha, \beta) = \mathrm{F}(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} \mathrm{F}(\xi, \alpha),$$

where $\mathrm{F}(\gamma, \gamma) = \{\gamma\}$ for all γ.

Clearly, $\mathrm{F}(\alpha, \beta) \supseteq \mathrm{L}(\alpha, \beta)$ but $\mathrm{F}(\alpha, \beta)$ is considerably larger than $\mathrm{L}(\alpha, \beta)$ as it includes also the traces of walks between any two ordinals $\leq \alpha$ that have ever been referred to during the walk from β to α. The following two properties of the full lower trace are straightforward to check (see [66]).

1.5 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\mathrm{F}(\alpha, \gamma) \subseteq \mathrm{F}(\alpha, \beta) \cup \mathrm{F}(\beta, \gamma)$,

(b) $\mathrm{F}(\alpha, \beta) \subseteq \mathrm{F}(\alpha, \gamma) \cup \mathrm{F}(\beta, \gamma)$.

1.6 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_0(\alpha, \beta) = \rho_0(\min(\mathrm{F}(\beta, \gamma) \setminus \alpha), \beta)^\frown \rho_0(\alpha, \min(\mathrm{F}(\beta, \gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha, \gamma) = \rho_0(\min(\mathrm{F}(\beta, \gamma) \setminus \alpha), \gamma)^\frown \rho_0(\alpha, \min(\mathrm{F}(\beta, \gamma) \setminus \alpha))$.

[2] Technically speaking, ρ_0 operates on $[\omega_1]^2$ so $\rho_0(\alpha, \alpha) = 0$ makes no sense. What we mean is that whenever the formal recursive definition of $\rho_0(\alpha, \beta)$ involves the term $\rho_0(\alpha, \alpha)$ we take it to be equal to 0. This will be applied frequently in this chapter, not always in explicit form.

1.7 Definition. The ordering $<_c$ on ω_1 is defined as follows:

$$\alpha <_c \beta \quad \text{iff} \quad \rho_0(\xi,\alpha) <_r \rho_0(\xi,\beta),$$

where $\xi = \Delta(\alpha,\beta) = \min\{\eta \leq \min\{\alpha,\beta\} : \rho_0(\eta,\alpha) \neq \rho_0(\eta,\beta)\}$. Here $<_r$ refers to the *right lexicographical ordering* on $\omega^{<\omega}$ defined by letting $s <_r t$ iff s is an end-extension of t or $s(i) < t(i)$ for $i = \min\{j : s(j) \neq t(j)\}$.

1.8 Lemma. *The Cartesian square of the total ordering $<_c$ of ω_1 is the union of countably many chains.*

Proof. It suffices to decompose the set of all pairs (α,β) where $\alpha < \beta$. To each such pair we associate a hereditarily finite set $p(\alpha,\beta)$ which codes the finite structure obtained from $\mathrm{F}(\alpha,\beta)\cup\{\beta\}$ by adding relations that describe the way ρ_0 acts on it. To show that this parametrization works, suppose we are given two pairs (α,β) and (γ,δ) such that

$$p(\alpha,\beta) = p(\gamma,\delta) = p \quad \text{and} \quad \alpha <_c \gamma.$$

We must show that $\beta \leq_c \delta$. Let

$$\begin{aligned} \xi_{\alpha\beta} &= \min(\mathrm{F}(\alpha,\beta) \setminus \Delta(\alpha,\gamma)) \quad \text{and} \\ \xi_{\gamma\delta} &= \min(\mathrm{F}(\gamma,\delta) \setminus \Delta(\alpha,\gamma)). \end{aligned}$$

Note that $\mathrm{F}(\alpha,\beta)\cap\Delta(\alpha,\gamma) = \mathrm{F}(\gamma,\delta)\cap\Delta(\alpha,\gamma)$ so $\xi_{\alpha\beta}$ and $\xi_{\gamma\delta}$ correspond to each other in the isomorphism of the (α,β) and (γ,δ) structures. It follows that:

$$\begin{aligned} \rho_0(\xi_{\alpha\beta},\alpha) &= \rho_0(\xi_{\gamma\delta},\gamma)(= t_{\alpha,\gamma}), \\ \rho_0(\xi_{\alpha\beta},\beta) &= \rho_0(\xi_{\gamma\delta},\delta)(= t_{\beta,\delta}). \end{aligned}$$

Applying Lemma 1.6 we get:

$$\begin{aligned} \rho_0(\Delta(\alpha,\gamma),\alpha) &= t_{\alpha\gamma}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}), \\ \rho_0(\Delta(\alpha,\gamma),\gamma) &= t_{\alpha\gamma}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta}). \end{aligned}$$

It follows that $\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}) \neq \rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta})$. Applying Lemma 1.6 for β and δ and the ordinal $\Delta(\alpha,\gamma)$ we get:

$$\rho_0(\Delta(\alpha,\gamma),\beta) = t_{\beta\delta}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\alpha\beta}), \tag{3.1}$$

$$\rho_0(\Delta(\alpha,\gamma),\delta) = t_{\beta\delta}{}^\frown\rho_0(\Delta(\alpha,\gamma),\xi_{\gamma\delta}). \tag{3.2}$$

It follows that $\rho_0(\Delta(\alpha,\gamma),\beta) \neq \rho_0(\Delta(\alpha,\gamma),\delta)$. This shows $\Delta(\alpha,\gamma) \geq \Delta(\beta,\delta)$. A symmetric argument shows the other inequality $\Delta(\beta,\delta) \geq \Delta(\alpha,\gamma)$. It follows that

$$\Delta(\alpha,\gamma) = \Delta(\beta,\delta)(= \bar{\xi}).$$

Our assumption is that $\rho_0(\bar{\xi},\alpha) <_r \rho_0(\bar{\xi},\gamma)$ and since these two sequences have $t_{\alpha\gamma}$ as common initial part, this reduces to

$$\rho_0(\bar{\xi},\xi_{\alpha\beta}) <_r \rho_0(\bar{\xi},\xi_{\gamma\delta}). \tag{3.3}$$

On the other hand $t_{\beta\delta}$ is a common initial part of $\rho_0(\bar{\xi},\beta)$ and $\rho_0(\bar{\xi},\delta)$, so their lexicographical relationship depends on their tails which by (3.1) and (3.2) are equal to $\rho_0(\bar{\xi},\xi_{\alpha\beta})$ and $\rho_0(\bar{\xi},\xi_{\gamma\delta})$ respectively. Referring to (3.3) we conclude that indeed $\rho_0(\bar{\xi},\beta) <_r \rho_0(\bar{\xi},\delta)$, i.e. $\beta <_c \delta$. ⊣

1.9 Notation. *Well-ordered sets of rationals.* The set $\omega^{<\omega}$ ordered by the right lexicographical ordering $<_r$ is a particular copy of the rationals of the interval $(0,1]$ which we are going to denote by $\mathbb{Q}_r$ or simply by $\mathbb{Q}$. The next lemma shows that for a fixed α, $\rho_0(\xi,\alpha)$ is a strictly increasing function of ξ from α into $\mathbb{Q}_r$. Let $(\rho_0)_\alpha$ denote this function which we identify with its range, i.e. view as a member of the tree $\sigma\mathbb{Q}_r$ of all well-ordered subsets of $\mathbb{Q}_r$, ordered by end-extension.

1.10 Lemma. *$\rho_0(\alpha,\gamma) <_r \rho_0(\beta,\gamma)$ whenever $\alpha<\beta<\gamma$.*

At this point we recall several standard concepts for trees of height ω_1, concepts that generally figure in what follows: A tree of height ω_1 is an *Aronszajn tree* if all of its levels and chains are countable. A tree of height ω_1 is a *special Aronszajn tree* if it is an Aronszajn tree that admits a decomposition into countably many antichains or, equivalently, admits a strictly increasing map into the rationals. Finally, a tree of height ω_1 is a *Souslin tree* if all of its chains and antichains countable.

The sequence $(\rho_0)_\alpha$ $(\alpha<\omega_1)$ of members of $\sigma\mathbb{Q}_r$ naturally determines the subtree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \omega_1\}.$$

Note that for a fixed α, the restriction $(\rho_0)_\beta\restriction\alpha$ is determined by the way $(\rho_0)_\beta$ acts on the finite set $\mathrm{F}(\alpha,\beta)$. This is the content of Lemma 1.6. Hence all levels of $T(\rho_0)$ are countable, and therefore $T(\rho_0)$ is a particular example of an Aronszajn tree. We shall now see that $T(\rho_0)$ is in fact a special Aronszajn tree. The proof of this will depend on the following straightforward fact.

1.11 Lemma. *$\{\xi<\beta : \rho_0(\xi,\beta)=\rho_0(\xi,\gamma)\}$ is a closed subset of β whenever $\beta<\gamma$.*

It follows that $T(\rho_0)$ does not branch at limit levels. From this we can conclude that $T(\rho_0)$ is a special subtree of $\sigma\mathbb{Q}$ since this is easily seen to be so for any subtree of $\sigma\mathbb{Q}$ which is finitely branching at limit nodes.

1.12 Definition. Identifying the power set of $\mathbb{Q}$ with the particular copy $2^{\mathbb{Q}}$ of the Cantor set, define for every countable ordinal α,

$$G_\alpha = \{x\in 2^{\mathbb{Q}} : x \text{ end-extends no } (\rho_0)_\beta\restriction\alpha \text{ for } \beta\ge\alpha\}.$$

1.13 Lemma. *G_α ($\alpha < \omega_1$) is an increasing sequence of proper G_δ-subsets of the Cantor set whose union is equal to the Cantor set.*

1.14 Lemma. *The set $X = \{(\rho_0)_\beta : \beta < \omega_1\}$ considered as a subset of the Cantor set $2^{\mathbb{Q}}$ has universal measure zero.*

Proof. Let μ be a given non-atomic Borel measure on $2^{\mathbb{Q}}$. For $t \in T(\rho_0)$, set

$$P_t = \{x \in 2^{\mathbb{Q}} : x \text{ end-extends } t\}.$$

Note that each P_t is a perfect subset of $2^{\mathbb{Q}}$ and therefore is μ-measurable. Let

$$S = \{t \in T(\rho_0) : \mu(P_t) > 0\}.$$

Then S is a downward closed subtree of $\sigma\mathbb{Q}$ with no uncountable antichains. By an old result of Kurepa (see [55]), no Souslin tree admits a strictly increasing map into the reals (as for example $\sigma\mathbb{Q}$ does). It follows that S must be countable and so we are done. ⊣

1.15 Definition. The *maximal weight* of the walk is the two-place function $\rho_1 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

where we stipulate that $\rho_1(\alpha, \alpha) = 0$ for all $\alpha < \omega_1$.[3] Thus $\rho_1(\alpha, \beta)$ is simply the maximal integer appearing in the sequence $\rho_0(\alpha, \beta)$.

1.16 Lemma. *For all $\alpha < \beta < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}$ is finite,

(b) $\{\xi \leq \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$ is finite.

Proof. The proof is by induction. To prove (a) it suffices to show that for every $n < \omega$ and every $A \subseteq \alpha$ of order-type ω there is a $\xi \in A$ such that $\rho_1(\xi, \alpha) > n$. Let $\eta = \sup(A)$. If $\eta = \alpha$ one chooses arbitrary $\xi \in A$ with the property that $|C_\alpha \cap \xi| > n$, so let us consider the case $\eta < \alpha$. Let $\alpha_1 = \min(C_\alpha \setminus \eta)$. By the inductive hypothesis there is a $\xi \in A$ such that:

$$\xi > \max(C_\alpha \cap \eta) \quad \text{and} \quad \rho_1(\xi, \alpha_1) > n.$$

Note that $\rho_0(\xi, \alpha) = \langle |C_\alpha \cap \eta| \rangle^\frown \rho_0(\xi, \alpha_1)$, and therefore

$$\rho_1(\xi, \alpha) \geq \rho_1(\xi, \alpha_1) > n.$$

To prove (b) we show by induction that for every $A \subseteq \alpha$ of order-type ω there exists a $\xi \in A$ such that $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$. Let $\eta = \sup(A)$ and let $\beta_1 = \min(C_\beta \setminus \eta)$. Let $n = |C_\beta \cap \eta|$ and let

$$B = \{\xi \in A : \xi > \max(C_\beta \cap \eta) \quad \text{and} \quad \rho_1(\xi, \beta_1) > n\}.$$

[3] This is another use of the convention $\rho_1(\alpha, \alpha) = 0$ that is necessary for the recursive definition to work.

Then B is infinite, so by the induction hypothesis we can find $\xi \in B$ such that $\rho_1(\xi,\alpha) = \rho_1(\xi,\beta_1)$. Then

$$\rho_1(\xi,\beta) = \max\{n, \rho_1(\xi,\beta_1)\} = \rho_1(\xi,\beta_1),$$

so we are done. $\dashv$

1.17 Remark. Define $(\rho_1)_\alpha$ from ρ_1 just as $(\rho_0)_\alpha$ was defined from ρ_0 above. It follows that the sequence

$$(\rho_1)_\alpha : \alpha \longrightarrow \omega \ (\alpha < \omega_1)$$

of finite-to-one functions is *coherent* in the sense that $(\rho_1)_\alpha =^* (\rho_1)_\beta \restriction \alpha$ whenever $\alpha \leq \beta$. (Here $=^*$ for functions denotes agreement on all but finitely many arguments.) The corresponding tree

$$T(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t =^* (\rho_1)_\alpha\}$$

is a homogeneous, special Aronszajn tree with many other interesting properties, some of which we are going to describe here. For example, we have the following fact whose proof (see [66]) is quite analogous to that of Lemma 1.8.

1.18 Lemma. *The Cartesian square of $T(\rho_1)$ ordered lexicographically is the union of countably many chains.*

1.19 Definition. Consider the following extension of $T(\rho_1)$:

$$\widetilde{T}(\rho_1) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t\restriction\xi \in T(\rho_1) \text{ for all } \xi < \alpha\}.$$

If we order $\widetilde{T}(\rho_1)$ by the right lexicographical ordering $<_r$ we get a complete linearly ordered set. It is not continuous, as it contains jumps of the form

$$[t^\frown\langle m\rangle, t^\frown\langle m+1\rangle^\frown\vec{0}\,],$$

where $t \in T(\rho_1)$ and $m < \omega$. Removing the right-hand points from all the jumps we get a linearly ordered continuum which we denote by $\widetilde{A}(\rho_1)$ For the proof of the following, see [66].

1.20 Lemma. *$\widetilde{A}(\rho_1)$ is a homogeneous non-reversible ordered continuum that can be represented as the union of an increasing ω_1-sequence of Cantor sets.*

1.21 Definition. The set $\widetilde{T}(\rho_1)$ has another natural structure, a topology generated by the family of sets of the form

$$\widetilde{V}_t = \{u \in \widetilde{T}(\rho_1) : t \subseteq u\},$$

for t a node of $T(\rho_1)$ of successor length as a clopen subbase. Let $T^0(\rho_1)$ denote the set of all nodes of $T(\rho_1)$ of successor length. Then $\widetilde{T}(\rho_1)$ can be regarded as the set of all downward closed chains of the tree $T^0(\rho_1)$ and the topology on $\widetilde{T}(\rho_1)$ is simply the topology one obtains from identifying the power set of $T^0(\rho_1)$ with the cube $\{0,1\}^{T^0(\rho_1)}$ with its Tychonoff topology.[4] $\widetilde{T}(\rho_1)$ being a closed subset of the cube is compact. In fact $\widetilde{T}(\rho_1)$ has some very strong topological properties such as the property that closed subsets of $\widetilde{T}(\rho_1)$ are its retracts.

1.22 Lemma. *$\widetilde{T}(\rho_1)$ is a homogeneous compactum whose function space $\mathcal{C}(\widetilde{T}(\rho_1))$ is generated by a weakly compact subset.*[5]

Proof. The proof that $\widetilde{T}(\rho_1)$ is homogeneous is quite similar to the corresponding part of the proof of the Lemma 1.20. To see that $\widetilde{T}(\rho_1)$ is an Eberlein compactum, i.e. that the function space $\mathcal{C}(\widetilde{T}(\rho_1))$ is weak compactly generated, let $\{X_n\}$ be a countable antichain decomposition of $T(\rho_1)$ and consider the set $K = \{2^{-n}\chi_{\widetilde{V}_t} : n < \omega, t \in X_n\} \cup \{\chi_\emptyset\}$. Note that K is a weakly compact subset of $\mathcal{C}(\widetilde{T}(\rho_1))$ which separates the points of $\widetilde{T}(\rho_1)$. $\dashv$

The coherent sequence $(\rho_1)_\alpha : \alpha \longrightarrow \omega$ $(\alpha < \omega_1)$ of finite-to-one maps can easily be turned into a coherent sequence of maps that are actually one-to-one. For example, one way to achieve this is via the following formula:

$$\bar{\rho}_1(\alpha, \beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho_1(\xi, \beta) = \rho_1(\alpha, \beta)\}| + 1).$$

Define $(\bar{\rho}_1)_\alpha$ from $\bar{\rho}_1$ just as $(\rho_1)_\alpha$ was defined from ρ_1; then the $(\bar{\rho}_1)_\alpha$'s are one-to-one. From ρ_1 one has a natural sequence r_α $(\alpha < \omega_1)$ of elements of ω^ω defined as follows:

$$r_\alpha(n) = |\{\xi \leq \alpha : \rho_1(\xi, \alpha) \leq n\}|.$$

Note that r_β eventually dominates r_α whenever $\alpha + \omega < \beta$.

1.23 Definition. The sequences $e_\alpha = (\bar{\rho}_1)_\alpha$ $(\alpha < \omega_1)$ and r_α $(\alpha < \omega_1)$ can be used in describing a functor $G \longmapsto G^*$, which to every graph G on ω_1 associates another graph G^* on ω_1 as follows:

$$\{\alpha, \beta\} \in G^* \quad \text{iff} \quad \{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in G$$

for all $l < \Delta(r_\alpha, r_\beta)$ for which these preimages are both defined and different.

The proof of the following lemma can be found in [66].

1.24 Lemma. *Suppose that every uncountable family $\mathcal{F}$ of pairwise disjoint finite subsets of ω_1 contains two sets A and B such that $A \otimes B \subseteq G$.*[6] *Then the same is true about G^* provided the uncountable family $\mathcal{F}$ consists of finite cliques*[7] *of G^*.*

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[4] This is done by identifying a subset V of $T^0(\rho_1)$ with its characteristic function $\chi_V : T^0(\rho_1) \longrightarrow 2$.

[5] Compacta K with this property of their function spaces $\mathcal{C}(K)$ are known in the literature under the name of *Eberlein compacta*.

[6] Here, $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha \neq \beta\}$.

[7] A *clique* of G^* is a subset C of ω_1 with the property that $[C]^2 \subseteq G^*$.

1.25 Lemma. *If there is an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$ then ω_1 can be decomposed into countably many sets Σ such that $[\Sigma]^2 \subseteq G$.*

Proof. Fix an uncountable $\Gamma \subseteq \omega_1$ such that $[\Gamma]^2 \subseteq G^*$. For a finite binary sequence s of length equal to some $l+1$, set

$$\Gamma_s = \{\xi < \omega_1 : e(\xi, \alpha) = l \text{ for some } \alpha \text{ in } \Gamma \text{ with } s \subseteq r_\alpha\}.$$

Then the sets Γ_s cover ω_1 and $[\Gamma_s]^2 \subseteq G$ for all s. ⊣

1.26 Remark. Let G be the comparability graph of some Souslin tree T. Then for every uncountable family $\mathcal{F}$ of pairwise disjoint cliques of G (finite chains of T) there exist $A \neq B$ in $\mathcal{F}$ such that $A \cup B$ is a clique of G (a chain of T). However, it is not hard to see that G^* fails to have this property (i.e. the conclusion of Lemma 1.24). This shows that some assumption on the graph G in Lemma 1.24 is necessary. There are indeed many graphs that satisfy the hypothesis of Lemma 1.24. Many examples appear when one is trying to apply Martin's Axiom to some Ramsey-theoretic problems. Note that the conclusion of Lemma 1.24 is simply saying that the poset of all finite cliques of G^* is c.c.c. while its hypothesis is a bit stronger than the fact that the poset of all finite cliques of G is c.c.c. in all of its finite powers. Applying Lemma 1.25 to the case when G is the incomparability graph of some Aronszajn tree, we see that the statement saying that all Aronszajn trees are special is a purely Ramsey-theoretic statement in the same way Souslin's Hypothesis, that there are no Souslin trees, is.

2. Subadditive Functions

In this section we describe a metric feature of the space ω_1 of countable ordinals. One first encounters this feature by analyzing properties of the following function.

2.1 Definition. The *rho-function* $\rho : [\omega_1]^2 \longrightarrow \omega$ is defined recursion follows:

$$\rho(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},$$

where we stipulate that $\rho(\alpha, \alpha) = 0$ for all $\alpha < \omega_1$.

2.2 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho(\xi, \alpha) \leq n\}$ *is finite,*

(b) $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,

(c) $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

Proof. Note that $\rho(\alpha,\beta) \geq \rho_1(\alpha,\beta)$, so (a) follows from the corresponding property of ρ_1. The proof of (b) and (c) is simultaneous by induction on α, β and γ:

To prove (b), consider $n = \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$, and let

$$\xi_\alpha = \min(C_\gamma \setminus \alpha) \quad \text{and} \quad \xi_\beta = \min(C_\gamma \setminus \beta).$$

We have to show that $\rho(\alpha,\gamma) \leq n$.

Case 1^b. $\xi_\alpha = \xi_\beta$. Then by the inductive hypothesis,

$$\rho(\alpha,\xi_\alpha) \leq \max\{\rho(\alpha,\beta), \rho(\beta,\xi_\beta)\}.$$

From the definition of $\rho(\beta,\gamma) \leq n$ we get that $\rho(\beta,\xi_\beta) \leq \rho(\beta,\gamma)$, so replacing $\rho(\beta,\xi_\beta)$ by $\rho(\beta,\gamma)$ in the above inequality we get $\rho(\alpha,\xi_\alpha) \leq n$. Consider a $\xi \in C_\gamma \cap \alpha = C_\gamma \cap \beta$. By the inductive hypothesis

$$\rho(\xi,\alpha) \leq \max\{\rho(\xi,\beta), \rho(\alpha,\beta)\}.$$

From the definition of $\rho(\beta,\gamma)$ we see that $\rho(\xi,\beta) \leq \rho(\beta,\gamma)$, so replacing $\rho(\xi,\beta)$ with $\rho(\beta,\gamma)$ in the last inequality we get that $\rho(\xi,\alpha) \leq n$. Since $|C_\gamma \cap \alpha| = |C_\gamma \cap \beta| \leq \rho(\beta,\gamma) \leq n$, referring to the definition of $\rho(\alpha,\gamma)$ we conclude that $\rho(\alpha,\gamma) \leq n$.

Case 2^b. $\xi_\alpha < \xi_\beta$. Then $\xi_\alpha \in C_\gamma \cap \beta$, so

$$\rho(\xi_\alpha,\beta) \leq \rho(\beta,\gamma) \leq n.$$

By the inductive hypothesis

$$\rho(\alpha,\xi_\alpha) \leq \max\{\rho(\alpha,\beta), \rho(\xi_\alpha,\beta)\} \leq n.$$

Similarly, for every $\xi \in C_\gamma \cap \alpha \subseteq C_\gamma \cap \beta$,

$$\rho(\xi,\alpha) \leq \max\{\rho(\xi,\beta), \rho(\alpha,\beta)\} \leq n.$$

Finally $|C_\gamma \cap \alpha| \leq |C_\gamma \cap \beta| \leq \rho(\beta,\gamma) \leq n$. Combining these inequalities we get the desired conclusion $\rho(\alpha,\gamma) \leq n$.

To prove (c), consider now $n = \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$. We have to show that $\rho(\alpha,\beta) \leq n$. Let ξ_α and ξ_β be as above and let us consider the same two cases as above.

Case 1^c. $\xi_\alpha = \xi_\beta = \bar{\xi}$. Then by the inductive hypothesis

$$\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\bar{\xi}), \rho(\beta,\bar{\xi})\}.$$

This gives the desired bound $\rho(\alpha,\beta) \leq n$, since $\rho(\alpha,\xi_\alpha) \leq \rho(\alpha,\gamma) \leq n$ and $\rho(\beta,\xi_\beta) \leq \rho(\beta,\gamma) \leq n$.

Case 2^c. $\xi_\alpha < \xi_\beta$. Applying the inductive hypothesis again we get

$$\rho(\alpha,\beta) \leq \max\{\rho(\alpha,\xi_\alpha), \rho(\xi_\alpha,\beta)\} \leq n.$$

This completes the proof. $\dashv$

The following simple consequence shows that the function ρ has a considerably finer coherence property than ρ_1.

2.3 Lemma. *If $\alpha < \beta < \gamma$ and $\rho(\alpha, \beta) > \rho(\beta, \gamma)$, then $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$.*

2.4 Definition. Define $\bar{\rho} : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\bar{\rho}(\alpha, \beta) = 2^{\rho(\alpha,\beta)} \cdot (2 \cdot |\{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}| + 1).$$

Using the properties of ρ one easily checks the following facts about its stretching $\bar{\rho}$.

2.5 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$,*

(a) $\bar{\rho}(\alpha, \gamma) \neq \bar{\rho}(\beta, \gamma)$,

(b) $\bar{\rho}(\alpha, \gamma) \leq \max\{\bar{\rho}(\alpha, \beta), \bar{\rho}(\beta, \gamma)\}$,

(c) $\bar{\rho}(\alpha, \beta) \leq \max\{\bar{\rho}(\alpha, \gamma), \bar{\rho}(\beta, \gamma)\}$.

The following property of $\bar{\rho}$ is also sometimes useful (see [66]).

2.6 Lemma. *Suppose $\eta_\alpha \neq \eta_\beta < \min\{\alpha, \beta\}$ and $\bar{\rho}(\eta_\alpha, \alpha) = \bar{\rho}(\eta_\beta, \beta) = n$. Then $\bar{\rho}(\eta_\alpha, \beta), \bar{\rho}(\eta_\beta, \alpha) > n$.*

2.7 Definition. For $p \in \omega^{<\omega}$ define a binary relation $<_p$ on ω_1 by letting $\alpha <_p \beta$ iff $\alpha < \beta$, $\bar{\rho}(\alpha, \beta) \in |p|$, and

$$p(\bar{\rho}(\xi, \alpha)) = p(\bar{\rho}(\xi, \beta))$$

for any $\xi < \alpha$ such that $\bar{\rho}(\xi, \alpha) < |p|$.

2.8 Lemma.

(a) $<_p$ *is a tree ordering on* ω_1 *of height* $\leq |p| + 1$,

(b) $p \subseteq q$ *implies* $<_p \subseteq <_q$.

Proof. This follows immediately from Lemma 2.5. ⊣

2.9 Definition. For $x \in \omega^\omega$, set

$$<_x = \bigcup\{<_{x \restriction n} : n < \omega\}.$$

The proof of the following lemma can also be found in [66].

2.10 Lemma. *For every $p \in \omega^{<\omega}$ there is a partition of ω_1 into finitely many pieces such that if $\alpha < \beta$ belong to the same piece then there is a $q \supseteq p$ in $\omega^{<\omega}$ such that $\alpha <_q \beta$.*

2.11 Theorem. *For every infinite subset $\Gamma \subseteq \omega_1$, the set*

$$G_\Gamma = \{x \in \omega^\omega : \alpha <_x \beta \text{ for some } \alpha, \beta \in \Gamma\}$$

is a dense open subset of the Baire space.

Proof. This is an immediate consequence of Lemma 2.10. ⊣

2.12 Definition. For $\alpha < \beta < \omega_1$, let $\alpha <_{\bar\rho} \beta$ denote the fact that $\bar\rho(\xi,\alpha) = \bar\rho(\xi,\beta)$ for all $\xi < \alpha$. Then $<_{\bar\rho}$ is a tree ordering on ω_1 obtained by identifying α with the member $\bar\rho(\cdot,\alpha)$ of the tree $T(\bar\rho)$. Note that $<_{\bar\rho} \subseteq <_x$ for all $x \in \omega^\omega$ and that there exists an $x \in \omega^\omega$ such that $<_x = <_{\bar\rho}$ (e.g., one such x is the identity map $\mathrm{id} : \omega \longrightarrow \omega$).

The following result is an analogue of Lemma 2.10 for the incomparability relation, though its proof is considerably simpler.

2.13 Lemma. *If Γ is an infinite $<_{\bar\rho}$-antichain, the set*

$$H_\Gamma = \{x \in \omega^\omega : \alpha \not<_x \beta \textit{ for some } \alpha < \beta \textit{ in } \Gamma\}$$

is a dense open subset of Baire space.

2.14 Definition. For a family $\mathcal{F}$ of infinite $<_{\bar\rho}$-antichains, we say that a real $x \in \omega^\omega$ is $\mathcal{F}$-*Cohen* if $x \in G_\Gamma \cap H_\Gamma$ for all $\Gamma \in \mathcal{F}$. We say that x is $\mathcal{F}$-*Souslin* if no member of $\mathcal{F}$ is a $<_x$-chain or a $<_x$-antichain. We say that a real $x \in \omega^\omega$ is *Souslin* if the tree ordering $<_x$ on ω_1 has no uncountable chains nor antichains, i.e. when x is $\mathcal{F}$-Souslin for $\mathcal{F}$ equal to the family of all uncountable subsets of ω_1.

Note that since every uncountable subset of ω_1 contains an uncountable $<_{\bar\rho}$-antichain, if a family $\mathcal{F}$ refines the family of all uncountable $<_{\bar\rho}$-antichains, then every $\mathcal{F}$-Souslin real is Souslin. The following fact summarizes Theorems 2.11 and 2.13 and connects the two kinds of reals.

2.15 Theorem. *If $\mathcal{F}$ is a family of infinite $<_{\bar\rho}$-antichains, then every $\mathcal{F}$-Cohen real is $\mathcal{F}$-Souslin.*

2.16 Corollary. *If the density of the family of all uncountable subsets of ω_1 is smaller than the number of nowhere dense sets needed to cover the real line, then there is a Souslin tree.*

2.17 Remark. Recall that the *density* of a family $\mathcal{F}$ of infinite subsets of some set S is the minimal size of a family $\mathcal{F}_0$ of infinite subsets of S with the property that every member of $\mathcal{F}$ is refined by a member of $\mathcal{F}_0$. A special case of Corollary 2.16, when the density of the family of all uncountable subsets of ω_1 is equal to $\aleph_1$, was first observed by Miyamoto (unpublished).

2.18 Corollary. *Every Cohen real is Souslin.*

Proof. Every uncountable subset of ω_1 in the Cohen extension contains an uncountable subset from the ground model. So it suffices to consider the family $\mathcal{F}$ of all infinite $<_{\bar\rho}$-antichains from the ground model. ⊣

If ω_1 is a successor cardinal in the constructible universe, then $\bar{\rho}$ can be chosen to be coanalytic and so the transformation $x \longmapsto <_x$ will transfer combinatorial notions of Souslin, Aronszajn or special Aronszajn trees into the corresponding classes of reals that lie in the third level of the projective hierarchy. This transformation has been explored in several places in the literature (see e.g. [3, 21]).

2.19 Remark. We have just seen how the combination of the subadditivity properties (2.5(b),(c)) of the coherent sequence $\bar{\rho}_\alpha : \alpha \longrightarrow \omega$ ($\alpha < \omega_1$) of one-to-one mappings can be used in controlling the finite disagreement between them. It turns out that in many contexts the coherence and the subadditivities are essentially equivalent restrictions on a given sequence $e_\alpha : \alpha \longrightarrow \omega$ ($\alpha < \omega_1$). For example, the following construction shows that this is so for any sequence of finite-to-one mappings $e_\alpha : \alpha \longrightarrow \omega$ ($\alpha < \omega_1$).

2.20 Definition. Given a coherent sequence $e_\alpha : \alpha \longrightarrow \omega$ ($\alpha < \omega_1$) of finite-to-one mappings, define $\tau_e : [\omega_1]^2 \longrightarrow \omega$ as follows

$$\tau_e(\alpha, \beta) = \max\{\max\{e(\xi, \alpha), e(\xi, \beta)\} : \xi \leq \alpha \text{ and } e(\xi, \alpha) \neq e(\xi, \beta)\}.$$

2.21 Lemma. *For every $\alpha < \beta < \gamma < \omega_1$,*

(a) $\tau_e(\alpha, \beta) \geq e(\alpha, \beta)$,

(b) $\tau_e(\alpha, \gamma) \leq \max\{\tau_e(\alpha, \beta), \tau_e(\beta, \gamma)\}$,

(c) $\tau_e(\alpha, \beta) \leq \max\{\tau_e(\alpha, \gamma), \tau_e(\beta, \gamma)\}$.

Proof. Since (a) is true if $e(\alpha, \beta) = 0$, let us assume $e(\alpha, \beta) > 0$. By our convention, $e(\alpha, \alpha) = 0$ and so $e(\alpha, \alpha) \neq e(\alpha, \beta) = 0$. It follows that $\tau_e(\alpha, \beta) \geq \max\{\max\{e(\alpha, \alpha), e(\alpha, \beta)\} = e(\alpha, \beta)$. This shows (a).

To show (b), let $n = \max\{\tau_e(\alpha, \beta), \tau_e(\beta, \gamma)\}$. Suppose $\tau_e(\alpha, \gamma) > n$. Then we can choose $\xi \leq \alpha$ such that $e(\xi, \alpha) > n$ or $e(\xi, \gamma) > n$. If $e(\xi, \alpha) > n$ then $e(\xi, \beta) = e(\xi, \alpha) > n$ and so $e(\xi, \beta) \neq e(\xi, \gamma)$. It follows that $\tau_e(\beta, \gamma) \geq e(\xi, \beta) > n$, a contradiction. If $e(\xi, \gamma) > n$ then $e(\xi, \beta) = e(\xi, \gamma) > n$. In particular, $e(\xi, \alpha) \neq e(\xi, \beta)$. It follows that $\tau_e(\beta, \gamma) \geq e(\xi, \beta) > n$, a contradiction.

The proof of (c) is similar. ⊣

2.22 Definition. A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is *transitive* if

$$a(\alpha, \gamma) \leq \max\{a(\alpha, \beta), a(\beta, \gamma)\}$$

for all $\alpha < \beta < \gamma < \omega_1$.

Transitive maps occur quite frequently in set-theoretic constructions. For example, given a sequence A_α ($\alpha < \omega_1$) of subsets of ω that increases relative to the ordering $\subseteq^*$ of inclusion modulo a finite set, the mapping $a : [\omega_1]^2 \longrightarrow \omega$ defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}$$

is a transitive map. The transitivity condition by itself is not nearly as useful as its combination with the other subadditivity property (2.5(c)). Fortunately, there is a general procedure that produces a subadditive dominant to every transitive map.

2.23 Definition. For a transitive $a : [\omega_1]^2 \longrightarrow \omega$ define $\rho_a : [\omega_1]^2 \longrightarrow \omega$ recursively as follows:

$$\rho_a(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, a(\min(C_\beta \setminus \alpha), \beta), \\ \rho_a(\alpha, \min(C_\beta \setminus \alpha)), \rho_a(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}.$$

2.24 Lemma. *For all $\alpha < \beta < \gamma < \omega_1$ and $n < \omega$,*

(a) $\{\xi \leq \alpha : \rho_a(\xi, \alpha) \leq n\}$ is finite,

(b) $\rho_a(\alpha, \gamma) \leq \max\{\rho_a(\alpha, \beta), \rho_a(\beta, \gamma)\}$,

(c) $\rho_a(\alpha, \beta) \leq \max\{\rho_a(\alpha, \gamma), \rho_a(\beta, \gamma)\}$,

(d) $\rho_a(\alpha, \beta) \geq a(\alpha, \beta)$.

Proof. The proof of (a), (b), (c) is quite similar to the corresponding part of the proof of Lemma 2.2. This comes of course from the fact that the definition of ρ and ρ_a are closely related. The occurrence of the factor $a(\min(C_\beta \setminus \alpha), \beta)$ complicates a bit the proof that ρ_a is subadditive, and the fact that a is transitive is quite helpful in getting rid of the additional difficulty. The details are left to the interested reader. Given $\alpha < \beta$, for every step $\beta_n \to \beta_{n+1}$ of the minimal walk $\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \alpha$, we have $\rho_a(\alpha, \beta) \geq \rho_a(\beta_n, \beta_{n+1}) \geq a(\beta_n, \beta_{n+1})$ by the very definition of ρ_a. Applying the transitivity of a to this path of inequalities we get the conclusion (d). $\dashv$

2.25 Lemma. *$\rho_a(\alpha, \beta) \geq \rho_a(\alpha + 1, \beta)$ whenever $0 < \alpha < \beta$ and α is a limit ordinal.*

Proof. Recall the assumption (c) about the fixed C-sequence C_ξ $(\xi < \omega_1)$ on which all our definitions are based: if ξ is a limit ordinal > 0, then no point of C_ξ is a limit ordinal. It follows that if $0 < \alpha < \beta$ and α is a limit ordinal, then the minimal walk $\beta \to \alpha$ must pass through $\alpha + 1$ and therefore $\rho_a(\alpha, \beta) \geq \rho_a(\alpha + 1, \beta)$. $\dashv$

Let us now give an application of ρ_a to a classical phenomenon of occurrence of gaps in the quotient algebra $\mathcal{P}(\omega)/\mathrm{fin}$.

2.26 Definition. A *Hausdorff gap* in $\mathcal{P}(\omega)/\mathrm{fin}$ is a pair of sequences A_α $(\alpha < \omega_1)$ and B_α $(\alpha < \omega_1)$ such that

(a) $A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$ whenever $\alpha < \beta$, but

(b) there is no C such that $A_\alpha \subseteq^* C \subseteq^* B_\alpha$ for all α.

The following straightforward reformulation shows that a Hausdorff gap is just another instance of a nontrivial coherent sequence

$$f_\alpha : A_\alpha \longrightarrow 2 \ \ (\alpha < \omega_1),$$

where the domain A_α of f_α is not the ordinal α itself but a subset of ω and that the corresponding sequence of domains A_α $(\alpha < \omega_1)$ is a realization of ω_1 inside $\mathcal{P}(\omega)/\text{fin}$ in the sense that $A_\alpha \subseteq^* A_\beta$ whenever $\alpha < \beta$.

2.27 Lemma. *A pair of ω_1-sequences A_α $(\alpha < \omega_1)$ and B_α $(\alpha < \omega_1)$ form a Hausdorff gap iff the pair of ω_1-sequences $\bar{A}_\alpha = A_\alpha \cup (\omega \setminus B_\alpha)$ $(\alpha < \omega_1)$ and $\bar{B}_\alpha = \omega \setminus B_\alpha$ $(\alpha < \omega_1)$ has the following three properties:*

(a) $\bar{A}_\alpha \subseteq^* \bar{A}_\beta$ whenever $\alpha < \beta$,

(b) $\bar{B}_\alpha =^* \bar{B}_\beta \cap \bar{A}_\alpha$ whenever $\alpha < \beta$,

(c) there is no B such that $\bar{B}_\alpha =^* B \cap \bar{A}_\alpha$ for all α.

From now on we fix a strictly $\subseteq^*$-increasing chain A_α $(\alpha < \omega_1)$ of infinite subsets of ω and let $a : [\omega_1]^2 \longrightarrow \omega$ be defined by

$$a(\alpha, \beta) = \min\{n : A_\alpha \setminus n \subseteq A_\beta\}.$$

Let $\rho_a : [\omega_1]^2 \longrightarrow \omega$ be the corresponding subadditive dominant of a defined above. For $\alpha < \omega_1$, set

$$D_\alpha = A_{\alpha+1} \setminus A_\alpha.$$

2.28 Lemma. *The sets $D_\alpha \setminus \rho_a(\alpha, \gamma)$ and $D_\beta \setminus \rho_a(\beta, \gamma)$ are disjoint whenever $0 < \alpha < \beta < \gamma$ and α and β are limit ordinals.*

Proof. This follows immediately from Lemmas 2.24 and 2.25. ⊣

We are in a position to define a partial mapping $m : [\omega_1]^2 \longrightarrow \omega$ by

$$m(\alpha, \beta) = \min(D_\alpha \setminus \rho_a(\alpha, \beta)),$$

whenever $\alpha < \beta$ and α is a limit ordinal.

2.29 Lemma. *The mapping m is coherent, i.e., $m(\alpha, \beta) = m(\alpha, \gamma)$ for all but finitely many limit ordinals $\alpha < \min\{\beta, \gamma\}$.*

Proof. This is by the coherence of ρ_a and the fact that $\rho_a(\alpha, \beta) = \rho_a(\alpha, \gamma)$ already implies $m(\alpha, \beta) = m(\alpha, \gamma)$. ⊣

2.30 Lemma. *$m(\alpha, \gamma) \neq m(\beta, \gamma)$ whenever $\alpha \neq \beta < \gamma$ and α, β are limit ordinals.*

Proof. This follows from Lemma 2.28. ⊣

For $\beta < \omega_1$, set

$$B_\beta = \{m(\alpha, \beta) : \alpha < \beta \text{ and } \alpha \text{ limit}\}.$$

2.31 Lemma. *$B_\beta =^* B_\gamma \cap A_\beta$ whenever $\beta < \gamma$.*

Proof. By the coherence of m. ⊣

Note the following immediate consequence of Lemma 2.28 and the definition of m.

2.32 Lemma. *$m(\alpha, \beta) = \max(B_\beta \cap D_\alpha)$ whenever $\alpha < \beta$ and α is a limit ordinal.*

2.33 Lemma. *There is no $B \subseteq \omega$ such that $B \cap A_\beta =^* B_\beta$ for all β.*

Proof. Suppose that such a B exists and for a limit ordinal α let us define $g(\alpha) = \max(B \cap D_\alpha)$. Then by Lemma 2.32, $g(\alpha) = m(\alpha, \beta)$ for all $\beta < \omega_1$ and all but finitely many limit ordinals $\alpha < \beta$. By Lemma 2.30, it follows that g is a finite-to-one map, a contradiction. ⊣

2.34 Theorem. *For every strictly $\subset^*$-increasing chain A_α $(\alpha < \omega_1)$ of subsets of ω, there is a sequence B_α $(\alpha < \omega_1)$ of subsets of ω such that:*

(a) $B_\alpha =^* B_\beta \cap A_\alpha$ whenever $\alpha < \beta$,

(b) there is no B such that $B_\alpha =^* B \cap A_\alpha$ for all α.

3. Steps and Coherence

3.1 Definition. The *number of steps* of the minimal walk is the function $\rho_2 : [\omega_1]^2 \longrightarrow \omega$ defined recursively by

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1,$$

with the convention that $\rho_2(\gamma, \gamma) = 0$ for all γ.

This is an interesting mapping which is particularly useful on higher cardinalities and especially in situations where the more informative mappings ρ_0, ρ_1 and ρ lack their usual coherence properties. Here is a typical property of this mapping which will be explained in much more general terms in later sections of this chapter.

3.2 Lemma. $\sup_{\xi<\alpha}|\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty$ *for all $\alpha < \beta < \omega_1$.*

In this section we use ρ_2 only to succinctly express the following mapping.

3.3 Definition. The *last step function* of the minimal walk is the map $\rho_3 : [\omega_1]^2 \longrightarrow 2$ defined by letting

$$\rho_3(\alpha, \beta) = 1 \quad \text{iff} \quad \rho_0(\alpha, \beta)(\rho_2(\alpha, \beta) - 1) = \rho_1(\alpha, \beta).$$

In other words, we let $\rho_3(\alpha, \beta) = 1$ just in case the last step of the walk $\beta \to \alpha$ comes with the maximal weight.

3.4 Lemma. $\{\xi < \alpha : \rho_3(\xi, \alpha) \neq \rho_3(\xi, \beta)\}$ *is finite for all* $\alpha < \beta < \omega_1$.

Proof. It suffices to show that for every infinite $\Gamma \subseteq \alpha$ there exists a $\xi \in \Gamma$ such that $\rho_3(\xi, \alpha) = \rho_3(\xi, \beta)$. Shrinking Γ we may assume that for some fixed $\bar{\alpha} \in F(\alpha, \beta)$ and all $\xi \in \Gamma$:

(1) $\bar{\alpha} = \min(F(\alpha, \beta) \setminus \xi)$,

(2) $\rho_1(\xi, \alpha) = \rho_1(\xi, \beta)$,

(3) $\rho_1(\xi, \alpha), \rho_1(\xi, \beta) > \max\{\rho_1(\bar{\alpha}, \alpha), \rho_1(\bar{\alpha}, \beta)\}$.

It follows (see Lemma 1.6) that for every $\xi \in \Gamma$:

$$\rho_0(\xi, \alpha) = \rho_0(\bar{\alpha}, \alpha)^\frown \rho_0(\xi, \bar{\alpha}),$$
$$\rho_0(\xi, \beta) = \rho_0(\bar{\alpha}, \beta)^\frown \rho_0(\xi, \bar{\alpha}).$$

So for any $\xi \in \Gamma$, $\rho_3(\xi, \alpha) = 1$ iff the last term of $\rho_0(\xi, \bar{\alpha})$ is its maximal term iff $\rho_3(\xi, \beta) = 1$. ⊣

The sequence $(\rho_3)_\alpha : \alpha \longrightarrow 2$ $(\alpha < \omega_1)$[8] is therefore coherent in the sense that $(\rho_3)_\alpha =^* (\rho_3)_\beta \restriction \alpha$ whenever $\alpha < \beta$. We need to show that the sequence is not trivial, i.e. that it cannot be uniformized by a single total map from ω_1 into 2. In other words, we need to show that ρ_3 still contains enough information about the C-sequence C_α $(\alpha < \omega_1)$ from which it is defined. For this it will be convenient to assume that C_α $(\alpha < \omega_1)$ satisfies the following natural condition:

(d) If α is a limit ordinal > 0 and if ξ occupies the nth place in the increasing enumeration of C_α (that starts with $\min(C_\alpha)$ on its 0th place), then $\xi = \lambda + n + 1$ for some limit ordinal λ (possibly 0).

3.5 Definition. Let Λ denote the set of all countable limit ordinals and for an integer $n \in \omega$, let $\Lambda + n = \{\lambda + n : \lambda \in \Lambda\}$.

The assumption (d) about the C-sequence is behind the following property of ρ_3.

3.6 Lemma. $\rho_3(\lambda + n, \beta) = 1$ *for all but finitely many* n *with* $\lambda + n < \beta$.

[8] Recall the way one always defines the fiber-functions from a two-variable function applied to the context of ρ_3: $(\rho_3)_\alpha(\xi) = \rho_3(\xi, \alpha)$.

3.7 Lemma. *For all $\beta < \omega_1$, $n < \omega$, the set $\{\lambda \in \Lambda : \lambda + n < \beta$ and $\rho_3(\lambda + n, \beta) = 1\}$ is finite.*

Proof. Given an infinite subset Γ of $(\Lambda+n)\cap\beta$ we need to find a $\lambda+n \in \Gamma$ such that $\rho_3(\lambda+n, \beta) = 0$. Shrinking Γ if necessary assume that $\rho_1(\lambda+n, \beta) > n+2$ for all $\lambda+n \in \Gamma$. So if $\rho_3(\lambda+n, \beta) = 1$ for some $\lambda+n \in \Gamma$ then the last step of $\beta \rightarrow \lambda+n$ would have to be of weight$> n+2$ which is impossible by our assumption (d) about C_α $(\alpha < \omega_1)$. $\dashv$

The meaning of these properties of ρ_3 perhaps is easier to comprehend if we reformulate them in a way that resembles the original formulation of the existence of Hausdorff gaps.

3.8 Lemma. *Let $B_\alpha = \{\xi < \alpha : \rho_3(\xi, \alpha) = 1\}$ for $\alpha < \omega_1$. Then:*

(1) $B_\alpha =^* B_\beta \cap \alpha$ for $\alpha < \beta$,

(2) $(\Lambda + n) \cap B_\beta$ is finite for all $n < \omega$ and $\beta < \omega_1$,

(3) $\{\lambda + n : n < \omega\} \subseteq^* B_\beta$ whenever $\lambda + \omega \leq \beta$.

In particular, there is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \beta \subseteq^* B_\beta$ for all $\beta < \omega_1$. On the other hand, the P-ideal[9] $\mathfrak{I}$ generated by B_β $(\beta < \omega_1)$ is large as it contains all intervals of the form $[\lambda, \lambda + \omega)$. The following general dichotomy about P-ideals shows that here indeed we have quite a canonical example of a P-ideal on ω_1.

3.9 Definition. *The P-ideal dichotomy:* For every P-ideal $\mathfrak{I}$ of countable subsets of some set S either:

(1) there is an uncountable $X \subseteq S$ such that $[X]^\omega \subseteq \mathfrak{I}$, or

(2) S can be decomposed into countably many sets orthogonal to $\mathfrak{I}$.

3.10 Remark. It is known that the P-ideal dichotomy is a consequence of the Proper Forcing Axiom and moreover that it does not contradict the Continuum Hypothesis (see [64]). This is an interesting dichotomy which will be used in this article for testing various notions of coherence as we encounter them. For example, let us consider the following notion of coherence, already encountered above at several places, and see how it is influenced by the P-ideal dichotomy.

3.11 Definition. A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is *coherent* if for every $\alpha < \beta < \omega_1$ there exist only finitely many $\xi < \alpha$ such that $a(\xi, \alpha) \neq a(\xi, \beta)$, or in other words, $a_\alpha =^* a_\beta \restriction \alpha$.[10] We say that a is *nontrivial* if there is no $h : \omega_1 \longrightarrow \omega$ such that $h \restriction \alpha =^* a_\alpha$ for all $\alpha < \omega_1$.

---

[9] Recall that an ideal $\mathfrak{I}$ of subsets of some set S is a *P-ideal* if for every sequence A_n $(n < \omega)$ of elements of $\mathfrak{I}$ there is a B in $\mathfrak{I}$ such that $A_n \setminus B$ is finite for all $n < \omega$. A set X is *orthogonal* to $\mathfrak{I}$ if $X \cap A$ is finite for all A in $\mathfrak{I}$.

[10] A mapping $a : [\omega_1]^2 \longrightarrow \omega$ is naturally identified with a sequence a_α $(\alpha < \omega_1)$, where $a_\alpha : \alpha \longrightarrow \omega$ is defined by $a_\alpha(\xi) = a(\xi, \alpha)$.

Note that the existence of a coherent and nontrivial $a : [\omega_1]^2 \longrightarrow 2$ (such as, for example, the function ρ_3 defined above) is something that corresponds to the notion of a Hausdorff gap (cf. the previous lemma) in this context. Notice moreover, that this notion is also closely related to the notion of an Aronszajn tree since

$$T(a) = \{t : \alpha \longrightarrow \omega : \alpha < \omega_1 \text{ and } t =^* a_\alpha\}$$

is such an Aronszajn tree whenever $a : [\omega_1]^2 \longrightarrow \omega$ is coherent and nontrivial.[11] In fact, we shall call an arbitrary Aronszajn tree T *coherent* if T is isomorphic to $T(a)$ for some coherent and nontrivial $a : [\omega_1]^2 \longrightarrow \omega$. In case the range of the map a is actually smaller than ω, e.g. equal to some integer k, then it is natural to let $T(a)$ be the collection of all $t : \alpha \longrightarrow k$ such that $\alpha < \omega_1$ and $t =^* a_\alpha$. This way, we have coherent binary, ternary, etc. Aronszajn trees rather than only ω-ary coherent Aronszajn trees.

3.12 Definition. The *support* of a map $a : [\omega_1]^2 \longrightarrow \omega$ is the sequence $\mathrm{supp}(a_\alpha) = \{\xi < \alpha : a(\xi, \alpha) \neq 0\}$ $(\alpha < \omega_1)$ of subsets of ω_1. A set Γ is *orthogonal* to a if $\mathrm{supp}(a_\alpha) \cap \Gamma$ is finite for all $\alpha < \omega_1$. We say that $a : [\omega_1]^2 \longrightarrow \omega$ is *nowhere dense* if there is no uncountable $\Gamma \subseteq \omega_1$ such that $\Gamma \cap \alpha \subseteq^* \mathrm{supp}(a_\alpha)$ for all $\alpha < \omega_1$.

Note that ρ_3 is an example of a nowhere dense coherent map for the simple reason that ω_1 can be covered by countably many sets $\Lambda + n$ $(n < \omega)$ that are orthogonal to ρ_3. The following immediate fact shows that ρ_3 is indeed a prototype of a nowhere dense and coherent map $a : [\omega_1]^2 \longrightarrow \omega$.

3.13 Proposition. *Under the P-ideal dichotomy, for every nowhere dense and coherent map $a : [\omega_1]^2 \longrightarrow \omega$ the domain ω_1 can be decomposed into countably many sets orthogonal to a.*

3.14 Notation. To every $a : [\omega_1]^2 \longrightarrow \omega$ associate the corresponding Δ-function $\Delta_a : [\omega_1]^2 \longrightarrow \omega$ as follows:

$$\Delta_a(\alpha, \beta) = \min\{\xi < \alpha : a(\xi, \alpha) \neq a(\xi, \beta)\}$$

with the convention that $\Delta_a(\alpha, \beta) = \alpha$ whenever $a(\xi, \alpha) = a(\xi, \beta)$ for all $\xi < \alpha$. Given this notation, it is natural to let

$$\Delta_a(\Gamma) = \{\Delta_a(\alpha, \beta) : \alpha, \beta \in \Gamma, \alpha < \beta\}$$

for an arbitrary set $\Gamma \subseteq \omega_1$.

The following simple fact reveals a crucial property of coherent trees.

[11] Similarities between the notion of a Hausdorff gap and the notion of an Aronszajn tree have been further explained recently in the two papers of Talayco [53, 54], where it is shown that they naturally correspond to first cohomology groups over a pair of very similar spaces.

3.15 Lemma. *Suppose that $a : [\omega_1]^2 \longrightarrow \omega$ is nontrivial and coherent and that every uncountable subset of $T(a)$ contains an uncountable antichain. Then for every pair Σ,Ω of uncountable subsets of ω_1 there exists an uncountable subset Γ of ω_1 such that $\Delta_a(\Gamma) \subseteq \Delta_a(\Sigma) \cap \Delta_a(\Omega)$.*

3.16 Notation. For $a : [\omega_1]^2 \longrightarrow \omega$, set

$$\mathcal{U}(a) = \{A \subseteq \omega_1 : A \supseteq \Delta_a(\Gamma) \text{ for some uncountable } \Gamma \subseteq \omega_1\}.$$

By Lemma 3.15, $\mathcal{U}(a)$ is a uniform filter on ω_1 for every nontrivial coherent $a : [\omega_1]^2 \longrightarrow \omega$ for which $T(a)$ contains no Souslin subtrees. It turns out that under some very mild assumption, $\mathcal{U}(a)$ is in fact a uniform ultrafilter on ω_1. The proof of this can be found in [66].

3.17 Theorem. *Under MA_{ω_1}, the filter $\mathcal{U}(a)$ is an ultrafilter for every nontrivial and coherent $a : [\omega_1]^2 \longrightarrow \omega$.*

3.18 Remark. One may find Theorem 3.17 a bit surprising in view of the fact that it gives us an ultrafilter $\mathcal{U}(a)$ on ω_1 that is Σ_1-definable over the structure $(H_{\omega_2}, \in)$. It is well-known that there is no ultrafilter on ω that is Σ_1-definable over the structure $(H_{\omega_1}, \in)$.

It turns out that the transformation $a \longmapsto \mathcal{U}(a)$ captures some of the essential properties of the corresponding and more obvious transformation $a \longmapsto T(a)$. To state this we need some standard definitions.

3.19 Definition. For two trees S and T, by $S \leq T$ we denote the fact that there is a strictly increasing map $f : S \longrightarrow T$. Let $S < T$ whenever $S \leq T$ and $T \not\leq S$ and let $S \equiv T$ whenever $S \leq T$ and $T \leq S$. In general, the equivalence relation $\equiv$ on trees is very far from the finer relation $\cong$, the isomorphism relation. However, the following fact shows that in the realm of trees $T(a)$, these two relations may coincide and moreover, that the mapping $T(a) \longmapsto \mathcal{U}(a)$ reduces $\equiv$ and $\cong$ to the equality relation among ultrafilters on ω_1 (see [65]).

The following fact reveals in particular that the class of coherent trees has the Schroeder-Bernstein property. Its proof can again be found in [66].

3.20 Theorem. *Assuming MA_{ω_1}, for every pair of coherent and nontrivial mappings $a : [\omega_1]^2 \longrightarrow \omega$ and $b : [\omega_1]^2 \longrightarrow \omega$, the trees $T(a)$ and $T(b)$ are isomorphic iff $T(a) \equiv T(b)$ iff $\mathcal{U}(a) = \mathcal{U}(b)$.*

3.21 Definition. The *shift* of $a : [\omega_1]^2 \longrightarrow \omega$ is defined to be the mapping $a^{(1)} : [\omega_1]^2 \longrightarrow \omega$ determined by the equation $a^{(1)}(\alpha, \beta) = a(\alpha + 1, \hat{\beta})$, where $\hat{\beta} = \min\{\lambda \in \Lambda : \lambda \geq \beta\}$. The n-fold iteration of the shift operation is defined recursively by the formula $a^{(n+1)} = (a^{(n)})^{(1)}$.

The following fact, whose proof can be found in [66], shows that Aronszajn trees are not well-quasi-ordered under the quasi-ordering $\leq$ (see also [65]).

3.22 Theorem. *If a is nontrivial, coherent and orthogonal to Λ, then $T(a) > T(a^{(1)})$.*

3.23 Corollary. *If a is nontrivial, coherent and orthogonal to $\Lambda + n$ for all $n < \omega$, then $T(a^{(n)}) > T(a^{(m)})$ whenever $n < m < \omega$.*

Proof. Note that if a is orthogonal to $\Lambda + n$ for all $n < \omega$, then so is every of its finite shifts $a^{(m)}$. ⊣

3.24 Corollary. $T(\rho_3^{(n)}) > T(\rho_3^{(m)})$ *whenever $n < m < \omega$.*

Somewhat unexpectedly, with very little extra assumptions we can say much more about $\leq$ in the domain of coherent Aronszajn trees (for proofs see [65] and [66]).

3.25 Theorem. *Under MA_{ω_1}, the family of coherent Aronszajn trees is totally ordered under $\leq$.*

3.26 Remark. While under MA_{ω_1}, the class of coherent Aronszajn trees is totally ordered by $\leq$, Corollary 3.24 gives us that this chain of trees is not well-ordered. This should be compared with an old result of Ohkuma [39] that the class of all scattered trees is well-ordered by $\leq$ (see also [32]). It turns out that the class of all Aronszajn trees is not totally ordered under $\leq$, i.e. there exist Aronszajn trees S and T such that $S \not\leq T$ and $T \not\leq S$. The reader is referred to [65] and [66] for more information on this and other related results that we chose not to reproduce here.

4. The Trace and the Square-Bracket Operation

Recall the notion of a *minimal walk* from a countable ordinal β to a smaller ordinal α along the fixed C-sequence C_ξ $(\xi < \omega_1)$: $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$ where $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$. Recall also the notion of a *trace*

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\},$$

the finite set of places visited in the minimal walk from β to α. The following simple fact about the trace lies at the heart of all known definitions of square-bracket operations not only on ω_1 but also at higher cardinalities.

4.1 Lemma. *For every uncountable subset Γ of ω_1 the union of $\mathrm{Tr}(\alpha, \beta)$ for $\alpha < \beta$ in Γ contains a closed and unbounded subset of ω_1.*

Proof. It suffices to show that the union of traces contains every countable limit ordinal δ such that $\sup(\Gamma \cap \delta) = \delta$. Pick an arbitrary $\beta \in \Gamma \setminus \delta$ and let

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta$$

be the minimal walk from β to δ. Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. By the choice of δ there is an $\alpha \in \Gamma \cap \delta$ above γ. Then the minimal walk from β to α starts as $\beta_0 > \beta_1 > \cdots > \beta_k$, so in particular δ belongs to $\mathrm{Tr}(\alpha, \beta)$. ⊣

We shall now see that it is possible to pick a single place $[\alpha\beta]$ in $\mathrm{Tr}(\alpha,\beta)$ so that Lemma 4.1 remains valid with $[\alpha\beta]$ in place of $\mathrm{Tr}(\alpha,\beta)$. Recall that by Lemma 1.11,

$$\Delta(\alpha,\beta) = \min\{\xi \leq \alpha : \rho_0(\xi,\alpha) \neq \rho_0(\xi,\beta)\}$$

is a successor ordinal. We shall be interested in its predecessor,

4.2 Definition. $\sigma(\alpha,\beta) = \Delta(\alpha,\beta) - 1$.

Thus, if $\xi = \sigma(\alpha,\beta)$, then $\rho_0(\xi,\alpha) = \rho_0(\xi,\beta)$ and so there is a natural isomorphism between $\mathrm{Tr}(\xi,\alpha)$ and $\mathrm{Tr}(\xi,\beta)$. We shall define $[\alpha\beta]$ by comparing the three sets $\mathrm{Tr}(\alpha,\beta), \mathrm{Tr}(\xi,\alpha)$ and $\mathrm{Tr}(\xi,\beta)$.

4.3 Definition. The *square-bracket operation* on ω_1 is defined as follows:

$$[\alpha\beta] = \min(\mathrm{Tr}(\alpha,\beta) \cap \mathrm{Tr}(\sigma(\alpha,\beta),\beta)) = \min(\mathrm{Tr}(\sigma(\alpha,\beta),\beta) \setminus \alpha).$$

Next, recall the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$ defined from the C-sequence C_ξ $(\xi < \omega_1)$ and the corresponding tree $T(\rho_0)$. For $\gamma < \omega_1$ let $(\rho_0)_\gamma$ be the fiber-mapping : $\gamma \to \omega^{<\omega}$ defined by $(\rho_0)_\gamma(\alpha) = \rho_0(\alpha,\gamma)$.

4.4 Lemma. *For every uncountable subset Γ of ω_1 the set of all ordinals of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ contains a closed and unbounded subset of ω_1.*

Proof. For $t \in T(\rho_0)$ let $\Gamma_t = \{\gamma \in \Gamma : (\rho_0)_\gamma \text{ end-extends } t\}$. Let S be the collection of all $t \in T(\rho_0)$ for which Γ_t is uncountable. Clearly, S is a downward closed uncountable subtree of T. The lemma is established once we prove that every countable limit ordinal $\delta > 0$ with the following two properties can be represented as $[\alpha\beta]$ for some $\alpha < \beta$ in Γ:

(1) $\sup(\Gamma_t \cap \delta) = \delta$ for every $t \in S$ of length $< \delta$,

(2) every $t \in S$ of length $< \delta$ has two incomparable successors in S both of length $< \delta$.

Fix such a δ and choose $\beta \in \Gamma \setminus \delta$ such that $(\rho_0)_\beta \restriction \delta \in S$ and consider the minimal walk from β to δ:

$$\beta = \beta_0 > \beta_1 > \cdots > \beta_k = \delta.$$

Let $\gamma < \delta$ be an upper bound of all sets of the form $C_{\beta_i} \cap \delta$ for $i < k$. Since the restriction $t = (\rho_0)_\beta \restriction \gamma$ belongs to S, by (2) we can find one of its end-extensions s in S which is incomparable with $(\rho_0)_\beta$. It follows that for $\alpha \in \Gamma_s$, the ordinal $\sigma(\alpha,\beta)$ has the fixed value

$$\xi = \min\{\xi < |s| : s(\xi) \neq \rho_0(\xi,\beta)\} - 1.$$

Note that $\xi \geq \gamma$, so the walk $\beta \to \delta$ is a common initial part of walks $\beta \to \xi$ and $\beta \to \alpha$ for every $\alpha \in \Gamma_s \cap \delta$. Hence if we choose $\alpha \in \Gamma_s \cap \delta$

above $\min(C_\delta \setminus \xi)$ (which we can by (1)), we get that the walks $\beta \to \xi$ and $\beta \to \alpha$ never meet after δ. In other words for any such α, the ordinal δ is the minimum of $\mathrm{Tr}(\alpha, \beta) \cap \mathrm{Tr}(\xi, \beta)$, or equivalently δ is the minimum of $\mathrm{Tr}(\xi, \beta) \setminus \alpha$. $\dashv$

It should be clear that the above argument can easily be adjusted to give us the following slightly more general fact about the square-bracket operation.

4.5 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, the set of all ordinals of the form $[a(1)b(1)] = [a(2)b(2)] = \cdots = [a(n)b(n)]$ for some $a \neq b$ in A contains a closed and unbounded subset of ω_1.*[12]

It turns out that the square-bracket operation can be used in constructions of various mathematical objects of complex behavior where all known previous constructions needed the Continuum Hypothesis or stronger enumeration principles. The usefulness of $[\cdot\cdot]$ in these constructions is based on the fact that $[\cdot\cdot]$ reduces the quantification over uncountable subsets of ω_1 to the quantification over closed unbounded subsets of ω_1. For example composing $[\cdot\cdot]$ with a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the values stationary many times one gets the following fact about the mapping $c(\alpha, \beta) = [\alpha\beta]^*$.

4.6 Theorem. *There is a mapping $c : [\omega_1]^2 \longrightarrow \omega_1$ which takes all the values from ω_1 on any square $[\Gamma]^2$ of some uncountable subset Γ of ω_1.*

Note that the basic C-sequence C_α ($\alpha < \omega_1$) which we have fixed at the beginning of this chapter can be used to actually define a unary operation $* : \omega_1 \longrightarrow \omega_1$ which takes each of the ordinals from ω_1 stationarily many times. So the projection $[\alpha\beta]^*$ can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. We are now at the point to see that our basic structure is actually rigid.

4.7 Lemma. *The algebraic structure $(\omega_1, [\cdot\cdot], *)$ has no nontrivial automorphisms.*

Proof. Let h be a given automorphism of $(\omega_1, [\cdot\cdot], *)$. If the set Γ of fixed points of h is uncountable, h must be the identity map. To see this, consider a $\xi < \omega_1$. By the property of the map $c(\alpha, \beta) = [\alpha\beta]^*$ stated in Theorem 4.6 there exists a $\gamma < \delta$ in Γ such that $[\gamma\delta]^* = \xi$. Applying h to this equation we get

$$h(\xi) = h([\gamma\delta]^*) = (h([\gamma\delta]))^* = [h(\gamma)h(\delta)]^* = [\gamma\delta]^* = \xi.$$

It follows that $\Delta = \{\delta < \omega_1 : h(\delta) \neq \delta\}$ is in particular uncountable. Shrinking Δ and replacing h by h^{-1}, if necessary, we may safely assume

[12] For a finite set x of ordinals of size n we use the notation $x(1), x(2), \ldots, x(n)$ or $x(0), x(1), \ldots, x(n-1)$, depending on the context, for the enumeration of x according to the natural ordering on the ordinals.

that $h(\delta) > \delta$ for all $\delta \in \Delta$. Consider a $\xi < \omega_1$ and let S_ξ be the set of all $\alpha < \omega_1$ such that $\alpha^* = \xi$. By our choice of $*$ the set S_ξ is stationary. By Lemma 4.5 applied to the family $A = \{\{\delta, h(\delta)\} : \delta \in \Delta\}$ we can find $\gamma < \delta$ in Δ such that $[\gamma\delta] = [h(\gamma)h(\delta)]$ belongs to S_ξ, or in other words,

$$[\gamma\delta]^* = [h(\gamma)h(\delta)]^* = \xi.$$

Since $[h(\gamma)h(\delta)]^* = h([\gamma\delta]^*)$ we conclude that $h(\xi) = \xi$. Since ξ was an arbitrary countable ordinal, this shows that h is the identity map. $\dashv$

We give now an application of this rigidity result to a problem in model theory about the quantifier Qx = "there exist uncountably many x" and its higher dimensional analogues $Q^n x_1 \cdots x_n$ = "there exist an uncountable n-cube many $x_1, \ldots, x_n$". By a result of Ebbinghaus and Flum [15] (see also [40]) every model of every sentence of $L(Q)$ has nontrivial automorphisms. However we shall now see that this is no longer true about the quantifier Q^2.

4.8 Example. *A sentence of $L(Q^2)$ with only rigid models.* The sentence ϕ will talk about one unary relation N, one binary relation $<$ and two binary functional symbols C and E. It is the conjunction of the following seven sentences

(ϕ_1) $Qx\ x = x$,

(ϕ_2) $\neg Qx\ N(x)$,

(ϕ_3) $<$ is a total ordering,

(ϕ_4) E is a symmetric binary operation,

(ϕ_5) $\forall x < y\ N(E(x, y))$,

(ϕ_6) $\forall x < y < z\ E(x, z) \neq E(y, z)$,

(ϕ_7) $\forall x \forall n \{N(n) \rightarrow \neg Q^2 uv[\exists u' < u \exists v' < v(u' \neq v' \wedge E(u', u) = E(v', v) = n) \wedge \forall u' < u\ \forall v' < v(E(u', u) = E(v', v) = n \rightarrow (C(u', v') \neq x \vee C(u, v) \neq x))]\}$.

The model of ϕ that we have in mind is the model $(\omega_1, \omega, <, c, e)$ where $c(\alpha, \beta) = [\alpha\beta]^*$ and $e : [\omega_1]^2 \longrightarrow \omega$ is any mapping such that $e(\alpha, \gamma) \neq e(\beta, \gamma)$ whenever $\alpha < \beta < \gamma$ (e.g. we can take $e = \bar{\rho}_1$ or $e = \bar{\rho}$). The sentence ϕ_7 is simply saying that for every $\xi < \omega_1$ and every uncountable family A of pairwise disjoint unordered pairs of countable ordinals there exist $a \neq b$ in A such that

$$c(\min a, \min b) = c(\max a, \max b) = \xi.$$

This is a consequence of Lemma 4.5 and the fact that $S_\xi = \{\alpha : \alpha^* = \xi\}$ is a stationary subset of ω_1. These are the properties of $[\cdot\cdot]$ and $*$ which we have used in the proof of Lemma 4.7 in order to prove that $(\omega_1, [\cdot\cdot], *)$ is a rigid structure. So a quite analogous proof will show that any model $(M, N, <, C, E)$ of ϕ must be rigid.

The crucial property of $[\cdot\cdot]$ stated in Lemma 4.5 can also be used to provide a negative answer to the basis problem for uncountable graphs by constructing a large family of pairwise orthogonal uncountable graphs.

4.9 Definition. For a subset Γ of ω_1, let $\mathcal{G}_\Gamma$ be the graph whose vertex-set is ω_1 and whose edge-set is equal to $\{\{\alpha,\beta\} : [\alpha\beta] \in \Gamma\}$.

4.10 Lemma. *If the symmetric difference between Γ and Δ is a stationary subset of ω_1, then the corresponding graphs $\mathcal{G}_\Gamma$ and $\mathcal{G}_\Delta$ are orthogonal to each other, i.e. they do not contain uncountable isomorphic subgraphs.*

We have seen above that comparing $[\cdot\cdot]$ with a map $\pi : \omega_1 \longrightarrow I$ where I is some set of mathematical objects/requirements in such a way that each object/requirement is given a stationary preimage, gives us a way to meet each of these objects/requirements in the square of any uncountable subset of ω_1. This observation is the basis of all known applications of the square-bracket operation. A careful choice of I and $\pi : \omega_1 \longrightarrow I$ gives us a projection of the square-bracket operation that can be quite useful. So let us illustrate this on yet another example.

4.11 Definition. Let $\mathcal{H}$ be the collection of all maps $h : 2^n \longrightarrow \omega_1$ where n is a positive integer denoted by $n(h)$. Choose a mapping $\pi : \omega_1 \longrightarrow \mathcal{H}$ which takes each value from $\mathcal{H}$ stationarily many times. Choose also a one-to-one sequence r_α $(\alpha < \omega_1)$ of elements of the Cantor set 2^ω. Note that both these objects can actually be defined in our basic structure $(\omega_1, \omega, \vec{C})$. Consider the following projection of the square-bracket operation:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta]))).$$

It is easily checked that the property of $[\cdot\cdot]$ stated in Lemma 4.5 corresponds to the following property of the projection $[\![\alpha\beta]\!]$:

4.12 Lemma. *For every uncountable family A of pairwise disjoint finite subsets of ω_1, all of the same size n, and for every n-sequence $\xi_1, \dots, \xi_n$ of countable ordinals there exist a and b in A such that $[\![a(i)b(i)]\!] = \xi_i$ for $i = 1, \dots, n$.*

This projection of $[\cdot\cdot]$ leads to an interesting example of a Banach space with "few" operators, which we will now describe.

4.13 Theorem. *There is a nonseparable reflexive Banach space E with the property that every bounded linear operator $T : E \longrightarrow E$ can be expressed as $T = \lambda I + S$ where λ is a scalar, I the identity operator of E, and S an operator with separable range.*

Proof. Let $I = 3 \times [\omega_1]^{<\omega}$ and let us identify the index-set I with ω_1, i.e. pretend that $[\![\cdot\cdot]\!]$ takes its values in I rather than ω_1. Let $[\![\cdot\cdot]\!]_0$ and $[\![\cdot\cdot]\!]_1$ be the two projections of $[\![\cdot\cdot]\!]$.

$$\mathcal{G} = \{G \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 0 \text{ for all } \{\alpha,\beta\} \in [G]^2\},$$
$$\mathcal{H} = \{H \in [\omega_1]^{<\omega} : [\![\alpha\beta]\!]_0 = 1 \text{ for all } \{\alpha,\beta\} \in [H]^2\}.$$

Let $\mathcal{K}$ be the collection of all finite sets $\{\{\alpha_i, \beta_i\} : i < k\}$ of pairs of countable ordinals such that for all $i < j < k$:

(i) $\max\{\alpha_i, \beta_i\} < \min\{\alpha_j, \beta_j\}$,

(ii) $[\![\alpha_i\alpha_j]\!]_0 = [\![\beta_i\beta_j]\!]_0 = 2$,

(iii) $[\![\alpha_i\alpha_j]\!]_1 = [\![\beta_i\beta_j]\!]_1 = \{\alpha_l : l < i\} \cup \{\beta_l : l < i\}$.

The following properties of $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ should be clear:

(1) $\mathcal{G}$ and $\mathcal{H}$ contain all the singletons, are closed under subsets and they are 1-orthogonal to each other in the sense that $\mathcal{G} \cap \mathcal{H}$ contains no doubleton.

(2) $\mathcal{G}$ and $\mathcal{H}$ are both 2-orthogonal to the family of the unions of members of $\mathcal{K}$.

(3) If K and L are two distinct members of $\mathcal{K}$, then there are no more than 5 ordinals α such that $\{\alpha, \beta\} \in K$ and $\{\alpha, \gamma\} \in L$ for some $\beta \neq \gamma$.

(4) For every sequence $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \omega_1)$ of pairwise disjoint pairs of countable ordinals there exist arbitrarily large finite sets $\Gamma, \Delta \subseteq \omega_1$ such that $\{\alpha_\xi : \xi \in \Gamma\} \in \mathcal{G}$, $\{\beta_\xi : \xi \in \Gamma\} \in \mathcal{H}$ and $\{\{\alpha_\xi, \beta_\xi\} : \xi \in \Delta\} \in \mathcal{K}$.

For a function x from ω_1 into $\mathbb{R}$, set

$$\|x\|_{\mathcal{H},2} = \sup\left\{ \left(\textstyle\sum_{\alpha \in H} x(\alpha)^2\right)^{\frac{1}{2}} : H \in \mathcal{H} \right\},$$
$$\|x\|_{\mathcal{K},2} = \sup\left\{ \left(\textstyle\sum_{\{\alpha,\beta\} \in K} (x(\alpha) - x(\beta))^2\right)^{\frac{1}{2s}} : K \in \mathcal{K} \right\}.$$

Let $\|\cdot\| = \max\{\|\cdot\|_\infty, \|\cdot\|_{\mathcal{H},2}, \|\cdot\|_{\mathcal{K},2}\}$ and define $\bar{E}_2 = \{x : \|x\| < \infty\}$. Let 1_α be the characteristic function of $\{\alpha\}$. Finally, let E_2 be the closure of the linear span of $\{1_\alpha : \alpha \in \omega_1\}$ inside $(\bar{E}_2, \|\cdot\|)$. The following facts about the norm $\|\cdot\|$ are easy to establish using the properties of the families $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ listed above.

(i) If x is supported by some $G \in \mathcal{G}$, then $\|x\| \leq 2 \cdot \|x\|_\infty$.

(ii) If x is supported by $\bigcup K$ for some K in $\mathcal{K}$, then $\|x\| \leq 10 \cdot \|x\|_\infty$.

The role of the seminorm $\|\cdot\|_{\mathcal{H},2}$ is to ensure that every bounded operator $T : E_2 \longrightarrow E_2$ can be expressed as $D + S$, where D is a diagonal operator relative to the basis[13] 1_α $(\alpha < \omega_1)$ and where S has separable range.

Note that $\|x\| \leq 2\|x\|_2$ for all $x \in \ell_2(\omega_1)$. It follows that $\ell_2(\omega_1) \subseteq E_2$ and the inclusion is a bounded linear operator. Note also that $\ell_2(\omega_1)$ is a dense subset of E_2. Therefore E_2 is a weak compactly generated space. For

[13] Indeed it can be shown that 1_α $(\alpha < \omega_1)$ is a "transfinite basis" of E_2 in the sense of [51]. So every vector x of E_2 has a unique representation as $\Sigma_{\alpha<\omega_1} x(\alpha)1_\alpha$ and the projection operators $P_\beta : E_2 \to E_2 \restriction \beta$ $(\beta < \omega_1)$ are uniformly bounded.

example, $W = \{x \in \ell_2(\omega_1) : \|x\|_2 \leq 1\}$ is a weakly compact subset of E_2 and its linear span is dense in E_2. To get a reflexive example out of E_2 one uses an interpolation method of Davis, Figiel, Johnson and Pelczynski [9] as follows. Let p_n be the Minkowski functional of the set $2^n W + 2^{-n}\mathrm{Ball}(E_2)$.[14] Let

$$E = \left\{x \in E_2 : \|x\|_E = \left(\textstyle\sum_{n=0}^{\infty} p_n(x)^2\right)^{\frac{1}{2}} < \infty\right\}.$$

By [9, Lemma 1], E is a reflexive Banach space and $\ell_2(\omega_1) \subseteq E \subseteq E_2$ are continuous inclusions. Note that $p_n(x) < r$ iff $x = y + z$ for some $y \in E_2$ and $z \in \ell_2(\omega_1)$ such that $\|y\| < 2^{-n} r$ and $\|z\|_2 < 2^n r$. Then the reflexive version of the space also has the property that every bounded operator $T : E \longrightarrow E$ has the form $\lambda I + S$. $\dashv$

4.14 Remark. The above example is reproduced from Wark [70] who based his example on a previous construction due to Shelah and Steprans [50]. The reader is referred to these sources and to [66] for more information.

We only mention yet another interesting application of the square-bracket operation, given recently by Erdős, Jackson and Mauldin [17]:

4.15 Theorem. *For every positive integer n there exist collections $\mathcal{H}$ and X of hyperplanes and points of $\mathbb{R}^n$, respectively, and a coloring $P : \mathcal{H} \longrightarrow \omega$ such that:*

(1) any n hyperplanes of distinct colors meet in at most one point,

(2) there is no coloring $Q : X \longrightarrow \omega$ such that for every $H \in \mathcal{H}$ there exists at most $n-1$ points x in $X \cap H$ such that $Q(x) = P(H)$.

Let us now introduce yet another projection of the square-bracket operation which has some universality properties.

4.16 Definition. Let $\mathcal{H}$ now be the collection of all maps $h : 2^n \times 2^n \longrightarrow \omega_1$ where $n = n(h) < \omega$ and let π be a mapping from ω_1 onto $\mathcal{H}$ that takes each of the values stationarily many times. Define a new operation on ω_1 by

$$|\alpha\beta| = \pi([\alpha\beta])(r_\alpha \restriction n(\pi([\alpha\beta])), r_\beta \restriction n(\pi([\alpha\beta]))).$$

4.17 Lemma. *For every positive integer n, every uncountable subset Γ of ω_1 and every symmetric $n \times n$-matrix M of countable ordinals there is a one-to-one $\phi : n \longrightarrow \Gamma$ such that $|\phi(i)\phi(j)| = M(i,j)$ for $i, j < n$.*

[14] I.e. $p_n(x) = \inf\{\lambda > 0 : x \in \lambda B\}$, where B denotes this set.

5. A Square-Bracket Operation on a Tree

In this section we try to show that the basic idea of the square-bracket operation on ω_1 can perhaps be more easily grasped by working on an arbitrary special Aronszajn tree rather than $T(\rho_0)$. So let $T = \langle T, <_T \rangle$ be a fixed special Aronszajn tree and let $a : T \longrightarrow \omega$ be a fixed map witnessing this, i.e. a mapping with the property that $a^{-1}(\{n\})$ is an antichain of T for all $n < \omega$. We shall assume that for every $s, t \in T$ the greatest lower bound $s \wedge t$ exists in T. For $t \in T$ and $n < \omega$, set

$$F_n(t) = \{s \leq_T t : s = t \text{ or } a(s) \leq n\}.$$

Finally, for $s, t \in T$ with $\mathrm{ht}(s) \leq \mathrm{ht}(t)$, let

$$[st]_T = \min\{v \in F_{a(s \wedge t)}(t) : \mathrm{ht}(v) \geq \mathrm{ht}(s)\}.$$

(If $\mathrm{ht}(s) \geq |ht(t)$ we let $[st]_T = [ts]_T$.)

The following fact corresponds to Lemma 4.4 when $T = T(\rho_0)$.

5.1 Lemma. *If X is an uncountable subset of T, the set of nodes of T of the form $[st]_T$ for some $s, t \in X$ intersects a closed and unbounded set of levels of T.*

We do not give a proof of this fact as it is almost identical to the proof of Lemma 4.4 which deals with the special case $T = T(\rho_0)$. But one can go further and show that $[\cdot\cdot]_T$ shares all the other properties of the square-bracket operation $[\cdot\cdot]$ described in the previous section. Some of these properties, however, are easier to visualize and prove in the general context. For example, consider the following fact which in the case $T = T(\rho_0)$ is the essence of Lemma 4.

5.2 Lemma. *Suppose $A \subseteq T$ is an uncountable antichain and that for each $t \in A$ be given a finite set F_t of its successors. Then for every stationary set $\Gamma \subseteq \omega_1$ there exists an arbitrarily large finite set $B \subseteq A$ such that the height of $[xy]_T$ belongs to Γ whenever $x \in F_s$ and $y \in F_t$ for some $s \neq t$ in B.*

Let us now examine in more detail the collection of graphs $\mathcal{G}_\Gamma(\Gamma \subseteq \omega_1)$ of 4.9 but in the present more general context.

5.3 Definition. For $\Gamma \subseteq \omega_1$, let $K_\Gamma = \{\{s, t\} \in [T]^2 : \mathrm{ht}([st]_T) \in \Gamma\}$.

Working as in Lemma 4.10 one shows that (T, K_Γ) and (T, K_Δ) have no isomorphic uncountable subgraph whenever the symmetric difference between Γ and Δ is a stationary subset of ω_1, i.e. whenever they represent different members of the quotient algebra $\mathcal{P}(\omega_1)/\mathrm{NS}$. In particular, K_Γ contains no square $[X]^2$ of an uncountable set $X \subseteq T$ whenever Γ contains no closed and unbounded subset of ω_1. The following fact is a sort of converse to this. Its proof can be found in [66].

5.4 Lemma. *If Γ contains a closed and unbounded subset of ω_1 then there is a proper forcing notion introducing an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$.*

5.5 Corollary. *The graph K_Γ contains the square of some uncountable subset of T in some ω_1-preserving forcing extension if and only if Γ is a stationary subset of ω_1.*

Proof. If Γ is disjoint from a closed and unbounded subset then in any ω_1-preserving forcing extension its complement $\Delta = \omega_1 \setminus \Gamma$ will be a stationary subset of ω_1. So by the basic property Lemma 5.1 of the square-bracket operation no such a forcing extension will contain an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. On the other hand, if Γ is a stationary subset of ω_1, going first to some standard ω_1-preserving forcing extension in which Γ contains a closed and unbounded subset of ω_1 and then applying Lemma 5.4, we get an ω_1-preserving forcing extension having an uncountable set $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. $\dashv$

5.6 Remark. Corollary 5.5 gives us a further indication of the extreme complexity of the class of graphs on the vertex-set ω_1. It also bears some relevance to the recent work of Woodin [74] who, working in his $\mathbb{P}_{\max}$-forcing extension, was able to associate a stationary subset of ω_1 to any partition of $[\omega_1]^2$ into two pieces. So one may view Corollary 5.5 as some sort of converse to this since in the $\mathbb{P}_{\max}$-extension one is able to get a sufficiently generic filter to the forcing notion $\mathbb{P} = \mathbb{P}_\Gamma$ of Lemma 5.4 that would give us an uncountable $X \subseteq T$ such that $[X]^2 \subseteq K_\Gamma$. In other words, under a bit of PFA or Woodin's axiom $(*)$, a set $\Gamma \subseteq \omega_1$ contains a closed and unbounded subset of ω_1 if and only if K_Γ contains $[X]^2$ for some uncountable $X \subseteq T$.

6. Special Trees and Mahlo Cardinals

One of the most basic questions frequently asked about set-theoretical trees is the question whether they contain any *cofinal branch*, a branch that intersects each level of the tree. The fundamental importance of this question has already been realized in the work of Kurepa [31] and then later in the works of Erdős and Tarski in their respective attempts to develop the theory of partition calculus and large cardinals (see [16]). A tree T of height equal to some regular cardinal θ may not have a cofinal branch for a very special reason as the following definition indicates.

6.1 Definition. For a tree $T = \langle T, <_T \rangle$, a function $f : T \to T$ is *regressive* if $f(t) <_T t$ for every $t \in T$ that is not a minimal node of T. A tree T of height θ is *special* if there is a regressive map $f : T \longrightarrow T$ with the property that the f-preimage of every point of T can be written as the union of $< \theta$ antichains of T.

This definition in case $\theta = \omega_1$ reduces indeed to the old definition of special tree, a tree that can be decomposed into countably many antichains. More generally we have the following:

6.2 Lemma. *If θ is a successor cardinal then a tree T of height θ is special if and only if T is the union of fewer than θ antichains.*

The new definition, however, seems to be the right notion of specialness as it makes sense even if θ is a limit cardinal.

6.3 Definition. A tree T of height θ is *Aronszajn* if T has no cofinal branches and if every level of T has size $< \theta$.

Recall the well-known characterization of weakly compact cardinals due to Tarski and his collaborators: a strongly inaccessible cardinal θ is weakly compact if and only if there are no Aronszajn trees of height θ. We supplement this with the following:

6.4 Theorem. *The following are equivalent for a strongly inaccessible cardinal θ:*

(1) θ is Mahlo,

(2) there are no special Aronszajn trees of height θ.

Proof. Suppose θ is a Mahlo cardinal and let T be a given tree of height θ all of whose levels have size $< \theta$. To show that T is not special let $f : T \longrightarrow T$ be a given regressive mapping. By our assumption of θ there is an elementary submodel M of some large enough structure H_κ such that $T, f \in M$ and $\lambda = M \cap \theta$ is a regular cardinal $< \theta$. Note that $T \restriction \lambda$ is a subset of M and since this tree of height λ is clearly not special, there is an $t \in T \restriction \lambda$ such that the preimage $f^{-1}(\{t\})$ is not the union of $< \lambda$ antichains. Using the elementarity of M we conclude that $f^{-1}(\{t\})$ is actually not the union of $< \theta$ antichains.

The proof that (2) implies (1) uses the method of minimal walks in a rather crucial way. So suppose to the contrary that our cardinal contains a closed and unbounded subset C consisting of singular strong limit cardinals. Using C, we choose a C-sequence C_α $(\alpha < \theta)$ such that: $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit such that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$ but if $\alpha = \sup(C \cap \alpha)$ then take C_α such that:

(a) $\mathrm{tp}(C_\alpha) = \mathrm{cf}(\alpha) < \min(C_\alpha)$,

(b) $\xi = \sup(C_\alpha \cap \xi)$ implies $\xi \in C$,

(c) $\xi \in C_\alpha$ and $\xi > \sup(C_\alpha \cap \xi)$ imply that $\xi = \eta + 1$ for some $\eta \in C$.

Given the C-sequence C_α $(\alpha < \theta)$ we have the notion of minimal walk along the sequence and various distance functions defined above. In this proof we are particularly interested in the function ρ_0 from $[\theta]^2$ into the set $\mathbb{Q}_\theta$ of all finite sequences of ordinals from θ:

$$\rho_0(\alpha,\beta) = \langle \mathrm{tp}(C_\beta \cap \alpha)\rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$$

where we stipulate that $\rho_0(\gamma,\gamma) = 0$ for all $\gamma < \theta$. We would like to show that the tree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}$$

is a special Aronszajn tree of height θ. Note that the size of the αth level $(T(\rho_0))_\alpha$ of $T(\rho_0)$ is controlled in the following way:

$$|(T(\rho_0))_\alpha| \le |\{C_\beta \cap \alpha : \alpha \le \beta < \theta\}| + |\alpha + \omega|. \tag{3.4}$$

So under the present assumption that θ is a strongly inaccessible cardinal, all levels of $T(\rho_0)$ do indeed have size $< \theta$. It remains to define the regressive map

$$f : T(\rho_0) \longrightarrow T(\rho_0)$$

that will witness specialness of $T(\rho_0)$. Note that it really suffices defining f on all levels whose index belong to our club C of singular cardinals. So let $t = (\rho_0)_\beta \restriction \alpha$ be a given node of T such that $\alpha \in C$ and $\alpha \le \beta < \theta$. Note that by our choice of the C-sequence every term of the finite sequence of ordinals $\rho_0(\alpha,\beta)$ is strictly smaller than α. So, if we let $f(t) = t \restriction \ulcorner \rho_0(\alpha,\beta) \urcorner$, where $\ulcorner \cdot \urcorner$ is a standard coding of finite sequences of ordinals by ordinals, we get a regressive map. To show that f is one-to-one on chains of $T(\rho_0)$, which would be more than sufficient, suppose $t_i = (\rho_0)_{\beta_i} \restriction \alpha_i$ $(i < 2)$ are two nodes such that $t_0 \subsetneqq t_1$. Our choice of the C-sequence allows us to deduce the following general fact about the corresponding ρ_0-function as in the case $\theta = \omega_1$ dealt with above in Lemma 1.11.

> If $\alpha \le \beta \le \gamma$, α is a limit ordinal, and if $\rho_0(\xi,\beta) = \rho_0(\xi,\gamma)$ for all $\xi < \alpha$, then $\rho_0(\alpha,\beta) = \rho_0(\alpha,\gamma)$.

Applying this to the triple of ordinals α_0, β_0 and β_1 we conclude that $\rho_0(\alpha_0,\beta_0) = \rho_0(\alpha_0,\beta_1)$. Now observe another fact about the ρ_0-function whose proof is identical to that of $\theta = \omega_1$ dealt with above in Lemma 1.10.

$$\text{If} \quad \alpha \le \beta \le \gamma \quad \text{then} \quad \rho_0(\alpha,\gamma) <_r \rho_0(\beta,\gamma).$$

Applying this to the triple $\alpha_0 < \alpha_1 \le \beta_1$ we in particular have that $\rho_0(\alpha_0,\beta_1) \ne \rho_0(\alpha_1,\beta_1)$. Combining this with the above equality gives us that $\rho_0(\alpha_0,\beta_0) \ne \rho_0(\alpha_1,\beta_1)$ and therefore that $f(t_0) \ne f(t_1)$. $\dashv$

A similar argument gives us the following characterization of Mahlo cardinals due to Hajnal, Kanamori and Shelah [20] which improves a bit an earlier characterization of this sort due to Schmerl [46]. Its proof can also be found in [66].

6.5 Theorem. *A cardinal θ is a Mahlo cardinal if and only if every regressive map f defined on a cube $[C]^3$ of a closed and unbounded subset of θ has an infinite min-homogeneous set $X \subseteq C$.*[15]

Starting from the case $n = 1$ one can now easily deduce the following characterization also due to Hajnal, Kanamori and Shelah [20].

6.6 Theorem. *The following are equivalent for an uncountable cardinal θ and a positive integer n:*

(1) θ is n-Mahlo,

(2) Every regressive map defined on $[C]^{n+2}$ for some closed and unbounded subset C of θ has an infinite min-homogeneous subset.

The proof of Theorem 6.4 gives us the following well-known fact, first established by Silver (see [36]) when θ is a successor of a regular cardinal, which we are going to reprove now.

6.7 Theorem. *If θ is a regular uncountable cardinal which is not Mahlo in the constructible universe, then there is a constructible special Aronszajn tree of height θ.*

Proof. Working in L we choose a closed and unbounded subset C of θ consisting of singular ordinals and a C-sequence C_α $(\alpha < \theta)$ such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ when $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, while if α is a limit point of C we take C_α to have the following properties:

(i) $\xi = \sup(C_\alpha \cap \xi)$ implies $\xi \in C$,

(ii) $\xi > \sup(C_\alpha \cap \xi)$ implies $\xi = \eta + 1$ for some $\eta \in C$.

We choose the C-sequence to also have the following crucial property:

(iii) $|\{C_\alpha \cap \xi : \xi \leq \alpha < \theta\}| \leq |\xi| + \aleph_0$ for all $\xi < \theta$.

It is clear then that the tree $T(\rho_0)$, where ρ_0 is the ρ_0-function of C_α $(\alpha < \theta)$, is a constructible special Aronszajn tree of height θ. ⊣

We are also in a position to deduce the following well-known fact.

6.8 Theorem. *The following are equivalent for a successor cardinal θ:*

(a) There is a special Aronszajn tree of height θ.

(b) There is a C-sequence C_α $(\alpha < \theta)$ such that $\mathrm{tp}(C_\alpha) \leq \theta^-$ for all α and such that $\{C_\alpha \cap \xi : \alpha < \theta\}$ has size $\leq \theta^-$ for all $\xi < \theta$.

[15] Recall, that X is min-homogeneous for f if $f(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma')$ for every pair $\alpha < \beta < \gamma$ and $< \alpha' < \beta' < \gamma'$ of triples of elements of X such that $\alpha = \alpha'$.

Proof. If C_α $(\alpha < \theta)$ is a C-sequence satisfying (b) and if ρ_0 is the associated ρ_0-function then $T(\rho_0)$ is a special Aronszajn tree of height θ. Suppose $<_T$ is a special Aronszajn tree ordering on θ such that $[\theta^- \cdot \alpha, \theta^- \cdot (\alpha+1))$ is its αth level. Let C be the club of ordinals $< \theta$ divisible by θ^-. Let $f : \theta \longrightarrow \theta^-$ be such that the f-preimage of every ordinal $< \theta^-$ is an antichain of the tree $(\theta, <_T)$. We choose a C-sequence C_α $(\alpha < \theta)$ such that $C_{\alpha+1} = \{\alpha\}$, $C_\alpha = (\bar{\alpha}, \alpha)$ for α limit with the property that $\bar{\alpha} = \sup(C \cap \alpha) < \alpha$, but if α is a limit point of C we take C_α more carefully as follows: $C_\alpha = \{\alpha_\xi : \xi < \eta\}$ where

$$\begin{aligned}
\alpha_\lambda = {} & \sup\{\alpha_\xi : \xi < \lambda\} \text{ for } \lambda \text{ limit} < \eta,\\
\alpha_0 = {} & \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image},\\
\alpha_{\xi+1} = {} & \text{the } <_T\text{-predecessor of } \alpha \text{ with minimal } f\text{-image subject}\\
& \text{to the requirement that } f(\alpha_{\xi+1}) > f(\alpha_{\zeta+1}) \text{ for all } \zeta < \xi,\\
\eta = {} & \text{the limit ordinal} \leq \theta^- \text{ where the process stops, i.e.}\\
& \sup\{f(\alpha_{\xi+1}) : \xi < \eta\} = \theta^-.
\end{aligned}$$

Note that if α and β are two limit points of C and if $\gamma <_T \alpha, \beta$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$. From this one concludes that the C-sequence is locally small, i.e. that $\{C_\alpha \cap \gamma : \gamma \leq \alpha < \theta\}$ has size $\leq \theta^-$ for all $\gamma < \theta$. ⊣

6.9 Corollary. *If $\theta^{<\theta} = \theta$ then there exists a special Aronszajn tree of height θ^+.*

6.10 Corollary. *In the constructible universe, special Aronszajn trees of any regular uncountable non-Mahlo height exist.*

6.11 Remark. In a large portion of the literature on this subject the notion of a special Aronszajn tree of height equal to some successor cardinal θ^+ is somewhat weaker, equivalent to the fact that the tree can be embedded inside the tree $\{f : \alpha \longrightarrow \theta : \alpha < \theta \ \&\ f \text{ is } 1-1\}$. One would get our notion of specialness by restricting the tree on successor ordinals losing thus the frequently useful property of a tree that different nodes of the same limit height have different sets of predecessors. The result 6.9 in this weaker form is due to Specker [52], while the result 6.10 is essentially due to Jensen [23].

7. The Weight Function on Successor Cardinals

In this section we assume that $\theta = \kappa^+$ and we fix a C-sequence C_α $(\alpha < \kappa^+)$ such that

$$\mathrm{tp}(C_\alpha) \leq \kappa \quad \text{for all } \alpha < \kappa^+.$$

Let $\rho_1 : [\kappa^+]^2 \longrightarrow \kappa$ be defined recursively by

$$\rho_1(\alpha, \beta) = \max\{\mathrm{tp}(C_\beta \cap \alpha), \rho_1(\alpha, \min(C_\beta \setminus \alpha))\},$$

where we stipulate that $\rho_1(\gamma,\gamma) = 0$ for all $\gamma < \kappa^+$.

7.1 Lemma. $|\{\xi \leq \alpha : \rho_1(\xi,\alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ *for all* $\alpha < \kappa^+$ *and* $\nu < \kappa$.

Proof. Let ν^+ be the first infinite cardinal above the ordinal ν. The proof of the conclusion is by induction on α. So let $\Gamma \subseteq \alpha$ be a given set of order-type ν^+. We need to find a $\xi \in \Gamma$ such that $\rho_1(\xi,\alpha) > \nu$. This will clearly be true if there is a $\xi \in \Gamma$ such that $\mathrm{tp}(C_\alpha \cap \xi) > \nu$. So, we may assume that $\mathrm{tp}(C_\alpha \cap \xi) \leq \nu$ for all $\xi \in \Gamma$. Then there must be an ordinal $\alpha_1 \in C_\alpha$ such that $\Gamma_1 = \{\xi \in \Gamma : \alpha_1 = \min(C_\alpha \setminus \xi)\}$ has size ν^+. By the inductive hypothesis there is a $\xi \in \Gamma_1$ such that, $\rho_1(\xi,\alpha_1) > \nu \geq \mathrm{tp}(C_\alpha \cap \xi)$. It follows that

$$\rho_1(\xi,\alpha) = \max\{\mathrm{tp}(C_\alpha \cap \xi), \rho_1(\xi,\alpha_1)\} = \rho_1(\xi,\alpha_1) > \nu.$$

This finishes the proof. ⊣

7.2 Lemma. *If* κ *is regular, then* $\{\xi \leq \alpha : \rho_1(\xi,\alpha) \neq \rho_1(\xi,\beta)\}$ *has size* $< \kappa$ *for all* $\alpha < \beta < \kappa^+$.

Proof. The proof is by induction on α and β. Let $\Gamma \subseteq \alpha$ be a given set of order-type κ. We need to find $\xi \in \Gamma$ such that $\rho_1(\xi,\alpha) = \rho_1(\xi,\beta)$. Let $\gamma = \sup(\Gamma)$, $\gamma_0 = \max(C_\beta \cap \gamma)$, and $\beta_0 = \min(C_\beta \setminus \gamma)$. Note that by our assumption on κ and the C-sequence, these two ordinals are well-defined and

$$\gamma_0 < \gamma \leq \beta_0 < \beta.$$

By Lemma 7.1 and the inductive hypothesis there is an ξ in $\Gamma \cap (\gamma_0,\gamma)$ such that

$$\rho_1(\xi,\alpha) = \rho_1(\xi,\beta_0) > \mathrm{tp}(C_\beta \cap \gamma).$$

It follows that $C_\beta \cap \gamma = C_\beta \cap \xi$ and $\beta_0 = \min(C_\beta \setminus \xi)$, and so

$$\rho_1(\xi,\beta) = \max\{\mathrm{tp}(C_\beta \cap \xi), \rho_1(\xi,\beta_0)\} = \rho_1(\xi,\beta_0) = \rho_1(\xi,\alpha).$$

⊣

7.3 Remark. The assumption about the regularity of κ in Lemma 7.2 is essential. For example, it can be seen (see [5, p. 72]) that the conclusion of this lemma fails if κ is a singular limit of supercompact cardinals.

7.4 Definition. For κ regular, define $\bar{\rho}_1 : [\kappa^+]^2 \longrightarrow \kappa$ by

$$\bar{\rho}_1(\alpha,\beta) = 2^{\rho_1(\alpha,\beta)} \cdot (2 \cdot \mathrm{tp}\{\xi \leq \alpha : \rho_1(\xi,\beta) = \rho_1(\alpha,\beta)\} + 1).$$

7.5 Lemma. *If* κ *is a regular cardinal then*

(a) $\bar{\rho}_1(\alpha,\gamma) \neq \bar{\rho}_1(\beta,\gamma)$ *whenever* $\alpha < \beta < \gamma < \kappa^+$,

(b) $|\{\xi \leq \alpha : \bar{\rho}_1(\xi,\alpha) \neq \bar{\rho}_1(\xi,\beta)\}| < \kappa$ *whenever* $\alpha < \beta < \kappa^+$.

7.6 Remark. Note that Lemma 7.5 gives an alternative proof of Corollary 6.9 since under the assumption $\kappa^{<\kappa} = \kappa$ the tree $T(\bar{\rho}_1)$ will have levels of size at most κ. It should be noted that the coherent sequence $(\bar{\rho})_\alpha$ $(\alpha < \kappa^+)$ of one-to-one mappings is an object of independent interest which can be particularly useful in stepping-up combinatorial properties of κ to κ^+. It is also an object that has interpretations in such areas as the theory of Čech-Stone compactifications of discrete spaces (see, e.g. [71, 8, 43, 30, 13]). We have already noted that if κ is singular then we may no longer have the coherence property of Lemma 7.2. To get this property, one needs to make some additional assumption on the C-sequence C_α $(\alpha < \kappa^+)$, an assumption about the coherence of the C-sequence. This will be subject of some of the following chapters where we will concentrate on the finer function ρ rather than ρ_1.

8. The Number of Steps

The purpose of this section is to isolate a condition on C-sequences C_α $(\alpha < \theta)$ on regular uncountable cardinals θ as weak as possible subject to a requirement that the corresponding function

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, \min(C_\beta \setminus \alpha)) + 1$$

is in some sense nontrivial, and in particular, far from being constant. Without doubt the C-sequence $C_\alpha = \alpha$ $(\alpha < \theta)$ is the most trivial choice and the corresponding ρ_0-function gives no information about the cardinal θ. The following notion of the triviality of a C-sequence on θ seems to be only marginally different.

8.1 Definition. A C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ is *trivial* if there is a closed and unbounded set $C \subseteq \theta$ such that for every $\alpha < \theta$ there is a $\beta \geq \alpha$ with $C \cap \alpha \subseteq C_\beta$.

The proof of the following fact can be found in [66].

8.2 Theorem. *The following are equivalent for any C-sequence C_α $(\alpha < \theta)$ on a regular uncountable cardinal θ and the corresponding function ρ_2:*

(i) C_α $(\alpha < \theta)$ is nontrivial.

(ii) For every family A of θ pairwise disjoint finite subsets of θ and every integer n there is a subfamily B of A of size θ such that $\rho_2(\alpha, \beta) > n$ for all $\alpha \in a$, $\beta \in b$ and $a \neq b$ in B.

8.3 Corollary. *Suppose that C_α $(\alpha < \theta)$ is a nontrivial C-sequence and let $T(\rho_0)$ be the corresponding tree (see Sect. 6 above). Then every subset of $T(\rho_0)$ of size θ contains an antichain of size θ.*

Proof. Consider a subset K of $[\theta]^2$ of size θ which gives us a subset of $T(\rho_0)$ of size θ as follows: $\{(\rho_0)_\beta\restriction(\alpha+1)\colon \{\alpha,\beta\} \in K\}$. Here, we are assuming without loss of generality that the set consists of successor nodes of $T(\rho_0)$. Clearly, we may also assume that the set takes at most one point from a given level of $T(\rho_0)$. Shrinking K further, we obtain that ρ_2 is constant on K. Let n be the constant value of $\rho_2\restriction K$. Applying Theorem 8.2(ii) to K and n, we get $K_0 \subseteq K$ of size θ such that $\rho_2(\alpha,\delta) > n$ for all $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ from K_0 with properties $\alpha < \beta$, $\gamma < \delta$ and $\alpha < \gamma$. Then $\{(\rho_0)_\beta\restriction(\alpha+1)\colon \{\alpha,\beta\} \in K_0\}$ is an antichain in $T(\rho_0)$. ⊣

8.4 Remark. It should be clear that nontrivial C-sequences exist on any successor cardinal. Indeed, with very little extra work one can show that nontrivial C-sequences exist for some inaccessible cardinals quite high in the Mahlo hierarchy. To show how close this is to the notion of weak compactness, we state an interesting characterization of it, proved in [66], which is of independent interest.[16]

8.5 Theorem. *The following are equivalent for an inaccessible cardinal θ:*

(i) θ is weakly compact.

(ii) For every C-sequence C_α $(\alpha < \theta)$ there is a closed and unbounded set $C \subseteq \theta$ such that for all $\alpha < \theta$ there is a $\beta \geq \alpha$ with $C_\beta \cap \alpha = C \cap \alpha$.

We have already remarked that every successor cardinal $\theta = \kappa^+$ admits a nontrivial C-sequence C_α $(\alpha < \theta)$. It suffices to take the C_α's to be all of order-type $\leq \kappa$. It turns out that for such a C-sequence the corresponding ρ_2-function has a property that is considerably stronger than Theorem 8.2(ii). The proof of this can again be found in [66].

8.6 Theorem. *For every infinite cardinal κ there is a C-sequence on κ^+ such that the corresponding ρ_2-function has the following unboundedness property: for every family A of κ^+ pairwise disjoint subsets of κ^+, all of size $< \kappa$, and for every $n < \omega$ there exists a $B \subseteq A$ of size κ^+ such that $\rho_2(\alpha,\beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \neq b$ in B.*

Theorems 8.2 and 8.6 admit the following variation.

8.7 Theorem. *Suppose that a regular uncountable cardinal θ supports a nontrivial C-sequence and let ρ_2 be the associated function. Then for every integer n and every pair of θ-sized families A_0 and A_1, where the members of A_0 are pairwise disjoint bounded subsets of θ and the members of A_1 are pairwise disjoint finite subsets of θ, there exist $B_0 \subseteq A_0$ and $B_1 \subseteq A_1$ of size θ such that $\rho_2(\alpha,\beta) > n$ whenever $\alpha \in a$ and $\beta \in b$ for some $a \in B_0$ and $b \in B_1$ such that* $\sup(a) < \min(b)$.

[16] It turns out that every C-sequence on θ being trivial is not quite as strong as the weak compactness of θ. As pointed out to us by Donder and König, one can show this using a model of Kunen [29, §3].

9. Square Sequences

9.1 Definition. A C-sequence C_α $(\alpha < \theta)$ is a *square sequence* if and only if it is *coherent*, i.e. it has the property that $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β.

Note that the nontriviality conditions appearing in Definition 8.1 and Theorem 8.5 coincide in the realm of square sequences:

9.2 Lemma. *A square sequence C_α $(\alpha < \theta)$ is trivial if and only if there is a closed and unbounded subset C of θ such that $C_\alpha = C \cap \alpha$ whenever α is a limit point of C.*

To a given square sequence C_α $(\alpha < \theta)$ one naturally associates a tree ordering $<^2$ on θ as follows by letting $\alpha <^2 \beta$ if and only if α is a limit point of C_β. The triviality of C_α $(\alpha < \theta)$ is then equivalent to the statement that the tree $(\theta, <^2)$ has a chain of size θ. In fact, one can characterize the tree orderings $<_T$ on θ for which there exists a square sequence C_α $(\alpha < \theta)$ such that for all $\alpha < \beta < \theta$,

$$\alpha <_T \beta \text{ if and only if } \alpha \text{ is a limit point of } C_\beta. \tag{3.5}$$

9.3 Lemma. *A tree ordering $<_T$ on θ admits a square sequence C_α $(\alpha < \theta)$ satisfying (3.5) if and only if*

(i) $\alpha <_T \beta$ can hold only for limit ordinals α and β such that $\alpha < \beta$,

(ii) $P_\beta = \{\alpha : \alpha <_T \beta\}$ is a closed subset of β, which is unbounded in β whenever $\mathrm{cf}(\beta) > \omega$ and

(iii) minimal as well as successor nodes of the tree $<_T$ on θ are ordinals of cofinality ω.

Proof. For each ordinal $\alpha < \theta$ of countable cofinality we fix a subset $S_\alpha \subseteq \alpha$ of order-type ω cofinal with α. Given a tree ordering $<_T$ on θ with properties (i)–(iii), for a limit ordinal $\beta < \theta$ let P^+_β be the set of all successor nodes from $P_\beta \cup \{\beta\}$ including the minimal one. For $\alpha \in P^+_\beta$ let α^- be its immediate predecessor in P_β. Finally, set

$$C_\beta = P_\beta \cup \bigcup\{S_\alpha \cap [\alpha^-, \alpha) : \alpha \in P^+_\beta\}.$$

It is easily checked that this defines a square sequence C_β $(\beta < \theta)$ with the property that $\alpha <_T \beta$ holds if and only if α is a limit point of C_β. ⊣

9.4 Remark. It should be clear that the proof of Lemma 9.3 shows that the exact analogue of this result is true for any cofinality $\kappa < \theta$ rather than just for the cofinality ω.

The most important result about square sequences is of course the following well-known result of Jensen [23].

9.5 Theorem. *If a regular uncountable cardinal θ is not weakly compact in the constructible universe then there is a nontrivial square sequence on θ which is moreover constructible.*

9.6 Corollary. *If a regular uncountable cardinal θ is not weakly compact in the constructible universe then there is a constructible Aronszajn tree on θ.*

Proof. Let C_α $(\alpha < \theta)$ be a fixed nontrivial square sequence which is constructible. Changing the C_α's a bit, we may assume that if β is a limit ordinal with $\alpha = \min C_\beta$ or if $\alpha \in C_\beta$ but $\sup(C_\beta \cap \alpha) < \alpha$ then α must be a successor ordinal in θ. Consider the corresponding function $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$

$$\rho_0(\alpha, \beta) = \mathrm{tp}(C_\beta \cap \alpha)^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha)),$$

where $\rho_0(\gamma, \gamma) = \emptyset$ for all $\gamma < \theta$. Consider the tree

$$T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}.$$

Clearly $T(\rho_0)$ is constructible. By (3.4) the αth level of $T(\rho_0)$ is bounded by the size of the set $\{C_\beta \cap \alpha : \beta \ge \alpha\}$. Since the intersection of the form $C_\beta \cap \alpha$ is determined by its maximal limit point modulo a finite subset of α, we conclude that the αth level of $T(\rho_0)$ has size $\le |\alpha| + \aleph_0$. Since the sequence C_α $(\alpha < \theta)$ is nontrivial, the proof of Theorem 8.5 shows that $T(\rho_0)$ has no cofinal branches. $\dashv$

9.7 Lemma. *Suppose C_α $(\alpha < \theta)$ is a square sequence on θ, $<^2$ the associated tree ordering on θ and $T(\rho_0) = \{(\rho_0)_\beta \restriction \alpha : \alpha \le \beta < \theta\}$ where $\rho_0 : [\theta]^2 \longrightarrow \mathbb{Q}_\theta$ is the associated ρ_0-function. Then $\alpha \longmapsto (\rho_0)_\alpha$ is a strictly increasing map from the tree $(\theta, <^2)$ into the tree $T(\rho_0)$.*

Proof. If α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$ so the walks $\alpha \to \xi$ and $\beta \to \xi$ for $\xi < \alpha$ get the same code $\rho_0(\xi, \alpha) = \rho_0(\xi, \beta)$. $\dashv$

The purpose of this section, however, is to analyze a family of ρ-functions associated with a square sequence C_α $(\alpha < \theta)$ on some regular uncountable cardinal θ, both fixed from now on. Recall that an ordinal α *divides* an ordinal γ if there is a β such that $\gamma = \alpha \cdot \beta$, i.e. γ can be written as the union of an increasing β-sequence of intervals of type α. Let $\kappa \le \theta$ be a fixed infinite regular cardinal. Let $\Lambda_\kappa : [\theta]^2 \longrightarrow \theta$ be defined by

$$\Lambda_\kappa(\alpha, \beta) = \max\{\xi \in C_\beta \cap (\alpha + 1) : \kappa \text{ divides } \mathrm{tp}(C_\beta \cap \xi)\}.$$

Finally, we are ready to define the main object of study in this section:

$$\rho_\kappa : [\theta]^2 \longrightarrow \kappa$$

defined recursively by

$$\rho_\kappa(\alpha, \beta) = \sup\{\mathrm{tp}(C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)), \rho_\kappa(\alpha, \min(C_\beta \setminus \alpha)),$$
$$\rho_\kappa(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda_\kappa(\alpha, \beta), \alpha)\},$$

where we stipulate that $\rho_\kappa(\gamma,\gamma)=0$ for all γ.

The following consequence of the coherence property of C_α $(\alpha<\theta)$ will be quite useful.

9.8 Lemma. *If α is a limit point of C_β then $\rho_\kappa(\xi,\alpha)=\rho_\kappa(\xi,\beta)$ for every $\xi<\alpha$.*

Note that ρ_κ is something that corresponds to the function $\rho:[\omega_1]^2\longrightarrow\omega$ considered in Definition 2.1 (see also Sect. 11) and that the ρ_κ's are simply various *local versions* of the key definition. It turns out that they all have the crucial subadditive properties (see [66]).

9.9 Lemma. *If $\alpha<\beta<\gamma<\theta$ then*

(a) $\rho_\kappa(\alpha,\gamma)\leq\max\{\rho_\kappa(\alpha,\beta),\rho_\kappa(\beta,\gamma)\}$,

(b) $\rho_\kappa(\alpha,\beta)\leq\max\{\rho_\kappa(\alpha,\gamma),\rho_\kappa(\beta,\gamma)\}$.

The following is an immediate consequence of the fact that the definition of ρ_κ is closely tied to the notion of a walk along the fixed square sequence.

9.10 Lemma. *$\rho_\kappa(\alpha,\gamma)\geq\rho_\kappa(\alpha,\beta)$ whenever $\alpha\leq\beta\leq\gamma$ and β belongs to the trace of the walk from γ to α.*

9.11 Lemma. *Suppose $\beta\leq\gamma<\theta$ and that β is a limit ordinal >0. Then $\rho_\kappa(\alpha,\gamma)\geq\rho_\kappa(\alpha,\beta)$ for coboundedly many $\alpha<\beta$.*

Proof. Let $\gamma=\gamma_0>\gamma_1>\cdots>\gamma_{n-1}>\gamma_n=\beta$ be the trace of the walk from γ to β. Let $\bar\gamma=\gamma_{n-1}$ if β is a limit point of $C_{\gamma_{n-1}}$, otherwise let $\bar\gamma=\beta$. Note that by Lemma 9.8, in any case we have that

$$\rho_\kappa(\alpha,\beta)=\rho_\kappa(\alpha,\bar\gamma)\quad\text{for all }\alpha<\beta.\tag{3.6}$$

Let $\bar\beta<\beta$ be an upper bound of all $C_{\gamma_i}\cap\beta$ $(i<n)$ which are bounded in β. Then $\bar\gamma$ is a member of the trace of any walk from γ to some ordinal α in the interval $[\bar\beta,\beta)$. Applying Lemma 9.10 to this fact gives us

$$\rho_\kappa(\alpha,\gamma)\geq\rho_\kappa(\alpha,\bar\gamma)\quad\text{for all }\alpha\in[\bar\beta,\beta).$$

Since $\rho_\kappa(\alpha,\bar\gamma)=\rho_\kappa(\alpha,\beta)$ for all $\alpha<\beta$ (see (3.6)), this gives us the conclusion of the lemma. ⊣

The proof of the following lemma can be found in [66].

9.12 Lemma. *The set $P^\kappa_\nu(\beta)=\{\xi<\beta:\rho_\kappa(\xi,\beta)\leq\nu\}$ is a closed subset of β for every $\beta<\theta$ and $\nu<\kappa$.*

For $\alpha<\beta<\theta$ and $\nu<\kappa$ set

$$\alpha<^\kappa_\nu\beta\quad\text{if and only if}\quad\rho_\kappa(\alpha,\beta)\leq\nu.$$

9.13 Lemma.

(1) $<^\kappa_\nu$ *is a tree ordering on* θ*,*

(2) $<^\kappa_\nu \subseteq <^\kappa_\mu$ *whenever* $\nu < \mu < \kappa$*,*

(3) $\in \restriction (\theta \times \theta) = \bigcup_{\nu<\kappa} <^\kappa_\nu$*.*

Proof. This follows immediately from Lemma 9.9. $\dashv$

Recall the notion of a special tree of height θ from Sect. 6, a tree T for which one can find a T-regressive map $f : T \longrightarrow T$ with the property that the preimage of any point is the union of $< \theta$ antichains. By a *tree on* θ we mean a tree of the form $(\theta, <_T)$ with the property that $\alpha <_T \beta$ implies $\alpha < \beta$.

9.14 Lemma. *If a tree* T *naturally placed on* θ *is special, then there is an ordinal-regressive map* $f : \theta \longrightarrow \theta$ *and a closed and unbounded set* $C \subseteq \theta$ *such that* f *is one-to-one on all chains separated by* C*.*

Proof. Let $g : \theta \longrightarrow \theta$ be a T-regressive map such that for each $\xi < \theta$ the preimage $g^{-1}(\{\xi\})$ can be written as a union of a sequence $A_\delta(\xi)$ $(\delta < \lambda_\xi)$ of antichains, where $\lambda_\xi < \theta$. Let C be the collection of all limit $\alpha < \theta$ with the property that $\lambda_\xi < \alpha$ for all $\xi < \alpha$. Choose an ordinal-regressive $f : \theta \longrightarrow \theta$ as follows. If there is a $\delta \in C$ such that $g(\alpha) < \delta \leq \alpha$, then $f(\alpha)$ is smaller than the minimal member of C above $g(\alpha)$, $f(\alpha)$ codes in some standard way the ordinal $g(\alpha)$ as well as the index δ of the antichain $A_\delta(g(\alpha))$ to which α belongs, and $f(\alpha) \notin C$. If no member of C separates $g(\alpha)$ and α, let $f(\alpha)$ be the maximal member of C that is smaller than α. $\dashv$

By Lemma 9.13 we have a sequence $<^\kappa_\nu$ $(\nu < \kappa)$ of tree orderings on θ. The following lemma tells us that they are frequently quite large orderings.

9.15 Lemma. *If* $\theta > \kappa$ *is not a successor of a cardinal of cofinality* κ *then there must be a* $\nu < \kappa$ *such that* $(\theta, <^\kappa_\nu)$ *is a nonspecial tree on* θ*.*

Proof. Suppose to the contrary that all trees are special. By Lemma 9.14 we may choose ordinal-regressive maps $f_\nu : \theta \longrightarrow \theta$ for all $\nu < \kappa$ and a single closed and unbounded set $C \subseteq \theta$ such that each of the maps f_ν is one-to-one on $<^\kappa_\nu$-chains separated by C. Using the Pressing Down Lemma we find a stationary set Γ of cofinality κ^+ ordinals $< \theta$ and $\lambda < \theta$ such that $f_\nu(\gamma) < \lambda$ for all $\gamma \in \Gamma$ and $\nu < \kappa$. If $|\lambda|^+ < \theta$, let $\Delta = \lambda$, $\Gamma = \Gamma_0$ and if $|\lambda|^+ = \theta$, represent λ as the increasing union of a sequence Δ_ξ $(\xi < \mathrm{cf}(|\lambda|))$ of sets of size $< |\lambda|$. Since $\kappa \neq \mathrm{cf}(|\lambda|)$ there is a $\bar{\xi} < \mathrm{cf}(|\lambda|)$ and a stationary $\Gamma_0 \subseteq \Gamma$ such that for all $\gamma \in \Gamma_0$, $f_\nu(\gamma) \in \Delta_{\bar{\xi}}$ for κ many $\nu < \kappa$. Let $\Delta = \Delta_{\bar{\xi}}$. This gives us subsets Δ and Γ_0 of θ such that

(1) $|\Delta|^+ < \theta$ and Γ_0 is stationary in θ,

(2) $\Sigma_\gamma = \{\nu < \kappa : f_\nu(\gamma) \in \Delta\}$ is unbounded in κ for all $\gamma \in \Gamma_0$.

Let $\bar\theta = \kappa^+ \cdot |\Delta|^+$. Then $\bar\theta < \theta$ and so we can find $\beta \in \Gamma_0$ such that $\Gamma_0 \cap C \cap \beta$ has size $\bar\theta$. Then there will be $\nu_0 < \kappa$ and $\Gamma_1 \subseteq \Gamma_0 \cap C \cap \beta$ of size $\bar\theta$ such that $\rho_\kappa(\alpha, \beta) \leq \nu_0$ for all $\alpha \in \Gamma_1$. By (2) we can find $\Gamma_2 \subseteq \Gamma_1$ of size $\bar\theta$ and $\nu_1 \geq \nu_0$ such that $f_{\nu_1}(\alpha) \in \Delta$ for all $\alpha \in \Gamma_2$. Note that Γ_2 is a $<^\kappa_{\nu_1}$-chain separated by C, so f_{ν_1} is one-to-one on Γ_2. However, this gives us the desired contradiction since the set Δ, in which f_{ν_1} embeds Γ_2 has size smaller than the size of Γ_2. This finishes the proof. ⊣

It is now natural to ask the following question: under which assumption on the square sequence C_α $(\alpha < \theta)$ can we conclude that neither of the trees $(\theta, <^\kappa_\nu)$ will have a branch of size θ?

9.16 Lemma. *If the set $\Gamma_\kappa = \{\alpha < \theta : \mathrm{tp}(C_\alpha) = \kappa\}$ is stationary in θ, then none of the trees $(\theta, <^\kappa_\nu)$ has a branch of size θ.*

Proof. Assume that B is a $<^\kappa_\nu$-branch of size θ. By Lemma 9.12, B is a closed and unbounded subset of θ. Pick a limit point β of B which belongs to Γ_κ. Pick $\alpha \in B \cap \beta$ such that $\mathrm{tp}(C_\beta \cap \alpha) > \nu$. By definition of $\rho_\kappa(\alpha, \beta)$ we have that $\rho_\kappa(\alpha, \beta) \geq \mathrm{tp}(C_\beta \cap \alpha) > \nu$ since clearly $\Lambda_\kappa(\alpha, \beta) = 0$. This contradicts the fact that $\alpha <^\kappa_\nu \beta$ and finishes the proof. ⊣

9.17 Definition. A square sequence on θ is *special* if the corresponding tree $(\theta, <^2)$ is special, i.e. there is a $<^2$-regressive map $f : \theta \longrightarrow \theta$ with the property that the f-preimage of every $\xi < \theta$ is the union of $< \theta$ antichains of $(\theta, <^2)$.

9.18 Theorem. *Suppose $\kappa < \theta$ are regular cardinals such that θ is not a successor of a cardinal of cofinality κ. Then to every square sequence C_α $(\alpha < \theta)$ for which there exist stationarily many α such that $\mathrm{tp}(C_\alpha) = \kappa$, one can associate a sequence $C_{\alpha\nu}$ $(\alpha < \theta, \nu < \kappa)$ such that:*

- *(i) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ for all α and $\nu \leq \mu$,*
- *(ii) $\alpha = \bigcup_{\nu<\kappa} C_{\alpha\nu}$ for all limit α,*
- *(iii) $C_{\alpha\nu}$ $(\alpha < \theta)$ is a nonspecial (and nontrivial) square sequence on θ for all $\nu < \kappa$.*

Proof. Fix $\nu < \kappa$ and define $C_{\alpha\nu}$ by induction on $\alpha < \theta$. So suppose β is a limit ordinal $< \theta$ and that $C_{\alpha\nu}$ is defined for all $\alpha < \beta$. If $P^\kappa_\nu(\beta)$ is bounded in β, let $\bar\beta$ be the maximal limit point of $P^\kappa_\nu(\beta)$ ($\bar\beta = 0$ if the set has no limit points) and let

$$C_{\beta\nu} = C_{\bar\beta\nu} \cup P^\kappa_\nu(\beta) \cup (C_\beta \cap [\max(P^\kappa_\nu(\beta)), \beta)).$$

If $P^\kappa_\nu(\beta)$ is unbounded in β, let

$$C_{\beta\nu} = P^\kappa_\nu(\beta) \cup \bigcup\{C_{\alpha\nu} : \alpha \in P^\kappa_\nu(\beta) \text{ and } \alpha = \sup(P^\kappa_\nu(\beta) \cap \alpha)\}.$$

By Lemmas 9.9 and 9.12, $C_{\beta\nu}$ $(\beta < \theta)$ is well defined and it forms a square sequence on θ. The properties (i) and (ii) are also immediate. To see that for each $\nu < \kappa$ the sequence $C_{\beta\nu}$ $(\beta < \theta)$ is nontrivial, one uses Lemma 9.16 and the fact that if α is a limit point of $C_{\beta\nu}$ occupying a place in $C_{\beta\nu}$ that is divisible by κ, then $\alpha <_\nu^\kappa \beta$. By Lemma 9.15, or rather its proof, we conclude that there is a $\bar{\nu} < \kappa$ such that $C_{\beta\nu}$ $(\beta < \theta)$ is nonspecial for all $\nu \geq \bar{\nu}$. This finishes the proof. ⊣

The following facts whose proof can be found in [66] gives us a square sequence satisfying the hypothesis of Lemma 9.18.

9.19 Lemma. *For every pair of regular cardinals $\kappa < \theta$, every special square sequence C_α $(\alpha < \theta)$ can be refined to a square sequence $\bar{C}_\alpha$ $(\alpha < \theta)$ with the property that* $\mathrm{tp}(\bar{C}_\alpha) = \kappa$ *for stationarily many $\alpha < \theta$.*

Finally we can state the main result of this section which follows from Theorem 9.18 and Lemma 9.19.

9.20 Theorem. *A regular uncountable cardinal $\theta \neq \omega_1$ carries a nontrivial square sequence iff it also carries such a sequence which is moreover nonspecial.*

9.21 Corollary. *If a regular uncountable cardinal $\theta \neq \omega_1$ is not weakly compact in the constructible universe then there is a nonspecial Aronszajn tree of height θ.*

Proof. By Theorem 9.5, θ carries a nontrivial square sequence C_α $(\alpha < \theta)$. By Theorem 9.20 we may assume that the sequence is moreover nonspecial. Let ρ_0 be the associated ρ_0-function and consider the tree $T(\rho_0)$. As in Corollary 9.6 we conclude that $T(\rho_0)$ is an Aronszajn tree of height θ. By Lemma 9.7 there is a strictly increasing map from $(\theta, <^2)$ into $T(\rho_0)$, so $T(\rho_0)$ must be nonspecial. ⊣

9.22 Remark. The assumption $\theta \neq \omega_1$ in Theorem 9.20 is essential as there is always a nontrivial square sequence on ω_1 but it is possible to have a situation where all Aronszajn trees on ω_1 are special. For example MA_{ω_1} implies this. In [33], Laver and Shelah have shown that any model with a weakly compact cardinal admits a forcing extension satisfying CH and the statement that all Aronszajn trees on ω_2 are special. A well-known open problem in this area asks whether one can have GCH rather than CH in a model where all Aronszajn trees on ω_2 are special.

10. The Full Lower Trace of a Square Sequence

In this section θ is a regular uncountable cardinal and C_α $(\alpha < \theta)$ is a nontrivial square sequence on θ. Recall the function $\Lambda = \Lambda_\omega : [\theta]^2 \longrightarrow \theta$:

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1).$$

($\Lambda(\alpha,\beta)=0$ if $C_\beta\cap(\alpha+1)$ has no limit points.)

The purpose of this section is to study the following recursive trace formula, describing a mapping $\mathrm{F}:[\theta]^2\longrightarrow[\theta]^{<\omega}$:

$$\mathrm{F}(\alpha,\beta)=\mathrm{F}(\alpha,\min(C_\beta\setminus\alpha))\cup\bigcup\{\mathrm{F}(\xi,\alpha):\xi\in C_\beta\cap[\Lambda(\alpha,\beta),\alpha)\},$$

where $\mathrm{F}(\gamma,\gamma)=\{\gamma\}$ for all γ.

As in the case $\theta=\omega_1$, the full lower trace has the following two properties (see [66]).

10.1 Lemma. *For all $\alpha\leq\beta\leq\gamma$,*

(a) $\mathrm{F}(\alpha,\gamma)\subseteq\mathrm{F}(\alpha,\beta)\cup\mathrm{F}(\beta,\gamma)$,

(b) $\mathrm{F}(\alpha,\beta)\subseteq\mathrm{F}(\alpha,\gamma)\cup\mathrm{F}(\beta,\gamma)$.

10.2 Lemma. *For all $\alpha\leq\beta\leq\gamma$,*

(a) $\rho_0(\alpha,\beta)=\rho_0(\min(\mathrm{F}(\beta,\gamma)\setminus\alpha),\beta)^\frown\rho_0(\alpha,\min(\mathrm{F}(\beta,\gamma)\setminus\alpha))$,

(b) $\rho_0(\alpha,\gamma)=\rho_0(\min(\mathrm{F}(\beta,\gamma)\setminus\alpha),\gamma)^\frown\rho_0(\alpha,\min(\mathrm{F}(\beta,\gamma)\setminus\alpha))$.

Recall the function $\rho_2:[\theta]^2\longrightarrow\omega$ which counts the number of steps in the walk along the fixed C-sequence C_α $(\alpha<\theta)$ which in this section is assumed to be moreover a square sequence:

$$\rho_2(\alpha,\beta)=\rho_2(\alpha,\min(C_\beta\setminus\alpha))+1,$$

where we let $\rho_2(\gamma,\gamma)=0$ for all γ. Thus $\rho_2(\alpha,\beta)+1$ is simply equal to the cardinality of the trace $\mathrm{Tr}(\alpha,\beta)$ of the minimal walk from β to α.

10.3 Lemma. $\sup_{\xi<\alpha}|\rho_2(\xi,\alpha)-\rho_2(\xi,\beta)|<\infty$ *for all $\alpha<\beta<\theta$.*

Proof. By Lemma 10.2, $\sup_{\xi<\alpha}|\rho_2(\xi,\alpha)-\rho_2(\xi,\beta)|$ is less than or equal than $\sup_{\xi\in F(\alpha,\beta)}|\rho_2(\xi,\alpha)-\rho_2(\xi,\beta)|$. ⊣

10.4 Definition. Set $\mathcal{I}$ to be the set of all countable $\Gamma\subseteq\theta$ such that $\sup_{\xi\in\Delta}\rho_2(\xi,\alpha)=\infty$ for all $\alpha<\theta$ and infinite $\Delta\subseteq\Gamma\cap\alpha$.

10.5 Lemma. *$\mathcal{I}$ is a P-ideal of countable subsets of θ.*

Proof. Let Γ_n $(n<\omega)$ be a given sequence of members of $\mathcal{I}$ and fix $\beta<\theta$ such that $\Gamma_n\subseteq\beta$ for all n. For $n<\omega$ set $\Gamma_n^*=\{\xi\in\Gamma_n:\rho_2(\xi,\beta)\geq n\}$. Since Γ_n belongs to $\mathcal{I}$, Γ_n^* is a cofinite subset of Γ_n. Let $\Gamma_\infty=\bigcup_{n<\omega}\Gamma_n^*$. Then Γ_∞ is a member of $\mathcal{I}$ such that $\Gamma_n\setminus\Gamma_\infty$ is finite for all n. ⊣

10.6 Theorem. *The P-ideal dichotomy implies that a nontrivial square sequence can exist only on $\theta=\omega_1$.*

Proof. Applying the P-ideal dichotomy on $\mathcal{I}$ from Definition 10.4 we get the two alternatives (see Definition 3.9):

(1) there is an uncountable $\Delta \subseteq \theta$ such that $[\Delta]^\omega \subseteq \mathcal{I}$, or

(2) there is a decomposition $\theta = \bigcup_{n<\omega} \Sigma_n$ such that $\Sigma_n \perp \mathcal{I}$ for all n.

By Lemma 10.3, if (1) holds, then $\Delta \cap \alpha$ must be countable for all $\alpha < \theta$ and so the cofinality of θ must be equal to ω_1. Since we are working only with regular uncountable cardinals, we see that (1) gives us that $\theta = \omega_1$ must hold. Suppose now (2) holds and pick $k < \omega$ such that Σ_k is unbounded in θ. Since $\Sigma_k \perp \mathcal{I}$ we have that $(\rho_2)_\alpha$ is bounded on $\Sigma_k \cap \alpha$ for all $\alpha < \theta$. So there is an unbounded set $\Gamma \subseteq \theta$ and an integer n such that for each $\alpha \in \Gamma$ the restriction of $(\rho_2)_\alpha$ on $\Sigma_k \cap \alpha$ is bounded by n. By Theorem 8.2 we conclude that the square sequence C_α $(\alpha < \theta)$ we started with must be trivial. ⊣

10.7 Definition. By S_θ we denote the *sequential fan* with θ edges, i.e. the space on $(\theta \times \omega) \cup \{*\}$ with $*$ as the only nonisolated point, while a typical neighborhood of $*$ has the form $\mathcal{U}_f = \{(\alpha, n) : n \geq f(\alpha)\} \cup \{*\}$ where $f : \theta \longrightarrow \omega$.

The *tightness* of a point x in a space X is equal to θ if θ is the minimal cardinal such that, if a set $W \subseteq X \setminus \{x\}$ accumulates to x, then there is a subset of W of size $\leq \theta$ that accumulates to x.

10.8 Theorem. *If there is a nontrivial square sequence on θ then the square of the sequential fan S_θ has tightness equal to θ.*

The proof will be given after a sequence of definitions and lemmas.

10.9 Definition. Given a square sequence C_α $(\alpha < \theta)$ and its number of steps function $\rho_2 : [\theta]^2 \longrightarrow \omega$ we define $d : [\theta]^2 \longrightarrow \omega$ by letting

$$d(\alpha, \beta) = \sup_{\xi \leq \alpha} |\rho_2(\xi, \alpha) - \rho_2(\xi, \beta)|.$$

10.10 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho_2(\alpha, \beta) \leq d(\alpha, \beta)$,

(b) $d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma)$,

(c) $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\beta, \gamma)$.

Proof. The conclusion (a) follows from the fact that we allow $\xi = \alpha$ in the definition of $d(\alpha, \beta)$. The conclusions (b) and (c) are consequences of the triangle inequalities of the ℓ_∞-norm and the fact that in both inequalities we have that the domain of functions on the left-hand side is included in the domain of functions on the right-hand side. ⊣

10.11 Definition. For $\gamma \leq \theta$, let

$$W_\gamma = \{((\alpha, d(\alpha, \beta)), (\beta, d(\alpha, \beta))) : \alpha < \beta < \gamma\}.$$

The following lemma establishes that the tightness of the point $(*,*)$ of S^2_θ is equal to θ, giving us the proof of Theorem 10.8.

10.12 Lemma. $(*,*) \in \bar{W}_\theta$ *but* $(*,*) \notin \bar{W}_\gamma$ *for all* $\gamma < \theta$.

Proof. To see that W_θ accumulates to $(*,*)$, let $\mathcal{U}^2_f$ be a given neighborhood of $(*,*)$. Fix an unbounded set $\Gamma \subseteq \theta$ on which f is constant. By Theorem 8.2 and Lemma 10.10(a) there exists an $\alpha < \beta$ in Γ such that $d(\alpha,\beta) \geq f(\alpha) = f(\beta)$. Then $((\alpha, d(\alpha,\beta)), (\beta, d(\alpha,\beta)))$ belongs to the intersection $W_\theta \cap \mathcal{U}^2_f$. To see that for a given $\gamma < \theta$ the set W_γ does not accumulate to $(*,*)$, choose $g : \theta \longrightarrow \omega$ such that

$$g(\alpha) = 2d(\alpha,\gamma) + 1 \quad \text{for } \alpha < \gamma.$$

Suppose $W_\gamma \cap \mathcal{U}^2_g$ is nonempty and choose $((\alpha, d(\alpha,\beta)), (\beta, d(\alpha,\beta)))$ from this set. Then

$$d(\alpha,\beta) \geq 2d(\alpha,\gamma) + 1 \quad \text{and} \quad d(\alpha,\beta) \geq 2d(\beta,\gamma) + 1,$$

and so, $d(\alpha,\beta) \geq d(\alpha,\gamma) + d(\beta,\gamma) + 1$, contradicting Lemma 10.10(c). $\dashv$

Since $\theta = \omega_1$ admits a nontrivial square sequence, Theorem 10.8 leads to the following result of Gruenhage and Tanaka [19].

10.13 Corollary. *The square of the sequential fan with* ω_1 *edges is not countably tight.*

10.14 Question. What is the tightness of the square of the sequential fan with ω_2 edges?

10.15 Corollary. *If a regular uncountable cardinal* θ *is not weakly compact in the constructible universe then the square of the sequential fan with* θ *edges has tightness equal to* θ.

11. Special Square Sequences

The following well-known result of Jensen [23] supplements the corresponding result for weakly compact cardinals listed above as Theorem 9.5.

11.1 Theorem. *If a regular uncountable cardinal* θ *is not Mahlo in the constructible universe then there is a special square sequence on* θ *which is moreover constructible.*

Today we know many more inner models with sufficient amount of fine structure necessary for building special square sequences. So the existence of special square sequences, especially at successors of strong-limit singular cardinals, is tied to the existence of some other large cardinal axioms. The reader is referred to the relevant chapters of this Handbook for the specific information. In this section we give the combinatorial analysis of walks along special square sequences and the corresponding distance functions. Let us start by restating some results of Sect. 9.

11.2 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $C_{\alpha\nu}$ $(\alpha < \theta, \nu < \kappa)$ such that:*

(1) $C_{\alpha\nu} \subseteq C_{\alpha\mu}$ for all α and $\nu < \mu$,

(2) $\alpha = \bigcup_{\nu<\kappa} C_\alpha$ for all limit α,

(3) $C_{\alpha\nu}$ $(\alpha < \theta)$ is a nontrivial square sequence on θ for all $\nu < \kappa$.

Moreover, if θ is not a successor of a cardinal of cofinality κ then each of the square sequences can be chosen to be nonspecial.

11.3 Theorem. *Suppose $\kappa < \theta$ are regular cardinals and that θ carries a special square sequence. Then there exist $<_\nu$ $(\nu < \kappa)$ such that:*

(i) $<_\nu$ is a closed tree ordering of θ for each $\nu < \kappa$,

(ii) $\in\restriction(\theta \times \theta) = \bigcup_{\nu<\kappa} <_\nu$,

(iii) no tree $(\theta, <_\nu)$ has a chain of size θ.

11.4 Lemma. *The following are equivalent when θ is a successor of some cardinal κ:*

(1) there is a special square sequence on θ,

(2) there is a square sequence C_α $(\alpha < \theta)$ such that $\mathrm{tp}(C_\alpha) \le \kappa$ for all $\alpha < \theta$.

Proof. Let D_α $(\alpha < \kappa^+)$ be a given special square sequence. By Lemma 6.2 the corresponding tree $(\kappa^+, <^2)$ can be decomposed into κ antichains so let $f : \kappa^+ \longrightarrow \kappa$ be a fixed map such that $f^{-1}(\{\xi\})$ is a $<^2$-antichain for all $\xi < \kappa$. Let $\alpha < \kappa^+$ be a given limit ordinal. If D_α has a maximal limit point $\bar{\alpha} < \alpha$, let $C_\alpha = D_\alpha \setminus \bar{\alpha}$. Suppose now that $\{\xi : \xi <^2 \alpha\}$ is unbounded in α and define a strictly increasing continuous sequence $c_\alpha(\xi)$ $(\xi < \nu(\alpha))$ of its elements as follows. Let $c_\alpha(0) = \min\{\xi : \xi <^2 \alpha\}$, $c_\alpha(\eta) = \sup_{\xi<\eta} c_\alpha(\xi)$ for η limit, and $c_\alpha(\xi+1)$ is the minimal $<^2$-predecessor γ of α such that $\gamma > c_\alpha(\xi)$ and has the minimal f-image among all $<^2$-predecessors that are $> c_\alpha(\xi)$. The ordinal $\nu(\alpha)$ is defined as the place where the process stops, i.e. when $\alpha = \sup_{\xi<\nu(\alpha)} c_\alpha(\xi)$. Let $C_\alpha = \{c_\alpha(\xi) : \xi < \nu(\alpha)\}$. It is easily checked that this gives a square sequence C_α $(\alpha < \kappa^+)$ with the property that $\mathrm{tp}(C_\alpha) \le \kappa$ for all $\alpha < \kappa^+$. $\dashv$

Square sequences C_α $(\alpha < \kappa^+)$ that have the property $\mathrm{tp}(C_\alpha) \le \kappa$ for all $\alpha < \kappa^+$ are usually called $\square_\kappa$-sequences. So let C_α $(\alpha < \kappa^+)$ be a $\square_\kappa$-sequence fixed from now on. Let

$$\Lambda(\alpha, \beta) = \text{maximal limit point of } C_\beta \cap (\alpha + 1)$$

when such a limit point exists; otherwise $\Lambda(\alpha,\beta)=0$. The purpose of this section is to analyze the following distance function:

$$\rho : [\kappa^+]^2 \longrightarrow \kappa$$

defined recursively by

$$\begin{aligned}\rho(\alpha,\beta) = \max\{&\mathrm{tp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)),\\ &\rho(\xi,\alpha) : \xi \in C_\beta \cap [\Lambda(\alpha,\beta),\alpha)\},\end{aligned}$$

where we stipulate that $\rho(\gamma,\gamma)=0$ for all $\gamma<\kappa^+$. Clearly $\rho(\alpha,\beta)\geq\rho_1(\alpha,\beta)$, so by Lemma 7.1 we have

11.5 Lemma. $|\{\xi\leq\alpha : \rho(\xi,\alpha)\leq\nu\}|\leq|\nu|+\aleph_0$ *for* $\alpha<\kappa^+$ *and* $\nu<\kappa$.

The following two crucial subadditive properties of ρ have proofs that are almost identical to the proofs of the corresponding properties of, say, the function ρ_ω discussed above in Sect. 9.

11.6 Lemma. *For all* $\alpha\leq\beta\leq\gamma$,

(a) $\rho(\alpha,\gamma)\leq\max\{\rho(\alpha,\beta),\rho(\beta,\gamma)\}$,

(b) $\rho(\alpha,\beta)\leq\max\{\rho(\alpha,\gamma),\rho(\beta,\gamma)\}$.

The following immediate fact will also be quite useful.

11.7 Lemma. *If* α *is a limit point of* C_β, *then* $\rho(\xi,\alpha)=\rho(\xi,\beta)$ *for every* $\xi<\alpha$.

The following as well is an immediate consequence of the fact that the definition of ρ is closely tied to the notion of a minimal walk along the square sequence.

11.8 Lemma. $\rho(\alpha,\gamma)\geq\max\{\rho(\alpha,\beta),\rho(\beta,\gamma)\}$ *whenever* $\alpha\leq\beta\leq\gamma$ *and* β *belongs to the trace of the walk from* γ *to* α.

Using Lemmas 11.7 and 11.8 one proves the following fact exactly as in the case of ρ_κ of Sect. 9 (the proof of Lemma 9.11).

11.9 Lemma. *If* $0<\beta\leq\gamma$ *and* β *is a limit ordinal, then there is a* $\bar\beta<\beta$ *such that* $\rho(\alpha,\gamma)\geq\rho(\alpha,\beta)$ *for all* α *in the interval* $[\bar\beta,\beta)$.

The proof of the following fact is also completely analogous to the proof of the corresponding fact for the local version ρ_κ considered above in Sect. 9 (the proof of Lemma 9.12).

11.10 Lemma. $P_\nu(\gamma)=\{\beta<\gamma : \rho(\beta,\gamma)\leq\nu\}$ *is a closed subset of* γ *for all* $\gamma<\kappa^+$ *and* $\nu<\kappa$.

The discussion of $\rho : [\kappa^+]^2 \longrightarrow \kappa$ now splits naturally into two cases depending on whether κ is a regular or a singular cardinal (with the case $\mathrm{cf}(\kappa)=\omega$ of special importance).

12. Successors of Regular Cardinals

In this section, κ is a fixed regular cardinal, C_α ($\alpha < \kappa^+$) a fixed $\square_\kappa$-sequence and $\rho : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding ρ-function. For $\nu < \kappa$ and $\alpha < \beta < \kappa^+$ set

$$\alpha <_\nu \beta \quad \text{if and only if} \quad \rho(\alpha, \beta) \leq \nu.$$

The following is an immediate consequence of the analysis of ρ given in the previous section.

12.1 Lemma.

(a) $<_\nu$ *is a closed tree ordering of* κ^+ *of height* $\leq \kappa$ *for all* $\nu < \kappa$*,*

(b) $<_\nu \subseteq <_\mu$ *whenever* $\nu \leq \mu < \kappa$*,*

(c) $\in \restriction (\kappa^+ \times \kappa^+) = \bigcup_{\nu<\kappa} <_\nu$*.*

The following result shows that these trees have some properties of smallness not covered by statements of Lemma 12.1.

12.2 Lemma. *If* $\kappa > \omega$*, then no tree* $(\kappa^+, <_\nu)$ *has a branch of size* κ*.*

Proof. Suppose towards a contradiction that some tree $(\kappa^+, <_\nu)$ does have a branch of size κ and let B be one such fixed branch (maximal chain). By Lemmas 11.5 and 11.10, if $\gamma = \sup(B)$ then B is a closed and unbounded subset of γ of order-type κ. Since κ is regular and uncountable, $C_\gamma \cap B$ is unbounded in C_γ, so in particular we can find $\alpha \in C_\gamma \cap B$ such that $\mathrm{tp}(C_\gamma \cap \alpha) > \nu$. Reading off the definition of $\rho(\alpha, \gamma)$ we conclude that $\rho(\alpha, \beta) = \mathrm{tp}(C_\gamma \cap \alpha) > \nu$. Similarly we can find a $\beta > \alpha$ belonging to the intersection of $\lim(C_\gamma)$ and B. Then $C_\beta = C_\gamma \cap \beta$ so $\alpha \in C_\beta$ and therefore $\rho(\alpha, \beta) = \mathrm{tp}(C_\beta \cap \alpha) > \nu$ contradicting the fact that $\alpha <_\nu \beta$. ⊣

A tree of height κ is *Souslin* if all of its chains and antichains are of cardinality less than κ.

12.3 Lemma. *If* $\kappa > \omega$*, then no tree* $(\kappa^+, <_\nu)$ *has a tree of height* κ *which is Souslin subtree.*

Proof. Forcing with subtree of $(\kappa^+, <_\nu)$ of height κ which is Souslin would produce an ordinal γ of cofinality κ and a closed and unbounded subset B of C_γ forming a chain of the tree $(\kappa^+, <_\nu)$. It is well-known that in this case B would contain a ground model subset of size κ, contradicting Lemma 12.2. ⊣

12.4 Lemma. *If* $\kappa > \omega$ *then for every* $\nu < \kappa$ *and every family* A *of* κ *pairwise disjoint finite subsets of* κ^+ *there exists an* $A_0 \subseteq A$ *of size* κ *such that for all* $a \neq b$ *in* A_0 *and all* $\alpha \in a$*,* $\beta \in b$ *we have* $\rho(\alpha, \beta) > \nu$*.*

Proof. We may assume that for some n and all $a \in A$ we have $|a| = n$. Let $a(0), \ldots, a(n-1)$ enumerate a given element a of A increasingly. By Lemma 12.3, shrinking A we may assume that $a(i)$ $(a \in A)$ is an antichain of $(\kappa^+, <_\nu)$ for all $i < n$. Going to a subfamily of A of equal size we may assume to have a well-ordering $<_w$ of A with the property that if $a <_w b$ then no node from a is above a node from b in the tree ordering $<_\nu$. Define $f : [A]^2 \longrightarrow \{0\} \cup (n \times n)$ by letting $f(a, b) = 0$ if $a \cup b$ is an $<_\nu$-antichain; otherwise, assuming $a <_w b$, let $f(a, b) = (i, j)$ where (i, j) is the minimal pair such that $a(i) <_\nu b(j)$. By the Dushnik-Miller partition theorem [14], either there exists an $A_0 \subseteq A$ of size κ such that f is constantly equal to 0 on $[A_0]^2$ or there exist $(i, j) \in n \times n$ and an infinite $A_1 \subseteq A$ such that f is constantly equal to (i, j) on $[A_1]^2$. The first alternative is what we want, so let us see that the second one is impossible. Otherwise, choose $a <_w b <_w c$ in A_1. Then $a(i)$ and $b(i)$ are both $<_\nu$-dominated by $c(j)$, so they must be $<_\nu$-comparable, contradicting our initial assumption about A. This completes the proof. ⊣

The unboundedness property of Lemma 12.4 can be quite useful in designing forcing notions satisfying good chain conditions. Having such applications in mind, we now state a further refinement of this kind of unboundedness property of the ρ-function. Its tedious proof can be found for example in [66].

12.5 Lemma. *Suppose $\kappa > 0$, let $\gamma < \kappa^+$ and let $\{\alpha_\xi, \beta_\xi\}$ $(\xi < \kappa)$ be a sequence of pairwise disjoint elements of $[\kappa^+]^{\leq 2}$. Then there is an unbounded set $\Gamma \subseteq \kappa$ such that $\rho\{\alpha_\xi, \beta_\eta\} \geq \min\{\rho\{\alpha_\xi, \gamma\}, \rho\{\beta_\eta, \gamma\}\}$ for all $\xi \neq \eta$ in Γ.*[17]

This lemma allows a further refinement as follows (see [66]). A cardinal κ is *λ-inaccessible* if $\nu^\tau < \kappa$ for all $\nu < \kappa$ and $\tau < \lambda$.

12.6 Lemma. *Suppose κ is λ-inaccessible for some $\lambda < \kappa$ and that A is a family of size κ of subsets of κ^+, all of size $< \lambda$. Then for every ordinal $\nu < \kappa$ there is a subfamily B of A of size κ such that for all a and b in B:*

(a) $\rho\{\alpha, \beta\} > \nu$ for all $\alpha \in a \setminus b$ and $\beta \in b \setminus a$.

(b) $\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}$ for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$.

12.7 Definition. The set-mapping $D : [\kappa^+]^2 \longrightarrow [\kappa^+]^{<\kappa}$ is defined by

$$D(\alpha, \beta) = \{\xi \leq \alpha : \rho(\xi, \alpha) \leq \rho(\alpha, \beta)\}.$$

(Note that $D(\alpha, \beta) = \{\xi \leq \alpha : \rho(\xi, \beta) \leq \rho(\alpha, \beta)\}$, so we could take the formula

$$D\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : \rho(\xi, \alpha) \leq \rho\{\alpha, \beta\}\}$$

as our definition of $D\{\alpha, \beta\}$ when there is no implicit assumption about the ordering between α and β as there is whenever we write $D(\alpha, \beta)$.)

[17] Here, and everywhere else later in this chapter, the convention is that, $\rho\{\alpha, \beta\}$ is meant to be equal to $\rho(\alpha, \beta)$ if $\alpha < \beta$, equal to $\rho(\beta, \alpha)$ if $\beta < \alpha$, and equal to 0 if $\alpha = \beta$.

12.8 Lemma. *If κ is λ-inaccessible for some $\lambda < \kappa$, then for every family A of size κ of subsets of κ^+, all of size $< \lambda$, there exists a $B \subseteq A$ of size κ such that for all a and b in B and all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$:*

$$
\begin{aligned}
&(a)\ \alpha, \beta > \gamma \implies D\{\alpha, \gamma\} \cup D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\},\\
&(b)\quad\ \ \beta > \gamma \implies D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\},\\
&(c)\quad\ \ \alpha > \gamma \implies D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\},\\
&(d)\ \gamma > \alpha, \beta \implies D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\} \text{ or } D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}.
\end{aligned}
$$

Proof. Choose $B \subseteq A$ of size κ satisfying the conclusion (b) of Lemma 12.6. Pick $a \neq b$ in B and consider $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$. By the conclusion of 12.6(b), we have

$$\rho\{\alpha, \beta\} \geq \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\}. \tag{3.7}$$

- **a.** Suppose $\alpha, \beta > \gamma$. Note that in this case a single inequality $\rho(\gamma, \alpha) \leq \rho\{\alpha, \beta\}$ or $\rho(\gamma, \beta) \leq \rho\{\alpha, \beta\}$ given to us by (3.7) implies that we actually have both inequalities simultaneously holding. The subadditivity of ρ gives us $\rho(\xi, \alpha) \leq \rho\{\alpha, \beta\}$, or equivalently $\rho(\xi, \beta) \leq \rho\{\alpha, \beta\}$ for any $\xi \leq \gamma$ with $\rho(\xi, \gamma) \leq \rho(\gamma, \alpha)$ or $\rho(\xi, \gamma) \leq \rho(\gamma, \beta)$. This is exactly the conclusion of Lemma 12.8(a).

- **b.** Suppose that $\beta > \gamma > \alpha$. Using the subadditivity of ρ we see that in both cases given to us by (3.7) we have that $\rho(\alpha, \gamma) \leq \rho\{\alpha, \beta\}$. So the inclusion $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ follows immediately.

- **c.** Suppose that $\alpha > \gamma > \beta$. The conclusion $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$ follows from the previous case by symmetry.

- **d.** Suppose that $\gamma > \alpha, \beta$. Then $\rho(\alpha, \gamma) \leq \rho\{\alpha, \beta\}$ gives $D\{\alpha, \gamma\} \subseteq D\{\alpha, \beta\}$ while $\rho(\beta, \gamma) \leq \rho\{\alpha, \beta\}$ gives us $D\{\beta, \gamma\} \subseteq D\{\alpha, \beta\}$.

This completes the proof. $\dashv$

12.9 Remark. Note that $\min\{x, y\} \in D\{x, y\}$ for every $\{x, y\} \in [\kappa^+]^2$, so the conclusion (a) of Lemma 12.8 in particular means that $\gamma < \min\{\alpha, \beta\}$ implies $\gamma \in D\{\alpha, \beta\}$. In applications, one usually needs this consequence of Lemma 12.8(a) rather than Lemma 12.8(a) itself.

12.10 Definition. The Δ-*function* of some family $\mathcal{F}$ of subsets of some ordinal κ (respectively, a family of functions with domain κ) is the function $\Delta : [\mathcal{F}]^2 \longrightarrow \kappa$ defined by $\Delta(f, g) = \min(f \triangle g)$, (respectively, $\Delta(f, g) = \min\{\xi : f(\xi) \neq g(\xi)\}$).

Note the following property of Δ:

12.11 Lemma. $\Delta(f, g) \geq \min\{\Delta(f, h), \Delta(g, h)\}$ *for all* $\{f, g, h\} \in [\mathcal{F}]^3$.

12.12 Remark. This property can be very useful when transferring objects that live on κ to objects on $\mathcal{F}$. This is especially interesting when $\mathcal{F}$ is of size larger than κ while all of its restrictions $\mathcal{F}\restriction\nu = \{f \cap \nu : f \in \mathcal{F}\}$ ($\nu < \kappa$) have size $< \kappa$, i.e. when $\mathcal{F}$ is a *Kurepa family* (see for example [10]). We shall now see that it is possible to have a Kurepa family $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\}$ whose Δ-function is dominated by ρ, i.e. $\Delta(f_\alpha, f_\beta) \leq \rho(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$.

12.13 Theorem. *If $\Box_\kappa$ holds and κ is λ-inaccessible then there is a λ-closed κ-c.c. forcing notion $\mathcal{P}$ that introduces a Kurepa family on κ.*

Proof. Put p in $\mathcal{P}$, if p is a one-to-one function from a subset of κ^+ of size $< \lambda$ into the family of all subsets of κ of size $< \lambda$ such that for all α and β in $\mathrm{dom}(p)$:

$$p(\alpha) \cap p(\beta) \text{ is an initial part of } p(\alpha) \text{ and of } p(\beta), \tag{3.8}$$

$$\Delta(p(\alpha), p(\beta)) \leq \rho(\alpha, \beta) \text{ provided that } \alpha \neq \beta. \tag{3.9}$$

Let $p \leq q$ whenever $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(\alpha) \supseteq q(\alpha)$ for all $\alpha \in \mathrm{dom}(q)$. Clearly $\mathcal{P}$ is a λ-closed forcing notion. The proof that $\mathcal{P}$ satisfies the κ-chain condition, depends heavily on the properties of the set-mapping D and can be found in [66]. $\dashv$

Recall that a poset satisfies *property K* (for Knaster) if every uncountable subset has a further uncountable subset consisting of pairwise compatible elements. Note that in the case $\kappa = \omega_1$ the proof of the previous theorem shows that the corresponding poset has the property K rather just the c.c.c.

12.14 Corollary. *If $\Box_{\omega_1}$ holds, and so in particular if ω_2 is not a Mahlo cardinal in the constructible universe, then there is a property K poset, forcing the Kurepa hypothesis.*

12.15 Remark. This is a variation on a result of Jensen, namely that under $\Box_{\omega_1}$ there is a c.c.c. poset forcing the Kurepa hypothesis. Veličković [67] was the first to use the function ρ to reprove Jensen's result though his proof works only in case $\kappa = \omega_1$ and produces only a c.c.c. poset rather than a property K poset. It should also be noted that Jensen also proved (see [24]) that in the Levy collapse of a Mahlo cardinal to ω_2 there is no c.c.c. poset forcing the Kurepa hypothesis. We shall now see that ρ provides sufficient ground for another well-known forcing construction, the forcing construction of Baumgartner and Shelah [4] of a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable.

12.16 Theorem. *If $\Box_{\omega_1}$ holds then there is a property K forcing notion that introduces a locally compact scattered topology on ω_2 all of whose Cantor-Bendixson ranks are countable.*

Proof. Let $\mathcal{P}$ be the set of all $p = \langle D_p, \leq_p, M_p\rangle$ where D_p is a finite subset of ω_2, where $\leq_p$ is a partial ordering of D_p compatible with the well-ordering and $M_p : [D_p]^2 \longrightarrow [\omega_2]^{<\omega}$ has the following properties:

(i) $M_p\{\alpha,\beta\} \subseteq D\{\alpha,\beta\} \cap D_p$,

(ii) $M_p\{\alpha,\beta\} = \{\alpha\}$ if $\alpha \leq_p \beta$ and $M_p\{\alpha,\beta\} = \{\beta\}$ if $\beta \leq_p \alpha$,

(iii) $\gamma \leq_p \alpha,\beta$ for all $\gamma \in M_p\{\alpha,\beta\}$,

(iv) for every $\delta \leq_p \alpha,\beta$ there is a $\gamma \in M_p\{\alpha,\beta\}$ such that $\delta \leq_p \gamma$.

We let $p \leq q$ if and only if $D_p \supseteq D_q$, $\leq_p \restriction D_q = \leq_q$ and $M_p \restriction [D_q]^2 = M_q$. To verify that $\mathcal{P}$ satisfies property K one again relies heavily on the properties of the function D. Full details about this can be found for example in [66]. $\dashv$

12.17 Remark. A function $f : [\omega_2]^2 \longrightarrow [\omega_2]^{\leq\omega}$ has *property* Δ if for every uncountable set A of finite subsets of ω_2 there exist a and b in A such that for all $\alpha \in a \setminus b$, $\beta \in b \setminus a$ and $\gamma \in a \cap b$, $\alpha, \beta > \gamma$ implies $\gamma \in f\{\alpha,\beta\}$, if $\beta > \gamma$ implies $f\{\alpha,\gamma\} \subseteq f\{\alpha,\beta\}$, and if $\alpha > \gamma$ implies $f\{\beta,\gamma\} \subseteq f\{\alpha,\beta\}$. This definition is due to Baumgartner and Shelah [4] who used it in their forcing construction of the scattered topology on ω_2. They were also able to force a function with the property Δ using a σ-closed ω_2-c.c. poset. This part of their result was reproved by Veličković (see [4, p.129]) who showed that the function $D\{\alpha,\beta\} = \{\xi \leq \min\{\alpha,\beta\} : \rho(\xi,\alpha) \leq \rho\{\alpha,\beta\}\}$ has property Δ. We have seen above that D has many more properties of independent interest which are likely to be needed in similar forcing constructions. The reader is referred to papers of Koszmider [27] and Rabus [41] for further work in this area.

13. Successors of Singular Cardinals

In the previous section we saw that the function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ defined from a $\Box_\kappa$-sequence C_α $(\alpha < \kappa^+)$ can be quite a useful tool in stepping-up objects from κ to κ^+. In this section we analyze the stepping-up power of ρ under the assumption that κ is a singular cardinal of cofinality ω. So let κ_n $(n < \omega)$ be a strictly increasing sequence of regular cardinals converging to κ fixed from now on. This immediately gives rise to a rather striking tree decomposition $<_n$ $(n < \omega)$ of the $\in$-relation on κ^+:

$$\alpha <_n \beta \quad \text{if and only if} \quad \rho(\alpha,\beta) \leq \kappa_n. \tag{3.10}$$

13.1 Lemma.

(1) $\in \restriction (\kappa^+ \times \kappa^+) = \bigcup_{n<\omega} <_n$,

(2) $<_n \subseteq <_{n+1}$,

(3) $(\kappa^+, <_n)$ *is a tree of height* $\leq \kappa_n^+$.

13.2 Definition. Let $F_n(\alpha) = \{\xi \leq \alpha : \rho(\xi,\alpha) \leq \kappa_n\}$, and let $f_\alpha(n) = \mathrm{tp}(F_n(\alpha))$ for $\alpha < \kappa^+$ and $n < \omega$. Let $L = \{f_\alpha : \alpha < \kappa^+\}$, considered as a linearly ordered set with the lexicographical ordering.

Since L is a subset of ${}^{\omega}\kappa$, it has an order-dense subset of size κ, so in particular it contains no well-ordered subset of size κ^+. The following result shows, however, that every subset of L of smaller size is the union of countably many well-ordered subsets.

13.3 Lemma. *For each $\beta < \kappa^+$, $L_\beta = \{f_\alpha : \alpha < \beta\}$ can be decomposed into countably many well-ordered subsets.*

Proof. Let $L_{\beta n} = \{f_\alpha : \alpha \in F_n(\beta)\}$ for $n < \omega$. Note that the projection $f \longmapsto f\restriction(n+1)$ is one-to-one on $L_{\beta n}$ so each $L_{\beta n}$ is lexicographically well-ordered. ⊣

13.4 Remark. Note that $\mathcal{K} = \{\{(n, f_\alpha(n)) : n < \omega\} : \alpha < \kappa^+\}$ is a family of countable subsets of $\omega \times \kappa$ which has the property that $\mathcal{K}\restriction X = \{K \cap X : K \in \mathcal{K}\}$ has size $\leq |X| + \aleph_0$ for every $X \subseteq \omega \times \kappa$ of size $< \kappa$. We shall now see that with a bit more work a considerably finer such a family can be constructed.

13.5 Definition. If a family $\mathcal{K} \subseteq [S]^\omega$ is at the same time locally countable and cofinal in $[S]^\omega$ then we call it a *cofinal Kurepa family* (*cofinal K-family* for short). Two cofinal K-families $\mathcal{H}$ and $\mathcal{K}$ are *compatible* if $H \cap K \in \mathcal{H} \cap \mathcal{K}$ for all $H \in \mathcal{H}$ and $K \in \mathcal{K}$. We say that $\mathcal{K}$ *extends* $\mathcal{H}$ if they are compatible and if $\mathcal{H} \subseteq \mathcal{K}$.

13.6 Remark. Note that the size of any cofinal K-family $\mathcal{K}$ on a set S is equal to the cofinality of $[S]^\omega$. Note also that for every $X \subseteq S$ there is a $Y \supseteq X$ of size $\mathrm{cf}([X]^\omega)$ such that $K \cap Y \in \mathcal{K}$ for all $K \in \mathcal{K}$.

13.7 Definition. Define $\mathrm{CK}(\theta)$ to be the statement that every sequence $\mathcal{K}_n$ $(n < \omega)$ of comparable cofinal K-families with domains included in θ which are closed under $\cup$, $\cap$ and $\setminus$ can be extended to a single cofinal K-family on θ, which is also closed under these three operations.

13.8 Lemma. $\mathrm{CK}(\omega_1)$ *is true and if* $\mathrm{CK}(\theta)$ *is true for some θ such that* $\mathrm{cf}([\theta]^\omega) = \theta$ *then* $\mathrm{CK}(\theta^+)$ *is also true.*

Proof. The easy proof of $\mathrm{CK}(\omega_1)$ is left to the reader.

Suppose $\mathrm{CK}(\theta)$ and let $\mathcal{K}_n$ $(n < \omega)$ be a given sequence of compatible cofinal K-families as in the hypothesis of $\mathrm{CK}(\theta^+)$. By Remark 13.6 there is a strictly increasing sequence δ_ξ $(\xi < \theta^+)$ of ordinals $< \theta$ such that $\mathcal{K}_n\restriction\delta_\xi \subseteq \mathcal{K}_n$ for all $\xi < \theta^+$ and $n < \omega$. Recursively on $\xi < \theta^+$ we construct a chain $\mathcal{H}_\xi$ $(\xi < \theta^+)$ of cofinal K-families as follows. If $\xi = 0$ or $\xi = \eta + 1$ for some η, using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi-1}$ and $\mathcal{K}_n\restriction\delta_\xi$ $(n < \omega)$. If ξ has uncountable cofinality then the union of $\bar{\mathcal{H}}_\xi = \bigcup_{\eta<\xi} \mathcal{H}_\eta$ is a cofinal K-family with domain included in δ_ξ, so using $\mathrm{CK}(\delta_\xi)$ we can find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\bar{\mathcal{H}}_\xi$ and $\mathcal{K}_n\restriction\delta_\xi$ $(n < \omega)$. If ξ has countable cofinality, pick a sequence $\{\xi_n\}$ converging to ξ and use $\mathrm{CK}(\delta_\xi)$ to find a cofinal K-family $\mathcal{H}_\xi$ on δ_ξ extending $\mathcal{H}_{\xi_n}$ $(n < \omega)$ and $\mathcal{K}_n$ $(n < \omega)$.

When the recursion is done, set $\mathcal{H} = \bigcup_{\xi<\theta^+} \mathcal{H}_\xi$. Then $\mathcal{H}$ is a cofinal K-family on θ^+ extending $\mathcal{K}_n$ $(n < \omega)$. $\dashv$

13.9 Corollary. *For each $n < \omega$ there is a cofinal Kurepa family on ω_n.*

13.10 Definition. Let κ be a cardinal of cofinality ω. A *Jensen matrix* on κ^+ is a matrix $\mathrm{J}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ of subsets of κ with the following properties, where κ_n $(n < \omega)$ is some increasing sequence of cardinals converging to κ:

(1) $|\mathrm{J}_{\alpha n}| \leq \kappa_n$ for all $\alpha < \kappa^+$ and $n < \omega$,

(2) for all $\alpha < \beta$ and $n < \omega$ there is an $m < \omega$ such that $\mathrm{J}_{\alpha n} \subseteq \mathrm{J}_{\beta m}$,

(3) $\bigcup_{n<\omega}[\mathrm{J}_{\beta n}]^\omega = \bigcup_{\alpha<\beta}\bigcup_{n<\omega}[\mathrm{J}_{\alpha n}]^\omega$ whenever $\mathrm{cf}(\beta) > \omega$,

(4) $[\kappa^+]^\omega = \bigcup_{\alpha<\kappa^+}\bigcup_{n<\omega}[\mathrm{J}_{\alpha n}]^\omega$.

13.11 Remark. The notion of a Jensen matrix is the combinatorial essence behind Silver's proof of Jensen's model-theoretic two-cardinal transfer theorem in the constructible universe (see [23, appendix]), so the matrix could equally well be called "Silver matrix". It has been implicitly or explicitly used in several places in the literature. The reader is referred to the paper of Foreman and Magidor [18] which gives quite a complete discussion of this notion and its occurrence in the literature.

13.12 Lemma. *Suppose some cardinal κ of countable cofinality carries a Jensen matrix $\mathrm{J}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ relative to some sequence of cardinals κ_n $(n < \omega)$ that converge to κ. If $\mathrm{CK}(\kappa_n)$ holds for all $n < \omega$ then $\mathrm{CK}(\kappa^+)$ is also true.*

Proof. Let $\mathcal{K}_n$ $(n < \omega)$ be a given sequence of compatible cofinal K-families with domains included in κ^+. Given $\mathrm{J}_{\alpha n}$, there is a natural continuous chain $\mathrm{J}^\xi_{\alpha n}$ $(\xi < \omega_1)$ of subsets of κ^+ of size $\leq \kappa_n$ such that $\mathrm{J}^0_{\alpha n} = \mathrm{J}_{\alpha n}$ and $\mathrm{J}^{\xi+1}_{\alpha n}$ equal to the union of all $K \in \bigcup_{n<\omega}\mathcal{K}_n$ which intersect $\mathrm{J}^\xi_{\alpha n}$. Let $\mathrm{J}^*_{\alpha n} = \bigcup_{\xi<\omega_1}\mathrm{J}^\xi_{\alpha n}$. It is easily seen that $\mathrm{J}^*_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ is also a Jensen matrix. By recursion on α and n we define a sequence $\mathcal{H}_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ of compatible cofinal K-families as follows. If $\alpha = \beta + 1$ or $\alpha = 0$ and $n < \omega$ using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ with domain $\mathrm{J}^*_{\alpha n}$ compatible with $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{H}_{(\alpha-1)m}$ $(m < \omega)$ and $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ $(m < \omega)$. If $\mathrm{cf}(\alpha) = \omega$ let α_n $(n < \omega)$ be an increasing sequence of ordinals converging to α. Using $\mathrm{CK}(\kappa_n)$ we can find a cofinal K-family $\mathcal{H}_{\alpha n}$ which extends $\mathcal{H}_{\alpha m}$ $(m < n)$, $\mathcal{K}_m \restriction \mathrm{J}^*_{\alpha n}$ $(m < \omega)$ and each of the families $\mathcal{H}_{\alpha_i k}$ $(i < \omega, k < \omega$ and $\mathrm{J}^*_{\alpha_i k} \subseteq \mathrm{J}^*_{\alpha n})$. Finally, suppose that $\mathrm{cf}(\alpha) > \omega$. For $n < \omega$, set

$$\mathcal{H}_{\alpha n} = [\mathrm{J}^*_{\alpha n}]^\omega \cap \big(\bigcup_{\xi<\alpha}\bigcup_{m<\omega}\mathcal{H}_{\xi m}\big).$$

Using the properties of the Jensen matrix (especially (3)) as well as the compatibility of $\mathcal{H}_{\xi m}$ $(\xi < \alpha, m < \omega)$ one easily checks that $\mathcal{H}_{\alpha n}$ is a cofinal

K-family with domain $\mathrm{J}^*_{\alpha n}$ which extends each member of $\mathcal{H}_{\alpha m}$ $(m < n)$ and $\mathcal{K}_m\lceil\mathrm{J}^*_{\alpha n}$ $(m < \omega)$ and which is compatible with all of the previously constructed families $\mathcal{H}_{\xi m}$ $(\xi < \alpha, m < \omega)$. When the recursion is done we set

$$\mathcal{H} = \bigcup_{\alpha<\kappa^+}\bigcup_{n<\omega}\mathcal{H}_{\alpha n}.$$

Using the property (4) of $\mathrm{J}^*_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$, it follows easily that $\mathcal{H}$ is a cofinal K-family on κ^+ extending $\mathcal{K}_n$ $(n < \omega)$. ⊣

13.13 Theorem. *If a Jensen matrix exists on any successor of a cardinal of cofinality ω, then a cofinal Kurepa family exists on any domain.*

The ρ-function $\rho : [\kappa^+]^2 \longrightarrow \kappa$ associated with a $\Box_\kappa$-sequence C_α $(\alpha < \kappa^+)$ for some singular cardinal κ of cofinality ω leads to the matrix

$$F_n(\alpha) = \{\xi < \alpha : \rho(\xi, \alpha) \leq n\}(\alpha < \kappa^+, n < \omega) \tag{3.11}$$

which has the properties (1)–(3) of Definition 13.10 as well as some other properties not captured by the definition of a Jensen matrix. If one additionally has a sequence a_α $(\alpha < \kappa^+)$ of countable subsets of κ^+ that is cofinal in $[\kappa^+]^\omega$ one can extend the matrix (3.11) as follows:

$$M_{\beta n} = \bigcup_{\alpha<_n\beta}(a_\alpha \cup \{\alpha\})\ (\beta < \kappa^+, n < \omega).$$

(Recall that $<_n$ is the tree ordering on κ^+ defined by the formula $\alpha <_n \beta$ iff $\rho(\alpha, \beta) \leq \kappa_n$ where κ_n is a fixed increasing sequence of cardinals converging to κ.) The matrix $M_{\beta n}$ $(\beta < \kappa^+, n < \omega)$ has properties not captured by Definition 13.10 that are of independent interest.

13.14 Lemma.

(1) $\alpha <_n \beta$ implies $M_{\alpha n} \subseteq M_{\beta n}$,

(2) $M_{\alpha m} \subseteq M_{\alpha n}$ whenever $m < n$,

(3) if $\beta = \sup\{\alpha : \alpha <_n \beta\}$ then $M_{\beta n} = \bigcup_{\alpha<_n\beta} M_{\alpha n}$,

(4) every countable subset of κ^+ is covered by some $M_{\beta n}$,

(5) $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ is a locally countable family if we have started with a locally countable $\mathcal{K} = \{a_\alpha : \alpha < \kappa^+\}$.

13.15 Remark. One can think of the matrix $\mathcal{M} = \{M_{\beta n} : \beta < \kappa^+, n < \omega\}$ as a version of a "morass" for the singular cardinal κ (see [68]). It would be interesting to see how far one can go in this analogy. We give a few applications just to illustrate the usefulness of the families we have constructed so far.

13.16 Definition. A *Bernstein decomposition* of a topological space X is a function $f : X \longrightarrow 2^{\mathbb{N}}$ with the property that f takes all the values from $2^{\mathbb{N}}$ on any subset of X homeomorphic to the Cantor set.

13.17 Remark. The classical construction of Bernstein [6] can be interpreted by saying that every space of size at most continuum admits a Bernstein decomposition. For larger spaces one must assume Hausdorff's separation axiom, a result of Nešetril and Rődl (see [38]). In this context Malykhin was able to extend Bernstein's result to all spaces of size $< \mathfrak{c}^{+\omega}$ (see [35]). To extend this to all Hausdorff spaces, some use of square sequences seems natural. In fact, the first Bernstein decompositions of an arbitrary Hausdorff space have been constructed using $\square_\kappa$ and $\kappa^\omega = \kappa^+$ for every $\kappa > \mathfrak{c}$ of cofinality ω by Weiss [72] and Wolfsdorf [73]. We shall now see that cofinal K-families are quite natural tools in constructions of Bernstein decompositions. The proof of this result can be found for example in [66].

13.18 Theorem. *Suppose every regular $\theta > \mathfrak{c}$ supports a cofinal Kurepa family of size θ. Then every Hausdorff space admits a Bernstein decomposition.*

It is interesting that various less pathological classes of spaces admit a local version of Theorem 13.18 (see [66]).

13.19 Theorem. *Every metric space that carries a cofinal Kurepa family admits a Bernstein decomposition.*

13.20 Definition. Recall the notion of a *coherent family* of partial functions indexed by some ideal $\mathfrak{I}$, a family of the form $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ with the property that $\{x \in a \cap b : f_a(x) \neq f_b(x)\}$ is finite for all $a, b \in \mathfrak{I}$.

It can be seen (see [64]) that the P-ideal dichotomy (see Definition 3.9) has a strong influence on such families provided $\mathfrak{I}$ is a P-ideal of countable subsets of some set Γ.

13.21 Theorem. *Assuming the P-ideal dichotomy, for every coherent family of functions $f_a : a \longrightarrow \omega$ $(a \in \mathfrak{I})$ indexed by some P-ideal $\mathfrak{I}$ of countable subsets of some set Γ, either*

(1) there is an uncountable $\Delta \subseteq \Gamma$ such that $f_a \restriction \Delta$ is finite-to-one for all $a \in \mathfrak{I}$, or

(2) there is a $g : \Gamma \longrightarrow \omega$ such that $g \restriction a =^* f_a$ for all $a \in \mathfrak{I}$.

Proof. Let $\mathfrak{L}$ be the family of all countable subsets b of Γ for which one can find an a in $\mathfrak{I}$ such that $b \setminus a$ is finite and f_a is finite-to-one on b. To see that $\mathfrak{L}$ is a P-ideal, let $\{b_n\}$ be a given sequence of members of $\mathfrak{L}$ and for each n fix a member a_n of $\mathfrak{I}$ such that f_{a_n} is finite-to-one on b_n. Since $\mathfrak{I}$ is a P-ideal, we can find $a \in \mathfrak{I}$ such that $a_n \setminus a$ is finite for all n. Note that for all n, $b_n \setminus a$ is finite and that f_a is finite-to-one on b_n. For $n < \omega$, let

$$b_n^* = \{\xi \in b_n \cap a : f_a(\xi) > n\}.$$

Then b_n^* is a cofinite subset of b_n for each n, so if we set b to be equal to the union of the b_n^*'s, we get a subset of a which almost includes each b_n and on which f_a is finite-to-one. It follows that b belongs to $\mathfrak{L}$. This completes the proof that $\mathfrak{L}$ is a P-ideal. Applying the P-ideal dichotomy to $\mathfrak{L}$, we get the two alternatives that translate into the alternatives (1) and (2) of the theorem. ⊣

This leads to the natural question whether for any set Γ one can construct a family $\{f_a : a \longrightarrow \omega\}$ of finite-to-one mappings indexed by $[\Gamma]^\omega$. This question was answered by Koszmider [26] using the notion of a Jensen matrix discussed above. We shall present Koszmider's result using the notion of a cofinal Kurepa family instead.

13.22 Theorem. *If Γ carries a cofinal Kurepa family then there is a coherent family $f_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ of finite-to-one mappings.*

Proof. Let $\mathcal{K}$ be a fixed well-founded cofinal K-family on Γ and let $<_w$ be a well-ordering of $\mathcal{K}$ compatible with $\subseteq$. It suffices to produce a coherent family of finite-to-one mappings indexed by $\mathcal{K}$. This is done by induction on $<_w$. Suppose $K \in \mathcal{K}$ and $f_H : H \longrightarrow \omega$ is determined for all $H \in \mathcal{K}$ with $H <_w K$. Let H_n $(n < \omega)$ be a sequence of elements of $\mathcal{K}$ that are $<_w K$ and have the property that for every $H \in \mathcal{K}$ with $H <_w K$ there is an $n < \omega$ such that $H \cap K =^* H_n \cap K$. So it suffices to construct a finite-to-one $f_K : K \longrightarrow \omega$ which coheres with each f_{H_n} $(n < \omega)$, a straightforward task. ⊣

13.23 Corollary. *For every nonnegative integer n there is a coherent family $f_a : a \longrightarrow \omega$ $(a \in [\omega_n]^\omega)$ of finite-to-one mappings.*

13.24 Remark. It is interesting that "finite-to-one" cannot be replaced by "one-to-one" in these results. For example, there is no coherent family of one-to-one mappings $f_a : a \longrightarrow \omega$ $(a \in [\mathfrak{c}^+]^\omega)$. We finish this section with a typical application of coherent families of finite-to-one mappings discovered by Scheepers [44].

13.25 Theorem. *If there is a coherent family $f_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ of finite-to-one mappings, then there is an $F : [[\Gamma]^\omega]^2 \longrightarrow [\Gamma]^{<\omega}$ with the property that for every strictly $\subseteq$-increasing sequence a_n $(n < \omega)$ of countable subsets of Γ, the union of $F(a_n, a_{n+1})$ $(n < \omega)$ covers the union of a_n $(n < \omega)$.*

Proof. For $a \in [\Gamma]^\omega$ let $x_a : \omega \longrightarrow \omega$ be defined by letting $x_a(n) = |\{\xi \in a : f_a(\xi) \leq n\}|$. Note that x_a is eventually dominated by x_b whenever a is a proper subset of b. Choose $\Phi : \omega^\omega \longrightarrow \omega^\omega$ with the property that $x <^* y$ implies $\Phi(y) <^* \Phi(x)$, where $<^*$ is the ordering of eventual dominance on ω^ω (i.e. $x <^* y$ if $x(n) < y(n)$ for all but finitely many n's). Define another family of functions $g_a : a \longrightarrow \omega$ $(a \in [\Gamma]^\omega)$ by letting

$$g_a(\xi) = \Phi(x_a)(f_a(\xi)).$$

Note the following interesting property of g_a $(a \in [\Gamma]^\omega)$:

$$F(a, b) = \{\xi \in a : g_b(\xi) \geq g_a(\xi)\} \text{ is finite for all } a \subsetneqq b \text{ in } [\Gamma]^\omega.$$

So if a_n $(n < \omega)$ is a strictly $\subseteq$-increasing sequence of countable subsets of Γ and $\bar{\xi}$ belongs to some $a_{\bar{n}}$ then the sequence of integers $g_{a_n}(\bar{\xi})$ $(\bar{n} \le n < \omega)$ must have some place $n \ge \bar{n}$ with the property that $g_{a_n}(\bar{\xi}) < g_{a_{n+1}}(\bar{\xi})$, i.e. a place $n \ge \bar{n}$ such that $\xi \in F(a_n, a_{n+1})$. $\dashv$

13.26 Remark. Note that if κ is a singular cardinal of cofinality ω with the property that $\mathrm{cf}([\theta]^\omega) < \kappa$ for all $\theta < \kappa$, then the existence of a cofinal Kurepa family on κ^+ implies the existence of a Jensen matrix on κ^+. So these two notions appear to be quite close to each other. The three basic properties of the function $\rho : [\kappa^+] \longrightarrow \kappa$ (Lemmas 11.5 and 11.6(a),(b)) seem much stronger in view of the fact that the linear ordering as in Lemma 13.3 cannot exist for κ above a supercompact cardinal and the fact that Foreman and Magidor [18] have produced a model with a supercompact cardinal that carries a Jensen matrix on any successor of a singular cardinal of cofinality ω. The "Chang's conjecture" $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$ is the model-theoretic transfer principle asserting that every structure of the form $(\kappa^+, \kappa, <, \ldots)$ with a countable signature has an uncountable elementary submodel B with the property that $B \cap \omega_1$ is countable. Note that $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega)$ for some singular κ of cofinality ω implies that every locally countable family $\mathcal{K} \subseteq [\kappa]^\omega$ must have size $\le \kappa$. So, one of the models of set theory that has no cofinal K-family on, say $\aleph_{\omega+1}$, is the model of Levinski, Magidor and Shelah [34], in which $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_1, \omega)$ holds. It seems still unknown whether the conclusion of Theorem 13.25 can be proved without additional set-theoretic assumptions.

14. The Oscillation Mapping

In what follows, θ will be a fixed regular infinite cardinal.

$$\mathrm{osc} : \mathcal{P}(\theta)^2 \longrightarrow \mathrm{Card}$$

is defined by

$$\mathrm{osc}(x, y) = |x \setminus (\sup(x \cap y) + 1)/ \sim |,$$

where $\sim$ is the equivalence relation on $x \setminus (\sup(x \cap y) + 1)$ defined by letting $\alpha \sim \beta$ iff the closed interval determined by α and β contains no point from y. So, if x and y are disjoint, $\mathrm{osc}(x, y)$ is simply the number of convex pieces the set x is split by the set y. The oscillation mapping has proven to be a useful device in various schemes for coding information. It usefulness in a given context depend very much of the corresponding "oscillation theory", a set of definitions and lemmas that disclose when it is possible to achieve a given number as oscillation between two sets x and y in a given family $\mathcal{X}$. The following definition reveals the notion of largeness relevant to the oscillation theory that we develop in this section.

14.1 Definition. A family $\mathcal{X} \subseteq \mathcal{P}(\theta)$ is *unbounded* if for every closed and unbounded subset C of θ there exist $x \in \mathcal{X}$ and an increasing sequence

$\{\delta_n : n < \omega\} \subseteq C$ such that $\sup(x \cap \delta_n) < \delta_n$ and $[\delta_n, \delta_{n+1}) \cap x \neq \emptyset$ for all $n < \omega$.

This notion of unboundedness has proven to be the key behind a number of results asserting the complex behavior of the oscillation mapping on $\mathcal{X}^2$. The case $\theta = \omega$ seems to contain the deeper part of the oscillation theory known so far (see [58], [59, §1] and [63]), though in this section we shall only consider the case $\theta > \omega$. We shall also restrict ourselves to the family $\mathcal{K}(\theta)$ of all closed bounded subsets of θ rather than the whole power-set of θ. Our next lemma is the basic result about the behavior of the oscillation mapping in this context. Its proof can again be found in [66].

14.2 Lemma. *If $\mathcal{X}$ is an unbounded subfamily of $\mathcal{K}(\theta)$ then for every positive integer n there exist x and y in $\mathcal{X}$ such that* $\operatorname{osc}(x, y) = n$.

Lemma 14.2 also has a rectangular form.

14.3 Lemma. *If $\mathcal{X}$ and $\mathcal{Y}$ are two unbounded subfamilies of $\mathcal{K}(\theta)$ then for all but finitely many positive integers n there exist $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that* $\operatorname{osc}(x, y) = n$.

Recall the notion of a nontrivial C-sequence C_α $(\alpha < \theta)$ on θ from Sect. 8, a C-sequence with the property that for every closed and unbounded subset C of θ there is a limit point δ of C such that $C \cap \delta \nsubseteq C_\alpha$ for all $\alpha < \theta$.

14.4 Definition. For a subset D of θ let $\lim(D)$ denote the set of all $\alpha < \theta$ with the property that $\alpha = \sup(D \cap \alpha)$. A subsequence C_α $(\alpha \in \Gamma)$ of some C-sequence C_α $(\alpha < \theta)$ is *stationary* if the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$ is a stationary subset of θ.

14.5 Lemma. *A stationary subsequence of a nontrivial C-sequence on θ is an unbounded family of subsets of θ.*

Proof. Let C_α $(\alpha \in \Gamma)$ be a given stationary subsequence of a nontrivial C-sequence on θ. Let C be a given closed and unbounded subset of θ. Let Δ be the union of all $\lim(C_\alpha)$ $(\alpha \in \Gamma)$. Then Δ is a stationary subset of θ. For $\xi \in \Delta$ choose $\alpha_\xi \in \Gamma$ such that $\xi \in \lim(C_{\alpha_\xi})$. Applying the assumption that C_α $(\alpha \in \Gamma)$ is a nontrivial C-subsequence, we can find a $\xi \in \Delta \cap \lim(C)$ such that

$$C \cap [\eta, \xi) \nsubseteq C_{\alpha_\xi} \quad \text{for all } \eta < \xi. \tag{3.12}$$

If such a ξ cannot be found using the stationarity of the set $\Delta \cap \lim(C)$ we would be able to use the Pressing Down Lemma on the regressive mapping that would give us an $\eta < \xi$ violating (3.12) and get that a tail of C trivializes C_α $(\alpha \in \Gamma)$. Using (3.12) and the fact that $C_{\alpha_\xi} \cap \xi$ is unbounded in ξ we can find a strictly increasing sequence δ_n $(n < \omega)$ of elements of $(C \cap \xi) \setminus C_{\alpha_\xi}$ such that $[\delta_n, \delta_{n+1}) \cap C_{\alpha_\xi} \neq \emptyset$ for all n. So the set C_{α_ξ} satisfies the conclusion of Definition 14.1 for the given closed and unbounded set C. ⊣

Recall that $\mathbb{Q}_\theta$ denotes the set of all finite sequences of ordinals $< \theta$ and that we consider it ordered by the right lexicographical ordering. We need the following two further orderings on $\mathbb{Q}_\theta$: $s \sqsubseteq t$ if and only if s is an initial segment of t, and $s \sqsubset t$ if and only if s is a proper initial part of t.

14.6 Definition. Given a C-sequence C_α $(\alpha < \theta)$ we can define an *action* $(\alpha, t) \longmapsto \alpha_t$ of $\mathbb{Q}_\theta$ on θ recursively on the ordering $\sqsubseteq$ of $\mathbb{Q}_\theta$ as follows: $\alpha_\emptyset = \alpha$, $\alpha_{\langle\xi\rangle}$ is equal to the ξth member of C_α if $\xi < \operatorname{tp}(C_\alpha)$; otherwise $\alpha_{\langle\xi\rangle} = \alpha$, and finally, $\alpha_{t^\frown\langle\xi\rangle} = (\alpha_t)_{\langle\xi\rangle}$.

14.7 Remark. Note that if $\rho_0(\alpha, \beta) = t$ for some $\alpha < \beta < \theta$ then $\beta_t = \alpha$. In fact, if $\beta = \beta_0 > \cdots > \beta_n = \alpha$ is the walk from β to α along the C-sequence, each member of the trace $\operatorname{Tr}(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\}$ has the form β_s where s is the uniquely determined initial part of t. Note, however, that in general $\beta_t = \alpha$ does not imply that $\rho_0(\alpha, \beta) = t$.

14.8 Notation. Given a C-sequence C_α $(\alpha < \theta)$ on θ we shall use $\operatorname{osc}(\alpha, \beta)$ to denote $\operatorname{osc}(C_\alpha, C_\beta)$.

The proof of the following result can be found in [66].

14.9 Theorem. *If C_α $(\alpha < \theta)$ is a nontrivial C-sequence on θ, then for every unbounded set $\Gamma \subseteq \theta$ and positive integer n there exist $\alpha < \beta$ in Γ and $t \sqsubseteq \rho_0(\alpha, \beta)$ such that $\operatorname{osc}(\alpha_t, \beta_t) = n$, but $\operatorname{osc}(\alpha_s, \beta_s) = 1$ for all $s \sqsubset t$.*

14.10 Corollary. *Suppose a regular uncountable cardinal θ carries a nontrivial C-sequence. Then there is an $f : [\theta]^2 \longrightarrow \omega$ which takes all the values from ω on any set of the form $[\Gamma]^2$ for an unbounded subset of θ.*

Proof. Given $\alpha < \beta < \theta$, if there is a $t \sqsubseteq \rho_0(\alpha, \beta)$ satisfying the conclusion of 14.9, put $f(\alpha, \beta) = \operatorname{osc}(\alpha_t, \beta_t) - 2$; otherwise put $f(\alpha, \beta) = 0$. ⊣

14.11 Remark. The class of all regular cardinals θ that carry a nontrivial C-sequence is quite extensive. It includes not only all successor cardinals but also some inaccessible as well as hyperinaccessible cardinals such as for example, the first inaccessible cardinal or the first Mahlo cardinal. In view of the well-known Ramsey-theoretic characterization of weak compactness, Corollary 14.10 leads us to the following natural question.

14.12 Question. Can the weak compactness of a strong limit regular uncountable cardinal be characterized by the fact that for every $f : [\theta]^2 \longrightarrow \omega$ there exists an unbounded set $\Gamma \subseteq \theta$ such that $f``[\Gamma]^2 \neq \omega$? This is true when ω is replaced by 2, but can any other number beside 2 be used in this characterization?

15. The Square-Bracket Operation

In this section we show that the basic idea of the square-bracket operation on ω_1 introduced in Definition 4.3 extends to a general setting on an arbitrary uncountable regular cardinal θ that carries a nontrivial C-sequence C_α

($\alpha < \theta$). The basic idea is based on the oscillation map defined in the previous section and, in particular, on the property of this map described in Theorem 14.9: for $\alpha < \beta < \theta$ we set

$$[\alpha\beta] = \beta_t, \text{ where } t \sqsubseteq \rho_0(\alpha,\beta) \text{ is such that } \operatorname{osc}(\alpha_t,\beta_t) \geq 2 \text{ but } \operatorname{osc}(\alpha_s,\beta_s) = 1 \text{ for all } s \sqsubset t\text{; if such a } t \text{ does not exist, we let } [\alpha\beta] = \alpha. \tag{3.13}$$

Thus, $[\alpha\beta]$ is the first place visited by β on its walk to α where a nontrivial oscillation with the corresponding step of α occurs. What Theorem 14.9 is telling us is that the nontrivial oscillation indeed happens most of the time. Results that would say that the set of values $\{[\alpha\beta] : \{\alpha,\beta\} \in [\Gamma]^2\}$ is in some sense large no matter how small the unbounded set $\Gamma \subseteq \theta$ is, would correspond to the results of Lemmas 4.4–4.5 about the square-bracket operation on ω_1. It turns out that this is indeed possible and to describe it we need the following definition.

15.1 Definition. A C-sequence C_α ($\alpha < \theta$) on θ *avoids* a given subset Δ of θ if $C_\alpha \cap \Delta = \emptyset$ for all limit ordinals $\alpha < \theta$.

The proof of the following lemma is quite similar to the proof of the corresponding fact in case $\theta = \omega_1$ considered above though its full proof can be found in [66].

15.2 Lemma. *Suppose C_α ($\alpha < \theta$) is a given C-sequence on θ that avoids a set $\Delta \subseteq \theta$. Then for every unbounded set $\Gamma \subseteq \theta$, the set of elements of Δ not of the form $[\alpha\beta]$ for some $\alpha < \beta$ in Γ is nonstationary in θ.*

A similar proof gives the following more general result.

15.3 Lemma. *Suppose C_α ($\alpha < \theta$) avoids $\Delta \subseteq \theta$ and let A be a family of size θ consisting of pairwise disjoint finite sets, all of some fixed size n. Then the set of all elements of Δ that are not of the form $[a(1)b(1)] = [a(2)b(2)] = \cdots = [a(n)b(n)]$ for some $a \neq b$ in A is nonstationary in θ.*

Since $[\alpha\beta]$ belongs to the trace $\operatorname{Tr}(\alpha,\beta)$ of the walk from β to α it is not surprising that $[\cdot\cdot]$ strongly depends on the behavior of Tr. The following is one of the results which brings this out.

15.4 Lemma. *The set $\Omega \setminus \{[\alpha\beta] : \{\alpha,\beta\} \in [\Gamma]^2\}$ is not stationary in θ if and only if the set $\Omega \setminus \bigcup\{\operatorname{Tr}(\alpha,\beta) : \{\alpha,\beta\} \in [\Gamma]^2\}$ is not stationary in θ.*

This fact suggests the following definition.

15.5 Definition. The *trace filter* of a given C-sequence C_α ($\alpha < \theta$) is the normal filter on θ generated by sets of the form $\bigcup\{\operatorname{Tr}(\alpha,\beta) : \{\alpha,\beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ.

15.6 Remark. Having a proper (i.e. $\neq \mathcal{P}(\theta)$) trace filter is a strengthening of the nontriviality requirement on a given C-sequence C_α $(\alpha < \theta)$. For example, if a C-sequence avoids a stationary set $\Omega \subseteq \theta$, then its trace filter is nontrivial and in fact no stationary subset of Γ is a member of it. Note the following analogue of Lemma 15.4: the trace filter of a given C-sequence is the normal filter generated by sets of the form $\{[\alpha\beta] : \{\alpha, \beta\} \in [\Gamma]^2\}$ where Γ is an unbounded subset of θ. So to obtain the analogues of the results of Sect. 4 about the square-bracket operation on ω_1 one needs a C-sequence C_α $(\alpha < \theta)$ on θ whose trace filter is not only nontrivial but also not θ-saturated, i.e. it allows a family of θ pairwise disjoint positive sets. It turns out that the hypothesis of Lemma 15.2 is sufficient for both of these conclusions.

15.7 Lemma. *If a C-sequence on θ avoids a stationary subset of θ, then there exist θ pairwise disjoint subsets of θ that are positive with respect to its trace filter.*[18]

Proof. This follows from the well-known fact (see [25]) that if there is a normal, nontrivial and θ-saturated filter on θ, then for every stationary $\Omega \subseteq \theta$ there exists a $\lambda < \theta$ such that $\Omega \cap \lambda$ is stationary in λ (and the fact that the stationary set which is avoided by the C-sequence does not reflect in this way). ⊣

15.8 Corollary. *If a regular cardinal θ admits a nonreflecting stationary subset then there is a $c : [\theta]^2 \longrightarrow \theta$ which takes all the values from θ on any set of the form $[\Gamma]^2$ for some unbounded set $\Gamma \subseteq \theta$.*

To get such a c, one composes the square-bracket operation of some C-sequence, that avoids a stationary subset of θ, with a mapping $* : \theta \longrightarrow \theta$ with the property that the $*$-preimage of each point from θ is positive with respect to the trace filter of the square sequence. In other words, c is equal to the composition of $[\cdot\cdot]$ and $*$, i.e. $c(\alpha, \beta) = [\alpha\beta]^*$. Note that, as in Sect. 4, the property of the square-bracket operation from Lemma 15.3 leads to the following rigidity result which corresponds to Lemma 4.7.

15.9 Lemma. *The algebraic structure $(\theta, [\cdot\cdot], *)$ has no nontrivial automorphisms.*

15.10 Remark. Note that every θ which is a successor of a regular cardinal κ admits a nonreflecting stationary set. For example, $\Omega = \{\delta < \theta : \mathrm{cf}(\delta) = \kappa\}$ is such a set. Thus any C-sequence on θ that avoids Ω leads to a square bracket operation which allows analogues of all the results from Sect. 4 about the square-bracket operation on ω_1. The reader is urged to examine these analogues.

Let us now introduce a useful projection of the square-bracket operation, the analogue of Definition 4.11 considered above. This concerns the case

[18] A subset A of the domain of some filter $\mathcal{F}$ is *positive* with respect to $\mathcal{F}$ if $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.

when θ is the successor of some regular cardinal κ and when the square-bracket operation is based on a fixed C-sequence C_α $(\alpha < \kappa^+)$ on κ^+ such that $\mathrm{tp}(C_\alpha) \leq \kappa$ for all α, or equivalently, such that C_α $(\alpha < \kappa^+)$ avoids the set $\Omega_\kappa = \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$. Let $[\cdot\cdot]$ be the corresponding square-bracket operation. Let λ be the minimal cardinal such that $2^\lambda \geq \kappa^+$. Choose a sequence r_ξ $(\xi < \kappa^+)$ of distinct subsets of λ. Let $\mathcal{H}$ be the collection of all maps $h : \mathcal{P}(D(h)) \longrightarrow \kappa^+$ where $D(h)$ is a finite subset of λ. Let $\pi : \kappa^+ \longrightarrow \mathcal{H}$ be a map with the property that $\pi^{-1}(\{h\}) \cap \Omega_\kappa$ is stationary for all $h \in \mathcal{H}$. Finally, define an operation $[\![\cdot\cdot]\!]$ on κ^+ as follows:

$$[\![\alpha\beta]\!] = \pi([\alpha\beta])(r_\alpha \cap D(\pi([\alpha\beta]))).$$

The following is a simple consequence of the property Lemma 15.3 of the square-bracket operation.

15.11 Lemma. *For every family A of size κ^+ consisting of pairwise disjoint finite subsets of κ^+ all of some fixed size n and every sequence $\xi_0, \ldots, \xi_{n-1}$ of ordinals $< \kappa^+$ there exist $a \neq b$ in A such that $[\![a(i)b(i)]\!] = \xi_i$ for all $i < n$.*

For sufficiently large cardinals θ we have the following variation on the theme first encountered above in Theorem 8.2 and the reader can find its full proof in [66].

15.12 Theorem. *Suppose θ is bigger than the continuum and carries a C-sequence avoiding a stationary set Γ of cofinality $> \omega$ ordinals in θ. Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n. Then for every stationary $\Gamma_0 \subseteq \Gamma$ there exist $s, t \in \omega^n$ and a positive integer k such that for every $l < \omega$ there exist $a < b$* [19] *in A and $\delta_0 > \delta_1 > \cdots > \delta_l$ in $\Gamma_0 \cap (\max(a), \min(b))$ such that:*

(1) $\rho_2(\delta_{i+1}, \delta_i) = k$ *for all* $i < l$,

(2) $\rho_0(a(i), b(j)) = \rho_0(\delta_0, b(j))^\frown \rho_0(\delta_1, \delta_0)^\frown \cdots {}^\frown \rho_0(\delta_l, \delta_{l-1})^\frown \rho_0(a(i), \delta_l)$ *for all* $i, j < n$,

(3) $\rho_2(\delta_0, b(j)) = t_j$ *and* $\rho_2(a(i), \delta_l) = s_i$ *for all* $i, j < n$.

From now on, θ is assumed to be a fixed cardinal satisfying the hypotheses of Theorem 15.12. It turns out that Theorem 15.12 gives us a way to define another square-bracket operation which has complex behavior not only on squares of unbounded subsets of θ but also on rectangles formed by two unbounded subsets of θ. To define this new operation we choose a mapping $h : \omega \longrightarrow \omega$ such that:

$$\text{for every } k, m, n, p < \omega \text{ and } s \in \omega^n \text{ there is an } l < \omega \text{ such that } h(m + l \cdot k + s(i)) = m + p \text{ for all } i < n. \tag{3.14}$$

[19] Recall that if a and b are two sets of ordinals, then the notation $a < b$ means that $\max(a) < \min(b)$.

15.13 Definition. $[\cdot\cdot]_h : [\theta]^2 \longrightarrow \theta$ is defined by letting $[\alpha\beta]_h = \beta_t$ where $t = \rho_0(\alpha,\beta)\restriction h(\rho_2(\alpha,\beta))$.

Thus, $[\alpha\beta]_h$ is the $h(\rho_2(\alpha,\beta))$th place that β visits on its walk to α. It is clear that Theorem 15.12 and the choice of h in (3.14) give us the following conclusion.

15.14 Lemma. *Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n, and let Ω be an unbounded subset of θ. Then almost every $\delta \in \Gamma$ has the form $[a(0)\beta]_h = [a(1)\beta]_h = \cdots = [a(n-1)\beta]_h$ for some $a \in A$, $\beta \in \Omega$, $a < \beta$.*[20]

In fact, one can get a projection of this square-bracket operation with seemingly even more complex behavior. Keeping the notation of Theorem 15.12, pick a function $\xi \longmapsto \xi^*$ from θ to ω such that $\{\xi \in \Gamma : \xi^* = n\}$ is stationary for all n. This gives us a way to consider the following projection of the trace function $\mathrm{Tr}^* : [\theta]^2 \longrightarrow \omega^{<\omega}$:

$$\mathrm{Tr}^*(\alpha,\beta) = \langle \min(C_\beta \setminus \alpha)^* \rangle^\frown \mathrm{Tr}^*(\alpha, \min(C_\beta \setminus \alpha)),$$

where we stipulate that $\mathrm{Tr}^*(\gamma,\gamma) = \langle \gamma^* \rangle$ for all $\gamma < \theta$. It is clear that the proof of Theorem 15.12 allows us to add the following conclusions:

15.12* Theorem. *Under the hypothesis of Theorem 15.12, its conclusion can be extended by adding the following two new statements:*

(4) $\mathrm{Tr}^*(\delta_1,\delta_0) = \cdots = \mathrm{Tr}^*(\delta_l,\delta_{l-1})$,

(5) The maximal term of the sequence $\mathrm{Tr}^*(\delta_1,\delta_0) = \cdots = \mathrm{Tr}^*(\delta_l,\delta_{l-1})$ *is bigger than the maximal term of any of the sequences* $\mathrm{Tr}^*(\delta_0, b(j))$ *or* $\mathrm{Tr}^*(a(i),\delta_l)$ *for $i, j < n$.*

15.15 Definition. For $\alpha < \beta < \theta$, let $[\alpha\beta]^* = \beta_t$ for t the minimal initial part of $\rho_0(\alpha,\beta)$ such that $\beta_t^* = \max(\mathrm{Tr}^*(\alpha,\beta))$.

Thus $[\alpha\beta]^*$ is the first place in the walk from β to α where the function $*$ reaches its maximum among all other places visited during the walk. Note that combining the conclusions (1)–(5) of Theorem 15.12$^{(*)}$ we get:

15.12** Theorem. *Under the hypothesis of Theorem 15.12, its conclusion can be extended by adding the following:*

(6) $[a(i)b(j)]^* = [\delta_1\delta_0]^*$ *for all $i, j < n$.*

Having in mind the property of $[\cdot\cdot]_h$ stated in Lemma 15.14, the following variation is now quite natural.

15.16 Definition. $[\alpha\beta]_h^* = [\alpha[\alpha\beta]^*]_h$ for $\alpha < \beta < \theta$.

[20] Here "almost every" is to be interpreted by "all except a nonstationary set".

Using Theorem 15.12$^{(**)}$(1)–(6) one easily gets the following conclusion.

15.17 Lemma. *Let A be a family of θ pairwise disjoint finite subsets of θ, all of some fixed size n. Then for all but nonstationarily many $\delta \in \Gamma$ one can find $a < b$ in A such that $[a(i)b(j)]^*_h = \delta$ for all $i, j < n$.*

15.18 Remark. Composing $[\cdot\cdot]^*_h$ with a mapping $\pi : \theta \longrightarrow \theta$ with the property that $\pi^{-1}(\{\xi\}) \cap \Gamma$ is stationary for all $\xi < \theta$, one gets a projection of $[\cdot\cdot]^*_h$ for which the conclusion of Lemma 15.17 is true for *all* $\delta < \kappa$. Assuming that θ is moreover a successor of a regular cardinal κ (of size at least continuum), in which case Γ can be taken to be $\{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$, and proceeding as in 15.11 above we get a projection $[\![\cdot\cdot]\!]^*_h$ with the following property:

15.19 Lemma. *For every family A of pairwise disjoint finite subsets of κ^+ all of some fixed size n and for every $q : n \times n \longrightarrow \kappa^+$ there exist $a < b$ in A such that $[\![a(i)b(j)]\!]^*_h = q(i, j)$ for all $i, j < n$.*

15.20 Remark. The first example of a cardinal with such a complex binary operation was given by the author [58] using the oscillation mapping described above in Sect. 14. It was the cardinal $\mathfrak{b}$, the minimal cardinality of an unbounded subset of ω^ω under the ordering of eventual dominance. The oscillation mapping restricted to some well-ordered unbounded subset W of ω^ω is perhaps still the most interesting example of this kind due to the fact that its properties are preserved in forcing extensions that do not change the unboundedness of W (although they can collapse cardinals and therefore destroy the properties of the square-bracket operations on them). This absoluteness of osc is the key feature behind its applications in various coding procedures (see e.g. [61]).

15.21 Theorem. *For every regular cardinal κ of size at least the continuum, the κ^+-chain condition is not productive, i.e. there exist two partially ordered sets $\mathcal{P}_0$ and $\mathcal{P}_1$ satisfying the κ^+-chain condition but their product $\mathcal{P}_0 \times \mathcal{P}_1$ fails to have this property.*

Proof. Fix two disjoint stationary subsets Γ_0 and Γ_1 of $\{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$. Let $\mathcal{P}_i$ be the collection of all finite subsets p of κ^+ with the property that $[\alpha\beta]^*_h \in \Gamma_i$ for all $\alpha < \beta$ in p. By Lemma 15.17, $\mathcal{P}_0$ and $\mathcal{P}_1$ are κ^+-c.c. posets. Their product $\mathcal{P}_0 \times \mathcal{P}_1$, however, contains a family $\langle\{\alpha\}, \{\alpha\}\rangle$ $(\alpha < \kappa^+)$ of pairwise incomparable conditions. $\dashv$

15.22 Remark. Theorem 15.21 is due to Shelah [48] who proved it using similar methods. The first ZFC examples of non-productiveness of the κ^+-chain condition were given by the author in [57] using what is today known under the name pcf theory. After the full development of pcf theory it became apparent that the basic construction from [57] applies to every successor of a singular cardinal [49]. A quite different class of cardinals θ with θ-c.c. non-productive was given by the author in [56]. For example, $\theta = \mathrm{cf}(\mathfrak{c})$ is one of these cardinals. For an overview of recent advances in this area, the reader is referred to [37]. The following problem seems still open:

15.23 Question. Suppose that θ is a regular strong limit cardinal and the θ-chain condition is productive. Is θ necessarily a weakly compact cardinal?

16. Unbounded Functions on Successors of Regular Cardinals

In this section, κ is a regular cardinal and C_α $(\alpha < \kappa^+)$ is a fixed sequence with $\mathrm{tp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$. Define $\rho^* : [\kappa^+]^2 \longrightarrow \kappa$ by

$$\rho^*(\alpha, \beta) = \sup\{\mathrm{tp}(C_\beta \cap \alpha), \rho^*(\alpha, \min(C_\beta \setminus \alpha)), \rho^*(\xi, \alpha) : \xi \in C_\beta \cap \alpha\}, \quad (3.15)$$

where we stipulate that $\rho^*(\gamma, \gamma) = 0$ for all $\gamma < \kappa^+$. Since $\rho^*(\alpha, \beta) \geq \rho_1(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$ by Lemma 7.1 we have the following:

16.1 Lemma. *For $\nu < \kappa$, $\alpha < \kappa^+$ the set $P_\nu(\alpha) = \{\xi \leq \alpha : \rho^*(\xi, \alpha) \leq \nu\}$ has size no more than $|\nu| + \aleph_0$.*

The proof of the following subadditivity properties of ρ^* is very similar to the proof of the corresponding fact for the function ρ from Sect. 11.

16.2 Lemma. *For all $\alpha \leq \beta \leq \gamma$,*

(a) $\rho^*(\alpha, \gamma) \leq \max\{\rho^*(\alpha, \beta), \rho^*(\beta, \gamma)\}$,

(b) $\rho^*(\alpha, \beta) \leq \max\{\rho^*(\alpha, \gamma), \rho^*(\beta, \gamma)\}$.

We mention a typical application of this function to the problem of the existence of partial square sequences which, for example, have some applications in pcf theory (see [7]).

16.3 Theorem. *For every regular uncountable cardinal $\lambda < \kappa$ and stationary $\Gamma \subseteq \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \lambda\}$, there is a stationary set $\Sigma \subseteq \Gamma$ and a sequence C_α $(\alpha \in \Sigma)$ such that:*

(1) C_α is a closed and unbounded subset of α,

(2) $C_\alpha \cap \xi = C_\beta \cap \xi$ for every $\xi \in C_\alpha \cap C_\beta$.

Proof. For each $\delta \in \Gamma$, choose $\nu = \nu(\delta) < \kappa$ such that the set $P_{<\nu}(\delta) = \{\xi < \delta : \rho^*(\xi, \delta) < \nu\}$ is unbounded in δ and closed under taking suprema of sequences of size $< \lambda$. Then there are $\bar{\nu}, \bar{\mu} < \kappa$ and stationary $\Sigma \subseteq \Gamma$ such that $\nu(\delta) = \bar{\nu}$ and $\mathrm{tp}(P_{<\bar{\nu}}(\delta)) = \bar{\mu}$ for all $\delta \in \Sigma$. Let C be a fixed closed and unbounded subset of $\bar{\mu}$ of order-type λ. Finally, for $\delta \in \Gamma$ set

$$C_\delta = \{\alpha \in P_{<\bar{\nu}}(\delta) : \mathrm{tp}(P_{<\bar{\nu}}(\alpha)) \in C\}.$$

Using Lemma 16.2, one easily checks that C_α $(\alpha \in \Sigma)$ satisfies the conditions (1) and (2). $\dashv$

Another application concerns the fact described above in Sect. 13, that the inequalities from Lemma 16.2(a),(b) are particularly useful when κ has cofinality ω. Also consider the well-known phenomenon first discovered by Prikry (see [25]), that in some cases, the cofinality of a regular cardinal κ can be changed to ω, while preserving all cardinals.

16.4 Theorem. *In any cardinal-preserving extension of the universe which has no new bounded subsets of κ, but in which κ has a cofinal ω-sequence diagonalizing the filter of closed and unbounded subsets of κ restricted to the ordinals of cofinality $> \omega$, there is a sequence $C_{\alpha n}(\alpha \in \lim(\kappa^+), n < \omega)$ such that for all $\alpha < \beta$ in $\lim(\kappa^+)$:*

(1) $C_{\alpha n}$ is a closed subset of α for all n,

(2) $C_{\alpha n} \subseteq C_{\alpha m}$, whenever $n \le m$,

(3) $\alpha = \bigcup_{n<\omega} C_{\alpha n}$,

(4) $\alpha \in \lim(C_{\beta n})$ implies $C_{\alpha n} = C_{\beta n} \cap \alpha$.

Proof. For $\alpha < \kappa^+$, let D_α be the collection of all $\nu < \kappa$ for which $P_{<\nu}(\alpha)$ is σ-closed, i.e. closed under suprema of bounded countable subsets. Clearly, D_α contains a closed unbounded subset of κ, restricted to cofinality $> \omega$ ordinals. Note that $\nu \in D_\beta$ and $\rho^*(\alpha, \beta) < \nu$ imply that $\nu \in D_\alpha$. In the extended universe, pick a strictly increasing sequence ν_n $(n < \omega)$ which converges to κ and has the property that for each $\alpha < \kappa^+$ there is an $n < \omega$ such that $\nu_m \in D_\alpha$ for all $m \ge n$. Let $n(\alpha)$ be the minimal integer n with this property.

Given $\alpha < \kappa^+$ and $n < \omega$, we define $C_{\alpha n}$ according to the following cases. If there is a $\gamma \ge \alpha$ such that $n \ge n(\gamma)$ and $\sup P_{\nu_n}(\gamma) \cap \alpha = \alpha$, let $\gamma(\alpha, n)$ be the minimal such γ and let $C_{\alpha n} = \overline{P_{\nu_n}(\gamma(\alpha, n))} \cap \alpha$. If there is no such $\gamma \ge \alpha$, we let $C_{\alpha n} = \emptyset$ for $n < n(\alpha)$ and $C_{\alpha n} = \overline{P_{\nu_n}(\alpha)} \cap \alpha$ for $n \ge n(\alpha)$.

Then one can easily verify that $C_{\alpha n}$ $(\alpha < \kappa^+, n < \omega)$ satisfies the conditions (1), (2), (3) and (4). Detailed checking of this, however, can be found in [66]. ⊣

16.5 Remark. The combinatorial principle appearing in the statement of Theorem 16.4 is a member of a family of square principles that has been studied systematically by Schimmerling and others (see e.g. [45]). It is definitely a principle sufficient for all of the applications of $\square_\kappa$ appearing in Sect. 13 above.

16.6 Definition. A function $f : [\kappa^+]^2 \longrightarrow \kappa$ is *unbounded* if $f``[\Gamma]^2$ is unbounded in κ for every $\Gamma \subseteq \kappa^+$ of size κ. We shall say that such an f is *strongly unbounded* if for every family A of size κ^+, consisting of pairwise disjoint finite subsets of κ^+, and every $\nu < \kappa$ there exists an $A_0 \subseteq A$ of size κ such that $f(\alpha, \beta) > \nu$ for all $\alpha \in a$, $\beta \in b$ and $a \ne b$ in A_0.

16.7 Lemma. *If $f : [\kappa^+]^2 \longrightarrow \kappa$ is unbounded and subadditive (i.e. it satisfies the two inequalities 16.2(a),(b)), then f is strongly unbounded.*

Proof. For $\alpha < \beta < \kappa^+$, set $\alpha <_\nu \beta$ if and only if $f(\alpha, \beta) \leq \nu$. Then our assumption about f satisfying Lemma 16.2(a) and (b) reduces to the fact that each $<_\nu$ is a tree ordering on κ^+ compatible with the usual ordering on κ^+. Note that the unboundedness property of f is preserved by any forcing notion satisfying the κ-chain condition, so in particular no tree $(\kappa^+, <_\nu)$ can contain a subtree of height κ which is Souslin. In the proof of Lemma 12.4 above we have seen that this property of $(\kappa^+, <_\nu)$ alone is sufficient to conclude that every family A of κ many pairwise disjoint subsets of κ^+ contains a subfamily A_0 of size κ such that for every $a \neq b$ in A_0 every $\alpha \in a$ is $<_\nu$-incomparable to every $\beta \in b$, which is exactly the conclusion of f being strongly unbounded. ⊣

The following useful facts whose proof can be found in [66] relates the notions of unboundedness and subadditivity.

16.8 Lemma. *The following are equivalent:*

(1) There is a structure $(\kappa^+, \kappa, <, R_n)_{n<\omega}$ with no substructure B of size κ such that $B \cap \kappa$ is bounded in κ.

(2) There is an unbounded function $f : [\kappa^+]^2 \longrightarrow \kappa$.

(3) There is a strongly unbounded, subadditive function $f : [\kappa^+]^2 \longrightarrow \kappa$.

16.9 Remark. Recall that *Chang's Conjecture* is the model-theoretic transfer principle asserting that every structure of the form $(\omega_2, \omega_1, <, \ldots)$ with a countable signature has an uncountable elementary submodel B with the property that $B \cap \omega_1$ is countable. This principle shows up in many considerations including the first two uncountable cardinals ω_1 and ω_2. For example, it is known that it is preserved by c.c.c. forcing extensions, that it holds in the Silver collapse of an ω_1-Erdős cardinal, and that it in turn implies that ω_2 is an ω_1-Erdős cardinal in the core model of Dodd and Jensen (see e.g. [12, 25]).

16.10 Corollary. *The negation of Chang's Conjecture is equivalent to the statement that there exists an $e : [\omega_2]^2 \longrightarrow \omega_1$ such that:*

(a) $e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(b) $e(\alpha, \beta) \leq \max\{e(\alpha, \gamma), e(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(c) For every uncountable family A of pairwise disjoint finite subsets of ω_2 and every $\nu < \omega_1$ there exists an uncountable $A_0 \subseteq A$ such that $e(\alpha, \beta) > \nu$ whenever $\alpha \in a$ and $\beta \in b$ for every $a \neq b \in A_0$.

16.11 Remark. Note that if a mapping $e : [\omega_2]^2 \longrightarrow \omega_1$ has properties (a), (b) and (c) of Corollary 16.10, then $D_e : [\omega_2]^2 \longrightarrow [\omega_2]^{\aleph_0}$ defined by $D_e\{\alpha, \beta\} = \{\xi \leq \min\{\alpha, \beta\} : e(\xi, \alpha) \leq e\{\alpha, \beta\}\}$ satisfies the weak form of the Baumgartner-Shelah definition of a Δ-function considered above, where only the first condition is kept. It could be shown, however, that all three properties of a Δ-function cannot be achieved assuming only the negation of Chang's Conjecture. This shows that the function ρ, based on a $\square_{\omega_1}$-sequence, is a considerably deeper object than an $e : [\omega_2]^2 \longrightarrow \omega_1$ satisfying Corollary 16.10(a),(b),(c).

Recall that the successor of the continuum is characterized as the minimal cardinal θ with the property that every $f : [\theta]^2 \longrightarrow \omega$ is constant on the square of some infinite set. We shall now see that in slightly weakening the partition property by replacing squares by rectangles one gets a characterization of a quite different sort. To see this, let us use the arrow notation

$$\binom{\theta}{\theta} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$$

to succinctly express the statement that for every map $f : \theta \times \theta \longrightarrow \omega$, there exist infinite sets $A, B \subseteq \theta$ such that f is constant on their product. Let θ_2 be the minimal θ which fails to satisfy this property. Note that $\omega_1 < \theta_2 \leq \mathfrak{c}^+$. The following result whose proof can be found in [66] shows that θ_2 can have the minimal possible value ω_2, as well as that θ_2 can be considerably smaller than the continuum.

16.12 Theorem. *Chang's Conjecture is equivalent to the statement that*

$$\binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$$

holds in every c.c.c. forcing extension.

16.13 Remark. The relative size of θ_2 (or its higher-dimensional analogues $\theta_3, \theta_4, \ldots$) in comparison to the sequence of cardinals $\omega_2, \omega_3, \omega_4, \ldots$ is of considerable interest, both in set theory and model theory (see e.g. [47, 60, 62]). On the other hand, even the following most simple questions, left open by Theorem 16.12, are still unanswered.

16.14 Question. Can one prove any of the bounds like $\theta_2 \leq \omega_3$, $\theta_3 \leq \omega_4$, $\theta_4 \leq \omega_5$, etc. without appealing to additional axioms?

Note that by Corollary 16.10, Chang's Conjecture is equivalent to the statement that within every decomposition of the usual ordering on ω_2 as an increasing chain of tree orderings, one of the trees has an uncountable chain. Is it possible to have decompositions of $\in\restriction(\omega_2 \times \omega_2)$ into an increasing ω_1-chain of tree orderings of countable heights? It turns out that the answer to this question is equivalent to a different well-known combinatorial statement about ω_2 rather than Chang's Conjecture itself. Recall that $f : [\kappa^+]^2 \longrightarrow \kappa$ is *transitive* if $f(\alpha, \gamma) \leq \max\{f(\alpha, \beta), f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$. Given a transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$, one defines $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ recursively on $\alpha \leq \beta < \kappa^+$ as follows

$$\rho_f(\alpha, \beta) = \sup\{f(\min(C_\beta \setminus \alpha), \beta), \mathrm{tp}(C_\beta \cap \alpha), \\ \rho_f(\alpha, \min(C_\beta \setminus \alpha)), \rho_f(\xi, \alpha) : \xi \in C_\beta \cap \alpha\},$$

where we stipulate that $\rho_f(\alpha, \alpha) = 0$ for all $\alpha < \kappa^+$.

16.15 Lemma. *For every transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$ the corresponding $\rho_f : [\kappa^+]^2 \longrightarrow \kappa$ has the following properties:*

(a) $\rho_f(\alpha, \gamma) \leq \max\{\rho_f(\alpha, \beta), \rho_f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(b) $\rho_f(\alpha, \beta) \leq \max\{\rho_f(\alpha, \gamma), \rho_f(\beta, \gamma)\}$ whenever $\alpha \leq \beta \leq \gamma$,

(c) $|\{\xi \leq \alpha : \rho_f(\xi, \alpha) \leq \nu\}| \leq |\nu| + \aleph_0$ for $\nu < \kappa$ and $\alpha < \kappa^+$,

(d) $\rho_f(\alpha, \beta) \geq f(\alpha, \beta)$ for all $\alpha < \beta < \kappa^+$.

Transitive maps are frequently used combinatorial objects, especially when one works with quotient structures. Adding the extra subadditivity condition Lemma 16.15(b), one obtains a considerably more subtle object which is much less understood. For example, let $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ be a given sequence of functions such that $f_\alpha <^* f_\beta$ whenever $\alpha < \beta$.[21] Then the corresponding transitive map $f : [\kappa^+]^2 \longrightarrow \kappa$ is defined by $f(\alpha, \beta) = \min\{\mu < \kappa : f_\alpha(\nu) < f_\beta(\nu)$ for all $\nu \geq \mu\}$. Let ρ_f be the corresponding ρ-function that dominates this particular f and for $\nu < \kappa$ let $<^f_\nu$ be the corresponding tree ordering of κ^+, i.e., $\alpha <^f_\nu \beta$ if and only if $\rho_f(\alpha, \beta) \leq \nu$.

16.16 Lemma. *Suppose $f_\alpha \leq g$ for all $\alpha < \kappa^+$ where $\leq$ is the ordering of everywhere dominance. Then for every $\nu < \kappa$ the tree $(\kappa^+, <^f_\nu)$ has height $\leq g(\nu)$.*

Proof. Let P be a maximal chain of $(\kappa^+, <^f_\nu)$. $f(\alpha, \beta) \leq \rho_f(\alpha, \beta) \leq \nu$ for every $\alpha < \beta$ in P. It follows that $f_\alpha(\nu) < f_\beta(\nu) \leq g(\nu)$ for all $\alpha < \beta$ in P. So P has order-type $\leq g(\nu)$. ⊣

Note that if we have a function $g : \kappa \longrightarrow \kappa$ which bounds the sequence f_α $(\alpha < \kappa^+)$ in the ordering $<^*$ of eventual dominance, then the new sequence $\bar{f}_\alpha = \min\{f_\alpha, g\}$ $(\alpha < \kappa^+)$ is still strictly $<^*$-increasing but now bounded by g even in the ordering of everywhere dominance. So this proves the following result of Galvin (see [22, 42]).

16.17 Corollary. *The following two conditions are equivalent for every regular cardinal κ.*

[21] Here, $f_\alpha <^* f_\beta$ whenever $\{\nu < \kappa : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is bounded in κ.

(1) There is a sequence $f_\alpha : \kappa \longrightarrow \kappa$ $(\alpha < \kappa^+)$ which is strictly increasing and bounded in the ordering of eventual dominance.

(2) The usual order-relation of κ^+ can be decomposed into an increasing κ-sequence of tree orderings of heights $< \kappa$.

16.18 Remark. The assertion that every strictly $<^*$-increasing κ^+-sequence of functions from κ to κ is $<^*$-unbounded is strictly weaker than Chang's Conjecture and in the literature it is usually referred to as *weak Chang's Conjecture*. This statement still has considerable large cardinal strength (see [11]). Also note the following consequence of Corollary 16.17 which can be deduced from Lemmas 16.1 and 16.2 above as well.

16.19 Corollary. *If κ is a regular limit cardinal (e.g. $\kappa = \omega$), then the usual order-relation of κ^+ can be decomposed into an increasing κ-sequence of tree orderings of heights $< \kappa$.*

17. Higher Dimensions

The reader must have noticed already that in this chapter so far, we have only considered functions of the form $f : [\theta]^2 \longrightarrow I$ or equivalently sequences $f_\alpha : \alpha \longrightarrow I$ $(\alpha < \theta)$ of one-place functions. To obtain analogous results about functions defined on higher-dimensional cubes $[\theta]^n$ one usually develops some form of *stepping-up procedure* that lifts a function of the form $f : [\theta]^n \longrightarrow I$ to a function of the form $g : [\theta^+]^{n+1} \longrightarrow I$. The basic idea seems quite simple. One starts with a coherent sequence $e_\alpha : \alpha \longrightarrow \theta$ $(\alpha < \theta^+)$ of one-to-one mappings and wishes to define $g : [\theta^+]^{n+1} \longrightarrow I$ as follows:

$$g(\alpha_0, \alpha_1, \ldots, \alpha_n) = f(e_{\alpha_n}(\alpha_0), \ldots, e_{\alpha_n}(\alpha_{n-1})). \tag{3.16}$$

In other words, we use e_{α_n} to send $\{\alpha_0, \ldots, \alpha_{n-1}\}$ to the domain of f and then apply f to the resulting n-tuple. The problem with such a simple-minded definition is that for a typical subset Γ of θ^+, the sequence of restrictions $e_\delta \restriction (\Gamma \cap \delta)$ $(\delta \in \Gamma)$ may not cohere, so we cannot produce a subset of θ that would correspond to Γ and on which we would like to apply some property of f. It turns out that the definition (3.16) is basically correct except that we need to replace e_{α_n} by $e_{\tau(\alpha_{n-2},\alpha_{n-1},\alpha_n)}$, where $\tau : [\theta^+]^3 \longrightarrow \theta^+$ is defined as follows (see Definition 14.6):

$$\tau(\alpha, \beta, \gamma) = \gamma_t, \quad \text{where } \ t = \rho_0(\alpha, \gamma) \cap \rho_0(\beta, \gamma). \tag{3.17}$$

The function ρ_0 to which (3.17) refers is of course based on some C-sequence C_α $(\alpha < \theta^+)$ on θ^+. The following result shows that if the C-sequence is carefully chosen, the function τ will serve as a stepping-up tool. The following lemma whose proof can be found in [66] gives the basic idea behind this.

17.1 Lemma. *Suppose ρ_0 and τ are based on some $\square_\theta$-sequence C_α $(\alpha < \theta^+)$ and let κ be a regular uncountable cardinal $\leq \theta$. Then every set $\Gamma \subseteq \theta^+$ of order-type κ contains a cofinal subset Δ such that, if $\varepsilon = \sup(\Gamma) = \sup(\Delta)$, then $\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma))$ for all $\xi < \alpha < \beta < \gamma$ in Δ.*

Recall that for a given C-sequence C_α $(\alpha < \theta^+)$ such that $\mathrm{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$, the range of ρ_0 is the collection of all finite sequences of ordinals $< \theta$. There is a natural way to identify $\mathbb{Q}_\theta$ with θ itself via the well-ordering of $\mathbb{Q}_\theta$ of length θ: $s <_w t$ if and only if $\max(s) < \max(t)$, or $\max(s) = \max(t)$ and $t \subseteq s$, or $\max(s) = \max(t)$ and $s(i) \neq t(i)$ for some i in the common domain of s and t and $s(i) < t(i)$ for the minimal such i. This identification gives us a way to define a lift-up of an arbitrary map $f : [\theta]^n \longrightarrow I$ (really, $f : [\mathbb{Q}_\theta]^n \longrightarrow I$) to a map $f^+ : [\theta^+]^{n+1} \longrightarrow I$ by the following formula:

$$f^+(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon)), \tag{3.18}$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$.

Let us examine how this stepping-up procedure works on a particular example, a combinatorial property of a function f which has been stepped up by Velleman [69] from $n = 3$ to $n = 4$ using his version of the gap-2 morass.

17.2 Theorem. *Suppose θ is an arbitrary cardinal for which $\square_\theta$ holds. Suppose further that for some regular $\kappa > \omega$ and integer $n \geq 2$ there is a map $f : [\theta]^n \longrightarrow [[\theta]^{<\kappa}]^{<\kappa}$ such that:*

(1) $A \subseteq \min(a)$ for all $a \in [\theta]^n$ and $A \in f(a)$.

(2) For all $\nu < \kappa$ and $\Gamma \subseteq \theta$ of size κ there exist $a \in [\Gamma]^n$ and $A \in f(a)$ such that $\mathrm{tp}(A) \geq \nu$ and $A \subseteq \Gamma$.

Then θ^+ and κ satisfy the same combinatorial property, but with $n+1$ in place of n.

Proof. Identifying $\mathbb{Q}_\theta$ with θ using the wellordering $<_w$ defined above, we assume that actually $f : [\mathbb{Q}_\theta]^n \longrightarrow [[\mathbb{Q}_\theta]^{<\kappa}]^{<\kappa}$. Apply the idea of (3.18) and define $g : [\theta^+]^n \longrightarrow [[\theta^+]^{<\kappa}]^{<\kappa}$ by the formula

$$g(\alpha_0, \ldots, \alpha_{n-1}, \alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0, \varepsilon), \ldots, \rho_0(\alpha_{n-1}, \varepsilon))),$$

where $\varepsilon = \tau(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)$ and where τ is based on a fixed $\square_\theta$-sequence.

Note that the transformation $(\rho_0)_\varepsilon^{-1}$ does not necessarily preserve (1), so we intersect each member of a given $g(a)$ with $\min(a)$ in order to satisfy this condition. To check (2), let $\Gamma \subseteq \theta^+$ be a given set of size κ. By Lemma 17.1, shrinking Γ we may assume that Γ has order-type κ and that if $\varepsilon = \sup(\Gamma)$, then

$$\rho_0(\xi, \varepsilon) = \rho_0(\xi, \tau(\alpha, \beta, \gamma)) \quad \text{for all } \alpha < \beta < \gamma \text{ in } \Gamma. \tag{3.19}$$

It follows that g restricted to $[\Gamma]^{n+1}$ satisfies the formula

$$g(\alpha_0,\dots,\alpha_{n-1},\alpha_n) = (\rho_0)_\varepsilon^{-1}(f(\rho_0(\alpha_0,\varepsilon),\dots,\rho_0(\alpha_{n-1},\varepsilon))). \tag{3.20}$$

Shrinking Γ further we assume that the mapping $(\rho_0)_\varepsilon : (\varepsilon,\in) \longrightarrow (\mathbb{Q}_\theta,<_w)$ is strictly increasing, when restricted to Γ. Given an ordinal $\nu<\kappa$, we apply (2) for f to the set $\Delta=\{\rho_0(\alpha,\varepsilon):\alpha\in\Gamma\}$ and find $a\in[\Delta]^n$ and $A\in f(a)$ such that $\mathrm{tp}(A)\geq\nu$ and $A\subseteq\Delta$. Let $\{\alpha_0,\dots,\alpha_{n-1}\}$ be the increasing enumeration of the preimage $(\rho_0)_\varepsilon^{-1}(a)$ and pick $\alpha_n\in\Gamma$ above α_{n-1}. Let B be the preimage $(\rho_0)_\varepsilon^{-1}(A)$. Then $B\in g(\alpha_0,\dots,\alpha_{n-1},\alpha_n)$, $\mathrm{tp}(B)\geq\nu$ and $B\subseteq\Gamma$. This completes the proof. ⊣

If we apply this stepping-up procedure to the projection $[\![\cdot\cdot]\!]$ of the square-bracket operation defined in Definition 4.11, one obtains analogues of families $\mathcal{G},\mathcal{H}$ and $\mathcal{K}$ of Theorem 4.13 for ω_2 instead of ω_1. This will give us the following result whose proof can be found in [66].

17.3 Theorem. *Assuming $\square_{\omega_1}$, there is a reflexive Banach space E with a transitive basis of type ω_2 with the property that every bounded operator $T:E\longrightarrow E$ can be written as a sum of an operator with a separable range and a diagonal operator (relative to the basis) with only countably many changes of constants.*

17.4 Remark. In [28], Koszmider has shown that such a space cannot be constructed on the basis of the usual axioms of set theory. We refer the reader to that paper for more details about these kinds of examples of Banach spaces.

For the rest of this section we shall examine the stepping-up method with fewer restrictions on the given C-sequence C_α $(\alpha<\theta^+)$ on which it is based.

17.5 Theorem. *The following are equivalent for a regular cardinal θ such that* $\log\theta^+=\theta$.[22]

(1) There is a substructure of the form $(\theta^{++},\theta^+,<,\dots)$ with no substructure B of size θ^+ with $B\cap\theta^+$ of size θ.

(2) There is an $f:[\theta^{++}]^3\longrightarrow\theta^+$ which takes all the possible values on the cube of any subset Γ of θ^{++} of size θ^+.

Proof. To prove the nontrivial direction from (1) to (2), we use Lemma 16.8 and choose a strongly unbounded and subadditive $e:[\theta^{++}]^2\longrightarrow\theta^+$. We also choose a C-sequence C_α $(\alpha<\theta^+)$ such that $\mathrm{tp}(C_\alpha)\leq\theta$ for all $\alpha<\theta^+$ and consider the corresponding function $\rho^*:[\theta^+]^2\longrightarrow\theta$ defined above in (3.15). Finally, we choose a one-to-one sequence r_α $(\alpha<\theta^{++})$ of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta:[\theta^{++}]^2\longrightarrow\theta^+$:

$$\Delta(\alpha,\beta)=\Delta(r_\alpha,r_\beta)=\min\{\nu:r_\alpha(\nu)\neq r_\beta(\nu)\}. \tag{3.21}$$

[22] $\log\kappa=\min\{\lambda:2^\lambda\geq\kappa\}$.

The definition of $f : [\theta^{++}]^3 \longrightarrow \theta$ is given according to the following two rules applied to a given triple $x = \{\alpha, \beta, \gamma\} \in [\theta^{++}]^3$ ($\alpha < \beta < \gamma$):

Rule 1. If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\text{lex}} r_\beta <_{\text{lex}} r_\gamma$ or $r_\alpha >_{\text{lex}} r_\beta >_{\text{lex}} r_\gamma$, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\Delta(\beta, \gamma)) \setminus \Delta(\alpha, \beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha, \beta) : \rho^*(\xi, \Delta(\alpha, \beta)) \neq \rho^*(\xi, \Delta(\beta, \gamma))\}, \Delta(\beta, \gamma))$.

Rule 2. If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x and if x does not fall under Rule 1, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(e(\beta, \gamma)) \setminus e(\alpha, \beta)),$$

where $\nu = \rho^*\{\Delta(\alpha, \beta), e(\beta, \gamma)\}$.

The proof of the theorem is complete once we show the following: for every stationary set Σ of cofinality θ ordinals $< \theta^+$ and every $\Gamma \subseteq \theta^{++}$ of size θ^+ there exist $\alpha < \beta < \gamma$ in Γ such that $f(\alpha, \beta, \gamma) \in \Sigma$. The details of this can again be found in [66]. ⊣

17.6 Theorem. *If θ is a regular strong limit cardinal carrying a nonreflecting stationary set, then there is an $f : [\theta^+]^3 \longrightarrow \theta$ which takes all the values from θ on the cube of any subset of θ^+ of size θ.*

Proof. This is really a corollary of the proof of Theorem 17.5, so let us only indicate the adjustments. By Corollary 16.19 and Lemma 16.7, we can choose a strongly unbounded subadditive map $e : [\theta^+]^2 \longrightarrow \theta$. By the assumption about θ we can choose a C-sequence C_α ($\alpha < \theta$) avoiding a stationary set $\Sigma \subseteq \theta$ and consider the corresponding notion of a walk, trace, ρ_0-function and the square-bracket operation $[\cdot\cdot]$ as defined in (3.13) in Sect. 15. As in the proof of Theorem 17.5, we choose a one-to-one sequence r_α ($\alpha < \theta^+$) of elements of $\{0, 1\}^\theta$ and consider the corresponding function $\Delta : [\theta^+]^2 \longrightarrow \theta$:

$$\Delta(\alpha, \beta) = \Delta(r_\alpha, r_\beta) = \min\{\nu < \theta : r_\alpha(\nu) \neq r_\beta(\nu)\}.$$

The definition of $f : [\theta^+]^3 \longrightarrow \theta$ is given according to the following rules, applied to a given $x \in [\theta^+]^3$.

Rule 1. If $x = \{\alpha < \beta < \gamma\}$, $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\text{lex}} r_\beta <_{\text{lex}} r_\gamma$, or $r_\alpha >_{\text{lex}} r_\beta >_{\text{lex}} r_\gamma$, let

$$f\{\alpha, \beta, \gamma\} = [\Delta(\alpha, \beta)\Delta(\beta, \gamma)].$$

Rule 2. If $\alpha \in x$ is such that r_α is lexicographically between the other two r_ξ's for $\xi \in x$, if $\beta \in x \setminus \{\alpha\}$ is such that $\Delta(r_\alpha, r_\beta) > \Delta(r_\alpha, r_\gamma)$, where γ is the remaining element of x, and they do not satisfy the conditions of Rule 1, set

$$f\{\alpha, \beta, \gamma\} = \min(\mathrm{Tr}(\Delta(\alpha, \beta), e\{\beta, \gamma\}) \setminus e\{\alpha, \beta\}),$$

i.e. $f\{\alpha,\beta,\gamma\}$ is the minimal point on the trace of the walk from $e\{\beta,\gamma\}$ to $\Delta(\alpha,\beta)$ above the ordinal $e\{\alpha,\beta\}$; if such a point does not exist, set $f\{\alpha,\beta,\gamma\}=0$.

Then it suffices to show that for every stationary $\Omega\subseteq\Sigma$ and every $\Gamma\subseteq\theta^+$ of size θ, there exists an $x\in[\Gamma]^3$ such that $f(x)\in\Omega$. The details of this are given in [66]. ⊣

Since $\log\omega_1=\omega$, we get the following consequence of Theorem 17.5.

17.7 Theorem. *Chang's Conjecture is equivalent to the statement that for every $f:[\omega_2]^3\longrightarrow\omega_1$ there is an uncountable $\Gamma\subseteq\omega_2$ such that $f``[\Gamma]^3\neq\omega_1$.*

17.8 Remark. Since this same statement is stronger for functions from higher dimensional cubes $[\omega_2]^n$ into ω_1 Theorem 17.7 shows that they are all equivalent to Chang's Conjecture. Note also that $n=3$ is the minimal dimension for which this equivalence holds, since the case $n=2$ follows from the Continuum Hypothesis, which has no relationship to Chang's Conjecture.

For the rest of this section we examine the stepping-up procedure without the assumption that some form of Chang's Conjecture is false. So let θ be a given regular uncountable cardinal and let C_α $(\alpha<\theta^+)$ be a fixed C-sequence such that $\mathrm{tp}(C_\alpha)\leq\kappa$ for all $\alpha<\theta^+$. Let $\rho^*:[\theta^+]^2\longrightarrow\theta$ be the ρ^*-function defined above in (3.15). Recall that, in case C_α $(\alpha<\theta^+)$ is a $\Box_\theta$-sequence, the key to our stepping-up procedure was the function $\tau:[\theta^+]^3\longrightarrow\theta^+$ defined by the formula (3.17). Without the assumption of C_α $(\alpha<\theta^+)$ being a $\Box_\theta$-sequence, the following related function turns out to be a good substitute: $\chi:[\theta^+]^3\longrightarrow\omega$ defined by

$$\chi(\alpha,\beta,\gamma)=|\rho_0(\alpha,\gamma)\cap\rho_0(\beta,\gamma)|.$$

Thus $\chi(\alpha,\beta,\gamma)$ is equal to the length of the common part of the walks $\gamma\to\alpha$ and $\gamma\to\beta$.

17.9 Definition. A subset Γ of θ^+ is *stable* if χ is bounded on $[\Gamma]^3$.

The following result whose proof can be found in [66] relates this notion to the unboundedness property of ρ^*.

17.10 Lemma. *Suppose that Γ is a stable subset of θ^+ of size θ. Then $\{\rho^*(\alpha,\beta):\{\alpha,\beta\}\in[\Omega]^2\}$ is unbounded in θ for every $\Omega\subseteq\Gamma$ of size θ.*

17.11 Definition. The 3-*dimensional version of the oscillation mapping*, $\mathrm{osc}:[\theta^+]^3\longrightarrow\omega$ is defined on the basis of the 2-dimensional version of Sect. 14 as follows

$$\mathrm{osc}(\alpha,\beta,\gamma)=\mathrm{osc}(C_{\beta_s}\setminus\alpha,C_{\gamma_t}\setminus\alpha),$$

where $s=\rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t=\rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$.

In other words, we let n be the length of the common part of the two walks $\gamma \to \alpha$ and $\gamma \to \beta$, then we consider the walks $\gamma = \gamma_0 > \cdots > \gamma_k = \alpha$ and $\beta = \beta_0 > \cdots > \beta_l = \alpha$ from γ to α and β to α respectively; if both k and l are bigger than n, i.e. if γ_n and β_n are both defined, we let $\operatorname{osc}(\alpha, \beta, \gamma)$ be equal to the oscillation of the two sets $C_{\beta_n} \setminus \alpha$ and $C_{\gamma_n} \setminus \alpha$. If $\min\{k, l\} < n$, we let $\operatorname{osc}(\alpha, \beta, \gamma) = 0$. The proof of the following basic fact about the three-dimensional oscillation mapping can again be found in [66].

17.12 Lemma. *Suppose that Γ is a subset of θ^+ of size κ, a regular uncountable cardinal, and that every subset of Γ of size κ is unstable. Then for every integer $n \geq 1$, there exist $\alpha < \beta < \gamma$ in Γ such that $\operatorname{osc}(\alpha, \beta, \gamma) = n$.*

Applying the last two lemmas to the subsets of θ^+ of size θ, we get an interesting dichotomy:

17.13 Lemma. *Every $\Gamma \subseteq \theta^+$ of size θ can be refined to a subset Ω of size θ such that either:*

(1) ρ^* is unbounded and therefore strongly unbounded on Ω, or

(2) the oscillation mapping takes all possible values on the cube of Ω.

We finish the section with a typical application of this dichotomy.

17.14 Theorem. *Suppose θ is a regular cardinal such that $\log \theta^+ = \theta$. Then there is an $f : [\theta^{++}]^3 \longrightarrow \omega$ which takes all the values from ω on the cube of any subset of θ^{++} of size θ^+.*

Proof. We choose two C-sequences C_α ($\alpha < \theta^+$) and C^+_α ($\alpha < \theta^{++}$) on θ^+ and θ^{++} respectively, such that $\operatorname{tp}(C_\alpha) \leq \theta$ for all $\alpha < \theta^+$ and $\operatorname{tp}(C^+_\alpha) \leq \theta^+$ for all $\alpha < \theta^{++}$. Let $\rho^* : [\theta^+]^2 \longrightarrow \theta$ and $\rho^{*+} : [\theta^{++}]^2 \longrightarrow \theta^+$ be the corresponding ρ^*-functions defined above in Lemma 3.15. Also choose a one-to-one sequence r_α ($\alpha < \theta^{++}$) of elements of $\{0,1\}^{\theta^+}$ and consider the corresponding function $\Delta : [\theta^{++}]^2 \longrightarrow \theta^+$ defined in (3.21). We define $f : [\theta^{++}]^3 \longrightarrow \theta^+$ according to the following two cases for a given triple $\alpha < \beta < \gamma$ of elements of θ^{++}.

Case 1. $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha \neq \emptyset$, where $s = \rho_0(\alpha, \beta) \restriction \chi(\alpha, \beta, \gamma)$ and $t = \rho_0(\alpha, \gamma) \restriction \chi(\alpha, \beta, \gamma)$ assuming of course that $\rho_0(\alpha, \beta)$ has length $\geq \chi(\alpha, \beta, \gamma)$.

Rule 1. If $\Delta(r_\alpha, r_\beta) < \Delta(r_\beta, r_\gamma)$ and $r_\alpha <_{\text{lex}} r_\beta <_{\text{lex}} r_\gamma$ or $r_\alpha >_{\text{lex}} r_\beta >_{\text{lex}} r_\gamma$, set

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\Delta(\beta, \gamma)) \setminus \Delta(\alpha, \beta)),$$

where $\nu = \rho^*(\min\{\xi \leq \Delta(\alpha, \beta) : \rho^*(\xi, \Delta(\alpha, \beta)) \neq \rho^*(\xi, \Delta(\beta, \gamma))\}, \Delta(\beta, \gamma))$.

Rule 2. If $\bar{\alpha} \in \{\alpha, \beta, \gamma\}$ is such that $r_{\bar{\alpha}}$ is lexicographically between the other two r_ξ's for $\xi \in \{\alpha, \beta, \gamma\}$, if $\bar{\beta} \in \{\alpha, \beta, \gamma\} \setminus \{\bar{\alpha}\}$ is such that $\Delta(r_{\bar{\alpha}}, r_{\bar{\beta}}) > \Delta(r_{\bar{\alpha}}, r_{\bar{\gamma}})$, where $\bar{\gamma}$ is the remaining member of $\{\alpha, \beta, \gamma\}$, and if $\{\alpha, \beta, \gamma\}$ does not fall under Rule 1, let

$$f(\alpha, \beta, \gamma) = \min(P_\nu(\rho^{*+}\{\bar{\beta}, \bar{\gamma}\}) \setminus \rho^{*+}(\alpha, \beta)),$$

where $\nu = \rho^*\{\Delta(\alpha,\beta), \rho^{*+}(\beta,\gamma)\}$.

Case 2. $(C_{\beta_s} \cap C_{\gamma_t}) \setminus \alpha = \emptyset$, where $s = \rho_0(\alpha,\beta)\restriction\chi(\alpha,\beta,\gamma)$ and $t = \rho_0(\alpha,\gamma)\restriction\chi(\alpha,\beta,\gamma)$ assuming of course that $\rho_0(\alpha,\beta)$ has length $\geq \chi(\alpha,\beta,\gamma)$. Let

$$f(\alpha,\beta,\gamma) = \mathrm{osc}(\alpha,\beta,\gamma).$$

If a given triple $\alpha < \beta < \gamma$ does not fall into one of these two cases, let $f(\alpha,\beta,\gamma) = 0$.

Then it suffices to show that for every $\Gamma \subseteq \theta^{++}$ of size θ^+, the image $f``[\Gamma]^3$ either contains all positive integers or almost all ordinals $< \theta^+$ of cofinality θ. The details of this are given in [66]. $\dashv$

17.15 Corollary. *There is an $f : [\omega_2]^3 \longrightarrow \omega$ which takes all the values on the cube of any uncountable subset of ω_2.*

17.16 Remark. Note that the dimension 3 in this corollary cannot be lowered to 2 as long as one does not use some additional axioms to construct such f. Note also that the range ω cannot be replaced by a set of bigger size, as this would contradict Chang's Conjecture. We have seen above that Chang's Conjecture is equivalent to the statement that for every $f : [\omega_2]^3 \longrightarrow \omega_1$ there is an uncountable set $\Gamma \subseteq \omega_2$ such that $f``[\Gamma]^3 \neq \omega_1$. Is there a similar reformulation of the Continuum Hypothesis? More precisely, one can ask the following question.

17.17 Question. Is CH equivalent to the statement that for every $f : [\omega_2]^2 \longrightarrow \omega$ there exists an uncountable $\Gamma \subseteq \omega_2$ with $f``[\Gamma]^2 \neq \omega$?

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4. Borel Equivalence Relations

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Contents

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This chapter is setting out to achieve an impossibility, namely to survey the rapidly exploding field of Borel equivalence relations as found in descriptive set theory and the connections with areas entirely outside logic. The choice of content and emphasis is inevitably molded by this author's own prejudices and research history; others may have a radically different vision of the subject. For instance I have somewhat arbitrarily chosen to say nothing about the study of equivalence relations arising in Borel ideals, as found in papers such as [64, 51, 11], or the parallel theory of Borel linear orderings as found in say [49, 30], or the still unpublished work of Hugh Woodin's and of Richard Ketchersid's on the cardinality of certain Borel equivalence relations under strong determinacy assumptions, the work on general $\underset{\sim}{\Sigma}^1_1$ equivalence relations as found in [4], or the topological Vaught conjecture as discussed in [60, 45, 5, 37]. Moreover the discussion of Borel equivalence relations is organized around the Borel reducibility order, $\leq_B$, rather than notions such as orbit equivalence, as found in say [22, 20, 23, 59, 40], or notions of isomorphism, as discussed in [9]. Without a super-human effort to the contrary, it is easy to slip in to discussing what I know best, which, regretfully perhaps, are the papers I have written. Finally I should admit to being much more conversant with the mathematics of the subject than the history, and since my main concern is to communicate the most vibrant ideas with a certain immediacy undoubtedly some important citations have been overlooked.

Thus I stand impeached with prejudice, ignorance, arbitrariness, arrogance, and discourtesy.

But as unfortunate as these failings may be, they are inevitable, and I say all of this simply so the reader will understand that this is a project doomed to at least partial failure and nevertheless worth pursuing in the hope of partial success. A similar but rather different point of view can be found [52]. The reader might also look at [6] for a closer examination of some of the issues surrounding actions induced by Polish group actions, or at [37] for a discussion of the Vaught conjecture, or [58] for orbit equivalence. Another survey is given in [39], but in fact I am unable to even point to any small set of papers which would be fully adequate.

1. Definitions

Before moving on to the theory of Borel equivalence relations it would be helpful to discuss Borel sets in general. A thorough and more complete account can be found in [50].

1.1 Definition. A *Polish space* is a separable topological space which admits a compatible complete metric. The *Borel* subsets of a Polish space are those appearing in the σ-algebra generated by the open sets—that is to say, if we begin with the open sets and continue applying the operations of countable union, countable intersection, and complementation, then the Borel sets are those appearing in the collection which thereby arises.

Inside the Borel sets we make further distinctions. Thus a set is $\underset{\sim}{\Sigma}^0_1$ if it is open, and after that we define recursively a set to be $\underset{\sim}{\Pi}^0_\alpha$ if its complement is $\underset{\sim}{\Sigma}^0_\alpha$, and to be $\underset{\sim}{\Sigma}^0_\alpha$ if it is a countable union of Borel sets each of which are $\underset{\sim}{\Pi}^0_\beta$ for some $\beta < \alpha$. It is easily shown that $\underset{\sim}{\Pi}^0_\alpha \subseteq \underset{\sim}{\Sigma}^0_{\alpha+1}$ and every Borel set will be $\underset{\sim}{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$. At the bottom level of this hierarchy there is alternate notations used by analysts—for instance a $\underset{\sim}{\Pi}^0_2$ set is also called G_δ and a $\underset{\sim}{\Sigma}^0_2$ set is also called F_σ.

For much of the time we are unconcerned with the topological structure of a Polish space, focusing instead on its Borel structure. Accordingly, a set X equipped with a σ-algebra $\mathcal{B}$ is a *standard Borel space* if there is some Polish topology on X which gives rise to $\mathcal{B}$ as the collection of Borel sets.

1.2 Examples.

1. $\mathbb{R}$ and $\mathbb{C}$ are Polish. Any compact metric space is Polish.

2. There is a natural way to think of the collection of all subsets of the natural numbers as a Polish space. By associating with each set its characteristic function, or indicator function, we can identify the power set of $\mathbb{N}$, denoted by $\mathcal{P}(\mathbb{N})$, with
$$\{0,1\}^{\mathbb{N}} =_{\mathrm{df}} 2^{\mathbb{N}},$$
which is a compact space in the product topology. Similarly $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ or $\mathcal{P}(S)$ any countable set S.

3. Any Borel subset of a Polish space is a standard Borel space in the inherited Borel structure (see [55, 13.4], [50]).

4. The Borel probability measures on a standard Borel space themselves again form a standard Borel space (this ultimately follows from the Riesz representation theorem; compare [50, 17.23]).

5. Consider the collection of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ which form the graph of a function
$$\bullet : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$$
providing a finite rank torsion-free abelian group structure on $\mathbb{N}$. This collection in the natural Borel structure is a standard Borel space, since it is a Borel subset of the Polish space $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

6. Given a countable language we can form two possible Polish topologies on the space of $\mathcal{L}$-structures on $\mathbb{N}$. For simplicity assume the language is relational, though the more general case of $\mathcal{L}$ having function symbols is only slightly more complicated. We let $\mathrm{Mod}(\mathcal{L})$ be the set of all $\mathcal{L}$-structures with the natural numbers as their underlying set.

The first of the topologies is τ_{qf}. For $\psi(\vec{x})$ a formula in $\mathcal{L}$ and $\vec{a} = (a_1, \ldots, a_n)$ a sequence in $\mathbb{N}$, we let $U(\psi(\vec{x}), \vec{a})$ be the collection of all $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ with

$$\mathcal{N} \models \psi(\vec{a}),$$

and then τ_{qf} is the collection the topology with basis consisting of all the sets of the form $U(\psi(\vec{x}), \vec{a})$, as ψ ranges over the quantifier-free formulas. If $\mathcal{L}$ consists of relations $R_1, R_2, \ldots$, each R_i and $n(i)$-ary relation, then it is easily seen that $(\mathrm{Mod}(\mathcal{L}), \tau_{qf})$ is isomorphic to $\prod_{i\in\mathbb{N}} 2^{\mathbb{N}^{n(i)}}$ in the product topology. Since the class of Polish spaces is closed under product (see [36, §2.1]) we have that $(\mathrm{Mod}(\mathcal{L}), \tau_{qf})$ is Polish.

The more subtle topology is τ_{fo}, with basis consisting of all $U(\psi(\vec{x}), \vec{a})$ as ψ ranges over first-order formulas. This is a again a Polish topology, though the proof of this, say as found at [36, 2.42], is less obvious.

While these are divergent choices in topology, there is really one reasonable choice of *Borel structure* on this space. Since $\tau_{fo} \supseteq \tau_{qf}$ and both are Polish topologies, they give rise to the same Borel structure. (This follows from [50, 18.10, 18.14].)

1.3 Definition. Given a Polish space X, we let $\mathcal{F}(X)$ be the collection of all closed subsets of X, equipped with the σ-algebra generated by sets of the form $\{F \in \mathcal{F}(X) : U \cap F \neq \emptyset\}$ for U open.

This is known as the *Effros Borel structure* on the closed subsets of X. It is not hard to show that $\mathcal{F}(X)$ equipped with this Borel structure is indeed a standard Borel space—see for instance [36, §2] or [50].

The reader should definitely consult [50] for a more reasonable introduction to the theory of Borel sets. This is the smallest sketch.

1.4 Definition. A function

$$f : X \to Y$$

is said to be *Borel* if $f^{-1}[B]$ is Borel for any Borel set $B \subseteq Y$.

An equivalence relation E on X is said to be *Borel* if it is Borel as subset of $X \times X$. We then use $[x]_E$ to denote the equivalence class of x for any $x \in X$—that is to say, the set $\{y \in X : x\, E\, y\}$.

1.5 Definition. Given E and F Borel equivalence relations on standard Borel X and Y, we write

$$E \leq_B F,$$

E is *Borel reducible to* F, if there is a Borel $f : X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1\, E\, x_2 \iff f(x_1)\, F\, f(x_2);$$

in other words, f pushes down to an injective map

$$\bar{f} : X/E \to Y/F$$

between the quotient spaces.

After this we naturally set $E <_B F$ if there is a Borel reduction from E to F but not F to E. We set $E \sim_B F$ if each is Borel reducible to the other. Note that the order $\leq_B$ is transitive, since the composition of two Borel functions is again Borel.

The above definition is only for Borel f, but one can of course consider more general classes of functions if this seems too stingy—for instance, C-measurable functions, projective functions, $L(\mathbb{R})$ functions. In virtually all cases this makes no difference—the failures of reducibility for Borel functions persist to these wider classes. Indeed, appropriately understood most of the theorems about Borel equivalence relations turn into theorems about cardinality in $L(\mathbb{R})$. See Sect. 4 below.

It should also be admitted that there are other ways in which we can compare Borel equivalence relations, for instance asking that there be isomorphisms of the underlying spaces that conjugate the relations, or in the presence of a measure we can ask for measure preserving isomorphisms between the spaces which conjugate the equivalence relations almost everywhere; indeed this second notion is the subject of extensive study in areas such as operator algebras (for instance [59]), geometric group theory (for instance [22, 62]), and the rigidity theory in the sense of Zimmer [70]. For the purposes of descriptive set theory, I incline to the view that $\leq_B$ is the central notion. Indeed some kind of defense of the philosophical significance of $\leq_B$ is given in Sect. 4.

1.6 Examples.

1. For X a Polish space, we let $\mathrm{id}(X)$ be the identity relation on the space X. Since any two uncountable standard Borel spaces are isomorphic [50, 15.6], it follows that for any uncountable X we have
$$\mathrm{id}(X) \leq_B \mathrm{id}(\mathbb{R}).$$
2. We let E_0 be the equivalence relation of eventual agreement on infinite binary sequences. Thus for $\vec{x} = (x_0, x_1, \ldots), \vec{y} = (y_0, y_1, \ldots) \in 2^{\mathbb{N}}$, we set
$$\vec{x} \, E_0 \, \vec{y}$$
if and only if there exists some $N \in \mathbb{N}$ with
$$\forall n > N (x_n = y_n).$$
Here we have
$$\mathrm{id}(2^{\mathbb{N}}) <_B E_0.$$
To see that there is a Borel reduction from $\mathrm{id}(2^{\mathbb{N}})$ to E_0 is routine. It follows simply because there is perfect set in Cantor space consisting of mutually generic reals.

The failure of reducibility in the other direction is more subtle. First one observes that E_0 is given by the continuous action of the countable group

$$\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$$

with $(g_0, g_2, \ldots) \cdot (x_0, x_1, \ldots) = (g_0 + x_0 \mod 2, g_1 + x_1 \mod 2, \ldots)$. Then given a supposed Borel reduction f of E_0 to the identity relation on $2^{\mathbb{N}}$ we can find a comeager set C on which f is continuous. Then we can take

$$\bigcap_{\vec{g} \in \bigoplus \mathbb{Z}/2\mathbb{Z}} \vec{g} \cdot C,$$

which will still be comeager and now invariant. Taking any $\vec{x}$ in this set we obtain that f will be constant on $[\vec{x}]_{E_0}$. Since this equivalence class is dense, f will constant on the entire set $\bigcap_{\vec{g}} \vec{g} \cdot C$. Since this set is uncountable, it contains many equivalence classes with a contradiction. (For a more detailed and general argument, see [36, 3.2].)

3. We let E_1 be the equivalence relation of eventual agreement on infinite sequences of reals. Here one has

$$E_0 <_B E_1.$$

(See [53], or even [36].)

4. Given a countable group Γ, we let 2^Γ be the collection of all functions

$$f : \Gamma \to \{0, 1\}$$

with the product topology. We let Γ act on 2^Γ with

$$\gamma \cdot f(\delta) = f(\gamma^{-1}\delta),$$

the left shift action. We then let E_G be the resulting equivalence relation.

In the case that we start with $G = \mathbb{F}_2$, the free group on two generators, one has

$$E_0 <_B E_{\mathbb{F}_2}.$$

(See for instance Appendix A of [44] for a survey of stronger and more general results one can prove in this direction.)

5. An example of historical importance is the *Vitali equivalence relation*, E_v. For $r, s \in \mathbb{R}$ set

$$r \, E_v \, s$$

if and only if

$$r - s \in \mathbb{Q}.$$

The classical argument that this equivalence relation has no measurable selector can be modified to show that $\mathrm{id}(\mathbb{R}) <_B E_v$; alternatively one may appeal to a variation of the Baire category argument at example 2 just above.

On the other hand, at the level of Borel reducibility there is no distinction between E_0 and E_v. (See for instance the argument after [35, 1.5] for a short proof that $E_0 \sim_B E_v$.)

6. Let X_2 be the space of all $h \in 2^{\mathbb{N}\times\mathbb{N}}$, where at each $n \neq m$ there exists a k with $h(n,k) \neq h(m,k)$. In other words, if we set $h_n \in 2^{\mathbb{N}}$ to be given by $h_n(k) = h(n,k)$ then for $n \neq m$ we have $h_n \neq h_m$.

   Then define $\mathcal{T}_2$ on X_2 by $h^1 \mathcal{T}_2 h^2$ if and only if the corresponding countable sets in $2^{\mathbb{N}}$ are equal—that is to say,

$$\{h^1_n : n \in \mathbb{N}\} = \{h^2_n : n \in \mathbb{N}\}.$$

   Thus if we let S_∞ be the group all permutations of $\mathbb{N}$, acting on X_2 by

$$\sigma \cdot h(n,k) = h(\sigma^{-1}(n), k),$$

   then $\mathcal{T}_2$ is the resulting equivalence relation.

   It is well-known that for any E_G as above, arising from a countable group G acting on 2^G, one has

$$E_G <_B \mathcal{T}_2.$$

   (See for instance [36, 2.64].)

1.7 Definition. An equivalence relation E on standard Borel X is said to be *smooth* or *tame* if $E \leq_B \mathrm{id}(\mathbb{R})$.

Smoothness amounts to asserting the existence of a countable algebra $\{B_n : n \in \mathbb{N}\}$ of E-invariant Borel sets which separates points—which is to say

$$x\,E\,y \quad\Longleftrightarrow\quad \forall n(x \in B_n \iff y \in B_n).$$

This in turn is equivalent to saying that $X/E = \{[x]_E : x \in X\}$ in the quotient Borel structure consisting of all E-invariant Borel sets is a subset of a standard Borel space. (See [29, 48].)

1.8 Definition. A Borel equivalence relation E on a standard Borel space X is said to be *countable* if every equivalence class is countable. It is *essentially countable* if there is some other countable Borel equivalence relation F with $E \leq_B F$.

1.9 Example. Let $\mathbb{F}_2$ be the free group on two generators and let E_∞ arise from action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. Then this is countable, since the responsible group is countable.

It is known from [48] that this equivalence relation is *universal* among the countable equivalence relations, in the sense that for every countable Borel equivalence relation E one has

$$E \leq_B E_\infty.$$

We will not get ensnared in the details of this argument here, but it might be helpful to point out that the proof depends on the Feldman-Moore theorem, which in some sense reduces to the study of countable Borel equivalence relations to the study of the orbit equivalence relations induced by countable groups.

1.10 Theorem (Feldman-Moore [15, 16]). *If E is a countable Borel equivalence relation on X, then there is a Borel group of automorphisms whose orbits equal the E-equivalence classes.*

Proof. With the Luzin-Novikov Uniformization Theorem [50, 18.10, 18.15] we can find a sequence of Borel functions $(f_n)_{n\in\mathbb{N}}$ such that $[x]_E$ always equals $\{f_n(x) : n \in \mathbb{N}\}$. We can find Borel sets B_n consisting of exactly the points on which $f_n(x) \neq x$. We can then find a partition of B_n into Borel sets $\{C_{n,i} : i \in \mathbb{N}\}$ with $f_n[C_{n,i}]$ disjoint from $C_{n,i}$ and $f|_{C_{n,i}}$ one-to-one. We then let $g_{n,i}$ be the function which is the identity off of $f_n[C_{n,i}] \cup C_{n,i}$, equal to f_n on $C_{n,i}$, and to f_n^{-1} on $f[C_{n,i}]$. It follows by Luzin-Novikov again that each $g_{n,i}$ is Borel.

Letting G be the countable group of automorphisms generated by $\{g_{n,i} : i, n \in \mathbb{N}\}$ we obtain $E = E_G$, the orbit equivalence relation induced by G. $\dashv$

1.11 Lemma. *A countable Borel equivalence relation is smooth if and only if it has a Borel selector—that is to say, a Borel set which meets every equivalence class in exactly one point.*

Proof. This follows rapidly from the Luzin-Novikov Uniformization Theorem. See for instance [48], or [7] for far more general results. $\dashv$

This lemma fails for general equivalence relations. For instance if we take $C \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ with the projection $\{x : \exists y((x, y) \in C)\}$ non-Borel, and set $(x, y)E(x', y')$ on C exactly when $x = x'$, then a Borel selector would result in the projection being Borel, since the one-to-one image of any Borel set under a Borel function is Borel.

1.12 Definition. An equivalence relation E is *hyperfinite* if there is a sequence of Borel equivalence relations, $(F_n)_{n\in\mathbb{N}}$, with each F_n having all its equivalence classes finite, each $F_n \subseteq F_{n+1}$, and $E = \bigcup_{n\in\mathbb{N}} F_n$.

1.13 Lemma (See [48]). *A countable Borel equivalence relation E on X is hyperfinite if and only if $E \leq_B E_0$.*

1.14 Definition. An equivalence relation E on a standard Borel space X is *treeable* if its classes form the components of a Borel treeing on X—that is to say, there is $\mathcal{T} \subseteq X \times X$ which is Borel with respect to the product Borel structure, is symmetric, acyclic, loopless, and for which we have $x\,E\,y$ if and only if there is a finite chain $x_0 = x, x_1, x_2, \ldots, x_n = y$, each $(x_i, x_{i+1}) \in \mathcal{T}$. (So here one has in mind the notion of tree prevalent in certain branches of combinatorics or computer science, rather than descriptive set theory—an acyclic graph, with no distinguished root in the various connected components).

1.15 Definition. Let $\mathbb{F}_2$ be the free group on two generators. Let $F(2^{\mathbb{F}_2})$ be the free part of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$—that is to say, the set of $h : \mathbb{F}_2 \to \{0, 1\}$ such that for all $\sigma \in \mathbb{F}_2$ there exists τ with $h(\sigma^{-1}\tau) \neq h(\tau)$, equipped with the action $(\sigma \cdot h)(\gamma) = h(\sigma^{-1}\gamma)$. We then let $E_{\mathcal{T}\infty}$ be the orbit equivalence arising from this action of $\mathbb{F}_2$ on $F(2^{\mathbb{F}_2})$.

In most cases treeable equivalence relations have been studied simply in the case that E is already countable. Here one has an analogue of Lemma 1.13.

1.16 Lemma (See [48] or [41]). *A countable equivalence relation is treeable if and only if it is Borel reducible to* $E_{\mathcal{T}\infty}$.

Although this survey is primarily concerned with Borel equivalence relations, one must naturally connect this study with the analysis of those lying just outside this class.

1.17 Definition. A subset of a standard Borel space is *analytic* or $\utilde{\Sigma}^1_1$ if it is the image under a Borel function of a Borel set.

1.18 Definition. A *Polish group* is a topological group which is Polish as a space. Given a Polish group G, a *Polish G-space* is a Polish space equipped with a continuous action of G; a *standard Borel G-space* is a standard Borel space equipped with a Borel action by G.

In either case, if X is the space we use E^X_G, or even just E_G when there is no doubt to X, to denote the resulting orbit equivalence relation, and $[x]_G$ to denote the orbit of a point x.

The equivalence relations arising from the Borel action of a Polish group are always $\utilde{\Sigma}^1_1$, but frequently non-Borel. As noted in [6], for any such E^X_G we can write X as an $\aleph_1$ union of invariant Borel sets,

$$X = \bigcup_{\alpha<\omega_1} X_\alpha,$$

and E^X_G restricted to each X_α Borel.

1.19 Examples.

1. Let S_∞ be the group of all infinite permutations of the natural numbers equipped with the topology of pointwise convergence—that is to say,

a basic open set has the form $\{\sigma \in S_\infty : \sigma(n_0) = k_0, \ldots, \sigma(n_\ell) = k_\ell\}$. For $\mathcal{L}$ a countable language we let S_∞ act on $\mathrm{Mod}(\mathcal{L})$ in the natural way:

$$(\sigma \cdot \mathcal{M}) \models R(n_0, \ldots, n_\ell)$$

if and only if

$$\mathcal{M} \models R(\sigma^{-1}(n_0), \ldots, \sigma^{-1}(n_\ell)).$$

The resulting equivalence relation is nothing other than isomorphism on this class of structure, and for any reasonably rich one has that E_{S_∞} is $\undertilde{\Sigma}^1_1$ but non-Borel.

2. Given $\mathcal{L}$ as above, and $\varphi \in \mathcal{L}_{\omega_1,\omega}$, a countably infinitary formula, we let $\mathrm{Mod}(\varphi)$ be the structures in $\mathrm{Mod}(\mathcal{L})$ which satisfy φ. Since this is a Borel subset of a Polish space, it is a standard Borel space in its own right, and in the induced action a standard Borel S_∞-space.

   Here one should see [6] for further details. The first paper to consider these examples seriously from the point of the view of the $\leq_B$ ordering is [18].

3. Given a Polish group G we can let it act on itself by conjugation

$$\sigma \cdot \rho = \sigma^{-1} \rho \sigma.$$

   In many cases this corresponds naturally to some kind of classification problem. For instance, if M_∞ is the group of all measure-preserving transformations of the unit interval considered up to agreement almost everywhere, then this group is indeed Polish in the natural topology and the equivalence relation arising from its conjugation action is the equivalence relation of isomorphism of measure-preserving transformations. Or if we let U_∞ be the unitary group on infinite dimensional Hilbert space, we are find ourselves confronted with the classification up to unitary equivalence of unitary operators. Or one may consider $\mathrm{Hom}([0,1])$, the homeomorphism group of the unit interval, for the classification problem for homeomorphisms of the unit interval up to topological similarity. All these examples are discussed at length in [36].

2. A Survey of Structure Theorems

2.1. Structure

At the base level of the $\leq_B$ there is a sharp structure. The first of these results is due to Jack Silver. Although it was proved sometime prior to the general study of Borel equivalence relations, it may be paraphrased in modern terminology as follows.

2.1 Theorem (Silver [63]). *If E is a Borel equivalence relation, then exactly one of the following holds:*

(I) $E \leq_B \mathrm{id}(\mathbb{N})$*; or*

(II) $\mathrm{id}(\mathbb{R}) \leq_B E$.

In some ways this belongs to the prehistory of the subject. Momentum developed only after the seminal dichotomy theorem of Leo Harrington, Alexander Kechris, and Alain Louveau. This is sometimes known as the *Glimm-Effros dichotomy*, since forerunners of this theorem were proved by Glimm and then later Effros, who had the result in the case that E is an F_σ equivalence relation induced by the continuous action of a Polish group.

2.2 Theorem (Harrington-Kechris-Louveau [29]). *If E is a Borel equivalence relation, then exactly one of the following holds:*

(I) $E \leq_B \mathrm{id}(\mathbb{R})$*; or*

(II) $E_0 \leq_B E$.

Sketch of Proof. We describe some of the combinatorics under the drastic assumption that E is a *countable* Borel equivalence relation. This is misleading as to the difficulty of the proof, since then the vast majority of the mathematical issues simply evaporate. It should also be pointed out that the theorem in this case was already known to Glimm [28] and Effros [10].

First we can appeal to Theorem 1.10 to obtain a countable group G acting by Borel transformations on the Polish space X with $E_G = E$. Applying a change of topology argument, as found in say [50, §13], we may assume G acts by homeomorphisms.

Now there are two possibilities.

First of all we may have that for any $x \neq y \in X$ we have the closures of their orbits, $\overline{[x]_G}, \overline{[y]_G}$, distinct. In that case one defines a map

$$X \to \mathcal{F}(X),$$

$$x \mapsto \overline{[x]_G},$$

from X to the space of all closed subsets of X with the standard Borel structure. It is easily seen that this map is Borel, and so we obtain a reduction of E to $\mathrm{id}(\mathcal{F}(X))$. Since any two uncountable Borel spaces are Borel isomorphic, this is as good as a reduction to $\mathrm{id}(\mathbb{R})$.

The other case is that there are distinct orbits with the same closure. We will now work entirely inside some $X_0 \subseteq X$ consisting of all y with $\overline{[y]_G} = C$ for some closed C. X_0 is a G_δ subset of X, and hence Polish in its own right. (See for instance [36, §2.1].) At this stage we are assuming that X_0 contains more than one orbit for the purpose of deriving a contradiction. We let $\{g_n : n \in \mathbb{N}\}$ enumerate the group G. Now note that E_G is F_σ, and our assumptions on X_0 imply that both it and its complement are dense in $X_0 \times X_0$.

We then construct non-empty open neighborhoods $(U_s)_{s \in 2^{<\mathbb{N}}}$ in X_0 and $(g_s)_{s \in 2^{<\mathbb{N}}}$ so that:

(i) $\overline{U_s} \subseteq U_t$ for s a strict extension of t;

(ii) with respect to a compatible complete metric on X_0, the diameter of each U_s is less than 2^{-n} for $s \in 2^n$;

(iii) $g_{s^\frown 0} = g_s$; $g_{s^\frown 1} = g_s g_{\langle 0,0,\ldots,0\rangle^\frown 1}$; $g_{\langle 0,0,\ldots,0\rangle} = e$, the identity of the group;

(iv) $g_s \cdot U_{\langle 0,0,\ldots,0\rangle} = U_s$, where $\langle 0,0,\ldots,0\rangle$ is the constantly zero sequence of the same length as s;

(v) if $s, t \in 2^{n+1}$ and $s(n) \neq t(n)$, then for all $i \leq n$ we have $g_i \cdot U_s \cap U_t = \emptyset$.

Assuming we can effect this elaborate arrangement, the conclusion is brief. (iv) will guarantee that for each $s, t \in 2^n$ we have

$$g_t g_s^{-1} \cdot U_s = U_t,$$

and then repeated applications of (iii) ensures for any $w \in 2^{<\mathbb{N}}$

$$g_t g_s^{-1} \cdot U_{s^\frown w} = U_{t^\frown w}.$$

Thus for any $x \in 2^{\mathbb{N}}$ there will be a unique point

$$\theta(x) \in \bigcap U_{x|_n},$$

and if $x(n) = y(n)$ all $n \geq N$, then

$$g_{x|_n} g_{y|_n}^{-1}(\theta(y)) = \theta(x),$$

whilst if $x(n) \neq y(n)$ for infinitely many n then (v) guarantees E_G-inequivalence.

So suppose we have done this up through $(U_s)_{s \in 2^n}, (g_s)_{s \in 2^n}$. By the density of the complement of E_G we can find some $x, y \in U_{\langle 0,0,\ldots,0\rangle}$ which are inequivalent. We can then let $x_s = g_s \cdot x$, $y_s = g_s \cdot y$. We form small enough open sets W, V around x, y so that if $W_s = g_s \cdot W$, $V_s = g_s \cdot V$, then $g_i \cdot V_s \cap W_t = \emptyset$, $g_i^{-1} \cdot V_s \cap W_t = \emptyset$.

We then find a point $x' \in [x]_G \cap V_{\langle 0,0,\ldots,0\rangle}$ by the density of the orbits. We let $g_{\langle 0,0,\ldots,0\rangle^\frown 1}$ be a group element moving x to x'. We let $g_{s^\frown 1}$ be $g_s g_{\langle 0,0,\ldots,0\rangle^\frown 1}$. We then build a sufficiently small set $U^* \subseteq W$ so that if we let $U_t = g_t \cdot U^*$ for each $t \in 2^{n+1}$ then for $i \leq n$ we have ensured (i), (ii), and (v). $\dashv$

The argument for general Borel E is far harder, and uses the Gandy-Harrington topology. There is no known proof of Theorem 2.2 which does not use ideas from logic.

Therefore at the base of the picture one obtains the Borel equivalence relations with finitely many classes, followed by any Borel equivalence relation with exactly $\aleph_0$ many classes, then the identity relation on the reals, and then E_0, or, equivalently when considered up to Borel reducibility, the Vitali equivalence relation.

It should be mentioned that Harrington-Kechris-Louveau implies Silver's theorem. If $f : 2^{\mathbb{N}} \to X$ witnesses $E_0 \leq_B E$ then we can find a comeager set on which this function is continuous, and then inside this comeager we can find a compact perfect set of E_0-inequivalent reals. On the other hand if $E \leq \mathrm{id}(\mathbb{R})$ then the equivalence classes of E can be identified with a $\undertilde{\Sigma}^1_1$ subset of a Polish space, whereupon Silver's theorem reduces to the perfect set theorem for $\undertilde{\Sigma}^1_1$ sets.

Immediately after this one obtains a splintering. It is known from [53] that there are no other Borel equivalence relations E such that for any other Borel F one has either $E \leq_B F$ or $F \leq_B E$. Instead we have:

2.3 Theorem (Kechris-Louveau [53]). *Let E be a Borel equivalence relation with $E \leq_B E_1$. Then exactly one of the following holds:*

(I) $E \leq_B E_0$; or

(II) $E_1 \leq_B E$.

2.4 Definition. Let $E_0^{\mathbb{N}}$ be the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by

$$\vec{x}\, E_0^{\mathbb{N}}\, \vec{y}$$

if and only if

$$\forall n (x_n\, E_0\, y_n);$$

that is to say, we have E_0-equivalence at every coordinate.

2.5 Theorem (Hjorth-Kechris [43]). *Let E be a Borel equivalence relation with $E \leq_B E_0^{\mathbb{N}}$. Then exactly one of the following holds:*

(I) $E \leq_B E_0$; or

(II) $E_0^{\mathbb{N}} \leq_B E$.

There are no other known immediate successors to E_0, but Ilijas Farah [12] has proposed a continuum of plausible examples which are inspired by ideas in Banach space theory. What he does show is that these examples are incomparable, and that any equivalence relation strictly below any of them is essentially countable. The gnawing technical difficulty at the end of his argument is our inability to determine which countable equivalence relations are hyperfinite. Very recently, in a spectacular and unpublished piece of work, Su Gao and Steve Jackson showed all equivalence relations arising from the Borel actions of countable abelian groups are hyperfinite, and on this basis we might hope for further clarification in the coming years.

Although we are at a loss to provide further global dichotomy theorems, there is something more that can be said if we restrict ourselves to the region of equivalence relations which are below isomorphism of countable structures.

2.6 Definition. For $\mathcal{L}$ a countable language, we let $\cong_{\mathrm{mod}(\mathcal{L})}$ be isomorphism on the standard Borel space of $\mathcal{L}$-structures with underlying set $\mathbb{N}$. For $\psi \in \mathcal{L}_{\omega_1,\omega}$ we let $\cong_{\mathrm{mod}(\psi)}$ be the restriction of this equivalence relation to models of ψ.

2.7 Lemma (Becker-Kechris [6]). *Let E be Borel. Then the following are equivalent:*

(I) $E \leq_B \cong_{\mathrm{mod}(\mathcal{L})}$ for some countable language $\mathcal{L}$.

(II) $E \leq_B E_{S_\infty}$, where E_{S_∞} arises from the continuous action of a S_∞ on a Polish space.

2.8 Definition. If either of these equivalent conditions hold, then we say that E *admits classification by countable structures.*

In the still unpublished [31] a dichotomy theorem was recently announced for equivalence relations admitting classification by countable structures. It should be emphasized that this proof is long and has not been refereed, and accordingly the result has a provisional status.

2.9 Theorem (Hjorth [31]). *Let E be a Borel equivalence relation admitting classification by countable structures. Then exactly one of the following holds:*

(I) $E \leq E_\infty$, that is to say, E is essentially countable; or

(II) $E_0^{\mathbb{N}} \leq_B E$.

2.2. Anti-Structure

The last section represented the good news. Now for the evil.

Refining an earlier result of Woodin's, Louveau and Boban Velickovic embedded $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into the Borel equivalence relations considered up to Borel reducibility.

2.10 Theorem (Louveau-Velickovic [56]). *There is an assignment*

$$A \mapsto E_A$$

of $\underset{\sim}{\Pi}^0_3$ equivalence relations to subsets of $\mathbb{N}$ such that

$$E_{A_1} \leq_B E_{A_2}$$

if and only if $A_2 \setminus A_1$ is finite.

Therefore there is no global structure theorem for the $\leq_B$ ordering.

Given the dichotomy theorems of [53] and [43] we might at least hope for some kind of basis of immediate successors to E_0, but even this seems unlikely. In [13] Farah obtains an infinite descending chain in $\leq_B$ which is *unlikely* to be above an immediate successor to E_0. I say unlikely, though it is hard to make a fast guess at this point. Literally Farah obtains the existence of a sequence $(F_n)_{n\in\mathbb{N}}$ with each $F_{n+1} <_B F_n$, none of them essentially countable, and with the further property that any equivalence relation $E \leq_B F_n$ at every n must be essentially countable. Since the F_n's arise in a very simple form, in particular the continuous action of an abelian Polish group, there is some grounds for thinking the only countable equivalence relations reducible to one of these will be reducible to E_0.

Farah's work also disproves the existence of a dichotomy theorem for being classifiable by countable structures in many dramatic ways. For instance he obtains an uncountable sequence of Borel equivalence relations, $(E_x)_{x\in 2^{\mathbb{N}}}$, which are not classifiable by countable structures, and such that for any $x \neq y$ and Borel E with

$$E \leq_B E_x, E_y$$

we have E classifiable by countable structures.

2.3. Beyond Good and Evil

There are slender few candidates for outright Borel dichotomy theorems in the style of Theorems 2.2 and 2.10 and the later results of Farah would seem to rush hopes for a global analysis of the $\leq_B$ ordering. On the other hand it still seems that there are results which would help us understand when equivalence relations fall on some side of a key divide.

A distinction of philosophical interest is when we can classify by countable structures, which has clear parallels in broader mathematical practice. Here one has in mind for instance certain branches of topology, where one seeks complete algebraic invariants, and in effect one is trying to classify a certain equivalence relation by a countable algebraic structure considered up to isomorphism. The introduction of [36] surveys several examples along these lines from a variety of different mathematical areas.

2.11 Definition. Let G be a Polish group and X a Polish G-space. For $U \subseteq G$ an open neighborhood of the identity, $V \subseteq X$ an open neighborhood of a point x, the *local U-V-orbit of* x, written $\mathcal{O}(x, U, V)$, is the set of all y such that there exists a finite sequence of points $x_0 = x, x_1, x_2, \ldots, x_n = y$ in U with each $x_{i+1} \in V \cdot x_i$—that is to say, we can move from x_i to x_{i+1} using some group element in V. Then we say that the action of G on X is *turbulent* if the following three things hold:

(i) every orbit $[x]_G$ is dense in X;

(ii) every orbit $[x]_G$ is meager in X;

(iii) given $x, y \in X$, U an open neighborhood of y, V an open neighborhood of the identity in G, there is some $x' \in [x]_G \cap U$ with y in the closure of the local orbit $\mathcal{O}(x', U, V)$.

2.12 Theorem (Hjorth [36]). *If E_G arises from a turbulent action of G on the Polish G-space X and $\mathcal{L}$ is a countable language, then E_G is not Borel reducible to $\cong_{\mathrm{mod}(\mathcal{L})}$.*

Sketch of Proof. We present the argument in a simple case with a number of simplifying assumptions. Consider the example of an S_∞ action given before, where the space is X_2, consisting of all $h \in 2^{\mathbb{N}\times\mathbb{N}}$, where at each $n \neq m$ there exists a k with $h(n,k) \neq h(m,k)$. Then define $\mathcal{T}_2$ on X_2 by $h^1 \mathcal{T}_2 h^2$ if and only if the corresponding countable sets in $2^{\mathbb{N}}$ are equal. This is the orbit equivalence relation arising from the S_∞ action

$$(\sigma \cdot h)(m,n) = h(\sigma^{-1}(m), n).$$

We assume toward a contradiction that $\theta : X \to X_2$ witnesses $E_G \leq_B \mathcal{T}_2$.

Every Borel function is continuous on a comeager set, so let us actually make the simplifying assumption that θ is continuous everywhere.

Claim: For any n there is a comeager collection of $x \in X$ which have some basic open neighborhood $V_{x,n}$ of the identity in S_∞ for which there is a comeager collection of $\sigma \in V_{x,n}$ having

$$(\theta(x))(n,\cdot) = (\theta(\sigma \cdot x))(n,\cdot).$$

Proof of Claim: For each individual point x and m we consider the Borel function

$$f_x^m : G \to \mathbb{N}$$

which assigns to $g \in G$ the natural number $f(g)$ with $\theta(x)(m,\cdot) = \theta(g \cdot x)(f(g),\cdot)$. Each Borel set has the property of Baire, hence we can find open sets $(O_n^m)_{n\in\mathbb{N}}$ with dense union in G such that $O_n \Delta f^{-1}(n)$ is meager at every n. Then for any

$$g \in \bigcap_m \bigcup_n O_n^m$$

we have at each ℓ some basic open $V \subseteq G$ such that for a comeager collection of $h \in V$

$$\theta(g \cdot x)(\ell,\cdot) = \theta(hg \cdot x)(\ell,\cdot).$$

Thus we have show that inside each orbit the collection of $g \in G$ for which $g \cdot x$ has the required property is comeager. Now the conclusion of the claim follows by the Kuratowski-Ulam Theorem (see [50]).

Let us make the additional and rather radical assumption that

$$x \mapsto V_{x,n}$$

is defined everywhere, and "continuous" in the sense that given any basic open V and n the collection of x with $V_{x,n} = V$ is open.

Let $x, y \in X$. Consider some n. It suffices to show that there is a representative of the orbit of x,

$$x' \in [x]_G,$$

which has $\theta(x')(n, \cdot) = \theta(y)(n, \cdot)$. By the above simplifying assumptions we can find a basic open neighborhood of U of y and V a basic open neighborhood of the identity in the group which has

$$(\theta(z))(n, \cdot) = (\theta(\sigma \cdot z))(n, \cdot)$$

all $z \in U, \sigma \in V$.

Now if we take $x' \in [x]_G$ with

$$y \in \overline{\mathcal{O}(x', U, V)}$$

then we can find a sequence of points $(z_i)_{i \in \mathbb{N}}$, each $z_i \in \mathcal{O}(x', U, V)$ with

$$z_i \to y.$$

By the assumptions on U and V we have $\theta(z_i)(n, \cdot) = \theta(x')(n, \cdot)$ at all n, and hence $\theta(x')(n, \cdot) = \theta(y)(n, \cdot)$, as required. ⊣

As a practical matter [36] suggests turbulence to be the key phenomena in determining whether an equivalence relation is classifiable by countable structures. This has been supported in various examples and practical investigations, such as [26, 54], as well as some partial results given in [36]. The correct theorem was not established until a few years later.

2.13 Theorem (Hjorth [38]). *Let G be a Polish group and X a Polish G-space. Suppose E_G^X is Borel. Then exactly one of the following hold:*

(I) E_G^X admits classification by countable structures; or

(II) there is a turbulent Polish G-space Y with $E_G^Y \leq E_G^X$.

3. Countable Borel Equivalence Relations

The subject of countable Borel equivalence relations is notable for its interactions with such diverse fields as geometric group theory, the ergodic theory of non-amenable groups, operator algebras, and superrigidity in the sense of Zimmer. There is a sense in which this interaction has been largely one way, since one finds logicians borrowing from these other fields rather than these areas being serviced by logic.

In the course of this period it is perhaps unsurprising that logicians have made a few notable contributions to the benefit of the other fields. However in almost every case the applications do not consist in applying deep ideas from logic, as found in say the work of Hrushovski [47], but rather the natural result of mathematicians from one field rethinking the problems from another and approaching with a different point of view.

3.1. The Global Structure

In some form or another almost every argument which distinguishes countable Borel equivalence relations in the $\leq_B$ ordering uses measure theory. At the very base level we can use Baire category arguments to show $\mathrm{id}(\mathbb{R}) <_B E_0$ but beyond this there is a fundamental obstruction.

3.1 Theorem (Sullivan-Weiss-Wright [65]). *Let E be a countable Borel equivalence relation on a Polish space X. Then there is a comeager set on which E is hyperfinite.*

Sketch of Proof. (This draws on Segal's thesis [61]; see also [58].) E is induced by a countable group G acting by Borel automorphisms. Let us make the simplifying assumptions that G acts by homeomorphisms and the space is zero-dimensional—these assumptions are harmless, but we will skip over this point in the interest of keeping the argument short.

Let $\mathcal{B}$ be a countable basis for X consisting of clopen sets. Let $(g_n)_{n\in\mathbb{N}}$ enumerate the countable group G.

At each n let Y_n be the collection of sequences

$$\vec{A} = (A_0, A_1, \ldots, A_n),$$

where each A_i is a finite subset of $\mathcal{B}$. Given such a sequence we let $R_{\vec{A}}$ be the graph on X given by

$$x \, R_{\vec{A}} \, x'$$

if there is some $i \leq n$, $V, V' \in A_i$, with $x \in V, x' \in V'$ and

$$g_i \cdot x = x'.$$

We let $E_{\vec{A}}$ be the equivalence relation arising from the connected components of $R_{\vec{A}}$.

We then let Y_n^* be the subset of Y_n for which the induced equivalence relation $E_{\vec{A}}$ has all its equivalence classes finite. Given $\vec{A} = (A_0, \ldots, A_n) \in Y_n^*$, $\vec{B} = (B_0, \ldots, B_m) \in Y_n^*$ where $m \geq n$, we say that $\vec{B}$ *extends* $\vec{A}$ if every at every $i \leq n$

$$A_i \subseteq B_i.$$

Claim: Given any x, $y = g_i \cdot x$, $n \geq i$ and

$$\vec{A} \in Y_n^*$$

we can find an extension $\vec{B} \in Y_n^*$ with

$$x \, R_{\vec{B}} \, y.$$

Proof of Claim: The point is that we can add to A_i a small enough open set W around x that will specify x with sufficient exactness to ensure that for any $x' \in W$, the $E_{\vec{A}}$ equivalence class of x consists of $\{h_i \cdot x' : i \leq \ell\}$ for some finite sequence $h_0, \ldots, h_{\ell-1}$ of group elements. (This is where we use zero-dimensionality.) Then we can find $V \in \mathcal{B}$ that specifies x with sufficient precision to ensure that for any $x' \in V$ and $z \, E_{\vec{A}} \, x'$ we have either $z = x'$ or $z \notin V$. Similarly we can do the same for y with some V'. Then if we were to take the simple extension $\vec{B}$ which has $B_i = A_i \cup \{V, V'\}$ but $B_j = A_j$ at $j \neq i$, then $E_{\vec{A}}$ has index at most two in $E_{\vec{B}}$. (Claim ⊣)

We then let Y be the space of all infinite sequences

$$(\vec{A}^n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} Y_n^*$$

where each $\vec{A}^{n+1}$ extends A_n. Any such $(\vec{A}^n)_{n\in\mathbb{N}} \in Y$ gives rise to an increasing sequence of Borel equivalence relations with finite classes, taking $F_n^{(\vec{A}^n)_{n\in\mathbb{N}}}$ to be $E_{\vec{A}^n}$. $\prod_{n\in\mathbb{N}} Y_n^*$ comes with a natural product topology, under which it is Polish. Y is a closed subset of this product space, and hence Polish in its own right. The above claim shows that for all $x \in X$ there is a comeager collection of $(\vec{A}^n)_{n\in\mathbb{N}} \in Y$ with

$$[x]_E = \bigcup_{n\in\mathbb{N}} [x]_{F_n^{(\vec{A}^n)_{n\in\mathbb{N}}}}.$$

Thus by Kuratowski-Ulam (see [50]), there is a comeager collection of $(\vec{A}^n)_{n\in\mathbb{N}} \in Y$ for which there is a comeager collection of x with $[x]_E = \bigcup_{n\in\mathbb{N}} [x]_{F_n^{(\vec{A}^n)_{n\in\mathbb{N}}}}$. Taking any such $(\vec{A}^n)_{n\in\mathbb{N}} \in Y$ and corresponding comeager set, we are done. ⊣

On the other hand the subject of countable Borel equivalence relations considered up to Borel reducibility might collapse into a kind of death by heat dispersal if all these examples were hyperfinite. It turns that in the presence of an invariant probability measure, *non-amenable* groups give rise to equivalence relations which are not hyperfinite.

3.2 Definition. A countable group Γ is *amenable* if for any finite $F \subseteq \Gamma$ and $\epsilon > 0$ there is a finite, non-empty $A \subseteq \Gamma$ such that for all $\sigma \in F$

$$\frac{|A \Delta \sigma A|}{|A|} < \epsilon.$$

As a word on the notation, we use $|B|$ to denote the cardinality of the set B and $B \Delta C$ to denote the symmetric difference of the sets B and C. Thus amenability amounts to the existence of something like "almost invariant" subsets of the group—given any finite collection of group elements, any tolerance ($\epsilon > 0$), we can find a finite set which when translated by any of the group elements differs from its original position on a relatively small number of elements.

3.3 Examples.

1. $\mathbb{Z}$ is amenable. Given $F = \{k_1, \ldots, k_\ell\}$ in the group and $\epsilon > 0$, we let $k = \max(|k_1|, \ldots, |k_\ell|)$, the max of the absolute values, and then let

$$N > \frac{2(k+1)}{\epsilon}.$$

The set $A = \{-N, -N+1, -N+2, \ldots, 0, 1, \ldots, N-1, N\}$ is as required.

2. On the other hand, $\mathbb{F}_2 = \langle a, b \rangle$, the free group on two generators, is most famously non-amenable. Just take $\epsilon = 1/10$, $F = \{a, b, a^{-1}, b^{-1}\}$. We can see this by dividing the non-identity elements of $\mathbb{F}_2$ into four regions, $C_a, C_b, C_{a^{-1}}, C_{b^{-1}}$, where a (reduced) word is in the region C_u if it begins with u.

Consider some putative set A that is trying to witness amenability for $1/10$ and $\{a, b, a^{-1}, b^{-1}\}$. For $u = a, b, a^{-1}$, or b^{-1}, $n \in \mathbb{N}$, if at least n of A's elements are *not* in $C_{a^{-1}}, C_{b^{-1}}, C_a$, or C_b, respectively, then at least n of $u \cdot A$'s elements are in $C_a, C_b, C_{a^{-1}}$, or $C_{b^{-1}}$, respectively. Therefore we can clearly find two distinct u_1, u_2 with at least three fifths of $u_i \cdot A$'s elements in C_{u_i}. One of these sets must differ from A by at least $\frac{1}{10}|A|$.

Note that there is nothing special here about the use of *almost invariant* sets in $\mathbb{F}_2$. We would obtain a similar contradiction if we considered almost invariant *functions* in $\ell^1(\mathbb{F}_2)$. Given $f \in \ell^1(\mathbb{F}_2)$ and $u \in \{a, b, a^{-1}, b^{-1}\}$ we could look at the norm of the corresponding f_u defined by $f_u(\sigma) = f(\sigma)$ if $\sigma \in C_u$, $f_u(\sigma) = 0$ otherwise.

3.4 Definition. A *standard Borel probability space* is a standard Borel space X equipped with an atomless σ-additive probability measure μ on its Borel sets.

3.5 Theorem. *Let Γ be a countable non-amenable group. Suppose Γ acts freely and by measure preserving transformations on a standard Borel probability space (X, μ). Then E_Γ is not hyperfinite.*

Proof. We sketch the argument just in the case that $G = \mathbb{F}_2$.

For a contradiction suppose $E_{\mathbb{F}_2} = \bigcup_{n \in \mathbb{N}} F_n$, where each F_n is finite, Borel, and has $F_n \subseteq F_{n+1}$. At each $x \in X$ we define

$$f_{n,x} : \mathbb{F}_2 \to \mathbb{R}$$

with

$$f_{n,x}(\sigma) = \frac{1}{|[x]_{F_n}|}$$

if $\sigma^{-1} \cdot x F_n x$,

$$f_{n,x}(\sigma) = 0$$

otherwise. Let

$$f_n(\sigma) = \int_X f_{n,x}(\sigma)\, d\mu.$$

For any x we have $\|f_{n,x}\|_{\ell^1} = 1$, and so certainly $\|f_n\|_{\ell^1} = 1$, and in fact for any measurable set $A \subseteq X$

$$\sum_\sigma \int_A f_{n,x}(\sigma)\, d\mu \leq \mu(A).$$

Claim: For any $\gamma \in \mathbb{F}_2$, as $n \to \infty$

$$\|f_n - \gamma \cdot f_n\|_{\ell^1} \to 0.$$

Proof of Claim: Note that if $xF_n\gamma \cdot x$ then

$$\gamma \cdot f_{n,x} = f_{n,\gamma\cdot x},$$

since for any σ we have

$$\gamma \cdot f_{n,x}(\sigma) = f_{n,x}(\gamma^{-1}\sigma) = \frac{1}{|[x]_{F_n}|}\left(= \frac{1}{|[\gamma \cdot x]_{F_n}|}\right)$$

if and only if $\sigma^{-1}\gamma \cdot xF_n x$, which, by the assumption $xF_n\gamma \cdot x$, amounts to saying if and only if

$$f_{n,\gamma\cdot x}(\sigma) = \frac{1}{|[\gamma \cdot x]_{F_n}|}.$$

Thus if we let $A_{n,\gamma} = \{x \in X : xF_n\gamma \cdot x\}$, then

$$\int_{A_{n,\gamma}} \gamma \cdot f_{n,x}(\sigma)\, d\mu = \int_{\gamma\cdot A_{n,\gamma}} f_{n,x}(\sigma)\, d\mu,$$

and hence

$$\begin{aligned}\|f_n - \gamma \cdot f_n\|_{\ell^1} &\leq 2\sum_\sigma \int_{X\setminus A_{n,\gamma}} f_{n,x}(\sigma) = 2\int_{X\setminus A_{n,\gamma}} \sum_\sigma f_{n,x}(\sigma) \\ &= 2\mu(X \setminus A_{n,\gamma}).\end{aligned}$$

Since $\mu(A_{n,\gamma}) \to 1$ as $n \to \infty$, we are done. (Claim $\dashv$)

Now we have established the existence of almost invariant functions in $\ell^1(\mathbb{F}_2)$, and this can be refuted by the same kind of argument as used to show the non-amenability of $\mathbb{F}_2$. $\dashv$

For the longest while there was only a small finite number of countable Borel equivalence relations known to be distinct in the $\leq_B$ ordering. It was a notorious open problem to establish even the existence of $\leq_B$-incomparable examples. This was finally settled by:

3.6 Theorem (Adams-Kechris [2]). *There exists an assignment of countable Borel equivalence relations to Borel subsets of* $\mathbb{R}$*,*

$$B \mapsto E_B,$$

such that $E_B \leq_B E_C$ *if and only if* $B \subseteq C$*.*

Formally their result relied on the superrigidity theory of Zimmer [70] for lattices in higher rank Lie groups, and thus in turn had connections with earlier work of Margulis and Mostow. I will say nothing about Zimmer's work as such, but instead try to describe some of the engine which drives the theory.

3.7 Definition. For a compact metric space K we let $M(K)$ denote the probability measures on K. By the Riesz representation theorem this can be identified with a closed subset of the dual of $C(K)$, and thus is a Polish space in its own right. Note that the homeomorphism group of K acts on $M(K)$ in a natural way:

$$(\psi \cdot \mu)(f) = \mu(\psi^{-1} \cdot f),$$

where $\psi^{-1} \cdot f$ is defined by $(\psi^{-1} \cdot f)(x) = f(\psi(x))$.

3.8 Definition. Given a group Γ acting on a space X and another group H we say that

$$\alpha : \Gamma \times X \to H$$

is a *cocycle* if for all $\gamma_1, \gamma_2 \in \Gamma, x \in X$

$$\alpha(\gamma_2, \gamma_1 \cdot x)\alpha(\gamma_1, x) = \alpha(\gamma_1\gamma_2, x).$$

Here is a typical situation in which a cocycle arises. Given Γ a group of Borel automorphisms of standard Borel space X, H a countable group acting freely and in a Borel manner on a standard Borel space Y, if

$$\theta : X \to Y$$

witnesses $E_\Gamma^X \leq_B E_H^Y$, then we obtain a Borel cocycle by letting $\alpha(\gamma, x)$ be unique $h \in H$ with

$$h \cdot \theta(x) = \theta(\gamma \cdot x).$$

3.9 Lemma (Furstenberg [21], Zimmer [70]). *If* Δ *is a countable amenable group acting in a Borel manner on a standard Borel probability space* (X, μ) *and*

$$\alpha : \Delta \times X \to \mathrm{Hom}(K)$$

is a cocycle into the homeomorphism group of a compact metric space K*, then we can find a measurable assignment of measures*

$$x \mapsto \nu_x,$$

$$X \to M(K),$$

which is almost everywhere equivariant, that is to say

$$\forall^\mu x \forall \gamma (\alpha(\gamma, x) \cdot \nu_x) = \nu_{\gamma \cdot x}.$$

The applications of this lemma and its forerunners are much too involved to be discussed here. Some understanding of what is going on can be given by the following simple lemma. In fact, the hypotheses of the lemma can never be realized, and indeed the real theorem is that equivalence relations of the form $E^X_{\Gamma\times\mathbb{Z}}$ are never treeable, but I simply want to give a short illustration of some key ideas.

3.10 Lemma. *Suppose* Γ *is a non-amenable countable group. Let* $X = \{0,1\}^{\Gamma\times\mathbb{Z}}$ *and let* μ *be the product measure on this space. Let* $\Gamma \times \mathbb{Z}$ *act in the natural way on this space:*

$$\sigma \cdot f(\tau) = f(\sigma^{-1}\tau)$$

for any $f \in X$, $\sigma, \tau \in \Gamma \times \mathbb{Z}$.

Suppose $\mathbb{F}_2$ *acts freely and by Borel automorphisms on standard Borel probability space* Y. *Suppose*

$$\theta : X \to Y$$

witnesses $E^X_{\Gamma\times\mathbb{Z}} \leq_B E^Y_{\mathbb{F}_2}$.

Then there is homomorphism $\rho : \Gamma \to \mathbb{F}_2$ *and an alternative reduction*

$$\hat{\theta} : X \to Y,$$

which is equivalent in the sense that

$$\hat{\theta}(x) \, E^Y_{\mathbb{F}_2} \, \theta(x)$$

for all $x \in X$ *and whose resulting cocycle accords with* ρ *almost everywhere, in the sense that*

$$\forall^\mu x \forall \gamma (\rho(\gamma) \cdot \hat{\theta}(x)) = \hat{\theta}(\gamma \cdot x).$$

Sketch of Proof. Let

$$\alpha : (\Gamma \times \mathbb{Z}) \times X \to \mathbb{F}_2$$

be the induced cocycle.

Let $\partial(\mathbb{F}_2)$ denote the *infinite* reduced words from $\{a, b, a^{-1}, b^{-1}\}$. This is a compact metric space on which $\mathbb{F}_2$ acts by homeomorphisms. (See [44, Appendix C].) Following Lemma 3.9 we can find a measurable assignment

$$X \to M(\partial(\mathbb{F}_2)),$$

$$x \mapsto \mu_x,$$

with

$$\alpha(e,\ell)\cdot\mu_x=\mu_{(e,\ell)\cdot x}$$

for all $\ell\in\mathbb{Z}$ and a.e. $x\in X$. (We will use e for the identity in Γ and 0 for the identity in $\mathbb{Z}$.)

Claim: We can choose this assignment so that the measures μ_x concentrate almost everywhere on more than two points.

Proof of Claim: (Sketch only; see [44, Appendix C] for a completely precise argument for a more general claim.) Suppose instead that every such $\mathbb{Z}$-equivariant assignment of measures concentrates on at most two points.

The key observation here is that if

$$x\mapsto\mu_x$$

is such an assignment of measures then for any $\gamma\in\Gamma$ so is

$$x\mapsto\alpha(\gamma,0)^{-1}\cdot\mu_{(\gamma,0)\cdot x}.$$

Thus to prevent a situation in which we could simply pile on more and more of these measures, passing from say $x\mapsto\mu_x$ to

$$x\mapsto\frac{\mu_x+\alpha(\gamma,0)^{-1}\cdot\mu_{(\gamma,0)\cdot x}}{2}$$

we must be able to obtain an assignment which is actually Γ-equivariant.

Using certain strong ergodicity properties of the shift action of Γ on X (see [44, Appendix A]) and the hyperfiniteness of the action of $\mathbb{F}_2$ on $\partial(\mathbb{F}_2)$ (see [44, Appendix C]) we obtain that there is a single measure ν_0 such that for almost all x we have some $\sigma_x\in\mathbb{F}_2$ with

$$\sigma_x\cdot\mu_x=\nu_0.$$

Thus replacing θ with the reduction

$$\hat\theta:x\mapsto\sigma_x\cdot\theta(x)$$

we obtain a reduction of $E^X_{\Gamma\times\mathbb{Z}}$ to E^X_H where H is the subgroup of $\mathbb{F}_2$ corresponding which stabilizes the measure ν_0. This subgroup will be amenable (see [44, Appendix C]), which in turn provides a contradiction to the non-amenability of $\Gamma\times\mathbb{Z}$ (see [44, Appendix A] again). (Claim $\dashv$)

So now we obtain an assignment of measures that concentrates on at least three points almost everywhere. Here an observation of Russ Lyons (the rough idea is that a measure on $\partial(\mathbb{F}_2)$ concentrating on more than three points can be assigned a *center* in an $\mathbb{F}_2$-equivariant manner—again, see [44, Appendix C]) gives us a measurable map

$$\eta:X\to\mathbb{F}_2$$

such that
$$\eta((e,\ell)\cdot x) = \alpha((e,\ell),x)\cdot\eta(x)$$
almost everywhere. Replacing $\theta : X \to Y$ by
$$\hat{\theta} : X \to Y,$$
$$x \mapsto \eta(x)^{-1}\cdot\theta(x)$$
we obtain a reduction with an induced cocycle
$$\hat{\alpha} : (\Gamma\times\mathbb{Z})\times X \to \mathbb{F}_2$$
with $\hat{\alpha}((e,\ell),x) = e$ almost everywhere.

Then, as at the proof of [44, 2.2], the ergodicity of the action of $\mathbb{Z}$ on X gives that for any $\gamma\in\Gamma$,
$$x \mapsto \hat{\alpha}((\gamma,0),x)$$
is a.e. invariant. From this it is easily seen that we obtain the required homomorphism into $\mathbb{F}_2$. ⊣

Arguments of this form can be found very clearly in papers by Scott Adams such as [1], though in truth the ideas trace back to Margulis and Mostow by way of Zimmer [70].

Hjorth and Kechris [44] give a self-contained proof of the existence of many $\le_B$-incomparable countable Borel equivalence relations using arguments along these lines, but something similar is implicit in the superrigidity results of [70] to which Adams and Kechris appeal in the course of proving Theorem 3.6. In that case one is dealing not with the compact space $\partial(\mathbb{F}_2)$ but certain compact quotients of an algebraic group (see for instance [70, p. 88]) or the measures on projective space over a locally compact field (see [70, 3.2.1]). The appearance of product group actions is more subtle, but present; superrigidity typically on works for groups of matrices of rank greater than two, when we can hope to find a subgroup which indeed has the form $\Gamma\times\mathbb{Z}$ for Γ non-amenable.

In passing it should also be mentioned that there are applications of Furstenberg's lemma to the theory of the homeomorphism group of the circle. In [27] the combinatorial properties of $M(S^1)$ play a role in understanding what kinds of homomorphisms are possible into $\mathrm{Homeo}(S^1)$.

3.2. Treeable Equivalence Relations

The situation with the countable Borel equivalence relations can in some sense be viewed as resolved. It is a giant mess, with many incomparable examples, but we know it to be the ghastly mess that it is. The situation with treeable countable Borel equivalence relations is far different.

It is well-known that not all treeable equivalence relations are hyperfinite. If one take the free part of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$ then the resulting equivalence relation, $E_{\mathcal{T}\infty}$, is treeable—we define a Borel treeing by $x\,\mathcal{T}\,x'$ if there is a generator of $\mathbb{F}_2$ which moves x to x'. The equivalence relation is not hyperfinite, as shown for instance in [44], since the product measure concentrates on the free part and is $\mathbb{F}_2$-invariant.

It is also known that there is a maximal countable treeable equivalence relation. [48] shows that for any treeable countable Borel equivalence relation E one has $E \leq_B E_{\mathcal{T}\infty}$.

After that precious little is known. Reference [40] gives the existence of a treeable equivalence relation E with $E_0 <_B E <_B E_{\mathcal{T}\infty}$, but at the time of writing it is still open whether there are exactly two non-hyperfinite treeable countable Borel equivalence relations up to Borel reducibility.

3.3. Hyperfiniteness

One of the enduring problems in this field is to determine which countable Borel equivalence relations are hyperfinite. Nowadays this is almost always asked in the Borel context, since the measure theoretic setting is completely understood.

3.11 Theorem (Connes-Feldman-Weiss [8]). *Let G be a countable amenable group acting by measure preserving transformations on a standard Borel probability space (X, μ). Then there is a conull set on which the orbit equivalence relation is hyperfinite.*

Equivalently, we can find a *measurable* reduction of E_G to E_0.

Even this result for very simple groups remains excruciatingly difficult in the Borel setting.

3.12 Theorem (Jackson-Kechris-Louveau [48]). *Let G be a finitely generated group with a nilpotent subgroup H with $[G : H]$, the index of H in G, finite. If G acts by Borel automorphisms on a standard Borel space X, then the resulting equivalence relation E_G is hyperfinite.*

There was a considerable pause until quite recently it was shown:

3.13 Theorem (Gao-Jackson [25]). *Let G be a countable abelian group. If G acts by Borel automorphisms on a standard Borel space X, then the resulting equivalence relation E_G is hyperfinite.*

To give an idea of how difficult these problems have proved, it was not until Theorem 3.13, despite considerable efforts, we even knew that the commensurability equivalence relation on $\mathbb{R} \setminus \{0\}$,

$$r \, E_c \, s \quad \Longleftrightarrow \quad r/s \in \mathbb{Q}$$

was hyperfinite.

It would be natural to conjecture all E_G's arising from the Borel action of an countable amenable group on a standard Borel space are hyperfinite. This conjecture is presently far out of reach, and actually has little in the way of supporting evidence.

4. Effective Cardinality

One way in which to look at the theory of Borel reducibility is as a kind of theory of *Borel cardinality*, and in this sense there are definitely roots in papers by Harvey Friedman such as [17]. If we were to truly embrace a mathematical ontology consisting solely of Borel objects, then we would also be naturally led to consider certain kinds of quotients arising from Borel equivalence relations and to make any comparison along the lines of *cardinality* it seems we would use something like Borel reducibility.

In this sense I am more inclined to consider the theory of $\leq_B$ as something like a theory of cardinality, as opposed to a theory of reducibility of information, as one finds in the theory of Turing reducibility. In fact there are close parallels between the structure of the $\leq_B$-ordering on Borel equivalence relations and the cardinality theory of $L(\mathbb{R})$ under suitable determinacy assumptions.

4.1 Definition. For $A, B \in L(\mathbb{R})$ write

$$|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})},$$

the $L(\mathbb{R})$ *cardinality of A does not exceed that of B*, if there is an injection

$$i : A \hookrightarrow B$$

in $L(\mathbb{R})$.

4.2 Lemma (Folklore, but see [32]). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let E, F be Borel equivalence relations on $\mathbb{R}$. If*

$$|\mathbb{R}/E|_{L(\mathbb{R})} \leq |\mathbb{R}/F|_{L(\mathbb{R})}$$

then there is a function

$$f : \mathbb{R} \to \mathbb{R}$$

in $L(\mathbb{R})$ with for all $x_1, x_2 \in \mathbb{R}$

$$x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2).$$

Then in parallel to Silver's theorem Woodin has shown:

4.3 Theorem (Woodin). *Let $A \in L(\mathbb{R})$. Then exactly one of the following holds:*

(I) There is an ordinal α with $|A|_{L(\mathbb{R})} \leq |\alpha|_{L(\mathbb{R})}$—in other words, A is well-orderable; or

(II) $|\mathbb{R}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

And for Harrington-Kechris-Louveau:

4.4 Theorem (Hjorth [33]). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* $A \in L(\mathbb{R})$. *Then exactly one of the following holds:*

(I) There is an ordinal α *with* $|A|_{L(\mathbb{R})} \leq |2^\alpha|_{L(\mathbb{R})}$*—in other words,* A *has a well-orderable separating family; or*

(II) $|\mathbb{R}/E_v|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

In (II) we can equivalently say that $\mathcal{P}(\omega)/\mathrm{Fin}$ embeds into A.

The theory of effective cardinality, whether we choose to explicate it using Borel functions and objects or the more joyously playful world of $L(\mathbb{R})$, can also be compared with the idea of *classification difficulty*. If the effective cardinality of A is below that of B, then any objects which succeed as complete invariants for B do as well for A. Here one could even begin certain kinds of wild speculations, to the effect that subconsciously part of the mathematical activity of vaguely searching for some kind of ill-defined *classification theorem* for a class of objects is in fact a query as to its effective cardinality.

Even if circumspection draws us back from grand fantasies along these lines, calculations of Borel cardinality undoubtedly say *something* about classification difficulty. A non-reduction result, saying E not Borel reducible to F, will inform as to which kinds of objects would be *insufficient*, in the Borel category at least, to act as complete invariants for E.

Finally, in the context of $L(\mathbb{R})$ there is a curious refinement of the usual Borel hierarchy theorem.

4.5 Theorem (Hjorth [34]). *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then for every* $\alpha < \beta < \omega_1$

$$|\utilde{\Pi}^0_\alpha|_{L(\mathbb{R})} < |\utilde{\Pi}^0_\beta|_{L(\mathbb{R})}.$$

Thus, not only is $\utilde{\Pi}^0_\alpha$ strictly included in $\utilde{\Pi}^0_\beta$, it is also smaller in effective cardinality. Sharper results are possible here. Andretta et al. [3] determines the exact levels of the Wadge degrees which provide new cardinals in $L(\mathbb{R})$.

5. Classification Problems

Until this point our discussion has largely been concerned with the *structural* properties of the $\leq_B$ ordering. However in the sum total of papers on the subject, the majority deal not with the abstract issues of this partial order, but instead with the specific problems of placing certain naturally occurring equivalence relations in this hierarchy.

Here one pictures specific levels of classification difficulty, and for an example "from the wild" we ask whether it is *smooth*, or *classifiable by countable structures*, or *universal for countable Borel equivalence relations*, and so on.

5.1. Smooth versus Non-Smooth

At the very base, we have the distinction between smooth and non-smooth. If $E \leq_B \mathrm{id}(\mathbb{R})$ then we can classify the E-classes by points in a concrete space.

The very first writings dealing at all with attempting to understand the classification difficulty of mathematical problems as a kind of science in and of itself are due to George Mackey, for instance in [57]. Very specifically he was concerned with understanding which groups have *smooth duals*. Here given $U(\mathcal{H})$, the unitary group of a separable Hilbert space, and irreducible representations

$$\sigma_1, \sigma_2 : \Gamma \to U(\mathcal{H}),$$

we set $\sigma_1 \sim \sigma_2$ if there is some unitary $T \in U(\mathcal{H})$ with

$$T^{-1} \circ \sigma_1(\gamma) \circ T = \sigma_2(\gamma)$$

all $\gamma \in \Gamma$. For Γ countable, the space of unitary representations is a Polish space, since it is a closed subset of

$$U(\mathcal{H})^{\Gamma},$$

and then the irreducible representations form a G_δ subset of those, and hence again Polish in the subspace topology (see for instance [10]). We say that the group Γ has *smooth dual* if the equivalence relation $\sim$ on the irreducibles is smooth. Ultimately it was determined in [66] that the countable groups with smooth duals are exactly the abelian by finite.

Another example is the equivalence relation of matrices over $\mathbb{C}$ considered up to similarity. As remarked in [29], this equivalence relation is smooth, since we can assign to a matrix its canonical Jordan form as a complete invariant.

Some authors following on from this have drawn out from Mackey's writings the entirely general view that in all branches of mathematics the dividing line between classifiable and non-classifiable is given by the smooth versus non-smooth distinction. Indeed the author of [10] appears to flirt with such a opinion. For a rather different take one might look at the introduction of [36]; indeed this work cites many apparent classification theorems—Baer's analysis of rank one torsion free abelian groups, the von Neumann-Halmos analysis of discrete spectrum transformations, the spectral theorem for infinite dimensional unitary operators—for equivalence relations which are non-smooth.

5.2. Universal for Polish Group Actions

It is a well-known result—probably first due to Leo Harrington, but for a proof see [46]—that there is no *universal* Borel equivalence relation: For every Borel equivalence relation E there is another Borel equivalence relation F which does not Borel reduce to E. On the other hand there is a universal

Σ^1_1 equivalence relation, which we might in some sense think of as sitting on the throne above all our examples.

As a practical matter almost all the equivalence relations which seem to have independent interest arise as, or are reducible to, orbit equivalence relations induced by the continuous action of a Polish group on a Polish space. *Among these* there is an uppermost example: $E^{X_\infty}_{G_\infty}$, induced by the continuous action of a Polish group G_∞ on a Polish space X_∞. Although we are stepping slightly outside the subject of Borel equivalence relations as such, $E^{X_\infty}_{G_\infty}$ can be viewed as a kind of extreme, at the opposite end to $\mathrm{id}(\mathbb{R})$, of an equivalence relation with maximal complexity.

In unpublished work Kechris and Solecki showed that in the natural Borel structure compact metric spaces considered up to homeomorphism are $\sim_B$ with, bi-Borel reducible with, $E^{X_\infty}_{G_\infty}$. A recent and published example is given by [26], where they show that complete separable metric spaces considered up to isometry is $\sim_B$ to $E^{X_\infty}_{G_\infty}$, as are closed subsets of the Urysohn space under the orbit equivalence relation induced by the isometry group of this space.

5.3. Universal for S_∞

As shown in [6], for each Polish group G there is a corresponding Polish G-space X with E^X_G universal for orbit equivalence relations of G, moreover in the case of $G = S_\infty$ we can take X to be $\mathrm{Mod}(\mathcal{L})$ for any $\mathcal{L}$ which contains at least one relation of arity two or higher. For future reference let us fix some such universal space for S_∞ orbit equivalence relations and denote the corresponding equivalence relation by $E_{\infty S_\infty}$. Again this is Σ^1_1 but not Borel.

The study of which equivalence relations lie at the level of $E_{\infty S_\infty}$ go right back to the first work of logicians on the $\leq_B$ ordering.[1] One finds $\leq_B$ defined independently, quite by accident with the exact same notation and terminology, in [29] and [18]. The first of these papers deals with the structural result of Theorem 2.2, and the second almost entirely with specific issues of which classes of isomorphism are $\sim_B$ with $E_{\infty S_\infty}$. More generally, given any E which admits classification by countable structures, we automatically have $E \leq_B E_{\infty S_\infty}$ and we can go on to ask when in fact the reverse $\geq_B$ holds as well.

Friedman and Stanley [18] demonstrate, among other examples, that isomorphism of countable groups, countable fields, and countable linear orderings, are all $\sim_B$ $E_{\infty S_\infty}$. (In general, given any $\psi \in \mathcal{L}_{\omega_1,\omega}$, the set of $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ satisfying ψ is Borel, and hence forms a standard Borel space in its own right.) More recently [24] show countable boolean algebras up to isomorphism are universal in this class.

[1] I choose these words warily. Although the notion of $\leq_B$ does not appear explicitly in the writings of Mackey, and was first formally isolated in the late 1980s, in some form the idea is already implicit, both in Mackey and in other authors such as [14].

In general most of the questions in this area which can be reasonably posed have been answered. However one wound remains with us from [18]:

Question (Friedman-Stanley). Is isomorphism on countable torsion free abelian groups $\sim_B$ to $E_{\infty S_\infty}$?

This question remains open despite several efforts to give it closure.

5.4. E_∞

If E is a countable equivalence relation then Theorem 1.10 shows E is induced by the Borel action of a countable group, G. Any countable group is realizable as a closed subgroup of S_∞, and so by [6] we have that E admits classification by countable structures.

Thus the countable Borel equivalence relations form a subclass of the equivalence relations admitting classification by countable structures, with a top most example E_∞. The subclass is proper, since $\mathcal{T}_2$ from our earlier examples, corresponding to equality on countable sets of reals, is $<_B$ above E_∞. In fact, the lesson we take away from say [46] is that the countable Borel equivalence relations are only a tiny part of the totality of equivalence relations admitting classification by countable structures.

Nevertheless many important examples lie in the class of countable Borel equivalence relations, and on the whole this has proved to be one of the hardest and most exciting parts of the discipline. In general terms a class of countable structures tends to be $\leq_B E_\infty$, that is to say, *essentially countable*, if there is some notion of finite rank.

Simon Thomas and Velickovic, [69], in answer to questions raised in [42], show that isomorphism on finitely generated groups and fields of finite transcendence degree are $\sim_B E_\infty$.

5.5. E_0

If the diagram has a well-described bottom, $\mathrm{id}(\mathbb{R})$, then it equally has a well-defined second rung, that of E_0. Implicitly in the classical literature on this subject there is a classification theorem in terms of hyperfiniteness.

5.1 Theorem (Baer, in effect; see [19]). *Isomorphism on rank one torsion free abelian groups is hyperfinite.*

In fact, Baer's original result dates back to the 1920s and is not stated in this form. Rather he explicitly associates to each rank one torsion free group a kind of *type*, consisting of a description of orders considered up to finite difference, and goes on to show that the type is a complete invariant. This has generally been considered a satisfactory classification of the rank one torsion free groups, and Fuchs in [19] asks the vague but earnest question whether the rank two torsion free abelian groups can be "classified".

5.2 Definition. For each n let S_n be the subgroups of $(\mathbb{Q}^n, +)$. Let $\cong_n$ be the isomorphism relation on these groups.

Every rank n torsion free abelian group appears as a subgroup of $\mathbb{Q}^n$, and in turn these subgroups all have rank $\leq n$. Moreover two subgroups of $\mathbb{Q}^n$ are isomorphic if and only if there is a linear transformation over the rationals which sends one to the other, and thus the equivalence relation $\cong_n$ has countable classes—and thus $\leq_B E_\infty$.

The authors of [42] observe that Baer had implicitly shown $\cong_1 \leq_B E_0$ and went on to suggest as a possible explication of Fuchs' question whether $\cong_2 \leq_B E_0$. With absolutely no evidence, and armed only with the arrogance of ignorance, [42] conjectured that $\cong_2 \sim_B E_\infty$.

This conjecture was refuted by Thomas in a spectacular sequence of papers. There are several papers around the subject of isomorphism on finite rank torsion free abelian groups, most of which are surveyed in [67].

5.3 Theorem (Thomas [68]). *At every n*

$$\cong_n <_B \cong_{n+1} .$$

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5. Proper Forcing

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Contents

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1. Introduction

It is sometimes the case that new concepts not only widen our horizons, but also bring difficult old results into main stream and within common knowledge. Properness, introduced and developed by Saharon Shelah, is such a concept and the wealth of results, both old and new, that it provides justifies its early introduction into advanced set theory courses. My aim is to provide an introductory exposition of the theory of proper forcing which will also give some of its interesting applications, to the point where the reader can continue with research papers and with the more advanced material in Shelah's book [15]. I assume that the reader has some knowledge of axiomatic set theory and is familiar with the basics of the forcing method, including some iterated forcing (the consistency proof of Martin's Axiom is sufficient).

We deal here with countable support iteration. This type of iteration appeared in Jensen's consistency proof of the Continuum Hypothesis with the Souslin Hypothesis, and it also appeared in Laver's work on Borel's Conjecture (see [3] and [10]). (These two outstanding results will not be treated here. They have now simpler proofs in which the theory of proper forcing is used to concentrate on the single step of the iteration.)

The chapter contains four parts. First: preservation of properness in countable support iteration. Second: preservation of the ${}^{\omega}\omega$-bounding property, and an application concerning non-isomorphism of ultrapowers of elementarily equivalent structures. Third: preservation of unboundedness, and an application concerning two cardinal invariants, the bounding number and the splitting number. Fourth: Dee-completeness, forcings that add no countable sets of ordinals. These results are all due to Shelah, and most of them appear in Chapters V and VI of his proper forcing book [15], but are presented here at a more concrete level, and sometimes with simpler proofs. (Theorem 5.8, concerning the chromatic number of Hajnal-Máté graphs, is due to the author.)

Our notations follow a standard usage, and the introduction describes them and reviews some elementary facts about forcing. Note that $a \leq b$ in a forcing poset denotes here that b carries more information than a. We say then that b extends a.

A *preorder* is a transitive and reflexive relation on a set (its domain). If $\leq$ is a preorder on A and $a \leq b$ then we say that b *extends* a. We say that $a, b \in A$ are *compatible* if there exists some $x \in A$ that extends a and b. Otherwise, a and b are *incompatible*, and a set of pairwise incompatible elements is called an *antichain*. A set $D \subseteq A$ is *dense* iff

$$\forall a \in A \exists d \in D \ (a \leq d),$$

and D is predense iff

$$\forall a \in A \exists d \in D \ (a \text{ and } d \text{ are compatible}).$$

A preorder is *separative* iff whenever b does not extend a there is an extension of b that is incompatible with a. We say that $P = (A, \leq)$ is a *forcing poset* iff $\leq$ is a separative preorder on A with a minimal element 0_P. We often write $\leq_P$ for the preordering of P. In the context of iteration, preorderings are more convenient than (antisymmetric) orderings, because this is what one naturally obtains. (The reader who prefers orderings can remain with posets of course, but at the price of taking quotients with their notational burden.)

Let $P = (A, \leq)$ be a forcing poset. We say that $G \subseteq A$ is a *filter* iff G is downwards closed: $\forall x \leq y\ (y \in G \longrightarrow x \in G)$, and any two members of G are compatible in G. We say that G is (V, P)*-generic* iff G is a filter over P that meets (has a non-empty intersection with) every dense set of P that lies in V. It is convenient to employ (V, P)-generic filters and to be able to speak about actual generic extensions $V[G]$. I assume that the reader knows how to avoid such ontological commitments.

V^P is the class of all P forcing names in V. If $a \in V^P$ and G is (V, P)-generic, then $a[G]$ (or a^G) denotes the interpretation of a in $V[G]$. It is convenient to define the interpretation of names in such a way that every set in V can be interpreted as a name: $a[G] = \{y[G] \mid \exists p \in G(\langle p, y\rangle \in a)\}$. Usually for a set $a \in V$, $\check{a}$ denotes the (canonical) name of a. However, often a rather than $\check{a}$ is written here in forcing formulas, for graphical clarity and since this is very rarely a source of confusion. If φ is a forcing formula then "φ holds in V^P" means that $0_P \Vdash_P \varphi$. Often $\Vdash_P \varphi$ is written instead of $0_P \Vdash_P \varphi$. Also, I seldom put quotation marks around forcing formulas. The canonical name of the (V, P)-generic filter is denoted $\underset{\sim}{G}$.

The reason for employing separative posets is the following characterization: P is separative iff for any $p, q \in P$, $p \Vdash_P q \in \underset{\sim}{G}$ implies $q \leq p$. Notice that $p, q \in P$ is written rather than the more accurate $p, q \in A$. A poset may be used as a name for its own universe.

The relation $P \lhd Q$ on forcing posets P and Q means that there is a function (projection) $\pi : Q \to P$ such that

1. π is order-preserving (that is, $q_1 \leq_Q q_2$ implies $\pi(q_1) \leq_P \pi(q_2)$), π is onto P, and $\pi(0_Q) = 0_P$.

2. For every $q \in Q$ and $p' \in P$ such that $p' \geq \pi(q)$ there is a $q' \geq q$ in Q such that $\pi(q') = p'$.

If $P \lhd Q$ and $D \subseteq P$ is dense, then $\pi^{-1}(D)$ is dense in Q. Hence, if H is a (V, Q)-generic filter, then its image π“H generates a (V, P)-generic filter.

We say that π is a *trivial* projection iff $\pi(q_1) = \pi(q_2)$ implies that q_1, q_2 are compatible in Q. It can be seen that π is trivial iff for any (V, P)-generic filter G, $\pi^{-1}(G)$ is a (V, Q)-generic filter.

In many applications the projection of $P \lhd Q$ satisfies a stronger property than 2: for any $p \in P$ and $q \in Q$, if $p \geq \pi(q)$ then q has an extension $q_1 \in Q$, denoted $p + q$, such that

1. $\pi(q_1) = p$, and

2. if $r \in Q$ is such that $r \geq q$ and $\pi(r) \geq p$, then $r \geq q_1$.

If this additional property holds then every $p \in P$ can be identified with $i(p) = p + 0_Q$. (That is, P can be assumed to be a subposet of Q.)

If $Q \in V^P$ is a forcing poset (that is, by our convention, forced by the zero condition to be a forcing poset), then $P * Q$, the two-step iteration, is the forcing poset defined as follows. First, some sufficiently large set V_α is chosen so that if $b \in V^P$ is any name then there is already a name $a \in V_\alpha$ such that for any $p \in P$, if $p \Vdash_P b \in Q$, then $p \Vdash_P b = a$. (V_α here is the set of all sets of rank $< \alpha$.) Now form $P * Q$ as the set of all pairs (p, q) such that $p \in P$, $q \in V_\alpha \cap V^P$ and $p \Vdash_P q \in Q$. The preordering $\leq = \leq_{P*Q}$ is defined by

$$(p, q) \leq (p', q') \quad \text{iff} \quad p \leq_P p' \quad \text{and} \quad p' \Vdash_P q \leq_Q q'.$$

The map $\pi(p, q) = p$ is a projection. Usually one does not bother to define the set V_α from which the names q are taken and $P * Q$ is presented as a class.

I will summarize (without proofs) some basic facts about two-step iterations. The two notions, projection and two-step iteration, are closely related. If $P \lhd R$ with projection $\pi : R \to P$, and if G_0 is a (V, P)-generic filter, then form in $V[G_0]$ the following set

$$Q = R/G_0 = \{r \in R \mid \pi(r) \in G_0\}$$

and define a partial, separative preorder $\leq = \leq_{R/G_0}$ on Q by

$$\begin{aligned} r_1 \leq r_2 \quad \text{iff} \quad & \text{every } \leq_R \text{ extension of } r_2 \text{ in } R/G_0 \text{ is} \\ & \leq_R\text{-compatible with } r_1 \text{ in } R/G_0. \end{aligned} \tag{5.1}$$

Equivalently,

$$r_1 \leq r_2 \quad \text{iff} \quad r_1 \leq_R g + r_2 \text{ for some } g \in G_0 \text{ with } \pi(r_2) \leq_P g. \tag{5.2}$$

It follows that $r_1 \leq r_2$ iff for any $g_1, g_2 \in G_0$, $g_1 + r_1 \leq g_2 + r_2$. Sometimes, we write $R/\underset{\sim}{G}_0$, or even R/P, for the V^P name of $Q = R/G_0$.

1.1 Lemma. *Assume $P \lhd R$ as above and G_0 is a (V, P)-generic filter. Suppose that in $V[G_0]$ we have a sequence $\langle r_i \mid i \in \omega\rangle$ such that $r_i \in R/G_0$ and $r_i \leq_{R/G_0} r_{i+1}$. Then there is a sequence $\langle s_i \mid i \in \omega\rangle$ such that each s_i has the form $s_i = g_i + r_i$ for some $g_i \in G_0$ and $s_i \leq_R s_{i+1}$.*

Proof. Suppose that $g_i \in G_0$ is defined. Since $r_i \leq_{R/G_0} r_{i+1}$, there exists a $g \in G_0$ such that $r_i \leq_R g + r_{i+1}$. We may take $g \geq g_i$. Since $\pi(g + r_{i+1}) = g$, we get $g_i + r_i \leq_R g + r_{i+1}$. So that $g_{i+1} = g$ works. $\dashv$

Now let $\underset{\sim}{Q}$ be the name of Q in V^P and form the two-step iteration $P * \underset{\sim}{Q}$. Then R with its original ordering $\leq_R$ is isomorphic to a dense subset of $P * \underset{\sim}{Q}$. Namely, the map $r \mapsto (\pi(r), \check{r})$ is that embedding (where $\check{r}$ is the V^P name of r). Thus R itself can be viewed as a two-step iteration: the projection P followed by the quotient poset whose name is denoted by $R/\underset{\sim}{G}_0$.

Suppose now that some forcing poset $Q \in V^P$ is given and $R = P * Q$ is formed. Then $P \lhd R$, with projection $\pi(p, q) = p$. The stronger property of projections holds here, and we can identify $p \in P$ with $(p, \underset{\sim}{0}) \in P * Q$. Let G_0 be (V, P)-generic, and form R/G_0 as above. $R/G_0 = \{(p, q) \in P * Q \mid p \in G_0\}$ ordered as in (5.1). Then the following holds for $(p_1, q_1), (p_2, q_2) \in R/G_0$

$$(p_1, q_1) \leq_{R/G_0} (p_2, q_2) \quad \text{iff} \quad \exists p \in G_0\ ((p_1, q_1) \leq_{P*Q} (p, q_2)).$$

In $V[G_0]$, both R/G_0 and $Q[G_0]$ can be formed. ($Q[G_0]$ is the interpretation of Q in $V[G_0]$.) These are essentially the same poset. That is, the map

$$i : (p, q) \mapsto q[G_0]$$

taking $(p, q) \in R/G_0$ into the interpretation of q is a trivial projection.

Let P and R be any posets such that $P \lhd R$. We have said that if H is a (V, R)-generic filter, then $G_0 = \pi``H$ is (V, P)-generic, and clearly $H \subseteq R/G_0$. In fact, H is a $(V[G_0], R/G_0)$-generic filter.

On the other hand, if G_0 is (V, P)-generic, and if G_1 is $(V[G_0], R/G_0)$-generic, then $G_1 \subseteq R$ is a (V, R) generic filter and $\pi``G_1 = G_0$.

Now, given a poset P, suppose that $Q \in V^P$ is a poset and $R = P * Q$ is formed. Let G_0 be (V, P)-generic, and G_1 be $(V[G_0], Q[G_0])$-generic, where $Q[G_0]$ is the interpretation of Q in $V[G_0]$. Define in $V[G_0]$ the trivial projection $i : R/G_0 \to Q[G_0]$ as above, and define $H = i^{-1}(G_1)$ in $V[G_0][G_1]$. That is,

$$H = \{(p, q) \in P * Q \mid p \in G_0 \wedge q[G_0] \in G_1\}.$$

Then H is an $(R/G_0, V[G_0])$-generic filter (the pre-image of a trivial projection). Hence (by the preceding paragraph) H is also $(V, P * Q)$-generic, and $\pi``H = G_0$. Abusing the product notation we write $H = G_0 * G_1$.

Let $R = P * Q$. I want to explain the equation $V^R = (V^P)^Q$. We shall define a function

$$\rho : V^R \to V^P$$

from the R-names into the P-names such that the following holds for every (V, R)-generic filter H and $G_0 = \pi``H$ its projection.

1. As we have seen, $G_0 = \pi``H = \{p \in P \mid (p, q) \in R \text{ for some } q \in H\}$ is (V, P)-generic, and H is $(V[G_0], R/G_0)$-generic.

2. For every $a \in V^R$, $a[H] = (\rho(a)[G_0])[H]$. That is, the interpretation $a[H]$ of a in $V[H]$ can be obtained in two steps: first interpret $\rho(a)$ in $V[G_0]$, as a name, and then interpret this name in the remaining forcing over R/G_0.

Usually, we write a/G_0 instead of $\rho(a)[G_0]$ (without mentioning ρ) and then $a/\underset{\sim}{G}_0$ is written for $\rho(a)$.

To define ρ let pair : $V^P \times V^P \to V^P$ be a (definable) map such that for any $\tau_1, \tau_2 \in V^P$, $\text{pair}(\tau_1, \tau_2)[G_0] = \langle \tau_1[G_0], \tau_2[G_0] \rangle$. In plain words, $\text{pair}(\tau_1, \tau_2)$ is a canonical name for the pair formed from (the interpretations of) τ_1 and τ_2. Define now ρ by rank induction so that for any $a \in V^R$

$$\rho(a) = \{\langle \pi(r), \text{pair}(\check{r}, \rho(y)) \rangle \mid r \in R \text{ and } \langle r, y \rangle \in a\},$$

where $\check{r}$ is the canonical P-name for r. Suppose now that H is (V, R)-generic and G_0 is (V, P)-generic through the projection $G_0 = \pi``H$. For any $a \in V^R$

$$\begin{aligned} \rho(a)[G_0] &= \{\text{pair}(\check{r}, \rho(y))[G_0] \mid (r, y) \in a \wedge \pi(r) \in G_0\} \\ &= \{\langle r, \rho(y)[G_0] \rangle \mid (r, y) \in a \wedge r \in R/G_0\}. \end{aligned}$$

Now the required equality

$$\rho(a)[G_0][H] = a[H]$$

is established by the following equalities:

$$\begin{aligned} \rho(a)[G_0][H] &= \{\rho(y)[G_0][H] \mid \exists r \in H((r, y) \in a \wedge r \in R/G_0)\} \\ &= \{y[H] \mid \exists r \in H\ (r, y) \in a\} \\ &= a[H]. \end{aligned}$$

The function ρ can be defined as above whenever $P \lhd R$.

The following lemma is often used.

1.2 Lemma. *Assume that $P \lhd R$, and $D \subseteq R$ is dense. Let G be a (V, P)-generic filter. Then the following holds in $V[G]$: Every $q \in R$ with $\pi(q) \in G$ has an extension $d \in D$ with $\pi(d) \in G$. In other words, $D \cap R/G$ is dense in R/G.*

Proof. Suppose that $p \in P$ forces that $\pi(q) \in \underset{\sim}{G}$. Then $\pi(q) \leq_P p$, and hence there is an extension q_1 of q with $\pi(q_1) = p$. Now extend further q_1 to a condition $d \in D$, and then $\pi(d)$ forces $d \in D \cap R/G$ as required. ⊣

We will encounter the following situation in Sect. 3.

1.3 Lemma. *$Q_0 \lhd Q_1 \lhd Q_2$ are posets with projections $\pi_{i,j} : Q_i \to Q_j$ for $0 \leq j < i \leq 2$. The projections commute: $\pi_{2,0} = \pi_{1,0} \circ \pi_{2,1}$. Suppose G_0 is a (V, Q_0)-generic filter, and form $Q_1' = Q_1/G_0$ and $Q_2' = Q_2/G_0$. Then $\pi_{2,1} : Q_2' \to Q_1'$ is a projection and the quotient Q_2'/Q_1' can be seen to be exactly Q_2/Q_1.*

Let $\underset{\sim}{G}_{(Q_1/G_0)}$ be the canonical name in $V[G_0]$ of the Q_1/G_0 generic filter. Then

$$(Q_2/G_0)/\underset{\sim}{G}_{(Q_1/G_0)} \in V[G_0]^{Q_1/G_0}.$$

If G_1 is a $(V[G_0], Q_1/G_0)$-generic filter, then the interpretation of that name, $(Q_2/G_0)/G_1$, is equal to Q_2/G_1. In addition, G_1 is (V, Q_1)-generic and if $\underset{\sim}{f} \in V^{Q_2}$ then $(\underset{\sim}{f}/G_0) \in V[G_0]^{Q_1/G_0}$ and $(\underset{\sim}{f}/G_0)/G_1$ is $\underset{\sim}{f}/G_1$.

1.1. Countable Support Iterations

We deal here only with countable support iterations. An iteration of length γ (an ordinal) is defined by induction. For this, one needs a scheme to produce the next poset in the iteration. Suppose that this scheme is given by some function F (a formula that defines a function) such that for every forcing poset P, $F(P) = Q$ is in V^P a forcing poset. Then the iteration $\langle P_\alpha \mid \alpha \leq \gamma \rangle$ is defined so that:

1. Members of P_α are functions defined on α.
2. If $\alpha < \gamma$ then $P_\alpha = \{f \restriction \alpha \mid f \in P_\gamma\}$.
3. Every $f \in P_\gamma$ has a countable support, which is a set $S_f = S \subseteq \gamma$ such that f is trivial outside of S, that is, $f(\xi)$ is the P_ξ name of the zero condition for $\xi \in \gamma \setminus S$.

The definition of the iteration is as follows.

1. P_0 is the trivial poset consisting of the minimal condition $\emptyset$ alone, and V^{P_0} is (or is isomorphic to) V. If P_α is already defined and $\alpha < \gamma$, then $P_{\alpha+1}$ is defined as the set of all functions f defined on $\alpha + 1$ such that $f \restriction \alpha \in P_\alpha$, $f(\alpha) \in V^{P_\alpha}$ and
$$f \restriction \alpha \Vdash_{P_\alpha} f(\alpha) \in Q,$$
where $Q = F(P_\alpha)$. We define $f_1 \leq f_2$ iff $f_1 \restriction \alpha \leq f_2 \restriction \alpha$ and $f_2 \restriction \alpha \Vdash_{P_\alpha} f_1(\alpha) \leq f_2(\alpha)$. It is evident that $P_{\alpha+1}$ is defined to be isomorphic to $P_\alpha * Q$. (Since we want $P_{\alpha+1}$ to be a set, we must limit the possible values of $f(\alpha)$.)
2. If $\delta \leq \gamma$ is a limit ordinal and P_i is defined for every $i < \delta$, then P_δ is the set of all countably supported functions f defined on δ and such that for every $\alpha < \delta$, $f \restriction \alpha \in P_\alpha$. Thus for every $\alpha < \delta$, $f(\alpha) \in V^{P_\alpha}$ and f has a countable support. Define $f_1 \leq f_2$ iff for every $\alpha < \delta$, $f_1 \restriction \alpha \leq_{P_\alpha} f_2 \restriction \alpha$.

We assume that the reader knows the basic properties of these countable support iterations: first, that they form forcing posets, and then, that if $\gamma_0 < \gamma$ then $\pi : P_\gamma \to P_{\gamma_0}$ defined by $\pi(p) = p \restriction \gamma_0$ is a projection of P_γ onto P_{γ_0}. If $q \in P_{\gamma_0}$ and $q \geq \pi(p)$, then $p_1 = q + p$ is defined in P_γ by the requirement that

$$p_1(\xi) = q(\xi) \quad \text{for } \xi < \gamma_0 \quad \text{and} \quad p_1(\xi) = p(\xi) \quad \text{for } \xi \geq \gamma_0. \tag{5.3}$$

Strictly speaking P_{γ_0} is not a subset of P_γ, but in practice we identify $f \in P_{\gamma_0}$ with $f + 0_\gamma$ which is the trivial extension of f on γ. The ordering on P_γ is denoted $\leq_{P_\gamma}$ or just $\leq_\gamma$ for clarity. We shall often write $\Vdash_\gamma$ instead of $\Vdash_{P_\gamma}$.

The poset name $F(P_\alpha) = Q_\alpha$ is called "the αth iterand". In complex consistency proofs, the exact definition of F, and even its existence, is often passed over in silence. The term "bookkeeping device" is often invoked to refer to that part of the construction which is omitted.

2. Properness and Its Iteration

In this section we define *properness* and prove that the countable support iteration of proper forcing is proper. A slightly different proof can be read in [4].

In discussing proper forcing, the phrase "let λ be a sufficiently large cardinal, and H_λ the collection of sets of cardinality hereditarily $< \lambda$" appears so often that it deserves a remark. The role of H_λ is to encapsulate enough of the universe of sets V to reflect the statements in which we are interested. So the exact meaning of "sufficiently large" depends on the circumstances, and other reflecting sets such as V_λ can replace H_λ (but less naturally). When dealing with a forcing poset of cardinality κ, any cardinal $\lambda > 2^\kappa$ is sufficiently large for our purposes. We will be interested in countable elementary substructures of $\langle H_\lambda, \in, <, \text{etc.}\rangle$, where $<$ is some fixed well-ordering of H_λ, and etc. may include the poset P, the forcing relation, and other relevant parameters. The role of the well-ordering $<$ is to allow for inductive constructions. For notational clarity we just write $M \prec H_\lambda$, and omit the $\in$ relation, the well-order and the other parameters. We often say "$M \prec H_\lambda$ is as usual" to indicate that H_λ refers to a richer structure and the reader may have to include in M all those parameters that are relevant.

So let P be a poset, $M \prec H_\lambda$ be countable, with $P \in M$, as usual, and let G be (V, P)-generic. Define $M[G] = \{a[G] \mid a \in M\}$. So $M[G]$ is the set of all interpretations of names that lie in M. Since the forcing relation is (definable) in M (by virtue of the largeness of λ) the Forcing Theorem implies that $M[G] \prec H_\lambda[G]$ (for details, see [15, Theorem 2.11]). However $M[G]$ is not necessarily a generic extension of M. What does it mean that $M[G]$ is a generic extension of M? This is a delicate question because of the special status of the members of M: on one account they are just points with no other meaning than that provided by the structure M itself, and on the other hand, they are bona fide members of H_λ and carry information of which M is not aware. By collapsing M onto a transitive structure only the local properties remain and we are no longer confused by this double role. So let $\pi : M \to \bar{M}$ be the transitive collapsing of M, and let $\pi``G = \bar{G}$ be the image of G. Then genericity of G over M means that $\bar{G}$ is $\bar{M}$-generic over $\pi(P)$. That is, $\bar{G}$ has a non empty intersection with every dense subset of $\pi(P)$ that lies in $\bar{M}$. I find that collapsing is illuminating, but of course one can give a more direct definition: G is (M, P)*-generic* iff for any $D \subseteq P$ dense in P such that $D \in M$, $G \cap D \neq \emptyset$.

Properness of P, as we shall see in a moment, ensures that this is the normal situation. A good example for a non-proper forcing is the forcing P that collapses ω_1 to ω. (The "conditions", the members of P, are finite functions from ω into ω_1 and the ordering is extension.) Here it is obvious that the generic function $g : \omega \to \omega_1$ onto ω_1 is not $\bar{M}$-generic, since it involves ordinals not in $\bar{M}$.

To define properness, we need the concept of an (M, P)-generic condition.

Let P be a poset and $M \prec H_\lambda$, with $P \in M$, be an elementary substructure. A condition $q \in P$ is said to be (M,P)-*generic* iff for every dense subset $D \subseteq P$ such that $D \in M$, $D \cap M$ is predense above q, i.e., for any $q_1 \geq q$, there is a $q_2 \geq q_1$ extending some $d \in D \cap M$. Sometimes, when the identity of P is clear from the context, we just say that q is an "M-generic" condition.

In the proof of the properness preservation theorem we shall employ the following

2.1 Lemma. *A condition q is (M,P)-generic iff for every $D \in M$ dense in P there is a name $\underset{\sim}{p} \in V^P$ such that*

$$q \Vdash_P \underset{\sim}{p} \in M \cap D \cap \underset{\sim}{G}.$$

2.2 Definition. A poset P is called *proper* iff for any $\lambda > 2^{|P|}$ and countable $M \prec H_\lambda$ with $P \in M$, every $p \in P \cap M$ has an extension $q \geq p$ that is an (M,P)-generic condition.

Properness, and genericity of a condition have the following equivalent property which is often used. A condition q is (M,P)-generic iff

$$q \Vdash_P M[\underset{\sim}{G}] \cap \mathrm{On} = M \cap \mathrm{On}.$$

Assuming a predicate V that denotes the ground model, we can replace this by

$$q \Vdash_P M[\underset{\sim}{G}] \cap V = M \cap V.$$

Thus P is proper iff for any $\lambda > 2^{|P|}$ and countable $M \prec H_\lambda$ with $P \in M$, every $p \in P \cap M$ has an extension $q \geq p$ such that for every $\tau \in M \cap V^P$ and every $q' \geq q$, if $q' \Vdash_P \tau \in V$, then $q' \Vdash_P \tau \in M$.

We will prove that in the definition of properness, the quantification "for any $\lambda > 2^{|P|}$ and every countable $M \prec H_\lambda$" can be weakened to "for $\lambda = (2^{|P|})^+$ (or, for some $\lambda > 2^{|P|}$) and for a closed unbounded set of $M \prec H_\lambda \ldots$", and the resulting definition is equivalent to the original. (See also [15, Chap. III].)

One of the first consequences of properness is the following. If P is proper, and G a (V,P)-generic filter, then $\aleph_1$ is not collapsed in $V[G]$. Moreover, every countable set of ordinals in $V[G]$ is included in an old countable set (from V). Indeed, if q is (M,P)-generic and $\tau \in M$ is a name for an ordinal, then q forces τ to be in M. (By the alternative definition, or argue as follows. The set D of conditions that determine the value of τ is dense in P and is in M, and hence $D \cap M$ is dense above q which implies that every extension of q can be further extended to force $\tau = \alpha$ for some $\alpha \in M$.) It follows from this observation that if P is proper and forcing with P introduces no new subsets of ω, then forcing with P adds no new countable sets of ordinals.

The simplest examples of proper forcing posets are the countably closed posets and the c.c.c. posets. It is illuminating to realize that despite the obvious difference between these two families of posets, they have (at some level of abstraction) the same reason for not collapsing ω_1—namely their properness. If P is c.c.c., then *any* condition is (M, P)-generic, and if P is countably closed then any upper bound of an (M, P)-complete sequence is (M, P)-generic. For an arbitrary proper poset, finding generic conditions is usually the main burden of the proof.

A large family of posets was defined by Baumgartner [1], and called Axiom A posets. It turns out that they are all proper.

2.3 Definition. A poset $(P, \leq)$ *satisfies* Axiom A iff there are partial orders $\langle \leq_i \mid i < \omega \rangle$ on P, with $\leq_0 = \leq$, such that:

1. For $i < j$, $\leq_j \subseteq \leq_i$.
2. For every $p \in P$, dense $D \subseteq P$, and $n < \omega$, there are $p' \in P$ and countable $D_0 \subseteq D$ such that $p \leq_n p'$ and D_0 is predense above p' (i.e., if $p'' \geq p'$ then p'' is compatible with some condition in D_0).
3. If $\langle p_i \in P \mid i \in \omega \rangle$ is a sequence such that $p_i \leq_i p_{i+1}$, then there is a $p \in P$ (called the *fusion* of the sequence) such that for every i, $p_i \leq_i p$.

A poset $(P, \leq)$ *satisfies* Axiom A^* iff in addition the D_0 above can be taken to be finite.

The Sacks-Spector conditions (subtrees of $2^{<\omega}$ with arbitrarily high splitting) satisfies Axiom A^*.

It is easy to prove that any Axiom A forcing is proper (see Baumgartner [1]). In fact, if $M \prec H_\lambda$ is countable with $P \in M$ and $p_0 \in P \cap M$, then for any i there is an (M, P)-generic condition p such that $p_0 \leq_i p$.

The projection of a proper poset is also proper. That is, if $P \lhd Q$ and Q is proper, then P is proper. In fact, if $M \prec H_\lambda$ and $q \in Q$ is (M, Q)-generic, then $\pi(q)$ is (M, P)-generic.

Another equivalent definition of when a condition is (M, P)-generic can be obtained from the following lemma.

2.4 Lemma. *Let P be a poset, $M \prec H_\lambda$ countable with $P \in M$ (where $\lambda > 2^{|P|}$ so that $\mathcal{P}(P) \in H_\lambda$) and suppose that $p \in P$ is some (M, P)-generic condition. If $x, y \in M \cap V^P$ are such that $p \Vdash x \in y$, then for some $p_1 \geq p$ and $(a, b) \in y \cap M$, where $a \in P$ and $b \in V^P$, $a \leq p_1$ and $p_1 \Vdash x = b$ hold.*

Proof. In order to illustrate in a simple setting two possible approaches, we give two proofs for this lemma. It follows immediately from the definition of forcing that there is an extension of p (denoted p_1) and a pair $(a, b) \in y$ such that $p_1 \geq a$ and $p_1 \Vdash x = b$. The point of the lemma, however, is to get such a pair (a, b) in M whenever p is (M, P)-generic. Consider the set

$$E = \{p_0 \in P \mid \exists (a, b) \in y \ (a \leq p_0 \text{ and } p_0 \Vdash x = b)\}.$$

Note that $E \in M$ is an open subset of P that is dense above p. Let $F \subseteq P$ be the set of all conditions that are in E or else are incompatible with every condition in E. Then $F \in M$ is dense (open) and so p is compatible with some $p_0 \in F \cap M$. Since E is dense above p, $p_0 \in E \cap M$ and hence the defining clause applies to p_0. But M is an elementary substructure, and hence there is a pair $(a, b) \in y \cap M$ with $a \le p_0$ and such that $p_0 \Vdash x = b$. This proves the lemma.

For the second proof, let $\pi_0 : M \to \bar{M}$ be the transitive collapse. Let G be an arbitrary (V, P)-generic filter containing p. Define $G_0 = \pi_0 “G \cap M$. Then G_0 is a $(\bar{M}, \pi_0(P))$-generic filter and π_0 can be extended to $\pi_1 : M[G] \to \bar{M}[G_0]$. For every $m \in M \cap V^P$, $\pi_1(m[G]) = \pi_1(m)[G_0]$. We have $M[G] \prec H_\lambda[G]$, so that π_1^{-1} is an elementary embedding of $\bar{M}[G_0]$ into $H_\lambda[G]$.

Since $p \Vdash x \in y$, $x[G] \in y[G]$. Hence $\pi_1(x[G]) \in \pi_1(y[G]) = \pi_1(y)[G_0]$. So there are $(a_0, b_0) \in \pi_1(y)$ such that $a_0 \in G_0$ and $\pi_1(x[G]) = b_0[G_0]$. Hence, for $(a, b) = \pi_1^{-1}(a_0, b_0)$, $(a, b) \in y \cap M$ and we have $a \in G$ and $x[G] = \pi_1^{-1}(b_0[G_0]) = b[G]$.

Back in V, let $p_1 \ge p$ be an extension that forces these facts about (a, b). Thus, $p_1 \Vdash a \in \utilde{G}$ (so that $p_1 \ge a$) and $p_1 \Vdash x = b$, as required. $\dashv$

The next lemma is used in the proof that the two-step iteration of proper posets is proper.

2.5 Lemma. *Let P be a forcing poset, and $Q \in V^P$ a forcing poset in V^P. Let $M \prec H_\lambda$ be countable with $P, Q \in M$, and suppose that λ is sufficiently large. Then $(p, q) \in P * Q$ is $(M, P * Q)$-generic iff*

$$p \text{ is } (M, P)\text{-generic}$$

and

$$p \Vdash_P q \text{ is } (M[\utilde{G}_0], Q)\text{-generic},$$

where $\utilde{G}_0$ is the canonical name for the (V, P)-generic filter.

Now we prove that the iteration of two proper posets is again proper, and in fact the following stronger claim holds which we establish for later use.

2.6 Lemma. *Suppose that P_0 is proper, and P_1 is proper in V^{P_0} (that is, P_1 is a P_0-name and 0_{P_0} forces it to be proper). Let $R = P_0 * P_1$ be the two-step iteration, and let $\pi : R \longrightarrow P_0$ be the projection defined by $\pi(p, \utilde{q}) = p$. Suppose that $M \prec H_\lambda$ is countable with $R \in M$. Then every $r \in R \cap M$ an M-generic extension. Moreover, the following holds: Suppose that $p_0 \in P_0$ is an (M, P_0)-generic condition. For any name $\utilde{r} \in V^{P_0}$ if*

$$p_0 \Vdash_{P_0} \utilde{r} \in M \cap R \quad \text{and} \quad \pi(\utilde{r}) \in \utilde{G}_0 \tag{5.4}$$

($\utilde{G}_0$ is the canonical name of the generic filter over P_0) then there is some $\utilde{p_1} \in V^{P_0}$ such that $(p_0, \utilde{p_1})$ is an M-generic condition and

$$(p_0, \utilde{p_1}) \Vdash_R \utilde{r} \in \utilde{G}.$$

(Being a P_0-name, $\utilde{r}$ is also an R-name and it may appear in R-forcing formulas. $\utilde{G}$ is the canonical name of the (V, R)-generic filter.)

Proof. Let G_0 be any (V, P_0)-generic filter containing p_0. The name $\underset{\sim}{r}$ is not necessarily in M, but by (5.4) it is interpreted as some condition r in $M \cap R$ such that $\pi(r) \in G_0$. Say $r = (r_0, \underset{\sim}{r_1})$ where $\underset{\sim}{r_1}$ is a P_0 name for a condition in P_1. In $M[G_0]$, $\underset{\sim}{r_1}$ is interpreted as a condition r_1 in (the G_0 interpretation of) P_1. Since P_1 is proper, there is an extension p_1 of r_1 that is $M[G_0]$-generic. Let $\underset{\sim}{p_1}$ be a name of p_1 forced to have all of these properties. In particular,

$$p_0 \Vdash_{P_0} \underset{\sim}{p_1} \text{ extends the second component of } \underset{\sim}{r} \text{ in } P_1.$$

Then $u = (p_0, \underset{\sim}{p_1})$ is as required. Firstly, Lemma 2.5 gives that

$$u \text{ is } (M, R)\text{-generic.}$$

Secondly,

$$u \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G} \tag{5.5}$$

is proved as follows. Observe that $\underset{\sim}{r}$ is not a condition, but a V^{P_0} name and hence a V^R name of a condition in R. However, any condition above p_0 can be extended to decide the value of $\underset{\sim}{r}$ as a condition in R. Suppose any $u' \in R$ that extends u and determines for some $r \in R$ that

$$u' \Vdash \underset{\sim}{r} = r,$$

where $r \in R$ is of the form $r = (r_0, \underset{\sim}{r_1})$. To prove (5.5) we will show that $u' \Vdash r \in \underset{\sim}{G}$. Assume that $u' = (u'_0, \underset{\sim}{u'_1})$. Since $u'_0 \Vdash_{P_0} \pi(\underset{\sim}{r}) \in \underset{\sim}{G}_0$, and as P_0 is separative, $r_0 \leq u'_0$. But

$$u'_0 \Vdash \underset{\sim}{p_1} \text{ extends } \underset{\sim}{r_1} \text{ (the second component of } \underset{\sim}{r}\text{)},$$

and this implies $r \leq u'$ in R. Thus $u' \Vdash r \in \underset{\sim}{G}$. Since u' is an arbitrary extension of u that identifies $\underset{\sim}{r}$, $u \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}$. ⊣

2.1. Preservation of Properness

We prove here that the countable support iteration of proper forcing posets is proper. The expression "$\langle P_\alpha \mid \alpha \leq \gamma \rangle$ is an iteration of posets that satisfy property X" (such as properness) means that each successor stage $P_{\alpha+1}$ is isomorphic to some $P_\alpha * Q_\alpha$ formed with an iterand $Q_\alpha \in V^{P_\alpha}$ that satisfies property X in V^{P_α}.

2.7 Theorem. *Let δ be a limit ordinal. Suppose that $\langle P_\alpha \mid \alpha \leq \delta \rangle$ is a countable support iteration of proper forcings. Then P_δ is proper.*

We assume that δ is a limit ordinal, since the successor case was handled by Lemma 2.6. The inductive proof of the theorem is a paradigm for all preservation theorems given here, but first an intuitive (yet incorrect) overview of the proof is given. Let be given a countable structure $M \prec H_\lambda$

with $P_\delta \in M$ and a specified condition $p_0 \in P_\delta \cap M$. We are required to extend p_0 to an (M, P_δ)-generic condition. This is done in ω steps. Fix an increasing ω-sequence $\gamma_i \in \delta \cap M$, unbounded in $\delta \cap M$. (The sequence itself is not assumed to be in M, only its members.) At the n-th step we want to define $q_n \in P_{\gamma_n}$ that is (M, P_{γ_n})-generic, and is an extension of $p_0 \restriction \gamma_n$. We also require that $q_{n+1} \restriction \gamma_n = q_n$. The final condition $q = \bigcup_{n<\omega} q_n$ is in P_δ, and it extends the given condition p_0 since each initial condition does. Now at the n-th step we must also take care of D_n, the n-th dense set of P_δ in M in some pre-fixed enumeration of all the dense subsets of P_δ that are in M. It follows that we need at this step an auxiliary condition $p_n \in P_\delta \cap M \cap D_{n-1}$ that extends p_{n-1} and such that q_n extends $p_n \restriction \gamma_n$. We will first extend p_n to some $p_{n+1} \in D_n$ and then commit all the following q_m's to extend $p_{n+1} \restriction \gamma_m$ as well. Surely we cannot succeed in such a construction, for if we do, then $q \in \bigcap_{n\in\omega} D_n$ and this is too much (unless no reals are added, but this is a different story). So where did we go astray? When we claimed that p_{n+1} with $p_{n+1} \restriction \gamma_n \le q_n$ can be found in D_n. We could do that only in case $\{r \in P_{\gamma_n} \cap M \mid r \le q_n\}$ is a generic filter over M. Otherwise, we may only have a *name* for such a p_{n+1}. It turns out that this is enough for the proof, but we must formulate a slightly more involved inductive assumption.

2.8 Lemma (The Properness Extension Lemma). *Let $\langle P_\alpha \mid \alpha \le \gamma\rangle$ be a countable support iteration of proper forcing posets. Let λ be a sufficiently large cardinal. Let $M \prec H_\lambda$ be countable, with $\gamma, P_\gamma \in M$ etc. For any $\gamma_0 \in \gamma \cap M$, and $q_0 \in P_{\gamma_0}$ that is (M, P_{γ_0})-generic the following holds. If $\underset{\sim}{p_0} \in V^{P_{\gamma_0}}$ is such that*

$$q_0 \Vdash_{P_{\gamma_0}} \underset{\sim}{p_0} \in P_\gamma \cap M \quad \textit{and} \quad \underset{\sim}{p_0} \restriction \gamma_0 \in \underset{\sim}{G}_0$$

where $\underset{\sim}{G}_0$ is the canonical name for the generic filter over P_{γ_0}, then there is an (M, P_γ)-generic condition q such that $q \restriction \gamma_0 = q_0$ and

$$q \Vdash_{P_\gamma} \underset{\sim}{p_0} \in \underset{\sim}{G} \tag{5.6}$$

where $\underset{\sim}{G}$ is the canonical name of the generic filter over P_γ, and we view $\underset{\sim}{p_0}$ as a name in V^{P_γ}.

An equivalent formulation of (5.6) is that for every $q' \ge q$, if q' identifies $\underset{\sim}{p_0}$ (that is, $q' \Vdash_{p_\gamma} \underset{\sim}{p_0} = p_0$, for some $p_0 \in P_\gamma$), then $p_0 \le q'$.

We emphasize that $\underset{\sim}{p_0}$ is not necessarily in M, but it is forced by q_0 to be a condition in $P_\gamma \cap M$.

Proof of Theorem 2.7. Given $M \prec H_\lambda$ and $p_0 \in P_\delta$, we apply the lemma with $\lambda_0 = 0$ and p_0 viewed as a name in the trivial poset $P_0 = \{\emptyset\}$. $\dashv$

Proof of Lemma 2.8. The proof of the lemma is by induction on γ. For γ a successor, the lemma was essentially stated as Lemma 2.6. (There are two subcases here. $\gamma_0 + 1 = \gamma$, and $\gamma_0 + 1 < \gamma$. The first subcase is essentially stated as Lemma 2.6, and the second subcase is reduced to the first by the inductive hypothesis.)

So assume that γ is a limit ordinal. Pick an increasing sequence $\langle \gamma_i \mid i \in \omega \rangle$ cofinal in $\gamma \cap M$, with $\gamma_i \in M$ and where γ_0 is the given ordinal. (Note that γ may well be uncountable but $\gamma \cap M$ is a countable set of ordinals.) Fix an enumeration $\{D_i \mid i \in \omega\}$ of all the dense subsets of P_γ that are in M.

We will define by induction on $n < \omega$ conditions $q_n \in P_{\gamma_n}$ and names $\underset{\sim}{p}_n$ in $V^{P_{\gamma_n}}$ such that:

1. $q_0 \in P_{\gamma_0}$ is the given condition; $q_n \in P_{\gamma_n}$ is (M, P_{γ_n})-generic; and $q_{n+1} \restriction \gamma_n = q_n$.

2. $\underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_n$ is a P_{γ_n}-name such that

$$q_n \Vdash_{P_{\gamma_n}} \quad \underset{\sim}{p}_n \text{ is a condition in } P_\gamma \cap M \text{ such that:}$$

(a) $\underset{\sim}{p}_n \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,

(b) $\underset{\sim}{p}_{n-1} \leq_\gamma \underset{\sim}{p}_n$,

(c) $\underset{\sim}{p}_n$ is in D_{n-1} (for $n > 0$).

(Here and subsequently $\Vdash_\gamma$ may be written instead of $\Vdash_{P_\gamma}$ and $\leq_\gamma$ instead of $\leq_{P_\gamma}$.) To see where we are going, suppose that q_n, $\underset{\sim}{p}_n$ have already been constructed for all $n \in \omega$. Then let $q = \bigcup_n q_n$. We claim that for every n:

$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma. \tag{5.7}$$

It can be seen that this claim implies that q is (M, P_γ)-generic (because $\underset{\sim}{p}_n$ is forced to be in $D_{n-1} \cap M$, and by Lemma 2.1).

To prove the claim in (5.7), note first that by 2(b), for every $n < m$,

$$q \Vdash_\gamma \underset{\sim}{p}_n \leq_\gamma \underset{\sim}{p}_m.$$

By 2(a), for every m,

$$q \Vdash_\gamma \underset{\sim}{p}_m \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m}.$$

Hence, for every m and n such that $m \geq n$

$$q \Vdash_\gamma \underset{\sim}{p}_n \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m}.$$

This implies that for every n

$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma.$$

Indeed, for any q' extending q in P_γ, if $q' \Vdash \underset{\sim}{p}_n = p$, for some $p \in P_\gamma$, then

$$q' \Vdash_\gamma p \in P_\gamma \cap M \quad \text{and} \quad p \restriction \gamma_m \in \underset{\sim}{G}_{\gamma_m}, \quad \text{for } m \geq n.$$

Since the posets P_{γ_n} are separative, it follows that $p\restriction\gamma_m \le q'$ for every m. Hence $p \le q'$, because $p \in M$ and thus $\mathrm{dom}(p) \subseteq \gamma \cap M$, so that $\mathrm{dom}(p) \subseteq \sup\{\gamma_m \mid m < \omega\}$. So $q' \Vdash p \in \underset{\sim}{G}_\gamma$. This holds for any extension q' of q that determines $\underset{\sim}{p_n}$, and hence $q \Vdash \underset{\sim}{p_n} \in \underset{\sim}{G}_\gamma$.

Returning now to the inductive construction, assume that q_n and $\underset{\sim}{p_n}$ have been constructed. We will first define $\underset{\sim}{p_{n+1}}$ in $V^{P_{\gamma_n}}$, and then q_{n+1}. Imagine a generic extension $V[G_n]$, made via P_{γ_n}, such that $q_n \in G_n$. Then $\underset{\sim}{p_n}[G_n]$, the realization of $\underset{\sim}{p_n}$, is some condition p_n in $P_\gamma \cap M$ such that $p_n\restriction\gamma_n \in G_n$. Since q_n is (M, P_{γ_n})-generic, $G_n \cap M$ intersects every dense subset of P_{γ_n} that lies in M. An easy density argument now gives a condition $p_{n+1} \in D_n \cap M$, extending p_n, such that $p_{n+1}\restriction\gamma_n \in G_n$. (Argue in the collapsed structure $\bar{M}$ and use Lemma 1.2.) We let $\underset{\sim}{p_{n+1}}$ be a P_{γ_n}-name of p_{n+1}, forced by q_n to satisfy all of these properties of p_{n+1}. That is,

$$q_n \Vdash_{\gamma_n} \underset{\sim}{p_{n+1}} \in D_n \cap M, \quad \underset{\sim}{p_n} \le_\gamma \underset{\sim}{p_{n+1}}, \quad \text{and} \quad \underset{\sim}{p_{n+1}}\restriction\gamma_n \in \underset{\sim}{G}_{\gamma_n}.$$

Now that $\underset{\sim}{p_{n+1}}$ is defined, apply the inductive assumption of this lemma to $\gamma_n < \gamma_{n+1}$, q_n, and $\underset{\sim}{p_{n+1}}\restriction\gamma_{n+1}$. This gives $q_{n+1} \in P_{\gamma_{n+1}}$ that satisfies the required inductive assumptions. ⊣

We draw two further conclusions from the Properness Extension Lemma.

2.9 Corollary. *Let $\langle P_i \mid i \le \delta\rangle$ be a countable support iteration of proper forcings.*

1. *Suppose that $\mathrm{cf}(\delta) > \omega$. Then any real (or countable set of ordinals) in V^{P_δ} already appears in V^{P_i} for some $i < \delta$.*
2. *Suppose that $\mathrm{cf}(\delta) = \omega$ and $\langle\gamma_i \mid i \in \omega\rangle$ is increasing and cofinal in δ. Then for every name $\underset{\sim}{f} \in V^{P_\delta}$ of a countable sequence of ordinals and every condition $p_0 \in P_\delta$, there is an extension $p \ge p_0$ such that for every $n \in w$, there is a P_{γ_n} name c_n such that $p \Vdash_\delta \underset{\sim}{f}(n) = c_n$.*

Proof. Let $\underset{\sim}{f} \in V^{P_\delta}$ be a real. Find a countable $M \prec H_\lambda$ with $\underset{\sim}{f} \in M$, and let $q \in P_\delta$ be some (M, P_δ)-generic condition (given by the lemma). If $i = \sup(M \cap \delta)$ then $i < \delta$. The support of any condition in $P_\delta \cap M$ is included in i and $\underset{\sim}{f} \cap M$ can be viewed as a V^{P_i} name forced by q to be equal to $\underset{\sim}{f}$.

For item 2, let $M \prec H_\lambda$ be countable, containing all relevant parameters (including the name of the real). Repeat the proof of the Extension Lemma with $\langle\gamma_i \mid i \in \omega\rangle$ as the cofinal sequence and instead of dealing with all dense sets let D_{n-1} be the set of conditions that determine $\underset{\sim}{f}(n-1)$. Let p be the resulting (M, P_δ)-generic condition. ⊣

2.2. The $\aleph_2$-Chain Condition

In almost all applications given here, we assume the Continuum Hypothesis (CH) in the ground model, and iterate ω_2 proper forcings, each of size $\aleph_1$.

To conclude that $\aleph_2$ and higher cardinals are not collapsed, the following theorem can be invoked. More general chain condition theorems (which deal with bigger posets) can be found in Shelah's book ([15, Chap. VIII] on the p.i.c. condition, for example) and in Sect. 5.4.

2.10 Theorem. *Assume* CH. *Let $\langle P_i \mid i \leq \delta\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of proper forcings of size $\aleph_1$. Then P_δ satisfies the $\aleph_2$-c.c.*

Proof. The assumption is that each iterand Q_α has size $\aleph_1$ in V^{P_α}, but the posets P_i themselves may be large ($2^{\aleph_1}$, because of the names involved). In any family $\{r_\xi \mid \xi \in \omega_2\} \subseteq P_\delta$, we must find two compatible conditions. Fixing a large λ, pick for every $\xi \in \omega_2$ a countable $M_\xi \prec H_\lambda$ such that $r_\xi \in M_\xi$. Look at the isomorphism types of the countable structures M_ξ. Since CH holds, there is a set $I \subseteq \omega_2$ of size $\aleph_2$ such that all M_ξ for $\xi \in I$ are pairwise isomorphic. But the transitive collapse is determined by the isomorphism type, and hence there is a single transitive structure $\bar{M}$ to which all M_ξ for $\xi \in I$ are collapsed. In addition, we may form a Δ-system for the countable sets $M_\xi \cap \omega_2$ (again by CH). This leads to the following assumptions on I.

1. For some fixed transitive structure $\bar{M}$, $h_\xi : M_\xi \to \bar{M}$, where h_ξ are the collapsing functions for $\xi \in I$.

2. The countable sets $M_\xi \cap \omega_2$ form a Δ-system: For some countable $C \subseteq \omega_2$,
$$M_{\zeta_1} \cap M_{\zeta_2} \cap \omega_2 = C \quad \text{for all } \zeta_1 \neq \zeta_2 \text{ in } I.$$
   Moreover, C is an initial segment of $M_\xi \cap \omega_2$, and there is no interleaving of the $M_\xi \cap \omega_2 \setminus C$ parts. That is, if we define $\mu_\xi = \min(M_\xi \cap \omega_2 \setminus C)$ then for $\xi_1 < \xi_2$ in I, $\sup(C) < \mu_{\xi_1}$, and $\sup(M_{\xi_1} \cap \omega_2) < \mu_{\xi_2}$. We also assume that $h_\xi(r_\xi)$ does not depend on ξ and is a fixed member of $\bar{M}$.

We now claim that any two conditions with indices in I are compatible. Given $\xi_1, \xi_2 \in I$, it suffices to show that $r_1 = r_{\xi_1} \restriction \mu_{\xi_1} = r_{\xi_1} \restriction \sup(C)$ and $r_2 = r_{\xi_2} \restriction \mu_{\xi_2}$ are compatible. Because then, if $r \in P_{\sup(C)}$ extends r_1 and r_2, then $p = r \cup r_{\xi_1} \restriction (\omega_2 \setminus C) \cup r_{\xi_2} \restriction (\omega_2 \setminus C)$ extends the two given conditions. The fact that C is an initial segment of $M_\xi \cap \delta$ and hence that p is a function in P_δ is used in the proof (if the iteration were of length $\delta > \omega_2$ then $\aleph_2$ may indeed be collapsed).

Let $\mu = \mu_{\xi_1}$, and let $h : M_{\xi_1} \to M_{\xi_2}$ be an isomorphism of the two structures. Then h is the identity on $\mu \cap M_{\xi_1} = h(\mu) \cap M_{\xi_2}$, and $h(r_1) = r_2$. We shall prove that if p is any (M_{ξ_1}, P_μ)-generic condition extending r_1, then p extends r_2, and hence r_1 and r_2 are compatible as required. The iterands are of power $\aleph_1$ and we may assume that their universe is always ω_1. Then for every α in its domain, $r_1(\alpha)$ is forced by $r_1 \restriction \alpha$ to be a countable ordinal. A similar statement holds for r_2. However, $r_1(\alpha)$ is not necessarily the same name as $r_2(\alpha)$.

The following lemma therefore suffices.

2.11 Lemma. *Let M_1 and M_2 be two isomorphic countable elementary substructures of H_λ. Let $h : M_1 \to M_2$ be an isomorphism, and $\mu \in M_1 \cap \omega_2$ be such that h is the identity on $\mu \cap M_1$. (We do not require that $h(\mu) \neq \mu$, but this is possible.) If $p \in P_\mu$ is any (M_1, P_μ)-generic condition then for any condition $r \in P_\mu \cap M_1$, $p \geq r$ implies $p \geq h(r)$. (Hence any (M_1, P_μ)-generic condition is also $(M_2, P_{h(\mu)})$-generic.)*

Proof. The proof is by induction on μ.

Case 1. μ is a limit ordinal. Note that, for any $r \in P_\mu \cap M_1$, r and $h(r)$ have the same support, since h is the identity on μ and the support is countable. Assume that $p \in P_\mu$ is (M_1, P_μ)-generic and $r \in P_\mu \cap M_1$, $p \geq r$ as in the lemma. In order to prove that $p \geq h(r)$, it suffices to prove for every $\mu' \in \mu \cap M_1$ that $p\restriction\mu' \geq h(r)\restriction\mu'$. This follows from the inductive assumption, because $p\restriction\mu'$ is $(M_1, P_{\mu'})$-generic, and $p\restriction\mu' \geq r\restriction\mu'$ implies that $p\restriction\mu' \geq h(r\restriction\mu') = h(r)\restriction\mu'$.

Case 2. $\mu = \mu' + 1$. As $p \geq r$ in $P_{\mu'+1}$ $(= P_{\mu'} * Q_{\mu'})$, $p\restriction\mu'$ extends $r\restriction\mu'$ and

$$p\restriction\mu' \Vdash p(\mu') \quad \text{extends} \quad r(\mu') \text{ in } Q_{\mu'}. \tag{5.8}$$

By Lemma 2.5, $p\restriction\mu'$ is $(M_1, P_{\mu'})$-generic, and the inductive condition implies that $p\restriction\mu'$ extends $h(r)\restriction\mu'$. We want to prove that

$$p\restriction\mu' \Vdash p(\mu') \quad \text{extends} \quad h(r)(\mu') \text{ in } Q_{\mu'}.$$

Consider any t in $P_{\mu'}$ that extends $p\restriction\mu'$, and we shall find an extension t' of t in $P_{\mu'}$ forcing $r(\mu') = h(r)(\mu')$. Thus

$$t' \Vdash p(\mu') \quad \text{extends} \quad h(r)(\mu')$$

follows from (5.8) as required. The set of conditions that "know" the value of $r(\mu')$ (as an ordinal in ω_1) is dense above $r\restriction\mu'$ and is in M_1. Hence, by genericity of t, t is compatible with some $s \in P_{\mu'} \cap M_1$, $s \geq r\restriction\mu'$, such that

$$s \Vdash_{\mu'} r(\mu') = \alpha$$

for some $\alpha < \omega_1$. Since $h(\mu') = \mu'$, and $h(\alpha) = \alpha$, this implies that

$$h(s) \Vdash_{\mu'} h(r)(\mu') = \alpha.$$

Let $t' \in P_{\mu'}$ be a common extension of t and s. By the inductive assumption, t' extends $h(s)$ as well and so $t' \Vdash_{\mu'} r(\mu') = \alpha = h(r)(\mu')$.

$\dashv$

This completes the proof of Theorem 2.10. $\dashv$

Knowing the $\aleph_2$-c.c. we can prove by induction on $\delta \le \omega_2$ that $|P_\delta| \le 2^{\aleph_1}$. This helps in defining iterations of length ω_2 of proper forcing posets of size $\aleph_1$ each, when we want to ensure that every $A \subseteq \omega_1$ in $V^{P_{\omega_2}}$ has had its chance to be considered at some successor stage of the iteration.

A useful consequence of the previous lemma is the following

2.12 Theorem. *Assume* CH. *Let $\langle P_i \mid i \le \delta\rangle$ be a countable support iteration of length $\delta < \omega_2$ of proper forcings of size $\aleph_1$. Then* CH *holds in V^{P_δ}.*

Proof. Write P for P_δ. Let $x \in V^P$ be a name forced by $p_0 \in P$ to be a function from ω_2 into $\mathcal{P}(\omega)$ (the power set of ω). We shall find an extension p_1 of p_0 and two indexes $\xi_0 \ne \xi_1$ such that $p_1 \Vdash x(\xi_0) = x(\xi_1)$.

As above, we define countable $M_\xi \prec H_\lambda$ with $P, x, \xi \in M_\xi$ for $\xi < \omega_2$. By CH, there are $\xi_1 \ne \xi_2$ with an isomorphism $h : M_{\xi_1} \to M_{\xi_1}$ taking ξ_1 to ξ_2 and such that $M_{\xi_1} \cap \delta = M_{\xi_2} \cap \delta$. Let p_2 be an (M_{ξ_1}, P)-generic condition extending p_1. Then p_2 is also (M_{ξ_2}, P)-generic by the lemma, and hence

$$p_2 \Vdash x(\xi_1) = x(\xi_2)$$

follows from $h(\xi_1) = \xi_2$. $\dashv$

2.3. Equivalent Formulations

Suppose that Q is a non-proper poset and P is a proper poset. Then, in V^P, Q remains non-proper. To prove this basic result we need an equivalent formulation of properness in terms of preservation of stationary subsets of $\mathcal{P}_{\aleph_1}(A)$. Here, $\mathcal{P}_{\aleph_1}(A) = [A]^{<\aleph_1}$ is the collection of all countable subsets of A. We refer the reader to Jech's chapter on stationary sets for basic properties of the closed unbounded filter on $\mathcal{P}_{\aleph_1}(A)$. We shall rely on the following facts for any uncountable set A. (1) The collection of closed unbounded sets generates a countably closed and normal filter over $\mathcal{P}_{\aleph_1}(A)$. (2) For any closed unbounded set $C \subseteq \mathcal{P}_{\aleph_1}(A)$ there is a function $f : [A]^{<\aleph_0} \to A$ such that if $x \in \mathcal{P}_{\aleph_1}(A)$ is closed under f, then $x \in C$. (3) If $A_1 \subseteq A_2$ and $C \subseteq \mathcal{P}_{\aleph_1}(A_2)$ is closed unbounded, then $\{x \cap A_1 \mid x \in C\}$ contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(A_1)$. (4) A subset S of $\mathcal{P}_{\aleph_1}(A)$ is said to be stationary if it has non-empty intersection with every closed unbounded set. If S is stationary and f is a function such that $f(x) \in x$ for every non-empty $x \in S$, then for some $a \in A$, the set $\{x \in S \mid f(x) = a\}$ is stationary.

Let P be a poset and $p \in P$ a condition. We say that $D \subseteq P$ is *pre-dense above* p if $\forall p' \ge p \exists d \in D$ (p' and d are compatible in P). Equivalently, D is pre-dense above p if the set D' of all extensions of members of D is dense above p.

Given any poset P, form $A = P \cup \mathcal{P}(P)$, a disjoint union of the poset universe with its power set. The "test-set" for P is the collection of all $a \in \mathcal{P}_{\aleph_1}(A)$ such that, for any $p_0 \in P \cap a$ there exists an extension $p \in P$ such that, for every $D \in a \cap \mathcal{P}(P)$, if D is dense in P then $D \cap a$ is pre-dense above p. We can say that p is "generic" for a, and then the test-set is the collection of all a, countable subsets of A, that have generic conditions extending each member of $P \cap a$.

If $T \subseteq \mathcal{P}_{\aleph_1}(A)$ is the test-set of P then its complement $\mathcal{P}_{\aleph_1}(A) \setminus T$ is called the "failure set" for P.

2.13 Theorem (Properness Equivalents). *For any poset P the following are equivalent.*

1. *P is proper (as in Definition 2.2).*
2. *For some $\lambda > 2^{|P|}$ and any countable $M \prec H_\lambda$ with $P \in M$, every $p_0 \in P \cap M$ has an extension $p \geq p_0$ that is an M-generic condition.*
3. *For every uncountable λ, P preserves stationary subsets of $\mathcal{P}_{\aleph_1}(\lambda)$. That is, if $S \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ is stationary, then it remains so in any generic extension via P.*
4. *For $\lambda_0 = 2^{|P|}$, P preserves stationary subsets of $\mathcal{P}_{\aleph_1}(\lambda_0)$.*
5. *The test set for P, as defined above, contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(A)$.*

Proof. Clearly, *1* $\Rightarrow$ *2*. We prove that *1* $\Rightarrow$ *3*. So assume that P is proper, and let $S \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ be a stationary set. To prove that S remains stationary in any extension via P, we take any $f \in V^P$ such that $p_0 \Vdash f : [\lambda]^{<\aleph_0} \to \lambda$, and we shall find an extension $p \in P$ so that some $x \in S$ is forced by p be closed under f. Pick a sufficiently large cardinal κ and a countable $M \prec H_\kappa$ with $\lambda, P, p_0, f \in M$ and such that $M \cap \lambda \in S$. We can find one since the collection of intersections $M \cap \lambda$ for structures M as above contains a closed unbounded subset of $\mathcal{P}_{\aleph_1}(\lambda)$. As P is assumed proper, there is an extension $p \geq p_0$ that is (M, P)-generic. The genericity of p then implies that $p \Vdash M \cap \lambda$ is closed under f.

3 $\Rightarrow$ *4* is trivial, and *2* $\Rightarrow$ *4* is just like *1* $\Rightarrow$ *3*. We prove now that *4* $\Rightarrow$ *5*.

Assume that P preserves stationarity of subsets of $\mathcal{P}_{\aleph_1}(\lambda_0)$ for $\lambda_0 = 2^{|P|}$. Define $A = P \cup \mathcal{P}(P)$. Then $|A| = \lambda_0$. Suppose that S, the failure set for P as defined above, is stationary, and we shall derive a contradiction. By normality, we may assume that the failure is due to the same $p_0 \in a$ for $a \in S$. Let $G \subseteq P$ be a (V, P)-generic filter containing p_0. Then S is stationary in $V[G]$ since P preserves stationarity. Define a function $g : A \to A$ so that if D is dense in P then $g(D) \in G$. Since S is stationary, there is an $x \in S$ closed under g. If $p \geq p_0$ forces this fact about x, then p shows that x is in fact in the test set for P.

Finally we prove *5* $\Rightarrow$ *1*. Suppose that $\lambda > 2^{|P|}$, $M \prec H_\lambda$ is countable, and $P \in M$. Then $A = P \cup \mathcal{P}_{\aleph_1}(P) \in H_\lambda$ and $A \in M$. The test set for P is also in M. Assuming that this set contains a closed unbounded set, we may find such a closed unbounded set C in M. This implies that $M \cap A \in C$, and hence there exists an (M, P)-generic condition above any condition in $M \cap P$. ⊣

3. Preservation of ${}^{\omega}\omega$-Boundedness

The set of functions from ω to ω is denoted ${}^{\omega}\omega$ (the "reals"). For $f,\ g \in {}^{\omega}\omega$ and $k < \omega$ define $f <_k g$ iff $\forall n \geq k(f(n) \leq g(n))$. $<^* = \bigcup_k <_k$ is the *bounding* (also called eventual bounding) relation: If $f <^* g$, then g *bounds* (or *dominates*) f, and if $f <_0 g$ then g *totally bounds* f. A basic fact is that any countable $F \subseteq {}^{\omega}\omega$ is bounded by some $g \in {}^{\omega}\omega$.

A forcing poset P is said to be ${}^{\omega}\omega$*-bounding* iff for every generic filter $G \subseteq P$, $V \cap {}^{\omega}\omega$ bounds $V[G] \cap {}^{\omega}\omega$, i.e., for every $g \in {}^{\omega}\omega \cap V[G]$ there is an $h \in {}^{\omega}\omega \cap V$ with $g <^* h$ (we could equivalently require $g <_0 h$). Our aim is to prove that the countable support iteration of proper ${}^{\omega}\omega$-bounding posets is ${}^{\omega}\omega$-bounding.

Let P be a poset and $\underset{\sim}{f} \in V^P$ a name of a real (i.e., a name forced by 0_P to be a real). We say that an increasing sequence $\bar{p} = \langle p_i \mid i \in \omega \rangle$ of conditions in P *interprets* $\underset{\sim}{f}$ *as* $f^* \in {}^{\omega}\omega$ iff for every $n < \omega$ p_n forces $\underset{\sim}{f} \restriction n = f^* \restriction n$. We write in this case $f^* = \mathrm{intp}(\bar{p}, \underset{\sim}{f})$.

3.1 Definition. Let P be a forcing poset and $\underset{\sim}{f} \in V^P$ a name of a real. Suppose that $\bar{p} = \langle p_i \mid i \in \omega \rangle$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$. We say that $\bar{p}$ *respects* $g \in {}^{\omega}\omega$ iff

$$\mathrm{intp}(\bar{p}, \underset{\sim}{f}) <_0 g. \tag{5.9}$$

The following surprising property turns out to be important for the preservation theorem.

3.2 Theorem. *If P is ${}^{\omega}\omega$-bounding, then P satisfies the following ostensibly stronger property: Let $\underset{\sim}{f} \in V^P$ be a name of a real and let $M \prec H_\kappa$ be countable, with $P, \underset{\sim}{f} \in M$. Suppose that $g \in {}^{\omega}\omega$ dominates all the reals of M, and $\bar{p} \in M$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$ and respects g. Then, for some $p \in P \cap M$ and $h \in M$, $h <_0 g$ and $p \Vdash_P \underset{\sim}{f} \leq_0 h$. So that*

$$p \Vdash_P \underset{\sim}{f} <_0 g.$$

Proof. The point of the theorem is this. As P is ${}^{\omega}\omega$-bounding, every condition in M can be extended to force that $\underset{\sim}{f}$ is bounded by some real in M and hence that $\underset{\sim}{f} <^* g$, but it takes the theorem to find $p \in M$ that forces $\underset{\sim}{f} <_0 g$.

Work in M. By assumption $\bar{p} = \langle p_i \mid i \in \omega \rangle$ is an increasing sequence of conditions in P that interprets $\underset{\sim}{f}$ as f^*, and $f^* <_0 g$. For each n, that P is ${}^{\omega}\omega$-bounding and $p_n \Vdash \underset{\sim}{f} \restriction n = f^* \restriction n$ implies that there is an extension p'_n of p_n and an $h_n \in {}^{\omega}\omega \cap M$ such that:

1. $p'_n \Vdash \underset{\sim}{f} \leq_0 h_n$.
2. $h_n \restriction n = f^* \restriction n$.

Let $u \in {}^{\omega}\omega$ be defined by

$$u(m) = \max\{h_i(m) \mid i \leq m\}.$$

Then $u \in M$, and is hence bounded by g; say $u <_\ell g$. This implies that $h_\ell <_0 g$ by the following argument. For $k < \ell$, $h_\ell(k) = f^*(k) < g(k)$, and for $k \geq \ell$ $h_\ell(k) \leq u(k) < g(k)$. Now p'_ℓ is as required: it forces $\underset{\sim}{f} <_0 g$ since it forces $\underset{\sim}{f} \leq_0 h_\ell$.

Remark that we can require that p extends any given condition in the sequence $\bar{p}$. ⊣

A main tool in the preservation proof is the notion of a *derived sequence*. Let Q_1 and Q_2 be two forcing posets such that $Q_1 \lhd Q_2$, and let $\pi : Q_2 \to Q_1$ be the related projection. Let $\underset{\sim}{f}$ be a Q_2-name forced by 0_{Q_2} to be a real, and let $\bar{r} = \langle r_i \mid i \in \omega \rangle$ be an increasing sequence of conditions in Q_2 that interprets $\underset{\sim}{f}$. Fix a well-ordering of Q_2. Suppose that G_1 is a (V, Q_1)-generic filter. Recall that $Q_2/G_1 = \{q \in Q_2 \mid \pi(q) \in G_1\}$. We shall define in $V[G_1]$ an increasing sequence $\bar{s} = \langle s_i \mid i \in \omega \rangle$ of conditions in Q_2/G_1 that interprets $\underset{\sim}{f}$ by the following induction.

1. If $\pi(r_i) \in G_1$ then $s_i = r_i$. (In this case $\pi(r_k) \in G_1$ for every $k < i$ and $s_k = r_k$.)

2. If $\pi(r_i) \notin G_1$, then let s_i be the first Q_2-extension of s_{i-1} that is in Q_2/G_1 and determines the value of $\underset{\sim}{f} \restriction i$.

Thus, if $\pi(r_i) \in G_1$ for all $i \in \omega_1$, then $s_i = r_i$ for all i, but if n is the first index such that $\pi(r_n) \notin G_1$, then $s_i = r_i$ for $i < n$, and for $i \geq n$ we define $s_i \in Q_2$ as the first extension of s_{i-1} with $\pi(s_i) \in G_1$ and such that s_i determines $\underset{\sim}{f} \restriction i$ in the forcing relation $\Vdash_{Q_2}$.

We say that the sequence $\bar{s}$ defined above in $V[G_1]$ is "derived" from $\bar{r}$, G_1, and $\underset{\sim}{f}$. We write $\delta_{G_1}(\bar{r}, \underset{\sim}{f})$ to denote this derived sequence in Q_2/G_1. The V^{Q_1}-name of the derived sequence is denoted $\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$.

3.3 Lemma. *Let $Q_1 \lhd Q_2$ be posets with projection $\pi : Q_2 \to Q_1$, and suppose that Q_1 is ${}^{\omega}\omega$-bounding (no such assumption is made about Q_2). Suppose that:*

1. $\underset{\sim}{f} \in V^{Q_2}$ is forced by every condition to be a real.

2. $\bar{r} = \langle r_i \mid i < \omega \rangle$ is an increasing sequence of conditions in Q_2 (above some given $p \in Q_2$) that interprets $\underset{\sim}{f}$.

3. $M \prec H_\kappa$ is countable with $Q_1, Q_2, \underset{\sim}{f}, \bar{r}, p \in M$.

4. $g \in {}^{\omega}\omega$ bounds all the reals of M, and $\mathrm{int}(\bar{r}, \underset{\sim}{f}) <_0 g$. *That is, $\bar{r}$ respects g in its interpretation of $\underset{\sim}{f}$.*

Then some condition $s \in Q_1 \cap M$ extends $\pi(p)$ and forces that the derived sequence $\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$ respects g in its interpretation of $\underset{\sim}{f}$ and is above p in the Q_2 ordering.

Proof. Let $\underset{\sim}{\delta} = \underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f})$ be the name of the derived sequence, $\underset{\sim}{\delta} \in V^{Q_1}$. Define a name of a real, $\underset{\sim}{h} \in V^{Q_1}$, by

$$\underset{\sim}{h} = \mathrm{int}(\underset{\sim}{\delta}, \underset{\sim}{f}).$$

That is, if G_1 is a (V, Q_1)-generic filter, let $\delta_{G_1}(\bar{r}, \underset{\sim}{f})$ be the resulting derived sequence, a Q_2-increasing sequence of conditions in Q_2/G_1 interpreting $\underset{\sim}{f}$, and let $\underset{\sim}{h}[G_1]$ be that interpretation.

Define $p_i = \pi(r_i) \in Q_1$ for $i \in \omega$. Then $\langle p_i \mid i \in \omega \rangle \in M$ and $p_i \Vdash_{Q_1} r_i = \underset{\sim}{\delta}(i)$. That is, as $p_i \Vdash_{Q_1} r_i \in Q_1/\underset{\sim}{G}_1$, p_i "knows" that r_i is the i-th member of the derived sequence. Consequently, p_i determines $\underset{\sim}{h} \restriction i$ (in Q_1 forcing) as r_i determines $\underset{\sim}{f} \restriction i$ (in Q_2 forcing). Namely

$$\mathrm{int}(\langle p_i \mid i \in \omega \rangle, \underset{\sim}{h}) = \mathrm{int}(\bar{r}, \underset{\sim}{f}).$$

Hence, $\mathrm{int}(\langle p_i \mid i \in \omega \rangle, \underset{\sim}{h}) <_0 g$, and so by the surprising theorem there exists a $s \geq p_0$ in $Q_1 \cap M$ so that $s \Vdash_{Q_1} \underset{\sim}{h} <_0 g$. That is,

$$s \Vdash_{Q_1} \mathrm{int}(\underset{\sim}{\delta}_{Q_1}(\bar{r}, \underset{\sim}{f}), \underset{\sim}{f}) <_0 g.$$

Since $s \geq p_0$, s forces that r_0 is the first member of the derived sequence, and hence that all members of the derived sequence extend p in Q_2. $\dashv$

In the following we shall apply the previous lemma in a slightly more complex situation in which the discussion is not in V, but rather in V^{Q_0}, where $Q_0 \lhd Q_1 \lhd Q_2$. We will use $\underset{\sim}{G}_0$ and $\underset{\sim}{G}_1$ as canonical names for the generic filters Q_0 and Q_1.

3.4 Lemma. *Suppose that $Q_0 \lhd Q_1 \lhd Q_2$ are posets with commuting projections $\pi_{i,j} : Q_i \to Q_j$, for $0 \leq j < i \leq 2$. Assume that Q_1 (and hence Q_0) is proper and ${}^\omega\omega$-bounding. Suppose that:*

1. *$\underset{\sim}{f} \in V^{Q_2}$ is a name of a real.*

2. *$M \prec H_\kappa$ is countable with $Q_0, Q_1, Q_2, \underset{\sim}{f} \in M$. Let $q_0 \in Q_0$ be an (M, Q_0)-generic condition. Assume that $g \in {}^\omega\omega$ bounds all the reals of M.*

3. *$\underset{\sim}{p} \in V^{Q_0}$ is given such that $q_0 \Vdash_{Q_0} \underset{\sim}{p} \in (Q_2/\underset{\sim}{G}_0) \cap M$.*

4. *q_0 forces that there is in $M[\underset{\sim}{G}_0]$ a Q_2-increasing sequence of conditions in $Q_2/\underset{\sim}{G}_0$ that interprets $\underset{\sim}{f}$, respects g, and is above $\underset{\sim}{p}$ in the Q_2 ordering.*

Then there is an (M, Q_1)-generic condition q_1 such that

(a) $\pi_{1,0}(q_1) = q_0$.

(b) $q_1 \Vdash_{Q_1} \pi_{2,1}(\underset{\sim}{p}) \in \underset{\sim}{G}_1$.

(c) *q_1 forces that there is in $M[\underset{\sim}{G}_1]$ a Q_2-increasing sequence of conditions in $Q_2/\underset{\sim}{G}_1$ that interprets $\underset{\sim}{f}$ and respects g, and is above $\underset{\sim}{p}$ in the Q_2 ordering.*

Proof. Observe that what the lemma does is to push the situation described in *4* from $\underset{\sim}{G}_0$ to $\underset{\sim}{G}_1$. That is, the basic object that interests us is a sequence of conditions in Q_2-increasing in the ordering of Q_2 and interpreting $\underset{\sim}{f}$ (in $\Vdash_{Q_2}$) as a function that respects g. The assumed sequence (in *4*) is compatible with $\underset{\sim}{G}_0$, and the resulting sequence is compatible with $\underset{\sim}{G}_1$.

Let G_0 be some (V, Q_0)-generic filter containing q_0. Work in $M[G_0] \prec H_\kappa[G_0]$ and apply the previous lemma as follows. Observe first that g dominates all the reals in $M[G_0]$, since q_0 is (M, Q_0)-generic. Observe also that $\pi_{2,1} : Q_2/G_0 \to Q_1/G_0$ is a projection (see Lemma 1.3). Then $\underset{\sim}{f}/G_0 \in V[G_0]^{Q_2/G_0}$ is a name of a real. Following *4*, let $\bar{r} \in M[G_0]$ be an increasing sequence of conditions in Q_2/G_0 that interprets $\underset{\sim}{f}/G_0$ (since it interprets $\underset{\sim}{f}$) and respects g (and is above $\underset{\sim}{p}[G_0]$ in $\leq_{Q_2}$). We apply the previous lemma to Q_2/G_0, Q_1/G_0, and $\underset{\sim}{f}/G_0$ in $V[G_0]$. Thus by that lemma some condition $s \in (Q_1/G_0) \cap M[G_0]$ exists which extends $\pi_{2,1}(\underset{\sim}{p}[G_0])$ and forces in Q_1/G_0 that the derived sequence $\underset{\sim}{\delta}_{Q_1/G_0}(\bar{r}, \underset{\sim}{f}/G_0)$ respects g in its interpretation of $\underset{\sim}{f}/G_0$, and is above $\underset{\sim}{p}[G_0]$ in Q_2/G_0.

Now let $\underset{\sim}{s} \in V^{Q_0}$ be a name of s, forced to have all the properties of s described above. (Observe that $\underset{\sim}{s}$ is not in M since its definition involves g, but it is forced to become a condition in M.) By the Properness Extension Lemma there is a $q_1 \in Q_1$ that is (M, Q_1)-generic such that $\pi_{1,0}(q_1) = q_0$ and $q_1 \Vdash_{Q_1} \underset{\sim}{s} \in \underset{\sim}{G}_1$. So $q_1 \Vdash_{Q_1} \pi_{2,1}(\underset{\sim}{p}) \in \underset{\sim}{G}_1$. We claim that q_1 is as required.

Let G be any (V, Q_1)-generic filter containing q_1. Then $G_0 = \pi_{1,0}$"G is (V, Q_0)-generic and $p = \underset{\sim}{p}[G_0]$, $s = \underset{\sim}{s}[G_0]$ can be formed. Since $q_1 \in G$ and $q_1 \Vdash \underset{\sim}{s} \in \underset{\sim}{G}_1$, $s \in G$. Now G is also $(V[G_0], Q_1/G_0)$-generic, and we write $G_1 = G$ to emphasize that $G_1 \subseteq Q_1/G_0$. Since $s \in G_1$, whatever s forces holds in $V[G_1]$. Namely, there is in $M[G_1]$ a Q_2/G_0 increasing sequence in $(Q_2/G_0)/G_1 = Q_2/G_1$ that respects g in its interpretation of $\underset{\sim}{f}/G_0$ and is above p in the ordering of Q_2/G_0. If p_n is the n-th member of this sequence, then for some finite function e we have $p_n \Vdash_{Q_2/G_0} (\underset{\sim}{f}/G_0)\restriction n = e$. So there is some $g \in G_0$ such that $g + p_n \Vdash_{Q_2} \underset{\sim}{f}\restriction n = e$. Using Lemma 1.1, we can amend the derived sequence (that is, replace conditions p with conditions of the form $g + p$ where $g \in G_0$) and obtain a sequence of conditions in Q_2/G_1 that is increasing in Q_2 and respects g in its interpretation of $\underset{\sim}{f}$ (in Q_2 forcing) and is above p in Q_2. $\dashv$

3.5 Theorem. *Let $\langle P_i \mid i \leq \delta \rangle$ be a countable support iteration of proper ${}^\omega\omega$-bounding posets. Then P_δ is (proper and) ${}^\omega\omega$-bounding.*

Proof. We know that P_δ is proper, and the ${}^\omega\omega$-bounding property of P_δ is proved by induction on δ. The successor case is obvious and so we assume that δ is a limit ordinal. Let $\underset{\sim}{f}$ be a P_δ-name of a real; we must find a condition (extending a given condition $p_0 \in P_\delta$) that forces $\underset{\sim}{f} <_0 g$ for some ground model $g \in {}^\omega\omega$. Pick $M \prec H_\kappa$, countable, with $P_\delta, \underset{\sim}{f}, p_0 \in M$. Let $\gamma_n \in \delta \cap M$ for $n \in \omega$ be increasing and cofinal in $M \cap \delta$. For simplicity, start with $\gamma_0 = 0$.

Find in M an increasing sequence $\bar{r} = \langle r_i \mid i \in \omega \rangle$ of conditions in P_δ that interprets $\underset{\sim}{f}$ as $f^* \in {}^\omega\omega$, starting with the given condition p_0. Find $g \in {}^\omega\omega$ that bounds the reals of M, and such that $f^* <_0 g$. To prove the theorem, we will find $q \in P_\delta$ (extending p_0) that forces $\underset{\sim}{f} <_0 g$.

We intend to define by induction on $n \in \omega$ conditions $q_n \in P_{\gamma_n}$, and names $\underset{\sim}{p_n} \in V^{P_{\gamma_n}}$, that satisfy the following four properties (the first two are as in the Properness Extension Lemma 2.8 but the dense sets are not needed).

1. $q_0 \in P_0$ is the trivial condition; $q_n \in P_{\gamma_n}$ is (M, P_{γ_n})-generic; and $q_{n+1} \restriction \gamma_n = q_n$.

2. $\underset{\sim}{p_0} = p_0$ is given in P_δ, and $\underset{\sim}{p_n}$ is a P_{γ_n}-name such that: $q_n \Vdash_{P_{\gamma_n}}$ "$\underset{\sim}{p_n}$ is a condition in $P_\delta \cap M$ with $\underset{\sim}{p_n} \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$ and $\underset{\sim}{p_{n-1}} \leq_\delta \underset{\sim}{p_n}$."

3. $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p_n}$ determines $\underset{\sim}{f} \restriction n$ in P_δ-forcing to be totally bounded by $g \restriction n$."

4. $q_n \Vdash_{\gamma_n}$ "some $r \in M[\underset{\sim}{G}_{\gamma_n}]$ is a P_δ-increasing sequence of conditions in $P_\delta / \underset{\sim}{G}_{\gamma_n}$ that interprets $\underset{\sim}{f}$, respects g, and is above $\underset{\sim}{p_n}$."

In words, this last asserts that if G is a (V, P_{γ_n})-generic filter containing q_n, then there is in $M[G]$ a P_δ-increasing sequence r of conditions in P_δ / G that interpret $\underset{\sim}{f}$ (in P_δ forcing) and respect g, and are all above $\underset{\sim}{p_n}[G]$ in P_δ.

If we succeed in this and define $q = \bigcup_n q_n$, then $q \Vdash_\delta \underset{\sim}{p_n} \in \underset{\sim}{G}_\delta$, as in (5.7). So 3. implies that $q \Vdash \underset{\sim}{f} <_0 g$.

To start the induction observe that $\bar{r}$ respects g so that 4. holds for $n = 0$. Suppose that $q_n \in P_{\gamma_n}$, and $\underset{\sim}{p_n} \in V^{P_{\gamma_n}}$ are defined. We first define $\underset{\sim}{p_{n+1}}$ and then q_{n+1}.

Let G be a (V, P_{γ_n})-generic filter containing q_n. Then by 4. there is in $M[G]$ a P_δ-increasing sequence r of conditions in P_δ / G that interpret $\underset{\sim}{f}$, respect g, and are above $\underset{\sim}{p_n}[G]$. Let p_{n+1} be $r(n+1)$, the $(n+1)$-th member of r, which determines $\underset{\sim}{f} \restriction (n+1) = v$ for some $v : n+1 \to \omega$ with $v <_0 g \restriction (n+1)$. Then $\underset{\sim}{p_{n+1}}$ is defined to be a $V^{P_{\gamma_n}}$ name of p_{n+1}.

Now for $Q_0 = P_{\gamma_n}$, $Q_1 = P_{\gamma_{n+1}}$, $Q_2 = P_\delta$, Lemma 3.4 can be applied to q_n (as $q_0 \in Q_0$) and $\underset{\sim}{p_{n+1}}$ (as $\underset{\sim}{p} \in V^{Q_0}$). So there exists some $q_{n+1} \in P_{\gamma_{n+1}}$ that is $(M, P_{\gamma_{n+1}})$-generic with $q_n = \pi(q_{n+1})$ and such that the required inductive assumptions hold. $\dashv$

3.1. Application: Non-Isomorphism of Ultrapowers

The significance of Theorem 3.7 proved in this section is clarified by comparing the following two theorems of Keisler and of Shelah concerning the notion of elementary equivalence (see the book by Comfort and Negrepontis [2]).

3.6 Theorem (Keisler [9]). *If $2^\lambda = \lambda^+$ and $\mathcal{A}, \mathcal{B}$ are structures of size $\leq \lambda^+$ in a language of size λ, then $\mathcal{A} \equiv \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic ultrapowers,*

$$\mathcal{A}^\lambda/p \cong \mathcal{B}^\lambda/p$$

obtained by some ultrafilter p on λ.

Keisler also showed that it is not possible to obtain this result if the language has size λ^+.

The following was proved by Shelah [12]:

> If $\mathcal{A} \equiv \mathcal{B}$ are both of size $\leq \kappa$, then $\mathcal{A}^\alpha/p \cong \mathcal{B}^\alpha/p$ for some ultrafilter p on $2^\kappa = \alpha$.

In particular, for countable elementarily equivalent structures, Shelah's theorem provides an ultrafilter on a set of size $2^{\aleph_0}$ that makes their ultrapowers isomorphic, and Keisler's theorem obtains an ultrafilter on ω, provided that $2^{\aleph_0} = \aleph_1$. The theorem proved in this section shows that CH is indeed necessary for obtaining the ultrafilter to be on ω (see Shelah [14]).

3.7 Theorem. *Assuming* CH*, there are two countable elementarily equivalent structures $\mathcal{A} \equiv \mathcal{B}$ and there is a generic extension in which $2^{\aleph_0} = \aleph_2$ and for any ultrafilters p, q on ω*

$$\mathcal{A}^\omega/p \ncong \mathcal{B}^\omega/q.$$

To prove this theorem, we will consider two propositions, P_1 and P_2, show that they imply the existence of $\mathcal{A} \equiv \mathcal{B}$ as in the theorem, and then establish their consistency.

Let DP (Diverging and Positive) denote the set of functions $h \in {}^\omega\omega$ diverging to infinity with $h(n) > 0$ for every n. If $\langle A_n \mid n \in \omega\rangle$ is a sequence of finite (non-empty) sets, then $\Pi_{n<\omega} A_n$ is the set of all functions f defined on ω with $f(n) \in A_n$ for every n.

(P_1) If $\langle A_n \mid n \in \omega\rangle$ is a sequence of finite sets, and $\{f_\alpha \mid \alpha \in \omega_1\} \subseteq \Pi_{n<\omega} A_n$, then, for every $h \in$ DP, there is a choice of subsets $H_n \subseteq A_n$, $n \in \omega$, such that

  (a) $|H_n| \leq h(n)$ for all n,

  (b) $\forall \alpha < \omega_1 \exists n_0 \forall n \geq n_0\ (f_\alpha(n) \in H_n)$.

(P_2) $({}^\omega\omega, <^*)$ has a cofinal sequence of length ω_1.

It is left as an exercise to prove that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies P_1 but negates P_2, and CH implies P_2 but negates P_1.

Our aim now is to prove the following theorem.

3.8 Theorem. *$P_1 \wedge P_2$ implies the existence of two countable elementarily equivalent structures that have no isomorphic ultrapowers with ultrafilters on ω.*

We first investigate a consequence of P_2 concerning the structure of ultrapowers of a certain type of graph. Let Δ be the bipartite graph obtained by taking U and V to be two copies of ω, with edges such that every $n \in V$ is connected exactly to those k in U such that $k \le n$. Let p be any nonprincipal ultrafilter over ω, and form the ultrapower $P = \Delta^\omega/p$. Consider the cofinal sequence $\langle f_\alpha \mid \alpha \in \omega_1 \rangle$ from P_2, and form for every f_α an element a_α of U^P obtained by viewing $f_\alpha(n) \in U$ (and taking the equivalence class of f_α). The fact that the sequence of functions is cofinal in ${}^\omega\omega$ implies the following property of P (expressed with U and V as predicates):

There is a sequence $a_i \in U$, $i \in \omega_1$ such that there is no $b \in V$ edge connected with every a_i. (5.10)

Indeed, any $[b] \in V^P$ is the equivalence class of some function $n \mapsto b(n)$, and there is some f_α that bounds b. Hence b is not connected to f_α in the ultrapower.

In fact, we can redo this argument even in the following slightly more general situation. Suppose that Δ is a bipartite graph built on two copies U and V of ω, just as before, but now we know that the following holds.

For every finite set $X \subseteq V$ there is some $u \in U$ such that no $x \in X$ is connected with u. (5.11)

Then define a sequence $u_n \in U$ by induction on n such that u_n is not connected to any one of the first n nodes of V. Here too (5.10) holds in any ultrapower. That is, in Δ^ω/p there is a set of ω_1 members of U such that there is no $b \in V$ that is connected to all of them. In fact, this result on ultrapowers can be generalized to ultraproducts of countable bipartite graphs that satisfy property (5.11) above. These ultraproducts must satisfy (5.10).

Define $\Gamma_{k,\ell}$ to be the finite bipartite graph with two disjoint sets of vertices U and V, where $|U| = k$, and the vertices in V are obtained by associating with every $x \subseteq U$ of size $\le \ell$ a vertex a_x in V that is edge connected exactly with the vertices of x.

We are particularly interested in graphs of the form $\Gamma_{n^2+1,n}$ because they have the following property: for any $X \subseteq V$ with cardinality $\le n$ there is some $a \in U$ that is not connected to any $x \in X$.

Let Γ be the disjoint union of the graphs $G_n = \Gamma_{n^2+1,n}$ for $2 \le n < \omega$. The language of the structure Γ includes not only the edge relation but also the predicates U and V, and a partial order $<_\Gamma$ that puts the vertices of G_n below those of G_m for $n < m$. The nodes in G_n are incomparable in $<_\Gamma$.

The connected components of the graph Γ are the copies of the $\Gamma_{n^2+1,n}$. (Because for $n \ge 2$, $\Gamma_{k,n}$ is connected. In fact, any two nodes that are in the same G_n are connected by a sequence of at most four edges.) So the connected components of Γ are exactly the maximal antichains of $<_\Gamma$ and this fact can be expressed by a single sentence.

Let Γ_{NS} be some countable nonstandard elementary extension of Γ. Then $\Gamma_{NS} \equiv \Gamma$, but Γ_{NS} also contains infinite connected components. The connected components of Γ_{NS} are, again, its $<_\Gamma$ antichains. Assuming P_1

and P_2, we are going to prove that Γ_{NS} and Γ have no isomorphic ultrapowers on ω. Observe that any non-standard (infinite) component of Γ_{NS} has property (5.11). Hence the following statement is true in any nonprincipal ultrapower $(\Gamma_{NS})^\omega/q$. The set of components C that satisfy the following property is cofinal in the ordering $<_\Gamma$.

There are $a_i \in C \cap U$ for $i \in \omega_1$ such that there is no $b \in V$ edge connected with every a_i. (5.12)

Indeed, the set of components that are in fact ultraproducts of nonstandard components of Γ_{NS} is such a cofinal set in the ordering $<_\Gamma$ of all components.

On the other hand no ultrapower of Γ over ω can satisfy this property, because the following holds in any ultrapower Γ^ω/p. There is a complement of an initial set of components C in the $<_\Gamma$ ordering for which:

If $a_i \in C \cap U$ for $i \in \omega_1$, then there is some $b \in V$ connected to all the a_i's. (5.13)

It is easy to establish (5.13) for Γ^ω/p once the following observation is made. Suppose that $h \in \mathrm{DP}$, p is a nonprincipal ultrafilter on ω, and $G = (\Pi_n G_{h(n)})/p$. Then (P_1) implies that G is a bipartite graph with the following property.

If $a_i \in G \cap U$, for $i \in \omega_1$, then there is a $b \in G \cap V$ connected to all the a_i's. (5.14)

This follows by applying P_1 to $A_n = h(n)^2 + 1$, $\{a_\alpha \mid \alpha \in \omega_1\}$, and h.

Now we turn to the consistency result itself.

3.9 Theorem. *Let $\langle A_n \mid n \in \omega\rangle$ be a sequence of finite sets. Let $h \in \mathrm{DP}$ (diverging to ∞ with $h(n) \geq 1$). Then there is a proper, ${}^\omega\omega$-bounding forcing poset P, of size $2^{\aleph_0}$, such that in any generic extension $V[G]$ via P the following holds.*

> *There is in $V[G]$ a sequence $\langle H_n \mid n \in \omega\rangle$, with $H_n \subseteq A_n$ and $|H_n| \leq h(n)$, that eventually bounds every ground-model $f \in \Pi_n A_n$. That is, if $f \in V \cap \Pi_{n<\omega} A_n$, then for some k and for all $n \geq k$, $f(n) \in H_n$.*

To obtain the desired model where P_1 and P_2 hold, assume $\mathrm{CH} + 2^{\aleph_1} = \aleph_2$ and iterate with countable support ω_2 many posets as in the theorem. By CH, the resulting poset satisfies the $\aleph_2$-c.c. (Theorem 2.10). This ensures P_1, since any parameters for P_1 appears in an initial segment of the iteration, so that a suitable bookkeeping device takes care of all possible sequences. P_2 is a consequence of the fact that the resulting poset is ${}^\omega\omega$-bounding and hence the ground reals give a bounding sequence of length ω_1 (by the preservation theorem). We turn to the proof of the theorem.

For any finite set A let $\mathcal{P}(A)$ be the power set of A, and $\mathcal{P}_m(A)$ be the collection of subsets of A of cardinality $\leq m$. We say that $E \subseteq \mathcal{P}(A)$ is a

k-cover (k a natural number) if for every $x \subseteq A$ of size $\leq k$, for some $e \in E$, $x \subseteq e$.

Referring to the given DP function h, define

$$S = \bigcup_{\ell<\omega} \prod_{0 \leq i < \ell} \mathcal{P}_{h(i)}(A_i).$$

That is, $\eta \in S$ iff η is a finite sequence such that for $i \in \mathrm{dom}(\eta)$, $\eta(i)$ is a subset of A_i of size $\leq h(i)$. S forms a tree under extension (inclusion). We will force with infinite subtrees of S that are good in the following sense.

Let $T \subseteq S$ be a subtree (that is, a collection of sequences in S closed under initial segments). We make the following definitions.

1. If $\eta \in T$ then the number $\mathrm{dom}(\eta)$ is also called the *height* of the node η. Let $T\restriction m$ be the collection of all nodes in T of height $< m$.

2. We say that $\eta \in T$ is the *stem* of T if η is comparable to all nodes of T (under inclusion), and η is maximal with this property.

3. A node $\eta \in T$ is said to be *k-covering in T* iff its (immediate) successors in T form a k-cover of the appropriate A_i ($i = \mathrm{dom}(\eta)$). That is, $\{\mu(i) \mid \mu \in T \text{ extends } \eta\}$ is a k-cover of A_i.

Let P be the poset (under inclusion) of all infinite subtrees $T \subseteq S$ that have a stem $\sigma(T)$, and each node $\eta \in T$ is at least 1-covering (except the nodes below the stem which are not 1-covering), and such that, for every k, except for finitely many nodes, all nodes of T are k-covering.

In any forcing extension via P, the generic sequence of stems provides a sequence $H_n \subseteq A_n$ with $|H_n| \leq h(n)$. A density argument shows that every ground model $f \in \Pi_n A_n$ is eventually bounded by the H_n's. In this argument, use the obvious remark that if $E \subseteq P_m(A)$ is a k-cover, and $a \in A$, then the collection of $e \in E$ such that $a \in e$ form a $(k-1)$-cover of A. Both properness and the ${}^\omega\omega$-bounding property follow once we prove that P satisfies Axiom A^* of Baumgartner (see Definition 2.3). For this we define relations $\leq_k$, for $0 \leq k < \omega$, on the trees in P.

$\leq_0$ is just the poset ordering (inverse inclusion).

Define $T_1 \leq_1 T_2$ iff T_2 is a pure extension of T_1: that is, T_2 extends T_1, and they have the same stem.

Define $T_1 \leq_k T_2$ for $k > 1$ iff $T_1 \leq_1 T_2$, $T_1\restriction k = T_2\restriction k$ and for every $i \leq k$, for any $\eta \in T_2$, if η is i-covering in T_1 then η remains i-covering in T_2. The following is a direct consequence of the definitions.

3.10 Lemma. *$\leq_k$ is transitive, and $k < \ell$ implies that $\leq_\ell \subseteq \leq_k$.*

3.11 Lemma. *If $T_1 \leq_1 T_2 \leq_2, \ldots, T_n \leq_n T_{n+1}, \ldots,$ then a fusion $T \in P$ can be defined such that $T_i \leq_i T$ for all i.*

Proof. Indeed, $T = \bigcup_{1 \leq i < \omega} T_i\restriction i$ works. $\dashv$

Given a name $\underset{\sim}{\tau}$ of an ordinal we must show that every $T \in P$ has a $\leq_k$ extension that decides $\underset{\sim}{\tau}$ up to finitely many possibilities. (This can be seen to be an equivalent formulation of item 2 in Definition 2.3.)

Say that a tree T is *m-covering* if every node in T (not below the stem) is m-covering. For any $T \in P$ and $\eta \in T$, let $T(\eta)$ be the subtree of T obtained by letting η to be the stem. The following lemma suffices to prove Axiom A^*.

3.12 Lemma. *Let $\underset{\sim}{\tau}$ be a P-name of an ordinal. If $m \geq 2k$ and T is m-covering, then T has a pure extension T' that is k-covering and such that for some finite set B of ordinals, $T' \Vdash \underset{\sim}{\tau} \in B$.*

Proof. A node η of T is *good* iff $T(\eta)$ has a pure, k-covering extension that decides $\underset{\sim}{\tau}$ up to finitely many possibilities. η is *bad* if it is not good. So, the lemma says that the stem of T is good.

Let X be a set of successors of some η in T; we say that X is a *majority set* if X is k-covering. More formally, X is a majority set if for $i = |\eta|$ the collection $\{\mu(i) \mid \mu \in X\}$ is a k-cover of A_i.

Observe that since any $\eta \in T$ is m-covering and $m \geq 2k$, if the set of successors of η is given as a union $X_1 \cup X_2$, then X_1 or X_2 is a majority set. (For otherwise there are sets $x_1, x_2 \subseteq A_i$ of size k each such that x_1 is not covered by the nodes of X_1 and x_2 is not covered by the nodes of X_2. But then $x_1 \cup x_2$ is of size $\leq m$ and is not covered by any successor of η!) Hence if $\eta \in T$ is bad, then the bad successors of η form a majority set.

For any trees T_1, T_2 we say that T_2 is a majority extension of T_1 if $T_2 \geq T_1$ is obtained by taking only majority sets of successors in T_1. Equivalently, T_2 is a pure extension of T_1 which is k-covering.

Now if the lemma does not hold and the stem is bad, then there is a majority extension T' of T consisting entirely of bad nodes. This is impossible: pick any $T'' \geq T'$ that decides $\underset{\sim}{\tau}$, and find in T'' a node η such that $T(\eta)$ is k-covering. Then η must be good (already in T). $\dashv$

4. Preservation of Unboundedness

This section is adapted from [13] (reworked in [15, Chap. VI]). A forcing poset P is said to be *weakly ${}^{\omega}\omega$-bounding* if the old reals are not bounded in the extension. That is, the following holds in every extension $V[G]$ via P: for any $f \in {}^{\omega}\omega \cap V[G]$ there is a $g \in V$ such that $\{n \mid f(n) \leq g(n)\}$ is infinite. For example, the Cohen-real forcing is weakly ${}^{\omega}\omega$-bounding. (Given $f \in V^P$, let $\{c_n \mid n \in \omega\}$ enumerate all Cohen conditions, and define $g(n)$ so that some extension of c_n forces that $g(n) = \underset{\sim}{f}(n)$.)

4.1 Theorem. *The weak ${}^{\omega}\omega$-bounding property is preserved by the limit of a countable support iteration of proper posets if each initial part of the iteration is weakly ${}^{\omega}\omega$-bounding.*

Thus, if δ is a limit and $\langle P_i \mid i \leq \delta\rangle$ is a countable support iteration of proper posets and every P_i for $i < \delta$ is weakly ${}^{\omega}\omega$-bounding, then P_δ is weakly ${}^{\omega}\omega$-bounding as well.

Observe the difference between the formulation of this theorem and that of Theorem 3.5: here we speak about initial parts of the iteration—not about the iterands. In the next subsection we will explain why the iteration of weakly ${}^{\omega}\omega$-bounding posets is not necessarily weakly ${}^{\omega}\omega$-bounding, and we will define the notion of almost bounding and show that the iteration of almost bounding posets is weakly ${}^{\omega}\omega$-bounding.

Theorem 4.1 is proved by induction on δ. Let $\underset{\sim}{f}$ be a name for a real in V^{P_δ}, and $p_0 \in P_\delta$ an arbitrary condition. Pick $M \prec H_\kappa$ countable, with $P_\delta, p_0, \underset{\sim}{f} \in M$ as usual. Fix an increasing sequence $\gamma_i \in \delta \cap M$ converging to $\sup(\delta \cap M)$. Let $g \in {}^{\omega}\omega$ $<^*$-dominate all the reals of M. We will find in P_δ an extension q of p_0 that forces

$$\{n \in \omega \mid \underset{\sim}{f}(n) \leq g(n)\} \quad \text{is infinite.}$$

As before, we define by induction conditions $q_n \in P_{\gamma_n}$ that are (M, P_{γ_n})-generic, and names $\underset{\sim}{p_n} \in V^{P_{\gamma_n}}$ such that:

1. $q_{n+1} \restriction \gamma_n = q_n$.
2. $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p_n}$ is in $P_\delta \cap M$ and extends $\underset{\sim}{p_{n-1}}$, and $\underset{\sim}{p_n} \restriction \gamma_n \in \underset{\sim}{G_n}$ (the generic filter over P_{γ_n})".
3. $q_n \Vdash_{\gamma_n}$ "$\underset{\sim}{p_n} \Vdash_\delta$ for some $k \geq n$, $\underset{\sim}{f}(k) \leq g(k)$".

When done, $q = \bigcup_n q_n$ is in P_δ, and for every n

$$q \Vdash_\delta \underset{\sim}{p_n} \in G_\delta$$

(we have seen that in proving the Properness Extension Lemma 2.8). Hence q "knows" what every interpretation of $\underset{\sim}{p_n}$ knows, i.e., $q \Vdash_\delta$ for some $k \geq n$, $\underset{\sim}{f}(k) \leq g(k)$. This holds for every n. Hence q is as required.

We now turn to the inductive definition. To begin with, $\underset{\sim}{p_0}$ is in fact a condition—the given p_0—and $q_0 \in P_{\gamma_0}$ is an (M, P_{γ_0})-generic condition extending $p_0 \restriction \gamma_0$.

Suppose that q_n and $\underset{\sim}{p_n}$ are defined; we shall obtain $\underset{\sim}{p_{n+1}}$ and then q_{n+1}. Imagine a generic extension $V[G_n]$, where $q_n \in G_n \subseteq P_{\gamma_n}$. Then $\underset{\sim}{p_n}$ is realized as some condition $p_n \in P_\delta \cap M$ such that $p_n \restriction \gamma_n \in G_n$.

In $M[G_n]$, define an increasing sequence $\langle r_i \mid i \in \omega\rangle$ beginning with $r_0 = p_n$, of conditions in P_δ that decide the values of $\underset{\sim}{f}$, and such that $r_i \restriction \gamma_n \in G_n$ (use Lemma 1.2). Let f^* be the real thus interpreted; so for every $k < \omega$, r_k forces (in P_δ forcing) that $\underset{\sim}{f} \restriction k = f^* \restriction k$. Obviously $f^* \in M[G_n]$.

Since P_{γ_n} is weakly ${}^{\omega}\omega$-bounding, for some $h \in {}^{\omega}\omega \cap M$, $h(i)$ is above $f^*(i)$ for infinitely many i's. But $h <^* g$, and hence for some $i_0 \geq n+1$, $f^*(i_0) < g(i_0)$. For $j = i_0 + 1$, r_j fixes the value of $\underset{\sim}{f}(i_0)$ to be $f^*(i_0)$. Now define $\underset{\sim}{p}_{n+1}$ to be a P_{γ_n}-name of r_j. Finally, q_{n+1} is defined by the Properness Extension Lemma to be a condition in $P_{\gamma_{n+1}}$ such that $q_{n+1}\restriction\gamma_n = q_n$ and

$$q_{n+1} \Vdash_{\gamma_{n+1}} \underset{\sim}{p}_{n+1}\restriction\gamma_{n+1} \in \underset{\sim}{G}_{n+1}.$$

The proof is finished, but it is worth remarking that the definition of $\underset{\sim}{p}_{n+1}$ depends on g, the function that dominates M, hence $\underset{\sim}{p}_{n+1}$ cannot be defined in M. But of course it is always realized as some condition in M.

4.1. The Almost Bounding Property

The successor case, which causes no problem for the ${}^\omega\omega$-bounding property, is not obvious at all for the weakly ${}^\omega\omega$-bounding property. In fact, it is possible to have Q_1 weakly ${}^\omega\omega$-bounding, $Q_2 \in V$ weakly ${}^\omega\omega$-bounding in V^{Q_1}, yet $Q_1 \times Q_2$ adds a dominating real. For example, add $\aleph_1$ many Cohen reals (this is Q_1), and then do the Hechler forcing with conditions from V. (Hechler [6] posets adds a generic function in ${}^\omega\omega$ by giving finite information on the generic function, and a function in ${}^\omega\omega$ which the generic must from now on dominate. See also Jech [7].) Now, though Q_2 adds a dominating real to V, it is an exercise to see that Q_2 is weakly ${}^\omega\omega$-bounding in V^{Q_1}, because the Q_2-name of any real is already in $V^{Q_1\restriction\alpha}$ for some countable α.

In order to tackle the successor stage, we introduce a notion that is of intermediate strength between weakly ${}^\omega\omega$-bounding and ${}^\omega\omega$-bounding—*almost* ${}^\omega\omega$-bounding.

4.2 Definition. A poset Q is called *almost ${}^\omega\omega$-bounding* iff for every Q-name, $\underset{\sim}{f} \in {}^\omega\omega$, and condition $q \in Q$, there exists $g \in {}^\omega\omega$ such that

For every infinite $A \subseteq \omega$, there is a $q' \geq q$ such that: $(*)$

$$q' \Vdash \text{for infinitely many } n \in A,\ \underset{\sim}{f}(n) \leq g(n). \tag{5.15}$$

Notice the order of quantification: $\exists g \in {}^\omega\omega \forall A \subseteq \omega$. If it is reversed, then (for proper posets) this property becomes the weak ${}^\omega\omega$-bounding property.

4.3 Lemma. *If P is weakly ${}^\omega\omega$-bounding, and $Q \in V^P$ is almost ${}^\omega\omega$-bounding (in V^P), then $P * Q$ is weakly ${}^\omega\omega$-bounding.*

Proof. Let $\underset{\sim}{f}$ be a $P * Q$-name, and $(p, q) \in P * Q$ a condition that forces $\underset{\sim}{f} \in {}^\omega\omega$. We will find a generic extension via $P * Q$ with a filter containing (p, q) in which $\underset{\sim}{f}$ is weakly bounded by some function in V. First take a (V, P)-generic G with $p \in G$. Working in $V[G]$, $\underset{\sim}{f}$ "becomes" a name in Q-forcing of a real, and we continue to denote this name by $\underset{\sim}{f}$. Q "is" now an almost ${}^\omega\omega$-bounding forcing poset, and $q \in Q$ a condition. By definition, there is in $V[G]$ a function $g \in {}^\omega\omega$ such that $(*)$ (in Definition 4.2) holds. Since P is weakly ${}^\omega\omega$-bounding, g is weakly bounded, say by $h \in V$, and

$$A = \{n \mid g(n) \leq h(n)\}$$

is infinite, in $V[G]$. So there is an extension q' of q in Q for which (5.15) of $(*)$ holds. If the second generic extension is done with q' in the generic filter, then for an infinite subset $A_0 \subseteq A$, $f(n) \leq g(n)$ holds for $n \in A_0$. Thus $\underset{\sim}{f}$ is weakly dominated by $h \in V$. $\dashv$

By combining Theorem 4.1 and Lemma 4.3 we get the following theorem.

4.4 Theorem. *The iteration of almost ${}^{\omega}\omega$-bounding, proper posets is weakly ${}^{\omega}\omega$-bounding.*

4.2. Application to Cardinal Invariants

This section deals with two cardinal invariants $\mathfrak{b}$ and $\mathfrak{s}$ of the continuum. For additional information on these cardinals the reader may consult Blass's chapter in this Handbook. Following [13] we will establish here the consistency of

$$\text{bounding number} < \text{splitting number}.$$

The bounding number $\mathfrak{b}$ is the smallest cardinality of an unbounded subset of ${}^{\omega}\omega$ (in the eventual dominance ordering $<^*$). In what follows, $[\omega]^\omega$ denotes the set of infinite subsets of ω, and $\subseteq^*$ between members of $[\omega]^\omega$ denotes eventual inclusion, i.e. $A \subseteq^* B$ *iff* $A \setminus B$ is finite.

The splitting number $\mathfrak{s}$ is the smallest cardinality of a "splitting" set $S \subseteq [\omega]^\omega$, where S is splitting iff for every infinite $A \subseteq \omega$, some $B \in S$ splits A, that is, both $A \cap B$ and $A \setminus B$ are infinite. In other words, say that $A \subseteq \omega$ *makes an ultrafilter on* $S \subseteq [\omega]^\omega$ if for every $B \in S$ either $A \subseteq^* B$ or $A \subseteq^* \omega \setminus B$. Thus S is splitting iff no A makes an ultrafilter on S.

4.5 Theorem. *Assume* CH. *There is a generic extension in which $2^{\aleph_0} = \aleph_2$, cardinals are not collapsed, and $\mathfrak{b} < \mathfrak{s}$.*

The general structure of the consistency proof for this theorem is to iterate ω_2 almost ${}^{\omega}\omega$-bounding proper forcings that "kill" the old reals as a splitting family. Finally, by Theorem 4.4, the reals of the ground model are still not dominated, and hence $\mathfrak{b} = \aleph_1$, but $s = \aleph_2$ because no set of size $\aleph_1$ can be splitting. This is so because every set of reals of size $\aleph_1$ is included in some stage $\gamma < \omega_2$ of the iteration and hence was "killed" at the following stage (by introducing some A that makes an ultrafilter on $\mathcal{P}(\omega) \cap V_\gamma$). Thus we only need the following.

4.6 Theorem. *There is a proper, almost ${}^{\omega}\omega$-bounding poset Q of size $2^{\aleph_0}$ such that in V^Q:*

> *There is an infinite set $A \subseteq \omega$ such that for every $B \subseteq \omega$ from V, $A \subseteq^* B$ or $A \subseteq^* \omega \setminus B$.*

Proof. The first forcing notion that comes to mind is Mathias forcing [11]. It consists of pairs (u, E) where u is a finite and E an infinite subset of ω.

Extension is defined by $(u_1, E_1) \leq (u_2, E_2)$ iff $E_2 \subseteq E_1$, u_2 is an end-extension of u_1, and $u_2 \setminus u_1 \subseteq E_1$. If G is a generic filter, then $U = \bigcup\{u \mid (u, E) \in G$ for some $E\}$ makes an ultrafilter on $\mathcal{P}(\omega) \cap V$ (it is not split by any subset of ω in the ground model). However, this forcing introduces a dominating real—the enumeration of the generic subset—and hence we must search for another solution.

The conditions in Q will be pairs (u, T) such that $u \subseteq \omega$ is finite, and $T = \langle t_i \mid i \in \omega \rangle$ is a sequence of "logarithmic measures". Each t_i consists of a finite subset s_i of ω, also denoted $\mathrm{int}(t_i)$, and a finite measure, specified below, defined on all subsets of $\mathrm{int}(t_i)$ and taking natural number values. We have that $\max(u) < \min(s_i) \leq \max(s_i) < \min(s_{i+1})$. Define $\mathrm{int}(T) = \bigcup_i \mathrm{int}(t_i)$; this is an infinite set of numbers in ω above u, and the order on Q will be such that if $(u_1, T_1) \leq (u_2, T_2)$ holds, then $(u_1, \mathrm{int}(T_1)) \leq (u_2, \mathrm{int}(T_2))$ as Mathias conditions. The reason that the reals in V do not split the generic real $U = \bigcup\{u \mid \exists T (u, T) \in G\}$ is the same as for the Mathias forcing: it will be shown that if $(u_1, T_1) \in Q$ then whenever $\mathrm{int}(T_1) = x \cup y$, there is an extension (u_2, T_2) of (u_1, T_1) in Q such that $\mathrm{int}(T_2) \subseteq x$ or $\mathrm{int}(T_2) \subseteq y$. To define Q we need first the notion of "logarithmic measure".

A *logarithmic measure* on $S \subseteq \omega$ (S is usually finite) is a function $h : \mathcal{P}_\omega(S) \to \omega$ (where $\mathcal{P}_\omega(S)$ is the collection of all *finite* subsets of S) and such that if $A \cup B \subseteq S$ is finite and $h(A \cup B) \geq \ell + 1$ then $h(A) \geq \ell$ or $h(B) \geq \ell$. It follows that if $h(A_0 \cup \cdots \cup A_{n-1}) > \ell$ then $h(A_j) \geq \ell - j$ for some $0 \leq j < n$.

When h is a logarithmic measure on S and S is finite, then $h(S)$ is called the level of h, and is denoted $\mathrm{level}(h)$.

Actually, our measures will all be induced by a collection of positive sets as follows: Given a collection $P \subseteq \mathcal{P}_\omega(S)$ that is closed upwards ($a \in P$ and $a \subseteq b$ imply $b \in P$), a logarithmic measure h *induced by* P is inductively defined as follows on $\mathcal{P}_\omega(S)$:

1. $h(e) \geq 0$ for every $e \in \mathcal{P}_\omega(S)$.
2. $h(e) > 0$ iff $e \in P$.
3. For $\ell \geq 1$, $h(e) \geq \ell + 1$ iff $|e| > 1$ and whenever $e = e_1 \cup e_2$ then $h(e_1) \geq \ell$ or $h(e_2) \geq \ell$.

Then $h(e) = \ell$ iff ℓ is the maximal natural number such that $h(e) \geq \ell$ (there has to be such ℓ and $h(e) < \infty$).

We have, for measures defined by positive sets, that if $h(e) = k$ and $e \subseteq a$ then $h(a) \geq k$.

For example, if the positive sets are those containing at least two points, then $h(X)$ is the least i for which $|X| \leq 2^i$. We will use the following observation.

4.7 Lemma. *Let $P \subseteq \mathcal{P}_\omega(\omega)$ be an upwards closed collection (of finite non-empty sets). The following condition implies that the measure h induced by P on $\mathcal{P}_\omega(\omega)$ has arbitrarily high values:*

For every decomposition $\omega = A_1 \cup \cdots \cup A_n$ into finitely many sets, for some i, $\mathcal{P}_\omega(A_i) \cap P \neq \emptyset$.

Assuming this condition, for every $k < \omega$ and decomposition $\omega = \bigcup_{i<n} A_i$ into $n < \omega$ sets, for some $i < n$, $h(e) \geq k$ for some $e \subseteq A_i$.

Proof. Suppose that P satisfies the condition of the lemma, and we shall prove by induction on $k < \omega$ the required conclusion.

For $k = 1$, this is just the assumed condition. Assume the claim for $k = \ell$, and let us prove it for $\ell + 1$. Let $\omega = \bigcup_{i<n} A_i$ be a decomposition such that (contrary to our lemma) for every $j < \omega$, for all $i < n$, $h(A_i \cap j) \not\geq \ell + 1$. Thus for every $i < n$ there are e_1 and e_2 such that $A_i \cap j = e_1 \cup e_2$ and both $h(e_1) \not\geq \ell$ and $h(e_2) \not\geq \ell$. Kőnig's lemma can be used to find a decomposition $A_i = E^i_1 \cup E^i_2$ such that there is no x with $h(x) \geq \ell$ included in E^i_1 or in E^i_2. Hence a decomposition of ω into $2n$ sets contradicts the inductive assumption for ℓ. ⊣

To prove Theorem 4.6, we define the poset Q which was informally described above. Q consists of all pairs $p = (u, T)$ where

1. $u \subseteq \omega$ is finite (called the *stem* of p) and
2. $T = \langle t_i \mid i \in \omega \rangle$ is a sequence of measures $t_i = (s_i, h_i)$ where h_i is a logarithmic measure on s_i, a finite subset of ω ($s_i = \mathrm{int}(t_i)$), such that
   (a) $\max(u) < \min(s_0)$.
   (b) $\max(s_i) < \min(s_{i+1})$.
   (c) The level of the measures diverges to infinity, and, moreover, $\mathrm{level}(h_i) < \mathrm{level}(h_{i+1})$. (We defined $\mathrm{level}(h_i) = h_i(s_i)$.)

Recall that $\mathrm{int}(T) = \bigcup\{s_i \mid i \in \omega\}$. For convenience, we write $p = (u, T)$ even when $\max(u)$ is not below $\min(s_0)$. This p refers then to the condition obtained by throwing away sufficiently many t_i's, that is, $p = (u, \langle t_i \mid i \geq k \rangle)$ where k is first such that $\max(u) < \min(s_k)$.

The extension relation for Q is defined by:

$$(u_1, T_1) \leq (u_2, T_2),$$

where $T_\ell = \langle t^\ell_i \mid i \in \omega \rangle, t^\ell_i = (s^\ell_i, h^\ell_i)$ for $\ell = 1, 2$, if and only if

1. u_2 is an end extension of u_1, and $u_2 \setminus u_1 \subseteq \mathrm{int}(T_1)$.
2. $\mathrm{int}(T_2) \subseteq \mathrm{int}(T_1)$. Moreover, there is a sequence of finite subsets of ω, $\langle B_i \mid i \in \omega \rangle$ with $\max(B_i) < \min(B_{i+1})$ and $\max(u_2) < \min(s^1_j)$ for $j = \min(B_0)$, such that $s^2_i \subseteq \bigcup\{s^1_j \mid j \in B_i\}$.
3. For every i: if $e \subseteq s^2_i$ is h^2_i-positive (i.e., $h^2_i(e) > 0$), then for some j, $e \cap s^1_j$ is h^1_j-positive.

The reader may check that this defines an order on Q. An extension that does not change the stem is called a *pure extension.* Observe that if (w, R) extends (v, T) then (w, R) extends (w, T) as well. That is, any extension can be formed by first extending the stem and then taking a pure extension. We shall prove that Q is proper and almost ${}^\omega\omega$-bounding.

For properness, Axiom A will be shown to hold. The $\leq_n$ relations needed are defined as follows. $<_0$ is the extension relation on Q just defined. For $n > 0$,

$$(u_1, T_1) \leq_n (u_2, T_2) \quad \text{iff} \quad \begin{array}{l}(u_2, T_2) \text{ is a pure extension of} \\ (u_1, T_1) \text{ and for } 0 \leq i < n-1,\ h_i^1 = h_i^2.\end{array}$$

That is, the stem and the first $n-1$ measures (and sets) are the same in both conditions. In particular $(u_1, T_1) \leq_1 (u_2, T_2)$ iff (u_2, T_2) is a pure extension of (u_1, T_1).

We can check the fusion property. Suppose that $p_0 \leq_1 p_1 \leq_2 \cdots \leq_{i-1} p_{i-1} \leq_i p_i \ldots$ is a fusion sequence, where $p_\ell = (u_\ell, \langle t_i^\ell \mid i \in \omega\rangle)$. Then set $p = (u, T)$ by $u = u_0$, and $T = \langle t_i \mid i \in \omega\rangle$ defined as $t_i = t_i^{i+1}$ (p takes the common stem u, and the measure t_i that is common to all the conditions with indices above i) then $p \in Q$ and $p_i \leq_i p$ for all i.

To verify Axiom A, we have to prove that for any $n \in \omega$, $p = (u, T)$, and dense open set D, there are a countable $D_0 \subseteq D$ and an extension $p \leq_n p_0$, such that D_0 is predense above p_0.

We say that a condition (u, T) (with $T = \langle t_i \mid i \in \omega\rangle$) is *preprocessed* for D and i iff for every $v \subseteq i$ that is an end extension of u, if $(v, \langle t_j \mid j \geq i\rangle)$ has a pure extension in D, then $(v, \langle t_j \mid j \geq i\rangle)$ is already in D. The following can be easily proved:

1. If (u, T) is preprocessed for D and i, then any extension is also preprocessed for D and i.

2. Any given condition has a $\leq_{i+1}$ extension that is preprocessed for D and i.

3. Hence by taking the fusion of a sequence, one may obtain an extension of any given condition that is preprocessed for every i.

Now if $p_0 = (u, T)$ is preprocessed for every i and D_0 is the set of all conditions in D of the form $(v, \langle t_j \mid j \geq i\rangle)$, then D_0 is predense above p_0. Thus Axiom A holds.

The almost ${}^\omega\omega$-bounding property of Q is of course the main point.

4.8 Lemma (Main Lemma). *Let $\underset{\sim}{f}$ be a Q-name for a function in ${}^\omega\omega$, and $q \in Q$ a condition. There is a pure extension $p \geq q$, $p = (u, T)$, $T = \langle t_i \mid i \in \omega\rangle$, with the following property:*

> *For any i and $s \subseteq \mathrm{int}(t_i)$ that is t_i-positive, if $v \subseteq i$ then for some $w \subseteq s$, $((v \cup w), T)$ determines the value of $\underset{\sim}{f}(i)$.*

We first show how this lemma implies the almost bounding property (Definition 4.2). Given $\underset{\sim}{f}$ and q, let $p \in Q$ be as in the lemma. For each i, define

$$g(i) = \max\{k \mid \text{for some } v \subseteq i \text{ and } w \subseteq \operatorname{int}(t_i),\ (v \cup w, T) \Vdash \underset{\sim}{f}(i) = k\}.$$

Now let $A \subseteq \omega$ be any infinite set. Put $p' = (u, \langle t_i \mid i \in A\rangle)$. Then p' extends p and

$$p' \Vdash \text{ for infinitely many } i \in A,\ \underset{\sim}{f}(i) \leq g(i).$$

To see this, let p'' be any extension of p' and k an arbitrarily high integer. Say $p'' = (v, R)$ where $R = \langle r_i \mid i \in \omega\rangle$. Find $i > k$, $i \in A$, such that $v \subseteq i$, and $\operatorname{int}(R) \cap \operatorname{int}(t_i)$ is t_i-positive (there is such an i by the definition of extension in Q). Using the property of the Main Lemma for $s = \operatorname{int}(R) \cap \operatorname{int}(t_i)$, let $w \subseteq s$ be such that $(v \cup w, T)$ decides the value of $\underset{\sim}{f}(i)$. Then $(v \cup w, R)$ extends p'' and makes the same decision because it is also an extension of $(v \cup w, T)$.

Now we turn to the proof of the Main Lemma. The required condition p is obtained as a fusion of a sequence defined inductively in ω steps. At the ith step we have a condition $p_i = (u, \langle t_j \mid j \in \omega\rangle)$ and we define p_{i+1} so that $p_i \leq_{i+1} p_{i+1}$. That is, u and $t_0, \dots, t_{i-1}$ are not touched in the extension. We start with $T = \langle t_j \mid j \geq i\rangle$ and apply the following lemma 2^i times, considering each $v \subseteq i$ in turn.

4.9 Lemma. *Let $(\emptyset, T)$ be a condition, $\underset{\sim}{f}$ a name for a function in ${}^{\omega}\omega$, and i any natural number. Fix $v \subseteq i$. There is a pure extension $(\emptyset, R)$ of $(\emptyset, T)$ with $R = \langle r_\ell \mid \ell \in \omega\rangle$ such that for every ℓ and r_ℓ-positive $s \subseteq \operatorname{int}(r_\ell)$, for some $w \subseteq s$, $(v \cup w, \langle r_m \mid m > \ell\rangle)$ determines the value of $\underset{\sim}{f}(i)$. (Observe that any further pure extension of $(\emptyset, R)$ retains this property.)*

Proof. We may assume that $(\emptyset, T)$ (and thence any extension) is preprocessed for $\underset{\sim}{f}(i)$: If an extension (w, R) determines the value of $\underset{\sim}{f}(i)$, then already (w, T) determines that value.

Define a measure h on $\operatorname{int}(T)$ induced by the following positive sets. A finite set $x \subseteq \operatorname{int}(T)$ is *positive* iff

1. For some l, $x \cap \operatorname{int}(t_l)$ is t_l-positive. (t_l are the measures composing T.)

2. For some $w \subseteq x$, $(v \cup w, T)$ determines the value of $\underset{\sim}{f}(i)$.

1. ensures that if $(\emptyset, R)$ is obtained by taking a sequence of subsets of $\operatorname{int}(T)$ with increasing h-measures, then $(\emptyset, R)$ is an extension of $(\emptyset, T)$. 2. ensures that this extension has the required properties.

It remains to check that the basic property required to obtained arbitrarily high values of h holds (Lemma 4.7). So let $\operatorname{int}(T) = A_0 \cup \cdots \cup A_{n-1}$ be a partition, and we will find some A_ℓ that contains a positive set. Because the measures t_i are logarithmic and increasing to infinity, for some $\ell < n$ there exists an infinite index set $I \subseteq \omega$ such that the t_i-measures of $A_\ell \cap \mathrm{int}(t_i)$ for $i \in I$ are diverging to infinity. (Otherwise, for every $\ell < n$ there is a finite bound on the t_i measures of $A_\ell \cap \mathrm{int}(t_i)$ for $i \in \omega$, and hence there is a bound k on the measures of $A_\ell \cap \mathrm{int}(t_i)$ where $\ell < n$ and $i \in \omega$. But this is impossible when the measure of t_i is greater than $k + n$.) We may thus find an extension of (v, T) of form (v, R) such that $\mathrm{int}(R) \subseteq A_\ell$. Now pick any extension $(v \cup w, R')$ of (v, R) that decides the value of $\underset{\sim}{f}(i)$. Then already $(v \cup w, T)$ decides that value. This shows the existence of a positive subset of A_ℓ, namely a finite union x of $\mathrm{int}(r_m)$'s such that $w \subseteq x$. $\dashv$

This completes the proof of Theorem 4.6. $\dashv$

5. No New Reals

This last section deals with proper posets that add no new reals, that is, introduce no new subsets of ω in any generic extension. (Such posets add no countable sequences of ordinals either, but the shorter expression is the customary description.) It follows from the work of Jensen and Johnsbråten [8] that the countable support iteration of forcing posets that add no new reals may well add a new real. This shows the need for more complex schemes for iterating posets that add no new reals, and the notion of Dee-completeness is simpler than any other scheme introduced by Shelah for that purpose. The preservation proof that we present here uses the notion of α-properness, and we therefore begin with this notion (following Shelah [15, Chap. V]). Our aim is to explain Dee-completeness by means of a simple example (in Sect. 5.2), and then to give a rather detailed proof of the Dee-completeness Iteration Theorem 5.17.

5.1. α-Properness

Let $\alpha > 0$ be a countable ordinal and $\bar{M} = \langle M_i \mid i < \alpha \rangle$ a sequence of countable, elementary substructures of H_λ (where λ is some fixed regular cardinal). We say that $\bar{M}$ is an *α-tower* iff

1. For every limit $\delta < \alpha$, $M_\delta = \bigcup_{i<\delta} M_i$.
2. For every $j < \alpha$, $\langle M_i \mid i \leq j \rangle \in M_{j+1}$.

Since λ is regular (or, at least, $\mathrm{cf}(\lambda) > \aleph_0$) if $M \subseteq H_\lambda$ is countable then $M \in H_\lambda$. Thus, for $j < \alpha$, $\langle M_i \mid i \leq j \rangle \in H_\lambda$ so that 2. makes sense.

5.1 Definition. Let $\alpha > 0$ be a countable ordinal. A forcing poset P is *α-proper* iff for sufficiently large λ and every α-tower $\bar{M} = \langle M_i \mid i < \alpha \rangle$ of countable, elementary substructures of H_λ such that $P \in M_0$, the following holds: Every $p \in P \cap M_0$ has an extension $q \geq p$ that is (M_i, P)-generic for every $i < \alpha$. We say that q is $(\bar{M}, P)$-generic in this case.

Clearly, properness is 1-properness. We say that P is *$<\omega_1$-proper* if it is α-proper for every countable ordinal α.

Any c.c.c. poset is $<\omega_1$-proper. Any countably closed poset is $<\omega_1$-proper. In proving this, one sees why each successor structure in the tower needs to contain the sequence of structures up to that point.

Another example is given by Axiom A posets (Definition 2.3). Let P be an Axiom A poset and prove by induction on $\alpha < \omega_1$ that P is α-proper. For $\alpha = \omega$ we argue as follows. Let $\bar{M} = \langle M_i \mid i < \omega \rangle$ be a tower of countable, elementary substructures of H_λ with $P \in M_0$, and let $p_0 \in P \cap M_0$ be a given condition. Construct by induction conditions $p_i \in P$ such that:

$$p_i \leq_i p_{i+1}, \quad \text{and} \quad p_{i+1} \in M_{i+1} \text{ is } (M_i, P)\text{-generic.}$$

Let q be the fusion condition, satisfying $p_i \leq_i q$ for all i. Then $p_i \leq q$ and q is thence (M_i, P)-generic.

It is not difficult to check that if P is α-proper then it is $(\alpha+1)$-proper. So properness implies n-properness for every $n < \omega$. It does not imply ω-properness. If $\alpha = \beta_1 + \beta_2$ is a sum of two smaller ordinals, then any poset that is both β_1 and β_2 proper is also α proper. So, for α-properness, the values that really count are indecomposable countable ordinals.

Equivalent Definition

As for properness, it is useful to know that if P and Q are posets, P is α-proper and Q is α-proper in V^P, then Q is α-proper already in V. A suitable notion of closed unbounded sets is introduced which is the basis for an equivalent definition of α-properness, from which that useful fact follows. Recall that $\mathcal{P}_{\aleph_1}(A)$ is the collection of all countable subsets of A.

5.2 Definition. Let A be an uncountable set and α a countable ordinal.

1. $P^\alpha_{\aleph_1}(A)$ is the set of all increasing and continuous sequences $\langle a_i \mid i < \alpha \rangle$ where $a_i \in \mathcal{P}_{\aleph_1}(A)$ for all $i < \alpha$. (The sequence is increasing if $a_i \subseteq a_j$ for $i < j$, and it is continuous if for limit $\delta < \alpha$, $a_\delta = \bigcup_{i<\delta} a_i$.)

2. Let $F : (\bigcup_{\beta<\alpha} P^\beta_{\aleph_1}(A)) \times [A]^{<\aleph_0} \to \mathcal{P}_{\aleph_1}(A)$ be given. We say that F is an *α-function.* A sequence $\langle a_i \mid i < \alpha \rangle \in P^\alpha_{\aleph_1}(A)$ is said to be *closed* under F if for every $\beta < \alpha$ that is a successor ordinal or zero, for every $x \in [a_\beta]^{<\aleph_0}$, $F(\langle a_i \mid i < \beta \rangle, x) \subseteq a_\beta$. So a_0 is closed under the function taking $x \in [a_0]^{<\aleph_0}$ to $F(\emptyset, x)$; a_1 is closed under the function taking $x \in [a_1]^{<\aleph_0}$ to $F(\langle a_0 \rangle, x)$, and so forth.

3. Let $G(F) \subseteq P^\alpha_{\aleph_1}(A)$ be the collection of all α-sequences that are closed under F. Then $\{G(F) \mid F \text{ is an } \alpha\text{-function}\}$ generates a countably closed filter on $P^\alpha_{\aleph_1}(A)$, which is denoted $\mathcal{D}^\alpha_{\aleph_1}(A)$.

4. We say that $S \subseteq P^\alpha_{\aleph_1}(A)$ is *stationary* if its complement is not in $\mathcal{D}^\alpha_{\aleph_1}(A)$.

Useful examples of $\mathcal{D}^{\alpha}_{\aleph_1}(A)$ sets are the following:

1. The collection of all α-towers $\langle M_i \mid i < \alpha\rangle$ of countable elementary substructures of H_λ. Here A is the set H_λ, and M_i refers to the universe of that structure.

2. For a closed unbounded set $C \subseteq \mathcal{P}_{\aleph_1}(A)$, collect all sequences $\langle a_i \mid i < \alpha\rangle \in P^{\alpha}_{\aleph_1}(A)$ such that $a_i \in C$ for all i.

In a sense, $\mathcal{D}^{\alpha}_{\aleph_1}(A)$ is normal. If $g : \mathcal{P}_{\aleph_1}(A) \to A$ is a choice function (namely $g(x) \in x$ whenever x is non-empty) and $S \subseteq P^{\alpha}_{\aleph_1}(A)$ is stationary, then for some fixed $v \in A$, $\{\langle a_i \mid i < \alpha\rangle \in S \mid g(a_0) = v\}$ is stationary.

The following is standard.

5.3 Lemma. *Suppose that $A_0 \subseteq A_1$ are uncountable and $C_1 \in \mathcal{D}^{\alpha}_{\aleph_1}(A_1)$. Define $C_0 = \{\langle a_i \cap A_0 \mid i < \alpha\rangle \mid \langle a_i \mid i < \alpha\rangle \in C_1\}$. Then $C_0 \in \mathcal{D}^{\alpha}_{\aleph_1}(A_0)$.*

The proof of the following theorem resembles that of the Properness Equivalents Theorem 2.13.

5.4 Theorem. *For any poset P and countable ordinal α the following are equivalent.*

1. *P is α-proper (as in Definition 5.1).*

2. *For some $\lambda > 2^{|P|}$, for every α-tower $\bar{M}$ of countable elementary substructures of H_λ, any condition in M_0 has an extension that is $(\bar{M}, P)$-generic.*

3. *For every uncountable λ, P preserves stationary subsets of $P^{\alpha}_{\aleph_1}(\lambda)$.*

4. *For $\lambda_0 = 2^{|P|}$, P preserves stationary subsets of $P^{\alpha}_{\aleph_1}(\lambda_0)$.*

5. *The α-test set for P, as defined below, is in $\mathcal{D}^{\alpha}_{\aleph_1}(A)$.*

   Form $A = P \cup \mathcal{P}(P)$. Then $\langle a_i \mid i < \alpha\rangle \in P^{\alpha}_{\aleph_1}(A)$ is in the α-test set for P iff for every $p_0 \in a_0 \cap P$ there is a $p \in P$ that is a_i-generic for every $i < \alpha$. (That is, for every $D \in a_i \cap \mathcal{P}(P)$, if D is dense in P, then D is pre-dense above p.)

Preservation of α-Properness

We shall prove the following.

5.5 Theorem. *Let $\alpha < \omega_1$ be a countable ordinal and $\langle P_i \mid i \leq \gamma\rangle$ a countable support iteration of α-proper posets. Then the limit P_γ is α-proper.*

The theorem is obtained as a particular case of the α-Extension Lemma which is proved by induction on α. As the case $\alpha = \omega$ involves almost all the essential ideas of the general case, the reader may concentrate on ω-properness.

5.6 Lemma (The α-Extension Lemma). *Let α be any countable ordinal and $\langle P_i \mid i \leq \gamma \rangle$ a countable support iteration of α-proper posets. Let λ be a sufficiently large cardinal, and let $\bar{M} = \langle M_\xi \mid \xi \leq \alpha \rangle$ be an $(\alpha+1)$-tower of countable elementary substructures of H_λ, with $\gamma, P_\gamma, \alpha \in M_0$. For every $\gamma_0 \in \gamma \cap M_0$ and $q_0 \in P_{\gamma_0}$ that is $(\bar{M}, P_{\gamma_0})$-generic, the following holds:*

If $\underset{\sim}{p_0} \in V^{P_{\gamma_0}}$ is such that

$$q_0 \Vdash_{\gamma_0} \underset{\sim}{p_0} \in P_\gamma \cap M_0 \wedge \underset{\sim}{p_0} \restriction \gamma_0 \in \underset{\sim}{G_0}$$

(where $\underset{\sim}{G_0}$ is the canonical name for the (V, P_{γ_0})-generic filter), then there is an $(\bar{M}, P_\gamma)$-generic condition q such that

$$q \restriction \gamma_0 = q_0 \quad \textit{and} \quad q \Vdash_\gamma \underset{\sim}{p_0} \in \underset{\sim}{G}$$

(where $\underset{\sim}{G}$ is the canonical name for the (V, P_γ)-generic filter, and the name $\underset{\sim}{p_0} \in V^{P_{\gamma_0}}$ is now viewed as member of V^{P_γ}).

Proof. The proof is by induction on $\alpha < \omega_1$ and for any fixed α by induction on γ. We begin with $\alpha = \alpha' + 1$ a successor ordinal. We are given a tower $\bar{M} = \langle M_\xi \mid \xi \leq \alpha' + 1 \rangle$, a condition q_0, and a name $\underset{\sim}{p_0}$ as in the lemma. We intend to define a name $\underset{\sim}{r} \in V^{P_{\gamma_0}}$ such that q_0 forces (in P_{γ_0}) the following sentences.

1. $\underset{\sim}{r} \in M_\alpha \cap P_\gamma$ is $\langle M_\xi \mid \xi \leq \alpha' \rangle$-generic.
2. $\underset{\sim}{r} \restriction \gamma_0 \in \underset{\sim}{G_0}$, and $\underset{\sim}{p_0} <_{P_\gamma} \underset{\sim}{r}$.

Then, using the Properness Extension Lemma 2.8, we can find a $q \in P_\gamma$ that is (M_α, P_γ)-generic and such that $q \restriction \gamma_0 = q_0$ and $q \Vdash \underset{\sim}{r} \in \underset{\sim}{G}$. It follows that q is as required.

To define $\underset{\sim}{r}$, let G_0 be a (V, P_{γ_0})-generic filter with $q_0 \in G_0$, and we shall describe $\underset{\sim}{r}[G_0]$. Let $p_0 \in P_\gamma \cap M_0$ be the interpretation of $\underset{\sim}{p_0}$. Then $p_0 \restriction \gamma_0 \in G_0$ and we can find a $q_0' \in G_0$ that extends both q_0 and $p_0 \restriction \gamma_0$. Now we can apply the inductive assumption on α' to the tower $\langle M_\xi \mid \xi \leq \alpha' \rangle$, to q_0' and to p_0, and we find a $q' \in P_\gamma$ such that $q' \restriction \gamma_0 = q_0'$, q' is $(\langle M_\xi \mid \xi \leq \alpha' \rangle, P_\gamma)$-generic, and $p_0 <_{P_\gamma} q'$. Since $M_\alpha[G_0] \prec H_\lambda[G_0]$, we can find a $q^* \in M_\alpha[G_0]$ with similar properties as q'. Namely, $q^* \in P_\gamma$ (and so $q^* \in M_\alpha$, as $M_\alpha[G_0] \cap V = M_\alpha$), $q^* \restriction \gamma_0 \in G_0$, $p_0 <_{P_\gamma} q^*$, and q^* is $(\langle M_\xi \mid \xi \leq \alpha' \rangle, P_\gamma)$-generic. Let $\underset{\sim}{r}$ be a name of q^* forced by q_0 to have these properties. Then $\underset{\sim}{r}$ is as required.

Assume now that α is a limit ordinal. In case γ is a successor ordinal, we can inductively apply the following two-step iteration lemma, whose proof is similar to the corresponding case of proper forcing (and is hence not given here).

5.7 Lemma. *Suppose that P_0 is α-proper and $P_1 \in V^{P_0}$ is α-proper in V^{P_0}. Then $R = P_0 * P_1$ is α-proper, and the following holds. Suppose that $\bar{M} \prec H_\lambda$ is an α-tower and $R \in M_0$. Let $\underset{\sim}{r} \in V^{P_0}$ be a name, and $p_0 \in P_0$ be an $(\bar{M}, P_0)$-generic condition such that*

$$p_0 \Vdash_{P_0} \underset{\sim}{r} \in M_0 \cap R \quad \textit{and} \quad \pi(\underset{\sim}{r}) \in \underset{\sim}{G_0}$$

where $\pi : R \to P_0$ is the projection (taking $(a,b) \in R$ to a), and $\underset{\sim}{G}_0$ is the canonical name for the (V, P_0)-generic filter. Then there is a name $\underset{\sim}{p}_1 \in V^{P_0}$ such that

1. *$(p_0, \underset{\sim}{p}_1)$ is $(\bar{M}, R)$-generic, and*
2. *$(p_0, \underset{\sim}{p}_1) \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}$*

where $\underset{\sim}{G}$ is the canonical name for the (V, R)-generic filter.

Continuing the proof of the α-Extension Lemma for α a limit ordinal, assume now that γ is a limit ordinal. We fix a sequence $\langle \alpha_n \mid n \in \omega \rangle$ increasing and cofinal in α, and a sequence $\langle \gamma_i \mid i \in \omega \rangle$ increasing in γ and such that $\gamma_n \in M_{\alpha_n}$ with γ_0 the given ordinal. (If $\mathrm{cf}(\gamma) = \omega$, let $\langle \gamma_i \mid i \in \omega \rangle$ be an increasing, cofinal in γ sequence in $M_0 \cap \gamma$, with γ_0 as given. If $\mathrm{cf}(\gamma) > \omega$, define $\gamma_n = \sup(\gamma \cap M_{\alpha_{n-1}})$ for $n \geq 1$.) We intend to define by induction on $n < \omega$ conditions $q_n \in P_{\gamma_n}$ and names $\underset{\sim}{p}_n \in V^{P_{\gamma_n}}$ such that:

1. $q_0 \in P_{\gamma_0}$ is the given condition. And for $n > 0$, $q_n \in P_{\gamma_n}$ is $(\langle M_\xi \mid \alpha_n < \xi \leq \alpha \rangle, P_{\gamma_n})$-generic and $q_{n+1} \restriction \gamma_n = q_n$. (In fact, q_n is generic for the complete tower, but this follows from item 2 below.)
2. $\underset{\sim}{p}_0$ is given. $\underset{\sim}{p}_n$ is a P_{γ_n}-name such that

$$q_n \Vdash_{\gamma_n} \underset{\sim}{p}_n \text{ is a condition in } P_\gamma \cap M_{\alpha_n+1} \text{ such that:}$$

   (a) $\underset{\sim}{p}_n \restriction \gamma_n \in \underset{\sim}{G}_{\gamma_n}$,

   (b) $\underset{\sim}{p}_{n-1} \leq_\gamma \underset{\sim}{p}_n$,

   (c) $\underset{\sim}{p}_n$ is an $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic condition (when $n > 0$).

When this sequence is defined, $q = \bigcup_n q_n$ is a condition in P_γ and

$$q \Vdash_\gamma \underset{\sim}{p}_n \in \underset{\sim}{G}_\gamma$$

as we have seen in the proof of the Properness Extension Lemma. But as q forces that $\underset{\sim}{p}_n$ is $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic, q itself is $(\langle M_\xi \mid \xi \leq \alpha_n \rangle, P_\gamma)$-generic for every $n \in \omega$, and the proof of the lemma is concluded since q is then $(\langle M_\xi \mid \xi < \alpha \rangle, P_\gamma)$ and hence $(\bar{M}, P_\gamma)$-generic (as α is a limit ordinal).

We turn now to the inductive construction. Suppose that q_n and $\underset{\sim}{p}_n$ are defined. As before, we shall first define $\underset{\sim}{p}_{n+1}$ and then q_{n+1}.

We define $\underset{\sim}{p}_{n+1}$ as a P_{γ_n}-name by the following requirements. If G is any (V, P_{γ_n})-generic filter containing q_n, form

$$M_{\alpha_{n+1}+1}[G] \prec H_\lambda[G], \tag{5.16}$$

and let $p \in P_\gamma$ be the interpretation of $\underset{\sim}{p}_n$. Then $p \in P_\gamma \cap M_{\alpha_n+1}$ and $p \restriction \gamma_n \in G$. As $q_n, p \restriction \gamma_n \in P_{\gamma_n}$ are in G, there is a $q_n' \in G$ that extends both q_n and $p\restriction\gamma_n$. By the α_{n+1}-Properness Extension Lemma applied to q'_n and p there is a $q^*_n \in P_\gamma$ such that $q^*_n\restriction\gamma_n = q'_n$, $p \leq_\gamma q^*_n$, and q^*_n is $(\langle M_\xi \mid \alpha_n+1 \leq \xi \leq \alpha_{n+1}\rangle, P_\gamma)$-generic. It follows from (5.16) that (similarly to q^*_n) there is in $M_{\alpha_{n+1}+1}$ a condition $q^* \in P_\gamma$ such that $q^*\restriction\gamma_n \in G$, $p \leq_\gamma q^*$, and q^* is $(\langle M_\xi \mid \alpha_n+1 \leq \xi \leq \alpha_{n+1}\rangle, P_\gamma)$-generic. Then we define the interpretation of $\underset{\sim}{p}_{n+1}$ in $V[G]$ to be q^*.

Clearly q_n forces that $\underset{\sim}{p}_{n+1}$ is in $P_\gamma \cap M_{\alpha_{n+1}+1}$ and

1. $\underset{\sim}{p}_{n+1}\restriction\gamma_n \in \underset{\sim}{G}_{\gamma_n}$,
2. $\underset{\sim}{p}_n \leq \underset{\sim}{p}_{n+1}$ in P_γ,
3. $\underset{\sim}{p}_{n+1}$ is $(\langle M_\xi \mid \alpha_n+1 \leq \xi \leq \alpha_{n+1}\rangle, P_\gamma)$-generic (and so by item 2 it is $(\langle M_\xi \mid \xi \leq \alpha_{n+1}\rangle, P_\gamma)$-generic).

Now $\underset{\sim}{p}_{n+1}\restriction\gamma_{n+1}$ is forced by q_n to be in $M_{\alpha_{n+1}+1}$ and the inductive assumption for γ_{n+1} can be applied to yield a condition $q_{n+1} \in P_{\gamma_{n+1}}$ that is $(\langle M_\xi \mid \alpha_{n+1} < \xi \leq \alpha\rangle, P_{\gamma_{n+1}})$-generic and such that

$$q_{n+1} \Vdash_{\gamma_{n+1}} \underset{\sim}{p}_{n+1}\restriction\gamma_{n+1} \in \underset{\sim}{G}_{\gamma_{n+1}}.$$

Hence q_{n+1} is also $(\langle M_\xi \mid \xi \leq \alpha_{n+1}\rangle, P_{\gamma_{n+1}})$-generic. This completes the proof of the α-Extension Lemma, and hence of the α-properness preservation theorem. $\dashv$

5.2. A Coloring Problem

The definitions needed for Dee-completeness are quite complex and they can be better understood with an example. Hence, before presenting the general definition of Dee-completeness (in Sect. 5.3) we discuss a particular case. The simplest that I know is a problem of Hajnal and Máté concerning the chromatic number of graphs in a certain family of graphs on ω_1 described below. (There is also a nostalgic reason for discussing this example: Theorem 5.8 is my first result in set theory.) The *chromatic number* of any (non-directed) graph $g = (V, E)$ is the least cardinal κ such that there is a function $f : V \to \kappa$ from the set of vertices V into κ such that for every $\alpha \neq \beta$ in V, if $\alpha\, E\, \beta$ then $f(\alpha) \neq f(\beta)$.

Hajnal and Máté investigated in [5] the following family of graphs $g = (V, E)$ with set of vertices $V = \omega_1$, and in which for any limit $\delta \in \omega_1$ the set $\chi^g_\delta = \{\alpha \mid \alpha \in \delta \text{ and } \alpha\, E\, \delta\}$ forms an ω-sequence cofinal in δ (and for non-limit $\beta \in \omega_1$ there is no $\alpha < \beta$ such that $\alpha\, E\, \beta$). We shall call such graphs *Hajnal-Máté graphs*. They had shown that if the diamond principle $\diamondsuit$ holds, then there is an Hajnal-Máté graph of chromatic number $\aleph_1$, and if MA + $2^{\aleph_0} > \aleph_1$ holds, then every Hajnal-Máté graph has countable chromatic number. They had suggested that Jensen's method [3] for proving the consistency of CH + "there are no Souslin trees" may lead to a consistency proof for CH + "the chromatic number of every Hajnal-Máté graph is $\aleph_0$". This turned out to be true.

5.8 Theorem. *Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. There is a generic extension that adds no new reals, collapses no cardinals, and such that every Hajnal-Máté graph in the extension has a countable chromatic number and $2^{\aleph_1} = \aleph_2$.*

Let $g = (V, E)$ be a Hajnal-Máté graph. Define P_g as the poset for making the chromatic number of g countable. That is, $h \in P_g$ iff for some countable ordinal γ, $h : \gamma + 1 \to \omega$ is such that whenever $\alpha < \beta \leq \gamma$ and $\alpha E \beta$ then $h(\alpha) \neq h(\beta)$. So the domain, $\mathrm{dom}(h)$, of a condition in P_g is always a countable successor ordinal.

The ordering on P_g is extension.

Clearly if $h \in P_g$ and $\gamma' < \gamma = \mathrm{dom}(h)$ is a successor ordinal then $h \restriction \gamma' \in P_g$. It is easy to check that any condition in P_g has extensions with arbitrarily high domain in ω_1.

If $h \in P_g$ and x is a finite function from $\mathrm{dom}(x) \subseteq \omega_1$ to ω, we say that x is compatible with h if $h \cup x$ is a function that assigns distinct values (in ω) to connected vertices.

5.9 Lemma. *If $h \in P_g$ and $\gamma + 1 = \mathrm{dom}(h)$, then for every countable μ above γ there is some condition $h' > h$ with $\mu + 1 = \mathrm{dom}(h')$. Moreover, for any finite x that is compatible with h there is an extension h' of $h \cup x$ in P_g.*

Thus if G is generic over P_g then $\bigcup G$ is defined on ω_1 and the chromatic number of g in the extension is countable.

We plan to prove that P_g is proper, α-proper for every countable α, and show how to iterate P_g forcings without adding new reals. Then $2^{\aleph_1} = \aleph_2$ implies that an iteration of length ω_2 suffices to ensure that all Hajnal-Máté graphs produced are taken care of, and the theorem can be established.

5.10 Lemma. *P_g is proper. Moreover, if $M \prec H_\lambda$ is countable, $P_g \in M$, and $h_0 \in P_g \cap M$, then for any finite x compatible with h_0 there is an (M, P_g)-generic condition h compatible with x. (x is not necessarily in M.)*

Proof. Given $h_0 \in P_g \cap M$ with $\mathrm{dom}(h_0) = \gamma_0 + 1$, list all dense sets $\langle D_i \mid i \in \omega \rangle$ that are in M. Let $\delta = M \cap \omega_1$ and let χ^g_δ be the ω-sequence of ordinals that are connected in g to δ. Our aim is to define an increasing sequence of conditions $h_n \in M$ so that $h_{n+1} \in D_n$ and then to define $h = \bigcup_{n \in \omega} h_n$ and to extend h on δ as well, in order to obtain an (M, P_g)-generic condition. The problem, of course, is that h may map χ^g_δ onto ω and leave no place for the value of δ, or that it assigns a value that is incompatible with x. The solution is based on the fact that δ "is" ω_1 for M.

We make an observation. Given a condition $f \in P_g$, and v, a finite function compatible with f, let $H_f(v)$ be the first (in some well-ordering) condition in P_g that extends $f \cup v$.

5.11 Claim. *Let $D \subseteq P_g$ be dense. For any $f \in P_g$ there is a closed unbounded set $C_f \subseteq \omega_1$ such that for every $\gamma \in C$ and finite function v defined on a subset of γ and compatible with f, if $h = H_f(v)$, then h has an extension $h' \in D$ such that $\mathrm{dom}(h') < \gamma$.*

This claim is not difficult to prove.

Consider the given finite function x compatible with h_0. We may assume that $\delta \in \mathrm{dom}(x)$ (or else extend x). Let $x_0 = x \cap M$ and $x_1 = x \setminus M$ be the lower and upper parts of x. By further extending h_0 in M we may assume that $x_0 \subseteq h_0$. We can also assume that if $\alpha < \delta$ and α is connected in the graph to some point in the domain of x_1 above δ then $\alpha \in \mathrm{dom}(h_0)$ (as there are only finitely many such αs). So, in fact, in defining h_n's we must only be careful to avoid $k = x(\delta)$ on χ^g_δ. This k is "reserved" as the value for δ. Assume that $h_n \in M$ is defined and $k \notin h_n$"χ^g_δ. For $f = h_n$ and $D = D_n$ there is an unbounded set $C_f \subseteq \omega_1$ as in the claim above. As C_f is definable, $C_f \in M$. Let γ be the first member of C_f above $\mathrm{dom}(h_n)$. Let $u = \chi^g_\delta \cap \gamma \setminus \mathrm{dom}(h_n)$. Then u is finite since $\gamma < \delta$. Let v be a function defined on u, compatible with h_n, and avoiding the value k. Then $h = H_f(v)$ has an extension $h' = h_{n+1} \in D_n$ that lives in γ, so that $\{\langle \delta, k \rangle\}$ is still compatible with h_{n+1}. Finally $h = \bigcup_{n \in \omega} h_n \cup \{\langle \delta, k \rangle\}$ is (M, P_g)-generic. This completes the properness proof for P_g. ⊣

The generic condition h thus obtained is a "completely generic" condition, which means that it actually defines an (M, P_g)-generic filter. This shows that P_g adds no new reals.

5.12 Definition. If $P \in M$ is a poset, then $q \in P$ is *completely* (M, P)*-generic* iff $\{p \in P \cap M \mid p \leq q\}$ is an (M, P)-generic filter. We say that G is *bounded by* q and also that q *induces* G.

A poset P is *completely proper* iff P is proper and the properness definition applies to P with "completely generic condition" replacing "generic condition".

Clearly, completely proper posets do not add new reals. In fact, P is completely proper iff P is proper and adds no new reals. (If the latter condition holds and M is as in the definition of properness, find q that is (M, P)-generic and then further extend it to some condition that determines $\underset{\sim}{G} \cap M$.)

Thus we know that P_g adds no new reals, and now we prove by induction on $\alpha < \omega_1$ that

5.13 Lemma. *P_g is α-proper. In fact, if $\langle M_\xi \mid \xi \leq \alpha \rangle$ is any α tower of countable elementary submodels of H_λ with $P_g \in M_0$ and $h_0 \in M_0$ is any condition, then for every finite x compatible with h_0 there is an extension $h \in P_g$ that is completely (M_ξ, P_g)-generic for every $\xi \leq \alpha$ and is compatible with x.*

Proof. The proof of the lemma is by induction on $\omega \leq \alpha < \omega_1$. Using the properness Lemma 5.10 we assume that α is a limit ordinal, and pick an increasing and cofinal sequence α_n. We define an increasing sequence h_n of conditions compatible with x such that $h_{n+1} \in M_{\alpha_{n+1}+1}$ is $\langle M_\xi \mid \xi \leq \alpha_{n+1} \rangle$. Then $\bigcup_{n \in \omega} h_n \cup \{\langle \delta, x(\delta) \rangle\}$ is as required. ⊣

Now we make a crucial observation which is the key to the proof that the iteration of P_g posets adds no new reals. Let $M \prec H_\lambda$ be a countable elementary substructure with $P_g \in M$, and let $p_0 \in P_g \cap M$ be given. Recall that $G^* \subseteq P_g \cap M$ is an (M, P_g)-generic filter if it is a filter over $P_g \cap M$ that intersects every dense set in M. Define

$$\mathrm{Gen}_{p_0}(M, P_g)$$

to be the set of all (M, P_g)-generic filters containing p_0. An (M, P_g)-generic G^* is *extendible* to a condition in P_g iff for some $q \in P_g$, $G^* = \{p \in P_g \cap M \mid p < q\}$. We say in this case that q bounds G^*. Clearly G^* is extendible to a condition in P_g iff its range of values on χ^g_δ (where $\delta = \omega_1 \cap M$) is not all of ω. For any ω-sequence x cofinal in δ, we shall say that G^* is *appropriate* for x iff $\bigcup G^* \restriction x$ omits at least one value. So G^* is extendible to a condition in P_g iff G^* is appropriate for χ^g_δ.

For an ω-sequence x cofinal in $\delta = \omega_1 \cap M$ define

$$A^{p_0}_x = \{G^* \in \mathrm{Gen}_{p_0}(M, P_g) \mid G^* \text{ is appropriate for } x\}.$$

Thus $A^{p_0}_{\chi^g_\delta}$ is the collection of all $G^* \in \mathrm{Gen}_{p_0}(M, P_g)$ extendible to a condition in P_g.

We claim that (for a fixed p_0) the collection

$$\{A^{p_0}_x \mid x \subseteq \delta \ \text{ is a cofinal } \omega \text{ sequence}\}$$

has the countable intersection property. That is, if $X = \{x_i \mid i \in \omega\}$ is some countable collection of ω-sequences cofinal in $\delta = \omega_1 \cap M$, then there is some G^* in $\bigcap_{i\in\omega} A^{p_0}_{x_i}$. To prove this, find some ω-sequence x converging to δ and such that $\mathrm{range}(x_i) \setminus \mathrm{range}(x)$ is finite for every i. It is easy to define such x that almost contains each x_i by induction. If G^* is some (M, P_g)-generic filter containing p_0 that omits infinitely many values on x (and now we can easily define such a filter), then $G^* \in A^{p_0}_{x_i}$ for every i.

We describe in general terms what is involved in proving that the iteration adds no new reals. In proving that the iteration of P_g-like forcings does not add any new real we will be asked to produce a completely generic condition for some $M \prec H_\lambda$ without knowing the value of χ^g_δ. This ω-sequence of ordinals connected to δ will only be given as a name. So instead of χ_δ we will be offered a countable collection $\{x_i \mid i \in \omega\}$ of ω-sequences with the assurance that χ^g_δ is forced to be among them. We will still be able to find G^* by taking into consideration all the x_i's as was shown above. The essence of this argument is embodied in the following lemma.

5.14 Lemma. *Suppose that $M_0 \prec M_1 \prec H_\lambda$ are countable elementary substructures with $M_0 \in M_1$. Suppose that $P \in M_0$ is a poset that adds no new reals, and that $g \in V^P \cap M_0$ is a name for some Hajnal-Máté graph. Let $G_0 \in M_1$ be some (M_0, P)-generic filter. Then there exists an $(M_0, P * P_g)$-generic filter G_1 extending G_0, and there exists a name $\underset{\sim}{r} \in V^P$ such that the following holds.*

If $q \in P$ is a (plain) (M_1, P)-generic condition that bounds G_0 (and is hence completely (M_0, P)-generic), then the condition $(q, \underset{\sim}{r}) \in P * P_g$ bounds G_1.

In fact, both the filter G_1 and the name $\underset{\sim}{r}$ are definable from parameters in M_1 and a countable enumeration of M_1. Thus, if $M_1 \in M_2 \prec H_\lambda$ then $G_1, \underset{\sim}{r} \in M_2$.

Proof. Let $\mu : M_0 \to N_0$ be the Mostowski collapsing function of M_0 onto a transitive structure N_0. Then $N_0 \in M_1$. Let $\mathcal{G}_0 = \mu$“$G_0 \subseteq \mu(P)$ be the image of G_0 under the collapsing map. Forming $N_0[\mathcal{G}_0]$ as a generic extension, $\mu(g)[\mathcal{G}_0]$ is a Hajnal-Máté graph there denoted h. This graph is clearly on $\delta = \omega_1 \cap M_0$. As $\mathcal{G}_0 \in M_1$, $N_0[\mathcal{G}_0] \in M_1$.

Let $X = \{x_i \mid i \in \omega\}$ be an enumeration (the least in some global well order) of all ω-sequences x in M_1 that are cofinal in δ. We know how to find an $(N_0[\mathcal{G}_0], P_h)$-generic filter H that is appropriate for every x_i. Now form $\mathcal{G}_1 = \mathcal{G}_0 * H$. Then $\mathcal{G}_1$ is an N_0-generic filter extending $\mathcal{G}_0$. The required filter G_1 is the μ pre-image of $\mathcal{G}_1$.

The name $\underset{\sim}{r} \in V^P$ is defined by the following requirement as a condition in P_g with domain $\delta + 1$. It is easier to describe the interpretation of $\underset{\sim}{r}$ in (V, P)-generic extensions $V[G]$. If $\bigcup H$ (which is a function on δ that lies in V) can be extended (by assigning a value to δ) to a condition in $P_{g[G]}$, then let $\underset{\sim}{r}[G]$ be that condition.

Assume now that $q \in P$ is as in the lemma, a bound of $G_0 = \mu^{-1}$“$\mathcal{G}_0$ that is also an (M_1, P)-generic condition. Since P adds no new countable sequences, there exists in M_1 a dense set of conditions in P that determine the value of χ^g_δ in V. Thus $q \Vdash \chi^g_\delta \in M_1$, and hence $q \Vdash \exists i \in \omega\ (\chi^g_\delta = x_i)$. Hence $q \Vdash H$ is bounded by $\underset{\sim}{r}$. ⊣

Observe that $\underset{\sim}{r}$ is a name of a function defined on $\delta + 1$. Although $\underset{\sim}{r} \upharpoonright \delta$ is, in a sense, $\bigcup H$, $\underset{\sim}{r}(\delta)$ is just a name and any specific value for $\underset{\sim}{r}(\delta)$ may conflict with some x_i. Only after determining (generically) χ^g_δ can we assign a compatible value to $\underset{\sim}{r}(\delta)$.

In applications we need a slightly stronger version of this lemma, in which a condition $(p_0, p_1) \in P * P_g \cap M_0$ is also given, and $p_0 < q$ is assumed. Then $\underset{\sim}{r}$ is also required to satisfy $q \Vdash_P \underset{\sim}{r} > p_1$, so that the given condition (p_0, q_0) is in the (M_0, P)-generic filter determined by $(q, \underset{\sim}{r})$.

The final model of CH + “every Hajnal-Máté graph has countable chromatic number” is obtained through an iteration with countable support of posets of the form P_g, done over a ground model in which CH holds. Since each P_g has size $\aleph_1$ the iteration satisfies the $\aleph_2$-c.c., and a suitable book-keeping device ensures that every possible Hajnal-Máté graph is dealt with.

As a preparation for the final iteration we prove here that P_ω (the iteration of the first $\aleph_0$ posets) adds no new reals. So let P_n be defined by $P_{n+1} \equiv P_n * P_{g_n}$ where g_n is a name for some Hajnal-Máté graph. Specifically, the members of P_n are functions defined on n and P_ω is the countable support limit.

Let $M_0 \prec H_\lambda$ be some countable elementary substructure with $P_\omega \in M_0$, and $p_0 \in P_\omega \cap M_0$ a given condition. It suffices to find an extension of p_0 that is completely (M_0, P_ω)-generic. Let $\langle D_n \mid n \in \omega \rangle$ be an enumeration of all dense subsets of P_ω that are in M_0.

Starting with M_0 build a tower $\langle M_n \mid n \in \omega \rangle$ of countable elementary substructures of H_λ. We plan to define a sequence of conditions $q_n \in P_n$ and $p_n \in P_\omega \cap M_0$ such that

1. $q_n \in P_n$ is completely (M_0, P_n)-generic, and it is (M_k, P_n)-generic for every $k \geq n$,
2. $p_n \restriction n \leq q_n$, and $q_n = q_{n+1} \restriction n$,
3. $p_n \leq p_{n+1}$ in P_ω, and $p_{n+1} \in D_n \cap M_0$.

When the construction is done, define $q = \bigcup_n q_n$. Then $q \geq p_k \restriction n$ for every n and hence $q \geq p_k$ for every k. This shows that $q \in P_\omega$ is completely (M_0, P_ω)-generic, because the p_k's visit every dense set in M_0.

Suppose that p_n and q_n are defined. The assumption that q_n is completely (M_0, P_n)-generic means that

$$G_0 = \{p \in P_n \cap M_0 \mid p \leq q_n\}$$

is (M_0, P_n)-generic. Use Lemma 1.2 to find $p_{n+1} \geq p_n$ with $p_{n+1} \in D_n \cap M_0$ and such that $p_{n+1} \restriction n \leq q_n$.

Notice that $G_0 \in M_n$ because the set D of conditions $q \in P_n$ that are completely (M_0, P_n)-generic is pre-dense above q_n. Since $D \in M_n$, q_n is compatible with a member q of $D \cap M_n$ and, in the definition of G_0, q can replace q_n.

The previous lemma is applicable to $M_0 \in M_n \in M_{n+1}$ and to $P = P_n$. So there is a name $\underset{\sim}{r} \in M_{n+1}$ such that $(q_n, \underset{\sim}{r}) \in P_{n+1}$ is completely (M_0, P_{n+1})-generic and $p_{n+1} \restriction n+1 \leq (q_n, \underset{\sim}{r})$. As P_{g_n} is (forced to be) ω-proper, there is a name $r' \in V^{P_n}$ such that

$$q_n \Vdash \underset{\sim}{r} < r' \,\&\, r' \text{ is } (\langle M_i[\underset{\sim}{G}_{P_n}] \mid i \geq n+1 \rangle, P_{g_n})\text{-generic.}$$

Define $q_{n+1} = q_n {}^\frown \langle r' \rangle$. Then $q_{n+1} \in P_{n+1}$ is tower generic for $\langle M_k \mid k \geq n+1 \rangle$, $q_n = q_{n+1} \restriction n$, and $q_{n+1} \geq p_{n+1} \restriction n+1$.

5.3. Dee-Completeness

The aim of Dee-completeness is to provide a framework for obtaining models of CH. It allows countable support iteration of a large family of posets that add no new countable sets. Our definitions here are slightly different from those originally given by Shelah [15, Chap. V], but we have kept the original names believing that our interpretation of the basic ideas is accurate.

A *completeness system* is a three-argument function $\mathbb{D}(N, P, p_0)$ defined when

1. N is a countable transitive model of ZFC$^-$ (ZFC minus the Power Set Axiom),

2. $P \in N$ is a forcing poset in N, and

3. $p_0 \in P$.

$\mathbb{D}(N, P, p_0)$ is consequently a non-empty collection of non-empty subsets of $\mathrm{Gen}_{p_0}(N, P)$. That is, if $A \in \mathbb{D}(N, P, p_0)$ then every $G \in A$ is a filter over P containing p_0 and intersecting every dense subset of P in N.

For example, if $g \in N$ is some Hajnal-Máté graph and $P = P_g$ is in N the poset for coloring g with countably many colors, then we define $\mathbb{D}(N, P, p_0) = \{A_X \mid X \subseteq N\}$ where $A_X \subseteq \mathrm{Gen}_{p_0}(N, P)$ is defined as follows. In case X is an ω-sequence cofinal in ω_1^N, then $G \in A_X$ iff $G \in \mathrm{Gen}_{p_0}(N, P)$ is such that $\bigcup G \restriction X$ omits infinitely many colors. If X is not as above, then $A_X = \mathrm{Gen}_{p_0}(N, P)$. It is reasonable to have g as a parameter, although in our case it is reconstructible from P.

We apply $\mathbb{D}$ to non-transitive structures as well: if $M \prec H_\lambda$ is a countable elementary submodel, $P \in M$ a poset, and $p_0 \in P \cap M$, then we let $\pi : M \to N$ be the transitive collapsing isomorphism and for each $X \in \mathbb{D}(N, \pi(P), \pi(p_0))$ we define $\pi^{-1}(X) = \{\pi^{-1}``G \mid G \in X\}$. This yields $\mathbb{D}(M, P, p_0) = \{\pi^{-1}(X) \mid X \in \mathbb{D}(N, \pi(P), \pi(p_0))\}$. In simple terms, $\mathbb{D}(M, P, p_0)$ is defined by viewing the argument (M, P, p_0) as a representation of its isomorphism type.

We say that a poset P is *Dee-complete* (or just *complete*, for brevity) *with respect to a completeness system* $\mathbb{D}$ if for sufficiently large λ, for every countable $M \prec H_\lambda$ with $P \in M$ and every $p_0 \in P \cap M$, there is an $X \in \mathbb{D}(M, P, p_0)$ such that every $G \in X$ is bounded in P. (Thus P is completely proper.)

Repeating this definition, now with transitive structures, we obtain that P is Dee-complete with respect to $\mathbb{D}$ if the following holds for every countable $M \prec H_\lambda$ with $P \in M$:

For any $p_0 \in P \cap M$, if $\pi : M \to N$ is the transitive collapse, there is an $X \in \mathbb{D}(N, \pi(P), \pi(p_0))$ such that (5.17)

$$\text{for every } \mathcal{G} \in X,\ \pi^{-1}``\mathcal{G} \text{ is bounded in } P.$$

We say that a completeness system is *countably complete* iff whenever $A_i \in \mathbb{D}(N, P, p)$ for $i \in \omega$ then $\bigcap_{i \in \omega} A_i \neq \emptyset$. We have seen that for every Hajnal-Máté graph g the system defined above is countably complete. Thus every P_g is Dee-complete with respect to a countably complete system.

It is sometimes convenient to add a parameter to $\mathbb{D}$. We shall say that $\mathbb{D}$ is a completeness system with a parameter if $\mathbb{D}$ is a four argument function: $\mathbb{D}(N, P, p_0, c)$ is defined when N, P, p_0 are as in the definition given above, and $c \in N$ is the parameter. As before, $\mathbb{D}(N, P, p_0, c)$ is a non-empty collection of non-empty subsets of $\mathrm{Gen}_{p_0}(N, P)$. We say that a poset P is

Dee-complete with respect to a completeness system $\mathbb{D}$ *with parameter* iff for sufficiently large λ there is a $c \in H_\lambda$ such that:

$$\text{for every countable } M \prec H_\lambda \text{ with } P, c \in M \text{ and every } p_0 \in P \cap M, \text{ if } \pi : M \to N \text{ is the transitive collapse, then there is an } X \in \mathbb{D}(N, \pi(P), \pi(p_0), \pi(c)) \text{ such that, for every } \mathcal{G} \in X,\ \pi^{-1}``\mathcal{G} \text{ is bounded in } P. \tag{5.18}$$

In fact, the parameters are dispensable by the following lemma. Recall that by H_λ we mean the structure $(H_\lambda, \in, <)$ where $<$ is a fixed well-ordering of H_λ.

5.15 Lemma. *Let* P *be a poset that is Dee-complete with respect to some countably complete completeness system* $\mathbb{D}$ *with parameter. Then* P *is also Dee-complete with respect to some three-argument countably complete completeness system* $\mathbb{D}'$*.*

Proof. Let λ be sufficiently large so that H_λ with parameter $c \in H_\lambda$ shows the completeness of P with respect to $\mathbb{D}$ (as in (5.18)). Then, for $\lambda' > \lambda^{<\lambda} + 2^{2^{\mathfrak{c}}}$, $H_\lambda \in H_{\lambda'}$ and $\mathbb{D} \in H_{\lambda'}$ (since $\mathbb{D}$ is a function from $H_{\aleph_1}$ to $\mathcal{P}\mathcal{P}(H_{\aleph_1})$). So $H_{\lambda'}$ satisfies the statement $\psi(P, \lambda, c, \mathbb{D})$ saying that λ is a cardinal with $P, c \in H_\lambda$ and $\mathbb{D}$ is a four-argument system such that (5.18) holds.

Using the assumed well-ordering of $H_{\lambda'}$, let λ_0, c_0, and $\mathbb{D}_0$ be minimal such objects for which $\psi(P, \lambda_0, c_0, \mathbb{D}_0)$ holds. If $M \prec H_{\lambda'}$ is any countable elementary substructure with $P \in M$, then $\lambda_0, c_0, \mathbb{D}_0 \in M$ since they are definable in $H_{\lambda'}$, and moreover, these objects are minimal in M to satisfy $\psi(P, \lambda_0, c_0, \mathbb{D}_0)$. Observe that $H^M_{\lambda_0} = M \cap H_{\lambda_0} \prec H_{\lambda_0}$, and also that if $D \in M$ is a subset of P then $D \in H^M_{\lambda_0}$.

Let $\pi : M \to N$ be the collapse onto a transitive structure. Then $\pi_0 = \pi \restriction H_{\lambda_0}$ is the collapse of a countable elementary substructure of H_{λ_0}, namely $H^M_{\lambda_0}$ onto $H^N_{\pi(\lambda_0)} = N_0$.

So $\mathbb{D}_0(N_0, \pi(P), \pi(p_0), \pi(c_0))$ has the required good properties, and in particular each $G \in X \in \mathbb{D}_0(N_0, \pi(P), \pi(p_0), \pi(c_0))$ is generic not only over N_0 but also over N. This leads to the following definition of $\mathbb{D}'$ as required by the lemma.

If N is any countable transitive structure, $R \in N$ a poset and $r \in R$, define $\mathbb{D}'(N, R, r)$ as follows. Look for λ'_0, c'_0, that are minimal to satisfy $\exists D \psi(R, \lambda'_0, c'_0, D)$ in N, and apply $\mathbb{D}_0(H^N_{\lambda'_0}, R, r, c'_0)$. If there is no such λ'_0, then $\mathbb{D}'(N, R, r)$ is arbitrarily defined. $\dashv$

The definition of Dee-completeness has the form "for λ sufficiently large etc.". It is not difficult to see that if there is a completeness system that works for one λ, there is one that works for all larger λ as well. We are going to argue now that it is always possible in this case to take $\lambda = (2^{|P|})^+$ (Shelah, personal communication).

5.16 Lemma. *If a poset P is complete with respect to a countably complete completeness system $\mathbb{D}$ then there is a countably complete (four-argument) completeness system $\mathbb{D}'$ so that already $\lambda_0 = (2^{|P|})^+$ suffices to demonstrate the completeness of P. That is, for some $c \in H_{\lambda_0}$, (5.18) holds.*

Proof. The definition of $\mathbb{D}'$ is simple. For any countable transitive structure N, poset $R \in N$, condition $r \in R$, and parameter $p \in N$, define

$$\mathbb{D}'(N, R, r, p) = \mathbb{D}(p, R, r)$$

if this makes sense, that is, if p is transitive, $R, r \in p$ and $\mathbb{D}(p, R, r)$ is indeed a collection of sets of (N, R)-generic filters as required. In case this definition does not make sense, let $\mathbb{D}'(N, R, r, p)$ be defined as an arbitrary collection of subsets of $\mathrm{Gen}_r(N, R)$ with the countable intersection property. We must define a good parameter $c \in H_{\lambda_0}$ that will work.

Let λ be sufficiently large so that for every countable $M \prec H_\lambda$ with $P \in H_\lambda$ (5.17) holds. Let $\kappa = |P|$ be the cardinality of P, and assume for simplicity that κ is the universe of P. We may assume that $\lambda > \lambda_0 = (2^\kappa)^+$ (or else there is nothing to show). Let K be an elementary substructure of H_λ of cardinality 2^κ (containing all subsets of κ). Clearly, every elementary substructure of K is also an elementary substructure of H_λ, so that if we let $\pi : K \to \bar{K}$ be the transitive collapse of K then, for every $M \prec \bar{K}$, the transitive collapse of M is the transitive collapse of an elementary substructure of K, namely the pre-image of M.

We claim that $c = \bar{K} \in H_{\lambda_0}$ works. Let $M \prec H_{\lambda_0}$ be countable with $P, \bar{K} \in M$, and $p_0 \in P \cap M$ be given. Then $M \cap \bar{K} \prec \bar{K}$. Let $\pi : M \to N$ be the collapsing function onto a transitive structure. $\pi(\bar{K})$ is the transitive collapse of $M \cap \bar{K}$, and $\mathbb{D}'(N, \pi(P), \pi(p_0), \pi(\bar{K})) = \mathbb{D}(\pi(\bar{K}), \pi(P), \pi(p_0))$. The point is that if $G \in X \in \mathbb{D}(\pi(\bar{K}), \pi(P), \pi(p_0))$, then G is not only $(\pi(P), \pi(\bar{K}))$-generic filter but also (N, P)-generic, since any subset of $\pi(\kappa)$ in N is already in $\pi(\bar{K})$ (as any subset of κ in M is already in $M \cap \bar{K}$). $\dashv$

Our aim is to prove the following

5.17 Theorem (Dee-Completeness Iteration Theorem). *The countable support iteration of any length γ of $<\omega_1$-proper posets, each Dee-complete with respect to some countably complete system in the ground model, does not add any new reals.*

Note the inductive character of this theorem. For $Q_i \in V^{P_i}$ to be Dee-complete with respect to a system that lies in V, one needs that P_i adds no new countable sets—so that every countable transitive set in V^{P_i} is in V.

To prove the theorem we shall first define for each countable $M \prec H_\lambda$ (with $P_\gamma \in M$) an (M, P_γ)-generic filter G_γ. Then we will prove that G_γ is bounded in P_γ. That is, we will find a condition in P_γ that is completely (M, P_γ)-generic. The definition of G_γ is by induction, and we shall actually have to define for every $\gamma_0 < \gamma$ and G_{γ_0} that is (M, P_{γ_0})-generic, a filter G_γ that extends G_{γ_0}. There will be two main cases in this definition: γ successor and γ limit, and likewise there will be two cases in the proof that G_γ is bounded. We start with what is needed for the successor case.

Two-Step Iteration

Let P be a poset and $\underset{\sim}{Q} \in V^P$ a name forced (by 0_P) to be a poset. Let λ be sufficiently large and $M_0 \prec H_\lambda$ be a countable elementary submodel such that $P, \underset{\sim}{Q} \in M_0$. We want to find a criterion for when a condition $(q_0, q_1) \in P * \underset{\sim}{Q}$ is completely $(M_0, P * \underset{\sim}{Q})$-generic. A first guess is: q_0 is completely (M_0, P)-generic and q_0 forces that q_1 is completely $(M_0[\underset{\sim}{G}], \underset{\sim}{Q})$-generic. But a moment's reflection reveals that this is not sufficient for (q_0, q_1) to determine, in V, an $(M_0, P * \underset{\sim}{Q})$-generic filter. So we need a finer criterion.

Let $\pi : M_0 \to N_0$ be the transitive collapsing map. Suppose that $q_0 \in P$ is completely (M_0, P)-generic and let $G_P \subseteq P \cap M_0$ be the (M_0, P)-generic filter induced by q_0. Then $\mathcal{G}_0 = \pi$“G_P is an $(N_0, \pi(P))$-generic filter and we can form the (transitive) extension $N_0^* = N_0[\mathcal{G}_0]$. In N_0, $\pi(\underset{\sim}{Q})$ is a name, and its interpretation $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$ is a poset in N_0^*.

Let $\underset{\sim}{G} \in V^P$ be the canonical name of the generic filter over P. If F is any (V, P)-generic filter containing q_0, then $M_0[F] \prec H_\lambda[F]$ can be formed and the collapsing map π on M_0 can be extended to collapse $M_0[F]$ onto N_0^*. Let $\underset{\sim}{\pi}$ be the name of this extended collapse. Then $q_0 \Vdash_P \underset{\sim}{\pi} : M_0[\underset{\sim}{G}] \to N_0^*$. We phrase now the desired criterion but omit the routine proof.

5.18 Lemma. *With the above notation,* (q_0, q_1) *is completely* $(M_0, P * \underset{\sim}{Q})$*-generic iff*

1. q_0 *is completely* (M_0, P)*-generic, and*

2. *for some* (N_0^*, Q_0^*)*-generic* $\mathcal{G}_1 \subseteq Q_0^*$, $q_0 \Vdash \underset{\sim}{\pi}^{-1}$“$\mathcal{G}_1$ *is bounded by* q_1.

In this case, the filter induced by (q_0, q_1) *over* $M_0 \cap P * \underset{\sim}{Q}$ *is* π^{-1}“$\mathcal{G}_0 * \mathcal{G}_1$.

Given a countable $M_0 \prec H_\lambda$ such that the two-step iteration $P * \underset{\sim}{Q}$ is in M_0, our aim (under some assumptions stated in the following definition) is to extend each (M_0, P)-generic filter G_0 to an $(M_0, P * \underset{\sim}{Q})$-generic filter. This definition depends not only on M_0, but also on another countable elementary submodel $M_1 \prec H_\lambda$ such that $M_0 \in M_1$. In addition, we assume some $p_0 \in P * \underset{\sim}{Q}$ which we want to include in the extended filter. All of this leads to a five place function $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ that we define now.

5.19 Definition. Let P be a poset that adds no new countable sets of ordinals, and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in V^P$ are such that

$$\Vdash_P \quad \begin{array}{l} \underset{\sim}{\mathbb{D}} \in V \text{ is a countably complete system} \\ \text{and } \underset{\sim}{Q} \text{ is Dee-complete with respect to } \underset{\sim}{\mathbb{D}}. \end{array}$$

Let λ be sufficiently large, and $M_0 \prec M_1 \prec H_\lambda$ be countable elementary submodels with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Let $G_0 \subseteq M_0 \cap P$ be (M_0, P)-generic and suppose that $G_0 \in M_1$. Let $p_0 \in (P * \underset{\sim}{Q}) \cap M_0$ be given, $p_0 = (a, \underset{\sim}{b})$ with $a \in G_0$. Then we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0), \tag{5.19}$$

an $(M_0, P * \underset{\sim}{Q})$-generic filter containing p_0, by the following procedure.

Let $\pi : M_0 \to N_0$ be the transitive collapse, and $\mathcal{G}_0 = \pi``G_0$. Form $N_0^* = N_0[\mathcal{G}_0]$. Observe that $N_0^* \in M_1$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$ because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably complete completeness system. Thus $\mathbb{D}(N_0^*, Q_0^*, b^*)$ is defined in M_1 where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is countable, $\exists \mathcal{G}_1 \in \bigcap(M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*))$. $\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$.

Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. Finally, set

$$\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = \pi^{-1}``\mathcal{G}.$$

This completes Definition 5.19.

In fact, we want to define a formula ψ so that

$$H_\lambda \models \psi(G, M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$$

iff (5.19) holds. That is, we want to define $\mathbb{E}$ in H_λ. We cannot take the above definition literally because it relies on the assumption that M_0 and M_1 are elementary substructures of H_λ, something which is not expressible in H_λ itself. So we redo that definition for any countable subsets M_0 and M_1 of H_λ (or models of ZF$^-$). Whenever Definition 5.19 above relies on some fact that happens not to hold, we let G have an arbitrary value. For example, if N_0^* is not in M_1 or if $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is empty, then we let G be some arbitrary fixed $(M_0, P * \underset{\sim}{Q})$-generic filter.

The following is a main lemma which exhibits the crux of the argument (compare with Lemma 5.14). It analyzes the iteration of two posets when the second is Dee-complete.

5.20 Lemma (The Gambit Lemma). *Let P be a poset and suppose that $\underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in V^P$ are such that*

$$\Vdash_P \quad \begin{array}{l} \underset{\sim}{\mathbb{D}} \in V \text{ is a countably complete system} \\ \text{and } \underset{\sim}{Q} \text{ is Dee-complete with respect to } \underset{\sim}{\mathbb{D}}. \end{array}$$

Let λ be sufficiently large, and $M_0 \prec M_1 \prec H_\lambda$ be countable elementary submodel with $M_0 \in M_1$ and $P, \underset{\sim}{Q}, \underset{\sim}{\mathbb{D}} \in M_0$. Suppose that $q_0 \in P$ is (M_1, P)-generic as well as completely (M_0, P)-generic, and let $G_0 \subseteq M_0 \cap P$ be the M_0 filter over $M_0 \cap P$ induced by q_0. Let $p_0 \in P * \underset{\sim}{Q}$, $p_0 \in M_0$ be given so that $p_0 = (a, \underset{\sim}{b})$ and $a \in G_0$. Then there is a $q_1 \in V^P$ so that (q_0, q_1) is completely $(M_0, P * \underset{\sim}{Q})$-generic and $p_0 < (q_0, q_1)$. In fact, (q_0, q_1) bounds $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$.

Proof. Notice that $G_0 \in M_1$ by the following argument. Let R be the collection of all conditions $r \in P$ that are completely (M_0, P)-generic. Then $R \in M_1$ and $q_0 \in R$. Since q_0 is (M_1, P)-generic it follows that it is compatible with some $r \in R \cap M_1$. But any two compatible conditions in R induce the same filter, and hence G_0 is the filter induced by r.

As in Definition 5.19, let $\pi : M_0 \to N_0$ be the transitive collapse, and $\mathcal{G}_0 = \pi``G_0$. We recall the definition of $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$. Form $N_0^* = N_0[\mathcal{G}_0]$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. $\mathbb{D}_0 \in N_0$ and $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably complete completeness system. Thus $\mathbb{D}(N_0^*, Q_0^*, b^*)$ is defined in M_1 where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*)$ is countable and non-empty, we were able to pick $\mathcal{G}_1 \in \bigcap(M_1 \cap \mathbb{D}(N_0^*, Q_0^*, b^*))$, (N_0^*, Q_0^*)-generic with $b^* \in \mathcal{G}_1$. We defined $\mathcal{G} = \mathcal{G}_0 * \mathcal{G}_1$, and defined $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ as $\pi^{-1}``\mathcal{G}$.

Let $\underset{\sim}{G} \in V^P$ be the canonical name of the generic filter over P. Then q_0 forces that π can be extended to a collapse $\underset{\sim}{\pi}$ which is onto N_0^*: that is,

$$q_0 \Vdash_P \underset{\sim}{\pi} : M_0[\underset{\sim}{G}] \to N_0^*.$$

The conclusion of our lemma follows if we show that

$$q_0 \Vdash_P \underset{\sim}{\pi}^{-1}``\mathcal{G}_1 \text{ is bounded in } \underset{\sim}{Q}. \tag{5.20}$$

In this case, if we define $q_1 \in V^P$ so that $q_0 \Vdash_P q_1$ bounds $\underset{\sim}{\pi}^{-1}``\mathcal{G}_1$, then the previous lemma (5.18) implies that the $(M_0, P * \underset{\sim}{Q})$-generic filter induced by $(q_0, q_1)$ is $\pi^{-1}``\mathcal{G}_0 * \mathcal{G}_1$.

So let F be (V, P)-generic with $q_0 \in F$. $\underset{\sim}{\pi}[F]$ collapses $M_0[F]$ onto N_0^*, and there is a set $X \in \mathbb{D}(N_0^*, Q_0^*, b^*)$ so that if $\mathcal{H} \in X$ is any filter then $\underset{\sim}{\pi}^{-1}``\mathcal{H}$ is bounded in $\underset{\sim}{Q}[F]$. As $M_1[F] \prec H_\lambda[F]$, we can have $X \in M_1[F]$. But since $\mathbb{D}$ is in the ground model, $X \in M_1$. Thus $\mathcal{G}_1 \in X$, where $\mathcal{G}_1$ is the filter defined above. This proves (5.20). $\dashv$

Proof of Theorem 5.17

Let P_γ be a countable support iteration of length γ, obtained by iterating $Q_i \in V^{P_i}$ as in the theorem. That is, each Q_i is Dee-complete in V^{P_i} for some countably complete system taken from V. Let λ be a sufficiently large cardinal. To prove the theorem we first describe a machinery for obtaining generic filters over countable submodels of H_λ. We define a function $\mathbb{E}$ that takes five arguments $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$, of the following types.

1. $M_0 \prec H_\lambda$ is countable, $P_\gamma \in M_0$ (so $\gamma \in M_0$), and $p_0 \in P_\gamma \cap M_0$.

2. For some $\gamma_0 \in M_0 \cap \gamma$, G_0 is an (M_0, P_{γ_0})-generic filter such that $p_0 \restriction \gamma_0 \in G_0$. We assume that $G_0 \in M_1$.

3. The order type of $M_0 \cap [\gamma_0, \gamma)$ is α.

4. $\bar{M} = \langle M_\xi \mid 1 \le \xi \le \alpha \rangle$ is a tower of countable elementary submodels of H_λ, and $M_0 \in M_1$. It will be clear later why we separate M_0 from the rest of the tower.

The value returned, $G_\gamma = \mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ is an (M_0, P_γ)-generic filter that extends G_0 and contains p_0. Formally, in saying that G_γ extends G_0 we mean that the restriction projection takes G_γ onto G_0. The definition of $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ is by induction on $\alpha < \omega_1$.

Assume that $\alpha = \alpha' + 1$ is a successor ordinal. Then $\gamma = \gamma' + 1$ is also a successor. Assume first that $\gamma_0 = \gamma'$. Then $\alpha = 1$ and we have only two structures: M_0 and M_1. Since P_γ is isomorphic to $P_{\gamma_0} * Q_{\gamma_0}$, we can define G_γ by Definition 5.19. So

$$G_\gamma = \mathbb{E}(M_0, M_1, P_{\gamma_0} * Q_{\gamma_0}, G_0, p_0).$$

Assume next that $\gamma_0 < \gamma'$. Then

$$G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi \mid 1 \le \xi \le \alpha' \rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma')$$

is defined and is an $(M_0, P_{\gamma'})$-generic filter that extends G_0 and contains $p_0 \restriction \gamma'$. Moreover, we assume that $G_{\gamma'} \in M_\alpha$, for otherwise the inductive definition stops. (When we finish this definition, it will be evident that it continues through every $\alpha < \omega_1$ since $M_\alpha \prec H_\lambda$ and the parameters are all in M_α.)

This brings us to the previous case and we define

$$G_\gamma = \mathbb{E}(M_0, M_\alpha, P_{\gamma'} * Q_{\gamma'}, G_{\gamma'}, p_0). \tag{5.21}$$

Now suppose that α is a limit ordinal, and let $\langle \alpha_n \mid n \in \omega \rangle$ be an increasing and cofinal sequence with $\alpha_0 = 0$. Let $\gamma_n \in M_0$ be the corresponding increasing and cofinal in γ sequence (so that α_n is the order-type of $M_0 \cap [\gamma_0, \gamma_n)$). Let $\langle D_n \mid n \in \omega \rangle$ be an enumeration of all dense subsets of P_γ that are in M_0.

We define $G_\gamma = \mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$ as follows. We define by induction on $n \in \omega$ a condition $p_n \in P_\gamma \cap M_0$ and an (M_0, P_{γ_n})-generic filter $G_n \in M_{\alpha_n + 1}$ such that:

1. G_0 and p_0 are given. $p_n \restriction \gamma_n \in G_n$.

2. $p_n \le p_{n+1}$ and $p_{n+1} \in D_n$.

Suppose that G_n and p_n are defined. First, we can easily find a $p_{n+1} \in D_n \cap M_0$ such that $p_{n+1} \restriction \gamma_n \in G_n$. Now define

$$G_{n+1} = \mathbb{E}(M_0, \langle M_\xi \mid \alpha_n + 1 \le \xi \le \alpha_{n+1} \rangle, P_{\gamma_{n+1}}, G_n, p_{n+1} \restriction \gamma_{n+1}). \tag{5.22}$$

Finally, let G_γ be the generic filter generated by $\{p_n \mid n \in \omega\}$. This completes the definition of $\mathbb{E}(M_0, \bar{M}, P_\gamma, G_0, p_0)$.

Theorem 5.17 is a direct consequence of the following lemma.

5.21 Lemma (Dee-Properness Extension Lemma). *Let $\langle P_i \mid i \leq \gamma\rangle$ be a countable support iteration of forcing posets (γ is any ordinal) where each iterand Q_i satisfies the following in V^{P_i}:*

1. *Q_i is α-proper for every countable α.*
2. *Q_i is Dee-complete with respect to some countably complete completeness system in the ground model V.*

Suppose that $M_0 \prec H_\lambda$ is countable, $P_\gamma \in M_0$ and $p_0 \in P_\gamma \cap M_0$. For any $\gamma_0 \in \gamma \cap M_0$, if α is the order-type of $M_0 \cap [\gamma_0, \gamma)$ and $\bar{M} = \langle M_k \mid k \leq \alpha\rangle$ is a tower of countable elementary substructures (starting with the given M_0) then the following holds. For any $q_0 \in P_{\gamma_0}$ that is completely (M_0, P_{γ_0})-generic as well as $(\bar{M}, P_{\gamma_0})$-generic, if $p_0 \restriction \gamma_0 < q_0$ then there is some $q \in P_\gamma$ such that $q_0 = q \restriction \gamma_0$, $p_0 < q$ and q is completely (M_0, P_γ)-generic. In fact, the filter induced by q is $\mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha\rangle, P_\gamma, G_0, p_0)$ where $G_0 \subseteq P_{\gamma_0} \cap M_0$ is the filter induced by q_0.

Proof. Let $G_0 \subseteq P_{\gamma_0} \cap M_0$ be the (M_0, P_{γ_0})-generic filter induced by q_0. Observe that $G_0 \in M_1$ follows from the assumption that q_0 is (also) M_1-generic. We shall prove by induction on α (the order-type of $M_0 \cap [\gamma_0, \gamma)$) that q can be found which bounds $G_\gamma = \mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha\rangle, P_\gamma, G_0, p_0)$.

Suppose first that $\alpha = \alpha' + 1$ and consequently $\gamma = \gamma' + 1$ are successor ordinals. Define, in M_α, $X \subseteq P_{\gamma_0}$ a maximal antichain of conditions r such that

1. r bounds G_0.
2. r is $\langle M_\xi \mid 1 \leq \xi \leq \alpha'\rangle$-generic.

Then $X \in M_\alpha$ is pre-dense above q_0. By our inductive assumption every $r_0 \in X$ has a prolongation $r_1 \in P_{\gamma'}$ that bounds $G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi \mid 1 \leq \xi \leq \alpha'\rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma')$. Since all the parameters are in M_α, we get that $G_{\gamma'} \in M_\alpha$. Since $M_\alpha \prec H_\lambda$, we can choose $r_1 \in M_\alpha$ whenever $r_0 \in X \cap M_\alpha$. This defines a name $\underset{\sim}{r_1} \in V^{P_{\gamma_0}}$, forced by q_0 to be in $M_\alpha \cap P_{\gamma'}$. Namely, if G is any (V, P_{γ_0})-generic filter containing q_0, then $X \cap G$ contains a unique condition r_0, and we let $\underset{\sim}{r_1}[G] = r_1$. By the Properness Extension Lemma we can find a $q_1 \in P_{\gamma'}$ with $q_1 \restriction \gamma_0 = q_0$ so that q_1 is $(M_\alpha, P_{\gamma'})$-generic, and $q_1 \Vdash \underset{\sim}{r_1}$ is in the generic filter. It follows that q_1 bounds $G_{\gamma'}$. We must define $q_2 \in P_\gamma$ such that $q_2 \restriction \gamma' = q_1$ and q_2 bounds G_γ. In order to define $q_2(\gamma')$ use Lemma 5.20 and (5.21).

Now assume that α is a limit ordinal. We follow the definition of G_γ (see (5.22)). Recall that we had an ω-sequence $\langle \alpha_n \mid n \in \omega\rangle$ cofinal in α, and we defined γ_n cofinal in γ as the resulting sequence. We defined by induction $p_n \in P_\gamma \cap M_0$ and filters $G_n \subseteq P_{\gamma_n}$, $G_n \in M_{\alpha_n+1}$, and defined G_γ as the filter generated by the p_n's. We shall define now $q_n \in P_{\gamma_n}$ by induction on $n \in \omega$ so that the following hold.

1. q_n bounds G_n.

2. $p_n \restriction \gamma_n < q_n$.

3. $q_n = q_{n+1} \restriction \gamma_n$.

4. q_n is $\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha \rangle$ generic over P_{γ_n}.

Thus, as q_n gains in length, it loses its status as an M_ξ generic condition for $0 < \xi \leq \alpha_n$. So, finally, $q = \bigcup_{n \in \omega} q_n$ is not M_ξ-generic for any $\xi > 0$. But these M_ξ's are not needed anymore as q gained its complete genericity over M_0.

Suppose that q_n is defined. Let X in $M_{\alpha_{n+1}+1}$ be a maximal antichain in P_{γ_n} of conditions r that induce G_n and are $\langle M_\xi \mid \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle$-generic over P_{γ_n}. Observe that X is pre-dense above q_n. For each $r_0 \in X$ find a $r_1 \in P_{\gamma_{n+1}}$ such that r_1 bounds G_{n+1}, $p_{n+1} \restriction \gamma_{n+1} < r_1$, and $r_1 \restriction \gamma_n = r_0$ (use the inductive assumption). If $r_0 \in X \cap M_{\alpha_{n+1}+1}$ then r_1 is taken from $M_{\alpha_{n+1}+1}$. Now view $\{r_1 \mid r_0 \in X\}$ as a name $\underset{\sim}{r}$ for a condition forced by q_n to lie in $M_{\alpha_{n+1}+1}$. By the α-Extension Lemma define q_{n+1} that satisfies 2–4 above and such that $q_{n+1} \Vdash \underset{\sim}{r} \in \underset{\sim}{G}$. Then q_{n+1} bounds G_{n+1} and is as required. $\dashv$

Simple Completeness Systems

An important chapter in the theory of forcing is the study of forcing axioms. These are axioms in the spirit of Martin's Axiom, and the best known among those related to proper forcing is probably the Proper Forcing Axiom (PFA) which uses a supercompact cardinal for its consistency. PFA is due to Baumgartner, and [1] contains many applications and a consistency proof. (The reader can also read a consistency proof in Cummings's chapter in this Handbook.) Chapters VII, VIII, XVII of [15] discuss many of these proper forcing axioms, and we shall restrict our attention here to just one axiom: a variant of Axiom II of Chap. VII which is used to obtain consistency results with CH. This will motivate the notion of *simple* completeness systems and will lead to the p.i.c.

Assume that κ is a supercompact cardinal in the ground model V. Define a countable support iteration P_γ, for $\gamma \leq \kappa$ of $<\omega_1$-proper posets, of cardinality $< \kappa$ each, that are Dee-complete for countably complete completeness systems from V. The actual choice of the iterand is done by some "Laver diamond" function.

Let $P = P_\kappa$ be the resulting countable support iteration. It follows by arguments that are very similar to those used for the PFA consistency result that if G is (V, P)-generic, then $V[G]$ satisfies the following axiom, formulated with the ground model V as a predicate:

> GCH holds. V is a ZFC subuniverse containing all reals, and such that the following holds. Let P be any poset such that
>
> 1. P is $<\omega_1$-proper.

2. P is Dee-complete with respect to some countably complete completeness system from V.

Then for any sequence $D_\alpha \subseteq P$ of dense subsets of P for $\alpha < \omega_1$, there is some filter $G \subseteq P$ such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$.

It can be maintained that an axiom should not relate to a subuniverse V in its formulation, and that a more local axiom is required. For this the notion of simplicity is introduced, and we bring here an axiom which is a variant of the original formulation of Shelah [15].

Let HC $= (H_{\aleph_1}, \in)$ be the structure consisting of the universe of all hereditarily countable sets together with the membership relation $\in$. We say that a completeness system $\mathbb{D}$ is *simple over* HC if there is a first-order formula $\psi(y_0, \dots, y_4)$ in the $\in$ language such that for every transitive and countable model N of ZFC$^-$, poset $P \in N$, and $p \in P$

$$\mathbb{D}(N, P, p) = \{A_X \mid X \in H_{\aleph_1}\}$$

where

$$A_X = \{G \in \mathrm{Gen}_p(N, P) \mid \mathrm{HC} \models \psi(G, X, N, P, p)\}. \tag{5.23}$$

For example, $\psi(G, X, N, P, p)$ could say the following.

Assume that:

1. In N, g is a Hajnal-Mátė graph, $P = P_g$, and $p \in P$.

2. X is an unbounded ω-sequence in ω_1^N.

Then $G \subseteq P$ is (N, P)-generic, $p \in G$, and the restriction of $\bigcup G$ to X omits infinitely many colors. If the above assumptions do not hold, then G is any (N, P)-generic filter containing p.

Now the axiom that can be used to obtain results that are consistent with CH is the following (PFA for countably complete simple completeness systems).

> CH holds. If P is any $<\omega_1$-proper poset that is Dee-complete for some countably complete completeness system that is simple over HC, and if $\{D_i \mid i \in \omega_1\}$ is a collection of dense subsets of P, then there is a filter $G \subseteq P$ that meets all of the D_i's. (5.24)

As mentioned, the consistency of this axiom relies on a supercompact cardinal in the ground model, yet in many of the specific applications of this axiom, the supercompact cardinal is not needed. For example, the Souslin hypothesis (which says there are no Souslin trees) is a consequence of the axiom, and in fact the axiom implies that every Aronszajn tree is special. For every Aronszajn tree T, there is a $<\omega_1$-proper poset which is Dee-complete with respect to some countably complete *simple* over HC system, and which specializes T (see Shelah [15, Chap. V]). Hence the axiom quoted above implies this strong form of the Souslin hypothesis. Yet, to get the consistency with CH of "every Aronszajn tree is special" no large cardinal is needed—this is the result of Jensen [3]. We may, without any large cardinal assumption, iterate such specializing posets (by the Dee-Completeness Iteration Theorem) and obtain the same consistency result. If we do so, we encounter a small difficulty which is discussed in the following subsection, namely that the specializing posets have size $2^{\aleph_1}$ each and so it is unclear, at this stage, that the iteration satisfies the $\aleph_2$-c.c. Although we shall not describe the specializing posets, it turns out that they satisfy the $\aleph_2$-p.i.c. (a strong form of the chain condition described below) and hence the $\aleph_2$-c.c of the iteration follows. Another use of the $\aleph_2$-p.i.c. (which is the one that will be exemplified) is to obtain extensions in which $2^{\aleph_1} > \aleph_2$.

5.4. The Properness Isomorphism Condition

Using the iteration scheme of the previous section we know how to obtain models of ZFC + CH + $2^{\aleph_1} = \aleph_2$ + "Every Hajnal-Máté graph has countable chromatic number". In this section we modify a little the construction in order to obtain such models with $2^{\aleph_1}$ arbitrarily large. To obtain this, Shelah uses the following simple idea. Starting with $2^{\aleph_1}$ already large, form a countable support *product* of all posets of the form P_g that are in the ground model. This takes care of all Hajnal-Máté graphs in V. Now iterate such large products ω_2 times, and obtain a model of ZFC + CH + $2^{\aleph_1}$ large + "Every Hajnal-Máté graph has countable chromatic number". The main technical problem in this approach is to prove that the iteration satisfies the $\aleph_2$-c.c. After we prove that this is the case, we will argue that every Hajnal-Máté graph in the extension already appears in some intermediate stage and hence acquired a countable chromatic number at the following stage. To establish the $\aleph_2$-c.c. we will use the condition named $\aleph_2$-p.i.c. (for Properness Isomorphism Condition), introduced in [15, Chap. VIII] exactly for such applications in mind.

We employ the following terminology. Suppose $M_0, M_1 \prec H_\lambda$ are countable, isomorphic, elementary submodels, and $\aleph_2 \leq \kappa < \lambda$ is a regular cardinal such that $\kappa \in M_0 \cap M_1$; typically $\kappa = \aleph_2$. We say that M_0 and M_1 are in *standard situation* (with respect to κ) iff

1. The sets $A = M_0 \cap M_1 \cap \kappa$, $B = (M_0 \setminus M_1) \cap \kappa$, and $C = (M_1 \setminus M_0) \cap \kappa$ are arranged $A < B < C$ (where $X < Y$ means that $\forall x \in X \forall y \in Y (x < y)$).

2. The isomorphism, denoted $h : M_0 \to M_1$, is the identity function on $M_0 \cap M_1$ (so in particular on A).

5.22 Definition. Let M_0 and M_1 be in standard situation with $h : M_0 \to M_1$ the isomorphism. Suppose that $P \in M_0 \cap M_1$ is a poset. We say that a condition $q \in P$ is *simultaneously* (M_0, P)*- and* (M_1, P)*-generic* iff

1. q is both (M_0, P)- and (M_1, P)-generic,
2. $$q \Vdash_P (\forall r \in M_0 \cap P)\ r \in \underset{\sim}{G} \quad \text{iff} \quad h(r) \in \underset{\sim}{G} \tag{5.25}$$
   where $\underset{\sim}{G}$ is the name of the P generic filter.

Equivalently, (5.25) can be stated as: for every $q' \geq q$ and $r \in M_0 \cap P$, if $r < q'$ then $h(r) < q'$. Yet another equivalent formulation is that p forces that h can be extended to an isomorphism of $M_0[\underset{\sim}{G}]$ onto $M_1[\underset{\sim}{G}]$.

Note that the requirement in 1. that q be (M_1, P)-generic is dispensable, since it follows from 2. when q is (M_0, P)-generic.

5.23 Definition. Let κ be an uncountable regular cardinal. A poset P *satisfies the* κ-p.i.c. if the following holds for sufficiently large cardinals λ and any two isomorphic countable elementary submodels $M_0, M_1 \prec H_\lambda$ with $P \in M_0 \cap M_1$ that are in standard situation. For any $p \in M_0 \cap P$ there is a $q > p$ in P that is simultaneously (M_0, P)- and (M_1, P)-generic. (Hence, in particular, $q > h(p)$.)

This definition is phrased so that the κ-p.i.c. of P implies its properness (take $M_0 = M_1$), but for clarity we shall use the expression "P is a proper κ-p.i.c. poset".

For example, any proper poset of size $\aleph_1$ is $\aleph_2$-p.i.c. because $M_0 \cap P = M_1 \cap P$. So our discussion here generalizes Sect. 2.2. In fact, if P is proper and $|P| < \kappa$ then P satisfies the κ-p.i.c. (see [15, Chap. VIII]), but $\mu^{\aleph_0} < \kappa$ for $\mu < \kappa$ is needed for the lemma that derives the κ chain condition (see below). The Cohen forcing poset for adding $\aleph_2$ reals (with finite conditions), while c.c.c., is not $\aleph_2$-p.i.c. In contrast, the poset for adding subsets of ω_1 with countable conditions is $\aleph_2$-p.i.c.

5.24 Lemma. *If κ is a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, then any κ-p.i.c. poset satisfies the usual κ-c.c.*

Proof. In fact, every collection $\{p_i \mid i < \kappa\} \subseteq P$ contains a subcollection of size κ of pairwise compatible conditions. For each $i < \kappa$ pick some countable $M_i \prec H_\lambda$ with $p_i \in M_i$. Since $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, a standard Δ-system argument yields a set $I \subseteq \kappa$ of cardinality κ such that $\{M_i \cap \kappa \mid i \in I\}$ form a Δ-system. So for $i, j \in I$ with $i < j$, M_i and M_j are in standard situation. We may also assume that (M_i, p_i) and (M_j, p_j) are isomorphic (so $h : M_i \to M_j$ is an isomorphism such that $h(p_i) = p_j$). Now the κ-p.i.c. implies that some $q \in P$ extends both p_i and p_j. ⊣

We shall prove now the main theorem using a short argument followed by a series of lemmas which are brought without proof or with a short proof since they resemble those of Sect. 2.

5.25 Theorem. *Suppose that κ is a regular cardinal and $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$.*

1. *If P_κ is a countable support iteration of length κ of proper κ-p.i.c. posets, then P_κ satisfies the κ-c.c.*

2. *If P_γ is a countable support iteration of length $\gamma < \kappa$ of proper κ-p.i.c. posets, then P_γ satisfies the κ-p.i.c. (For this we do not need the assumption on $\mu^{\aleph_0}$.)*

Proof. To prove the first part, consider a collection $\{p_i \mid i < \kappa\} \subseteq P_\kappa$. Use Fodor's Theorem to fix a bound $i_0 < \kappa$ on $\sup(i \cap \mathrm{dom}(p_i))$ on a stationary set of indices i's with uncountable cofinality, so that $\mathrm{dom}(p_i)$ form a Δ-system. This shows that it suffices to prove that P_{i_0}, the iteration of the first $i_0 < \kappa$ posets, is κ-c.c. In fact it is κ-p.i.c. as the second part of the theorem shows. The proof of this part is in the following sequence of lemmas. ⊣

5.26 Lemma. *Let P be a poset and $Q \in V^P$ a name of a poset. Form $R = P * Q$. Then for any countable $M_0, M_1 \prec H_\lambda$ in standard situation and such that $R \in M_0 \cap M_1$ we have the following characterization: $(p, q) \in R$ is simultaneously (M_0, R)- and (M_1, R)-generic iff*

1. *$p \in P$ is simultaneously (M_0, P)- and (M_1, P)-generic; and*

2. *$p \Vdash_P q$ is simultaneously $(M_0[\underset{\sim}{G}_P], Q)$and $(M_1[\underset{\sim}{G}_P], Q)$-generic.*

For the proof, use Lemma 2.5 and the equivalent statement following (5.25).

5.27 Lemma. *Suppose that P is a κ-p.i.c. poset and $Q \in V^P$ is (forced to be) κ-p.i.c. there. Then $R = P * Q$ is also κ-p.i.c. and the following stronger form of Lemma 2.5 holds.*

Suppose that

1. *$M_0, M_1 \prec H_\lambda$ with $R \in M_0 \cap M_1$ are countable elementary submodels in standard situation.*

2. *$p_0 \in P$ is a simultaneously (M_0, P)- and (M_1, P)-generic.*

3. *$\underset{\sim}{r} \in V^P$ is a name such that*

$$p_0 \Vdash_P \underset{\sim}{r} \in M_0 \cap R \quad \textit{and} \quad \pi(\underset{\sim}{r}) \in \underset{\sim}{G}_0$$

where $\pi : P * Q \to P$ is the projection, and $\underset{\sim}{G}_0$ is the canonical name for the P-generic filter.

Then there is some $q_0 \in V^P$ such that (p_0, q_0) is simultaneously (M_0, R)- and (M_1, R)-generic and

$$(p_0, q_0) \Vdash_R \underset{\sim}{r} \in \underset{\sim}{G}.$$

So also

$$(p_0, q_0) \Vdash_R h(\underset{\sim}{r}) \in \underset{\sim}{G}.$$

The following lemma completes the proof of the second item of Theorem 5.25.

5.28 Lemma (Extension of p.i.c.). *Let P_γ be a countable support iteration of length $\gamma < \kappa$ of proper posets that are κ-p.i.c. Let λ be sufficiently large and $M_0, M_1 \prec H_\lambda$ be countable with $P_\gamma \in M_0 \cap M_1$ and that are in standard situation. For any $\gamma_0 \in \gamma \cap M_0$ and $q_0 \in P_{\gamma_0}$ that is simultaneously (M_0, P_{γ_0})- and (M_1, P_{γ_0})-generic the following holds. If $\underset{\sim}{p_0} \in V^{P_{\gamma_0}}$ is such that*

$$q_0 \Vdash_{\gamma_0} \underset{\sim}{p_0} \in P_\gamma \cap M_0 \quad \textit{and} \quad \underset{\sim}{p_0} \restriction \gamma_0 \in \underset{\sim}{G}_0$$

(where $\underset{\sim}{G}_0$ is the name of the P_{γ_0} generic filter) then there is a condition $q \in P_\gamma$ such that $q \restriction \gamma_0 = q_0$, q is simultaneously (M_0, P_γ)- and (M_1, P_γ)-generic, and $q \Vdash_\gamma \underset{\sim}{p_0} \in \underset{\sim}{G}$. (Thus also $q \Vdash_\gamma h(\underset{\sim}{p_0}) \in \underset{\sim}{G}$.)

The proof follows the same steps of the Properness Extension Lemma 2.8.

The following discussion can help to clarify some of the definitions and statements described above. Let HM be the collection of all Hajnal-Máté graphs. Recall that for any $g \in HM$, P_g is the poset for making the chromatic number of g countable. Let $P = \Pi^{\aleph_0}_{g \in HM} P_g$ be the countable support product of all ground model P_g's. That is, $f \in P$ iff f is a function defined on HM with $f(g) \in P_g$ and such that $f(g) = \emptyset$ for all but countably many g's. The ordering is coordinate wise extension.

The reader can go over the corresponding steps for P_g and prove that P is

1. proper,
2. α-proper for every $\alpha < \omega_1$,
3. Dee-complete for a countably complete completeness system $\mathbb{D}$,
4. an $\aleph_2$-p.i.c. poset.

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6. Combinatorial Cardinal Characteristics of the Continuum

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Contents

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1. Introduction

The first theorem about cardinal characteristics of the continuum is Cantor's classical result [38] that the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum is strictly larger than the cardinality $\aleph_0$ of a countably infinite set. The distinction between $\aleph_0$ and $\mathfrak{c}$ was soon put to good use, especially in real analysis, where countable sets were shown to have many useful properties that cannot be extended to sets of cardinality $\mathfrak{c}$. Here are a few familiar examples; more examples are implicit throughout this chapter.

- Countably many nowhere dense sets cannot cover the real line. (The Baire Category Theorem.)
- If countably many sets each have Lebesgue measure zero then so does their union.
- Given countably many sequences of real numbers, there is a single sequence that eventually dominates each of the given ones.
- Let countably many bounded sequences $S_k = \langle x_{k,n} \rangle_{n\in\omega}$ of real numbers be given. There is an infinite subset A of ω such that all the corresponding subsequences $S_k \restriction A = \langle x_{k,n} \rangle_{n\in A}$ converge.

Each of these results becomes trivially false if the hypothesis of countability is weakened to allow cardinality $\mathfrak{c}$. It is natural to ask whether the hypothesis can be weakened at all and, if so, by how much. For which uncountable cardinals, if any, do these results remain correct?

If the Continuum Hypothesis (CH) is assumed, the answer is trivial. The results are false already for $\aleph_1$ because $\aleph_1 = \mathfrak{c}$. But the Continuum Hypothesis, though not refutable from the usual (ZFC) axioms of set theory, is also not provable from them, so one can reasonably ask what happens if CH is false. Then there are cardinals strictly between $\aleph_0$ and $\mathfrak{c}$, and it is not evident whether the results cited above remain valid when "countable" is replaced by one of these cardinals.

Not only is it not evident, but it is not decidable in ZFC. For example, it is consistent with ZFC that $\mathfrak{c} = \aleph_2$ and all the cited results remain correct for $\aleph_1$, but it is also consistent that $\mathfrak{c} = \aleph_2$ and all the cited results fail for $\aleph_1$. It may seem that this undecidability prevents us from saying anything useful about extending the results above to higher cardinals. Fortunately, though little can be said about extending any one of these results, there are surprising and deep connections between extensions of different results. For example, if the Lebesgue measure result quoted above remains true for a cardinal κ, then so do the results about Baire category and about eventual domination.

A major goal of the theory of cardinal characteristics of the continuum is to understand relationships of this sort, either by proving implications like the one just cited or by showing that other implications are unprovable in ZFC. The cardinal characteristics are simply the smallest cardinals for which various results, true for $\aleph_0$, become false. (The characteristics corresponding to the four results cited above are called $\mathbf{cov}(\mathcal{B})$, $\mathbf{add}(\mathcal{L})$, $\mathfrak{b}$, and $\mathfrak{s}$, respectively, so the implication at the end of the preceding paragraph would be expressed by the inequalities $\mathbf{add}(\mathcal{L}) \leq \mathbf{cov}(\mathcal{B})$ and $\mathbf{add}(\mathcal{L}) \leq \mathfrak{b}$.) We shall be concerned here only with results about $\aleph_0$ that are false for $\mathfrak{c}$, so the characteristics we consider lie in the interval from $\aleph_1$ to $\mathfrak{c}$, inclusive.

A second goal of the theory, which we touch on only briefly here, is to find situations, in set theory or other branches of mathematics, where cardinal characteristics arise naturally. Wherever a result involves a countability hypothesis, one can ask whether it extends to some uncountable cardinals. Quite often, one can extend it to all cardinals below some previously studied characteristic. (Of course, if the result fails for $\mathfrak{c}$, one can simply use it to define a new characteristic, but this is of little value unless one can relate it to more familiar characteristics or at least give a simple, combinatorial description of it.) Such applications are fairly common in set-theoretic topology—notice that the two standard survey articles on cardinal characteristics, [42] and [111], appeared in topology books. They are becoming more common in other branches of mathematics as these branches come up against set-theoretic independence results.

We digress for a moment to comment on the meaning of "continuum" in the name of our subject. In principle, "continuum" refers to the real line $\mathbb{R}$ or to an interval like $[0, 1]$ in $\mathbb{R}$, regarded as a topological space. It is, however, common practice in set theory to apply the word also to spaces like ${}^\omega 2$, ${}^\omega\omega$ and $[\omega]^\omega$. Here ${}^\omega X$ means the space of ω-sequences of elements of X, topologized as a product of discrete spaces. Thus, ${}^\omega 2$ consists of sequences of zeros and ones; it may be identified with the power set $\mathcal{P}\omega$ of ω. $[\omega]^\omega$ is the subspace of $\mathcal{P}\omega$ consisting of the infinite sets. All these spaces are equivalent for many purposes, since any two become homeomorphic after removal of suitable countable subsets. We remark in particular that there is a continuous bijection from ${}^\omega\omega$ to $[0, 1)$, whose inverse is continuous except at dyadic rationals. This bijection, which takes the sequence $(a_0, a_1, \dots) \in {}^\omega\omega$ to the number whose binary expansion is a_0 ones, a zero, a_1 ones, a zero, …, also behaves nicely with respect to measure. Lebesgue measure on $[0, 1)$ corresponds to the product measure on ${}^\omega\omega$ obtained from the measure on ω giving each point n the measure 2^{-n-1}. Similarly, the obvious "binary notation" map from ${}^\omega 2$ onto $[0, 1]$, which fails to be one-to-one only over the dyadic rationals, makes Lebesgue measure correspond to the product measure on ${}^\omega 2$ obtained from the uniform measure on 2. In view of correspondences like these, we shall, without further explanation, apply cardinal characteristics like $\mathbf{cov}(\mathcal{B})$ and $\mathbf{add}(\mathcal{L})$ to all these versions of the continuum (with the corresponding measures), although they were defined in terms of $\mathbb{R}$ (with Lebesgue measure).

Another aspect of our subject's name also deserves a brief digression. Are these cardinals really characteristics of the continuum, or do they depend on more of the set-theoretic universe? Of course they depend on the class of cardinals; a characteristic that ceases to be a cardinal in some forcing extension will obviously cease to be a characteristic there also. So a more reasonable question would be whether the characteristics are determined by the continuum and the cardinals. More specifically, can cardinal characteristics of the continuum be changed by a forcing that neither adds reals nor collapses cardinals? Mildenberger [78] has shown that, for certain characteristics, such changes are possible but only in the presence of inner models with large cardinals.

As a final comment on the name of the subject, we mention that the traditional terminology was "invariants" rather than "characteristics"; see for example [99]. The alternative name "characteristics" was introduced because the invariants varied; indeed, much of the theory is about what sorts of variation are possible. Nevertheless, "invariant" is still in very common use—for example in Bartoszyński's chapter in this Handbook and [115].

We adopt the following standard notations for dealing with "modulo finite" notions on the natural numbers. First, $\forall^\infty x$ means "for all but finitely many x"; here x will always range over natural numbers, so the quantifier is equivalent to "for all sufficiently large x". Similarly $\exists^\infty x$ means "for infinitely many x" or equivalently "there exist arbitrarily large x such that". Notice that these quantifiers stand in the same duality relation as simple $\forall$ and $\exists$, namely $\neg\forall^\infty x$ is equivalent to $\exists^\infty x\neg$. An asterisk is often used to indicate a weakening from "for all" to "for all but finitely many". In particular, for subsets X and Y of ω, we write $X \subseteq^* Y$ to mean that X is almost included (or included modulo finite) in Y, i.e., $\forall^\infty x\ (x \in X \implies x \in Y)$. Similarly, for functions $f, g \in {}^\omega\omega$, we write $f \leq^* g$ to mean $\forall^\infty x\ (f(x) \leq g(x))$. We often use "almost" to mean modulo finite sets. For example, an almost decreasing sequence of sets is one where $X_m \supseteq^* X_n$ whenever $m < n$.

We use the standard abbreviations (some already mentioned above): ZFC for Zermelo-Fraenkel set theory including the axiom of choice, CH for the Continuum Hypothesis ($\mathfrak{c} = \aleph_1$), and GCH for the Generalized Continuum Hypothesis ($2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all cardinals $\aleph_\alpha$).

2. Growth of Functions

The ordering $\leq^*$ on ${}^\omega\omega$ provides two simple but frequently useful cardinal characteristics, the dominating and (un)bounding numbers.

2.1 Definition. A family $\mathcal{D} \subseteq {}^\omega\omega$ is *dominating* if for each $f \in {}^\omega\omega$ there is $g \in \mathcal{D}$ with $f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest cardinality of any dominating family, $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ dominating}\}$.

2.2 Definition. A family $\mathcal{B} \subseteq {}^\omega\omega$ is *unbounded* if there is no single $f \in {}^\omega\omega$ such that $g \leq^* f$ for all $g \in \mathcal{B}$. The *bounding number* $\mathfrak{b}$ (sometimes called the *unbounding number*) is the smallest cardinality of any unbounded family.

2.3 Remark. Had we used the "everywhere" ordering ($f \leq g$ if $\forall x\,(f(x) \leq g(x))$) instead of the "almost everywhere" ordering, $\mathfrak{d}$ would be unchanged, as any dominating $\mathcal{D}$ could be made dominating in the new sense by adding all finite modifications of its members. But $\mathfrak{b}$ would drop down to $\aleph_0$, as the constant functions form an unbounded family in the new sense.

Both $\mathfrak{b}$ and $\mathfrak{d}$ would be unchanged if in their definitions we replaced ${}^\omega\omega$ with ${}^\omega\mathbb{R}$ or with the set of sequences from any linear ordering of cofinality ω.

The following theorem gives all the constraints on $\mathfrak{b}$ and $\mathfrak{d}$ that are provable in ZFC.

2.4 Theorem. $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.

Proof. That $\aleph_1 \leq \mathfrak{b}$ means that, for every countably many functions $g_n : \omega \to \omega$, there is a single $f \geq^* $ all of them. Such an f is given by $f(x) = \max_{n \leq x} g_n(x)$.

To prove that $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$, let $\mathcal{D}$ be a dominating family of size $\mathfrak{d}$, and let it be decomposed into the union of $\mathrm{cf}(\mathfrak{d})$ subfamilies $\mathcal{D}_\xi$ of cardinalities $< \mathfrak{d}$. So there is, for each ξ, some f_ξ not dominated by any $g \in \mathcal{D}_\xi$. There can be no f dominating all the f_ξ, for such an f would not be dominated by any $g \in \mathcal{D}$. So $\{f_\xi : \xi < \mathrm{cf}(\mathfrak{d})\}$ is unbounded.

The proof that $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$ is similar, and the rest of the theorem is obvious. $\dashv$

Hechler [57] has shown that, if P is a partially ordered set in which every countable subset has an upper bound, then P can consistently be isomorphic to a cofinal subset of $({}^\omega\omega, \leq^*)$. More precisely, given any such P, Hechler constructs a c.c.c. forcing extension of the universe where there is a strictly order-preserving, cofinal embedding of P into $({}^\omega\omega, \leq^*)$. (Hechler's proof, done soon after the invention of forcing, has been reworked, using a more modern formulation, by Talayco in [109, Chap. 4] and by Burke in [36].) Hechler's result implies that the preceding theorem is optimal in the following sense.

2.5 Theorem. *Assume* GCH, *and let* $\mathfrak{b}'$, $\mathfrak{d}'$, *and* $\mathfrak{c}'$ *be any three cardinals satisfying*

$$\aleph_1 \leq \mathrm{cf}(\mathfrak{b}') = \mathfrak{b}' \leq \mathrm{cf}(\mathfrak{d}') \leq \mathfrak{d}' \leq \mathfrak{c}'$$

and $\mathrm{cf}(\mathfrak{c}') > \aleph_0$. *Then there is a c.c.c. forcing extension of the universe satisfying* $\mathfrak{b} = \mathfrak{b}'$, $\mathfrak{d} = \mathfrak{d}'$, *and* $\mathfrak{c} = \mathfrak{c}'$.

Proof. Apply Hechler's theorem to $P = [\mathfrak{d}']^{<\mathfrak{b}'}$ partially ordered by inclusion. The regularity of $\mathfrak{b}'$ implies that any $< \mathfrak{b}'$ elements in P have an upper bound, but some $\mathfrak{b}'$ elements (e.g., distinct singletons) do not. From $\mathrm{cf}(\mathfrak{d}') \geq \mathfrak{b}'$ and GCH we get that $|P| = \mathfrak{d}'$. Fewer than $\mathfrak{d}'$ elements of P cannot be cofinal, for their union (as sets) has cardinality smaller than $\mathfrak{d}'$. These observations imply that $\mathfrak{b} = \mathfrak{b}'$ and $\mathfrak{d} = \mathfrak{d}'$ in the forcing extension given by Hechler's theorem. Finally, to get $\mathfrak{c} = \mathfrak{c}'$, adjoin $\mathfrak{c}'$ random reals; these will not damage

$\mathfrak{b}$ or $\mathfrak{d}$, as the ground model's ${}^\omega\omega$ is cofinal in the ${}^\omega\omega$ of any random real extension. ⊣

To see that $\mathfrak{b} < \mathfrak{d}$ is consistent, it is not necessary to invoke Hechler's theorem. The original Cohen models [40] for the negation of CH have $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$. In fact, if one adjoins $\kappa \geq \aleph_1$ Cohen reals (by the usual product forcing) to any model of set theory, then the resulting model has $\mathfrak{b} = \aleph_1$ while $\mathfrak{d}$ becomes at least κ.

The contrary situation, that $\mathfrak{b} = \mathfrak{d}$, has the following useful characterization.

2.6 Theorem. $\mathfrak{b} = \mathfrak{d}$ *if and only if there is a scale in* ${}^\omega\omega$, *i.e., a dominating family well-ordered by* $\leq^*$.

Proof. If $\mathcal{D} = \{f_\xi : \xi < \mathfrak{b}\}$ is a dominating family of size $\mathfrak{b}$, then we obtain a scale $\{g_\xi : \xi < \mathfrak{b}\}$ by choosing each g_ξ to dominate f_ξ and all previous g_η ($\eta < \xi$); this can be done because we need to dominate fewer than $\mathfrak{b}$ functions at a time.

Conversely, if there is a scale, choose one and let $\mathcal{B}$ be an unbounded family of size $\mathfrak{b}$. By increasing each element of $\mathcal{B}$ if necessary, we can arrange for $\mathcal{B}$ to be a subset of our scale. But then, being unbounded, it must be cofinal in the well-ordering $\leq^*$ of the scale. Therefore it is a dominating family. ⊣

There are several alternative ways of looking at $\mathfrak{b}$ and $\mathfrak{d}$. We present two of them here and refer to [42, 56, 58] for others.

The first of these involves the "standard" characteristics of an ideal, defined as follows.

2.7 Definition. Let $\mathcal{I}$ be a proper ideal of subsets of a set X, containing all singletons from X.

- The *additivity* of $\mathcal{I}$, $\mathbf{add}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union not in $\mathcal{I}$.
- The *covering number* of $\mathcal{I}$, $\mathbf{cov}(\mathcal{I})$, is the smallest number of sets in $\mathcal{I}$ with union X.
- The *uniformity* of $\mathcal{I}$, $\mathbf{non}(\mathcal{I})$, is the smallest cardinality of any subset of X not in $\mathcal{I}$.
- The *cofinality* of $\mathcal{I}$, $\mathbf{cof}(\mathcal{I})$ is the smallest cardinality of any subset $\mathcal{B}$ of $\mathcal{I}$ such that every element of $\mathcal{I}$ is a subset of an element of $\mathcal{B}$. Such a $\mathcal{B}$ is called a *basis* for $\mathcal{I}$.

It is easy to check that both $\mathbf{cov}(\mathcal{I})$ and $\mathbf{non}(\mathcal{I})$ are $\geq \mathbf{add}(\mathcal{I})$ and $\leq \mathbf{cof}(\mathcal{I})$. In fact, $\mathbf{add}(\mathcal{I})$ is a lower bound for the cofinalities $\mathrm{cf}(\mathbf{non}(\mathcal{I}))$ and $\mathrm{cf}(\mathbf{cof}(\mathcal{I}))$ also. In this chapter, $\mathcal{I}$ will always be a σ-ideal, so its additivity (and therefore the other three characteristics) will be uncountable. Furthermore, $\mathcal{I}$ will have a basis consisting of Borel sets; since there are only

$\mathfrak{c}$ Borel sets, the cofinality (and therefore the other three characteristics) will be $\leq \mathfrak{c}$. (That the other three characteristics are $\leq \mathfrak{c}$ follows already from the simpler fact that the underlying set X is the continuum.)

The ideal relevant to the present section is the σ-ideal $\mathcal{K}_\sigma$ generated by the compact subsets of ${}^\omega\omega$, i.e., the ideal of sets coverable by countably many compact sets. Its connection with $\leq^*$ was pointed out by Rothberger in [91].

2.8 Theorem. $\mathbf{add}(\mathcal{K}_\sigma) = \mathbf{non}(\mathcal{K}_\sigma) = \mathfrak{b}$ *and* $\mathbf{cov}(\mathcal{K}_\sigma) = \mathbf{cof}(\mathcal{K}_\sigma) = \mathfrak{d}$.

Proof. Since a subset of the discrete space ω is compact if and only if it is finite, the Tychonoff theorem implies that a subset of ${}^\omega\omega$ is compact if and only if it is closed and included in a product of finite subsets of ω. There is no loss of generality in taking the finite subsets to be initial segments, so we find that all sets of the form

$$\{f \in {}^\omega\omega : f \leq g\} = \textstyle\prod_{n\in\omega}[0, g(n)]$$

are compact and every compact set is included in one of this form. It follows that all sets of the form $\{f \in {}^\omega\omega : f \leq^* g\}$ (with $\leq^*$ instead of $\leq$) are in $\mathcal{K}_\sigma$ and every set in $\mathcal{K}_\sigma$ is a subset of one of these. (The last uses that $\mathfrak{b} \geq \aleph_1$ to show that countably many bounds g for countably many compact sets are all $\leq^*$ a single bound.)

This connection between $\mathcal{K}_\sigma$ and $\leq^*$ easily implies the theorem. $\dashv$

Recalling that ${}^\omega\omega$ is homeomorphic, via continued fraction expansions, to the space of irrational numbers $\mathbb{R}-\mathbb{Q}$ (topologized as a subspace of $\mathbb{R}$), we see that the theorem remains valid if we interpret $\mathcal{K}_\sigma$ as the σ-ideal generated by the compact subsets of $\mathbb{R} - \mathbb{Q}$. In particular, $\mathfrak{d}$ is characterized as the minimum number of compact sets whose union is $\mathbb{R} - \mathbb{Q}$. (Here the choice of "continuum" is important. The corresponding cardinals for the spaces ${}^\omega 2$, $[0, 1]$, and $\mathbb{R}$ are clearly 1, 1, and $\aleph_0$, respectively.)

Yet another way of looking at the ordering $\leq^*$ and the associated cardinals $\mathfrak{b}$ and $\mathfrak{d}$ involves partitions of ω into finite intervals. (The earliest reference I know for this idea is Solomon's [103].)

2.9 Definition. An *interval partition* is a partition of ω into (infinitely many) finite intervals I_n $(n \in \omega)$. We always assume that the intervals are numbered in the natural order, so that, if i_n is the left endpoint of I_n then $i_0 = 0$ and $I_n = [i_n, i_{n+1})$. We say that the interval partition $\{I_n : n \in \omega\}$ *dominates* another interval partition $\{J_n : n \in \omega\}$ if $\forall^\infty n\, \exists k\ (J_k \subseteq I_n)$. We write IP for the set of all interval partitions.

2.10 Theorem. $\mathfrak{d}$ *is the smallest cardinality of any family of interval partitions dominating all interval partitions.* $\mathfrak{b}$ *is the smallest cardinality of any family of interval partitions not all dominated by a single interval partition.*

Proof. We prove only the first statement, as the second can be proved similarly or deduced from the proof of the first using the duality machinery of Sect. 4.

Suppose first that we have a family $\mathcal{F}$ of interval partitions dominating all interval partitions. To each of the partitions $\{I_n = [i_n, i_{n+1}) : n \in \omega\}$ in $\mathcal{F}$, associate the function $f : \omega \to \omega$ defined by letting $f(x)$ be the right endpoint of the interval after the one containing x; thus if $x \in I_n$ then $f(x) = i_{n+2} - 1$. We shall show that these functions f form a dominating family, so $\mathfrak{d} \leq |\mathcal{F}|$. Given any $g \in {}^\omega\omega$, the required f dominating g is obtained as follows. Form an interval partition $\{J_n = [j_n, j_{n+1}) : n \in \omega\}$ such that whenever $x \leq j_n$ then $g(x) < j_{n+1}$; it is trivial to do this by choosing the j_n inductively. Let $\{I_n = [i_n, i_{n+1}) : n \in \omega\}$ in $\mathcal{F}$ dominate this $\{J_n : n \in \omega\}$, and let f be the function associated to $\{I_n : n \in \omega\}$. To see that $g(x) \leq f(x)$ for all sufficiently large x, we chase through the definitions as follows. Let n be the index such that $x \in I_n$ and let (since x is sufficiently large) k be an index such that $J_k \subseteq I_{n+1}$. Then as $x \leq j_k$, we have $g(x) \leq j_{k+1} - 1 \leq i_{n+2} - 1 = f(x)$. This completes the proof that $\mathfrak{d} \leq |\mathcal{F}|$.

To produce a dominating family of interval partitions of cardinality $\mathfrak{d}$, we begin with a dominating family $\mathcal{D}$ of cardinality $\mathfrak{d}$ in ${}^\omega\omega$, and we associate to each $g \in \mathcal{D}$ an interval partition $\{J_n = [j_n, j_{n+1}) : n \in \omega\}$ exactly as in the preceding paragraph. To show that the resulting family of $\mathfrak{d}$ interval partitions dominates all interval partitions, let an arbitrary interval partition $\{I_n = [i_n, i_{n+1}) : n \in \omega\}$ be given, associate to it an $f \in {}^\omega\omega$ as in the preceding paragraph, and let $g \in \mathcal{D}$ be $\geq^* f$. We shall show that the $\{J_n : n \in \omega\}$ associated to this g dominates $\{I_n : n \in \omega\}$. For any sufficiently large n, we have $f(j_n) \leq g(j_n) \leq j_{n+1} - 1$. By virtue of the definition of f, this means that the next I_k after the one containing j_n lies entirely in J_n. ⊣

3. Splitting and Homogeneity

In this section, we treat several characteristics related to the "competition" between partitions trying to split sets and sets trying to be homogeneous for partitions. We begin with a combinatorial definition of a characteristic already mentioned, from an analytic point of view, in the introduction.

3.1 Definition. A set $X \subseteq \omega$ *splits* an infinite set $Y \subseteq \omega$ if both $Y \cap X$ and $Y - X$ are infinite. A *splitting family* is a family $\mathcal{S}$ of subsets of ω such that each infinite $Y \subseteq \omega$ is split by at least one $X \in \mathcal{S}$. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of any splitting family.

Having defined $\mathfrak{s}$ differently in the introduction, we hasten to point out that the definitions are equivalent.

3.2 Theorem. *$\mathfrak{s}$ is the minimum cardinality of any family of bounded ω-sequences $S_\xi = \langle x_{\xi,n} \rangle_{n\in\omega}$ of real numbers such that for no infinite $Y \subseteq \omega$ do all the corresponding subsequences $S_\xi \restriction Y = \langle x_{\xi,n} \rangle_{n\in Y}$ converge. The same is true if we consider only sequences consisting of just zeros and ones.*

Proof. The second assertion, where all S_ξ are in ${}^\omega 2$, is a trivial rephrasing of the definition of $\mathfrak{s}$; just regard the sequences S_ξ as the characteristic functions of the sets in a splitting family. The key point is that, for the characteristic function of X, convergence means eventual constancy, and so convergence of its restriction to Y means that Y is not split by X.

Half of the first assertion follows immediately from the second. To prove the other half of the first assertion, use the fact that a bounded sequence of real numbers converges if (though not quite only if) for each k the sequence of kth binary digits converges. $\dashv$

The last part of the preceding proof implicitly used the fact that $\mathfrak{s}$ is uncountable. We omit the easy, direct proof of this, because it will also follow from results to be proved later ($\aleph_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s}$; see Sect. 6).

Theorem 2.10 makes it easy to relate $\mathfrak{s}$ to $\mathfrak{d}$.

3.3 Theorem. $\mathfrak{s} \leq \mathfrak{d}$.

Proof. By Theorem 2.10, fix a family of $\mathfrak{d}$ interval partitions dominating all interval partitions. To each partition $\Pi = \{I_n : n \in \omega\}$ in this family, associate the union $\varphi(\Pi) = \bigcup_n I_{2n}$ of its even-numbered intervals. We shall show that these $\mathfrak{d}$ sets $\varphi(\Pi)$ constitute a splitting family. To this end, consider an arbitrary infinite subset X of ω. Associate to it an interval partition $\psi(X)$ in which every interval contains at least one member of X. Our dominating family of interval partitions contains a Π that dominates $\psi(X)$. But then each interval of Π, except for finitely many, includes an interval of $\psi(X)$ and therefore contains a point of X. It follows immediately that both $\varphi(\Pi)$ and its complement (the union of the odd-numbered intervals) contain infinitely many points of X. So $\varphi(\Pi)$ splits X. $\dashv$

We record for future reference the basic property of the constructions φ and ψ that makes the preceding proof work: For any interval partition Π and any infinite $X \subseteq \omega$,

$$\Pi \text{ dominates } \psi(X) \quad \Longrightarrow \quad \varphi(\Pi) \text{ splits } X.$$

The inequality in the theorem can consistently be strict. For example, if one adds $\kappa > \aleph_0$ Cohen reals to a model of set theory, then in the resulting model $\mathfrak{d} \geq \kappa$ (as remarked earlier) while $\mathfrak{s} = \aleph_1$ because any $\aleph_1$ of the added Cohen reals constitute a splitting family.

The splitting number is the simplest of a family of characteristics defined in terms of structures that are not simultaneously homogeneous (modulo finite) on any one infinite set. For $\mathfrak{s}$, the "structures" are two-valued functions and "homogeneous" simply means constant. Other notions of structure and homogeneity are suggested by various partition theorems. We shall characterize the analog of $\mathfrak{s}$ arising from Ramsey's Theorem and briefly mention a few other analogs afterward.

3.4 Definition. A set $H \subseteq \omega$ is *homogeneous* for a function $f : [\omega]^n \to k$ (a partition of $[\omega]^n$ into k pieces) if f is constant on $[H]^n$. H is *almost* *homogeneous* for f if there is a finite set F such that $H - F$ is homogeneous for f. $\mathfrak{par}_n$ is the smallest cardinality of any family of partitions of $[\omega]^n$ into two pieces such that no single infinite set is almost homogeneous for all of them simultaneously.

We note that $\mathfrak{par}_1$ is simply $\mathfrak{s}$ and that the definition of $\mathfrak{par}_n$ would be unchanged if we allowed partitions into any finite number of pieces (for any such partition could be replaced with the finitely many coarser partitions into two pieces). We note also that the use of *almost* homogeneity in the definition is essential; it is easy to produce countably many partitions with no common infinite homogeneous set.

3.5 Theorem. *For all integers $n \geq 2$, $\mathfrak{par}_n = \min\{\mathfrak{b}, \mathfrak{s}\}$.*

Proof. Notice first that $\mathfrak{par}_n \leq \mathfrak{par}_m$ if $n \geq m$, because any partition $[\omega]^m \to 2$ can be regarded as a partition of $[\omega]^n$ ignoring the last $n - m$ elements of its input. In particular, we have $\mathfrak{par}_n \leq \mathfrak{s}$, and if we show $\mathfrak{par}_2 \leq \mathfrak{b}$ then the $\leq$ direction of the theorem will be proved. For the $\geq$ direction, we must consider arbitrary n, but in fact we shall confine attention to $n = 2$ since the general case is longer but not harder.

To show $\mathfrak{par}_2 \leq \mathfrak{b}$, let $\mathcal{B} \subseteq {}^\omega\omega$ be an unbounded family of size $\mathfrak{b}$, assume without loss of generality that each $g \in \mathcal{B}$ is monotone increasing, and associate to each such g the partition of $[\omega]^2$ that puts a pair $\{x < y\}$ into class 0 if $g(x) < y$ and into class 1 otherwise. We shall show that no infinite $H \subseteq \omega$ is almost homogeneous for all these partitions simultaneously. Notice first that a homogeneous set of class 1 must be finite since, if x is its first element, then all the other elements are majorized by $g(x)$. So suppose, toward a contradiction, that H is infinite and almost homogeneous of class 0 for all the partitions associated to the functions $g \in \mathcal{B}$. Consider the function h sending each natural number x to the second member of H above x. For each x, we have $x < y < h(x)$ with both y and $h(x)$ in H. By almost homogeneity of H, we have, for each $g \in \mathcal{B}$ and for all sufficiently large x, $g(y) < h(x)$ and thus, by monotonicity of g, $g(x) < h(x)$. Thus, $g \leq^* h$ for all $g \in \mathcal{B}$, contrary to our choice of $\mathcal{B}$.

To show $\mathfrak{par}_2 \geq \min\{\mathfrak{b}, \mathfrak{s}\}$, suppose we are given a family of $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$ partitions $f_\xi : [\omega]^2 \to 2$; we must find an infinite set almost homogeneous for all of them. First, consider the functions

$$f_{\xi,n} : \omega \to 2 : x \mapsto f_\xi\{n, x\}.$$

(This is undefined for $x = n$; define it arbitrarily there.) Since the number of these functions is $\kappa \cdot \aleph_0 < \mathfrak{s}$, there is an infinite $A \subseteq \omega$ on which they are almost constant; say $f_{\xi,n}(x) = j_\xi(n)$ for all $x \geq g_\xi(n)$ in A. Furthermore, since $\kappa < \mathfrak{s}$ we can find an infinite $B \subseteq A$ on which each j_ξ is almost constant, say $j_\xi(n) = i_\xi$ for all $n \geq b_\xi$ in B. And since $\kappa < \mathfrak{b}$ we have a function h majorizing each g_ξ from some integer c_ξ on. Let $H = \{x_0 < x_1 < \cdots\}$ be an infinite subset of B chosen so that $h(x_n) < x_{n+1}$ for all n. Then this H is almost homogeneous for each f_ξ. Indeed, if $x < y$ are elements of H larger than b_ξ and c_ξ, then $y > h(x) \geq g_\xi(x)$ and so $f_\xi(\{x,y\}) = f_{\xi,x}(y) = j_\xi(x) = i_\xi$. ⊣

One can define characteristics analogous to $\mathfrak{par}_n$ using stronger partition theorems in place of Ramsey's Theorem, for example Hindman's finite sums theorem [59] or the Galvin-Prikry theorem [49] and its extension to analytic sets by Silver [102]. It is not difficult to see that these characteristics are bounded above by $\min\{\mathfrak{b}, \mathfrak{s}\}$. The Silver and (a fortiori) the Galvin-Prikry variants of $\mathfrak{par}$ are easily seen to be bounded below by the characteristic $\mathfrak{h}$ defined in Sect. 6. Eisworth has also obtained (private communication) a lower bound of the form $\min\{\mathfrak{b}, \mathfrak{s}'\}$, where $\mathfrak{s}'$ is the following variant of $\mathfrak{s}$. A cardinal κ is $< \mathfrak{s}'$ if, for any κ reals, there exist

1. a transitive model N of enough of ZFC containing the given reals,
2. $\mathcal{U} \in N$ such that N satisfies "$\mathcal{U}$ is a non-principal ultrafilter on ω", and
3. an infinite $a \subseteq \omega$ almost included in every member of $\mathcal{U}$.

Eisworth's proof uses forcing techniques from [61], but a direct combinatorial proof can be based on [17, Theorem 4]. Note that, if we weakened requirement (2) in the definition of $\mathfrak{s}'$ to say only that $\mathcal{U}$ is a non-principal ultrafilter in the Boolean algebra of subsets of ω in N (but $\mathcal{U}$ need not be in N), then the cardinal defined would be simply $\mathfrak{s}$. It is not known whether $\mathfrak{s}' < \mathfrak{s}$ is consistent.

For the variant of $\mathfrak{par}$ based on Hindman's theorem, the best lower bound known to me is the characteristic $\mathfrak{p}$ defined in Sect. 6. The proof that this is a lower bound uses the construction from Martin's Axiom mentioned in [16, p. 93], the observation that Martin's Axiom is applied here to a σ-centered poset, and Bell's theorem (Theorem 7.12 below).

One can also consider weaker sorts of homogeneity. For example, define $\mathfrak{par}_{1,c}$ to be the smallest cardinality of a family $\mathcal{F}$ of functions $f : \omega \to \omega$ such that there is no single infinite set $A \subseteq \omega$ on which all the functions from $\mathcal{F}$ are almost one-to-one or almost constant, where "almost" means, as usual, except at finitely many points in A. (The subscript $1, c$ refers to the canonical partition theorem for sets of size 1.) Each function f gives rise to a partition $f' : [\omega]^2 \to 2$, where $f'(\{x,y\}) = 0$ just when $f(x) = f(y)$. The sets where f is one-to-one or constant are the homogeneous sets of f', so $\mathfrak{par}_{1,c} \geq \mathfrak{par}_2$. In fact equality holds here, because $\mathfrak{par}_{1,c}$ is $\leq$ both $\mathfrak{s}$ and $\mathfrak{b}$. To see the former, associate to each set X from a splitting family its characteristic function. To see the latter, fix a family of $\mathfrak{b}$ interval partitions not dominated by any single interval partition (by Theorem 2.10) and associate to each of these partitions a function f constant on exactly the intervals of the partition. Since such an f is not constant on any infinite set, it suffices to show that there is no infinite A on which each f is almost one-to-one. But if there were such an A, then we could build an interval partition in which each interval contains at least three elements of A, and this partition would dominate all the partitions in our chosen, allegedly undominated family.

We now shift our focus from counting partitions to counting candidates for homogeneous sets.

3.6 Definition. A family $\mathcal{R}$ of infinite subsets of ω is *unsplittable* if no single set splits all members of $\mathcal{R}$. It is *σ-unsplittable* if no countably many sets suffice to split all members of $\mathcal{R}$. The *unsplitting number* $\mathfrak{r}$, also called the *refining* or *reaping number*, is the smallest cardinality of any unsplittable family. The *σ-unsplitting number* $\mathfrak{r}_\sigma$ is the smallest cardinality of any σ-unsplittable family.

Obviously, $\mathfrak{r} \leq \mathfrak{r}_\sigma$. It is not known whether strict inequality here is consistent with ZFC.

We omit the proof of the following theorem since it involves nothing beyond what went into the proof of Theorem 3.2.

3.7 Theorem. $\mathfrak{r}_\sigma$ *is the minimum cardinality of any family of infinite sets $Y \subseteq \omega$ such that, for each bounded sequence $\langle x_n \rangle_{n\in\omega}$ of real numbers, the restriction $\langle x_n \rangle_{n\in Y}$ to some Y in the family converges. If we consider only sequences of zeros and ones, then the corresponding minimum cardinality is* $\mathfrak{r}$.

We emphasize that, although in Theorem 3.2 the cardinal was the same for real-valued sequences as for two-valued sequences, the analogous equality in the present theorem is an open problem.

3.8 Theorem. $\mathfrak{b} \leq \mathfrak{r}$.

Proof. As in the proof of Theorem 3.3, let φ be the operation sending any interval partition to the union of its even-numbered intervals, and let ψ be an operation sending any infinite subset X of ω to an interval partition in which every interval contains at least one member of X. Let $\mathcal{R}$ be an unsplittable family of $\mathfrak{r}$ infinite subsets of ω; thanks to Theorem 2.10, we can complete the proof by showing that no interval partition Π dominates all the partitions $\psi(X)$ for $X \in \mathcal{R}$. But, as we showed in the proof of Theorem 3.3 and recorded for reference immediately thereafter, if Π dominated all these $\psi(X)$, then $\varphi(\Pi)$ would split every $X \in \mathcal{R}$, contrary to the choice of $\mathcal{R}$. $\dashv$

We next introduce the homogeneity cardinals associated to Ramsey's Theorem and the "one-to-one or constant" theorem. As in the discussion of partition counting, we could define homogeneity cardinals from Hindman's theorem, the Galvin-Prikry theorem, etc., but (as there) not much could be said about them.

3.9 Definition. $\mathfrak{hom}_n$ is the smallest size of any family $\mathcal{H}$ of infinite subsets of ω such that every partition of $[\omega]^n$ into two pieces has an almost homogeneous set in $\mathcal{H}$. $\mathfrak{hom}_{1,c}$ is the smallest size of any family $\mathcal{H}$ of infinite subsets of ω such that every function $f : \omega \to \omega$ is almost one-to-one or almost constant on some set in $\mathcal{H}$.

This definition would be unchanged if we deleted "almost", for we could put into $\mathcal{H}$ all finite modifications of its members. Notice that $\mathfrak{hom}_1 = \mathfrak{r}$ and that $\mathfrak{hom}_n \geq \mathfrak{hom}_m$ if $n \geq m$ (the reverse of the corresponding inequality for $\mathfrak{par}$).

3.10 Theorem. *For all integers $n \geq 2$, $\mathfrak{hom}_n = \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$. In addition, $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{hom}_{1,c} \leq \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.*

Proof. Although this proof contains only one idea not already in the proof of Theorem 3.5 and the subsequent discussion of $\mathfrak{par}_{1,c}$, we repeat some of the earlier ideas to clarify why we now have $\mathfrak{r}$ in one assertion and $\mathfrak{r}_\sigma$ elsewhere.

To show that $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{hom}_{1,c}$, we assume that $\mathcal{H}$ is as in the definition of $\mathfrak{hom}_{1,c}$, and we show that its cardinality is $\geq$ both $\mathfrak{r}$ and $\mathfrak{d}$. For the former, we find that $\mathcal{H}$ is unsplittable because if X splits H then the characteristic function of X is neither almost one-to-one nor almost constant on H. For the comparison with $\mathfrak{d}$, associate to each $H \in \mathcal{H}$ an interval partition Π_H such that each of its intervals contains at least three members of H. By Theorem 2.10, we need only check that every interval partition Θ is dominated by such a Π_H. Given Θ, let f be constant on exactly its intervals, and find $H \in \mathcal{H}$ on which f is almost one-to-one (as f is not constant on any infinite set). But then any interval of Π_H (except for finitely many) contains three points from H, all from different intervals of Θ, so it must contain a whole interval of Θ. So we have the required domination.

Next, we show that $\mathfrak{hom}_2 \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ by constructing an $\mathcal{H}$ of size $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ with the homogeneity property required in the definition of $\mathfrak{hom}_2$. (Note the similarity of this construction with the argument proving $\mathfrak{par}_2 \geq \min\{\mathfrak{b}, \mathfrak{s}\}$.) Let $\mathcal{D} \subseteq {}^\omega\omega$ be a dominating family of size $\mathfrak{d}$. Let $\mathcal{R}$ be a σ-unsplittable family of size $\mathfrak{r}_\sigma$. For each $A \in \mathcal{R}$, let $\mathcal{R}_A$ be an unsplittable family of $\mathfrak{r}$ subsets of A. For each $h \in \mathcal{D}$, each $A \in \mathcal{R}$, and each $B \in \mathcal{R}_A$, let $H = H(h, A, B)$ be an infinite subset of B such that, for any $x < y$ in H, $h(x) < y$. The family $\mathcal{H}$ of all these sets $H(h, A, B)$ has size at most $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$, and we shall now show that it contains an almost homogeneous set for every partition $f : [\omega]^2 \to 2$. Given f, define (as in the proof of Theorem 3.5) $f_n : \omega \to 2 : x \mapsto f\{n, x\}$. As $\mathcal{R}$ is σ-unsplittable, it contains an A on which each f_n is almost constant, say $f_n(x) = j(n)$ for all $x \geq g(n)$ in A. The function $j : A \to 2$ is almost constant on some B in the unsplittable family $\mathcal{R}_A$, say $j(n) = i$ for all $n \geq b$ in B. And $\mathcal{D}$ contains an h dominating g, say $h(x) \geq g(x)$ for all $x \geq c$. It is now routine to check (as in the proof of Theorem 3.5) that f is constant with value i on all pairs of elements larger than b and c in $H(h, A, B)$.

The proof that $\mathfrak{hom}_n \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ for $n > 2$ is similar to the preceding but uses n rather than two nestings of σ-unsplittable families (with no σ needed for the last one). We omit the details.

The preceding arguments, along with the observation that "one-to-one or constant" is a special case of homogeneity for partitions of pairs, establish that

$$\max\{\mathfrak{r}, \mathfrak{d}\} \le \mathfrak{hom}_{1,c} \le \mathfrak{hom}_2 \le \mathfrak{hom}_3 \le \cdots \le \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}.$$

All that remains to be proved is that $\mathfrak{r}_\sigma \le \mathfrak{hom}_2$, and this requires a method not involved in Theorem 3.5. The following argument is due to Brendle [31]. (Shelah had previously established the corresponding result for $\mathfrak{hom}_3$.)

Let $\mathcal{H}$ be as in the definition of $\mathfrak{hom}_2$, and let countably many functions $f_n : \omega \to 2$ be given. We seek a set in $\mathcal{H}$ on which each f_n is almost constant. Define, for each $x \in \omega$, the sequence of zeros and ones $\widehat{x} = \langle f_n(x) \rangle_{n\in\omega}$, so $\widehat{x}_n = f_n(x)$. Then define a partition of $[\omega]^2$ by putting $\{x < y\}$ into class 0 if $\widehat{x}$ lexicographically precedes $\widehat{y}$ and into class 1 otherwise. Let $H \in \mathcal{H}$ be almost homogeneous for this partition, let H' be a homogeneous set obtained by removing finitely many elements from H, and from now on let x and y range only over elements of H'. Suppose H' is homogeneous for class 0. (The case of class 1 is analogous.) Then as x increases, $\widehat{x}_0$ can only increase. That is, if the value of $f_0(x)$ ever changes, then it changes from 0 to 1 and remains constant forever after. Once $\widehat{x}_0$ has stabilized, $\widehat{x}_1$ can only increase and must therefore stabilize. Continuing in this way, we see that, as x increases through values in H', each $\widehat{x}_n$ eventually stabilizes. This means that each $f_n(x)$ is almost constant on H' and therefore on H, as required. $\dashv$

3.11 Remark. The last paragraph of this proof is similar to the proof that cardinals κ satisfying the partition relation $\kappa \longrightarrow (\kappa)^2_2$ are strong limit cardinals. The nature of the stabilization, where each component moves at most once after all its predecessors have stabilized, is also reminiscent of the proof that all requirements are eventually satisfied in a finite-injury priority argument.

4. Galois-Tukey Connections and Duality

We interrupt the description and discussion of particular cardinal characteristics in order to set up some machinery that is useful for describing many (though not all) of the characteristics and the relationships between them. This machinery was isolated by Vojtáš [112] under the name of "generalized Galois-Tukey connections"; the basic ideas had been used, but neither isolated nor named, in earlier work of Fremlin [47] and Miller (unpublished). The definitions of many cardinal characteristics have the form "the smallest cardinality of any set Y (of objects of a specified sort) such that every object x (of a possibly different sort) is related to some $y \in Y$ in a specified way." And many proofs of inequalities between such cardinals involve the construction of maps between the various sorts of objects involved in the definitions. This is formalized as follows.

4.1 Definition. A triple $\mathbf{A} = (A_-, A_+, A)$ consisting of two sets $A_\pm$ and a binary relation $A \subseteq A_- \times A_+$ will be called simply a *relation*. In connection with such a relation, we call A_- the set of *challenges* and A_+ the set of *responses*; we read xAy (meaning $(x, y) \in A$) as "response y *meets* challenge x".

4.2 Definition. The *norm* $\|\mathbf{A}\|$ of a relation $\mathbf{A} = (A_-, A_+, A)$ is the smallest cardinality of any subset Y of A_+ such that every $x \in A_-$ is related by A to at least one $y \in Y$. That is, it is the minimum number of responses needed to meet all challenges.

The definitions of cardinal characteristics in the preceding sections (as well as many others) amount to norms of relations. Furthermore, characteristics tend to come in pairs whose relations are dual to each other in the following sense.

4.3 Definition. If $\mathbf{A} = (A_-, A_+, A)$ then the *dual* of $\mathbf{A}$ is the relation $\mathbf{A}^\perp = (A_+, A_-, \neg\breve{A})$ where $\neg$ means complement and $\breve{A}$ is the converse of A; thus $(x, y) \in \neg\breve{A}$ if and only if $(y, x) \notin A$.

4.4 Example. Let $\mathfrak{D}$ be the relation $({}^\omega\omega, {}^\omega\omega, <^*)$. Then $\|\mathfrak{D}\| = \mathfrak{d}$ and $\|\mathfrak{D}^\perp\| = \|({}^\omega\omega, {}^\omega\omega, \not>^*)\| = \mathfrak{b}$. By Theorem 2.10, the same equations hold if we replace $\mathfrak{D}$ with $\mathfrak{D}' = (\mathrm{IP}, \mathrm{IP}, \text{is dominated by})$. (Recall from Definition 2.9 that IP is the set of interval partitions.)

Let $\mathfrak{R}$ be the relation $(\mathcal{P}(\omega), [\omega]^\omega, \text{does not split})$. Then $\|\mathfrak{R}\| = \mathfrak{r}$ and $\|\mathfrak{R}^\perp\| = \mathfrak{s}$.

Let $\mathfrak{Hom}_n$ be the relation $(P, [\omega]^\omega, H)$ where P is the set of partitions $f : [\omega]^n \to 2$ and where fHX means that X is almost homogeneous for f. Then $\|\mathfrak{Hom}_n\| = \mathfrak{hom}_n$ and $\|\mathfrak{Hom}_n{}^\perp\| = \mathfrak{par}_n$.

Let $\mathcal{I}$ be an ideal of subsets of X. Let $\mathbf{Cov}(\mathcal{I})$ be the relation $(X, \mathcal{I}, \in)$ and let $\mathbf{Cof}(\mathcal{I})$ be the relation $(\mathcal{I}, \mathcal{I}, \subseteq)$. Then we have $\|\mathbf{Cov}(\mathcal{I})\| = \mathbf{cov}(\mathcal{I})$, $\|\mathbf{Cov}(\mathcal{I})^\perp\| = \mathbf{non}(\mathcal{I})$, $\|\mathbf{Cof}(\mathcal{I})\| = \mathbf{cof}(\mathcal{I})$, and $\|\mathbf{Cof}(\mathcal{I})^\perp\| = \mathbf{add}(\mathcal{I})$.

In general, we name the relation corresponding to a characteristic by capitalizing the name of the characteristic, except when another name is readily available, e.g., as the dual of a previously defined relation.

4.5 Remark. We remarked earlier that the definition of $\mathfrak{d}$ would be unaffected if we replaced $\le^*$ by $\le$. That is, $\mathfrak{d}$ is the norm not only of the $\mathfrak{D}$ defined above but also of $({}^\omega\omega, {}^\omega\omega, \le)$. The dual of this last relation, however, has norm $\aleph_0$, not $\mathfrak{b}$.

Similar remarks apply to $\mathfrak{R}$ and $\mathfrak{Hom}$. It was for the sake of duality that we used "modulo finite" even in definitions where it could have been left out.

The following example indicates another situation where a change in a relation does not affect its norm but might affect the norm of the dual.

4.6 Example. Let $\mathfrak{R}_\sigma$ be the relation $({}^\omega\mathcal{P}(\omega), [\omega]^\omega, \text{does not split})$, where an ω-sequence of sets is said to split X if at least one term in the sequence splits X. Then $\|\mathfrak{R}_\sigma\| = \mathfrak{r}_\sigma$ and $\|\mathfrak{R}_\sigma{}^\perp\| = \mathfrak{s}$.

Thus, both $\mathfrak{r}$ and $\mathfrak{r}_\sigma$ can be regarded as duals of $\mathfrak{s}$. Duality is well-defined on relations but in general not on characteristics.

4.7 Remark. For any relation $\mathbf{A}$, one can define a relation $\mathbf{A}_\sigma$ that is related to $\mathbf{A}$ as $\mathfrak{R}_\sigma$ in the preceding example is related to $\mathfrak{R}$. That is,

$$\mathbf{A}_\sigma = ({}^\omega A_-, A_+, A_\sigma)$$

where $f A_\sigma a$ means that $f(n) A a$ for all $n \in \omega$. Thus, $\|\mathbf{A}_\sigma\|$, also written $\|\mathbf{A}\|_\sigma$, is the minimum number of responses needed so that every countably many challenges can be met simultaneously by a single one of these responses. For some relations, the σ construction produces nothing new; for example, $\mathfrak{d}_\sigma = \mathfrak{d}$. But for other relations, interesting new characteristics arise in this way. We already mentioned $\mathfrak{r}_\sigma$ above; $\mathfrak{s}_\sigma$ is studied in, for example, [64] and [73].

Clearly, $\|\mathbf{A}_\sigma\| \geq \|\mathbf{A}\|$. Whether the reverse inequality is provable in ZFC or whether strict inequality is consistent is, as we mentioned above, an open problem for $\mathbf{A} = \mathfrak{R}$. It is also open for $\mathbf{A} = \mathfrak{R}^\perp$; that is, it is not known whether $\mathfrak{s}_\sigma > \mathfrak{s}$ is consistent. On the other hand, it is known that $\mathbf{cov}(\mathcal{L})_\sigma > \mathbf{cov}(\mathcal{L})$ is consistent. See Bartoszyński's chapter in this Handbook for a proof that $\mathbf{cov}(\mathcal{L})$ can consistently have countable cofinality; it is easy to see that no $\|\mathbf{A}_\sigma\|$ can have countable cofinality.

Notice that the transformation $\mathbf{A} \mapsto \mathbf{A}_\sigma$ does not commute with duality. Indeed, in all non-trivial cases, $(\mathbf{A}_\sigma)^\perp$ has the same norm as $\mathbf{A}^\perp$, whereas, as indicated above, $(\mathbf{A}^\perp)_\sigma$ may well have a different norm.

The next definition captures the construction used in the proofs of many cardinal characteristic inequalities.

4.8 Definition. A *morphism* from one relation $\mathbf{A} = (A_-, A_+, A)$ to another $\mathbf{B} = (B_-, B_+, B)$ is a pair $\varphi = (\varphi_-, \varphi_+)$ of functions such that

- $\varphi_- : B_- \to A_-$,
- $\varphi_+ : A_+ \to B_+$,
- for all $b \in B_-$ and $a \in A_+$, if $\varphi_-(b) A a$ then $b B \varphi_+(a)$.

We use "morphism" instead of Vojtáš's "generalized Galois-Tukey connection" partly for brevity and partly because our convention differs from his as to direction. A morphism from $\mathbf{A}$ to $\mathbf{B}$ is a generalized Galois-Tukey connection from $\mathbf{B}$ to $\mathbf{A}$.

It is clear from the definitions that if $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$ then $\varphi^\perp = (\varphi_+, \varphi_-)$ is a morphism from $\mathbf{B}^\perp$ to $\mathbf{A}^\perp$.

Relations and morphisms form (as the name "morphism" suggests) a category in an obvious way, and we shall use the notation $\varphi : \mathbf{A} \to \mathbf{B}$ for morphisms. The category has products and coproducts, but these seem to be of little relevance to cardinal characteristics. Duality is a contravariant involution.

4.9 Theorem. *If there is a morphism $\varphi : \mathbf{A} \to \mathbf{B}$ then $\|\mathbf{A}\| \geq \|\mathbf{B}\|$ and $\|\mathbf{A}^\perp\| \leq \|\mathbf{B}^\perp\|$.*

Proof. It suffices to prove the first inequality, as the second follows by applying the first to the dual morphism $\varphi^\perp$.

Let $X \subseteq A_+$ have cardinality $\|\mathbf{A}\|$ and contain responses meeting all challenges in A_-. Then $Y = \varphi_+(X) \subseteq B_+$ has cardinality $\leq \|\mathbf{A}\|$, so we need only check that it contains responses meeting all challenges from B_-. Given $b \in B_-$, find in X a response x meeting $\varphi_-(b)$. Then $\varphi_+(x)$ is in Y and meets b because, by definition of morphism, $\varphi_-(b)Ax$ implies $bB\varphi_+(x)$. ⊣

Morphisms and Theorem 4.9 were implicit in several proofs of inequalities in the preceding sections. For example, the proof of Theorem 2.10 exhibits morphisms in both directions between $\mathfrak{D}$ and $\mathfrak{D}' = (\mathrm{IP}, \mathrm{IP}, \text{is dominated by})$, where IP is the set of all interval partitions. Both morphisms consist of the same two maps (in opposite order). One map sends any interval partition to the function sending any natural number x to the right endpoint of the next interval of the partition after the interval containing x. The other sends any function $f \in {}^\omega\omega$ to an interval partition $\{[j_n, j_{n+1}) : n \in \omega\}$ such that $f(x) < j_{n+1}$ for all $x \leq j_n$. The existence of this pair of morphisms implies not only that $\mathfrak{d} = \|\mathfrak{D}'\|$, but also, by duality, $\mathfrak{b} = \|\mathfrak{D}'^\perp\|$. The latter is the second assertion of Theorem 2.10, whose proof we omitted earlier.

The preceding example is somewhat atypical in that the same maps give morphisms in both directions between the same relations. Usually, one has a morphism in only one direction, and therefore an inequality rather than equality between cardinal characteristics. For example, the essential point in the proof of $\mathfrak{s} \leq \mathfrak{d}$ (Theorem 3.3), can be expressed by saying that the functions φ and ψ defined in that proof constitute a morphism $(\psi, \varphi) : \mathfrak{D}' \to \mathfrak{R}^\perp$. It follows that they also constitute a morphism $(\varphi, \psi) : \mathfrak{R} \to \mathfrak{D}'^\perp$, so we have $\mathfrak{b} \leq \mathfrak{r}$ (Theorem 3.8). Morphisms, duality, and Theorem 4.9 codify the observation that Theorems 3.3 and 3.8 have "essentially the same proof."

If $\mathcal{I}$ is an ideal on X containing all singletons, then in view of Example 4.4, the inequalities $\mathbf{add}(\mathcal{I}) \leq \mathbf{cov}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I})$ and $\mathbf{add}(\mathcal{I}) \leq \mathbf{non}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I})$ follow from the existence of morphisms from $\mathbf{Cof}(\mathcal{I}) = (\mathcal{I}, \mathcal{I}, \subseteq)$ to both $\mathbf{Cov}(\mathcal{I}) = (X, \mathcal{I}, \in)$ and its dual $\mathbf{Cov}(\mathcal{I})^\perp = (\mathcal{I}, X, \not\ni)$. The first of these can be taken to be (S, id), where S is the singleton map $x \mapsto \{x\}$ and id is the identity map. The second can be taken to be (id, N), where N sends each $I \in \mathcal{I}$ to some element of $X - I$.

The inequalities $\|\mathbf{A}_\sigma\| \geq \|\mathbf{A}\|$, for all $\mathbf{A}$, also arise from morphisms $\mathbf{A}_\sigma \to \mathbf{A}$. The map on challenges sends each $a \in A_-$ to the constant function $\omega \to A_-$ with value a, and the map on responses is the identity function.

The inequalities $\mathfrak{par}_n \leq \mathfrak{b}$ and $\mathfrak{par}_n \leq \mathfrak{s}$ in Theorem 3.5 and their duals $\mathfrak{hom}_n \geq \mathfrak{d}$ and $\mathfrak{hom}_n \geq \mathfrak{r}$ in Theorem 3.10 are also given by morphisms, as an inspection of the proofs will show. The same goes for Brendle's improvement of the last of these inequalities, with $\mathfrak{r}_\sigma$ in place of $\mathfrak{r}$, and the same goes for the analogous inequalities for $\mathfrak{par}_{1,c}$ and $\mathfrak{hom}_{1,c}$.

But the same cannot be said (yet) for the reverse inequalities, $\mathfrak{par}_n \geq \min\{\mathfrak{b}, \mathfrak{s}\}$ and its dual $\mathfrak{hom}_n \leq \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$, simply because the minimum and maximum here are not (yet) realized as the norms of natural relations. There are, fortunately, several ways to combine two relations into a third whose norm is the maximum (or the minimum) of the norms of the first two. Two of these provide what we need in order to present in terms of morphisms the proofs of the inequalities just cited; we present a third combination along with these two because of its category-theoretic naturality.

To avoid trivial exceptions, we assume in the following that, in the relations (A_-, A_+, A) under consideration, the sets $A_\pm$ are not empty. We also adopt the convention of using a boldface letter for the relation whose components are denoted by the corresponding lightface letter; thus $\mathbf{A} = (A_-, A_+, A)$.

4.10 Definition. The *categorical product* $\mathbf{A} \times \mathbf{B}$ is $(A_- \sqcup B_-, A_+ \times B_+, C)$, where $\sqcup$ means disjoint union and where $x\,C\,(a, b)$ means $x\,A\,a$ if $x \in A_-$ and $x\,B\,b$ if $x \in B_-$.

The *conjunction* $\mathbf{A} \wedge \mathbf{B}$ is $(A_- \times B_-, A_+ \times B_+, K)$, where $(x, y)\,K\,(a, b)$ means $x\,A\,a$ and $y\,B\,b$.

The *sequential composition* $\mathbf{A}; \mathbf{B}$ is $(A_- \times {}^{A_+}B_-, A_+ \times B_+, S)$, where the superscript means a set of functions and where $(x, f)\,S\,(a, b)$ means $x\,A\,a$ and $f(a)\,B\,b$.

The dual operations are the *categorical coproduct* $\mathbf{A} + \mathbf{B} = (\mathbf{A}^\perp \times \mathbf{B}^\perp)^\perp$, the *disjunction* $\mathbf{A} \vee \mathbf{B} = (\mathbf{A}^\perp \wedge \mathbf{B}^\perp)^\perp$, and the *dual sequential composition* $\mathbf{A} \check{;} \mathbf{B} = (\mathbf{A}^\perp; \mathbf{B}^\perp)^\perp$.

The two categorical operations are, as their names suggest, the product and coproduct in the category of relations and morphisms.

The conjunction was called the product in a preprint version of [112] and has therefore sometimes been called the *old product*. It is a sort of parallel composition. A challenge consists of separate challenges in both components and a (correct) response consists of (correct) responses in both components separately.

Sequential composition describes a two-inning game between the challenger and the responder. The first inning consists of a challenge x in $\mathbf{A}$ followed by a response a there; the second inning consists of a challenge $f(a)$ in $\mathbf{B}$, which may depend on the previous response a, followed by a response b there. To model this in a single inning, we regard the whole function f as part of the challenge. As in the case of conjunction, a correct response in the sequential composition must be correct in both components.

Notice that one can obtain a description of disjunction by simply changing the last "and" to "or" in the definition of conjunction. The dualization of sequential composition is more complicated; not only does "and" become "or" but the functional dependence changes so that the response in $\mathbf{B}$ can depend on the challenge in $\mathbf{A}$.

The following theorem describes the effect of these operations on norms. Its proof is quite straightforward and therefore omitted.

4.11 Theorem.

1. $\|\mathbf{A} \times \mathbf{B}\| = \max\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$.

2. $\max\{\|\mathbf{A}\|, \|\mathbf{B}\|\} \leq \|\mathbf{A} \wedge \mathbf{B}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$.

3. $\|\mathbf{A}; \mathbf{B}\| = \|\mathbf{A}\| \cdot \|\mathbf{B}\|$.

4. $\|\mathbf{A} + \mathbf{B}\| = \min\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$.

5. $\|\mathbf{A} \vee \mathbf{B}\| = \min\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$.

6. $\|\mathbf{A} \breve{;} \mathbf{B}\| = \min\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$.

When the norms are infinite, maxima and products are the same, so the second and third items in the theorem simplify to $\|\mathbf{A} \wedge \mathbf{B}\| = \|\mathbf{A}; \mathbf{B}\| = \max\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$. (In the finite case there is no such simplification. Both of the inequalities involving $\|\mathbf{A} \wedge \mathbf{B}\|$ can be strict; consider $\mathbf{A} = \mathbf{B} = (3, 3, \neq)$.)

4.12 Example. In the proof of Theorem 3.10, the part showing that $\mathfrak{hom}_2 \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ actually gives a morphism from $\mathfrak{R}_\sigma; (\mathfrak{R} \wedge \mathfrak{D})$ to $\mathfrak{Hom}_2$, as detailed below. By Theorems 4.9 and 4.11, the existence of such a morphism implies both $\mathfrak{hom}_2 \leq \max\{\mathfrak{r}_\sigma, \mathfrak{r}, \mathfrak{d}\} = \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ and $\mathfrak{par}_2 \geq \min\{\mathfrak{s}, \mathfrak{b}\}$ (the part of Theorem 3.5 that really involves all three cardinals simultaneously).

To exhibit the morphism implicit in the proof of Theorem 3.10, we first describe $\mathfrak{R}_\sigma; (\mathfrak{R} \wedge \mathfrak{D})$. Following the definitions, we find that a challenge here amounts to a triple (S, F, G) where S is an ω-sequence of subsets S_n of ω, F is a function assigning to each infinite $A \subseteq \omega$ a subset $F(A)$ of ω, and G is a function assigning to each such A a function $G(A) \in {}^\omega\omega$. A response is a triple (A, B, h) where A and B are infinite subsets of ω and $h \in {}^\omega\omega$. The response (A, B, h) meets the challenge (S, F, G) if (1) A is not split by any component S_n of S, (2) B is not split by $F(A)$, and (3) $G(A) <^* h$. Using the notation (f_n, j, g, H) of the proof of Theorem 3.10 and the notation e_A for the increasing enumeration of an infinite $A \subseteq \omega$, we can describe the morphism from $\mathfrak{R}_\sigma; (\mathfrak{R} \wedge \mathfrak{D})$ to $\mathfrak{Hom}_2$ as follows. The "challenge" part sends any partition $f : [\omega]^2 \to 2$ to (S, F, G), where S_n has characteristic function f_n, where $F(A)$ has characteristic function $j \circ e_A$, and where $G(A) = g$. (The j and g in the proof of Theorem 3.10 depend on A.) The "response" part of the morphism sends a triple (A, B, h) to $H(h, A, e_A(B))$. The verification that these two operations constitute a morphism is as in Theorems 3.5 and 3.10. (The need for e_A in the present discussion but not in the earlier proofs results from our tacit use, in the earlier proofs, of the equivalence between splitting phenomena in ω and the analogous phenomena in any infinite subset A. e_A serves to make the equivalence explicit.)

We remark that the formal structure, $\mathfrak{R}_\sigma; (\mathfrak{R} \wedge \mathfrak{D})$, reflects the intuitive structure of the proof of Theorem 3.5. That proof invoked the hypothesis $\kappa < \mathfrak{s}$ twice (corresponding to $\mathfrak{R}_\sigma$ and $\mathfrak{R}$) and $\kappa < \mathfrak{b}$ once (corresponding to $\mathfrak{D}$). The first use of $\kappa < \mathfrak{s}$ logically precedes the other two (corresponding to sequential composition) because the unsplit set A obtained at the first step is used to produce the j and g for the other two steps. The second use of $\kappa < \mathfrak{s}$ and the use of $\kappa < \mathfrak{b}$ can proceed in parallel, as neither depends on the other (corresponding to conjunction).

4.13 Example. Sequential composition also occurs naturally in much simpler situations. Consider, for example, the following variant of unsplitting: $\mathfrak{R}_3 = ({}^{\omega}3, [\omega]^{\omega}, \text{is almost constant on})$. Its norm $\mathfrak{r}_3$ is the minimum number of infinite subsets of ω not all split by a single partition of ω into three pieces. This cardinal is easily seen to be equal to $\mathfrak{r}$, but one direction of the proof involves a sequential composition. A "3-unsplittable" family is obtained by starting with an unsplittable family and then forming, within each of its sets, a further unsplittable family. The union of the latter families is then 3-unsplittable (and even 4-unsplittable). In terms of morphisms, one obtains $\mathfrak{R}; \mathfrak{R} \to \mathfrak{R}_3$ (as well as the trivial $\mathfrak{R}_3 \to \mathfrak{R}$).

Equipped with the concept of morphism, we can address an issue that had been glossed over in the introduction. If one believes the Continuum Hypothesis (CH), then the theory of cardinal characteristics becomes trivial, for they are all equal to $\aleph_1$. Nevertheless, there is non-trivial combinatorial content in proofs like those of Theorems 2.10 and 3.3, even if CH holds and makes the theorems themselves trivial. That combinatorial content is used to construct the morphisms $\mathfrak{D} \leftrightarrow \mathfrak{D}' \to \mathfrak{R}^{\perp}$, so one might hope that the existence of such morphisms is what the argument "really" proves, a non-trivial result even in the presence of CH. Yiparaki [114] showed that this hope is not justified; CH implies not only the equality of all our cardinal characteristics but also the existence of morphisms in both directions between the corresponding relations. The last part of the following theorem embodies this result.

4.14 Theorem. *Let $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ be two relations and let κ be an infinite cardinal.*

1. *$\|\mathbf{A}\| \le \kappa$ if and only if there is a morphism from $(\kappa, \kappa, =)$ to $\mathbf{A}$.*
2. *If $\|\mathbf{A}\| = |A_+| = \kappa$, then there is a morphism from $\mathbf{A}$ to $(\kappa, \kappa, <)$.*
3. *If $\|\mathbf{A}^{\perp}\| = |A_-| = \kappa$, then there is a morphism from $(\kappa, \kappa, <)$ to $\mathbf{A}$.*
4. *If $\|\mathbf{A}\| = |A_+| = \|\mathbf{B}^{\perp}\| = |B_-| \ge \aleph_0$, then there is a morphism from $\mathbf{A}$ to $\mathbf{B}$.*

Proof. The "if" direction of 1 is immediate from Theorem 4.9 and the fact that $\|(\kappa, \kappa, =)\| = \kappa$. For the "only if" direction, let $\varphi_+ : \kappa \to A_+$ enumerate a set of at most κ responses meeting all challenges; then for each challenge $a \in A_-$ let $\varphi_-(a)$ be any $\alpha < \kappa$ such that $\varphi_+(\alpha)$ meets a.

For 2, let $\varphi_+ : A_+ \to \kappa$ be any one-to-one map. Then, for any $\alpha < \kappa$, the set $\{a \in A_+ : \varphi_+(a) \le \alpha\}$ has cardinality smaller than $\kappa = \|\mathbf{A}\|$, so some challenge in A_- has no correct response in this set. Let $\varphi_-(\alpha)$ be any such challenge.

Note that 2 remains true if we replace $(\kappa, \kappa, <)$ with $(\kappa, \kappa, \leq)$. Then dualization gives 3.

Finally, to prove 4, just compose the morphisms $\mathbf{A} \to (\kappa, \kappa, <) \to \mathbf{B}$ given by 2 and 3. ⊣

If CH holds, then part 4 of this theorem applies to most of the relations in Example 4.4 above, for the cardinals involved are $\aleph_1$. The only exceptions are $\mathbf{Cov}(\mathcal{I})$ and $\mathbf{Cof}(\mathcal{I})$, but even here we can (indirectly) apply part 4 when $\mathcal{I}$ is the ideal of measure zero sets or the ideal of meager sets in $\mathbb{R}$ or a similar ideal. More precisely, if $\mathcal{I}$ is an ideal on $\mathbb{R}$ and $\mathcal{I}$ has a cofinal subset $\mathcal{I}_0$ of size $\leq \mathfrak{c}$, then part 4 applies directly to variants of $\mathbf{Cov}(\mathcal{I})$ and $\mathbf{Cof}(\mathcal{I})$ with $\mathcal{I}$ replaced by $\mathcal{I}_0$. But it is trivial to check that there are morphisms in both directions between these variants and the original relations. In effect then, part 4 provides morphisms in both directions between any two of the relations we are considering; CH trivializes not only the inequalities between cardinal characteristics but also the morphisms between the corresponding relations.

Nevertheless, there is still some hope of using morphisms to describe the combinatorial content of the theory in a way that makes good sense even when CH holds. This hope is based on the observation that the morphisms given by Theorem 4.14 are highly non-constructive; they involve well-orderings of the continuum (and similar sets). By contrast, the morphisms given by the proofs of cardinal characteristic inequalities are much better behaved. They consist of Borel maps with respect to the usual topologies on the sets involved (like ${}^\omega\omega$ and $\mathcal{P}(\omega)$). Two clarifications are in order here. One is that, when the sets involved are bases $\mathcal{I}_0$ for some ideals, as in the preceding paragraph, then the sets in $\mathcal{I}_0$ should be coded by reals in some standard way. For example, if $\mathcal{I}$ is the ideal of meager (resp. measure zero) sets in $\mathbb{R}$, then $\mathcal{I}_0$ can be taken to consist of the F_σ (resp. G_δ) members of $\mathcal{I}$, and there are well-known ways of coding such sets (or arbitrary Borel sets) by reals. The second clarification is that Pawlikowski and Recław have shown [85] that, with suitable coding, the morphisms can be taken to consist of continuous maps; nevertheless, we shall continue to use “Borel” as our main criterion of simplicity.

The existence of Borel morphisms seems to serve well as a codification of the combinatorial content of proofs of cardinal characteristic inequalities. On the one hand, the usual proofs provide Borel morphisms. On the other hand, when an inequality is not provable then, although it may hold in specific models and even have morphisms attesting to it (e.g., in models of CH), there will never be Borel morphisms attesting to it. The following theorem establishes this last fact for the particular unprovable inequality $\mathfrak{d} \leq \mathfrak{s}$. Similar arguments can be given for other unprovable inequalities, but they usually involve notions of forcing more complicated than the Cohen forcing used here. We remark that the theorem proves a bit more than was claimed above; a morphism φ attesting to $\mathfrak{d} \leq \mathfrak{s}$ cannot have even one of its two constituent functions $\varphi_\pm$ Borel. (The weaker result that $\varphi_\pm$ cannot both be Borel in this situation was established in [22].)

4.15 Theorem. *If φ is a morphism $\mathfrak{R}^\perp \to \mathfrak{D}$, then neither φ_+ nor φ_- is a Borel function.*

Proof. Recalling the definitions of $\mathfrak{R}$ and $\mathfrak{D}$, we see that

$$\begin{array}{rcl} \varphi_- : {}^\omega\omega & \to & [\omega]^\omega, \\ \varphi_+ : \mathcal{P}(\omega) & \to & {}^\omega\omega, \quad \text{and} \\ \varphi_-(a) \text{ is split by } b & \Longrightarrow & a <^* \varphi_+(b). \end{array}$$

Suppose first that φ_- were a Borel function, with code p (in a standard coding system for Borel sets and functions). Adjoin to the universe a Cohen-generic function $c : \omega \to \omega$, and define $d = \tilde{\varphi}_-(c)$, where $\tilde{\varphi}_-$ is the Borel function coded by p in $V[c]$. Thus $d \in [\omega]^\omega$ in $V[c]$. The ground model reals form a splitting family in the Cohen extension $V[c]$ (because they form a non-meager family there; see Sect. 11.3 and the proof of Theorem 5.19 below). So there is a real $r \in V \cap \mathcal{P}(\omega)$ that splits d. In the ground model V, let $g = \varphi_+(r)$ and notice that, because φ is a morphism,

$$\forall x \in {}^\omega\omega\, [\varphi_-(x) \text{ is split by } r \implies x <^* g].$$

This is a Π^1_1 statement about r, g, and the code p of φ_-. So it remains true in $V[c]$, where p codes $\tilde{\varphi}_-$ and where x can take c as a value. Thus we find, in $V[c]$, since $\tilde{\varphi}_-(c) = d$ is split by r, that $c <^* g$. But this is absurd; a Cohen-generic $c \in {}^\omega\omega$ cannot be dominated by a g from the ground model. This contradiction shows that φ_- cannot be a Borel map.

Now suppose instead that φ_+ were a Borel function, with Borel code p. Let $c \in \mathcal{P}(\omega)$ be Cohen-generic and let $e = \tilde{\varphi}_+(c)$, where $\tilde{\varphi}_+$ is the Borel function coded by p in $V[c]$. Thus $e \in {}^\omega\omega$ in $V[c]$. The ground model reals are unbounded in ${}^\omega\omega$ in a Cohen extension, so fix $r \in V \cap {}^\omega\omega$ with $r \not<^* e$. Let $q = \varphi_-(r)$, an infinite subset of ω in V. Because φ is a morphism,

$$\forall x \in \mathcal{P}(\omega)\, [q \text{ is split by } x \implies r <^* \varphi_+(x)].$$

As before, this is a Π^1_1 statement about q, r, and p, so it remains true in $V[c]$. There c is a possible value of x and p codes $\tilde{\varphi}_+$, so from $r \not<^* e = \tilde{\varphi}_+(c)$ we can infer that q is not split by c. This is absurd, as every infinite subset of ω in the ground model V is split by the Cohen subset c of ω. ⊣

The use of Borel morphisms can also clarify the need for sequential (and other) composition operations on relations. Specifically, a forcing argument is used in [22] to show that some naturally occurring morphisms involving sequential compositions (e.g., the proof of Theorem 5.6 below) cannot be simplified to use conjunctions or products or even sequential composition in a different order. Mildenberger [77] and Spinas [107] have obtained similar results by combinatorial methods in some cases where the forcing method of [22] does not apply. A forcing argument in [24] shows that the sequential composition $\mathfrak{R}_\sigma; (\mathfrak{R} \wedge \mathfrak{D})$ used in the proof of $\mathfrak{hom}_2 \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ cannot be replaced by simply $\mathfrak{R}_\sigma \wedge \mathfrak{D}$. But other potential simplifications in this problem and similar simplifications in other problems, though they seem unlikely, have not been proved impossible.

4.16 Remark. Let $\mathbf{A}$ and $\mathbf{B}$ be relations where $A_\pm$ and $B_\pm$ are sets of reals. Call a morphism $\varphi : \mathbf{A} \to \mathbf{B}$ *semi-Borel* (on the positive side) if φ_+ is a Borel function. Thus, Theorem 4.15 asserts that certain morphisms cannot be semi-Borel.

Call a set X of reals *small with respect to* $\mathbf{A}$ if there is no semi-Borel morphism from $(X, X, =)$ to $\mathbf{A}$. Without "semi-Borel", this definition would say simply that $|X| < \|\mathbf{A}\|$, by the first part of Theorem 4.14. With "semi-Borel," smallness is a weaker notion, related to the topological (or Borel) structure of X, not just to its cardinality. It can be expressed as "no image of X under a Borel function to A_+ contains responses meeting all challenges from A_-."

The smallness properties associated in this way to the relations involved in Cichoń's diagram (see the end of Sect. 5) were introduced and studied by Pawlikowski and Recław [85], who connected them with various classical smallness properties of sets of reals. Bartoszyński's chapter in this Handbook contains extensive information about this topic.

5. Category and Measure

Despite their origins in real analysis, Baire category and Lebesgue measure are, to a large extent, combinatorial notions. As such, they have close ties with some of the objects discussed in the preceding sections. We give here a rather cursory presentation of some of these combinatorial aspects of category and measure. For a more complete treatment, see Bartoszyński's chapter in this Handbook and the book [5] of Bartoszyński and Judah.

Recall Definition 2.7 of the four cardinal characteristics **add**, **cov**, **non**, **cof** associated to any proper ideal (containing all singletons) on any set. We shall be interested in these and in the corresponding relations ($\mathbf{Cof}^\perp$, $\mathbf{Cov}$, $\mathbf{Cov}^\perp$, and $\mathbf{Cof}$, respectively, from Example 4.4) when the ideal is either the σ-ideal of meager (also called first category) sets or the σ-ideal of sets of Lebesgue measure zero (also called null sets). We use $\mathcal{B}$ and $\mathcal{L}$ respectively to denote these two ideals. (The notation stands for "Baire" and "Lebesgue"; other authors have used $\mathcal{C}$ for "Category," $\mathcal{K}$ for "Kategorie," $\mathcal{M}$ for "meager," $\mathcal{M}$ for "measure," and $\mathcal{N}$ for "null".) As indicated in the introduction, we do not distinguish notationally between the meager ideals on various versions of the continuum, $\mathbb{R}$, ${}^\omega 2$, ${}^\omega\omega$, etc., and similarly for measure.

The various versions of each cardinal characteristic are equal; the various versions of each relation admit morphisms in both directions. We tolerate an additional, equally innocuous ambiguity by not distinguishing between an ideal and a basis for it. Thus, we may pretend that $\mathcal{B}$ consists of meager F_σ sets and that $\mathcal{L}$ consists of G_δ null sets. If we discuss Borel morphisms, we further identify F_σ and G_δ sets with some standard encoding as reals.

We begin our treatment of Baire category by giving a convenient combinatorial description of meagerness in the space ${}^\omega 2$. This idea was introduced in a more specialized context by Talagrand [108].

5.1 Definition. A *chopped real* is a pair (x, Π), where $x \in {}^\omega 2$ and Π is an interval partition of ω. Recall that we introduced the notation IP for the set of all interval partitions; we write CR for the set ${}^\omega 2 \times \mathrm{IP}$ of chopped reals. A real $y \in {}^\omega 2$ *matches* a chopped real (x, Π) if $x \restriction I = y \restriction I$ for infinitely many intervals $I \in \Pi$.

5.2 Theorem. *A subset M of ${}^\omega 2$ is meager if and only if there is a chopped real that no member of M matches.*

Proof. The set of reals y that match a given chopped real $(x, \{I_n : n \in \omega\})$ is

$$\mathrm{Match}(x, \{I_n : n \in \omega\}) = \bigcap_k \bigcup_{n \geq k} \{y : x \restriction I_n = y \restriction I_n\},$$

the intersection of countably many dense open sets. So $\mathrm{Match}(x, \Pi)$ is comeager, and the "if" part of the theorem follows.

To prove "only if," suppose M is meager, and fix a countable sequence of nowhere dense sets F_n that cover M. Note that, for the standard (product) topology on ${}^\omega 2$, to say that a set F is nowhere dense means that for every finite sequence $s \in {}^{<\omega} 2$ there is an extension $t \in {}^{<\omega} 2$ such that no $y \in F$ extends t. Note also that the union of finitely many nowhere dense sets is nowhere dense, so we can and do arrange that $F_n \subseteq F_{n+1}$ for all n. Then we can complete the proof by constructing a chopped real $(x, \{I_n : n \in \omega\})$ such that, for each n, no real in F_n agrees with x on I_n. This suffices because then any y that matches $(x, \{I_n : n \in \omega\})$ will be outside infinitely many F_n, hence outside them all by monotonicity, and hence outside M.

To define I_n and $x \restriction I_n$, suppose the earlier I_k $(k < n)$ are already defined and are contiguous intervals. So we know the point m where I_n should start. I_n will be the union of 2^m contiguous subintervals J_i $(i < 2^m)$ defined as follows. List all the functions $m \to 2$ as u_i $(i < 2^m)$. By induction on i, choose J_i and $x \restriction J_i$ so that no element of F_n is an extension of $u_i \cup \bigcup_{j \leq i} (x \restriction J_j)$. These choices are possible because F_n is nowhere dense. Finally, let $I_n = \bigcup_{j < 2^m} J_i$; having already defined each $x \restriction J_i$, we have determined $x \restriction I_n$.

If y agrees with x on I_n, then y extends $u_i \cup \bigcup_{j \leq i} (x \restriction J_j)$ for some i, namely the i such that $u_i = y \restriction m$. Therefore, $y \notin F_n$, as required. $\dashv$

The theorem shows that the sets $\mathrm{Match}(x, \Pi)$ form a base for the filter of comeager sets and so their complements form a base for the ideal $\mathcal{B}$. We may therefore confine attention to these complements when discussing the cardinal characteristics of $\mathcal{B}$ and the associated relations. In this connection, it is useful to have the following combinatorial formulation of the inclusion relation between these sets; we leave the straightforward proof to the reader.

5.3 Proposition. $\mathrm{Match}(x, \Pi) \subseteq \mathrm{Match}(x', \Pi')$ *if and only if for all but finitely many intervals* $I \in \Pi$ *there exists an interval* $J \in \Pi'$ *such that* $J \subseteq I$ *and* $x' \upharpoonright J = x \upharpoonright J$.

We shall say that (x, Π) *engulfs* (x', Π') when the equivalent conditions in the proposition hold.

Thus, we have morphisms in both directions between $\mathbf{Cof}(\mathcal{B})$ and

$$\mathbf{Cof}'(\mathcal{B}) = (\mathrm{CR}, \mathrm{CR}, \text{is engulfed by}),$$

as well as morphisms in both directions between $\mathbf{Cov}(\mathcal{B})$ and

$$\mathbf{Cov}'(\mathcal{B}) = ({}^{\omega}2, \mathrm{CR}, \text{does not match}).$$

Notice that if (x, Π) engulfs (x', Π') then Π dominates Π'. Combining this with the characterization of $\mathfrak{d}$ and $\mathfrak{b}$ in Theorem 2.10 and the characterization of $\mathbf{add}(\mathcal{B})$ and $\mathbf{cof}(\mathcal{B})$ in Example 4.4, we obtain the following inequalities.

5.4 Corollary. $\mathbf{add}(\mathcal{B}) \le \mathfrak{b}$ *and* $\mathfrak{d} \le \mathbf{cof}(\mathcal{B})$.

Another relation between the characteristics from Sect. 2 and the characteristics of Baire category follows from Theorem 2.8.

5.5 Proposition. $\mathfrak{b} \le \mathbf{non}(\mathcal{B})$ *and* $\mathbf{cov}(\mathcal{B}) \le \mathfrak{d}$.

Proof. In ${}^{\omega}\omega$, any set of the form $\{f : f \le g\}$ is clearly nowhere dense (because every finite sequence in ${}^{<\omega}\omega$ has an extension in ${}^{<\omega}\omega$ with some values greater than the corresponding values of g). The proof of Theorem 2.8 shows, therefore, that all compact sets in ${}^{\omega}\omega$ are nowhere dense and therefore $\mathcal{K}_\sigma \subseteq \mathcal{B}$. That immediately implies $\mathbf{cov}(\mathcal{K}_\sigma) \ge \mathbf{cov}(\mathcal{B})$ and $\mathbf{non}(\mathcal{K}_\sigma) \le \mathbf{non}(\mathcal{B})$. (Indeed, whenever $\mathcal{I} \subseteq \mathcal{J}$ are ideals, we have a morphism $\mathbf{Cov}(\mathcal{I}) \to \mathbf{Cov}(\mathcal{J})$ given by the identity map on challenges and the inclusion map on responses.) Now Theorem 2.8 completes the proof. ⊣

All ZFC-provable inequalities among $\mathfrak{b}$, $\mathfrak{d}$, and the four characteristics of $\mathcal{B}$ are obtainable by transitivity from the preceding corollary and proposition and the general facts that $\mathbf{add} \le \mathbf{cov} \le \mathbf{cof}$ and $\mathbf{add} \le \mathbf{non} \le \mathbf{cof}$ for any nontrivial ideal. There are, however, two additional relations due to Miller [80] and Truss [110], each involving three of these cardinals.

5.6 Theorem.

1. *There is a morphism from* $(\mathbf{Cov}'(\mathcal{B}))^{\perp}; \mathfrak{D}'$ *to* $\mathbf{Cof}'(\mathcal{B})$.
2. $\mathbf{cof}(\mathcal{B}) = \max\{\mathbf{non}(\mathcal{B}), \mathfrak{d}\}$.

3. $\mathbf{add}(\mathcal{B}) = \min\{\mathbf{cov}(\mathcal{B}), \mathfrak{b}\}$.

Proof. Recall that $\mathfrak{D}' = (\mathrm{IP}, \mathrm{IP}, \text{is dominated by})$ where IP is the set of interval partitions, that $\|\mathfrak{D}'\| = \mathfrak{d}$, and that $\|\mathfrak{D}'^{\perp}\| = \mathfrak{b}$. Thus, if we prove part 1 of the theorem, then the $\leq$ half of part 2 and the $\geq$ half of part 3 will follow by Theorems 4.9 and 4.11. The other halves of parts 2 and 3 were already established, so we need only prove part 1.

A morphism φ as claimed in part 1 would consist of a function φ_- from the set CR of chopped reals to $\mathrm{CR} \times {}^{({}^\omega 2)}\mathrm{IP}$ and a function φ_+ from ${}^\omega 2 \times \mathrm{IP}$ to CR, satisfying an implication to be exhibited after we simplify notation a bit. As a map into a product, φ_- consists of two maps, $\alpha : \mathrm{CR} \to \mathrm{CR}$ and $\beta : \mathrm{CR} \to {}^{({}^\omega 2)}\mathrm{IP}$. We shall take α and φ_+ to be identity maps. (Recall that $\mathrm{CR} = {}^\omega 2 \times \mathrm{IP}$, so this makes sense.) It remains to define β so as to satisfy the required implication, which now reads: For all $x \in \mathrm{CR}$, all $y \in {}^\omega 2$, and all $\Pi \in \mathrm{IP}$,

$$[y \text{ matches } x \text{ and } \Pi \text{ dominates } \beta(x)(y)] \quad \Longrightarrow \quad [(y, \Pi) \text{ engulfs } x].$$

It does not matter how we define $\beta(x)(y)$ when y does not match x. If y does match x, i.e., if there are infinitely many intervals I in the partition component of the chopped real x on which x and y agree, then we define $\beta(x)(y)$ to be some interval partition each of whose intervals includes at least one such I. ⊣

5.7 Remark. It is easy to specify the β in the last part of the proof more explicitly so that $\beta(x)(y)$ is a Borel function of x and y; since the other components of φ are trivial, we can say that part 1 of the theorem is witnessed by a Borel morphism. It is shown in [22] that one cannot get a Borel morphism in part 1 if one replaces the sequential product there with the categorical product, or the conjunction, or the sequential product in the other order.

Before turning from category to measure, we give an elegant, combinatorial description of $\mathbf{cov}(\mathcal{B})$, due to Bartoszyński [4].

5.8 Definition. Call two functions $x, y \in {}^\omega\omega$ *infinitely equal* if $\exists^\infty n\,(x(n) = y(n))$ and *eventually different* otherwise, i.e., if $\forall^\infty n\,(x(n) \neq y(n))$.

5.9 Theorem.

1. $\mathbf{cov}(\mathcal{B}) = \|({}^\omega\omega, {}^\omega\omega, \text{eventually different})\|$.

2. $\mathbf{non}(\mathcal{B}) = \|({}^\omega\omega, {}^\omega\omega, \text{infinitely equal})\|$.

Proof. We prove only part 1 as part 2 is dual to it. The $\leq$ direction is clear once one observes that, for any $x \in {}^\omega\omega$, the set of $y \in {}^\omega\omega$ eventually different from x is meager. (In fact, sending x to this set defines half of a morphism from the relation on the right of part 1 to $\mathbf{Cov}(\mathcal{B})$ (when the reals are taken to be ${}^\omega\omega$); the other half of the morphism is the identity map.)

To prove the $\geq$ direction of part 1, we show how to match, with a single real y, all the chopped reals in a family $\{(x_\alpha, \Pi_\alpha) : \alpha < \kappa\}$, where

$$\kappa < \|({}^\omega\omega, {}^\omega\omega, \text{eventually different})\|.$$

Note that the norm here is trivially $\leq \mathfrak{d}$ (there's a morphism from $\mathfrak{D}$ consisting of the identity map in both directions). So by Theorem 2.10 there is an interval partition Θ not dominated by any Π_α.

Temporarily fix an arbitrary $\alpha < \kappa$. Non-domination means that Π_α has infinitely many intervals that include no interval of Θ and are therefore covered by two adjacent intervals of Θ. Call a pair of adjacent intervals of Θ good if they cover an interval of Π_α; so there are infinitely many good pairs.

Define a function f_α on ω as follows. $f_\alpha(n)$ is obtained by taking $2n+1$ disjoint good pairs, taking the union of the two intervals in each pair to obtain $2n+1$ intervals $J_0, \ldots, J_{2n}$, and then forming the set of restrictions of x_α to these intervals:

$$f_\alpha(n) = \{x_\alpha \restriction J_0, \ldots, x_\alpha \restriction J_{2n}\}.$$

Note that, although the values of f_α are not natural numbers, they can be coded as natural numbers.

Now un-fix α. By our hypothesis on κ, find a function g infinitely equal to each f_α. Without harming this property of g, we can arrange that, for each n, $g(n)$ is a set of $2n+1$ functions, each mapping an interval of ω to 2. Furthermore, we can arrange that these $2n+1$ intervals are disjoint and each of them is the union of two adjacent intervals of Θ. (Any n for which $g(n)$ is not of this form could not contribute to the agreement between g and any f_α, so we are free to modify $g(n)$ arbitrarily.)

We define a function $y : \omega \to 2$ by recursion, where at each stage we specify the restriction of y to a certain pair of adjacent intervals in Θ. After stages 0 through $n-1$ are completed, y is defined on only $2n$ intervals of Θ, so at least one of the $2n+1$ members of $g(n)$, say $z(n)$, has its domain J disjoint from where y is already defined. Extend y to agree with $z(n)$ on J. This completes the recursion; if there are places where y never gets defined, define it arbitrarily there.

To complete the proof, we show that y matches every (x_α, Π_α). Consider any α and any one of the infinitely many n for which $g(n) = f_\alpha(n)$. At stage n of the construction of y, we ensured that y extends some $z(n) \in g(n) = f_\alpha(n)$. But the construction of $f_\alpha(n)$ ensures that $z(n)$ is the restriction of x_α to an interval (the union of a good pair of intervals from Θ) that includes an interval of Π_α. Thus y agrees with x_α on that interval of Π_α. Since this happens for infinitely many n, y matches (x_α, Π_α). $\dashv$

5.10 Remark. The preceding proof exhibits a morphism from $\mathbf{Cov}'(\mathcal{B})$ to $\mathfrak{D}' \breve{;} ({}^\omega\omega, {}^\omega\omega, \text{eventually different})$. Ignoring the coding needed to make f_α and g functions into ω, we can say that the "challenge" half of the morphism is the construction of y from Θ and g and the "response" half of the morphism sends any (x, Π) (where we omit the α subscripts needed in the proof but not here) to the pair consisting of Π and the function that maps any Θ not dominated by Π to the f as in the proof (and maps Θ's that are dominated by Π arbitrarily).

It is an open problem whether one can omit the "$\mathfrak{D}$;" part, i.e., whether there is a Borel morphism from $\mathbf{Cov}'(\mathcal{B})$ to $({}^\omega\omega, {}^\omega\omega, \text{eventually different})$. An essentially equivalent question is whether any forcing that adds a real (in ${}^\omega\omega$) infinitely equal to all ground model reals (called a "half-Cohen" real) must add a Cohen real. The proof above shows that if one first adds an unbounded real and then a half-Cohen real over the resulting model, the final model contains a Cohen real over the ground model.

We now turn to Lebesgue measure (and equivalent measures on ${}^\omega 2$, ${}^\omega\omega$, etc.) and its connections with Baire category. The first such connection was given by Rothberger [90].

5.11 Theorem. $\mathbf{cov}(\mathcal{B}) \le \mathbf{non}(\mathcal{L})$ *and* $\mathbf{cov}(\mathcal{L}) \le \mathbf{non}(\mathcal{B})$.

Proof. Let Π be the interval partition whose nth interval I_n has $n+1$ elements for all n. Define a binary relation R on ${}^\omega 2$ by letting $x\,R\,y$ mean that $x\restriction I_n = y\restriction I_n$ for infinitely many n, i.e., that y matches the chopped real (x, Π). Notice that R is symmetric and, for every x, the set $R_x = \{y : x\,R\,y\}$ is a comeager set of measure zero. ("Comeager" was proved in Theorem 5.2. The calculation for "measure zero" consists of noticing that, once x is fixed, the y's that agree with it on I_n form a set of measure $2^{-(n+1)}$, so the y's that agree with x on at least one I_n beyond I_k form a set of measure at most 2^{-k}, and so the y's that do this for all k form a set of measure zero.)

Thus, letting $\mathbf{R} = ({}^\omega 2, {}^\omega 2, R)$, we have morphisms $\varphi : \mathbf{R} \to \mathbf{Cov}(\mathcal{L})$ and $\psi : \mathbf{R}^\perp \to \mathbf{Cov}(\mathcal{B})$, where φ_+ and ψ_+ send x to R_x and ${}^\omega 2 - R_x$ respectively and where both φ_- and ψ_- are the identity on ${}^\omega 2$. Composing each of these morphisms with the dual of the other, we get morphisms $\mathbf{Cov}(\mathcal{B})^\perp \to \mathbf{Cov}(\mathcal{L})$ and $\mathbf{Cov}(\mathcal{L})^\perp \to \mathbf{Cov}(\mathcal{B})$. Since $\mathbf{cov} = \|\mathbf{Cov}\|$ and $\mathbf{non} = \|\mathbf{Cov}^\perp\|$ for both ideals, the theorem follows. ⊣

5.12 Remark. The relation R in the preceding proof could be replaced by any relation of the form "$x \oplus y \in M$" where $\oplus$ is addition modulo 2 and M is any comeager set of measure zero. For example, M could be the set of 0–1 sequences in which the density of 1's in initial segments does not approach 1/2.

In this form, the proof generalizes to any pair of translation-invariant (with respect to $\oplus$) ideals that concentrate on disjoint sets.

The rest of our discussion of measure characteristics is based on a combinatorial characterization, due to Bartoszyński [3], of $\mathbf{add}(\mathcal{L})$. To formulate it, we need the following terminology.

5.13 Definition. A *slalom* is a function S assigning to each $n \in \omega$ a set $S(n) \subseteq \omega$ of cardinality n. We say that a real $x \in {}^\omega\omega$ *goes through* slalom S if $\forall^\infty n\,(x(n) \in S(n))$.

5.14 Theorem. $\mathbf{add}(\mathcal{L})$ *is the smallest cardinality of any family* $\mathcal{F} \subseteq {}^\omega\omega$ *such that there is no single slalom through which all the members of* $\mathcal{F}$ *go.*

For the proof, we refer to Bartoszyński's original paper [3], his chapter in this Handbook, his book with Judah [5, Theorem 2.3.9], or Fremlin's article [47].

5.15 Remark. The theorem would remain true if we modified the definition of "slalom" by requiring $S(n)$ to have cardinality $f(n)$ instead of n; here f can be any function $\omega \to \omega$ that grows without bound. We refer to this modified notion of slalom as an f-slalom (or $f(n)$-slalom). Suppose, for example, that κ is a cardinal such that every κ functions in ${}^\omega\omega$ go through a single f-slalom. To show that any κ functions x_α go through a single slalom in the original sense, partition ω into intervals such that the nth interval starts at or after $f(n)$. Let $y_\alpha(n)$ be (or code) the restriction of x_α to the nth interval. From an f-slalom through which all the y_α go, one easily gets a slalom in the original sense through which all the x_α go.

Despite this observation, it is not true that one could simply omit the cardinality bound (n or $f(n)$) in the definition of slalom and merely require each $S(n)$ to be finite. Indeed, with this weakening, the cardinal described in the theorem would be simply $\mathfrak{b}$, which can be strictly larger than $\mathbf{add}(\mathcal{L})$.

As indicated for example in Oxtoby's book [83], there are a great many similarities between Baire category and Lebesgue measure. The following inequality, due to Bartoszyński [3] and independently but a bit later to Raisonnier and Stern [89], was an early indication that the symmetry is not so extensive as one might have thought. (The first indication of this was Shelah's proof [98] that ZF (without choice) plus "all sets of reals have the Baire property" is consistent if ZF is, whereas the consistency of ZF plus "all sets of reals are Lebesgue measurable" requires the consistency of an inaccessible cardinal.) The theme of measure-category asymmetry is developed much further in the book [5].

5.16 Theorem. $\mathbf{add}(\mathcal{L}) \leq \mathbf{add}(\mathcal{B})$*.*

Proof. In view of Theorem 5.6, it suffices to prove that $\mathbf{add}(\mathcal{L}) \leq \mathfrak{b}$ and $\mathbf{add}(\mathcal{L}) \leq \mathbf{cov}(\mathcal{B})$. The former is immediate, in view of Theorem 5.14, for a family of reals going through a single slalom is obviously bounded. (It should be mentioned that the inequality $\mathbf{add}(\mathcal{L}) \leq \mathfrak{b}$ was originally proved by Miller [81] before Theorem 5.14 was known.) For the second inequality, we use Theorem 5.9.

If $\kappa < \mathbf{add}(\mathcal{L})$ and if we are given κ functions $x_\alpha \in {}^\omega\omega$, we must find a single function y infinitely equal to them all. Fix an interval partition Π whose nth interval I_n has cardinality $\geq n$. To each x_α associate the function $x'_\alpha \in {}^\omega\omega$ where $x'_\alpha(n)$ codes (in some standard way) $x_\alpha \restriction I_n$. Let S be a slalom through which all the x'_α go. We may assume that all n elements of $S(n)$ code functions $I_n \to \omega$, for any other elements can be replaced with such codes without harming the fact that all x'_α go through S. For each n, choose a function $y_n : I_n \to \omega$ that agrees at least once with each of the n members of $S(n)$; this is trivial to arrange, since $|I_n| \geq n$. Then the union of all the y_n is the desired y. Indeed, every x_α agrees with y at least once in each I_n except for finitely many. ⊣

5.17 Remark. The preceding proof, though short, has a defect from the point of view of morphisms between relations. Because it relies on Theorems 5.6 and 5.9, it involves sequential compositions. In fact, it provides a Borel morphism from $\mathbf{Cof}(\mathcal{L});\mathbf{Cof}(\mathcal{L});\mathbf{Cof}(\mathcal{L})$ to $\mathbf{Cof}(\mathcal{B})$. The presence of these sequential compositions is an artifact of the proof. Bartoszyński's chapter in this Handbook contains a different proof, giving a Borel morphism from $\mathbf{Cof}(\mathcal{L})$ to $\mathbf{Cof}(\mathcal{B})$.

Since the proof gave a morphism, we also have the dual result.

5.18 Corollary. $\mathbf{cof}(\mathcal{B}) \leq \mathbf{cof}(\mathcal{L})$.

Our discussion of the four standard characteristics (**add**, **cov**, **non**, and **cof**) of measure and category, along with $\mathfrak{b}$ and $\mathfrak{d}$, is now complete, in the following strong sense. If one assigns to each of these ten characteristics one of the values $\aleph_1$ and $\aleph_2$, and if the assignment is consistent with the equations and inequalities proved above, then that assignment is realized in some model of ZFC. We shall comment on a few of these models in Sect. 11 below, but we refer to [6] or [5, Chap. 7] for all the details.

The inequalities between these ten cardinal characteristics are summarized in the following picture, known as *Cichoń's diagram*, in which one goes from larger to smaller cardinals by moving down or to the left along the arrows. (A 45° counterclockwise rotation would produce a Hasse diagram in the customary orientation. We've drawn the arrows in the direction of the morphisms between the corresponding relations, hence from larger to smaller characteristics.)

$$\begin{array}{ccccccc}
\mathbf{cov}(\mathcal{L}) & \longleftarrow & \mathbf{non}(\mathcal{B}) & \longleftarrow & \mathbf{cof}(\mathcal{B}) & \longleftarrow & \mathbf{cof}(\mathcal{L}) \\
\Big\downarrow & & \downarrow & & \downarrow & & \Big\downarrow \\
 & & \mathfrak{b} & \longleftarrow & \mathfrak{d} & & \\
 & & \downarrow & & \downarrow & & \\
\mathbf{add}(\mathcal{L}) & \longleftarrow & \mathbf{add}(\mathcal{B}) & \longleftarrow & \mathbf{cov}(\mathcal{B}) & \longleftarrow & \mathbf{non}(\mathcal{L})
\end{array}$$

To conclude this section, we point out an elementary connection between the covering and uniformity numbers studied here and the splitting and refining numbers from Sect. 3.

5.19 Theorem. $\mathfrak{s} \leq \mathbf{non}(\mathcal{B}), \mathbf{non}(\mathcal{L})$ *and* $\mathfrak{r} \geq \mathbf{cov}(\mathcal{B}), \mathbf{cov}(\mathcal{L})$.

Proof. For any infinite $A \subseteq \omega$, the sets $X \subseteq \omega$ that fail to split A form a meager, measure-zero set U_A. Then the function $A \mapsto U_A$ and the identity function on $\mathcal{P}(\omega)$ constitute a morphism from $\mathfrak{R}$ to $\mathbf{Cov}(\mathcal{B})$ and also to $\mathbf{Cov}(\mathcal{L})$. ⊣

6. Sparse Sets of Integers

This section is primarily about two characteristics, $\mathfrak{t}$ and $\mathfrak{h}$, related to the idea of thinning out infinite subsets of ω, i.e., replacing them by subsets, usually so as to achieve some useful property like homogeneity for some partition. $\mathfrak{t}$ is concerned with the (transfinite) thinning process itself; $\mathfrak{h}$ focuses on what can be achieved by iterated thinning. We shall also briefly consider two characteristics, $\mathfrak{p}$ and $\mathfrak{g}$, whose definitions resemble those of $\mathfrak{t}$ and $\mathfrak{h}$, though their most significant properties are treated only in later sections.

We begin with the definition and simplest properties of $\mathfrak{t}$.

6.1 Definition. A *pseudointersection* of a family $\mathcal{F}$ of sets is an infinite set that is $\subseteq^*$ every member of $\mathcal{F}$.

6.2 Definition. A *tower* is an ordinal-indexed sequence $\langle T_\alpha : \alpha < \lambda \rangle$ such that:

1. Each T_α is an infinite subset of ω.

2. $T_\beta \subseteq^* T_\alpha$ whenever $\alpha < \beta < \lambda$.

3. $\{T_\alpha : \alpha < \lambda\}$ has no pseudointersection.

The *tower number* $\mathfrak{t}$ is the smallest λ that is the length of a tower.

6.3 Remark. Hechler [55] has constructed a model where many regular cardinals occur as the lengths of towers.

Some authors define "tower" using only the first two clauses in the definition above, i.e., an almost decreasing sequence in $[\omega]^\omega$; what we call a tower, they would call an inextendible tower. Also, some authors take towers to be almost increasing sequences of co-infinite subsets of ω rather than almost decreasing sequences of infinite sets.

We shall not always be as careful as we were in clause 3 of the definition about the distinction between a sequence like $\langle T_\alpha : \alpha < \lambda \rangle$ and the set $\{T_\alpha : \alpha < \lambda\}$ of its terms.

6.4 Proposition. $\mathfrak{t}$ *is a regular uncountable cardinal.*

Proof. Regularity is clear since any cofinal subsequence of a tower is a tower. To show that there can be no tower $\langle T_n : n \in \omega \rangle$ of length ω, note that we could form an infinite set X by taking any element of T_0, any different element of $T_0 \cap T_1$, any different element of $T_0 \cap T_1 \cap T_2$, etc., since all these sets are infinite. This X would be a pseudointersection, violating requirement 3 in the definition of tower. $\dashv$

Before continuing with further properties of $\mathfrak{t}$, we introduce $\mathfrak{h}$, its basic properties, and its connection with $\mathfrak{t}$.

6.5 Definition. A family $\mathcal{D} \subseteq [\omega]^\omega$ is *open* if it is closed under almost subsets. It is *dense* if every $X \in [\omega]^\omega$ has a subset in $\mathcal{D}$. The *distributivity number* $\mathfrak{h}$ is the smallest number of dense open families with empty intersection.

6.6 Remark. The open sets as defined here constitute a topology on $[\omega]^\omega$, which we call the *lower topology*. Density as defined here agrees with topological density in the lower topology as long as $\mathcal{D}$ is closed under finite modifications (for example if it is open). Analogous definitions can be made for any pre-ordered set in place of $([\omega]^\omega, \subseteq^*)$.

The name "distributivity number" comes from viewing $([\omega]^\omega, \subseteq)$ as a notion of forcing and asking how distributive the associated complete Boolean algebra is. Standard techniques from forcing theory show that the answer is given by $\mathfrak{h}$. More precisely, Boolean meets of fewer than $\mathfrak{h}$ terms distribute over arbitrary (finite or infinite) joins, but meets of $\mathfrak{h}$ terms need not distribute even over binary joins. Equivalently, in a forcing extension by $([\omega]^\omega, \subseteq)$, $\mathfrak{h}$ has new subsets but smaller ordinals do not (not even new functions into the ordinals). We shall see later (6.20) that this forcing extension collapses $\mathfrak{c}$ to $\mathfrak{h}$ if $\mathfrak{h} < \mathfrak{c}$.

6.7 Proposition. *The intersection of any fewer than* $\mathfrak{h}$ *dense open families is dense open.* $\mathfrak{h}$ *is a regular cardinal.*

Proof. The second sentence follows immediately from the first. (It also follows from the remark about distributivity.) To show that the intersection of fewer than $\mathfrak{h}$ dense open families $\mathcal{D}_\alpha$ is dense open, note first that it is obviously open. As for density, let X be any infinite subset of ω and consider the families $\mathcal{D}'_\alpha = \{Y \in \mathcal{D}_\alpha : Y \subseteq X\}$. These are fewer than $\mathfrak{h}$ dense open families of subsets of X, so they have a common member Y. That is, $Y \subseteq X$ and $Y \in \bigcap_\alpha \mathcal{D}_\alpha$. ⊣

The definition of $\mathfrak{t}$ is essentially about the process of thinning out infinite subsets of ω by repeatedly passing to (almost) subsets. If one attempts to iterate such a thinning process transfinitely, the definition of $\mathfrak{t}$ ensures that one will not get stuck at limit stages of cofinality $< \mathfrak{t}$.

The definition of $\mathfrak{h}$ addresses the same idea from the point of view of what such thinning can achieve. A dense open family is one that one can get into, from an arbitrary infinite subset of ω, by passing to a subset (and subsequent passages to further (almost) subsets will not undo this achievement). The next proposition is just the result of comparing these intuitions that stand behind $\mathfrak{t}$ and $\mathfrak{h}$.

6.8 Proposition. $\mathfrak{t} \le \mathfrak{h}$.

Proof. Suppose $\kappa < \mathfrak{t}$, and let κ dense open families $\mathcal{D}_\alpha$ $(\alpha < \kappa)$ be given; we must find a set in their intersection. Define an almost decreasing sequence $\langle T_\alpha : \alpha \le \kappa \rangle$ by the following recursion. $T_0 = \omega$. $T_{\alpha+1}$ is any subset of T_α

that is in $\mathcal{D}_\alpha$; this exists because $\mathcal{D}_\alpha$ is dense. If $\lambda \leq \kappa$ is a limit ordinal, then T_λ is any pseudointersection of $\{T_\alpha : \alpha < \lambda\}$; this exists because $\kappa < \mathfrak{t}$ so $\{T_\alpha : \alpha < \lambda\}$ cannot be a tower, yet the previous steps ensured that it satisfies the first two requirements for a tower. Since $T_\kappa \subseteq^* T_{\alpha+1}$ for all $\alpha < \kappa$, we have, thanks to openness, that T_κ is in all the families $\mathcal{D}_\alpha$. $\dashv$

It is consistent with ZFC to have $\mathfrak{t} < \mathfrak{h}$. In fact, Dordal [41] built a model where $\mathfrak{h} = \aleph_2 = \mathfrak{c}$ but there are no towers of length ω_2.

Upper bounds for $\mathfrak{h}$, and therefore also for $\mathfrak{t}$, can be obtained by considering specific examples of dense open families. One family of examples consists of $\{X \in [\omega]^\omega : X \text{ is not split by } Y\}$ for arbitrary Y. Another consists of $\{X \in [\omega]^\omega : \forall^\infty x \in X \, \forall y \in X \, (\text{if } x < y \text{ then } f(x) < y)\}$ for arbitrary $f : \omega \to \omega$. Using these, one easily obtains the following proposition, but we give another proof to suggest another class of examples.

6.9 Theorem. $\mathfrak{h} \leq \mathfrak{b}, \mathfrak{s}$.

Proof. By Theorem 3.5 it suffices to show $\mathfrak{h} \leq \mathfrak{par}_2$. So let $\kappa < \mathfrak{h}$ partitions f_α of $[\omega]^2$ be given; we must find an infinite set almost homogeneous for them all. For each α, let $\mathcal{D}_\alpha$ be the family of all infinite subsets of ω that are almost homogeneous for f_α. Then $\mathcal{D}_\alpha$ is dense open, thanks to Ramsey's Theorem. So there is a set H common to all the $\mathcal{D}_\alpha$. $\dashv$

6.10 Remark. The same proof shows that one can get simultaneous almost homogeneity for fewer than $\mathfrak{h}$ partitions of more complicated sorts, provided one has the analog of Ramsey's Theorem to ensure density. Thus, for example, Silver's partition theorem for analytic sets [102] implies that any $< \mathfrak{h}$ partitions of $[\omega]^\omega$ into an analytic and a coanalytic piece have a common infinite almost homogeneous set.

By Proposition 6.8, the upper bounds on $\mathfrak{h}$ apply also to $\mathfrak{t}$, but for $\mathfrak{t}$ we can improve $\mathfrak{b}$ to $\mathbf{add}(\mathcal{B})$. In order to prove this, we need the following lemma, in which $\mathbb{Q}$ denotes the set of rational numbers and "dense" has its usual topological (or order-theoretic) meaning for subsets of $\mathbb{Q}$. Both the lemma and the subsequent theorem are from [93] (stated for special cases but the proofs work in general); they were rediscovered in [86].

6.11 Lemma. *Suppose $\lambda < \mathfrak{t}$ and $\langle T_\alpha : \alpha < \lambda \rangle$ is an almost decreasing sequence of dense subsets of $\mathbb{Q}$. Then there exists a dense $X \subseteq \mathbb{Q}$ that is almost included in every T_α.*

Proof. In each interval I with rational endpoints, consider the almost decreasing sequence $\langle T_\alpha \cap I : \alpha < \lambda \rangle$ of infinite subsets of I. As it is too short to be a tower, there is an infinite $Y_I \subseteq I$ almost included in all the T_α. (The union of all the Y_I is dense, but it need not be $\subseteq^* T_\alpha$, so we must work a bit harder to get X.) Enumerate each Y_I as an ω-sequence $\langle y_{I,n} \rangle$. For each α let $f_\alpha(I) \in \omega$ be an upper bound for the finitely many n such that $y_{I,n} \notin T_\alpha$.

Since $\lambda < \mathfrak{t} \leq \mathfrak{b}$ (and the set of intervals I is countable), let g be a function to ω from the set of rational intervals such that g dominates all the f_α's. Then

$$X = \bigcup_I \{y_{I,n} : n > g(I)\}$$

is dense in $\mathbb{Q}$ (because it contains almost all of each Y_I) and is almost included in each T_α (for $X - T_\alpha$ consists of finitely many elements from each of the finitely many Y_I where $g(I) < f_\alpha(I)$). $\dashv$

6.12 Theorem. $\mathfrak{t} \leq \mathbf{add}(\mathcal{B})$.

Proof. We must show that if $\kappa < \mathfrak{t}$ then the intersection of any κ dense open subsets G_α ($\alpha < \kappa$) of $\mathbb{R}$ is comeager. We begin by defining an almost decreasing sequence $\langle T_\alpha : \alpha \leq \kappa \rangle$ of dense subsets of $\mathbb{Q}$. Start with $T_0 = \mathbb{Q}$. At limit stages, apply the lemma. At successor stages, set $T_{\alpha+1} = T_\alpha \cap G_\alpha$; this is dense because it is the intersection of two dense sets one of which is open. Note that T_κ, being $\subseteq^*$ each $T_{\alpha+1}$ ($\alpha < \kappa$) is also $\subseteq^*$ each G_α.

For $t \in T_\kappa$ and $\alpha < \kappa$, define $f_\alpha(t) \in \omega$ to be some n such that $(t - \frac{1}{n}, t + \frac{1}{n}) \subseteq G_\alpha$ if $t \in G_\alpha$, and 0 otherwise. Since T_κ is countable and $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there is a $g : T_\kappa \to \omega$ dominating all the f_α's.

For each finite $F \subseteq T_\kappa$, let

$$U_F = \bigcup_{t \in T_\kappa - F} \left(t - \frac{1}{g(n)}, t + \frac{1}{g(n)} \right).$$

Then U_F is dense, because it almost includes T_κ, and it is obviously open; since there are only countably many F's, $\bigcap_F U_F$ is comeager, and it remains only to prove that this intersection is included in the intersection of the G_α's. In fact, each G_α includes one of the U_F's; given α just take F to contain the finitely many $t \in T_\kappa - G_\alpha$ and the finitely many t where $g(t) < f_\alpha(t)$. $\dashv$

6.13 Remark. By a countable support iteration of Mathias forcing over a model of CH, one obtains a model where $\mathfrak{h} = \aleph_2$ but $\mathbf{cov}(\mathcal{B})$ and therefore $\mathbf{add}(\mathcal{B})$ are only $\aleph_1$ (as no Cohen reals are produced). Thus, the preceding theorem cannot be improved by putting $\mathfrak{h}$ in place of $\mathfrak{t}$.

The next theorem can be viewed as another upper bound on $\mathfrak{t}$.

6.14 Theorem. *If* $\aleph_0 \leq \kappa < \mathfrak{t}$ *then* $2^\kappa = \mathfrak{c}$.

Proof. We need only check that $2^\kappa \leq \mathfrak{c}$, and we do this by building a complete binary tree of $\kappa + 1$ levels, whose nodes are distinct subsets of ω. More precisely, we associate to every sequence $\eta \in {}^{\leq\kappa}2$ an infinite subset T_η of ω in such a way that:

1. If η is an initial segment of θ, then $T_\theta \subseteq^* T_\eta$.
2. If neither of η and θ is an initial segment of the other, then $T_\eta \cap T_\theta$ is finite.

The construction is by recursion on the length of η, starting with $T_\emptyset = \omega$. At successor stages, we define $T_{\eta^\frown\langle 0\rangle}$ and $T_{\eta^\frown\langle 1\rangle}$ to be any two disjoint, infinite subsets of T_η. Finally, for θ of limit length λ, we observe that $\langle T_{\theta\restriction\alpha} : \alpha < \lambda\rangle$ is an almost decreasing sequence but cannot be a tower because $\lambda \le \kappa < \mathfrak{t}$. So there is an infinite X almost included in all these $T_{\theta\restriction\alpha}$; any such X can serve as T_θ.

It is immediate that the construction has the desired properties. In particular, the 2^κ sets T_η, for all η of length κ, are infinite and almost disjoint and therefore certainly distinct. $\dashv$

6.15 Corollary. $\mathfrak{t} \le \mathrm{cf}(\mathfrak{c})$.

Proof. If $\kappa < \mathfrak{t}$ then, by Theorem 6.14 and König's theorem, $\mathrm{cf}(\mathfrak{c}) = \mathrm{cf}(2^\kappa) > \kappa$. $\dashv$

Returning to consider $\mathfrak{h}$ in more detail, we first give an alternative way to view dense open families of subsets of ω.

6.16 Definition. An *almost disjoint* family is a family of infinite sets, every two of which have finite intersection. A *maximal almost disjoint* (MAD) family is an infinite almost disjoint family of subsets of ω, maximal with respect to inclusion.

6.17 Remark. Note that MAD families are required to be infinite; in the absence of this requirement, any partition of ω into finitely many infinite sets would count as MAD. Note also that, if $\mathcal{A}$ is MAD and X is any infinite subset of ω, then $X \cap A$ is infinite for at least one $A \in \mathcal{A}$.

6.18 Proposition. *If $\mathcal{A}$ is a* MAD *family, then $\mathcal{A}{\downarrow} = \{X \in [\omega]^\omega : \exists A \in \mathcal{A}\ (X \subseteq^* A)\}$ is dense open. Every dense open family includes one of this form.*

Proof. The first statement is proved by routine checking of definitions. For the second, let $\mathcal{D}$ be dense open, and let $\mathcal{A}_0$ be an infinite, almost disjoint subfamily of $\mathcal{D}$; for example, take some $X \in \mathcal{D}$ and partition it into infinitely many infinite pieces. By Zorn's Lemma, let $\mathcal{A} \supseteq \mathcal{A}_0$ be an almost disjoint family included in $\mathcal{D}$ and maximal among such families. We claim that $\mathcal{A}$ is maximal among all almost disjoint families, not just those included in $\mathcal{D}$. Once we establish this claim, we will have $\mathcal{A}$ MAD and $\mathcal{A}{\downarrow} \subseteq \mathcal{D}$ as required.

To establish maximality, consider any $X \in [\omega]^\omega$. As $\mathcal{D}$ is dense, it contains a subset Y of X. As $\mathcal{A}$ is maximal among almost disjoint subfamilies of $\mathcal{D}$, it contains a set A that has infinite intersection with Y and therefore also with X. $\dashv$

6.19 Corollary. $\mathfrak{h}$ *is the minimum number of* MAD *families such that, for each $X \in [\omega]^\omega$, one of these families contains at least two sets whose intersections with X are infinite.*

Proof. X has infinite intersection with at least two sets from a MAD family $\mathcal{A}$ if and only if $X \notin \mathcal{A}{\downarrow}$. With this observation, the corollary follows immediately from the proposition and the definition of $\mathfrak{h}$. ⊣

The following theorem of Balcar, Pelant, and Simon [2] was the original motivation for the introduction of $\mathfrak{h}$. A tree of the sort described by this theorem is called a *base matrix tree* (for $[\omega]^\omega$). The theorem would become false if $\mathfrak{h}$ were replaced by any smaller cardinal. The symbol $\mathfrak{h}$ was chosen to refer to the "height" of the base matrix tree.

6.20 Theorem. *There is a family $\mathcal{T} \subseteq [\omega]^\omega$ with the following properties.*

1. *Ordered by reverse almost inclusion, $\mathcal{T}$ is a tree of height $\mathfrak{h}$ with root ω.*
2. *Each level of $\mathcal{T}$, except for the root, is a* MAD *family.*
3. *Every $X \in [\omega]^\omega$ has a subset in $\mathcal{T}$.*

Proof. Let $\mathcal{D}_\alpha$ for $\alpha < \mathfrak{h}$ be dense open families with no common member. We define the levels $\mathcal{T}_\alpha$ of the desired tree inductively as follows. At level 0, put ω. At a limit level $\lambda < \mathfrak{h}$, use Proposition 6.7 to obtain a dense open family included in all $\mathcal{T}_\alpha{\downarrow}$ for $\alpha < \lambda$. By Proposition 6.18, shrink this to a dense open family of the form $\mathcal{A}{\downarrow}$, and let that $\mathcal{A}$ be $\mathcal{T}_\lambda$.

At an odd-numbered successor stage, say $2\alpha + 1$, choose $\mathcal{T}_{2\alpha+1}$ as a MAD family included in both $\mathcal{T}_{2\alpha}{\downarrow}$ and $\mathcal{D}_\alpha$. This can be done by Propositions 6.7 and 6.18.

At an even-numbered successor stage, say $2\alpha+2$, proceed as follows. Call a set $X \in [\omega]^\omega$ *active* at this stage if it has infinite intersection with $\mathfrak{c}$ members of $\mathcal{T}_{2\alpha+1}$. Assign to each active X some $\psi(X) \in \mathcal{T}_{2\alpha+1}$ that has infinite intersection with X, and do this in such a way that ψ is one-to-one. This is easily done by arranging all the active X's in a well-ordered sequence, of length $\le \mathfrak{c}$, and defining ψ by recursion along this ordering. At each stage of the recursion, there are $\mathfrak{c}$ elements of $\mathcal{T}_{2\alpha+1}$ that have infinite intersection with the current X and fewer than $\mathfrak{c}$ of them have already been assigned as earlier values of ψ, so there are plenty of candidates left to serve as $\psi(X)$. Once ψ has been defined, partition each $Y \in \mathcal{T}_{2\alpha+1}$ into two infinite pieces Y' and Y'', subject to the requirement that if $Y = \psi(X)$ for some (unique) X then $Y' \subseteq X$. Then let $\mathcal{T}_{2\alpha+2}$ consist of these sets Y' and Y'' for all $Y \in \mathcal{T}_{2\alpha+1}$.

This completes the construction of $\mathcal{T}$. The first two parts of the theorem are clear, and the third will be clear once we show that every $X \in [\omega]^\omega$ is active at some stage $2\alpha+2$. To this end, we consider a fixed X and we build a binary subtree of $\mathcal{T}$, of height ω, as follows. Its root is the root ω of $\mathcal{T}$. After its nth level has been constructed, consisting of 2^n nodes Z of $\mathcal{T}$, all at the same level of $\mathcal{T}$, say level α_n, and all having infinite intersection with X, we produce the next level as follows. For each node Z of level n in our subtree, $Z \cap X$ is an infinite set and cannot be in all the dense open families $\mathcal{D}_\xi$ as these were chosen to have no common member. Since $\mathcal{T}_{2\xi+1} \subseteq \mathcal{D}_\xi$, $Z \cap X$ must meet at least two sets in $\mathcal{T}_\beta$ for all sufficiently large β. Choose a β that is sufficiently large in this sense for all 2^n Z's; call it α_{n+1}, and let the successors of each Z at this level be two nodes that meet $Z \cap X$ infinitely. Note that these are necessarily $\subseteq^* Z$ (for otherwise they would be almost disjoint from Z), so we are getting a subtree of $\mathcal{T}$. After the subtree has been constructed, use the fact that $\mathfrak{h}$ is uncountable and regular to see that the supremum, say γ, of the α_n's is still $< \mathfrak{h}$, so there are a γth and a $(\gamma+1)$st level of $\mathcal{T}$.

For each path p through our subtree, the nodes along that path, intersected with X, form an almost-decreasing ω-sequence, so there is an infinite $X' \subseteq X$ almost included in all of them (as $\mathfrak{t} > \omega$). That X' has infinite intersection with some node Y in level $\gamma + 1$ of $\mathcal{T}$, because the level is a MAD family. This Y has infinite intersection with each of the nodes Z along the path p, so Y is almost included in each of these Z's (because $\mathcal{T}$ is a tree). Since distinct nodes at the same level are almost disjoint, distinct paths p must lead to distinct nodes Y. So we have $\mathfrak{c}$ nodes Y at level $\gamma + 1$, all meeting X infinitely. Since γ is a limit ordinal, $\gamma = 2\gamma$ and X is active at stage $2\gamma + 2$. ⊣

6.21 Remark. Clause 3 of the theorem implies that forcing with the poset $([\omega]^\omega, \subseteq^*)$ is equivalent to forcing with the tree $(\mathcal{T}, \subseteq^*)$. It is not difficult to modify the construction of the base matrix tree so that each node has $\mathfrak{c}$ immediate successors. Then this forcing clearly adjoins a function from $\mathfrak{h}$ onto $\mathfrak{c}$. Since $\mathfrak{h}$ is not collapsed and no reals are added (because of the distributivity), we find that $\mathfrak{h}$ is the cardinality of the continuum in the forcing extension by $([\omega]^\omega, \subseteq^*)$.

We introduce a cardinal $\mathfrak{p}$, a slight modification of $\mathfrak{t}$, that is often useful because of its connection with forcing; see Theorem 7.12 below. Notice that it makes sense to ask about pseudointersections of families more general than towers. An obvious necessary condition for a family to have a pseudointersection is the strong finite intersection property defined below; $\mathfrak{p}$ measures the extent to which this necessary condition is also sufficient.

6.22 Definition. A family $\mathcal{F}$ of infinite sets has the *strong finite intersection property* if every finite subfamily has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the smallest cardinality of any $\mathcal{F} \subseteq [\omega]^\omega$ with SFIP but with no pseudointersection.

Since a tower is a family with SFIP and no pseudointersection, we immediately get half of the following proposition. The other half, that $\mathfrak{p}$ is uncountable, is proved exactly as for $\mathfrak{t}$ (and is improved in Proposition 6.24 below).

6.23 Proposition. $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t}$.

It is not known whether $\mathfrak{p}$ can be strictly smaller than $\mathfrak{t}$, but the next theorem shows that, for this to happen, $\mathfrak{p}$ would have to be at least $\aleph_2$ and (therefore) $\mathfrak{t}$ at least $\aleph_3$. To prove the theorem, we need a proposition that will be useful again later (in the proofs of Theorems 8.13 and 9.25) and is of some interest in its own right as it can serve as a characterization of $\mathfrak{d}$ (the $\mathfrak{d}$ in the hypothesis is easily seen to be optimal). The theorem and a version of the proposition are in [92]; a form of the proposition closer to the present one is in [66].

6.24 Proposition. *Suppose $\langle C_n : n \in \omega \rangle$ is a decreasing (or almost decreasing) sequence of infinite subsets of ω, and suppose $\mathcal{A}$ is a family of fewer than $\mathfrak{d}$ subsets of ω such that each set in $\mathcal{A}$ has infinite intersection with each C_n. Then $\{C_n : n \in \omega\}$ has a pseudointersection B that has infinite intersection with every set in $\mathcal{A}$.*

Proof. We may assume $\langle C_n : n \in \omega \rangle$ is decreasing, for if it is only almost decreasing then we can replace each C_n with $\bigcap_{k \le n} C_k$ without affecting the other hypotheses or the conclusion, as each new C_n differs only finitely from the old.

For any $h \in {}^\omega\omega$, let $B_h = \bigcup_{n \in \omega} (C_n \cap h(n))$. Each C_n includes all but the first n terms of this union, and these terms are finite, so B_h is a pseudointersection of the C_n's. It remains to choose h so that $A \cap B_h$ is infinite for all $A \in \mathcal{A}$.

For each such A, let $f_A(n)$ denote the nth element of the infinite set $A \cap C_n$. Observe that, if $h(n) > f_A(n)$ for some A and n, then $A \cap B_h$ has cardinality at least n, for it contains the first n elements of $A \cap C_n$. So B_h can serve as the B in the proposition provided $\forall A \in \mathcal{A}\, \exists^\infty n\, (h(n) > f_A(n))$. But there are fewer than $\mathfrak{d}$ functions f_A, so there is an h not dominated by any of them. $\dashv$

6.25 Theorem. *If $\mathfrak{p} = \aleph_1$, then $\mathfrak{t} = \aleph_1$.*

Proof. Since $\mathfrak{t} \le \mathfrak{h} \le \mathfrak{b} \le \mathfrak{d}$, the result is immediate if $\mathfrak{d} = \aleph_1$. So we assume for the rest of the proof that $\mathfrak{d} > \aleph_1$.

By hypothesis we have a family $\mathcal{A} = \{A_\alpha : \alpha < \aleph_1\}$ with SFIP but with no pseudointersection, and we may assume that it is closed under finite intersections. We build a tower $\langle T_\alpha : \alpha < \aleph_1 \rangle$ of length $\aleph_1$ by recursion, ensuring at each stage that T_α has infinite intersection with each $A \in \mathcal{A}$ and that $T_{\alpha+1} \subseteq A_\alpha$. We start with $T_0 = \omega$, and at countable limit stages λ we continue the tower by applying the proposition (with $\langle C_n : n \in \omega \rangle$ being a cofinal subsequence of $\langle T_\alpha : \alpha < \lambda \rangle$). At successor stages we set $T_{\alpha+1} = T_\alpha \cap A_\alpha$. It is easy to verify that this defines an almost decreasing sequence with the claimed properties. It is a tower, because any pseudointersection of the T_α's would also be a pseudointersection of the A_α's. $\dashv$

We close this section by discussing the groupwise density number $\mathfrak{g}$, a close relative of $\mathfrak{h}$. More information about it, including the motivation for its definition, is in Sect. 8.

6.26 Definition. A family $\mathcal{G} \subseteq [\omega]^\omega$ is *groupwise dense* if it is open in the lower topology (i.e., closed under almost subsets) and, for every interval partition Π, some union of (infinitely many) intervals of Π belongs to $\mathcal{G}$. The *groupwise density number* $\mathfrak{g}$ is the smallest number of groupwise dense families with empty intersection.

It is conventional, though perhaps unnatural, to include closure under almost subsets in the definition of "groupwise dense" even though it is not in the definition of "dense". The first part of the following proposition gives a convenient synonym for "groupwise dense", namely "nonmeager open" where "non-meager" refers to the usual topology of $[\omega]^\omega$ (as a subspace of $\mathcal{P}\omega \cong {}^\omega 2$) whereas "open" refers to the lower topology.

6.27 Proposition.

1. *A family $\mathcal{G} \subseteq [\omega]^\omega$ is groupwise dense if and only if it is closed under almost subsets and nonmeager in the standard topology.*
2. *The intersection of any fewer than $\mathfrak{g}$ groupwise dense families is groupwise dense.*
3. *$\mathfrak{g}$ is regular.*
4. *$\mathfrak{h} \leq \mathfrak{g} \leq \mathfrak{d}$.*

Proof. Identify $[\omega]^\omega$ with a cocountable subset of ${}^\omega 2$ via characteristic functions. Let $\mathcal{G} \subseteq [\omega]^\omega$ be closed under almost subsets. By Theorem 5.2, it is nonmeager if and only if it contains enough reals to match each chopped real (x, Π). Thanks to closure under subsets, it suffices to match those chopped reals whose first component x is the identically 1 function. But matching these chopped reals is precisely what the definition of groupwise density requires.

For part 2, suppose $\mathcal{G}_\alpha$ are fewer than $\mathfrak{g}$ groupwise dense families. Their intersection $\mathcal{G}$ is clearly closed under almost subsets, so consider an arbitrary interval partition $\Pi = \{I_n : n \in \omega\}$. We must find an infinite set $X \subseteq \omega$ such that $\bigcup_{n\in X} I_n \in \mathcal{G}$. That is, we must find an X common to the families $\mathcal{H}_\alpha = \{X \in [\omega]^\omega : \bigcup_{n\in X} I_n \in \mathcal{G}_\alpha\}$. Since there are fewer than $\mathfrak{g}$ of these families, it suffices to prove that each of them is groupwise dense. This is a routine verification; to see that $\mathcal{H}_\alpha$ contains a union of intervals from a partition Θ, use that $\mathcal{G}_\alpha$ contains a union of intervals from the partition $\{\bigcup_{n\in J} I_n : J \in \Theta\}$.

The regularity of $\mathfrak{g}$ follows immediately from part 2.

Every groupwise dense family $\mathcal{G}$ is dense. Indeed, given any infinite $X \subseteq \omega$, we can form an interval partition in which each interval contains a member of X. Then $\mathcal{G}$ contains a union of such intervals and therefore, being closed under subsets, contains an infinite subset of X. This observation immediately gives $\mathfrak{h} \leq \mathfrak{g}$.

Finally, to show that $\mathfrak{g} \leq \mathfrak{d}$, let $\mathcal{D}$ be a dominating family of size $\mathfrak{d}$, and associate to each $f \in \mathcal{D}$ the set

$$\mathcal{G}_f = \{X \in [\omega]^\omega : \exists^\infty n\ (X \cap [n, f(n)) = \emptyset)\}.$$

Then $\mathcal{G}_f$ is groupwise dense; given any interval partition Π, we can form an element of $\mathcal{G}_f$ by taking infinitely many intervals from Π, separated by gaps so long that each gap includes $[n, f(n))$ for some n. But there can be no X common to all the $\mathcal{G}_f$, for if there were then the function sending each $n \in \omega$ to the next larger member of X would not be dominated by any $f \in \mathcal{D}$. So we have $\mathfrak{d}$ groupwise dense families with empty intersection. $\dashv$

6.28 Remark. This proposition shows that, in the lattice of subsets of $[\omega]^\omega$ closed downward with respect to $\subseteq^*$, the non-meager sets form a filter, indeed a $<\mathfrak{g}$-complete filter. This may be somewhat surprising, since in the lattice (Boolean algebra) of all subsets of $[\omega]^\omega$, two non-meager sets can be disjoint; in fact there are $\mathfrak{c}$ pairwise disjoint non-meager sets.

6.29 Remark. The characteristics studied in this section, as well as $\mathfrak{s}$ and $\mathfrak{r}$ from Sect. 3 above and $\mathfrak{a}$ from Sect. 8 below, have interesting analogs in structures other than $[\omega]^\omega$. One example is the system of dense subsets of $\mathbb{Q}$, ordered by $\subseteq^*$. Little is known about these characteristics, but Lemma 6.11 says that the tower number in this situation is no smaller than the ordinary tower number.

Another example is the set of partitions of ω into infinitely many pieces, ordered by "coarser than modulo finite". Several characteristics of this sort have been studied by Krawczyk (unpublished).

7. Forcing Axioms

Forcing axioms are combinatorial statements designed to express what is achieved by certain sorts of iterated forcing constructions. They serve to hide such constructions in a "black box"; instead of showing that a statement of interest can be forced by such a construction, one derives it from the combinatorial principle. The oldest and still the most frequently used of these principles is Martin's Axiom, introduced in [74]. To state it, we need some terminology from forcing theory.

7.1 Definition. Let $(P, \leq)$ be a nonempty partially ordered set. Two elements $p, q \in P$ are *compatible* if they have a common lower bound and *incompatible* otherwise. An *antichain* is a set of pairwise incompatible elements. P satisfies the *countable chain condition* (c.c.c.) or *countable antichain condition* if all its antichains are countable. More generally, P satisfies the $<\kappa$-chain condition if all its antichains have cardinalities $< \kappa$.

A subset $D \subseteq P$ is *dense* if every element of P is $\geq$ an element of D. If $\mathcal{D}$ is a family of dense subsets of P, then $G \subseteq P$ is $\mathcal{D}$-*generic* if it is closed upward and directed downward (every two members have a common lower bound) and intersects every $D \in \mathcal{D}$.

7.2 Definition. *Martin's Axiom* (MA) is the statement that, if $\mathcal{D}$ is a family of fewer than $\mathfrak{c}$ dense subsets of a partial order P with c.c.c., then there is a $\mathcal{D}$-generic filter $G \subseteq P$. More generally, if κ is a cardinal and $\mathcal{K}$ is a class of nonempty partial orders, then we write $\mathrm{MA}_\kappa(\mathcal{K})$ for the statement that every family $\mathcal{D}$ of κ dense subsets in a member P of $\mathcal{K}$ admits a $\mathcal{D}$-generic $G \subseteq P$. $\mathrm{MA}_{<\kappa}(\mathcal{K})$ is defined analogously. One omits the subscript when it is "$<\mathfrak{c}$" and one omits the class $\mathcal{K}$ when it is the class of c.c.c. posets.

Thus, MA is $\mathrm{MA}_{<\mathfrak{c}}$(c.c.c.). Some authors write MA_κ to mean what we have called $\mathrm{MA}_{<\kappa}$.

MA describes the model obtained by a finite support forcing iteration, of length some uncountable $\kappa = \kappa^{<\kappa}$, in which all c.c.c. posets (of the extension) of size $< \kappa$ are used as forcing conditions during the iteration. This iteration, which is itself a c.c.c. forcing, produces a model of MA and $\mathfrak{c} = \kappa$. Thus, MA is consistent with the continuum being arbitrarily large. Although only small (smaller than κ) posets were used during the iteration, a reflection argument (essentially the Löwenheim-Skolem Theorem; see the second preliminary simplification in the proof of Theorem 7.12 below) shows that all c.c.c. posets, not only the small ones, acquire generic sets with respect to small families of dense subsets. For details about this, see [106] or [68, Sect. VIII.6].

For orientation, we mention that:

- MA_ω(all posets) is provable in ZFC, and therefore CH implies MA.
- $\mathrm{MA}_{\aleph_1}$(all posets) is refutable. Take P to be ${}^{<\omega}\aleph_1$ ordered by reverse inclusion, take $D_\alpha = \{p \in P : \alpha \in \mathrm{ran}(p)\}$, and observe that a generic G would give a map $\bigcup G$ of ω onto $\aleph_1$.
- $\mathrm{MA}_{\mathfrak{c}}$(Cohen) is refutable, where "Cohen" refers to the single poset ${}^{<\omega}2$ ordered by reverse inclusion. For each $f : \omega \to 2$, let $D_f = \{p : p \not\subseteq f\}$ and observe that a generic G would give a function $\bigcup G : \omega \to 2$ different from every f.

The last two of these observations indicate why MA refers only to c.c.c. posets and only to $< \mathfrak{c}$ dense sets.

The effect of MA on cardinal characteristics of the continuum is to make them large, as the next two theorems and their corollaries show. These results are from [74].

7.3 Theorem. MA *implies* $\mathbf{add}(\mathcal{L}) = \mathfrak{c}$.

Proof. Suppose $\kappa < \mathfrak{c}$ and we are given κ sets $N_\alpha \subseteq \mathbb{R}$ $(\alpha < \kappa)$ of measure zero. We must show, assuming MA, that their union has measure 0. It suffices to find, for each positive ε, a set of measure $\leq \varepsilon$ that includes all the N_α as subsets.

Given ε, let P be the set of open subsets of $\mathbb{R}$ having measure $< \varepsilon$, and order P by reverse inclusion. In order to apply MA to this P, we first verify the c.c.c. Let uncountably many elements p of P be given. Inside each of these open sets, find a finite union $q(p)$ of open intervals with rational endpoints, large enough so that $\mu(p - q(p)) < \varepsilon - \mu(p)$. Notice that this implies $\mu(p - q(p)) < \frac{1}{2}(\varepsilon - \mu(q(p)))$. There are only countably many possibilities for $q(p)$, so two (in fact uncountably many) of the $q(p)$ must be the same q. But then the union of the two corresponding p's has measure $< \varepsilon$ (because it consists of q plus the two remainders $p - q$, and each remainder has measure less than half of $\varepsilon - \mu(q)$), so it is in P and is a common lower bound for those two p's. Thus, an uncountable family of p's cannot be an antichain.

For each of the given N_α's, let $D_\alpha = \{p \in P : N_\alpha \subseteq p\}$, and notice that this is a dense subset of P (because a set of measure zero is included in open sets of arbitrarily small measure). Since $\kappa < \mathfrak{c}$, MA provides a generic G meeting all the D_α. Then $\bigcup G$ includes all the N_α. Furthermore, as a directed union of open sets of measure $< \varepsilon$, this $\bigcup G$ has measure $\leq \varepsilon$. $\dashv$

7.4 Corollary. MA *implies that all the cardinals in Cichoń's diagram are equal to* $\mathfrak{c}$ *and that* $\mathfrak{r} = \mathfrak{c}$.

7.5 Remark. The partial ordering used in the proof of the theorem is called the *amoeba* order. To understand the name, visualize the open sets in three dimensional space instead of $\mathbb{R}$, and visualize the proof of density of D_α as extruding a tentacle[1] from a given open set to engulf N_α.

The proof of c.c.c. for the amoeba actually establishes the stronger property of being σ-linked in the sense of the following definition.

7.6 Definition. In a partial order, a subset S is called *linked* if every two of its elements are compatible. It is *n-linked* if every n of its members have a common lower bound. It is *centered* if every finitely many of its members have a common lower bound. *σ-linked* means the union of countably many linked subsets. *σ-n-linked* and *σ-centered* are defined analogously.

Clearly, σ-centered implies σ-linked, which in turn implies c.c.c. σ-n-linked becomes stronger as n increases, but still remains weaker than σ-centered.

In the proof of Theorem 7.3, we essentially showed that the amoeba is σ-linked, as witnessed by the countably many sets $\{p : q(p) = q\}$, where q ranges over finite unions of rational intervals and where $q(p)$ is defined for all p as it was defined in the proof above for p in the supposed antichain. A similar argument shows that the amoeba is σ-n-linked for all n. But it is not σ-centered.

Instead of working directly with sets of measure zero, one can prove the preceding theorem by using Theorem 5.14, which described $\mathbf{add}(\mathcal{L})$ in terms of slaloms. Given fewer than $\mathfrak{c}$ functions $\omega \to \omega$, one needs a slalom through

[1] It has been pointed out to me that an amoeba has pseudopodia, not tentacles. But it seems easier to visualize tentacles.

which all of them go. This is obtainable by applying MA to a poset P consisting of pieces of slaloms. Specifically, a member p of P is a function on ω assigning to each n a finite set of natural numbers, such that, for some k, $|p(n)|$ is n for $n < k$ and k for $n \geq k$. The ordering is componentwise reverse inclusion, and the relevant dense sets are $\{p : \forall n > k\, (f(n) \in p(n))\}$, where k witnesses that $p \in P$ and where f is one of the given functions that should go through our slalom. This forcing is called localization forcing in [5, Sect. 3.1].

7.7 Theorem. MA *implies* $\mathfrak{p} = \mathfrak{c}$.

Proof. Suppose $\mathcal{F} \subseteq [\omega]^\omega$ has the strong finite intersection property and $|\mathcal{F}| < \mathfrak{c}$. To find a pseudointersection X for $\mathcal{F}$, we apply MA to the following poset P. A member of P is a pair (s, F) where s is a finite subset of ω and F is a finite subset of $\mathcal{F}$. (The "meaning" of (s, F) is that the desired X should include s and should, except for s, be included in each $A \in F$.) The ordering puts $(s', F') \leq (s, F)$ if

$$s \text{ is an initial segment of } s', \quad F' \supseteq F, \quad \text{and} \quad \forall A \in F\ (s' - s \subseteq A).$$

Any two pairs with the same first component are compatible, as one can just take the union of the second components. (In fact, any finitely many pairs with the same first component have a common lower bound. So this ordering is σ-centered.) For each $A \in \mathcal{F}$, the set $D_A = \{(s, F) \in P : A \in F\}$ is dense. So is $D_n = \{(s, F) \in P : |s| > n\}$ because of the SFIP of $\mathcal{F}$. MA provides a generic G meeting all these dense sets. Let $X = \bigcup_{(s,F)\in G} s$. This is infinite because G meets each D_n. To see that it is almost included in each $A \in \mathcal{F}$, use that G and D_A have a common member (s_0, F_0). That means $A \in F_0$, and we shall show that $X - s_0 \subseteq A$. Any member k of $X - s_0$ is in $s - s_0$ for some (s, F) in G, and, as G is directed downward, it contains some $(s', F') \leq$ both (s, F) and (s_0, F_0). Then $k \in s - s_0 \subseteq s' - s_0 \subseteq A$, as required. $\dashv$

7.8 Remark. The forcing used in the preceding proof is called *Mathias forcing with respect to* $\mathcal{F}$. One can equivalently view it as consisting of pairs (s, A) where A is the intersection of finitely many sets from $\mathcal{F}$; in this form, the ordering $(s', A') \leq (s, A)$ is defined by

$$s \text{ is an initial segment of } s', \quad A' \subseteq A, \quad \text{and} \quad s' - s \subseteq A.$$

Mathias forcing (without respect to any $\mathcal{F}$) means the similarly defined poset where the second components A can be arbitrary infinite subsets of ω. In contrast to Mathias forcing with respect to an $\mathcal{F}$ with SFIP, this Mathias forcing does not satisfy the c.c.c. It can be viewed as a two-step forcing iteration, where the first step is forcing with $([\omega]^\omega, \subseteq^*)$, which adjoins a generic ultrafilter $\mathcal{U}$ on ω, and the second step is Mathias forcing with respect to $\mathcal{U}$.

7.9 Corollary. MA *implies* $\mathfrak{p} = \mathfrak{t} = \mathfrak{h} = \mathfrak{g} = \mathfrak{s} = \mathfrak{c}$.

Thus, all the characteristics we have discussed are equal to $\mathfrak{c}$ if MA holds. The proofs actually show a bit more, if we introduce new characteristics related directly to MA.

7.10 Definition. For any class $\mathcal{K}$ of posets, $\mathfrak{m}(\mathcal{K})$ is the smallest κ for which $\mathrm{MA}_\kappa(\mathcal{K})$ is false. If $\mathcal{K}$ is the class of c.c.c. posets, we omit mention of it and write simply $\mathfrak{m}$.

Thus MA is the statement $\mathfrak{m} = \mathfrak{c}$. Clearly,

$$\mathfrak{m} \le \mathfrak{m}(\sigma\text{-linked}) \le \mathfrak{m}(\sigma\text{-3-linked}) \le \cdots \le \mathfrak{m}(\sigma\text{-centered}) \le \mathfrak{m}(\text{Cohen}).$$

See [69] for a model where $\mathfrak{m} < \mathfrak{m}(\sigma\text{-linked})$; similar techniques can be used to get strict inequalities between other such variants of $\mathfrak{m}$.

The proofs of the last two theorems and our remarks about the σ-linked and σ-centered properties of the posets in the proofs establish the following inequalities.

7.11 Corollary. $\mathfrak{m}(\sigma\text{-linked}) \le \mathbf{add}(\mathcal{L})$ *and* $\mathfrak{m}(\sigma\text{-centered}) \le \mathfrak{p}$.

Of course, one could be even more specific about the posets used; for example the proof above shows that $\mathfrak{m}(\text{amoeba}) \le \mathbf{add}(\mathcal{L})$. In fact, equality holds here; see [5, Theorem 3.4.17].

The second half of the last corollary can also be improved to an equality, Bell's Theorem [12].

7.12 Theorem. $\mathfrak{m}(\sigma\text{-centered}) = \mathfrak{p}$.

Proof. In view of Corollary 7.11, it suffices to consider an arbitrary σ-centered poset P, say the union of centered parts C_n, and to find a $\mathcal{D}$-generic G for a prescribed family $\mathcal{D}$ of fewer than $\mathfrak{p}$ dense subsets of P. It is convenient to begin with several simplifications of the problem.

First, we may assume that each $D \in \mathcal{D}$ is closed downward, because closing the dense sets will not affect $\mathcal{D}$-genericity.

Second, we may assume that $|P| < \mathfrak{p}$. Indeed, suppose the theorem were proved in this case, and suppose we are given the situation above with $|P| \ge \mathfrak{p}$. By the Löwenheim-Skolem Theorem, the structure $(P, \le, C_n, D)_{n\in\omega, D\in\mathcal{D}}$ has an elementary substructure P' of cardinality $< \mathfrak{p}$. Then P' is σ-centered and $\mathcal{D}' = \{P' \cap D : D \in \mathcal{D}\}$ is a family of $< \mathfrak{p}$ dense subsets, so there is a $\mathcal{D}'$-generic $G' \subseteq P'$. The upward closure of G' in P is then $\mathcal{D}$-generic, as required.

Third, instead of producing a $\mathcal{D}$-generic G, it suffices to produce a linked L meeting every $D \in \mathcal{D}$. Indeed, suppose we could always do this. Then, given P and $\mathcal{D}$ as above, we enlarge $\mathcal{D}$ by adjoining the sets

$$D_{p,q} = \{r \in P : r \text{ is incompatible with } p \text{ or with } q, \text{ or } r \le p, q\},$$

which are easily seen to be dense. If L is linked and meets all the sets in $\mathcal{D}$ and all the $D_{p,q}$, then the upward closure G of L is $\mathcal{D}$-generic. The only thing to check is that it is directed downward. To find a common lower bound for any $p, q \in G$, we may, by lowering both, assume that p and q are in L. Let $r \in L \cap D_{p,q}$. Then r cannot be incompatible with p or with q as L is linked; so $r \leq p, q$, as required.

Fourth, we may assume that, for each $n \in \omega$, there is some $D_n \in \mathcal{D}$ disjoint from C_n. Otherwise, C_n could serve as the required L.

Fifth, we may assume that $\mathcal{D}$ is closed under finite intersections. Closing it in this way does no harm, because the cardinality $|\mathcal{D}|$ will not be increased (unless it was finite—a trivial case) and the intersection of any finitely many dense, downward-closed sets is again dense and downward closed.

After all these simplifications, we begin the real work of the proof. For each $p \in P$ and each $D \in \mathcal{D}$, let $A(p, D)$ be the set of those $n \in \omega$ such that some member of $C_n \cap D$ is $\leq p$.

I claim that, for each $k \in \omega$, the family $\mathcal{F}_k = \{A(p, D) : p \in C_k \text{ and } D \in \mathcal{D}\}$ has the strong finite intersection property. By our fourth simplification, it suffices to show that each finite subfamily $\mathcal{F}_k^0$ of $\mathcal{F}_k$ has nonempty intersection, for we could always include in $\mathcal{F}_k^0$ sets of the form $A(p, D_n)$ for any finitely many of the D_n and so keep any finitely many n's out of the intersection. By our fifth simplification, we may assume that the sets in $\mathcal{F}_k^0$ are $A(p_i, D)$ for various $p_i \in C_k$ but just one $D \in \mathcal{D}$, for different D's could be replaced with their intersection. As C_k is centered, the p_i's have a lower bound p, and below that we can find a member q of the dense set D. If $q \in C_n$ then $n \in \bigcap \mathcal{F}_k^0$. This completes the verification of the claim.

Since $|\mathcal{F}_k| \leq |P| \cdot |\mathcal{D}| < \mathfrak{p}$ by our second simplification, $\mathcal{F}_k$ has a pseudointersection A_k.

Next, we define several labellings of the ω-branching tree ${}^{<\omega}\omega$ of height ω, a primary labeling by natural numbers and, for each $D \in \mathcal{D}$, a secondary labeling by members of P. In the primary labeling, the label of the root is (arbitrarily chosen as) 0, and if a node has label k then the labels of its immediate successors are the numbers in A_k (once each). The secondary labeling associated to a particular $D \in \mathcal{D}$ is defined as follows. The secondary label of the root is an arbitrary element of C_0. If a node has been given secondary label p and if an immediate successor of it has primary label n, then the secondary label of that successor is to be an element of $C_n \cap D$ that is $\leq p$ in P, provided such an element exists, i.e., provided $n \in A(p, D)$—in this case we call that successor node "good" for D. If no such element exists, then the secondary label of that successor node is chosen arbitrarily from C_n and the node is called "bad" for D. Notice that, whether a node is good or bad, its secondary label is always in C_n where n is its primary label.

Because A_k is a pseudointersection for $\mathcal{F}_k$, all but finitely many of the immediate successors of any node are good for any particular $D \in \mathcal{D}$. For each node s and each $D \in \mathcal{D}$, let $f_D(s)$ be a number so large that all the nodes $s^\frown\langle m\rangle$ for $m \geq f_D(s)$ are good for D. Since $|\mathcal{D}| < \mathfrak{p} \leq \mathfrak{b}$ (and since the tree has only countably many nodes), there is $g : {}^{<\omega}\omega \to \omega$ that is $>^*$ all the f_D.

Using g, we define a path X through the tree ${}^{<\omega}\omega$ by starting at the root and, after reaching a node s, proceeding to $s^\frown\langle g(s)\rangle$. Our choice of g ensures that, for each $D \in \mathcal{D}$, all but finitely many nodes along the path X are good for D. Choose, for each D, a node s_D on X such that it and all later nodes on X are good for D, and let p_D be its secondary label associated to D. Thus $p_D \in D$. This guarantees that $L = \{p_D : D \in \mathcal{D}\}$ meets every $D \in \mathcal{D}$.

To complete the proof, we verify that L is linked. Consider any two elements $p_D, p_{D'} \in L$. If $s_D = s_{D'}$ then both of $p_D, p_{D'}$ are in the same C_n, where n is the primary label of s_D, so they are compatible because C_n is centered. Suppose therefore that s_D occurs before $s_{D'}$ along the path X. By choice of s_D, all the nodes along the path X from s_D to $s_{D'}$ are good for D, so the secondary labeling associated to D puts at the node $s_{D'}$ a label q that is $\leq p_D$. But, being in the same C_n, q and $p_{D'}$ are compatible. Therefore so are p_D and $p_{D'}$. ⊣

There is a similar (but easier) result about countable partial orders.

7.13 Theorem. $\mathfrak{m}(\text{Cohen}) = \mathfrak{m}(\text{countable}) = \mathbf{cov}(\mathcal{B})$.

Proof. Since Cohen forcing is a countable poset, MA_κ(countable) implies MA_κ(Cohen). We shall complete the proof by showing that MA_κ(Cohen) implies $\kappa < \mathbf{cov}(\mathcal{B})$ and that this in turn implies MA_κ(countable).

Assume MA_κ(Cohen) and let κ nowhere dense subsets X_α of ${}^\omega 2$ be given. We must show that the X_α do not cover ${}^\omega 2$. For each α, let D_α be the set of those $s \in {}^{<\omega}2$ that have no extensions in X_α. Because X_α is nowhere dense (in the topological sense), D_α is dense (in the partial order sense). So MA_κ(Cohen) gives us a generic $G \subseteq {}^\omega 2$ meeting every D_α. Then $\bigcup G \in {}^\omega 2$ is in none of the X_α.

Finally, assume $\kappa < \mathbf{cov}(\mathcal{B})$, and let κ dense subsets D_α of a countable poset P be given. Let $T : {}^\omega P \to {}^\omega P$ be the transformation that turns any sequence $x \in {}^\omega P$ into a (weakly) decreasing sequence $T(x)$ in a greedy way; that is, $T(x)(0) = x(0)$, and $T(x)(n+1) = x(n+1)$ if this is $\leq T(x)(n)$ in P, and otherwise $T(x)(n+1) = T(x)(n)$. We similarly define T on finite sequences instead of infinite ones.

The sets $U_\alpha = \{x \in {}^\omega P : \exists n\,(T(x)(n) \in D_\alpha)\}$ are dense open subsets of ${}^\omega P$ (where P has the discrete topology and ${}^\omega P$ has the product topology so it is homeomorphic to ${}^\omega\omega$). To verify density, consider any nonempty $s \in {}^{<\omega}P$, let p be the last term of $T(s)$, and let $q \leq p$ be in D_α. Then every extension of $s^\frown\langle q\rangle$ is in U_α.

As $\kappa < \mathbf{cov}(\mathcal{B})$, there is an x in the intersection of all the U_α. Then the range of the decreasing sequence $T(x)$ meets every D_α, and the upward closure of this range is therefore the desired generic set. ⊣

As an application of Bell's Theorem 7.12, we give an analog of Proposition 6.24, weakening the hypothesis of countability (of the list of C's) and strengthening the hypothesis of cardinality $< \mathfrak{d}$ (for $\mathcal{A}$) by replacing both with the hypothesis of cardinality $< \mathfrak{p}$.

7.14 Theorem. *Suppose $\mathcal{C}$ and $\mathcal{A}$ are families of $< \mathfrak{p}$ subsets of ω, and suppose every intersection of finitely many sets from $\mathcal{C}$ and one set from $\mathcal{A}$ is infinite. Then $\mathcal{C}$ has a pseudointersection B that has infinite intersection with each set in $\mathcal{A}$.*

Proof. Let P be Mathias forcing with respect to $\mathcal{C}$, as defined in the proof of Theorem 7.7 and the remark following it. As shown there, this is σ-centered, and for each $C \in \mathcal{C}$ the set $D_C = \{(s, F) \in P : C \in F\}$ is dense. Furthermore, for each $A \in \mathcal{A}$ and each $n \in \omega$, the set $D_{A,n} = \{(s, F) \in P : |s \cap A| > n\}$ is dense because each intersection of finitely many sets from $\mathcal{C}$ and one set from $\mathcal{A}$ is infinite.

As both $|\mathcal{C}|$ and $|\mathcal{A}|$ are $< \mathfrak{p} = \mathfrak{m}(\sigma\text{-centered})$, P has a generic subset G meeting all these D_C and $D_{A,n}$. As in the proof of Theorem 7.7, we define $B = \bigcup_{(s,F)\in G} s$ and we find that this is a pseudointersection of $\mathcal{C}$. Furthermore, it has infinite intersection with each $A \in \mathcal{A}$ because G meets each $D_{A,n}$. ⊣

As a consequence, we obtain that $\mathfrak{p}$, like its relatives $\mathfrak{t}$, $\mathfrak{h}$, and $\mathfrak{g}$, is regular, but the proof is trickier than for the relatives. This proof is taken from [48, Sect. 21], where it is attributed to Szymański.

7.15 Theorem. $\mathfrak{p}$ *is regular.*

Proof. Suppose $\mathfrak{p}$ were singular with cofinality $\lambda < \mathfrak{p}$. Let $\mathcal{A}$ be a family of $\mathfrak{p}$ subsets of ω having the strong finite intersection property but having no pseudointersection. Express $\mathcal{A}$ as the union of an increasing λ-sequence of subfamilies $\mathcal{A}_\alpha$, each of cardinality $< \mathfrak{p}$. To simplify later considerations, we assume without loss of generality that $\mathcal{A}$ and all the $\mathcal{A}_\alpha$ are closed under finite intersections.

In this situation, we have the following improvement of Theorem 7.14. If $\mathcal{C}$ is any family of fewer than $\mathfrak{p}$ sets such that every intersection of finitely many sets from $\mathcal{C}$ and one set from $\mathcal{A}$ is infinite, then $\mathcal{C}$ has a pseudointersection B whose intersection with each set from $\mathcal{A}$ is infinite. (The improvement is that $\mathcal{A}$ has size $\mathfrak{p}$ rather than strictly smaller size.) To prove this, note first that each $\mathcal{C} \cup \mathcal{A}_\alpha$ has the SFIP and has size $< \mathfrak{p}$, so it has a pseudointersection Z_α. Then apply Theorem 7.14 with $\{Z_\alpha : \alpha < \lambda\}$ in the role of $\mathcal{A}$.

We intend to build an almost decreasing $\lambda + 1$-sequence $\langle C_\alpha : \alpha \le \lambda \rangle$ such that each C_α for $\alpha < \lambda$ is a pseudointersection of $\mathcal{A}_\alpha$. If we can do this then, because the C_α are almost decreasing and the $\mathcal{A}_\alpha$ are increasing and cover $\mathcal{A}$, C_λ will be a pseudointersection of $\mathcal{A}$, a contradiction.

We define the C_α by recursion. To make the recursion work, we carry along the additional requirement that each C_α must have infinite intersection with every member of $\mathcal{A}$.

Suppose $\alpha \le \lambda$ and C_β is already defined for all $\beta < \alpha$ in such a way that our requirements are satisfied. We need to define C_α so that it is $\subseteq^*$ each previous C_β, it is $\subseteq^*$ each member of $\mathcal{A}_\alpha$, and it has infinite intersection with every member of $\mathcal{A}$. Such a set is produced by applying the improvement above of Theorem 7.14 with $\mathcal{C} = \{C_\beta : \beta < \alpha\} \cup \mathcal{A}_\alpha$, provided the hypothesis of that improvement is satisfied. So we need only check that every intersection X of finitely many C_β's ($\beta < \alpha$) and finitely many members of $\mathcal{A}_\alpha$ and one member of $\mathcal{A}$ is infinite. Since the C_β's are almost decreasing, since $\mathcal{A}_\alpha \subseteq \mathcal{A}$, and since $\mathcal{A}$ is closed under finite intersection, such an X almost includes $C_\beta \cap A$ for some $\beta < \alpha$ and some $A \in \mathcal{A}$. The induction hypothesis guarantees that $C_\beta \cap A$ and therefore X are infinite. ⊣

7.16 Remark. This section has dealt almost exclusively with forcing axioms for the class of c.c.c. posets and subclasses, because these are the forcing axioms most relevant to cardinal characteristics. To avoid giving a completely unbalanced picture, however, we should at least mention that numerous other forcing axioms have been considered. The most popular of these is the *Proper Forcing Axiom* PFA, which is $\mathrm{MA}_{\aleph_1}$(proper). Proper forcing was defined by Shelah [97, Chap. III], who showed that it permits countable support iterations without collapsing $\aleph_1$; see Abraham's chapter in this Handbook. PFA summarizes the result of a countable support iteration of all small proper posets. Unlike the construction of a model for MA, where the improvement from small c.c.c. posets to all c.c.c. posets was handled by a Löwenheim-Skolem argument, the construction of a model for PFA uses a supercompact cardinal in the ground model to get the necessary reflection property for the corresponding improvement from small to all.

8. Almost Disjoint and Independent Families

This section is devoted to families of subsets of ω with various special properties, and particularly to those families that are maximal with respect to these properties.

Recall from Sect. 6 that an almost disjoint family is a family of infinite sets whose pairwise intersections are finite, and that the phrase "maximal almost disjoint (MAD) family" refers to an *infinite* family of subsets of ω maximal with respect to almost disjointness.

Although a set of size κ clearly cannot support a family of more than κ disjoint sets, the situation for almost disjoint sets is quite different.

8.1 Proposition. *On any countably infinite set, there is a family of $\mathfrak{c}$ almost disjoint subsets.*

Proof. It clearly does not matter which countably infinite set we consider. Choosing the binary tree ${}^{<\omega}2$ as the ambient set, we can use its $\mathfrak{c}$ branches as the almost disjoint family. ⊣

8.2 Remark. There are at least two other similar and equally easy proofs of this proposition. One uses the set of rationals as the ambient set and assigns to every real r a sequence of distinct rationals converging to r; sequences with different limits are clearly almost disjoint. Another uses $\omega \times \omega$ as the ambient set and assigns to each positive real r the set $\{(n, \lfloor rn \rfloor) : n \in \omega\}$.

The proposition and Zorn's lemma imply the existence of a MAD family of cardinality $\mathfrak{c}$, but there may also be smaller MAD families. For example, it is shown in [68, Theorem VIII.2.3] that if one adds any number of Cohen reals to a model of CH, then the resulting model has a MAD family of size $\aleph_1$; see also Sect. 11. Hechler [55] gives a model with MAD families of many different cardinalities.

8.3 Definition. The *almost disjointness number* $\mathfrak{a}$ is the smallest cardinality of any MAD family.

8.4 Proposition. $\mathfrak{b} \leq \mathfrak{a}$.

Proof. Let $\mathcal{A}$ be a MAD family of size $\mathfrak{a}$, let C_n $(n \in \omega)$ be any countably many members of it, and let $\mathcal{A}'$ be the rest of $\mathcal{A}$. By making finite changes to each C_n, we can arrange that these sets are really disjoint, not just almost disjoint, and that they partition ω. By a suitable bijection, identify ω with $\omega \times \omega$ in such a way that C_n is the column $\{n\} \times \omega$. Each $A \in \mathcal{A}'$ has only finitely many elements per column, so we can define $f_A : \omega \to \omega$ to be the function whose graph is the upper boundary of A. If there were a function $g : \omega \to \omega$ that is $>^*$ all the f_A, then its graph would be almost disjoint from all $A \in \mathcal{A}'$ and all C_n, contrary to maximality of $\mathcal{A}$. So the f_A's constitute an unbounded family of size $\mathfrak{a}$. ⊣

Shelah [99] showed that $\mathfrak{b} < \mathfrak{a}$ is consistent. He also showed there that, if we define $\mathfrak{a}_s$ like $\mathfrak{a}$ except that we use $\omega \times \omega$ as the ambient set and require the MAD family to consist of graphs of partial functions, then $\mathfrak{a} < \mathfrak{a}_s$ is consistent. Brendle has pointed out the following alternative proof of the consistency of $\mathfrak{a} < \mathfrak{a}_s$. By part 2 of Theorem 5.9, we have $\mathbf{non}(\mathcal{B}) \leq \mathfrak{a}_s$. We shall see in Sect. 11 that the random real model (obtained by forcing with a large measure algebra over a model of GCH) has $\mathbf{non}(\mathcal{B}) = \mathfrak{c} > \aleph_1$ and $\mathfrak{a} = \aleph_1$. Therefore it has $\mathfrak{a} < \mathfrak{a}_s$.

Little else is known about connections between $\mathfrak{a}$ and other cardinal characteristics, but Shelah has shown in [100] that $\mathfrak{a} > \mathfrak{d}$ is consistent.

8.5 Remark. Proposition 8.1 can be used to evaluate the "dual" of $\mathfrak{h}$. Unlike the definitions of $\mathfrak{t}$ and $\mathfrak{p}$, that of $\mathfrak{h}$ fits the "norm of relation" format discussed in Sect. 4. Indeed, $\mathfrak{h} = \|([\omega]^\omega, \mathrm{DO}, \notin)\|$ where DO is the family of dense open subsets of $[\omega]^\omega$. (There is an important difference between this relation and those associated to cardinal characteristics in Sects. 4 and 5. The elements of DO cannot be coded by reals, nor does DO possess a nice base whose elements can be coded by reals.) It is natural to ask about the norm of the dual relation, i.e., the minimum size of a family $\mathcal{X} \subseteq [\omega]^\omega$ such that every dense open family $\mathcal{D}$ intersects $\mathcal{X}$. It follows from Proposition 8.1 that this cardinal is $\mathfrak{c}$. In fact, the same also holds for the dual of $\mathfrak{g}$, by nearly the same proof.

8.6 Theorem. *The minimum number of sets in $[\omega]^\omega$ meeting every dense open family, or even every groupwise dense family, is $\mathfrak{c}$.*

Proof. Suppose $\mathcal{X} \subseteq [\omega]^\omega$ has cardinality $< \mathfrak{c}$; we shall find a groupwise dense (hence dense open) $\mathcal{D} \subseteq [\omega]^\omega$ disjoint from $\mathcal{X}$. Let $\mathcal{D} = \{Y \in [\omega]^\omega : \forall X \in \mathcal{X}\ (X \not\subseteq^* Y)\}$. This $\mathcal{D}$ is clearly disjoint from $\mathcal{X}$ and closed under almost subsets, so we need only check that, for any interval partition $\{I_n : n \in \omega\}$, the union of some infinitely many of its intervals is in $\mathcal{D}$. Let $\mathcal{A}$ be a family of $\mathfrak{c}$ almost disjoint subsets of ω, and for each $A \in \mathcal{A}$ let $A' = \bigcup_{n \in A} I_n$. Then the A' are also almost disjoint, so no two of them can almost include the same $X \in \mathcal{X}$. Since there are more A''s than X's, some A' must not almost include any X, i.e., some A' must be in $\mathcal{D}$. ⊣

8.7 Corollary. $\mathrm{cf}(\mathfrak{c}) \geq \mathfrak{g}$.

Proof. Let $[\omega]^\omega = \bigcup_{\alpha < \mathrm{cf}(\mathfrak{c})} \mathcal{X}_\alpha$, where each $|\mathcal{X}_\alpha| < \mathfrak{c}$. By the theorem, there are groupwise dense families $\mathcal{D}_\alpha$ each disjoint from the corresponding $\mathcal{X}_\alpha$. No set can belong to all the $\mathcal{D}_\alpha$, for it would then belong to no $\mathcal{X}_\alpha$. So we have $\mathrm{cf}(\mathfrak{c})$ groupwise dense families with empty intersection. ⊣

Notice that this corollary subsumes Corollary 6.15. The intermediate result that $\mathrm{cf}(\mathfrak{c}) \geq \mathfrak{h}$ was proved in [2]. Among the familiar cardinal characteristics of the continuum, $\mathfrak{g}$ is the largest one known (to me) to be a lower bound for $\mathrm{cf}(\mathfrak{c})$. In particular, it is consistent that $\mathfrak{b} > \mathrm{cf}(\mathfrak{c})$ and it is consistent that $\mathfrak{s} > \mathrm{cf}(\mathfrak{c})$. For the former, start with a model satisfying MA and $\mathfrak{c} = \aleph_2$ and GCH at all larger cardinals, and adjoin $\aleph_{\aleph_1}$ random reals. Then $\mathfrak{c} = \aleph_{\aleph_1}$ and $\mathfrak{b}$, unaffected by the random reals, is $\aleph_2 > \mathrm{cf}(\mathfrak{c})$. For the latter, start with a model of $\mathfrak{c} = \aleph_{\aleph_1}$, and do an $\aleph_2$-stage, finite support iteration of Mathias forcing with respect to (arbitrarily chosen) ultrafilters. The finite support iteration automatically adds Cohen reals at limit stages of cofinality ω and choosing one of them at each stage provides a splitting family of size $\aleph_2$. There is no smaller splitting family, because any $\aleph_1$ reals lie in an intermediate extension and fail to split the subsequently added Mathias reals.

8.8 Definition. A family $\mathcal{I}$ of subsets of ω is *independent* if the intersection of any finitely many members of $\mathcal{I}$ and the complements of any finitely many other members of $\mathcal{I}$ is infinite.

The "infinite" at the end of the definition could be equivalently replaced with "nonempty" if we assumed that $\mathcal{I}$ is infinite.

8.9 Proposition. *There is an independent family of cardinality $\mathfrak{c}$.*

Proof. Let C be the set of finite subsets of $\mathbb{Q}$. Since C is countably infinite, it suffices to find $\mathfrak{c}$ independent subsets of C. For each real r, let

$$E_r = \{F \in C : |F \cap (-\infty, r)| \text{ is even}\}.$$

To see that these sets E_r are independent, let any finitely many distinct reals $r_1, \dots, r_k, s_1, \dots, s_l$ be given. We must find an $F \in C$ that belongs to all the E_{r_i} and none of the E_{s_j}. But this is easy; F consists of 0 or 1 rationals from each of the $(k+l+1)$ intervals into which the r's and s's partition $\mathbb{R}$, the choice of 0 or 1 being made so as to get the right parities. ⊣

8.10 Remark. The preceding proposition is due to Fichtenholz and Kantorovich [46]. It was generalized by Hausdorff [54] who showed that any infinite cardinal κ has 2^κ independent subsets.

Hausdorff's construction (for $\kappa = \aleph_0$) uses the countable set C of pairs (a, B) where a ranges over finite subsets of ω and B ranges over subsets of $\mathcal{P}(a)$. To each $X \subseteq \omega$ associate the subset $\{(a, B) \in C : a \cap X \in B\}$ of C. It is easy to verify that all these subsets are independent.

The corresponding generalization of Proposition 8.1 fails. Baumgartner [8, Theorem 5.6] showed that $\aleph_1$ need not have $2^{\aleph_1}$ uncountable subsets with pairwise countable intersections.

The proposition and Zorn's lemma provide a maximal independent family of size $\mathfrak{c}$, but there may be smaller maximal independent families.

8.11 Definition. The *independence number* $\mathfrak{i}$ is the smallest cardinality of any maximal independent family of subsets of ω.

No upper bounds (except for the trivial $\mathfrak{c}$) are known for $\mathfrak{i}$, but there are two lower bounds.

8.12 Proposition. $\mathfrak{r} \leq \mathfrak{i}$.

Proof. Let $\mathcal{I}$ be a maximal independent family, and let $\mathcal{R}$ consist of all the sets obtainable by intersecting finitely many sets from $\mathcal{I}$ and the complements of finitely many others. The definition of independence ensures that $\mathcal{R} \subseteq [\omega]^\omega$, and $\mathcal{R}$ must be unsplittable because if X were to split all its members then $\mathcal{I} \cup \{X\}$ would be independent, contrary to the maximality of $\mathcal{I}$. So $|\mathcal{R}| \geq \mathfrak{r}$, from which it follows that $|\mathcal{I}| \geq \mathfrak{r}$. ⊣

The following more difficult estimate of $\mathfrak{i}$ is due to Shelah [111, Appendix by Shelah]. The proof we give, a simplification of Shelah's, is from [20]; the simplification was found independently by Bill Weiss.

8.13 Theorem. $\mathfrak{d} \leq \mathfrak{i}$.

Proof. Suppose $\mathcal{I}$ is an independent family of cardinality $< \mathfrak{d}$; we shall show that it is not maximal. Throughout the proof, we let $\mathcal{X}$ and $\mathcal{Y}$ stand for finite, disjoint subfamilies of $\mathcal{I}$; thus, the independence of $\mathcal{I}$ means that $\bigcap \mathcal{X} - \bigcup \mathcal{Y}$ is always infinite, and our goal is to find Z such that each $\bigcap \mathcal{X} - \bigcup \mathcal{Y}$ meets both Z and $\omega - Z$ in an infinite set.

Select any countably many sets $D_n \in \mathcal{I}$, and let $\mathcal{I}'$ be the rest of $\mathcal{I}$. Write D_n^0 for D_n and write D_n^1 for $\omega - D_n$. For each $x : \omega \to 2$, apply Proposition 6.24 with

$$C_n = \bigcap\nolimits_{k<n} D_k^{x(k)}$$

and

$$\mathcal{A} = \left\{\bigcap\mathcal{X} - \bigcup\mathcal{Y} : \mathcal{X}, \mathcal{Y} \text{ finite disjoint subfamilies of } \mathcal{I}'\right\}.$$

Independence of $\mathcal{I}$ gives the hypotheses of the proposition. So we get $B_x \subseteq \omega$ with:

1. $B_x \subseteq^* \bigcap_{k<n} D_k^{x(k)}$ for all n.
2. B_x has infinite intersection with each $\bigcap\mathcal{X} - \bigcup\mathcal{Y} \in \mathcal{A}$.

It follows from 1 that the B_x's for distinct x are almost disjoint.

Fix two disjoint, countable, dense (in the usual topology) subsets Q and Q' of ${}^\omega 2$. Removing finitely many elements from B_x for each $x \in Q \cup Q'$, we can arrange that these countably many B_x's are really disjoint, not just almost disjoint. Set

$$Z = \bigcup_{x\in Q} B_x \quad \text{and} \quad Z' = \bigcup_{x\in Q'} B_x.$$

So Z and Z' are disjoint. We shall show that Z has infinite intersection with every $\bigcap\mathcal{X} - \bigcup\mathcal{Y}$; the same argument applies to Z', so $\omega - Z$ will also have infinite intersection with every $\bigcap\mathcal{X} - \bigcup\mathcal{Y}$, and so the proof will be complete.

Let finite, disjoint $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{I}$ be given, and let $\mathcal{X}'$ and $\mathcal{Y}'$ be their intersections with $\mathcal{I}'$. Fix n so large that, if D_k is in $\mathcal{X}$ or $\mathcal{Y}$ then $k < n$. Using the density of Q, fix $x \in Q$ such that if D_k is in $\mathcal{X}$ or $\mathcal{Y}$ then $x(k)$ is 0 or 1, respectively. Thus,

$$\begin{aligned}
\bigcap\mathcal{X} - \bigcup\mathcal{Y} &= \left(\bigcap\mathcal{X}' - \bigcup\mathcal{Y}'\right) \cap \bigcap_{k:D_k\in\mathcal{X}\cup\mathcal{Y}} D_k^{x(k)} \\
&\supseteq \left(\bigcap\mathcal{X}' - \bigcup\mathcal{Y}'\right) \cap \bigcap_{k<n} D_k^{x(k)} \\
&\supseteq^* \left(\bigcap\mathcal{X}' - \bigcup\mathcal{Y}'\right) \cap B_x.
\end{aligned}$$

The last intersection here is infinite, by property 2 of B_x. It is included in Z because $x \in Q$. So we have an infinite set almost included in $Z \cap (\bigcap\mathcal{X} - \bigcup\mathcal{Y})$, and the proof is complete. ⊣

9. Filters and Ultrafilters

This section is devoted to filters and ultrafilters on ω. We begin by summarizing the terminology we use. Note that we require all filters to contain the cofinite sets, so all our ultrafilters are non-trivial.

9.1 Definition. A *filter* (on ω) is a family $\mathcal{F} \subseteq \mathcal{P}\omega$ that contains all cofinite sets but not the empty set, is closed under supersets, and is closed under finite intersections. An *ultrafilter* (on ω) is a filter with the additional property that, for any $X \subseteq \omega$, either X or its complement belongs to $\mathcal{F}$. A *base* for a filter $\mathcal{F}$ is a subfamily of $\mathcal{F}$ containing subsets of all the sets in $\mathcal{F}$.

We occasionally stretch the meaning of "base" of $\mathcal{F}$ to apply to a family $\mathcal{B}$ such that for every $F \in \mathcal{F}$ there is $B \in \mathcal{B}$ with $B \subseteq^* F$ (rather than $B \subseteq F$). This stretching will make no essential difference but will simplify a few statements.

We shall need the following well-known consequences of the definition. A subset $\mathcal{X}$ of $\mathcal{P}\omega$ is included in a filter if and only if it has the strong finite intersection property, and then the smallest filter including $\mathcal{X}$ consists of the almost supersets of intersections of finite subfamilies of $\mathcal{X}$. We say that $\mathcal{X}$ generates this filter.

An ultrafilter is the same thing as a maximal filter; thus by Zorn's lemma every family with SFIP is included in an ultrafilter. Since an ultrafilter contains a set $X \subseteq \omega$ if and only if it does not contain $\omega - X$, it follows that a family $\mathcal{Y} \subseteq \mathcal{P}\omega$ is disjoint from some ultrafilter if and only if no finitely many members of $\mathcal{Y}$ almost cover ω.

9.2 Definition. Let $\mathcal{F}$ be a subset of $\mathcal{P}\omega$ (usually a filter, but the definition makes sense in general) and let $f : \omega \to \omega$. Then $f(\mathcal{F})$ is defined to be $\{X \subseteq \omega : f^{-1}(X) \in \mathcal{F}\}$.

If $\mathcal{F}$ is a filter or an ultrafilter, then so is $f(\mathcal{F})$ provided it contains all cofinite sets. This proviso is automatically satisfied if f is finite-to-one, which will usually be the case in what follows.

9.3 Definition. A filter $\mathcal{F}$ is *feeble* if, for some finite-to-one $f : \omega \to \omega$, $f(\mathcal{F})$ consists of only the cofinite sets.

The cofinite sets constitute the smallest filter, so feeble filters should also be thought of as small. They are at the opposite extreme from ultrafilters.

9.4 Proposition. *The following are equivalent for any filter $\mathcal{F}$ on ω.*

1. *$\mathcal{F}$ is feeble.*
2. *There is a partition of ω into finite sets such that every set in $\mathcal{F}$ intersects all but finitely many pieces of the partition.*
3. *There is an interval partition as in 2 above.*
4. *$\{\omega - X : X \in \mathcal{F}\}$ is not groupwise dense.*
5. *$\mathcal{F}$ is meager (in the usual topology on $\mathcal{P}\omega \cong {}^{\omega}2$).*

Proof. The equivalence of 1 and 2 is immediate if one views the pieces of a partition (as in 2) as the sets on which a finite-to-one function (as in 1) is constant. If a partition Π is as in 2, then we can find an interval partition Θ, each of whose intervals includes at least one piece of Π; then Θ works in 3. The equivalence of 3 and 4 is just the definition of groupwise density. Finally, the equivalence of 4 and 5 follows from Proposition 6.27 because complementation ($X \mapsto \omega - X$) is a homeomorphism from $\mathcal{P}\omega$ to itself and thus preserves meagerness. $\dashv$

We next consider how many sets are needed to generate a large filter, where "large" can have a strong interpretation—ultrafilter—or a weak one—non-feeble filter. The former gives a new cardinal characteristic, while the latter gives a new view of an old characteristic. Notice that any infinite generating set for a filter yields a base of the same cardinality just by closing under finite intersections. So we can equivalently ask about cardinalities of bases.

We begin with a result similar to Propositions 8.1 and 8.9, namely that ultrafilter bases can be large. Of course, any ultrafilter is a base for itself and has cardinality $\mathfrak{c}$; the following proposition, due to Pospíšil [88], shows that for some ultrafilters there are no smaller bases.

9.5 Proposition. *There is an ultrafilter on ω every base of which has cardinality $\mathfrak{c}$.*

Proof. Let $\mathcal{I}$ be an independent family of size $\mathfrak{c}$, by Proposition 8.9, and let $\mathcal{X}$ consist of

- all sets in $\mathcal{I}$ and
- the complements of all sets of the form $\bigcap \mathcal{C}$ with $\mathcal{C}$ an infinite subset of $\mathcal{I}$.

The independence of $\mathcal{I}$ easily implies that $\mathcal{X}$ has the SFIP, so there is an ultrafilter $\mathcal{U} \supseteq \mathcal{X}$. Suppose, toward a contradiction, that $\mathcal{U}$ had a base $\mathcal{Y}$ of cardinality $< \mathfrak{c}$. As each set in $\mathcal{I}$ has a subset in $\mathcal{Y}$ and $|\mathcal{I}| > |\mathcal{Y}|$, there must be infinitely many sets in $\mathcal{I}$ all including the same $Y \in \mathcal{Y}$. Then the intersection of these infinitely many sets from $\mathcal{I}$ is in $\mathcal{U}$ (because it includes Y), but its complement is in $\mathcal{X}$ and thus also in $\mathcal{U}$. This contradiction completes the proof. ⊣

Nevertheless, it is consistent that some ultrafilters have bases of cardinality smaller than $\mathfrak{c}$.

9.6 Definition. $\mathfrak{u}$, sometimes called the *ultrafilter number*, is the minimum cardinality of any ultrafilter base.

Kunen [68, Chap. 8, Ex. A10] built a finite support iterated forcing model where $\mathfrak{c} = \aleph_{\aleph_1}$ but $\mathfrak{u} = \aleph_1$. Baumgartner and Laver [11] showed that an $\aleph_2$-step, countable support iteration of Sacks forcing (over a model of GCH) produces a model where certain ultrafilters in the ground model (the selective ones) generate ultrafilters in the extension. Thus, their model has $\mathfrak{u} = \aleph_1$ while $\mathfrak{c} = \aleph_2$.

An ultrafilter base is an unsplittable family, for if X were to split it then neither X nor $\omega - X$ could be in the ultrafilter it generates. Thus, we immediately have the following inequality.

9.7 Proposition. $\mathfrak{r} \leq \mathfrak{u}$.

In most known models, $\mathfrak{r} = \mathfrak{u}$, but Goldstern and Shelah [52] showed that the inequality can be strict. A stronger connection between $\mathfrak{r}$ and ultrafilters is given by the following result of Balcar and Simon [1].

9.8 Definition. A *pseudobase* or *π-base* for a filter $\mathcal{F}$ on ω is a family $\mathcal{X} \subseteq [\omega]^\omega$ such that every set in $\mathcal{F}$ has a subset in $\mathcal{X}$.

This differs from the notion of base only in that $\mathcal{X}$ need not be a subfamily of $\mathcal{F}$.

9.9 Proposition. *$\mathfrak{r}$ is the minimum cardinality of any ultrafilter pseudobase.*

Proof. A family $\mathcal{X} \subseteq [\omega]^\omega$ is an ultrafilter pseudobase if and only if there is an ultrafilter disjoint from

$$\mathcal{Y} = \{Y \subseteq \omega : Y \text{ has no subset in } \mathcal{X}\}.$$

As mentioned above, this is equivalent to saying that ω is not almost covered by finitely many sets from $\mathcal{Y}$. Equivalently, whenever ω is partitioned into finitely many pieces, one of the pieces must have an almost subset in $\mathcal{X}$. This means that $\mathcal{X}$ must be unsplittable, 3-unsplittable (in the sense of Example 4.13), ..., n-unsplittable for all finite n. On the one hand, mere unsplittability requires $\mathcal{X}$ to have cardinality at least $\mathfrak{r}$. On the other hand we can, as in Example 4.13, produce an n-unsplittable family of size $\mathfrak{r}$ for each n and then take the union of these families to obtain an $\mathcal{X}$ as above of size $\mathfrak{r}$. ⊣

We now consider what is needed to generate a non-feeble filter. The first part of the following theorem is essentially due to Solomon [103]; the second part is an unpublished result of Petr Simon.

9.10 Theorem. *Every filter on ω generated by fewer than $\mathfrak{b}$ sets is feeble, but there is a non-feeble filter generated by $\mathfrak{b}$ sets.*

Proof. Consider first a filter with a base of fewer than $\mathfrak{b}$ sets, and associate to each set A in this base an interval partition Π_A chosen so that each interval in the partition contains an element of A. By Theorem 2.10, there is a single interval partition dominating all these Π_A's. It clearly satisfies statement 3 in Proposition 9.4, so our filter is feeble.

To produce a non-feeble filter generated by $\mathfrak{b}$ sets, we distinguish two cases, according to whether $\mathfrak{b} = \mathfrak{d}$.

If $\mathfrak{b} = \mathfrak{d}$, invoke Theorem 2.10 to get a $\mathfrak{b}$-indexed family of interval partitions Π_α ($\alpha < \mathfrak{b}$) dominating all interval partitions. We build the desired filter and a generating family $\mathcal{X}$ for it by a recursion of length $\mathfrak{b}$, starting with the family of cofinite sets, and adding at most one new set to $\mathcal{X}$ at each stage. At stage α, see whether the filter $\mathcal{F}_\alpha$ generated by the sets already put into $\mathcal{X}$ contains a set disjoint from infinitely many intervals of Π_α. If so, do nothing at stage α. If not, put into $\mathcal{X}$ the union of the even-numbered intervals of Π_α, and note that the SFIP of $\mathcal{X}$ is preserved. In either case, our final filter will contain a set missing infinitely many intervals of Π_α. After all $\mathfrak{b}$ steps have been completed, we have a filter that is not feeble because any interval partition as in statement 3 of Proposition 9.4 could not be dominated by any Π_α.

There remains the case that $\mathfrak{b} < \mathfrak{d}$. Let $\mathcal{B}$ be an unbounded family of size $\mathfrak{b}$ in ${}^\omega\omega$; without loss of generality, assume it is closed under forming the pointwise maximum of two functions and assume each function $g \in \mathcal{B}$ is non-decreasing. Since $|\mathcal{B}| < \mathfrak{d}$, let $f \in {}^\omega\omega$ be non-decreasing and dominated by no member of $\mathcal{B}$. Thus, the sets

$$X_g = \{n \in \omega : g(n) < f(n)\} \quad (g \in \mathcal{B})$$

are infinite. The family $\{X_g : g \in \mathcal{B}\}$ is closed under finite intersections (because $\mathcal{B}$ is closed under maxima), so it is a base for a filter $\mathcal{F}$. To complete the proof, we suppose that $\mathcal{F}$ is feeble and we deduce a contradiction.

Suppose therefore that $\{I_n : n \in \omega\}$ is an interval partition such that each set in $\mathcal{F}$ meets all but finitely many I_n's. Define $f' : \omega \to \omega$ by letting $f'(k)$ be the value of f at the right endpoint of the next I_n after the one containing k. Consider an arbitrary $g \in \mathcal{B}$ and a k so large that X_g, being in $\mathcal{F}$, meets the next interval I_n after the one containing k. Calling that interval $[a, b]$ and letting c be in its intersection with X_g, we have, since f and g are non-decreasing,

$$g(k) \le g(c) < f(c) \le f(b) = f'(k).$$

Thus, $g <^* f'$; since g was an arbitrary element of $\mathcal{B}$, we have a contradiction to the fact that $\mathcal{B}$ is unbounded. ⊣

9.11 Remark. The first part of the preceding proof actually shows that a filter with a pseudobase of size $< \mathfrak{b}$ must be feeble.

It is easy to see that every filter $\mathcal{F}$ is the intersection of some ultrafilters, in fact of at most $\mathfrak{c}$ ultrafilters. Indeed, for each $A \in \mathcal{P}\omega - \mathcal{F}$, the family $\mathcal{F} \cup \{\omega - A\}$ has the SFIP and is therefore included in an ultrafilter $\mathcal{U}_A$. The intersection of these $\mathcal{U}_A$'s is $\mathcal{F}$.

The next two propositions contain information about how many ultrafilters must be intersected in order to get filters that are small in one or another sense. The first one, due to Plewik [87], is another application of Proposition 8.1.

9.12 Proposition. *The intersection of fewer than $\mathfrak{c}$ ultrafilters is not feeble.*

Proof. Suppose the feeble filter $\mathcal{F}$ is the intersection of ultrafilters $\mathcal{U}_\alpha$. Let f be a finite-to-one function such that $f(\mathcal{F})$ consists only of the cofinite sets. Let $\mathcal{A}$ be a family of $\mathfrak{c}$ almost disjoint subsets of ω. For each $A \in \mathcal{A}$, we have $\omega - A \notin f(\mathcal{F})$ (as A is infinite), so $\omega - f^{-1}(A) = f^{-1}(\omega - A) \notin \mathcal{F}$, so $\omega - f^{-1}(A) \notin \mathcal{U}_\alpha$ for at least one α, and so $f^{-1}(A) \in \mathcal{U}_\alpha$. But the sets

$f^{-1}(A)$ are almost disjoint because f is finite-to-one. So no two can be in the same $\mathcal{U}_\alpha$. Therefore there must be at least as many $\mathcal{U}_\alpha$'s as there are A's, namely $\mathfrak{c}$. ⊣

9.13 Proposition. *There are $\mathfrak{d}$ ultrafilters whose intersection is not sent to an ultrafilter by any finite-to-one function.*

Proof. By Theorem 2.10, choose a family of $\mathfrak{d}$ interval partitions dominating all interval partitions, and associate to each $\Pi = \{I_n : n \in \omega\}$ in this family two ultrafilters $\mathcal{U}_\Pi$ and $\mathcal{V}_\Pi$ such that one contains $A_\Pi = \bigcup_n I_{8n}$ and the other contains $B_\Pi = \bigcup_n I_{8n+4}$. We shall show that the $\mathfrak{d}$ ultrafilters $\mathcal{U}_\Pi$ and $\mathcal{V}_\Pi$ are as required.

Suppose, to the contrary, that their intersection $\mathcal{F}$ is mapped to an ultrafilter by a finite-to-one map f. Let Θ be an interval partition such that each of the finite fibers $f^{-1}(\{n\})$ is included in the union of two adjacent intervals of Θ. (Simply build Θ inductively so that the right end of each interval is greater than all elements of all fibers whose left ends were in the previous interval.) Let Π be an interval partition in our originally chosen family that dominates Θ. Then each interval of Θ, except for finitely many, is included in the union of two consecutive intervals of Π. It follows that each fiber of f, except for finitely many, is covered by four consecutive intervals of Π and therefore cannot meet both A_Π and B_Π. So $f(A_\Pi)$ and $f(B_\Pi)$ are almost disjoint and $f(\mathcal{F})$, being an ultrafilter, must contain the complement of one of them, say $\omega - f(A_\Pi)$. But then this complement would be in $f(\mathcal{U}_\Pi)$, which is absurd as $A_\Pi \in \mathcal{U}_\Pi$. ⊣

We shall next present some consequences of the inequality $\mathfrak{u} < \mathfrak{g}$. This inequality was introduced in [23] (where $\mathfrak{g}$ was first defined) as a “black box” summary of the crucial properties of the models, due to Shelah [25, 26], in which every two ultrafilters have a common finite-to-one image. Since then, numerous additional consequences and reformulations of $\mathfrak{u} < \mathfrak{g}$ have been found, and we present some of them here.

9.14 Definition. For any family $\mathcal{F} \subseteq [\omega]^\omega$, we write $\sim\!\mathcal{F}$ for its complement and $\mathcal{F}\!\sim$ for the family of complements of its members.

$$\sim\!\mathcal{F} = [\omega]^\omega - \mathcal{F} \quad \text{and} \quad \mathcal{F}\!\sim = \{\omega - X : X \in \mathcal{F}\}.$$

We write $\breve{\mathcal{F}}$ for the dual family $\sim\!\mathcal{F}\!\sim = \{X \in [\omega]^\omega : \omega - X \notin \mathcal{F}\}$.

If $\mathcal{F}$ is closed under supersets then $\breve{\mathcal{F}}$ consists of just those $X \in [\omega]^\omega$ that intersect every member of $\mathcal{F}$. If $\mathcal{F}$ is a filter then $\mathcal{F} \subseteq \breve{\mathcal{F}}$, with equality holding exactly when $\mathcal{F}$ is an ultrafilter.

9.15 Lemma. *Suppose that $\mathcal{X}, \mathcal{Y} \subseteq [\omega]^\omega$, that $|\mathcal{X}| < \mathfrak{g}$, and that $\mathcal{Y}\!\sim$ is groupwise dense. Then there is a finite-to-one $f : \omega \to \omega$ such that*

$$\forall X \in \mathcal{X}\, \exists Y \in \mathcal{Y}\ (f(Y) \subseteq f(X)).$$

Proof. For each $X \in \mathcal{X}$ define

$$\mathcal{G}_X = \{Z \in [\omega]^\omega : \exists Y \in \mathcal{Y}\, \forall a, b \in Z \text{ (if } [a,b) \cap Y \neq \emptyset \text{ then } [a,b) \cap X \neq \emptyset)\}.$$

We verify that $\mathcal{G}_X$ is groupwise dense. $\mathcal{G}_X$ is clearly closed under subsets, and it is closed under finite modifications because $\mathcal{Y}$ is. Now suppose Π is any interval partition. Coarsening it, we may assume that each of its intervals contains an element of X. As $\mathcal{Y}\!\sim$ is groupwise dense, it contains a union of infinitely many intervals of Π. Call that union Z, and call its complement, which is in $\mathcal{Y}$, Y. We show that $Z \in \mathcal{G}_X$, witnessed by Y. Suppose $a < b$ are in Z and there is an element of Y in $[a, b)$. That means that a whole interval of Π must lie between a and b, and that interval contains a member of X. This completes the proof that $\mathcal{G}_X$ is groupwise dense.

Since there are fewer than $\mathfrak{g}$ X's in $\mathcal{X}$, there is a Z common to all the $\mathcal{G}_X$'s. Fix such a Z and define $f : \omega \to \omega$ by letting $f(n)$ be the number of elements of Z that are $\leq n$. Thus f is finite-to-one, being constant on the intervals $[a, b)$ where $a < b$ are consecutive in Z. For each $X \in \mathcal{X}$, the fact that $Z \in \mathcal{G}_X$ implies that there is $Y \in \mathcal{Y}$ with $f(Y) \subseteq f(X)$, as required. $\dashv$

9.16 Theorem. *Assume* $\mathfrak{u} < \mathfrak{g}$*. For any filter* $\mathcal{F}$ *on* ω *either* $\mathcal{F}$ *is feeble or there is a finite-to-one* f *such that* $f(\mathcal{F})$ *is an ultrafilter.*

Proof. Apply the lemma with $\mathcal{X}$ being an ultrafilter base of cardinality $< \mathfrak{g}$ and $\mathcal{Y}$ being $\mathcal{F}$. If $\mathcal{F}$ is not feeble, then $\mathcal{Y}\!\sim$ is groupwise dense by Proposition 9.4, so the lemma provides a finite-to-one f such that $f(X) \in f(\mathcal{F})$ for all $X \in \mathcal{X}$ and therefore for all X in the ultrafilter $\mathcal{U}$ generated by $\mathcal{X}$. Thus, the ultrafilter $f(\mathcal{U})$ is included in the filter $f(\mathcal{F})$. Since ultrafilters are maximal filters, the inclusion cannot be proper, and $f(\mathcal{F})$ is an ultrafilter. $\dashv$

9.17 Remark. The conclusion of this theorem is called the principle of *filter dichotomy*. It is not known whether it implies $\mathfrak{u} < \mathfrak{g}$.

The hypothesis of the theorem can be replaced by the apparently weaker $\mathfrak{r} < \mathfrak{g}$. Indeed, if $\mathcal{X}$ is not an ultrafilter base but merely unsplittable, the proof above provides a finite-to-one f such that $f(\mathcal{F})$ is also unsplittable. But an unsplittable filter is an ultrafilter.

The improvement is, however, illusory, for Mildenberger has shown that the inequalities $\mathfrak{u} < \mathfrak{g}$ and $\mathfrak{r} < \mathfrak{g}$ are equivalent. In fact, she proved $\mathfrak{r} \geq \min\{\mathfrak{u}, \mathfrak{g}\}$.

9.18 Corollary. *Assume* $\mathfrak{u} < \mathfrak{g}$ *(or just filter dichotomy). For every two ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$ *on* ω*, there is a finite-to-one function* f *with* $f(\mathcal{U}) = f(\mathcal{V})$*.*

Proof. Apply filter dichotomy to the filter $\mathcal{U} \cap \mathcal{V}$. It is not feeble, and any f that maps it to an ultrafilter must map both $\mathcal{U}$ and $\mathcal{V}$ to the same ultrafilter. $\dashv$

9.19 Remark. The conclusion of this corollary is called the principle of *near coherence of filters* (NCF). The name refers to the easily equivalent formulation: For any two filters $\mathcal{F}$ and $\mathcal{G}$ on ω, there is a finite-to-one function f such that $f(\mathcal{F})$ and $f(\mathcal{G})$ are coherent in the sense that their union generates a filter.

NCF implies $\mathfrak{u} < \mathfrak{d}$, but it is not known whether it implies $\mathfrak{u} < \mathfrak{g}$ or even filter dichotomy.

Corollary 9.18 can be improved to handle not just two ultrafilters but any number $< \mathfrak{c}$, by essentially the same proof, using Proposition 9.12 to ensure that the intersection filter is not feeble. NCF alone implies the improvement to $< \mathfrak{d}$ ultrafilters. It is also equivalent to the statement that every ultrafilter has a finite-to-one image that is generated by $< \mathfrak{d}$ sets. See [15] for these results and more information on NCF.

9.20 Corollary. *If* $\mathfrak{u} < \mathfrak{g}$ *(or just filter dichotomy) then* $\mathfrak{b} = \mathfrak{u}$ *and* $\mathfrak{d} = \mathfrak{c}$.

Proof. Without any hypothesis, we have $\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{u}$ and $\mathfrak{d} \leq \mathfrak{c}$. If we assume filter dichotomy then Proposition 9.13 provides a feeble filter that is the intersection of $\mathfrak{d}$ ultrafilters, and then Proposition 9.12 says that $\mathfrak{d} \geq \mathfrak{c}$.

Theorem 9.10 gives a non-feeble filter with a basis of $\mathfrak{b}$ sets. By filter dichotomy, some image of this filter, which also has a basis of $\mathfrak{b}$ sets (the images of the sets in the previous basis), is an ultrafilter. So $\mathfrak{u} \leq \mathfrak{b}$. $\dashv$

9.21 Remark. The conclusion $\mathfrak{d} = \mathfrak{c}$ can be strengthened to $\mathfrak{g} = \mathfrak{c}$ under the hypothesis $\mathfrak{u} < \mathfrak{g}$; see [19].

The following result of Laflamme [70] extends the preceding dichotomy to a trichotomy for all upward-closed families. Its conclusion is in fact equivalent to $\mathfrak{u} < \mathfrak{g}$ but we omit the proof of this; see [19].

9.22 Theorem. *Assume* $\mathfrak{u} < \mathfrak{g}$. *For any family* $\mathcal{Y} \subseteq [\omega]^\omega$ *that is closed under almost supersets, there is a finite-to-one* $f : \omega \to \omega$ *such that one of the following holds:*

- $f(\mathcal{Y})$ *contains only cofinite sets.*
- $f(\mathcal{Y}) = [\omega]^\omega$.
- $f(\mathcal{Y})$ *is an ultrafilter.*

Proof. Let $\mathcal{Y}$ be as in the theorem and let $\mathcal{X}$ be an ultrafilter base of cardinality $< \mathfrak{g}$. If $\mathcal{Y}\sim$ is not groupwise dense, then we have (by definition of groupwise dense) the first alternative in the theorem, and if $\sim\mathcal{Y}$ is not groupwise dense, then we have the second alternative. So we assume that both $\mathcal{Y}\sim$ and $\sim\mathcal{Y} = \breve{\mathcal{Y}}\sim$ are groupwise dense. The former lets us apply Lemma 9.15 to obtain a finite-to-one g such that each $g(X)$ with $X \in \mathcal{X}$ includes some $g(Y)$ with $Y \in \mathcal{Y}$. If $\mathcal{U}$ is the ultrafilter with base $\mathcal{X}$, then we have that $g(\mathcal{U}) \subseteq g(\mathcal{Y})$. Since finite-to-one images preserve groupwise density and commute with complementation, we also have that $g(\breve{\mathcal{Y}})\sim$ is groupwise dense, so we can apply the lemma with the base $\{g(X) : X \in \mathcal{X}\}$ of $g(\mathcal{U})$ in the role of $\mathcal{X}$ and with $g(\breve{\mathcal{Y}})$ in the role of $\mathcal{Y}$. We obtain a finite-to-one h such that $hg(\mathcal{U}) \subseteq hg(\breve{\mathcal{Y}}) = (hg(\mathcal{Y}))\breve{}$. Since dualization $(\breve{\ })$ reverses inclusions and fixes ultrafilters, we get $hg(\mathcal{U}) \supseteq hg(\mathcal{Y})$. The reverse inequality follows from $g(\mathcal{U}) \subseteq g(\mathcal{Y})$. So the finite-to-one map hg sends $\mathcal{Y}$ to an ultrafilter. $\dashv$

We conclude this section with a brief discussion of some special sorts of ultrafilters. The theory of these ultrafilters is quite extensive, but we shall consider only those aspects that directly involve some of the cardinal characteristics defined earlier.

9.23 Definition. An ultrafilter $\mathcal{U}$ on ω is *selective* if every function $f : \omega \to \omega$ becomes either one-to-one or constant when restricted to some set in $\mathcal{U}$. It is a *P-point ultrafilter* if every function $f : \omega \to \omega$ becomes either finite-to-one or constant when restricted to some set in $\mathcal{U}$. It is a *Q-point* if every finite-to-one function $f : \omega \to \omega$ becomes one-to-one when restricted to some set in $\mathcal{U}$.

9.24 Remark. Clearly, an ultrafilter is selective if and only if it is both a P-point and a Q-point.

The name "selective" refers to the fact that, when ω is partitioned into pieces that are not in $\mathcal{U}$ then some set in $\mathcal{U}$ selects one element per piece. Selective ultrafilters are also called Ramsey ultrafilters because Kunen showed (see [28]) that, if $\mathcal{U}$ is selective and $f : [\omega]^k \to 2$, then some set in $\mathcal{U}$ is homogeneous for f. Thus, any pseudobase for a selective ultrafilter must have cardinality at least $\mathfrak{hom} = \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$. Selective ultrafilters are also called RK-minimal, for they are minimal in the Rudin-Keisler ordering defined by putting $f(\mathcal{U}) \leq \mathcal{U}$ for all ultrafilters $\mathcal{U}$ and all mappings f.

An ultrafilter $\mathcal{U}$ is a P-point if and only if every decreasing (or almost-decreasing) ω-sequence of sets from $\mathcal{U}$ has a pseudo-intersection in $\mathcal{U}$. To prove the equivalence of this with the definition above, just arrange that $f(n)$ is constant exactly on the differences of consecutive sets in the decreasing sequence. (One can assume without loss of generality that the sequence begins with ω and that its intersection is empty.) There is a general topological concept of P-point (see for example [95, 50]), namely a point (in a topological space) such that every countable intersection of open neighborhoods of it includes another open neighborhood of it. When applied to the topological space $\beta\omega - \omega$, the Stone-Čech remainder of the discrete space ω, whose points are naturally identified with (non-trivial) ultrafilters on ω, this topological notion becomes the concept defined above. The "P" in "P-point" refers to prime ideals (in rings of functions); see [50, Exercises 4J and 4L].

The "Q" in "Q-point" was chosen because it is next to "P" in the alphabet. Q-points are also called rare ultrafilters.

There are ultrafilters that are neither P-points nor Q-points. Indeed, if $\mathcal{U}$ is any ultrafilter on ω then

$$\mathcal{V} = \{X \subseteq \omega^2 : \{a : \{b : \langle a, b\rangle \in X\} \in \mathcal{U}\} \in \mathcal{U}\}$$

is an ultrafilter on ω^2. It is not a P-point because the first projection $\omega^2 \to \omega$ is neither finite-to-one nor constant on any set in $\mathcal{V}$. It is not a Q-point because the second projection is finite-to-one on a set in $\mathcal{V}$, namely $\{\langle a, b\rangle : a < b\}$, but not one-to-one on any set in $\mathcal{V}$.

The existence of P-points, Q-points, and selective ultrafilters is more problematic. W. Rudin [95] showed that implies the existence of P-points, and other existence results followed, with the hypothesis weakened to MA or even to $\mathfrak{p} = \mathfrak{c}$ once these axioms had been formulated; see for example [28, 13, 14, 75, 94].

But some hypotheses beyond ZFC are needed for such existence results. Kunen [67] showed that adding $\aleph_2$ random reals to a model of GCH produces a model with no selective ultrafilters. Miller [79] showed that an $\aleph_2$-step, countable support iteration of Laver forcing over a model of GCH produces a model with no Q-points. And Shelah [97, Sect. VI.4], [113] produced a model with no P-points by iterating a product of Grigorieff forcings.

We shall be concerned here with conditions for the existence of these special ultrafilters. It turns out that cardinal characteristics can be used to give necessary and sufficient conditions for the extendibility, to special ultrafilters, of all filters with sufficiently small bases. Thus, they provide sufficient, though not necessary, conditions for the mere existence of special ultrafilters. The first result of this sort is due to Ketonen [66], who showed that $\mathfrak{c} = \mathfrak{d}$ implies the existence of P-points, by a proof that essentially gives the following result.

9.25 Theorem.

1. *If $\mathfrak{c} = \mathfrak{d}$ then every filter generated by fewer than $\mathfrak{c}$ sets is included in some P-point.*
2. *There is a filter generated by $\mathfrak{d}$ sets that is not included in any P-point.*
3. *Every ultrafilter generated by fewer than $\mathfrak{d}$ sets is a P-point.*

Proof. For part 1, assume $\mathfrak{c} = \mathfrak{d}$, let $\mathcal{F}$ be a filter generated by fewer than $\mathfrak{c}$ sets, and let $\langle S^\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of all decreasing ω-sequences of infinite subsets of ω, $S^\alpha = \langle S^\alpha_0 \supseteq S^\alpha_1 \supseteq \cdots\rangle$. We shall define an increasing sequence of filters $\langle \mathcal{F}^\alpha : \alpha \le \mathfrak{c}\rangle$, starting with $\mathcal{F}^0 = \mathcal{F}$, taking unions at limit stages, and at successor stages adding one new generator to the filter in such a way that either the new generator is a pseudointersection of S^α or it is the complement of some S^α_n. Of course, we must make sure that the newly added generator at stage $\alpha + 1$ has infinite intersection with every set in $\mathcal{F}^\alpha$, so that $\mathcal{F}^{\alpha+1}$ will be a filter. But this is not difficult. If, for some n, $S^\alpha_n \notin \mathcal{F}^\alpha$, then $\omega - S^\alpha_n$ can be added. If, on the other hand, $S^\alpha_n \in \mathcal{F}^\alpha$ for all n, then, because $\mathcal{F}^\alpha$ is generated by fewer than $\mathfrak{d}$ sets, Proposition 6.24 provides a pseudointersection of S^α that has infinite intersection with every finite intersection of the generators of $\mathcal{F}^\alpha$ and hence with every set in $\mathcal{F}^\alpha$. That pseudointersection can serve as the new generator for $\mathcal{F}^{\alpha+1}$. Thus, the construction of the sequence of filters can be carried out, and it clearly ensures that any ultrafilter extending $\mathcal{F}^{\mathfrak{c}}$ is a P-point.

For part 2, consider the filter on ω^2 generated by the sets $\{\langle a,b\rangle : a \geq n\}$ for $n \in \omega$ and the sets $\{\langle a,b\rangle : b > f(a)\}$ for f in a dominating family $\mathcal{D} \subseteq {}^{\omega}\omega$ of cardinality $\mathfrak{d}$. An ultrafilter extending this filter cannot be a P-point, for any set on which the first projection $\omega^2 \to \omega$ is constant or finite-to-one is disjoint from a set in the filter and is therefore not in the ultrafilter.

For part 3, let $\mathcal{U}$ be an ultrafilter generated by fewer than $\mathfrak{d}$ sets and let $S = \langle S_n \rangle$ be a decreasing sequence of sets from $\mathcal{U}$. As in the proof of part 1, Proposition 6.24 provides a pseudointersection of S that meets every finite intersection of generators of $\mathcal{U}$. But as $\mathcal{U}$ is an ultrafilter, it follows that this pseudointersection is in $\mathcal{U}$. ⊣

Canjar [37] proved the following analogous result for selective ultrafilters. It was also found independently by Bartoszyński and Judah; see [5, Sect. 4.5.B].

9.26 Theorem.

1. *If* $\mathfrak{c} = \mathbf{cov}(\mathcal{B})$ *then every filter generated by fewer than* $\mathfrak{c}$ *sets is included in some selective ultrafilter.*

2. *There is a filter generated by* $\mathbf{cov}(\mathcal{B})$ *sets that is not included in any selective ultrafilter.*

Proof. For part 1, we proceed as in the corresponding proof for P-points, using an enumeration $\langle f^\alpha : \alpha < \mathfrak{c}\rangle$ of ${}^{\omega}\omega$ in place of the enumeration of decreasing sequences S^α. At stage α we have a filter $\mathcal{F}^\alpha$ with a basis $\mathcal{X}$ of fewer than $\mathbf{cov}(\mathcal{B})$ sets and we wish to form $\mathcal{F}^{\alpha+1}$ by adding one new generator, a set on which f^α is one-to-one or constant. If some set of the form $(f^\alpha)^{-1}(\{n\})$ (on which f is constant) has infinite intersection with every set in $\mathcal{F}^\alpha$, then it can serve as the new generator. So from now on we assume that this is not the case. We intend to find a "selector" $g \in \prod_{n\in R}(f^\alpha)^{-1}(\{n\})$, where $R = \mathrm{ran}(f^\alpha)$, such that for each set X in the basis $\mathcal{X}$ of $\mathcal{F}^\alpha$ we have $\exists^\infty n\,(g(n) \in X)$. Once we have such a g, its range can clearly serve as the new generator for $\mathcal{F}^{\alpha+1}$. To obtain g, notice first that the space $\prod_{n\in R}(f^\alpha)^{-1}(\{n\})$ from which we want to choose it is a product of countable (possibly finite) discrete sets, so it is not covered by fewer than $\mathbf{cov}(\mathcal{B})$ meager sets. But for each $X \in \mathcal{X}$, those g that fail to have infinitely many values in X form a meager set. So, since $|\mathcal{X}| < \mathbf{cov}(\mathcal{B})$, the desired g exists.

Part 2 is immediate from part 2 of the preceding theorem if $\mathbf{cov}(\mathcal{B}) = \mathfrak{d}$, so we may assume for the rest of the proof that $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$ (recall Proposition 5.5). By Theorem 5.2, fix a family of $\mathbf{cov}(\mathcal{B})$ chopped reals (x_α, Π_α) such that no single real matches them all. Assume without loss of generality that every finitely many of these chopped reals are engulfed by another chopped real from the chosen family, i.e., the family is directed upward with respect to the engulfing relation. Since we are assuming $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$, there is an interval partition $\Theta = \{J_n : n \in \omega\}$ not dominated by any of the Π_α. This implies that each Π_α has infinitely many intervals I_k that do not include any interval of Θ; such an I_k is covered by $J_n \cup J_{n+1}$ for some n.

Let Z be the set of functions z whose domains are unions of two consecutive intervals of Θ and whose values are 0's and 1's. For $z \in Z$, let $p(z)$ be the n such that $\mathrm{dom}(z) = J_n \cup J_{n+1}$. Thus, $p : Z \to \omega$ is finite-to-one. Let $\mathcal{F}$ be the filter on Z generated by the sets $\{z \in Z : p(z) > n\}$ for all $n \in \omega$ and the sets

$$A_\alpha = \{z \in Z : \exists I \in \Pi_\alpha\, (I \subseteq \mathrm{dom}(z) \text{ and } z{\restriction}I = x_\alpha{\restriction}I)\}$$

for all α. We must check that these sets have the SFIP, so consider any finitely many of them. We may assume only one of them is of the form $\{z \in Z : p(z) > n\}$; if there are more, keep only the one with the largest n as it is a subset of the others. Thanks to our assumption that any finitely many (x_α, Π_α) are engulfed by another, we may also assume that only one A_α is involved, for if (x_β, Π_β) engulfs certain (x_α, Π_α)'s, then the corresponding A_β is almost included in the corresponding A_α's. So our task is simply to check that each A_α contains z's with arbitrarily large $p(z)$. But this follows immediately from the fact that infinitely many intervals of Π_α are included in sets of the form $J_n \cup J_{n+1}$.

So $\mathcal{F}$ is a filter generated by $\mathbf{cov}(\mathcal{B})$ sets. Let $\mathcal{U}$ be any ultrafilter extending $\mathcal{F}$. p is a finite-to-one function, so it is certainly not constant on any set in $\mathcal{U}$. Suppose it were one-to-one on some set $X \in \mathcal{U}$. One of $X_0 = \{x \in X : p(x) \text{ even}\}$ and $X_1 = \{x \in X : p(x) \text{ odd}\}$ is in $\mathcal{U}$; say it is X_i. Then the union g of all the members of X_i is a partial function from ω to 2 such that each Π_α contains infinitely many intervals on which g agrees with x_α (because X_i meets all sets in $\mathcal{F}$). Any extension of g to a total function $\omega \to 2$ therefore matches all the (x_α, Π_α), contrary to our choice of these chopped reals. So p is not one-to-one on any set in $\mathcal{U}$. ⊣

By analogy with part 3 of Theorem 9.25, one might expect Theorem 9.26 to assert that every ultrafilter generated by fewer then $\mathbf{cov}(\mathcal{B})$ sets is selective. Though true, that assertion is vacuous, since Theorem 5.19 and Proposition 9.7 give $\mathbf{cov}(\mathcal{B}) \leq \mathfrak{r} \leq \mathfrak{u}$.

Canjar [37] also obtained an analogous result for Q-points.

9.27 Theorem.

1. *If* $\mathbf{cov}(\mathcal{B}) = \mathfrak{d}$ *then every filter generated by fewer than* $\mathfrak{d}$ *sets can be extended to a Q-point.*

2. *If* $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$ *then there is a filter generated by* $\mathbf{cov}(\mathcal{B})$ *sets that is not included in any Q-point.*

The proof of Theorem 9.26 also establishes part 2 of the present theorem, and part 1 is established similarly to parts 1 of Theorems 9.25 and 9.26.

10. Evasion and Prediction

The terminology of prediction and evasion and the evasion number $\mathfrak{e}$ were introduced in [21] on the basis of motivation from algebra. Since then, several variants have been studied, particularly in [30, 33], but we begin with the original version.

10.1 Definition. A *predictor* is a pair $\pi = (D, \langle \pi_n : n \in D \rangle)$ where $D \in [\omega]^\omega$ and where each $\pi_n : {}^n\omega \to \omega$. This predictor π *predicts* a function $x \in {}^\omega\omega$ if, for all but finitely many $n \in D$, $\pi_n(x \restriction n) = x(n)$. Otherwise, x *evades* π. The *evasion number* $\mathfrak{e}$ is the smallest cardinality of any family $\mathcal{E} \subseteq {}^\omega\omega$ such that no single predictor predicts all members of $\mathcal{E}$.

We may identify a predictor $(D, \langle \pi_n : n \in D \rangle)$ with $\bigcup_{n \in D} \pi_n$, a partial function from ${}^{<\omega}\omega$ to ω.

The idea behind the definition is that the values $x(n)$ of an unknown $x \in {}^\omega\omega$ are being revealed one at a time (in order) and we are trying to guess some of these values just before they are revealed. A predictor $(D, \langle \pi_n : n \in D \rangle)$ is a strategy for guessing $x(n)$, for each $n \in D$, after we have seen $x \restriction n$, and it predicts x if it is successful in the sense that almost all of its guesses about x are correct.

Clearly, it would make no difference if we defined predictors with $\pi_n : {}^nC \to C$ and used them to predict functions in ${}^\omega C$ for any countably infinite set C.

What was directly relevant to the algebraic subject of [21] was not $\mathfrak{e}$ but a variant, the *linear evasion number* $\mathfrak{e}_l$, whose definition is similar except that the components of a predictor are *linear* functions $\pi_n : \mathbb{Z}^n \to \mathbb{Q}$ and the functions being predicted are in ${}^\omega\mathbb{Z}$. Thus a remnant of algebra (linearity) was mixed with the combinatorics. Fortunately, it is proved in [33] that $\mathfrak{e}_l = \min\{\mathfrak{e}, \mathfrak{b}\}$, so the algebra can be eliminated in favor of pure combinatorics.

Several additional variants were defined in [30] by restricting the possible values of the functions being predicted, as follows.

10.2 Definition. Let $f : \omega \to \omega - \{0, 1\}$. Let $\mathfrak{e}_f$ be the smallest cardinality of any family $\mathcal{E} \subseteq \prod_{n \in \omega} f(n)$ such that no single predictor predicts all members of $\mathcal{E}$. When f is the constant function with value $n \geq 2$, we write $\mathfrak{e}_n$ instead of $\mathfrak{e}_f$. The *unbounded evasion number* $\mathfrak{e}_{\text{ubd}}$ is the minimum of $\mathfrak{e}_f$ over all functions f as above.

Clearly, $\mathfrak{e}_f \geq \mathfrak{e}_g$ whenever $f \leq g$, and $\mathfrak{e}_{\text{ubd}} \geq \mathfrak{e}$. The following theorem from [30] summarizes relationships between these variants and the original $\mathfrak{e}$ (as well as $\mathfrak{b}$ and $\mathfrak{s}$).

10.3 Theorem.

1. *$\mathfrak{e}_n = \mathfrak{e}_2$ for all $n \geq 2$.*

2. *$\mathfrak{e}_2 \geq \mathfrak{s}$.*

3. $\mathfrak{e} \geq \min\{\mathfrak{e}_{\mathrm{ubd}}, \mathfrak{b}\}$.

4. It is consistent that $\mathfrak{e} < \mathfrak{e}_{\mathrm{ubd}}$.

5. It is consistent that $\mathfrak{e}_{\mathrm{ubd}} < \mathfrak{e}_2$.

Proof. We only sketch the proofs, referring to [30] for details.

For part 1, the idea is to predict a function $x : \omega \to n$ (where $n \geq 3$) by predicting the two functions $k \mapsto x(k) \bmod 2$ and $k \mapsto \lfloor x(k)/2 \rfloor$, whose ranges are smaller than n. More precisely, after predicting the former on some D, one predicts (on some $D' \subseteq D$) the restriction of the latter to D.

For part 2, we show that a family $\mathcal{E} \subseteq {}^{\omega}2$ that is not splitting (when viewed in $\mathcal{P}\omega$) can be predicted. If X is an infinite set on which each $x \in \mathcal{E}$ is almost constant, then let π be the predictor, with domain $D = X - \{\min X\}$, predicting that x will take, at any point of D, the same value that it took at the last previous member of X. This guess is right almost always, for every $x \in \mathcal{E}$.

For part 3, the idea is that any fewer than $\min\{\mathfrak{e}_{\mathrm{ubd}}, \mathfrak{b}\}$ functions can be predicted by first dominating them with some f (as there are fewer than $\mathfrak{b}$ of them) and then regarding them as functions in $\prod_{n\in\omega} f(n)$, where they can be predicted (as there are fewer than $\mathfrak{e}_{\mathrm{ubd}} \leq \mathfrak{e}_f$ of them). Some care is needed as each function is below f only almost everywhere.

Part 4 is proved by an iterated forcing argument, where each step is a σ-centered forcing adding a predictor that predicts all ground-model elements of $\prod_{n\in\omega} f(n)$ for some f. A condition consists of a finite part of the desired predictor plus a promise to predict correctly all later values of finitely many functions. A finite support iteration of this clearly makes $\mathfrak{e}_{\mathrm{ubd}}$ large in the extension. We omit the hard part of the proof, namely showing that $\mathfrak{e}$ does not become large.

For part 5, iterate Mathias forcing with countable supports for $\aleph_2$ steps over a model of GCH. The resulting model has $\mathfrak{h} = \mathfrak{c} = \aleph_2$, so both $\mathfrak{b}$ and $\mathfrak{s}$ are $\aleph_2$. By part 2, we have $\mathfrak{e}_2 = \aleph_2$. On the other hand, the forcing adds no Cohen reals, so $\mathbf{cov}(\mathcal{B}) = \aleph_1$. We shall see below (Table 2 and its explanation) that $\mathfrak{e} \leq \mathbf{cov}(\mathcal{B})$. So by part 3 we have $\min\{\mathfrak{e}_{\mathrm{ubd}}, \mathfrak{b}\} \leq \aleph_1$. Since $\mathfrak{b} = \aleph_2$, we must have $\mathfrak{e}_{\mathrm{ubd}} = \aleph_1$. ⊣

Returning from the discussion of these variants to the original $\mathfrak{e}$, we have the following results.

10.4 Theorem.

1. $\mathbf{add}(\mathcal{L}), \mathfrak{p} \leq \mathfrak{e} \leq \mathbf{non}(\mathcal{B}), \mathbf{cov}(\mathcal{B})$.

2. It is consistent that $\mathfrak{e} < \mathbf{add}(\mathcal{B})$.

3. It is consistent that $\mathfrak{b} < \mathfrak{e}$.

In part 1, the inequality involving $\mathbf{cov}(\mathcal{B})$ is due to Kada [63], and the rest of part 1 is from [21]. Parts 2 and 3 are from [30] and [33] respectively.

Proof. The upper bound of $\mathbf{cov}(\mathcal{B})$ will follow from Tables 2 and 3 and their justification below. The upper bound of $\mathbf{non}(\mathcal{B})$ follows from the observation that any predictor can predict only a meager set of functions. (The set of functions predicted by any π also has measure zero in the standard measure, described in the introduction, on ${}^\omega\omega$. So $\mathbf{non}(\mathcal{L})$ is also an upper bound, but this is a weaker bound than $\mathbf{cov}(\mathcal{B})$ by Theorem 5.11.)

To establish the lower bound of $\mathfrak{p}$, we use Theorem 7.12. We assume MA_κ(σ-centered) and show that any family $\mathcal{H}$ of κ functions can be predicted by some predictor (D, π). Let P be the set of triples (d, p, F) where d is a finite subset of ω, p is a finite partial function into ω whose domain consists of sequences from ${}^n\omega$ for $n \in d$, and F is a finite subset of $\mathcal{H}$. (The "meaning" of (d, p, F) is that d is an initial segment of D, p is a finite part of π, and the functions in F will be predicted correctly at all points of $D - d$.) Partially order P by putting $(d', p', F') \le (d, p, F)$ if d is an initial segment of d', $p \subseteq p'$, $F \subseteq F'$, and whenever $n \in d' - d$ and $x \in F$ then $p'(x\restriction n)$ is defined and equal to $x(n)$. Any finitely many elements with the same first and second components have a lower bound, obtained by taking the union of the third components. So MA_κ(σ-centered) provides $G \subseteq P$ generic with respect to the dense sets $\{(d, p, F) \in P : x \in F\}$ for all $x \in \mathcal{H}$, $\{(d, p, F) \in P : s \in \mathrm{dom}(p) \text{ or } n \notin d, n < \max d\}$ for all $n \in \omega$, $s \in {}^n\omega$, and $\{(d, p, F) \in P : |d| \ge n\}$ for all $n \in \omega$. (For proving the density of the last of these, the idea is that, starting with any (d, p, F), we can enlarge d by choosing m so large that all the $x\restriction m$ for $x \in F$ are distinct and then adjoining m to d and enlarging p as required by the definition of $\le$. The choice of m ensures that the required enlargements of p do not conflict.) Then by letting D and π be the unions of the first components and second components, respectively, of the triples in G, we obtain a predictor predicting all the functions in $\mathcal{H}$.

To prove the lower bound of $\mathbf{add}(\mathcal{L})$, suppose we are given a family $\mathcal{H}$ of fewer than $\mathbf{add}(\mathcal{L})$ functions $x : \omega \to \omega$. Let $\{I_n : n \in \omega\}$ be the interval partition where $|I_n| = n+1$. To each $x \in \mathcal{H}$ associate the function defined by $x'(n) = x\restriction I_n$. By Theorem 5.14, we can assign to each n a set $S(n)$ consisting of n functions $I_n \to \omega$ in such a way that $\forall x \in \mathcal{H}\, \forall^\infty n\ (x'(n) \in S(n))$. Any n functions produce at most $n-1$ branching points, i.e., points k where two of the functions first differ. So there is some $i_n \in I_n$ that is not a branching point for any of the n functions in $S(n)$. So we can define a predictor with $D = \{i_n : n \in \omega\}$ by setting $\pi(s) = z(i_n)$ if s has length i_n and $z \in S(n)$ and s agrees with z on $I_n \cap i_n$. (Extend p arbitrarily to those s whose length is in D but which admit no such z.) This π predicts all $x \in \mathcal{H}$ because the associated x' have almost all their values in $S(n)$.

This completes (modulo Tables 2 and 3) the proof of part 1. For parts 2 and 3, we only indicate the forcings used, referring to [30, 33] for the hard parts of the proofs.

Part 2 is proved by a finite support iteration of Hechler forcing. Since this adds Cohen reals and dominating reals, both $\mathbf{cov}(\mathcal{B})$ and $\mathfrak{b}$ and therefore also their minimum $\mathbf{add}(\mathcal{B})$ are large in the extension. The hard part of the proof is to show that $\mathfrak{e}$ remains small.

Part 3 is proved by a finite support iteration where each step adds a predictor that predicts all ground model reals. As in the proof of $\mathfrak{p} \leq \mathfrak{e}$ above, a condition consists of a finite part of the desired predictor together with finitely many functions that are to be predicted correctly at all later points. This forcing clearly makes $\mathfrak{e}$ large; the hard part is to prove that $\mathfrak{b}$ remains small. $\dashv$

10.5 Remark. Laflamme has improved the inequality $\mathfrak{p} \leq \mathfrak{e}$ in Theorem 10.4 to $\mathfrak{t} \leq \mathfrak{e}$. In [71, Proposition 2.3] he shows that $\mathfrak{t} \leq \mathfrak{e}_{\mathrm{ubd}}$, and he mentions that $\mathfrak{t} \leq \mathfrak{e}$ follows via part 3 of Theorem 10.3.

We turn next to some additional variations on the theme of prediction and evasion. These variations turn out to be closely connected to cardinals studied in previous sections. We consider three sorts of variations, singly and in combination.

First, the predictor could guess less information than the exact value of the $x(n)$ being predicted. Thus, we consider predictors $(D, \langle \pi_n : n \in D \rangle)$ where each $\pi_n : {}^n\omega \to \mathcal{P}\omega$, and we consider that $x \in {}^\omega\omega$ is predicted by such a π if $\forall^\infty n\ (x(n) \in \pi_n(x \restriction n))$. To avoid trivialities, the sets that occur as values of π_n must be small in some sense. (The predictor whose values are all equal to ω predicts every x.) We shall consider the following six possibilities for the values of π_n.

- Singletons. (This is the case considered above.)
- Sets of cardinality k for some fixed $k \in \omega$.
- Sets of cardinality $f(n)$, where f is a function $\omega \to \omega$ that tends to infinity.
- Finite sets.
- Co-infinite sets.
- Proper subsets of ω.

Thus, we shall refer to "single-valued" predictors, "k-valued" predictors, etc. Each type of predictor gives rise to an evasion number, namely the minimum number of functions not all predicted by a single predictor of that type.

Clearly, as the predictor's guesses become less specific (as we go down the list above), prediction becomes easier, evasion harder, and the evasion number larger.

Notice also that we could replace "finite sets" as values for π with "initial segments of ω" without affecting the evasion number, for given any predictor π of one sort we can trivially produce a predictor π' of the other sort predicting all the functions predicted by π. For the same reason, we can replace "proper subsets of ω" with "co-singletons".

The next variation concerns which values of x a predictor must guess correctly in order to predict x; it was also considered by Kada [63]. The definitions above permit the predictor to specify an infinite set D and guess $x(n)$ only for $n \in D$; it predicts x if almost all of these guesses are correct. We can make the definition more restrictive by requiring $D = \omega$. This variation will be called *global* prediction, and the original version will, when we want to emphasize the difference, be called *local* prediction.

Alternatively, we can make the definition less restrictive by saying that π predicts x if infinitely many (rather than almost all) of the guesses are right. We refer to this as *infinite* prediction. Notice that in this situation one might as well take $D = \omega$, because extending a predictor to a larger D can only increase the collection of functions it predicts. Thus, for both global and infinite prediction, we usually regard a predictor as either a sequence $\langle \pi_n \rangle_{n\in\omega}$ or as the union of such a sequence, $\pi : {}^{<\omega}\omega \to \omega$.

Clearly, as we move from global to local to infinite prediction, prediction becomes easier, evasion harder, and the evasion number larger.

The final variation that we consider here is to make $\pi_n(s)$ independent of s. In other words, the predictor is not allowed to see $x \restriction n$ but only knows n when guessing $x(n)$. Thus, the predictor is essentially just a function π on ω or D, taking "small" values in one of the senses above. We refer to such predictors as *non-adaptive* while predictors of the original sort are *adaptive*. Clearly, adaptive prediction is easier than non-adaptive prediction, evasion harder, and the evasion number larger.

The six choices for "small", the three choices global or local or infinite, and the two choices non-adaptive or adaptive give 36 evasion numbers, one of which (singleton, local, adaptive) is $\mathfrak{e}$. Many of the others coincide with cardinals discussed earlier, and for the rest there are bounds in terms of such cardinals. This information is summarized in the following tables. The first column of each table lists the six species of smallness, with G representing a typical guess for $x(n)$.

Our remarks above imply that the entries in each table increase (weakly) from top to bottom and from left to right; also, as we go from one table to the next (global to local to infinite), the entries in any single position increase (weakly). We shall usually refer to these facts as "monotonicity" without going into any more detail.

The question marks in four of the entries indicate that I do not know the values of these evasion numbers but only the indicated bounds and the result of Mildenberger (unpublished) that the following three cardinals are equal:

- $\mathfrak{e}$,
- the smaller of $\mathfrak{e}_2$ and the question mark in the "$|G| = k$" line of Table 2,
- the smaller of $\mathfrak{e}_{\mathrm{ubd}}$ and the question mark in the "$|G|$ finite" line of Table 2.

Table 1: Evasion numbers for global prediction

	Non-adaptive	Adaptive
$\lvert G\rvert = 1$	2	$\aleph_1$
$\lvert G\rvert = k$	$k+1$	$\mathfrak{m}(\sigma\text{-}k\text{-linked}) \leq ? \leq \mathbf{add}(\mathcal{L})$
$\lvert G\rvert = f(n)$	$\mathbf{add}(\mathcal{L})$	$\mathbf{add}(\mathcal{L})$
G finite	$\mathfrak{b}$	$\mathfrak{b}$
$\omega - G$ infinite	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$
$G \subsetneqq \omega$	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$

Table 2: Evasion numbers for local prediction

	Non-adaptive	Adaptive
$\lvert G\rvert = 1$	2	$\mathfrak{e}$
$\lvert G\rvert = k$	$k+1$	$\mathfrak{e} \leq ? \leq \mathbf{cov}(\mathcal{B}), \mathbf{non}(\mathcal{B})$
$\lvert G\rvert = f(n)$	$\min\{\mathfrak{e}, \mathfrak{b}\}$	$\mathfrak{e} \leq ? \leq \mathbf{cov}(\mathcal{B}), \mathbf{non}(\mathcal{B})$
G finite	$\mathfrak{b}$	$\mathfrak{e}, \mathfrak{b} \leq ? \leq \mathfrak{d}, \mathbf{non}(\mathcal{B})$
$\omega - G$ infinite	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$
$G \subsetneqq \omega$	$\mathbf{non}(\mathcal{B})$	$\mathbf{non}(\mathcal{B})$

One could regard these entries with question marks as defining four more cardinal characteristics. On the other hand, one might regard the entry $\mathfrak{e}$ in Table 2 as a euphemism for a question mark with the bounds given in Theorem 10.4. The difference between $\mathfrak{e}$ and the question marks is that the former has been studied enough to indicate that it differs from the previously studied characteristics, while the question marks might well reduce to something simpler. It is, however, known [34, p. 359] that, in the global prediction table, neither of the inequalities around the question mark can be improved to a provable equality.

In the following paragraphs, we give reasons for the table entries, leaving some details to the reader.

$\boxed{2 \text{ and } k+1}$ For both global and local prediction, $k+1$ distinct constant functions evade any non-adaptive predictor of k-element sets. And any k functions clearly can be predicted.

$\boxed{\mathbf{add}(\mathcal{L})}$ In the non-adaptive column of Table 1, the occurrence of $\mathbf{add}(\mathcal{L})$ expresses Theorem 5.14 and the remark following it. The occurrence in the adaptive column comes from the fact that an adaptive $f(n)$-valued predictor π gives rise to a non-adaptive $f'(n)$-valued predictor π' (for a larger f') such that all functions globally predicted by π are also globally predicted by π'. Given π, associate to each $s \in {}^{<\omega}\omega$ and each natural number n the set $\pi_s(n)$ of all possible values of $x(n)$ for functions $x \in {}^{\omega}\omega$ that start with s and are correctly predicted by π thereafter. (That is, $x(k)$ is $s(k)$ for $k < \text{length}(s)$ and $\pi(x{\upharpoonright}k)$ for all larger k.) Also fix an enumeration of ${}^{<\omega}\omega$ in an ω-sequence. Let $\pi'(n)$ be the union of the sets $\pi_s(n)$ as s ranges over the first n elements

Table 3: Evasion numbers for infinite prediction

	Non-adaptive	Adaptive
$\lvert G\rvert = 1$	$\mathbf{cov}(\mathcal{B})$	$\mathbf{cov}(\mathcal{B})$
$\lvert G\rvert = k$	$\mathbf{cov}(\mathcal{B})$	$\mathbf{cov}(\mathcal{B})$
$\lvert G\rvert = f(n)$	$\mathbf{cov}(\mathcal{B})$	$\mathbf{cov}(\mathcal{B})$
G finite	$\mathfrak{d}$	$\mathfrak{d}$
$\omega - G$ infinite	$\mathfrak{c}$	$\mathfrak{c}$
$G \subsetneq \omega$	$\mathfrak{c}$	$\mathfrak{c}$

of ${}^{<\omega}\omega$. It is easy to find an appropriate f' depending only on f and to verify that π' globally predicts everything that π does.

$\boxed{\mathfrak{b}}$ We may take a finite-valued predictor's guesses to be initial segments of ω, i.e., natural numbers, a guess being correct if it is greater than the actual value of the function being guessed. In this light, the occurrence of $\mathfrak{b}$ in the non-adaptive column of Table 1 expresses just the definition of $\mathfrak{b}$. The occurrence in the adaptive column is justified by an argument analogous to that in the discussion of $\mathbf{add}(\mathcal{L})$ above.

As for the occurrence in Table 2, consider any unbounded family $\mathcal{E}$ of $\mathfrak{b}$ non-decreasing functions. We shall see that they evade local prediction by any non-adaptive, finite-valued predictor (π, D). As above, we assume the values of π are natural numbers. Define $\pi' : \omega \to \omega$ by letting $\pi'(n)$ be the value of π at the next member of D after n. By our choice of $\mathcal{E}$, it contains a member x not dominated by π'. Since x is non-decreasing, if (π, D) locally predicted it then for all sufficiently large $n \in \omega$ we would have, letting k be the next element of D after n,

$$x(n) \leq x(k) < \pi(k) = \pi'(n).$$

This contradicts the choice of x, so x evades (π, D).

$\boxed{\mathbf{non}(\mathcal{B})}$ Let us consider first the bottom row in Tables 1 and 2, where the guesses are proper subsets of ω. Without loss of generality, we may assume that the guesses are complements of singletons. We'll write $\tilde{\pi}(n)$ for the number absent from $\pi(n)$ (and similarly for $\tilde{\pi}(s)$ in the adaptive case). Part 2 of Theorem 5.9 says that the bottom entry in the non-adaptive column of Table 1 is $\mathbf{non}(\mathcal{B})$. By monotonicity, the other entries in the bottom row of Tables 1 and 2 are no smaller. They are no larger because any predictor predicts globally or locally only a meager subset of ${}^{\omega}\omega$.

To justify the next-to-bottom row in Tables 1 and 2, where the guesses are co-infinite, it suffices, thanks to monotonicity, to show that a family $\mathcal{F}$ of fewer than $\mathbf{non}(\mathcal{B})$ functions cannot evade global prediction by non-adaptive co-infinite predictors. Fix a map $p : \omega \to \omega$ such that every $p^{-1}\{n\}$ is infinite. The fewer than $\mathbf{non}(\mathcal{B})$ functions $p \circ f$ for $f \in \mathcal{F}$ are globally predicted by a predictor π of proper subsets of ω (by Theorem 5.9); so the functions in $\mathcal{F}$ are predicted by $p^{-1} \circ \pi$, whose values are co-infinite.

$\boxed{\mathfrak{c}}$ Clearly, all evasion numbers are $\leq \mathfrak{c}$, since any predictor, even an adaptive predictor of co-singletons, can be completely evaded by some function. On the other hand, to evade infinite prediction even by non-adaptive predictors π of co-singletons requires $\mathfrak{c}$ functions, because one needs functions eventually equal to any prescribed $f : \omega \to \omega$ (giving the values omitted by the predictor). To obtain the same result with "co-infinite" in place of "co-singleton", use the same "compose with p" trick as in the discussion of $\mathbf{non}(\mathcal{B})$ above.

$\boxed{\mathfrak{d}}$ As in the discussion of $\mathfrak{b}$ above, we may assume that predictors give natural numbers, intended as upper bounds for the values to be guessed. Then the $\mathfrak{d}$ in the non-adaptive column of Table 3 is justified by the definition of $\mathfrak{d}$. To see that $\mathfrak{d}$ functions suffice to evade even adaptive prediction, take a family of $\mathfrak{d}$ adaptive predictors that dominate all the adaptive predictors, and choose for each of these predictors some function evading it.

$\boxed{\mathbf{cov}(\mathcal{B})}$ Of the six occurrences of $\mathbf{cov}(\mathcal{B})$ in Table 3, the top one in the non-adaptive column expresses Part 1 of Theorem 5.9. To justify the rest, it suffices by monotonicity to check that $\mathbf{cov}(\mathcal{B})$ functions suffice to evade infinite prediction by adaptive predictors whose guesses at n have cardinality $f(n)$. We do this first for non-adaptive predictors, by a modification of the argument for Theorem 5.9, and then we show how to extend the result to the adaptive case. We may assume $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$, for otherwise the desired information follows by monotonicity from the $\mathfrak{d}$'s in the next row of Table 3.

Fix $\mathbf{cov}(\mathcal{B})$ chopped reals (x_α, Π_α) with no single $y \in {}^\omega 2$ matching them all (by Theorem 5.2). Since $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$, fix an interval partition Θ not dominated by any of the Π_α. As in the proof of Theorem 5.9, this means that every Π_α contains infinitely many intervals each covered by two consecutive intervals of Θ.

Define $g(n) = 2 \cdot \sum_{k \leq n} f(k) - 1$. For each α and each n, we define a set $q_\alpha(n)$ as follows. Find $g(n)$ disjoint pairs of consecutive Θ-intervals, each pair covering a Π_α-interval; let the unions of these pairs be $J_0, \dots, J_{g(n)-1}$. Then let $q_\alpha(n) = \{x_\alpha \restriction J_0, \dots, x_\alpha \restriction J_{g(n)-1}\}$. So each $q_\alpha(n)$ is a set of $g(n)$ functions into 2, each having as domain the union of two consecutive Θ-intervals, and such that the domains of different members of $q_\alpha(n)$ are disjoint.

By coding their values as natural numbers, we can regard the q_α as functions $\omega \to \omega$. We claim that these $\mathbf{cov}(\mathcal{B})$ functions evade infinite prediction by any non-adaptive, $f(n)$-valued predictor, i.e., any f-slalom.

Suppose this failed. So there is a function S assigning to each $n \in \omega$ a set $S(n)$ of $f(n)$ elements such that for each α we have $\exists^\infty n\ (q_\alpha(n) \in S(n))$. Without loss of generality, each element s of $S(n)$ is a set of $g(n)$ functions into 2, each having as domain the union of two consecutive Θ-intervals, and such that the domains of different members of s are disjoint. Now define $y \in {}^\omega 2$ by the following recursion, defining y on $2 \cdot f(n)$ Θ-intervals at step n.

Suppose steps 0 through $n-1$ have been completed, so y is already defined on $2 \cdot \sum_{k<n} f(k)$ Θ-intervals. From each $s \in S(n)$, remove those partial

functions whose domains overlap the set where y is already defined. That removes at most $2 \cdot \sum_{k<n} f(k)$, so at least $2 \cdot f(n) - 1$ are left, since s had cardinality $g(n)$. Go through the $f(n)$ sets so obtained (one from each $s \in S(n)$) in some order, picking one function from each, making sure that the domain of each chosen function is disjoint from the domains of the previously chosen functions. Since each of the domains is the union of two consecutive Θ-intervals, each domain can overlap at most two others. Thus, there are at least $2 \cdot f(n) - 1$ options for the first choice, at least $2 \cdot f(n) - 3$ for the second, and so on down to at least 1 option for the $f(n)$th choice. So all the choices can be made. Then extend y to agree with each of the chosen functions on its domain. This completes step n of the recursion. After all steps are completed, if y is not defined on all of ω, extend it arbitrarily.

For each α, there are infinitely many n with $q_\alpha(n) \in S(n)$, so $q_\alpha(n)$ is one of the s's considered at step n in the definition of y. So some element z of $q_\alpha(n)$ becomes part of y. But that z is $x_\alpha \restriction J$ for some J that includes an interval of Π_α. So y matches each (x_α, Π_α), contrary to our choice of these chopped reals. This contradiction shows that the (coded) q_α are evasive as claimed.

It remains to extend the result to adaptive predictors whose guesses have size $f(n)$. Such a predictor is a function $\pi : {}^{<\omega}\omega \to [\omega]^{<\omega}$ (with $\pi(s) \in [\omega]^{f(n)}$ if $s \in {}^n\omega$). Identifying the domain ${}^{<\omega}\omega$ with ω via some bijective coding, we can view every such π as a non-adaptive predictor whose guesses have size $f'(n)$ for a certain f' (that depends on f and the coding). Applying the preceding argument to these non-adaptive predictors, and then reversing the coding process, we get a family $\mathcal{E}$ of $\mathbf{cov}(\mathcal{B})$ functions ${}^{<\omega}\omega \to \omega$ such that, for every adaptive predictor π as above,

$$\exists z \in \mathcal{E}\, \forall^\infty s \in {}^{<\omega}\omega\ (z(s) \notin \pi(s)).$$

Use each $z \in \mathcal{E}$ to recursively define a $z' : \omega \to \omega$ by $z'(n) = z(z' \restriction n)$. Then the family $\mathcal{E}' = \{z' : z \in \mathcal{E}\}$ evades infinite prediction by any π as above. Indeed, with π and z as above, we have, for all but finitely many n, that

$$z'(n) = z(z' \restriction n) \notin \pi(z' \restriction n).$$

$\boxed{\mathfrak{e}}$ The entry $\mathfrak{e}$ in Table 2 is just the definition of $\mathfrak{e}$. In view of what we just proved about $\mathbf{cov}(\mathcal{B})$, we get $\mathfrak{e} \leq \mathbf{cov}(\mathcal{B})$ by monotonicity. This completes the proof of Theorem 10.4 above.

$\boxed{\aleph_1}$ Any countably many functions $h_i : \omega \to \omega$ are globally predicted by the adaptive predictor defined by requiring $\pi(s) = h_i(n)$ if s has length n and i is the first index with $h_i \restriction n = s$. Such a π predicts each h_i accurately at all n beyond the points where h_i first differs from the earlier h_j's.

On the other hand, any adaptive predictor whose values are singletons can globally predict only countably many functions. Indeed, a function h globally predicted by such a π is completely determined by the finite part of h consisting of the values not correctly guessed by π.

$\boxed{\mathfrak{m}(\sigma\text{-}k\text{-linked}) \leq ? \leq \mathbf{add}(\mathcal{L})}$ Monotonicity gives us the upper bound of $\mathbf{add}(\mathcal{L})$. To establish the lower bound, we assume $\mathrm{MA}_\kappa(\sigma\text{-}k\text{-linked})$ and we prove that any family $\mathcal{H}$ of κ functions $\omega \to \omega$ can be globally predicted by an adaptive predictor π with k-element guesses, i.e., $\pi : {}^{<\omega}\omega \to [\omega]^k$. Let P be the set of pairs (p, F) where p is a finite partial map from ${}^{<\omega}\omega$ to $[\omega]^k$ and F is a finite subset of $\mathcal{H}$ with the following "branching restriction": For any two functions in F, if s is their longest common initial segment, then $s \in \mathrm{dom}(p)$. (The "meaning" of (p, F) is that p is part of the desired predictor and that each function in F will be guessed correctly except possibly at those places where p is defined.) Partially order P by putting $(p', F') \leq (p, F)$ if $p \subseteq p'$, $F \subseteq F'$, and, whenever $f \in F$ and $f \restriction n \in \mathrm{dom}(p') - \mathrm{dom}(p)$ then $f(n) \in p'(f \restriction n)$.

This partial ordering is σ-k-linked because any k elements (p, F_i) with the same first component have a common lower bound, constructed as follows. First form $(p, \bigcup_i F_i)$. If this is not the desired lower bound, it is because the branching restriction is violated. So there are some $s \notin \mathrm{dom}(p)$ that are the largest common initial segments of some $f \in F_i$ and $g \in F_j$. Then $i \neq j$ because each of the (p, F_i) satisfied the branching restriction. So any such s, say of length n, is an initial segment of at most k members of $\bigcup_i F_i$ that have different values at n. But then we can extend p by defining $p(s)$ to be a k-set containing the values at n of those $\leq k$ members of $\bigcup_i F_i$. Doing this for each such s, we get the desired lower bound.

Applying $\mathrm{MA}_\kappa(\sigma\text{-}k\text{-linked})$, we get a set $G \subseteq P$ generic with respect to the dense sets $\{(p, F) \in P : s \in \mathrm{dom}(p)\}$ for all $s \in {}^{<\omega}\omega$ and $\{(p, F) \in P : f \in F\}$ for all $f \in \mathcal{H}$. (The former is dense thanks to the branching restriction. To verify the density of the latter, given any (p, F) and any $f \in \mathcal{H} - F$, first form $(p, F \cup \{f\})$. If the branching restriction is violated, extend p so as to be defined at the new branching locations. Here we need that $k \geq 2$.) Let π be the union of all the first components of the pairs $(p, F) \in G$. It is routine to check (as in the proof of Theorem 7.7) that this π is an adaptive predictor with k-set guesses, globally predicting every function from $\mathcal{H}$.

$\boxed{\mathfrak{e} \leq ? \leq \mathbf{cov}(\mathcal{B}), \mathbf{non}(\mathcal{B})}$ Monotonicity implies all three inequalities.

$\boxed{\mathfrak{e}, \mathfrak{b} \leq ? \leq \mathfrak{d}, \mathbf{non}(\mathcal{B})}$ Again, monotonicity implies all four inequalities.

$\boxed{\min\{\mathfrak{e}, \mathfrak{b}\}}$ This is [33, Lemma 2.5]. Monotonicity gives the upper bound $\mathfrak{b}$. The proof that $\mathfrak{e}$ is also an upper bound is essentially the same as the proof of $\mathbf{add}(\mathcal{L}) \leq \mathfrak{e}$ in Theorem 10.4. The only difference is that here we are dealing with "partial slaloms", i.e., functions S defined on some infinite $D \subseteq \omega$ and satisfying $|S(n)| = f(n)$ for all $n \in D$. Instead of predicting at all i_n, as in the earlier argument, we now predict at i_n for $n \in D$.

To prove that $\min\{\mathfrak{e}, \mathfrak{b}\}$ is also a lower bound, let $\mathcal{H}$ be a family of fewer than $\min\{\mathfrak{e}, \mathfrak{b}\}$ functions; we must find a partial slalom (in the sense defined above) such that each $h \in \mathcal{H}$ satisfies $\forall^\infty n \in D\,(h(n) \in S(n))$. Since there are fewer than $\mathfrak{b}$ functions in $\mathcal{H}$, we can find a single, strictly increasing

$g : \omega \to \omega$ that dominates them all. Let $\{I_n : n \in \omega\}$ be an interval partition such that, if a is the left endpoint of any I_n, then there is some $i_n \in I_n$ with $f(i_n) \geq g(a)^a$. To each $h \in \mathcal{H}$ associate the function defined by $h'(n) = h \restriction I_n$. Since the number of such h' is $< \mathfrak{e}$, there is an adaptive predictor of singletons (D', π') that locally predicts all the h'; that is,

$$\forall h \in \mathcal{H}\, \forall^\infty n \in D' \ (h'(n) = \pi'(h' \restriction n)).$$

Do the following for each $n \in D'$. Let a be the left endpoint of I_n, and recall that our interval partition was chosen so that $g(a)^a \leq f(i_n)$ for some $i_n \in I_n$. Consider all functions s from a into $g(a)$; there are exactly $g(a)^a$, and thus no more than $f(i_n)$, of them. Each gives an s' by $s'(m) = s \restriction I_m$ for $m < n$. Then $\pi'(s')$ is some function $I_n \to \omega$; evaluate it at i_n. Doing this for each s gives no more than $f(i_n)$ numbers; let $S(i_n)$ be the set of these numbers.

Doing this for all $n \in D'$, we get a partial slalom defined on $D = \{i_n : n \in D'\}$. For each $h \in \mathcal{H}$, if π' predicted $h'(n)$ correctly (where $n \in D'$), then $h(i_n) \in S(i_n)$. So we have the desired partial slalom.

10.6 Remark. The variants of evasion discussed at the beginning of this section ($\mathfrak{e}_g$ and $\mathfrak{e}_{\mathrm{ubd}}$) can be combined with some of the variants in Tables 1 to 3. Finite and co-infinite predictors no longer make sense. When, as in the case of $\mathfrak{e}_g$, the functions to be predicted are bounded by a fixed g, we need to pay attention to the function f in the $|G| = f(n)$ lines of the tables; it is no longer the case that any function tending to infinity is equivalent to any other. Also, in this situation, the co-singleton case becomes a special case of $|G| = f(n)$ with $f(n) = g(n) - 1$. Thus, we would have three-line tables for these variants. We omit any further discussion of these, since little is known about them beyond carrying over some of the arguments presented above.

Another variation, lying between global and local, was introduced by Kamo [65]. Say that a function $\pi : {}^{<\omega}\omega \to \omega$ *constantly predicts* $x : \omega \to \omega$ if there is $n \in \omega$ such that, with finitely many exceptions, any interval $[m, m+n)$ of length n contains some k such that $x(k) = \pi(x \restriction k)$. This concept has been studied further by Kamo, Kada, and Brendle; see for example [32] and the references there.

Finally, all the evasion cardinals considered in this section have duals of the form: the smallest number of predictors needed to predict all functions. These too have been little studied, but there is one remarkable result concerning the number of f-slaloms needed to globally predict all members of $\prod_n g(n)$. Goldstern and Shelah [53] showed that this cardinal can vary with f and g and in fact that in some models of set theory uncountably many cardinals are of this form (infinitely many with recursive f and g).

11. Forcing

In this final section, we describe the effect of various forcing constructions on cardinal characteristics. We shall discuss only the most commonly used forcing notions and their most natural iterations; for a far more extensive discussion, see [5, Chaps. 3, 6, and 7].

Most of the forcing notions we consider are designed to add a real with some prescribed properties, and the properties are often closely connected with some Borel relation $\mathbf{A} = (A_-, A_+, A)$ (where we use the notation of Sect. 4). Specifically, we say that a real x in a forcing extension *solves* $\mathbf{A}$ (over the ground model) if $x \in \tilde{A}_+$ and $(a, x) \in \tilde{A}$ for all $a \in A_-$ in the ground model. Here $\tilde{A}$ denotes the relation in the extension having the same Borel code as A has in the ground model, and similarly for $\tilde{A}_+$ etc., but we shall often omit the tilde since no confusion will result.

If there is a morphism $\varphi : \mathbf{A} \to \mathbf{B}$ whose φ_+ component is Borel, so that $\tilde{\varphi}_+$ makes sense, and if x solves $\mathbf{A}$ then $\tilde{\varphi}_+(x)$ solves $\mathbf{B}$. Indeed, given any $b \in B_-$ in the ground model, let $a = \varphi_-(b) \in A_-$. The statement

$$\forall u \in A_+ \, (aAu \implies bB\varphi_+(u))$$

is true in V and absolute when expressed in terms of the Borel codes of A_+, A, B, and φ_+. Thus it is true in any forcing extension that

$$\forall u \in \tilde{A}_+ \, (a\tilde{A}u \implies b\tilde{B}\tilde{\varphi}_+(u)).$$

Since a is in the ground model, we have $a\tilde{A}x$ and therefore $b\tilde{B}\tilde{\varphi}_+(x)$ as claimed.

Notice that we do not need A_-, B_- or φ_- to be Borel in the preceding discussion.

It is easy to check that if x solves $\mathbf{A}$ and y solves $\mathbf{B}$ then (x, y) solves the conjunction $\mathbf{A} \wedge \mathbf{B}$ and the product $\mathbf{A} \times \mathbf{B}$. For sequential composition, the situation is more complicated, because even if $\mathbf{A}$ and $\mathbf{B}$ are Borel, the set of challenges in $\mathbf{A}; \mathbf{B}$ is of higher type, so this relation cannot be Borel. However, if we have a morphism $\varphi : \mathbf{A}; \mathbf{B} \to \mathbf{C}$ then under suitable Borelness hypotheses we can conclude, by a proof very similar to that above, that if $V \subseteq V' \subseteq V''$, if $x \in V'$ solves $\mathbf{A}$ over V, and if $y \in V''$ solves $\mathbf{B}$ over V' then $\tilde{\varphi}_+(x, y)$ solves $\mathbf{C}$ over V. Most of the "suitable Borelness hypotheses" are the ones obviously needed for the statement to make sense: A_+, A, B_+, B, B_-, and φ_+ must be Borel. (B_-, unlike A_-, must be Borel so that solving $\mathbf{B}$ over V', not over V, makes sense.) But one additional Borelness hypothesis is needed for the proof. If we regard $\varphi_- : C_- \to A_- \times {}^{A_+}B_-$ as a pair of functions $\alpha : C_- \to A_-$ and $\beta : C_- \to {}^{A_+}B_-$, and if we regard β as $\beta' : C_- \times A_+ \to B_-$ (where $\beta'(c, a) = \beta(c)(a)$), then we need that β' is Borel. We leave the details to the reader.

Most of the iterations we consider will be either finite support iterations of c.c.c. forcing notions or countable support iterations of proper forcing notions. For general information about iterations, see Abraham's chapter in this Handbook or [62, 9, 97]. All the proper forcing notions considered below satisfy Baumgartner's Axiom A [9], which is stronger and usually easier to check than properness. We usually write V for the ground model and V_α for the model obtained after α stages of an iteration.

11.1. Finite Support Iteration and Martin's Axiom

A finite support iteration of c.c.c. forcing is equivalent to a single c.c.c. forcing [106] and therefore preserves cardinals. Also, if the length λ of the iteration has uncountable cofinality, then every real in the final extension V_λ is already in an intermediate extension V_α, $\alpha < \lambda$. If, cofinally often in such an iteration, one adjoins a real solving $\mathbf{A}^\perp$ over the previous model, then in V_λ the norm $\|\mathbf{A}\|$ will be at least $\mathrm{cf}(\lambda)$. Indeed, given any fewer than $\mathrm{cf}(\lambda)$ members of A_+ in V_λ, we can find an $\alpha < \lambda$ such that all these reals are in V_α; increasing α if necessary, we can, by hypothesis, arrange that $V_{\alpha+1}$ contains a real $x \in A_-$ solving $\mathbf{A}^\perp$ over V_α. But that means in particular that x is A-related to none of our given fewer than $\mathrm{cf}(\lambda)$ reals.

The preceding remarks indicate a way to make a characteristic $\|\mathbf{A}\|$ large, namely iterate a c.c.c. forcing that solves $\mathbf{A}^\perp$, with finite support, for λ stages, where λ is regular and large.

Applying this method with all c.c.c. forcings of size $< \lambda$ (in all the intermediate models) suitably interleaved, one obtains a model of MA and $\mathfrak{c} = \lambda$ provided GCH held in the ground model. If one uses only σ-centered posets in the iteration, then one obtains a model of MA(σ-centered), i.e., $\mathfrak{p} = \mathfrak{c}$ (see Theorem 7.12), but MA fails and in fact $\mathbf{cov}(\mathcal{L}) = \aleph_1$ (see [5, Sect. 6.5D]). Similar constructions give models satisfying various fragments of MA while violating others; see Appendix B1 of [48] and the references therein.

To prove independence results in the theory of cardinal characteristics, one needs techniques for making one characteristic large while keeping another small. As indicated above, it is not difficult to make a chosen characteristic large, but it is usually difficult to prove that another characteristic remains small. In fact, some characteristics cannot be kept small in a non-trivial finite support iteration. The reason is that such an iteration always introduces Cohen reals at all limit stages of cofinality ω. Cohen reals solve various Borel relations (see below), notably $\mathbf{Cov}(\mathcal{B})^\perp$, and therefore finite support iterations cannot avoid making certain characteristics, notably $\mathbf{cov}(\mathcal{B})$, large.

11.2. Countable Support Proper Iteration

A countable support iteration of proper forcing is equivalent to a single proper forcing [97, Theorem 3.2] and therefore preserves $\aleph_1$. For our purposes, it will be important to also preserve larger cardinals, and this is usually ensured by an appeal to [97, Theorem 4.1], which gives the $<\aleph_2$-chain condition provided (1) CH holds in the ground model, (2) the forcing notion used to produce $V_{\alpha+1}$ from V_α has cardinality at most $\mathfrak{c}$ in V_α, and (3) the length of the iteration is at most ω_2. (See Abraham's chapter in this Handbook.) The first two of these provisos will be satisfied automatically in the situations we are interested in, but the third is a real impediment. This limitation on the length of the iteration prevents us from making the continuum arbitrarily large with countable support iterations; only $\mathfrak{c} = \aleph_2$ can be achieved. It is shown in [11] that iterating Sacks forcing (which is proper) with countable support for ω_2+1 steps collapses $\aleph_2$. Also, it is pointed out in [51, Remark 0.3] that a countable support ω_1-stage iteration of any non-trivial forcings will collapse $\mathfrak{c}$ to $\aleph_1$.

Our inability to produce larger values of $\mathfrak{c}$ with the kind of detailed control available for countable support iterations has prevented the solution of several problems. For example, although we have models with no P-points and models with no Q-points (both obtained by countable support proper iterations), we do not know how to achieve both simultaneously. By Theorems 9.25 and 9.27, such a model would need to have $\mathbf{cov}(\mathcal{B}) < \mathfrak{d} < \mathfrak{c}$ and therefore $\mathfrak{c} \geq \aleph_3$. Similarly, we have no model for $\mathfrak{p} < \mathfrak{t}$; by Theorem 6.25, such a model would need to have $\aleph_2 \leq \mathfrak{p} < \mathfrak{t}$ and therefore $\mathfrak{c} \geq \aleph_3$. (Brendle has pointed out, however, that there is no a priori reason why a model of $\mathfrak{p} < \mathfrak{t}$ could not be produced by finite support iteration. This contrasts with the situation for producing a model with neither P-points nor Q-points; here finite support iteration has no chance because the Cohen reals it introduces make $\mathbf{cov}(\mathcal{B})$ large, and then Theorem 9.26 produces a selective ultrafilter.)

The "$\aleph_3$ barrier" is widely regarded as merely a technical problem. It has, however, resisted our efforts long enough to suggest that perhaps our inability to produce certain models is caused not by our technical deficiencies but by the non-existence of the models.

In the rest of this section, countable support iterations will always be of the sort discussed above; that is, GCH will hold in the ground model, each step will be a proper forcing notion of cardinality at most $\mathfrak{c}$, and the length of the iteration will be ω_2. Thus, all cardinals are preserved. Furthermore, every real in the final model V_{ω_2} is already in some intermediate model V_α, $\alpha < \omega_2$. Thus, as with finite support iterations, we can increase a characteristic $\|\mathbf{A}\|$ (but only up to $\aleph_2$) by cofinally often adding reals that solve $\mathbf{A}^\perp$. To prove independence results, we want to simultaneously keep some other characteristic small, and for this purpose there are a large number of powerful preservation theorems; see [97, 51, 45]. For example, in a countable support proper iteration, if each $V_\alpha \cap {}^\omega\omega$ is a dominating family in $V_{\alpha+1}$ then $V \cap {}^\omega\omega$ is dominating in V_λ. In other words, if $V_{\alpha+1}$ never contains a real solving $\mathfrak{D}^\perp$ over V_α, then $\mathfrak{d}$ remains $\aleph_1$ in the final model.

11.1 Remark. Zapletal [115] has shown that, under a strong large cardinal assumption (a proper class of measurable Woodin cardinals), many cardinal characteristics $\mathfrak{y}$ admit an optimal notion of forcing $P_\mathfrak{y}$ to make them large. Optimality means that, if $\mathfrak{x}$ is any tame characteristic and $\mathfrak{x} < \mathfrak{y}$ can be forced by some set forcing notion, then it is forced by $P_\mathfrak{y}$. The notion of tameness used here is somewhat more general than being the norm of a projective relation, in that it permits some additional restrictions on the set $Y \subseteq A_+$ in Definition 4.1 of norms. All norms of Borel relations are tame, and so are, for example, $\mathfrak{p}$, $\mathfrak{t}$, and $\mathfrak{u}$, but not, for example, $\mathfrak{g}$.

Zapletal gives the following specific examples (among others) of optimal forcings for certain characteristics. See the following subsections for descriptions of these forcings. Cohen forcing is optimal for $\mathbf{cov}(\mathcal{B})$. Random forcing is optimal for $\mathbf{cov}(\mathcal{L})$. Sacks forcing is optimal for $\mathfrak{c}$. Laver forcing is optimal for $\mathfrak{b}$. Mathias forcing is optimal for $\mathfrak{h}$. Miller forcing is optimal for $\mathfrak{d}$. For a somewhat more extensive description of this work of Zapletal, see the final section of Bartoszyński's chapter of this Handbook, and for the details see [116].

11.3. Cohen Reals

The Cohen forcing poset, ${}^{<\omega}2$ ordered by reverse inclusion, adjoins a real $c : \omega \to 2$ (namely the union of the conditions in the generic set) that matches every chopped real (x, Π) from the ground model. Indeed, for each (x, Π) and each $n \in \omega$, the forcing conditions that agree with x on at least one interval of Π beyond n form a dense set in the ground model, so by genericity one of them must be included in c. Thus, a Cohen real solves $\mathbf{Cov}(\mathcal{B})^{\perp}$. (In fact, this characterizes Cohen reals.)

The usual way to iterate Cohen forcing is with finite support. Since the forcing poset is absolute, finite support iteration and finite support product are equivalent. The resulting model (when the ground model satisfies GCH) is usually called "the Cohen model" independently of the number λ of factors; for more precision, one says "the λ Cohen real model". This is the model used by Cohen [40] for his proof of the independence of GCH. Because of the c.c.c., every real in the Cohen model is already in the intermediate model generated by (the restriction of the generic filter to) some countable subproduct. Such a countable product (indeed, any countable atomless forcing notion) is equivalent to the single forcing ${}^{<\omega}2$. Thus any real in the Cohen model is in a submodel generated by a single Cohen real.

Since a Cohen real solves $\mathbf{Cov}(\mathcal{B})^{\perp}$, the λ Cohen real model (for any uncountable regular λ) has $\mathbf{cov}(\mathcal{B}) = \lambda = \mathfrak{c}$. It follows that all cardinals in the right half of Cichoń's diagram equal λ in this model. Furthermore, since

$$\mathbf{cov}(\mathcal{B}) \leq \mathfrak{r} \leq \mathfrak{u}, \mathfrak{i},$$

all these cardinals also equal $\mathfrak{c}$ in the Cohen model. (One can also see directly that a Cohen real splits all ground model reals, so $\mathfrak{r} = \lambda$.)

On the other hand, $\mathbf{non}(\mathcal{B}) = \aleph_1$ in the Cohen model, the set of ground model reals being non-meager. To prove this, we must show that every chopped real (x, Π) in the extension is matched by some ground model real. By our remarks above, we may assume that (x, Π) is in the forcing extension by a single Cohen real. In the ground model, we construct a real y such that for no condition $p \in {}^{<\omega}2$ and natural number n can p force "y does not agree with x on any interval of Π beyond n". Such a y is easily built by a recursion of length ω in which each step defines $y(k)$ for finitely many k and takes care of one pair (p, n). Taking care of (p, n) means to proceed as follows. Extend p to a condition q deciding a particular value for the restriction of x to the first interval $I \in \Pi$ whose left endpoint is greater than n and greater than all points already in the domain of y. Then extend y to agree with that restriction of x. Thus, q forces that y and x agree on an interval of Π beyond

n, so p cannot force the contrary. (Note that this proof shows more than claimed. Not only the set of all ground model reals but any non-meager set in the ground model remains non-meager in a Cohen extension.)

In fact, $\mathbf{non}(\mathcal{B}) = \aleph_1$ holds in any model obtained by adjoining at least $\aleph_1$ Cohen reals to any ground model whatsoever. The reason is that $\aleph_1$ Cohen reals constitute a non-meager set.

From $\mathbf{non}(\mathcal{B}) = \aleph_1$, it immediately follows that all cardinals in the left half of Cichoń's diagram are $\aleph_1$. Furthermore, we have

$$\mathbf{non}(\mathcal{B}) \geq \mathfrak{b} \geq \mathfrak{h} \geq \mathfrak{t} \geq \mathfrak{p} \geq \mathfrak{m}, \quad \mathbf{non}(\mathcal{B}) \geq \mathfrak{s}, \quad \text{and} \quad \mathbf{non}(\mathcal{B}) \geq \mathfrak{e},$$

so all these cardinals are also $\aleph_1$ in the Cohen model.

Kunen showed [68, Theorem VIII.2.3] that $\mathfrak{a} = \aleph_1$ in the Cohen model. The idea is to construct, by transfinite induction in the ground model (where CH is available) a MAD family that remains MAD when one adds a Cohen real to the universe. It therefore remains MAD in any Cohen extension, since a failure to remain MAD would be witnessed by a single real. We omit the construction, since a similar one is given in the discussion of random reals below.

Finally, we cite from [18] the result that $\mathfrak{g} = \aleph_1$ in the Cohen model (or indeed in any model obtained by adjoining at least $\aleph_1$ Cohen reals to any model at all).

11.4. Random Reals

The notion of forcing to add one random real is the Boolean algebra of Borel sets modulo sets of Lebesgue measure zero (in any of $[0,1]$, $\mathbb{R}$, ${}^\omega 2$, ${}^\omega\omega$; they are all equivalent). (Here and in general, when one refers to a Boolean algebra as a notion of forcing, one means the algebra minus its zero element.) Random forcing was introduced by Solovay [104, 105]. A generic G determines a real r, called "random", such that, if B is any Borel set in the ground model, then $r \in \tilde{B}$ if and only if $[B] \in G$. (For basic intervals B, this is the definition of r; for other B it is a theorem.) Thus, r solves $\mathbf{Cov}(\mathcal{L})^\perp$. This property characterizes random reals.

Although random forcing can be iterated with finite support or with countable support (being c.c.c. and therefore proper), the most common way to add many random reals uses a large measure algebra, namely the algebra of Borel subsets modulo measure zero sets in ${}^I 2$ for large I. The measure here is the product measure induced by the uniform measure on 2. This forcing adds a random function $f : I \to 2$ whose restrictions to countable subsets of I in V amount to random reals. One often starts with a ground model satisfying GCH, takes $I = \lambda \times \omega$, and regards the forcing as adding the λ random reals $r_\alpha : \omega \to 2 : n \mapsto f(\alpha, n)$. Any real in this λ random reals model is in the submodel generated by countably many of the r_α, and this submodel is equivalent to one obtained by adjoining a single random real to the ground model.

Because a random real solves $\mathbf{Cov}(\mathcal{L})^\perp$, the λ random reals model has (for uncountable regular λ) $\mathbf{cov}(\mathcal{L}) = \lambda = \mathfrak{c}$. Therefore, all the cardinals in the top row of Cichoń's diagram equal $\mathfrak{c}$ in this model, and so do $\mathfrak{r}$, $\mathfrak{u}$, and $\mathfrak{i}$.

On the other hand, $\mathfrak{d}$ and $\mathbf{non}(\mathcal{L})$ are both $\aleph_1$, as in the ground model. More generally, if uncountably many random reals are added (with the usual measure algebra forcing) to any ground model, then in the extension $\mathfrak{d}$ will have the same value as in the ground model while $\mathbf{non}(\mathcal{L})$ will be $\aleph_1$. The former follows from the fact that all reals in a random extension are majorized by ground model reals. The latter follows from the fact that any $\aleph_1$ of the added random reals form a set of positive outer measure. (A measure-zero Borel set, or rather its code, depends on only countably many of the added random reals; all the rest of the added random reals, being random over an intermediate model containing the code, must be outside that Borel set.)

It follows that all the cardinals in the middle and bottom rows of Cichoń's diagram are $\aleph_1$, and therefore so are $\mathfrak{s}$, $\mathfrak{e}$, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{m}$.

Finally, we show, adapting Kunen's proof for the Cohen model, that $\mathfrak{a} = \aleph_1$ in the random model. Since every real in the random model is in a submodel that can be generated by a single random real, it suffices to construct a family $\mathcal{A}$ in the ground model that is MAD and remains so when one random real is adjoined to the universe. We proceed as follows in the ground model. Because the forcing notion to adjoin one random real has cardinality $\mathfrak{c}$ and satisfies the c.c.c., there are only $\mathfrak{c} = \aleph_1$ essentially different names for subsets of ω; enumerate them as $\langle x_\alpha : \alpha < \omega_1 \rangle$. We construct $\mathcal{A}$ by a recursion of length $\aleph_1$, starting with a partition of ω into $\aleph_0$ infinite pieces, and adding one set a_α to $\mathcal{A}$ at each step. This set will be chosen so as to be almost disjoint from the previous a_β's and to have infinite intersection with the denotation (with respect to every generic set) of x_α (unless some earlier a_β already does or x_α is finite). That will ensure that $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ remains MAD in the random extension. Let $[B]$ be the Boolean truth value of "x_α is not almost included in the union of finitely many $\check{a}_\beta$ with $\beta < \check{\alpha}$". We shall make sure that the truth value of "$x_\alpha \cap \check{a}_\alpha$ is infinite" is at least $[B]$. Equivalently, since we are dealing with a measure algebra, we shall make sure that for every n the Boolean truth value of "$x_\alpha \cap \check{a}_\alpha$ has a member $> n$" intersected with $[B]$ has measure at least $\mu[B] - \frac{1}{n}$. And of course we must ensure that a_α is almost disjoint from the earlier a_β's. We define a_α as follows.

Let the earlier a_β's be enumerated in an ω-sequence as a'_n. We shall construct a_α by a recursion of length ω, adding finitely many elements at each stage, and ensuring at stage n that the measure requirement at the end of the last paragraph is satisfied for n. To ensure almost disjointness, we shall not add any elements of a'_k after stage k. We now describe stage n. Let $v = \bigcup_{k<n} a'_k$, whose elements are no longer to be added to a_α. With truth value at least $[B]$, $x_\alpha - v$ is infinite. So $[B]$ is the Boolean sum of (countably many) pairwise incompatible conditions $[B_i]$ each forcing a specific value for the first element z of $x_\alpha - v$ that is $> n$. Since the measures of all the $[B_i]$ add up to the measure of $[B]$, finitely many of them come to within $\frac{1}{n}$ of that total. Put the corresponding finitely many z's into a_α. This completes the construction of a_α; we omit the routine verification that it does what was required.

11.5. Sacks Reals

The Sacks forcing notion, introduced in [96] and also called perfect set forcing, consists of perfect subtrees of ${}^{<\omega}2$, i.e., nonempty subtrees that have branching beyond each node; the partial ordering is inclusion. This is a proper forcing that adjoins a real s, namely the unique common path through all the trees in a generic set G.

The forcing extension $V[s]$ enjoys the *Sacks property*: For every function $f : \omega \to V$ in $V[s]$, there is a function $g : \omega \to V$ in V such that for all $n \in \omega$ we have $f(n) \in g(n)$ and $|g(n)| \leq 2^n$. To prove this, suppose we are given a name $\dot{f}$ for f and a condition p. Working in V, we prune the tree p in ω steps to produce a perfect subtree q forcing that a certain g is as required; by genericity, this will suffice. Begin by choosing $p_0 \leq p$ deciding a specific value for $\dot{f}(0)$. This value will be the unique element of $g(0)$. The first branching node a of p_0 will be the first branching node of the final q; i.e., neither a nor its immediate successors $a^\frown\langle 0\rangle$ and $a^\frown\langle 1\rangle$ will be pruned away later. Regard p_0 as the union of two perfect subtrees, one consisting of the nodes comparable with $a^\frown\langle 0\rangle$ and the other of the nodes comparable with $a^\frown\langle 1\rangle$. In each of these, find a perfect subtree deciding $\dot{f}(1)$ (possibly different decisions for the two subtrees). Reuniting these two subtrees, we get a perfect subtree p_1 of p_0, where a is still a branching node, and such that p_1 forces $\dot{f}(1)$ to have one of just two specific values. Those values will be the elements of $g(1)$. All later steps will preserve the two second-level branching nodes of p_1. Regard p_1 as the union of four perfect subtrees, one through each of the immediate successors of those nodes. Shrink each of the four to decide a (possibly different) value for $\dot{f}(2)$; and reunite them to get p_2. Continuing in this way, we finally obtain a tree $q = \bigcap_{n\in\omega} p_n$ that is perfect because we retain more and more branching as the construction progresses. q is an extension of p forcing each $\dot{f}(n)$ to have one of 2^n specific values known in V, so the desired g exists in V. (Although 2^n emerges naturally from the proof as the bound for $|g(n)|$, we could, as in Remark 5.15, replace 2^n by any function tending to ∞.)

This sort of construction, repeatedly pruning a tree but retaining more and more branching, is referred to as *fusion*. It can also be used to prove that adjoining a Sacks real produces a minimal extension in the sense that if $x \in V[s] - V$ is a set of ordinals then $V[x] = V[s]$.

The usual way to iterate Sacks forcing is with countable support for $\aleph_2$ steps, starting with a model of GCH. The resulting model is often called the *Sacks model*. Properness of Sacks forcing implies that cardinals are preserved. Furthermore, the Sacks model has the Sacks property, because this property is preserved by countable support proper iterations; see [5, Sect. 6.3.F], [97, Sects. VI.1–2], or [51]. It follows, by the dual of Theorem 5.14, that $\mathbf{cof}(\mathcal{L}) = \aleph_1$ in the Sacks model. Therefore, all cardinals in Cichoń's diagram as well as $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{s}$, $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{m}$ are equal to $\aleph_1$ in this model. Baumgartner and Laver [11] showed that selective ultrafilters in the ground model, which exist since GCH holds there, generate ultrafilters in the Sacks model. (In fact, the same is true of P-points.) Therefore the Sacks model has $\mathfrak{u} = \mathfrak{r} = \aleph_1$.

Spinas has shown (private communication) that the Sacks model satisfies $\mathfrak{a} = \aleph_1$. In outline, his argument is as follows. By general properties of Souslin proper forcing (see [60], [51, Sect. 7], and [101]), it suffices to find, in the ground model, a MAD family $\mathcal{A}$ that remains MAD in the extension obtained by iterating Sacks forcing for ω_1 steps with countable support. List in an ω_1-sequence all pairs (τ, p) where p is a condition in this iteration and τ is a name forced by p to denote an infinite subset of ω. We define the desired $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ by induction in the ground model, ensuring at step α that for the αth pair (τ, p) some extension of p either forces (a) "$\tau \cap \check{A}_\alpha$ is infinite" or forces (b) "τ is almost included in $\check{A}_{\beta_1} \cup \cdots \cup \check{A}_{\beta_r}$" for some finitely many $\beta_1, \ldots, \beta_r < \alpha$. Either way, p cannot force $\mathcal{A} \cup \{\tau\}$ to be almost disjoint with $\tau \notin \mathcal{A}$, so the maximality is preserved. To define A_α, assume the previous A_β's are already defined; modifying them finitely and re-numbering them (see the proof of Proposition 8.4), we can pretend that the ω we are working in is $\omega \times \omega$ and that these earlier A_β's are the columns $\{n\} \times \omega$. We can also assume that p forces τ to meet infinitely many of these columns, as otherwise we already have alternative (b) above. We shall take A_α to be $\{(a, b) : b < f(a)\}$ for a suitably large $f : \omega \to \omega$. Then clearly A_α is almost disjoint from the previous A_β's (the columns). To obtain alternative (a) and thus complete the proof, we need only choose f large enough. Specifically, use the name τ to produce a name D for the set of n such that the nth column meets τ and a name g for a function $D \to \omega$ such that p forces "D is infinite and, for each $d \in D$, τ contains an element (d, b) with $b < g(d)$". Then, thanks to the Sacks property, p also forces "some ground model function $f : \omega \to \omega$ majorizes g". Choosing an extension of p that decides what f is, we obtain alternative (a), and the proof is complete.

Finally, Eisworth and Shelah (unpublished) have shown that $\mathfrak{i} = \aleph_1$ in the Sacks model.

For many cardinal characteristics, a recent result of Shelah gives a uniform reason why they are $\aleph_1$ in the Sacks model. Shelah has shown that a countable support proper iteration of forcings that individually add no reals can, at limit stages of cofinality ω, introduce Sacks reals. But there are numerous iteration theorems (see [97, 51, 45]) saying that certain properties of a ground model will be unchanged by a countable support proper forcing iteration provided they are unchanged by the individual steps. These properties, then, are not changed by adding Sacks reals.

Another explanation for the smallness of many cardinal characteristics in the Sacks model is that countable support iteration of Sacks forcing is the optimal forcing for increasing $\mathfrak{c}$, in the sense of Zapletal [115]; see Remark 11.1.

Thus, all tame cardinal characteristics that can be forced to remains small when $\mathfrak{c}$ is increased by some set forcing in fact remain small in the Sacks model, provided there is a proper class of measurable Woodin cardinals.

11.6. Hechler Reals

Introduced by Hechler [57] for his proof of Theorem 2.5, Hechler forcing, also called dominating forcing, is the set of pairs (s, f) where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$. (The "meaning" of (s, f) is that the generic real in ${}^{\omega}\omega$ has s as an initial segment and thereafter majorizes f.) The ordering puts $(s', f') \leq (s, f)$ if s is an initial segment of s', $f \leq f'$, and $s'(n) \geq f(n)$ for all $n \in \mathrm{dom}(s') - \mathrm{dom}(s)$. This forcing satisfies c.c.c.; in fact it is σ-centered, since any finitely many conditions with the same first component have a lower bound. A Hechler-generic set G determines a function $g : \omega \to \omega$, namely the union of the first components of the members of G. Such a g is called a Hechler real. Genericity implies that it dominates all ground model functions $\omega \to \omega$, i.e., g solves $\mathfrak{D}$. ("Dominating real" is sometimes used as a synonym for "Hechler real" and sometimes to mean any real that dominates all ground model reals.) Replacing each of the values of g by its parity, we obtain a Cohen real, g mod 2.

By "the Hechler model" we mean the result of a finite support iteration of Hechler forcing over a model of GCH, where the number of steps is some regular uncountable cardinal λ. One can also consider countable support iterations (for up to ω_2 stages, as usual) but we shall not do so here. Hechler's original use of Hechler forcing [57] amounted to a combination of finite support iteration and product constructions.

Since a Hechler real solves $\mathfrak{D}$ and its parity solves $\mathbf{Cov}(\mathcal{B})^{\perp}$, the Hechler model satisfies $\mathbf{cov}(\mathcal{B}) = \mathfrak{b} = \lambda = \mathfrak{c}$. By Theorem 5.6, it satisfies $\mathbf{add}(\mathcal{B}) = \mathfrak{c}$. Thus, in this model, the cardinals in the second through fourth columns of Cichoń's diagram equal $\mathfrak{c}$. Those in the first column, on the other hand, equal $\aleph_1$ since this forcing adds no random reals [5, second model in 7.6.9]. Since $\mathfrak{b}$ is large, so are $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{a}$, and $\mathfrak{i}$. Baumgartner and Dordal showed in [10] that $\mathfrak{s}$ in the Hechler model is $\aleph_1$, and therefore so are $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{m}$. Brendle [30, Theorem 10.4] showed that $\mathfrak{e} = \aleph_1$ in the Hechler model.

The value of $\mathfrak{g}$ in the Hechler model should be $\aleph_1$. Brendle has shown (private communication) that it is $\aleph_1$ if Hechler forcing is iterated for only ω_2 steps. Shelah has sketched a proof that it is $\aleph_1$ in general, but so far as I know this proof has yet to be written down carefully and checked (private communication from Eisworth).

Pawlikowski [84] showed that, although $\mathbf{add}(\mathcal{B})$ is large in the Hechler model, adjoining a single Hechler real to the ground model does not produce any real solving $\mathbf{Cof}(\mathcal{B})$. Such a real appears, however, when two Hechler reals are added iteratively. This last fact follows from part 1 of Theorem 5.6, which says that $\mathbf{Cof}(\mathcal{B})$ admits a morphism from a sequential composition of two relations each of which is solved when a single Hechler real is adjoined.

11.7. Laver Reals

Conditions in Laver forcing are trees $p \subseteq {}^{<\omega}\omega$ in which there is a node s, called the stem, such that all nodes are comparable with s and every node beyond s has infinitely many immediate successors. (So, starting at the root of p, one finds no branching until one reaches s and then infinite branching everywhere thereafter.) The ordering is inclusion. A generic set G determines a function $g : \omega \to \omega$ called a Laver real, namely the union of the stems of all the conditions in G, or equivalently the unique common path through all members of G. Laver forcing is proper. Genericity implies that a Laver real dominates all ground model functions $\omega \to \omega$.

The Laver model is obtained by an ω_2-stage countable support iteration of Laver forcing over a model of GCH. (Historically, Laver forcing and countable support iteration were introduced together in [72]. For the purpose of that paper, producing a model of the Borel conjecture, one needs to dominate all ground model reals, but one must not introduce Cohen reals, so neither Hechler forcing nor a finite support iteration can be used.) Since a Laver real solves $\mathfrak{D}$, the Laver model has $\mathfrak{b} = \aleph_2 = \mathfrak{c}$. It follows that the cardinals in all but the left column and bottom row of Cichon's diagram are $\aleph_2$, and so are $\mathfrak{r}$, $\mathfrak{i}$, $\mathfrak{u}$, and $\mathfrak{a}$.

Like Hechler forcing, Laver forcing even when iterated does not produce random reals, but unlike Hechler forcing it does not produce Cohen reals either. In fact, the set of ground model reals does not have measure zero in the extension. See [5, Sect. 7.3.D] for proofs of these facts. It follows that $\mathbf{cov}(\mathcal{L})$ and $\mathbf{non}(\mathcal{L})$ are both $\aleph_1$ in the Laver model, and therefore so are $\mathbf{add}(\mathcal{L})$, $\mathbf{add}(\mathcal{B})$, $\mathbf{cov}(\mathcal{B})$, $\mathfrak{e}$, $\mathfrak{s}$, $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{m}$.

Finally, Brendle has pointed out that the proof of $\mathfrak{g} = \mathfrak{c}$ for the Miller model [26, 18] applies also to the Laver model. The same argument was used for a slightly different purpose in [45, Lemma 4.3.5].

11.8. Mathias Reals

Mathias forcing was described in Remark 7.8. It consists of pairs (s, A) with $s \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$ ("meaning" that the generic subset of ω has s as an initial segment and otherwise is included in A). The ordering, defined in Remark 7.8, is based on this meaning, and the resulting forcing is proper. A generic filter G determines an infinite subset X of ω called a Mathias real, namely the union of the first components of all the members of G. Mathias forcing was used in [74] and was studied in detail in [76].

The essential property of a Mathias real X is that, if $\mathcal{D} \subseteq [\omega]^{\omega}$ is any dense open family in the ground model, then X is included in some member of $\mathcal{D}$. To prove this, consider an arbitrary condition (s, A) and use the density of $\mathcal{D}$ to extend it to (s, A') with $A' \in \mathcal{D}$. Then (s, A') forces the generic real X to be almost included in A' and therefore included in some member of $\mathcal{D}$ since dense open families are closed under finite modifications.

By the Mathias model, we mean the result of an ω_2-stage countable support iteration of Mathias forcing over a model of GCH. The preceding paragraph together with a reflection argument implies that $\mathfrak{h} = \aleph_2 = \mathfrak{c}$ in this model. Specifically, given any $\aleph_1$ dense open families $\mathcal{D}_\xi$, we can find a common member as follows. Using the $<\aleph_2$-chain condition, we obtain an $\alpha < \omega_2$ (in fact an ω_1-closed unbounded set of such α's) such that each $\mathcal{D}_\xi \cap V_\alpha$ is a member of V_α and is a dense open set in the sense of V_α. Then the Mathias real X adjoined in going from V_α to $V_{\alpha+1}$ has, by the preceding paragraph, supersets in each $\mathcal{D}_\xi \cap V_\alpha$ and therefore belongs to each $\mathcal{D}_\xi$.

Because $\mathfrak{h}$ is large, so are $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{d}$, $\mathfrak{a}$, $\mathfrak{u}$, $\mathfrak{i}$, and **non** and **cof** of both category and measure.

On the other hand, both $\mathbf{cov}(\mathcal{B})$ and $\mathbf{cov}(\mathcal{L})$ are only $\aleph_1$ because neither Cohen nor random reals are added. See [5, Sect. 7.4.A] for the proof. It follows that $\mathbf{add}(\mathcal{L})$, $\mathbf{add}(\mathcal{B})$, $\mathfrak{e}$, $\mathfrak{t}$, $\mathfrak{p}$, and $\mathfrak{m}$ are also $\aleph_1$.

11.9. Miller Reals

The Miller forcing notion, introduced in [82], consists of superperfect trees (also called rational perfect trees), i.e., subtrees of ${}^{<\omega}\omega$ in which beyond every node there is one with infinitely many immediate successors. The order is inclusion. As with other such tree forcings, this is proper, and a generic set G determines a real $g : \omega \to \omega$, namely the union of the stems of the members of G or equivalently the unique path through all members of G. It is sometimes convenient to replace the Miller forcing notion with the isomorphic one in which the nodes of the trees are strictly increasing finite sequences from ω. Then the generic g is an increasing map $\omega \to \omega$, the enumeration of an infinite $X \subseteq \omega$. Either g or X can be called a Miller real or a superperfect real.

The Miller model is the result of an ω_2-stage countable support iteration of Miller forcing over a model of GCH. It is shown in [26, 18] that a Miller real X has supersets in all groupwise dense families from the ground model. This and a reflection argument show, just as in the discussion of Mathias forcing above, that $\mathfrak{g} = \aleph_2 = \mathfrak{c}$ in the Miller model. It follows that $\mathfrak{d}$, $\mathfrak{i}$, $\mathbf{cof}(\mathcal{B})$, and $\mathbf{cof}(\mathcal{L})$ are also $\aleph_2$.

On the other hand, it is shown in [5, 7.3.E] that both $\mathbf{non}(\mathcal{L})$ and $\mathbf{non}(\mathcal{B})$ are $\aleph_1$ in the Miller model. Therefore so are $\mathfrak{s}$, $\mathfrak{e}$, $\mathfrak{b}$, $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{m}$, and all the cardinals in Cichoń's diagram except $\mathfrak{d}$ and the two cofinalities.

It is also shown in [26] that every P-point in the ground model generates an ultrafilter in the Miller model. Therefore this model satisfies $\mathfrak{u} = \mathfrak{r} = \aleph_1$.

Finally, the proof that $\mathfrak{a} = \aleph_1$ in the Sacks model can, as Spinas pointed out, be transferred to the Miller model with only a minor modification. At the end of the proof, instead of using the Sacks property (which fails in the Miller model), one uses the fact that the ground model is an unbounded family in ${}^\omega\omega$ to show that p forces the function g in the extension to be majorized on an infinite subset of D by an f from the ground model. Another proof that $\mathfrak{a} = \aleph_1$ in the Miller model is given in [44, Proposition 8.24]. Eisworth

Table 4: Cardinal characteristics in iterated forcing models

	MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
$\mathfrak{a}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{b}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{d}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{e}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{g}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{h}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{i}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{m}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{p}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{r}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{s}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{t}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{u}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathbf{add}(\mathcal{L})$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathbf{cov}(\mathcal{L})$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathbf{non}(\mathcal{L})$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathbf{cof}(\mathcal{L})$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathbf{add}(\mathcal{B})$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathbf{cov}(\mathcal{B})$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathbf{non}(\mathcal{B})$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$
$\mathbf{cof}(\mathcal{B})$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$

pointed out (private communication) that the same argument applies to the Sacks model.

11.10. Summary of Iterated Forcing Results

Table 4 summarizes the preceding results concerning the values of cardinal characteristics in the iterated forcing models described above. Remember that in the countable support models, i.e., in the Sacks, Laver, Mathias, and Miller columns of the table, $\mathfrak{c}$ is just $\aleph_2$.

Figure 1 is a Hasse diagram of the main cardinal characteristics discussed in this chapter, except for the characteristics of the measure and category ideals. A line joining two characteristics in the figure means that the lower one is provably $\leq$ the upper one.

11.11. Other Forcing Iterations

The preceding sections cover only a few of the many kinds of iterated forcing, over models of GCH, that have been used in the theory of cardinal characteristics. There are other kinds of reals that one can adjoin, for example infinitely equal reals, Prikry-Silver reals, Matet reals, Grigorieff reals. Ex-

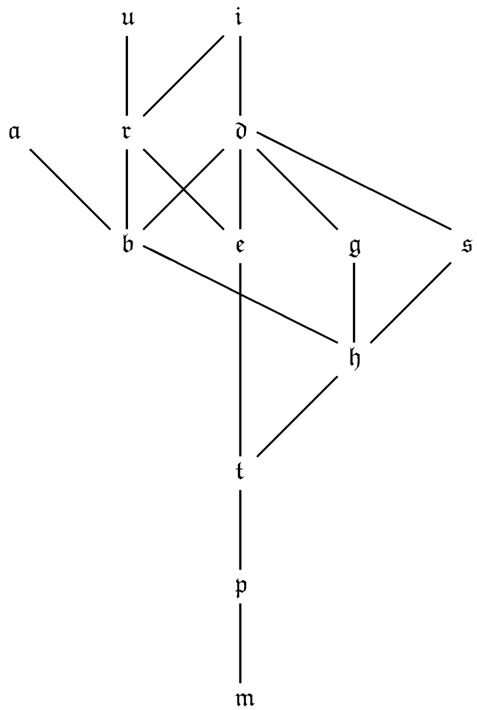

Figure 1: Hasse diagram of combinatorial characteristics

cept for Matet reals, which are defined in the last section of [18], these and many others can be found in [5] or [62]. Most of these forcing notions do not satisfy the c.c.c., so they are iterated with countable support and therefore one enlarges $\mathfrak{c}$ only to $\aleph_2$.

A model constructed in [25, Sect. 2] involves iterating a forcing that looks less natural than those discussed in the preceding sections or mentioned in the preceding paragraph, but we list its cardinal characteristics here because they are somewhat unusual, e.g., $\mathfrak{u} < \mathfrak{s}$. The model has $\mathfrak{u} = \aleph_1$ and therefore all of $\mathfrak{r}$, $\mathfrak{e}$, $\mathfrak{b}$, $\mathfrak{h}$, $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{m}$, and the covering numbers and additivities for both category and measure are $\aleph_1$. On the other hand, it has $\mathfrak{s} = \mathfrak{c} = \aleph_2$ and therefore all of $\mathfrak{d}$, $\mathfrak{i}$, and the uniformities and cofinalities of both measure and category are $\aleph_2$. (See [25, Theorem 5.2].) In addition, this model, designed to satisfy NCF, has $\mathfrak{g} = \aleph_2$, as was shown in [23, Theorem 2]. It also has $\mathfrak{a} = \aleph_1$ by the same Souslin-forcing argument used above for Sacks and Miller reals.

A frequently useful sort of iterated forcing is one where two or more different forcings are used alternately. Numerous examples of this can be found in [5, Chap. 7]. Dow's paper [43] describes, among other things, the models obtained by alternating Laver and Mathias forcings; it turns out to make a difference which forcing one uses at limit ordinals.

Dordal [41] uses a mixed support iteration of Mathias forcings. Viewing Mathias forcing as a two-step iteration, where one first adjoins an ultrafilter generically and then does Mathias forcing with respect to this ultrafilter (see Remark 7.8), he defines an iteration in which the adjunctions of ultrafilters are done with countable support while the interleaved Mathias forcings with respect to these ultrafilters are done with finite support.

All the preceding forcing iterations began with a ground model satisfying GCH. Thus, all cardinal characteristics are $\aleph_1$ in the ground model, and the iterations are designed to raise some characteristics while leaving others small. An alternative approach is to begin with a model where $\mathfrak{c}$ and some other characteristics are already large (e.g., a model of MA) and to do an iteration, usually of small length, to lower some characteristics while leaving others large. We briefly describe two examples; many more can be found in [5, Chap. 7].

Start with a model of MA + $\neg$CH (so all the characteristics we have discussed are large) and adjoin $\aleph_1$ random reals. Since the ${}^\omega\omega$ of a random extension is dominated by that of the ground model, we obtain a model where $\mathfrak{b}$ has the same large value that it had in the ground model of MA. On the other hand, $\mathfrak{s}$ is only $\aleph_1$ in the extension, and in fact so is $\mathbf{non}(\mathcal{L})$, since the $\aleph_1$ random reals form a set of positive outer measure and thus a splitting family. This proof for the consistency of $\mathfrak{b} > \mathfrak{s}$, due to Balcar and Simon, is easier than either of the ones obtainable from Table 4 (the Hechler and Laver models).

Another application of forcing over a model with large continuum is the construction in [27] of a model where $\mathfrak{u} < \mathfrak{d}$. This model, which predates the ones in [25, 26] that establish the stronger $\mathfrak{u} < \mathfrak{g}$, has the advantage that $\mathfrak{u} < \mathfrak{d}$ can be any prescribed uncountable regular cardinals. It begins with a Cohen model, where $\mathfrak{d}$ has the desired value, and extends it by a finite support iteration of Mathias forcings with respect to carefully chosen ultrafilters. The length of the iteration is the prescribed $\mathfrak{u}$. The easier part of "carefully chosen" is that each ultrafilter contains the previously adjoined Mathias reals, so that the sequence of Mathias reals is almost decreasing and generates an ultrafilter in the final model. Thus $\mathfrak{u}$ will be small. The hard part of "carefully chosen," which we omit here, is to keep $\mathfrak{d}$ large.

11.12. Adding One Real

In this subsection, we briefly summarize some results about the effect on cardinal characteristics of adjoining one real to a model of ZFC. Here the ground model will not satisfy CH, for the single-real forcings we consider would preserve CH and leave all characteristics at $\aleph_1$. We consider situations where some characteristics are large in the ground model and we ask how adding a single real affects them. Most of what is known about this concerns the cardinals from Cichoń's diagram. The results summarized here are from

[29, 35, 7, 39, 84].

Adding a Cohen real to any model of ZFC makes $\mathbf{add}(\mathcal{L}) = \mathbf{cov}(\mathcal{L}) = \aleph_1$ and $\mathbf{non}(\mathcal{L}) = \mathbf{cof}(\mathcal{L}) = \mathfrak{c}$. The values of $\mathbf{add}(\mathcal{B})$, $\mathbf{non}(\mathcal{B})$, and $\mathfrak{b}$ in the extension are the $\mathbf{add}(\mathcal{B})$ of the ground model, and dually the values of $\mathbf{cof}(\mathcal{B})$, $\mathbf{cov}(\mathcal{B})$, and $\mathfrak{d}$ in the extension are the $\mathbf{cof}(\mathcal{B})$ of the ground model.

Adding a random real produces a value for $\mathbf{cov}(\mathcal{L})$ that is no smaller than $\max\{\mathbf{cov}(\mathcal{L}), \mathfrak{b}\}$ of the ground model, and may be strictly larger. Dually, the extension's $\mathbf{non}(\mathcal{L})$ is at most $\min\{\mathbf{non}(\mathcal{L}), \mathfrak{d}\}$ and may be strictly smaller. Except for $\mathbf{cov}(\mathcal{L})$ and $\mathbf{non}(\mathcal{L})$, the cardinals in Cichoń's diagram remain unchanged.

Adding one Hechler real makes all cardinals in the left half of Cichoń's diagram $\aleph_1$ and all those in the right half $\mathfrak{c}$. It also makes $\mathfrak{a} = \aleph_1$.

Adding one Laver or Mathias real makes the $\mathfrak{d}$ of the extension $\aleph_1$. These forcings also collapse $\mathfrak{c}$ to $\mathfrak{h}$. Since $\mathfrak{h} \leq \mathfrak{d}$, it follows that a two-step iteration of these forcings produces a model of CH.

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7. Invariants of Measure and Category

Tomek Bartoszynski

Contents

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1. Introduction

The purpose of this chapter is to discuss various results concerning the relationship between measure and category. We are mostly interested in set-theoretic properties of the associated ideals, particularly, their cardinal invariants (called characteristics in [11]). This is a very large area, and it was necessary to make some choices. We decided to present several new results and new approaches to old problems. In most cases we do not present the optimal result, but a simpler theorem that still carries most of the weight of that original result. For example, we construct Borel morphisms in the Cichoń diagram while continuous ones can be constructed. We believe however that the reader should have no problems upgrading the material presented here to the current state of the art. The standard reference for this subject is [8], and this chapter updates it as most of the material presented here was proved after [8] was published.

Measure and category have been studied for about a century. The beautiful book [36] contains a lot of classical results, mostly from analysis and topology, that involve these notions. The roles played by Lebesgue measure and Baire category in these results are more or less identical. There are, of course, classical theorems indicating lack of complete symmetry, but the difference do not seem very significant. For example, Kuratowski's theorem (cf. Theorem 3.7) asserts that for every Borel function $f : {}^{\omega}2 \longrightarrow {}^{\omega}2$ there exists a meager set $F \subseteq {}^{\omega}2$ such that $f{\restriction}({}^{\omega}2 - F)$ is continuous. The dual proposition stating that for every Borel function $f : {}^{\omega}2 \longrightarrow {}^{\omega}2$ there exists a measure one set $G \subseteq {}^{\omega}2$ such that $f{\restriction}G$ is continuous is false. We only have a theorem of Luzin which guarantees that such G's can have measure arbitrarily close to one.

The last 15 years have brought a wealth of results indicating that hypotheses relating to measure are often stronger than the analogous ones relating to category. This chapter contains several examples of this phenomenon. Before we delve into this subject let us give a little historical background. The first result of this kind is due to Shelah [47]. He showed that

- If all projective sets are measurable then there exists an inner model with an inaccessible cardinal.
- It is consistent relative to ZFC that all projective sets have the property of Baire.

In 1984 Bartoszynski [3] and Raisonnier and Stern [40] showed that additivity of measure is not greater than additivity of category, whereas Miller [33] had shown that it can be strictly smaller. In subsequent years several more results of that kind were found. Let us mention one more (cf. [10]) concerning filters on ω (treated as subsets of ${}^{\omega}2$):

- There exists a measurable filter that does not have the Baire property. In fact, every filter that has measure zero can be extended to a measure zero filter that does not have the Baire property.

- It is consistent with ZFC that every filter that has the Baire property is measurable.

All these results as well as many others concerning measurability and the Baire property of projective sets, connections with forcing and others can be found in [8].

2. Tukey Connections

The starting point for our considerations is the following list of cardinal invariants of an ideal. For a proper ideal $\mathcal{J}$ of subsets of a set X which contains singletons (i.e. $\{a\} \in \mathcal{J}$ for $a \in X$) define

1. $\mathbf{add}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} \notin \mathcal{J}\}$,
2. $\mathbf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} = X\}$,
3. $\mathbf{non}(\mathcal{J}) = \min\{|Y| : Y \subseteq X \ \& \ Y \notin \mathcal{J}\}$,
4. $\mathbf{cof}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \forall B \in \mathcal{J} \ \exists A \in \mathcal{A} \ B \subseteq A\}$.

2.1 Definition. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are partial orderings. We say that $\mathcal{P} \preceq \mathcal{Q}$ if there is function $f : \mathcal{P} \longrightarrow \mathcal{Q}$ such that for every bounded set $X \subseteq \mathcal{Q}$, $f^{-1}(X)$ is bounded in $\mathcal{P}$. Such a function f is called a *Tukey embedding*. Define $\mathcal{P} \equiv \mathcal{Q}$ if $\mathcal{P} \preceq \mathcal{Q}$ and $\mathcal{Q} \preceq \mathcal{P}$.

Note that if $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a Tukey embedding then there is an associated function $f^\star : \mathcal{Q} \longrightarrow \mathcal{P}$ defined so that $f^\star(q)$ is a bound of the set $f^{-1}(\{p : p \leq q\})$. Observe that f maps every set unbounded in $\mathcal{P}$ onto a set unbounded in $\mathcal{Q}$ and $f^\star$ maps every set cofinal in $\mathcal{Q}$ onto a set cofinal in $\mathcal{P}$.

2.2 Lemma. *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals. If $\mathcal{I} \preceq \mathcal{J}$, then $\mathbf{add}(\mathcal{I}) \geq \mathbf{add}(\mathcal{J})$ and $\mathbf{cof}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{J})$.*

Proof. Suppose that $f : \mathcal{I} \longrightarrow \mathcal{J}$ is a Tukey function.

Let $\mathcal{A} \subseteq \mathcal{I}$ be a family of size $< \mathbf{add}(\mathcal{J})$. Find a set $B \in \mathcal{J}$ such that $\bigcup_{A \in \mathcal{A}} f(A) \subseteq B$. It follows that $\bigcup \mathcal{A} \subseteq f^\star(B)$.

Similarly, if $\mathcal{B} \subseteq \mathcal{J}$ is a basis for $\mathcal{J}$, then $\{f^\star(B) : B \in \mathcal{B}\}$ is a basis for $\mathcal{I}$. $\dashv$

We will need a slightly stronger definition which will encompass both cardinal invariants and Tukey embeddings.

2.3 Definition. Suppose that $\mathbf{A} = (A_-, A_+, A)$, where A is a binary relation between A_- and A_+. Let

$$\begin{aligned}
\mathfrak{d}(\mathbf{A}) &= \{Z : Z \subseteq A_+ \ \& \ \forall x \in A_- \ \exists z \in Z \ A(x,z)\},\\
\mathfrak{b}(\mathbf{A}) &= \{Z : Z \subseteq A_- \ \& \ \forall y \in A_+ \ \exists z \in Z \ \neg A(z,y)\},\\
\|\mathbf{A}\| &= \min\{|Z| : Z \in \mathfrak{d}(\mathbf{A})\}.
\end{aligned}$$

Define $\mathbf{A}^\perp = (A_+, A_-, A^\perp)$, where $A^\perp = \{(z, x) : \neg A(x, z)\}$. Note that $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$.

Note that $\|\mathbf{A}\|$ is the smallest size of the "dominating" family in A_+ and $\|\mathbf{A}^\perp\|$ is the smallest size of the "unbounded" family in A_-. With some notable exceptions such as $\mathfrak{p}, \mathfrak{t}, \mathfrak{g}, \mathfrak{h}, \mathfrak{a}, \mathfrak{u}$ (see Blass's chapter [11] for the definitions), virtually all cardinal characteristics, commonly called "invariants" of the continuum, can be expressed in this framework. For an ideal $\mathcal{J}$ of subsets of X we have:

- $\mathbf{cof}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)\|$,
- $\mathbf{add}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)^\perp\| = \|(\mathcal{J}, \mathcal{J}, \not\supseteq)\|$,
- $\mathbf{cov}(\mathcal{J}) = \|(X, \mathcal{J}, \in)\|$,
- $\mathbf{non}(\mathcal{J}) = \|(X, \mathcal{J}, \in)^\perp\| = \|(\mathcal{J}, X, \not\ni)\|$.

For $f, g \in {}^\omega\omega$ we define $f \leq^\star g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Let

- $\mathfrak{d} = \|({}^\omega\omega, {}^\omega\omega, \leq^\star)\|$,
- $\mathfrak{b} = \|({}^\omega\omega, {}^\omega\omega, \leq^\star)^\perp\| = \|({}^\omega\omega, {}^\omega\omega, \not\geq^\star)\|$.

The notion of Tukey embedding generalizes to the following:

2.4 Definition. A *morphism* φ between $\mathbf{A}$ and $\mathbf{B}$ is a pair of functions $\varphi_- : A_- \longrightarrow B_-$ and $\varphi_+ : B_+ \longrightarrow A_+$ such that for each $a \in A_-$ and $b \in B_+$,

$$A(a, \varphi_+(b)), \quad \text{whenever } B(\varphi_-(a), b).$$

If there is a morphism between $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A} \preceq \mathbf{B}$.

Note that if a pair of functions $f, f^\star$ witnesses that $\mathcal{P} \preceq \mathcal{Q}$, then $\varphi = (f, f^\star)$ is a morphism between $(\mathcal{P}, \mathcal{P}, \leq)$ and $(\mathcal{Q}, \mathcal{Q}, \leq)$.

2.5 Lemma.

(1) $\mathbf{A} \preceq \mathbf{B} \iff \mathbf{A}^\perp \succeq \mathbf{B}^\perp$,

(2) If $\mathbf{A} \preceq \mathbf{B}$, *then* $\|\mathbf{A}\| \leq \|\mathbf{B}\|$ *and* $\|\mathbf{A}^\perp\| \geq \|\mathbf{B}^\perp\|$.

Proof. (1) If $\varphi = (\varphi_-, \varphi_+)$ is a morphism between $\mathbf{A}$ and $\mathbf{B}$, then $\varphi^\perp = (\varphi_+, \varphi_-)$ is a morphism between $\mathbf{B}^\perp$ and $\mathbf{A}^\perp$.

(2) Suppose that $Z \in \mathfrak{d}(\mathbf{B})$ is such that $|Z| = \|\mathbf{B}\|$. Then $\{\varphi_+(z) : z \in Z\}$ is cofinal in A_+. In other words, $\|\mathbf{A}\| \leq |Z|$. $\dashv$

For two Polish spaces X, Y (i.e. complete, metric, separable with no isolated points) define $\mathsf{BOREL}(X, Y)$ to be the space of all Borel functions from X to Y.

Given relation $\mathbf{A}$ and assuming that both A_- and A_+ are Polish spaces we define families of small sets of reals as:

$$\mathsf{D}(\mathbf{A}) = \{X \subseteq \mathbb{R} : \forall f \in \mathsf{BOREL}(\mathbb{R}, A_+)\ f``X \notin \mathfrak{d}(\mathbf{A})\}$$

and

$$\mathsf{B}(\mathbf{A}) = \{X \subseteq \mathbb{R} : \forall f \in \mathsf{BOREL}(\mathbb{R}, A_-)\ f``X \notin \mathfrak{b}(\mathbf{A})\}.$$

In other words, $\mathsf{D}(\mathbf{A})$ consists of sets of reals whose Borel images are not “dominating” and $\mathsf{B}(\mathbf{A})$ consists of sets whose Borel images are “bounded”.

2.6 Lemma.

(1) $\mathbf{non}(\mathsf{D}(\mathbf{A})) = \|\mathbf{A}\|$ *and* $\mathbf{non}(\mathsf{B}(\mathbf{A})) = \|\mathbf{A}^\perp\|$.

(2) If there exists a Borel morphism from $\mathbf{A}$ *to* $\mathbf{B}$*, then* $\mathsf{B}(\mathbf{B}) \subseteq \mathsf{B}(\mathbf{A})$ *and* $\mathsf{D}(\mathbf{A}) \subseteq \mathsf{D}(\mathbf{B})$.

Proof. (1) Clearly $\mathbf{non}(\mathsf{D}(\mathbf{A})) \geq \|\mathbf{A}\|$. To show the other inequality notice that there is a Borel function from $\mathbb{R}$ onto A_+.

(2) Suppose that $X \notin \mathsf{B}(\mathbf{A})$ and let $f : \mathbb{R} \longrightarrow A_-$ be a Borel function such that $f``X \in \mathfrak{b}(\mathbf{A})$. It follows that $\varphi_- \circ f``X \in \mathfrak{b}(\mathbf{B})$. Since $\varphi_- \circ f$ is a Borel function it follows that $X \notin \mathsf{B}(\mathbf{B})$. $\dashv$

For cardinals $\kappa = \|\mathbf{A}\|$ and $\lambda = \|\mathbf{B}\|$ the question whether the inequality $\kappa \leq \lambda$ is provable in ZFC leads naturally to the question whether $\mathbf{A} \preceq \mathbf{B}$ and $\mathsf{D}(\mathbf{A}) \subseteq \mathsf{D}(\mathbf{B})$. Even though these questions are more general, in most cases the proof that $\kappa \leq \lambda$ yields $\mathbf{A} \preceq \mathbf{B}$. Moreover, the existence of a Borel morphism witnessing that $\mathbf{A} \preceq \mathbf{B}$ uncovers the combinatorial aspects of these problems.

Historical Remarks. Tukey embeddings were defined in [59] and further studied in [24]. In context of the orderings considered here see [18, 19, 32].

The framework used in Definition 2.3 is due to Vojtáš [60]; the particular formulation used here comes from [12].

3. Inequalities Provable in ZFC

The notions defined in the previous section are quite general. The focus of this chapter is on the ideals of meager sets ($\mathcal{B}$) and of measure zero (null) sets ($\mathcal{L}$) with respect to the standard product measure on μ on ${}^\omega 2$ or the Lebesgue measure μ on $\mathbb{R}$.

For an ideal $\mathcal{J}$, by a *Borel mapping* $H : \mathbb{R} \longrightarrow \mathcal{J}$ we mean a Borel set $H \subseteq \mathbb{R} \times \mathbb{R}$ such that, with $(H)_x = \{y : (x, y) \in H\}$, H is a *Borel* $\mathcal{J}$*-set*, i.e. $(H)_x \in \mathcal{J}$ for all $x \in \mathbb{R}$.

Using this terminology we define the following classes of small sets:

- $\mathsf{COF}(\mathcal{L}) = \mathsf{D}(\mathcal{L}, \mathcal{L}, \subseteq) = \{X \subseteq \mathbb{R}$: for every Borel $\mathcal{L}$-set H, $\{(H)_x : x \in X\}$ is not a basis of $\mathcal{L}\}$,

- $\mathsf{ADD}(\mathcal{L}) = \mathsf{B}(\mathcal{L}, \mathcal{L}, \subseteq) = \{X \subseteq \mathbb{R} :$ for every Borel $\mathcal{L}$-set H, $\bigcup_{x \in X} (H)_x \in \mathcal{L}\}$,
- $\mathsf{COV}(\mathcal{L}) = \mathsf{D}(\mathbb{R}, \mathcal{L}, \in) = \{X \subseteq \mathbb{R} :$ for every Borel $\mathcal{L}$-set H, $\bigcup_{x \in X} (H)_x \neq \mathbb{R}\}$,
- $\mathsf{NON}(\mathcal{L}) = \mathsf{B}(\mathbb{R}, \mathcal{L}, \in) = \{X \subseteq \mathbb{R} :$ every image of X by a Borel function is in $\mathcal{L}\}$.

In the same way we define $\mathsf{ADD}(\mathcal{B})$, $\mathsf{COV}(\mathcal{B})$, etc.

Finally, let

- $\mathsf{D} = \mathsf{D}({}^\omega\omega, {}^\omega\omega, \leq^\star)$,
- $\mathsf{B} = \mathsf{B}({}^\omega\omega, {}^\omega\omega, \leq^\star)$.

Instead of dealing with all null and meager sets we need to consider only suitably chosen cofinal families.

1. $A \in \mathcal{L}$ if and only if there exists a family of basic open sets $\{C_n : n \in \omega\}$ such that $\sum_{n=0}^{\infty} \mu(C_n) < \infty$ and $A \subseteq \bigcap_{n \in \omega} \bigcup_{m > n} C_m$,
2. $A \in \mathcal{B}$ if and only if there is a family of $\{F_n : n \in \omega\}$ of closed nowhere dense sets such that $A \subseteq \bigcup_{n \in \omega} F_n$.

In particular every null set can be covered by a null set of type G_δ and every meager set can be covered by a meager set of type F_σ.

For every $t \in {}^{<\omega}2$ let $[t] = \{x \in {}^\omega 2 : t \subseteq x\}$, and note that the family $\{[t] : t \in {}^{<\omega}2\}$ forms a standard basis for ${}^\omega 2$. Let $\mathbb{C}$ be the collection of clopen subsets of ${}^\omega 2$.

3.1 Definition. Let $\{C^n_m : n, m \in \omega\}$ be a family of clopen subsets of ${}^\omega 2$ such that

1. $\mathbb{C} = \{C^n_m : n, m \in \omega\}$,
2. $\mu(C^n_m) \leq 2^{-n}$ for each $m, n < \omega$.

3.2 Lemma. $A \in \mathcal{L} \iff \exists f \in {}^\omega\omega\, (A \subseteq \bigcap_m \bigcup_{n > m} C^n_{f(n)})$.

Proof. ($\Longleftarrow$) Note that the set $\bigcup_{n>m} C^n_{f(n)}$ has measure at most 2^{-m}.

($\Longrightarrow$) For an open set $U \subseteq {}^\omega 2$ let

$$\widetilde{U} = \{t \in {}^{<\omega}2 : [t] \subseteq U \;\&\; \forall s \subsetneq t\ ([s] \not\subseteq U)\}.$$

Note that $\widetilde{U}$ gives a canonical representation of U as a union of disjoint basic intervals.

Find open sets $\{G_n : n \in \omega\}$ covering A such that $\mu(G_n) \leq 2^{-n}$. Let $\{t_n : n \in \omega\}$ be the lexicographic enumeration of $\bigcup_{n\in\omega} \widetilde{G}_n$. Define for $n \in \omega$,

$$h(n+1) = \min\left\{k > h(n) : \sum_{j=k}^{\infty} \mu\big([t_j]\big) \leq 2^{-n}\right\},$$

and let

$$D_n = \bigcup_{j=h(n)}^{h(n+1)} [t_j].$$

Let $f \in {}^\omega\omega$ be such that $D_n = C^n_{f(n)}$ for each n. ⊣

We will need an analogous characterization of meager subsets of ${}^{<\omega}2$.

3.3 Definition. Let $\{U_n : n \in \omega\}$ be a basis for ${}^\omega 2$ and let $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ be any family of clopen sets. We say that $\mathcal{S}$ is *good* if

1. $S^n_m \cap U_n \neq \emptyset$ for $n, m \in \omega$,
2. For any open dense set $U \subseteq {}^\omega 2$ and $n \in \omega$ there is m such that $S^n_m \subseteq U$.

3.4 Lemma. *Suppose that the family $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ is good. Then*

$$A \in \mathcal{B} \quad \iff \quad \exists f \in {}^\omega\omega\big(A \subseteq {}^\omega 2 - \textstyle\bigcap_m \bigcup_{n>m} S^n_{f(n)}\big).$$

Proof. ($\Longleftarrow$) Note that the set $\bigcup_{n>m} S^n_{f(n)}$ is open and dense for every m.

($\Longrightarrow$) Let $\langle F_n : n \in \omega\rangle$ be an increasing sequence of closed nowhere dense sets covering A. For each n let

$$f(n) = \min\{m : S^n_m \cap F_n = \emptyset\}.$$

It is clear that $\bigcup_n F_n \cap \bigcap_m \bigcup_{n>m} S^n_{f(n)} = \emptyset$. ⊣

Define *master sets* $N, M \subseteq {}^\omega\omega \times {}^\omega 2$ by

$$N = \textstyle\bigcap_m \bigcup_{n>m} \bigcup_{f\in{}^\omega\omega} \{f\} \times C^n_{f(n)}, \quad \text{and}$$
$$M = ({}^\omega\omega \times {}^\omega 2) - \textstyle\bigcap_m \bigcup_{n>m} \bigcup_{f\in{}^\omega\omega} \{f\} \times S^n_{f(n)}.$$

Note that N is a G_δ set while M is an F_σ set. Moreover, $\{(N)_f : f \in {}^\omega\omega\}$ is cofinal in $\mathcal{L}$ and $\{(M)_f : f \in {}^\omega\omega\}$ is cofinal in $\mathcal{B}$.

In the sequel we will need to find a good family with some additional properties. The following lemma shows that the representation of meager sets does not depend on the choice of good family:

3.5 Lemma. *Suppose that $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ and $\mathcal{T} = \{T^n_m : n, m \in \omega\}$ are good and M and $\overline{M}$ are associated master sets. Then there are Borel functions $\varphi_-, \varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ such that*

$$(M)_f \subseteq (M)_{\varphi_+(g)}, \quad \textit{whenever } (\overline{M})_{\varphi_-(f)} \subseteq (\overline{M})_g.$$

Proof. For $f, g \in {}^\omega\omega$ and $n \in \omega$ define

$$\varphi_-(f)(n) = \min\bigl\{k : T^n_k \subseteq \bigcup_{m\geq n} S^m_{f(m)}\bigr\}, \quad \text{and}$$
$$\varphi_+(g)(n) = \min\bigl\{k : S^n_k \subseteq \bigcup_{m\geq n} T^m_{g(m)}\bigr\}.$$

We leave it to the reader to verify that these functions have the required properties. ⊣

The following two theorems will be helpful in many subsequent constructions.

3.6 Theorem. *Suppose that $H \subseteq {}^\omega 2 \times {}^\omega 2$ is a Borel set.*

(1) $\{x : (H)_x \in \mathcal{L}\}$ is Borel,

(2) $\{x : (H)_x \in \mathcal{B}\}$ is Borel,

(3) If U is open and $(H)_x$ is compact for every x, then $\{x : U \cap (H)_x = \emptyset\}$ is Borel,

(4) If for every x, $(H)_x$ is "large", where large is either "of positive measure" or "nonmeager", then there exists a Borel function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ such that for every x, $f(x) \in (H)_x$.

Proof. See [29] 16.A for (1) and (2), 18.B for (4). For (3) note that $\{x : U \cap (H)_x = \emptyset\} = \{x : (H)_x \subseteq {}^\omega 2 - U\}$ and inclusion between the compact sets does not involve quantifiers over the reals. ⊣

3.7 Theorem. *If X and Y are Polish spaces and $f : X \longrightarrow Y$ is a Borel mapping, then there is a dense G_δ set $G \subseteq X$ such that $f{\restriction}G$ is continuous.*

Proof. This is a special case of a theorem of Kuratowski; see [29, 8.I]. ⊣

Lemmas 3.2 and 3.4 have their two-dimensional analogs.

3.8 Lemma. *The following conditions are equivalent for a Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$:*

(1) $\forall x\,((H)_x \in \mathcal{L})$,

(2) For every $\varepsilon > 0$ there exists a Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ such that

(a) $H \subseteq B$,

(b) for every x, $(B)_x$ is an open set of measure $< \varepsilon$.

(3) There exists a Borel function $x \rightsquigarrow f_x$ such that

$$\forall x((H)_x \subseteq (N)_{f_x}).$$

Proof. (2) → (3) Let $\{B_n : n \in \omega\}$ be a family of Borel sets such that

1. $H \subseteq \bigcap_n B_n$,
2. For every x, $(B_n)_x$ is an open set of measure $< 2^{-n}$.

Look at the proof of Lemma 3.2 to see that for each x, $(B)_x = (N)_{f_x}$ and the function $x \rightsquigarrow f_x$ is Borel.

(3) $\to$ (1) is obvious.

(1) $\to$ (2) By induction on complexity we show that for every $\varepsilon > 0$ and every Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$ there exists a Borel set $B \supseteq H$ such that for every x, $(B)_x$ is open and $\mu((B - H)_x) < \varepsilon$. The only nontrivial part is to show that if the theorem holds for sets in Σ^0_α, then it holds for any set $A \in \Pi^0_\alpha$. To see this write $A = \bigcap_n A_n$ where $\langle A_n : n \in \omega\rangle$ is a descending sequence of sets in Σ^0_α. For each n let B_n be the set obtained from the induction hypothesis for A_n and $\varepsilon/2$. Let $K^n = \{x : \mu((A_n - A)_x) < \varepsilon/2\}$. Each set K^n is Borel. Now define

$$B = B_0 \cap (K^0 \times {}^\omega 2) \cup \bigcup_{n\in\omega} B_{n+1} \cap ((K^{n+1} - K^n) \times {}^\omega 2).$$

$\dashv$

3.9 Lemma. *The following conditions are equivalent for a Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$:*

(1) $\forall x\ ((H)_x \in \mathcal{B})$,

(2) There exists a family of Borel sets $\{G_n : n \in \omega\} \subseteq {}^\omega 2 \times {}^\omega 2$ such that

(a) $(G_n)_x$ is a closed nowhere dense set for all $x \in {}^\omega 2$,

(b) $H \subseteq \bigcup_{n\in\omega} G_n$.

(3) There exists a Borel function $x \rightsquigarrow f_x$ such that

$$\forall x\ ((H)_x \subseteq (M)_{f_x}).$$

Proof. (1) $\to$ (2) By induction on complexity we show that for any Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$ there are Borel sets B and $\{F_n : n \in \omega\}$ such that

1. $(B)_x$ is open for every x,
2. $(F_n)_x$ is closed nowhere dense for every x and n,
3. $H \triangle B \subseteq \bigcup_n F_n$.

As before the nontrivial part is to show the theorem for the class Π^0_α given that it holds for Σ^0_α. Suppose that $A \in \Sigma^0_\alpha$ and B is the set obtained by applying the inductive hypothesis to A. Let $\langle U_n : n \in \omega\rangle$ be an enumeration of basic sets in ${}^\omega 2$. Define for $n \in \omega$,

$$Z_n = \{x : U_n \cap (B)_x = \emptyset\}.$$

Note that sets Z_n are Borel. Let $B' = \bigcup_n Z_n \times U_n$. The vertical sections of the set $F = {}^\omega 2 \times {}^\omega 2 - (B \cup B')$ are closed nowhere dense and $({}^\omega 2 \times {}^\omega 2 - A) \triangle B' \subseteq F$, which completes the proof.

(2) $\rightarrow$ (3) For $x \in {}^\omega 2$ let

$$f_x(n) = \min\{k : \forall i \leq n\ (S^n_k \cap (G_i)_x = \emptyset)\}.$$

$\dashv$

From these two lemmas it follows that:

3.10 Lemma. *Let $\mathcal{I}$ be $\mathcal{L}$ or $\mathcal{B}$ and let I be the associated master set. Then for $X \subseteq \mathbb{R}$:*

(1) $X \in \mathsf{ADD}(\mathcal{I}) \iff$
$\forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f \in {}^\omega\omega\ \forall x \in X\ ((I)_{F(x)} \subseteq (I)_f)$,

(2) $X \in \mathsf{COF}(\mathcal{I}) \iff$
$\forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f \in {}^\omega\omega\ \forall x \in X\ ((I)_f \not\subseteq (I)_{F(x)})$,

(3) $X \in \mathsf{COV}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists z\ \forall x \in X\ (z \notin (I)_{F(x)})$,

(4) $X \in \mathsf{NON}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f\ \forall x \in X\ (F(x) \in (I)_f)$.

The goal of this section is to establish:

3.11 Theorem.

$$\begin{tikzcd}
(\mathbb{R},\mathcal{L},\in) \arrow[r, "\preceq"] & (\mathcal{B},\mathbb{R},\not\ni) \arrow[r, "\preceq"] & (\mathcal{B},\mathcal{B},\subseteq) \arrow[r, "\preceq"] & (\mathcal{L},\mathcal{L},\subseteq) \\
 & ({}^\omega\omega,{}^\omega\omega,\not\geq^*) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & ({}^\omega\omega,{}^\omega\omega,\leq^*) \arrow[u, "\preceq"] & \\
(\mathcal{L},\mathcal{L},\not\supseteq) \arrow[uu, "\preceq"] \arrow[r, "\preceq"] & (\mathcal{B},\mathcal{B},\not\supseteq) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & (\mathbb{R},\mathcal{B},\in) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & (\mathcal{L},\mathbb{R},\not\ni) \arrow[uu, "\preceq"]
\end{tikzcd}$$

As a consequence of the fact that the above morphisms turn out to be Borel we get the following two diagrams:

$$\begin{tikzcd}
\mathsf{COV}(\mathcal{L}) \arrow[r, "\subseteq"] & \mathsf{NON}(\mathcal{B}) \arrow[r, "\subseteq"] & \mathsf{COF}(\mathcal{B}) \arrow[r, "\subseteq"] & \mathsf{COF}(\mathcal{L}) \\
 & \mathsf{B} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{D} \arrow[u, "\subseteq"] & \\
\mathsf{ADD}(\mathcal{L}) \arrow[uu, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{ADD}(\mathcal{B}) \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{COV}(\mathcal{B}) \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{NON}(\mathcal{L}) \arrow[uu, "\subseteq"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{cov}(\mathcal{L}) \arrow[r, "\leq"] & \mathbf{non}(\mathcal{B}) \arrow[r, "\leq"] & \mathbf{cof}(\mathcal{B}) \arrow[r, "\leq"] & \mathbf{cof}(\mathcal{L}) \\
 & \mathfrak{b} \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathfrak{d} \arrow[u, "\leq"] & \\
\mathbf{add}(\mathcal{L}) \arrow[uu, "\leq"] \arrow[r, "\leq"] & \mathbf{add}(\mathcal{B}) \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathbf{cov}(\mathcal{B}) \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathbf{non}(\mathcal{L}) \arrow[uu, "\leq"]
\end{tikzcd}$$

The last of these diagrams is called the Cichoń diagram.

It is enough to find the following morphisms:

1. $(\mathbb{R}, \mathcal{L}, \in) \preceq (\mathcal{B}, \mathbb{R}, \not\ni)$,
2. $(\mathcal{B}, \mathcal{B}, \subseteq) \preceq (\mathcal{L}, \mathcal{L}, \subseteq)$,
3. $(\mathcal{B}, \mathcal{B}, \not\supseteq) \preceq ({}^{\omega}\omega, {}^{\omega}\omega, \not\geq^{\star})$,
4. $({}^{\omega}\omega, {}^{\omega}\omega, \not\geq^{\star}) \preceq (\mathcal{B}, \mathbb{R}, \not\ni)$,
5. $(\mathcal{L}, \mathcal{L}, \not\supseteq) \preceq (\mathbb{R}, \mathcal{L}, \in)$.

The remaining morphisms are dual to those listed above. In each case we will find a Borel morphism. Note that thanks to the master sets M and N defined earlier, Borel morphisms between these structures can be interpreted as the automorphisms of the index set i.e. ${}^{\omega}\omega$.

3.12 Theorem. *$\mathcal{B} \preceq \mathcal{L}$; there are two Borel functions $\varphi_-, \varphi_+ : {}^{\omega}\omega \longrightarrow {}^{\omega}\omega$ such that*

$$(M)_f \subseteq (M)_{\varphi_+(g)}, \quad \textit{whenever } (N)_{\varphi_-(f)} \subseteq (N)_g.$$

Thus, $\mathsf{ADD}(\mathcal{L}) \subseteq \mathsf{ADD}(\mathcal{B})$ *and* $\mathsf{COF}(\mathcal{B}) \subseteq \mathsf{COF}(\mathcal{L})$, $\mathbf{add}(\mathcal{L}) \leq \mathbf{add}(\mathcal{B})$ *and* $\mathbf{cof}(\mathcal{B}) \leq \mathbf{cof}(\mathcal{L})$.

Proof. Let

$$\mathcal{C} = \left\{ S \in {}^{\omega}({}^{<\omega}[\omega]) : \sum_{n=1}^{\infty} \frac{|S(n)|}{2^n} < \infty \right\}.$$

For $S, S' \in \mathcal{C}$ define $S \subseteq^{\star} S'$ if for all but finitely many n, $S(n) \subseteq S'(n)$.

3.13 Lemma. *$\mathcal{L} \equiv \mathcal{C}$.*

Proof. To see that $\mathcal{L} \preceq \mathcal{C}$ define $\varphi_- : {}^{\omega}\omega \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow {}^{\omega}\omega$ such that for $f \in {}^{\omega}\omega$ and $S \in \mathcal{C}$ we have

$$(N)_f \subseteq (N)_{\varphi_+(S)}, \quad \text{whenever } \varphi_-(f) \subseteq^{\star} S.$$

For $f \in {}^\omega\omega$ put $\varphi_-(f) = h$, where $h(n) = \{f(2n), f(2n+1)\}$. If $S \in \mathcal{C}$ let $\varphi_+(S) = g \in {}^\omega\omega$ be such that

$$C^n_{g(n)} = \bigcup_{k \in S(n)} C^{2n}_k \cup \bigcup_{k \in S(n)} C^{2n+1}_k.$$

Note that this formula defines $g(n)$ for all but finitely many n. The verification that these functions have the required properties is straightforward.

To show that $\mathcal{C} \preceq \mathcal{L}$ we will find Borel functions $\varphi_- : \mathcal{C} \longrightarrow {}^\omega\omega$ and $\varphi_+ : {}^\omega\omega \longrightarrow \mathcal{C}$ such that for $S \in \mathcal{C}$ and $f \in {}^\omega\omega$,

$$S \subseteq^\star \varphi_+(f), \quad \text{whenever } (N)_{\varphi_-(S)} \subseteq (N)_f.$$

Let $\{G^n_m : n, m \in \omega\}$ be a family of clopen probabilistically independent sets such that $\mu(G^n_m) = 2^{-n}$. For $S \in \mathcal{C}$ define $\varphi_-(S) = f \in {}^\omega\omega$ such that

$$\bigcap_{m \in \omega} \bigcup_{n > m} \bigcup_{k \in S(n)} G^n_k \subseteq (N)_f.$$

To do this, first consider $H' \subseteq \mathcal{C} \times {}^\omega 2$ defined by

$$(H')_S = \bigcap_{m \in \omega} \bigcup_{n > m} \bigcup_{k \in S(n)} G^n_k$$

for $S \in \mathcal{C}$. Note that H' is a Borel set and $(H')_S$ has measure zero for every S. Fix a Borel isomorphism $a : \mathcal{C} \longrightarrow {}^\omega\omega$ and let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be defined as $(H)_{a(S)} = (H')_S$ for $S \in \mathcal{C}$. Apply Lemma 3.8 to find a Borel function $x \rightsquigarrow f_x$ such that $(H)_x \subseteq (N)_{f_x}$ and define $\varphi_-(S) = f_{a(S)}$.

To define $\varphi_+ : {}^\omega\omega \longrightarrow \mathcal{C}$ we proceed as follows. Find a Borel set $K \subseteq {}^\omega\omega \times {}^\omega 2$ such that

1. $(K)_f$ is a compact set of measure $\geq 1/2$ for all $f \in {}^\omega\omega$.
2. $N \cap K = \emptyset$.
3. For any basic open set $U \subseteq {}^\omega 2$ and $f \in {}^\omega\omega$, if $U \cap (K)_f \neq \emptyset$ then $U \cap (K)_f$ has positive measure.

To do this, first use Lemma 3.8 to find a set K' satisfying the first two conditions. Let $\langle U_j : j \in \omega \rangle$ be an enumeration of basic open sets in ${}^\omega 2$. For each j let $Z_j = \{f : \mu(U_j \cap (K')_f) = 0\}$. By Theorem 3.6, the sets Z_j are Borel for each j. Define $K = K' - \bigcup_j (Z_j \times U_j)$.

For $f \in {}^\omega\omega, j, n \in \omega$ define

$$S^f_j(n) = \{i \in \omega : (K)_f \cap U_j \neq \emptyset \;\&\; (K)_f \cap U_j \cap G^n_i = \emptyset\}.$$

Note that

$$0 < \mu((K)_f \cap U_j) \leq \prod_n \prod_{i \in S^f_j(n)} \mu({}^\omega 2 - G^n_i).$$

Thus

$$0 < \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right)^{|S^f_j(n)|}.$$

It follows that
$$\sum_{n=1}^{\infty} \frac{|S_j^f(n)|}{2^n} < \infty,$$
so $S_j^f \in \mathcal{C}$ for each j. Moreover, the function $f \rightsquigarrow \langle S_j^f : j \in \omega\rangle \in {}^\omega\mathcal{C}$ is Borel (by Theorem 3.6(3)). Fix a Borel function from ${}^\omega\mathcal{C}$ to $\mathcal{C}$ with $\langle S_j^f : j \in \omega\rangle \rightsquigarrow S_\infty^f$ such that
$$\forall j\ \forall^\infty n\ S_j^f(n) \subseteq S_\infty^f(n).$$

Finally define φ_+ by the formula:
$$\varphi_+(f)(n) = S_\infty^f(n).$$

Suppose that for some $S \in \mathcal{C}$, $(N)_{\varphi_-(S)} \subseteq (N)_f$. It follows that,
$$(K)_f \cap \bigcap_m \bigcup_{n>m} \bigcup_{k\in S(n)} G_k^n = \emptyset.$$

By the Baire Category Theorem, there is a basic open set U_j and $m_0 \in \omega$ such that $U_j \cap (K)_f \neq \emptyset$ but
$$(K)_f \cap U_j \cap \bigcup_{n>m_0} \bigcup_{k\in S(n)} G_k^n = \emptyset.$$

Therefore
$$\forall^\infty n\ S(n) \subseteq S_j^f(n) \subseteq S_\infty^f(n) = \varphi_+(f)(n),$$
which finishes the proof. ⊣

3.14 Lemma. *$\mathcal{B} \preceq \mathcal{C}$; there are Borel functions $\varphi_- : {}^\omega\omega \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow {}^\omega\omega$ such that for any $f \in {}^\omega\omega$ and $S \in \mathcal{C}$,*
$$(M)_f \subseteq (M)_{\varphi_+(S)}, \quad \textit{whenever } \varphi_-(f) \subseteq^\star S.$$

Proof. We will need the following lemma:

3.15 Lemma. *There exists a good family $\{S_m^n : n, m \in \omega\}$ such that for each n,*
$$\forall X \in [\omega]^{\leq 2^n} \quad \left(\bigcap_{j\in X} S_j^n \neq \emptyset\right).$$

Proof. Fix $n \in \omega$. Let $\langle C_m : m \in \omega\rangle$ be an enumeration of all clopen sets. For $k \in \omega$ define
$$A_k = \left\{l > k : C_l \cap \bigcap_{i\in I} C_i \cap U_n \neq \emptyset \text{ when } I \subseteq k+1 \text{ and } U_n \cap \bigcap_{i\in I} C_i \neq \emptyset\right\}.$$

Consider the family
$$\mathcal{S}_n = \left\{\bigcup_{i\leq 2^n} C_{m_i} : m_0 \in \omega \text{ and } \ m_{i+1} \in A_{m_i} \text{ for } i \leq 2^n\right\}.$$

We have to check that $\mathcal{S} = \bigcup_n \mathcal{S}_n$ satisfies the two conditions of Definition 3.3 with $\mathcal{S}_n$ in the role of $\{\mathcal{S}_m^n : m \in \omega\}$.

The first condition, that each member of $\mathcal{S}_n$ intersects U_n, follows immediately from the definition of $\mathcal{S}_n$.

For the second condition, let U be a dense open subset of ${}^\omega 2$. Note that, by density of U, $A_k \cap \{l \in \omega : U_n \cap C_l \subseteq U\} \neq \emptyset$ for every $k \in \omega$. Now define by induction a sequence $\{m_i : i \leq 2^n\}$ such that $C_{m_i} \subseteq U$ and $m_{i+1} \in A_{m_i}$ for $i < 2^n$. Clearly $U \supseteq \bigcup_{i \leq 2^n} C_{m_i} \in \mathcal{S}_n$.

Suppose that $V_1, V_2, \ldots, V_{2^n} \in \mathcal{S}_n$. For any $j \leq 2^n$, $V_j = \bigcup_{i \leq 2^n} C_{m_i^j}$, where $m_i^j \in A_{m_i^j}$ for $i, j \leq 2^n$. Order the sets V_j in such a way that $m_i^i \leq m_i^j$ for $i \leq j \leq 2^n$. It is easy to show by induction that $\bigcap_{j \leq 2^n} V_j \supseteq \bigcap_{j \leq 2^n} C_{m_j^j} \neq \emptyset$. ⊣

Returning to the proof of Lemma 3.14, let $\mathcal{S} = \bigcup_n \mathcal{S}_n = \{S_m^n : n, m \in \omega\}$ be the family constructed above. For $f \in {}^\omega\omega$ define $\varphi_-(f) = f \in \mathcal{C}$. For $S \in \mathcal{C}$ let $\varphi_+(S) = f \in {}^\omega\omega$ be such that

$$(M)_f \supseteq {}^\omega 2 - \bigcap_{m \in \omega} \bigcup_{n > m} \bigcap_{i \in S(n)} S_i^n.$$

Since $|S(n)| \leq 2^n$ for all but finitely many n, by Lemma 3.15,

$$\emptyset \neq U_n \cap \bigcap_{i \in S(n)} S_i^n.$$

Now suppose that $\varphi_-(f) \subseteq^\star S$. This assumption means that there exists an $n_0 \in \omega$ such that $f(m) \in S(m)$ for $m \geq n_0$. It follows that

$$(M)_{\varphi_+(S)} \supseteq {}^\omega 2 - \bigcap_{m \in \omega} \bigcup_{n > m} \bigcap_{i \in S(n)} S_i^n \supseteq {}^\omega 2 - \bigcap_{m \in \omega} \bigcup_{n > m} S_{f(n)}^n.$$

⊣

Theorem 3.12 follows immediately; compose the morphisms constructed in Lemmas 3.13 and 3.14. ⊣

3.16 Theorem. $(\mathbb{R}, \mathcal{L}, \in) \preceq (\mathcal{B}, \mathbb{R}, \not\ni)$; *there are Borel functions* $\varphi_-, \varphi_+ : \mathbb{R} \longrightarrow {}^\omega\omega$ *such that for* $x, y \in \mathbb{R}$,

$$x \in (N)_{\varphi_+(y)}, \quad \textit{whenever } y \notin (M)_{\varphi_-(x)}.$$

Thus, $\mathsf{COV}(\mathcal{L}) \subseteq \mathsf{NON}(\mathcal{B})$ *and* $\mathsf{COV}(\mathcal{B}) \subseteq \mathsf{NON}(\mathcal{L})$, $\mathbf{cov}(\mathcal{L}) \leq \mathbf{non}(\mathcal{B})$ *and* $\mathbf{cov}(\mathcal{B}) \leq \mathbf{non}(\mathcal{L})$.

Proof. Let B be a G_δ null set whose complement is meager. Use Theorem 3.6(3) and Theorem 3.7 to find Borel functions $\varphi_-, \varphi_+ : \mathbb{R} \longrightarrow {}^\omega\omega$ such that

$$\forall x \; (B + x \subseteq (N)_{\varphi_-(x)}) \quad \text{and} \quad \forall y \; ({}^\omega 2 - (B + y) \subseteq (M)_{\varphi_+(y)}).$$

Note that in this context $+$ is the coordinatewise addition modulo 2.

If $y \notin (N)_{\varphi_-(x)}$ then $y \notin B + x$. It follows that $x \in {}^\omega 2 - (B + y) \subseteq (M)_{\varphi_+(y)}$. ⊣

3.17 Theorem. $(\mathcal{B}, \mathcal{B}, \not\supseteq) \preceq ({}^{\omega}\omega, {}^{\omega}\omega, {}^{\star}\not\geq)$*; there are Borel functions* φ_-, φ_+: ${}^{\omega}\omega \longrightarrow {}^{\omega}\omega$ *such that for* $f, g \in {}^{\omega}\omega$

$$(M)_f \not\supseteq (M)_{\varphi_+(g)}, \quad \textit{whenever } g \not\leq^{\star} \varphi_-(f).$$

In particular, $\mathsf{D} \subseteq \mathsf{COF}(\mathcal{B})$ *and* $\mathsf{ADD}(\mathcal{B}) \subseteq \mathsf{B}$, $\mathfrak{d} \leq \mathbf{cof}(\mathcal{B})$ *and* $\mathbf{add}(\mathcal{B}) \leq \mathfrak{b}$.

Proof. Let $\mathcal{S}_n$ be the family of clopen sets C such that there exists a $k > n$ and $s \in {}^{[n,k)}2$ such that

$$C = \{x \in {}^{\omega}2 : x{\restriction}[n,k) = s\}.$$

Note that the family $\mathcal{S} = \bigcup_n \mathcal{S}_n$ is good (given the appropriate choice of the sequence $\{U_n : n \in \omega\}$).

For $f \in {}^{\omega}\omega$ let $\varphi_-(f)(n) = k$ if and only if $\mathsf{dom}(S^n_{f(n)}) = [n,k)$.

For a strictly increasing function $f \in {}^{\omega}\omega$ define $\varphi_+(f) = h \in {}^{\omega}\omega$ such that

$$(M)_h = \{x \in {}^{\omega}2 : \forall^{\infty} n\ \exists i \in [n, f(n))\ (x(i) \neq 0)\}.$$

Note that the image of ${}^{\omega}\omega$ under φ_+ is rather small, $\varphi_+ {}^{``\omega}\omega$ is not even cofinal in $\mathcal{B}$.

To finish the proof it is enough to show that if $\varphi_-(f)(n) < g(n)$ for infinitely many n, then

$$\begin{aligned}&\{x \in {}^{\omega}2 : \forall^{\infty} n\ \exists i \in [n, g(n))\ x(i) \neq 0\}\\ &\quad \not\subseteq \{x \in {}^{\omega}2 : \forall^{\infty} n\ x{\restriction}[n, \varphi_-(f)(n)) \neq S^n_{f(n)}\}.\end{aligned}$$

Find a sequence $\langle n_k : k \in \omega\rangle$ such that for all k,

$$n_k < \varphi_-(f)(n_k) < g(n_k) < n_{k+1}.$$

Construct a real z such that $z{\restriction}[n_k, \varphi_-(f)(n_k)) = S^{n_k}_{f(n_k)}$. Thus $z \in (M)_f$ but $z{\restriction}[n, g(n)) \not\equiv 0$ for all n, so $z \in \{x \in {}^{\omega}2 : \forall^{\infty} n\ \exists i \in [n, f(n))\ (x(i) \neq 0)\}$. ⊣

3.18 Theorem. $({}^{\omega}\omega, {}^{\omega}\omega, \not\geq) \preceq (\mathcal{B}, \mathbb{R}, \not\ni)$*; there are Borel functions* $\varphi_- :$ ${}^{\omega}\omega \longrightarrow {}^{\omega}\omega$ *and* $\varphi_+ : \mathbb{R} \longrightarrow {}^{\omega}\omega$ *such that for* $f \in {}^{\omega}\omega$ *and* $y \in \mathbb{R}$,

$$f \not\geq^{\star} \varphi_+(y), \quad \textit{whenever } y \notin (M)_{\varphi_-(f)}.$$

In particular, $\mathsf{COV}(\mathcal{B}) \subseteq \mathsf{D}$ *and* $\mathsf{B} \subseteq \mathsf{NON}(\mathcal{B})$, $\mathbf{cov}(\mathcal{B}) \leq \mathfrak{d}$ *and* $\mathfrak{b} \leq \mathbf{non}(\mathcal{B})$.

Proof. Identify $\mathbb{R} - \mathbb{Q}$ with ${}^{\omega}\omega$ and define $\varphi_-(f) = h$ such that

$$(M)_h = \{z \in {}^{\omega}\omega : \forall^{\infty} n\ (z(n) \leq f(n))\}$$

and $\varphi_+(y) = y$. ⊣

3.19 Theorem. *$(\mathcal{L},\mathcal{L},\not\supseteq) \preceq (\mathcal{L},\mathbb{R},\in)$; there are Borel functions $\varphi_- : {}^\omega\omega \longrightarrow \mathbb{R}$ and $\varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ such that for $f,g \in {}^\omega\omega$,*

$$(N)_f \not\supseteq (N)_{\varphi_+(g)}, \quad \textit{whenever } \varphi_-(f) \in (N)_g.$$

The same is true if we replace $\mathcal{L}$ by $\mathcal{B}$.

Proof. Let $\varphi_- : {}^\omega\omega \longrightarrow \mathbb{R}$ be any Borel function such that for $f \in {}^\omega\omega$, $\varphi_-(f) \notin (N)_f$ (see Theorem 3.6(4)) and let $\varphi_+(g) = g$ for $g \in {}^\omega\omega$. Verification that both functions have the required properties is straightforward. ⊣

We conclude this section with some remarks concerning Luzin sets.

3.20 Definition. Given $\mathbf{A} = (A_-, A_+, A)$ and two cardinals $\kappa \leq \lambda$ we call a set $X \subseteq A_-$ a (κ,λ)-*Luzin set for* $\mathbf{A}$ if $|X| \geq \lambda$ and for every $Y \subseteq X$, $|Y| = \kappa$, $Y \in \mathfrak{b}(\mathbf{A})$.

When $\mathbf{A} = (\mathbb{R},\mathcal{B},\in)$, $\kappa = \aleph_1$ and $\lambda > \aleph_0$ then we get the original Luzin set. The set given by $(\mathbb{R},\mathcal{L},\in)$, $\kappa = \aleph_1$ and $\lambda > \aleph_0$ is usually called a Sierpiński set.

3.21 Lemma. *Suppose that X is a (κ,λ)-Luzin set for $\mathbf{A}$ and $\kappa \leq \lambda$. Then $\|\mathbf{A}\| \geq \lambda$ and $\|\mathbf{A}^\perp\| \leq \kappa$.*

Proof. Since every set $Y \subseteq X$, $|Y| = \kappa$ belongs to $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$, we get the second inequality.

For the first inequality note that if $y \in A_+$ then $\{x \in X \cap A_- : A(x,y)\}$ has size $< \kappa \leq |X|$. Thus any family that dominates X has to have a size at least $|X| \geq \lambda$. ⊣

Morphisms preserve Luzin sets.

3.22 Lemma. *Suppose that $\mathbf{A} \leq \mathbf{B}$ and X is a (κ,λ)-Luzin set for $\mathbf{A}$. Then φ_-"X is a (κ,λ)-Luzin set for $\mathbf{B}$.*

Proof. Clearly every subset of size κ of φ_-"X is unbounded. Moreover, for every $b \in B_-$, $\varphi_-^{-1}(b) \cap X$ has size $< \kappa$. Thus $|\varphi_-$"$X| \geq \lambda$. ⊣

Historical Remarks. Families of small sets as defined here appeared in various contexts. Recław [42] suggested considering small sets rather than cardinal characteristics.

Many people contributed to the proof of Theorem 3.11. In the last diagram:

- Rothberger [45] showed that $\mathbf{cov}(\mathcal{B}) \leq \mathbf{non}(\mathcal{L})$ and $\mathbf{cov}(\mathcal{L}) \leq \mathbf{non}(\mathcal{B})$.
- Miller [33] and Truss [57] showed that $\mathbf{add}(\mathcal{B}) = \min\{\mathfrak{b}, \mathbf{cov}(\mathcal{B})\}$ and Fremlin showed that $\mathbf{cof}(\mathcal{B}) = \max\{\mathfrak{d}, \mathbf{non}(\mathcal{B})\}$.

- Bartoszynski [3] and Raisonnier and Stern [40] showed that $\mathbf{add}(\mathcal{L}) \leq \mathbf{add}(\mathcal{B})$ and $\mathbf{cof}(\mathcal{B}) \leq \mathbf{cof}(\mathcal{L})$. Different proofs of these inequalities have been found, in particular a forcing proof by Judah and Repický in [25].

Fremlin [17] first realized that Tukey embeddings are responsible for the inequalities in the Cichoń diagram. Pawlikowski [37] proved Lemma 3.14, which was the crucial step in the proof of $\mathcal{B} \preceq \mathcal{L}$.

The first diagram of Theorem 3.11:

- Vojtáš [60] proved it with arbitrary morphisms,
- Recław [42] proved a version with Borel morphisms (which gives the second diagram),
- Pawlikowski and Recław [39] proved the existence of continuous morphisms.

Lemma 3.22 was proved in [15].

4. Combinatorial Characterizations

This section is devoted to the combinatorics associated with the cardinal invariants of the Cichoń diagram. We will find the combinatorial equivalents of most of the invariants as well as characterize membership in the corresponding classes of small sets. We conclude the section with a characterization of the ideal $(\mathcal{L}, \subseteq)$ as maximal in the sense of Tukey connections among a large class of partial orderings.

4.1 Theorem. *The following are equivalent:*

(1) $X \in \mathsf{COV}(\mathcal{B})$*,*

(2) for every Borel function $x \rightsquigarrow f^x \in {}^\omega\omega$ *there exists a function* $g \in {}^\omega\omega$ *such that*
$$\forall x \in X\ \exists^\infty n\ (f^x(n) = g(n)).$$

Proof. (1) $\to$ (2). Suppose that $x \rightsquigarrow f^x \in {}^\omega\omega$ is a Borel function. Let $H = \{\langle x, h\rangle \in {}^\omega 2 \times {}^\omega\omega : \forall^\infty n\ (h(n) \neq f^x(n))\}$. Clearly H is a Borel set with all $(H)_x$ meager and if $g \notin \bigcup_{x\in X}(H)_x$ then g has required properties.

(2) $\to$ (1). We will need several lemmas. To avoid repetitions let us define:

4.2 Definition. Suppose that $X \subseteq {}^\omega 2$. X is *nice* if for every Borel function $x \rightsquigarrow f^x \in {}^\omega\omega$ there exists a function $g \in {}^\omega\omega$ such that
$$\forall x \in X\ \exists^\infty n\ (f^x(n) = g(n)).$$

4.3 Lemma. *Suppose that X is nice. Then for every Borel function $x \rightsquigarrow \langle Y^x, f^x\rangle \in {}^\omega[\omega] \times {}^\omega\omega$ there exists a $g \in {}^\omega\omega$ such that*

$$\forall x \in X\ \exists^\infty n \in Y^x\ (f^x(n) = g(n)).$$

Proof. Suppose that a Borel function $x \rightsquigarrow \langle Y^x, f^x\rangle$ is given. Let y^x_n denote the n-th element of Y^x for $x \in {}^\omega 2$. For every $x \in {}^\omega 2$ define a function h^x as follows:

$$h^x(n) = f^x{\restriction}\{y^x_0,\ y^x_1, \dots, y^x_n\} \quad \text{for } n \in \omega.$$

Since the function $x \rightsquigarrow h^x$ is Borel and functions h^x can be coded as elements of ${}^\omega\omega$ there is a function h such that

$$\forall x \in X\ \exists^\infty n\ (h^x(n) = h(n)).$$

Without loss of generality we can assume that $h(n)$ is a function from an $n+1$-element subset of ω into ω.

Define $g \in {}^\omega\omega$ in the following way. Recursively choose

$$z_n \in \mathsf{dom}\big(h(n)\big) - \{z_0, z_1, \dots, z_{n-1}\} \quad \text{for } n \in \omega.$$

Then let g be any function such that $g(z_n) = h(n)(z_n)$ for $n \in \omega$.

We show that the function g has the required properties. Suppose that $x \in X$. Notice that the equality $h^x(n) = h(n)$ implies that

$$f^x(z_n) = g(z_n) \quad \text{and} \quad z_n \in Y^x.$$

That finishes the proof since $h^x(n) = h(n)$ for infinitely many $n \in \omega$. ⊣

4.4 Lemma. *Suppose that X is nice. Then for every Borel function $x \rightsquigarrow f^x \in {}^\omega\omega$ there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ such that*

$$\forall x \in X\ \exists^\infty k\ (f^x(n_k) < n_{k+1}).$$

Proof. Suppose that the lemma is not true and let $x \rightsquigarrow f^x$ be a counterexample. Without loss of generality we can assume that f^x is increasing for all $x \in X$. To get a contradiction we will define a Borel function $x \rightsquigarrow g^x \in {}^\omega\omega$ such that $\{g^x : x \in X\}$ is a dominating family. That will contradict the assumption that X is nice.

Define for $n \in \omega$,

$$g^x(n) = \underbrace{f^x \circ f^x \circ \cdots \circ f^x}_{n+1 \text{ times}}(n).$$

Suppose that $h \in {}^\omega\omega$ is an increasing function. By the assumption there exist $x \in X$ and k_0 such that

$$\forall k \geq k_0\ (f^x(h(k)) \geq h(k+1)).$$

In particular, for $k \geq h(k_0)$,

$$\forall k \geq h(k_0)\ (h(k) \leq g^x(k))$$

which finishes the proof. ⊣

We now return to the proof of (2) → (1) for 4.1. Let $x \rightsquigarrow f^x \in {}^\omega\omega$ be a Borel function. We want to show that $\bigcup_{x\in X}(M)_{f^x} \neq {}^\omega 2$.

Without loss of generality we can assume that M is the set built using the family from the proof of Lemma 3.17. For each x let $g^x \in {}^\omega\omega$ and $\{s^x_n : n \in \omega\}$ be such that $S^n_{f^x(n)} = \{x \in {}^\omega 2 : x\restriction[n, g^x(n)) = s^x_n\}$.

By Lemma 4.4, there exists a sequence $\langle n_k : k \in \omega\rangle$ such that

1. $n_{k+1} > \sum_{i=0}^k n_i$, for all k,
2. $\forall x \in X\ \exists^\infty n\ (g^x(n_k) < n_{k+1})$.

For $x \in X$ let $Z^x = \{k : g^x(n_k) < n_{k+1}\}$. By Lemma 4.3, there exists a sequence $\langle s_k : k \in \omega\rangle$ such that

$$\forall x \in X\ \exists^\infty k \in Z^x\ (s^x_{n_k} = s_k).$$

Without loss of generality we can assume that $s_k : [n_k, m_k) \longrightarrow 2$, where $m_k < n_{k+1}$. Choose $z \in {}^\omega 2$ such that $s_k \subseteq z$ for all k. It follows that $z \notin (M)_{f^x}$ for every $x \in X$. ⊣

As a corollary we have:

4.5 Theorem. *The following are equivalent:*

(1) $\mathbf{cov}(\mathcal{B}) > \kappa$,

(2) $\forall F \subseteq [{}^\omega\omega]^\kappa\ \exists g \in {}^\omega\omega\ \forall f \in F\ \exists^\infty n\ (f(n) = g(n))$.

The above proof can be dualized to give:

4.6 Theorem. *The following conditions are equivalent:*

(1) $X \times X \in \mathsf{NON}(\mathcal{B})$,

(2) for every Borel function $x \rightsquigarrow f^x \in {}^\omega\omega$ *there exists a function* $g \in {}^\omega\omega$ *such that*

$$\forall x \in X\ \forall^\infty n\ f^x(n) \neq g(n).$$

We only explain why we have $X \times X$ in (1) rather than X. If we analyze the proof of Theorem 4.1, we see that in order to produce a real z such that $z \notin \bigcup_{x\in X}(M)_{f_x}$ we had to diagonalize (find an infinitely often equal real) twice.

A similar situation arises here; each element of X produces two functions, and a real that avoids a given meager set is constructed from two such functions, each coming from a different point of X.

As a corollary we get:

4.7 Theorem. $\mathbf{non}(\mathcal{M})$ *is the size of the smallest family* $F \subseteq {}^\omega\omega$ *such that*

$$\forall g \in {}^\omega\omega\ \exists f \in F\ \exists^\infty n\ (f(n) = g(n)).$$

4.8 Theorem. $\mathsf{ADD}(\mathcal{B}) = \mathsf{B} \cap \mathsf{COV}(\mathcal{B})$. *In particular,* $\mathbf{add}(\mathcal{B}) = \min\{\mathfrak{b}, \mathbf{cov}(\mathcal{B})\}$.

Proof. The inclusion $\subseteq$ follows immediately from Theorem 3.11.

Suppose that $X \in \mathsf{B} \cap \mathsf{COV}(\mathcal{B})$. Let $x \rightsquigarrow f_x \in {}^\omega\omega$ be a Borel function. Since $X \in \mathsf{COV}(\mathcal{B})$ there is a real z such that $z \notin \bigcup_{x \in X} (M)_{f_x}$. For $x \in X$ define for $n \in \omega$,

$$g_x(n) = \min\bigl\{l : \forall t \in {}^n2 \ \bigl([t^\frown z\restriction[n, l)] \subseteq \bigcup_{m>n} S^m_{f_x(m)}\bigr)\bigr\}.$$

The function $x \rightsquigarrow g_x$ is also Borel. Since $X \in \mathsf{B}$, it follows that there is an increasing function $h \in {}^\omega\omega$ such that

$$\forall x \in X \ \forall^\infty n \ (g_x(n) \leq h(n)).$$

Consider the set

$$G = \bigcap_n \bigcup_{m>n} \bigcup\{[t^\frown z\restriction\bigl[m, h(m))] : t \in {}^m2\}.$$

Clearly G is a dense G_δ set. Moreover, for every $x \in X$ there is n such that

$$\bigcup_{m>n} \bigcup\{[t^\frown z\restriction\bigl[m, h(m))] : t \in {}^m2\} \subseteq \bigcup_{m>n} S^m_{f_x(m)}.$$

It follows that

$$\bigcup_{x \in X} (M)_{f_x} \subseteq {}^\omega2 - G,$$

which finishes the proof. ⊣

From Theorem 3.11 it follows that $\mathsf{D} \cup \mathsf{NON}(\mathcal{B}) \subseteq \mathsf{COF}(\mathcal{B})$. The other inclusion does not hold. We only have the following result dual to Theorem 4.8.

4.9 Theorem. *If* $X \notin \mathsf{D}$ *and* $Y \notin \mathsf{NON}(\mathcal{B})$ *then* $X \times Y \notin \mathsf{COF}(\mathcal{B})$. *In particular,* $\mathbf{cof}(\mathcal{B}) = \max\{\mathbf{non}(\mathcal{B}), \mathfrak{d}\}$.

4.10 Definition. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and define

$$\ell^1 = \left\{f \in {}^\omega\mathbb{R}_+ : \sum_{n=1}^{\infty} f(n) < \infty\right\}.$$

For $f, g \in \ell^1$, $f \leq^\star g$ if $f(n) \leq g(n)$ holds for all but finitely many n.

4.11 Theorem. *The following are equivalent:*

(1) $X \in \mathsf{ADD}(\mathcal{L})$,

(2) for every Borel function $x \rightsquigarrow S^x \in \mathcal{C}$ *there exists an* $S \in \mathcal{C}$ *such that*

$$\forall x \in X \ \forall^\infty n \ (S^x(n) \subseteq S(n)).$$

(3) *for every Borel function* $x \rightsquigarrow f^x \in \ell^1$ *there exists a function* $f \in \ell^1$ *such that*

$$\forall x \in X\ (f^x \leq^\star f).$$

In particular, the following conditions are equivalent:

(a) $\mathbf{add}(\mathcal{L}) > \kappa$,

(b) *for every family* $F \subseteq \mathcal{C}$ *of size* κ *there exists an* $\bar{S} \in \mathcal{C}$ *such that*

$$\forall S \in F\ \forall^\infty n\ (S(n) \subseteq \bar{S}(n)),$$

(c) *for every family* $F \subseteq \ell^1$ *of size* κ *there exists a* $g \in \ell^1$ *such that*

$$\forall f \in F\ \forall^\infty n\ (f \leq^\star g).$$

Proof. We will establish the equivalence of (1) and (2). Suppose that $X \in \mathsf{ADD}(\mathcal{L})$ and $x \rightsquigarrow S_x$ is a Borel function. Consider the morphism (φ_-, φ_+) witnessing that $\mathcal{C} \preceq \mathcal{L}$. Let f be such that $\bigcup_{x \in X} (N)_{\varphi_-(S_x)} \subseteq (N)_f$. Then $\varphi_+(f) \in \mathcal{C}$ is the object we are looking for.

Suppose that $X \notin \mathsf{ADD}(\mathcal{L})$. Let $F : X \longrightarrow {}^\omega\omega$ be a Borel function such that $\bigcup_{x \in X} (N)_{F(x)} \not\subseteq (N)_f$ for $f \in {}^\omega\omega$. Consider the morphism (φ_-, φ_+) witnessing that $\mathcal{L} \preceq \mathcal{C}$. It follows that there is no $S \in \mathcal{C}$ such that

$$\forall x \in X\ \forall^\infty n\ (\varphi_-(F(x))(n) \subseteq^\star S(n)).$$

Equivalence of (2) and (3) follows from:

4.12 Lemma. $\mathcal{C} \equiv \ell^1$.

Proof. To show that $\ell^1 \preceq \mathcal{C}$ define $\varphi_- : \ell^1 \longrightarrow \mathcal{C}$ as

$$\varphi_-(f)(n) = \{k : 2^{-n} > f(k) \geq 2^{-n-1}\}.$$

Similarly, define $\varphi_+ : \mathcal{C} \longrightarrow \ell^1$ by: $\varphi_+(S)(n) = \max\{2^{-k} : n \in S(k)\}$. It is easy to see that these functions have the required properties.

To show that $\mathcal{C} \preceq \ell^1$ identify $\omega \times \omega$ with ω via functions $L, K \in {}^\omega\omega$. For $S \in \mathcal{C}$ let

$$\varphi_-(S)(n) = \begin{cases} 2^{-n} & \text{if } K(n) \in S(L(n)), \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \ell^1$ let

$$\varphi_+(f)(n) = \left\{k : \frac{1}{2^{n-1}} > f(k) \geq \frac{1}{2^n}\right\}.$$

⊣

The second part of Theorem 4.11 follows readily from the first. ⊣

The dual version yields:

4.13 Theorem. *The following are equivalent:*

(1) $X \in \mathsf{COF}(\mathcal{L})$,

(2) for every Borel function $x \rightsquigarrow S^x \in \mathcal{C}$ *there exists an* $S \in \mathcal{C}$ *such that*

$$\forall x \in X\ \exists^\infty n\ (S(n) \not\subseteq S^x(n)),$$

(3) for every Borel function $x \rightsquigarrow f^x \in \ell^1$ *there exists a function* $f \in \ell^1$ *such that*

$$\forall x \in X\ \exists^\infty n\ (f^x(n) \leq f(n)).$$

In particular, the following are equivalent:

(a) $\mathbf{cof}(\mathcal{L}) < \kappa$,

(b) for every family $F \subseteq \mathcal{C}$ *of size* κ *there exists an* $\bar{S} \in \mathcal{C}$ *such that*

$$\forall S \in F\ \exists^\infty n\ (S(n) \not\subseteq \bar{S}(n)),$$

(c) for every family $F \subseteq \ell^1$ *of size* κ *there exists a* $g \in \ell^1$ *such that*

$$\forall f \in F\ \exists^\infty n\ (f(n) \leq g(n)).$$

Additivity of measure, $\mathbf{add}(\mathcal{L})$, has a special place among cardinal invariants of the continuum as being provably smaller than a large number of them. It has been conjectured (wrongly in [2]) that this is because additivity of measure is equivalent to Martin's Axiom for a large class of forcing notions (Suslin c.c.c.). Only very recently has this phenomenon been explained as being directly related to the combinatorial complexity of the measure ideal.

4.14 Definition. We say that an ideal $\mathcal{J} \subseteq P(\omega)$ is a *P-ideal* if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{J}$ there is a $X \in \mathcal{J}$ such that $X_n \subseteq^\star X$ for $n \in \omega$.
Define

$$\mathbf{add}^\star(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}\ \&\ \neg\exists Y \in \mathcal{J}\ \forall X \in \mathcal{A}\ X \subseteq^\star Y\}.$$

It is easy to see that the $\mathbf{cof}^\star$ defined analogously is equivalent to the ordinary cofinality.

Many ideals of Borel subsets of $\mathbb{R}$ are Tukey equivalent to analytic ($\mathbf{\Sigma}^1_1$) ideals of subsets of ω. For example:

- Ideal of null sets $\mathcal{L}$. Work with $\mathcal{C}$ instead of $\mathcal{L}$. For $S \in \mathcal{C}$ let $A_S = \{(n,k) \in \omega \times \omega : k \in S(n)\}$. The family $\{A_S : S \in \mathcal{C}\}$ generates an analytic P-ideal on $\omega \times \omega \simeq \omega$.

- $({}^\omega\omega, \leq^\star)$. This is very similar. For $f \in {}^\omega\omega$ let $A_f = \{(n,k) : k \leq f(n)\}$.

- ℓ^1 is Tukey equivalent to the P-ideal of summable sets

$$\left\{X \subseteq \omega : \sum_{n \in X} \frac{1}{n} < \infty\right\}.$$

Moreover, in all these cases the additivity of the ideal is equal to the *additivity of the associated ideal on ω. For example, $\mathbf{add}(\mathcal{L}) = \mathbf{add}^\star(\ell^1) = \mathbf{add}^\star(\mathcal{C})$, etc. In the remainder of this section we will show for a (nontrivial) analytic P-ideal $\mathcal{J}$ on ω we have

$${}^\omega\omega \preceq \mathcal{J} \preceq \mathcal{L}.$$

We need a few general facts about analytic P-ideals. To simplify the notation let us identify ${}^\omega 2$ with $P(\omega)$ via characteristic functions.

Let $\mathsf{K}({}^\omega 2)$ be the collection of compact subsets of ${}^\omega 2$ with Hausdorff metric d_H defined as follows. For two nonempty compact sets $K, L \subseteq {}^\omega 2$ let

$$d_H(K, L) = \max(\rho(K, L), \rho(L, K)),$$

where $\rho(K, L) = \max_{x \in L} d(x, K)$ (d is the usual metric in ${}^\omega 2$).

Let $\mathsf{M} \subseteq \mathsf{K}({}^\omega 2)$ be the collection of compact subsets of ${}^\omega 2$ which are downward closed. We will use the following well known facts:

4.15 Lemma.

(1) $\mathsf{K}({}^\omega 2)$ *is a compact Polish space,*

(2) M *is a closed subspace of* $\mathsf{K}({}^\omega 2)$.

Proof. See [29, 4.F]. $\dashv$

Let $\mathcal{J}$ be an analytic P-ideal on ω. Define

$$\mathsf{F} = \{K \in \mathsf{M} : \forall X \in \mathcal{J}\ \exists n\ (X - n \in K)\}.$$

It is clear that F is a filter.

4.16 Lemma. *Suppose that $H \subseteq \mathsf{F}$ is a closed set. There exists a nonempty relatively clopen set $U \subseteq H$ such that*

$$\textstyle\bigcap_{K \in U} K \in \mathsf{F}.$$

In particular, $H = \bigcup_{n \in \omega} H_n$, where for each n, $\bigcap_{K \in H_n} K \in \mathsf{F}$.

Proof. Let $\langle U_n : n \in \omega\rangle$ be an enumeration of clopen subsets of $\mathsf{K}({}^\omega 2)$.

For $X \in \mathcal{J}$ and n define $H_n(X) = \{K \in H : X - n \in K\}$. The sets $H_n(X)$ are closed and $H = \bigcup_{n \in \omega} H_n(X)$ for every $X \in \mathcal{J}$. By the Baire Category Theorem for each X there is a pair $(n(X), m(X)) \in \omega \times \omega$ such that

$$H_{n(X)}(X) \cap U_{m(X)} = H \cap U_{m(X)}.$$

Since $\mathcal{J}$ is a P-ideal, we can find (n, m) such that

$$\{X : n(X) = n \;\&\; m(X) = m\} \text{ is cofinal in } \mathcal{J}.$$

It follows that $\bigcap_{K \in U_m \cap H} K$ contains $X - n$ for cofinally many X and thus belongs to F. That finishes the proof of the first part.

To prove the second part build by induction a sequence $\langle H_\alpha : \alpha < \omega_1 \rangle$ such that

1. $H_0 = H$,
2. $H_\lambda = \bigcap_{\beta<\lambda} H_\beta$ for limit λ,
3. $H_{\alpha+1} = H_\alpha - U_\alpha$, where U_α is like in the part that is already proved.

The construction has to terminate after $\alpha_0 < \omega_1$ steps with $H_{\alpha_0} = \emptyset$. ⊣

4.17 Lemma. F *is* F_σ *in* M.

Proof. Consider $\mathsf{G} = \mathsf{M} - \mathsf{F}$. Note that for $K \in \mathsf{M}$ we have

$$K \in \mathsf{G} \iff \exists X \in \mathcal{J}\ \forall n\ (X - n \notin K).$$

It follows that G is an analytic ideal. Moreover, G is a σ-ideal; if $\{K_n : n \in \omega\} \subseteq \mathsf{G}$ and $K \subseteq \bigcup_n K_n$ then $K \in \mathsf{G}$. To see this let X_n witness that $K_n \in \mathsf{G}$. Find $X \in \mathcal{J}$ such that $X_n \subseteq^\star X$ for all n. Clearly, $X - n \notin K$ for all n. Now the lemma follows immediately from the following:

4.18 Theorem. *Let $\mathcal{I}$ be an analytic σ-ideal of compact sets in a compact metrizable space E. Then $\mathcal{I}$ is actually G_δ.*

Proof. See [14], [41] or [30]. ⊣

This completes the proof of Lemma 4.17. ⊣

4.19 Lemma. F *is countably generated.*

Proof. Using Lemma 4.17 represent $\mathsf{F} = \bigcup_n H_n$, where each H_n is closed. Apply Lemma 4.16 to write for $n \in \omega$, $H_n = \bigcup_{m\in\omega} H^n_m$, where $G^n_m = \bigcap_{K \in H^n_m} K \in \mathsf{F}$. It is clear that $\{G^n_m : n, m \in \omega\}$ generates F. ⊣

Let $\langle G_n : n \in \omega \rangle$ be a descending sequence generating F. The following lemma provides a simple ($F_{\sigma\delta}$) description of $\mathcal{J}$ in terms of $\langle G_n : n \in \omega \rangle$.

4.20 Lemma. $X \in \mathcal{J} \iff \forall n\ \exists m\ (X - m \in G_n)$.

Proof. The implication ($\Longrightarrow$) is obvious.

($\Longleftarrow$) We will use the following result.

4.21 Theorem. *Suppose that $\mathcal{I} \subseteq P(\omega)$ is an ideal containing all finite sets. The following conditions are equivalent:*

(1) $\mathcal{I}$ has the Baire property,

(2) $\mathcal{I}$ is meager,

(3) there exists a partition $\{I_n : n \in \omega\}$ of ω into disjoint intervals such that
$$\forall X \in \mathcal{I}\ \forall^\infty n\ (I_n \not\subseteq X).$$

Proof. See [54] or [8]. $\dashv$

Suppose that $X \notin \mathcal{J}$. The ideal $\mathcal{J}{\restriction}X = \{Y \cap X : Y \in \mathcal{J}\} \subseteq P(X)$ is analytic and hence has the Baire property. By Theorem 4.21(3) there exists a partition $\{I_n : n \in \omega\}$ of X into finite sets such that
$$\forall Z \in \mathcal{J}\ \forall^\infty n\ (I_n \not\subseteq Z).$$

Consider the set
$$K = \{Y : \forall n\ (I_n \not\subseteq Y)\} \in \mathsf{F}.$$

Let k be such that $G_k \subseteq K$. It follows that for every $m \in \omega$,
$$X - m =^\star \bigcup_{n\in\omega} I_n \notin G_k,$$

which finishes the proof of Lemma 4.20. $\dashv$

For $K, L \in \mathsf{K}({}^\omega 2)$ define $K \oplus L = \{X \cup Y : X \in K,\ Y \in L\}$. ($\cup$ is in $P(\omega)$ the same as coordinate-wise maximum in ${}^\omega 2$).

Let $\langle G_n : n \in \omega\rangle$ continue to be a descending sequence generating F.

4.22 Lemma. *For every $K \in \mathsf{F}$ there exists an m such that $G_m \oplus G_m \subseteq K$.*

Proof. Fix $X \in \mathcal{J}$ and using the fact that $\mathcal{J}$ is a P-ideal find k such that the set
$$H_X = \{Y : (X - k) \cup Y \in K\} \in \mathsf{F}.$$

Let $n(X)$ be such that $G_{n(X)} \subseteq H_X$. We have
$$\{X - n(X)\} \oplus G_{n(X)} \subseteq K.$$

Choose an n such that $\{X : n(X) = n\}$ is cofinal in $\mathcal{J}$. The set $L = \{X : \{X - n\} \oplus G_n \subseteq K\} \in \mathsf{F}$. Let $m \geq n$ be such that $G_m \subseteq L$. It follows that $G_m \oplus G_m \subseteq K$. $\dashv$

We are ready to formulate the first result.

4.23 Theorem. *Suppose that $\mathcal{J}$ is an analytic P-ideal on ω. Then $\mathcal{J} \preceq \ell^1$. In particular, $\mathbf{add}^\star(\mathcal{J}) \geq \mathbf{add}(\mathcal{L})$ and $\mathbf{cof}(\mathcal{J}) \leq \mathbf{cof}(\mathcal{L})$.*

Proof. Use Lemma 4.22 to find a descending sequence $\langle G_n : n \in \omega \rangle$ generating F such that for each n,

$$\underbrace{G_{n+1} \oplus \cdots \oplus G_{n+1}}_{2^{2n+1} \text{ times}} \subseteq G_n.$$

For $X \in \mathcal{J}$ let $\langle k_n(X) : n \in \omega \rangle$ be an increasing sequence such that

$$\forall n \; (X - k_n(X) \in G_{n+2}).$$

Identify ω with $[\omega]^{<\omega}$ and define $\varphi_- : \mathcal{J} \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow \mathcal{J}$ such that

$$X \subseteq^\star \varphi_+(S), \quad \text{whenever } \varphi_-(X) \subseteq^\star S.$$

Since $\mathcal{C} \equiv \ell^1 \equiv \mathcal{L}$ this will finish the proof. For $X \in \mathcal{J}$ and $n \in \omega$ define

$$\varphi_-(X)(n) = X \cap k_n(X) \in [\omega]^{<\omega} \simeq \omega.$$

The function φ_+ will be defined as follows. Suppose that $S \in \mathcal{C}$ is given (with $S(n) \subseteq [\omega]^{<\omega}$). For $n \in \omega$ let

$$Z_n = \{(t, s) \in S(n+1) \times S(n) : s \subseteq t \;\&\; t - \max(s) \in G_{n+2}\}.$$

Now define

$$v_n = \bigcup\nolimits_{(t,s) \in Z_n} t - \max(s).$$

Note that for sufficiently large n, v_n is a sum of at most 2^{2n} terms, each belonging to G_{n+2}, and so, $v_n \in G_{n+1}$.

The motivation for this definition is following: if $\varphi_-(X)(n) = X \cap k_n(X) \in S(n)$ and $\varphi_-(X)(n+1) = X \cap k_{n+1}(X) \in S(n+1)$, then

$$X \cap k_{n+1}(X) - \max\big(X \cap k_n(X)\big) = X \cap \big[k_n(X), k_{n+1}(X)\big) \subseteq v_n.$$

The requirements of the definition describe this situation and filter out "background noise" coming with S.

Finally define

$$\varphi_+(S) = Y = \bigcup\nolimits_n v_n.$$

By the remarks above it is clear that if $X \in \mathcal{J}$ and $S \in \mathcal{C}$ then from the fact that

$$\forall^\infty n \; (\varphi_-(X)(n) \in S(n))$$

it follows that $X \subseteq^\star Y = \varphi_+(S)$. To finish the proof it remains to show that the range of φ^+ is contained in $\mathcal{J}$.

Let $\varphi_+(S) = Y = \bigcup_n v_n$ be defined as above. For $j \in \omega$, let $Y_j = \bigcup_{n \geq j} v_n$. Since $Y - Y_j$ is finite for every j, by the lemma above, in order to show that $Y \in \mathcal{J}$ it would suffice to show that $Y_j \in G_j$.

4.24 Lemma. *For each $l \in \omega$,*

$$v_n \cup v_{n+1} \cup \cdots \cup v_{n+l} \in G_n.$$

Proof. We prove this by induction on l. For each n, $v_n \in G_{n+1}$ so the lemma is true for $l = 0$. Suppose it holds for l and all n. We have

$$v_n \cup v_{n+1} \cup v_{n+l+1} = v_n \cup (v_{n+1} \cup \ldots v_{n+1+l}) \in G_{n+1} \oplus G_{n+1} \subseteq G_n,$$

which finishes the proof. $\dashv$

Since the sets G_n are closed, we conclude that $Y_j \in G_j$. In particular, by Lemma 4.20, $Y = Y_0 \in \mathcal{J}$. This completes the proof of Theorem 4.23. $\dashv$

The last theorem gave us a lower bound for $\mathbf{add}^\star(\mathcal{J})$. The next theorem gives us an upper bound.

Suppose that $\mathcal{J} \subseteq P(\omega)$ is an ideal. We say that $\mathcal{J}$ is *atomic* if there is a $Z \in \mathcal{J}$ such that $\mathcal{J} = \{X \subseteq \omega : X \subseteq^\star Z\}$. It is clear that $\mathbf{add}^\star(\mathcal{J})$ is undefined (or equal to ∞) for an atomic ideal.

4.25 Theorem. *Suppose that $\mathcal{J}$ is an analytic P-ideal which is not atomic. Then ${}^\omega\omega \preceq \mathcal{J}$. In particular, $\mathbf{add}^\star(\mathcal{J}) \leq \mathfrak{b}$ and $\mathbf{cof}(\mathcal{J}) \geq \mathfrak{d}$.*

Proof. Let $\mathcal{J}$ be an analytic P-ideal. Use Lemma 4.22 to find a descending sequence $\langle G_n : n \in \omega\rangle$ generating F such that for each n, $G_{n+1} \oplus G_{n+1} \subseteq G_n$. To show that ${}^\omega\omega \preceq \mathcal{J}$ it suffices (by duality), to check that $(\mathcal{J}, \mathcal{J}, \not\supseteq^\star) \preceq (\omega^\omega, \omega^\omega, \not\geq^\star)$. Thus we need to find a function $\varphi_- : \mathcal{J} \longrightarrow \omega^\omega$ such that φ_-"$\mathcal{J}$ is cofinal in ω^ω.

For $X \in \mathcal{J}$ define

$$\varphi_-(X)(n) = \min\{j \geq n : X - j \in G_n\} \quad \text{for } n \in \omega.$$

Note that if $X \subseteq^\star Y$ then $\varphi_-(X) \leq^\star \varphi_-(Y)$.

Let $g \in \omega^\omega$ be an increasing function. For $X \in \mathcal{J}$ let

$$Z^g_X = \{n : X - g(n) \notin G_n\}.$$

Observe that $Z^g_{X\cup Y} \supseteq Z^g_X \cup Z^g_Y$ for $X, Y \in \mathcal{J}$, hence the family $\{Z^g_X : X \in \mathcal{J}\}$ generates an ideal which we call $\mathcal{I}^g$.

Note that we are trying to show that

$$\forall g \in \omega^\omega \ \exists X \in \mathcal{J} \ (|\omega - Z^g_X| < \aleph_0),$$

which means that all ideals $\mathcal{I}^g$ are trivial.

Suppose that for some $g \in \omega^\omega$, $\mathcal{I}^g$ is a proper ideal. We will show that $\mathcal{J}$ is an atomic ideal.

Since $\mathcal{I}^g$ is a continuous image of $\mathcal{J}$, $\mathcal{I}^g$ is an analytic ideal so it has the Baire property. By Theorem 4.21, there exists a sequence of disjoint intervals $\langle I_n : n \in \omega\rangle$ such that

$$\forall X \in \mathcal{J} \ \forall^\infty n \ (I_n \not\subseteq Z^g_X).$$

Let $h(n) = \max(I_n)$ for $n \in \omega$. It follows that

$$\forall X \in \mathcal{J}\ \forall^\infty n\ (X - g(h(n)) \in G_n).$$

For $n \in \omega$ let

$$U_n = \bigcup\{Y : Y \subseteq [g(h(n)), g(h(n+1))) : Y \in G_n\},$$

and let $U = \bigcup_n U_n$. By the choice of h, for every $X \in \mathcal{J}$, $X \cap [g(h(n)), g(h(n+1))) \subseteq U_n$ holds for all but finitely many n. Thus, $X \subseteq^\star U$ for every $X \in \mathcal{J}$. Therefore, to finish the proof it is enough to check that $U \in \mathcal{J}$.

4.26 Lemma. *$\forall^\infty n\ (U_n \in G_n)$.*

Proof. For $k \in \omega$, if $U_{k+1} \in G_{k+1}$ set $U'_{k+1} = \emptyset$. Otherwise, let $U'_{k+1} \subseteq U_{k+1}$ be such that $U'_{k+1} \in G_k - G_{k+1}$. Note that since $G_{n+1} \oplus G_{n+1} \subseteq G_n$ for every n, and U_{k+1} is a union of sets in G_{k+1}, such a set can be found. Moreover, for every $m > k$, $U'_{k+1} \cup U'_{k+2} \cup \cdots \cup U'_m \in G_k$. Thus, by compactness, $\bigcup_{l>k} U'_l \in G_k$. It follows from Lemma 4.20 that $X = \bigcup_k U'_k \in \mathcal{J}$. On the other hand, if X is infinite then

$$\exists^\infty n\ \left(X - g(h(n)) \notin G_n\right),$$

which contradicts the choice of h. ⊣

Suppose that for $n > n_0$, $U_n \in G_n$ and define $U^k = \bigcup_{j>k+n_0} U_j$, for $k \in \omega$. As above, $U =^\star U^k \in G_k$ for $k \in \omega$. Therefore, by Lemma 4.20, $U \in \mathcal{J}$. This completes the proof of Theorem 4.25. ⊣

Historical Remarks. Theorem 4.1 was proved in [39] and [7]. Theorem 4.6 is due to Pawlikowski and Recław in [39]. Theorems 4.5 and 4.7 were proved in [4]. Theorem 4.8 was proved in [39]. The second part is due to Miller [33]. The first part of Theorem 4.11 was proved in [39] and the second in [3]. Todorcevic [56] proved Theorem 4.25. Theorem 4.23 is due to Todorcevic [55] and Louveau and Velickovic [32]. Methods used in the proof, in particular Lemmas 4.17 and 4.22, are due to Solecki [50, 51]. Similar ideas were already present in [56] and earlier in [28]. Theorem 4.18 is due to Christensen and Saint Raymond. It was generalized in [30]. Theorem 4.21 was proved by Talagrand.

5. Cofinality of $\mathbf{cov}(\mathcal{J})$ and $\mathsf{COV}(\mathcal{J})$

It is clear that cardinal invariants **add**, **non** and **cof** have uncountable cofinality and families ADD, NON and COF are σ-ideals. It this section we investigate **cov** and COV for both ideals $\mathcal{B}$ and $\mathcal{L}$.

5.1 Theorem. *$\mathsf{COV}(\mathcal{B})$ is a σ-ideal. In particular,* $\mathrm{cf}(\mathbf{cov}(\mathcal{B})) > \aleph_0$.

Proof. Suppose that $\{X_n : n \in \omega\} \subseteq \mathsf{COV}(\mathcal{B})$. Let $x \rightsquigarrow f_x \in {}^\omega\omega$ be a Borel function. It is enough to find a $g \in {}^\omega\omega$ such that

$$\forall n\ \forall x \in X_n\ \exists^\infty m\ (g(m) = f_x(m)).$$

Let $\{A_k : k \in \omega\}$ be a partition of ω into infinitely many infinite pieces. For each n consider the function $x \rightsquigarrow f_x{\restriction}A_n$ and find a $g_n \in {}^{A_n}\omega$ such that

$$\forall x \in X_n\ \exists^\infty k \in A_n\ (f_x(k) = g_n(k)).$$

Then $g = \bigcup_n g_n$ is as required. $\dashv$

In the presence of many dominating reals we have a similar result for the measure ideal.

5.2 Theorem. *If* $\mathbf{cov}(\mathcal{L}) \leq \mathfrak{b}$ *then* $\mathrm{cf}(\mathbf{cov}(\mathcal{L})) > \aleph_0$.

Proof. See [5] or [8]. $\dashv$

The following surprising result of Shelah shows that without any additional assumptions it is not possible to show that $\mathbf{cov}(\mathcal{L})$ has uncountable cofinality.

5.3 Theorem. *It is consistent with* ZFC *that* $\mathsf{COV}(\mathcal{L})$ *is not a* σ*-ideal and* $\mathrm{cf}(\mathbf{cov}(\mathcal{L})) = \aleph_0$.

The proof of this theorem will occupy the rest of this section. The model will be obtained by a two-step finite support iteration. We start with a suitably chosen model V_0 satisfying $2^{\aleph_0} = \aleph_1$ and add $\aleph_\omega$ Cohen reals followed by a finite support iteration of subalgebras of the random algebra $\mathbf{B}$. We start by developing various tools needed for the construction.

The Random Real Algebra

Recall that the random real algebra can be represented as

$$\mathbf{B} = \{P \subseteq {}^\omega 2 : \mu(P) > 0 \text{ and } P \text{ is closed}\}.$$

For $P_1, P_2 \in \mathbf{B}$, $P_1 \leq P_2$ if $P_1 \subseteq P_2$. Elements of $\mathbf{B}$ can be coded by reals in the following way. Let $\widetilde{P} \in V_0$ be a universal closed set, i.e. $\widetilde{P} \subseteq {}^\omega 2 \times {}^\omega 2$ is closed and for every closed set $P \subseteq {}^\omega 2$ there is x such that $P = (\widetilde{P})_x$. Let $H = \{x : \mu((\widetilde{P})_x) > 0\}$. By Theorem 3.6(1), H is a Borel set. Define $\widetilde{B} = (H \times {}^\omega 2) \cap \widetilde{P}$. If M is a model of ZFC then we define

$$\mathbf{B}^M = \{P \in \mathbf{B} : \exists x \in M \cap {}^\omega 2\ (P = (\widetilde{B})_x)\}.$$

Δ-systems

The following concepts will be crucial for the construction of the model.

5.4 Definition. Let $\mathcal{R} \in V_0$ be a forcing notion Suppose that $\bar{p} = \langle p_n : n \in \omega\rangle$ is a sequence of conditions in $\mathcal{R}$. Let $\dot{X}_{\bar{p}}$ be the $\mathcal{R}$-name for the set $\{n : p_n \in \dot{G}_{\mathcal{R}}\}$. In other words, for every n, $p_n = [\![n \in \dot{X}_{\bar{p}}]\!]$.

At the moment we will be concerned with the case when $\mathcal{R} = \mathbf{C}_{\aleph_{\omega+1}}$ is the forcing notion adding $\aleph_{\omega+1}$ Cohen reals. For the definition below, neither the representation of $\mathbf{C}$ nor the ordering on $\mathbf{C}$ matters, so for simplicity, we will identify $\mathbf{C}$ with ω.

5.5 Definition. Let $\Delta \subseteq (\mathbf{C}_{\aleph_{\omega+1}})^\omega$ be the collection of all sequences $\bar{p} = \{p_n : n \in \omega\}$ such that there exist a $k, l \in \omega$ and $g \in {}^{l\times\omega}\omega$, $s \in {}^k\omega$ such that

1. $\mathsf{dom}(p_n) = \{\beta_1, \dots, \beta_k\} \dot\cup \{\alpha_1^n, \dots, \alpha_l^n\}$, with $\beta_1 < \cdots < \beta_k$ and $\alpha_1^n < \cdots < \alpha_l^n$ for $n \in \omega$,
2. $\alpha_l^n < \alpha_1^{n+1}$ for $n \in \omega$, (so the $\mathsf{dom}(p_n)$'s form a Δ-system with root $\{\beta_1, \dots, \beta_k\}$),
3. $p_n(\alpha_i^n) = g(i, n)$ for every $i \le l, n \in \omega$,
4. $p_n(\beta_i) = s(i)$ for $i \le k, n \in \omega$.

Let $p_{\bar{p}} = p_0 \restriction \{\beta_1, \dots, \beta_k\}$.

Note that if $\bar{p} \in \Delta$ then $f_{\bar{p}} = \bigcup_{n\in\omega} p_n$ is a function. Moreover, $p_{\bar{p}} = f_{\bar{p}} \restriction \{\beta_1, \dots, \beta_k\}$ and $p_{\bar{p}} \Vdash_{\mathbf{C}_{\aleph_{\omega+1}}} X_{\bar{p}}$ is infinite.

5.6 Definition. A subset $\Delta' \subseteq \Delta$ is *filter-like* if for any $\bar{p}^1, \dots, \bar{p}^n \in \Delta'$ there exists a q such that

$$q \Vdash_{\mathbf{C}_{\aleph_{\omega+1}}} \bigcap_{i\le n} X_{\bar{p}^i} \text{ is infinite.}$$

5.7 Theorem. *Suppose that $V \models 2^{\aleph_0} = \aleph_1$ & $2^{\aleph_1} = \aleph_{\omega+1}$. Then Δ is the union of $\aleph_1$ filter-like sets.*

Proof. Let T be the collection of $\langle k, l, v, \{f_{i,n}, g_j : i \le l, j \le k, n \in \omega\}, g, s\rangle$ such that

1. $k, l \in \omega$,
2. $v \in [\aleph_1]^{\le\aleph_0}$,
3. $g_j, f_{i,n} \in {}^v\omega$ are pairwise different for $i \le l, j \le k, n \in \omega$,
4. $g \in {}^{l\times\omega}\omega$,
5. $s \in {}^k\omega$.

From the assumption about the cardinal arithmetic in V it follows that $V \models \aleph_n^{\aleph_0} = \aleph_n$ for $n \ge 1$. In particular $V \models |T| = \aleph_1$. Moreover, since $V \models 2^{\aleph_1} = \aleph_{\omega+1}$ we can find in V an enumeration $\langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of ${}^{\aleph_1}\omega$.

Given $t = \langle k, l, v, \{f_{i,n}, g_j : i \le l, j \le k, n \in \omega\}, g, s\rangle \in T$ define $\Delta_t \subseteq \Delta$ to be the collection of all $\bar{p} = \langle p_n : n \in \omega\rangle$ such that

1. $\mathsf{dom}(p_n) = \{\beta_1, \dots, \beta_k\} \dot\cup \{\alpha_1^n, \dots, \alpha_l^n\}$, with $\beta_1 < \cdots < \beta_k$,

2. $p_n(\alpha_i^n) = g(i,n)$,

3. $p_n(\beta_i) = s(i)$,

4. $\forall i \leq l \ (h_{\alpha_i^n}\restriction v = f_{i,n})$,

5. $\forall j \leq k \ (h_{\beta_j}\restriction v = g_j)$.

5.8 Lemma. Δ_t *is filter-like for every* $t \in T$.

Proof. For simplicity, suppose that $\bar{p}^1, \bar{p}^2 \in \Delta_t$ (the proof is the same when a larger number of $\bar{p}$s is involved). First we show that $f_{\bar{p}^1} \cup f_{\bar{p}^2}$ is a function. Suppose that $\alpha \in \mathsf{dom}(f_{\bar{p}^1}) \cap \mathsf{dom}(f_{\bar{p}^2})$. Consider the function h_α and note that exactly one of the following possibilities happens:

1. there exists exactly one pair (n,i) such that $h_\alpha\restriction v = f_{i,n}$. In this case $f_{\bar{p}^1}, f_{\bar{p}^2}$ agree on α with the value $g(i,n)$,

2. there exists exactly one $j \leq k$ such that $h_\alpha\restriction v = g_j$ (so $f_{\bar{p}^1}(\alpha) = f_{\bar{p}^2}(\alpha) = s(j)$).

Now, put $q = p_{\bar{p}_1} \cup p_{\bar{p}_2}$ and note that q has the required property. $\dashv$

To finish the proof of Theorem 5.7 note that $\Delta = \bigcup_{t \in T} \Delta_t$. Suppose that $\bar{p} = \langle p_n : n \in \omega \rangle \in \Delta$. Let k, l, g and s be as in Definition 5.5, and put v to be a countable set such that $h_{\alpha_i^n}\restriction v$ and $h_{\beta_j}\restriction v$ are pairwise different. $\dashv$

Finitely Additive Measures on ω

5.9 Definition. A set $\mathcal{A} \subseteq P(\omega)$ is an *algebra* if

1. $X \cup Y \in \mathcal{A}$ whenever $X, Y \in \mathcal{A}$,

2. $\omega - X \in \mathcal{A}$ whenever $X \in \mathcal{A}$,

3. $\emptyset, \omega \in \mathcal{A}$, $\{n\} \in \mathcal{A}$ for $n \in \omega$.

Given an algebra $\mathcal{A}$, a function $m : \mathcal{A} \longrightarrow [0,1]$ is a *finitely additive measure* if

1. $m(\omega) = 1$ and $m(\emptyset) = m(\{n\}) = 0$ for every n,

2. if $X, Y \subseteq \omega$ are disjoint, then $m(X \cup Y) = m(X) + m(Y)$.

We say that m is atomless if for every set $A \in \mathcal{A}$, $m(A) > 0$ there exists a $B \subseteq A$, $B \in \mathcal{A}$ such that $0 < m(B) < m(A)$.

Any non-principal filter on ω corresponds to a finitely additive 2-valued measure and any ultrafilter is a maximal such measure. In the sequel we will work with measures defined on $\mathcal{A} = P(\omega)$.

5.10 Definition. For a real valued function $f : \omega \longrightarrow [0,1]$ and any finitely additive measure m define

$$\int_\omega f\, dm = \lim_{n\to\infty} \sum_{k=0}^{2^n} \frac{k}{2^n} \cdot m(A_k),$$

where

$$A_k = \left\{ n : \frac{k}{2^n} \le f(n) < \frac{k+1}{2^n} \right\}.$$

We leave it to the reader to verify that integration with respect to m has its usual properties.

The following is a special case of the Hahn-Banach theorem.

5.11 Theorem (Hahn-Banach). *Suppose that m is a finitely additive measure on an algebra $\mathcal{A}$, and $X \notin \mathcal{A}$. Let $a \in [0,1]$ be such that*

$$\sup\{m(A) : A \subseteq X \;\&\; A \in \mathcal{A}\} \le a \le \inf\{m(B) : X \subseteq B \;\&\; B \in \mathcal{A}\}.$$

Then there exists a measure $\bar{m}$ on $P(\omega)$ extending m such that $\bar{m}(X) = a$.

We will need several results concerning the existence of measures in forcing extensions.

5.12 Lemma. *Let $m_0 \in V$ be a finitely additive measure on $P(\omega)$. For $i = 1,2$ let $\mathcal{R}_i$ be a forcing notion and let $\dot{m}_i$ be an $\mathcal{R}_i$-name for a finitely additive measure on $V^{\mathcal{R}_i} \cap P(\omega)$ extending m_0. Then there exists a $\mathcal{R}_1 \times \mathcal{R}_2$-name for a measure $\dot{m}_3$ extending both $\dot{m}_1$ and $\dot{m}_2$.*

Proof. We extend the measures using the Hahn-Banach theorem and we only need to check that the requirements are consistent. Suppose that we have $\mathcal{R}_1$-name $\dot{X}$ and $\mathcal{R}_2$-name $\dot{Y}$ such that $\Vdash_{\mathcal{R}_1 \times \mathcal{R}_2} \dot{X} \subseteq^\star \dot{Y}$. A necessary and sufficient condition for both measures to have a common extension is that in such a case $m_1(\dot{X}) \le m_2(\dot{Y})$. Let $(\bar{p}, \bar{q}) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $\bar{n}$ be such that

$$(\bar{p}, \bar{q}) \Vdash_{\mathcal{R}_1 \times \mathcal{R}_2} \dot{X} - \bar{n} \subseteq \dot{Y}.$$

Let

$$Z = \{n > \bar{n} : \exists p \in \mathcal{R}_1 \; (p \le \bar{p} \;\&\; p \Vdash_{\mathcal{R}_1} n \in \dot{X})\}.$$

Set Z belongs to V and $\bar{p} \Vdash_{\mathcal{R}_1} \dot{X} - \bar{n} \subseteq Z$. Similarly $\bar{q} \Vdash_{\mathcal{R}_2} Z \subseteq \dot{Y}$. In particular,

$$(\bar{p}, \bar{q}) \Vdash_{\mathcal{R}_1 \times \mathcal{R}_2} \dot{m}_1(\dot{X}) \le \dot{m}_1(Z) = m_0(Z) = \dot{m}_2(Z) \le \dot{m}_2(\dot{Y}).$$

$\dashv$

We will need the following theorem.

5.13 Theorem. *Suppose that* $m \in V$ *is a finitely additive atomless measure on* $P(\omega)$ *and* $v \in \mathbf{B}$. *For a* $\mathbf{B}$*-name* $\dot{X}$ *for an element of* $[\omega]^\omega$ *define*

$$\dot{m}^v_{\mathbf{B}}(\dot{X}) = \sup\left\{\inf\left\{\int_\omega \frac{\mu(q \cap [\![n \in \dot{X}]\!])}{\mu(q)}\, dm : q \leq p\right\} : p \leq v,\ p \in \dot{G}_{\mathbf{B}}\right\}.$$

The name $\dot{m}^v_{\mathbf{B}}$ *has the following properties:*

(1) $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}} : P(\omega) \longrightarrow [0,1]$,

(2) $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}}$ *is a finitely additive atomless measure,*

(3) for $X \in V \cap P(\omega)$ $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}}(X) = m(X)$,

(4) if $\dot{X}$ *is a* $\mathbf{B}$*-name for a subset of* ω *and* $\mu([\![n \in \dot{X}]\!]_{\mathbf{B}} \cap v)/\mu(v) = a > 0$ *for all* n, *then there is a condition* $p \in \mathbf{B}$, $p \leq v$, *such that* $p \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}}(\dot{X}) \geq a$.

Proof. Without loss of generality we can assume that $v = 2^\omega$ and therefore we will drop the superscript v altogether.

(1) is clear.

(2) For a $\mathbf{B}$-name $\dot{X}$ for a subset of ω and $p \in \mathbf{B}$ let

$$m_p(\dot{X}) = \int_\omega \frac{\mu(p \cap [\![n \in \dot{X}]\!])}{\mu(p)}\, dm$$

and

$$m^\star_p(\dot{X}) = \inf\{m_q(\dot{X}) : q \leq p\}.$$

Clearly, $\dot{m}_{\mathbf{B}}(\dot{X}) = \sup_{p \in \dot{G}} \inf_{q \leq p} m_q(\dot{X}) = \sup_{p \in \dot{G}} m^\star_p(\dot{X})$. Note that if $p \Vdash_{\mathbf{B}} \dot{X} \subseteq \dot{Y}$ then $p \cap [\![n \in \dot{X}]\!] \subseteq p \cap [\![n \in \dot{Y}]\!]$ for every n. It follows that $m_p(\dot{X}) \leq m_p(\dot{Y})$ and $m^\star_p(\dot{X}) \leq m^\star_p(\dot{Y})$.

Similarly, if $p \Vdash_{\mathbf{B}} \dot{X} \cap \dot{Y} = \emptyset$ and $\dot{Z}$ is a name for $\dot{X} \cup \dot{Y}$ then $m_p(\dot{X}) + m_p(\dot{Y}) = m_p(\dot{Z})$ and $m^\star_p(\dot{X}) + m^\star_p(\dot{Y}) \leq m^\star_p(\dot{Z})$.

5.14 Lemma. $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) \geq r \iff m^\star_p(\dot{X}) \geq r$.

Proof. ($\Longleftarrow$) This is obvious.

($\Longrightarrow$) Suppose that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) \geq r$. Fix a rational $t < r$ and $p' \leq p$. It follows that

$$D = \{q \leq p' : m^\star_q(\dot{X}) \geq t\}$$

is dense below p'. Let $\{q_n : n \in \omega\}$ be a maximal antichain in D. We have

$m_{q_n}(\dot{X}) \geq m^\star_{q_n}(\dot{X}) \geq t$. For every $k \in \omega$,

$$\begin{aligned} m_{\bigcup_{j\leq k} q_i}(\dot{X}) &= \int_\omega \frac{\mu(\bigcup_{j\leq k} q_i \cap [\![n \in \dot{X}]\!])}{\mu(\bigcup_{j\leq k} q_i)}\, dm \\ &= \int_\omega \frac{\sum_{j\leq k} \mu(q_j \cap [\![n \in \dot{X}]\!])}{\sum_{j\leq n} \mu(q_j)}\, dm \\ &= \sum_{j\leq k} \frac{\mu(q_j)}{\sum_{i\leq k}\mu(q_i)} \int_\omega \frac{\mu(q_j \cap [\![n \in \dot{X}]\!])}{\mu(q_i)}\, dm \\ &\geq t \sum_{j\leq k} \frac{\mu(q_j)}{\sum_{i\leq k}\mu(q_i)} = t. \end{aligned}$$

We leave it to the reader to check that by passing to the limit we get that $m_{p'}(\dot{X}) \geq t$, and since t and p' were arbitrary, that $m^\star_p(\dot{X}) \geq r$. $\dashv$

Now we show that $\dot{m}_{\mathbf{B}}$ is a finitely additive measure.

Suppose that $\Vdash_{\mathbf{B}} \dot{X} \subseteq \dot{Y}$. Suppose that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) > \dot{m}_{\mathbf{B}}(\dot{Y})$. Let $q \leq p$ and r be such that $q \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) > r \geq \dot{m}_{\mathbf{B}}(\dot{Y})$. Then $m^\star_q(\dot{X}) \geq r$ and $m^\star_q(\dot{Y}) < r$—contradiction.

Suppose that $\Vdash_{\mathbf{B}} \dot{X} \cap \dot{Y} = \emptyset$ and let $\dot{Z}$ be a name for $\dot{X} \cup \dot{Y}$. Let p, r_1, r_2 be such that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) \geq r_1$ and $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{Y}) \geq r_1$. It follows that $m^\star_p(\dot{X}) \geq r_1$ and $m^\star_p(\dot{Y}) \geq r_2$. Thus $m^\star_p(\dot{Z}) \geq r_1 + r_2$, so $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{Z}) \geq \dot{m}_{\mathbf{B}}(\dot{X}) + \dot{m}_{\mathbf{B}}(\dot{Y})$.

Suppose that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{Z}) > \dot{m}_{\mathbf{B}}(\dot{X}) + \dot{m}_{\mathbf{B}}(\dot{Y})$. There are reals r_1, r_2 and $q \leq p$ such that $q \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{X}) < r_1$, $q \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{Y}) < r_2$ and $q \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(\dot{Z}) > r_1 + r_2$. Use Lemma 5.14, to find $q' \leq q$ such that $m_{q'}(\dot{X}) < r_1$ and $m_{q'}(\dot{Y}) < r_2$. By the lemma, $m_{q'}(\dot{Z}) \geq m^\star_{q'}(\dot{Z}) \geq r_1 + r_2$. On the other hand, since $m_{q'}$ is additive, $m_{q'}(\dot{Z}) < r_1 + r_2$—contradiction.

(3) Suppose that $\dot{X}$ is a $\mathbf{B}$-name and for some $p \in \mathbf{B}$ and $X \in V \cap P(\omega)$, $p \Vdash_{\mathbf{B}} \dot{X} = X$. That means that for every $q \leq p$,

$$\frac{\mu(q \cap [\![n \in \dot{X}]\!])}{\mu(q)} = \begin{cases} 1 & \text{if } n \in X, \\ 0 & \text{if } n \notin X. \end{cases}$$

It follows that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(X) \geq m(X)$. Since $\dot{m}_{\mathbf{B}}$ is a measure, by looking at the complements we get, $p \Vdash_{\mathbf{B}} 1 - \dot{m}_{\mathbf{B}}(X) \geq 1 - m(X)$, hence $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(X) = m(X)$.

(4) Suppose that $\mu([\![n \in \dot{X}]\!]_{\mathbf{B}}) = a > 0$ for $n \in \omega$. Let

$$D = \{p : \exists \varepsilon > 0\ m_p(\dot{X}) \leq (1 - \varepsilon) \cdot a\}.$$

If D is not dense in $\mathbf{B}$, then the condition witnessing that has the required property.

So suppose that D is dense and work towards a contradiction. Let $\{q_n : n \in \omega\}$ be a maximal antichain in D. Clearly $\sum_{n=0}^{\infty} \mu(q_n) = 1$. Let $\varepsilon_0 > 0$ be such that $m_{q_0}(\dot{X}) \leq (1-\varepsilon_0) \cdot a$, which means that

$$\int_\omega \mu(q_0 \cap [\![n \in \dot{X}]\!])\, dm \leq (1-\varepsilon_0) \cdot a \cdot \mu(q_0).$$

Similarly for $n > 0$,

$$\int_\omega \mu(q_n \cap [\![n \in \dot{X}]\!])\, dm \leq a \cdot \mu(q_n).$$

Let $q = \bigcup_{i \leq n} q_n$. We have

$$\int_\omega \mu(q \cap [\![n \in \dot{X}]\!])\, dm \leq (1-\varepsilon_0) \cdot a \cdot \mu(q_0) + \sum_{j=1}^{n} a \cdot \mu(q_j) = a \cdot \mu(q) - \varepsilon_0 \cdot a \cdot \mu(q_0).$$

This is a contradiction since

$$\lim_{\mu(q) \to 1} \int_\omega \mu(q \cap [\![n \in \dot{X}]\!])\, dm = a.$$

$\dashv$

The Iteration

Let V_0 be a model satisfying $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = 2^{\aleph_2} = \cdots = \aleph_{\omega+1}$. In V_0 we will define the following objects:

1. A finite support iteration $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \aleph_{\omega+1} \rangle$.
2. A sequence $\langle A_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1} \rangle$.
3. A sequence $\langle \dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1}, \xi < \aleph_1 \rangle$ such that
   (a) $\dot{m}^\xi_\alpha$ is a $\mathcal{P}_\alpha$-name for a finitely additive measure on ω,
   (b) $\dot{m}^\xi_\alpha$ extends $\bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$. In particular, if $\mathrm{cf}(\gamma) > \aleph_0$ then $\dot{m}^\xi_\gamma = \bigcup_{\beta<\gamma} \dot{m}^\xi_\beta$.

The definition is inductive. Formally, given $\mathcal{P}_\alpha$, $\{\dot{m}^\xi_\alpha : \xi < \aleph_1\}$ and A_α we define $\{\dot{m}^\xi_{\alpha+1} : \xi < \aleph_l\}$ followed by $A_{\alpha+1}$ and then $\mathcal{P}_{\alpha+1} = \mathcal{P}_\alpha \star \dot{\mathcal{Q}}_\alpha$.

For limit α, $\mathcal{P}_\alpha$ and $\{\dot{m}^\xi_\alpha : \xi < \aleph_1\}$ will be defined by the previous values and $A_\alpha = \emptyset$. Since the definition of $\dot{m}^\xi_\alpha$ is most complicated it is more natural to proceed in the reverse order by making commitments about the defined objects as we go along.

We will use the following notation: suppose that $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta \rangle$ is a finite support iteration and $A \subseteq \delta$, $A \in V_0$. Let $\mathcal{P}(A)$ be the subalgebra generated by $\dot{G}{\restriction}A$ and let $V_0[\dot{G}{\restriction}A]$ denote model $V_0[\dot{G} \cap \mathcal{P}(A)]$. As we are going to iterate c.c.c. forcing notions of size $2^{\aleph_0}$, it follows that if $|A| = \aleph_n$, $n > 0$ then $V_0[\dot{G}{\restriction}A] \models 2^{\aleph_0} = \aleph_n$.

To define the iteration we require that:

A0. $A_\alpha \subseteq \alpha$ for $\alpha < \aleph_{\omega+1}$.

Let $\langle \mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be a finite support iteration such that

$$\Vdash_\alpha \mathcal{Q}_\alpha = \begin{cases} \mathbf{C} & \text{if } \alpha < \aleph_\omega, \\ \mathbf{B}^{V_0[\dot{G}\restriction A_\alpha]} & \text{if } \alpha \geq \aleph_\omega. \end{cases}$$

5.15 Lemma. *Suppose that G is $\mathcal{P}_\alpha$-generic over V_0 and $x \in V_0[G] \cap P(\omega)$. Then x can be computed from countably many generic reals. In other words, there exists a countable set $\{\alpha_n : n \in \omega\} \subseteq \alpha$, $\{\alpha_n : n \in \omega\} \in V_0$ and a Borel function $f \in {}^\omega({}^\omega 2) \longrightarrow {}^\omega 2$, $f \in V_0$ such that $x = f(\dot{G}(\alpha_1), \dots, \dot{G}(\alpha_n), \dots)$.*

Proof. Induction on α.

Case 1. $\alpha = \beta + 1$. Let $G \subseteq \mathcal{P}_\alpha$ be a generic filter and let $x \in V_0[G]$. Work in the model $V_0[G\restriction\beta]$. Since $\mathcal{P}_\alpha = \mathcal{P}_\beta \star \mathbf{B}^{V_0[\dot{G}\restriction A_\beta]}$ there exists a Borel function $\tilde{f} \in V_0[G \cap \mathcal{P}_\beta]$ such that

$$V_0[G \cap \mathcal{P}_\alpha] \models \tilde{f}(G(\beta)) = x.$$

Since $\tilde{f}$ is coded by a real, there exists a set $\{\alpha_n : n \in \omega\} \subseteq \beta$ and a function $f \in V_0$ such that

$$\tilde{f} = f(G(\alpha_1), \dots, G(\alpha_n), \dots).$$

The required function is constructed from f and the set $\{\alpha_n : n \in \omega\} \cup \{\beta\}$.

Case 2. $\mathrm{cf}(\alpha) = \aleph_0$. Fix an increasing sequence $\langle \alpha_n : n \in \omega\rangle$ such that $\sup_n \alpha_n = \alpha$ and suppose that x is a $\mathcal{P}_\alpha$-name for a real number (i.e. a set of countably many antichains). Let x_n be a $\mathcal{P}_{\alpha_n}$-name for a real obtained by taking those conditions in these antichains that belong to $\mathcal{P}_{\alpha_n}$. Observe that typically $x_n(i)$ is defined only for finitely many values of i, and that only below various conditions in $\mathcal{P}_{\alpha_n}$. So, formally, we need to extend this definition (arbitrarily) so that every condition in $\mathcal{P}_{\alpha_n}$ forces that x_n is a real. Note that $\Vdash_{\mathcal{P}_\alpha} \lim_n x_n = x$. Apply the induction hypothesis to x_n's to get Borel functions f_n and countable sets A_n. Let $A = \bigcup_n A_n$ and let $f : {}^{\omega\times\omega}({}^\omega 2) \longrightarrow {}^\omega 2$ be defined as

$$f(\dots, x^n_m, \dots) = \lim_n f_n(\dots, x^n_m, \dots).$$

Case 3. $\mathrm{cf}(\alpha) > \aleph_0$. Since no reals are added at the step α there is nothing to prove. ⊣

Furthermore, we will require that

A1. $|A_\alpha| < \aleph_\omega$ for any $\aleph_\omega \leq \alpha < \aleph_{\omega+1}$.

A2. For every set $A \in [\aleph_{\omega+1}]^{<\aleph_\omega} \cap V_0$ there are cofinally many α with $A \subseteq A_\alpha$.

To state the next requirement we will need the following notation: suppose that $A \subseteq \aleph_{\omega+1}$. Let $\mathcal{P}\restriction A = \{p \in \mathcal{P} : \mathsf{dom}(p) \subseteq A\}$. Suppose that $\dot{f} \subseteq {}^\omega 2 \times {}^\omega 2$ is a name for an arbitrary function from ${}^\omega 2$ to ${}^\omega 2$ (not necessarily Borel). Then $\dot{f}\restriction A = \{(\dot{x}, \dot{y}) \in \dot{f} : \dot{x}, \dot{y}$ are $\mathcal{P}\restriction A$-names$\}$.

A3. $\mathsf{dom}(\dot{m}^\xi_\alpha \restriction A_\beta) = P(\omega) \cap V_0[\dot{G}\restriction A_\beta]$ for every $\xi < \aleph_1$ and $\aleph_\omega \leq \beta \leq \alpha < \aleph_{\omega+1}$. In other words, $\dot{m}^\xi_\alpha\restriction A_\beta$ is a name for finitely additive measure on $P(\omega) \cap V_0[\dot{G}\restriction A_\beta]$.

Suppose that the measures $\{\dot{m}^\xi_{\alpha+1} : \xi < \aleph_1\}$ are given, and suppose that in order to meet the requirement A2 we have to cover certain set A of size $\aleph_n$. Define a sequence $\langle A^\gamma_{\alpha+1} : \gamma < \omega_1\rangle$ such that

1. $A^0_{\alpha+1} = A$,
2. $A^\beta_{\alpha+1} \subseteq A^\delta_{\alpha+1}$ for $\beta \leq \delta$,
3. $A^\delta_{\alpha+1} = \bigcup_{\beta<\delta} A^\beta_{\alpha+1}$ for limit δ,
4. for every set $X \in V_0[\dot{G}\restriction A^\beta_{\alpha+1}]$ and $\xi < \aleph_1$, $\dot{m}^\xi_{\alpha+1}(X) \in V_0[\dot{G}\restriction A^{\beta+1}_{\alpha+1}]$,
5. $|A^\gamma_{\alpha+1}| = \aleph_n + \aleph_1$ for all γ.

Note that since $V_0[\dot{G}\restriction A^\beta_{\alpha+1}] \models 2^{\aleph_0} = \aleph_n$, in order to produce $A^{\beta+1}_{\alpha+1}$ we have to add to $A^\beta_{\alpha+1}$ at most $\aleph_n + \aleph_1$ countable sets. Finally let $A_{\alpha+1} = \bigcup_{\gamma<\omega_1} A^\gamma_{\alpha+1}$. It is clear that $A_{\alpha+1}$ is as required.

If δ is limit then we put $A_\delta = \emptyset$. Note that in both cases condition A3 is satisfied by the induction hypothesis and the fact that $\dot{m}^\xi_\delta$ extends $\bigcup_{\alpha<\delta} \dot{m}^\xi_\alpha$.

In order to finish the construction we have to define measures $\{\dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1}\}$.

We start with the definition of a certain dense subset of $\mathcal{P}$ and from now on use only conditions belonging to this subset. Let $D \subseteq \mathcal{P}$ be a subset such that $p \in D$ if

1. $\mathsf{dom}(p) \in [\aleph_{\omega+1}]^{<\omega}$,
2. $p(\alpha) \in {}^{<\omega}\omega \simeq \mathbf{C}$, for $\alpha \in \mathsf{dom}(p) \cap \aleph_\omega$,
3. for each $\alpha \in \mathsf{dom}(p) - \aleph_\omega$,
   (a) $\Vdash_\alpha p(\alpha) \in \mathbf{B}^{V_0[\dot{G}\restriction A_\alpha]}$,
   (b) there is a clopen set $C_\alpha \subseteq {}^\omega 2$ such that
   $$\Vdash_\alpha \frac{\mu(C_\alpha \cap p(\alpha))}{\mu(C_\alpha)} \geq 1 - \frac{1}{2^{n-j+5}},$$
   where $n = |\mathsf{dom}(p) - \aleph_\omega|$ and $j = |\alpha \cap (\mathsf{dom}(p) - \aleph_\omega)|$.

5.16 Lemma. *D is dense in $\mathcal{P}$.*

Proof. Induction on $\max(\mathsf{dom}(p))$. ⊣

Let $\mathbb{C}$ be the collection of clopen subsets of 2^ω. Represent $\mathbf{C}_{\aleph_{\omega+1}}$ as the collection of functions q such that $\mathsf{dom}(q) \in [\aleph_{\omega+1}]^{<\omega}$ and $q(\alpha) \in \mathbf{C}$ for $\alpha < \aleph_\omega$ and $q(\alpha) \in \mathbb{C}$ for $\alpha \geq \aleph_\omega$.

Note that there is a natural projection π from D to $\mathbf{C}_{\aleph_{\omega+1}}$ defined as

$$\pi(p)(\alpha) = \begin{cases} p(\alpha) & \text{if } \alpha < \aleph_\omega, \\ C_\alpha & \text{if } \alpha \geq \aleph_\omega. \end{cases}$$

For a sequence $\bar{p} = \langle p_n : n \in \omega \rangle$ let $\pi(\bar{p}) = \langle \pi(p_n) : n \in \omega \rangle$. Suppose that $\bar{p}$ is such that $\pi(\bar{p}) \in \Delta$, as defined in Definition 5.5. We will define a condition $p_{\bar{p}}$ in the following way; $\mathsf{dom}(p_{\bar{p}}) = \widetilde{\Delta}$, where $\widetilde{\Delta}$ is the root of the Δ-system $\{\mathsf{dom}(p_n) : n \in \omega\}$.

Case 1. $\alpha \in \widetilde{\Delta} \cap \aleph_\omega$. Let $p_{\bar{p}}(\alpha)$ be the common value of $p_n(\alpha)$ for $n \in \omega$.

Case 2. $\alpha \in \widetilde{\Delta} - \aleph_\omega$. Work in the model $V = V_0[\dot{G}\restriction A_\alpha]$ and let $C = \pi(p_n(\alpha))$. Clearly $V \models C \in \mathbf{B}$. It follows that for some $k > 0$ and every $n \in \omega$,

$$V \models \frac{\mu(C \cap p_n(\alpha))}{\mu(C)} \geq 1 - \frac{1}{2^k}.$$

Let $\dot{X}$ be a $\mathbf{B}$-name such that $[\![n \in \dot{X}]\!] = C \cap p_n(\alpha)$. Apply, Theorem 5.13 in V, to find a condition $r \in \mathbf{B}, r \leq C$ such that

$$r \Vdash_{\mathbf{B}} \dot{m}^C_{\mathbf{B}}(\dot{X}_{\bar{p}}) \geq 1 - \frac{1}{2^k}.$$

Let $p_{\bar{p}}(\alpha) = r$.

Now we turn our attention to the sequence $\langle \dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1} \rangle$. By Theorem 5.7, Δ is a union of $\aleph_1$ filter-like sets and each of these sets will yield one of the measures $\dot{m}^\xi$. Specifically, let $\Delta = \bigcup_{\xi<\aleph_1} \Delta_\xi$ be the decomposition into as in Theorem 5.7. For $\xi < \aleph_1$ let

$$\Delta^\xi = \{\bar{p} \in [\mathcal{P}]^\omega : \pi(\bar{p}) \in \Delta_\xi\}.$$

The measure $\dot{m}^\xi_\alpha$ will be first defined on the set

$$\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap [\mathcal{P}_\alpha]^\omega\}.$$

We will do it in such a way that for $\bar{p} \in \Delta^\xi \cap [\mathcal{P}_\alpha]^\omega$

$$p_{\bar{p}} \Vdash_\alpha \dot{m}^\xi_\alpha(\dot{X}_{\bar{p}}) > 0,$$

where $p_{\bar{p}}$ is the condition defined above. Next $\dot{m}^\xi_\alpha$ will be extended arbitrarily to the set $P(\omega) \cap V_0^{\mathcal{P}_\alpha}$.

Here are the details; fix $\xi < \aleph_1$ and define $\dot{m}^\xi_\alpha$ as follows:

CASE 1. $\alpha = \aleph_\omega$. Consider the family

$$\dot{\mathcal{H}}_\xi = \{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap [\mathcal{P}_{\aleph_\omega}]^\omega,\ p_{\bar{p}} \in \dot{G}_{\mathcal{P}}\}.$$

It is easy to see that $\dot{\mathcal{H}}_\xi$ is a $\mathcal{P}_{\aleph_\omega}$-name for a filter base. Let $\dot{\mathcal{F}}_\xi$ be any $\mathcal{P}$-name for an ultrafilter extending $\dot{\mathcal{H}}_\xi$ and let $\dot{m}^\xi_{\aleph_\omega}$ be the corresponding measure. In other words, for $\dot{X} \in \dot{\mathcal{H}}_\xi$,

$$\Vdash_{\aleph_\omega} \dot{m}^\xi_{\aleph_\omega}(\dot{X}) = 1.$$

CASE 2. $\alpha > \aleph_\omega$ and $\mathrm{cf}(\alpha) = \aleph_0$. Since $\dot{m}^\xi_\alpha$ extends $\bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$, we have to define $\dot{m}^\xi_\alpha$ on the set

$$\left\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap \left([\mathcal{P}_\alpha]^\omega - \textstyle\bigcup_{\beta<\alpha} [\mathcal{P}_\beta]^\omega\right)\right\}.$$

Put $\mathcal{A} = \Delta^\xi \cap ([\mathcal{P}_\alpha]^\omega - \bigcup_{\beta<\alpha}[\mathcal{P}_\beta]^\omega)$ and for $\bar{p} \in \mathcal{A}$ let $j = j_{\bar{p}} \in \omega$ be such that

$$\beta = \sup_{n\in\omega} \alpha^n_{j-1} < \sup_{n\in\omega} \alpha^n_j = \alpha,$$

where α^n_i is the i'th element of $\mathsf{dom}(p_n)$. Consider sequences $\bar{p}^- = \langle p_n \restriction \alpha^n_j : n \in \omega\rangle$ and $\bar{p}^+ = \langle p_n \restriction [\alpha^n_j, \alpha) : n \in \omega\rangle$. Let $\dot{\mathcal{H}}_\xi$ be a $\mathcal{P}_\alpha$-name for the family $\{\dot{X}_{\bar{p}^+} : \bar{p} \in \mathcal{A}\}$. Note that

1. $\Vdash_\alpha \dot{\mathcal{H}}_\xi$ is a filter base,
2. $\forall \dot{X} \in \dot{\mathcal{H}}_\xi\ \forall \beta < \alpha\ \forall \dot{Y} \in [\omega]^\omega \cap V_0^{\mathcal{P}_\beta}\ \Vdash_\alpha \dot{X} \cap \dot{Y}$ is infinite.

Suppose that $\bar{p} \in \mathcal{A}$ and note that

$$p_{\bar{p}^-} \Vdash_\beta \dot{m}^\xi_\beta(\dot{X}_{\bar{p}^-}) = a > 0.$$

By the remarks made above, we can set $\dot{m}_\alpha(\dot{X}_{\bar{p}^+}) = 1$ and $\dot{m}_\alpha(\dot{X}_{\bar{p}}) = a$. Finally note that the value a is forced by $p_{\bar{p}}$.

CASE 3. α is a limit and $\mathrm{cf}(\alpha) > \aleph_0$. Let $\dot{m}^\xi_\alpha = \bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$. This definition is correct since no subsets of ω are added at the step α.

CASE 4. $\alpha = \delta + 1$. As before we have to define $\dot{m}^\xi_\alpha$ on

$$\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap ([\mathcal{P}_\alpha]^\omega - [\mathcal{P}_\delta]^\omega)\}.$$

Set $\mathcal{A} = \Delta^\xi \cap ([\mathcal{P}_\alpha]^\omega - [\mathcal{P}_\delta]^\omega)$ and note that if $\bar{p} \in \mathcal{A}$ then $\delta \in \bigcap_{n\in\omega} \mathsf{dom}(p_n)$. Thus, let C be a clopen set such that $\pi(p_n(\delta)) = C$ for $n \in \omega$. Let $V = V_0[\dot{G}\restriction A_\delta]$. Find a forcing notion $\mathcal{R}$ such that $\mathcal{P}_\delta = (\mathcal{P}_\delta \restriction A_\delta) \star \mathcal{R}$. It follows that $V_0^{\mathcal{P}_\alpha} = V_0^{\mathcal{P}_{\delta+1}} = V^{\mathcal{R}\times\mathbf{B}}$. By the induction hypothesis $m = \dot{m}^\xi_\delta \restriction A_\delta$ is a finitely additive measure. In other words $m \in V$ is a finitely additive measure defined on $P(\omega) \cap V$. Clearly $\dot{m}^\xi_\delta$ is an extension of m to $V^{\mathcal{R}} \cap P(\omega)$. On the other hand let $\dot{m}^C_{\mathbf{B}}$ be an extension of m to $V^{\mathbf{B}}$ as given by Theorem 5.13. Let $\dot{m}^\xi_\alpha = \dot{m}^\xi_{\delta+1}$ be the common extension of $\dot{m}^\xi_\delta$ and $\dot{m}^C_{\mathbf{B}}$ guaranteed by Lemma 5.12. It is clear that $\dot{m}^\xi_\alpha$ has the required properties.

Finally let $\dot{m}^\xi = \bigcup_{\aleph_\omega \leq \alpha < \aleph_{\omega+1}} \dot{m}^\xi_\alpha$. Note that each $\dot{m}^\xi$ is a $\mathcal{P}$-name for a finitely additive measure on $P(\omega) \cap V_0^{\mathcal{P}}$.

Proof of Theorem 5.3

We are ready now for the proof of the main theorem. The following lemma gives the lower bound for $\mathbf{cov}(\mathcal{L})$.

5.17 Lemma. *$V_0^{\mathcal{P}} \models \mathbf{cov}(\mathcal{L}) \geq \aleph_\omega$. In particular, $[\mathbb{R}]^{<\aleph_\omega} \subseteq \mathsf{COV}(\mathcal{L})$.*

Proof. Suppose that $\{H_\alpha : \alpha < \kappa < \aleph_\omega\}$ is a family of measure zero sets in $V_0^{\mathcal{P}}$. Let N be a master set for $\mathcal{L}$ defined earlier. Without loss of generality we can assume that for some $f_\alpha \in {}^\omega\omega$, $H_\alpha = (N)_{f_\alpha}$, and let $\dot{f}_\alpha$ be a $\mathcal{P}$-name for f_α. As in Lemma 5.15, let $K_\alpha \in [\aleph_{\omega+1}]^{\aleph_0} \cap V_0$ be a countable set such that $f_\alpha \in V_0[\dot{G}\restriction K_\alpha]$. Find β such that $\bigcup_{\alpha<\kappa} K_\alpha \subseteq A_\beta$. The random real added by $\mathbf{B}^{V_0[\dot{G}\restriction A_\beta]}$ avoids all null sets coded in $V_0[\dot{G}\restriction A_\beta]$, in particular, all H_α's. ⊣

It remains to be checked that $\mathbf{cov}(\mathcal{L}) \leq \aleph_\omega$ in the extension.

Let $X = \{f_\alpha : \alpha < \aleph_\omega\} = \dot{G}\restriction\aleph_\omega$ be the sequence of the first $\aleph_\omega$ Cohen reals added by $\mathcal{P}$. Our intention is to show that $X \notin \mathsf{COV}(\mathcal{L})$. In fact we will show that

$$\bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha} = {}^\omega 2,$$

where N is the master set defined in the previous section. That will finish the proof since X is a countable union of sets of smaller size (so they are all in $\mathsf{COV}(\mathcal{L})$) and thus X witnesses that $\mathsf{COV}(\mathcal{L})$ is not a σ-ideal and that $\mathbf{cov}(\mathcal{L}) \leq \aleph_\omega$.

Suppose the opposite and let z be such that

$$V_0^{\mathcal{P}} \models z \notin \bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha}.$$

5.18 Lemma. *There exists a $\mathcal{P}$-name $\dot{Y}$ for a subset of $\aleph_\omega$ and $\bar{n} \in \omega$ such that*

1. *$\Vdash_{\mathcal{P}} \dot{Y} \in [\aleph_\omega]^{\aleph_1}$,*
2. *$\Vdash_{\mathcal{P}} {}^\omega 2 - \bigcup_{\alpha\in\dot{Y}} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}$ is uncountable.*

Proof. Denote by $\dot{z}$ a $\mathcal{P}$-name for z and let $\delta < \aleph_{\omega+1}$ be the least ordinal such that $\dot{z}$ is a $\mathcal{P}_\delta$-name. We have the following two cases:

Case 1. $\delta = \lambda + 1$ is a successor ordinal. Suppose first that $\delta > \aleph_\omega$. Work in $V = V_0{}^{\mathcal{P}_\lambda}$ and let $\mathbf{B}_\lambda = \mathbf{B}^{V_0[\dot{G}\restriction A_\lambda]}$. For each $\alpha < \aleph_\omega$ choose $q_\alpha \in \mathbf{B}_\lambda$ and

$n_\alpha \in \omega$ such that $V \models q_\alpha \Vdash_{\mathbf{B}_\lambda} \dot{z} \notin \bigcup_{n>n_\alpha} C^n_{f_\alpha(n)}$. Since $\mathbf{B}_\lambda$ has a dense subset of size $< \aleph_\omega$, we can find $q \in \mathbf{B}_\lambda$ and $\bar{n} \in \omega$ such that the set

$$Y = \{\alpha : q_\alpha = q \;\&\; n_\alpha = \bar{n}\}$$

is uncountable. Consider the set $C = {}^\omega 2 - \bigcup_{\alpha\in Y} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}$ in V. Observe that C is a closed set and if it was countable then all its elements would be in V. However, $V^{\mathbf{B}_\lambda} \models z \in C$ and $z \notin V$.

If $\delta < \aleph_\omega$, then the argument is identical except that we use $\mathbf{C}$ instead of $\mathbf{B}_\lambda$. In fact one can show that

$$V_0^{\mathcal{P}} \cap {}^\omega 2 \subseteq \bigcup_{\alpha<\omega_1} (N)_{f_\alpha} \subseteq \bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha}.$$

Case 2. δ is a limit and $\mathrm{cf}(\delta) = \aleph_0$. In $V_0^{\mathcal{P}_\delta}$ we can find an $\bar{n} \in \omega$ and an uncountable set $Z \subseteq \aleph_\omega$ such that

$$V_0^{\mathcal{P}_\delta} \models z \notin \bigcup_{\alpha\in Z} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}.$$

Let $\dot{Z}$ be a $\mathcal{P}_\delta$-name for Z. Suppose that $G \subseteq \mathcal{P}_\delta$ is a generic filter over V_0. For each $\alpha < \omega_1$ choose $p_\alpha \in \mathcal{P}_\delta \cap G$ and η_α such that $p_\alpha \Vdash_{\mathcal{P}_\delta} \dot{Z}(\alpha) = \eta_\alpha$, where $\dot{Z}(\alpha)$ is a $\mathcal{P}$-name for the α-th element of Z.

There is an uncountable set $I \subseteq \omega_1$, and $\lambda < \delta$ such that $p_\alpha \in \mathcal{P}_\lambda \cap G$ for $\alpha \in I$. Let $Y = \{\eta_\alpha : \alpha \in I\}$ and let $\dot{Y}$ be a $\mathcal{P}_\lambda$-name for Y. As in the previous case, consider the set $C = {}^\omega 2 - \bigcup_{\alpha\in Y} \bigcup_{n>\bar{n}} C^n_{f_{\eta_\alpha}(n)}$ in $V_0^{\mathcal{P}_\lambda}$. We see that C is uncountable because it contains an element which does not belong to $V_0^{\mathcal{P}_\lambda}$. $\dashv$

Find different ordinals $\{\eta_\alpha : \alpha < \omega_1\}$ and conditions $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}$ such that $p_\alpha \Vdash_{\mathcal{P}} \eta_\alpha \in \dot{Y}$. Using the Δ-lemma we can assume that there are $\widetilde{k}, \widetilde{l} \in \omega$, $s \in {}^{\widetilde{k}}\omega$ and clopen sets $\{C_j : j \le \widetilde{l}\}$ such that

1. $\mathsf{dom}(p_\alpha)$ form a Δ-system,
2. $\mathsf{dom}(p_\alpha) = \{\gamma_1^\alpha < \cdots < \gamma_{\widetilde{k}}^\alpha < \aleph_\omega \le \delta_1^\alpha < \cdots < \delta_{\widetilde{l}}^\alpha\}$,
3. $\forall\alpha\, \forall j \le \widetilde{k}\, (p_\alpha(\gamma_j^\alpha) = s(j))$,
4. for all $j \le \widetilde{l}$
$$\Vdash_{\alpha_j} \frac{\mu(C_j \cap p_\alpha(\delta_j^\alpha))}{\mu(C_j)} \ge 1 - \frac{1}{2^{\tilde{l}-j+5}}.$$

Without loss of generality we can assume that $\eta_\alpha \in \mathsf{dom}(p_\alpha)$. Furthermore we can assume that for some $j_0 \le \widetilde{k}$, $\eta_\alpha = \gamma_{j_0}^\alpha$ and that $s(j_0) = s^\star$ with $|s^\star| = n^\star$.

Consider the first ω conditions $\bar{p} = \{p_n : n \in \omega\}$. Our next step is to extend the p_n's slightly to get a new sequence $\bar{p}^\star$. We will need the following definition.

5.19 Definition. For a clopen set $C \subseteq {}^\omega 2$ define $\mathsf{supp}(C)$ to be the smallest set $F \subseteq \omega$ such that $C = (C \cap {}^F 2) \times {}^{\omega - F} 2$. Thus, the support of C is the set of coordinates that carry information about C.

Let $K_n = \{m : \mathsf{supp}(C^{n^\star}_m) \subseteq n\}$ and let $\{J_n : n \in \omega\}$ be a partition of ω such that $|J_n| = |K_n|$ for each n. Fix a function $o \in {}^\omega\omega$ such that $o``J_n = K_n$ for every n. Define

$$p^\star_n = \begin{cases} p_n(\alpha) & \text{if } \alpha \neq \eta_n, \\ s^\star {}^\frown (n^\star, o(n)) & \text{if } \alpha = \eta_n. \end{cases}$$

Observe that there is $\xi < \aleph_1$ such that $\bar{p}^\star = \{p^\star_n : n \in \omega\} \in \Delta^\xi$. This is being witnessed by the $\widetilde{k}, \widetilde{l}$, $s \in {}^{\widetilde{k}}\omega$, clopen sets $\{C_j : j \leq \widetilde{l}\}$ and function g defined as

$$g(i, n) = \begin{cases} s(i) & \text{if } i \leq \widetilde{k},\ i \neq j_0, \\ s^\star {}^\frown (n^\star, o(n)) & \text{if } i = j_0. \end{cases}$$

Our goal is to show:

5.20 Theorem. *There exists a condition $p^{\star\star}$ and $\varepsilon > 0$ such that*

$$p^{\star\star} \Vdash_{\mathcal{P}} \exists^\infty n \ \frac{|\{m \in J_n : p^\star_m \in \dot{G}_{\mathcal{P}}\}|}{|J_n|} \geq \varepsilon.$$

Before we prove this theorem let us see that Theorem 5.3 follows readily from it. Recall that in Lemma 5.18 we showed that $\Vdash_{\mathcal{P}} {}^\omega 2 - \bigcup_{\alpha \in \dot{Y}} \bigcup_{n > \bar{n}} C^n_{f_\alpha(n)}$ is uncountable. Since this set is closed, there is a $\mathcal{P}$-name for a perfect tree $\dot{T}$ such that $\Vdash_{\mathcal{P}} \bigcup_{\alpha \in \dot{Y}} \bigcup_{n > \bar{n}} C^n_{f_\alpha(n)} \cap [\dot{T}] = \emptyset$. Let $\dot{Z}_n = \{m \in J_n : p^\star_m \in \dot{G}_{\mathcal{P}}\}$ for $n \in \omega$. It follows that for every n,

$$p^{\star\star} \Vdash_{\mathcal{P}} \left(\bigcup_{k \in \dot{Z}_n} C^{n^\star}_k\right) {\restriction} n \cap \dot{T} {\restriction} n = \emptyset.$$

This is because for a clopen set C and a tree T, if $C \cap [T] = \emptyset$ then $(C {\restriction} \mathsf{supp}(C)) \cap (T {\restriction} \mathsf{supp}(C)) = \emptyset$. Fix $n \in \omega$ and suppose that $|\dot{T} {\restriction} n| = m$. The size of the set J_n is equal to $\binom{2^n}{2^{n-n^\star}}$. On the other hand the number of sets $C^{n^\star}_k$ which are disjoint with $\dot{T} {\restriction} n$ is at most $\binom{2^n - m}{2^{n-n^\star}}$. Put $2^{-n^\star} = \epsilon$. It follows, after some elementary calculations, that for some constant $a \geq 1$:

$$\frac{|\dot{Z}_n|}{|J_n|} \leq \frac{\binom{2^n - m}{2^{n-n^\star}}}{\binom{2^n}{2^{n-n^\star}}} = \prod_{j=1}^{m} \left(1 - \frac{2^{n-n^\star}}{2^n - m + j}\right) \leq a \cdot e^{-\epsilon \cdot m}.$$

Thus

$$\frac{|\dot{Z}_n|}{|J_n|} \leq a \cdot e^{-\epsilon \cdot |\dot{T} {\restriction} n|}.$$

Since $p^{\star\star} \Vdash_{\mathcal{P}} \limsup_n \frac{|\dot{Z}_n|}{|J_n|} \geq \varepsilon$ we get that $p^{\star\star} \Vdash_{\mathcal{P}} \lim_n |\dot{T} {\restriction} n| < \infty$ (the size of $T {\restriction} n$ increases with n). In particular,

$$p^{\star\star} \Vdash_{\mathcal{P}} \dot{T} \text{ is not perfect,}$$

which gives a contradiction.

Proof of Theorem 5.3: Conclusion

In order to complete the proof of Theorem 5.3 we have to prove Theorem 5.20. We will need one more modification of the sequence $\bar{p}^\star$ and we will require the construction described below.

5.21 Lemma. *Let $\widetilde{\Delta}$ be a finite subset of $\aleph_{\omega+1} - \aleph_\omega$. Suppose that $\{q_i : i \leq N\}$ is a sequence of conditions in $\mathcal{P}$ such that*

(1) $\mathsf{dom}(q_i) = \widetilde{\Delta}$,

(2) $\forall \alpha \in \widetilde{\Delta}\ \exists a_\alpha\ \forall i \leq N\ \Vdash_\alpha \mu(q_i(\alpha)) = a_\alpha > 3/4$.

Then there exists a condition $q^\star$ such that

(a) $\mathsf{dom}(q^\star) = \widetilde{\Delta}$,

(b) $q^\star \in \mathcal{P}$,

(c) $\forall \alpha \in \widetilde{\Delta}\ \Vdash_\alpha \mu(q^\star(\alpha)) \geq 2a_\alpha - 1$,

(d) $q^\star \Vdash_{\mathcal{P}} \{k \leq N : \forall \alpha \in \widetilde{\Delta}\ q^\star{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}$ *has at least* $2^{-|\widetilde{\Delta}|} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta}} a_\alpha$ *elements, where $\dot{x}_\alpha$ is the generic real added by $\dot{G}(\alpha)$.*

Proof. If $\widetilde{\Delta} = \emptyset$, then there is nothing to prove.

We proceed by induction on $\max(\widetilde{\Delta})$ which we denote by β. Let $q'_k = q_k{\restriction}\beta$ for $k \leq N$. Apply the induction hypothesis to get a condition q' such that

1. $\mathsf{dom}(q') = \widetilde{\Delta} - \{\beta\}$,
2. $q' \in \mathcal{P}$,
3. $\forall \alpha \in \widetilde{\Delta} - \{\beta\}\ \Vdash_\alpha \mu(q'(\alpha)) \geq 2a_\alpha - 1$,
4. $q' \Vdash_{\mathcal{P}} \{k \leq N : \forall \alpha \in \widetilde{\Delta} - \{\beta\}\ q'{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}$ has at least $2^{-|\widetilde{\Delta}|+1} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta} - \{\beta\}} a_\alpha$ elements.

Let $\dot{W}$ be a $\mathcal{P}$-name for the set

$$\{k \leq N : \forall \alpha \in \widetilde{\Delta} - \{\beta\}\ q'{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}.$$

Let $\{W^i : i \leq \ell\}$ be a list of subsets of N of size at least $2^{-|\widetilde{\Delta}|} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta} - \{\beta\}} a_\alpha$ and $\{q^i : i \leq \ell\}$ a maximal antichain below q' such that $q^i \Vdash_{\mathcal{P}} \dot{W} = W^i$ for $i \leq \ell$.

We will need the following easy observation.

5.22 Lemma. *Suppose that $\{A_n : n < N\}$ is a family of subsets of ${}^\omega 2$ of measure $a > 0$. Let*

$$B = \left\{x \in {}^\omega 2 : x \textit{ belongs to at least } \frac{N \cdot a}{2} \textit{ sets } A_i\right\}.$$

Then $\mu(B) \geq \max\{a/2, 2a - 1\}$.

Proof. Let χ_{A_i} be the characteristic function of the set A_i for $i \leq N$. It follows that $\int \sum_{i \leq N} \chi_{A_i} = N \cdot a$. On the other hand, estimation of this integral yields,

$$N \cdot \mu(B) + \frac{N \cdot a}{2}(1 - \mu(B)) \geq N \cdot a$$

and after simple computations we get $\mu(B) \geq \frac{a/2}{1-a/2}$. It follows that we get the following estimates:

$$\mu(B) \geq \frac{a/2}{1 - a/2} \geq \max\{a/2, 2a - 1\} = \begin{cases} a/2 & \text{if } a < \frac{2}{3}, \\ 2a - 1 & \text{if } a \geq \frac{2}{3}. \end{cases}$$

$\dashv$

Work in $V^{\mathcal{P}_\beta}$ and for each $i \leq \ell$ apply Lemma 5.22 to the family $\{q_k(\beta) : k \in W^i\}$ and obtain a condition $r^i \in \mathbf{B}^{V_0[\dot{G} \restriction A_\beta]}$ such that

$$r^i \Vdash \{k \in W^i : \dot{x}_\beta \in q_k(\beta)\} \text{ has at least } \frac{|W^i|}{2} \cdot a_\beta \text{ elements,}$$

and $\Vdash_\beta \mu(r^i) \geq 2a_\beta - 1$.

Finally, define $q^\star$ to be a $\mathcal{P}$-name such that for $i \leq \ell$, $q^i \Vdash q^\star(\beta) = r^i$. It is easy to see that $q^\star$ is as required. $\dashv$

Let $q_k = p_k^\star \restriction \widetilde{\Delta}$, where $\widetilde{\Delta} = \{\alpha_1 < \cdots < \alpha_\ell\}$ is the root of the Δ-system $\{\mathsf{dom}(p_k^\star \restriction [\aleph_\omega, \aleph_{\omega+1}) : k \in \omega\}$.

For each n apply Lemma 5.21 to the family $\{q_k : k \in J_n\}$ to get a condition $q_n^\star$ such that

1. $\mathsf{dom}(q_n^\star) = \widetilde{\Delta}$,
2. $\forall i \leq \ell \ \Vdash_{\alpha_i} \frac{\mu(q^\star(\alpha_i) \cap C_{\alpha_i})}{\mu(C_{\alpha_i})} \geq 2(1 - \frac{1}{2^{\ell-i+5}}) - 1 = \frac{1}{2^{\ell-i+4}}$,
3. $q_n^\star \Vdash_{\mathcal{P}} |\{k \leq N : \forall \alpha \in \widetilde{\Delta}\ (q_n^\star \restriction \alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha))\}| \geq \frac{|J_n|}{2^{\ell+1}}$.

Define for $k \in \omega$,

$$p_k^{\star\star}(\alpha) = \begin{cases} p_k^\star(\alpha) \cap q_n^\star & \text{if } \alpha \in \widetilde{\Delta},\ k \in J_n, \\ p_k^\star(\alpha) & \text{otherwise.} \end{cases}$$

Let $\bar{p}^{\star\star} = \{p^{\star\star}_n : n \in \omega\}$. Find $\xi < \aleph_1$ such that $\bar{p}^{\star\star} \in \Delta^\xi$. According to our definitions,

$$p^{\star\star} = p_{\bar{p}^{\star\star}} \Vdash_{\mathcal{P}} \dot{m}^\xi(\dot{X}_{\bar{p}^{\star\star}}) > 0.$$

In particular,

$$p^{\star\star} \Vdash_{\mathcal{P}} \dot{X}_{\bar{p}^{\star\star}} = \{n : p^{\star\star}_n \in \dot{G}_{\mathcal{P}}\} \text{ is infinite.}$$

Let $\varepsilon = 2^{-\ell-1}$ and note that

$$p^{\star\star}_n \Vdash_{\mathcal{P}} \frac{|\{k \in J_n : p^\star_k \in \dot{G}_{\mathcal{P}}\}|}{|J_n|} \geq \varepsilon.$$

It follows that $p^{\star\star}$ is the condition required in Theorem 5.20.

Historical Remarks. Theorem 5.1 was proved by Miller [34]. Better estimates are true (see [7] and [6] or [8]). Theorem 5.2 was proved in [5] (see [8]). Theorem 5.3 is due to Shelah. His [49] contains a more a general construction, where in addition $\mathbf{MA}_{\aleph_1}$ holds.

6. Consistency Results and Counterexamples

This section is devoted to the consistency results involving cardinal invariants of the Cichoń diagram and non-inclusion between the corresponding classes of small sets. We will describe several such constructions in detail.

Suppose that $\mathcal{P}$ is a forcing notion. Let $\mathcal{D}(\mathcal{P})$ denote the family of all dense subsets of $\mathcal{P}$ and $\mathcal{G}(\mathcal{P})$ the family of all filters on $\mathcal{P}$. With $\mathcal{P}$ we can associate the following cardinal invariants:

1. $\mathfrak{ma}(\mathcal{P}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{D}(\mathcal{P})\ \&\ \neg\exists G \in \mathcal{G}(\mathcal{P})\ \forall D \in \mathcal{A}\ (G \cap D \neq \emptyset)\}$,
2. $\mathfrak{am}(\mathcal{P}) = \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{G}(\mathcal{P})$ & for every countable sequence $\{D_n : n \in \omega\} \subseteq \mathcal{D}(\mathcal{P})\ \exists G \in \mathcal{G}(\mathcal{P})\ \forall n\ (G \cap D_n \neq \emptyset)\}$.

In other words, $\mathfrak{ma}(\mathcal{P})$ is the size of the smallest family of dense subsets of $\mathcal{P}$ for which there is no filter intersecting all of them and $\mathfrak{am}(\mathcal{P})$ is the size of the smallest family of filters such that for every countable family of dense subsets of $\mathcal{P}$ there is a filter in the family that intersects all of them.

Consider the forcing notions:

- Amoeba forcing $\mathbf{A} = \{U \subseteq {}^\omega 2 : U \text{ is open and } \mu(U) < 1/2\}$. For $U, V \in \mathbf{A}$, $U \leq V$ if $U \supseteq V$.
- Random real forcing $\mathbf{B} = \{P \subseteq {}^\omega 2 : P$ is a closed set of positive measure$\}$.
- Cohen forcing $\mathbf{C}$.
- Dominating real forcing $\mathbf{D} = \{\langle n, f\rangle : n \in \omega\ \&\ f \in {}^\omega\omega\}$. For $\langle n, f\rangle$, $\langle m, g\rangle \in \mathbf{D}$, $\langle n, f\rangle \leq \langle m, g\rangle$ if $n \geq m\ \&\ f{\restriction}m = g{\restriction}m\ \&\ \forall k\ f(k) \geq g(k)$.

We have the following result (see [8] for the proof):

6.1 Theorem.

1. $\mathbf{add}(\mathcal{L}) = \mathfrak{ma}(\mathbf{A})$ *and* $\mathbf{cof}(\mathcal{L}) = \mathfrak{am}(\mathbf{A})$.
2. $\mathbf{cov}(\mathcal{L}) = \mathfrak{ma}(\mathbf{B})$ *and* $\mathbf{non}(\mathcal{L}) = \mathfrak{am}(\mathbf{B})$.
3. $\mathbf{cov}(\mathcal{B}) = \mathfrak{ma}(\mathbf{C})$ *and* $\mathbf{non}(\mathcal{B}) = \mathfrak{am}(\mathbf{C})$.
4. $\mathbf{add}(\mathcal{B}) = \mathfrak{ma}(\mathbf{D})$ *and* $\mathbf{cof}(\mathcal{B}) = \mathfrak{am}(\mathbf{D})$.

This description is particularly well suited to use with the finite support iteration. If $\mathcal{P}$ is a c.c.c. forcing notion having "nice" definition and $\mathcal{P}_\kappa$ is a finite support iteration of $\mathcal{P}$ of length κ then

1. If $V \models 2^{\aleph_0} = \aleph_1$ then $V^{\mathcal{P}_{\omega_2}} \models \mathfrak{ma}(\mathcal{P}) = \aleph_2$.
2. If $V \models 2^{\aleph_0} = \aleph_2$ then $V^{\mathcal{P}_{\omega_1}} \models \mathfrak{am}(\mathcal{P}) = \aleph_1$.

This example motivates the following definition: a pair of models V and V' is dual if

$$V \models \mathfrak{ma}(\mathcal{P}) = 2^{\aleph_0} \quad \iff \quad V' \models \mathfrak{am}(\mathcal{P}) < 2^{\aleph_0}.$$

For our purpose we restrict our attention to the coefficients of the Cichoń diagram and define that V is dual to V' if all of the following hold for $\mathcal{J} = \mathcal{B}$ and for $\mathcal{J} = \mathcal{L}$:

1. $V \models \mathbf{cov}(\mathcal{J}) = 2^{\aleph_0} \iff V' \models \mathbf{non}(\mathcal{J}) < 2^{\aleph_0}$,
2. $V \models \mathbf{add}(\mathcal{J}) = 2^{\aleph_0} \iff V' \models \mathbf{cof}(\mathcal{J}) < 2^{\aleph_0}$,
3. $V \models \mathbf{non}(\mathcal{J}) = 2^{\aleph_0} \iff V' \models \mathbf{cov}(\mathcal{J}) < 2^{\aleph_0}$,
4. $V \models \mathbf{cof}(\mathcal{J}) = 2^{\aleph_0} \iff V' \models \mathbf{add}(\mathcal{J}) < 2^{\aleph_0}$,
5. $V \models \mathfrak{b} = 2^{\aleph_0} \iff V' \models \mathfrak{d} < 2^{\aleph_0}$,
6. $V \models \mathfrak{d} = 2^{\aleph_0} \iff V' \models \mathfrak{b} < 2^{\aleph_0}$.

To illustrate this consider the following theories:

$$\mathsf{ZFC} + \mathbf{add}(\mathcal{B}) = \mathbf{cov}(\mathcal{L}) = \aleph_2 + \mathbf{add}(\mathcal{L}) = \aleph_1$$

and

$$\mathsf{ZFC} + \mathbf{cof}(\mathcal{L}) = \aleph_2 + \mathbf{cof}(\mathcal{B}) = \mathbf{non}(\mathcal{L}) = \aleph_1.$$

A model for the first of these theories can be obtained by a finite support iteration of $\mathbf{B} \star \mathbf{D}$ of length $\aleph_2$ over a model for CH and the second by iteration of $\mathbf{B} \star \mathbf{D}$ of length $\aleph_1$ over a model for $2^{\aleph_0} = \aleph_2$. It is clear that $\mathbf{add}(\mathcal{B})$, $\mathbf{cov}(\mathcal{L})$ and $\mathbf{cof}(\mathcal{B})$ and $\mathbf{non}(\mathcal{L})$ have the required values. What is less obvious is that $\mathbf{add}(\mathcal{L}) = \aleph_1$ in the first and $\mathbf{cof}(\mathcal{L}) = \aleph_2$ in the second case. To check that we need a preservation result which ensures that the iteration which we use does not change the value of these invariants. Such theorems were proved in [27, 43, 8].

We will not study these examples any further because this method has one fundamental weakness: it can give us only some of the models we need. This is because the finite support iteration adds Cohen reals. We will use however the notion of duality outlined above. From now on we will focus on obtaining the models using countable support iteration. To this end we will associate with every cardinal invariant of the Cichoń diagram a proper forcing notion and a "preservation theorem" as follows:

- $\mathbf{add}(\mathcal{L}) \leftrightsquigarrow$ Amoeba forcing $\mathbf{A}$, preservation of "not adding amoeba reals".
- $\mathbf{cov}(\mathcal{L}) \leftrightsquigarrow$ random real forcing $\mathbf{B}$, preservation of "not adding random reals".
- $\mathbf{cov}(\mathcal{B}) \leftrightsquigarrow$ Cohen forcing $\mathbf{C}$, preservation of "not adding Cohen reals".
- $\mathbf{non}(\mathcal{B}) \leftrightsquigarrow$ forcing $\mathbf{PT}_{f,g}$, preservation of non-meager sets.
- $\mathfrak{b} \leftrightsquigarrow$ Laver forcing $\mathbf{LT}$, preservation of "not adding unbounded reals".
- $\mathfrak{d} \leftrightsquigarrow$ rational perfect set forcing $\mathbf{PT}$, preservation of "not adding dominating reals".
- $2^{\aleph_0} \leftrightsquigarrow$ Sacks forcing $\mathbf{S}$, preservation of Sacks property.
- $\mathbf{cof}(\mathcal{L}) \leftrightsquigarrow$ forcing $\mathbf{S}_2$, preservation of non-meager sets, and preservation of "not adding unbounded reals".
- $\mathbf{non}(\mathcal{L}) \leftrightsquigarrow$ forcing $\mathbf{S}_{g,g^\star}$, preservation of positive outer measure.

We do not assign anything to $\mathbf{add}(\mathcal{B})$ and $\mathbf{cof}(\mathcal{B})$ because they are expressible using the remaining invariants. We refer the reader to [8] for the definitions of all these forcing notions and the formulation of the preservation theorems. We will illustrate the problems with the following examples.

6.2 Example. Dominating number $\mathfrak{d}$. Rational perfect set forcing $\mathbf{PT}$ associated with $\mathfrak{d}$ is one of the forcing notions that increase $\mathfrak{d}$ without affecting other characteristics in the Cichoń diagram (except those bigger than $\mathfrak{d}$).

The preservation theorem can be stated as follows. We say that a proper forcing notion $\mathcal{P}$ is *${}^\omega\omega$-bounding* if

$$\forall f \in V^{\mathcal{P}} \cap {}^\omega\omega \ \exists g \in V \cap {}^\omega\omega \ \forall n \ (f(n) \leq g(n)).$$

It is clear that $\mathcal{P}$ is ${}^\omega\omega$-bounding if and only if $\mathcal{P}$ preserves dominating families. See Abraham's chapter [1] for the proof of the following theorem.

6.3 Theorem. *The countable support iteration of proper ${}^\omega\omega$-bounding forcing notions is ${}^\omega\omega$-bounding.*

This is the ideal situation—no matter what forcing notion we assign to $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{B})$, $\mathbf{cof}(\mathcal{L})$ and $\mathbf{non}(\mathcal{L})$ it has to be ${}^\omega\omega$-bounding and this property is preserved under countable support iteration.

6.4 Example. Covering numbers $\mathbf{cov}(\mathcal{B})$ and $\mathbf{cov}(\mathcal{L})$. The choice of forcing notions that we assign to these invariants is determined by Theorem 6.1; it has to be equivalent to Cohen and random real forcing respectively.

The preservation theorem could be stated as follows (see [26] or [8]).

6.5 Theorem. *Suppose that $\mathcal{P}_\delta = \lim_{\alpha<\delta} \mathcal{P}_\alpha$ (δ a limit) is a countable support iteration of proper forcing notions such that for every $\alpha < \delta$, $\mathcal{P}_\alpha$ does not add random reals. Then $\mathcal{P}_\delta$ does not add random reals.*

The question whether this theorem remains true if we replace words "random" by "Cohen" is open. However, even if the preservation theorem for not adding Cohen reals is true, both results cover only limit stages of the iteration. For the successor steps we do not have an analog of Theorem 6.3, and indeed we can find two c.c.c. forcing notions $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P}$ does not add random reals, and $\Vdash_{\mathcal{P}}$ "$\mathcal{Q}$ does not add random reals" but $\mathcal{P} \star \mathcal{Q}$ adds random reals. Similarly for Cohen reals.

These facts impose the following requirements:

- any iteration of finite length of forcing notions assigned to $\mathfrak{b}$, $\mathbf{non}(\mathcal{B})$, $\mathfrak{d}$, $\mathbf{cov}(\mathcal{B})$, $\mathbf{non}(\mathcal{L})$ and $\mathbf{cof}(\mathcal{L})$ does not add random reals,
- iteration of any length of forcing notions assigned to $\mathfrak{b}$, $\mathbf{cov}(\mathcal{L})$, $\mathbf{non}(\mathcal{B})$, $\mathbf{non}(\mathcal{L})$ and $\mathbf{cof}(\mathcal{L})$ does not add Cohen reals.

It is easy to verify that each of the forcing notions chosen for these invariants have the required properties. However, the reasons why they, for example, do not add Cohen reals are different in each case. Thus, the preservation theorems are often difficult, technical and at the same time not very general.

The full proof that the construction outlined above is possible can be found in [8]. A preservation theorem for not adding Cohen reals that covers the cases we are interested in can be found in [44].

We will take all these constructions for granted and present some applications.

Let us consider the following examples:

6.6 Theorem. *It is consistent with* ZFC *that*

$$\mathfrak{b} = \aleph_2 + \mathbf{cov}(\mathcal{L}) = \mathbf{non}(\mathcal{L}) = \aleph_1.$$

Proof. Recall that for any tree T, $\mathsf{stem}(T)$ is the longest node of T such that for all $t \in T$, $t \subseteq \mathsf{stem}(T)$ or $\mathsf{stem}(T) \subseteq t$ and for $s \in T$, $\mathsf{succ}_T(s) = \{t : s \subseteq t \ \& \ |t| = |s| + 1\}$.

Laver forcing **LT** is the following forcing notion:

$$T \in \mathbf{LT} \iff \begin{array}{l} T \subseteq {}^{<\omega}\omega \text{ is a tree } \& \\ \forall s \in T\ (|s| \geq \mathsf{stem}(T) \rightarrow |\mathsf{succ}_T(s)| = \aleph_0). \end{array}$$

For $T, T' \in \mathbf{LT}$, $T \leq T'$ if $T \subseteq T'$.

6.7 Lemma.

(1) $V^{\mathbf{LT}} \models V \cap {}^\omega\omega$ *is bounded in* ${}^\omega\omega$.

(2) $V^{\mathbf{LT}} \models V \cap {}^\omega 2 \notin \mathcal{L}$.

(3) **LT** *does not add random reals.*

Moreover (2) *and* (3) *hold for the countable support iteration of Laver forcing as well.*

Proof. See [8]. ⊣

Let $\mathcal{P}_{\omega_2}$ be a countable support iteration of length $\aleph_2$ of Laver forcing. It follows from Lemma 6.7 that $\mathfrak{b} = \aleph_2$ in $V^{\mathcal{P}_{\omega_2}}$, while both $\mathbf{cov}(\mathcal{L})$ and $\mathbf{non}(\mathcal{L})$ are equal to $\aleph_1$. ⊣

6.8 Theorem. *It is consistent with* ZFC *that*

$$\mathfrak{d} = \aleph_1 + \mathbf{cov}(\mathcal{L}) = \mathbf{non}(\mathcal{L}) = \aleph_2.$$

Proof. We will use forcing notion **EE** defined below rather than $\mathbf{S}_{g,g^\star}$; it has a much simpler definition and has the required properties (but difficulties appear when unbounded reals are added).

The infinitely equal forcing notion **EE** is defined as follows: $p \in \mathbf{EE}$ if the following conditions are satisfied:

1. $\mathsf{dom}(p) \subseteq \omega,\ |\omega - \mathsf{dom}(p)| = \aleph_0$.
2. $p : \mathsf{dom}(p) \longrightarrow {}^{<\omega}2$.
3. $p(n) \in {}^n2$ for all $n \in \mathsf{dom}(p)$, and

for $p, q \in \mathbf{EE}$ we define $p \leq q$ if $p \supseteq q$.

6.9 Lemma. *Forcing* **EE** *has the following properties:*

1. $V^{\mathcal{P}} \models V \cap {}^\omega 2 \in \mathcal{L}$. *In fact,*

$$\forall x \in V \cap {}^\omega 2\ \exists^\infty n\ (x{\restriction}n = f_G(n)),$$

where f_G *is a generic real.*

2. $\mathcal{P}$ *does not add random reals.*
3. $\mathcal{P}$ *is* ${}^\omega\omega$*-bounding.*

Proof. See [8]. ⊣

Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ be a countable support iteration such that for every $\alpha < \omega_2$,

1. $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{EE}$ if α is even, and

2. $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{B}$ if α is odd.

Let G be a $\mathcal{P}_{\omega_2}$-generic filter over $V \models \mathsf{CH}$.

It is clear that $V[G] \models \mathbf{non}(\mathcal{L}) = \mathbf{cov}(\mathcal{L}) = \aleph_2$. To see that $\mathfrak{d} = \aleph_1$ in the extension note that both forcing notions $\mathbf{B}$ and $\mathbf{EE}$ are ${}^\omega\omega$-bounding and use Theorem 6.3. ⊣

Now consider the corresponding problem concerning the families of small sets. The question is whether the models constructed for the Cichoń diagram yield the sets witnessing the strict inclusion between the corresponding classes of sets. For example, $\mathbf{add}(\mathcal{L}) < \mathbf{cov}(\mathcal{L})$ is consistent. Is this construction of any help if we want to construct a set $X \in \mathsf{COV}(\mathcal{L}) - \mathsf{ADD}(\mathcal{L})$? It is clear that we cannot show that in ZFC alone. For example, it is consistent that $\mathsf{ADD}(\mathcal{L}) = \mathsf{COV}(\mathcal{L}) = [\mathbb{R}]^{\leq\aleph_0}$ (a model for Dual Borel Conjecture, see [8]).

However, the theory $\mathsf{ZFC}+\mathsf{CH}$ provides a sufficiently rich universe in which such constructions can be carried out. Moreover $<$-results about invariants **add**, **cov**, etc. in a natural way yield $\subsetneq$ results about ADD, COV, etc.

We will describe here several such constructions in detail. First consider those that involve only forcing notions satisfying c.c.c.

6.10 Theorem ($\mathsf{ZFC} + \mathsf{CH}$). *There is a set $X \subseteq \mathbb{R}$ such that $X \in \mathsf{D}$ and $X \notin \mathsf{NON}(\mathcal{L}) \cup \mathsf{NON}(\mathcal{B})$.*

Proof. The construction is canonical. Set the cardinal invariants corresponding to the families that X belongs to $\aleph_2$ and the other ones to $\aleph_1$. In our case $\mathfrak{d} = \aleph_2$ and $\mathbf{non}(\mathcal{L}) = \mathbf{non}(\mathcal{B}) = \aleph_1$. Now consider the forcing notion that produces the model for the dual setup, i.e. $\mathfrak{b} = \aleph_1$ and $\mathbf{cov}(\mathcal{L}) = \mathbf{cov}(\mathcal{B}) = \aleph_2$. According to our table it is the iteration of Cohen and random forcings, $\mathbf{C} \star \mathbf{B}$. Let $\langle M_\alpha : \alpha < \aleph_1\rangle$ be an increasing sequence of countable submodels of $H(\lambda)$ (here and elsewhere in this chapter $H(\lambda)$ denotes the collection of sets whose transitive closure has size $< \lambda$)) such that

1. ${}^\omega 2 \subseteq \bigcup_{\alpha<\omega_1} M_\alpha$,

2. for every $\alpha < \omega_1$, $M_{\alpha+1} \models M_\alpha$ is countable,

3. $\langle M_\beta : \beta \leq \alpha\rangle \in M_{\alpha+1}$.

For each α choose a pair $(c_\alpha, r_\alpha) \in M_{\alpha+1}$ such that (c_α, r_α) is $\mathbf{C}\star\mathbf{B}$-generic over M_α. Note that such a pair will also be generic over M_β for $\beta < \alpha$. Let z_α encode (c_α, r_α) as

$$z_\alpha(n) = \begin{cases} c_\alpha(k) & \text{if } n = 2k, \\ r_\alpha(k) & \text{if } n = 2k+1. \end{cases}$$

Let $X = \{z_\alpha : \alpha < \omega_1\}$. We will show that X has the required properties.

To show that $X \in \mathsf{D}$ fix a Borel function $F : \mathbb{R} \longrightarrow {}^\omega\omega$ and find α_0 such that F is coded in M_{α_0}. Let f be any function which dominates $M_{\alpha_0} \cap {}^\omega\omega$. For any $\alpha < \omega_1$, $F(z_\alpha) \in M_{\alpha_0}^{\mathbf{C}\star\mathbf{B}}$. Since $\mathbf{C} \star \mathbf{B}$ does not add dominating reals it follows that for every α there is a function $g \in M_{\alpha_0} \cap {}^\omega\omega$ such that $g \not\leq^\star F(z_\alpha)$. Since g is dominated by f we conclude that $f \not\leq^\star F(z_\alpha)$ for every $\alpha < \omega_1$.

To see that $X \notin \mathsf{NON}(\mathcal{B}) \cup \mathsf{NON}(\mathcal{L})$ let $Y = \{c_\alpha : \alpha < \omega_1\}$. Observe that Y is a continuous image of X. Moreover, if $F \in M_{\alpha_0}$ is a meager set then $c_\alpha \notin F$ for $\alpha > \alpha_0$ since c_α is a Cohen real over M_{α_0}. The argument that $X \notin \mathsf{NON}(\mathcal{L})$ is analogous. $\dashv$

Observe that the crucial point of the above construction is that the real z_α defined at the step α is generic not only over model M_α but also over models M_β for $\beta < \alpha$. To illustrate this point suppose that $\mathcal{P}$ is a forcing notion, $M \subseteq N$ are two submodels of $H(\lambda)$ and $\mathcal{P} \in M$. Let $\mathcal{A} \in M$ be a maximal antichain in $\mathcal{P}$. If $\mathcal{P}$ satisfies c.c.c. then $\mathcal{A} \subseteq M$, as a range of a function on ω. If $\mathcal{P}$ is absolutely c.c.c. then $N \models \mathcal{A}$ is an maximal antichain, so a $\mathcal{P}$-generic real over N is also $\mathcal{P}$-generic over M. If $\mathcal{P}$ is not absolutely c.c.c. then we no longer know if $\mathcal{A}$ is a maximal antichain in N. In fact, we do not know if $\mathcal{A}$ is an antichain at all, if the incompatibility relation is not absolute between M and N. However, if both M and N are elementary submodels of $H(\lambda)$, then $N \models \mathcal{A}$ is a maximal antichain. Finally, if $\mathcal{P}$ does not satisfy c.c.c., then it is no longer true that $\mathcal{A} \subseteq M$, so a $\mathcal{P}$-generic real over V may not be generic over M. Recall that a condition $p \in \mathcal{P}$ is $(M, \mathcal{P})$-generic if p forces that the above situation does not happen. If for every countable $M \prec H(\lambda)$ the collection of $(M, \mathcal{P})$-generic conditions is dense in $\mathcal{P}$, then $\mathcal{P}$ is proper.

The following strengthenings of properness will allow us to carry out the construction from the proof of Theorem 6.10 for non-c.c.c. posets. See Abraham's chapter [1] for more on these concepts.

6.11 Definition. Suppose that $\mathcal{P}$ is a forcing notion and $\alpha < \omega_1$ is an ordinal. We say that $\mathcal{P}$ is *α-proper* if for every sequence $\langle M_\beta : \beta \leq \alpha \rangle$ such that

1. for every β, M_β is a countable elementary submodel of $H(\lambda)$,

2. $\{M_\gamma : \gamma \leq \beta\} \in M_{\beta+1}$,

3. $M_{\beta+1} \models M_\beta$ is countable,

4. $M_\lambda = \bigcup_{\beta<\lambda} M_\beta$ for limit λ,

5. $\mathcal{P} \in M_0$,

and for every $p \in \mathcal{P} \cap M_0$, there exists a $q \leq p$ which is $(M_\beta, \mathcal{P})$-generic for $\beta \leq \alpha$.

6.12 Definition. A forcing notion $\mathcal{P}$ satisfies *Axiom* A if there exists a sequence $\langle \leq_n : n \in \omega \rangle$ of orderings on $\mathcal{P}$ (not necessarily transitive) such that

1. if $p \leq_{n+1} q$, then $p \leq_n q$ and $p \leq q$ for $p, q \in \mathcal{P}$,

2. if $\langle p_n : n \in \omega \rangle$ is a sequence of conditions such that $p_{n+1} \leq_n p_n$, then there exists a $p \in \mathcal{P}$ such that $p \leq_n p_n$ for all n, and

3. if $\mathcal{A} \subseteq \mathcal{P}$ is an antichain, then for every $p \in \mathcal{P}$ and $n \in \omega$ there exists a $q \leq_n p$ such that $\{r \in \mathcal{A} : q \text{ is compatible with } r\}$ is countable.

All forcing notions assigned to the cardinal invariants from Cichoń diagram satisfy Axiom A.

6.13 Lemma. *If $\mathcal{P}$ satisfies Axiom A, then $\mathcal{P}$ is α-proper for every $\alpha < \omega_1$.*

Proof. Proceed by induction on α. Let $\langle M_\beta : \beta \leq \alpha \rangle$ be a sequence of models having the required properties. Fix $p \in \mathcal{P} \cap M_0$ and $n \in \omega$. We will find a $q \leq_n p$ which is M_β-generic for $\beta \leq \alpha$. If $\alpha = 0$, then it is the usual proof that Axiom A implies properness. If $\alpha = \gamma + 1$, then first find a $q' \leq_n p$ which is M_δ-generic for $\delta \leq \gamma$ and then use properness of $\mathcal{P}$ to get $q \leq_n q'$ which is M_α-generic. If α is limit, then fix an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ such that $\sup_n \alpha_n = \alpha$. Use the induction hypothesis to find conditions $\{p_k : k \in \omega\}$ such that

1. $p_{k+1} \in M_{\alpha_k+1}$,

2. p_k is M_γ-generic for $\gamma < \alpha_k$,

3. $p_{k+1} \leq_{n+k} p_k$ for each k.

Let q be such that $q \leq_{n+k} p_k$ for each k. It is the condition we are looking for. $\dashv$

6.14 Theorem ($\mathsf{ZFC} + \mathsf{CH}$). *There is a set $X \subseteq \mathbb{R}$ such that $X \in \mathsf{B}$ and $X \notin \mathsf{COV}(\mathcal{L}) \cup \mathsf{NON}(\mathcal{L})$.*

Proof. In terms of cardinal invariants the statement of the theorem corresponds to the dual to the model for $\mathfrak{b} = \aleph_2$ and $\mathbf{cov}(\mathcal{L}) = \mathbf{non}(\mathcal{L}) = \aleph_1$, that is, the one where $\mathfrak{d} = \aleph_1$ and $\mathbf{cov}(\mathcal{L}) = \mathbf{non}(\mathcal{L}) = \aleph_2$. The set we are looking for is defined using the forcing notion used to construct that model (cf. Theorem 6.8).

Let $\langle f_\alpha : \alpha < \omega \rangle$ be an enumeration of $\mathbb{R}$. Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a sequence of countable elementary submodels of $H(\lambda)$ such that

1. $f_\alpha \in M_\alpha$,
2. $\langle M_\beta : \beta \le \alpha\rangle \in M_{\alpha+1}$, and $M_{\alpha+1} \models M_\alpha$ is countable,
3. $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$ for limit γ.

Note that from (2) it follows that for every $\beta < \alpha$, $M_\alpha \models$ "M_β is countable." Let $\langle e_\alpha, r_\alpha : \alpha < \omega_1\rangle$ be a sequence of reals such that

1. $e_\alpha, r_\alpha \in M_{\alpha+1}$,
2. e_α is **EE**-generic over M_β for $\beta \le \alpha$,
3. r_α is **B**-generic over $M_\beta[e_\alpha]$ for $\beta \le \alpha$.

For $\alpha < \omega_1$ define

$$z_\alpha(n) = \begin{cases} e_\alpha(k) & \text{if } n = 2k, \\ r_\alpha(k) & \text{if } n = 2k+1. \end{cases}$$

Let $Z = \{z_\alpha : \alpha < \omega_1\}$.

$Z \notin \mathsf{NON}(\mathcal{L})$. The set $X = \{r_\alpha : \alpha \in \omega_1\}$ is a Borel image of Z. Given $f \in {}^\omega\omega$ find an α such that $f = f_\alpha$. Notice that $r_\beta \notin (N)_f$ for $\beta > \alpha$. In fact, this proof shows that no uncountable subset of Z is in $\mathsf{NON}(\mathcal{L})$.

$Z \notin \mathsf{COV}(\mathcal{L})$. Consider the set $Y = \{e_\alpha : \alpha < \omega_1\}$ which is a Borel image of X. Let $\overline{P} = \{f \in {}^\omega([\omega]^{<\omega}) : \forall n\ (f(n) \in {}^n2)\}$. Let

$$\widetilde{H} = \{(f, x) : f \in \overline{P}, x \in {}^\omega2 \ \&\ \exists^\infty n\ x{\restriction}n = f(n)\}.$$

It is easy to see that $\widetilde{H}$ is a Borel set in $\overline{P} \times {}^\omega2$ and $(\widetilde{H})_f \in \mathcal{L}$ for every f. Suppose that $x \in {}^\omega2$. Find an α such that $x \in M_\alpha$ and note that for $\beta > \alpha$, $x \in (\widetilde{H})_{e_\beta}$. It follows that no uncountable subset of Z is in $\mathsf{COV}(\mathcal{L})$.

$X \in \mathsf{D}$. Let $F : X \longrightarrow {}^\omega\omega$ be a Borel function. Find an α such that F is coded in M_α. Let $f \in {}^\omega\omega$ be such that for every $g \in M_\alpha \cap {}^\omega\omega$, $g \le^\star f$. Since M_α is countable, such an f exists. Since both **B** and **EE** are ${}^\omega\omega$-bounding (and therefore so is $\mathbf{EE}\star\mathbf{B}$) for every $\beta > \alpha$, there exists a $g \in M_\alpha$ such that $F(z_\alpha) \le^\star g \le^\star f$. ⊣

6.15 Theorem ($\mathsf{ZFC} + \mathsf{CH}$). *There is a set $X \subseteq \mathbb{R}$ such that $X \in \mathsf{COV}(\mathcal{L}) \cap \mathsf{NON}(\mathcal{L})$ and $X \notin \mathsf{D}$.*

Proof. Let $\langle M_\alpha : \alpha < \omega_1\rangle$ be a sequence of countable elementary submodels of $H(\lambda)$ as in the previous proof.

In this case we use the Laver forcing from Theorem 6.6. The only difference is that in order to ensure that the constructed set belongs to $\mathsf{COV}(\mathcal{L})$ we construct a set of witnesses for that.

Let $\langle l_\alpha, r_\alpha : \alpha < \omega_1\rangle$ be a sequence of reals such that

1. $l_\alpha, r_\alpha \in M_{\alpha+1}$,

2. l_α is **LT**-generic over M_β for $\beta \leq \alpha$,

3. r_α is **B**-generic over $M_\alpha[l_\beta]$ for all $\beta < \omega_1$.

To meet the condition (3) we need the following result:

6.16 Theorem. *Suppose that $N \prec H(\lambda)$ is a countable model of* ZFC*. Let $S \in N \cap \mathbf{LT}$ and let x be a random real over N. There exists a $T \leq S$ such that T is N-generic and $T \Vdash_{\mathbf{LT}} x$ is random over $N[\dot{G}]$.*

Proof. See [27], [38] or [8]. ⊣

Let $X = \{l_\alpha : \alpha < \omega_1\}$. The difference between this and the previous construction is that we define the set of witnesses $\{r_\alpha : \alpha < \omega_1\}$ that $X \in \mathsf{COV}(\mathcal{L})$.

$X \in \mathsf{COV}(\mathcal{L})$. Let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be a Borel set with null sections. Find α such that $H \in M_\alpha$. Note that

$$r_\alpha \notin \bigcup_{\beta<\omega_1} (H)_{l_\beta},$$

since r_α is random over $M_\alpha[l_\beta]$ for all β and $(H)_{l_\beta} \in M_\alpha[l_\beta]$.

$X \in \mathsf{NON}(\mathcal{L})$. Let $F : X \longrightarrow {}^\omega 2$ be a Borel function. Find α such that F is coded in M_α. Let $B = \bigcup\{A : A \in \mathcal{L} \cap M_\alpha\}$. Since M_α is countable, B is a null set. By Lemma 6.7(3) for every $\beta > \alpha$, $F(l_\alpha) \in B$.

$X \notin \mathsf{D}$. This is obvious, since by Lemma 6.7(1), for every α

$$\forall f \in M_\alpha \cap {}^\omega\omega \ \forall^\infty n \ (f(n) < l_\alpha(n)).$$

⊣

The method of constructing counterexamples to the Cichoń diagram described above is very elegant and effective but assumes a rather large body of knowledge involving forcing, preservation theorems, and so forth. We will conclude this section with a sketch of an alternative method of constructing examples of small sets which is also quite general but more direct. Along the way we translate the forcing results that we have used into statements about sets of reals.

Suppose that $\mathcal{P}$ is a forcing notion and conditions of $\mathcal{P}$ are sets of reals. Note that all forcing notions associated with the Cichoń Diagram are (or can be taken to be) of this form. For a description of a much larger class of forcing notions of that kind see [44].

Let

$$I_{\mathcal{P}} = \{X \subseteq \mathbb{R} : \forall p \in \mathcal{P} \ \exists q \leq p \ (q \cap X = \emptyset)\}.$$

The following lemma lists the obvious observations about $I_{\mathcal{P}}$.

6.17 Lemma.

(1) $I_{\mathcal{P}}$ is an ideal,

(2) *$X \in I_{\mathcal{P}}$ iff there exists a maximal antichain $\mathcal{A} \subseteq \mathcal{P}$ such that $X \cap \bigcup \mathcal{A} = \emptyset$,*

(3) *$\forall p \in \mathcal{P}\ (p \notin I_{\mathcal{P}})$.*

Suppose that $\mathcal{P}$ is a forcing notion satisfying Axiom A. Let

$$I^{\omega}_{\mathcal{P}} = \{X \subseteq \mathbb{R} : \forall p \in \mathcal{P}\ \forall n \in \omega\ \exists q \leq_n p\ (q \cap X = \emptyset)\}.$$

Note that $I^{\omega}_{\mathcal{P}}$ is a σ-ideal contained in $I_{\mathcal{P}}$.

If $\mathcal{P}$ satisfies c.c.c., then we can witness that $\mathcal{P}$ satisfies Axiom A by putting $p \leq_0 q$ if $p \leq q$, and for $n > 0$, $p \leq_n q$ if $p = q$. In this case $I^{\omega}_{\mathcal{P}} = \{\emptyset\}$. However, for non-c.c.c. forcings as well as some c.c.c. posets (like the random real algebra $\mathbf{B}$) we can define $\leq_n$'s in such a way that $I^{\omega}_{\mathcal{P}} = I_{\mathcal{P}}$.

First we will describe how to translate the forcing theorems.

6.18 Lemma. *Suppose that $\mathcal{P}$ is a forcing notion such that*

(1) *$\mathcal{P}$ is proper,*

(2) *for every V-generic filter $G \subseteq \mathcal{P}$ there exists a real x_G such that $V[G] = V[x_G]$,*

(3) *conditions of $\mathcal{P}$ are Borel sets of reals, ordered by inclusion, and*

(4) *every countable antichain in $\mathcal{P}$ can be represented by a countable family of pairwise disjoint elements of $\mathcal{P}$.*

Then for every $\mathcal{P}$-name $\dot{x}$ such that $\Vdash_{\mathcal{P}} \dot{x} \in {}^{\omega}2$ and $p \in \mathcal{P}$ there exists a Borel function $F \in V$, $F : {}^{\omega}2 \longrightarrow {}^{\omega}2$ and a $q \leq p$ such that $q \Vdash_{\mathcal{P}} \dot{x} = F(x_{\dot{G}})$.

Proof. Fix $\dot{x}$ and let $\mathcal{A}_n$ be a maximal antichain of conditions deciding $\dot{x}\restriction n$. Use properness to find a $q \leq p$ such that each $\mathcal{A}'_n = \{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$ is countable. By the assumption we can assume that elements of $\mathcal{A}'_n$ are pairwise disjoint. Define $F_n : q \longrightarrow 2^n$ as

$$F_n(x) = s \quad \text{if } x \in r \in \mathcal{A}'_n \text{ and } r \Vdash_{\mathcal{P}} \dot{x}\restriction n = s.$$

Note that $F = \lim_n F_n$ is the function we are looking for. ⊣

Let $\mathcal{P}$ be a forcing notion satisfying the assumptions of the above lemma.

- $\mathcal{P}$ does not add random reals if for every $\mathcal{P}$-name $\dot{x}$ for an element of ${}^{\omega}2$ and every $p \in \mathcal{P}$ there is a $q \leq p$ and an $H \in V \cap \mathcal{L}$ such that $q \Vdash_{\mathcal{P}} \dot{x} \in H$.

- $\mathcal{P}$ is ${}^{\omega}\omega$-bounding if for every $\mathcal{P}$-name $\dot{f}$ for an element of ${}^{\omega}\omega$ and every $p \in \mathcal{P}$ there is a $q \leq p$ and a $g \in V \cap {}^{\omega}\omega$ such that $q \Vdash_{\mathcal{P}} \dot{f} \leq^{\star} g$.

- $\mathcal{P}$ preserves outer measure if for every set of positive outer measure $X \subseteq {}^{\omega}2$, $X \in V$ and every $\dot{F}$, a $\mathcal{P}$-name for a Borel function from ${}^{\omega}2$ to ${}^{\omega}\omega$ and $p \in \mathcal{P}$ there is a $q \leq p$ such that $q \Vdash_{\mathcal{P}} X - (N)_{\dot{F}(x_{\dot{G}})}) \neq \emptyset$.

These statements translate as:

- (not adding random reals) For every Borel function $F : {}^\omega 2 \longrightarrow {}^\omega 2$ and $p \in \mathcal{P}$ there exist a set $H \in \mathcal{L}$, $q \le p$ and $A \in I_{\mathcal{P}}$ such that $F``(q - A) \subseteq H$.
- ($\mathcal{P}$ is ${}^\omega\omega$-bounding) For every Borel function $F : {}^\omega 2 \longrightarrow {}^\omega\omega$ and $p \in \mathcal{P}$ there is a function $f \in {}^\omega\omega$, $q \le p$ and $A \in I_{\mathcal{P}}$ such that $F``(q-A) \le^\star f$.
- ($\mathcal{P}$ preserves outer measure) For every set of positive outer measure $X \subseteq {}^\omega 2$, and every Borel function $F : {}^\omega 2 \longrightarrow {}^\omega\omega$ and $p \in \mathcal{P}$ there is $q \le p$ and $A \in I_{\mathcal{P}}$ such that $X - \bigcup_{x \in q-A} (N)_{F(x)} \neq \emptyset$.

If in addition $\mathcal{P}$ satisfies Axiom A and $I_{\mathcal{P}} = I^\omega_{\mathcal{P}}$, then we can put $A = \emptyset$.

Second proof of Theorem 6.14. For $p, q \in \mathbf{EE}$ and $n \in \omega$ we define $p \le_n q$ if $p \le q$ and first n elements of $\omega - \mathsf{dom}(p)$ and $\omega - \mathsf{dom}(q)$ are the same.

For $p, q \in \mathbf{B}$ and $n \in \omega$ let $p \le_n q$ if $p \le q$ and $\mu(q - p) \le 2^{-n-1} \cdot \mu(q)$.

The forcing notions $\mathbf{EE}$, $\mathbf{B}$ (and the remaining ones as well) can be represented as collections of perfect subsets of ${}^\omega 2$ (or ${}^\omega\omega$). This is not critical for the construction, but it makes it more natural.

In case of $\mathbf{EE}$ for $n \in \omega$ let $k_n = 2^{n+1} - 1$. Consider sets $P \subseteq {}^\omega 2$ of form $\bigcap_{n\in\omega}[C_n]$, where $\{C_n : n \in \omega\}$ satisfies the following conditions:

1. $C_n \subseteq {}^{[k_n, k_{n+1})}2$,
2. for every n, $|C_n| = 1$ or $|C_n| = 2^n$ (so $C_n = {}^{[k_n, k_{n+1})}2$),
3. $\exists^\infty n \; (|C_n| = 2^n)$.

It is clear that every condition $p \in \mathbf{EE}$ corresponds to a set P as above and vice versa. Therefore from now on we identify $\mathbf{EE}$ with these sets.

Let $\mathbf{B} \star \mathbf{EE}$ be the collection of subsets $H \subseteq 2^\omega \times 2^\omega$ such that

1. H is Borel and $\mathsf{dom}(H) = \{x : (H)_x \neq \emptyset\} \in \mathbf{B}$,
2. $\forall x \; ((H)_x \neq \emptyset \rightarrow (H)_x \in \mathbf{EE})$.

The elements of $\mathbf{B} \star \mathbf{EE}$ are $\mathbf{B}$-names for the elements of $\mathbf{EE}$. Thus, the set $\mathbf{B}\star\mathbf{EE}$ indeed corresponds to the iteration of $\mathbf{B}$ and $\mathbf{EE}$. For $H_1, H_2 \in \mathbf{B}\star\mathbf{EE}$ and $n \in \omega$ let $H_1 \le H_2$ mean that

1. $\mathsf{dom}(H_1) \le_n \mathsf{dom}(H_2)$,
2. $\forall x \in \mathsf{dom}(H_1) \; ((H_1)_x \le_n (H_2)_x)$.

Note that $\le_n$ on $\mathbf{B} \star \mathbf{EE}$ witnesses that it satisfies Axiom A.

Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of ${}^\omega 2$, $\langle F_\alpha : \alpha < \omega_1 \rangle$ an enumeration of $\mathsf{BOREL}({}^\omega 2 \times {}^\omega 2, {}^\omega\omega)$, and $\langle f_\alpha : \alpha < \omega_1 \rangle$ an enumeration of ${}^\omega\omega$. We will build an ω_1-tree $\mathbf{A}$ of elements of $\mathbf{B}\star\mathbf{EE}$. Let $\mathbf{A}_\alpha$ denote the α-th level of $\mathbf{A}$. The tree $\mathbf{A}$ satisfies the following recursive conditions:

1. $\forall \beta > \alpha\ \forall n\ \forall H \in \mathbf{A}_\alpha\ \exists H' \in \mathbf{A}_\beta\ (H' \leq_n H)$,
2. $\exists f \in {}^\omega\omega\ \forall \alpha\ \forall H \in \mathbf{A}_{\alpha+1}\ (F_\alpha``H \leq^\star f)$,
3. $\forall H \in \mathbf{A}_{\alpha+1}\ (\mathsf{dom}(H) \cap (N)_{f_\alpha} = \emptyset)$,
4. $\forall H \in \mathbf{A}_{\alpha+1}\ \forall x \in \mathsf{dom}(H)\ \exists^\infty n\ (|C^x_n| = 1\ \&\ C^x_n \subseteq x_\alpha)$, where $(H)_x = \bigcap_n [C^x_n]$.

Case 1. $\alpha = \beta + 1$. We will describe how to build a set of immediate successors of an element $H \in \mathbf{A}_\beta$. Given $H \in \mathbf{A}_\beta$ and $n \in \omega$ find an $H'_n \leq_n H$ satisfying conditions (3) and (4). By further shrinking we can ensure that (2) holds as well. Condition (2) follows from the statement that the iteration of **EE** and **B** is ${}^\omega\omega$-bounding.

Case 2. α is limit. Suppose that $H \in \mathbf{A}_{\beta_0}$ for some $\beta_0 \in \omega$ and that $n \in \omega$ is given. Fix an increasing sequence $\langle \beta_k : k \in \omega\rangle$ such that $\beta_k \to \alpha$. Choose a sequence $\langle H_k : k \in \omega\rangle$ such that

1. $H_0 = H_1 = \cdots = H_n = H$,
2. for $k \geq 0$, $H_{n+k+1} \leq_n H_{n+k}$,
3. $H_{k+n} \in \mathbf{A}_{\beta_k}$.

Use Axiom A to find an H' such that $H' \leq_k H_k$. Level $\mathbf{A}_\alpha$ will consist of elements selected in this way.

Let $X = \{(x_p, y_p) : p \in \mathbf{A}\}$ be a selector from elements of $\mathbf{A}$. Note that $\pi_1(X) = \{x_p : (x_p, y_p) \in X\} \notin \mathsf{NON}(\mathcal{L})$ (by (3)), $\pi_2(X) = \{y_p : (x_p, y_p) \in X\} \notin \mathsf{COV}(\mathcal{L})$ (by (4)) and $X \in \mathsf{D}$ (by (2)). $\dashv$

Now let us look at the set constructed in Theorem 6.15.

Second proof of Theorem 6.15. For every $T \in \mathbf{LT}$ and $s \in {}^{<\omega}\omega$ define a node $T(s)$ in the following way: $T(\emptyset) = \mathsf{stem}(T)$ and for $n \in \omega$ let $T(s^\frown n)$ be the n-th element of $\mathsf{succ}_T(T(s))$.

For $T, T' \in \mathbf{LT}$ and $n \in \omega$ define $T \leq_n T'$ if $T \leq T'\ \&\ \forall s \in n^{\leq n}\ (T(s) = T'(s))$. In particular, $T \leq_0 T'$ is equivalent to $T \leq T'$ and $\mathsf{stem}(T) = \mathsf{stem}(T')$. It is easy to check that Laver forcing satisfies Axiom A.

Suppose that

1. $\langle f_\alpha : \alpha < \omega_1\rangle$ is an enumeration of ${}^\omega\omega$,
2. $\langle F_\alpha : \alpha < \omega_1\rangle$ is an enumeration of $\mathsf{BOREL}({}^\omega\omega, {}^\omega 2)$,
3. $\langle G_\alpha : \alpha < \omega_1\rangle$ is an enumeration of $\mathsf{BOREL}({}^\omega\omega, {}^\omega\omega)$.

We build an ω_1-tree $\mathbf{A}$ satisfying the following inductive conditions:

1. $\forall \beta > \alpha\ \forall n\ \forall T \in \mathbf{A}_\alpha\ \exists S \in \mathbf{A}_\beta\ (S \leq_n T)$,

2. $\forall T \in \mathbf{A}_{\alpha+1}\ \forall x \in [T]\ (f_\alpha \leq^\star x)$ (**LT** adds a dominating real (Lemma 6.7(1))),

3. for every $T \in \mathbf{A}_{\alpha+1}$, F_α"T has measure zero (**LT** does not add random reals (Lemma 6.7(3))),

4. $\forall T \in \mathbf{A}_{\alpha+1}$ (${}^\omega 2 - \bigcup_{x\in[T]}(N)_{G_\alpha(x)}$ is uncountable) (**LT** preserves outer measure (Lemma 6.7(2))).

Next we want to chose a selector X from elements of $\mathbf{A}$. Condition (2) will guarantee that $X \notin \mathsf{D}$ and (3) that $X \in \mathsf{NON}(\mathcal{L})$. Unfortunately (4) does not suffice to show that $X \in \mathsf{COV}(\mathcal{L})$. It is conceivable that $2^\omega = \bigcup_{T\in\mathbf{A}_{\alpha+1}} \bigcup_{x\in[T]}(N)_{G_\alpha(x)}$, because $\mathsf{COV}(\mathcal{L})$ is not a σ-ideal. Therefore we need stronger property:

4′. For every Borel function $F : {}^\omega\omega \longrightarrow {}^\omega\omega$ and every sequence $\langle T_n : n \in \omega\rangle$ of conditions in **LT** there exists an uncountable set $Y \subseteq 2^\omega$ such that for each $x \in Y$we can find sequence a $\langle S^n_k : n, k \in \omega\rangle$ such that $S^n_k \leq_k T_n$ and $y \notin \bigcup_{n,k} \bigcup_{x\in[S^n_k]}(N)_{F(n)}$.

Property (4′) is a translation of Theorem 6.16.

Now we construct X along with $\mathbf{A}$. At the step α we have X_α and $\mathbf{A}_\alpha$. Let $\mathbf{A}_\alpha = \langle T_n : n \in \omega\rangle$ and pick $y \notin X_\alpha$ together with $\langle S^n_k : n, k \in \omega\rangle = \mathbf{A}_{\alpha+1}$ as in (4′). $\dashv$

Historical Remarks. Parts (1) and (4) of Theorem 6.1 are due to Truss [57] and [58] and parts (2) and (3) to Solovay [52]. Theorem 6.3 and other preservation results are due to Shelah [46]. Various presentations of these results appear in [22, 26] and most generally in [48]. Models for the Cichoń diagram were constructed by Miller in [33], more in [27] and the latest ones in [9]. Theorem 6.7(2) is due to Judah and Shelah [27], the remaining parts are due to Laver [31]. The forcing **EE** and Lemma 6.9 are due to Miller [33]. Brendle [13] constructed the counterexamples for the $\subsetneq$ for the families of small sets. Constructions of this type were already considered in [20]. The technique of "Aronszajn tree of perfect sets" was invented by Todorcevic (see [21]). Theorem 6.16 is due to Judah and Shelah.

7. Further Reading

There are new developments that occurred after the previous sections were written that to some extent relate to the subject of this chapter, and to the theory of cardinal invariants in general. We will discuss these results briefly and point the reader to the relevant publications. We will start with a brief account of Zapletal's theory. The main reference here is [62].

In the previous section we described a canonical way of building models where cardinal invariants related to measure and category have various values. Similarly, we also described a way to use forcing notions to construct sets that belong to some families of small sets but not to others. The key to these constructions was the ability to pair each cardinal invariant with a forcing notion that increased it and nothing else. The second feature was the duality, that is the ability to obtain the results in pairs. The choice of the appropriate forcing notions was dictated by the heuristics. Zapletal's theory puts those results in a more general context and explains that in most cases these choices are in fact canonical. Moreover, the constructions of the previous section are special cases of a more general phenomenon. To large extent, Zapletal's theory explains the duality as well.

7.1 Definition. A cardinal invariant $\mathfrak{y}$ is *tame* if $\mathfrak{y}$ is the minimum size of a set of reals A with properties $\phi(A)$ and such that $\forall x \in \mathbb{R}\ \exists y \in A\ \theta(x, y)$, where θ is a projective formula that does not mention A and ϕ quantifies only over ω and A. The set A is a *witness* for $\mathfrak{y}$.

It is easy to see that all cardinal invariants considered in this section are tame.

7.2 Definition. We say that a cardinal invariant $\mathfrak{inv}$ can be *isolated* if there is a forcing notion $\mathbb{P}_{\mathfrak{inv}}$ such that for every tame cardinal invariant $\mathfrak{y}$, if $\mathfrak{r} < \mathfrak{inv}$ holds in some set forcing extension then it holds in a $\mathbb{P}_{\mathfrak{inv}}$ extension of a model for CH.

Let LC stand for an unspecified large cardinal assumption; typically these are needed to show regularity properties for various families of sets.

7.3 Theorem (Zapletal [61, 62]). *Assume* LC. *The following cardinal invariants can be isolated:* $\mathfrak{c}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathbf{cov}(\mathcal{B})$, *and* $\mathbf{cov}(\mathcal{L})$. *The isolating forcing notion is the countable support iteration of length* $\aleph_2$ *of the corresponding forcing notion defined in the previous section.*

While some cardinal invariants cannot be isolated ($\mathbf{cof}(\mathcal{B})$ and $\mathbf{non}(\mathcal{B})$ for example), and some are not tame ($\mathfrak{h}$ and $\mathfrak{g}$ for example) this is a very powerful theorem which gives a lot of structure to the theory of cardinal invariants.

The duality heuristic is explained by the following result:

7.4 Theorem (Zapletal [62]). *Let* I *be an analytic* σ*-ideal of subsets of* $\mathbb{R}$. *If* ZFC *proves* $\mathbf{cov}(I) = \mathfrak{c}$ *then* ZFC *proves* $\mathbf{non}(I) \leq \aleph_2$.

Let I *be a projective* σ*-ideal of subsets of* $\mathbb{R}$. *If* $\mathsf{ZFC}+\mathsf{LC}$ *proves* $\mathbf{cov}(I) = \mathfrak{c}$ *then* $\mathsf{ZFC} + \mathsf{LC}$ *proves* $\mathbf{non}(I) \leq \aleph_2$.

The subject of Zapletal's theory is the study of posets of form $\mathbb{P}_I = \mathsf{Borel}(\mathbb{R})/I$ where I is a σ-ideal. The cardinal invariants mentioned in Theorem 7.3 are the covering numbers of various σ ideals and the isolating forcing notions are the iterations of forcings $\mathbb{P}_I$. For example, let the σ-ideal generated by bounded subsets of ${}^{\omega}\omega$ be called $\mathtt{bounded}$. Then $\mathbf{cov}(\mathtt{bounded}) = \mathfrak{d}$ and $\mathbb{P}_{\mathfrak{d}}$ is the countable support iteration of length ω_2 of $\mathbb{P}_{\mathtt{bounded}}$. Furthermore, $\mathbb{P}_{\mathtt{bounded}}$ has a dense subset isomorphic to Miller forcing.

The models that are obtained by using forcing notions that isolate cardinal invariants are very interesting because of their strong combinatorial properties as exhibited by Zapletal's theorem. His work in [61], further expanded in [62], leads to axioms that describe combinatorial properties of these models. Historically, the first such attempt was undertaken by Ciesielski and Pawlikowski in [16]. They formulated a family of axioms of varying strength capturing the combinatorics of the Sacks model. Their axioms have a form:

$\mathsf{CPA} \iff \mathfrak{c} = \aleph_2$ and for any *appropriately dense* subset $\mathcal{A}$ of Sacks forcing there is a subfamily $\mathcal{A}_0$ of size $\aleph_1$ such that $|\mathbb{R} \setminus \bigcup \mathcal{A}_0| \leq \aleph_1$.

The CPA axioms of Ciesielski and Pawlikowski are geared towards deriving the maximum number of properties of the Sacks model, which is the goal of [16]. Zapletal's work leads to more abstractly formulated family of axioms $\mathsf{CPA}(I)$, which imply the Ciesielski and Pawlikowski axioms in case of Sacks forcing. The starting point is the analysis of the forcing notions of the form $\mathbb{P}_I$, and their countable iterations. It turns out that for a large class of ideals I, for a countable ordinal α, a countable support iteration of length α of $\mathbb{P}_I$ is isomorphic with $\mathsf{Borel}(\mathbb{R}^\alpha)/I^\alpha$, where I^α is a Fubini-power of I.

7.5 Definition. Let $\alpha < \omega_1$. We say that a set $B \subseteq {}^\alpha\mathbb{R}$ is *I-perfect* if

1. for $\beta < \alpha$ and $s \in B\restriction\beta = \{u\restriction\beta : u \in B\}$, the set $\{r \in \mathbb{R} : s^\frown r \in B\restriction\beta+1\}$ is I-positive,
2. for every increasing sequence $\langle \beta_n : n \in \omega\rangle$ and $s_n \in B\restriction\beta_n$ such that $s_n \subseteq \beta_{n+1}$, $\bigcup_n s_n \in B\restriction \bigcup_n \beta_n$.

Let I^α be the ideal on ${}^\alpha\mathbb{R}$ defined as the collection of all sets $A \subseteq {}^\alpha\mathbb{R}$ such that Player I has a winning strategy in the following game $G(A)$.

The game $G(A)$ lasts α rounds; at the round $\beta < \alpha$ Player I plays $B_\beta \in I$ and Player II responds with $r_\beta \in \mathbb{R} - B_\beta$. Player II wins if $\langle r_\beta : \beta < \alpha\rangle \in A$. Otherwise, Player I wins.

It is not hard to see that Player II has a winning strategy in $G(A)$ iff A contains an I-positive set.

Now we can define $\mathsf{CPA}(I)$. Let G be a game consisting of ω_1 moves that is played as follows: at the stage $\beta < \omega_1$ Player I plays $\alpha_\beta < \omega_1$, and I-perfect set $B_\beta \subseteq {}^{\alpha_\beta}\mathbb{R}$ and a Borel function $f_\beta : B_\beta \longrightarrow \mathbb{R}$. Player II responds with a Borel I-positive set $C_\beta \subseteq B_\beta$. Player I wins iff $\bigcup_{\beta<\omega_1} f_\beta``C_\beta = \mathbb{R}$.

$\mathsf{CPA}(I)$ is the statement $\mathbf{cov}(I) > \aleph_1$ and Player II does not have a winning strategy in G.

For a large class of ideals I, and very tame cardinal invariants $\mathfrak{y}$ (a small technical modification of tame invariants) we have:

7.6 Theorem (Zapletal [62]). $\mathsf{ZFC}+\mathsf{LC}$ *proves that* $\mathfrak{y} < \mathbf{cov}(I)$ *can be forced if and only if* $\mathsf{ZFC} + \mathsf{LC} + \mathsf{CPA}(I)$ *proves that* $\mathfrak{x} < \mathbf{cov}(I)$.

A different approach to exploring the combinatorial strength of cardinal invariants is presented in [35]. Recall that $\diamondsuit$ stands for the principle:

There is a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ such that $S_\alpha \subseteq \alpha$ and for all $X \subseteq \omega_1$, $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is stationary.

Hrušák [23] initiated a study of modifications of $\diamondsuit$ that relate to cardinal invariants. This theory was further developed in [35].

7.7 Definition. Let $\mathbf{A} = (A_-, A_+, A)$ be a relation as considered in Definition 2.3. Let $\Phi(\mathbf{A})$ stand for the statement:

For every function $F : {}^{<\omega_1}2 \longrightarrow A_-$ there exists a function $g : \omega_1 \longrightarrow A_+$ such that for all $f : \omega_1 \longrightarrow 2$, $\{\alpha : F(f{\restriction}\alpha) = g(\alpha)\}$ is stationary.

We call a function $F : {}^{<\omega_1}2 \longrightarrow X$ (where X is a Polish space) *Borel* if for every $\alpha < \omega_1$, $F{\restriction}{}^{<\alpha}2 : {}^{<\alpha}2 \longrightarrow X$ is Borel. Note that ${}^{<\alpha}2$ is homeomorphic to ${}^{\omega}2$.

For a Borel $\mathbf{A}$ let $\diamondsuit(\mathbf{A})$ stand for the statement:

For every Borel function $F : {}^{<\omega_1}2 \longrightarrow A_-$ there exists a function $g : \omega_1 \longrightarrow A_+$ such that for all $f : \omega_1 \longrightarrow 2$, $\{\alpha : F(f{\restriction}\alpha) = g(\alpha)\}$ is stationary.

The function g is called a *$\diamondsuit$-sequence for F*. Moreover, we often write $\diamondsuit(\|\mathbf{A}\|)$ rather than $\diamondsuit(\mathbf{A})$. For example $\diamondsuit(\mathbf{non}(\mathcal{B}))$ stands for $\diamondsuit(\mathcal{B}, \mathbb{R}, \not\ni)$.

7.8 Lemma (Moore-Hrušák-Dzamonja [35]).

1. *$\Phi(\mathbf{A})$ implies $\diamondsuit(\mathbf{A})$ whenever $\mathbf{A}$ is Borel,*
2. *$\diamondsuit$ implies $\Phi(\mathbf{A})$,*
3. *$\diamondsuit(\mathbf{A})$ implies $\|\mathbf{A}\| = \aleph_1$,*
4. *If $\mathbf{A} \preceq \mathbf{B}$ then $\Phi(\mathbf{B})$ implies $\Phi(\mathbf{A})$,*
5. *If $\mathbf{A}, \mathbf{B}$ are Borel and $\mathbf{A} \preceq \mathbf{B}$ is witnessed by a Borel morphism then $\diamondsuit(\mathbf{B})$ implies $\diamondsuit(\mathbf{A})$.*

Proof. (1) is obvious. To see (2) let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence (for elements ${}^{\omega_1}2$). Put $g(\alpha) \in A_+$ such that $F(S_\alpha) A g(\alpha)$. Then g is a $\diamondsuit(\mathbf{A})$-sequence since $\{\alpha : f{\restriction}\alpha = A_\alpha\} \subseteq \{\alpha : F(f{\restriction}\alpha) A g(\alpha)\}$.

(3) Let $F_0 : {}^{\omega}2 \longrightarrow A_-$ be a (Borel) surjection. Extend F_0 to F by putting, for an infinite α and $s \in {}^{\alpha}2$, $F(s) = F_0(s{\restriction}\omega)$. If g is a $\diamondsuit$-sequence for F then the range of g witnesses that $\|\mathbf{A}\| = \aleph_1$.

(4) and (5) are easy to verify. $\dashv$

The diamond principles $\diamondsuit(\mathbf{A})$ are forms of anti-Martin's Axiom $\|\mathbf{A}\| = \aleph_1$. Typically $\diamondsuit(\mathbf{A})$ is stronger than $\|\mathbf{A}\| = \aleph_1$, for example $\diamondsuit(\mathfrak{d})$ implies that ${}^{\omega}\omega$ can be partitioned into $\aleph_1$ compact sets, while $\mathfrak{d} = \aleph_1$ does not [35, 53]. Similarly, in most cases $\diamondsuit(\mathbf{A})$ is consistent with the negation of CH, and $\diamondsuit(\mathbf{A})$ holds in many "natural" models for $\|\mathbf{A}\| = \aleph_1$ including all minimal models for the tame cardinal invariants considered earlier. For example, $\diamondsuit(\mathfrak{b})$, $\diamondsuit(\mathbf{non}(\mathcal{B}))$, $\diamondsuit(\mathbf{non}(\mathcal{L}))$, $\diamondsuit(\mathbf{cov}(\mathcal{B}))$, $\diamondsuit(\mathbf{cov}(\mathcal{L}))$ hold in the Miller model, and similarly for the Laver, random, Cohen and Sacks models.

We will conclude with some consequences of the $\diamondsuit$-principles from [35].

7.9 Theorem.

1. *$\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{a} = \aleph_1$.*
2. *$\diamondsuit(\mathbf{non}(\mathcal{B}))$ implies that there exists a Suslin tree.*
3. *$\diamondsuit(\mathbf{non}(\mathcal{L}))$ does not imply that there exists a Suslin tree.*
4. *$\diamondsuit(\mathbf{non}(\mathcal{L})) + \diamondsuit(\mathbf{non}(\mathcal{B}))$ do not imply that there exists a Sierpiński or Luzin set.*

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8. Constructibility and Class Forcing

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Contents

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The method of forcing has had great success in demonstrating the relative consistency and independence of set-theoretic problems with respect to the traditional ZFC axioms, or to extensions of these axioms asserting the existence of large cardinals. One begins with a model M, selects a partial ordering $P \in M$ and shows that statements of interest hold in extensions of M of the form $M[G]$, when G is P-generic over M.

However, forcing can play another role in set theory. Not only is it a tool for establishing relative consistency and independence results, it is also a tool for *proving theorems*. This theorem-proving role of forcing in set theory did not become fully apparent until the development of *class forcing*.

In class forcing, the partial ordering P is no longer assumed to be an element of M, but instead a *class* in M. Section 2 below introduces the necessary definitions. We can nevertheless in this introduction explain the special role of class forcing in set theory by beginning with the basic question:

Do P-generic classes exist?

This question never arises in the traditional use of forcing to establish consistency results, for the simple reason that, thanks to the Löwenheim-Skolem Theorem, one can assume that the model M is countable. This assumption ensures an easy construction of a P-generic class. Without the countability assumption, our question becomes a serious one, in light of the following, where L is Gödel's universe of constructible sets, and where "L-definable" means "L-definable without parameters":

There exist L-definable class forcings P_0 and P_1 such that whenever G_0, G_1 are P_0-generic, P_1-generic over L, respectively:

(a) ZFC holds in $\langle L[G_0], G_0\rangle$ and in $\langle L[G_1], G_1\rangle$.

(b) ZFC (indeed Replacement) fails in $\langle L[G_0, G_1], G_0, G_1\rangle$.

This result forces us to make a choice: we cannot preserve ZFC and have generics for all ZFC preserving L-definable class forcings.

The Silver-Solovay theory of $0^\#$ provides a useful criterion for selecting the L-definable forcings which "should" have generics. We say that L is *rigid* if there is no elementary embedding from $\langle L, \in\rangle$ to itself, other than the identity.

L is rigid in class-generic extensions of L. If L is not rigid, then there is a smallest inner model in which L is not rigid, and this inner model is $L[0^\#]$, where $0^\#$ is a real.

Now we say that an L-definable forcing P is *relevant* if there is a class which is P-generic over L and which is definable in the inner model $L[0^\#]$. If P_0 and P_1 are relevant forcings then clearly generics for P_0 and for P_1 can coexist, as they both exist definably over $L[0^\#]$. Moreover, by adopting the base theory ZFC + "$0^\#$ exists", we can hope to use the theory of relevant forcing to prove new theorems, by constructing objects which actually exist (in the inner model $L[0^\#]$) rather than objects which may exist only in a generic extension of the universe.

In this article we discuss the basic theory and applications of class forcing, with an emphasis on three problems posed by Solovay which can be resolved using it. As class forcing, unlike traditional set forcing, does not in general preserve ZFC, we first isolate the first-order property of *tameness*, necessary and sufficient for this preservation. After mentioning four basic examples, we discuss the question of the relevance of class forcing before turning to the most important technique in the subject, the technique of *Jensen coding*. Armed with these ideas we then proceed to describe the solutions to the Solovay problems. We next discuss *generic saturation*, a concept which helps to explain the special role of $0^\#$ in this theory. We end by briefly describing some other applications.

For the deeper study of class forcing, including the many proofs omitted here, we refer the reader to [5]. Two examples of class forcing not treated in this paper are due to Woodin. These are the class forcing versions of the stationary tower forcing, discussed in [20], and the extender algebra forcing, found in [13]. We also mention the paper [21], where a theory of "locally set-generic" class forcings is presented in terms of a strengthening of tameness.

1. Three Problems of Solovay

Solovay's three problems each demand the existence of a real that neither constructs $0^\#$, nor is attainable by set forcing over L.

1.1 Definition. If x, y are sets of ordinals then we write $x \leq_L y$ for $x \in L[y]$; $x <_L y$ for $x \leq_L y$ and $y \not\leq_L x$; and $x =_L y$ for $x \leq_L y$ and $y \leq_L x$.

Genericity Problem. Does there exist a real $R <_L 0^\#$ such that R does not belong to any generic extension of L?

It was to answer this question affirmatively (when "generic" is interpreted to mean "set-generic") that Jensen proved his Coding Theorem. Roughly speaking, he showed that if G is generic for *Easton forcing at successors*, the standard L-definable class forcing that adds a κ-Cohen subset to κ for each L-successor cardinal κ, then there is a real $R <_L 0^\#$, obtained by class forcing over $\langle L[G], G\rangle$, such that $L[G] \subseteq L[R]$ and G is definable over $L[R]$. Then R does not belong to a set-generic extension of L as $L[G]$ is not included in any such extension.

Solovay's second problem concerns definability of reals.

1.2 Definition. R is an *absolute singleton* if for some formula φ, R is the unique solution to φ in every inner model containing R.

Shoenfield's Absoluteness Theorem states that if φ is Π^1_2 (i.e., of the form $\forall x \exists y \psi$ where x and y vary over reals and ψ is arithmetical) then $\varphi(R) \longleftrightarrow M \vDash \varphi(R)$ where M is any inner model containing R. Thus any Π^1_2-singleton (i.e., unique solution to a Π^1_2 formula) is an absolute singleton; $0^\#$ is an example. Also 0 is trivially an example. Solovay asked if there are any less evident examples.

Π^1_2-Singleton Problem. Does there exists a real R, $0 <_L R <_L 0^\#$ such that R is a Π^1_2-singleton?

Suppose that R is set-generic over L. Then it can be shown that R belongs to a P-generic extension of L, where there are only countably many constructible subsets of P, and therefore we can build a P-generic containing any condition in P. So we conclude that if R is nonconstructible and set-generic over L, then R cannot be a Π^1_2-singleton, as there must be other P-generic extensions with reals $R' \neq R$ satisfying any given Π^1_2 formula satisfied by R. This is why the Π^1_2-singleton problem requires Jensen's method: an affirmative answer to the Π^1_2-singleton problem implies an affirmative answer to the genericity problem (for set-genericity).

Solovay's third problem concerns admissibility spectra. Let T be a subtheory of ZFC and R a real. The *T-spectrum* of R, $\Lambda_T(R)$, is the class of all ordinals α such that $L_\alpha[R] \vDash T$. A general problem is to characterize the possible T-spectra of reals for various theories T. An important special case is where $T = T_0 =$ (ZFC without the Power Set Axiom and with Replacement restricted to Σ_1 formulas). We may refer to this as "admissibility theory", as an ordinal α is R-admissible if and only if it is either ω or belongs to the T_0-spectrum, or *admissibility spectrum*, of R. We denote the latter by $\Lambda(R)$.

There are some basic facts that limit the possibilities for $\Lambda(R)$: First, if R belongs to a set-generic extension of L, then $\Lambda(R)$ contains $\Lambda - \beta$ for some ordinal β, where $\Lambda = \Lambda(0)$. This is because if $\alpha \in \Lambda$ and $P \in L_\alpha$, then $L_\alpha[G] \vDash T_0$ for P-generic G. Second, if $0^\# \leq_L R$ then $\Lambda(R) - \beta \subseteq L$-inaccessibles for some β. This is because if $0^\# \in L_\beta[R]$ then every α in $\Lambda(R) - \beta$ is in $\Lambda(0^\#)$ and hence is a "Silver indiscernible", an ordinal which is very large (and in particular inaccessible) in L.

Thus to get a nontrivial admissibility spectrum for R without $0^\#$ we need Jensen's methods. An ordinal is *recursively inaccessible* if it is admissible and also the limit of admissibles.

Admissibility Spectrum Problem. Does there exist a real $R <_L 0^\#$ such that $\Lambda(R) =$ the recursively inaccessible ordinals?

Before we can say more about the solutions to the Solovay problems, we must first develop the basic theory of class forcing, to which we turn next.

2. Tameness

We want our class forcings to preserve ZFC. First we isolate a first-order condition that guarantees this.

2.1 Definition. A *ground model* is a structure $\langle M, A\rangle$ where:

(a) $\langle M, A\rangle$ is a transitive model of ZFC; i.e., M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate.

(b) $M \vDash V = L(A) = \bigcup\{L(A \cap V_\alpha) \mid \alpha \in \mathrm{On}\}$, the smallest inner model containing each of the sets $A \cap V_\alpha$, $\alpha \in \mathrm{On}$.

Property (a) implies that $\langle M, A\rangle$ is *amenable*: for x in M, $A \cap x$ also belongs to M. Property (b) guarantees that if $M \subseteq N \vDash \mathrm{ZFC}$, then M is definable over $\langle N, A\rangle$.

From now on, $\langle M, A\rangle$ will always denote a ground model. Suppose that $G \subseteq P$ where $P \subseteq M$ is an $\langle M, A\rangle$*-forcing*, i.e., a pre-ordering (reflexive, transitive relation) with greatest element 1^P, definable over $\langle M, A\rangle$. G is P*-generic over* $\langle M, A\rangle$ if G is compatible, upward-closed and $G \cap D \neq \emptyset$ whenever $D \subseteq P$ is dense and $\langle M, A\rangle$-definable.

For any $G \subseteq M$ we define $M[G]$ as follows: A *name* is a set $\sigma \in M$ whose elements are of the form $\langle \tau, a\rangle$, τ a name and $a \in M$ (defined recursively). Interpret names by: $\sigma^G = \{\tau^G \mid \langle \tau, a\rangle \in \sigma$ for some $a \in G\}$. Then $M[G] = \{\sigma^G \mid \sigma$ a name$\}$. A P*-generic extension of* $\langle M, A\rangle$ is a model $\langle M[G], A, G\rangle$ where G is P-generic over $\langle M, A\rangle$. P is an M*-forcing* if it is an $\langle M, A\rangle$-forcing for some A. A *generic extension of* M is a model $\langle M[G], A, G\rangle$ for some choice of A, P and of G P-generic over $\langle M, A\rangle$. $X \subseteq M$ is *generic over* M if X is definable in a generic extension of M.

Set forcings always preserve ZFC but class forcings in general do not. Fix a ground model $\langle M, A\rangle$ and $\langle M, A\rangle$-forcing P. P is ZFC *preserving* if $\langle M[G], A, G\rangle$ is a model of ZFC for all G which are P-generic over $\langle M, A\rangle$. For countable M there is a useful first-order property equivalent to ZFC preservation, called *tameness*, which we now describe. First we consider $\mathrm{ZFC}^- = \mathrm{ZFC}$ without the Power Set Axiom:

2.2 Definition. $D \subseteq P$ is *predense* $\leq p \in P$ if every $q \leq p$ is compatible with an element of D. $q \in P$ *meets* D if q extends an element of D. P is *pretame* if whenever $p \in P$ and $\langle D_i \mid i \in a\rangle$, $a \in M$, is an $\langle M, A\rangle$-definable sequence of classes predense $\leq p$ there exists a $q \leq p$ and a $\langle d_i \mid i \in a\rangle \in M$ such that for each $i \in a$, $d_i \subseteq D_i$ and d_i is predense $\leq q$.

2.3 Proposition. *Suppose that M is countable and P is* ZFC^- *preserving. Then P is pretame.*

Proof. Given $\langle D_i \mid i \in a\rangle$ and p as in the statement of pretameness choose G such that $p \in G$, G P-generic over $\langle M, A\rangle$ and consider $f(i) =$ least rank of an element of $G \cap D_i$. If pretameness failed for $p, \langle D_i \mid i \in a\rangle$ then for every $q \le p$ and $\alpha \in \mathrm{On}(M)$ there would be an $r \le q$ and $i \in a$ with r incompatible with each element of $D_i \cap V_\alpha$. But then by genericity, no ordinal of M can bound the range of f, so Replacement fails in $\langle M[G], A, G, M\rangle$. As $\langle M, A\rangle$ is a ground model, Replacement fails in $\langle M[G], A, G\rangle$. ⊣

The forcing relation $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ is defined by: $\langle M[G], A, G\rangle \models \varphi(\sigma_1^G, \ldots, \sigma_n^G)$ for each G which is P-generic over $\langle M, A\rangle$.

2.4 Proposition. *Supose that P is pretame, P-forcing is definable (i.e., for each formula φ, the relation $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ of $p, \sigma_1, \ldots, \sigma_n$ is $\langle M, A\rangle$-definable) and the Truth Lemma holds for P-forcing (i.e., for G P-generic over $\langle M, A\rangle$, $\langle M[G], A, G\rangle \models \varphi(\sigma_1^G, \ldots, \sigma_n^G)$ iff $\exists p \in G, p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$). Then P is* ZFC$^-$ *preserving.*

Proof. Suppose that G is P-generic over M. As $M[G]$ is transitive and contains ω, it is a model of all axioms of ZFC$^-$ with the possible exception of Pairing, Union and Replacement.

For Pairing, given σ_1^G, σ_2^G consider $\sigma = \{\langle \sigma_1, 1^P\rangle, \langle \sigma_2, 1^P\rangle\}$. Then $\sigma^G = \{\sigma_1^G, \sigma_2^G\}$.

For Replacement, suppose that $f : \sigma^G \longrightarrow M[G]$, f definable (with parameters) in $\langle M[G], A, G\rangle$ and by the Truth Lemma choose $p \in G$, $p \Vdash f$ is a total function on σ. Then for each σ_0 of rank $<$ rank σ, $D(\sigma_0) = \{q \mid$ for some τ, $q \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is dense $\le p$. Thus by the definability of P-forcing and pretameness we get that for each $q \le p$ there is an $r \le q$ and $\alpha \in \mathrm{On}(M)$ such that $D_\alpha(\sigma_0) = \{s \mid s \in V_\alpha$ and for some τ of rank $< \alpha$, $s \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is predense $\le r$ for each σ_0 of rank $<$ rank σ. By genericity there is a $q \in G$ and $\alpha \in \mathrm{On}(M)$ such that $q \le p$ and $D_\alpha(\sigma_0)$ is predense $\le q$ for each σ_0 of rank $<$ rank σ. Thus $\mathrm{ran}(f) = \pi^G$ where $\pi = \{\langle \tau, r\rangle \mid \mathrm{rank}\,\tau < \alpha, r \in V_\alpha, r \Vdash \tau \in \mathrm{ran}(f)\}$. So $\mathrm{ran}(f) \in M[G]$.

For Union, given σ^G consider $\pi = \{\langle \tau, p\rangle \mid p \Vdash \tau \in \bigcup \sigma\}$. This is not a set, but for each α we may consider $\pi_\alpha = \pi \cap V_\alpha^M$. By Replacement in $\langle M[G], A, G\rangle$, π_α^G is constant for sufficiently large $\alpha \in \mathrm{On}(M)$. For such α we have $\pi_\alpha^G = \bigcup \sigma^G$. ⊣

Thus the work in establishing the equivalence (for countable M) of ZFC$^-$ preservation with pretameness resides in:

2.5 Lemma (Main Lemma). *If P is pretame and M is countable, then P-forcing is definable and the Truth Lemma holds for P-forcing.*

Proof. We define a relation $\Vdash^*$, prove the lemma for $\Vdash^*$, and finally show that $\Vdash$ and $\Vdash^*$ are the same.

2.6 Definition (of $\Vdash^*$). We say that $D \subseteq P$ is dense $\le p$ if $\forall q \le p \exists r$ ($r \le q$ and r belongs to D).

(a) $p \Vdash^* \sigma \in \tau$ iff $\{q \mid \exists \langle \pi, r\rangle \in \tau$ such that $q \le r, q \Vdash^* \sigma = \pi\}$ is dense $\le p$.

(b) $p \Vdash^* \sigma = \tau$ iff for all $\langle \pi, r\rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \longleftrightarrow \pi \in \tau)$.

(c) $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

(d) $p \Vdash^* \sim\varphi$ iff $\forall q \leq p(\sim q \Vdash^* \varphi)$.

(e) $p \Vdash^* \forall x\varphi$ iff for all names $\sigma, p \Vdash^* \varphi(\sigma)$.

Note that circularity is avoided in (a), (b) as $\max(\operatorname{rank}\sigma, \operatorname{rank}\tau)$ goes down (in at most three steps) when these definitions are applied.

2.7 Sublemma.

(a) If $p \Vdash^* \varphi$ and $q \leq p$, then $q \Vdash^* \varphi$.

(b) If $\{q \mid q \Vdash^* \varphi\}$ is dense $\leq p$ then $p \Vdash^* \varphi$.

(c) If $\sim p \Vdash^* \varphi$ then there is $q \leq p$ such that $q \Vdash^* \sim\varphi$.

Proof of Sublemma 2.7.
(a) Clear, by induction on φ, as dense $\leq p \longrightarrow$ dense $\leq q$.
(b) Again by induction on φ. The proof uses the following facts: if $\{q \mid D$ is dense $\leq q\}$ is dense $\leq p$ then D is dense $\leq p$; if $\{q \mid q \Vdash^* \sim\varphi\}$ is dense $\leq p$ then $\forall q \leq p(\sim q \Vdash^* \varphi)$, using (a).
(c) Immediate by (b). ⊣

2.8 Sublemma (Definability of $\Vdash^*$). *For each formula φ, the relation $p \Vdash^* \varphi(\sigma_1 \cdots \sigma_n)$ of $p, \sigma_1, \ldots, \sigma_n$ is $\langle M, A\rangle$-definable.*

Proof of Sublemma 2.8. It suffices to show that the relations $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma = \tau$ are $\langle M, A\rangle$-definable. Note that by modifying A if necessary, we may assume that the relations "$x = V_\alpha^M$", "p, q are compatible", "d is predense below p", as well as $(P, \leq)$, are Δ_1-definable over $\langle M, A\rangle$.

Using pretameness we shall define a function F from pairs $(p, \sigma \in \tau)$, $(p, \sigma = \tau)$ into M such that:

(a) $F(p, \sigma \in \tau) = (i, d)$ where $\emptyset \neq d \in M$, $d \subseteq P$, $q \in d \longrightarrow q \leq p$ and either ($i = 1$ and $q \Vdash^* \sigma \in \tau$ for all $q \in d$) or ($i = 0$ and $q \Vdash^* \sigma \notin \tau$ for all $q \in d$).

(b) The same holds for $\sigma = \tau$, $\sigma \neq \tau$ instead of $\sigma \in \tau, \sigma \notin \tau$.

(c) F is Σ_1-definable over $\langle M, A\rangle$.

Given this we can define $p \Vdash^* \sigma \in \tau$ by: $p \Vdash^* \sigma \in \tau$ iff for all $q \leq p$, $F(q, \sigma \in \tau) = (1, d)$ for some d. This definition is correct because Lemma 2.7 gives us that $p \Vdash^* \sigma \in \tau \longleftrightarrow \{q \mid q \Vdash^* \sigma \in \tau\}$ is dense $\leq p$. Similarly for $p \Vdash^* \sigma = \tau$.

Now define F by recursion on $\sigma \in \tau$, $\sigma = \tau$. We consider the cases separately.

$\underline{\sigma \in \tau}$ Given p, search for $\langle\pi, r\rangle \in \tau$ and $q \leq p$, $q \leq r$ such that $F(q, \sigma = \pi) = (1, d)$ for some d. If such exist, let $F(p, \sigma \in \tau) = (1, e)$ where e is the union of all such d which appear by the least possible stage α (i.e., this Σ_1 property is true in $\langle V_\alpha^M, A \cap V_\alpha^M\rangle$, α least). If not, then $\bigcup\{d \mid \text{for some } q \leq r,\ F(q, \sigma = \pi) = (0, d)\} \cup \{q \mid q \text{ is incompatible with } r\} = D(\pi, r)$ is dense below p for each $\langle\pi, r\rangle \in \tau$. So also search for $\langle d(\pi, r) \mid \langle\pi, r\rangle \in \tau\rangle \in M$ and $q \leq p$ such that $d(\pi, r) \subseteq D(\pi, r)$ for each $\langle\pi, r\rangle$ and each $d(\pi, r)$ is predense $\leq q$; if this latter search terminates then set $F(p, \sigma \in \tau) = (0, e)$, where e consists of all such q witnessed by the least possible stage α. One of these searches must terminate (by pretameness) and hence $F(p, \sigma \in \tau)$ is defined and either of the form $(1, e)$ where $q \in e \longrightarrow q \leq p$, $q \Vdash^* \sigma \in \tau$, or of the form $(0, e)$ where $q \in e \longrightarrow q \leq p$, $q \Vdash^* \sigma \notin \tau$.

$\underline{\sigma = \tau}$ Given p, search for $\langle\pi, r\rangle \in \sigma \cup \tau$ and $q \leq p$, r such that $F(q, \pi \in \sigma) = (i, d)$, $q' \in d$, $F(q', \pi \in \tau) = (1 - i, e)$ and if this search terminates then set $F(p, \sigma = \tau) = (0, f)$ where f is the union of all such e which appear by the least possible stage α. If this search fails then for each $\langle\pi, r\rangle \in \sigma \cup \tau$, $D(\pi, r) = \bigcup\{e \mid \text{for some } q \leq p, \text{ some } q', d, i,\ F(q, \pi \in \sigma) = (i, d),\ q' \in d,\ F(q', \pi \in \tau) = (i, e)\} \cup \{q \mid q \text{ is incompatible with } r\}$ is dense $\leq p$. So also search for $\langle d(\pi, r) \mid \langle\pi, r\rangle \in \sigma \cup \tau\rangle \in M$ and $q \leq p$ such that for each $\langle\pi, r\rangle \in \sigma \cup \tau$, $d(\pi, r) \subseteq D(\pi, r)$ and $d(\pi, r)$ is predense $\leq q$. If this latter search terminates then $q \Vdash^* \sigma = \tau$ for all such q and let $F(p, \sigma = \tau) = (1, f)$, where f consists of all such q witnessed to obey the above by the least stage α. One of these searches must terminate (by pretameness) and hence $F(p, \sigma = \tau)$ is defined and either of the form $(0, f)$ where $q \in f \to q \leq p$, $q \Vdash^* \sigma \neq \tau$, or of the the form $(1, f)$ where $q \in f \to q \leq p$, $q \Vdash^* \sigma = \tau$.

⊣

Now that we have the definability of $\Vdash^*$ we can prove:

2.9 Sublemma. *For G P-generic over M:*

$$\langle M[G], A, G\rangle \vDash \varphi(\sigma_1^G, \ldots, \sigma_n^G) \quad \text{iff for some } p \in G, \quad p \Vdash^* \varphi(\sigma_1, \ldots, \sigma_n).$$

Proof of Sublemma 2.9. By induction on φ.

$\underline{\sigma \in \tau}$ $(\longrightarrow)$ If $\sigma^G \in \tau^G$ then choose a $\langle\pi, r\rangle \in \tau$ such that $\sigma^G = \pi^G$ and $r \in G$. By induction we can choose $p \in G$, $p \leq r$, $p \Vdash^* \sigma = \pi$. Then $p \Vdash^* \sigma \in \tau$. $(\longleftarrow)$ If $p \in G, \{q \mid \exists\langle\pi, r\rangle \in \tau \text{ such that } q \leq r,\ q \Vdash^* \sigma = \pi\} = D$ is dense $\leq p$ then by genericity we can choose $q \in G$, $\langle\pi, r\rangle \in \tau$ such that $q \leq r$, $q \Vdash^* \sigma = \pi$; then by induction $\sigma^G = \pi^G$ and as $r \geq q \in G$ we get $r \in G$ and hence by definition of $\tau^G, \pi^G \in \tau^G$. So $\sigma^G \in \tau^G$.

$\underline{\sigma = \tau}$ $(\longrightarrow)$ Suppose that $\sigma^G = \tau^G$. Consider $D = \{p \mid \text{either } p \Vdash^* \sigma = \tau \text{ or for some } \langle\pi, r\rangle \in \sigma \cup \tau,\ p \Vdash^* \sim(\pi \in \sigma \longleftrightarrow \pi \in \tau)\}$. Then

D is dense, using the definition of $p \Vdash^* \sigma = \tau$ and Lemma 2.7(c). By genericity there is a $p \in G \cap D$ and by induction it must be that $p \Vdash^* \sigma = \tau$. $(\longleftarrow)$ Suppose that $p \in G$, $p \Vdash^* \sigma = \tau$. Then by induction, $\pi^G \in \sigma^G \longleftrightarrow \pi^G \in \tau^G$ for all $\langle \pi, r \rangle \in \sigma \cup \tau$. So $\sigma^G = \tau^G$.

$\underline{\varphi \wedge \psi}$ Clear by induction, using the fact that if P and q belong to G then for some r in G, $r \le p$ and $r \le q$.

$p, q \in G \longrightarrow \exists r \in G$ $(r \le p$ and $r \le q)$.

$\underline{\sim \varphi}$ Clear by induction, using the density of $\{p \mid p \Vdash^* \varphi$ or $p \Vdash^* \sim \varphi\}$.

$\underline{\forall x \varphi}$ $(\longrightarrow)$ Suppose that $M[G] \vDash \forall x \varphi$. As in the proof of $(\longrightarrow)$ for $\sigma = \tau$, there is a $p \in G$ such that either $p \Vdash^* \forall x \varphi$ or for some σ, $p \Vdash^* \sim \varphi(\sigma)$. By induction the latter is impossible so $p \Vdash^* \forall x \varphi$. $(\longleftarrow)$ Clear by induction.

$\dashv$

2.10 Sublemma. *The relations* $\Vdash^*$ *and* $\Vdash$ *are the same.*

Proof of Sublemma 2.10. By Sublemma 2.9, $p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n) \longrightarrow p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$. And $\sim p \Vdash^* \varphi(\sigma_1, \dots, \sigma_n) \longrightarrow q \Vdash^* \sim \varphi(\sigma_1, \dots, \sigma_n)$ for some $q \le p$ (by Sublemma 2.7(c)) $\longrightarrow \sim p \Vdash \varphi(\sigma_1, \dots, \sigma_n)$ using the countability of M to obtain a generic G, $p \in G$. $\dashv$

This completes the proof of Lemma 2.5. $\dashv$

P is *tame* if P is pretame and in addition $1^P \Vdash$ "Power Set Axiom". The latter is first-order for pretame P as pretameness yields the definability of P-forcing. By the Truth Lemma for P-forcing we get:

2.11 Theorem (Stanley [19], Friedman [5]; Tameness Theorem). *Suppose that M is countable. Then P is* ZFC *preserving iff P is tame.*

3. Examples

We next discuss the four basic examples of tame class forcings, which serve as prototypes for more complex examples, such as Jensen coding. In each of these basic examples we take the ground model to be $\langle L, \emptyset \rangle$. We shall show that these forcings *preserve cofinalities* (i.e., for any ordinal α, $\mathrm{cf}(\alpha)^L = \mathrm{cf}(\alpha)^{L[G]}$, for P-generic G) and *preserve* GCH (i.e., P-generic extensions satisfy GCH).

Easton Forcing

A condition in P is a function $p : \alpha(p) \to L$ where $\alpha(p) \in \mathrm{On}$ and $p(\alpha) = \emptyset$ unless α is infinite and regular, in which case $p(\alpha) \in 2^{<\alpha} = \{f \mid f : \beta \to 2$ for some $\beta < \alpha\}$. We also require *Easton support* which means that $\{\beta <$

$\alpha \mid p(\beta) \neq \emptyset\}$ is bounded in α for inaccessible α. For any α, and $p \in P$, $p(\leq\alpha)$ denotes $p\restriction[0,\alpha]$ and $p(>\alpha)$ denotes $p\restriction(\alpha,\alpha(p))$. Also, for $X \subseteq P$, $X(\leq\alpha) = \{p(\leq\alpha) \mid p \in X\}$ and $X(>\alpha) = \{p(>\alpha) \mid p \in X\}$.

3.1 Proposition. *P is tame and preserves both cofinalities and* GCH.

Proof. First we verify pretameness. Suppose that $p \in P$, $\langle D_i \mid i < \kappa\rangle$ is an L-definable sequence of classes predense $\leq p$ and κ is regular. Let $\langle q_i \mid i < \kappa\rangle$ list all elements of $P(\leq\kappa) = \{q(\leq\kappa) \mid q \in P\}$, using the Easton support requirement. View each $i < \kappa$ as a pair $\langle i_0, i_1\rangle$ and define $p_0 = p$; $p_{i+1} =$ least $r \leq p_i$ such that $r(\leq\kappa) = p_i(\leq\kappa)$ and $q_{i_0} \cup r(>\kappa)$ is a condition meeting some $r_i \in D_{i_1}$, if possible ($p_{i+1} = p_i$ otherwise); $p_\lambda = \bigcup\{p_i \mid i < \lambda\}$ for limit $\lambda \leq \kappa$. Then $p^* = p_\kappa \leq p$ has the property: if $r \leq p^*$ meets D_i then r extends r_j for some $j < \kappa$. Thus $d_i = \{r_j \mid r_j \in D_i\}$ is predense $\leq p^*$ for each i, proving pretameness.

To verify the remaining properties we may use:

3.2 Lemma (Product Lemma). *Suppose that $P = P_0 \times P_1$ where P_0 and P_1 are $\langle M, A\rangle$-definable.*

(a) If G_0 is P_0-generic over $\langle M, A\rangle$ and G_1 is P_1-generic over $\langle M[G_0], A, G_0\rangle$, then $G_0 \times G_1$ is P-generic over $\langle M, A\rangle$.

(b) If G is P-generic over $\langle M, A\rangle$, then $G = G_0 \times G_1$ where G_0 is P_0-generic over $\langle M, A\rangle$. If in addition P_0-forcing is definable, then G_1 is P_1-generic over $\langle M[G_0], A, G_0\rangle$.

Proof. (a) Suppose that $D \subseteq P$ is dense and $\langle M, A\rangle$-definable. Then $D_1 = \{p_1 \mid$ for some p_0 in G_0, (p_0, p_1) meets $D\}$ is $\langle M[G_0], A, G_0\rangle$-definable. We claim that it is dense on P_1: given $p_1 \in P_1$ form $D_0(p_1) = \{p_0 \mid (p_0, p_1')$ meets D for some $p_1' \leq p_1\}$. Then $D_0(p_1)$ is dense since D is, so $G_0 \cap D_0(p_1) \neq \emptyset$. Thus (p_0, p_1') meets D for some $p_0 \in G_0$ and some $p_1' \leq p_1$, and therefore p_1' is an extension of p_1 in D_1.

As D_1 is dense we can choose a $p_1 \in G_1 \cap D_1$ and so we get $(p_0, p_1) \in G_0 \times G_1$ with (p_0, p_1) meeting D. As $G_0 \times G_1$ is compatible and closed upwards (since G_0, G_1 are) we have shown that $G_0 \times G_1$ is P-generic over $\langle M, A\rangle$.

(b) Let $G_0 = \{p_0 \in P_0 \mid (p_0, p_1)$ belongs to G for some $p_1\}$, $G_1 = \{p_1 \mid (p_0, p_1)$ belongs to G for some $p_0\}$. Clearly $G \subseteq G_0 \times G_1$ and conversely if $(p_0, p_1) \in G_0 \times G_1$ then (p_0, p_1) is compatible with every element of G, and hence by the genericity of G, (p_0, p_1) belongs to G. If $D_0 \subseteq P_0$ is dense and $\langle M, A\rangle$-definable then $D = \{(p_0, p_1) \mid p_0$ belongs to $D_0\} \subseteq P$ is dense and $\langle M, A\rangle$-definable, and since G meets D we get that G_0 meets D_0. So G_0 is P_0-generic over $\langle M, A\rangle$, as compatibility and upward closure for G_0 follow from these properties for G.

Suppose that $D_1 \subseteq P_1$ is $\langle M[G_0], A, G_0\rangle$-definable and dense. Then $D = \{(p_0, p_1) \mid p_0 \Vdash p_1 \in D_1\}$ is $\langle M, A\rangle$-definable by the definability of P_0-forcing (where "$p_1 \in D_1$" is expressed using a defining formula for D_1). Also D is dense $\leq (p_0, p_1)$ provided $p_0 \Vdash D_1$ is dense. As G_0 is P_0-generic over $\langle M, A\rangle$ we can choose a $p_0 \in G_0$ so that $p_0 \Vdash D_1$ is dense, and then the genericity of G over $\langle M, A\rangle$ produces $(p_0', p_1) \in G$ such that $p_0' \Vdash \hat{p}_1 \in D_1$; then $p_1 \in G_1 \cap D_1$ and as compatibility, upward closure for G_1 are clear, we have shown that G_1 is P_1-generic over $\langle M[G_0], A, G_0\rangle$. ⊣

In the case of Easton forcing, $P \simeq P(>\kappa) \times P(\leq\kappa)$ and if G is P-generic, then $L[G] = L[G(>\kappa)][G(\leq\kappa)]$; (b) applies as $P(>\kappa)$ is pretame and hence $P(>\kappa)$-forcing is definable. As $P(>\kappa)$ is $\leq\kappa$-closed and $P(\leq\kappa)$ has cardinality κ for regular κ (by Easton support), we get the preservation of "cofinality $> \kappa$" for regular κ and hence all cofinalities are preserved. And we have that for regular κ any subset of κ in $L[G]$ belongs to $L[G(\leq\kappa)]$. As $G(\leq\kappa)$ is equivalent to a subset of κ, GCH follows at regular κ. For singular κ we get $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^{L[G(\leq\kappa^+)]}$ and hence $2^\kappa = (2^\kappa)^{L[G(\leq\kappa^+)]} = \kappa^+$. ⊣

Long Easton Forcing

We drop the Easton support requirement. For *successor* cardinals κ we still have that $P(\leq\kappa)$ has cardinality κ, $P(>\kappa)$ is $\leq\kappa$-closed, and so the previous arguments show us that P is tame, "cofinality $> \kappa$" is preserved for *successor* cardinals κ and GCH is preserved. But not all cardinals need be preserved. Recall that a cardinal κ is *Mahlo* if it is inaccessible and in addition $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary in κ.

3.3 Theorem. *If κ is Mahlo, then κ^+ is collapsed by P; otherwise κ^+ is preserved.*

Proof. Let $G = \langle G_\alpha \mid \alpha \text{ infinite, regular}\rangle$ be P-generic. For each $\alpha < \kappa$ consider $A_\alpha \subseteq \kappa$ defined by: $\beta \in A_\alpha$ iff $\alpha \in G_\beta$.

3.4 Claim. *Suppose that κ is Mahlo. Then $\{A_\alpha \mid \alpha < \kappa\} \subseteq L$ but for no $\gamma < (\kappa^+)^L$ do we have $\{A_\alpha \mid \alpha < \kappa\} \subseteq L_\gamma$.*

Proof of Claim. For any $\alpha < \kappa$ and condition p, we can extend p to q so that $\alpha < \bar\kappa < \kappa$, $\bar\kappa$ regular implies that $p(\bar\kappa)$ has length greater than α. Thus A_α is forced to belong to L.

Given $\gamma < (\kappa^+)^L$ and a condition p, define $f(\bar\kappa) = \text{length}(p(\bar\kappa))$ for regular $\bar\kappa < \kappa$. As κ is Mahlo, f has stationary domain and hence by Fodor's Theorem we may choose $\alpha < \kappa$ such that $\text{length}(p(\bar\kappa))$ is less than α for stationary many regular $\bar\kappa < \kappa$. Then p can be extended so that A_α is guaranteed to be distinct from the κ-many subsets of κ in L_γ. ⊣

Thus κ^+ is collapsed if κ is Mahlo. Conversely, if κ is not Mahlo, then choose a closed unbounded $C \subseteq \kappa$ consisting of cardinals which are not inaccessible (we may assume that κ is a limit cardinal). Suppose that $\langle D_\alpha \mid \alpha \in C\rangle$ is a definable sequence of dense classes. Given p we can successively extend $p(\geq \alpha^+)$ for α in C so that $\{q \leq p \mid q \text{ and } p \text{ agree at or above } \alpha^+$

and q belongs to $D_\alpha\}$ is predense $\leq p$. There is no difficulty in obtaining a condition at a limit stage *less than* κ precisely because conditions are trivial at limit points of C. Thus we have shown that $P({<}\kappa) \times P({>}\kappa)$ preserves κ^+ as κ-many dense classes can be *simultaneously reduced* to predense subsets of size $< \kappa$ (i.e., for any p in P and definable sequence $\langle D_\alpha \mid \alpha < \kappa\rangle$ of dense classes there are $q \leq p$ and $\langle d_\alpha \mid \alpha < \kappa\rangle$ such that $d_\alpha \subseteq D_\alpha$, $\mathrm{Card}(d_\alpha) < \kappa$ and d_α is predense $\leq q$ for each α). Finally $P \simeq P({<}\kappa) \times P({>}\kappa) \times P(\kappa)$ and $P(\kappa)$ preserves κ^+ as it has size κ. $\dashv$

The previous proof shows that full cofinality preservation is obtained if we consider *long Easton forcing at successors*, where κ-Cohen sets are added only for infinite *successor* cardinals κ. We shall consider this and other variants of long Easton forcing in the next section, on relevant forcing.

Reverse Easton Forcing

We consider the iteration defined by: $P(0) = \{\emptyset\}$, the trivial forcing; $P({\leq}\alpha)$ is the two-step iteration $P({<}\alpha) * P(\alpha)$, where $P(\alpha)$ is the trivial forcing unless α is regular, in which case $P(\alpha) = 2^{<\alpha} = \alpha$-Cohen forcing; for limit λ, $P({<}\lambda) =$ Direct Limit $\langle P({<}\alpha) \mid \alpha < \lambda\rangle$ if λ is regular and $P({<}\lambda) =$ Inverse Limit $\langle P({<}\alpha) \mid \alpha < \lambda\rangle$ if λ is singular. (Thus conditions in $P({<}\lambda)$ are trivial on a final segment of λ if λ is regular, and are unrestricted otherwise. That is, Easton supports are being used.) Let P be Direct Limit $\langle P({<}\alpha) \mid \alpha \in \mathrm{On}\rangle$.

3.5 Proposition (See Sect. 2.3 of [5]).

(a) If κ is regular, then $P({\leq}\kappa)$ has a dense suborder of size κ.

(b) For $\alpha < \beta$, $P({<}\beta) \simeq P({\leq}\alpha) * P(\alpha, \beta)$ where $P(\alpha, \beta)$ is the natural reverse Easton iteration of γ-Cohen forcings, $\alpha < \gamma < \beta$, defined in $L[G({\leq}\alpha)]$.

(c) If κ is regular, then $P({\leq}\kappa) \Vdash P(\kappa, \mathrm{On})$ is ${\leq}\kappa$-closed, where $P(\kappa, \mathrm{On})$ is the direct limit of $\langle P(\kappa, \alpha) \mid \kappa \leq \alpha \in \mathrm{On}\rangle$.

It follows that $P =$ Direct Limit $\langle P({<}\alpha) \mid \alpha \in \mathrm{On}\rangle$ is tame and preserves cofinalities and GCH.

Amenable Forcing

In amenable forcing, the generic class G is *amenable* to the ground model, in the sense that $G \cap V_\alpha$ belongs to the ground model for every ordinal α. The basic example P of amenable forcing over L consists of all $p : \alpha \to 2$, ordered by extension. P is ${\leq}\kappa$-closed for all κ and hence tame. Cofinality and GCH preservation are trivial as P adds no new sets.

4. Relevance

We now address the issues raised in the introduction by discussing when generic classes exist for L-definable forcings. For this purpose we shall recall the Silver-Solovay theory of $0^\#$.

4.1 Proposition. *There exist tame L-definable forcings P_0 and P_1 such that not both P_0 and P_1 have generics.*

Proof. For any ordinal α, let $n(\alpha)$ be the least n such that L_α is not a model of Σ_n-Replacement, if such an n exists. Let $S_0 = \{\alpha \mid n(\alpha)$ exists and is even$\}$. P_0 consists of all closed p such that $p \subseteq S_0$, ordered by $p \leq q$ iff q is an initial segment of p.

Note that S_0 is unbounded in On: Given α, let β be least such that $\beta > \alpha$ and $L_\beta \vDash \Sigma_1$-Replacement; then $n(\beta) = 2$ so $\beta \in S_0$. If $G_0 \subseteq P_0$ is P_0-generic over L then $\bigcup G_0$ is therefore a closed unbounded subclass of On included in S_0. To show that P_0 is tame, it suffices to show that it is κ^+-distributive for every L-regular κ : If $\langle D_i \mid i < \kappa \rangle$ is an L-definable sequence of classes dense in P_0 and $p \in P_0$, then choose n odd so that $\langle D_i \mid i < \kappa \rangle$ is Σ_n definable and choose $\langle \alpha_i \mid i < \kappa \rangle$ to be the first κ-many α such that L_α is Σ_n-elementary in L and $\kappa, p, x \in L_\alpha$ where x is the defining parameter for $\langle D_i \mid i < \kappa \rangle$. We can define $p \geq p_0 \geq p_1 \geq \cdots$ so that p_{i+1} meets D_i and $\bigcup p_i = \alpha_i$, using the Σ_n-elementarity of L_{α_i} in L. As $n(\alpha_i) = n + 1$ and $n + 1$ is even, we have no problem in defining p_λ to be $\bigcup\{p_i \mid i < \lambda\} \cup \{\alpha_\lambda\}$ for limit $\lambda \leq \kappa$ and we see that $q = p_\kappa \leq p$ meets each D_i.

Now define P_1 in the same way, but using $S_1 = \{\alpha \mid n(\alpha)$ is defined and odd$\}$. Then P_1 is also tame yet if G_0 and G_1 are P_0 and P_1-generic over L respectively, then $\bigcup G_0$ and $\bigcup G_1$ are disjoint closed unbounded subclasses of On. ⊣

Thus we need a criterion for choosing L-definable forcings for which we can have a generic. Our approach is to isolate a "property of transcendence" (#) such that:

(a) In tame class-generic extensions of L, (#) fails.

(b) If (#) is true in V, then there is a least inner model $L(\#)$ satisfying (#).

Then our criterion for generic class existence is: P has a generic iff it has one definable over $L(\#)$.

4.2 Definition. An amenable $\langle L, A \rangle$ is *rigid* if there is no nontrivial elementary embedding $\langle L, A \rangle \to \langle L, A \rangle$. L is *rigid* if $\langle L, \emptyset \rangle$ is rigid.

We take (#) to be: L is not rigid. First we discuss property (b) above, i.e., that there is a least inner model in which L is not rigid (if there is one at all).

4.3 Theorem (Kunen; Silver [16], Solovay [17]). *Suppose that L is not rigid. Then there is a unique closed unbounded class $I = \{i_\alpha \mid \alpha \in \mathrm{On}\}$ of L-indiscernibles whose Skolem hull is L. Moreover, I is unbounded in every uncountable cardinal and if $0^\#$ = first-order theory of $\langle L, \in, i_0, i_1, \ldots\rangle$ (where the first ω elements $i_0, i_1, \ldots$ of I are introduced as constants) then we have the following:*

(a) $0^\# \in L[I]$, I is $\Delta_1(L[0^\#])$ in the parameter $0^\#$ and I is unbounded in α whenever $L_\alpha[0^\#] \models \Sigma_1$-Replacement.

(b) $0^\#$, viewed as a real, is the unique solution to a Π^1_2 formula (i.e., a formula of the form $\forall x \exists y \psi$ where x, y vary over reals and ψ is arithmetical).

(c) If $f : I \to I$ is increasing and not the identity, then there is a unique $j : L \to L$ extending f with critical point in I, and every $j : L \to L$ is of this form.

(d) If $\langle L, A\rangle$ is amenable, then A is $\Delta_1(L[0^\#])$, $\langle L, A\rangle$ is not rigid and a final segment of I is a class of $\langle L, A\rangle$-indiscernibles.

4.4 Remark.

(i) As I is closed and unbounded in every uncountable cardinal, it follows that every uncountable cardinal belongs to I and $0^\#$ = first-order theory of $\langle L, \in, \omega_1, \omega_2, \ldots\rangle$.

(ii) The Σ^1_2-absoluteness of L ([15]) implies that the unique solution to a Σ^1_2 formula is constructible; so in a sense (b) is best possible.

(iii) I is a class of *strong* indiscernibles: if $\vec{i}, \vec{j}$ are increasing tuples from I of the same length and $x < \min(\vec{i}), \min(\vec{j})$, then for any φ, $L \models \varphi(x, \vec{i}) \longleftrightarrow \varphi(x, \vec{j})$.

In case the conclusion of Theorem 4.3 holds (i.e. in case L is not rigid) we say that "$0^\#$ exists" and refer to I as the *Silver indiscernibles.* Note that Theorem 4.3 implies that if L is not rigid, then $L[0^\#]$ is the smallest inner model in which L is not rigid, verifying that "L is not rigid" obeys condition (b) of our property of transcendence (#).

Before turning to condition (a) of property (#) we mention Jensen's Covering Theorem and some of its consequences. A set X is *covered in L* if there is a constructible Y such that $X \subseteq Y$, $\mathrm{Card}(Y) = \mathrm{Card}(X)$.

4.5 Theorem (Jensen [4]). *Suppose that there exists an uncountable set of ordinals which is not covered in L. Then $0^\#$ exists.*

For proofs of Theorems 4.3, 4.5 see [5, Sect. 3.1].

Using the Covering Theorem, we see that the existence of $0^\#$ takes many equivalent forms.

4.6 Theorem. *Each of the following is equivalent to the existence of* $0^\#$*:*

(a) L *is not rigid.*

(b) $\langle L, A\rangle$ *is not rigid whenever* $\langle L, A\rangle$ *is amenable (i.e., whenever* $A \subseteq L$ *and* $A \cap L_\alpha \in L$ *for every ordinal* α*).*

(c) Some uncountable set of ordinals is not covered in L*.*

(d) Some singular cardinal is regular in L*.*

(e) $\kappa^+ \neq (\kappa^+)^L$ *for some singular cardinal* κ*.*

(f) Every constructible subset of ω_1 *either contains or is disjoint from a closed, unbounded subset of* ω_1*.*

(g) $\{\alpha \mid \alpha$ *is an* L*-cardinal*$\}$ *is* Δ_1*-definable with parameters.*

(h) There exists a $j : L_\alpha \to L_\beta$, $\mathrm{crit}(j) = \kappa, \kappa^+ \leq \alpha$.

(i) There exists a $j : L_\alpha \to L_\beta$, $\mathrm{crit}(j) = \kappa, (\kappa^+)^L \leq \alpha$, $\alpha \geq \omega_2$.

Proof. It is straightforward to show that these all follow from the existence of $0^\#$; using Theorem 4.3. Also (a), (b) imply the existence of $0^\#$ by Theorem 4.3. Conditions (d), (e) each easily imply (c), and we get $0^\#$ from (c) via Theorem 4.5. Condition (f) implies (a), since we get an elementary embedding $L \to L \simeq \mathrm{Ult}(L, U)$ = ultrapower of L by U, where U consists of all constructible subsets of ω_1 containing a closed unbounded subset. (g) implies that $(\kappa^+)^L < \kappa^+$ for κ a sufficiently large cardinal; by taking κ singular we get $0^\#$ via condition (e). To see that (h) implies the existence of $0^\#$, define an ultrafilter U on constructible subsets of κ by: $X \in U$ iff $\kappa \in j(X)$. Then $\mathrm{Ult}(L, U)$ is well-founded, for if not then by Löwenheim-Skolem there would be an infinite descending chain in $\mathrm{Ult}(L_{\kappa^+}, U)$ which contradicts $\kappa^+ \leq \alpha$.

Finally we show that (i) implies the existence of $0^\#$. Define U as before by: $X \in U$ iff $\kappa \in j(X)$. First suppose that κ is at least ω_2. We shall argue that U is *countably complete*, i.e. that if $\{X_n \mid n \in \omega\} \subseteq U$, then $\bigcap\{X_n \mid n \in \omega\} \neq \emptyset$. (This gives $0^\#$ as it implies that $\mathrm{Ult}(L, U)$ is well-founded.) By the Covering Theorem 4.5, there is an $F \in L$ of cardinality ω_1 such that $X_n \in F$ for each n. Then as we have assumed that $\kappa \geq \omega_2$, F has L-cardinality less than κ. We may assume that F is a subset of $\mathcal{P}(\kappa) \cap L$, and hence as α is an L-cardinal, F belongs to L_α and there is a bijection $h : F \longleftrightarrow \gamma$ for some $\gamma < \kappa$, $h \in L_\alpha$. But then $F^* = \{X \in F \mid \kappa \in j(X)\}$ belongs to L_α as $X \in F^* \longleftrightarrow \kappa \in j(h^{-1})(h(X))$ and F^* has nonempty intersection as $j(F^*) = \mathrm{ran}(j \restriction F^*)$ and $\kappa \in \bigcap j(F^*)$. Thus $\{X_n \mid n \in \omega\}$ has nonempty intersection since it is a subset of F^*. If κ is less than ω_2, then we have $\alpha \geq \omega_2 \geq \kappa^+$ so we have a special case of (h). ⊣

The next theorem verifies (a) of transcendence property (#).

4.7 Theorem (Beller [1], Friedman [5]). *Suppose that G is P-generic over $\langle L, A\rangle$ and P is tame. Then $L[G] \models 0^\#$ does not exist.*

Proof. Suppose that $p_0 \in P$, $p_0 \Vdash$ "I, the class of Silver indiscernibles, is a closed unbounded class satisfying $i < j$ in $I \rightarrow L_i \prec L_j$". Suppose that $p \leq p_0$ and $p \Vdash \alpha \in I$. Then $L_\alpha \prec L$ in any P-generic extension $\langle L[G], A, G\rangle$ with $p \in G$. (By Löwenheim-Skolem we can assume that such a G exists for the sake of this argument.) Thus, an L-Satisfaction predicate is definable over $\langle L, A\rangle$ as $L \models \varphi(x)$ iff for some $p \in P$ below p_0, some α with $x \in L_\alpha$, $p \Vdash \varphi(x)$ is true in L_α. This is a contradiction if $A = \emptyset$, for then L-satisfaction would be L-definable. But note that for any A such that $\langle L, A\rangle$ is amenable we can apply the same argument to get the $\langle L, A\rangle$-definability of $\langle L, A\rangle$-satisfaction, using the fact that by Theorem 4.3(d), $\langle L_\alpha, A \cap L_\alpha\rangle \prec \langle L, A\rangle$ for α in a final segment of I. ⊣

4.8 Definition. A forcing P defined over a ground model $\langle L, A\rangle$ is *relevant* if there is a G P-generic over $\langle L, A\rangle$ which is definable (with parameters) over $L[0^\#]$.

Examples of Relevance

Assume that $0^\#$ exists. Then any $L[0^\#]$-*countable* $P \in L$ is relevant, as there are only countably many constructible subsets of P (using the fact that $\omega_1^{L[0^\#]}$ is inaccessible in L). Note that this includes the case of any forcing $P \in L$ definable in L without parameters.

The situation is far less clear for uncountable $P \in L$. The next result treats the case of κ-Cohen forcing.

4.9 Proposition. *Suppose that κ is L-regular and let $P(\kappa)$ denote κ-Cohen forcing in L: conditions are constructible $p : \alpha \rightarrow 2$, $\alpha < \kappa$ and $p \leq q$ iff p extends q.*

(a) If κ has cofinality ω in $L[0^\#]$, then $P(\kappa)$ is relevant.

(b) If κ has uncountable cofinality in $L[0^\#]$, then $P(\kappa)$ is not relevant.

Proof. Let j_n denote the first n Silver indiscernibles $\geq \kappa$.

(a) We use the fact that $P(\kappa)$ is κ-distributive in L. Let $\kappa_0 < \kappa_1 < \cdots$ be an ω-sequence in $L[0^\#]$ cofinal in κ. Then any $D \subseteq P(\kappa)$ in L belongs to $\mathrm{Hull}(\kappa_n \cup j_n)$ for some n, where Hull denotes Skolem hull in L. As $\mathrm{Hull}(\kappa_n \cup j_n)$ is constructible of L-cardinality $< \kappa$ we can use the κ-distributivity of $P(\kappa)$ to choose $p_0 \geq p_1 \geq \cdots$ successively below any $p \in P(\kappa)$ to meet all dense $D \subseteq P(\kappa)$ in L.

(b) Note that in this case $\kappa \in \mathrm{Lim}\, I$, as otherwise $\kappa = \bigcup\{\kappa_n \mid n \in \omega\}$ where $\kappa_n = \bigcup(\kappa \cap \mathrm{Hull}(\bar\kappa + 1 \cup j_n)) < \kappa$, $\bar\kappa = \max(I \cap \kappa)$, and hence κ has $L[0^\#]$-cofinality ω. Suppose that $G \subseteq P(\kappa)$ were $P(\kappa)$-generic over L. For any $p \in P(\kappa)$ let $\alpha(p)$ denote the domain of p. Define $p_0 \geq p_1 \geq \cdots$ in G so that $\alpha(p_{n+1}) \in I$ and p_{n+1} meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha(p_n) \cup j_n)$.

Then $p = \bigcup\{p_n \mid n \in \omega\}$ meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha \cup j)$ where $\alpha = \bigcup\{\alpha(p_n) \mid n \in \omega\} \in I$ and $j = \bigcup\{j_n \mid n \in \omega\}$. But then p is $P(\alpha)$-generic over L, as every constructible dense $\bar{D} \subseteq P(\alpha)$ is of the form $D \cap P(\alpha)$ for some D as above. So p is not constructible, contradicting $p \in G$. $\dashv$

As a consequence of Proposition 4.9(b) we see that the basic class forcing examples of Easton and long Easton forcing are not relevant. However, we can recover relevance for these forcings by restricting to successor cardinals, thereby not adding κ-Cohen sets for κ of uncountable $L[0^\#]$-cofinality. *Easton forcing at successors* is defined as follows: Conditions are constructible $p : \alpha(p) \to L$ where $p(\alpha) = \emptyset$ unless α is a successor cardinal of L, in which case $p(\alpha) \in \alpha$-Cohen forcing; we also require that if α is L-inaccessible then $\{\beta < \alpha \mid p(\beta) \neq \emptyset\}$ is bounded in α and define $p \leq q$ iff $p(\alpha)$ extends $q(\alpha)$ for each $\alpha < \alpha(q)$.

4.10 Theorem. *Let P be Easton forcing at successors. Then P is relevant.*

Proof. By recursion on $i \in I$, the class of Silver indiscernibles, we define $G(<i)$ to be $P(<i)$-generic over L, where $P(<i)$ is Easton forcing at successors restricted to L_i. For $i = \min I$ take $G(<i)$ to be any $P(<i)$-generic (note that $P(<i)$ is countable in $L[0^\#]$). If $G(<i)$ has been defined, we now define $G(<i^*)$ as follows (where $i < i^*$ are adjacent in I) : $P(<i^*)$ factors as $P(<i) \times P(i, i^*)$ where $P(i, i^*)$ is i^+-closed in L, so it suffices to define a $P(i, i^*)$-generic $G(i, i^*)$, for then $G(<i^*) = G(<i) \times G(i, i^*)$ is $P(<i^*)$-generic. To obtain $G(i, i^*)$, successively choose $p_0 \geq p_1 \geq \cdots$ in $G(i, i^*)$ so that p_{n+1} meets all dense $D \subseteq P(i, i^*)$ in $\mathrm{Hull}(i \cup j_n)$ where $j_n =$ first n Silver indiscernibles $\geq i$. Then set $G(i, i^*) = \{p \mid p \geq p_n \text{ for some } n\}$.

Finally if $i \in \mathrm{Lim}\, I$, let $G(<i) = \bigcup\{G(<j) \mid j \in I \cap i\}$. Note that if $D \subseteq P(<i)$ is dense and constructible then for some $j \in I \cap i$, $D \cap P(<j)$ is dense and constructible and hence is met by $G(<j) \subseteq G(<i)$. So $G(<i)$ is $P(<i)$-generic. Similarly, $G = \bigcup\{G(<i) \mid i \in I\}$ is P-generic over L (and in fact meets all L-amenable dense $D \subseteq P$). $\dashv$

Reverse Easton forcing is relevant, without restriction.

4.11 Theorem. *Let P be the basic example of reverse Easton forcing defined in the last section. Then P is relevant.*

Proof. Recall that $P(<\alpha)$ has a dense subset of L-cardinality $\leq (\alpha^+)^L$ for each α. By recursion on $i \in I$ we define $G(\leq i) = G(<i) * G(i)$ to be $P(\leq i)$-generic over L, where $P(\leq i) = P(<i) * P(i)$, the first $i + 1$ stages in the iteration defining P. We will have that $i \leq j$ in I implies $G(j)$ extends $G(i)$; this will enable us to get through limit stages. For $i = \min I$, take $G(\leq i)$ to be any $P(\leq i)$-generic in $L[0^\#]$. If $G(\leq i)$ has been defined and $i^* = I$-successor to i, then write $P(<i^*)$ as $P(\leq i) * P[i+1, i^*)$ and as $P(\leq i) \Vdash P[i+1, i^*)$ is i^+-closed we can select $G[i+1, i^*)$ to be $P[i+1, i^*)^{G(\leq i)}$-generic over $L[G(\leq i)]$ (the collection of dense sets that must be met is the countable union of

subcollections of size i in $L[G(\leq i)]$, using the Hull$(i \cup j_n)$'s as in the previous proof). Then $G(<i^*) = G(\leq i) * G[i+1, i^*)$ is $P(<i^*)$-generic over L. We also choose $G(i^*)$ to be $P(i^*)^{G(<i^*)}$-generic over $L[G(< i^*)]$, extending the condition $G(i)$ in this forcing.

For $i \in \operatorname{Lim} I$ take $G(<i)$ to be $\bigcup\{G(<j) \mid j \in I \cap i\}$; as in the previous proof $G(<i)$ is $P(<i)$-generic over L. And we take $G(i) = \bigcup\{G(j) \mid j \in I \cap i\}$, which by our construction extends each $G(j)$ for $j \in I \cap i$. Again we get genericity for $G(\leq i)$ from that of $G(\leq j)$ for $j \in I \cap i$, as $G(<i)$ and $G(i)$ extend $G(<j)$ and $G(j)$ respectively for each $j \in I \cap i$. $\dashv$

Before turning to *long Easton forcing at successors* (obtained from Easton forcing at successors by dropping the support condition that $\{\beta < \alpha \mid p(\beta) \neq \emptyset\}$ be bounded in α for L-inaccessible α), we establish the relevance of *thin Easton forcing at successors*. The latter is obtained by weakening the support condition in Easton forcing at successors to: $\{\beta < \alpha \mid p(\beta^+) \neq \emptyset\}$ is nonstationary in α for L-inaccessible α.

4.12 Theorem. *Let P be thin Easton forcing at successors. Then P is relevant.*

Proof. This proof uses, as do later proofs, the notion of *reduction of dense sets*. For any L-cardinal δ, P can be factored as $P(\leq\delta) \times P(>\delta)$. If α is an L-cardinal greater than δ and $D \subseteq P(\leq\alpha)$ is open dense, then we say that $p \in P(\leq\alpha)$ *reduces D below δ* if for some successor L-cardinal $\bar{\delta} \leq \delta$, any extension q of p can be extended into D without changing q above $\bar{\delta}$. (In case δ is itself a successor L-cardinal, then we can of course take $\bar{\delta}$ to be δ itself.)

Now let i be any indiscernible and for any n let j_n be the first n indiscernibles $\geq i$. We can define $p_0^i \geq p_1^i \geq \cdots$ in $P(\leq i^+)$ such that if $D \subseteq P(\leq i^+)$ is open dense and belongs to Hull$(\gamma^+ \cup j_n)$ then p_{n+1}^i reduces D below γ^+ for any L-cardinal $\gamma < i$. This is possible by successively extending on $[\gamma^{++}, i^+]$ (without violating the nonstationary support requirement). Let $G_0^i = \{p \in P(\leq i^+) \mid p \geq p_n^i \text{ for some } n\}$.

G_0^i is *not* $P(\leq i^+)$-generic over L as $p \in G_0^i \to p(j^+) = \emptyset$ for all $j \in I \cap i$. Notice that for $i_0 < i_1 < \cdots < i_n \leq i$ in I, $G_0^{i_0} \cup \cdots \cup G_0^{i_n}$ is a compatible set of conditions. We take $G(\leq i^+) = \{p \in P(\leq i^+) \mid p \geq q_0 \wedge \cdots \wedge q_n$ for some $q_l \in G_0^{i_l}$, where $i_0 < \cdots < i_n \leq i$ in $I\}$. Now we claim that $G(\leq i^+)$ is $P(\leq i^+)$- generic over L. Indeed if $D \subseteq P(\leq i^+)$ is dense and belongs to Hull$(\{k_0, \ldots, k_m\} \cup j_n)$ with $k_0 < \cdots < k_m < i$ in I, then p_{n+1}^i reduces D below k_m^+, $p_{n+1}^i \wedge p_{n+2}^{k_m}$ reduces D below $k_{m-1}^+, \ldots$ and eventually we get $p_{n+1}^i \wedge p_{n+2}^{k_m} \wedge \cdots \wedge p_{n+m+2}^{k_0}$ in $G(\leq i^+)$ meeting D.

Now note that in the above we could have chosen our initial $p_0^i \in P(\leq i^+)$ to reduce every dense $D \subseteq P \cap L_i$ in Hull$(\gamma^+ \cup \{i\})$ below γ^+, for any $\gamma < i$. Thus the resulting generic $G(\leq i^+)$ meets every dense $D \subseteq P \cap L_i$ definable over L_i. Now let $G = \bigcup\{G(\leq i^+) \mid i \in I\}$ and we see that G is P-generic over L. $\dashv$

In the above proof we use thin supports to guarantee that for $i < j$ in I, the "pre-generics" G_0^i and G_0^j agree at i^+ (indeed they equal $\emptyset$ at i^+). A less severe restriction is to require coherence on a closed unbounded set:

4.13 Definition. Let P denote long Easton forcing at successors and suppose that p belongs to $P(\leq\kappa^+)$, where κ is L-regular. For $\xi \in [\kappa, \kappa^+)$ let f_ξ be the L-least 1-1 function from κ onto ξ. For $s \in P(\kappa^+) = \kappa^+$-Cohen forcing and $\alpha < \kappa$ define s_α as follows: If $\xi = \text{length}(s) \leq \kappa$ or $\alpha \neq \kappa \cap f_\xi[\alpha]$, then $s_\alpha = \emptyset$. Otherwise s_α has domain $[\alpha, \bar{\xi})$ where $\bar{\xi} = \text{ordertype}\, f_\xi[\alpha]$ and $s_\alpha(\delta) = s(f_\xi(\delta))$. We say that p is *coherent* at κ if $p(\kappa^+)_\alpha$ and $p(\alpha^+)$ are compatible for closed unboundedly many $\alpha < \kappa$. A condition p in P is *coherent* if for each L-inaccessible κ in the domain of p, p is coherent at κ. Coherent Easton forcing at successors is the forcing whose conditions are the coherent conditions in long Easton forcing at successors.

4.14 Theorem. *Let P be coherent Easton forcing at successors. Then P is relevant.*

Proof. Follow the proof of the previous Theorem. The only new observation is that by virtue of coherence at indiscernibles, we again have the compatibility of G_0^i and G_0^j for $i < j$ in I. ⊣

4.15 Remark. Thin Easton forcing at successors and coherent Easton forcing at successors serve as prototypes for Jensen coding, introduced in the next section. In Jensen coding, conditions are sequences of pairs (p_α, p_α^*) where coherence is used on the "coding strings" p_α and thinness is used on the "restraints" p_α^*.

Finally we turn to long Easton forcing at successors.

4.16 Theorem. *Let P be long Easton forcing at successors. Then P is relevant.*

Proof. Suppose that p belongs to P and i is a Silver indiscernible. We say that p is *coherent at i* if $p(i^+)$ and $\pi(p)(i^+)$ are compatible, where $\pi : L \to L$ is an elementary embedding with critical point i. Equivalently: $p(i^+)_\alpha$ and $p(\alpha^+)$ are compatible for all α in a set X belonging to the "L-ultrafilter" derived from the embedding π (cf. the proof of Theorem 4.6). It suffices to show that if p belongs to $P(\leq i^+)$ and is coherent at indiscernibles $\leq i$ and $D \subseteq P(\leq i^+)$ where $D \in L$ is L-definable from indiscernibles $\geq i$, then p has an extension meeting D which is coherent at indiscernibles $\leq i$. For then, we can repeat the proof of Theorem 4.12, using conditions which are coherent at indiscernibles $\leq i$ to construct G_0^i, and therefore again obtain the compatibility of G_0^i and G_0^j for $i < j$ in I.

Given p and D as above, recursively extend $p(\alpha^+)$ for $\alpha < i$ an L-limit cardinal to $q(\alpha^+)$ as follows: if $q{\restriction}\alpha$ has been defined, then let $q(\alpha^+)$ be least so that for some least $r_\alpha \in P(< \alpha)$, $r_\alpha \cup \{q(\alpha^+)\}$ extends $q{\restriction}\alpha \cup \{p(\alpha^+)\}$ and meets D. Now choose X in the ultrafilter derived from π (X containing all indiscernibles $< i$) such that the r_α cohere for α in X to a condition r in $P(<i)$. Also define $r(i^+)$ to be $\pi(r)(i^+)$. Then r extends p, is coherent at indiscernibles $\leq i$, and meets D. $\dashv$

Indiscernible Preservation

Though we have shown a number of variants of Easton forcing to be relevant, we can ask for more, namely that our generic classes preserve indiscernibles. This will be important in the next section, where Jensen coding is introduced, as we can only code a class by a real (in $L[0^\#]$) if the class preserves (a periodic subclass of) the class I of Silver indiscernibles.

4.17 Definition. A class $A \subseteq L$ *preserves indiscernibles* if I is a class of indiscernibles for the structure $\langle L[A], A\rangle$.

4.18 Theorem. *For each of Easton at successors, reverse Easton, thin Easton at successors, coherent Easton at successors and long Easton at successors there is a generic class G that preserves indiscernibles.*

Proof. The generic classes built earlier for thin Easton at successors, coherent Easton at successors, and long Easton at successors preserve indiscernibles. We now treat the case of reverse Easton forcing. It suffices, for i_ω the ωth indiscernible, to build an $H \subseteq L_{i_\omega}$ which is $P(<i_\omega)$-generic over L_{i_ω} and such that $t(j_1, \dots, j_n) \in H$ iff $t(j'_1 \dots j'_n) \in H$ whenever $j_1 < \dots < j_n$ and $j'_1 < \dots < j'_n$ belong to $I \cap i_\omega$. For then define G by: $t(k_1, \dots, k_n) \in G$ for $k_1 < \dots < k_n$ in I iff $t(i_1, \dots, i_n) \in H$ for $i_1 < \dots < i_n$ the first n indiscernibles. This is well-defined using the above property of H. And G is P-generic over L: it suffices to consider predense $D \in L$ as P has the ∞-chain condition. Write $D \in L$ as $s(l_1, \dots, l_m)$ where $l_1 < \dots < l_m$ in I; then $\overline{D} = s(i_1, \dots, i_m)$ is predense on $P(<i_\omega)$. If $\bar{p} = t(i_1, \dots, i_n) \in H$ meets $\overline{D}$, then $p = t(l_1, \dots, l_m, l_{m+1}, \dots, l_n)$ meets D, where $l_m < l_{m+1} < \dots < l_n$ belong to I. Also $p \in G$ by definition of G. Finally, note that if $k_1 < \dots < k_m < l_1 < \dots < l_m$, $k_1, \dots, k_m$ in I and $l_1, \dots, l_m$ are limit members of I, then for any φ, $\langle L[G], G\rangle \vDash \varphi(k_1, \dots, k_m) \longleftrightarrow \varphi(l_1, \dots, l_m)$ by the Truth Lemma and the fact that G obeys the same invariance property that characterized H. So I is a class of indiscernibles for $\langle L[G], G\rangle$.

Now we build H. Let $H_2 \subseteq P(<i_2)$ be a $P(<i_2)$-generic in $L[0^\#]$ and $H_1 = H_2 \cap P(<i_1)$. We must now define $H_3 \subseteq P(<i_3)$ to be $P(<i_3)$-generic so that $t(i_1, \vec{j}\,) \in H_2$ iff $t(i_2, \vec{j}\,) \in H_3$, where $\vec{j}$ is an increasing sequence from $I - i_\omega$. Note that $H_2(i_1)$, a subset of i_1 generic over $L[H_1]$, is a condition in the i_2-Cohen forcing defined over $L[H_2]$; choose $H_3(i_2)$ to be a generic for this forcing extending $H_2(i_1)$. Now note that for each n there is a $t_n(i_1, \vec{j}_{n^*}) = p_n \in H_2$ which reduces all predense $D \subseteq P(<i_2)$ in $\mathrm{Hull}(i_1 \cup \{i_1, k_1, \dots, k_n\})$ below i_1, where $i_\omega \leq k_1 < \dots < k_n$ belong to I, using the i_1^+-distributivity of $P(>i_1)^{H_2(\leq i_1)}$ in $L[H_2(\leq i_1)]$. So if we define $H'_3 = \{t_n(i_2, \vec{j}_n) \mid n \in \omega\}$ we have that H'_3 reduces all predense $D \subseteq P(<i_3)$ with $D \in L$ below i_2. So, the desired H_3 can be defined by $H_3 = \{p \in P(<i_3) \mid p(\leq i_2) \in H_3(\leq i_2)$

and p compatible with $H_3'\}$. By construction, $t(i_1, \vec{j}\,) \in H_2$ iff $t(i_2, \vec{j}\,) \in H_3$. Note that H_3 was uniquely determined by this last condition, once a choice of $H_3(i_2)$ was made.

H_4 is uniquely determined by $P({<}i_4)$-genericity and the condition that $t(i_1, i_2, \vec{j}\,)$ belongs to H_3 iff $t(i_2, i_3, \vec{j}\,)$ belongs to H_4, as the forcing to add $H_3(i_2)$ is i_1^+-distributive (and the forcing to add $H_3({>}i_2)$ is i_2^+-distributive). We must check that $t(i_1, i_3, \vec{j}\,) \in H_4$ iff $t(i_2, i_3, \vec{j}\,) \in H_4$. Now any condition in H_4 is extended by one of the form $p = (p_0, p_1)$ where $p_0 \in H_4({\le}i_3)$ and $p_1 = t(i_3, \vec{j}\,)$, as such p reduce all dense $D \subseteq P({<}i_4)$ with $D \in L$ below i_3. So, it suffices to show that $t(i_1, i_3, \vec{j}\,) \in H_4({\le}i_3)$ iff $t(i_2, i_3, \vec{j}\,) \in H_4({\le}i_3)$. By definition of H_4 we have $t(i_2, i_3, \vec{j}\,) \in H_4({\le}i_3)$ iff $t(i_1, i_2, \vec{j}\,) \in H_3({\le}i_2)$. But the latter implies that $t(i_1, i_2, \vec{j}\,) = t(i_1, i_3, \vec{j}\,)$, and as $H_3({\le}i_2)$ extends $H_2({\le}i_1)$, we have that $H_4({\le}i_3)$ extends $H_3({\le}i_2)$. So $t(i_1, i_2, \vec{j}\,) \in H_3({\le}i_2)$ iff $t(i_1, i_2, \vec{j}\,) \in H_4({\le}i_3)$ iff $t(i_1, i_3, \vec{j}\,) \in H_4({\le}i_3)$.

In general, define H_{m+3} by the condition that $t(i_m, i_{m+1}, \vec{j}\,)$ belong to H_{m+2} iff $t(i_{m+1}, i_{m+2}, \vec{j}\,)$ belongs to H_{m+3}. As above we get that H_{m+3} is $P({<}i_{m+3})$-generic and $t(i_1, \dots, i_{m+1}, \vec{j}\,) \in H_{m+2}$ iff $t(i_1, \dots, i_m, i_{m+2}, \vec{j}\,) \in H_{m+3}$. Finally let $H = \bigcup\{H_m \mid m \in \omega\}$. Then H is $P({<}i_\omega)$-generic over L and for any $k_1 < \dots < k_{l+2} < \vec{j}$ in $I, k_{l+2} < i_\omega \le \vec{j}$ we have $t(k_1, \dots, k_{l+1}, \vec{j}\,) \in H$ iff $t(k_1, \dots, k_l, k_{l+2}, \vec{j}\,) \in H$. This is enough to imply that $t(\vec{k}_0) \in H$ iff $t(\vec{k}_1) \in H$ whenever $\vec{k}_0$ and $\vec{k}_1$ are increasing sequences from $I \cap i_\omega$. This completes the proof in the case of reverse Easton forcing.

Easton forcing at successors can be handled in the same way without need to consider $H(i)$ for $i \in I$, as $H(\alpha)$ is nontrivial only when α is a successor L-cardinal. (Indeed, without the latter restriction the construction fails as there is no available choice for $H(i_2)$.) $\dashv$

5. The Coding Theorem

Class forcing became an important tool in set theory as a result of the following theorem of Jensen (see [1]):

5.1 Theorem (Coding Theorem). *Suppose that $\langle M, A\rangle$ is a ground model. Then there is an $\langle M, A\rangle$-definable class forcing P such that if $G \subseteq P$ is P-generic over $\langle M, A\rangle$, then:*

(a) $\langle M[G], A, G\rangle \vDash \mathrm{ZFC}$.

(b) For some $R \subseteq \omega$, $M[G] \vDash V = L[R]$, and $\langle M[G], A, G\rangle \vDash A, G$ are definable from the parameter R.

Before discussing the proof of this theorem, we mention the following corollary, which constitutes a partial positive solution to Solovay's genericity problem (for set-genericity):

5.2 Corollary. *There is an L-definable class forcing for producing a real R which is not set-generic over L.*

Proof. Let P_0 be Easton forcing, G_0 P_0-generic over L, and $P_0 * P_1 = P$ a two-step iteration where P_1 adds a real R as in Theorem 5.1 such that G_0 is definable over $L[R]$. Then in $L[R]$ there are κ-Cohen sets for every L-regular κ. Thus R is not set-generic over L as no forcing of size κ can add a κ^+-Cohen set. $\dashv$

In fact R as in Corollary 5.2 can be chosen to satisfy $R <_L 0^\#$, but this property makes use of the relevance of Jensen coding, a topic to be discussed later.

The proof of the Coding Theorem is far easier if one makes the further assumption that $0^\# \notin M$. Indeed, with this extra hypothesis there is a proof, which we provide below, making no use of Jensen's fine structure theory. Instead one uses the following consequence of Jensen's Covering Theorem (Theorem 4.5), expressed by Theorem 4.6(i):

5.3 Proposition. *Suppose that $0^\#$ does not exist, and $j : L_\alpha \to L_\beta$ is Σ_1-elementary with $\alpha \geq \omega_2$. If $\kappa = \mathrm{crit}(j)$ then $\alpha < (\kappa^+)^L$.*

We now give a brief introduction to the coding proof, assuming $0^\# \notin M$. We may assume that $M \vDash \mathrm{GCH}$, as this can be easily arranged by a preliminary class forcing. Moreover, we need not code into a real R; it suffices to code into a *reshaped* subset of ω_1:

5.4 Definition. $b \subseteq \omega_1$ is *reshaped* if for any $\xi < \omega_1$, ξ is countable in $L[b \cap \xi]$.

The following result of [9] provides one of the key ideas in the proof.

5.5 Proposition. *Suppose that $V = L[b]$, b a reshaped subset of ω_1. Then there is a c.c.c. forcing R^b for adding a real R such that $b \in L[R]$.*

Proof. Using the fact that b is reshaped we may define $\langle R_\xi \mid \xi < \omega_1\rangle$ by: $R_\xi = L[b \cap \xi]$-least real distinct from the $R_{\xi'}$ for $\xi' < \xi$. Separate the R_ξ's by setting $R^*_\xi = \{n \mid n \text{ codes a finite initial segment of } R_\xi\}$.

A condition in R^b is $p = (s(p), s^*(p))$ where $s(p)$ is a finite subset of ω, $s^*(p)$ is a finite subset of b. Extension is defined by: $p \leq q$ iff $s(p) \supseteq s(q)$, $s^*(p) \supseteq s^*(q)$ and $\xi \in s^*(q)$ implies $s(p) - s(q)$ is disjoint from R^*_ξ. This is c.c.c. as $s(p) = s(q)$ implies p and q are compatible. If G is R^b-generic, then let $R = \bigcup\{s(p) \mid p \in G\}$. We get:

$$\xi \in b \longleftrightarrow R \cap R^*_\xi \quad \text{finite.}$$

Thus, given R we can test "$\xi \in b$" if we know R_ξ; as R_ξ is computable in $L[b \cap \xi]$ this gives an inductive calculation of $b \cap \xi$ from R. $\dashv$

There is a perfectly analogous notion of *reshaped subset of* κ^+ for any infinite cardinal κ and if κ is an infinite *successor* cardinal, an analogous forcing R^b for b a reshaped subset of κ^+.

Now we do not necessarily have reshaped sets in our ground model; instead we must force them. A *reshaped string at* κ is a function $s : \alpha \to 2$ for some $\alpha < \kappa^+$ such that $\xi \leq \alpha \to L[s{\restriction}\xi] \vDash \mathrm{Card}(\xi) \leq \kappa$. Reshaped strings at κ of arbitrary length $\alpha < \kappa^+$ do exist and serve to approximate the desired reshaped subsets of κ^+.

We now give a rough description of the forcing conditions. P consists of sequences $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in \mathrm{Card}, \alpha \leq \alpha(p)\rangle$ where $\alpha(p) \in \mathrm{Card}$ and:

(a) $p_{\alpha(p)}$ is a reshaped string at $\alpha(p)$ and $p^*_{\alpha(p)} = \emptyset$.

(b) For $\alpha \in \mathrm{Card} \cap \alpha(p)$, $(p_\alpha, p^*_\alpha) \in R^{p_{\alpha^+}}$, the forcing for coding p_{α^+}, $A \cap \alpha^+$ by a subset of α^+ using reshaped strings at α.

(c) For α a limit cardinal $\leq \alpha(p)$, $p{\restriction}\alpha$ "exactly codes" p_α.

(d) For α inaccessible $\leq \alpha(p)$ there is a closed unbounded $C \subseteq \alpha$ such that $\beta \in C$ implies $p^*_\beta = \emptyset$.

Clause (d) is over-simplified in that "inaccessible" should really be (something like) "$L[p_\alpha] \vDash \alpha$ is inaccessible" and C should be required to belong to (something like) $L[p_\alpha]$. Clause (c) refers to the limit coding, as yet undefined. The key idea that enables one to carry out a fine structure-free proof of the Coding Theorem (assuming $0^\#$ does not exist) is the use of *coding delays* in the limit coding. The details are supplied in the proof below.

The two main properties of P that must be demonstrated are:

(Extendibility) Suppose that $p \in P$ and $f : \alpha \to \alpha$ with $f(\beta) < \beta^+$ for successor cardinals $\beta < \alpha$. Then there exists a $q \leq p$ with length $q_\beta \geq f(\beta)$ for each successor cardinal $\beta < \alpha$.

(Distributivity) Suppose that D_i is i^+*-dense on* P for each $i < \alpha$, i.e. for all p there is a $q \leq p$ with $q \in D_i$ satisfying $(q_\beta, q^*_\beta) = (p_\beta, p^*_\beta)$ for all $\beta \leq i$. Then for all p there is $q \leq p$, q meets each D_i.

Proposition 5.3 is used to facilitate the proof of Distributivity. Extendibility is not difficult, taking advantage of the coding delays.

Proof of Theorem 5.1, assuming $0^\# \notin M$. We make the following assumption about the predicate A: If H_α, with α an infinite $L[A]$-cardinal, denotes $\{x \in L[A] \mid \text{transitive closure } (x) \text{ has } L[A]\text{-cardinality} < \alpha\}$ then $H_\alpha = L_\alpha[A]$. This is easily arranged using the fact that GCH holds in $L[A]$.

Let Card denote all infinite $L[A]$-cardinals. Also $\mathrm{Card}^+ = \{\alpha^+ \mid \alpha \in \mathrm{Card}\}$ and $\mathrm{Card}' =$ all uncountable limit cardinals.

Let α belong to Card.

5.6 Definition (Strings). S_α consists of all $s : [\alpha, |s|) \longrightarrow 2$, $\alpha \leq |s| < \alpha^+$ such that $|s|$ is a multiple of α and for all $\eta \leq |s|$, $L_\delta[A \cap \alpha, s \restriction \eta] \vDash \mathrm{Card}(\eta) \leq \alpha$ for some $\delta < (\eta^+)^L \cup \omega_2$.

Thus for $\alpha = \omega$ or ω_1, elements of S_α are "reshaped" in the natural sense mentioned above, but for $\alpha \geq \omega_2$ we insist that $s \in S_\alpha$ be "quickly reshaped" in that $\eta \leq |s|$ is collapsed relative to $A \cap \alpha$, $s \restriction \eta$ before the next L-cardinal. This will be important when we use the nonexistence of $0^\#$ to establish cardinal-preservation, via Proposition 5.3. Elements of S_α are called "strings". Note that we allow the empty string $\emptyset_\alpha \in S_\alpha$, where $|\emptyset_\alpha| = \alpha$. For $s, t \in S_\alpha$ write $s \leq t$ for $s \subseteq t$ and $s < t$ for $s \leq t$, $s \neq t$.

5.7 Definition (Coding Structures). For $s \in S_\alpha$ define $\mu^{<s}$, μ^s recursively by: $\mu^{<\emptyset_\alpha} = \alpha$, $\mu^{<s} = \bigcup\{\mu^t \mid t < s\}$ for $s \neq \emptyset_\alpha$ and $\mu^s =$ least $\mu > \mu^{<s}$ which is a limit of multiples of α such that $L_\mu[A \cap \alpha, s] \vDash s \in S_\alpha$. And $\mathcal{A}^s = L_{\mu^s}[A \cap \alpha, s]$.

Thus by definition there is $\delta < \mu^s$ such that $L_\delta[A \cap \alpha, s] \vDash \mathrm{Card}(|s|) \leq \alpha$ and $L_{\mu^s} \vDash \mathrm{Card}(\delta) \leq |s|$, when $\alpha \geq \omega_2$.

5.8 Definition (Coding Apparatus). For $\alpha > \omega$, $s \in S_\alpha$, $i < \alpha$ let $H^s(i) =$ Σ_1 Skolem hull of $i \cup \{A \cap \alpha, s\}$ in $\mathcal{A}^s$ and $f^s(i) =$ ordertype $(H^s(i) \cap \mathrm{On})$. For $\alpha \in \mathrm{Card}^+, b^s = \mathrm{ran}(f^s \restriction B^s)$ where $B^s =$ the successor elements of $\{i < \alpha \mid i = H^s(i) \cap \alpha\}$.

Using the above we will construct a tame, cofinality-preserving forcing P for coding $\langle L[A], A\rangle$ by a subset G_ω of ω_1 which is reshaped in the sense that proper initial segments of (the characteristic function of) G_ω belong to S_ω.

5.9 Definition (Partition of the Ordinals). Let B, C, D, and E denote the classes of ordinals congruent to $0, 1, 2$, and 3 mod 4, respectively. Also for any ordinal α and $X = B, C, D$ or E, we write α^X for the α^{th} element of X (when X is listed in increasing order). If Y is a set of ordinals then $Y^X = \{\alpha^X \mid \alpha \in Y\}$.

5.10 Definition (The Successor Coding). Suppose that $\alpha \in \mathrm{Card}$ $s \in S_{\alpha^+}$. A *condition* in R^s is a pair (t, t^*) where $t \in S_\alpha$, $t^* \subseteq \{b^{s \restriction \eta} \mid \eta \in [\alpha^+, |s|)\} \cup |t|$, $\mathrm{Card}(t^*) \leq \alpha$. Extension of conditions is defined by: $(t_0, t_0^*) \leq (t_1, t_1^*)$ iff $t_1 \leq t_0, t_1^* \subseteq t_0^*$ and:

(a) $|t_1| \leq \gamma^B < |t_0|$, $\gamma \in b^{s \restriction \eta} \in t_1^* \longrightarrow t_0(\gamma^B) = 0$ or $s(\eta)$.

(b) $|t_1| \leq \gamma^C < |t_0|$, $\gamma = \langle \gamma_0, \gamma_1\rangle$, $\gamma_0 \in A \cap t_1^* \longrightarrow t_0(\gamma^C) = 0$.

In (b), $\langle \cdot, \cdot \rangle$ is an L-definable pairing function on On so that $\mathrm{Card}(\langle \gamma_0, \gamma_1 \rangle) = \mathrm{Card}(\gamma_0) + \mathrm{Card}(\gamma_1)$ in L for infinite γ_0, γ_1. An R^s-generic over $\mathcal{A}^s$ is determined by a function $T : \alpha^+ \longrightarrow 2$ such that $s(\eta) = 0$ iff $T(\gamma^B) = 0$ for sufficiently large $\gamma \in b^{s \restriction \eta}$ and such that for $\gamma_0 < \alpha^+ : \gamma_0 \in A$ iff $T(\langle \gamma_0, \gamma_1 \rangle^C) = 0$ for sufficiently large $\gamma_1 < \alpha^+$. Note that R^s is an element of $\mathcal{A}^s$.

Now we come to the definition of the Limit Coding, which incorporates the idea of "coding delays". Suppose that $s \in S_\alpha$, $\alpha \in \mathrm{Card}'$ and $p = \langle (p_\beta, p^*_\beta) \mid \beta \in \mathrm{Card} \cap \alpha \rangle$ where $p_\beta \in S_\beta$ for each $\beta \in \mathrm{Card} \cap \alpha$. A natural definition of "p codes s" would be: for $\eta < |s|$, $p_\beta(f^{s \restriction \eta}(\beta)) = s(\eta)$ for sufficiently large $\beta \in \mathrm{Card} \cap \alpha$. There are a number of problems with this definition however. First, to avoid conflict with the Successor Coding we should use $f^{s\restriction\eta}(\beta)^D$ instead of $f^{s\restriction\eta}(\beta)$. Second, to lessen conflict with codings at $\beta \in \mathrm{Card}' \cap \alpha$ we only require the above for $\beta \in \mathrm{Card}^+ \cap \alpha$. However there are still serious problems in making sure that the coding of s is consistent with the coding of p_β by $p\restriction\beta$ for $\beta \in \mathrm{Card}' \cap \alpha$.

We introduce coding delays to facilitate extendibility of conditions. The rough idea is to code not using $f^{s\restriction\eta}(\beta)^D$, but instead to code just after the least ordinal $\geq f^{s\restriction\eta}(\beta)^D$ where p_β takes the value 1. In addition, we "precode" s by a subset of α, which is then coded with delays by $\langle p_\beta \mid \beta \in \mathrm{Card} \cap \alpha\rangle$; this "indirect" coding further facilitates extendibility of conditions.

5.11 Definition. Suppose that $\alpha \in \mathrm{Card}$, $X \subseteq \alpha$, $s \in S_\alpha$. Let $\tilde{\mu}^s$ be defined just as we defined μ^s but with the requirement "limit of multiples of α" replaced by the weaker condition "multiple of α". Then note that $\tilde{\mathcal{A}}^s = L_{\tilde{\mu}^s}[A \cap \alpha, s]$ belongs to $\mathcal{A}^s$, contains s and Σ_1 Hull$(\alpha \cup \{A \cap \alpha, s\})$ in $\tilde{\mathcal{A}}^s = \tilde{\mathcal{A}}^s$. Now X precodes s if X is the Σ_1 theory of $\tilde{\mathcal{A}}_s$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$ (viewed as a subset of α).

5.12 Definition (Limit Coding). Suppose that $s \in S_\alpha$, $\alpha \in \mathrm{Card}'$ and $p = \langle (p_\beta, p^*_\beta) \mid \beta \in \mathrm{Card} \cap \alpha\rangle$ where $p_\beta \in S_\beta$ for each $\beta \in \mathrm{Card} \cap \alpha$. We wish to define "$p$ codes s". First we define a sequence $\langle s_\gamma \mid \gamma \leq \gamma_0\rangle$ of elements of S_α as follows. Let $s_0 = \emptyset_\alpha$. For limit $\gamma \leq \gamma_0$, $s_\gamma = \bigcup\{s_\delta \mid \delta < \gamma\}$. Now suppose that s_γ is defined and let $f^{s_\gamma}_p(\beta) =$ least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^D) = 1$, if such a δ exists. If for cofinally many $\beta \in \mathrm{Card}^+ \cap \alpha$, $f^{s_\gamma}_p(\beta)$ is undefined, then set $\gamma_0 = \gamma$. Otherwise define $X \subseteq \alpha$ by: $\delta \in X$ iff $p_\beta((f^{s_\gamma}_p(\beta) + 1 + \delta)^D) = 1$ for sufficiently large $\beta \in \mathrm{Card}^+ \cap \alpha$. If Even $(X) = \{\delta \mid 2\delta \in X\}$ precodes an element t of S_α extending s_γ such that $f^{s_\gamma}_p, X \in \mathcal{A}^t$ then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * X^E$ (the concatenation of s_γ with the characteristic function of X^E), if this results in $f^{s_\gamma}_p \in \mathcal{A}^{s_{\gamma+1}}$; if not, then $\gamma_0 = \gamma$. Now p exactly codes s if $s = s_\gamma$ for some $\gamma \leq \gamma_0$ and p codes s if $s \leq s_\gamma$ for some $\gamma \leq \gamma_0$.

Note that the Successor Coding only restrains p_β from taking certain nonzero values, so there is no conflict between the Successor Coding and these delays. The advantage of delays is that they give us more control over *where* the Limit Coding takes place, thereby enabling us to avoid conflict between the Limit Codings at different cardinals.

5.13 Definition (The Conditions). A *condition* in P is a sequence $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in \mathrm{Card}, \alpha \leq \alpha(p)\rangle$ where $\alpha(p) \in \mathrm{Card}$ and:

(a) $p_{\alpha(p)}$ belongs to $S_{\alpha(p)}$ and $p^*_{\alpha(p)} = \emptyset$.

(b) For $\alpha \in \text{Card} \cap \alpha(p)$, (p_α, p^*_α) belongs to $R^{p_{\alpha^+}}$.

(c) For $\alpha \in \text{Card}'$, $\alpha \le \alpha(p)$, $p \restriction \alpha$ belongs to $\mathcal{A}^{p_\alpha}$ and exactly codes p_α.

(d) For $\alpha \in \text{Card}'$, $\alpha \le \alpha(p)$, α inaccessible in $\mathcal{A}^{p_\alpha}$ there exists a closed unbounded $C \subseteq \alpha$ such that C belongs to $\mathcal{A}^{p_\alpha}$ and $p^*_\beta = \emptyset$ for β in C.

For $\alpha \in \text{Card}, P^{<\alpha}$ denotes the set of all conditions p such that $\alpha(p) < \alpha$. Conditions are ordered by: $p \le q$ iff $\alpha(p) \ge \alpha(q)$, $p(\alpha) \le q(\alpha)$ in $R^{p_{\alpha^+}}$ for $\alpha \in \text{Card} \cap \alpha(p) \cap (\alpha(q)+1)$ and $p_{\alpha(p)}$ extends $q_{\alpha(p)}$ if $\alpha(q) = \alpha(p)$. Also for $s \in S_\alpha$, $\omega < \alpha \in \text{Card}$, P^s denotes $P^{<\alpha}$ together with all $p \restriction \alpha$ for conditions p such that $\alpha(p) = \alpha$, $p_{\alpha(p)} \le s$. P^s is an element of $\tilde{\mathcal{A}}^s$. To order conditions in P^s, define $p^+ = p$ for $p \in P^{<\alpha}$ and for $p \in P^s - P^{<\alpha}$, $p^+ \restriction \alpha = p$ and $p^+(\alpha) = (s \restriction \eta, \emptyset)$ where η is least such that $p \in P^{s \restriction \eta}$; then $p \le q$ iff $p^+ \le q^+$ as conditions in P. Finally, $P^{<s} = \bigcup\{P^{s\restriction\eta} \mid \eta < |s|\} \cup P^{<\alpha}$.

It is worth noting that (c) above implies that f^{p_α} dominates the coding of p_α by $p \restriction \alpha$, in the sense that f^{p_α} strictly dominates each $f^{p_\alpha \restriction \eta}_{p \restriction \alpha}$, $\eta < |p_\alpha|$ on a tail of $\text{Card}^+ \cap \alpha$. The purpose of (d) is to guarantee that extendibility of conditions at (local) inaccessibles is not hindered by the Successor Coding (see the proof of Extendibility below).

We now embark on a series of lemmas which together show that P preserves cofinalities and if G is P-generic over $\langle L[A], A\rangle$ then for some reshaped $X \subseteq \omega_1$, $L[A, G] = L[X]$ and A is $L[X]$-definable from the parameter X. Then X can be coded by a real via a c.c.c. forcing using the Solovay method described earlier.

5.14 Lemma (Distributivity for R^s). *Suppose that $\alpha \in \text{Card}$ and $s \in S_{\alpha^+}$. Then R^s is α^+-distributive in $\mathcal{A}^s$: if $\langle D_i \mid i < \alpha\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of R^s and $p \in R^s$ then there is a $q \le p$ such that q meets each D_i.*

Proof. Choose $\mu < \mu^s$ to be a large enough limit ordinal such that $p, \langle D_i \mid i < \alpha\rangle, \mu^{<s} \in \mathcal{A} = L_\mu[A \cap \alpha^+, s]$. Let $\langle \alpha_i \mid i < \alpha\rangle$ enumerate the first α elements of $\{\beta < \alpha^+ \mid \beta = \alpha^+ \cap \Sigma_1 \text{ Hull of } (\beta \cup \{p, \langle D_i \mid i < \alpha\rangle, \mu^{<s}\}) \text{ in } \mathcal{A}\}$.

Now write p as (t_0, t^*_0) and successively extend to (t_i, t^*_i) for $i \le \alpha$ as follows: (t_{i+1}, t^*_{i+1}) is the least extension of (t_i, t^*_i) meeting D_i such that t^*_{i+1} contains $\{b^{s\restriction\eta} \mid \eta \in H_i \cap [\alpha^+, |s|)\}$ where $H_i = \Sigma_1$ Hull of $\alpha_i \cup \{p, \langle D_i \mid i < \alpha\rangle, \mu^{<s}\}$ in $\mathcal{A}$ and: (a) If $b^{s\restriction\eta} \in t^*_i, s(\eta) = 1$ then $t_{i+1}(\gamma^B) = 1$ for some $\gamma \in b^{s\restriction\eta}, \gamma > |t_i|$. (b) If $\gamma_0 \notin A, \gamma_0 < |t_i|$ then $t_{i+1}(\langle \gamma_0, \gamma_1\rangle^C) = 1$ for some $\gamma_1 > |t_i|$.

The lemma reduces to:

5.15 Claim. *(t_λ, t^*_λ), the greatest lower bound to $\langle (t_i, t^*_i) \mid i < \lambda\rangle$, exists for limit $\lambda \le \alpha$.*

Proof of Claim. We must show that $t_\lambda = \bigcup\{t_i \mid i < \lambda\}$ belongs to S_α. Note that $\langle t_i \mid i < \lambda\rangle$ is definable over $\overline{H}_\lambda$ = transitive collapse of H_λ and by construction, t_λ codes $\overline{H}_\lambda$ definably over $L_{\bar{\mu}_\lambda}[t_\lambda]$, where $\bar{\mu}_\lambda$ = height of $\overline{H}_\lambda$. So t_λ is reshaped, as $|t_\lambda|$ is singular, definably over $L_{\bar{\mu}_\lambda}[t_\lambda]$. By Proposition 5.3, $\bar{\mu}_\lambda < (|t_\lambda|^+)^L$ if $\alpha \geq \omega_2$. So t_λ belongs to S_α. ⊣

⊣

The next lemma illustrates the use of coding delays.

5.16 Lemma (Extendibility for P^s). *Suppose that α is a limit cardinal, $p \in P^s$, $s \in S_\alpha$, $X \subseteq \alpha$ and $X \in \mathcal{A}^s$. Then there exists a $q \leq p$ such that $X \cap \beta \in \mathcal{A}^{q_\beta}$ for each $\beta \in \mathrm{Card} \cap \alpha$.*

Proof. Let $Y \subseteq \alpha$ be chosen so that $\mathrm{Even}(Y)$ precodes s and $\mathrm{Odd}(Y)$ is the Σ_1 theory of $\mathcal{A}$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$, where $\mathcal{A}$ is an initial segment of $\mathcal{A}^s$ of limit height large enough to extend $\widetilde{\mathcal{A}}^s$ and contain X, p. For $\beta \in \mathrm{Card} \cap \alpha$ let $\overline{\mathcal{A}}_\beta$ =transitive collapse of $\Sigma_1\,\mathrm{Hull}(\beta \cup \{A \cap \alpha, s\})$ in $\mathcal{A}$. Then for sufficiently large $\beta \in \mathrm{Card}' \cap \alpha$, either Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$ where s_β = pre-image of s under the natural embedding $\overline{\mathcal{A}}_\beta \longrightarrow \mathcal{A}$, or $|p_\beta| < (\beta^+)^{\overline{\mathcal{A}}_\beta}$ in which case f^{p_β} is dominated by the function $g(\gamma) = (\gamma^+)^{\overline{\mathcal{A}}_\gamma}$ on a final segment of $\mathrm{Card}^+ \cap \beta$.

Define q as follows: $q_\beta = s_\beta$ if Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$. For other $\beta \in \mathrm{Card}' \cap \alpha$, $q_\beta = p_\beta * (Y \cap \beta)^E$, the concatenation of p_β with the characteristic function of $(Y \cap \beta)^E$. For $\beta \in \mathrm{Card}^+ \cap \alpha$, $q_\beta = p_\beta * \vec{0} * 1 * (Y \cap \beta)^D$ where $\vec{0}$ has length $g(\beta)$.

As $g \restriction \beta, Y \cap \beta$ are definable over $\overline{\mathcal{A}}_\beta$ for $\beta \in \mathrm{Card}' \cap \alpha$ we get $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{s_\beta}$ when Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$. Also $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{q_\beta}$ for other $\beta \in \mathrm{Card}' \cap \alpha$ as Odd $(Y \cap \beta)$ codes $\overline{\mathcal{A}}_\beta$. And note that for all $\beta \in \mathrm{Card}' \cap \alpha$, $g \restriction \beta$ dominates f^{p_β} on a final segment of $\mathrm{Card}^+ \cap \alpha$ (and hence $q \restriction \beta$ exactly codes q_β), unless Even $(Y \cap \beta)$ precodes s_β and $s_\beta = p_\beta$, in which case $q \restriction \beta$ exactly codes $q_\beta = s_\beta$ because $p \restriction \beta$ does.

So we conclude that for sufficiently large $\beta \in \mathrm{Card}' \cap \alpha$, $q \restriction \beta$ exactly codes q_β and $X \cap \beta \in \mathcal{A}^{q_\beta}$. Apply induction on α to obtain this for all $\beta \in \mathrm{Card}' \cap \alpha$. Finally, note that the only problem in verifying $q \leq p$ is that the restraint p^*_β may prevent us from making the extension q_β of p_β when $q_\beta = s_\beta$, Even $(Y \cap \beta)$ precodes s_β. Note that this case can be avoided for sufficiently large $\beta < \alpha$ if α is not inaccessible in $\mathcal{A}^s$, by enlarging $\mathcal{A}$. So assume that α is inaccessible in $\mathcal{A}^s$. But property (d) in the definition of condition guarantees that $p^*_\beta = \emptyset$ for β in a closed unbounded $C \subseteq \alpha$, $C \in \mathcal{A}^s$. We may assume that $C \in \mathcal{A}$ and hence for sufficiently large β as above we get $\beta \in C$ and hence $p^*_\beta = \emptyset$. So $q \leq p$ on a final segment of $\mathrm{Card} \cap \alpha$, and we may again apply induction to get $q \leq p$ everywhere. ⊣

The key idea of Jensen's proof lies in the verification of distributivity for P^s. Before we can state and prove distributivity we need some definitions.

5.17 Definition. Suppose that $i < \beta \in \mathrm{Card}$ and $D \subseteq P^s$, $s \in S_{\beta^+}$. D is i^+-predense on P^s if $\forall p \in P^s \exists q \in P^s$ $(q \leq p, q$ meets D and $q \restriction i^+ = p \restriction i^+)$. $X \subseteq \mathrm{Card} \cap \beta^+$ is thin if for each inaccessible $\gamma \leq \beta$, $X \cap \gamma$ is not stationary in γ. A function $f : \mathrm{Card} \cap \beta^+ \longrightarrow V$ is small if for each $\gamma \in \mathrm{Card} \cap \beta^+$,

$\mathrm{Card}(f(\gamma)) \leq \gamma$ and Support $(f) = \{\gamma \in \mathrm{Card} \cap \beta^+ \mid f(\gamma) \neq \emptyset\}$ is thin. If $D \subseteq P^s$ is predense and $p \in P^s$, $\gamma \in \mathrm{Card} \cap \beta^+$ we say that p reduces D below γ if for some $\delta \in \mathrm{Card}^+$ with $\delta \leq \gamma$, $q \leq p \longrightarrow \exists r \leq q$ (r meets D and $r \restriction [\delta, \beta] = q \restriction [\delta, \beta]$). Finally, for $p \in P^s$, f small, $f \in \mathcal{A}^s$ we define $\Sigma^p_f =$ all $q \leq p$ in P^s such that whenever $\gamma \in \mathrm{Card} \cap \beta^+$, $D \in f(\gamma), D$ predense on $P^{p_{\gamma^+}}$, we have that q reduces D below γ.

5.18 Lemma (Distributivity for P^s). *Suppose that $s \in S_{\beta^+}$ where $\beta \in$ Card.*

(a) If $\langle D_i \mid i < \beta \rangle$ belongs to $\mathcal{A}^s$, D_i is i^+-dense on P^s for each $i < \beta$ and p belongs to P^s, then there is a $q \leq p$ such that q meets each D_i.

(b) If p belongs to P^s and f is small in $\mathcal{A}^s$ then there exists a $q \leq p$ such that q belongs to Σ^p_f.

Proof. We demonstrate (a) and (b) by a simultaneous induction on β. If $\beta = \omega$ or belongs to Card^+ then by induction, (a) and (b) reduce to the following: If S is a collection of β-many predense subsets of P^s, $S \in \mathcal{A}^s$ then $\{q \in P^s \mid q$ reduces each $D \in S$ below $\beta\}$ is dense on P^s. The latter follows from Lemma 5.14, since P^s factors as $R^s * Q$ where $1^{R^s} \Vdash Q$ is β^+-c.c., and hence any $p \in P^s$ can be extended to a $q \in P^s$ such that $D^q = \{r \mid r \cup q(\beta)$ meets $D\}$ is predense $\leq q \restriction \beta$ for each $D \in S$.

Now suppose that β is inaccessible. We first show that (b) holds for f, provided $f(\beta) = \emptyset$. First select a closed unbounded $C \subseteq \beta$ in $\mathcal{A}^s$ such that $\gamma \in C \rightarrow f(\gamma) = \emptyset$ and extend p so that $f \restriction \gamma$, $C \cap \gamma$ belong to $\mathcal{A}^{p_\gamma}$ for each $\gamma \in \mathrm{Card} \cap \beta^+$. Then we can successively extend p on $[\beta_i^+, \beta_{i+1}]$ in the least way so as to meet Σ^p_f on $[\beta_i^+, \beta_{i+1}]$, where $\langle \beta_i \mid i < \beta \rangle$ is the increasing enumeration of C. At limit stages λ, we still have a condition, as the sequence of first λ extensions belongs to $\mathcal{A}^{p_{\beta_\lambda}}$. The final condition, after β steps, is an extension of p in Σ^p_f.

Now we prove (a) in this case. Suppose that $p \in P^s$ and $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$, D_i is i^+-dense on P^s for each $i < \beta$. Let $\mu_0 < \mu^s$ be a large enough limit ordinal so that $\langle D_i \mid i < \beta \rangle$, p, $\tilde{\mu}^s \in L_{\mu_0}[A \cap \beta^+, s]$ and for $i < \beta$ let $\mu_i = \mu_0 + \omega \cdot i < \mu^s$. For any X we let $H_i(X)$ denote $\Sigma_1 \,\mathrm{Hull}(X \cup \{\langle D_i \mid i < \beta \rangle, p, \tilde{\mu}^s, s, A \cap \beta^+\})$ in $L_{\mu_i}[A \cap \beta^+, s]$.

Let $f_i : \mathrm{Card} \cap \beta \rightarrow V$ be defined by: $f_i(\gamma) = H_i(\gamma)$ if $i < \gamma \in H_i(\gamma)$ and $f_i(\gamma) = \emptyset$ otherwise. Then each f_i is small in $\mathcal{A}^s$ and we recursively define $p = p^0 \geq p^1 \geq \cdots$ in P^s as follows: $p^{i+1} =$ least $q \leq p^i$ such that:

(a) $q(\beta)$ meets all predense $D \subseteq R^s$, $D \in H_i(\beta)$,

(b) q meets $\Sigma^{p^i}_{f_i}$ and D_i,

(c) $q \restriction i^+ = p^i \restriction i^+$.

For limit $\lambda \leq \beta$ we take p^λ to be the greatest lower bound to $\langle p^i \mid i < \lambda \rangle$, if it exists.

5.19 Claim. *p^λ is a condition in P^s, where for each $\gamma \in \mathrm{Card} \cap \beta^+$,*

$$p^\lambda(\gamma) = \left(\bigcup\{p^i_\gamma \mid i < \lambda\}, \bigcup\{{p^i_\gamma}^* \mid i < \lambda\}\right).$$

Suppose that γ belongs to $H_\lambda(\gamma) \cap \beta$. First we verify that $p^\lambda_\gamma = \bigcup\{p^i_\gamma \mid i < \lambda\}$ belongs to S_γ. Let $\bar{H}_\lambda(\gamma)$ be the transitive collapse of $H_\lambda(\gamma)$ and write $\bar{H}_\lambda(\gamma)$ as $L_{\bar{\mu}}[\bar{A}, \bar{s}]$, $\bar{P}$ = image of $P^s \cap H_\lambda(\gamma)$ under transitive collapse, $\bar{\beta}$ = image of β under collapse. Also write $\bar{P}$ as $\bar{R}^{\bar{s}} * P^{\bar{G}_{\bar{\beta}}}$ where $\bar{G}$ denotes an $\bar{R}^{\bar{s}}$-generic (just as P^s factors as $R^s * P^{G_\beta}$, G_β denoting an R^s-generic).

Now the construction of the p^i's (see conditions (a), (b)) was designed to guarantee: (i) $\bar{G}_{\bar{\beta}} = \{\bar{p} \in R^{\bar{s}} \mid \bar{p}$ is extended by some $\bar{p}^i(\bar{\beta}), i < \lambda\}$ is $R^{\bar{s}}$-generic over $\bar{H}_\lambda(\gamma)$, where $\bar{p}^i$ = image of p^i under collapse, and (ii) for each $\bar{\delta}$ in (Card^+ of $\bar{H}_\lambda(\gamma)$), $\gamma < \bar{\delta} < \bar{\beta}$, $\{\bar{p} \mid \bar{p}$ is extended by some $\bar{p}^i\restriction[\gamma, \bar{\delta})$ in $\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma\}$ is $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma$-generic over $\mathcal{A}^{\bar{G}_{\bar{\delta}}} = \bigcup\{\mathcal{A}^{\bar{p}^i_{\bar{\delta}}} \mid i < \lambda\}$, where $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma = \bigcup\{\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma \mid i < \lambda\}$ and $\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma$ denotes the image under collapse of $P^{p^i_\delta}_\gamma = \{q\restriction[\gamma, \delta) \mid q \in P^{p^i_\delta}\}$, $\bar{\delta}$ = image of δ under collapse.

5.20 Remark. We do not necessarily have property (ii) above for $\bar{\delta} = \bar{\beta}$, and this is the source of our need for the nonexistence of $0^\#$ in this proof.

By induction, we have the distributivity of P^t for $t \in S_\delta$, $\delta \in \mathrm{Card}^+ \cap \beta$, and hence that of $\bar{P}^{\bar{t}}$ for $\bar{t} \in \bar{S}_{\bar{\delta}}$, $\bar{\delta} \in$ (Card^+ of $\bar{H}_\lambda(\gamma)$), $\bar{\delta} < \bar{\beta}$. So the "weak" genericity of the preceding paragraph implies that:

(d) $L_{\bar{\beta}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is a cardinal.

Also:

(e) $L_{\bar{\mu}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is Σ_1-singular.

Thus $p^\lambda_\gamma \in S_\gamma$ (by (e)) provided we can show that when $\gamma \geq \omega_2$, $\bar{\mu} < (|p^\lambda_\gamma|^+)^L$. But $\bar{H}_\lambda(\gamma) \xrightarrow{\sim} H_\lambda(\gamma)$ gives a Σ_1-elementary embedding with critical point $|p^\lambda_\gamma|$, so by Proposition 5.3, this is true. Also note that we now get $p^\lambda\restriction\gamma \in \mathcal{A}^{p^\lambda_\gamma}$ as well, since $p^\lambda\restriction\gamma$ is definable over $\bar{H}_\lambda(\gamma)$ and we defined $\mathcal{A}^{p^\lambda_\gamma}$ to be large enough to contain $\bar{H}_\lambda(\gamma)$, since $L_{\bar{\beta}} \vDash |p^\lambda_\gamma|$ is a cardinal by (d) and $\bar{\beta}$ is a cardinal of $L_{\bar{\mu}}$.

The previous argument applies also if $\gamma = \beta$, using the distributivity of R^s, or if $\gamma = \beta \cap H_\lambda(\gamma)$, using the fact that p^λ_β collapses to p^λ_γ. If $\gamma < \gamma^* = \min(H_\lambda(\gamma) \cap [\gamma, \beta))$ then we can apply the first argument to get the result for γ^*, and then the second argument to get the result for γ.

Finally, to prove the Claim we must verify the restraint condition (d) in the definition of P. Suppose that γ is inaccessible and for $i < \lambda$ let C^i be the least closed unbounded subset of γ in $\mathcal{A}^{p^i_\gamma}$ disjoint from $\{\bar{\gamma} < \gamma \mid {p^i_{\bar{\gamma}}}^* \neq \emptyset\}$. If $\lambda < \gamma$ then $\bigcap\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ, if $\gamma < \lambda$ then the restraint condition for p^λ at γ follows by induction on λ and if $\gamma = \lambda$ then $\Delta\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ, where Δ denotes diagonal intersection.

Thus the Claim and therefore (a) is proved in case β is inaccessible. To verify (b) in this case, note that as we have already proved (b) when $f(\beta) = \emptyset$ it suffices to show: if $\langle D_i \mid i < \beta\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of P^s then $\forall p \exists q \leq p$ (q reduces each D_i below β). But using distributivity we see that $D_i^* = \{q \mid q$ reduces D_i below $i^+\}$ is i^+-dense for each $i < \beta$, so again by distributivity there is $q \leq p$ reducing D_i below i^+ for each i.

We are now left with the case where β is singular. The proof of (a) can be handled using the ideas from the inaccessible case as follows. Choose $\langle \beta_i \mid i < \lambda_0\rangle$ to be a continuous and cofinal sequence of cardinals $< \beta$, $\lambda_0 < \beta_0$. First we argue that $p \in P^s$ can be extended to meet Σ^p_f for any f small in $\mathcal{A}^s$ provided $f(\beta) = \emptyset$: extend p if necessary so that for each $\gamma \in \mathrm{Card} \cap \beta^+$, $f\restriction\gamma$ and $\{\beta_i \mid \beta_i < \gamma\}$ belong to $\mathcal{A}^{p_\gamma}$. Now perform a construction like the one used to prove distributivity in the inaccessible case, extending p successively on $[\beta_0, {\beta_i}^+]$ so as to meet Σ^p_f on $[\beta_0, {\beta_i}^+]$ as well as appropriate $\Sigma^{p^i}_{f_i}$'s defined on $[\beta_0, {\beta_i}^+]$ to guarantee that p^λ is a condition for limit $\lambda \leq \lambda_0$. Note that each extension is made on a bounded initial segment of $[\beta_0, \beta)$ and therefore by induction $\Sigma^p_f, \Sigma^{p^i}_{f_i}$ can be met on these intervals. The result is that p can be extended to meet Σ^p_f on a final segment of $\mathrm{Card} \cap \beta$ and therefore by induction can be extended to meet Σ^p_f. Second, use the density of Σ^p_f when $f(\beta) = \emptyset$ to carry out the distributivity proof as we did in the inaccessible case. And again, (b) follows from (a). This complete the proof of Lemma 5.18. ⊣

Theorem 5.1 now follows, as the argument of the previous lemma also shows:

5.21 Lemma (Distributivity for P). *If $\langle D_i \mid i < \kappa\rangle$ is $\langle M, A\rangle$-definable where D_i is i^+-dense for each $i < \kappa$ and $p \in P$ then there exists a $q \leq p$ such that q meets each D_i.*

Thus P is tame and preserves cofinalities. ⊣

The proof of Theorem 5.1 in the general case is far more difficult; we refer the reader to [5, Sect. 4.3].

The forcing used to prove the Coding Theorem preserves a number of large cardinal properties consistent with $V = L[R]$, $R \subseteq \omega$, such as the Mahlo and α-Erdős properties. In addition for any m, n a predicate A^* can be adjoined to $\langle M, A\rangle$ so that if κ is Σ^n_m-indescribable then κ is Σ^n_m-indescribable relative to A^*, and then A^* can be coded by a real, via a modification of the forcing described above, so as to preserve Σ^n_m-indescribability (see [5, Sect. 4.4]). Preservation of Π^n_m-indescribability for $n > 1$ is an open problem.

When considering the relevance of Jensen coding, we see the importance of indiscernible preservation:

5.22 Proposition. *Suppose that $A \subseteq L$ preserves indiscernibles. Then there is a real $R \in L[A, 0^\#]$ generic over $\langle L[A], A\rangle$ such that A is definable in $L[R]$. Moreover, R preserves indiscernibles.*

The following proof of Proposition 5.22 is reminiscent of the proof of relevance for coherent Easton forcing at successors.

Proof. First assume that $A = \emptyset$. For any indiscernible i let j_n be the first n indiscernibles $\geq i$. Then define $s_n \in S^{i^+}$ and $p^n \in P^{s_n}$ recursively, meeting the following conditions: $s_0 = \emptyset$, $p^0 =$ the trivial condition. $s_{n+1} = \pi_i(p^n)_{i^+}$ where $\pi_i : L \to L$ is an elementary embedding with critical point i, $p^{n+1} =$ least $q \leq p^n$ in P^{s_n} meeting $\Sigma^{p^n}_{f_n}$ where $f_n(\beta) = \mathrm{Hull}(\beta \cup j_n)$ if $\beta \in \mathrm{Hull}(\beta \cup j_n)$, $f_n(\beta) = \emptyset$ otherwise. (β ranges over $\mathrm{Card} \cap i^+$ and when $\beta = i$ we take $p^n_{\beta+}$ to be s_n.) Let $G^i_0 = \{p \mid p \text{ is extended by some } p^n\}$.

G^i_0 is *not* P^{s_n}-generic over $\mathcal{A}^{s_n}$ in general as all conditions in G^i_0 have empty restraint at indiscernibles $< i$. But notice that for $i_0 < i_1 < \cdots < i_n \leq i$ in I, $G^{i_0}_0 \cup \cdots \cup G^{i_n}_0$ is a compatible set of conditions. We take G^i to be $\{p \mid p \text{ is extended by } q_0 \wedge \cdots \wedge q_n \text{ for some } q_l \in G^{i_l}_0,\ i_0 < \cdots < i_n \leq i \text{ in } I\}$. Now we claim that G^i is P^{s_n}-generic over $\mathcal{A}^{s_n}$ for each n. Indeed, if D is predense on P^{s_n} and belongs to $\mathcal{A}^{s_n}$, $D \in \mathrm{Hull}(\{k_0, \ldots, k_m\} \cup j_n)$ with $k_0 < \cdots < k_m < i$ in I then p^{n+1} reduces D below k^+_m, p^{n+2} reduces D below $k^+_{m-1}, \ldots$ and eventually we get p^{n+m+2} in G^i meeting D.

It follows that $G^i(<i) = G^i \cap P^i$ is generic over L_i (for L_i-definable dense sets) and hence G is P-generic over L where $G = \bigcup\{G^i(<i) \mid i \in I\}$. Clearly G preserves indiscernibles.

If $A \neq \emptyset$ then first force to obtain GCH, preserving indiscernibles, and then apply the above argument. ⊣

5.23 Corollary (Jensen). *There is a real $R <_L 0^\#$, R not set-generic over L. Hence the genericity problem has an affirmative solution when "generic" is interpreted to mean "set-generic".*

Not every $A \subseteq L$ can be coded generically by a real, in the presence of $0^\#$, as a result of Paris's work on "patterns of indiscernibles":

5.24 Definition. For $\alpha, \beta \in \mathrm{On}$, $\beta \neq 0$ let $I_{\alpha,\beta} = \{i_{\alpha+\beta\gamma} \mid \gamma \in \mathrm{On}\}$ where $\langle i_\alpha \mid \alpha \in \mathrm{On}\rangle$ is the increasing enumeration of I.

For any real R with $L[R]$ not rigid, the *Silver indiscernibles for* $L[R]$ are defined just like the Silver indiscernibles for L, replacing L by $L[R]$. In this case we say that "$R^\#$ exists".

5.25 Theorem (Paris [14]). *If $R \subseteq \omega$ and $0^\# \notin L[R]$, then for some $\alpha, \beta < \omega_1$, $I_{\alpha,\beta} =$ the Silver indiscernibles for $L[R]$.*

There exist classes $A \subseteq L$ which are generic over L, yet relative to which $I_{\alpha,\beta}$ is *not* a class of indiscernibles for any pair α, β. (For any $B \subseteq I$, there is A generic over L such that $A \cap I = B$; but B can be chosen to split each

$I_{\alpha,\beta}$.) It follows that A cannot be generically coded by a real R, as any such R satisfies the hypothesis of Paris's theorem. However this is the only restriction.

5.26 Theorem. *If $I_{\alpha,\beta}$ is a class of indiscernibles for $\langle L[A], A\rangle$ for some $\alpha, \beta < \omega_1$, then there is a real $R \in L[A, 0^{\#}]$ generic over $\langle L[A], A\rangle$ such that A is definable in $L[R]$. Moreover, $I_{\alpha,\beta}$ is a class of indiscernibles for $L[R]$.*

In addition:

5.27 Theorem. *For any $\alpha, \beta < \omega_1$ there exists a real R such that $I_{\alpha,\beta} =$ the Silver indiscernibles for $L[R]$.*

Theorems 5.26 and 5.27 are proved by first using reverse Easton methods to create $A^* \subseteq L$ such that $I_{\alpha,\beta}$ is a *generating* class of indiscernibles for $\langle L[A^*], A^*\rangle$ and then using the method of Proposition 5.22 to code A^* by a real, preserving the indiscernibility of $I_{\alpha,\beta}$.

6. The Solovay Problems

We are now prepared to discuss the solutions to the three problems posed in Sect. 1. For a full treatment of this material, we refer the reader to Chaps. 5, 6, 7 of [5].

The Genericity Problem

We show that there is a real $R <_L 0^{\#}$ which is not class-generic over L. First recall the statement of the Truth Lemma, which holds for all tame L-forcings:

Truth Lemma. If G is P-generic over $\langle L, A\rangle$ then

$$\langle L[G], A, G\rangle \vDash \varphi(\sigma_1^G, \ldots, \sigma_n^G) \quad \text{iff for some } p \in G, \quad p \Vdash \varphi(\sigma_1 \ldots \sigma_n).$$

We also have:

Uniform Definability Lemma. The relation "$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$" is definable as a relation of $p, \varphi, \langle\sigma_1, \ldots, \sigma_n\rangle$ over $\langle L, \mathrm{Sat}\langle L, A\rangle\rangle$ where $\mathrm{Sat}\langle L, A\rangle$ denotes the Satisfaction relation for $\langle L, A\rangle$.

6.1 Remark. $\langle L, \mathrm{Sat}\langle L, A\rangle\rangle$ is amenable, as $\langle L, A\rangle$ amenable implies that $\langle L_i, A \cap L_i\rangle \prec \langle L, A\rangle$ for sufficiently large $i \in I$.

A consequence is the following:

If G is P-generic over $\langle L, A\rangle$, then $\mathrm{Sat}\langle L[G], A, G\rangle$ is definable over the structure $\langle L[G], \mathrm{Sat}\langle L, A\rangle, G\rangle$.

Using this, we can see a strategy for producing a real R *not* generic over L: If $R \in L[G]$ where G is P-generic over $\langle L, A\rangle$, then by the above and Tarski's Undefinability of Satisfaction, $\mathrm{Sat}\langle L, A\rangle$ *cannot* be definable over $\langle L[G], A, G\rangle$ and hence *cannot* be definable over $\langle L[R], A\rangle$. Thus:

6.2 Proposition. *If $R \subseteq \omega$ is generic over L, then for some amenable $\langle L, A\rangle$, $\mathrm{Sat}\langle L, A\rangle$ is not definable over $\langle L[R], A\rangle$.*

6.3 Theorem. *There is a real $R <_L 0^\#$ such that $\mathrm{Sat}\langle L, A\rangle$ is definable over $\langle L[R], A\rangle$ for every amenable $\langle L, A\rangle$.*

To prove Theorem 6.3 we define for each $i \in I$ a forcing $P_i \subseteq L_{i^+}$ for producing $X_i \subseteq i$ such that for each constructible $A \subseteq i$, $\mathrm{Sat}\langle L_i, A\rangle$ is definable over $\langle L_i[X_i], A, X_i\rangle$. This forcing P_i is of the Easton variety and hence preserves cofinalities. The main part of the proof consists in showing that there is a single $X \subseteq \mathrm{On}$ definable in $L[0^\#]$ such that $X \cap i$ is P_i-generic for all $i \in I$ simultaneously, and such that X preserves indiscernibles. Then for each amenable $\langle L, A\rangle$, $\mathrm{Sat}\langle L, A\rangle$ is definable over $\langle L[X], A, X\rangle$ and X can be coded by a real $R <_L 0^\#$ with the same property, using the fact that X preserves indiscernibles and Proposition 5.22.

The proof is not special to the Sat operator and can be used to prove:

6.4 Theorem. *Suppose that $F : \mathcal{P}^L(\omega_1) \to \mathcal{P}^L(\omega_1)$ is constructible, where $\mathcal{P}^L(\omega_1)$ is the set of constructible subsets of ω_1. Then there is a real $R <_L 0^\#$ such that $F(A)$ is definable over $\langle L_{\omega_1}[R], A\rangle$ for all $A \in \mathcal{P}^L(\omega_1)$.*

The Π^1_2-Singleton Problem

The following result gives an affirmative solution to this problem:

6.5 Theorem. *There is a real R such that $0 <_L R <_L 0^\#$ and R is the unique solution to a Π^1_2 formula.*

The heart of the matter is to build an L-definable forcing with a unique generic, in the form of a real. To guarantee uniqueness we design our forcing so as to make our generic "guess" at which ordinals belong to I, the class of Silver indiscernibles. Of course no generic can correctly answer this question, but we arrange that only one generic does a reasonable job of guessing, in the sense that other potential generics would in fact produce closed unbounded classes disjoint from I, an impossibility. More precisely, a generic consists of a real R and a class A such that:

(a) R codes A as in Jensen coding.

(b) There is a $\Sigma_1(L)$ procedure $(i_1, \dots, i_n) \mapsto p(i_1, \dots, i_n)$ such that the generic corresponding to (R, A) is $\{p(i_1, \dots, i_n) \mid i_1 < \cdots < i_n \text{ in } I\}$.

(c) A adds closed unbounded sets so as to "kill" any $(i_1, \dots, i_n)$ such that $p(i_1, \dots, i_n)$ disagrees with R (in the sense that any generic containing $p(i_1, \dots, i_n)$ corresponds to (R', A') for some real R' different from R).

(d) No $(i_1, \dots, i_n) \in I^n$ can be killed.

It follows that $\{p(i_1, \ldots, i_n) \mid i_1 < \cdots < i_n \text{ in } I\}$ is the *only* generic, as by (c) another generic R' would kill $(i_1, \ldots, i_n) \in I^n$ such that $p(i_1, \ldots, i_n)$ disagrees with R', an impossibility by (d).

Of course there is a circularity here, as to design P we need the procedure in (b), which is defined assuming that we know P. This is resolved using the Recursion Theorem.

The killing method above involves forcing of the reverse Easton variety and the coding of A by R uses Jensen coding, a variety of coherent Easton forcing at successors. Thus unlike the solution to the genericity problem, here we must mix the relevance arguments for two different types of class forcing together, to obtain a generic in $L[0^\#]$ for P.

The Admissibility Spectrum Problem

We first describe the proof of:

6.6 Theorem (David [3], Friedman [5]). *There is a real $R <_L 0^\#$ such that $\Lambda(R) \subseteq$ the recursively inaccessible ordinals.*

We wish to arrange that R-admissibles be recursively inaccessible. Suppose that we have a $D \subseteq \omega_1$ such that D-admissibles are recursively inaccessible. (α is *D-admissible* if $L_\alpha[D]$ obeys ZFC^-, with Replacement restricted to formulas which are Σ_1 and mention D as a predicate.) Then we may hope to code D by a real R with the same property. However, if we code D by R in the usual way (with almost disjoint forcing) we only obtain:

$$\alpha \text{ is } R\text{-admissible} \to \alpha \text{ is } D \cap \omega_1^{L_\alpha}\text{-admissible}.$$

The reason is that to decode D from R we need to know the almost disjoint coding reals R_ξ and it is only for $\xi < \omega_1^{L_\alpha}$ that we have $R_\xi \in L_\alpha$. Thus the recovery of D from R is not "fast enough". On the other hand we would be in good shape if D were to have the following stronger properties:

($*$) If α is $D \cap \xi$-admissible and $L_\alpha[D \cap \xi] \vDash \xi = \omega_1$, then α is recursively inaccessible.

($**$) If α is D-admissible and $L_\alpha[D] \vDash \omega_1$ does not exist, then α is recursively inaccessible.

For then we need only recover $D \cap \omega_1^{L_\alpha}$ inside $L_\alpha[R]$ to guarantee that α be recursively inaccessible (or inadmissible relative to R), a recovery that can be successfully made.

The question is how to obtain $D \subseteq \omega_1$ obeying ($*$), ($**$). The natural thing to do is to force with conditions d which are bounded subsets of ω_1 obeying ($*$), ($**$) for $\xi \leq \sup(d)$, ordered by end extension. We now come to the key part of the argument, which is contained in the following two observations:

(a) Extendibility for this forcing is trivial because given d and $\xi > \sup(d)$ we are free to extend d to length ξ by killing all admissibles between $\sup(d)$ and ξ. It is important for this argument that we are only concerned with killing admissibility, not with preserving it.

(b) Distributivity for this forcing is easily established assuming the following: There exists a $D' \subseteq \omega_2$ such that:

$(*')$ If α is $D' \cap \xi$-admissible and $L_\alpha[D' \cap \xi] \vDash \xi = \omega_2$ then α is recursively inaccessible.

$(**')$ If α is D'-admissible and $L_\alpha[D'] \vDash \omega_2$ does not exist, then α is recursively inaccessible.

Thus, we are faced with the original difficulty, but one cardinal higher! However note that we need not already have all of D' before we can start building D; thus the idea of the proof (as in other Jensen coding constructions) is to build $R, D, D', D'', \ldots$ simultaneously and check distributivity for any final segment of the forcing.

To solve the admissibility spectrum problem we must introduce the requirement of admissibility *preservation* into the above. This requires the method of *strong coding*.

6.7 Theorem. *There is a real $R <_L 0^\#$ such that $\Lambda(R) =$ the recursively inaccessible ordinals.*

We approach the problem as in the previous proof. Of course the Extendibility property is more difficult to establish (Distributivity is approximately the same). Indeed the desired extension of d to d' of length $\geq \xi$ must be made so as to preserve the admissibility of recursively inaccessible ordinals. Thus our conditions must be constructed out of sets which are generic for "local" versions of the full forcing. In fact we construct a strong coding forcing $P^\beta \subseteq L_\beta$ at each admissible β and then inductively build P^β out of sets which are generic for the various $P^{\beta'}$ for $\beta' < \beta$.

The main difficulty is in showing that the desired locally generic sets actually exist; note that we want a P^β-generic over L_β to exist where β may be uncountable. The proof of local generic existence is by a simultaneous induction with the proofs of Extendibility and Distributivity and requires a substantial use of the kind of fine structure theory used in the construction of higher gap morasses.

7. Generic Saturation

Suppose that P is an L-forcing which has a generic; need it have a generic definable in $L[0^\#]$? Not necessarily, as the forcing P could produce a real R that guarantees the countability of $\omega_1^{L[0^\#]}$, and clearly no such real can exist in $L[0^\#]$. However, we can weaken this slightly to obtain a positive result:

7.1 Definition. Suppose that $M \subseteq N$ are inner models of ZFC. We say that N is *generically saturated over* M if whenever an M-forcing has a generic, then it has one definable in a set-generic extension of N.

With a mild assumption about On = the class of all ordinals, it can be shown that $L[0^\#]$ is generically saturated over L. This assumption involves the concept of an *Erdős cardinal*.

7.2 Definition. A cardinal κ is α-*Erdős* if whenever $A \subseteq \kappa$ and C is closed unbounded in κ, there exists an $X \subseteq C$ such that ordertype $X = \alpha$ and $\gamma \in X$ implies $X - \gamma$ is a set of indiscernibles for $\langle L[A], A, \delta\rangle_{\delta<\gamma}$. We say that On is α-Erdős if this holds where κ is replaced by On and indiscernibility is only required for Σ_1 formulas.

7.3 Theorem. *Suppose that* On *is* $\omega+\omega$*-Erdős. Then* $L[0^\#]$ *is generically saturated over* L.

Theorem 7.3 is proved by starting with G P-generic over $\langle L, A\rangle$ and using $\omega+\omega$ indiscernibles for $\langle L[G, 0^\#], A, G\rangle$ to produce another P-generic G^*, which is "periodic". The latter means that for some $\alpha \in$ On and $0 < \beta \in$ On, $I_{\alpha,\beta} = \{i_{\alpha+\beta\gamma} \mid \gamma \in \text{On}\}$ is a class of indiscernibles for $\langle L[G^*], A, G^*\rangle$, where $I = \langle i_\alpha \mid \alpha \in \text{On}\rangle$ is the increasing enumeration of I. Then by an absoluteness argument, such a G^* may be defined in a set-generic extension of $L[0^\#]$ in which α and β are countable.

Proof of Theorem 7.3. Suppose that $G \subseteq P$ is P-generic over $\langle L, A\rangle$. We shall construct another P-generic G^* (in a set-generic extension of V) such that G^* has periodic indiscernibles.

Let X be a set of indiscernibles for $\langle L[0^\#, G], G, A\rangle$ of ordertype $\omega+\omega$ such that $\alpha \in X \to \alpha$ is Σ_1-stable in $0^\#, G, A$. The latter means that $\langle L_\alpha[0^\#, G\cap L_\alpha], G\cap L_\alpha, A\cap L_\alpha\rangle$ is Σ_1-elementary in $\langle L[0^\#, G], G, A\rangle$. We can obtain X as $C = \{\alpha \mid \alpha \text{ is } \Sigma_1\text{-stable in } 0^\#, G, A\}$ is closed unbounded.

Choose $\langle D(\alpha_1, \dots, \alpha_n) \mid \alpha_1 < \dots < \alpha_n \text{ in On}\rangle$ such that each $\langle L, A\rangle$-definable open dense $D \subseteq P$ is of the form $D(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1 < \dots < \alpha_n$ in I. Also assume that this sequence is $\Delta_1\langle L, \text{Sat}\langle L, A\rangle\rangle$. Let $D^*(\alpha_1, \dots, \alpha_n) = \bigcap\{D(\vec\beta) \mid \vec\beta \text{ a subsequence of } \langle\alpha_1, \dots, \alpha_n\rangle\}$.

For $j_0 \in X$ choose $t_{j_0}(\vec{k_0}(j_0), j_0, \vec{k_1}(j_0))$ to be least in $D(j_0) \cap G$. By the choice of the indiscernibles X, we can write this as $t_0(\vec{k_0}, j_0, \vec{k_1}(j_0))$, and in addition $\vec{k_1}(j_0) < j_1$ for $j_0 < j_1$ in X.

Next for $j_0 < j_1$ in X choose $t_{j_0,j_1}(\vec{k_0^1}(j_0, j_1), j_0, \vec{k_1^1}(j_0, j_1), j_1, \vec{k_2^1}(j_0, j_1))$ to be least in $D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1)) \cap G$. By the choice of X we can write this as $t_1(\vec{k_0^1}, j_0, \vec{k_1^1}(j_0), j_1, \vec{k_2^1}(j_0, j_1))$, and by Σ_1-stability this is less than j_2 whenever $j_1 < j_2$ in X. But we want to argue that in fact $\vec{k_2^1}(j_0, j_1)$ can be chosen *independently of* j_0.

Assuming the latter, we have $t_1(\vec{k_0^1}, j_0, \vec{k_1^1}(j_0), j_1, \vec{k_2^1}(j_1))$ belongs to $D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1)) \cap G$ for $j_0 < j_1$ in X. By modifying t_1 we can guarantee that $\vec{k^1_1}(j_0) = \vec{k^1_2}(j_0)$ for all $j_0 \in X$, $j_0 \neq \min X$. Also we can arrange that $\vec{k_0} \subseteq \vec{k^1_0}$, $\vec{k_1}(j_0) \subseteq \vec{k^1_1}(j_0)$ for $j_0 \in X$. By indiscernibility, the structure $\langle \vec{k^1_1}(j_0), < \rangle$ with a unary predicate for $\vec{k_1}(j_0)$ has isomorphism type independent of the choice of $j_0 \in X$.

Similarly choose an element $t_2(\vec{k^2_0}, j_0, \vec{k^2_1}(j_0), j_1, \vec{k^2_1}(j_1), j_2, \vec{k^2_1}(j_2))$ of $D^*(\vec{k^1_0}, j_0, \vec{k^1_1}(j_0), j_1, \vec{k^1_1}(j_1), j_2, \vec{k^1_1}(j_2)) \cap G$ so that $\vec{k^1_0} \subseteq \vec{k^2_0}$ and for $j_0 \in X$, $\vec{k^1_1}(j_0) \subseteq \vec{k^2_1}(j_0)$ with the isomorphism type of $\langle \vec{k^2_1}(j_0), < \rangle$ with unary predicates for $\vec{k_1}(j_0), \vec{k^1_1}(j_0)$ independent of j_0. Continue with $t_3, t_4, \ldots$.

Let i_α be the minimum of X and β be the ordertype of $\bigcup\{\vec{k^n_1}(j_0) \mid n \in \omega\}$, an ordinal independent of the choice of $j_0 \in X$. In a generic extension where α is countable we may also arrange that $\bigcup\{\vec{k^n_0} \mid n \in \omega\} = I \cap i_\alpha$.

For any indiscernible i_γ define $\vec{k^n_1}(i_\gamma) \subseteq I \cap (i_\gamma, i_{\gamma+\beta})$ so that we have that $\langle I \cap (i_\gamma, i_{\gamma+\beta}), < \rangle$ with a predicate for $\vec{k^n_1}(i_\gamma)$ is isomorphic to $\langle \bigcup\{\vec{k^n_1}(j_0) \mid n \in \omega\}, < \rangle$ with a predicate for $\vec{k^n_1}(j_0)$, for $j_0 \in X$. Define: $G^* = \{p \in P \mid p$ is extended by some $t_n(\vec{k^n_0}, i_{\alpha_1}, \vec{k^n_1}(i_{\alpha_1}), \ldots, i_{\alpha_n}, \vec{k^n_1}(i_{\alpha_n}))$ where $\alpha \leq \alpha_1 < \cdots < \alpha_n$ are of the form $\alpha + \beta\gamma$ for some $\gamma \in \mathrm{On}\}$. Using the indiscernibility of $I - i_\alpha$ in $\langle L, A \rangle$, G^* is compatible and meets every $\langle L, A \rangle$-definable open dense subclass of P. Thus G^* is P-generic and $I_{\alpha,\beta}$ is a class of indiscernibles for $\langle L[G^*], A, G^* \rangle$.

To complete the proof we return to the problem of making $\vec{k^1_2}(j_0, j_1)$ independent of j_0. First a lemma:

7.4 Lemma. *Let $x < y$ by the maximum difference order on finite sets of ordinals: $x < y$ iff $\alpha \in y$ where α is the greatest element of the symmetric difference of x and y. For any $j_0 < j_1$ in X and any open dense D definable in $\langle L, A \rangle$ there exists a $t(\vec{\ell_0}, j_0, \vec{\ell_1}, j_1, \vec{\ell_2}, \vec{\ell}) \in L_{\min(\vec{\ell})} \cap D \cap G$ such that $\vec{\ell_0} < j_0 < \vec{\ell_1} < j_1 < \vec{\ell_2} < \vec{\ell}$ belong to I and $\vec{\ell_0} \cup \vec{\ell_1} \cup \vec{\ell_2}$ is the $<$-least finite set of ordinals (not necessarily indiscernibles) x such that $t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell})$ belongs to $L_{\min(\vec{\ell})} \cap D \cap G$.*

Proof. Let x be $<$-least such that for some t and indiscernibles $\vec{\ell} > \max(x)$, $t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell}) \in L_{\min(\vec{\ell})} \cap D \cap G$. If some $\alpha \in x$ were not in I then there would be a $t^*(x^* \cap j_0, j_0, x^* \cap (j_0, j_1), j_1, x^* - j_1, \vec{\ell^*}) = t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell})$ with $\vec{\ell}$ an initial segment of $\vec{\ell^*}$ and $x^* - \alpha = x^* - (\alpha + 1)$, as α is L-definable from indiscernibles $< \alpha$ and indiscernibles $> \vec{\ell}$. So let $\vec{\ell_0}, \vec{\ell_1}, \vec{\ell_2}$ be $x \cap j_0, x \cap (j_0, j_1), x - j_1$. ⊣

For $j_0 < j_1$ in X choose

$$t_{j_0,j_1}(\vec{k^1_0}(j_0, j_1), j_0, \vec{k^1_1}(j_0, j_1), j_1, \vec{k^1_{2,0}}(j_0, j_1), \vec{k^1_{2,1}}(j_0, j_1))$$

to be least satisfying Lemma 7.4 with $D = D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1))$, and

$\vec{\ell}$ denoted by $k^{\vec{1}}_{2,1}(j_0, j_1)$. By the choice of X we can write this as

$$t_1(k^{\vec{1}}_0, j_0, k^{\vec{1}}_1(j_0), j_1, k^{\vec{1}}_{2,0}(j_0, j_1), \vec{\infty}),$$

where $\vec{\infty}$ denotes an arbitrary sequence of large indiscernibles (of the appropriate length). Note that $\langle k^{\vec{1}}_0, k^{\vec{1}}_1(j_0), k^{\vec{1}}_{2,0}(j_0, j_1)\rangle$ is definable in $\langle L[G], A, G\rangle$ from $\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1), \vec{\infty}$ and so $k^{\vec{1}}_{2,0}(j_0, j_1)$ is definable in $\langle L[G], A, G\rangle$ from $\vec{k_1}(j_1), \vec{\infty}$ and ordinals $\leq j_1$.

7.5 Claim. $k^{\vec{1}}_{2,0}(j_0, j_1)$ *is independent of* j_0.

Proof. Let $j_0 < j_1 < \cdots < j$ be the first $\omega + 1$ elements of X and for any n, m let $\vec{k}(j_n, j)(m) = m$th element of $k^{\vec{1}}_{2,0}(j_n, j)$. If the Claim fails then for some fixed m, $\vec{k}(j_0, j)(m) < \vec{k}(j_1, j)(m) < \cdots$ is an increasing sequence of indiscernibles with supremum $\ell \in I$ (using the fact that $X - j$ has ordertype $>$ length$(\vec{\infty})$). As these ordinals are definable in $\langle L[G], A, G\rangle$ from ordinals in $(j+1) \cup \vec{k_1}(j) \cup \vec{\infty}$ we get that ℓ has cofinality $\leq j$ in $L[G]$. But $0^\# \notin L[G]$ (as G is generic over L) so by Jensen's Covering Theorem, ℓ has L-cofinality $< (j^+$ in $L[G])$. As $\ell \in I$, ℓ is L-regular and hence j^+ in $L < j^+$ in $L[G]$.

But then in $L[G]$ there is a closed unbounded $C \subseteq j$ such that $D \subseteq j$, D closed unbounded, $D \in L \to C \subseteq D \cup \alpha$ for some $\alpha < j$. Now $I \cap j$ is the intersection of countably many such D's and therefore as j has uncountable cofinality (in $L[G, 0^\#]$) we get $C \subseteq I \cup \alpha$ for some $\alpha < j$. This yields $0^\# \in L[G]$, contradiction.

This proves the claim. ⊣

With the claim we see that there is a P-generic G^* (in a set-generic extension of V) such that $\langle L[G^*], A, G^*\rangle$ has a periodic class of indiscernibles $I_{\alpha,\beta}$. It now follows by absoluteness that there is such a G^* definable in a set-generic extension of $L[0^\#]$ in which α and β are countable. This completes the proof of Theorem 7.3. ⊣

It can be shown that there can be no countable bound on the α and β of the previous proof, using the solution to the Π^1_2-singleton problem. (See [5, Sect. 8.2].)

8. Further Results

The material below is discussed in [5, Chap. 8].

Strict Genericity

In set forcing, one may show that an inner model of a generic extension is itself a generic extension. This can fail for class forcing.

8.1 Definition. Let M be a an inner model of ZFC. A real R is *generic* over M if it belongs to a generic extension of M (via a forcing defined over a ground model of the form $\langle M, A\rangle$). R is *strictly generic* over M if for some ground model $\langle M, A\rangle$, some forcing P definable over $\langle M, A\rangle$ and some G P-generic over $\langle M, A\rangle$, R belongs to $M[G]$ and G is definable over $\langle M[R], A\rangle$.

8.2 Theorem. *There is a real $R <_L 0^\#$ such that R is generic over L (for an L-definable forcing) but R is not strictly generic over L.*

As with the solution to the genericity problem, Theorem 8.2 is reduced to the violation of a definability property: If R is strictly generic over L then for some A amenable to L, $\mathrm{Sat}\langle L[R], \emptyset\rangle$ is definable over $\langle L[R], A\rangle$. The latter can be violated using class forcing.

Minimal Universes

The minimal model of $V = L[0^\#]$ can be "minimized" by a class which does *not* construct $0^\#$:

8.3 Theorem. *Suppose that for no α is $L_\alpha[0^\#]$ a model of* ZFC*. Then there is an $A \subseteq \mathrm{On}$ definable in $L[0^\#]$ such that $0^\# \notin L[A]$ and for no α is $\langle L_\alpha[A], A \cap \alpha\rangle$ elementary in $\langle L[A], A\rangle$.*

This result is partial evidence for the conjecture that $0^\#$ is generic over some proper inner model of $L[0^\#]$.

Countable Π^1_2 Sets

Assume that $R^\#$ exists for every real R (i.e. that $L[R]$ is not rigid, for every real R). Kechris and Woodin [10] showed that a nonempty countable Π^1_2 set must have an ordinal-definable element; we show that in a sense their result is optimal. First some definitions.

8.4 Definition. A set of reals X is *n-absolute* if for some formula φ, $R \in X \leftrightarrow L[R] \models \varphi(R, \omega_1, \ldots, \omega_n)$, where ω_k denotes the ω_k of V. An n-absolute singleton is a real R such that $\{R\}$ is n-absolute. We say *absolute* for 0-absolute, and *absolute singleton* for 0-absolute singleton.

8.5 Theorem (Kechris-Woodin [10]). *Assume $R^\#$ exists for every real R. A nonempty countable Π^1_2 set contains an n-absolute singleton for some n.*

Our next result demonstrates the optimality of the previous theorem.

8.6 Theorem. *For each n there is a countable Π^1_2 set X_n such that $R \in X_n$ implies that R is not an n-absolute singleton.*

Not all elements of countable Π^1_2 sets are n-absolute singletons for some n:

8.7 Theorem. *There exists a countable Π^1_2 set X and $R \in X$ such that for all n, R is not an n-absolute singleton.*

Not every absolute singleton belongs to a countable Π^1_2 set: If a set is Σ^1_2 (with a constructible parameter) and contains a non-constructible real, then it has a constructibly-coded perfect closed subset, and a code for this perfect closed set can be computed as a Σ^1_2 function applied to an index $n \in \omega$ for the given Σ^1_2 set X_n. Moreover, $\{n \mid X_n$ has a perfect closed subset$\}$ is Σ^1_2. It follows that in L there is a perfect closed set C, with code recursive in the complete Σ^1_2 subset of ω, such that $R \in C$ implies R does not belong to any Π^1_2 set whose complement contains a non-constructible real. In particular $R \in C$ implies R does not belong to a countable Π^1_2 set. As the set C has code recursive in the complete Σ^1_2 set, it contains elements which are Δ^1_3 in L, and hence which are absolute singletons.

An open problem is to provide a revealing characterization of the reals which belong to a countable Π^1_2 set.

In [8] it is proved: If X is a nonempty Π^1_2 set then X has an element R such that either $R \leq_L 0^\#$ or $0^\# \leq_L R$. Our next result implies that $0^\#$ has least nonzero L-degree among reals with this property, even when X is restricted to have a unique element.

8.8 Theorem. *There exists a sequence $\langle (R^n_0, R^n_1) \mid n \in \omega \rangle$ of pairs of reals such that:*

(a) If a real $R \leq_L R^n_0$ and $R \leq_L R^n_1$, then $R \in L$.

(b) $\{\langle R, n, i\rangle \mid R = R^n_i\}$ is Π^1_2.

(c) $n \in 0^\# \longleftrightarrow n \in R^n_0 \longleftrightarrow n \in R^n_1$.

8.9 Corollary. *Suppose that R is a non-constructible real and every Π^1_2-singleton is $\leq_L$-comparable with R. Then $0^\# \leq_L R$.*

Thus $0^\#$ is the least "canonical" Π^1_2-singleton.

New Σ^1_3 Facts

If M is an inner model with $0^\# \notin M$, then of course there is a true Σ^1_3 sentence not holding in M, namely the sentence asserting the existence of $0^\#$; can this effect be achieved by forcing over M?

8.10 Theorem. *There exists an ω-sequence of Σ^1_3 sentences $\langle \varphi_n \mid n \in \omega\rangle$ such that if M is an inner model, $0^\# \notin M$:*

(a) φ_n is false in M for some n.

(b) For each n, some generic extension of M satisfies φ_n.

Moreover, if $M = L[R]$ for some real R, then the generic extensions in (b) can be taken to be inner models of $L[R, 0^\#]$.

The proof is based on the following, which may be of independent interest.

8.11 Theorem. *There exists an L-definable function $n : L\text{-Singulars} \to \omega$ such that if M is an inner model with $0^\# \notin M$:*

(a) For some n, $M \models \{\alpha \mid n(\alpha) \le n\}$ is stationary.

(b) For each n there is a generic extension of M in which $0^\#$ does not exist and $\{\alpha \mid n(\alpha) \le n\}$ is non-stationary.

In (a) of the previous theorem, we intend that whenever $C \subseteq \mathrm{On}$ is closed unbounded and M-definable, then there is an $\alpha \in C$, $n(\alpha) \le n$. In (b) we intend that the generic extension satisfy ZFC and have a definable closed unbounded class $C \subseteq \mathrm{On}$ such that $\alpha \in C \to n(\alpha) > n$.

Killing Admissibles Revisited

8.12 Definition. α is *quasi R-admissible* if every well-ordering in $L_\alpha[R]$ has ordertype less than α.

R-admissibility implies quasi R-admissibility, but not conversely, as the limit of the first ω R-admissibles is quasi R-admissible but not R-admissible. Let $\Lambda^*(R)$ denote $\{\alpha > \omega \mid \alpha \text{ is quasi } R\text{-admissible}\}$, a closed unbounded class of ordinals containing $\Lambda(R)$.

8.13 Theorem. *Suppose that φ is Σ_1 and $L \models \varphi(\kappa)$ whenever κ is an L-cardinal. Then there is a real $R <_L 0^\#$ such that $\Lambda^*(R) \subseteq \{\alpha \mid L \models \varphi(\alpha)\}$.*

8.14 Corollary (Beller [1]). *Suppose that α is countable, $L_\alpha \models \mathrm{ZF}$. Then for some real R, α is the least ordinal such that $L_\alpha[R] \models \mathrm{ZF}$.*

8.15 Corollary. *There is a real $R <_L 0^\#$ such that $\Lambda^*(R) \subseteq \{\alpha \mid L_\alpha \models \mathrm{ZFC}^-\}$.*

Non-Characterizability of Admissibility Spectra

There cannot be a simple characterization of admissibility spectra, by virtue of the following result.

8.16 Theorem. *Let $X = \{A \subseteq \omega_1^L \mid A \in L$ and for some real R, $\omega_1^{L[R]} = \omega_1^L$ and $\Lambda(R) \cap \omega_1^L = A\}$. Then $X =_L 0^\#$.*

Δ_1-Coding

The results described here (with the exception of Theorem 8.27) are taken from [6]. A real R Δ_1*-codes* a class $A \subseteq \mathrm{On}$ iff A is Δ_1-definable over $L[R]$. Every L-amenable class A is Δ_1-coded by $0^\#$. The next result provides a converse to this result.

8.17 Proposition. *Suppose that L-Card $= \{\alpha \mid \alpha$ is a cardinal of $L\}$ is Σ_1 over $L[R]$, for a real R. Then $0^\# \le_L R$.*

Proof. Suppose that the Σ_1 definition has parameters less than κ, where κ is a singular cardinal. As κ^+ is an L-cardinal, by reflection there must be unboundedly many $\alpha < \kappa^+$ such that $\alpha \in L$-Card. But then $(\kappa^+)^L < \kappa^+$, which implies that $0^\#$ exists. As this argument can be carried out in $L[R]$, in fact $0^\# \leq_L R$. ⊣

We introduce a sufficient condition for an L-amenable class to be Δ_1-coded by a real which is class-generic over L. To motivate it we first indicate a necessary condition for Δ_1-codability:

8.18 Definition. Suppose that x is an extensional set (i.e., $\langle x, \in\rangle$ satisfies the Axiom of Extensionality). Let $\bar{x}$ denote the transitive collapse of x. For $A \subseteq \mathrm{On}$ we say that x *preserves* A if $\langle \bar{x}, \in, A \cap \bar{x}\rangle$ is isomorphic to $\langle x, \in, A \cap x\rangle$.

8.19 Definition. For a set x and ordinal δ, $x[\delta]$ denotes $\{f(\gamma) \mid \gamma < \delta,$ $f \in x$, f a function whose domain contains $\gamma\}$. We say that x strongly preserves $A \subseteq \mathrm{On}$ if $x[\delta]$ is extensional and preserves A for each cardinal δ. A sequence of extensional sets $t_0 \subseteq t_1 \subseteq \cdots$ is tight if it is continuous (i.e., $t_\lambda = \bigcup\{t_i \mid i < \lambda\}$ for limit λ) and for each i: $t_i = t_{i+1}$ or $t_i \in t_{i+1}$, $\langle \bar{t}_j \mid j < i\rangle$ belongs to the least ZFC^- model containing $\bar{t}_i$ as an element which correctly computes $\mathrm{Card}(\bar{t}_i)$.

Condensation Condition. Suppose that t is transitive, κ is regular, $\kappa \in t$ and $x \in t$. Then:

(a) There is a tight κ-sequence $t_0 \prec t_1 \prec \cdots \prec t$ such that $x \in t_0$ and for each $i < \kappa$: $\mathrm{Card}(t_i) = \kappa$ and t_i strongly preserves A.

(b) If κ is inaccessible, then there exists a $t_0 \prec t_1 \prec \cdots \prec t$ as above, but where each $\mathrm{Card}(t_i) = \omega_i$.

8.20 Theorem (Δ_1-Coding Theorem). *Suppose that A is L-amenable and obeys the Condensation Condition in L. Then A is Δ_1-coded in a tame class-generic extension of $\langle L, A\rangle$ by a real R such that $L, L[R]$ have the same cofinalities.*

8.21 Corollary. *Suppose that A is L-amenable, obeys the Condensation Condition in L and preserves indiscernibles. Then A is Δ_1-definable over $L[R]$ for some indiscernible preserving real R such that L and $L[R]$ have the same cofinalities.*

We can apply the above to show that $L\text{-Cof}(\omega) = \{\alpha \mid \alpha \text{ has } L\text{-cofinality } \omega\}$ is Δ_1-definable in $L[R]$, where R is a real not constructing $0^\#$.

8.22 Lemma. *There is a real R_0, class-generic over L, such that $R_0 <_L 0^\#$, the cardinals of $L[R_0]$ are those of L, excluding ω_1^L, and the Condensation Condition holds for $A = L\text{-Cof}(\omega)$ in $L[R_0]$.*

8.23 Corollary. *There is a real $R <_L 0^\#$ such that R is class-generic over L, R preserves indiscernibles, the cardinals of $L[R]$ are those of L, excluding ω_1^L, and L-$\mathrm{Cof}(\omega)$ is Δ_1 over $L[R]$.*

8.24 Corollary. *There is a real $R <_L 0^\#$ such that every quasi R-admissible has uncountable L-cofinality.*

8.25 Corollary. *There is a real $R <_L 0^\#$ such that the function $f(\alpha) = [\alpha]^\omega \cap L$ is Δ_1 over $L[R]$.*

An *immune partition* is $F : \mathrm{On} \to 2$ such that neither $\{\alpha \mid F(\alpha) = 0\}$ nor $\{\alpha \mid F(\alpha) = 1\}$ contains an infinite constructible set.

8.26 Corollary. *There is a real $R <_L 0^\#$ such that some immune partition is $\Delta_1(L[R])$.*

We consider the "characterization problem" for Δ_1-definability in a real: Is there an exact constructible criterion for a subset of an L-cardinal κ to be the intersection with κ of a predicate which is Δ_1-definable in $L[R]$ for some real R that preserves L-cardinals? The answer is "No" when κ is ω_3^L.

8.27 Theorem. *Let $S = \{X \subseteq \omega_3^L \mid X = \omega_3^L \cap A$ for some $A \subseteq \mathrm{On}$, A Δ_1-definable in $L[R]$ for some real R such that $L[R]$ and L have the same cardinals$\}$. Then $S =_L 0^\#$.*

Theorem 8.27 rules out any simple characterization of when an L-amenable predicate can be Δ_1-definable in a real not constructing $0^\#$.

Minimal Coding

We have the following strengthening of the Coding Theorem.

8.28 Theorem. *Suppose that $A \subseteq \mathrm{On}$ and $\langle L[A], A\rangle$ is a model of $\mathrm{ZFC} + \mathrm{GCH}$. Then there is an $\langle L[A], A\rangle$-definable class forcing P such that if $G \subseteq P$ is P-generic over $\langle L[A], A\rangle$:*

(a) $\langle L[A, G], A, G\rangle$ is a model of $\mathrm{ZFC} + \mathrm{GCH}$.

(b) $L[A, G] = L[R]$ for some real R and A, G are definable over $L[R]$ from the parameter R.

(c) $L[A]$ and $L[R]$ have the same cofinalities.

(d) R is minimal over $L[A]$: if $x \in L[R]$, then either $x \in L[A]$ or $R \in L[A, x]$.

Thus a universe obeying GCH can be "coded minimally" by a real. Note that in clause (d) of the Theorem, x is any set constructible from R, not necessarily a real.

Further Applications to Descriptive Set Theory

Solovay [70] established the consistency of a number of regularity properties for projective sets of reals, using a natural model in which ω_1 *is inaccessible to reals* (i.e., ω_1 is an inaccessible cardinal in $L[R]$ for each real R). In this section we construct other models with this property, which can be applied to the study of regularity properties for projective sets and projective prewellorderings.

Recall that a set of reals is $\mathbf{\Sigma^1_1}$ if it is the continuous image of a Borel set and is $\mathbf{\Pi^1_1}$ if its complement is $\mathbf{\Sigma^1_1}$. It is $\mathbf{\Sigma^1_{n+1}}$ if it is the continuous image of a $\mathbf{\Pi^1_n}$ set and is $\mathbf{\Pi^1_{n+1}}$ if its complement is $\mathbf{\Sigma^1_{n+1}}$. A set of reals is $\mathbf{\Delta^1_n}$ if both it and its complement are $\mathbf{\Sigma^1_n}$. Similar definitions apply to k-ary relations on the reals. It a set of reals (or k-ary relation in reals) is $\mathbf{\Sigma^1_n}$ for some n then we say that it is *projective*.

Regularity Properties

8.29 Definition. Measure ($\mathbf{\Sigma^1_n}$) is the assertion that every $\mathbf{\Sigma^1_n}$ set of reals is Lebesgue Measurable. Category ($\mathbf{\Sigma^1_n}$) is the assertion that every $\mathbf{\Sigma^1_n}$ set of reals has the Baire Property, i.e., has meager symmetric difference with some Borel set. Perfect ($\mathbf{\Sigma^1_n}$) is the assertion that any uncountable $\mathbf{\Sigma^1_n}$ set of reals contains a perfect closed subset. Similar definitions apply to $\mathbf{\Pi^1_n}, \mathbf{\Delta^1_n}$.

In ZFC one may prove Measure ($\mathbf{\Sigma^1_1}$), Category ($\mathbf{\Sigma^1_1}$), Perfect ($\mathbf{\Sigma^1_1}$). In Gödel's model L one has $\sim$Measure ($\mathbf{\Delta^1_2}$), $\sim$Category ($\mathbf{\Delta^1_2}$), $\sim$Perfect ($\mathbf{\Pi^1_1}$) using the fact that in L there is a $\mathbf{\Delta^1_2}$ wellordering of the reals (and the Kondo-Addison Uniformization Theorem for $\mathbf{\Pi^1_1}$). By extending ZFC slightly we get:

8.30 Theorem (Solovay [18]). *Assume that ω_1 is inaccessible to reals. Then the following hold: Measure ($\mathbf{\Sigma^1_2}$), Category ($\mathbf{\Sigma^1_2}$), Perfect ($\mathbf{\Sigma^1_2}$).*

Our next result implies that the previous Theorem is optimal. The proof is based on [2].

8.31 Theorem. *Assume the consistency of an inaccessible cardinal. Then there is a model in which:*

(a) ω_1 is inaccessible to reals.

(b) There is a Δ^1_3 wellordering of the reals, and hence $\sim$Measure (Δ^1_3), $\sim$Category (Δ^1_3).

(c) $\sim$Perfect (Π^1_2).

8.32 Remark. We use $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ to denote the "effective" versions of $\mathbf{\Sigma^1_n}$, $\mathbf{\Pi^1_n}$, $\mathbf{\Delta^1_n}$; see [12] for details.

Another axiom with consequences for regularity properties of projective sets is Martin's Axiom (MA). (We take MA to include the hypothesis $\sim$ CH.)

8.33 Theorem. MA *implies Measure* $(\mathbf{\Sigma^1_2})$, *Category* $(\mathbf{\Sigma^1_2})$.

Again this is optimal.

8.34 Theorem. *This is a model of* MA *in which:*

(a) $\omega_1 = \omega_1^L$.

(b) There is a Δ^1_3 *wellordering of the reals.*

8.35 Remark. Perfect (Π^1_1) fails in the above model, as this property implies that ω_1^L is countable. It is not known if (a) can be replaced by "ω_1 is inaccessible to reals" in the previous theorem (assuming the consistency of a weakly compact cardinal; this is a necessary assumption for the consistency of MA + "ω_1 is inaccessible to reals").

Theorem 8.31 generalizes to higher levels of the projective hierarchy. Recall again that κ is *Mahlo* if κ is inaccessible and $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary.

8.36 Theorem. *Assume the consistency of a Mahlo cardinal. Then there is a model in which:*

(a) Measure $(\mathbf{\Sigma^1_3})$, *Category* $(\mathbf{\Sigma^1_3})$. *Perfect* $(\mathbf{\Sigma^1_3})$.

(b) There is a Δ^1_4 *wellordering of the reals.*

(c) $\sim$*Perfect* (Π^1_3).

8.37 Remark. To go further, one must replace L by a sufficiently $\mathbf{\Sigma^1_3}$ correct model. Thus, assuming the consistency of a Mahlo cardinal κ, together with "$x^\#$ exists for every bounded subset x of κ", one obtains a model of Measure $(\mathbf{\Sigma^1_4})$, Category $(\mathbf{\Sigma^1_4})$, Perfect $(\mathbf{\Sigma^1_4})$, $\sim$Perfect (Π^1_4) with a Δ^1_5 wellordering of the reals. However the author does not know if this use of #'s is necessary.

Prewellorderings

A *prewellordering* is a reflexive, transitive well-founded relation. A wellordering is obtained by identifying two elements a, b when $a \leq b$, $b \leq a$; the *length* of the prewellordering is the ordertype of its associated wellordering.

$\delta^1_{\mathbf{n}}$ denotes the supremum of the lengths of all $\mathbf{\Delta^1_n}$ prewellorderings of the reals.

8.38 Theorem (Classical). $\delta^1_{\mathbf{1}} = \omega_{\mathbf{1}}$.

Kunen and Martin showed that $\delta^1_{\mathbf{2}}$ is at most ω_2 (see [11]). The next result shows that this result is the best possible.

8.39 Theorem. *It is consistent with* ZFC *that* $\delta^1_{\mathbf{2}} = \omega_{\mathbf{2}}$.

Using the Condensation Condition, we can simultaneously have ω_1 inaccessible to reals:

8.40 Theorem (Friedman-Woodin [7]). *Assuming the consistency of an inaccessible, there is a model in which $\boldsymbol{\delta^1_2} = \omega_2$ and ω_1 is inaccessible to reals.*

There is no explicit bound on $\boldsymbol{\delta^1_3}$ provable in ZFC, even with the added hypothesis that ω_1 is inaccessible to reals.

8.41 Theorem (Sect. 8.4 of Friedman [5]). *Assuming the consistency of an inaccessible, there is a model in which ω_1 is inaccessible to reals and there is a Π^1_2 wellordering of some set of reals of length κ, for any pre-chosen L-definable cardinal κ (and hence $\boldsymbol{\delta^1_3} \geq \kappa$).*

9. Some Open Problems

1. Can one code a class by a real preserving Π^n_m-indescribability for $n > 1$?
2. Define *n-generic over L* as follows: R is *0-generic over L* iff R is generic over L. R is *$n+1$-generic over L* iff R is generic over an inner model of $L[S]$, where S is n-generic over L. Does $n+1$-genericity imply n-genericity for some n? Is there a real $R <_L 0^\#$ which is *not* n-generic over L for any n?
3. Is $0^\#$ generic over some proper inner model of $L[0^\#]$?
4. Can one prove that $L[0^\#]$ is generically saturated over L in the theory ZFC + "$0^\#$ exists"?
5. Is $L[0^\#]$ the *least* inner model which is generically saturated over L?
6. Is there a reasonable notion of "forcing" with the property that every real either constructs $0^\#$ or can be obtained by "forcing" over L?
7. Is there a real R such that $0 <_L R <_L 0^\#$ which is the unique solution to a Π^1_2 formula φ which *provably in* ZFC has at most one solution?
8. Is there a simple characterization of the reals which belong to a countable Π^1_2 set?
9. Assuming only the consistency of an inaccessible cardinal, is it consistent for each n that all $\boldsymbol{\Sigma^1_n}$ sets of reals be Lebesgue Measurable and have the Baire and Perfect Set properties, while there is a Δ^1_{n+1} wellordering of the reals?
10. Assuming only the consistency of a weakly compact cardinal, is it consistent to have Martin's Axiom, ω_1 inaccessible to reals, and a Δ^1_3 wellordering of the reals?
11. Is it consistent for Δ^1_3-reducibility and L-reducibility to coincide?

12. Assuming only the consistency of an inaccessible cardinal, is it consistent for Post's problem to fail in HC = the hereditarily countable sets?

13. Is there a *remarkable real*; i.e., a real $R <_L 0^{\#}$ such that R is not generic over L, R is a Π^1_2-singleton, $\Lambda(R) =$ the recursively inaccessible ordinals and R has minimal L-degree? It has not yet been shown that there is a real $R <_L 0^{\#}$ which has more than one of these properties simultaneously.

14. Is it consistent that any parameter-free Σ^1_3 sentence true in a class forcing extension of V, be already true in V?

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9. Fine Structure

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Contents

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Fine structure theory is an in-depth study of definability over levels of constructible hierarchies. It was invented by Ronald B. Jensen (cf. [3]), and later pursued by Jensen, Mitchell, Steel, and others (cf. for instance [7] and [4]). Our aim here is to give a self-contained introduction to this theory.

Fine structure theory is a necessary tool for a detailed analysis of Gödel's L and of more complicated constructible models; in fact, it is unavoidable even for the construction of an important class of such models, the so-called core models. The present chapter is thus intended as an introduction to chapters [5], [9], and [12], where core model theory is developed and applied. It may also be read as an introduction to [7], [4], or [15].

An important result of [3] is the Σ_n-uniformization theorem (cf. [3, Theorem 3.1]), which implies that for any ordinal α and for any positive integer n there is a Σ_n-Skolem function for J_α, i.e., a Skolem function for Σ_n relations over J_α which is itself Σ_n-definable over J_α. The naïve approach for obtaining such a Skolem function only works for $n = 1$; for $n > 1$, fine structure theory is called for.

Classical applications of the fine structure theory are to establish Jensen's results that $\square_\kappa$ holds in L for every infinite cardinal κ (cf. [3, Theorem 5.2]) and his Covering Lemma: If $0^\#$ does not exist, then every uncountable set of ordinals can be covered by a set in L of the same size (cf. [2]). We shall prove $L \models \square_\kappa$ as well as a slight weakening of Jensen's Covering Lemma in the final section of this chapter (cf. [5] on a complete proof of the Covering Lemma); they have been generalized by recent research (cf. [10] and [8]; cf. also [9]).

The present chapter will discuss the "pure" part of fine structure theory, the part which is not linked to any particular kind of constructible model one might have in mind. We shall discuss Jensen's classical version of this theory. We shall not, however, deal with Jensen's Σ^* theory (which may be found in [15, Sects. 1.6–1.8] or in [14]), and we shall also ignore other variants of the fine structure theory which have been created. What we shall deal with here is tantamount to what is presented in (parts of) [7, §2 and 4].

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1. Acceptable J-structures

An inner model is a transitive proper class model of ZF. If A is a set or a proper class, then $L[A]$ is the least inner model which is closed under the operation $x \mapsto A \cap x$. An important example is $L = L[\emptyset]$, Gödel's constructible universe. V itself, the universe of all sets, is of the form $L[A]$

in a class forcing extension that does not add any new sets.[1]

Any model of the form $L[A]$ may be stratified in two ways: into levels of the L-hierarchy and into levels of the J-hierarchy. The former approach was Gödel's original one, but it turned out that the latter one (which was introduced by Jensen in [3]) is more useful.

In order to define the J-hierarchy we need the concept of rudimentary functions (cf. [3, p. 233]).

1.1 Definition. Let A be a set or a proper class. A function $f : V^k \to V$, where $k < \omega$, is called *rudimentary in A* (or, rud_A) if it is generated by the following schemata:

$$\begin{aligned}
f(\langle x_1, \ldots, x_k\rangle) &= x_i, \\
f(\langle x_1, \ldots, x_k\rangle) &= x_i \setminus x_j, \\
f(\langle x_1, \ldots, x_k\rangle) &= \{x_i, x_j\}, \\
f(\langle x_1, \ldots, x_k\rangle) &= h(g_1(\langle x_1, \ldots, x_k\rangle), \ldots, g_\ell(\langle x_1, \ldots, x_k\rangle)), \\
f(\langle x_1, \ldots, x_k\rangle) &= \textstyle\bigcup_{y \in x_1} g(\langle y, x_2, \ldots, x_k\rangle), \\
f(x) &= x \cap A
\end{aligned}$$

f is called *rudimentary* (or, rud) if f is $\mathrm{rud}_\emptyset$.

Let us write $\vec{x}$ for $\langle x_1, \ldots, x_k\rangle$. It is easy to verify that for instance the following functions are rudimentary: $f(\vec{x}) = \bigcup x_i$, $f(\vec{x}) = x_i \cup x_j$, $f(\vec{x}) = \{x_1, \ldots, x_k\}$, and $f(\vec{x}) = \langle x_1, \ldots, x_k\rangle$. Proposition 1.3 below will provide more information.

If U is a set and A is a set or a proper class then we shall denote by $\mathrm{rud}_A(U)$ the rud_A closure of U,[2] i.e., the set

$$U \cup \{f(\langle x_1, \ldots, x_k\rangle) \mid f \text{ is } \mathrm{rud}_A \text{ and } x_1, \ldots, x_k \in U\}.$$

It is not hard to verify that if U is transitive, then so is $\mathrm{rud}_A(U \cup \{U\})$. We shall now be interested in $\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\})$ (cf. Lemma 1.4 below).

1.2 Definition. Let A be a set or a proper class. A relation $R \subseteq V^k$, where $k < \omega$, is called *rudimentary in A* (or, rud_A) if there is a rud_A function $f : V^k \to V$ such that $R = \{\vec{x} \mid f(\vec{x}) \neq \emptyset\}$. R is called *rudimentary* (or, rud) if R is $\mathrm{rud}_\emptyset$.

1.3 Proposition. *Let A be a set or a proper class.*

(a) The relation $\notin$ *is* rud.

(b) Let f, R *be* rud_A. *Let* $g(\vec{x}) = f(\vec{x})$ *if* $R(\vec{x})$ *holds, and* $g(\vec{x}) = \emptyset$ *if not. Then* g *is* rud_A.

---

[1] This class forcing extension is obtained simply by forcing with enumerations $p : \alpha \to V$, ordered by end-extension.

[2] This is in contrast to [3, p. 238], where $\mathrm{rud}_A(U)$ stands for the rud_A closure of $U \cup \{U\}$.

(c) If R, S are rud_A*, then so is* $R \cap S$*.*

(d) Membership in A is rud_A*.*

(e) If R is rud_A*, then so is its characteristic function* χ_R*.*

(f) R is rud_A *iff* $\neg R$ *is* rud_A*.*

(g) Let R be rud_A*. Let* $f(\langle y, \vec{x}\,\rangle) = y \cap \{z; R(\langle z, \vec{x}\,\rangle)\}$*. Then f is* rud_A*.*

(h) If $R(\langle y, \vec{x}\,\rangle)$ *is* rud_A *then so is* $\exists z \in y R(\langle z, \vec{x}\,\rangle)$*.*

Proof. (a) $x \notin y$ iff $\{x\} \setminus y \neq \emptyset$. (b) If $R(\vec{x}) \leftrightarrow r(\vec{x}) \neq \emptyset$, where r is rud_A, then $g(\vec{x}) = \bigcup_{y \in r(\vec{x})} f(\vec{x})$. (c) Let $R(\vec{x}) \leftrightarrow f(\vec{x}) \neq \emptyset$, where f is rud_A. Let $g(\vec{x}) = f(\vec{x})$ if $S(\vec{x})$ holds, and $g(\vec{x}) = \emptyset$ if not. g is rud_A by (b), and thus g witnesses that $R \cap S$ is rud_A. (d) $x \in A$ iff $\{x\} \cap A \neq \emptyset$. (e): by (b). (f) $\chi_{\neg R}(\vec{x}) = 1 \setminus \chi_R(\vec{x})$. (g) Let $g(\langle z, \vec{x}\rangle) = \{z\}$ if $R(\langle z, \vec{x}\rangle)$ holds, and $g(\langle z, \vec{x}\rangle) = \emptyset$ if not. We have that g is rud_A by (b), and $f(\langle y, \vec{x}\rangle) = \bigcup_{z \in y} g(z, \vec{x})$. (h) Set $f(y, \vec{x}) = y \cap \{z; R(\langle z, \vec{x}\rangle)\}$. f is rud_A by (g), and thus f witnesses that $\exists z \in y R(\langle z, \vec{x}\rangle)$ is rud_A. $\dashv$

We shall be concerned here with structures of the form $\langle U, \in, A_0, \ldots, A_m\rangle$, where U is transitive. (By $\langle U, \in, A_0, \ldots, A_m\rangle$ we shall mean the structure $\langle U, \in\restriction U, A_0 \cap U, \ldots, A_m \cap U\rangle$.) Each such structure comes with a language $\mathcal{L}_{\dot{A}_0, \ldots, \dot{A}_m}$ with predicates $\dot{\in}$, $\dot{A}_0, \ldots, \dot{A}_m$. We shall restrict ourselves to the case where $m = 0$ or $m = 1$.

If $M = \langle |M|, \ldots\rangle$ is a structure, $X \subseteq |M|$, and $n < \omega$ then we let $\Sigma_n^M(X)$ denote the set of all relations which are Σ_n-definable over M from parameters in X. We shall also write $\mathbf{\Sigma}_n^M$ for $\Sigma_n^M(M)$, and we shall write $\mathbf{\Sigma}_\omega^M$ for $\bigcup_{n<\omega} \mathbf{\Sigma}_n^M$. Further, we shall write Σ_n^M for $\Sigma_n^M(\emptyset)$, where $n \leq \omega$.

The following lemma says that $\mathrm{rud}_A(U \cup \{U\})$ is just the result of "stretching" $\mathbf{\Sigma}_\omega^{\langle U, \in, A\rangle}$ without introducing additional elements of $\mathcal{P}(U)$.

1.4 Lemma. *Let U be a transitive set, and let A be a set or proper class such that* $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$*. Then* $\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\}) = \mathcal{P}(U) \cap \mathbf{\Sigma}_\omega^{\langle U, \in, A\rangle}$*.*

Proof. Notice that $\mathcal{P}(U) \cap \mathbf{\Sigma}_\omega^{\langle U, \in, A\rangle} = \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U \cup \{U\}, \in, A \cap U\rangle}$, so that we have to prove that

$$\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\}) = \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U \cup \{U\}, \in, A\rangle}$$

"$\supseteq$": By Proposition 1.3 (a) and (d), $\notin$ and membership in A are both rud_A. By Proposition 1.3 (f), (c), and (h), the collection of rud_A relations is closed under complement, intersection, and bounded quantification. Therefore we get inductively that every relation which is Σ_0 in the language $\mathcal{L}_{\dot{A}}$ with $\dot{\in}$ and $\dot{A}$ is also rud_A.

Now let $x \in \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U \cup \{U\}, \in, A\rangle}$. There is then some rud_A relation R and there are $x_1, \ldots, x_k \in U \cup \{U\}$ such that $y \in x$ iff $y \in U$ and $R(\langle y, x_1, \ldots, x_k\rangle)$

holds. But then $x = U \cap \{y; R(\langle y, x_1, \ldots, x_k\rangle)\} \in \mathrm{rud}_A(U \cup \{U\})$ by Proposition 1.3 (g).

"$\subseteq$": Call a function $f : V^k \to V$, where $k < \omega$, *simple* iff the following holds true: if $\varphi(v_0, v_1, \ldots, v_m)$ is Σ_0 in the language $\mathcal{L}_{\dot{A}}$ with $\dot{\in}$ and $\dot{A}$, then $\varphi(f(v'_1, \ldots, v'_k), v_1, \ldots, v_m)$ is equivalent over transitive rud_A closed structures to a Σ_0 formula in the same language. It is not hard to verify inductively that every rud_A function is simple. (Here we use the hypothesis that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$ which ensures that in this situation quantifying over A is tantamount to quantifying over $A \cap U$.)

Now let $x \in \mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\})$, say $x = f(\langle x_1, \ldots, x_k\rangle)$, where $x_1, \ldots, x_k \in U \cup \{U\}$ and f is rud_A. Then "$v_0 \in f(\langle v_1, \ldots, v_k\rangle)$" is (equivalent over $\mathrm{rud}_A(U \cup \{U\})$ to) a Σ_0 formula in the language $\mathcal{L}_{\dot{A}}$, and hence $x = \{y \in U \mid y \in f(\langle x_1, \ldots, x_n\rangle)\}$ is in $\Sigma_0^{\langle U \cup \{U\}, \in, A\rangle}(\{x_1, \ldots, x_n\})$. $\dashv$

Of course Lemma 1.4 also holds with $\mathcal{P}(U)$ being replaced by the set of all relations on U. The hypothesis that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$ in Lemma 1.4 is needed to avoid pathologies; it is always met in the construction of fine structural inner models.

Let U be rud_A closed, and let $x \in U$ be transitive. Suppose that $B \in \Sigma_0^{\langle U, \in, A\rangle}(\{x_1, \ldots, x_k\})$, where $x_1, \ldots, x_k \in x$. Then $B \cap x \in \mathbf{\Sigma}_0^{\langle x, \in, A\rangle}$, and hence $B \cap x \in \mathrm{rud}_A(x \cup \{x\})$ by Lemma 1.4. But $\mathrm{rud}_A(x \cup \{x\}) \subseteq U$, and therefore $B \cap x \in U$. We have shown the following.

1.5 Lemma. *Let U be a transitive set such that for every $x \in U$ there is some transitive $y \in U$ with $x \in y$, let A be a set or a proper class, and suppose that U is* rud_A *closed. Then $\langle U, \in, A\rangle$ is a model of Σ_0 Comprehension in the sense that if $B \in \mathbf{\Sigma}_0^{\langle U, \in, A\rangle}$ and $x \in U$ is transitive then $B \cap x \in U$.*

In the next section we shall start with studying possible failures of Σ_1 Comprehension in rud_A closed structures. Lemma 1.5 provides the key element for proving that (all but two of) the structures we are now about to define are models of "basic set theory" (cf. [1, p. 36]), a theory that consists of Σ_0 Comprehension together with Extensionality, Foundation, Pairing, Union, Infinity, and the statement that Cartesian products exist.[3]

We may now define the J^A_α hierarchy as follows. For later purposes it is convenient to index this hierarchy by limit ordinals.[4]

1.6 Definition. Let A be a set or a proper class.

$$\begin{aligned} J^A_0 &= \emptyset, \\ J^A_{\alpha+\omega} &= \mathrm{rud}_A(J^A_\alpha \cup \{J^A_\alpha\}), \\ J^A_{\omega\lambda} &= \textstyle\bigcup_{\alpha<\lambda} J^A_{\omega\alpha} \quad \text{for limit } \lambda, \\ L[A] &= \textstyle\bigcup_{\alpha\in\mathrm{On}} J^A_{\omega\alpha}. \end{aligned}$$

[3] Said structures will also be models of "the transitive closure of any set exists", a statement which—despite of a claim made in [1]—is not provable even in Zermelo's set theory.
[4] This is again in contrast with [3].

Every J^A_α is rud_A closed and transitive. We shall also denote by J^A_α the structure $\langle J^A_\alpha, \in\restriction J^A_\alpha, A \cap J^A_\alpha\rangle$.

An important special case is obtained by letting $A = \emptyset$ in Definition 1.6. We write J_α for $J^\emptyset_\alpha$, and L for $L[\emptyset]$. L is Gödel's constructible universe; it will be studied in the last section of this chapter. Other important examples are obtained by letting A code a (carefully chosen) sequence of extenders; such models are discussed in [5, 9, 12].

The following is an immediate consequence of Lemma 1.4.

1.7 Lemma. *Let A be a set or proper class such that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$, and let α be a limit ordinal. Then $\mathcal{P}(J^A_\alpha) \cap J^A_{\alpha+\omega} = \mathcal{P}(J^A_\alpha) \cap \boldsymbol{\Sigma}^{J^A_\alpha}_\omega$.*

It is often necessary to work with the auxiliary hierarchy S^A_α of [3, p. 244] which is defined as follows:

$$\begin{aligned}
S^A_0 &= \emptyset,\\
S^A_{\alpha+1} &= \mathbf{S}^A(S^A_\alpha),\\
S^A_\lambda &= \textstyle\bigcup_{\xi<\lambda} S^A_\xi \quad \text{for limit } \lambda
\end{aligned}$$

where $\mathbf{S}^A$ is an operator which, applied to a set U, adds images of members of $U \cup \{U\}$ under rud_A functions from a certain carefully chosen fixed *finite* list. We may set

$$\mathbf{S}^A(U) = \textstyle\bigcup_{i=0}^{15} F_i``(U \cup \{U\})^2,$$

where

$$\begin{aligned}
F_0(x,y) &= \{x,y\},\\
F_1(x,y) &= x \setminus y,\\
F_2(x,y) &= x \times y,\\
F_3(x,y) &= \{\langle u,z,v\rangle \mid z \in x \wedge \langle u,v\rangle \in y\},\\
F_4(x,y) &= \{\langle u,v,z\rangle \mid z \in x \wedge \langle u,v\rangle \in y\},\\
F_5(x,y) &= \textstyle\bigcup x,\\
F_6(x,y) &= \mathrm{dom}(x),\\
F_7(x,y) &= \in \cap (x \times x),\\
F_8(x,y) &= \{x``\{z\} \mid z \in y\},\\
F_9(x,y) &= \langle x,y\rangle,\\
F_{10}(x,y) &= x``\{y\},\\
F_{11}(x,y) &= \langle \mathrm{left}(y), x, \mathrm{right}(y)\rangle,\\
F_{12}(x,y) &= \langle \mathrm{left}(y), \mathrm{right}(y), x\rangle,\\
F_{13}(x,y) &= \{\mathrm{left}(y), \langle \mathrm{right}(y), x\rangle\},\\
F_{14}(x,y) &= \{\mathrm{left}(y), \langle x, \mathrm{right}(y)\rangle\},\\
F_{15}(x,y) &= A \cap x.
\end{aligned}$$

(Here, $\langle x_1, x_2, \ldots, x_n\rangle = \langle x_1, \langle x_2, \ldots, x_n\rangle\rangle$, and $\mathrm{left}(y) = u$ and $\mathrm{right}(y) = v$ if $y = \langle u, v\rangle$ and $\mathrm{left}(y) = 0 = \mathrm{right}(y)$ if y is not an ordered pair.) It is not difficult to show that each F_i, $0 \leq i \leq 15$, is rud_A. A little bit more work is necessary to show that every rud_A function can be generated by using functions from this list. The functions F_i, $0 \leq i \leq 15$, are therefore a *basis* for the set of rud_A functions (cf. [3, Lemma 1.8]).

Every S^A_α is transitive,[5] and moreover

$$J^A_\alpha = S^A_\alpha \tag{9.1}$$

for all limit ordinals α. It is easy to see that there is only a finite jump in rank from S^A_α to $S^A_{\alpha+1}$. A straightforward induction shows that $J^A_\alpha \cap \mathrm{On} = \alpha$ for all limit ordinals α.

Recall that a structure $\langle U, \in, A_1, \ldots, A_m\rangle$ is called *amenable* if and only if $A_i \cap x \in U$ whenever $1 \leq i \leq m$ and $x \in U$. Lemma 1.5 together with (9.1) readily gives the following.

1.8 Lemma. *Let A be a set or proper class, and let α be a limit ordinal. Let $B \in \mathbf{\Sigma}_0^{J^A_\alpha}$. Then $\langle J^A_\alpha, B\rangle$ is amenable, i.e., J^A_α is a model of Σ_0 Comprehension in the language $\mathcal{L}_{\dot{A},\dot{B}}$ with $\dot{\in}$, $\dot{A}$ and $\dot{B}$.*

1.9 Definition. A *J-structure* is an amenable structure of the form $\langle J^A_\alpha, B\rangle$ for a limit ordinal α and predicates A, B.

Here, $\langle J^A_\alpha, B\rangle$ denotes the structure $\langle J^A_\alpha, \in\restriction J^A_\alpha, A \cap J^A_\alpha, B \cap J^A_\alpha\rangle$. Any J^A_α is a J-structure.

1.10 Lemma. *Let J^A_α be a J-structure.*

(1) For all $\beta < \alpha$, $\langle S^A_\gamma \mid \gamma < \beta\rangle \in J^A_\alpha$. In particular, $S^A_\beta \in J^A_\alpha$ for all $\beta < \alpha$.

(2) $\langle S^A_\gamma \mid \gamma < \alpha\rangle$ is uniformly $\Sigma_1^{J^A_\alpha}$. That is, "$x = S^A_\gamma$" is Σ_1 over J^A_α as witnessed by a formula that does not depend on α.

Proof. (1) and (2) are shown simultaneously by induction on $\langle \alpha, \beta\rangle$, ordered lexicographically. Fix α and $\beta < \alpha$. If β is a limit ordinal then inductively by (2), $\langle S^A_\gamma \mid \gamma < \beta\rangle$ is $\Sigma_1^{J^A_\beta}$, and hence $\langle S^A_\gamma \mid \gamma < \beta\rangle \in J^A_\alpha$ by Lemma 1.7. If $\beta = \delta + 1$ then inductively by (1), $\langle S^A_\gamma \mid \gamma < \delta\rangle \in J^A_\alpha$. If δ is a limit ordinal then $S^A_\delta = \bigcup_{\gamma<\delta} S^A_\gamma \in J^A_\alpha$, and if $\delta = \bar{\delta} + 1$ then $S^A_\delta = S^A_{\bar{\delta}} \cup \mathbf{S}^A(S^A_{\bar{\delta}}) \in J^A_\alpha$ as well. It follows that $\langle S^A_\gamma \mid \gamma < \beta\rangle \in J^A_\alpha$. (2) is then not hard to verify. ⊣

We may recursively define a well-ordering $<^A_\beta$ of S^A_β as follows. If β is a limit ordinal then we let $<^A_\beta = \bigcup_{\gamma<\beta} <^A_\gamma$. Now suppose that $\beta = \bar{\beta} + 1$. The

[5] The above list in fact contains more functions than the list from [3, Lemma 1.8]; this enlargement yields the transitivity of each S^A_α.

order $<^A_\beta$ induces a lexicographical order, call it $<^A_{\bar\beta,\mathrm{lex}}$, of $16 \times S^A_{\bar\beta} \times S^A_{\bar\beta}$. We may then set

$$x <^A_\beta y \quad\Longleftrightarrow\quad \begin{cases} x, y \in S^A_{\bar\beta} \quad \text{and} \quad x <^A_{\bar\beta} y, \quad \text{or else} \\ x \in S^A_{\bar\beta} \wedge y \notin S^A_{\bar\beta}, \quad \text{or else} \\ x, y \notin S^A_{\bar\beta} \quad \text{and} \quad (i, u_x, v_x) <^A_{\bar\beta,\mathrm{lex}} (j, u_y, v_y) \\ \quad \text{where } (i, u_x, v_x) \text{ is } <^A_{\bar\beta,\mathrm{lex}}\text{-minimal with } x = F_i(u_x, v_x) \\ \quad \text{and } (j, u_y, v_y) \text{ is } <^A_{\bar\beta,\mathrm{lex}}\text{-minimal with } y = F_j(u_y, v_y). \end{cases}$$

The following is easy to prove.

1.11 Lemma. *Let J^A_α be a J-structure.*

(1) For all $\beta < \alpha$, $\langle <^A_\gamma \mid \gamma < \beta\rangle \in J^A_\alpha$. In particular, $<^A_\beta \in J^A_\alpha$ for all $\beta < \alpha$.

(2) $\langle <^A_\gamma \mid \gamma < \alpha\rangle$ is uniformly $\Sigma_1^{J^A_\alpha}$. That is, "$x = <^A_\gamma$" is Σ_1 over J^A_α as witnessed by a formula that does not depend on α.

If $M = J^A_\alpha$, then we shall also write $<_M$ for $<^A_\alpha$.

We shall now start working towards showing that J-structures have Σ_1-definable Σ_1-Skolem functions.

In what follows we shall fix a recursive enumeration $\langle \varphi_i ; i \in \omega\rangle$ of all Σ_1 formulae of the language $\mathcal{L}_{\dot A}$. (What we shall say easily generalizes to $\mathcal{L}_{\dot A_1,\ldots,\dot A_m}$.) We shall denote by $\ulcorner\varphi\urcorner$ the Gödel number of φ, i.e., $\ulcorner\varphi\urcorner = i$ iff $\varphi = \varphi_i$. We may and shall assume that if $\bar\varphi$ is a proper subformula of φ then $\ulcorner\bar\varphi\urcorner < \ulcorner\varphi\urcorner$. We shall write $v(i)$ for the set of free variables of φ_i. Recall that all the relevant syntactical concepts are representable in (weak fragments of) Peano arithmetic, so that the representability of these concepts is immediate.

Let M be a structure for $\mathcal{L}_{\dot A}$. We shall express by $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$ the fact that $\mathsf{a} : v(i) \to M$, i.e., a assigns elements of M to the free variables of φ_i, and φ_i holds true in M under this assignment. We shall also write $\models^{\Sigma_1}_M$ for the set of $\langle i, \mathsf{a}\rangle$ such that $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$. We shall express by $\models^{\Sigma_0}_M \varphi_i[\mathsf{a}]$ the fact that $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$ holds, but with φ_i being a Σ_0 formula, and we shall write $\models^{\Sigma_0}_M$ for the set of $\langle i, \mathsf{a}\rangle$ such that $\models^{\Sigma_0}_M \varphi_i[\mathsf{a}]$.

It turns out that once we have verified that $\models^{\Sigma_0}_M$ is uniformly Δ_1 over J-structures M (which are structures of $\mathcal{L}_{\dot A}$), we easily get that $\models^{\Sigma_1}_M$ is Σ_1-definable over such structures and that these structures admit Σ_1-definable Σ_1-Skolem functions. R on M is Δ_1 iff R, $\neg R$ are both Σ_1.

Let us fix a J-structure $M = J^A_\alpha$, a structure for $\mathcal{L}_{\dot A}$.

1.12 Proposition. *Let $N \in M$ be transitive. For each $n < \omega$, there is a unique $f = f^N_n \in M$ such that* $\mathrm{dom}(f) = n$ *and for all $i < n$, if φ_i is not a Σ_0 formula then $f(i) = \emptyset$, and if φ_i is a Σ_0 formula then*

$$f(i) = \{\mathsf{a} \in {}^{v(i)}N \mid \models^{\Sigma_0}_N \varphi_i[\mathsf{a}]\}.$$

Proof. As uniqueness is clear, let us verify inductively that $f_n^N \in M$. Well, $f_0^N = \emptyset \in M$. Now suppose that $f_n^N \in M$. If φ_n is not Σ_0, then $f_{n+1}^N = f_n^N \cup \{\langle n, \emptyset\rangle\} \in M$. Now let φ_i be Σ_0. We have that ${}^{v(n)}N \in M$, and if

$$T = \{\mathsf{a} \in {}^{v(n)}N \mid \models_N^{\Sigma_0} \varphi_n[\mathsf{a}]\}$$

then $T \in \mathcal{P}({}^{v(n)}N) \cap \mathbf{\Sigma}_0^M$, and thus $T \in M$ by Lemma 1.8. Therefore,

$$f_{n+1}^N = f_n^N \cup \{\langle n, T\rangle\} \in M.$$

$\dashv$

Now let $\Theta(f, N, n)$ denote the following formula.

$$\begin{aligned}
&N \text{ is transitive} \wedge f : n \to N \wedge \forall i < n \\
&\quad [(i = \ulcorner v_{i_0} \in v_{i_1} \urcorner, \text{ some } v_{i_0}, v_{i_1} \to f(i) = \{\mathsf{a} \in {}^{v(i)}N \mid \mathsf{a}(v_{i_0}) \in \mathsf{a}(v_{i_1})\}) \\
&\quad \wedge (i = \ulcorner \dot{A}(v_{i_0}) \urcorner, \text{ some } v_{i_0} \to f(i) = \{\mathsf{a} \in {}^{v(i)}N \mid \mathsf{a}(v_{i_0}) \in A\}) \\
&\quad \wedge (i = \ulcorner \psi_0 \wedge \psi_1 \urcorner, \text{ some } \psi_0, \psi_1 \\
&\quad\ \to f(i) = \{\mathsf{a} \in {}^{v(i)}N \mid \mathsf{a}{\restriction} v(\ulcorner \psi_0 \urcorner) \in f(\ulcorner \psi_0 \urcorner) \wedge \mathsf{a}{\restriction} v(\ulcorner \psi_1 \urcorner) \in f(\ulcorner \psi_1 \urcorner)\}) \\
&\quad \wedge (i = \ulcorner \exists v_{i_0} \in v_{i_1} \psi \urcorner, \text{ some } v_{i_0}, v_{i_1}, \psi, \text{ where } \psi \text{ is } \Sigma_0 \\
&\quad\ \to f(i) = \{\mathsf{a} \in {}^{v(i)}N \mid \exists x \in \mathsf{a}(v_{i_1})(\mathsf{a} \cup \{\langle v_{i_0}, x\rangle\}){\restriction} v(\ulcorner \psi \urcorner) \in f(\ulcorner \psi \urcorner)\}) \\
&\quad \wedge (i = \ulcorner \varphi \urcorner, \text{ some } \varphi, \text{ where } \varphi \text{ is not } \Sigma_0 \to f(i) = \emptyset)].
\end{aligned}$$

It is straightforward to check that $\Theta(f, N, n)$ holds (in M) if and only if $f = f_n^N$. Now Proposition 1.12 and the fact that every element of M is contained in a transitive element of M (for instance in some S_β^A; cf. Lemma 1.10) immediately gives the following.

1.13 Proposition. *Let φ_i be Σ_0, and let $\mathsf{a} : v(i) \to M$. Then $\models_M^{\Sigma_0} \varphi_i[\mathsf{a}]$ if and only if*

$$M \models \exists f\, \exists N\, (\mathrm{ran}(\mathsf{a}) \subseteq N \wedge \Theta(f, N, i+1) \wedge \mathsf{a} \in f(i)),$$

if and only if

$$M \models \forall f\, \forall N\, ((\mathrm{ran}(\mathsf{a}) \subseteq N \wedge \Theta(f, N, i+1)) \to \mathsf{a} \in f(i)).$$

In particular, the relation $\models_M^{\Sigma_0}$ is Δ_1^M.

We are now ready to prove two important results.

1.14 Theorem. *Let M be a J-structure. The Σ_1-satisfaction relation $\models_M^{\Sigma_1}$ is then uniformly Σ_1^M.*

If M is a structure then h is a Σ_1-*Skolem function* for M if

$$h : \textstyle\bigcup_{i<\omega} (\{i\} \times {}^{v(i)}|M|) \to |M|,$$

where h may be partial, and whenever $\varphi_i = \exists v_{i_0} \varphi_j$ and $\mathsf{a} : v(i) \to |M|$,

$$\begin{aligned}
&\exists y \in |M| \models_M^{\Sigma_1} \varphi_j[(\mathsf{a} \cup \{\langle v_{i_0}, y\rangle\}){\restriction} v(j)] \\
&\quad \Longrightarrow \quad \models_M^{\Sigma_1} \varphi_j[(\mathsf{a} \cup \{\langle v_{i_0}, h(i, \mathsf{a})\rangle\}){\restriction} v(j)].
\end{aligned}$$

1.15 Theorem. *Let M be a J-structure. There is a Σ_1-Skolem function h_M which is uniformly Σ_1^M.*

The above two theorems are to be understood as follows. There are Σ_1 formulae Φ, Ψ such that whenever M is a J-structure,

(a) Φ defines $\models_M^{\Sigma_1}$, i.e. $\models_M^{\Sigma_1} \varphi_i[\mathsf{a}] \iff M \models \Phi(i, \mathsf{a})$, and

(b) Ψ defines h_M, i.e. $y = h_M(i, \mathsf{a}) \iff M \models \Psi(i, \mathsf{a}, y)$.

Proof of Theorem 1.14. We have that $\models_M^{\Sigma_1} \varphi_i[\mathsf{a}]$ iff

$$\begin{aligned}&\exists \mathsf{b} \in M\, \exists \langle v_{i_0}, \ldots, v_{i_k}, j, \rangle, \text{ some } v_{i_0}, \ldots, v_{i_k},\\ &\quad j[i = \ulcorner \exists v_{i_0} \cdots \exists v_{i_k} \varphi_j \urcorner \wedge \varphi_j \text{ is } \Sigma_0 \;\wedge\; \mathsf{a}, \mathsf{b} \text{ are functions}\\ &\quad \wedge \operatorname{dom}(\mathsf{a}) = v(i) \wedge \operatorname{dom}(\mathsf{b}) = v(j) \wedge \mathsf{a} = \mathsf{b} \restriction v(i) \wedge \models_M^{\Sigma_0} \varphi_j[\mathsf{b}]].\end{aligned}$$

Here, $\models_M^{\Sigma_0}$ is uniformly Δ_1^M by Proposition 1.13. The rest follows. ⊣

Proof of Theorem 1.15. The idea here is to let $y = h_M(i, \mathsf{a})$ be the "first component" of a minimal witness to the Σ_1 statement in question (rather than letting y be minimal itself). We may let $y = h_M(i, \mathsf{a})$ iff

$$\begin{aligned}&\exists N\, \exists \beta\, \exists R\, \exists \mathsf{b}, \text{ all in } M, \exists \langle v_{i_0}, \ldots, v_{i_k}, j \rangle, \text{ some } v_{i_0}, \ldots, v_{i_k}, j\\ &\quad [N = S_\beta^A \wedge R = <_\beta^A \wedge i = \ulcorner \exists v_{i_0} \cdots \exists v_{i_k} \varphi_j \urcorner \wedge \varphi_j \text{ is } \Sigma_0\\ &\quad \wedge \mathsf{a}, \mathsf{b} \text{ are functions } \wedge \operatorname{dom}(\mathsf{a}) = v(i) \wedge \operatorname{dom}(\mathsf{b}) = v(j) \wedge \mathsf{a} = \mathsf{b} \restriction v(i)\\ &\quad \wedge \operatorname{ran}(\mathsf{b}) \subseteq N \wedge \models_M^{\Sigma_0} \varphi_j[\mathsf{b}] \wedge \forall \bar{\mathsf{b}} \in N((\bar{\mathsf{b}} \text{ is a function} \wedge \operatorname{dom}(\bar{\mathsf{b}}) = v(j)\\ &\quad \wedge \mathsf{a} = \bar{\mathsf{b}} \restriction v(i) \wedge \operatorname{ran}(\bar{\mathsf{b}}) \subseteq N \wedge \bar{\mathsf{b}}\ R\ \mathsf{b}) \rightarrow \neg \models_M^{\Sigma_0} \varphi_j[\bar{\mathsf{b}}]) \wedge y = \mathsf{b}(v_{i_0})].\end{aligned}$$

Here, "$N = S_\beta^A$" and "$R = <_\beta^A$" are uniformly Σ_1^M by Lemmata 1.10 (2) and 1.11 (2), and $\models_M^{\Sigma_0}$ is uniformly Δ_1^M by Proposition 1.13. Therefore, the rest follows. ⊣

If we were to define a Σ_2-Skolem function for M in the same manner then we would end up with a Σ_3 definition. Jensen solved this problem by showing that under favorable circumstances Σ_n over M can be viewed as Σ_1 over a "reduct" of M. Reducts will be introduced in the fifth section of this chapter.

Another useful fact is the so-called *Condensation Lemma.*[6]

1.16 Theorem. *Let $M = \langle J_\alpha^A, B \rangle$ be a J-structure, and let $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$ where $\bar{M}$ is transitive. Then $\bar{M}$ is a J-structure, i.e., there are $\bar{\alpha} \leq \alpha$, $\bar{A}$, and $\bar{B}$ such that $\bar{M} = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{B} \rangle$.*

[6] For $n < \omega$, $X \prec_{\Sigma_n} M$ means that Σ_n formulae with parameters taken from X are absolute between X and M. To have $\pi : \bar{M} \xrightarrow[\Sigma_n]{} M$ means that $\operatorname{ran}(\pi) \prec_{\Sigma_n} M$.

Proof. Set $\bar{\alpha} = \bar{M} \cap \mathrm{On} \le \alpha$, $\bar{A} = \pi^{-1}``A$, and $\bar{B} = \pi^{-1}``B$. We claim that $\bar{M} = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{B} \rangle$.

Well, Lemma 1.10 easily gives that $S_{\beta}^{\bar{A}} \in \bar{M}$ whenever $\beta < \bar{\alpha}$. Therefore, $J_{\bar{\alpha}}^{A} \subseteq \bar{M}$. On the other hand, let $x \in \bar{M}$. Then $\pi(x) \in S_{\beta}^{A}$ for some $\beta < \alpha$, and hence $x \in S_{\beta}^{\bar{A}}$ for some $\beta < \bar{\alpha}$. ⊣

We also want to write $h_M(X)$ for the closure of X under h_M, more precisely:

Convention. Let $M = J_{\alpha}^{A}$ be a J-structure, and let $X \subseteq |M|$. We shall write $h_M(X)$ for $h_M``(\bigcup_{i<\omega}(\{i\} \times {}^{v(i)}X))$.

Using Theorem 1.15, it is easy to verify that $h_M(X) \prec_{\Sigma_1} M$. There will be no danger of confusing the two usages of "h_M". $[X]^{<\omega}$ denotes the set of all finite subsets of X.

1.17 Lemma. *Let $M = J_{\alpha}^{A}$ be a J-structure. There is then some surjective $f : [\alpha]^{<\omega} \to M$ which is Σ_1^M.*

If α is closed under the Gödel pairing function, then there is a surjection $g : \alpha \to J_{\alpha}^{A}$ which is Σ_1^M. For an arbitrary α, there is a surjection $h : \alpha \to J_{\alpha}^{A}$ which is $\mathbf{\Sigma}_1^M$.

Proof. We have that $h_M(\alpha) \prec_{\Sigma_1} M$, and hence $h_M(\alpha) = M$. But it is straightforward to construct a surjective $g' : [\alpha]^{<\omega} \to \bigcup_{i<\omega}(\{i\} \times {}^{v(i)}\alpha)$ which is Σ_1^M. We may then set $f = h_M \circ g'$.

As to the existence of g, let $\Phi : \mathrm{otp}(<_{\alpha}^{A}) \to J_{\alpha}^{A}$ denote the enumeration of J_{α}^{A} according to $<_{\alpha}^{A}$. It is not hard to verify by a simultaneous induction (cf. the proof of Lemma 1.10) that for all limit ordinals $\beta \le \alpha$, $\Phi \restriction \beta$ is $\Sigma_1^{J_{\beta}^{A}}$ and for all $\gamma < \beta$, $\Phi \restriction \gamma \in J_{\beta}^{A}$. But now if α is closed under the Gödel pairing function then $\mathrm{otp}(<_{\alpha}^{A}) = \alpha$.

The existence of h is established by [3, Lemma 2.10]. ⊣

In the following we describe a useful class of formulae, which lies somewhere between Σ_1 and Π_2. It turns out that many notions can be expressed by statements belonging to this class.

1.18 Definition. We say that φ is a *Q-formula* iff φ is of the form

$$\forall u\, \exists v \supseteq u\ \psi(v),$$

where ψ is Σ_1 and does not contain u. Instead of $\forall u\, \exists v \supseteq u$ we write briefly Qv. The above formula then has the form $Qv\ \psi(v)$ and we read "for cofinally many v, $\psi(v)$". A map π that preserves Q formulae is called *Q-preserving* and we write

$$\pi : \bar{M} \underset{Q}{\longrightarrow} M.$$

A property characterized by a Q-formula is also called a *Q-condition*.

1.19 Definition. Let U, V be transitive structures. A map $\sigma : U \to V$ is *cofinal* iff for all $y \in V$ there is some $x \in U$ such that $y \subseteq \sigma(x)$.

Let $\sigma : U \underset{\Sigma_1}{\longrightarrow} V$, where U, V are transitive structures, and let φ be a Q-formula. It is easy to see that

(a) φ is preserved downwards,

(b) if σ is cofinal, then φ is preserved upwards.

Note also that Q-formulae are closed under $\wedge$ and $\vee$ (modulo the "basic set theory" of [1, p. 36]).

We now introduce the notion of acceptability which is fundamental for the general fine structure theory. As will follow from the definition, acceptability can be considered as a strong version of GCH.

1.20 Definition. A J-structure $M = \langle J^A_\alpha, B\rangle$ is *acceptable* iff the following holds: Whenever $\xi < \alpha$ is a limit ordinal and $\mathcal{P}(\tau) \cap J^A_{\xi+\omega} \not\subseteq J^A_\xi$ for some $\tau < \xi$, there is a surjective map $f : \tau \to \xi$ in $J^A_{\xi+\omega}$. (This means that $\mathrm{Card}(\xi) \le \tau$ in $J^A_{\xi+\omega}$.)

1.21 Lemma. *Being an acceptable J-structure is a Q-property. More precisely: There is a fixed Q-sentence Ψ such that for any $M = \langle |M|, A, B\rangle$ which is transitive and closed under pairing, M is an acceptable J-structure iff $M \models \Psi$.*

Proof. The statement $\langle |M|, A\rangle = J^A_\alpha$ is a Q-condition for $\langle |M|, A\rangle$, as we may write this as

$$Qu\,\exists\beta\; u = S^A_\beta.$$

Here, "$u = S^A_\beta$" is the Σ_1 formula from Lemma 1.10 (2). Amenability can be expressed by

$$Qu\,\exists z\; z = B \cap u.$$

It only remains to prove that the fact that we collapse ξ whenever we add a new bounded subset is expressible in a Q-fashion. We note first that a J-structure M is acceptable iff the following holds in M:

$$\begin{aligned}&\forall \text{ limit ordinals } \xi\ \exists n \in \omega\ \forall m \ge n\ \forall \tau < \xi\\ &\qquad [\mathcal{P}(\tau) \cap S^A_{\xi+m} \not\subseteq S^A_\xi \to \exists f \in S^A_{\xi+m}\ f : \tau \xrightarrow{\text{onto}} \xi].\end{aligned} \tag{9.2}$$

Denote the sentence (9.2) by ψ. It is easy to see that if M satisfies ψ then M is acceptable. To see the converse, fix a limit ordinal ξ and suppose that there is a $\tau < \xi$ such that $\mathcal{P}(\tau) \cap S^A_{\xi+\omega} \not\subseteq S^A_\xi$. Let τ be minimal with this property. By acceptability of M, there is an $n \in \omega$ such that $S^A_{\xi+n}$ contains a function f mapping τ onto ξ. Using f, it is easy to construct a surjective map $f_{\tau'} : \tau' \to \xi$ for any $\tau' < \xi$ whose power set in $S^A_{\xi+\omega}$ is larger than that in S^A_ξ (since $\tau' \ge \tau$) and it follows immediately that $f_{\tau'} \in S^A_{\xi+n+k}$ for some $k < \omega$; so we have a uniform bound for all such functions.

If the height of M is $\omega\alpha$ for some limit α, then acceptability is equivalent to the statement

$$Q\xi\ S_\xi^A \models \psi.$$

Otherwise, we have to state (9.2) explicitly for the last level. Hence, the desired condition is then

$$\begin{aligned} &\langle |M|, A, B\rangle \text{ is an amenable } J\text{-structure} \\ &\quad \wedge Q\zeta\ (\lim(\zeta) \to S_\zeta^A \models \psi) \wedge [Q\zeta\ (\zeta \text{ is a limit}) \vee \varphi] \end{aligned} \tag{9.3}$$

where φ is the sentence

$$Q\zeta\exists\beta < \zeta\ [\lim(\beta) \wedge \forall\eta < \zeta\ (\eta > \beta \to \mathrm{succ}(\eta)) \wedge \varphi'(\beta,\zeta)]$$

and $\varphi'(\beta,\zeta)$ is the formula

$$\forall\tau < \beta\ [\exists u \in S_\zeta^A\ (u \notin S_\beta^A \wedge u \subseteq \tau) \to \exists f \in S_\zeta^A f : \tau \xrightarrow{\text{onto}} \beta].$$

φ is clearly a Q-sentence, hence (9.3) is a Q-sentence as well. Denote this formula by Ψ. Then M is an acceptable J-structure iff $M \models \Psi$. ⊣

1.22 Corollary.

(a) *If $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$ and M is acceptable, then so is $\bar{M}$.*

(b) *If $\pi : \bar{M} \xrightarrow[Q]{} M$ and $\bar{M}$ is acceptable, then so is M. This holds in particular if π is a Σ_0 preserving cofinal map.*

1.23 Lemma. *Let $M = J_\alpha^A$ be acceptable and let $\rho \in M$ be an infinite cardinal in M. Given $u \in J_\rho^A$, any $a \in M$ which is a subset of u is in fact an element of J_ρ^A.*

Proof. Since $u \in J_\rho^A$, there is some $\tau < \rho$ and a surjective map $g : \tau \to u$ in J_ρ^A (cf. Lemma 1.17). Set $\bar{a} = g^{-1}``a$. Then $\bar{a} \subseteq \tau$ and $a \in J_\rho^A \iff \bar{a} \in J_\rho^A$. But if $\bar{a} \notin J_\rho^A$, then there is an $f : \tau \xrightarrow{\text{onto}} \xi$, where ξ is such that $\bar{a} \in J_{\xi+\omega}^A \setminus J_\xi^A$; hence $\xi \geq \rho$. This contradicts the fact that ρ is a cardinal in J_α^A. Consequently, $a \in J_\rho^A$. ⊣

1.24 Lemma. *Let M be as above and ρ an infinite successor cardinal in M. Let $a \subseteq J_\rho^A$ be such that $a \in M$ and* Card$(a) < \rho$ *in M. Then $a \in J_\rho^A$.*

Proof. Let $\gamma < \rho$ and $g \in M$ be such that $g : \gamma \xrightarrow{\text{onto}} a$. Define $f : \gamma \to \rho$ by

$$\zeta = f(\xi) \iff g(\xi) \in S_{\zeta+\omega}^A \setminus S_\zeta^A.$$

Then $f \in M$.

Claim. f is bounded in ρ.

Suppose that this Claim holds. Let $\delta < \rho$ be such that $\mathrm{ran}(f) \subseteq \delta$. Then $a \subseteq S^A_\delta \in J^A_\rho$ and, by Lemma 1.23, $a \in J^A_\rho$. Hence it suffices to prove the Claim.

Suppose that f is unbounded in ρ. Define $G : \rho \to M$ by

$$G(\eta) = \text{ the } <^A_\alpha\text{-least function of } \gamma \text{ onto } \eta.$$

This is possible since ρ is a successor cardinal in M, so we can pick γ large enough that all we have done so far goes through. Clearly G is definable over J^A_ρ, hence $G \in M$. Now define $F : \gamma \times \gamma \xrightarrow{\text{onto}} \rho$ by

$$F(\xi, \eta) = G(f(\xi))(\eta).$$

Then $F \in M$. By Lemma 1.17, there is some surjection $g : \gamma \to \gamma \times \gamma$ which is $\mathbf{\Sigma}_1^{J^A_\gamma}$; hence $g \in M$. But then $F \circ g \in M$ witnesses that ρ is not a cardinal in M. This contradiction yields the Claim. ⊣

1.25 Corollary. *Let M, ρ be as in Lemma 1.24. Then $J^A_\rho \models \mathrm{ZFC}^-$.*

Proof. The point here is to verify the Separation and Replacement Schemata in J^A_ρ, since the rest of the axioms hold in J^A_ρ automatically. The former follows from Lemma 1.23 and the latter from Lemma 1.24 in a straightforward way. ⊣

1.26 Corollary. *Let M be as above where $\rho > \omega$ is a limit cardinal in M. Then $J^A_\rho \models \mathrm{ZC}$ (where ZC is Zermelo set theory with choice).*

Proof. We only have to verify the power set axiom; the rest goes through as before. Let $a \in J^A_\rho$. Pick $\gamma < \rho$ such that γ is a cardinal in J^A_ρ and $a \in J^A_\gamma$. Then for every $x \in \mathcal{P}(a) \cap J^A_\rho$, $x \in J^A_\gamma$. Hence $\mathcal{P}(a) \cap J^A_\rho \in J^A_\rho$. ⊣

1.27 Corollary. *Let M, ρ be as above. Then*

$$|J^A_\rho| = H^M_\rho \stackrel{\text{def}}{=} \{x \in M \mid \mathrm{Card}\,(\mathrm{tc}(x)) < \rho \textit{ in } M\}.$$

Proof. Clearly $|J^A_\rho| \subseteq H^M_\rho$. So it is sufficient to prove the converse. Suppose that it fails. Let ρ be the least counterexample. Then ρ is a successor cardinal in M. Let $x \in H^M_\rho$ be $\in$-minimal such that $x \notin J^A_\rho$. Then $x \subseteq J^A_\rho$. Since $\mathrm{card}(x) < \rho$ in M, $x \in J^A_\rho$ by Lemma 1.23. Contradiction. ⊣

2. The First Projectum

We now introduce the central notions of fine structure theory—the notions *projectum, standard code* and *good parameter.* We stress that we are working with arbitrary J-structures and that these structures have, in general, very few closure properties. This means that there might be bounded definable subsets of these structures (in a precise sense) failing to be *elements.*

However, each J-structure has an initial segment which is "firm" in the sense that it does contain all sets reasonably definable over the whole structure. The height of this "firm" segment is called the *projectum. Standard codes* are (boldface) definable relations computing truth and *good parameters* are parameters that occur in the definitions of standard codes.

2.1 Definition. The Σ_1 *projectum* (or, *first projectum*) $\rho(M)$ of an acceptable J-structure $M = J_\alpha^A$ is defined by

$$\rho(M) = \text{ the least } \rho \in \mathrm{On} \text{ such that } \mathcal{P}(\rho) \cap \boldsymbol{\Sigma}_1^M \not\subseteq M.$$

2.2 Lemma. *Let M be as above. If $\rho(M) \in M$, then $\rho(M)$ is a cardinal in M.*

Proof. Suppose not. Set $\rho = \rho(M)$. Let $f \in M$ be such that $f : \gamma \xrightarrow{\text{onto}} \rho$ for some $\gamma < \rho$ and $A \in \boldsymbol{\Sigma}_1^M$ be such that $a = A \cap \rho \notin M$. Let $\bar{a} = f^{-1}``a$. Then $\bar{a} \notin M$, since otherwise $a = f``\bar{a} \in M$. On the other hand, $\bar{a} \in M$ by the definition of ρ, since $\bar{a} \subseteq \gamma$ and $\bar{a} \in \boldsymbol{\Sigma}_1^M$. Contradiction. ⊣

2.3 Lemma. *Let M be as above and $\rho = \rho(M)$. Then ρ is a Σ_1 cardinal in M (i.e. there is no $\boldsymbol{\Sigma}_1^M$ partial map from some $\gamma < \rho$ onto ρ).*

Proof. Suppose that there is such a map, say $f : \gamma \xrightarrow{\text{onto}} \rho$. We know that there is a $\boldsymbol{\Sigma}_1^{J_\rho^A}$ map of ρ onto J_ρ^A (cf. Lemma 1.17). Hence there is a $\boldsymbol{\Sigma}_1^M$ map $g : \gamma \xrightarrow{\text{onto}} J_\rho^A$. Define a set b by

$$\xi \in b \iff \xi \notin g(\xi).$$

Then b is clearly $\boldsymbol{\Sigma}_1^M$ and $b \subseteq \gamma$. Moreover, $b \notin J_\rho^A$ by a diagonal argument: if $b \in J_\rho^A$ then $b = g(\xi_0)$ for some $\xi_0 < \gamma$ which would give $\xi_0 \in b = g(\xi_0)$ iff $\xi_0 \notin g(\xi_0)$. Hence $b \notin M$: this follows from Lemma 1.23 if $\rho \in M$ and is immediate otherwise. On the other hand, $\gamma < \rho$, and therefore $b \in M$. Contradiction! ⊣

Lemma 1.17 and Lemma 1.23 now immediately give the following.

2.4 Corollary. *Let M be acceptable and $\rho = \rho(M)$.*

(a) If $B \subseteq J_\rho^A$ is $\boldsymbol{\Sigma}_1^M$, then $\langle J_\rho^A, B\rangle$ is amenable.

(b) $|J_\rho^A| = H_\rho^M$.

Recall that we fixed a recursive enumeration $\langle \varphi_i \mid i < \omega\rangle$ of all Σ_1 formulae. In what follows it will be convenient to pretend that each φ_i has exactly *one* free variable. For instance, if $\varphi_i = \varphi_i(v_{i_1}, \ldots, v_{i_\ell})$ with all free variables shown then we might confuse φ_i with

$$\exists v_{i_1} \ldots \exists v_{i_\ell}(u = \langle v_{i_1}, \ldots, v_{i_\ell}\rangle \wedge \varphi_i(v_{i_1}, \ldots, v_{i_\ell})).$$

To make things even worse, we shall nevertheless often write $\varphi_i(x_1,\ldots,x_\ell)$ instead of $\varphi_i(\langle x_1,\ldots,x_\ell\rangle)$. If $\mathsf{a} : v(i) \to V$ assigns values to the free variable(s) $v_{i_1},\ldots,v_{i_\ell}$ of φ_i then, setting $x_1 = \mathsf{a}(v_{i_1}),\ldots,x_\ell = \mathsf{a}(v_{i_\ell})$ we shall use the more suggestive $M \models \varphi_i(x_1,\ldots,x_\ell)$ rather than $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$ in what follows. We shall also write $h_M(i,\vec{x})$ instead of $h_M(i,\mathsf{a})$.

2.5 Definition. Let $M = \langle J^B_\alpha, D\rangle$ be an acceptable J-structure, $\rho = \rho(M)$ and $p \in M$. We define

$$A^p_M = \{\langle i,x\rangle \in \omega \times H^M_\rho \mid M \models \varphi_i(x,p)\}.$$

A^p_M is called the *standard code* determined by p. Let us stress that A^p_M is the intersection of $\omega \times H^M_\rho$ with a set $\tilde{A}^p_M$ (defined in an obvious way) which is $\Sigma^M_1(\{p\})$. We shall often write $A^p_M(i,x)$ instead of $\langle i,x\rangle \in A^p_M$. The structure

$$M^p = \langle J^B_\rho, A^p_M\rangle$$

is called the *reduct* determined by p. If $\delta = \rho$ or $\delta < \rho$ where δ is a cardinal in M, we also set

$$A^{p,\delta}_M = A^p \cap J^B_\delta,$$
$$M^{p,\delta} = \langle J^B_\delta, A^{p,\delta}_M\rangle.$$

We shall omit the subscript $_M$ whenever there is no danger of confusion.

2.6 Definition. Let M be acceptable and $\rho = \rho(M)$.

$$\begin{aligned} P_M = {} & \text{the set of all } p \in [\rho(M), M \cap \mathrm{On})^{<\omega} \text{ for which} \\ & \text{there is a } B \in \Sigma^M_1(\{p\}) \text{ such that } B \cap \rho \notin M. \end{aligned}$$

The elements of P_M are called *good parameters*.

2.7 Lemma. *Let M be as before, $p \in M$ and $A = A^p_M$. Then*

$$p \in P_M \iff A \cap (\omega \times \rho(M)) \notin M.$$

Proof. ($\Longrightarrow$) Pick B which witnesses that $p \in P_M$. Suppose that B is defined by φ_i. Hence $B(\xi) \leftrightarrow \langle i,\xi\rangle \in A$, which means that if $A \cap (\omega \times \rho(M))$ is in M, then so is $B \cap \rho(M)$. Hence the former is not an element of M.

($\Longleftarrow$) Suppose that $A \cap (\omega \times \rho(M)) \notin M$. Let $f : \omega \times \rho(M) \longrightarrow \rho(M)$ be defined by $f(i, \omega\xi + j) = \omega\xi + 2^i{\cdot}3^j$. Clearly f is Σ^M_1 uniformly and if $\rho(M) \in M$ then $f \in M$. Let $B = f``A$. Then $B$ is $\Sigma^M_1(\{p\})$ and $B \cap \rho(M) \notin M$: if $\rho(M) \in M$, this follows from the fact that $f \in M$ and $A \cap (\omega \times \rho(M)) = f^{-1}``(B \cap \rho(M))$; if $\rho(M) \notin M$, it follows from the fact that B is cofinal in $\rho(M)$. $\dashv$

2.8 Definition. Let M be acceptable and $\rho = \rho(M)$. We set

$$R_M = \text{ the set of all } r \in [\rho(M), M \cap \mathrm{On})^{<\omega} \text{ such that } h_M(\rho \cup \{r\}) = |M|.$$

The elements of R_M are called *very good parameters*.

2.9 Lemma. *$R_M \subseteq P_M \neq \emptyset$.*

Proof. By the proof of Lemma 1.17, $h_M(M \cap \mathrm{On}) = M$ for any J-structure M. Given an acceptable structure M, if A is a relation which is $\Sigma_1^M(\{p\})$ for some $p \in M$ then A is therefore also $\Sigma_1^M(\{q\})$ for some $q \in [M \cap \mathrm{On}]^{<\omega}$. As $\rho(M)$ is closed under the Gödel pairing function (if $\rho(M) < M \cap \mathrm{On}$), the inequality easily follows. As to the inclusion, define the set $a \subseteq \omega \times M \cap \mathrm{On}$ by

$$\langle i, \xi \rangle \in a \quad \Longleftrightarrow \quad \langle i, \xi \rangle \notin h_M(i, \langle \xi, p \rangle)$$

for a $p \in R_M$. By a diagonal argument, $a \cap \omega \times \rho(M) \notin M$ and a is $\Sigma_1(M)$ in p. Using the map f from the proof of Lemma 2.7 it is easy to turn the set a into some $b \subseteq M \cap \mathrm{On}$ such that b is Σ_1^M in p and $b \cap \rho(M) \notin M$. ⊣

We remark that P_M and R_M are often defined differently so as to include arbitrary elements of M rather than just finite sequences of ordinals in the half-open interval $[\rho(M), M \cap \mathrm{On})$.

3. Downward Extension of Embeddings

Given a Σ_0 preserving map between the reducts of two acceptable structures, the question naturally arises whether the map can be extended to a map between the original structures. It turns out that this is possible. The conjunction of the following three lemmas is called the *Downward Extension of Embeddings Lemma.*

3.1 Lemma. *Let $\pi : \bar{M}^{\bar{p}} \underset{\Sigma_0}{\longrightarrow} M^p$, where $\bar{p} \in R_{\bar{M}}$. Then there is a unique $\tilde{\pi} : \bar{M} \underset{\Sigma_0}{\longrightarrow} M$ such that $\tilde{\pi} \supseteq \pi$ and $\tilde{\pi}(\bar{p}) = p$. Moreover, $\tilde{\pi} : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$.*

Proof. Uniqueness: Assume that $\tilde{\pi}$ has the above properties. Let $x \in \bar{M}$. Then $x = h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)$ for some $i \in \omega$ and $\xi < \rho(M)$. Let $\bar{H}$ be $\Sigma_0^{\bar{M}}$ such that $\exists z\, \bar{H}(z, x, i, \xi, \bar{p})$ defines the Skolem function $h_{\bar{M}}$ (this involves a slight abuse of notation). $\bar{H}$ has a uniform definition, so let H have the same definition over M. Pick z such that $\bar{H}(z, x, i, \xi, \bar{p})$. Since $\tilde{\pi}$ is Σ_0 preserving, we have $H(\tilde{\pi}(z), \tilde{\pi}(x), i, \tilde{\pi}(\xi), p)$, i.e. $\tilde{\pi}(x) = h_M(i, \langle \tilde{\pi}(\xi), p \rangle) = h_M(i, \langle \pi(\xi), p \rangle)$. Hence, there is at most one such $\tilde{\pi}$.

Existence: The above proof of uniqueness suggests how to define the extension $\tilde{\pi}$. Here we show that such a definition is correct. We first observe:

Claim. Suppose that $\varphi(v_1, \dots, v_\ell)$ is a Σ_1 formula. Let $\bar{x}_i = h_{\bar{M}}(j_i, \langle \bar{\xi}_i, \bar{p} \rangle)$ for some $j_i \in \omega$, $\bar{\xi}_i < \rho(M)$ and $x_i = h_M(j_i, \langle \xi_i, p \rangle)$ where $\xi_i = \pi(\bar{\xi}_i)$ $(i = 1, \dots, \ell)$. Then

$$\bar{M} \models \varphi(\bar{x}_1, \dots, \bar{x}_\ell) \quad \Longleftrightarrow \quad M \models \varphi(x_1, \dots, x_\ell).$$

Proof. $\bar{M} \models \varphi(\bar{x}_1, \ldots, \bar{x}_\ell)$ is equivalent to

$$\bar{M} \models \varphi(h_{\bar{M}}(j_1, \langle \bar{\xi}_1, \bar{p} \rangle), \ldots, h_{\bar{M}}(j_\ell, \langle \bar{\xi}_\ell, \bar{p} \rangle)).$$

Since $h_{\bar{M}}$ has a uniform Σ_1 definition over $\bar{M}$, there is a Σ_1 formula ψ such that the above is equivalent to

$$\bar{M} \models \psi(\bar{\xi}_1, \ldots, \bar{\xi}_\ell, \bar{p}). \tag{9.4}$$

The formula ψ clearly does not depend on the actual structure in question, so the statement $M \models \varphi(x_1, \ldots, x_\ell)$ is equivalent to

$$M \models \psi(\xi_1, \ldots, \xi_\ell, p). \tag{9.5}$$

Now suppose that $\psi(\xi_1, \ldots, \xi_\ell, p) \iff \varphi_k(\langle \xi_1, \ldots, \xi_\ell \rangle, p)$ in our fixed recursive enumeration of Σ_1 formulae, hence (9.4) is equivalent to

$$A_{\bar{M}}^{\bar{p}}(k, \langle \bar{\xi}_1, \ldots, \bar{\xi}_\ell \rangle) \tag{9.6}$$

and (9.5) is equivalent to

$$A_M^p(k, \langle \xi_1, \ldots, \xi_\ell \rangle). \tag{9.7}$$

Since π is Σ_0 preserving, (9.6) and (9.7) are equivalent. $\dashv$

Now define $\tilde{\pi}$ by

$$\tilde{\pi}(h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)) \simeq h_M(i, \langle \pi(\xi), p \rangle)$$

for $i \in \omega$ and $\xi < \rho(\bar{M})$. We have to verify several facts:

- $\tilde{\pi}$ is well defined.
  Let $h_{\bar{M}}(j_1, \langle \bar{\xi}_1, \bar{p} \rangle) = h_{\bar{M}}(j_2, \langle \bar{\xi}_2, \bar{p} \rangle)$ for some $j_1, j_2 \in \omega$ and $\bar{\xi}_1, \bar{\xi}_2 < \rho(\bar{M})$. We have to show that $h_M(j_1, \langle \xi_1, p \rangle) = h_M(j_2, \langle \xi_2, p \rangle)$, where $\xi_i = \pi(\bar{\xi}_i)$ $(i = 1, 2)$. By the above claim, this follows immediately.

- $\tilde{\pi}$ is Σ_1 preserving.
  Since $\tilde{\pi}$ is a well defined map, this follows immediately from the above claim.

- $\tilde{\pi} \supseteq \pi$.
  There is an $i \in \omega$ such that the equality $x = h_M(i, \langle x, q \rangle)$ holds uniformly and independently of q. In particular, for $\xi < \rho(\bar{M})$ we have $\xi = h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)$, so

  $$\tilde{\pi}(\xi) = h_M(i, \langle \pi(\xi), p \rangle) = \pi(\xi).$$

- $\tilde{\pi}(\bar{p}) = p$.
  Similarly as above, there is an $i \in \omega$ such that $q = h(i, \langle x, q \rangle)$ uniformly. Hence $\bar{p} = h_{\bar{M}}(i, \langle 0, \bar{p} \rangle)$ and $\tilde{\pi}(\bar{p}) = h_M(i, \langle 0, p \rangle) = p$.

$\dashv$

3.2 Lemma. *Let $\bar{M}$, M, $\bar{p}$, p, π, $\tilde{\pi}$ be as above. Suppose moreover that $p \in R_M$. Let $\pi : \bar{M}^{\bar{p}} \underset{\Sigma_n}{\longrightarrow} M^p$. Then*

$$\tilde{\pi} : \bar{M} \underset{\Sigma_{n+1}}{\longrightarrow} M.$$

Proof. We shall proceed by induction. Suppose that the lemma holds for n. Suppose that we have a Σ_{n+1} formula φ which is of the form

$$\exists z_1 \, \forall z_2 \ldots \exists/\forall z_{n+1} \, \bar{\varphi}(z_1, \ldots, z_{n+1}, x_1, \ldots, x_\ell), \tag{9.8}$$

where $\bar{\varphi}$ is Σ_0. For notational simplicity, assume that $n = 2$ and $\ell = 1$. Given a J-structure N and an arbitrary $q \in R_N$, the structure N satisfies (9.8) iff it satisfies the formula

$$\begin{aligned} &\exists \xi_1 < \rho(N) \exists i_1 \forall \xi_2 < \rho(N) \forall i_2 [\exists y \, y = h_N(i_2, \langle \xi_2, q \rangle) \\ &\quad \to \exists y_1 \exists y_2 \exists x \exists z_3 (y_1 = h_N(i_1, \langle \xi_1, q \rangle) \wedge y_2 = h_N(i_2, \langle \xi_2, q \rangle) \wedge \\ &\qquad\qquad x = h_N(j_1, \langle \zeta_1, q \rangle) \wedge \bar{\varphi}(y_1, y_2, z_3, x))] \end{aligned} \tag{9.9}$$

where $x_1 = h_N(j_1, \langle \zeta_1, q \rangle)$. (For an arbitrary n, the analogous fact can be verified by a straightforward induction on n using that q is a very good parameter.) Notice that the matrix in (9.9) is of the form $\psi_1 \to \psi_2$, where both ψ_1 and ψ_2 are Σ_1 via a uniform transformation, i.e., there are k_1, $k_2 \in \omega$ depending only on $\bar{\varphi}$ such that (9.9) can be expressed in a Σ_2-fashion over N^q in the following way:

$$\begin{aligned} &\exists \xi_1 \, \exists i_1 \, \forall \xi_2 \, \forall i_2 \, [A^q_N(k_1, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle) \\ &\quad \to A^q_N(k_2, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle)]. \end{aligned}$$

Then, if in fact $x_1 = h_{\bar{M}}(j_1, \langle \zeta_1, \bar{p} \rangle)$,

$$\begin{aligned} \bar{M} \models \varphi(x_1) \quad &\iff \quad \exists \xi_1 \, \exists i_1 \, \forall \xi_2 \, \forall i_2 \, [A^{\bar{p}}_{\bar{M}}(k_1, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle) \\ &\qquad\quad \to A^{\bar{p}}_{\bar{M}}(k_2, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle)] \\ &\iff \quad \exists \xi_1 \, \exists i_1 \, \forall \xi_2 \, \forall i_2 \, [A^p_M(k_1, \langle \xi_1, i_1, \xi_2, i_2, \pi(\zeta_1), j_1 \rangle) \\ &\qquad\quad \to A^p_M(k_2, \langle \xi_1, i_1, \xi_2, i_2, \pi(\zeta_1), j_1 \rangle)] \\ &\iff \quad M \models \varphi(\tilde{\pi}(x_1)). \end{aligned}$$

Similar, but slightly more complicated reductions can be done for arbitrary n; we leave this to the reader. ⊣

3.3 Lemma. *Let $\pi : N \underset{\Sigma_0}{\longrightarrow} M^p$, where N is a J-structure and $p \in R_M$. Then there are unique $\bar{M}$, $\bar{p}$ such that $\bar{p} \in R_{\bar{M}}$ and $N = \bar{M}^p$.*

Proof. Let $M = \langle J^B_\alpha, D \rangle$, $M^p = \langle J^B_\rho, A \rangle$ and $N = \langle J^{\bar{B}'}_{\bar{\rho}}, \bar{A} \rangle$. (In particular, $A = A^p_M$.) Let also $X = \mathrm{ran}(\pi)$, $Y = h_M(X \cup \{p\})$, let $\bar{M}$ be the transitive collapse of Y and let $\tilde{\pi} : \bar{M} \to Y$ be the inverse of the Mostowski collapse.

The map $\tilde{\pi}$ is clearly Σ_1-preserving, hence $\bar{M}$ is of the form $\langle J^{\bar{B}}_{\bar{\alpha}}, \bar{D}\rangle$. We shall show that $\bar{M}$ is the desired structure. We prove first

$$\tilde{\pi} \supseteq \pi \quad \text{and} \quad J^{\bar{B}'}_{\bar{\rho}} = J^{\bar{B}}_{\bar{\rho}}, \tag{9.10}$$

which follows immediately from

$$\text{if} \quad x \in X,\ y \in Y \text{ and } y \in x, \quad \text{then} \quad y \in X \tag{9.11}$$

since the latter tells us that the collapsing map for Y restricted to X coincides with π^{-1}.

So suppose that $y = h_M(i, \langle z, p\rangle)$ for some $z \in X$. Since both $y, z \in M^p$, this can be equivalently expressed in the form $A(k, \langle y, z\rangle)$ for some $k \in \omega$. Thus we have

$$\exists v \in x\ A(k, \langle v, z\rangle)$$

which is preserved by a Σ_0 map, so

$$\exists v \in \bar{x}\ \bar{A}(k, \langle v, \bar{z}\rangle)$$

where $(\bar{x}, \bar{z}) = \pi^{-1}(x, z)$. Let $\bar{y} \in \bar{x}$ be such that $A(k, \langle \bar{y}, \bar{z}\rangle)$. Such a $\bar{y}$ is uniquely determined: if $\bar{y}_1$ were another one, we would have

$$A(k, \langle \pi(\bar{y}), z\rangle) \wedge A(k, \langle \pi(\bar{y}_1), z\rangle),$$

which means that $\pi(\bar{y}) = h_M(i, \langle z, p\rangle) = \pi(\bar{y}_1)$, hence $\bar{y} = \bar{y}_1$. This argument also shows $\pi(\bar{y}) = y$. Hence $y \in X$.

We have shown (9.10) and (9.11).

Now let

$$\bar{p} = \tilde{\pi}^{-1}(p).$$

Note that there is a $\mathbf{\Sigma}^{\bar{M}}_1$ map of $\bar{\rho}$ onto $\bar{M}$; this follows immediately from the fact that $\bar{M} = h_{\bar{M}}(J^{\bar{B}}_{\bar{\rho}} \cup \{\bar{p}\})$ and that there is a $\mathbf{\Sigma}^{\bar{M}}_1$ map of $\bar{\rho}$ onto $J^{\bar{B}}_{\bar{\rho}}$. So we have

$$\rho(\bar{M}) \leq \bar{\rho}. \tag{9.12}$$

Note also that if $i \in \omega$ and $x \in N$, then

$$\bar{A}(i, x) \iff A(i, \tilde{\pi}(x)) \iff M \models \varphi_i(\tilde{\pi}(x), p) \iff \bar{M} \models \varphi_i(x, \bar{p}) \tag{9.13}$$

where $\langle \varphi_i\rangle_i$ is the fixed recursive enumeration of the Σ_1 formulae. Our aim is to show

$$\rho(\bar{M}) = \bar{\rho} \tag{9.14}$$

which reduces to proving the inequality $\bar{\rho} \leq \rho(\bar{M})$. Let P be $\mathbf{\Sigma}^{\bar{M}}_1(\{\bar{q}\})$. By the fact that $h_{\bar{M}}(J^{\bar{B}}_{\bar{\rho}} \cup \{\bar{p}\}) = \bar{M}$ we can find an R which is $\Sigma^{\bar{M}}_1$ and such that

$$P(z) \iff R(z, x, \bar{p})$$

for some fixed $x \in N$. By (9.13), there is an $i \in \omega$ such that

$$P(z) \iff \bar{A}(i, \langle z, x\rangle).$$

Hence if $\gamma < \bar{\rho}$ then $P \cap \gamma$ is a projection of $\bar{A} \cap (\{i\} \times \gamma \times \{x\}) \in N \subseteq \bar{M}$, thus it is itself in $\bar{M}$. This proves (9.14).

As an immediate consequence of (9.13) and (9.14) we get

$$\bar{A} = A^{\bar{p}}_{\bar{M}}. \tag{9.15}$$

It only remains to prove that

$$\bar{p} \in R_{\bar{M}}. \tag{9.16}$$

If $\bar{M} = N$ this is trivial. Otherwise there is a Σ_1^M map of $\bar{\rho}$ onto $J^{\bar{B}}_{\bar{\rho}}$. Since $h_{\bar{M}}(J^{\bar{B}}_{\bar{\rho}} \cup \{\bar{p}\}) = \bar{M}$, the proof is complete. ⊣

4. Upward Extension of Embeddings

In this section we present a method which gives a solution to the problem dual to that from the previous section, where we formed the extension of an embedding from the reduct of a J-structure to the whole structure: namely, we now aim to build a target model that can serve as the codomain of an extended embedding. This problem is a bit more delicate than the previous one, since such an extension need not always exist; therefore we have to strengthen our requirements on the embeddings we intend to extend. The difference between forming downward and upward extensions lies in the fact that the former ones are related to taking hulls and collapsing them, which is always possible, whilst the latter ones are related to forming ultrapowers, which have transitive isomorphs only if they are well-founded.

4.1 Definition. $\pi : \bar{M} \to M$ is a *strong embedding* iff

(a) $\pi : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$.

(b) For any $\bar{R}$, R such that $\bar{R}$ is rudimentary over $\bar{M}$ and R is rudimentary over M by the same definition the following holds:

If $\bar{R}$ is well-founded, then so is R.

The *Upward Extension of Embeddings Lemma* is the conjunction of the following lemma together with Lemmata 3.1 and 3.2.

4.2 Lemma. *Let $\pi : \bar{M}^{\bar{p}} \to N$ be a strong embedding, where N is acceptable and $\bar{p} \in R_{\bar{M}}$. Then there are unique M, p such that $N = M^p$ and $p \in R_M$. Moreover, $\tilde{\pi}$ is strong, where $\tilde{\pi} \supseteq \pi$, $\tilde{\pi} : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$ and $\tilde{\pi}(\bar{p}) = p$.*

Proof. Uniqueness. Suppose that $\tilde{\pi}_1 : \bar{M} \to M_1$ and $\tilde{\pi}_2 : \bar{M} \to M_2$ are two extensions of π satisfying the conclusions of the lemma and that p_1, p_2 are the corresponding parameters. Then $A^{p_1}_{M_1} = A^{p_2}_{M_2}$, call it A, and every $x \in M_k$ is of the form $h_{M_k}(i, \langle \xi, p_k \rangle)$ for some $i \in \omega$ and $\xi \in N \cap \mathrm{On}$ $(k = 1, 2)$. Let $\sigma : M_1 \to M_2$ be the map sending $h_{M_1}(i, \langle \xi, p_1 \rangle)$ to $h_{M_2}(i, \langle \xi, p_2 \rangle)$. Then σ is well defined since

$$\exists z\, z = h_{M_1}(i, \langle \xi, p_1 \rangle) \iff A(j, \langle i, \xi \rangle) \iff \exists z\, z = h_{M_2}(i, \langle \xi, p_2 \rangle) \tag{9.17}$$

for an appropriate j (i.e., j is such that $\exists z\, z = h_{M_k}(i, \langle \xi, p_k \rangle)$ can be expressed as $M_k \models \varphi_j(\langle i, \xi \rangle)$ for $k = 1, 2$). Also, σ is Σ_1-preserving, since given any Σ_1 formula $\psi(v_1, \ldots, v_\ell)$ and $x_s = h_{M_1}(i_s, \langle \xi_s, p_1 \rangle)$ $(s = 1, \ldots, \ell)$,

$$\begin{aligned} M_1 \models \psi(x_1, \ldots, x_\ell) &\iff M_1 \models \psi(h_{M_1}(i_1, \langle \xi_1, p_1 \rangle), \ldots, h_{M_1}(i_\ell, \langle \xi_\ell, p_1 \rangle)) \\ &\iff A(j, \langle \langle i_1, \xi_1 \rangle, \ldots, \langle i_\ell, \xi_\ell \rangle \rangle) \\ &\iff M_2 \models \psi(h_{M_2}(i_1, \langle \xi_1, p_2 \rangle), \ldots, h_{M_2}(i_\ell, \langle \xi_\ell, p_2 \rangle)) \\ &\iff M_2 \models \psi(\sigma(x_1), \ldots, \sigma(x_\ell)) \end{aligned}$$

for a suitable j (which only depends on ψ). It is then easy to see that σ is structure preserving and $\sigma \circ \tilde{\pi}_1 = \tilde{\pi}_2$. Furthermore, (9.17) implies that $\mathrm{ran}(\sigma) = M_2$. Thus, $M_1 = M_2$, $\tilde{\pi}_1 = \tilde{\pi}_2$ and $p_1 = p_2$.

Existence. The idea of the construction is simple: using the fact that $\bar{p} \in R_{\bar{M}}$, we encode the whole structure $\bar{M}$ and its satisfaction relation in a rudimentary fashion over $\bar{M}^{\bar{p}}$. The preservation properties of π then guarantee that the corresponding relations with the same rudimentary definitions over N encode the required structure M; the process of decoding will also yield the extension $\tilde{\pi}$. However, the verification of all details is somewhat technical.

Suppose that $\bar{M} = \langle J^{\bar{B}}_{\bar{\alpha}}, \bar{D} \rangle$. Let $\bar{k}(\langle i, z \rangle) \simeq h_{\bar{M}}(i, \langle z, \bar{p} \rangle)$ and $\bar{d} = \mathrm{dom}(\bar{k})$. Then membership in $\bar{d}$ is expressible by a Σ_1 statement over $\bar{M}$ in $\bar{p}$ as

$$x \in \bar{d} \iff \exists i\, \exists z\ (i \in \omega \wedge x = \langle i, z \rangle \wedge \exists y\ y = h_{\bar{M}}(i, \langle z, \bar{p} \rangle).$$

So there is an $i_0 \in \omega$ such that for every $x \in \bar{M}^{\bar{p}}$ we have $x \in \bar{d}$ iff $A^{\bar{p}}_{\bar{M}}(i_0, x)$. Note that the latter is a rudimentary relation over $\bar{M}^{\bar{p}}$. Similarly, the identity and membership relations as well as the membership in $\bar{B}$ and $\bar{D}$ can be expressed in a Σ_1 fashion over $\bar{M}$ in $\bar{p}$, and therefore in a rudimentary fashion over $\bar{M}^{\bar{p}}$. More precisely, we introduce relations $\bar{I}, \bar{E}, \bar{B}^*$ and $\bar{D}^*$ over $\bar{d}$ as follows

$$\begin{aligned} x\, \bar{I}\, y &\iff \bar{k}(x) = \bar{k}(y), \\ x\, \bar{E}\, y &\iff \bar{k}(x) \in \bar{k}(y), \\ \bar{B}^*(x) &\iff \bar{B}(\bar{k}(x)), \\ \bar{D}^*(x) &\iff \bar{D}(\bar{k}(x)) \end{aligned}$$

and set

$$\bar{\mathbb{D}} = \langle \bar{d}, \bar{I}, \bar{E}, \bar{B}^*, \bar{D}^* \rangle.$$

The symbol for $=$ is interpreted in $\bar{\mathbb{D}}$ as $\bar{I}$, the symbol for $\in$ as $\bar{E}$, and the symbols for $\bar{B}, \bar{D}$ as $\bar{B}^*$ and $\bar{D}^*$, respectively. Thus $\bar{\mathbb{D}}$ encodes the structure $\bar{M}$: $\bar{I}$ is a congruence relation on $\bar{\mathbb{D}}$, $\bar{E}$ represents the membership relation and $\bar{k}$ is the Mostowski collapsing isomorphism between $\bar{\mathbb{D}}/\bar{I}$ and $\bar{M}$. We denote the Σ_1-satisfaction relation for $\bar{\mathbb{D}}$ by $\bar{T}$. More precisely, for $x_1, \ldots, x_\ell \in \bar{d}$ and $i \in \omega$ we have

$$\bar{T}(i, \langle x_1, \ldots, x_\ell \rangle) \quad \Longleftrightarrow \quad \bar{\mathbb{D}} \models \varphi_i(x_1, \ldots, x_\ell)$$

where $\langle \varphi_i; i \in \omega \rangle$ is our fixed recursive enumeration of Σ_1 formulae (remember that $\langle x_1, \ldots, x_\ell \rangle = \langle x_1, \langle x_2, \ldots, x_\ell \rangle \rangle$ and that we write $\varphi_i(x_1, \ldots, x_\ell)$ instead of $\varphi_i(\langle x_1, \ldots, x_\ell \rangle)$). One can easily show the following equivalence by induction:

$$\bar{T}(i, \langle x_1, \ldots, x_\ell \rangle) \quad \Longleftrightarrow \quad \bar{M} \models \varphi_i(\bar{k}(x_1), \ldots, \bar{k}(x_\ell)).$$

For atomic formulae this follows immediately from the definitions of the relations $\bar{I}, \bar{E}, \bar{B}^*$ and $\bar{D}^*$. To illustrate how the induction steps go we show the induction step for the formula $\varphi_j(v)$ of the form $\exists z \in \operatorname{left}(v)\ \varphi_i(z, v)$ (recall that $\operatorname{left}(v) = v_1$ and $\operatorname{right}(v) = v_2$ if $v = \langle v_1, v_2 \rangle$ and undefined otherwise). Then

$$\begin{aligned}
\bar{\mathbb{D}} \models \varphi_j(x_1, \ldots, x_\ell) \quad & \Longleftrightarrow \quad \exists w \in \bar{d}[w \bar{E} x_1 \wedge \bar{\mathbb{D}} \models \varphi_i(w, x_1, x_2, \ldots, x_\ell)] \\
& \Longleftrightarrow \quad \exists w[\bar{k}(w) \in \bar{k}(x_1) \\
& \qquad\quad \wedge \bar{M} \models \varphi_i(\bar{k}(w), \bar{k}(x_1), \bar{k}(x_2), \ldots, \bar{k}(x_\ell))] \\
& \Longleftrightarrow \quad \bar{M} \models \varphi_j(\bar{k}(x_1), \ldots, \bar{k}(x_\ell)).
\end{aligned}$$

The middle equivalence follows by the induction hypothesis, the last one by the fact that if there is a z witnessing the bottom formula, then such a z is always of the form $\bar{k}(w)$ for some w.

Let d, I, E, B^* and D^* be rudimentary over N by the same rudimentary definitions as their counterparts over $\bar{M}^{\bar{p}}$. It follows from the above that $\bar{T}$ is $\Sigma_1^{\bar{M}}$ in $\bar{p}$ and therefore rudimentary over $\bar{M}^{\bar{p}}$. Let

$$\mathbb{D} = \langle d, I, E, B^*, D^* \rangle$$

and T be a relation which is rudimentary over N by the same rudimentary definition as $\bar{T}$. We show that T is the Σ_1-satisfaction predicate for $\mathbb{D}$. Strictly speaking, we must show that the following equivalences hold, where $\varphi_i(v)$

has the form indicated on the left hand side:

$$\begin{array}{lrcl}
\text{left}(v) = \text{right}(v) & T(i, \langle x, y \rangle) & \iff & x\, I\, y \\
\text{left}(v) \in \text{right}(v) & T(i, \langle x, y \rangle) & \iff & x\, E\, y \\
B(v) & T(i, x) & \iff & B^*(x) \\
D(v) & T(i, x) & \iff & D^*(x) \\
\varphi_{i_1}(v) \wedge \varphi_{i_2}(v) & T(i, \vec{x}\,) & \iff & T(i_1, \vec{x}\,) \wedge T(i_2, \vec{x}\,) \\
\neg\varphi_j(v) & T(i, \vec{x}\,) & \iff & \neg T(j, \vec{x}\,) \\
\exists z \in \text{left}(v)\ \varphi_j(z, v) & T(i, \langle x, \vec{y}\,\rangle) & \iff & \exists z \in d\ (z\, E\, x \wedge T(j, \langle z, x, \vec{y}\,\rangle)) \\
\forall z \in \text{left}(v)\ \varphi_j(z, v) & T(i, \langle x, \vec{y}\,\rangle) & \iff & \forall z \in d\ (z\, E\, x \wedge T(j, \langle z, x, \vec{y}\,\rangle)) \\
\exists z\ \varphi_j(z, v) & T(i, \vec{x}\,) & \iff & \exists z \in d\ T(j, \langle z, \vec{x}\,\rangle).
\end{array}$$

Here $\vec{x}$ stands for $\langle x_1, \ldots, x_\ell \rangle$. We again proceed by induction on formulae. Suppose first that φ_i is an atomic formula, say the formula $\text{left}(v) = \text{right}(v)$. Then

$$\forall x, y \in \bar{d}\ (\bar{T}(i, \langle x, y \rangle) \iff x\, \bar{I}\, y).$$

This is a Π_1 statement over $\bar{M}^{\bar{p}}$, since the predicates $\bar{d}, \bar{T}$ and $\bar{I}$ are rudimentary. Since π is Σ_1-preserving, $T(i, \langle x, y \rangle)$ iff $x\, I\, y$ iff $\mathbb{D} \models \varphi_i(x, y)$ for all $x, y \in d$.

Now suppose that $\varphi_i(v)$ is the formula $\varphi_{i_1}(v) \wedge \varphi_{i_2}(v)$. Then

$$\begin{aligned}
&\forall x_1, \ldots, x_\ell \in \bar{d} \\
&\qquad [\bar{T}(i, \langle x_1, \ldots, x_\ell \rangle) \iff \bar{T}(i_1, \langle x_1, \ldots, x_\ell \rangle) \wedge \bar{T}(i_2, \langle x_1, \ldots, x_\ell \rangle)].
\end{aligned}$$

This is again a Π_1-statement over $\bar{M}^{\bar{p}}$, so we obtain

$$\begin{aligned}
T(i, \langle x_1, \ldots, x_\ell \rangle) &\iff T(i_1, \langle x_1, \ldots, x_\ell \rangle) \wedge T(i_2, \langle x_1, \ldots, x_\ell \rangle) \\
&\iff \mathbb{D} \models \varphi_{i_1}(x_1, \ldots, x_\ell) \wedge \varphi_{i_2}(x_1, \ldots, x_\ell) \\
&\iff \mathbb{D} \models \varphi_i(x_1, \ldots, x_\ell);
\end{aligned}$$

the second equivalence follows from the induction hypothesis. We proceed similarly if $\varphi_i(v)$ is of the form $\neg\varphi_j(v)$.

Finally, suppose that $\varphi_i(v)$ introduces a quantifier; say $\varphi_i(v)$ is of the form $\exists z \in \text{left}(v)\, \varphi_j(z, v)$. The implication ($\Longleftarrow$) follows easily: Since $\bar{T}$ is a satisfaction relation for $\bar{\mathbb{D}}$, we have

$$\forall x, y\, [\exists z\, (z\, \bar{E}\, x \wedge \bar{T}(j, \langle z, x, y \rangle)) \to \bar{T}(i, \langle x, y \rangle)].$$

This is a Π_1-statement over $\bar{M}^{\bar{p}}$ and is therefore preserved upwards by π. To see the converse, let $\bar{g}$ be a $\Sigma_1^{\bar{M}}$-function in $\bar{p}$ uniformizing the relation

$$z \in \bar{k}(x) \wedge \varphi_j(z, \bar{k}(x), \bar{k}(y)).$$

Then $\bar{g}(u) \simeq h_{\bar{M}}(m, \langle u, \bar{p} \rangle) \simeq \bar{k}(\langle m, u \rangle)$ for an appropriate $m \in \omega$; hence,

$$\begin{aligned}
&\varphi_i(\bar{k}(x), \bar{k}(y)) \to \langle m, x, y \rangle \in \text{dom}(\bar{k}) \wedge \bar{k}(\langle m, x, y \rangle) \in \bar{k}(x) \\
&\quad \wedge \varphi_j(\bar{k}(\langle m, x, y \rangle), \bar{k}(x), \bar{k}(y))
\end{aligned}$$

holds in $\bar{\mathbb{D}}$ for all x, y in $\bar{M}^{\bar{p}}$. Translating this into the language of $\bar{\mathbb{D}}$ we obtain

$$\forall x, y\ [\bar{T}(i, \langle x, y\rangle) \to \langle m, x, y\rangle \in \bar{d} \wedge \langle m, x, y\rangle \bar{E} x \wedge \bar{T}(j, \langle\langle m, x, y\rangle, x, y\rangle)],$$

which is again a Π_1-statement over $\bar{M}^{\bar{p}}$. The required implication for $\mathbb{D}$ then follows immediately.

One consequence of the fact that T is a satisfaction relation for $\mathbb{D}$ is:

$$\pi : \bar{\mathbb{D}} \underset{\Sigma_2}{\longrightarrow} \mathbb{D},$$

as follows immediately by the fact that $\bar{T}, T$ are rudimentary over $\bar{M}^{\bar{p}}, N$ respectively by the same rudimentary definition. This implies that I is a congruence relation on $\mathbb{D}$ and E is extensional modulo this congruence relation; the map π simply carries both properties from $\bar{\mathbb{D}}$ over to $\mathbb{D}$ (Extensionality being Π_2). Note also that $\bar{E}$ is well-founded modulo the congruence relation $\bar{I}$ (in other words, the relation $x\bar{E}y \wedge \neg(x\bar{I}y)$ is well-founded). Hence E is well-founded modulo the congruence relation I by the strongness of π.

Let M be the transitive collapse of $\mathbb{D}$ and k be the collapsing map. Define $\tilde{\pi} : \bar{M} \to M$ by

$$\tilde{\pi}(\bar{k}(x)) = k(\pi(x)) \quad \text{for all } x \in \bar{d}.$$

It follows immediately that

$$\tilde{\pi} : \bar{M} \underset{\Sigma_2}{\longrightarrow} M.$$

In the following we show that M has all the required properties. Note first that M is a J-structure, say $M = \langle J_\alpha^B, D\rangle$. For the rest of the proof fix $i, i^* \in \omega$ so that

$$\begin{aligned}
&x = h_Q(i, \langle x, q\rangle) \quad \text{holds uniformly over any } J\text{-structure } Q,\\
&\bar{p} = h_{\bar{M}}(i^*, \langle 0, \bar{p}\rangle).
\end{aligned}$$

Then

$$x = \bar{k}(\langle i, x\rangle) \quad \text{for all } x \in \bar{M}^{\bar{p}} \text{ and } \bar{p} = \bar{k}(\langle i^*, 0\rangle).$$

We first observe that $|N| \subseteq M$. Given any $x \in \bar{M}^{\bar{p}}$, $\bar{k}(\langle j, z\rangle) \in x = \bar{k}(\langle i, x\rangle)$ iff $\bar{k}(\langle j, z\rangle) = w = \bar{k}(\langle i, w\rangle)$ for some $w \in x$; in other words,

$$\langle j, z\rangle \in \bar{d} \quad \Longrightarrow \quad [\langle j, z\rangle \bar{E} \langle i, x\rangle \iff \exists w \in x\, (\langle j, z\rangle \bar{I} \langle i, w\rangle)]$$

holds for all $x, y \in \bar{M}^{\bar{p}}$ and $j \in \omega$. Quantifying over $\bar{M}^{\bar{p}}$ we obtain a Π_1-statement, which is preserved under π. This allows us to conclude: Whenever $x \in N$ (note: $\langle i, x\rangle \in d$ by the fact that π is Σ_1-preserving and the "same" holds for $\bar{M}^{\bar{p}}$),

$$k(\langle i, x\rangle) = \{k(\langle i, w\rangle); w \in x\}.$$

It follows by $\in$-induction that $k(\langle i,x\rangle)=x$ for all $x\in N$. Furthermore, given any $x\in\bar{M}^{\bar{p}}$,

$$\tilde{\pi}(x)=\tilde{\pi}(\bar{k}(\langle i,x\rangle))=k(\pi(\langle i,x\rangle))=k(\langle i,\pi(x)\rangle)=\pi(x),$$

and hence $\tilde{\pi}\supseteq\pi$.

Let B', A and ρ be such that $N=\langle J_\rho^{B'},A\rangle$ (recall that, by our assumption, N is an acceptable structure). Set $p=k(\langle i^*,0\rangle)$. It is easy to check that $\tilde{\pi}(\bar{p})=p$. We now prove that $A=A_M^p$. Given an $x\in\bar{M}^{\bar{p}}$ and $j\in\omega$, we have $A_{\bar{M}}^{\bar{p}}(j,x)\iff\bar{M}\models\varphi_j(x,\bar{p})\iff\bar{M}\models\varphi_j(\bar{k}(\langle i,x\rangle),\bar{k}(\langle i^*,0\rangle))\iff\bar{T}(j,\langle\langle i,x\rangle,\langle i^*,0\rangle\rangle)$. Hence, the Π_1-statement

$$\forall x\,\forall j\in\omega\;[A_{\bar{M}}^{\bar{p}}(j,x)\leftrightarrow\bar{T}(j,\langle\langle i,x\rangle,\langle i^*,0\rangle\rangle)]$$

is preserved by π, which means that

$$A(j,x)\quad\iff\quad T(j,\langle\langle i,x\rangle,\langle i^*,0\rangle\rangle)\quad\iff\quad M\models\varphi_j(x,p)\tag{9.18}$$

for every $x\in N$ and $j\in\omega$.

This equivalence easily yields that $B'=B\cap N$ and $N=\langle J_\rho^B,A\rangle$ for ρ as above. Pick a j such that $\varphi_j(u,v)$ is the formula "$B^*(u)$" for any acceptable structure of the form $\langle J_{\alpha^*}^{B^*},D^*\rangle$. Since $\bar{B}(x)\iff A_{\bar{M}}^{\bar{p}}(j,x)$ holds for every $x\in\bar{M}^{\bar{p}}$, the preservation properties of π guarantee that $A(j,x)\iff B'(x)$. However, $A(j,x)$ is equivalent to $B(x)$ for all $x\in N$, as follows from (9.18).

To see that $A=A_M^p$ it suffices to show that $\rho=\rho(M)$, as $|N|=H_\rho^M$ then follows immediately (recall that $\rho=N\cap\mathrm{On}$). The computation below will also yield that $p\in R_M$. Notice that

$$\bar{k}(\langle j,x\rangle)=h_{\bar{M}}(j,\langle x,\bar{p}\rangle)=h_{\bar{M}}(\bar{k}(\langle i,j\rangle),\langle\bar{k}(\langle i,x\rangle),\bar{k}(\langle i^*,0\rangle)\rangle),$$

where the left equality is simply the definition of $\bar{k}$. Leaving out the middle term we obtain a Σ_1 statement, so it can be represented by some $m\in\omega$. More precisely, $\bar{T}(m,\langle\langle j,x\rangle,\langle i,j\rangle,\langle i,x\rangle,\langle i^*,0\rangle\rangle)$ holds for all $x\in\bar{M}^{\bar{p}}$ and $j\in\omega$. As above we apply π to obtain the corresponding statement for T and all $x\in N$. Using the fact that the Σ_1-Skolem functions have uniform definitions we infer that $k(\langle j,x\rangle)=h_M(k(\langle i,j\rangle,\langle k(\langle i,x\rangle),k(\langle i^*,0\rangle)\rangle))=h_M(j,\langle x,p\rangle)$. Note that there is a *lightface* Σ_1^M map[7] from ρ onto $|N|$; since the values $k(\langle j,x\rangle)$ range over all of M, we have

$$M=h_M(\rho\cup\{p\})\quad\text{and}\quad\rho(M)\leq\rho.$$

The latter is obviously a consequence of the former. On the other hand, given any $r\in M$ we can pick a $\xi\in N\cap\mathrm{On}$ such that $r=h_M(j,\langle\xi,p\rangle)$ for some j. For any Σ_1 formula ψ, $M\models\psi(\eta,r)$ iff $M\models\varphi_m(\eta,\xi,p)$ for a suitable m; the latter is equivalent to $A(m,\langle\eta,\xi\rangle)$ by (9.18). Taken together, $M\models\psi(\eta,r)$ can be expressed in a rudimentary fashion over N. Since N is

[7] "Lightface" means that no parameters are needed in the definition of such a function.

amenable, every bounded $\mathbf{\Sigma}_1^M$ subset of ρ is an element of N, which means that $\rho \leq \rho(M)$. Thus,

$$\rho = \rho(M) \quad \text{and} \quad p \in R_M.$$

It only remains to show that $\tilde{\pi}$ is strong. Let $\bar{R}, R$ be binary relations which are rudimentary over $\bar{M}, M$ respectively by the same rudimentary definition. Define $\bar{R}^*, R^*$ as follows

$$\begin{aligned} x\,\bar{R}^*\,y \quad &\Longleftrightarrow \quad x, y \in \bar{d} \wedge \bar{k}(x)\,\bar{R}\,\bar{k}(y) \\ x\,R^*\,y \quad &\Longleftrightarrow \quad x, y \in d \wedge k(x)\,R\,k(y). \end{aligned}$$

Then $\bar{R}^*$ is well-founded since $\bar{R}$ is—any decreasing chain $x_0, x_1, \ldots, x_n, \ldots$ in $\bar{R}^*$ yields a decreasing chain $\bar{k}(x_0), \bar{k}(x_1), \ldots, \bar{k}(x_n), \ldots$ in $\bar{R}$, hence no such chain can be infinite. Furthermore, $\bar{R}^*, R^*$ are rudimentary over $\bar{M}^{\bar{p}}, N$ respectively by the same rudimentary definition. As π is strong, R^* must be well-founded. Hence R must be well-founded as well, since every infinite decreasing chain $x_0, x_1, \ldots, x_n, \ldots$ in R has form $k(z_0), k(z_1), \ldots, k(z_n), \ldots$ for some $z_0, z_1, \ldots, z_n, \ldots$ and the latter would be an infinite decreasing chain in R^*. $\dashv$

5. Iterated Projecta

In this section we shall show how to iterate the process of defining a projectum and forming a standard code, and we shall introduce the notion of nth projectum, nth standard code, and nth reduct.

5.1 Definition. Let $M = \langle J_\beta^B, D\rangle$ be an acceptable J-structure. For $n < \omega$ we recursively define the *nth projectum* $\rho_n(M)$, the *nth standard code* $A_M^{n,p}$, and the *nth reduct* $M^{n,p}$ as follows:

$$\begin{aligned} \rho_0(M) &= \beta, \qquad \Gamma_M^0 = \{\emptyset\}, \qquad A_M^{0,\emptyset} = \emptyset, \qquad M^{0,\emptyset} = M, \\ \rho_{n+1}(M) &= \min\{\rho(M^{n,p}); p \in \Gamma_M^n\}, \\ \Gamma_M^{n+1} &= \textstyle\prod_{i \in n+1} [\rho_{i+1}(M), \rho_i(M))^{<\omega}, \end{aligned}$$

and for $p \in \Gamma_M^{n+1}$,

$$\begin{aligned} A_M^{n+1,p} &= A_{M^{n,p\restriction n}}^{p(n),\rho_{n+1}(M)} \quad \text{and} \\ M^{n+1,p} &= (M^{n,p\restriction n})^{p(n),\rho_{n+1}(M)}. \end{aligned}$$

We also set $\rho_\omega(M) = \min\{\rho_n(M) \mid n < \omega\}$. The ordinal $\rho_\omega(M)$ is called the *ultimate projectum* of M.

The reader will gladly verify that $\rho_1(M) = \rho(M)$. On the other hand, if M is not 1-sound (cf. Definition 5.7 below) then it need not be the case that $\rho_2(M)$ is the least ρ such that $\mathcal{P}(\rho) \cap \mathbf{\Sigma}_2^M \not\subseteq M$.

Supposing that we know $\rho_n(M) \leq \cdots \leq \rho_1(M)$ we may identify $p = \langle p(0), \ldots, p(n)\rangle \in \Gamma_M^{n+1}$ with the (finite) set $\bigcup \mathrm{ran}(p)$ of ordinals; this will play a role in the next section.

5.2 Definition. We define $P_M^n, R_M^n \subseteq \Gamma_M^n$ as follows:

$$\begin{aligned} P_M^0 &= \{\emptyset\} \\ P_M^{n+1} &= \{p \in \Gamma_M^{n+1}; p{\restriction}n \in P_M^n \wedge \rho(M^{n,p\restriction n}) = \rho_{n+1}(M) \wedge p(n) \in P_{M^{n,p\restriction n}}\} \end{aligned}$$

R_M^n is defined in the same way but with $R_{M^{n,p\restriction n}}$ in place of $P_{M^{n,p\restriction n}}$.

As before, we call the elements of P_M^n *good parameters* and the elements of R_M^n *very good parameters*.

5.3 Lemma. *Let M be acceptable.*

(a) $R_M^n \subseteq P_M^n \neq \emptyset$.

(b) Let $p \in R_M^n$. If $q \in \Gamma_M^n$ then $A^{n,q}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$.

(c) Let $p \in R_M^n$. Then $\rho(M^{n,p}) = \rho_{n+1}(M)$.

(d) $p \in P_M^n \Longrightarrow \forall i \in n\ p(i) \in P_{M^{i,p\restriction i}}$, and similarly for R_M^n. Moreover, if $p{\restriction}(n-1) \in R_M^{n-1}$ then equivalence holds.

Proof. (a) This is easily shown inductively by using Lemma 2.9 and by amalgamating parameters.

(b) By induction on $n < \omega$. The case $n = 0$ is trivial. Now let $n > 0$, and suppose that (b) holds for $n - 1$. Write $m = n - 1$. Let $p \in R_M^n$ and $q \in \Gamma_M^n$. We have to show that $A_{M^{m,q\restriction m}}^{q(m),\rho_n(M)}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$. Inductively, $M^{m,q\restriction m}$ is $\mathrm{rud}_{M^{m,p\restriction m}}$ in a parameter $t \in M^{n,p}$. As $p(m) \in R_{M^{m,p\restriction m}}$, there are e_0 and e_1 and $z \in M^{n,p}$ such that

$$q(m) = h_{M^{m,p\restriction m}}(e_0, \langle z, p(m)\rangle)$$

and

$$t = h_{M^{m,p\restriction m}}(e_1, \langle z, p(m)\rangle).$$

For $i < \omega$ and $x \in M^{n,p}$, we have that

$$\begin{aligned} \langle i, x\rangle \in A_{M^{m,q\restriction m}}^{q(m),\rho_n(M)} &\iff M^{m,q\restriction m} \models \varphi_i(x, q(m)) \\ &\iff M^{m,q\restriction m} \models \varphi_i(x, h_{M^{m,p\restriction m}}(e_0, \langle z, p(m)\rangle)) \\ &\iff M^{m,p\restriction m} \models \varphi_j(\langle x, z\rangle, p(m)) \\ &\iff \langle j, \langle x, z\rangle\rangle \in A_{M^{m,p\restriction m}}^{p(m)}, \end{aligned}$$

for some j which is recursively computable from i, as $M^{m,q\restriction m}$ is $\mathrm{rud}_{M^{m,p\restriction m}}$ in the parameter $t = h_{M^{m,p\restriction m}}(e_1, \langle z, p(m)\rangle)$. Thus, $A^{q(m),\rho_n(M)}_{M^{m,q\restriction m}}$ is $\mathrm{rud}_{A^{p(m)}_{M^{m,p\restriction m}}}$ in the parameter z.

(c) Let $\rho_{n+1}(M) = \rho(M^{n,q})$, where $q \in \Gamma^n_M$. By (b), $M^{n,q}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$, which implies that $\boldsymbol{\Sigma}_1^{M^{n,q}} \subseteq \boldsymbol{\Sigma}_1^{M^{n,p}}$. But then $\rho(M^{n,p}) \leq \rho(M^{n,q}) = \rho_{n+1}(M)$, and hence $\rho(M^{n,p}) = \rho_{n+1}(M)$.

(d) This is shown inductively by using (c). ⊣

The following is given just by the definition of R^{n+1}_M. Let M be acceptable, and let $p \in R^{n+1}_M$. Then

$$M = h_M(h_{M^{1,p\restriction 1}}(\cdots h_{M^{n,p\restriction n}}(\rho_{n+1}(M) \cup \{p(n)\}) \cdots) \cup \{p(0)\}).$$

We thus can, uniformly over M, define a function h^{n+1}_M basically as the iterated composition of the Σ_1-Skolem functions of the i^{th} reducts of M, $0 \leq i \leq n$, such that M is the h^{n+1}_M-hull of $\rho_{n+1}(M) \cup \{p\}$ whenever $p \in R^{n+1}_M$. The precise definition of the (partial) function $h^{n+1}_M : {}^{<\omega}\omega \times {}^{<\omega}|M^{n+1,p}| \to |M|$ is by recursion on $n < \omega$; we set $h^1_M = h_M$ and

$$\begin{aligned} &h^{n+1}_M(\langle \vec{i}, i_0, \ldots, i_k\rangle, \langle \vec{x}_{i_0}, \ldots, \vec{x}_{i_k}\rangle) \\ &\quad = h^n_M(\vec{i}, \langle h_{M^{n,p\restriction n}}(i_0, \vec{x}_{i_0}), \ldots, h_{M^{n,p\restriction n}}(i_k, \vec{x}_{i_k})\rangle). \end{aligned}$$

5.4 Lemma. *Let $n < \omega$, and let M be acceptable. Then h^{n+1}_M is in Σ^M_ω, and*

$$M = h^{n+1}_M {}``(\rho_{n+1}(M) \cup \{p\}),$$

whenever $p \in R^{n+1}_M$.

5.5 Lemma. *Let $0 < n < \omega$. Let M be acceptable, and let $p \in R^n_M$. Then $\boldsymbol{\Sigma}^M_\omega \cap \mathcal{P}(M^{n,p}) = \boldsymbol{\Sigma}^{M^{n,p}}_\omega$.*

Proof. It is easy to verify that $\boldsymbol{\Sigma}^{M^{n,p}}_\omega \subseteq \boldsymbol{\Sigma}^M_\omega \cap \mathcal{P}(M^{n,p})$. Let us prove the other direction.

It is straightforward to verify by induction on $m \leq n$ that if φ is Σ_0 and x, $y \in M^{m,p\restriction m}$, then

$$\varphi(x, h^m_M(y)) \text{ is uniformly } \Delta_1^{M^{m-1,p\restriction(m-1)}}(x, y). \tag{9.19}$$

Now let $A \in \boldsymbol{\Sigma}^M_\omega \cap \mathcal{P}(M^{n,p})$, say

$$x \in A \iff M \models \exists x_1\, \forall x_2 \cdots \exists/\forall x_k\; \varphi(x, y, x_1, x_2, \ldots, x_k),$$

where φ is Σ_0 and $y \in M$. By Lemma 5.4, we may write

$$\begin{aligned} x \in A \iff\; &\exists x'_1 \in M^{n,p}\, \forall x'_2 \in M^{n,p} \cdots \exists/\forall x_k \in M^{n,p} \\ &\varphi(x, h^n_M(y'), h^n_M(x'_1), h^n_M(x'_2), \ldots, h^n_M(x'_k)), \end{aligned}$$

where $y' \in M^{n,p}$. But then $A \in \boldsymbol{\Sigma}^{M^{n,p}}_\omega$, with the help of (9.19) for $m = n$. ⊣

A more careful look at the proofs of Lemmata 5.4 and 5.5 shows the following.

5.6 Lemma. *Let $n < \omega$. Let M be acceptable and $p \in R^n_M$. Let $A \subseteq M^{n,p}$ be $\mathbf{\Sigma}^M_{n+1}$. Then A is $\mathbf{\Sigma}_1^{M^{n,p}}$.*

5.7 Definition. M is *n-sound* iff $R^n_M = P^n_M$. M is *sound* iff M is n-sound for all $n < \omega$.

We shall prove later (cf. Lemma 9.2) that every J_α is sound. In fact, a key requirement on initial segments of a core model is that they be sound.

We can now formulate a general downward extension of embeddings lemma as the conjunction of the following three lemmata which, in turn, are immediate consequences of the corresponding lemmata for the first projectum.

5.8 Lemma. *Let $\bar{M}$, M be acceptable and $\pi : \bar{M}^{n,\bar{p}} \underset{\Sigma_0}{\longrightarrow} M^{n,p}$, where $\bar{p} \in R^n_{\bar{M}}$. Then there is a unique map $\tilde{\pi} \supseteq \pi$ such that $\mathrm{dom}(\tilde{\pi}) = \bar{M}$, $\tilde{\pi}(\bar{p}) = p$ and, setting $\tilde{\pi}_i = \tilde{\pi} \restriction H^i_{\bar{M}}$,*

$$\tilde{\pi}_i : \bar{M}^{i,\bar{p}\restriction i} \underset{\Sigma_0}{\longrightarrow} M^{i,p\restriction i} \quad \textit{for } i \leq n.$$

The map $\tilde{\pi}_i$ is in fact Σ_1-preserving for $i \in n$.

5.9 Lemma. *Suppose that $\bar{M}$, M, $\bar{p}$, p, π, $\tilde{\pi}$, $\tilde{\pi}_i$, $i \leq n$, are as above and $p \in R^n_M$. Let $\pi : \bar{M}^{n,\bar{p}} \underset{\Sigma_\ell}{\longrightarrow} M^{n,p}$ where $\ell \in \omega$. Then*

$$\tilde{\pi}_i : \bar{M}^{i,\bar{p}\restriction i} \underset{\Sigma_{\ell+n-i}}{\longrightarrow} M^{i,p\restriction i} \quad \textit{for } i \leq n.$$

Hence, $\tilde{\pi}_0 : \bar{M} \underset{\Sigma_{\ell+n}}{\longrightarrow} M$.

5.10 Lemma. *Let $\pi : N \underset{\Sigma_0}{\longrightarrow} M^{n,p}$, where M is as above. Then there are unique $\bar{M}$, $\bar{p}$ such that $\bar{p} \in R^n_{\bar{M}}$ and $N = \bar{M}^{n,\bar{p}}$.*

The general upward extension of embeddings lemma is the conjunction of the following lemma together with Lemmata 5.8 and 5.9.

5.11 Lemma. *Let $\pi : \bar{M}^{n,\bar{p}} \to N$ be strong, where $\bar{M}$ is an acceptable J-structure and $\bar{p} \in R^n_{\bar{M}}$. Then there are unique M, p such that M is acceptable, $p \in R^n_M$ and $M^{n,p} = N$. Moreover, if $\tilde{\pi}$ is as in Lemma 5.8, then $\tilde{\pi}$ is strong.*

If π and $\tilde{\pi}$ are as in Lemma 5.8 then $\tilde{\pi}$ is often called the *n-completion* of π.

If we take $\pi : \bar{M}^{n,\bar{p}} \to M^{n,p}$ as in Lemma 5.8 and form the corresponding extension $\tilde{\pi}$, we can in fact do better than stated there. It is easy to see that for every appropriate $\bar{q}$ and $q = \tilde{\pi}(\bar{q})$,

$$\tilde{\pi}_i : \bar{M}^{i,\bar{q}\restriction i} \underset{\Sigma_1}{\longrightarrow} M^{i,q\restriction i} \quad \text{for } i \in n. \tag{9.20}$$

This suggests the general notion of a $\Sigma_\ell^{(n)}$-preserving embedding, where n indicates preservation at the nth level, i.e. if $\tilde{\pi}(\bar{q}) = q$ then (9.20) and

$$\tilde{\pi}_n : \bar{M}^{n,\bar{q}} \underset{\Sigma_\ell}{\longrightarrow} M^{n,q}. \tag{9.21}$$

It turns out that there is a canonical class of formulae, the so called $\Sigma_\ell^{(n)}$-formulae, such that the above embeddings are exactly those which are elementary with respect to this class. This idea leads towards Jensen's elegant Σ^* theory which is dealt with in [15, Sects. 1.6 ff.].

Following [7, §2], though, we shall call $\Sigma_1^{(n)}$ elementary maps $r\Sigma_{n+1}$ elementary. Here is our official definition, which presupposes that the structures involved possess very good parameters; it will play a role in the last two sections.

5.12 Definition. Let M, N be acceptable, let $\pi : M \to N$, and let $n < \omega$. Then π is called $r\Sigma_{n+1}$ *elementary* provided that there is $p \in R_M^n$ with $\pi(p) \in R_N^n$, and for all $i \leq n$,

$$\pi \restriction H_{\rho_i(M)}^M : M^{i,p\restriction i} \underset{\Sigma_1}{\longrightarrow} N^{i,\pi(p)\restriction i}. \tag{9.22}$$

The map π is called *weakly* $r\Sigma_{n+1}$ *elementary* provided that there is $p \in R_M^n$ with $\pi(p) \in R_N^n$, and for all $i < n$, (9.22) holds, and

$$\pi \restriction H_{\rho_n(M)}^M : M^{n,p} \underset{\Sigma_0}{\longrightarrow} N^{n,\pi(p)}.$$

If $\pi : M \to N$ is (weakly) $r\Sigma_{n+1}$ elementary then typically both M and N will be n-sound; however, neither M nor N has to be $(n+1)$-sound. It is possible to generalize this definition so as to not assume that very good parameters exist (cf. [7, §2]).

Lemma 5.8 therefore says that the map π can be extended to its n-completion $\tilde{\pi}$ which is weakly $r\Sigma_{n+1}$ elementary, and Lemma 5.9 says that if π is Σ_1 elementary to begin with then the n-completion $\tilde{\pi}$ will end up being $r\Sigma_{n+1}$ elementary.

Moreover, if a map $\pi : M \to N$ is $r\Sigma_{n+1}$ elementary then it respects h^{n+1} by Theorem 1.15, i.e.:

5.13 Lemma. *Let $n < \omega$, and let M, N be acceptable. Let $\pi : M \to N$ be $r\Sigma_{n+1}$ elementary. Then for all appropriate x,*

$$\pi(h_M^{n+1}(x)) = h_N^{n+1}(\pi(x)).$$

6. Standard Parameters

Finite sets of ordinals are well-ordered in a simple canonical way.

6.1 Definition. Let $a, b \in [\mathrm{On}]^{<\omega}$. Set

$$a <^* b \iff \exists \alpha \in b\ (a \setminus (\alpha + 1) = b \setminus (\alpha + 1) \wedge \alpha \notin a).[8]$$

The ordering $<^*$ has a rudimentary definition, therefore it is absolute for transitive rudimentarily closed structures and is also preserved under embeddings which are Σ_0 elementary. If we view finite sets of ordinals as finite decreasing sequences, $a <^* b$ precisely when a precedes b lexicographically. Moreover, we easily get the following.

6.2 Lemma. $[\mathrm{On}]^{<\omega}$ *is well-ordered by* $<^*$.

Let M be acceptable, and $n < \omega$. The well-ordering $<^*$ induces a well-ordering of Γ^n_M by confusing $p \in \Gamma^n_M$ with $\bigcup \mathrm{ran}(p)$ (i.e., by identifying p with the obvious set of ordinals; cf. above). We shall denote this latter well-ordering also by $<^*$.

6.3 Definition. Let M be acceptable. The $<^*$-least $p \in P^n_M$ is called the *nth standard parameter of M* and is denoted by $p_n(M)$. We shall write M^n for $M^{n,p_n(M)}$; M^n is called the nth *standard reduct of M*.

6.4 Lemma. *Let $p \in R^n_M$. Then p can be lengthened to some $p' \in P^{n+1}_M$, i.e., there is some $p' \in P^{n+1}_M$ with $p' \restriction n = p$.*

Proof. This follows immediately from Lemma 5.3 (c). ⊣

6.5 Corollary. *Let $n > 0$ and M be n-sound. Then*

$$p_{n-1}(M) = p_n(M) \restriction (n-1)$$

Proof. By Lemma 6.4. ⊣

6.6 Definition. Let M be acceptable. Suppose that for all $n < \omega$, $p_n(M) = p_{n+1}(M) \restriction n$. Then we set $p(M) = \bigcup_{n<\omega} p_n(M)$. $p(M)$ is called the *standard parameter of M*.

We shall often confuse $p(M)$ with $\bigcup \mathrm{ran}(p(M))$.

6.7 Corollary. *Let M be sound. Then $p(M)$ exists, i.e., for all $n < \omega$, $p_n(M) = p_{n+1}(M) \restriction n$.*

Proof. By Corollary 6.5. ⊣

6.8 Lemma. *M is sound iff $p_n(M) \in R^n_M$ for all $n \in \omega$.*

[8] I.e., $\max(a \triangle b) \in b$.

Proof. We shall prove the non-trivial direction ($\Longleftarrow$). For each $n > 0$ we shall prove

$$p_n(M) \in R^n_M \quad \Longrightarrow \quad R^n_M = P^n_M. \tag{9.23}$$

This holds trivially for $n = 0$. Now suppose that $n > 0$ is least such that the statement (9.23) fails. Hence $P^n_M \setminus R^n_M \neq \emptyset$ (cf. Lemma 5.3 (a)). Let q be the $<^*$-least element of $P^n_M \setminus R^n_M$. This means that $p <^* q$, where $p = p_n(M)$. By Lemma 5.3, we may let $i < n$ be least such that $q(i) \notin R_{M^{i,q\restriction i}}$.

Let us first consider the case $n = 1$.

Then, of course, $p(0) <^* q(0)$. Using the downward extension of embeddings lemma, we may let $\bar{M}$, $\bar{q}$, π be unique such that

$$\begin{aligned} &\bar{q} \in R_{\bar{M}} \\ &\pi : \bar{M} \to M \text{ is } \Sigma_1 \text{ elementary} \\ &\pi(\bar{q}) = q(0) \\ &\pi \restriction H^M_{\rho_1(M)} = \text{id}. \end{aligned} \tag{9.24}$$

As $q(0) \notin R_M$, $\pi \neq \text{id}$. Because $p(0) \in R_M$, there are e and $z \in [\rho_1(M), M \cap \text{On})^{<\omega}$ such that

$$q(0) = h_M(e, \langle z, p(0) \rangle).$$

As $p(0) <^* q(0)$, by elementarity we get that

$$\bar{M} \models \exists p' <^* \bar{q} \; (\bar{q} = h(e, \langle z, p' \rangle)).$$

Letting p^* be a witness, we may conclude that by elementarity again

$$\pi(p^*) <^* q(0) \wedge q(0) = h_M(e, \langle z, \pi(p^*) \rangle), \tag{9.25}$$

where we may and shall assume that $\pi(p^*) \in [\rho_1(M), M \cap \text{On})^{<\omega}$. We have that $\pi(p^*) \in P_M$ by (9.25). But $\pi(p^*) \in \text{ran}(\pi)$ and $\pi \neq \text{id}$, so that we must also have that $\pi(p^*) \notin R_M$. Because $\pi(p^*) <^* q(0)$, we have a contradiction to the choice of q.

Now let us consider the case $n > 1$.

If $p \restriction (n-1) = q \restriction (n-1)$ then we may apply the above argument to the reduct $M^{n-1, p\restriction(n-1)}$. We may thus assume that $p \restriction (n-1) <^* q \restriction (n-1)$. Let $i < n$ be least such that $p \restriction i <^* q \restriction i$. We shall assume that $i = 1$ and $n = 2$ for notational convenience. The general case is similar to this special case and is left to the reader.

As $p_2(M) \in R^2_M$, Lemma 5.3 (d) yields that $P^1_M = R^1_M$. As $p(0) <^* q(0)$, there is some $j \in \omega$ and $z \in [\rho_1(M), M \cap \text{On})^{<\omega}$ such that

$$\exists p' \; (p' <^* q(0) \wedge q(0) = h_M(j, \langle z, p' \rangle)).$$

If i is the Gödel number of this Σ_1 formula, we thus have that $A^{q(0)}_M(i, z)$ holds true. Let X be the smallest Σ_1 submodel of the (first) reduct $M^{q(0)}$. There is then some $z_0 \in X$ such that $A^{q(0)}_M(i, z_0)$ holds true. Therefore,

$$\exists p' \; (p' <^* q(0) \wedge q(0) = h_M(j, \langle z_0, p' \rangle)). \tag{9.26}$$

So there is some p', call it $\bar{q}(0)$, witnessing (9.26) which is an element of the smallest Σ_1 substructure of M which contains both $q(0)$ and z_0. There is then also some k with $\bar{q}(0) = h_M(k, \langle z_0, q(0)\rangle)$.

Now set $\bar{q} = \bar{q}(0) \cup q(1)$. Then $\bar{q} \in P^2_M$, because $q(0) = h_M(j, \langle z_0, \bar{q}(0)\rangle)$ and $q \in P^2_M$. We will now show that $\bar{q} \notin R^2_M$. As $\bar{q}(0) <^* q(0)$ (and hence $\bar{q} <^* q$) this will contradict the choice of q and finish the proof.

Well, to see that $\bar{q} \notin R^2_M$ it suffices to verify that if

$$Y = h_{M^{\bar{q}(0)}}(\rho_2(M) \cup \{q(1)\})$$

then $Y \neq M^{\bar{q}(0)}$. As $\bar{q}(0) = h_M(k, \langle z_0, q(0)\rangle)$, we can find a recursive $f : \omega \to \omega$ such that for all $\ell \in \omega$ and for all x,

$$A^{\bar{q}(0)}_M(\ell, x) \iff A^{q(0)}_M(f(\ell), \langle x, z_0\rangle).$$

Therefore, as $z_0 \in X$,

$$Y \subseteq h_{M^{q(0)}}(\rho_2(M) \cup \{q(1)\}).$$

But $q(1) \notin R^{M^{q(0)}}$, and so $Y \neq M^{\bar{q}(0)}$. ⊣

There is a class of structures, for which the above characterization of soundness has a particularly nice form. This class comprises all of the structures J_α where α is a limit ordinal. Moreover, the same applies to sufficiently iterable premice, which are the building blocks of core models.

7. Solidity Witnesses

Solidity witnesses are witnesses to the fact that a given ordinal is a member of the standard parameter. The key fact will be that being a witness is preserved under Σ_1 elementary maps, so that witnesses can be used for showing that standard parameters are mapped to standard parameters.

Whereas the pure theory of witnesses is easy to grasp, it is one of the deepest results of inner model theory that the structures considered there (viz., iterable premice) do contain witnesses.

7.1 Definition. Let M be an acceptable structure, let $p \in [M \cap \mathrm{On}]^{<\omega}$, and let $\nu \in p$. Let W be another acceptable structure with $\nu \subseteq W$, and let $r \in [W \cap \mathrm{On}]^{<\omega}$. We say that (W, r) is a *witness for $\nu \in p$ w.r.t. M, p* iff for every Σ_1 formula $\varphi(v_0, \ldots, v_{l+1})$ and for all $\xi_0, \ldots, \xi_l < \nu$

$$M \models \varphi(\xi_0, \ldots, \xi_l, p \setminus (\nu + 1)) \implies W \models \varphi(\xi_0, \ldots, \xi_l, r). \tag{9.27}$$

In this situation, we shall often suppress r and call W a witness. The proof of Lemma 7.2 will show that if a witness exists then there is also one where $\Longrightarrow$ may be replaced by $\Longleftrightarrow$ in (9.27).

7.2 Lemma. *Let M be an acceptable structure, and let $p \in P_M$. Suppose that for each $\nu \in p$ there is a witness W for $\nu \in p$ w.r.t. M, p such that $W \in M$. Then $p = p_1(M)$.*

Proof. Suppose not. Then $p_1(M) <^* p$, and we may let $\nu \in p \setminus p_1(M)$ be such that $p \setminus (\nu+1) = p_1(M) \setminus (\nu+1)$. Let us write q for $p \setminus (\nu+1) = p_1(M) \setminus (\nu+1)$. Let $(W, r) \in M$ be a witness for $\nu \in p$ w.r.t. M, p. Let $A \in \Sigma_1^M(\{p_1(M)\})$ be such that $A \cap \rho_1(M) \notin M$.

Let k be the number of elements of $p \cap \nu$, and if $k > 0$ then let $\xi_1 < \cdots < \xi_k$ be such that $p_1(M) \cap \nu = \{\xi_1, \ldots, \xi_k\}$. There is a Σ_1 formula $\varphi(v_0, \ldots, v_{k+1})$ such that

$$\xi \in A \iff M \models \varphi(\xi, \xi_1, \ldots, \xi_k, q).$$

Because $(W, r) \in M$ is a witness for $\nu \in p$ w.r.t. M, p, we have that

$$M \models \bar{\varphi}(\xi, \xi_1, \ldots, \xi_k, q) \implies W \models \bar{\varphi}(\xi, \xi_1, \ldots, \xi_k, r)$$

for every $\xi < \rho_1(M) \leq \nu$ and every $\bar{\varphi}$ which is Σ_1.

Let $\alpha = \sup(h_W(\nu \cup \{r\}) \cap \mathrm{On})$, and let $\bar{W} = J_\alpha^B$ (where $W = J_\beta^B$, some $\beta \geq \alpha$). By looking at the canonical elementary embedding from $h_M(\nu \cup \{q\})$ into $\bar{W}$, which is Σ_0 elementary and cofinal (and hence Σ_1 elementary) we get that

$$M \models \bar{\varphi}(\xi, \xi_1, \ldots, \xi_k, q) \iff \bar{W} \models \bar{\varphi}(\xi, \xi_1, \ldots, \xi_k, r) \tag{9.28}$$

for every $\xi < \rho_1(M) \leq \nu$ and every $\bar{\varphi}$ which is Σ_1. In particular, (9.28) holds with $\bar{\varphi}$ replaced by φ and every $\xi < \rho_1(M) \leq \nu$. As $\bar{W} \in M$, this shows that in fact $A \cap \rho_1(M) \in M$. Contradiction! ⊣

7.3 Definition. Let M be acceptable, let $p \in [M \cap \mathrm{On}]^{<\omega}$, and let $\nu \in p$. We denote by $W_M^{\nu,p}$ the transitive collapse of $h_M(\nu \cup (p \setminus (\nu+1)))$. We call $W_M^{\nu,p}$ the *standard witness for $\nu \in p$ w.r.t. M, p.*

7.4 Lemma. *Let M be acceptable, and let $\nu \in p \in P_M$. The following are equivalent.*

(1) $W_M^{\nu,p} \in M$.

(2) There is a witness W for $\nu \in p$ w.r.t. M, p such that $W \in M$.

Proof. We have to show (2) $\implies$ (1). Let $\sigma : W_M^{\nu,p} \to M$ be the inverse of the transitive collapse. We may also let $\sigma^* : W_M^{\nu,p} \to W$ be defined by $h_{W_M^{\nu,p}}(\xi, \sigma^{-1}(p \setminus \nu+1)) \mapsto h_W(\xi, r)$, $\xi < \nu$. Set $\alpha = \sup(h_W(\nu \cup \{r\}) \cap \mathrm{On})$, and let $\bar{W} = J_\alpha^A$ (where $W = J_\beta^A$, some $\beta \geq \alpha$). Again, $\sigma^* : W_M^{\nu,p} \to \bar{W}$ is Σ_1 elementary.

Let us assume without loss of generality that $W_M^{\nu,p}$ is not an initial segment of M.

Now if $\sigma(\nu) = \nu$ then a witness to $\rho_1(M)$ is definable over $W_M^{\nu,p}$, and hence over $\bar{W}$. But as $\bar{W} \in M$, this witness to $\rho_1(M)$ would then be in M. Contradiction!

We thus have that ν is the critical point of σ. Thus, if $M = J^B_\gamma$, we know that $\sigma(\nu)$ is regular in M and so $J^B_{\sigma(\nu)} \models \mathsf{ZFC}^-$. We may code $W^{\nu,p}_M$ by some $a \subseteq \nu$, Σ_1-definably over $W^{\nu,p}_M$. Using σ^*, a is definable over $\bar{W}$, so that $a \in M$. In fact, $a \in J^B_{\sigma(\nu)}$ by acceptability. We can thus decode a in $J^B_{\sigma(\nu)}$, which gives $W^{\nu,p}_M \in J^B_{\sigma(\nu)} \subseteq M$. ⊣

7.5 Definition. Let M be an acceptable structure. We say that M is 1-*solid* iff

$$W_M^{\nu,p_1(M)} \in M$$

for every $\nu \in p_1(M)$.

7.6 Lemma. *Let $\bar{M}$, M be acceptable structures, and let $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$. Let $\bar{\nu} \in \bar{p} \in [\bar{M} \cap \mathrm{On}]^{<\omega}$, and set $\nu = \pi(\bar{\nu})$ and $p = \pi(\bar{p})$. Let $(\bar{W}, \bar{r})$ be a witness for $\bar{\nu}$ w.r.t. $\bar{M}$, $\bar{p}$ such that $\bar{W} \in \bar{M}$, and set $W = \pi(\bar{W})$ and $r = \pi(\bar{r})$. Then (W, r) is a witness for ν w.r.t. M, p.*

Proof. Let $\varphi(v_0, \ldots, v_{l+1})$ be an arbitrary Σ_1 formula. We know that

$$\bar{M} \models \forall \xi_0, \ldots, \xi_l < \bar{\nu}[\varphi(\xi_0, \ldots, \xi_l, \bar{p} \setminus (\bar{\nu}+1)) \Longrightarrow \bar{W} \models \varphi(\xi_0, \ldots, \xi_l, \bar{r})].$$

As π is Π_1 elementary, this yields that

$$M \models \forall \xi_0, \ldots, \xi_l < \nu[\varphi(\xi_0, \ldots, \xi_l, p \setminus (\nu+1)) \Longrightarrow W \models \varphi(\xi_0, \ldots, \xi_l, r)].$$

We thus conclude that (W, r) is a witness for ν w.r.t. M, p. ⊣

7.7 Corollary. *Let $\bar{M}$, M be acceptable structures, and let $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$. Suppose that $\bar{M}$ is 1-solid and $\pi(p_1(\bar{M})) \in P_M$. Then $p_1(M) = \pi(p_1(\bar{M}))$, and M is 1-solid.*

The proof of the following lemma is virtually the same as the proof of Lemma 7.6.

7.8 Lemma. *Let $\bar{M}$, M be acceptable structures, and let $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$. Let $\bar{\nu} \in \bar{p} \in [On \cap \bar{M}]^{<\omega}$, and set $\nu = \pi(\bar{\nu})$ and $p = \pi(\bar{p})$. Let $(\bar{W}, \bar{r}) \in \bar{M}$ be such that, setting $W = \pi(\bar{W})$ and $r = \pi(\bar{r})$, (W, r) is a witness for ν w.r.t. M, p. Then $(\bar{W}, \bar{r})$ is a witness for $\bar{\nu} \in \bar{p}$ w.r.t. $\bar{M}$, $\bar{p}$.*

7.9 Corollary. *Let $\bar{M}$, M be acceptable structures, and let $\pi : \bar{M} \xrightarrow[\Sigma_1]{} M$. Suppose that M is 1-solid, and that in fact $W_M^{\nu,p_1(M)} \in \mathrm{ran}(\pi)$ for every $\nu \in p_1(M)$. Then $p_1(\bar{M}) = \pi^{-1}(p_1(M))$, and $\bar{M}$ is 1-solid.*

The following definition just extends Definition 7.5.

7.10 Definition. Let M be an acceptable structure. If $0 < n < \omega$ then we say that M is *n-solid* if for every $k < n$, $p_1(M^k) = p_{k+1}(M)(k) = p_n(M)(k)$ and M^k is 1-solid, i.e., if

$$W_{M^k}^{\nu,p_1(M^k)} \in M^k$$

for every $\nu \in p_1(M^k)$.

7.11 Lemma. *Let $\bar{M}$, M be acceptable, let $n > 0$, and let $\pi : \bar{M} \to M$ be $r\Sigma_n$ elementary as being witnessed by $p_{n-1}(M)$. If $\bar{M}$ is n-solid and $\pi(p_1(\bar{M}^{n-1})) \in P_{M^{n-1}}$ then $p_n(M) = \pi(p_n(\bar{M}))$ and M is n-solid.*

7.12 Lemma. *Let $\bar{M}$, M be acceptable, let $n > 0$, and let $\pi : \bar{M} \to M$ be $r\Sigma_n$ elementary as being witnessed by $\pi^{-1}(p_{n-1}(M))$. Suppose that M is n-solid, and in fact $W_{M^k}^{\nu,p_1(M^k)} \in \operatorname{ran}(\pi)$ for every $k < n$. If $\pi^{-1}(p_{n-1}(M)) \in P_{\bar{M}}^{n-1}$ then $p_n(\bar{M}) = \pi^{-1}(p_n(\bar{M}))$ and $\bar{M}$ is n-solid.*

The ultrapower maps we shall construct in the next section shall be elementary in the sense of the following definition. (Cf. [7, Definition 2.8.4].)

7.13 Definition. Let both M and N be acceptable, let $\pi : M \to N$, and let $n < \omega$. Then π is called an *n-embedding* if the following hold:

(1) Both M and N are n-sound,

(2) π is $r\Sigma_{n+1}$ elementary,

(3) $\pi(p_k(M)) = p_k(N)$ for every $k \leq n$, and

(4) $\rho_n(N) = \sup(\pi``\rho_n(M))$.

Other examples for n-embeddings are typically obtained as follows. Let M be acceptable, and let, for $n \in \omega$, $\mathfrak{C}_n(M)$ denote the transitive collapse of $h_M^n``(\rho_n(M) \cup \{p_n(M)\})$. $\mathfrak{C}_n(M)$ is called the nth *core of M*. The natural map from $\mathfrak{C}_{n+1}(M)$ to $\mathfrak{C}_n(M)$ will be an n-embedding under favorable circumstances.

8. Fine Ultrapowers

This section deals with the construction of "fine structure preserving" embeddings. Inner model theory is in need of such maps in two main contexts: first, in "lift up arguments" which are crucial for instance in the proof of the Covering Lemma for L or higher core models and in the proof of $\square_\kappa$ in such models (cf. [5] and the next section), and second, in performing iterations of premice (cf. [5, 9, 12]). This section will deal with the construction of such embeddings from an abstract point of view. The combinatorial objects which are used for defining such maps are called "extenders".

The following definition makes use of notational conventions which are stated right after it.

8.1 Definition. Let M be acceptable. Then $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ is called a *(κ,ν)-extender over M with critical points* $\langle \mu_a \mid a \in [\nu]^{<\omega}\rangle$ provided the following hold:

(1) (Ultrafilter property) For $a \in [\nu]^{<\omega}$ we have that E_a is an ultrafilter on the set $\mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ which is κ-complete with respect to sequences in M; moreover, μ_a is the least μ such that $[\mu]^{\mathrm{Card}(a)} \in E_a$.

(2) (Coherence) For $a, b \in [\nu]^{<\omega}$ with $a \subseteq b$ and for $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ we have that $X \in E_a \Longleftrightarrow X^{ab} \in E_b$.

(3) (Uniformity) $\mu_{\{\kappa\}} = \kappa$.

(4) (Normality) Let $a \in [\nu]^{<\omega}$ and $f : [\mu_a]^{\mathrm{Card}(a)} \to \mu_a$ with $f \in M$. If

$$\{u \in [\mu_a]^{\mathrm{Card}(a)} \mid f(u) < \max(u)\} \in E_a$$

then there is some $\beta < \max(a)$ such that

$$\{u \in [\mu_a]^{\mathrm{Card}(a\cup\{\beta\})} \mid f^{a,a\cup\{\beta\}}(u) = u_\beta^{a\cup\{\beta\}}\} \in E_{a\cup\{\beta\}}.$$

We write $\sigma(E) = \sup\{\mu_a + 1 \mid a \in [\nu]^{<\omega}\}$. The extender E is called *short* if $\sigma(E) = \kappa + 1$; otherwise E is called *long*.

Let $b = \{\beta_1 < \cdots < \beta_n\}$, and let $a = \{\beta_{j_1} < \cdots < \beta_{j_m}\} \subseteq b$. If $u = \{\xi_1 < \cdots < \xi_n\}$ then we write u_a^b for $\{\xi_{j_1} < \cdots < \xi_{j_m}\}$; we also write $u_{\beta_i}^b$ for ξ_i. If $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)})$, then we write X^{ab} for $\{u \in [\mu_b]^{\mathrm{Card}(b)} \mid u_a^b \in X\}$. Finally, if f has domain $[\mu_a]^{\mathrm{Card}(a)}$ then we write $f^{a,b}$ for that g with domain $[\mu_b]^{\mathrm{Card}(b)}$ such that $g(u) = f(u_a^b)$. Finally, we write pr for the function which maps $\{\beta\}$ to β (i.e., $\mathrm{pr} = \bigcup$).

Notice that if E is a (κ,ν)-extender over the acceptable J-model M with critical points μ_a, and if N is another acceptable J-model with $\mathcal{P}(\mu_a) \cap N = \mathcal{P}(\mu_a) \cap M$ for all $a \in [\nu]^{<\omega}$, then E is also an extender over N.

The currently known core models are built with just short extenders on their sequence (cf. [5, 9, 12]). On the other hand, already the proof of the Covering Lemma for L has to make use of long extenders.

The following is easy to verify.

8.2 Theorem. *Let M and N be acceptable, and let $\pi : M \underset{\Sigma_0}{\longrightarrow} N$ cofinally with critical point κ. Let $\nu \leq N \cap \mathrm{On}$. For each $a \in [\nu]^{<\omega}$ let μ_a be the least $\mu \leq M \cap \mathrm{On}$ such that $a \subseteq \pi(\mu)$, and set*

$$E_a = \{X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M \mid a \in \pi(X)\}.$$

Then $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ is a (κ,ν)-extender over M.

8.3 Definition. If $\pi : M \to N$, E, κ, and ν are as in the statement of Theorem 8.2 then E is called the *(κ,ν)-extender derived from* π.

8.4 Theorem. *Let $M = \langle J_\alpha^A, B\rangle$ be acceptable, and let $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ be a (κ,ν)-extender over M. There are then N and π such that the following hold:*

(a) $\pi : M \xrightarrow[\Sigma_0]{} N$ cofinally with critical point κ,

(b) the well-founded part $\mathrm{wfp}(N)$ of N is transitive and $\nu \subseteq \mathrm{wfp}(N)$,

(c) $N = \{\pi(f)(a) \mid a \in [\nu]^{<\omega}, f : [\mu_a]^{\mathrm{Card}(a)} \to M, f \in M\}$, and

(d) for $a \in [\nu]^{<\omega}$ we have that $X \in E_a$ if and only if $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ and $a \in \pi(X)$.

Moreover, N and π are unique up to isomorphism.

Proof. We do not construe (c) in the statement of this theorem to presuppose that N be well-founded; in fact, this statement makes perfect sense even if N is *not* well-founded.

Let us first argue that N and π are unique up to isomorphism. Suppose that N, π and N', π' are both as in the statement of the Theorem. We claim that

$$\pi(f)(a) \mapsto \pi'(f)(a)$$

defines an isomorphism between N and N'. Note for example that $\pi(f)(a) \in \pi(g)(b)$ if and only if, setting $c = a \cup b$,

$$c \in \pi(\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\}),$$

which by (d) yields that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\} \in E_c,$$

and hence by (d) once more that

$$c \in \pi'(\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\},$$

i.e., $\pi'(f)(a) \in \pi'(g)(b)$.

The existence is shown by an ultrapower construction. Let us set

$$D = \{\langle a, f\rangle \mid a \in [\nu]^{<\omega}, f : [\mu_a]^{\mathrm{Card}(a)} \to M, f \in M\}.$$

For $\langle a, f\rangle$, $\langle b, g\rangle \in D$ let us write

$$\langle a, f\rangle \sim \langle b, g\rangle \iff \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) = g^{b,c}(u)\} \in E_c, \quad \text{for } c = a \cup b.$$

We may easily use (1) and (2) of Definition 8.1 to see that $\sim$ is an equivalence relation on D. If $\langle a, f\rangle \in D$ then let us write $[a, f] = [a, f]_E^M$ for the equivalence class $\{\langle b, g\rangle \in D \mid \langle a, f\rangle \sim \langle b, g\rangle\}$, and let us set

$$\tilde{D} = \{[a, f] \mid \langle a, f\rangle \in D\}.$$

Let us also define, for $[a,f]$, $[b,g] \in \tilde{D}$,

$$\begin{aligned} [a,f] \mathrel{\tilde{\in}} [b,g] \quad &\Longleftrightarrow \quad \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\} \in E_c, \\ &\qquad\qquad \text{for } c = a \cup b, \\ \tilde{A}([a,f]) \quad &\Longleftrightarrow \quad \{u \in [\mu_a]^{\mathrm{Card}(a)} \mid f(u) \in A\} \in E_a, \\ \tilde{B}([a,f]) \quad &\Longleftrightarrow \quad \{u \in [\mu_a]^{\mathrm{Card}(a)} \mid f(u) \in B\} \in E_a. \end{aligned}$$

Notice that the relevant sets are members of M, as M is rud_A-closed and amenable. Moreover, by (1) and (2) of Definition 8.1, $\tilde{\in}$, $\tilde{A}$, and $\tilde{B}$ are well-defined. Let us set

$$N = \langle \tilde{D}, \tilde{\in}, \tilde{A}, \tilde{B} \rangle.$$

Claim 1. (Łoś's Theorem) Let $\varphi(v_1, \ldots, v_k)$ be a Σ_0 formula, and let $\langle a_1, f_1 \rangle$, $\ldots, \langle a_k, f_k \rangle \in D$. Then

$$\begin{aligned} &N \models \varphi([a_1, f_1], \ldots, [a_k, f_k]) \\ &\quad \Longleftrightarrow \quad \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models \varphi(f_1^{a_1,c}(u), \ldots, f_k^{a_k,c}(u))\} \in E_c \\ &\qquad\qquad \text{for } c = a_1 \cup \cdots \cup a_k. \end{aligned}$$

Notice again that the relevant sets are members of M. Claim 1 is shown by induction on the complexity of φ, by exploiting (1) and (2) of Definition 8.1. Let us illustrate this by verifying the direction from right to left in the case that, say, $\varphi \equiv \exists v_0 \in v_1 \; \psi$ for some Σ_0 formula ψ.

We assume that, setting $c = a_1 \cup \cdots \cup a_k$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models \exists v_0 \in v_1 \; \psi(f_1^{a_1,c}(u), \ldots, f_k^{a_k,c}(u))\} \in E_c.$$

Let us define $f_0 : [\mu_c]^{\mathrm{Card}(c)} \to \mathrm{ran}(f_1)$ as follows.

$$f_0(u) = \begin{cases} \text{the } <_M\text{-smallest } x \in \mathrm{ran}(f_1) \text{ with} & \\ \quad M \models \psi(x, f_1^{a_1,c}, \ldots, f_k^{a_k,c}(u)) & \text{if some such } x \text{ exists,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The point is that $f_0 \in M$, because M is rud_A-closed and amenable. But we then have that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models f_0(u) \in f_1^{a_1,c}(u) \wedge \psi(f_1^{a_1,c}(u), \ldots, f_k^{a_k,c}(u))\} \in E_c,$$

which inductively implies that

$$N \models [c, f_0] \in [a_1, f_1] \wedge \psi([a_1, f_1], \ldots, [a_k, f_k]),$$

and hence that

$$N \models \exists v_0 \in v_1 \psi([a_1, f_1], \ldots, [a_k, f_k]).$$

Given Claim 1, we may and shall from now on identify, via the Mostowski collapse, the well-founded part $\mathrm{wfp}(N)$ of N with a transitive structure. In particular, if $[a, f] \in \mathrm{wfp}(N)$ then we identify the equivalence class $[a, f]$ with its image under the Mostowski collapse.

Let us now define $\pi : M \to N$ by

$$\pi(x) = [0, c_x], \quad \text{where } c_x : [\mu_0]^0 \to \{x\}.$$

We aim to verify that N, π satisfy (a), (b), (c), and (d) from the statement of Theorem 8.4.

Claim 2. If $\alpha < \nu$ and $[a, f] \,\tilde{\in}\, [\{\alpha\}, \mathrm{pr}]$ then $[a, f] = [\{\beta\}, \mathrm{pr}]$ for some $\beta < \alpha$.

In order to prove Claim 2, let $[a, f] \,\tilde{\in}\, [\{\alpha\}, \mathrm{pr}]$. Set $b = a \cup \{\alpha\}$. By Łoś's Theorem,

$$\{u \in [\mu_b]^{\mathrm{Card}(b)} \mid f^{a,b}(u) \in \mathrm{pr}^{\{\alpha\},b}(u)\} \in E_b.$$

By (4) of Definition 8.1, there is some $\beta < \alpha$ such that, setting $c = b \cup \{\beta\}$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{a,c}(u) = \mathrm{pr}^{\{\beta\},c}(u)\} \in E_c,$$

and hence, by Łoś's Theorem again,

$$[a, f] = [\{\beta\}, \mathrm{pr}].$$

Claim 2 implies, via a straightforward induction, that

$$[\{\alpha\}, \mathrm{pr}] = \alpha \quad \text{for } \alpha < \nu. \tag{9.29}$$

In particular, (b) from the statement of Theorem 8.4 holds.

Claim 3. If $a \in [\nu]^{<\omega}$ then $[a, \mathrm{id}] = a$.

If $[b, f] \,\tilde{\in}\, [a, \mathrm{id}]$ then by Łoś's Theorem, setting $c = a \cup b$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{b,c}(u) \in u^c_a\} \in E_c.$$

However, as E_c is an ultrafilter, there must then be some $\alpha \in a$ such that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)} \mid f^{b,c}(u) = u^c_\alpha\} \in E_c,$$

and hence by Łoś's Theorem and (9.29)

$$[b, f] = [\{\alpha\}, \mathrm{pr}] = \alpha.$$

On the other hand, if $\alpha \in a$ then it is easy to see that $\alpha \in [a, \mathrm{id}]$. This shows Claim 3.

Claim 4. $[a, f] = \pi(f)(a)$.

Notice that this statement makes sense even if $[a, f] \notin \mathrm{wfp}(N)$.

Let $b = a \cup \{0\}$. We have that

$$\{u \in [\mu_b]^{\mathrm{Card}(b)} \mid f^{a,b}(u) = ((c_f)^{\{0\},b}(u))(\mathrm{id}^{a,b}(u))\} = [\mu_b]^{\mathrm{Card}(b)} \in E_b,$$

by (1) of Definition 8.1, and therefore by Łoś's Theorem and Claim 3,

$$[a, f] = [0, c_f]([a, \mathrm{id}]) = \pi(f)(a).$$

Claim 4 readily implies (c) from the statement of Theorem 8.4.

Claim 5. $\kappa = \mathrm{crit}(\pi)$.

Let us first show that $\pi \restriction \kappa = \mathrm{id}$. We prove that $\pi(\xi) = \xi$ for all $\xi < \kappa$ by induction on ξ.

Let $\xi < \kappa$. Suppose that $[a, f] \,\tilde{\in}\, \pi(\xi) = [0, c_\xi]$. Set $b = a \cup \{\xi\}$. Then

$$\{u \in [\mu_b]^{\mathrm{Card}(b)} \mid f^{a,b}(u) < \xi\} \in E_b.$$

As E_b is κ-complete with respect to sequences in M (cf. (1) of Definition 8.1), there is hence some $\bar{\xi} < \xi$ such that

$$\{u \in [\mu_b]^{\mathrm{Card}(b)} \mid f^{a,b}(u) = \bar{\xi}\} \in E_b,$$

and therefore $[a, f] = \pi(\bar{\xi})$ which is $\bar{\xi}$ by the inductive hypothesis. Hence $\pi(\xi) \subseteq \xi$. It is clear that $\xi \subseteq \pi(\xi)$.

We now prove that $\pi(\kappa) > \kappa$ (if $\pi(\kappa) \notin \mathrm{wfp}(N)$ we mean that $\kappa \,\tilde{\in}\, \pi(\kappa)$) which will establish Claim 5. Well, $\mu_{\{\kappa\}} = \kappa$, and

$$\{u \in [\kappa]^1 \mid \mathrm{pr}(u) < \kappa\} = [\kappa]^1 \in E_{\{\kappa\}},$$

from which it follows, using Łoś's Theorem, that $\kappa = [\{\kappa\}, \mathrm{pr}] < [0, c_\kappa] = \pi(\kappa)$.

The following, together with Claim 1 and Claim 5, will establish (a) from the statement of Theorem 8.4.

Claim 6. For all $[a, f] \in N$ there is some $y \in M$ with $[a, f] \,\tilde{\in}\, \pi(y)$.

To verify Claim 6, it is easy to see that we can just take $y = \mathrm{ran}(f)$.

It remains to prove (d) from the statement of Theorem 8.4.

Let $X \in E_a$. By (1) of Definition 8.1,

$$X = \{u \in [\mu_a]^{\mathrm{Card}(a)} \mid u \in X\} \in E_a,$$

which, by Łoś's Theorem and Claim 3, gives that $a = [a, \mathrm{id}] \,\tilde{\in}\, [0, c_X] = \pi(X)$.

On the other hand, suppose that $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ and $a \in \pi(X)$. Then by Claim 3, $[a, \mathrm{id}] = a \in \pi(X) = [0, c_X]$, and thus by Łoś's Theorem

$$X = \{u \in [\mu_a]^{\mathrm{Card}(a)} \mid u \in X\} \in E_a.$$

We have shown Theorem 8.4. ⊣

8.5 Definition. Let M, E, N, and π be as in the statement of Theorem 8.4. We shall denote N by $\mathrm{Ult}_0(M; E)$ and call it the Σ_0 ultrapower of M by E, and we call $\pi : M \to N$ the Σ_0 *ultrapower map* (*given by* E). We shall also write π_E for π.

8.6 Definition. Let M be acceptable, and let E be a (κ, ν)-extender over M. Let $n < \omega$ be such that $\rho_n(M) > \sigma(E)$. Suppose that M is n-sound, and set $p = p_n(M)$. Let

$$\pi : M^{n,p} \to \bar{N}$$

be the Σ_0 ultrapower map given by E. Suppose that

$$\tilde{\pi} : M \to N$$

is as given by the proof of Lemmata 4.2 and 5.11. Then we write $\mathrm{Ult}_n(M; E)$ for N and call it the $r\Sigma_{n+1}$ *ultrapower of* M *by* E, and we call $\tilde{\pi}$ the $r\Sigma_{n+1}$ *ultrapower map* (*given by* E).

A comment is in order here. Lemmata 4.2 and 5.11 presuppose that π is strong (cf. Definition 4.1). However, the construction of the term model in Sect. 4 does not require π to be strong, nor does it even require the target model $\bar{N}$ to be well-founded. Consequently, we can make sense of $\mathrm{Ult}_n(M; E)$ even if π is not strong or $\bar{N}$ is not well-founded. This is why we have "the proof of Lemmata 4.2 and 5.11" in the statement of Definition 8.6. We shall of course primarily be interested in situations where $\mathrm{Ult}_n(M; E)$ is well-founded after all. In any event, we shall identify the well-founded part of $\mathrm{Ult}_n(M; E)$ with its transitive collapse.

One can also construct $r\Sigma_{n+1}$ ultrapower maps without assuming that the model one takes the ultrapower of is n-sound; this is done by pointwise lifting up a directed system converging to the model in question. However, the construction of Definition 8.6 seems to be broad enough for most applications.

Recall Definition 5.12. It is clear in the light of the Upward Extension of Embeddings Lemma that any $r\Sigma_{n+1}$ ultrapower map is $r\Sigma_{n+1}$ elementary (and hence the name). The following will give more information.

8.7 Theorem. *Let M be acceptable, and let E be a (κ, ν)-extender over M. Let $n < \omega$ be such that $\rho_n(M) > \sigma(E)$. Suppose that M is n-sound and $(n + 1)$-solid. Let*

$$\pi : M \to \mathrm{Ult}_n(M; E)$$

be the $r\Sigma_{n+1}$ ultrapower map given by E. Assume that $\mathrm{Ult}_n(M; E)$ is transitive, and that $\pi(p_{n+1}(M)) \in P_N^{n+1}$.

Then π is an n-embedding, $\mathrm{Ult}_n(M; E)$ is $(n + 1)$-solid, and $\pi(p_{n+1}(M)) = p_{n+1}(N)$.

Proof. Set $N = \mathrm{Ult}_n(M; E)$. That N is n-sound follows from the Upward Extension of Embeddings Lemma. N is $(n + 1)$-solid and $\pi(p_{n+1}(M)) =$

$p_{n+1}(N)$ by Lemma 7.11. By construction we have that π is the upward Extension of

$$\pi\restriction M^n : M^n \to \mathrm{Ult}_0(M^n; E),$$

so that by the Upward Extension of Embeddings Lemma we shall now have that $N^n = \mathrm{Ult}_0(M^n; E)$, and therefore $\rho_n(N) = N^n \cap \mathrm{On} = \mathrm{Ult}_0(M^n; E) \cap \mathrm{On}$; however, $\pi\restriction M^n$ is cofinal in $\mathrm{Ult}_0(M^n; E)$ by Theorem 8.4, and thus $\rho_n(N) = \sup(\pi``\rho_n(M))$. The Upward Extension of Embeddings Lemma also implies that $\pi(\rho_k(M)) = \rho_k(N)$ for all $k < n$. ⊣

The following is sometimes called the "Interpolation Lemma." We leave the (easy) proof to the reader.

8.8 Lemma. *Let $n < \omega$. Let $\bar{M}$, M be acceptable, and let*

$$\pi : \bar{M} \longrightarrow M$$

be $r\Sigma_{n+1}$ elementary. Let $\nu \leq M \cap \mathrm{On}$, and let E be the (κ, ν)-extender derived from π.

There is then a weakly $r\Sigma_{n+1}$ elementary embedding

$$\sigma : \mathrm{Ult}_n(\bar{M}; E) \to M$$

such that $\sigma\restriction\nu = \mathrm{id}$ and $\sigma \circ \pi_E = \pi$.

If π is as in Theorem 8.7 then it is often crucial to know that $\rho_{n+1}(M) = \rho_{n+1}(\mathrm{Ult}_n(M; E))$. In order to be able to prove this we need that $\langle M, E\rangle$ satisfies additional hypotheses.

8.9 Definition. Let M be acceptable, and let $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Then E is *close to M* if for every $a \in [\nu]^{<\omega}$,

(1) E_a is $\Sigma_1^M(\{q\})$ for some $q \in M$, and

(2) if $Y \in M$, $M \models \mathrm{Card}(Y) \leq \kappa$, then $E_a \cap Y \in M$.

The following theorem is the key tool for proving the preservation of the standard parameter in iterations of mice.

8.10 Theorem. *Let M be acceptable, and let $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ be a short (κ, ν)-extender over M which is close to M. Suppose that $n < \omega$ is such that $\rho_{n+1}(M) \leq \kappa < \rho_n(M)$, M is n-sound, and $\mathrm{Ult}_n(M; E)$ is transitive. Then*

$$\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap \mathrm{Ult}_n(M; E), \quad \text{and}$$
$$\rho_{n+1}(M) = \rho_{n+1}(\mathrm{Ult}_n(M; E)).$$

In particular, if M is $(n+1)$-solid then $\mathrm{Ult}_n(M; E)$ is $(n+1)$-solid and $\pi(p_{n+1}(M)) = p_{n+1}(\mathrm{Ult}_n(M; E))$.

Proof. Set $N = \mathrm{Ult}_n(M; E)$. Trivially, $\mathcal{P}(\kappa) \cap M \subseteq \mathcal{P}(\kappa) \cap N$. To show that $\mathcal{P}(\kappa) \cap N \subseteq \mathcal{P}(\kappa) \cap M$, let $X \in \mathcal{P}(\kappa) \cap N$. Let $X = [a, f]_E^{M^n}$. As E is short, $\mu_a \leq \kappa$; in fact, without loss of generality, $\mu_a = \kappa$. Hence for $\xi < \kappa$, if we set

$$X_\xi = \{u \in [\kappa]^{\mathrm{Card}(a)} \mid \xi \in f(u)\}$$

then $\{X_\xi \mid \xi < \kappa\} \in M^n \subseteq M$. But $\xi \in X \iff X_\xi \in E_a$ by Łoś's Theorem, and since E is close to M we get that $X \in M$.

If $q \in M^n$ and $A \in \Sigma_1^{M^n}(\{q\})$, $A \cap \rho_{n+1}(M) \in \Sigma_1^{N^n}(\{\pi(q), \rho_{n+1}(M)\})$. Because $\mathcal{P}(\kappa) \cap N \subseteq \mathcal{P}(\kappa) \cap M$, we thus have that $\rho_{n+1}(N) \leq \rho_{n+1}(M)$.

To show that $\rho_{n+1}(M) \leq \rho_{n+1}(N)$, let $A \in \Sigma_1^{N^n}(\{q\})$ for some $q \in N^n$. Let $q = [a, f]_E^{M^n}$, and let

$$x \in A \quad \iff \quad N^n \models \varphi(x, [a, f]),$$

where φ is Σ_1. If $M^n = \langle J_\alpha^B, A\rangle$ and $\delta < \alpha$ then we shall write $M^{n,\delta}$ for $\langle J_\delta^B, A \cap J_\delta^B\rangle$. For $\delta < N^n \cap \mathrm{On}$, $N^{n,\delta}$ is defined similarly. Because $\pi_E : M^n \to N^n$ is cofinal, we have that

$$x \in A \quad \iff \quad \exists \delta < M^n \cap \mathrm{On}\ N^{n,\pi_E(\delta)} \models \varphi(x, [a, f]).$$

By Łoś's Theorem, we may deduce that for $\xi < \kappa$,

$$\xi \in A \quad \iff \quad \exists \delta < M^n \cap \mathrm{On}\ \{u \in [\kappa]^{\mathrm{Card}(a)} \mid M^{n,\delta} \models \varphi(\xi, f(u))\} \in E_a.$$

But now E is close to M, so that E_a is $\Sigma_1^{M^n}(\{q'\})$ for some $q' \in M^n$, which implies that $A \cap \kappa$ is $\Sigma_1^{M^n}(\{q', \kappa, f\})$.

We now finally have that $\pi(p_{n+1}(M)) \in P_M^{n+1}$. Therefore, if M is $(n+1)$-solid then $\mathrm{Ult}_n(M; E)$ is $(n+1)$-solid and $\pi(p_{n+1}(M)) = p_{n+1}(\mathrm{Ult}_n(M; E))$ by Lemma 7.12. ⊣

We now turn towards criteria for $\mathrm{Ult}_n(M; E)$ being well-founded.

8.11 Definition. Let M be acceptable, and let $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Let $\lambda < \mathrm{Card}(\kappa)$ be an infinite cardinal (in V). Then E is called λ-*complete* provided the following holds true. Suppose that $\langle\langle a_i, X_i\rangle \mid i < \lambda\rangle$ is such that $X_i \in E_{a_i}$ for all $i < \lambda$. Then there is some order-preserving map $\tau : \bigcup_{i<\lambda} a_i \to \sigma(E)$ such that $\tau``a_i \in X_i$ for every $i < \lambda$.

8.12 Lemma. *Let M be acceptable, and let $E = \langle E_a \mid a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Let $\lambda < \mathrm{Card}(\kappa)$ be an infinite cardinal. Then E is λ-complete if and only if for every $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M; E)$ of size λ there is some $\varphi : U \underset{\Sigma_0}{\longrightarrow} M$ such that $\varphi \circ \pi_E(x) = x$ whenever $\pi_E(x) \in U$.*

Proof. ($\Longrightarrow$) Let $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M; E)$ be of size λ. Write $U = \{[a, f] \mid \langle a, f\rangle \in \bar{U}\}$ for some $\bar{U}$ of size λ. Let $\langle\langle a_i, X_i\rangle \mid i < \lambda\rangle$ be an enumeration of all pairs $\langle c, X\rangle$ such that there is a Σ_0 formula ψ and there are $\langle a^1, f_1\rangle, \ldots,$ $\langle a^k, f_k\rangle \in \bar{U}$ with $c = a^1 \cup \cdots \cup a^k$ and

$$X = \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models \psi(f_1^{a^1,c}(u), \ldots, f_k^{a^k,c})\} \in E_c.$$

Let $\tau : \bigcup_{i<\lambda} a_i \to \sigma(E)$ be order-preserving such that $\tau``a_i \in X_i$ for every $i < \lambda$. Let us define $\varphi : U \to M$ by setting $\varphi([a, f]) = f(\tau``(a))$ for $\langle a, f\rangle \in \bar{U}$.

We get that φ is well-defined and Σ_0 elementary by the following reasoning. Let $\psi(v_1, \ldots, v_k)$ be Σ_0, and let $\langle a^j, f_j\rangle \in U$, $1 \leq j \leq k$. Set $c = a^1 \cup \cdots \cup a^k$. We then get that

$$\begin{aligned}
&U \models \psi([a^1, f_1], \ldots, [a^k, f_k])\\
&\quad\iff \quad \mathrm{Ult}_0(M; E) \models \psi([a^1, f_1], \ldots, [a^k, f_k])\\
&\quad\iff \quad \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models \psi(f_1^{a^1,c}(u), \ldots, f_k^{a^k,c}(u))\} \in E_c\\
&\quad\iff \quad \tau``c \in \{u \in [\mu_c]^{\mathrm{Card}(c)} \mid M \models \psi(f_1^{a^1,c}(u), \ldots, f_k^{a^k,c}(u))\}\\
&\quad\iff \quad M \models \psi(f_1(\tau" a^1), \ldots, f_k(\tau``a^k)).
\end{aligned}$$

We also get that $\varphi \circ \pi_E(x) = \varphi([\emptyset, c_x]) = c_x(\emptyset) = x$.

($\Longleftarrow$) Let $\langle\langle a_i, X_i\rangle \mid i < \lambda\rangle$ be such that $X_i \in E_{a_i}$ for all $i < \lambda$. Pick $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M; E)$ with $\{a_i, X_i \mid i < \lambda\} \subseteq U$, $\mathrm{Card}(U) = \lambda$, and let $\varphi :$ $U \underset{\Sigma_0}{\longrightarrow} M$ be such that $\varphi \circ \pi_E(x) = x$ whenever $\pi_E(x) \in U$. Set $\tau = \varphi \restriction \bigcup_{i<\lambda} a_i$. Then $\tau``a_i = \varphi(a_i) \in \varphi \circ \pi_E(X_i) = X_i$ for all $i < \lambda$. Clearly, $\mathrm{ran}(\tau) \subseteq \sigma(E)$. ⊣

8.13 Corollary. *Let M be acceptable, and let E be an $\aleph_0$-complete (κ, ν)-extender over M. Then $\mathrm{Ult}_0(M; E)$ is well-founded. In fact, if $n < \omega$ is such that $\rho_n(M) \geq \sigma(E)$ then $\mathrm{Ult}_n(M; E)$ is well-founded.*

The concept of $\aleph_0$-completeness is relevant for constructing inner models below the "sharp" for an inner model with a proper class of strong cardinals (cf. [11]). There are strengthenings of the concept of $\aleph_0$-completeness which are needed for the construction of inner models beyond the "sharp" for an inner model with a proper class of strong cardinals (cf. for instance [13, Definition 1.2], [6, Definition 1.6]).

8.14 Lemma. *Let λ be an infinite cardinal, and let θ be regular. Let $\pi :$ $\bar{H} \to H_\theta$, where $\bar{H}$ is transitive and ${}^\lambda\bar{H} \subseteq \bar{H}$. Suppose that $\pi \neq \mathrm{id}$, and set $\kappa = \mathrm{crit}(\pi)$. Let M be acceptable, let ρ be regular in M, and suppose that $H^M_\rho \subseteq \bar{H}$. Set $\nu = \sup(\pi``\rho)$, and let E be the (κ, ν)-extender derived from $\pi \restriction H^M_\rho$. Then E is λ-complete.*

Proof. Let $\langle\langle a_i, X_i\rangle \mid i < \lambda\rangle$ be such that $X_i \in E_{a_i}$, and hence $a_i \in \pi(X_i)$, for all $i < \lambda$. As ${}^\lambda\bar{H} \subseteq \bar{H}$, $\langle X_i \mid i < \lambda\rangle \in \bar{H}$. Let $\sigma : \mathrm{otp}(\bigcup_{i<\lambda} a_i) \cong \gamma$ be the transitive collapse; notice that $\gamma < \lambda^+ < \kappa$. For each $i < \lambda$ let $\bar{a}_i = \sigma``a_i$. We have that $\langle \bar{a}_i \mid i < \lambda \rangle \in \bar{H}$. But now

$$H_\theta \models \exists \tau : \gamma \tilde{\to} \mathrm{On}\ \forall i < \lambda\ \tau``\bar{a}_i \in \pi(\langle X_j \mid j < \lambda \rangle)(i),$$

as witnessed by σ. Therefore,

$$\bar{H} \models \exists \tau : \gamma \tilde{\to} \mathrm{On}\ \forall i < \lambda\ \tau``\bar{a}_i \in X_i.$$

Hence, if $\tau \in \bar{H}$ is a witness to this fact then $\tau \circ \sigma : \bigcup_{i<\lambda} a_i \to \mathrm{On}$ is such that $\tau \circ \sigma``a_i \in X_i$ for every $i < \lambda$. ⊣

We leave it to the reader to find variants of this result. For instance, extenders derived from canonical ultrapower maps witnessing that a given cardinal κ is measurable are λ-complete for every $\lambda < \kappa$.

9. Applications to L

In this final section we shall illustrate how to use the above machinery in the simplest case—in the constructible universe L. The theory developed above is, however, general enough so that it can be used for all the currently known core models.

We shall first prove two important lemmata. Recall that we index the J-hierarchy with limit ordinals.

9.1 Lemma. *For each limit ordinal α, J_α is acceptable.*

9.2 Lemma. *For each limit ordinal α, J_α is sound.*

We shall prove these two lemmata simultaneously. The proof goes by induction on α in a zig-zag way in the sense that we use soundness of J_α to prove the acceptability of $J_{\alpha+\omega}$ and then, knowing this, its soundness.

Proof. The case $\alpha = \omega$ is trivial. Now suppose that both lemmata hold for all limit ordinals $\beta < \alpha$.

Claim 1. J_α is acceptable.

This is trivial for α being a limit of limit ordinals. For $\alpha = \beta + \omega$ it is clear that the only thing we have to prove is the following:

$$\begin{aligned} &\text{If there is a } \tau < \beta \text{ and an } a \subseteq \tau \text{ such that } a \in J_{\beta+\omega} \setminus J_\beta, \\ &\qquad\qquad \text{then there is an } f \in J_{\beta+\omega} \text{ such that } f : \tau \xrightarrow{\text{onto}} \beta. \end{aligned} \tag{9.30}$$

We prove (9.30). Suppose that there are such τ, a and take τ to be the least one such that there is a as above. Then

$$\tau = \rho_\omega(J_\beta). \tag{9.31}$$

To see (9.31) note first that if n is such that $\rho_n(J_\beta) = \rho_\omega(J_\beta)$, then we have a new $\mathbf{\Sigma}_1^{J_\beta^n}$ subset of $\rho_\omega(J_\beta)$. Such a set is $\mathbf{\Sigma}_\omega^{J_\beta}$ by Lemma 5.5, and is hence in $J_\beta + \omega \setminus J_\beta$ by Lemma 1.7. Therefore, $\tau \leq \rho_\omega(J_\beta)$.

Let $a \subseteq \tau$ such that $a \in J_{\beta+\omega} \setminus J_\beta$. Then $a \in \mathbf{\Sigma}_n^{J_\beta}$ for some $n \in \omega$ by Lemma 1.7. By the above inequality, $a \subseteq \rho_n(J_\beta)$. Lemma 5.6 then yields that a is $\mathbf{\Sigma}_1^{J_\beta^{n-1}}$, since, by the induction hypothesis, J_β is sound. Consequently, $\rho_\omega(J_\beta) \leq \tau$. This proves (9.31).

Now we use the induction hypothesis once again to verify (9.30). By the soundness of J_β and by Lemmata 5.4 and 5.5, there is some $f \in \mathbf{\Sigma}_\omega^{J_\beta}$ such that $f : \rho_\omega(J_\beta) \to J_\beta$ is surjective. By Lemma 1.7, $f \in J_{\beta+\omega}$. This shows (9.30) and therefore also Claim 1.

Claim 2. J_α is sound.

We shall make use of Lemma 6.8 here. Hence, for $n < \omega$ we prove

$$p_n(J_\alpha) \in R^n_{J_\alpha}. \tag{9.32}$$

Suppose that this is false. Pick the first n such that $p = p_n(J_\alpha) \notin R^n_{J_\alpha}$. Let a be $\Sigma_1^{J_\alpha^{n-1}}(\{p\})$ such that $a \cap \rho_n(J_\alpha) \notin J_\alpha$. Using the Downward Extension of Embeddings Lemma we construct unique $J_{\bar\alpha}$, $\bar p$, π such that

$$\begin{aligned}
&\bar p \in R^n_{J_{\bar\alpha}},\\
&\pi : J_{\bar\alpha}^{n-1,\bar p\restriction(n-1)} \to J_\alpha^{n-1,p\restriction(n-1)} \text{ is } \Sigma_1 \text{ elementary},\\
&\pi(\bar p(n-1)) = p(n-1),\\
&\pi\restriction J_{\rho_n(J_\alpha)} = \mathrm{id}.
\end{aligned} \tag{9.33}$$

Hence $a \cap \rho_n(J_\alpha) = \bar a \cap \rho_n(J_{\bar\alpha})$ where $\bar a$ is $\Sigma_1^{J_{\bar\alpha}^{n-1}}(\{\bar p(n-1)\})$ by the same definition. Hence $\bar\alpha$ cannot be less than α, since otherwise $a \cap \rho_n(J_\alpha) \in J_\alpha$. Consequently, $\bar\alpha = \alpha$. It is also clear by the construction that $\bar p \leq^* p$. But $p \leq^* \bar p$ since p is the standard parameter. Hence, $p = \bar p$. But this means $p \in R^n_{J_\alpha}$. Contradiction. ⊣

Classical applications of the fine structure theory include Jensen's results that $\Diamond$ and $\Box$ hold in L and that L satisfies the Covering Lemma. The following is Jensen's Covering Lemma for L.

9.3 Theorem. *Suppose that $0^\#$ does not exist. Let X be a set of ordinals. Then there is a $Y \in L$ with $Y \supseteq X$ and* $\mathrm{Card}(Y) \leq \mathrm{Card}(X) \cdot \aleph_1$.

This result is shown in [2] (cf. also [5]). In order to illustrate the fine structural techniques we have developed we shall now give a proof of a corollary to Theorem 9.3. Recall that a cardinal κ is called *countably closed* if $\lambda^{\aleph_0} < \kappa$ whenever $\lambda < \kappa$.

9.4 Corollary. *Let κ be a countably closed singular cardinal. If $0^{\#}$ does not exist then $\kappa^{+L} = \kappa^{+}$.*

Proof. We shall use the fact that the existence of $0^{\#}$ is equivalent with the existence of a non-trivial elementary embedding $\pi : L \to L$. Suppose that $0^{\#}$ does not exist and κ is a countably closed singular cardinal such that $\kappa^{+L} < \kappa^{+}$. We aim to derive a contradiction.

Let $X \subseteq \kappa^{+L}$ be cofinal with $\mathrm{otp}(X) < \kappa$. We may pick an elementary embedding

$$\pi : \bar{H} \to H_{\kappa^+}$$

such that $\bar{H}$ is transitive, ${}^{\omega}\bar{H} \subseteq \bar{H}$, $X \subseteq \mathrm{ran}(\pi)$, and $\mathrm{Card}(\bar{H}) = \mathrm{otp}(X)^{\aleph_0}$. As κ is countably closed, $\mathrm{Card}(\bar{H}) < \kappa$, which implies that $\pi \neq \mathrm{id}$. Set $\lambda = \pi^{-1}(\kappa^{+L})$, and let E be the $(\kappa, \pi(\lambda))$-extender over J_λ derived from $\pi \restriction J_\lambda$.

By Lemma 8.14, E is $\aleph_0$-complete. By Corollary 8.13, this implies the following.

Claim. Let $\alpha \geq \lambda$, $\alpha \in \mathrm{On} \cup \{\mathrm{On}\}$. Suppose that λ is a cardinal in J_α (which implies that E is an extender over J_α). Suppose that $\rho_n(J_\alpha) \geq \lambda$. Then $\mathrm{Ult}_n(J_\alpha; E)$ is transitive, and therefore $\mathrm{Ult}_n(J_\alpha; E) = J_\beta$ for some $\beta \in \mathrm{On} \cup \{\mathrm{On}\}$. (If $\alpha = \mathrm{On}$ then by J_α we mean L, and we want $n = 0$; we shall then have $J_\beta = L$ as well.)

Now because $0^{\#}$ does not exist, we cannot have that $\alpha = \mathrm{On}$ satisfies the hypothesis of the Claim. Let $\alpha \in \mathrm{On} \setminus \lambda$ be largest such that λ is a cardinal in J_α. Let $n < \omega$ be such that $\rho_{n+1}(J_\alpha) < \lambda \leq \rho_n(J_\alpha)$. By Lemma 9.2, we have that

$$J_\alpha = h_{J_\alpha}^{n+1}``(\rho_{n+1}(J_\alpha) \cup \{p\}),$$

where $p = p_{n+1}(J_\alpha)$. (Cf. Lemma 5.4.) Because π_E is $r\Sigma_{n+1}$ elementary by Theorem 8.7, Lemma 5.13 implies that

$$X \subseteq \pi``J_\alpha \subseteq h_{J_\beta}^{n+1}``(\pi(\rho_{n+1}(J_\alpha)) \cup \{\pi(p)\}).$$

But $\pi(\rho_{n+1}) \leq \kappa$, so that in particular

$$J_{\beta+\omega} \models \pi(\lambda) \text{ is not a cardinal.}$$

However, $\pi(\lambda) = \kappa^{+L}$. Contradiction! ⊣

We finally aim to prove $\square_\kappa$ in L. This is the combinatorial principle the proof of which most heavily exploits the fine structure theory.

Let κ be an infinite cardinal. Recall that we say that $\square_\kappa$ holds if and only if there is a sequence $\langle C_\nu \mid \nu < \kappa^+ \rangle$ such that if ν is a limit ordinal, $\kappa < \nu < \kappa^+$, then C_ν is a club subset of ν with $\mathrm{otp}(C_\nu) \leq \kappa$ and whenever $\bar{\nu}$ is a limit point of C_ν then $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$.

9.5 Theorem. *Suppose that $V = L$. Let $\kappa \geq \aleph_1$ be a cardinal. Then $\square_\kappa$ holds.*

Proof. We shall verify that there is a club $C \subseteq \kappa^+$ and some $\langle C_\nu \mid \nu \in C \wedge \mathrm{cf}(\nu) > \omega\rangle$ such that if ν is a limit ordinal, $\kappa < \nu < \kappa^+$, then C_ν is a club subset of ν with $\mathrm{otp}(C_\nu) \leq \kappa$ and whenever $\bar{\nu}$ is a limit point of C_ν then $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$. It is not hard to verify that this implies $\square_\kappa$ (cf. [1, pp. 158ff.]).

Let $C = \{\nu < \kappa^+ \mid J_\nu \prec_{\Sigma_\omega} J_{\kappa^+}\}$, a closed unbounded subset of κ^+.

Let $\nu \in C$. Obviously, κ is the largest cardinal of J_ν. We may let $\alpha(\nu)$ be the largest $\alpha \geq \nu$ such that either $\alpha = \nu$ or ν is a cardinal in J_α. By Lemma 1.7, $\rho_\omega(J_{\alpha(\nu)}) = \kappa$. Let $n(\nu)$ be that $n < \omega$ such that $\kappa = \rho_{n+1}(J_{\alpha(\nu)}) < \nu \leq \rho_n(J_{\alpha(\nu)})$.

If $\nu \in C$, then we define D_ν as follows. D_ν consists of all $\bar{\nu} \in C \cap \nu$ such that $n(\bar{\nu}) = n(\nu)$, and there is a weakly $r\Sigma_{n(\nu)+1}$ elementary embedding

$$\sigma : J_{\alpha(\bar{\nu})} \longrightarrow J_{\alpha(\nu)}$$

such that $\sigma{\restriction}\bar{\nu} = \mathrm{id}$, $\sigma(p_{n(\bar{\nu})+1}(J_{\alpha(\bar{\nu})})) = p_{n(\nu)+1}(J_{\alpha(\nu)})$, and if $\bar{\nu} \in J_{\alpha(\bar{\nu})}$ then $\nu \in J_{\alpha(\nu)}$ and $\sigma(\bar{\nu}) = \nu$. It is easy to see that if $\bar{\nu} \in D_\nu$ then there is exactly one map σ witnessing this, namely the one with

$$\sigma(h^{n(\bar{\nu})+1}_{J_{\alpha(\bar{\nu})}}(\xi, p_{n(\bar{\nu})+1}(J_{\alpha(\bar{\nu})}))) = h^{n(\nu)+1}_{J_{\alpha(\nu)}}(\xi, p_{n(\nu)+1}(J_{\alpha(\nu)}))$$

$\xi < \kappa$; we shall denote this map by $\sigma_{\bar{\nu},\nu}$.

Claim 1. Let $\nu \in C$. The following hold:

(a) D_ν is closed.

(b) If $\mathrm{cf}(\nu) > \omega$ then D_ν is unbounded.

(c) If $\bar{\nu} \in D_\nu$ then $D_\nu \cap \bar{\nu} = D_{\bar{\nu}}$.

Proof of Claim 1. (a) and (c) are easy. Let us show (b). Suppose that $\mathrm{cf}(\nu) > \omega$. Set $\alpha = \alpha(\nu)$ and $n = n(\nu)$. Let $\beta < \nu$. We aim to show that $D_\nu \setminus \beta \neq \emptyset$.

Let $\pi : J_{\bar{\alpha}} \underset{\Sigma_{n+1}}{\longrightarrow} J_\alpha$ be such that $\bar{\alpha}$ is countable, $\beta \in \mathrm{ran}(\pi)$, and

$$\{W^{\nu, p_1(J^k_\alpha)}_{J^k_\alpha} \mid \nu \in p_1(J^k_\alpha), k \leq n\} \subseteq \mathrm{ran}(\pi).$$

Let $\bar{\nu} = \pi^{-1}(\nu)$ (if $\nu = \alpha$, we mean $\bar{\nu} = \bar{\alpha}$). Let

$$\pi' = \pi_{E_{\pi\restriction J_{\bar{\nu}}}} : J_{\bar{\alpha}} \underset{r\Sigma_{n+1}}{\longrightarrow} \mathrm{Ult}_n(J_{\bar{\alpha}}; E_{\pi\restriction J_{\bar{\nu}}}).$$

Write $J_{\alpha'} = \mathrm{Ult}_n(J_{\bar{\alpha}}; E_{\pi\restriction J_{\bar{\nu}}})$. By Lemma 8.8, we may define a weakly $r\Sigma_{n+1}$ elementary embedding

$$k : J_{\alpha'} \longrightarrow J_\alpha$$

with $k \circ \pi' = \pi$. As $\beta \in \mathrm{ran}(\pi)$, $k^{-1}(\nu) > \beta$. Moreover, $k^{-1}(\nu) = \sup(\pi``\bar{\nu}) < \nu$, as $\mathrm{cf}(\nu) > \omega$. Therefore $\beta < k^{-1}(\nu) \in D_\nu$. $\dashv$

Now let $\nu \in C$. We aim to define C_ν. Set $\alpha = \alpha(\nu)$, and $n = n(\nu)$. Recursively, we define sequences $\langle \nu_i \mid i \leq \theta(\nu)\rangle$ and $\langle \xi_i \mid i < \theta(\nu)\rangle$ as follows. Set $\nu_0 = \min(D_\nu)$. Given ν_i with $\nu_i < \nu$, we let ξ_i be the least $\xi < \kappa$ such that

$$h^{n+1}_{J_\alpha}(\xi, p_{n+1}(J_\alpha)) \setminus \mathrm{ran}(\sigma_{\nu_i,\nu}) \neq \emptyset.$$

Given ξ_i, we let ν_{i+1} be the least $\bar{\nu} \in D_\nu$ such that

$$h^{n+1}_{J_\alpha}(\xi_i, p_{n+1}(J_\alpha)) \in \mathrm{ran}(\sigma_{\bar{\nu},\nu}).$$

Finally, given $\langle \nu_i \mid i < \lambda\rangle$, where λ is a limit ordinal, we set $\nu_\lambda = \sup(\{\nu_i \mid i < \lambda\}$. Naturally, $\theta(\nu)$ will be the least i such that $\nu_i = \nu$. We set $C_\nu = \{\nu_i \mid i < \theta(\nu)\}$.

The following is now easy to verify.

Claim 3. Let $\nu \in C$. Then The following hold:

(a) $\langle \xi_i \mid i < \theta(\nu)\rangle$ is strictly increasing.

(b) $\mathrm{otp}(C_\nu) = \theta(\nu) \leq \kappa$.

(c) C_γ is closed.

(d) If $\bar{\nu} \in C_\nu$ then $C_\nu \cap \bar{\nu} = C_{\bar{\nu}}$.

(e) If D_ν is unbounded in ν then so is C_ν.

We have shown that $\Box_\kappa$ holds. ⊣

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10. Σ^* Fine Structure

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1. Σ^* Fine Structure

Why is fine structure necessary? One answer is that, although the constructible hierarchy is an extremely uniform creation, it is not *that* uniform. We have for any level of our J_α hierarchy uniform Σ_1-Skolem functions. In attempting to generalise this and prove the existence of Σ_n-uniformising functions one runs immediately into the difficulty that these cannot be given a completely uniform definition. The following example taken from [17] shows why (see also the related [8, pp. 106–107]).

For any $\tau < \omega_1$, $\{\zeta \mid J_{\omega_1+\tau} \models$ "$\omega_1+\zeta$ does not exist"$\}$ is a Π_1-relation over $J_{\omega_1+\tau}$ whose least member, i.e. τ itself, has a Σ_2 definition via applications of a posited uniformly definable Σ_2-Skolem function. For such τ we should have a resulting Σ_2 definition, which, by Fodor's Lemma, on a stationary set $D \subseteq \omega_1$ of τ, would be given by the same Σ_2 formula over J_τ, $\exists u \forall v \Phi(u, v, t)$ say. By Fodor again we may shrink D further to a stationary E so that for some constant δ, and some fixed $u_0 \in J_{\omega_1+\delta}$, and taking any $\eta < \omega_1$, if η is in $E \setminus (\delta+1)$ then $(\forall v \Phi(u_0, v, \eta))$ holds in $J_{\omega_1+\eta}$. In particular for any other $\gamma \in E \setminus (\delta+1)$ below η this Π_1 fact about η also holds in $J_{\omega_1+\gamma}$, where γ is supposed to be the unique solution. This is absurd.

We deduce from this that parameters must be involved in defining Σ_2-Skolem functions. Jensen's solution in [24] is to reduce the problem of Σ_2-uniformisation over J_τ to that of Σ_1-uniformisation over a *projectum structure* $\langle J_{\rho_\tau}, A_\tau \rangle$ with A_τ a Σ_1-*mastercode* essentially coding up Σ_1 truth over J_τ. $\Sigma_2(J_\tau) \cap J_{\rho_\tau}$ relations are transformed to $\Sigma_1(\langle J_{\rho_\tau}, A_\tau \rangle)$ relations, which can be Σ_1-uniformised, and such a uniformising function can be translated back up to a Σ_2 function over J_τ. This is because ρ_τ has been chosen so that there is a uniform (in τ) Σ_1-Skolem function mapping (a subset of) J_{ρ_τ} onto J_τ (although uniform, a *parameter* is inserted here dependent on τ). In this, admittedly very scantily sketched, manner we can effectively Σ_2-uniformise all of J_τ. This machinery can be generalised for all $n \geq 2$, and thus prove Jensen's Σ_n-*Uniformisation Theorem*: *Any $\Sigma_n(J_\tau)$ relation can be uniformised by a $\Sigma_n(J_\tau)$ function.* However as we have observed, there can be no uniform way of performing Σ_2-uniformisations: there was the notion of the projectum, and the parameter in the above, which will vary between J_τ's. All the objects concerned (the projects, the parameters etc.) nevertheless have definitions over J_τ; however these definitions are not at the same level of complexity in the Levy hierarchy of formulae as the level we are working at, and at which we would wish. The Σ^* hierarchy of formulae replaces the Levy hierarchy and seeks to encapsulate the idea that somehow Σ_{n+1} is "Σ_1-in-Σ_n" much as the above talked about Σ_2 (with $2 = n+1$) as "Σ_1-in-Σ_n" over the *first reduct* or projectum structure $\langle J_{\rho_\tau}, A_\tau \rangle$. Then we think of $\Sigma_1^{(n+1)}$ as Σ_1 in $\Sigma_1^{(n)}$—and that is why $\Sigma_1^{(n)}$ plays a part in the "atomic" clause definition of $\Sigma_1^{(n+1)}$ below. $\Sigma_1^{(n)}$ formulae then refer to nth reducts only indirectly through the type of the variables.

The Σ^* hierarchy then, is not at all necessary for an exposition of a fine structure theory for L: we have chosen on illustrative grounds to rework the proofs of, say, global $\Box$ for L below, using it. Although for a structure M the $\Sigma_1^{(n)}(M)$ hierarchy of relations is different from the usual $\Sigma_n(M)$ hierarchy *qua* hierarchy, the totality of relations $\boldsymbol{\Sigma^*}(M) = \bigcup_{n<\omega} \boldsymbol{\Sigma_1^{(n)}}(M)$ are the same: $\boldsymbol{\Sigma_\omega}(M)$. The gain in the Σ^* hierarchy comes chiefly in the development of the theory of fine structure suitable for mice, for fine-structural ultrapowers, and $L[E]$ models. One should further make the remark that much of the work in defining the notions of projects, good and very good parameters (the sets P_M and R_M), nth codes, and reducts etc., is prior to the notion of $\Sigma_1^{(n)}$ relation: we shall exploit this here by referring back to the chapter of Schindler and Zeman in this Handbook [46] for these concepts.

Each section consists of one or more fine structural constructions followed by some discussion on variants, extensions, and a necessarily brief list of some sample applications. Thus this first section will continue after these introductory remarks to introduce the $\Sigma_1^{(n)}$ hierarchy of formulae, and the accompanying relations on acceptable structures. Although these pages look notationally complex the proofs are essentially elementary. The notion of $\Sigma_k^{(n)}$*-preserving embeddings* as those embeddings that preserve formulae of the correct type, is of course natural and by Lemma 1.15 we can restate the Downward Extension of Embeddings Lemma and Jensen's uniformisation result as a $\Sigma_1^{(n)}$-Uniformisation Theorem. Notions of $\Sigma_1^{(n)}$-Skolem function and of the Condensation Lemma for L in a strong form follow.

In Sect. 1.1 variant fine structures are discussed: the original form of fine structure for the Dodd-Jensen core model [10]; the use of Skolem hulls to potentially replace parts of the fine structure apparatus, and the hyperfine structure theory. In Sect. 1.2 the important theory of Σ^* ultrapowers is developed. This is a natural notion of ultrapower based on the $\Sigma_1^{(n)}$ hierarchy. Section 1.3 reworks this material for so-called pseudo-ultrapowers, otherwise known as long extender ultrapowers. Section 2 is devoted to using this apparatus to a proof of Global $\Box$ in L. As remarked later, we do not actually need the $\Sigma_1^{(n)}$ hierarchy and the corresponding analysis of the constructible hierarchy to effect this. However it is illustrative of the kind of arguments one uses in this arena. Section 2.1 gives some variants of $\Box$ principles. We only very briefly mention some applications, but do discuss consequences of the failure of $\Box_\kappa$. The fascinating question of whether it can be proven that $\Box$ holds in fine structural inner models is discussed in Sect. 2.2. Recent work identifies for which κ in certain classes of $L[E]$ models we can have $\Box_\kappa$. Section 3 discusses morasses and gives constructions of both a gap-1 morass and the simplifying variant of a coarse gap-1 morass in L.

We shall not discuss in any great depth the two major applications of contemporary fine structure: the Covering Lemma and $L[E]$-Inner Models. Although we shall want to refer to these later, these two major topics are properly covered elsewhere in the Handbook, in the chapters by Mitchell and by Schimmerling.

Acknowledgements

Although this chapter is an expansion of a brief outline in [56] that was based on Jensen's original [28], the authoritative published reference to the $\Sigma_1^{(n)}$ theory is Zeman's [62]. We are very much indebted to his volume for much of the early part of the exposition here, as well as to the authors of [18] and [32], and would like to thank them. We should like to thank those who have commented helpfully on previous versions of this chapter, and again we are particularly grateful to Martin Zeman in this regard.

By "Lim" ("Sing") we mean the class of limit ordinals (singular limit ordinals, respectively), by "Card" we mean the class of cardinals, and by "Reg", we shall mean the class of regular cardinals. For h_M a Σ_1-Skolem function (see [46, 1.15]) for a structure M, we shall denote by $h_M(X)$ the Skolem hull inside M generated from the set $X \subseteq M$; more properly we should have written $h_M``(\omega \times [X]^{<\omega})$, where in turn $[X]^{<\omega}$ denotes the class of finite subsets of X. We use $\vec{y}$ to denote the list $y_0, \ldots, y_h$, and $\langle \vec{y} \rangle =_{\mathrm{df}} \langle y_0, \ldots, y_h \rangle$ is the ordered sequence, i.e. a finite function. Order type is abbreviated as ot. If $X \subseteq \mathrm{On}$ then X^* denotes the class of limit points of X. We use the following rudimentarily defined well-order on $[\mathrm{On}]^{<\omega}$: $a <^* b \leftrightarrow \max(a \triangle b) \in b$.

In what follows we assume that $M = \langle J_\alpha^A, \in, A, B \rangle$ is an acceptable J-structure in the sense of [46, 1.20] with J_α^A a level of a relativised J-hierarchy. That is, it is a structure satisfying the *axiom of acceptability*: $\forall \tau < \alpha \forall \xi < \tau$:

$$\mathcal{P}(\xi) \cap J_{\tau+1}^A \backslash J_\tau^A \neq \varnothing \longrightarrow (\exists f \in J_{\tau+1}^A)\,(f : \xi \overset{\text{onto}}{\to} \tau).$$

We further recall from there that being acceptable can be expressed as a *Q-property* (see [46, 1.18]). We shall write $\rho_M = \rho(M)$ as usual for the Σ_1-projectum of M. ([46] prefers the notation $\rho(M)$ here; we have also adhered to Jensen's notation, and that of [62], in writing here "J_α" for what [46] would name "$J_{\omega\alpha}$".) Similarly we shall write for the $(n+1)$*st projectum* $\rho_M^{n+1} =_{\mathrm{df}} \min\{\rho_{M^{n,p}} \mid p \in \Gamma_M^n\}$. ([46] would write $\rho_{n+1}(M)$ for ρ_M^{n+1} here, and similarly $\rho(M^{n,p})$ for $\rho_{M^{n,p}}$; as here, we keep in general to the notation of [62].) We can, and do, assume that parameters are finite sets of ordinals (cf. the comment of [46] before 6.3). This applies as well to the *nth-standard parameter* and the *standard parameter* [46, 6.3, 6.6] denoted here p_M^n, p_M respectively for a structure M as above. For the notions of *soundness* and *n-soundness* the reader can refer to [46] Definition 5.2—and see Lemma 6.8.

The Σ^* Hierarchy of Formulae

Notions of "fine-structural" preserving maps (and ultimately ultrapowers) can be smoothly presented in terms of Jensen's Σ^* hierarchy of formulae. As already intimated, this is a hierarchy of definability over a J-structure, with a different order of stratification than the usual Levy hierarchy of formulae. Although the relations Σ^*-definable over M are also the usual Σ_ω-definable relations, the intermediate levels of Σ_n-definability do not in general correspond to those within Σ^*. More complex initially, the Σ^* hierarchy possesses nice properties the Levy hierarchy lacks: again for example, over L, Jensen's Σ_n-Uniformisation Theorem states that such relations may be Σ_n-uniformised albeit *in a parameter*. When given the Σ^* analysis, $\Sigma_1^{(n)}(J_\alpha)$ relations enjoy *uniform* $\Sigma_1^{(n)}(J_\alpha)$-uniformising functions. Moreover the hierarchy encapsulates the notion of elementarity that the reader of [46] Sects. 3 and 4 on downward and upward extensions of embeddings will have already seen expressed by those results. As mentioned above, the hierarchy and all the results here of this first section were first exposited in [28] by Jensen.

We start by defining the language $\mathcal{L}^* = \mathcal{L}^*_{\{\dot{\in},\dot{=},\dot{A},\dot{B}\}}$. This has *variables* v^i_j $(i,j<\omega)$ of *type* i. The *atomic formulae* are those of the form $v^i_j \in \dot{A}$, $v^i_j \in \dot{B}$, $v^i_j \in v^k_l$, $v^i_j = v^k_l$ $(i,j,k,l<\omega)$. The *formulae* are those obtained from the atomic formulae by closing under $\neg, \wedge$ and *typed quantification* $\exists v^i_j, \forall v^i_j$. An $\mathcal{L}^*$-*structure* is then a structure of the form $\mathfrak{H} = \langle H, A, B, \in, H^0, H^1, \ldots\rangle$ with $H = H^0$ and $H^i \supseteq H^j \neq \varnothing$ for $i \leq j < \omega$, all H^j being transitive. The variables v^n_m are intended to range over H^n. An acceptable J-model $M = \langle J^A_\alpha, \in, A, B\rangle$ can be viewed as the $\mathcal{L}^*$-structure $M = \langle J^A_\alpha, \in, A, B, H^0_M, H^1_M, \ldots\rangle$, which amounts to the *standard interpretation.* Here the variables v^n_m are intended to range over $H^n_M =_{\mathrm{df}} H^M_{\omega\rho^n_M}$. Note that $n \leq m$ implies $\rho^n_M \geq \rho^m_M$, so this makes sense. In what follows we shall use extra variable symbols such x^n, u^m ad lib.

The stratification we referred to earlier is as follows:

1.1 Definition.

(a) The $\Sigma_0^{(n)}$ *formulae* are the formulae in the smallest class $\Sigma \subseteq \mathcal{L}^*$ such that:

  (i) Σ contains the atomic formulae, and the formulae $\Sigma_1^{(m)}$ for any $m < n$;

  (ii) Σ is closed under $\neg, \wedge$, and quantification which binds variables of type n by a higher type; that is if $\varphi \in \Sigma$, so are $\forall x^n \in y^m \varphi$, $\exists x^n \in y^m \varphi$, where $m \geq n$;

(b) $\Sigma_k^{(n)}$ *formulae* are obtained by alternating k blocks of quantifiers of type n: $\exists \vec{x}^n_1 \forall \vec{x}^n_2 \ldots \mathfrak{Q} \vec{x}^n_k \varphi$ where $\varphi \in \Sigma_0^{(n)}$ (and $\mathfrak{Q}$ is $\forall$ or $\exists$ depending on whether k is even or odd).

(c) $\Pi_0^{(n)} = \Sigma_0^{(n)}$, and $\Pi_k^{(n)}$ *formulae* as those of the form: $\forall \vec{x}^n_1 \exists \vec{x}^n_2 \ldots \mathfrak{Q} \vec{x}^n_k \varphi$ for the appropriate $\mathfrak{Q}$; and we define:

$$\Sigma^* = \bigcup\nolimits_n \Sigma_0^{(n)} (= \bigcup\nolimits_n \Sigma_1^{(n)}).$$

The notions of $\Sigma_k^{(n)}$ relation and function are defined fairly straightforwardly, using these formulae. We need to analyse such relations and functions, and we need a notion of *good* $\Sigma_1^{(n)}$ *functions* as those functions that may be substituted into a $\Sigma_1^{(n)}$ relation and still yield a $\Sigma_1^{(n)}$ relation. These will be defined below.

Over a model M $\Sigma_k^{(n)}$ relations involve a defining $\Sigma_k^{(n)}$ formula whose variables are of a certain *type*.

1.2 Definition.

(i) Let M be a J-structure; $R(x_1^{i_1}, \ldots, x_k^{i_k})$ is a $\Sigma_k^{(n)}(M)$ *relation of type* $\langle i_1, \ldots i_k\rangle$ *in parameters* $\vec{q} \in M^{<\omega}$, if and only if R is defined over M by a $\Sigma_k^{(n)}$ formula $\varphi(v_1^{i_1}, \ldots, v_k^{i_k}, \vec{q})$. (If we wish to mention the parameters we shall say that R is "$\Sigma_k^{(n)}(M)$ in the parameters $\vec{q}$.")

(ii) R is $\boldsymbol{\Sigma}_k^{(n)}(M)$ if and only if it is $\Sigma_k^{(n)}(M)$ in some parameters $\vec{q}$.

(iii) *The* $\Sigma^*(M)$ *relations* (also written $\Sigma_1^{(\omega)}(M)$), are those $\Sigma_0^{(n)}(M)$ for some n, and $\boldsymbol{\Sigma}^*(M)$ relations are defined analogously.

A relation more formally speaking is thus a pair: $\langle R, \langle i_1, \ldots, i_k\rangle\rangle$, consisting of the actual graph of the relation as a subset of $H^{i_1} \times \cdots \times H^{i_k}$ together with its type: $\langle i_1, \ldots, i_k\rangle$.

1.3 Definition. A function F is a $\Sigma_k^{(n)}(M)$ *function to* H_M^i *of argument type* $\langle i_1, \ldots, i_k\rangle$ if and only "$y^i = F(x_1^{i_1}, \ldots, x_k^{i_k})$" is a $\Sigma_k^{(n)}(M)$ relation of type $\langle i, i_1, \ldots, i_k\rangle$.

Part of our analysis will show that although, for example, $\Sigma_k^{(n)}(M)$ relations are not just graphs but come with the additional baggage of a type, for most intents and purposes, we shall not have to worry about the type. As the domains $H_M^i \supseteq H_M^j$ for $j \geq i$ are decreasing, replacing a variable in a formula defining a relation, by one of higher type, just reduces the domain specified, and so the resulting relation may be regarded as a *specialisation* of the original: if $\overline{R}(x_1^{j_1}, \ldots, x_k^{j_k}), R(x_1^{i_1}, \ldots, x_k^{i_k})$ are relations on H, then $\overline{R}$ is a *specialisation* of R, if $j_l \geq i_l$ $(0 < l \leq k)$ and $\overline{R} = R \cap H^{j_1} \times \cdots \times H^{j_k}$.

1.4 Lemma. *If* $R(x_1^i, \vec{x}\,)$ *is* $\Sigma_k^{(n)}(M)$*, and* $j \geq i$*, then so is the specialisation* $R(x_1^j, \vec{x}\,)$*.*

1.5 Lemma. *If* $R(x_1^j, \vec{x}\,)$ *is* $\Sigma_k^{(n)}(M)$ *(in* $\vec{q}$*) and* $j \geq i \geq n$*, then* R *is a specialisation of a* $\Sigma_k^{(n)}(M)$ *(in* $\vec{q}$*) relation* $R(x_1^i, \vec{x}\,)$*.*

The proofs of the above are both straightforward inductions on the structure of the defining formula for R. Notice that this implies that any $\Sigma_k^{(n)}(M)$ relation can be considered a specialisation of a $\Sigma_k^{(n)}(M)$ relation whose arguments are all of type less than or equal to n. Note also that trivial operations like permutation of variables, or insertion of dummy ones, in a $\Sigma_k^{(n)}(M)$ relation leave it in the same class of definability.

The next lemma tells how $\Sigma_k^{(n+1)}(M)$ relations R can be expressed as Σ_k relations on a suitably extended $(n+1)$st reduct domain. It is important for the reader to note the uniformities implicit in the statement of the lemma. Firstly note that although the lemma talks about $\Sigma_k^{(n+1)}(M)$ relations, the structure M really does not enter into consideration and the result is independent of it. (The reader should be aware of this kind of ostensible dependence on a structure M, of which it is in reality independent, as this reoccurs in several places in the development below.) The result is really a purely *syntactic* one about the defining formula for R. This is the second uniformity: the relations S^i that appear in the conclusion of the lemma are derived entirely from the matrix of this defining formula, and depend only on it.

1.6 Lemma. *$R(\vec{x}^{\,n+1}, \dots, \vec{x}^{\,0})$ is $\Sigma_k^{(n+1)}(M)$ if and only if there are $\Sigma_1^{(n)}(M)$ relations S^i $(i \le m)$ such that for all $\overrightarrow{x} = \vec{x}^{\,n}, \dots, \vec{x}^{\,0} \in M$:*

$R_{\overrightarrow{x}} =_{\mathrm{df}} \{\langle \vec{x}^{\,n+1}\rangle \mid R(\vec{x}^{\,n+1}, \overrightarrow{x})\}$ is uniformly $\Sigma_k(\langle H_M^{n+1}, \in, Q^0_{\overrightarrow{x}}, \dots, Q^m_{\overrightarrow{x}}\rangle)$,

where each $Q^i_{\overrightarrow{x}}$ (for $i \le m$) has the form

$$Q^i_{\overrightarrow{x}} = \{\langle \vec{y}^{\,n+1}\rangle \mid S^i(\vec{y}^{\,n+1}, \overrightarrow{x})\}.$$

Proof. We should first note that the "uniformly" here (a further uniformity) refers to the fact that the same Σ_k formula ψ of the conclusion works for every choice of $\vec{x}$. Although notationally complicated the lemma is actually rather simple: the relations S^i correspond to the fact that $\Sigma_1^{(n)}$ formulae play a role of atomic formulae in the definition of $\Sigma_0^{(n+1)}$ and are in fact the relations defined by those components. To continue in the proof of $(\Longrightarrow)$, we may write the formula $\varphi(\vec{x}^{\,n+1}, \dots, \vec{x}^{\,0})$ defining R in prenex form:

$$\begin{aligned} &\exists v_1^{n+1} \forall v_2^{n+1} \cdots (\exists w_1^{n+1} \in u_1^{n+1})(\forall w_2^{n+1} \in u_2^{n+1}) \\ &\quad \cdots \psi(\vec{v}^{\,n+1}, \vec{w}^{\,n+1}, \vec{x}^{\,n+1}, \vec{x}\,) \end{aligned} \tag{+}$$

(the bounding variables u_j^{n+1} being amongst the $\vec{v}^{\,n+1}, \vec{x}^{\,n+1}$). As intimated ψ is a propositional combination of $\Sigma_1^{(n)}$ formulae $\varphi_0, \dots, \varphi_m$ which define for us (relabelling the type $n+1$ variables as $\vec{y}^{\,n+1}$) the $S^i(\vec{y}^{\,n+1}, \vec{x}\,)$ $(i \le m)$. Let $Q(\vec{y}^{\,n+1}, \vec{x}\,)$ be the relation corresponding to that propositional combination of the S^i mirroring the structure ψ. Then $R_{\overrightarrow{x}}(\vec{t}\,)$ can be seen to be expressible as:

$$\exists v_1 \forall v_2 \cdots (\exists w_1 \in u_1)(\forall w_2 \in u_2) \cdots Q_{\overrightarrow{x}}(\vec{v}, \vec{w}, \vec{t}\,) \tag{$*$}$$

where now the bounding variables u_j are amongst the $\vec{v}, \vec{t}$.

For the converse, if $R_{\overrightarrow{x}}$ is $\Sigma_k(\langle H_M^{n+1}, \in, Q^0_{\overrightarrow{x}}, \dots, Q^m_{\overrightarrow{x}}\rangle)$, then it has the form of $(*)$. We now just unwind the previous process, thinking of each $Q^j_{\overrightarrow{x}}(\vec{v}, \vec{w}, \vec{t}\,)$ as a propositional combination of $Q^i_{\overrightarrow{x}}$; we now replace $Q^i_{\overrightarrow{x}}$, $\vec{v}$, $\vec{w}$, $\vec{t}$, by $\varphi_i, \vec{v}^{\,n+1}$, $\vec{w}^{n+1}$, $\vec{x}^{\,n+1}$ and obtain the $\Sigma_k^{(n+1)}$ form of $(+)$. $\dashv$

The next lemma relates the last to the reduct structures $M^{n,p(\vec{x})}$.

1.7 Lemma.

(i) *Let* $0 < k$. $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ *is* $\Sigma_k^{(n)}(M)$ *if and only if the relation* $R_{\overrightarrow{x}} = \{\langle \vec{x}^{\,n}\rangle \mid R(\vec{x}^{\,n}, \overrightarrow{x})\}$ *of the last lemma is uniformly* $\Sigma_k(M^{n,p(\vec{x})})$, *where* $p(\overrightarrow{x}) =_{\mathrm{df}} \langle\langle \vec{x}^{\,0}\rangle, \ldots, \langle \vec{x}^{\,n-1}\rangle\rangle$.

(ii) *If* $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ *is* $\Sigma_0^{(n)}(M)$ *then* $R_{\overrightarrow{x}}$ *is uniformly rudimentary in* $M^{n,p(\vec{x})}$; *conversely if* $R_{\overrightarrow{x}}$ *is* $\Sigma_0(M^{n,p(\vec{x})})$ *then* $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ *is* $\Sigma_0^{(n)}(M)$.

Proof. By an induction on n. For $n = 0$ it is trivial. For (i), we suppose this is true for m, and we shall prove it for $n = m + 1$.

For the forward direction, let $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ be $\Sigma_k^{(n)}(M)$. By appealing to, and using the notation of, the last lemma, there are $\Sigma_1^{(m)}(M)$ relations $S^0, \ldots, S^t$ so that for $\overrightarrow{x} = \vec{x}^{\,m}, \ldots, \vec{x}^{\,0} \in M$, $R_{\overrightarrow{x}}$ is uniformly $\Sigma_k(\langle H_M^n, \in, Q^0_{\overrightarrow{x}}, \ldots, Q^t_{\overrightarrow{x}}\rangle)$. However each $S^i(\vec{z}^{\,m+1}, \overrightarrow{x})$ can be considered a specialisation of a $\Sigma_1^{(m)}(M)$ relation $\widetilde{S}^i(\vec{z}^{\,m}, \overrightarrow{x})$. We use the notation $\overrightarrow{y} = \vec{x}^{\,m-1}, \ldots, \vec{x}^{\,0}$ and $p(\vec{y}) =_{\mathrm{df}} \langle\langle \vec{x}^{\,0}\rangle, \ldots, \langle \vec{x}^{\,m-1}\rangle\rangle$. Then we can rewrite: $\widetilde{S}^i(\vec{z}^{\,m}, \overrightarrow{x})$ as $\widetilde{S}^i(\vec{z}^{\,m}, \vec{x}^{\,m}, \overrightarrow{y})$. The inductive hypothesis implies that

$$Q^i_{\overrightarrow{y}} =_{\mathrm{df}} \{\langle \vec{z}^{\,m}, \vec{x}^{\,m}\rangle \mid \widetilde{S}^i(\vec{z}^{\,m}, \vec{x}^{\,m}, \overrightarrow{y})\}$$

is uniformly $\Sigma_1(M^{m,p(\vec{y})})$. There is thus a fixed Σ_1-formula $\varphi_{j(i)}$ such that

$$\begin{aligned} \langle \vec{z}^{\,m}, \vec{x}^{\,m}\rangle \in Q^i_{\overrightarrow{y}} \quad &\iff \quad M^{m,p(\vec{y})} \models \varphi_{j(i)}(\vec{z}^{\,m}, \vec{x}^{\,m}) \\ &\iff \quad \langle j(i), \vec{z}^{\,m}\rangle \in A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}} \end{aligned}$$

where the latter is of course the standard code predicate occurring in the nth reduct determined by $p(\vec{x})$: $M^{n,p(\vec{x})}$ (cf. [46, 5.1]). However $\langle \vec{z}^{\,m}\rangle \in Q^i_{\overrightarrow{x}} \iff \langle \vec{z}^{\,m}, \vec{x}^{\,m}\rangle \in Q^i_{\overrightarrow{y}}$, so if we replace every occurrence of "$\langle \vec{z}^{\,m}\rangle \in Q^i_{\overrightarrow{x}}$" by "$\langle j(i), \vec{z}^{\,m}\rangle \in A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}}$" in the $\Sigma_k(\langle H_M^n, \in, Q^0_{\overrightarrow{x}}, \ldots, Q^t_{\overrightarrow{x}}\rangle)$ definition of $R_{\overrightarrow{x}}$, we obtain a $\Sigma_k(M^{n,p(\vec{x})})$ definition for it. These substitutions and translations did not depend on $\overrightarrow{x}$ or M, but only on the definition of R, and so are themselves uniform and effective.

For the converse, suppose that $R_{\vec{x}}$ is uniformly $\Sigma_k(M^{n,p(\overrightarrow{x})})$-definable by some formula Φ say. This will contain atomic components of the form "$\langle j, \langle \vec{x}^{\,n}, \vec{z}^{\,n}\rangle\rangle \in A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}}$" (continuing with the same notation as from the first part). If we can show that these atomic formulae are expressible as $\Sigma_1^{(m)}$ formulae (in variables $j, \vec{z}^{\,n}, \vec{x}^{\,n}, \ldots, \vec{x}^{\,0}$) then we may effectively transform Φ into a $\Sigma_k^{(m)}$ formula by substituting these $\Sigma_1^{(m)}$ formulae for the atomic components. We have however:

$$\langle j, \langle \vec{x}^{\,n}, \vec{z}^{\,n}\rangle\rangle \in A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}} \quad \iff \quad M^{m,p(\vec{y})} \models \varphi_j(\langle \vec{x}^{\,n}, \vec{z}^{\,n}\rangle, \vec{x}^{\,m}).$$

We consider the right side here a specialisation of the relation: $M^{m,p(\vec{y})} \models \varphi_j(\langle \vec{w}^{\,m}, \vec{z}^{\,m}\rangle, \vec{x}^{\,m})$ and the latter is (since Σ_1 satisfaction over J-structures is uniformly Σ_1-definable) $\Sigma_1(M^{m,p(\vec{y})})$ in the variables $j, \vec{w}^{\,m}, \vec{z}^{\,m}, \vec{x}^{\,m}$ uniformly for all $\vec{y}$, and by the inductive hypothesis it is $\Sigma_1^{(m)}(M)$. Hence by Lemma 1.4 the atomic component is $\Sigma_1^{(m)}(M)$ also, This suffices for (i).

For (ii), the converse direction works as above, but in the forwards direction, when we perform the replacement "$\langle \vec{z}^{\,m}\rangle \in Q^i_{\overrightarrow{x}}$" by "$\langle j(i), \vec{z}^{\,m}\rangle \in A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}}$" as the latter is in general not Σ_0 in the predicate $A^{\vec{x}^{\,m}}_{M^{m,p(\vec{y})}}$ we obtain only that it is rudimentary in $M^{n,p(\vec{x})}$—but that is the only difference. ⊣

We can argue similarly to the above that $(n+1)$st standard codes are themselves uniformly $\Sigma_1^{(n)}$-definable:

1.8 Lemma. *There is an $A^*(\vec{x}^{\,n+1}, \ldots, \vec{x}^{\,0})$ which is $\Sigma_1^{(n)}(M)$ uniformly for all J-structures M such that (again with $p(\vec{x})$ as in the last lemma, and $\overrightarrow{x} = \vec{x}^{\,n}, \ldots, \vec{x}^{\,0}$)*

$$\langle \vec{x}^{\,n+1}\rangle \in A_M^{n+1,p(\vec{x})} \quad \iff \quad A^*(\vec{x}^{\,n+1}, \ldots, \vec{x}^{\,0});$$

i.e. $A_M^{n+1,p(\vec{x})} = A^*_{\vec{x}}$.

In general $\Sigma_1^{(n)}(M)$ relations are not closed under substitution of $\Sigma_1^{(n)}(M)$ functions. We shall need to define a class of functions, the *good functions* which do permit this kind of substitutability. As a preliminary:

1.9 Lemma. *Let $m \le n$, $0 < k$. Let $R(\vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ be $\Sigma_k^{(n)}(M)$. Let $\vec{F}^{\,n}, \ldots, \vec{F}^{\,0}$ be such that each $F_j^m(\vec{z}^{\,0}, \ldots, \vec{z}^{\,m})$ is a (possibly partial) $\Sigma_1^{(m)}(M)$ function to H_M^m. Then $R(\overrightarrow{F^i(\overrightarrow{z})})$ is uniformly $\Sigma_k^{(n)}(M)$.*

Proof. By induction on n. We assume it holds for $l < n$. For the sake of brevity, we consider just a single $F^m(\vec{z})$ of value type $m \le n$, this illustrates the idea and the reader will see that the rest is merely complication of notation. We can consider $R(F^m(\vec{z}), \vec{x})$ as defined by:

$$M \models \exists x^m (x^m = F^m(\vec{z}) \wedge R(x^m, \vec{x})).$$

If $m = n$ this already shows that $R(F^m(\vec{z}), \vec{x})$ is $\Sigma_k^{(n)}(M)$. So suppose $m < n$ and hence $0 < n$. Write out a prenex form of the definition of $R(x^m, \vec{x})$ with a $\Sigma_0^{(n)}$ matrix φ as:

$$M \models \exists \vec{v}_1^{\,n} \forall \vec{v}_2^{\,n} \cdots Q \vec{v}_k^{\,n} \varphi(\vec{v}, x^m, \vec{x}).$$

The free variable x^m only occurs in the $\Sigma_1^{(n-1)}$ "atomic constituents" of the $\Sigma_0^{(n)}$ formula φ. We shall apply the induction hypotheses to these constituents: they can be effectively listed, and if $\psi(x^m, \vec{y})$ is a typical member of this list, then apply the inductive hypothesis to $\psi(F^m(\vec{z}), \vec{y})$ to yield some $\Sigma_0^{(n)}$ formula. If this is done throughout φ the resultant formula $\widetilde{\varphi}(\vec{v}, \vec{z}, \vec{x})$ is itself now $\Sigma_0^{(n)}$. Hence $R(F^m(\vec{z}), \vec{x})$ is definable over M by

$$\exists \vec{v}_1^n \forall \vec{v}_2^n \cdots Q\vec{v}_k^n (\exists x^m = F^m(\vec{z}) \wedge \widetilde{\varphi}(\vec{v}, \vec{z}, \vec{x})).$$

The process is clearly effective and independent of M. ⊣

1.10 Definition. The *good* $\Sigma_1^{(n)}(M)$ *functions* form the smallest class $\mathfrak{F}$, such that, taking $i, j_1, \ldots, j_k \leq n$,

(i) each partial $\Sigma_1^{(i)}(M)$ function to H_M^i of the form $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in \mathfrak{F}$;

(ii) if $F(x_k^{j_k}, \ldots, x_1^{j_1}) = x^i \in \mathfrak{F}$ and $G_i(\vec{y}) = z^{j_i} \in \mathfrak{F}$ (and the $\vec{y}$ all have type $\leq n$), then $F(G_k(\vec{y}), \ldots, G_1(\vec{y})) \in \mathfrak{F}$.

The previous lemma together with an induction on the scheme generating the good $\Sigma_1^{(n)}$ functions proves:

1.11 Lemma. *Let $R(x^i, \vec{x})$ be $\Sigma_k^{(n)}(M)$, $k \geq 1$, $n \geq i$. Let $F^i(\vec{y})$ be a good $\Sigma_1^{(n)}(M)$ function of value type i. Then $R(F^i(\vec{y}), \vec{x})$ is uniformly $\Sigma_k^{(n)}(M)$.*

Again we should remark that the $\Sigma_k^{(n)}$ definition of the resultant relation is uniformly obtained from the scheme generating $\Sigma_1^{(n)}$ good functions, and the definition of $R(x^i, \vec{x})$. It has nothing to do with M (or $\vec{x}$). The following corollary then shows that $\Sigma_1^{(n)}$ relations are after all characterisable by their graphs alone.

1.12 Corollary.

(i) *Let $R(x_1^{i_1}, \ldots, x_k^{i_k})$ and $R(y_1^{j_1}, \ldots, y_k^{j_k})$ have the same graph. Then*

$$R(x_1^{i_1}, \ldots, x_k^{i_k}) \in \Sigma_1^{(n)}(M) \iff R(y_1^{j_1}, \ldots, y_k^{j_k}) \in \Sigma_1^{(n)}(M).$$

(ii) *In particular, if $R(x_1^{i_1}, \ldots, x_k^{i_k}) \in \Sigma_1^{(n)}(M)$, then it is a specialisation of $R'(x_1^0, \ldots, x_k^0) \in \Sigma_1^{(n)}(M)$ with the same graph, and all of whose arguments are of value type* 0.

Proof. To see (i) we may substitute into $R(x_1^{i_1}, \ldots, x_k^{i_k})$ the good $\Sigma_1^{(n)}(M)$-projection functions $x_1^{i_1} = y_1^{j_1}, \ldots, x_k^{i_k} = y_1^{j_k}$. (ii) is then a special case of this. ⊣

$\Sigma_k^{(n)}$-Preserving Embeddings

We may define $\Sigma_k^{(n)}$-preserving embeddings in a natural way: Let $\bar{M}, M$ be J-structures, $n, l < \omega$.

(i) $\pi : \bar{M} \longrightarrow_{\Sigma_l^{(n)}} M$ iff $\pi : \bar{M} \longrightarrow M$ and whenever $\varphi(v_1^{j_1}, \ldots, v_m^{j_m}) \in \Sigma_l^{(n)}$, $x_i \in H_{\bar{M}}^{j_i}$ $(1 \leq i \leq m)$ then $\pi(x_i) \in H_M^{j_i}$, and

$$\bar{M} \models \varphi(\vec{x}\,) \quad \Longleftrightarrow \quad M \models \varphi(\overrightarrow{\pi(x)});$$

(ii) $\pi : \bar{M} \longrightarrow_{\Sigma^*} M$ iff $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ for all $n < \omega$;

(iii) $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ *cofinally* iff $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ and $H_M^n = \bigcup \pi``H_{\bar{M}}^n$.

If F is a good $\Sigma_1^{(n)}(\bar{M})$-definable function, then one may show that F has such a definition that is "functionally absolute"; i.e. a definition that defines a function over any acceptable structure N, and thus is robust under $\Sigma_1^{(n)}$-preserving maps. (See [62, 1.8.10].) The key to this is that the canonical Σ_1-Skolem function has such a definition over any acceptable J-structure, in particular for those of the form $M^{n,p}$. Thence the same holds for any other $\Sigma_1^{(n)}(M)$ function to H_M^n also: briefly if $f(\overrightarrow{x})$ where $\overrightarrow{x} = \vec{x}^{\,n}, \ldots, \vec{x}^{\,0}$ is a $\Sigma_1^{(n)}(M)$ function to H_M^n defined by some formula φ, we may define $g_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq f(\overrightarrow{x})$ and $p(\vec{x}\,) =_{\mathrm{df}} \langle\langle \vec{x}^{\,0} \rangle, \ldots, \langle \vec{x}^{\,n-1} \rangle\rangle$. This makes the definition of $g_{\overrightarrow{x}}(\vec{x}^{\,n})$ $\Sigma_1(M^{n,p(\vec{x}\,)})$. However this depends only on φ and not $\overrightarrow{x}$. Hence there is a single fixed i so that $g_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq h_{M^{n,p(\vec{x}\,)}}(i, \vec{x}^{\,n})$ and the latter is a uniform functionally absolute definition. In short, concerning embeddings, any $\Sigma_1^{(n)}(\bar{M})$ functionally absolute definition applied over $\bar{M}$ where $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ also yields a $\Sigma_1^{(n)}$ function over M.

1.13 Lemma. *Let $\bar{M}, M$ be acceptable J-structures. Then: $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$ iff $\pi : \bar{M} \longrightarrow M$ and whenever $\bar{p} \in \Gamma_{\bar{M}}^n$ then $p = \pi(\bar{p}) \in \Gamma_M^n$, and $\pi{\restriction}H_{\bar{M}}^n : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$.*

Proof. This is a direct consequence of Lemma 1.7. ⊣

The following two lemmata correspond to [46, 5.8, 5.9] expressed in our language.

1.14 Lemma. *Let $\bar{M}$, M be acceptable J-structures. Let $\pi : \bar{M} \longrightarrow M$ be the n-completion of $\pi{\restriction}H_{\bar{M}}^n : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$ where $\bar{p} \in R_{\bar{M}}^n$ and $p = \pi(\bar{p}) \in \Gamma_M^n$. Then $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$.*

Proof. Let $\bar{q} \in \Gamma_{\bar{M}}^n$ be arbitrary. As $\bar{p} \in R_{\bar{M}}^n$ we have that for any $q = \pi(\bar{q}) \in \Gamma_M^n$ that $A_{\bar{M}}^{n,\bar{q}}$ is rudimentary in $A_{\bar{M}}^{n,\bar{p}}$ in some parameter r say. (See the proof of [46, 5.3(b)].) Then $A_M^{n,q}$ is rudimentary in $A_M^{n,p}$ in $\pi(r) \in H_M^n$ by the same rudimentary definition. So $\pi{\restriction}H_{\bar{M}}^n : \bar{M}^{n,\bar{q}} \longrightarrow_{\Sigma_k} M^{n,q}$. As $\bar{q}$ was arbitrary, the previous lemma shows $\pi : \bar{M} \longrightarrow_{\Sigma_k^{(n)}} M$. ⊣

1.15 Lemma (Downward Extension of Embeddings Lemma). *Let $\bar{M}, M$ be acceptable J-structures. Let $\pi\restriction H^n_{\bar{M}} : \bar{M}^{n,\bar{p}} \longrightarrow_{\Sigma_k} M^{n,p}$ where $\bar{p} \in R^n_{\bar{M}}$. Then there is a unique $\widetilde{\pi} \supseteq \pi$ such that $\widetilde{\pi}(\bar{p}) = p$ and $\widetilde{\pi} : \bar{M} \longrightarrow_{\Sigma^{(n)}_k} M$.*

Some of the fruits of this analysis are seen in the following two lemmas, the first of which is the analogue of Jensen's classical Σ_n-Uniformisation Lemma.

1.16 Lemma ($\Sigma^{(n)}_1$-Uniformisation Theorem). *Let $R(y^n, x_1^{j_1}, \ldots, x_m^{j_m})$ be $\Sigma^{(n)}_1(M)$ with $j_i \leq n$. Then there is a $\Sigma^{(n)}_1(M)$ function F (uniformly definable with respect to any such M) into H^n_M (which is thus good) uniformising R. Namely:*

(a) $\mathrm{dom}(F) = \{\langle x_1^{j_1}, \ldots, x_m^{j_m}\rangle \in M \mid \exists y^n \in M(R(y^n, \vec{x}\,))\}$ *(where $\vec{x} = x_1^{j_1}, \ldots, x_m^{j_m}$);*

(b) $\forall \vec{x} = x_1^{j_1}, \ldots, x_m^{j_m} \exists y^n (R(y^n, \vec{x}\,) \longleftrightarrow R(F(\vec{x}\,), \vec{x}\,))$.

Proof. Writing R in the form $R(y^n, \vec{x}^{\,n}, \ldots, \vec{x}^{\,0})$ and setting again $\overrightarrow{x} = \vec{x}^{\,n-1}, \ldots, \vec{x}^{\,0}$, we have that if $R_{\overrightarrow{x}} =_{\mathrm{df}} \{\langle y^n, \vec{x}^{\,n}\rangle \mid R(y^n, \vec{x}^{\,n}, \overrightarrow{x})\}$, then Lemma 1.7(i) shows that $R_{\overrightarrow{x}}$ is uniformly $\Sigma_1(M^{n,p(\vec{x}\,)})$, where, as before, $p(\vec{x}\,) =_{\mathrm{df}} \langle\langle \vec{x}^{\,0}\rangle, \ldots, \vec{x}^{\,n-1}\rangle\rangle$. If φ_i is a Σ_1 formula yielding this definition, then we may define the partial function

$$F(\vec{x}^{\,n}, \overrightarrow{x}) = F_{\overrightarrow{x}}(\vec{x}^{\,n}) \simeq h_{M^{n,p(\vec{x}\,)}}(i, \langle \vec{x}^{\,n}\rangle).$$

Then $F_{\overrightarrow{x}}$ is a uniformly $\Sigma_1(M^{n,p(\vec{x}\,)})$-definable Skolem function for $R_{\overrightarrow{x}}$, and by Lemma 1.7(i) F is $\Sigma^{(n)}_1$ with value type n, and will do the job. ⊣

1.17 Lemma. *Let $n < \omega$. There is a $\Sigma^{(n)}_1$ formula defining a good $\Sigma^{(n)}_1(M)$ function $F_n(u, v)$ into M (definable uniformly with respect to any J-structure M), so that*

$$\forall p(p \in R^{n+1}_M \longrightarrow |M| = F_n``H^{n+1}_M \times \{p\}).$$

Proof. By induction on n. For $n = 0$, we have that $p(0) \in R_M$ and thus $|M| = h_M``(\omega \times (H^1_M \times \{p(0)\}))$. Thus we can take $F_0(\langle i, v\rangle, w) = h_M(i, \langle v, w\rangle)$. Suppose that the lemma holds for $n-1$ as witnessed by $F_{n-1}(u^{n-1}, w^0)$ and we prove it for n. Let $p \in R^{n+1}_M$. Then $p\restriction n \in R^n_M$. Let $x \in |M|$ be arbitrary; then $x = F_{n-1}(z, p\restriction n)$ for some $z \in H^n_M$; z in turn equals $h_{M^{n,p\restriction n}}(j, \langle y, p(n)\rangle$ for some $y \in H^{n+1}_M$ since $p(n) \in R_{M^{n,p\restriction n}}$ as well. Note that $F_{n-1}(u^n, w^0)$ is also good $\Sigma^{(n-1)}_1(M)$ (as it is obtained from the good $\Sigma^{(n-1)}_1(M)$ function $F_{n-1}(u^{n-1}, w^0)$ by specialising the first variable u^{n-1} to u^n; it is a general fact goodness is preserved by specialisation). Hence we may define

$$F_n(u^n, w^0) \simeq F_{n-1}(h_{M^{n,w^0\restriction n}}((u^n)_0, \langle (u^n)_1, w^0(n)\rangle), w^0\restriction n).$$

As we are substituting good $\Sigma^{(n)}_1(M)$ functions (with value type $\leq n$) into $F_{n-1}(u^n, w^0)$ we end up with a good $\Sigma^{(n)}_1(M)$ function F_n. ⊣

To bring out the nature of the above argument, we may define

$$g_n(\langle j, y^{n+1}\rangle, p) = h_{M^{n,p\restriction n}}(j, \langle y^{n+1}, p(n)\rangle)$$

and more generally

$$g_i(\langle j, y^{i+1}\rangle, p) = h_{M^{i,p\restriction i}}(j, \langle y^{i+1}, p(i)\rangle)$$

and this is uniformly lightface $\Sigma_1^{(i)}(M)$ in the two variables $w^{i+1} = \langle j, y^{i+1}\rangle$ and $x^0 = p$ for $i \leq n$. We may compose these functions and if $p \in R_M^{n+1}$ then any $x \in |M|$ is the value of such an iterated composition. This is expressed below at part (ii) of the next definition and the fact that follows it.

1.18 Definition ($\Sigma_1^{(n)}$-Skolem Functions). Let M be an acceptable J-structure, and let $p \in \Gamma_M^n$.

(i) $h_M^{n,p} = h_{M^{n,p}}$;

(ii) $\widetilde{h}_M^n(w^n, x^0) = g_0(g_1(\cdots g_{n-1}((w^n)_0, \langle (w^n)_1, x^0(n-1)\rangle) \cdots x^0(0)\rangle)$.

Thus $\widetilde{h}_M^n$ is $\Sigma_1^{(n-1)}$ uniformly over all M. The Σ_1 hull of a set $X \subseteq M^{n,p}$ we shall denote by $h_M^{n,p}(X)$ (and is thus the set $\{h_M^{n,p}(i,x)) \mid i \in \omega, x \in X\}$). Note that $\widetilde{h}_M^1(\langle j, y^0\rangle, p(0)) = g_0(j, \langle y, p(0)\rangle) = h_M(j, \langle y, p(0)\rangle)$. If $p \in R_M^n$ then every $x \in M$ is of the form $\widetilde{h}_M^n(z,p)$ for some $z \in H_M^n$. We may similarly form hulls using $\widetilde{h}_M^n$: again if $X \subseteq M^{n,p}$ say, and $q \in M$ then the $\Sigma_1^{(n-1)}$ hull of $X \cup \{q\}$ is the set $\{\widetilde{h}_M^n(x,q)) \mid x \in X\}$). The following states some of these facts and are now easy to establish (see [62, p. 29]):

Fact. Let M be acceptable, and $p \in R_M^n$.

(i) if $\omega\rho_M^n \in M$ and $p \in R_M^n$ then $\widetilde{h}_M^n$ is a good, uniformly defined, $\Sigma_1^{(n-1)}(M)$ function mapping H_M^n onto M: $M = \{\widetilde{h}_M^n(u^n, p) \mid u^n \in H_M^n\}$.

(ii) (a) every $A \subseteq H_M^n$ which is $\mathbf{\Sigma}_1^{(n)}(M)$ is $\mathbf{\Sigma_1}(M^{n,p})$;
(b) $\rho_M^{n+1} = \rho_{M^{n,p}}$.

1.19 Lemma. *Let M be an acceptable J-structure.*

(i) $\mathbf{\Sigma}^*(M) \subseteq \mathbf{\Sigma}_\omega(M)$;

(ii) *If M is sound then* $\mathbf{\Sigma}^*(M) = \mathbf{\Sigma}_{\boldsymbol{\omega}}(M)$.

Proof. For (i) we just have to see that any typed variable v^n can be replaced by $v \in H_M^n$ and the latter is definable (not necessarily at the nth level of the Levy-hierarchy of complexity!). For (ii) we prove ($\supseteq$). Suppose by induction we have shown that for every $\Sigma_m(M)$ formula $\varphi(\vec{y})$ there is a $\mathbf{\Sigma}_1^{(\boldsymbol{m})}(M)$ formula $\widetilde{\varphi}(\vec{y}^0)$ such that $(\forall \vec{x} \in M)((\varphi(\vec{x}))_M \iff (\widetilde{\varphi}(\vec{x}))_M$. Let $n = m+1$

and suppose $p \in R^n_M$ (as M is sound). Let $\widetilde{h}^n_M$ be as at (i) of the last Fact. This is a good $\Sigma_1^{(m)}$ function of value type 0 (and hence good $\Sigma_1^{(n)}$ also) such that

$$\begin{aligned}(\exists x \psi(x, \vec{y}))_M \quad &\Longleftrightarrow \quad (\exists u \in H^n_M)(\psi(\widetilde{h}^n_M(u,p), \vec{y}))_M \\ &\Longleftrightarrow \quad (\exists u \in H^n_M)(\widetilde{\psi}(\widetilde{h}^n_M(u,p), \vec{y}))_M\end{aligned}$$

where $\widetilde{\psi}$ is the $\boldsymbol{\Pi_1^{(m)}}(M)$-formula given by the inductive hypothesis. Thence $(\widetilde{\psi}(\widetilde{h}^n_M(u,p), \vec{y}))_M$ is a $\boldsymbol{\Sigma_1^{(n)}}(M)$ property and $(\exists u^n \widetilde{\psi}(\widetilde{h}^n_M(u,p), \vec{y}))_M$ is a $\boldsymbol{\Sigma_1^{(n)}}(M)$ relation. ⊣

The following is a standard result. For its proof see [62, 1.11.2].

1.20 Lemma. *Suppose $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ and is such that* (i) $\pi \restriction \omega\rho^{n+1}_{\bar{M}} = \mathrm{id} \restriction \omega\rho^{n+1}_{\bar{M}}$ *and* (ii) $\mathrm{ran}(\pi) \cap P^*_M \neq \varnothing$. *Then π is Σ^*-preserving.*

Solidity Witnesses

Naturally the notion of *solidity witness* (see [46, Sect. 7]) can be defined in this context. We shall not re-enter a discussion of these notions, but simply give the definition.

1.21 Definition. Let M be an acceptable J-structure, $p \subseteq M$ a finite set of ordinals, and $\nu \in M$. The *standard witness for ν with respect to M*, p is the J-structure $W = W^{\nu,p}_M$ where, if n is such that $\omega\rho^{n+1}_M \leq \nu < \omega\rho^n_M$:

$\sigma : W \cong X$ and $X = \widetilde{h}^{n+1}_M(\nu \cup p \backslash (\nu+1))$ and σ is the inverse of the transitive collapse.

The notion of generalised witness has an analogous definition mutatis mutandis. The properties of witnesses, the definitions of n-solidity etc, then all go through. It is easy to check for the L-hierarchy that if $M = J_\beta$, $p = p_M$, the standard parameter, and $\omega\rho^{n+1}_M \leq \nu < \omega\rho^n_M$, then $\nu \in p \Longleftrightarrow W^{\nu,p}_M \in M$. We have for the L-hierarchy a strong form of condensation. We shall use the fact that the L-hierarchy is sound (cf. [46, 9.2]). Note as always for the pure J-hierarchy, the usual condensation property that hulls of a J_β always transitivise to some J_δ for a $\delta \leq \beta$.

1.22 Lemma (Condensation Lemma). *Let $M = J_\beta$, $\pi : J_\delta \longrightarrow_{\Sigma_1^{(n)}} J_\beta$, $\omega\rho^{n+1}_M \leq \alpha < \omega\rho^n_M$, $\pi(\bar{\alpha}) = \alpha$, and $\pi(\bar{p}) = p_{J_\beta} \backslash \alpha$. Then $\bar{p} = p_{J_\delta} \backslash \bar{\alpha}$.*

Proof. As π is $\Sigma_1^{(n)}$-preserving (and $\widetilde{h}^{n+1}_M(u^{n+1} \cup \{p_M\})$ is good $\Sigma_1^{(n)}(M)$) $J_\delta = \widetilde{h}^{n+1}_{J_\delta}(\bar{\alpha} \cup \{\bar{p}\})$. Thence follows that $\omega\rho^{n+1}_{J_\delta} \leq \bar{\alpha}$: for if this failed we should have a $\boldsymbol{\Sigma_1^{(n)}}(J_\delta)$ definable map of $\bar{\alpha} < \omega\rho^{n+1}_{J_\delta}$ onto the whole of J_δ. However this is impossible as the usual simple diagonalisation argument would yield a $\boldsymbol{\Sigma_1^{(n)}}(J_\delta)$-definable subset of $\bar{\alpha}$ which is not in J_δ. This would contradict the definition of $\omega\rho^{n+1}_{J_\delta}$. That $\bar\alpha < \omega\rho^{n}_{J_\delta}$ follows from the $\Sigma_1^{(n)}$ statement "$\exists u^n(u^n = \pi(\bar\alpha))$" and the $\Sigma_1^{(n)}$-preservation property of π.

As $J_\delta = \widetilde{h}^{n+1}_{J_\delta}(\bar\alpha \cup \{\bar p\})$, we have that $\bar p \in R^n_{J_\delta}$; this means that it can be lengthened to a parameter $p' \in P^{n+1}_{J_\delta}$ (cf. [46, 6.4]). By the minimality properties of the standard parameter [46, 6.3] we have then that $p_{J_\delta}\backslash\bar\alpha \leq^* \bar p$. If we had $<^*$ instead of $\leq^*$ then we should have that $\bar p \in \widetilde{h}^{n+1}_{J_\delta}(\bar\alpha \cup \{p_{J_\delta}\backslash\bar\alpha\})$. Applying the $\Sigma_1^{(n)}$-preserving π we'd conclude that $p_M\backslash\alpha \in \widetilde{h}^{n+1}_M(\alpha \cup \{\pi(p_{J_\delta}\backslash\bar\alpha)\})$ with $\pi(p_{J_\delta}\backslash\bar\alpha) = \pi(p_{J_\delta})\backslash\alpha <^* p_M\backslash\alpha$. A contradiction. ⊣

1.1. Variant Fine Structures

Although we may consider the $\Sigma_1^{(n)}$ hierarchy as emerging finally out of the original fine structure of [24], the historical line of development is not so direct. From the time of Jensen's proof of the Covering Lemma for L some attempts at simplifications were made by several people. Silver's development of *Silver machines* built the L-hierarchy in a quite different manner by using a very much slowed down version of construction based on creating hulls and using a different collection of Skolem functions. As opposed to steps in the L-hierarchy, very little happens in the transition from M^δ to $M^{\delta+1}$ in this machine hierarchy. This is expressed by a *finiteness property*: the hull in $M^{\delta+1}$ of a set A can be obtained by taking the hull in M^δ of $A \cap \delta$ together with just finitely many more ordinals less than δ (and lastly adding the point δ). This relatively simple apparatus enjoyed sufficient condensation properties that proofs of $\square$ and Jensen's Covering Lemma for L could be attained (see [8, Chap. IX] for an outline here, and [1] for a machine proof of $\square$). Once the theory of core models came to the fore it was not apparent that this mechanism could be used as a substitute for the fine structure of mice with measures that was then emerging.

The fine structure for mice has also not had a direct development. We mention here some of the history of this fine structure, and the various uses of the word "acceptable" in the literature. The fine structure for the mice of the Dodd-Jensen core model [10] and [9], although used for later core models with sequences of measures [35], and even in manuscript form of Dodd's for models with extenders, would seem to be far from amenable to the type of approximation hierarchy of Silver. The Dodd-Jensen fine structure was based on previous unpublished work of Solovay who had built on Jensen's fine structure for L, to do the same for $L[U]$; the order of set construction was that of relative constructibility in the traditional "macro" sense, and two notions of "acceptability" and "strong acceptability" occurred here. The latter is now closer to the current, and by now standard, use of the word "acceptable". This detailed how sets appeared in such J^U_α hierarchies, and which expressed a strong and uniform version of GCH. However they bore little resemblance to the current ordering of $L[E]$ hierarchies, and were becoming extremely difficult to work with, and, past a strong cardinal, particularly hair-raising.

Various papers from the time were published using this older fine structure, and for very thin core models, that are inner models of K^{DJ}, the vestigial notions of "Q-structure" that were a part of the old theory still play a role (e.g. [58]). However with the very successful reorganising of hierarchies due to Baldwin and Mitchell this form of fine structure became defunct.

Magidor replaced fine structure (in the sense of eliminating the use of projecta and mastercodes) by the use of more general Σ_n hulls and Skolem functions for Σ_n formulae that, albeit not Σ_n-definable, were preserved under condensations in reproving the Covering Lemma for L (this is the proof given in [8]) using the J-hierarchy, and even using the L-hierarchy alone [34].

One of the chief advantages of the Baldwin-Mitchell reorganisation of the hierarchy of construction from sequences of measures or extenders, was that every level of the models to be constructed was sound. Mitchell and Steel published an account [36] of a fine structure together with a model construction, of extenders whose comparison iteration required iteration trees (see the chapter by Steel in this Handbook). This was the first time that an account of fine structure for mice whose comparison required trees of iterations, allied with the Baldwin-Mitchell organisation of hierarchies, was published. The fine structure used there was inspired by the nature of the predicates that were being used. The notion of $r\Sigma_{n+1}$ formula is analogous to $\Sigma_1^{(n)}$ used here (and the notion of $r\Sigma_{n+1}$ embedding between structures with very good parameters is employed in [46] at 5.12). The success of this fine structure, together with the ability to build much larger models, has ensured its widespread use and represented a leap forward in the production of models that could contain many Woodin cardinals (cf. [50]). Jensen in a series of circulated manuscripts developed the theory for similarly large models using the Σ^* language [27, 25]. Another auxiliary variety of acceptability occurs in [14], where Feng and Jensen develop a theory of mice with some overlapping of extenders in the Σ^* language, in order to build a core model without assuming any 'technical hypothesis' in the form of a large cardinal Ω in the universe (see [41, Sect. 1]). In that paper a J-structure $M = \langle J^A_\alpha, \in, A\rangle$ is *strongly acceptable* if $\forall\tau < \alpha\forall\vec{\xi} < \tau\forall\varphi(v_0) \in \Sigma_1$:

$$J^A_{\tau+1} \models \varphi[\vec{\xi}] \wedge J^A_\tau \models \neg\varphi[\vec{\xi}] \Longrightarrow (J^A_{\tau+1} \models \mathrm{Card}(\tau) \leq \max\{\omega, \vec{\xi}\})$$

holds in M. Despite its name the notion is supplemental to acceptability (which it does not in general imply).

We shall lastly mention the *hyperfine structure* of Friedman-Koepke [19]. This is an attractive elongation of the usual L_α-hierarchy by interspersing infinitely many stages between the usual L_α and $L_{\alpha+1}$. A crucial feature, as will be seen below, for many fine-structural arguments is the notion of *singularisation* of an ordinal, the place where an ordinal ν is first seen to definably singularised by some formula. If $\exists x < \alpha\ \varphi(\xi, x, p)$ defines over L_β a cofinal subset of ξ's forming $C \subseteq \nu$ (for some least $\beta = \beta(\nu)$ and according to some $<_{L_\beta}$-least finite parameter sequence of ordinals p) we can think of the triple $(\beta, \varphi, (p^\frown\alpha))$ as a minimal *location* where ν is singularised. If we define $S_\varphi(q, x) =$ the $<_L$-least ξ so that $\varphi(\xi, x, q)$, as being the term for the Skolem function for φ, we have that $C = S_\varphi^{L_\beta}$"$\{(p, x) \mid x < \alpha\}$. We thus focus on structures of the form

$$L_{(\beta,\varphi_n,(p^\frown\alpha))} =_{\mathrm{df}} (L_\beta, \in, <_L, N, I, S, S_{\varphi_0}^{L_\beta}, S_{\varphi_1}^{L_\beta}, \ldots, S_{\varphi_n}^{L_\beta} \restriction \{w <_{\mathrm{lex}} p^\frown\alpha\}, \ldots)$$

(where $\varphi = \varphi_n$, the nth formula in some standard recursive enumeration, with all subformulae of φ_n being some φ_i for an $i \le n$). We can think of this as the singularising structure for ν in this context, as ν is singularised by the last object in this structure: $S_{\varphi_n}^{L_\beta} \restriction \{w <_{\mathrm{lex}} p^\frown\alpha\}$. The functions N, I, S are included, where the first two are functions for naming and interpreting objects respectively, and S is a general Skolem function. Properly construed one may form hulls and, importantly, the finiteness property of Silver's machines is valid here too; one may prove condensation—that hulls inside locations transitively collapse to other locations. Ultimately the theory allows for a particularly elegant and short proof of Global $\square$. This hyperfine structure can be extended to consider premice with a single measure—thus sufficient for forming the Dodd-Jensen K^{DJ}, but like other alternatives to "true" fine structure there seem to be real technical difficulties to going beyond that.

1.2. Σ^* Ultrapowers

We give an account of the formation of a fine-structure preserving ultrapower. [46, 8.4–8.5] gives the definition and construction of the Σ_0 ultrapower. We shall see here how to extend the usual notion of ultrapower that uses functions within the model M, to one using $\Sigma^*(M)$-definable functions f. Such a function f is not in general an element of the domain structure M, but with the correct assumptions, M has enough information about the measure or extender being used to form the ultrapower, that this can be sensibly done. The resulting target ultrapower structure N may contain more objects than the ordinary Σ_0 ultrapower, in particular more ordinals (if we are starting with a set model M). We shall develop this theory for ultrapowers by short extenders that are weakly amenable with respect to the models M concerned. This is more than one needs for dealing with premice with measures of order 0. However, this greater generality will enable us to quickly dispose of "pseudo-ultrapowers" in the next subsection. In any case it is still basically the account of such ultrapowers from [28]. Suppose that $M = \langle J_\alpha^A, A, B\rangle$ is an acceptable J-structure and E is a (κ, ν) extender over M with a single critical point $\mathrm{crit}(E) = \kappa < \mathrm{On} \cap M$ and $E = \langle E_a \mid \alpha \in [\nu]^{<\omega}\rangle$. We shall recall here for later the definition of *weakly amenability* of an extender E with respect to M. This is defined as for any $a \in [\nu]^{<\omega}$, $\langle X_\alpha \mid \alpha < \kappa\rangle \in M$, that $\{\alpha \mid X_\alpha \in E_a\} \in M$. This can be shown equivalent to saying that $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap \mathrm{Ult}_0(M, E)$ (the latter denoting the usual "ordinary" Σ_0 ultrapower of M by E). At some point we shall additionally assume that E is Σ_1-amenable: this in fact is just clause (1) of [46, 8.9]. Thus such E will be "close to M" in that terminology.

Assuming only that E is an extender, we shall write $\pi_E : M \longrightarrow_E N = \mathrm{Ult}_0(M, E)$ for the ordinary ("coarse") ultrapower embedding. The map π_E is a Σ_0 and cofinal embedding and hence Σ_1-preserving. In general this is as much definable preservation that one could hope for. The notion of Σ^* ultrapower involves using functions that lie outside of M but that are definable classes using formulae of the language $\mathcal{L}^*$. Let M and E be as above.

1.23 Definition. Suppose that E is a (κ, ν) extender over M. The relation $\pi : M \longrightarrow^*_E N$ holds iff the following conditions are met:

(i) Whenever $\rho^k_M > \kappa$, $\pi : M \longrightarrow_{\Sigma_0^{(k)}} N$; where N is transitive.

(ii) Let $\rho^n_M = \min\{\rho^m_M | \rho^m_M > \kappa\}$, and $H = \bigcup_{x \in H^n_M} \pi(x)$. Then $\pi{\restriction}H^n_M : H^n_M \longrightarrow_E H$.

(iii) Whenever $\rho^{k+1}_M > \kappa$, N is the closure of $H \cup \mathrm{ran}(\pi)$ under good $\Sigma_1^{(k)}(N)$ functions.

1.24 Remark. (a) (ii) requires that $\pi{\restriction}H^n_M$ be the usual Σ_0 ultrapower map into H. Thus at this level we have the familiar ultrapower with functions taken from H^n_M. Hence $\mathrm{crit}(\pi) = \mathrm{crit}(E)$.

(b) If $\omega\rho^\omega_M > \kappa$ then $\pi : M \longrightarrow^*_E N$ implies $\pi : M \longrightarrow_{\Sigma^*} N$. If we define a good $\Sigma_1^{(-1)}$ function to be a function in M then clause (iii) makes sense even if $\rho^1._M \leq \kappa$.

In most situations we shall want N to be well-founded (hence the definition), but sometimes it is convenient (although we shall not be concerned with this here) to ask only that the well-founded core of N (which will be assumed transitive) contains ν. The definition straightforwardly implies that if $\omega\rho^1_M < \kappa$ then $\pi : M \longrightarrow_E N$ if $\pi : M \longrightarrow^*_E N$. (We'll see that under this hypothesis the converse is also true.) We shall see that the existence of such a π, and an N with $\pi : M \longrightarrow^*_E N$ satisfying the above, implies that such π and N are unique. Why is this plausible? Suppose such a π and N exist. Suppose $\kappa < \omega\rho^{m+1}_M$ for consideration as in (iii) above. Then for every $z \in N$ there is a good $\Sigma_1^{(m)}(N)$ function F with $z = F(u, \pi(x))$ for some $u \in H$ and $x \in M$ for some m. Let $y \in H^n_M$ be such that $u \in \pi(y)$. Let $\bar{F}$ have the same functionally absolute definition over M as F does over N. Then $\bar{F}$ is a good $\Sigma_1^{(m)}(M)$ function. Let $\varphi(v_0)$ be any Σ_0 formula. Then $M \models \varphi(\bar{F}(u, x))$ is a $\Sigma_1^{(m)}(M)$ property in u, x. Let $w = \{v \in y \mid M \models \varphi(\bar{F}(v, x))\}$. Then, as $w \in H^n_M$ (since $y \in H^n_M$ and we may assume $n > m$):

$$M \models (\forall v^n \in y)(\varphi(\bar{F}(v^n, x)) \longleftrightarrow v^{n+1} \in w).$$

This is $\Pi_0^{(n)}$, and as π is Σ^*-preserving we have:

$$N \models (\forall v^n \in \pi(y))(\varphi(F(v^n, \pi(x))) \longleftrightarrow v^n \in \pi(w));$$

in short, for u as above:

$$N \models \varphi(F(u,\pi(x)) \;\longleftrightarrow\; u \in \pi(w).$$

This is tantamount to saying that Σ_0 facts about $(F(u,\pi(x)))_N$ are determined by checking whether $u \in \pi(w)$ for a $w \in H^n_M$ suitably formed as above. (And similarly for Σ_0 facts such as "$(F(u',\pi(x')) \in F(u,\pi(x)))_N$".) However $\pi\restriction H^n_M : H^n_M \longrightarrow_E H$ is the usual ultrapower of H^n_M by E, and this map is determined by M and by E. And thus so are the Σ_0 facts about N mentioned above. If the $\in$-diagram of N is determined then the whole map $\pi : M \longrightarrow N$ is determined—and thus unique—if it can be shown to exist. To put it another way, the above argument shows that there is an $\in$-isomorphism between any two such ultrapowers N, N'; however if the N, N' are taken as transitive, then this isomorphism is the identity.

We have implicitly in the above assumed that we can map across such good functions in a well-defined way. We justify this more formally in a moment.

We now proceed to describe ultrapowers formed by taking these extra class (over M) functions.

1.25 Definition. $\Gamma = \Gamma(\kappa, M)$ is the set of functions f with $\mathrm{dom}(f) = [\kappa]^k$ (some $k < \omega$) and either $f \in M$ or f is a good $\boldsymbol{\Sigma_1^{(n)}}(M)$ function for some n with $\omega\rho^{n+1}_M \geq \kappa$.

The extension of π to elements of Γ is effected as follows: if $f \in \Gamma$ and $\mathrm{dom}(f) = \kappa$, and f with $\mathrm{dom}(f) = \kappa$ has a functionally absolute $\Sigma_1^{(m)}(M)$ definition in a parameter r, where $\omega\rho^{m+1}_M > \kappa$, then we set $\pi(f)$ to be the function with $\mathrm{dom}(\pi(\kappa))$ defined over N by the same $\Sigma_1^{(m)}$ definition using $\pi(r)$. We need to argue that this is well-defined. Suppose $F^M(v^m, u^0)$ and $G^M(v^m, u^0)$ are two good $\Sigma_1^{(m)}(M)$ definitions in the parameters q, r respectively of the function f. Then $(\forall\xi < \kappa)(F^M(\xi, q) = G^M(\xi, r))$ holds over M. However this can be expressed as $\boldsymbol{\Pi_0^{(m+1)}}(M)$:

$$(\forall x^{m+1} < \kappa)(F^M(x^{m+1}, q) = G^M(x^{m+1}, r)).$$

As π is Π_0^{m+1} preserving, the same statement is true of $\pi(\kappa), \pi(q), \pi(r), F^N$, G^N over N. So $\pi(f)$ is indeed independent of the choice of its functionally absolute definition.

We shall form a suitable domain and a term model more or less exactly following the pattern for Σ_0 ultrapowers as is done in [46, 8.4], although we shall incorporate the functions from $\Gamma_M = \Gamma(\kappa, M)$, and here we only have the single relevant critical point κ (thus we set $\mu_a = \kappa$ for all $a \in \nu$ from their definition). Thus our domain will be:

$$D = \{\langle a, f\rangle \mid f \in \Gamma_M, a \in [\nu]^{<\omega}, f : [\kappa]^{\mathrm{card}(a)} \longrightarrow M\}.$$

The definitions of the $\sim$ and $\dot{\in}$ relations are then unaltered, and we use the same notation "$f^{a,b}$" where $\mathrm{dom}(f) = [\kappa]^n$, $a \subseteq b$, and $a = (u_{i_1}, \ldots, u_{i_n}) \subseteq$

$b = (u_1, \ldots, u_m)$ to denote the function with domain $[\kappa]^m$ given by $f^{a,b}(v) = f(v_{i_1}, \ldots, v_{i_n})$.

$$\begin{aligned}\langle a, f\rangle \sim \langle b, g\rangle &\iff \{u \in [\kappa]^{\mathrm{card}(c)} \mid f^{a,c}(u) = g^{b,c}(u)\} \in E_c \quad \text{for } c = a \cup b,\\ \langle a, f\rangle \mathrel{\dot\in} \langle b, g\rangle &\iff \{u \in [\kappa]^{\mathrm{card}(c)} \mid f^{a,c}(u) \in g^{b,c}(u)\} \in E_c \quad \text{for } c = a \cup b,\\ \dot C(\langle a, f\rangle) &\iff \{u \mid C(f(u)) \mid\in E_a\} \quad (\text{for } C = A, B).\end{aligned}$$

1.26 Definition. The term model $\mathbb{D}$ is defined as: $\mathbb{D} = \langle D, \sim, \dot\in, \dot A, \dot B\rangle$.

We observe:

1.27 Lemma. *Let $\varphi(v_0^{i_0}, \ldots, v_k^{i_k})$ be either a $\Sigma_1^{(n)}$ formula where $\rho_M^{n+1} > \kappa$, or else a $\Sigma_0^{(n)}$ formula where only $\rho_M^n > \kappa$ is assumed. Let $\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle \in D$, and let $b \in [\nu]^{<\omega}$ be such that $b \supseteq a_0 \cup \cdots \cup a_k$. Assume that for $j \le k$, f_j is a function to $H_M^{i_j}$.*

Then $\{u \mid M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))\} \in M$.

Proof. Assume first $\rho_M^{n+1} > \kappa$ and that φ is $\Sigma_1^{(n)}$. As we have assumed the value types of the defining formulae of the $f_j \in \Gamma_M$ satisfy $i_j \le n$ for $j \le k$. Then $M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))$ defines a $\boldsymbol{\Sigma_1^{(n)}}(M)$ relation. Now if φ is $\Sigma_0^{(n)}$ and if $\rho_M^{n+1} > \kappa$ still holds we have nothing to prove. However if $\rho_M^n > \kappa \ge \rho_M^{n+1}$ we have in this case that either the $f_i \in M$ or they are good $\Sigma_1^{(n-1)}$ and the set under consideration is a $\boldsymbol{\Sigma_0^{(n)}}$ subset of κ, and the result is immediate. ⊣

The following lemma is a kind of "uniformisation lemma" that we shall need in order to prove Łoś's theorems.

1.28 Lemma. *Let $R(y^m, x^{i_0}, \ldots, x^{i_k})$ be a $\Sigma_1^{(m)}$ relation with $i_0, \ldots, i_k \le m$. Let $m \le n$ be such that $\rho_M^{n+1} > \kappa$, and let $f_0, \ldots, f_k \in \Gamma$ be good $\boldsymbol{\Sigma_1^{(n)}}(M)$ functions with $f_j : [\kappa]^l \longrightarrow H_M^{i_j}$. Then there is a good $\boldsymbol{\Sigma_1^{(n)}}(M)$ function $g : [\kappa]^l \longrightarrow H_M^m$, $g \in \Gamma$, such that for any $u \in [\kappa]^l$:*

$$M \models \exists y^m R(y^m, f_0(u), \ldots, f_k(u)) \iff M \models R(g(u), f_0(u), \ldots, f_k(u)).$$

Proof. By Lemma 1.16 there is a good $\boldsymbol{\Sigma_1^{(m)}}(M)$ function $F(x^{i_0}, \ldots, x^{i_k}) = y^m$ such that:

$$\begin{aligned}&M \models \exists y^m R(y^m, x^{i_0}, \ldots, x^{i_k})\\ &\quad\iff M \models R(F(x^{i_0}, \ldots, x^{i_k}), x^{i_0}, \ldots, x^{i_k}).\end{aligned}$$

So set $g'(u) \simeq F(f_0(u), \ldots, f_k(u))$. g' is then a good $\boldsymbol{\Sigma_1^{(n)}}(M)$ function. $\mathrm{dom}(g') \subseteq [\kappa]^l$ and by Lemma 1.27, we see that $a =_{\mathrm{df}} \mathrm{dom}(g') \in M$. Now define $G(v^0, w^0)$ by

$$y = G(v^0, w^0) \iff (y = v^0 \wedge w^0 \in a) \vee (y = 0 \wedge w^0 \notin a).$$

Then G is good, $\boldsymbol{\Sigma}_0(M)$, and hence $g(u) =_{\mathrm{df}} G(g'(u), u)$ is good $\boldsymbol{\Sigma_1^{(n)}}(M)$. ⊣

Before proving a Łoś's theorem later for $\Sigma_1^{(n)}$, we state one for Σ_0:

1.29 Lemma. *Let $\varphi(v_0, \ldots, v_k)$ be a Σ_0 formula. Let $\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle \in D$, and let $b \in [\nu]^{<\omega}$ be such that $b \supseteq a_0 \cup \cdots \cup a_k$. Then*

$$\begin{aligned}&\mathbb{D} \models \varphi(\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle)\\ &\quad\Longleftrightarrow\quad \{u \mid M \models \varphi(f_0^{a_0 b}(u), \ldots, f_k^{a_k b}(u))\} \in E_b.\end{aligned}$$

This is an induction on the structure of the Σ_0 formula φ using the last two lemmas. We shall give more detail in the $\Sigma_1^{(n)}$ Łoś's theorem which follows.

Using the Σ_0 Łoś's theorem 1.29 one has that if $\pi : M \longrightarrow_E^* N$ then the map from $\mathbb{D}$ to N given by $\langle a, f\rangle \mapsto \pi(f)(a)$ is structure preserving, in particular on $\in$: $\langle a, f\rangle \mathrel{\dot\in} \langle b, g\rangle \Longleftrightarrow \pi(f)(a) \in \pi(g)(b)$. We then have that $\mathbb{D}$ is a model of Extensionality, and $\sim$ is a congruence relation. We thus, as for the Σ_0 ultrapower, form the equivalence classes, written as $[a, f]$, for $\langle a, f\rangle \in \mathbb{D}$. We shall assume from now on that $\dot\in$ is well founded and we thus have an onto factor map $[\,] : \mathbb{D} \longrightarrow \langle N, \in, A', B'\rangle$ satisfying $[x] = / \in [y]$ iff $x \sim / \mathrel{\dot\in} y$ and $\dot C(x) \Longleftrightarrow C'([x])$. Using Lemma 1.29 again we have the map $\pi : M \longrightarrow_{\Sigma_0} N$ is defined by the usual constant functions c_x (with $\mathrm{dom}(c_x) = [\kappa]^0$): $\pi(x) = [0, c_x]$.

It will be useful to have some notation to stratify $\Gamma(\kappa, M)$.

1.30 Definition. Suppose $f \in \Gamma(\kappa, M)$.

$$\Gamma = \Gamma_n(\kappa, M) =_{\mathrm{df}} \begin{cases}\{f \in \Gamma(\kappa, M) \mid \mathrm{ran}(f) \subseteq H_M^n\} & \text{if } \rho_M^{n+1} > \kappa;\\ \{f \in \Gamma(\kappa, M) \mid \mathrm{ran}(f) \in H_M^n\} & \text{if } \omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n.\end{cases}$$

1.31 Lemma. *Suppose n is such that $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Set $\overline{H} = H_M^n$, and $H = \bigcup \pi``\overline{H}$ with π as above. Then:*

(i) *$\pi\restriction\overline{H} : \overline{H} \longrightarrow_E H$, i.e. $\pi\restriction\overline{H}$ is the coarse ultrapower map;*

(ii) *$\mathrm{crit}(\pi) = \kappa$, and $[a, f] = \pi(f)(a)$ for $a \in [\nu]^{<\omega}$ and $f \in {}^\kappa\overline{H} \cap \overline{H}$;*

(iii) *$\mathcal{P}(\kappa) \cap \overline{H} = \mathcal{P}(\kappa) \cap H$.*

Proof. (i) Let $x \in H$; then $x = [a, f]$ for some $\langle a, f\rangle \in \mathbb{D}$. Suppose $f \in \Gamma_m(\kappa, M)$. We just need to know that f could have been chosen from $\overline{H}$, i.e. as $\pi\restriction\overline{H}$ is cofinal into H we can find $y \in \overline{H}$ with $x \in \pi(y)$. We can assume that $\mathrm{ran}(f) \subseteq y$, by intersecting f with $\kappa \times y$ if need be, and now the latter is essentially a $\mathbf{\Sigma}_1^{(n-1)}(M)$ bounded subset of ρ_M^n, and thus is in M. By acceptability it must be in $\overline{H}$. (ii) is the usual argument for such ultrapowers. For (iii) ($\subseteq$) is straightforward. For ($\supseteq$) suppose $x \subseteq \kappa, x = [a, f] \in N$. Then $x = \{\xi < \kappa \mid \xi \in \pi(f)(a)\} = \{\xi < \kappa \mid \{u \mid \xi \in f(u)\} \in E_a\}$. Amenability says precisely that then $x \in H$. $\dashv$

We thus have some sort of agreement with the notion of a Σ_0 ultrapower at the "crossing structure" $\overline{H}$ where the projectum crosses over the measurable cardinal. It is more work to establish that we are getting the correct kind of embedding to ensure Σ^*-elementarity.

1.32 Theorem (Fine Structural Ultrapower Theorem).

(i) *N is an acceptable J-structure.*

(ii) (a) $\pi : M \longrightarrow_{\Sigma_0^{(n)}} N$ *if* $\omega\rho_M^n > \kappa$.

(b) $\pi : M \longrightarrow_{\Sigma_2^{(n)}} N$ *if* $\omega\rho_M^{n+1} > \kappa$.

(iii) (*Łoś's Theorem*) *let φ be in $\Sigma_0^{(n)}$ (or in $\Sigma_1^{(n)}$, if $\kappa < \omega\rho_M^{n+1}$) and $b \supseteq \bigcup_i a_i$. Then*

$$N \models \varphi(\pi(f_1)(a_1), \ldots, \pi(f_n)(a_n)) \quad \Longleftrightarrow \quad \{u \mid M \models \varphi(\overrightarrow{f_i^{a_i b}(u)})\} \in E_b.$$

Additionally if we assume E is close to M:

(iv) $\pi : M \longrightarrow_{\Sigma^*} N$,

(v) $\mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(M) = \mathcal{P}(\kappa) \cap \mathbf{\Sigma}^*(N)$.

Proof. The proof of the theorem is in two stages. The main difficulty is in showing that the maps are between the relevant reducts. What this amounts to is showing that the elements in the natural "strata" that one defines from the functions in Γ are actually those obtained by using the iterated definition of projectum over N. The "strata" referred to arise as the H_n in the following definition: Set $\omega\rho_n = H_n \cap \mathrm{On}$ where:

$$H_n = \begin{cases} \{[a, f] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\} & \text{if } \rho_M^n > \kappa, \\ H_M^n & \text{otherwise.} \end{cases}$$

(1) *H_n is transitive.*

Proof of (1). Trivial if $\rho_M^n < \kappa$. Suppose $[b, g] \in [a, f] \in H_n$. If $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$ then transitivity follows in the usual manner for Σ_0 ultrapowers. We may assume then that $g \in \Gamma_m$ for an $m \leq n$ and $\rho_M^{n+1} > \kappa$. Now show there is a $[b', g'] \in H_n$ with $[b', g'] = [b, g]$ by appealing to Lemma 1.28. $\dashv$ (1)

We thus have $N = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$ and that if $x \in H_M^n$, then $\pi(x) \in H_n$; hence if we interpret on the N-side the variables v_j^i as varying over H_i, then π respects types. Thus $\langle N, \in, A', B', H_0, H_1, H_2, \ldots\rangle$ is an $\mathcal{L}^*$-structure, the *pseudo-interpretation* of $\mathcal{L}^*$. Our task is then two-fold: to show that $\pi : M \longrightarrow_{\Sigma^*} N$ in this pseudo-interpretation, and then to show that it is actually the *correct* interpretation, that is, $H_n = H_N^n$ for any n.

(2) (Łoś's Theorem for Pseudo-Interpretations). *Let* $\varphi(v^{i_1}, \ldots, v^{i_k})$ *be a* $\Sigma_1^{(n)}$ *formula, where* $\omega\rho_M^{n+1} > \kappa$, *or a* $\Sigma_0^{(n)}$ *formula if* $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. *Let* $[a, f_j] \in H_{i_j}$ *with* $f_j \in \Gamma_{i_j}$, *with* $i_j \leq n$ *for* $1 \leq j \leq k$. *Then*

$$N \models \varphi(\overrightarrow{[a, f_j]}) \quad \Longleftrightarrow \quad \{u \mid M \models \varphi(f_1(u), \ldots, f_k(u))\} \in E_a.$$

Proof of (2).

Case 1. $\rho_M^{n+1} > \kappa$.

By induction on the complexity of φ. Note that $\{u \mid M \models \varphi(f_1(u), \ldots, f_k(u))\} \in M$ by Lemma 1.27. We shall only check here the quantifier step $\varphi \equiv \exists v^m\, \psi(v^m, v^{i_1}, \ldots, v^{i_k})$ where $m \leq n$, $\psi \in \Sigma_0^{(m)}$ and (2) is assumed to hold for ψ. The forward direction is quite straightforward: we assume $N \models \varphi(\overrightarrow{[a, f_j]})$; then there is a $[b, f_0] \in H_m$ with $a \subseteq b$, $f_0 \in \Gamma_m$ and $N \models \psi([b, f_0], [b, f_1^{ab}], \ldots, [b, f_k^{ab}])$. By the inductive hypothesis:

$$\{u \mid M \models \psi(f_0^{ab}(u), f_1^{ab}(u), \ldots, f_k^{ab}(u))\} \in E_b.$$

As $\operatorname{ran}(f_0) \subseteq H_M^m$ we have:

$$\{u \mid M \models \exists v^m \psi(v^m, f_1^{ab}(u), \ldots, f_k^{ab}(u))\} \in E_b$$

and so:

$$\{u \mid M \models \exists v^m \psi(v^m, f_1(u), \ldots, f_k(u))\} \in E_a.$$

Conversely, assume that $\{u \mid M \models \exists v^m \psi(v^m, f_1(u), \ldots, f_k(u))\} \in E_a$. By Lemma 1.28 there is a $g \in \Gamma$, $g : [\kappa]^{\operatorname{card}(a)} \longrightarrow H_M^m$ such that:

$$\{u \mid M \models \psi(g(u), f_1(u), \ldots, f_k(u))\} \in E_a.$$

By the inductive hypothesis, $N \models \psi([a, g], \overrightarrow{[a, f_j]})$. As $\omega\rho_M^{m+1} > \kappa$ and $\operatorname{ran}(g) \subseteq H_M^m$ we have an $[a, g] \in H_m$ and hence $N \models \exists v^m \psi(v^m, \overrightarrow{[a, f_j]})$. $\dashv$ *Case 1*

Case 2. $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$.

Note that the only difficulty in *Case 1* was in the converse direction, when we had to appeal to Lemma 1.28. Again we look only at a representative (now bounded) quantifier step: $\varphi \equiv \exists v^n \in u^n \psi(v^n, u^n, v^{i_1}, \ldots, v^{i_k})$ with (2) assumed proven for ψ. The forward direction is just as in the previous case, so we omit it. So suppose for some $f_0 \in \Gamma_n$

$$\{u \mid M \models \exists v^n \in f_0(u) \psi(v^n, f_0(u), f_1(u), \ldots, f_k(u))\} \in E_a.$$

We define a witnessing function g another way, making use of the fact, that by the definition of Γ_n in this case, that $f_0 \in H_M^n$:

$$g(u) = \begin{cases} \text{the } <_M\text{-least } w \in f_0(u) \text{ so that } (\psi(w, f_0(u), f_1(u), \ldots, f_k(u)))_M, \\ 0 \quad \text{if this } w \text{ does not exist.} \end{cases}$$

We may assume $n > 0$ (for otherwise (2) is trivially true in this case). Then, using the fact that the canonical well-order of M is Δ_1^M, we have that g is a $\mathbf{\Sigma}_0^{(n)}(M)$ subset of $\kappa \times \bigcup \mathrm{ran}(f_0) \in H_M^n$, and thus is an element of H_M^n. (This is true even if $n = 0$.) Then for some $X \in H_M^n$ we have $\{u \mid M \models \psi(g(u), f_0(u), f_1(u), \ldots, f_k(u))\} \supseteq X \in E_a$ and we can finish as before. $\dashv$ (2)

(3) (a) *If* $\rho_M^{n+1} > \kappa$ *then* $\pi : M \longrightarrow_{\Sigma_2^{(n)}} N$ *in the pseudo-interpretation.*
(b) *If* $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$ *then* $\pi : M \longrightarrow_{\Sigma_1^{(n)}} N$ *in the pseudo-interpretation.*

Proof of (3). Suppose $\psi \equiv \exists v^n \varphi(v^n, \vec{u}\,)$ where φ is $\Sigma_0^{(n)}$ (or $\Pi_1^{(n)}$ for part (a)). Let $\vec{x} \in M$, and assume that $N \models \exists v^n \varphi(v^n, \pi(\vec{x}\,))$. Let then $[a, f] \in H_n$, $f \in \Gamma_n$ be such that $N \models \varphi([a, f], \vec{x}\,)$. By *Case 1* (or *2* for (b)) above $\{u \mid M \models \varphi(f(u), \vec{x}\,)\} \supseteq X \in E_a$. Hence $M \models \exists v^n \varphi(v^n, \vec{x}\,)$ as $\mathrm{ran}(f) \subseteq H_M^n$. $\dashv$ (3)

We now briefly consider what happens below the critical point κ.

(4) (i) *Let* $\kappa \geq \rho_M^n$. *Then* $\pi : M \longrightarrow_{\Sigma_\omega^{(n)}} N$ *in the pseudo-interpretation.* (ii) $\pi : M \longrightarrow_{\Sigma^*} N$ *in the pseudo-interpretation.*

Proof of (4). (i) This is essentially because we have defined $H_n = H_M^n$ for such n, and $\pi{\restriction}H_M^n = \mathrm{id}{\restriction}H_M^n$ and thus the variables of type n are interpreted in the same domains. Suppose (4) failed for some least such n, as witnessed by some $\Sigma_k^{(n)}$ formula φ of least complexity. Note that φ is not the 'atomic' $\Sigma_1^{(n-1)}$ part of a $\Sigma_0^{(n)}$ formula (by definition of n, or else by (3)(b) if $\omega\rho_M^{n-1} > \kappa$). Elementary considerations show we can assume φ is then not atomic, nor of the form $\neg\psi, (\psi \wedge \chi)$ but is $\varphi \equiv \exists v^n \psi(v^n, \vec{u}\,)$. Suppose $N \models \exists v^n \psi(v^n, \pi(\vec{x}\,))$ for some $\vec{x} \in H_M^n$. Let $y \in H_n = H_M^n$ witness this. $N \models \psi(y, \pi(\vec{x}\,))$. As ψ is a simpler formula, and $\pi(y) = y$, we have $M \models \psi(y, \vec{x}\,)$. (ii) then follows from (i) and (3). $\dashv$ (4)

(5) *N is an acceptable J-structure, π is Q-preserving, and $H_n = J_{\rho_n}^{A'}$ for all $n < \omega$.*

Proof of (5). The property of being an acceptable J-structure is a Q-condition (cf. [46, 1.21]). If $\pi : M \longrightarrow N$ is a standard Σ_0 ultrapower, then in fact π is Σ_0 and cofinal into N and such maps preserve Q properties. However otherwise π is at least $\Sigma_2^{(0)}$ (and *a fortiori* Σ_2)-preserving by (3)(a) which suffices. The same reasoning works level-by-level: if $\rho_M^{n+1} > \kappa$ then $\pi{\restriction}H_M^n : \langle H_m^n, H_m^n \cap A, H_m^n \cap B\rangle \longrightarrow_{\Sigma_2} \langle H_n, A' \cap H_n, B' \cap H_n\rangle$; thus H_n is a J-structure constructed from A'. If $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$ then $\pi{\restriction}H_M^n$: $\langle H_m^n, H_m^n \cap A, H_m^n \cap B\rangle \longrightarrow_{\Sigma_0} \langle H_n, A' \cap H_n, B' \cap H_n\rangle$ which is moreover cofinal, again, as it is a standard Σ_0 ultrapower map (Lemma 1.31), so the same is true at this level. For $\rho_M^n \leq \kappa$ it is trivial. $\dashv$ (5)

If we can now show the pseudo-interpretation is the correct one, we shall have fulfilled all the clauses (i)–(iv) of the theorem. The following computation of the size of the relevant ordinals finishes the task: it only remains to show that the projecta of N correspond to the ordinals ρ_n. We divide into the cases: above and below the measurable κ.

(6) $\rho_m = \rho_N^m$ *if* $\kappa \leq \omega\rho_M^m$.

Proof of (6). By induction on m. This is trivial for $m = 0$.

(6a) $\langle H_m, B\rangle$ *is amenable for* $B \in P(\omega\rho_m) \cap \mathbf{\Sigma}_1^{(m-1)}(N)$. *Hence* $\rho_m \leq \rho_N^m$.

Proof of (6a). Suppose $B \subseteq \omega\rho_m$ and is $\Sigma_1^{(m-1)}(N)$ in the parameter $[a, f]$, say $B(x) \Longleftrightarrow N \models \varphi(x, [a, f])$ with $\varphi \in \Sigma_1^{(m-1)}$. Let $w = [b, g] \in H_m$. (Without loss of generality we shall assume $a = b$ here.) We require $B \cap w \in H_m$. Define h by $h(y^m, v^0) = \{t^m \in y^m \mid M \models \varphi(t^m, v^0)\}$. Thus $h(y^m, v^0)$ is a $\Sigma_0^{(m)}$ function of value type m and thus is good. Hence it is in Γ. h is defined for all y^m, v^0 and $k : [\kappa]^{\mathrm{card}(a)} \longrightarrow H_M^m$ where $k(u) = h(g(u), f(u))$ (note that $\mathrm{ran}(k)$ is indeed contained in H_M^m, as each element of the form $h(y^m, v^0)$ is *bounded* $\Sigma_0^{(m)}$). k is in Γ being a composition of such.

If $\kappa < \omega\rho_M^{m+1}$ we may conclude that $k \in \Gamma_m$ as $\mathrm{ran}(k) \subseteq H_M^m$. If $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$ we need to see that $k \in H_M^m$ to infer this. Notice that $k(u) \subseteq g(u)$ and thus $K = \{\langle u, z\rangle \mid u \in k(z)\}$ is a $\Sigma_0^{(m)}(M)$ of $\{\langle u, z\rangle \mid u \in g(z)\}$ and the latter is in H_M^n. Hence so is K and we have that k is thus rudimentary over $M^{m,p}$ where p is a suitable choice of parameters including those used to define f, g. Hence $k \in H_M^m$. As

$$[\kappa]^{\mathrm{card}(a)} = \{u \mid M \models \forall t^m \in g(u)[\varphi(t^m, f(u)) \longleftrightarrow t^m \in k(u)]\} \in E_a$$

by Łoś we have $N \models \forall t^m \in [a, g](\varphi(t^m, [a, f]) \longleftrightarrow t^m \in [a, k])$. Hence

$$[a, k] = [a, g] \cap \{t^m \mid N \models \varphi(t^m, [a, f])\} = w \cap A \in H_m. \qquad \dashv \text{(6a)}$$

(6b) *There is an* $A \subseteq \rho_m \cap \mathbf{\Sigma}_1^{(m-1)}(N)$ *such that* $A \notin N$. *Hence* $\rho_m \geq \rho_N^m$.

Proof of (6b).

Case 1. $\kappa < \omega\rho_M^{m+1}$.

Suppose $\bar{A}$ is $\Sigma_1^{(m-1)}(M)$-definable in a parameter $\overline{p}$, but with $\bar{A} \cap \rho_M^m \notin M$, for example we can find such using $\bar{p} \in P_M^m$.

Let A be $\Sigma_1^{(m-1)}(N)$ in $\pi(\bar{p}) = p$ by the same formula φ say. We show that $A \cap \omega\rho_m \notin M$. Suppose otherwise. Let $A \cap \omega\rho_m = [a, f] \in N$. Then $N \models \forall x^m(x^m \in [a, f] \longleftrightarrow \varphi(x^m, p))$.

This is a $\Pi_1^{(m)}$, in p, formula, and by Łoś's Theorem (2):

$$\{u \mid M \models \forall x^n(x^n \in f(u) \longleftrightarrow \varphi(x^n, p))\} \in E_a.$$

So there is a $u \in [\kappa]^{\mathrm{card}(a)}$ with $f(u) \cap \rho_M^n = \overline{A} \cap \rho_M^n$—a contradiction.

Case 2. $\omega\rho_M^{m+1} \leq \kappa < \omega\rho_M^m$.

We alert the reader to this more problematic case. Here we explicitly use the weak amenability property of the extender to ensure that the map is *cofinal* at the level of this crossing projectum structure.

Let $p \in P_M^{m+1}$, and set $\overline{A} = A^{p(m-1)}_{M^{m-1,p\restriction m-1}}$. Then $\bar{A}$ is $\Sigma_1^{(m-1)}(M)$ in p; suppose that it is defined by φ. Let $\widetilde{A}$ be defined over N in the same way using $\pi(p)$. $M^{m,p\restriction m} = \langle H, \overline{A}\rangle$ is amenable and $\pi\restriction H_M^m$ is Σ_0 and cofinal into H_m. As π is $\Sigma_0^{(m)}$-preserving into N, we conclude that for any $x \in H_M^m$ $\pi(x \cap \overline{A}) = \pi(x) \cap \widetilde{A}$. Hence:

(i) $\langle H_m, \widetilde{A}\rangle$ is amenable; (ii) $\pi\restriction H_M^m : M^{m,p\restriction m} \longrightarrow_{\Sigma_0} \langle H_m, \widetilde{A}\rangle$ cofinally, and hence is Σ_1-elementary.

However now we see that ρ_m must equal ρ_N^m; for suppose $\rho_m < \rho_N^m$: then $\widetilde{A} \cap H_m \in N$. As $p \in P_M^{m+1}$ we can pick $B \in \mathcal{P}(\omega\rho_M^{m+1}) \cap \Sigma_1(M^{m,p\restriction m})$ in $p(m)$, with $B \notin M$. As $\pi\restriction\kappa = \mathrm{id}\restriction\kappa$ we have that $B \in \mathcal{P}(\kappa) \cap \boldsymbol{\Sigma}_1(\langle H_m, \widetilde{A}\rangle)$. Since $\langle H_m, \widetilde{A}\rangle \in N$, then so is B. By the amenability of our extender E (and acceptability of our structures) $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$; hence $B \in M$—a contradiction. $\dashv$ (6b)&(6)

In the course of proving (6b) we also showed:

(7) $\pi``P_M^m \subseteq P_N^m$ if $\omega\rho_M^m > \kappa$.

1.33 Remark. The argument of *Case 2*, where $\omega\rho_M^{m+1} \le \kappa < \omega\rho_M^m$, shows that the weak amenability of E implies that $\omega\rho_N^{m+1} \le \rho_{m+1}$. Without the requirement of amenability we should only be able to prove the inequality $\omega\rho_N^m \ge \rho_m$ and not its reverse. (Further only that (7) holds for those m with $\omega\rho_M^{m+1} > \kappa$.) The assumption of amenability then ensures the $\Sigma_0^{(m)}$-*cofinality* of the ultrapower map (and hence that it is in fact $\Sigma_1^{(n)}$-preserving). The assumption of Σ_1-amenability of M will enable us to prove (i) the equality $\omega\rho_N^m = \rho_m$; (ii) as well as that of the projecta below the measurable; (iii) finally, the full Σ^*-elementarity of the map π.

1.34 Remark. Still considering the above *Case 2* where $\omega\rho_M^{m+1} \le \kappa < \omega\rho_M^m$, we can vary assumptions on E, or M and still get that $\pi : M \longrightarrow_{\Sigma_0^{(n)}} N$ cofinally: if we assume E is Σ_1-amenable this suffices (see [62, 3.2.6]); similarly if $R_M^m \neq \varnothing$ then we can get the same conclusion (as well as the bonus that $\pi``R_M^m \subseteq R_N^m$ see [62, 3.2.4]).

First we shall need a preliminary lemma:

1.35 Lemma. *Suppose E is close to M, i.e., E is weakly amenable and Σ_1-amenable over M, and $\omega\rho_M^{n+1} \le \kappa < \omega\rho_M^n$. Let $B \in \boldsymbol{\Sigma}_1^n(N) \cap \mathcal{P}(\kappa)$. Then $B \in \boldsymbol{\Sigma}_1^n(M)$.*

Proof. Suppose for some $\varphi \in \Sigma_0^{(n)}$ and all ξ,

$$\xi \in B \iff \exists v^n \varphi(v^n, \xi, \pi(f)(a)).$$

By weak amenability,

$$\xi \in B \iff \exists u \in H^n_M \ \exists v^n \in \pi(u)\varphi(v^n, \xi, \pi(f)(a)),$$

and by Łoś for $\Sigma_0^{(n)}$,

$$\xi \in B \iff \exists u \in H^n_M(\{w \mid \exists v^n \in u\varphi(v^n, \xi, f(w))\} \in E_a).$$

Notice that the set $X_\xi = \{w \mid \exists v^n \in u\ \varphi(v^n, \xi, f(w))\}$ is a $\Sigma_0^{(n)}$ subset of $\mathrm{dom}(f) \in H^n_M$ (whichever n satisfies $f \in \Gamma_n$) and hence $X_\xi \in H^n_M$. By the Σ_1-amenability assumption, "$X \in E_a$" is $\mathbf{\Sigma}_1(M)$. Thus $B \in \mathbf{\Sigma}_1^{(n)}(M)$. ⊣

(8) *Suppose $\kappa \geq \omega\rho^m_M$. Then* (i) $\rho_m = \rho^m_N$, *and hence $H^m_M = H^m_N$; and* (ii) $\mathbf{\Sigma}_1^{(m)}(M) \cap \mathcal{P}(H^m_M) = \mathbf{\Sigma}_1^{(m)}(N) \cap \mathcal{P}(H^m_N)$.

Proof of (8). Let n be such that $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$. We prove (8) by induction on $m \geq n+1$. By the last lemma if $m = n+1$, or the inductive hypothesis otherwise:

$$\mathbf{\Sigma}_1^{(m-1)}(M) \cap \mathcal{P}(\omega\rho^m_M) = \mathbf{\Sigma}_1^{(m-1)}(N) \cap \mathcal{P}(\omega\rho^m_M).$$

For (i), that $\rho_m = \rho^m_N$, now follows directly from the fact that (a) $\forall\gamma < \omega\rho^m_M a \in \mathbf{\Sigma}_1^{(m-1)}(N) \cap \mathcal{P}(\gamma)$ then $a \in H^m_M = H^m_N$; and (b) if $A \in \mathbf{\Sigma}_1^{(m-1)}(M) \cap \omega\rho^m_M, A \notin M$ then $A \notin N$.

For (ii), by Lemma 1.6 we have, substituting either M or N for T, that $C \in \mathbf{\Sigma}_1^{(m)}(T) \cap \mathcal{P}(H^m_T)$ if and only if $C \in \mathbf{\Sigma}_1(\langle H^m_T, Q\rangle)$ for some $Q \in \mathbf{\Sigma}_1^{(m-1)}(T) \cap \mathcal{P}(H^m_T)$. By the inductive hypothesis (or again Lemma 1.35 if $m = n+1$), we have such a $Q \in \mathbf{\Sigma}_1^{(m-1)}(M) \cap \mathbf{\Sigma}_1^{(m-1)}(N)$. Since (i) shows $H^m_M = H^m_N$, we have, for $C \subseteq H^m_M$:

$$C \in \mathbf{\Sigma}_1^{(m)}(M) \iff C \in \mathbf{\Sigma}_1(\langle H^m_M, Q\rangle) \iff C \in \mathbf{\Sigma}_1^{(m)}(N).$$

⊣ (6&8)

We can now conclude that the ultrapower embedding is properly Σ^*-preserving.

(9) $\pi : M \longrightarrow_{\Sigma^*} N$ *in the standard interpretation.*

Proof of (9). (4) shows that $\pi : M \longrightarrow_{\Sigma^*} N$ in the pseudo-interpretation. (6) and 8(i) together show that this interpretation is the correct one. ⊣ (9)

1.36 Lemma. π"$P^*_M \subseteq P^*_N$.

Proof. Given (7) it suffices to show π"$P^m_M \subseteq P^m_N$ if $\omega\rho^m_M \leq \kappa$: however if $p \in P^m_M$ is chosen, there is a $C \in \Sigma_1^{(m-1)}(M)$ in p with $C \cap \omega\rho^m_M \notin M$. We have that π is Σ^*-elementary, and $\pi{\restriction}\kappa = \mathrm{id}{\restriction}\kappa$, hence if $\widetilde{C}$ has the same $\Sigma_1^{(m-1)}$ definition over N in $\pi(p)$, we shall have $C \cap \omega\rho^m_M = \widetilde{C} \cap \omega\rho^m_N \notin N$. Hence $\pi(p) \in P^m_N$. ⊣

(10) *For* $\langle a, f\rangle \in D, [a,f] = \pi(f)(a)$.

Proof of (10). All that we have left to show is (iii) in Definition 1.23, that N is the closure of $\mathrm{ran}(\pi) \cup \nu$ under good $\Sigma_1^{(n)}$ functions, for $\omega\rho_M^{n+1} > \kappa$. We've extended π to such functions so it is enough to show $[a,f] = \pi(f)(a)$, for $f \in \Gamma$. If $f \in M$ this would be a standard argument (see, e.g., [46, 8.4, Claim 4]. So suppose $f \in \Gamma_n$, with $\mathrm{dom}(f) = [\kappa]^k$, for some n with $\rho_M^{n+1} > \kappa$, and let G be a functionally absolute $\Sigma_1^{(n)}$ definition of the graph of f in some parameter $p \in M$. Let $\widetilde{G}$ be given by the same definition over N in $\pi(p)$. Then $[\kappa]^k = \{u \mid G(f(u), u, p)\} \in E_a$. Hence $\widetilde{G}([a,f], a, \pi(p))$. Then $\pi(f)(a) = [a,f]$ as $\widetilde{G}$ is the graph of $\pi(f)$. $\dashv$ (10)

We are only left with proving the last clause (v) of the theorem that $\mathcal{P}(\kappa)\cap\boldsymbol{\Sigma}^*(M) = \mathcal{P}(\kappa)\cap\boldsymbol{\Sigma}^*(N)$. That $\subseteq$ holds is given by (9). For $\supseteq$ we argue as follows: if for all n $\rho_M^n > \kappa$ then $\mathcal{P}(\kappa) \cap \boldsymbol{\Sigma}^*(N) \subseteq \mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap M$. Otherwise for some n we have $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Lemma 1.35 is not quite sufficient, as we need to prove a result for further $m \geq n+1$.

1.37 Lemma. *Let n be such that $\omega\rho_M^{n+1} \leq \kappa < \omega\rho_M^n$. Let $\varphi(v^m, v^{m-1}, \ldots, v^n, v^{n-1}, \ldots, v^0)$ be a $\Sigma_1^{(m)}$ formula for some $m \geq n$. Let $a \in [\nu]^{<\omega}$ and $f^{n-1}, \ldots, f^0$ be such that $f^j \in \Gamma_j, f^j : [\kappa]^{|a|} \longrightarrow H_M^j$ for $j < n$. Then there is a $\Sigma_1^{(m)}$ formula $\varphi'(v^m, v^{m-1}, \ldots, v^n)$ which is effectively obtainable from φ, the $\boldsymbol{\Sigma_1}(M)$ definition of E_a, and the $\boldsymbol{\Sigma_1^{(n-1)}}(M)$ definitions of the f^j, with the property that $(\forall x^m \cdots \forall x^{n+1} \in M)$ $(\forall f \in \Gamma_n$, with $\mathrm{dom}(f) = [\kappa]^{|a|})$*

$$\begin{aligned} &N \models \varphi(x^m, \ldots, x^{n+1}, [a,f], [a, f^{n-1}], \ldots, [a, f^0]) \\ &\iff \quad M \models \varphi'(x^m, \ldots, x^{n+1}, f). \end{aligned}$$

Proof. This is by induction on $m \geq n$. To simplify notation, we shall assume the $\boldsymbol{\Sigma_1^{(n-1)}}(M)$ definitions of the f^j, and the $\boldsymbol{\Sigma_1}(M)$ definition of E_a, are in face lightface, so parameter free. (It is routine to carry these parameters along otherwise.) We first consider the case that $m = n$. (This means that the variables $v^m, \ldots, v^{n+1}$ are absent from φ.) Let $\varphi(v^n, v^{n-1}, \ldots, v^0) \equiv \exists w^n \psi(w^n, v^n, v^{n-1}, \ldots, v^0)$, where $\psi \in \Sigma_0^{(n)}$. Now for the requisite kind of f and f^j:

$$\begin{aligned} &N \models \exists w^n \psi(w^n, [a,f], [a, f^{n-1}], \ldots, [a, f^0]) \\ &\iff \quad (\exists y^n \in H_M^n) N \models \exists w^n \in \pi(y^n) \\ &\qquad\qquad \psi(w^n, [a,f], [a, f^{n-1}], \ldots, [a, f^0]) \end{aligned}$$

(as $\pi{\restriction}H_M^n$ is cofinal into H_n); now by Łoś we have that:

$$\begin{aligned} &(\exists y^n \in H_M^n)\{u \mid M \models \exists w^n \in y^n \psi(w^n, f(u), f^{n-1}(u), \ldots, f^0(u))\} \in E_a \\ &\iff \quad (\exists y^n \in H_M^n)(\exists z^n \in H_M^n)(z^n \in E_a \wedge \\ &\qquad\qquad z^n = \{u \mid M \models \exists w^n \in y^n \psi(w^n, f(u), f^{n-1}(u), \ldots, f^0(u))\}). \end{aligned}$$

This last line can be seen to be a $\Sigma_1^{(n)}$ formula after substituting the $\Sigma_1^{(n-1)}$ definitions of the good $f^j(u)$ for v^j, and is our φ'.

For $m \geq n+1$ the argument is similar, but simpler: let $\varphi \equiv \exists w^m \psi$ with $\psi \in \Sigma_0^{(m)}$. We suppose that we have proven the effectiveness of the translation procedure for $\Sigma_1^{(n-1)}$ and hence for the atomic components of ψ; further that the result works inductively through the connectives, trivially enough. The argument for bounded quantifiers is similar to that for unbounded; we thus concentrate here on this one step, and assume as inductive hypothesis that the result holds for ψ. Now, again assuming we have the requisite functions f and f^j and:

$$\begin{aligned} & N \models \varphi(x^m, \ldots, x^{n+1}, [a,f], [a,f^{n-1}], \ldots, [a,f^0]) \\ \Longleftrightarrow \quad & (\exists w^m \in H_M^m) N \models \psi(w^m, x^m, \ldots, x^{n+1}, [a,f], [a,f^{n-1}], \ldots, [a,f^0]) \\ \Longleftrightarrow \quad & (\exists w^m \in H_M^m) M \models \psi'(w^m, x^m, \ldots, x^{n+1}, f) \\ \Longleftrightarrow \quad & M \models \exists w^m \psi'(w^m, x^m, \ldots, x^{n+1}, f), \end{aligned}$$

using that $H_M^n = H_n$ at the first equivalence, and the inductive hypothesis at the second. $\dashv$

This now can be used to show that $\mathcal{P}(\kappa) \cap \boldsymbol{\Sigma_1^{(m)}}(M) \supseteq \mathcal{P}(\kappa) \cap \boldsymbol{\Sigma_1^{(m)}}(N)$. Hence the last clause (v) is now proven and the proof of the Ultrapower Theorem is complete. $\dashv$ 1.32.

1.3. Pseudo-Ultrapowers

We develop a theory of *pseudo-ultrapowers*. Ordinary ultrapowers (both coarse and fine) can be considered as special types of pseudo-ultrapowers. The latter are used in many combinatorial constructions, and particularly come into their own in the proof of the Covering Lemma for fine structural inner models, or for Global Square, $\square$, or indeed many areas where combinatorial arguments involve fine structure.

One can think of pseudo-ultrapowers as an attempt to develop an answer to the following question. Suppose we have acceptable structures $\overline{Q}$, Q with a map $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$ cofinally between them. Suppose further that $\overline{Q} = J_{\overline{\nu}}^{\overline{A}}$, $Q = J_\nu^A$and moreover that $\overline{Q}$ is extended by an acceptable $\overline{M} = \langle J_{\overline{\beta}}^{\overline{A}}, \overline{B} \rangle$ where either $\overline{\nu} = \overline{\beta}$ or else $\overline{\nu}$ is a cardinal in $\overline{M}$. Can we find a structure M and an extension of σ to a structure preserving map $\widetilde{\sigma} \supseteq \sigma, \widetilde{\sigma} : \overline{M} \longrightarrow M$, where $M = \langle J_\beta^A, B \rangle$? We should hope that the extended map $\widetilde{\sigma}$ would be sufficiently fine-structural. What we should like is that $\widetilde{\sigma}$ should be $\Sigma_1^{(n)}$ for any n with $\omega\rho_{\overline{M}}^n \geq \overline{\nu}$. This may not be always possible for every such n, but with certain conditions on $\overline{M}$ we can achieve this. Weakening these conditions could allow us to weaken the conclusion: for example, we could have this level of elementarity if $\omega\rho_{\overline{M}}^{n+1} \geq \overline{\nu}$.

Requiring additionally that $\overline{\nu}$ be regular in $\overline{M}$ will ensure that the extended map takes $\overline{\nu}$ cofinally into its image. This requirement is a feature of the Global $\square$ proof where we use a pseudo-ultrapower that takes just enough functions to preserve the regularity of $\overline{\nu}$ in the sense of $\overline{M}$ whilst at the same time preserving the fact that it is $\boldsymbol{\Sigma}_1^{(n)}(\overline{M})$ singularised for some $n < \omega$.

1.38 Definition. $\Gamma = \Gamma_{\overline{M},\overline{\nu}} =_{\mathrm{df}}$

$$\begin{aligned}&\{f \mid (\mathrm{dom}(f) \in \overline{Q} \wedge \mathrm{ran}(f) \subseteq \overline{M})\\ &\quad \wedge((f \in \boldsymbol{\Sigma}_1^{(n)}(\overline{M}) \wedge \omega\rho_{\overline{M}}^{n+1} \geq \overline{\nu}) \vee (f \in H_{\overline{M}}^{n} \wedge \omega\rho_{\overline{M}}^{n} \geq \overline{\nu}))\}.\end{aligned}$$

We now define a term model, whose domain is:

$$D = \{\langle a, f\rangle \mid f \in \Gamma_{\overline{M},\overline{\nu}} \wedge a \in \sigma(\mathrm{dom}(f))\}.$$

We define a pseudo-epsilon relation e and a congruence I by:

$$\begin{aligned}\langle a, f\rangle e \langle b, g\rangle &\iff \langle a, b\rangle \in \sigma(\{\langle u, v\rangle \mid f(u) \in g(v)\}),\\ \langle a, f\rangle I \langle b, g\rangle &\iff \langle a, b\rangle \in \sigma(\{\langle u, v\rangle \mid f(u) = g(v)\}),\\ \dot{C}(\langle a, f\rangle) &\iff a \in \sigma(\{u \mid f(u) \in C\}) \text{ for } C = \overline{A}, \overline{B}.\end{aligned}$$

We set $\mathbb{D} = \mathbb{D}_{\bar{Q},\bar{\sigma},\bar{M}} = \langle D, I, e, \dot{A}, \dot{B}\rangle$.

1.39 Lemma. *Let $\varphi(v_0, \ldots, v_k)$ be a Σ_0 formula. Let $\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle \in D$, and let $b = \langle a_0, \ldots a_k\rangle$. Then*

$$\begin{aligned}&\mathbb{D} \models \varphi(\langle a_0, f_0\rangle, \ldots, \langle a_k, f_k\rangle)\\ &\quad \iff \quad b \in \sigma(\{\langle u_0, \ldots u_k\rangle \mid M \models \varphi(f_0(u_0), \ldots, f_k(u_k))\}).\end{aligned}$$

There is very little difference between the proof of this and the previous Lemma 1.29. By this, we can see that I is an identity for $\mathbb{D}$ and $\mathbb{D}$ models Extensionality. We shall for the purposes of this development assume that e is well-founded. We thus can define a transitivisation map

$$[\,] : D \longrightarrow M;$$

$$[x] = [y] \iff x\, I\, y; \qquad [x] \in [y] \iff x\, e\, y;$$

$$A_M([x]) \iff \dot{A}(x); \qquad B_M([x]) \iff \dot{B}(x).$$

With Σ_0-elementarity guaranteed by the last lemma we define:

1.40 Definition. $\widetilde{\sigma}$, the canonical extension of σ, is defined by: $\widetilde{\sigma}(x) = [\langle 0, \{\langle 0, x\rangle\}\rangle]$; M is the pseudo-ultrapower of $\overline{M}$ by σ.

1.41 Lemma. $\widetilde{\sigma}{\restriction}\overline{Q} = \sigma$.

Proof. Set $\widetilde{Q} = \bigcup \widetilde{\sigma}``\overline{Q}$. Then $\widetilde{Q}$ is transitive. We shall verify: $\widetilde{Q} = \{[\langle a, f\rangle] \mid f \in \overline{Q}\}$. For ($\supseteq$), pick a, f as specified, and set $b = \mathrm{ran}(f)$. As $f \in \overline{Q}$, so is b. Then $[a, f] \in [0, \langle 0, b\rangle]$ as can be directly verified from the Łoś lemma above. For ($\subseteq$), let $x \in \widetilde{\sigma}(y)$ with $y \in \overline{Q}$. If $x = [a, f]$ with $f \in \Gamma$, $a \in \sigma(\mathrm{dom}(f))$, then $a \in \sigma(\{u \in \mathrm{dom}(f) \mid f(u) \in y\})$. However if we set $f' =_{\mathrm{df}} f \cap (\mathrm{dom}(f) \times y)$, (where $\mathrm{dom}(f) \times y$ is a member of $\overline{Q}$) when f itself is not in $\overline{M}$, then this is at most $\boldsymbol{\Sigma}_1^{(n)}(\overline{M})$ where $\omega\rho_{\overline{M}}^{n+1} \geq \bar{\nu}$; it thus lies in $\overline{M}$ and hence by acceptability, in $\overline{Q}$. It is easy to verify by Łoś again that $[a, f] = [a, f']$, and we have thus verified ($\subseteq$).

Define $\sigma' : \widetilde{Q} \longrightarrow Q$ by $\sigma'([a, f]) = \sigma(f)(a)$. We have just shown that $\mathrm{dom}(\sigma') = \widetilde{Q}$. σ' is well-defined:

$$\begin{aligned}
[a, f] = [b, g] \quad &\Longleftrightarrow \quad \langle a, b\rangle \in \sigma(\{\langle u_0, u_1\rangle \mid f(u_0) = g(u_1)\}) \\
&\qquad\quad = \{\langle u_0, u_1\rangle \in \sigma(\mathrm{dom}(f) \times \mathrm{dom}(g)) \mid \sigma(f)(u_0) \\
&\qquad\quad = \sigma(g)(u_1)\} \\
&\Longleftrightarrow \quad \sigma(f)(a) = \sigma(g)(b).
\end{aligned}$$

A similar pair of equivalences show that $[a, f] \in [b, g] \iff \sigma(f)(a) \in \sigma(g)(b)$, and hence σ' is $\in$-preserving.

σ' is onto Q: this is clear, since if $v \in Q$ is arbitrary and $v \in \sigma(x)$ for some $x \in \overline{Q}$ (as σ is cofinal) then let $f \in \overline{Q}$ be the constant function $f : \gamma \longrightarrow x$ with constant value x with $\gamma < \bar{\nu}$; then clearly $v = \sigma(f)(\alpha)$ for some $\alpha < \sigma(\gamma)$. Thus σ' can only be the identity and $\widetilde{\sigma}(x) = [0, \{\langle 0, x\rangle\}] = \sigma(x)$. ⊣

The following theorem gives the basic preservation properties of pseudo-ultrapowers. The reader should compare this with Theorem 1.32.

1.42 Theorem (Pseudo-Ultrapower Theorem). *Let $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ be the canonical extension of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. Then*

(i) *$\widetilde{\sigma}$ is Q-preserving, M is an acceptable end extension of Q, and $M = \{\widetilde{\sigma}(f)(u) \mid u \in \sigma(\mathrm{dom}(f)), f \in \Gamma\}$.*

(ii) (a) *$\widetilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving for n with $\omega\rho_{\bar{M}}^{n} \geq \bar{\nu}$;*

(b) *$\widetilde{\sigma}$ is $\Sigma_2^{(n)}$-preserving for n with $\omega\rho_{\bar{M}}^{n+1} \geq \bar{\nu}$.*

Proof. We stratify the functions that will be responsible for the various projectum levels of the target structure M, which we shall name as H_n at first, later demonstrating that they have the correct interpretation. The reader will thus see that the tactic is analogous to that of the previous fine-structural ultrapower result.

Define

$$\begin{aligned}
\Gamma_n &= \begin{cases} \{f \in \Gamma \mid \mathrm{ran}(f) \subseteq H_{\bar{M}}^{n}\} & \text{if } \omega\rho_{\bar{M}}^{n+1} \geq \bar{\nu}, \\ \{f \in \Gamma \mid \mathrm{ran}(f) \in H_{\bar{M}}^{n}\} & \text{if } \omega\rho_{\bar{M}}^{n+1} < \bar{\nu} \leq \omega\rho_{\bar{M}}^{n}, \end{cases} \\
H_n &= \{[\langle a, f\rangle] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\} \quad \text{if } \Gamma_n \text{ is defined}, \\
\omega\rho_n &= H_n \cap \mathrm{On}.
\end{aligned}$$

Note (i) If $\omega\rho_{\bar{M}}^{n+1} < \bar{\nu} \leq \omega\rho_{\bar{M}}^{n}$ and $f \in \Gamma_n$ then $f \in \bar{M}$.
Then as for standard ultrapowers:

(1) *H_n is transitive.*

(2) Let $\overrightarrow{\langle\langle a_i, f_i\rangle\rangle} \in {}^m D$, where $f_i \in \Gamma_{j_i}$ and $j_i \leq n$, let $\varphi(v_1^{j_1}, \ldots v_m^{j_m}) \in \Sigma_0^{(n)}$ where $\omega\rho_{\bar{M}}^{n} \geq \bar{\nu}$ (or $\Sigma_1^{(n)}$ if $\omega\rho_{\bar{M}}^{n+1} \geq \bar{\nu}$). Then

$$M \models \varphi[\overrightarrow{[a_i, f_i]}] \iff \vec{a_i} \in \sigma(\{\vec{u} \mid M \models \varphi[\overrightarrow{f_i(\vec{u})}]\}).$$

This directly yields the following as for fine ultrapowers (but only in the pseudo-interpretation):

(3) *$\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$ for any n with $\omega\rho_{\bar{M}}^{n} \geq \bar{\nu}$, and $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_2^{(n)}} M$ for any n with $\omega\rho_{\bar{M}}^{n+1} \geq \overline{\nu}$.*

(4) M is acceptable. $\widetilde{\sigma}$ is Q-preserving, and $H_n = |J_{\rho_n}^{A'}|$ where $|M| = |J_{\alpha'}^{A'}|$, for n with $\omega\rho_{\bar{M}}^{n} \geq \bar{\nu}$.

Proof of (4). If $\rho_{\bar{M}}^1 \geq \bar{\nu}$ then by (3) $\widetilde{\sigma}$ is at least Σ_2-preserving; if $\rho_{\bar{M}}^1 < \bar{\nu}$ then $\widetilde{\sigma}$ is actually a cofinal map (it is using only functions f *in*: M see the Note above). In either case Q properties are preserved, and acceptability is such. The last part follows just as in (5) of Theorem 1.32, again noting that $\widetilde{\sigma}$ is cofinal into H_n if $\omega\rho_{\bar{M}}^{n} \geq \overline{\nu} > \omega\rho_{\bar{M}}^{n+1}$, since only functions $f \in \Gamma_n \subseteq |\bar{M}|$ are used. $\dashv$ (4)

(5) $\rho_m = \rho_M^m$ if $\bar{\nu} \leq \omega\rho_{\bar{M}}^{m+1}$; $\rho_m \leq \rho_M^m$ if $\nu \leq \omega\rho_{\bar{M}}^{m}$.

Proof of (5). This is by induction on m; for $m = 0$ it is trivial. We imitate the corresponding proofs of (6) in Theorem 1.32.

(5a) *$\langle H_m, A\rangle$ is amenable for $A \in P(\omega\rho_m) \cap \mathbf{\Sigma}_1^{(m-1)}(M)$. Hence $\rho_m \leq \rho_M^m$.*

Proof of (5a). As before suppose $A \subseteq \omega\rho_m$ and is $\Sigma_1^{(m-1)}(M)$ in the parameter $[a, f]$, say $A(x) \iff N \models \varphi(x, [a, f])$ with $\varphi \in \Sigma_1^{(m-1)}$. Let $w = [b, g] \in H_m$ (again without loss of generality we shall assume $a = b$ here, and similarly by enlarging domains if need be, that $\mathrm{dom}(f) = \mathrm{dom}(g)$ (it suffices given f, g, a, b as here to define functions f', g' with $\mathrm{dom}(f') = \mathrm{dom}(g') = \mathrm{dom}(f) \times \mathrm{dom}(g)$ and further amalgamate to get $c = \langle a, b\rangle$ etc.). We require $A \cap w \in H_m$. This time define

$$h(y^m, v^0) = \{t^m \in y^m \mid \bar{M} \models \varphi(t^m, v^0)\}.$$

Thus $h(y^m, v^0)$ is a $\Sigma_0^{(m)}$ function of value type m and thus is in Γ, being good. We may now define $k : \mathrm{dom}(g) \longrightarrow H_M^m$ where $k(u) = h(g(u), f(u))$.

If $\bar{\nu} \leq \omega\rho_M^{m+1}$ we may conclude that $k \in \Gamma_m$. If $\omega\rho_M^{m+1} < \bar{\nu} \leq \omega\rho_M^m$ we again have that k is $\Sigma_0^{(m)}$-definable in parameters (note now that $g \in H_M^m$) and we conclude that $k \in H_M^m$ for the same reasons. If

$$\mathrm{dom}(g) = \{u \mid \overline{M} \models \forall t^m \in g(u)[\varphi(t^m, f(u)) \longleftrightarrow t^m \in k(u)]\}$$

and as $a \in \sigma(\mathrm{dom}(g))$, by Łoś's theorem we have:

$$M \models \forall t^m \in [a, g](\varphi(t^m, [a, f]) \longleftrightarrow t^m \in [a, k]).$$

Hence $[a, k] \cap w = [a, g] \cap \{t^m \mid M \models \varphi(t^m, [a, f])\} = A \cap w \in H_m$. $\dashv$ (5a)

(5b) *Assume $\bar{\nu} \leq \omega\rho_{\bar{M}}^{m+1}$. Then there is an $A \subseteq \omega\rho_m \cap \mathbf{\Sigma}_1^{(m-1)}(M)$ such that $A \notin M$. Hence $\rho_m \geq \rho_M^m$.*

Proof of (5b). Suppose $\bar{A}$ is $\Sigma_1^{(m-1)}(\overline{M})$-definable in a parameter $\overline{p}$, but with $\bar{A} \cap \omega\rho_{\overline{M}}^m \notin \overline{M}$, for example we can find such using $\bar{p} \in P_{\overline{M}}^m$. Let A be $\Sigma_1^{(m-1)}(M)$ in $\pi(\bar{p}) = p$ by the same formula φ, say. We show that $A \cap \omega\rho_m \notin M$. Suppose otherwise. Let $A \cap \omega\rho_m = [a, f] \in M$. Then, in the pseudo-interpretation, $M \models \forall x^m(x^m \in [a, f] \longleftrightarrow \varphi(x^m, p))$. By Łoś's theorem $a \in \sigma(w)$ where

$$w = \{u \mid \overline{M} \models \forall x^m(x^m \in f(u) \longleftrightarrow \varphi(x^m, \bar{p}))\}.$$

So $w \neq \varnothing$ so there is a u with $f(u) = \bar{A} \cap \rho_{\overline{M}}^m = A \cap \rho_{\overline{M}}^m$—a contradiction.
$\dashv$ (5)&1.42

It is possible to improve the elementarity of the embedding where $\omega\rho_{\bar{M}}^n \geq \bar{\nu} > \omega\rho_{\bar{M}}^{n+1}$, under certain conditions.

1.43 Lemma. *$\widetilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving and cofinal (hence $\rho_n = \rho_M^n$), and thus $\Sigma_1^{(n)}$-preserving, if* (a) *n satisfies $\omega\rho_{\bar{M}}^n \geq \bar{\nu} > \omega\rho_{\bar{M}}^{n+1}$, and* (b) *$R_{\bar{M}}^n \neq \varnothing$. Further $\widetilde{\sigma}$ "$R_{\bar{M}}^n \subseteq R_M^n$.*

For a proof of this we refer the reader to [62, 3.2.4]. For our purposes in proving Global $\square$, we shall need a refinement of the above construction. We should like the canonical extension map $\widetilde{\sigma}$ to be sufficiently fine structure preserving to ensure that if $\bar{\nu}$ is regular in $\bar{M}$, but $\Sigma_1^{(n)}$ singularised over $\bar{M}$ (by this we mean that for some $\beta < \bar{\nu}$ there is a good $\Sigma_1^{(n)}$ map of a subset of β cofinally into $\bar{\nu}$), then whilst M still thinks that $\nu(= \widetilde{\sigma}(\bar{\nu}))$ is regular, there is nevertheless also a good $\Sigma_1^{(n)}(M)$-singularising map. This may require us to take fewer functions than Definition 1.38 allows.

We now describe the small modification to do this. Assume then that we are in the situation under discussion:

1.44 Definition. Suppose that $\bar{\nu}$ is regular in $\bar{M}$, but is $\Sigma_1^{(k)}$-singularised over $\bar{M}$ for some k: we define $k = k(\bar{M})$ to be least $k < \omega$ for which this is true.

Note that this implies that $\omega\rho_{\bar{M}}^{k+1} \leq \bar{\nu}$. Moreover:

1.45 Lemma.

$$\omega\rho_{\bar{M}}^{k+1} \leq \bar{\nu} \leq \omega\rho_{\bar{M}}^{k}.$$

Proof. By the last remark we have to show that $\bar{\nu} \leq \omega\rho_{\bar{M}}^{k}$. Suppose this failed. Then this would imply that $k = m+1$ for some $m \in \omega$. Let $r \in \bar{M}$ be such that there is a good $\Sigma_1^{(k)}(\bar{M})$ function f singularising $\bar{\nu}$ with domain of f contained in some $\gamma < \bar{\nu}$ and with $\omega\rho_{\bar{M}}^{k} < \gamma$. Let $X = \widetilde{h}_{\bar{M}}^{m+1}(\gamma \cup \{r, p\})$ be the $\Sigma_1^{(m)}(\bar{M})$ hull of the objects displayed where p has been chosen from $P_{\bar{M}}^{*}$. This is of a lower degree of elementarity, and hence $X \cap \bar{\nu}$ is bounded in $\bar{\nu}$. Let $\pi : N \longrightarrow \bar{M}$ be the inverse of the transitive collapse map on X. As $\pi{\restriction}\omega\rho_{\bar{M}}^{k+1} = \mathrm{id}{\restriction}\omega\rho_{\bar{M}}^{k+1}$ and $p \in \mathrm{ran}(\pi) \cap P_{\bar{M}}^{*}$ we map appeal to Lemma 1.20 and see that π is Σ^*-preserving. However the parameter r is in the $\mathrm{ran}(\pi)$ and thus $\mathrm{ran}(f) \subseteq \mathrm{ran}(\pi) = X$. This is absurd. ⊣

We now simply restrict the functions used in forming the canonical extension to those $\Sigma_1^{(n)}(\bar{M})$-definable functions, for an $n < k$.

1.46 Definition. Let $\bar{M}, \bar{\nu}, k = k(\bar{M})$ be as above. Define $\Gamma_{\bar{M},\bar{\nu}}^{k} =_{\mathrm{df}}$:

$$\{f \mid \mathrm{dom}(f) \in \overline{Q} \wedge \mathrm{ran}(f) \subseteq \overline{M}) \wedge (n < k \wedge f \in \mathbf{\Sigma}_1^{(n)}(\overline{M}) \wedge \omega\rho_{\overline{M}}^{n+1} \geq \bar{\nu})\}.$$

1.47 Definition. Suppose we construct the pseudo-ultrapower using functions from $\Gamma_{\bar{M},\bar{\nu}}^{k}$ where $k = k(\bar{M})$. Then the resulting map $\tilde{\sigma}$ is the *canonical k-extension* of σ and M the *canonical k-pseudo-ultrapower.*

1.48 Theorem (k-Pseudo-Ultrapower Theorem). *Let $\bar{M}, \bar{\nu}, k = k(\bar{M})$ be as above. Let $\widetilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ be the canonical k-extension of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. Then*

(i) *$\widetilde{\sigma}$ is Q-preserving, M is an acceptable end extension of Q, and $M = \{\widetilde{\sigma}(f)(u) \mid u \in \sigma(\mathrm{dom}(f)), f \in \Gamma\}$.*

(ii) (a) *$\widetilde{\sigma}$ is $\Sigma_2^{(n)}$-preserving for $n < k$;*

(b) *$\widetilde{\sigma}$ is $\Sigma_0^{(k)}$-preserving.*

1.49 Lemma. *Suppose $\bar{\nu}$, $\bar{M}$, $n = k(\bar{M})$, $\widetilde{\sigma}$, M are as above. Then*

(i) *$\rho_n = \rho_M^n$, and $\widetilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving and cofinal (thus $\Sigma_1^{(n)}$-preserving);*

(ii) *$\widetilde{\sigma}(\bar{\nu}) = \nu$ and the latter is regular in M;*

(iii) *n is least so that there is a $\mathbf{\Sigma_1^{(n)}}(M)$ map cofinalising ν.*

Note: As in Lemma 1.45 the hypothesis implies here too that $\omega\rho^n_{\bar M} \geq \bar\nu \geq \omega\rho^{n+1}_{\bar M}$.

Proof. From what we have already done, (i) will follow if we can show $\rho_n \geq \rho^n_M$. We first note that $\widetilde\sigma(\bar\nu)$ is regular in M. For $\widetilde\sigma(\bar\nu)$ this is either because $n = 0$ and $\widetilde\sigma$ is then Σ_0 and cofinal—hence Σ_1-preserving, or because $n > 0$ and by the above $\widetilde\sigma$ is Σ_2-preserving. As one would hope $\widetilde\sigma(\bar\nu) = \nu$: suppose $\eta = \widetilde\sigma(f)(a) < \widetilde\sigma(\bar\nu)$ for some $f \in \Gamma^{k(\bar M)}$. Then $f \in \bar M$. Set $\delta = \sup(\mathrm{ran}(f) \cap \bar\nu)$. As $\mathrm{dom}(f) \in \bar M$ and $\bar\nu$ is regular in $\bar M$, $\delta < \bar\nu$. Then $\mathrm{ran}(\widetilde\sigma(f)) \subset \widetilde\sigma(\delta)$. Thus $\eta < \widetilde\sigma(\delta)$. This proves (ii).

By hypothesis there is $\bar f$ a good function $\Sigma^{(n)}_1(\bar M)$ in a parameter $\bar p$ say, that maps a subset of some $\gamma < \bar\nu$ cofinally into $\bar\nu$. (We'll assume $\mathrm{ran}(f) \subseteq \bar\nu$.) Let $p = \widetilde\sigma(\bar p)$. Let $\widetilde A =_{\mathrm{df}} A^{n,p}_M \cap J^{A^{n,p}_M}_{\rho_n}$; $\widetilde M = \langle J^{\widetilde A}_{\rho_n}, \widetilde A\rangle$. Then $\widetilde\sigma{\restriction}\bar M^{n,\bar p} : \bar M^{n,\bar p} \longrightarrow_{\Sigma_0} \widetilde M$ cofinally. $\bar f$ is rudimentary over $\bar M^{n,\bar p}$ and we can let f be rudimentary over $\widetilde M$ by the same definition. Then $\mathrm{dom}(f)$ is bounded in ν. As $\widetilde\sigma{\restriction}\bar\nu$ is cofinal into ν (because σ is) we have $\widetilde\sigma``\mathrm{ran}(\bar f) \subseteq \mathrm{ran}(f)$ and the latter is thus cofinal in ν and hence f singularises ν. However if $\rho_n < \rho^n_M$ we should have $\widetilde M$, and thus f, both members of M. This contradicts the regularity of ν in M which we have just remarked upon above.

Thus $\widetilde\sigma$ is $\Sigma^{(n)}_0$-preserving and cofinal, hence $\Sigma^{(n)}_1$-preserving; $\widetilde\sigma(\bar f)$ is thus well-defined, and is a good $\boldsymbol{\Sigma}^{(n)}_1(M)$ map cofinalising ν. ⊣

1.50 Remark. We can conclude from the above existence of the good cofinalising map that $\omega\rho^n_M \geq \nu \geq \omega\rho^{n+1}_M$ but *not* that $\nu > \omega\rho^{n+1}_M$ even if $\bar\nu > \omega\rho^{n+1}_{\bar M}$.

In the above the $\in$-relation, e, of the term model $\mathbb{D}$ was assumed well-founded. The Interpolation Lemma to follow is again not in the most general form, but will be useful later. The hypotheses imply that $\omega\rho^n_{\bar M} \geq \bar\nu \geq \omega\rho^{n+1}_{\bar M}$ for the named n; we thus may form the term model using the functions in Γ_n. Since $\overline M$ is $\Sigma^{(n)}_0$-embeddable into the well-founded model M by the map g below, we can then embed the term model into M *via* $g'([a,G]) = g(G)(a)$ and g' is then a $\Sigma^{(n)}_0$ map. Hence it is $\in$-preserving on its domain, and so demonstrates that e is well-founded. The structure $\widetilde M$ is that formed by the pseudo-ultrapower construction as the canonical $k(\bar M)$-extension of $\sigma = g{\restriction}J^{\overline A}_{\bar\nu} : J^{\overline A}_{\bar\nu} \longrightarrow J^A_\nu$ from above.

1.51 Lemma (Interpolation Lemma). *Suppose $\overline M = \langle J^{\overline A}_{\bar\beta}, \overline B\rangle$ is a structure such that $\bar\nu$ is regular in $\overline M$, but that $k(\bar M)$ is defined, i.e. for some least $n < \omega$, there is a $\boldsymbol{\Sigma}^{(n)}_1(\overline M)$ function singularising $\bar\nu$. Suppose further that $g : \overline M \longrightarrow_{\Sigma^{(n)}_1} M = \langle J^A_\beta, B\rangle$. Let $\widetilde\nu = \sup g``\bar\nu$. Let $\sigma_1 =_{\mathrm{df}} \mathrm{id}{\restriction}J^{A'}_{\widetilde\nu}$. Then there is a structure $M' = \langle J^{A'}_{\beta'}, B'\rangle$, a map $\widetilde g : \overline M \longrightarrow M'$ with* (a) *$\widetilde g \supseteq g{\restriction}J^{\overline A}_{\bar\nu}$ and* (b) *$\widetilde g$ is $\Sigma^{(n)}_0$ and cofinal at the nth level (and hence $\Sigma^{(n)}_1$-preserving); further there is a unique $g' : M' \longrightarrow_{\Sigma^{(n)}_0} M$, with $g = g' \circ \widetilde g$ and $g'{\restriction}\widetilde\nu = \sigma_1$.*

$$
\begin{tikzcd}
\langle J^{\bar{A}}_{\bar{\beta}}, \bar{B}\rangle \arrow[r, "\widetilde{\sigma} = \widetilde{g} \quad \Sigma_1^{(n)}"'] \arrow[rr, bend left, "g \quad \Sigma_1^{(n)}"] & \langle J^{A'}_{\beta'}, B'\rangle \arrow[r, "g' \quad \Sigma_0^{(n)}"'] & \langle J^{A}_{\beta}, B\rangle \\
\langle J^{\widetilde{A}}_{\widetilde{\nu}}, \widetilde{B}\rangle \arrow[u, hook] \arrow[r, "\sigma = g\restriction J^{\bar{A}}_{\bar{\nu}}"] & \langle J^{A'}_{\widetilde{\nu}}, B'\rangle \arrow[u, hook] \arrow[r, "g'\restriction J_{\widetilde{\nu}} = \sigma_1"] & \langle J^{A}_{\nu}, B\rangle \arrow[u, hook]
\end{tikzcd}
$$

Figure 10.1: The Interpolation Lemmata

Proof. As intimated, M' and $\widetilde{g}$ come from the construction of this section as the canonical $k(\bar{M})$-extension of the cofinal mapping $g\restriction J^{\overline{A}}_{\overline{\nu}} : J^{\overline{A}}_{\overline{\nu}} \longrightarrow_{\Sigma_0} J^{A}_{\widetilde{\nu}}$ playing the role of $\sigma : \overline{Q} \longrightarrow_{\Sigma_0} Q$. That $\widetilde{g}$ has the requisite properties has been established. g' is defined through: $g'(\widetilde{g}(G)(a)) = g(G)(a)$. Then g' has to be the unique such map with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \mathrm{id}\restriction\widetilde{\nu}$, since $g'(\widetilde{g}(G)(a)) = g' \circ \widetilde{g}(G)(g'(a)) = g(G)(a)$. (Note that trivially $g'\restriction\widetilde{\nu} = \mathrm{id}\restriction\widetilde{\nu} \longleftrightarrow g'\restriction J^{\bar{A}}_{\widetilde{\nu}} = \mathrm{id}\restriction J^{\bar{A}}_{\widetilde{\nu}}$.) ⊣

A slightly more generalised form of the above will be useful. This allows σ_1 to be different from the identity function.

1.52 Lemma (Generalised Interpolation Lemma). *Suppose $\overline{M} = \langle J^{\overline{A}}_{\overline{\beta}}, \overline{B}\rangle$ is a structure such that $\overline{\nu}$ is regular in $\overline{M}$, but that $k(\bar{M})$ is defined, i.e. for some least $n < \omega$, there is a $\boldsymbol{\Sigma_1^{(n)}}(\overline{M})$ function singularising $\overline{\nu}$. Suppose further for $n = k(\bar{M})$ that $g : \overline{M} \longrightarrow_{\Sigma_1^{(n)}} M = \langle J^{A}_{\beta}, B\rangle$ and that $\sigma : J^{\overline{A}}_{\overline{\nu}} \longrightarrow J^{A'}_{\widetilde{\nu}}$, and $\sigma_1 : J^{A'}_{\widetilde{\nu}} : J^{A'}_{\widetilde{\nu}} \longrightarrow J^{A}_{\nu}$ are both Σ_0 with σ cofinal, and finally that $\sigma_1 \circ \sigma = g\restriction J^{\overline{A}}_{\overline{\nu}}$.*

Then there is a structure $M' = \langle J^{A'}_{\beta'}, B'\rangle$, a map $\widetilde{g} : \overline{M} \longrightarrow M'$ with (a) *$\widetilde{g} \supseteq \sigma$ and* (b) *$\widetilde{g}$ is $\Sigma_0^{(n)}$ cofinal at the nth level (and so $\Sigma_1^{(n)}$-preserving); further there is a unique $g' : M' \longrightarrow_{\Sigma_0^{(n)}} M$, with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \sigma_1$.*

Proof. The argument is just as before except now g' is defined through: $g'(\widetilde{g}(G)(a)) = g(G)(\sigma_1(a))$. Again g' has to be the unique such map with $g = g' \circ \widetilde{g}$ and $g'\restriction\widetilde{\nu} = \sigma_1\restriction\widetilde{\nu}$, since $g'(\widetilde{g}(G)(a)) = g' \circ \widetilde{g}(G)(g'(a)) = g(G)(\sigma_1(a))$. ⊣

2. Global □

We derive a global □ sequence in L, the constructible hierarchy. We assume now $V = L$. Our template is basically that of the proof of Jensen [2] here, although we use ideas from [30]. We shall build our C_ν sequence along a closed and unbounded class $S \subseteq \mathrm{Sing}$ of singular ordinals. The definition of the C_ν sets themselves we shall present is extremely uniform, and quite simple. They will be preserved by Σ_0 embeddings between levels of the J-hierarchy of a sufficient ordinal height to see that ν is singular. The choice of the closed and unbounded class S is also rather flexible: we require S again be simply defined so that membership in it is preserved downwards into Σ_1-elementary substructures. To this end we could take S to be the class of all limits of admissible ordinals. (Recall that α is *admissible* if $\langle L_\alpha, \in\rangle$ is a model of Kripke-Platek set theory.) Although this is fine from a technical point of view, it appears to require more closure on the structures J_α than it really needs. For most purposes it will be useful to assume that $\omega\nu = \nu$ as a minimum, (this will have the effect of tying up C_ν with the ordinal height of J_ν), and that $\omega\nu$ is sufficiently closed under some further basic ordinal theoretic operations. We shall accordingly take S to be the class of *primitive recursively (p.r.) closed ordinals.*

2.1 Definition.

(i) The *primitive recursive set functions* are obtained by adding to the rudimentary functions the following schema:

$$f(\vec{x}, y) = g(\vec{x}, y, \langle f(\vec{x}, z) \mid z \in y\rangle).$$

f is *primitive recursive (p.r.) in* $A \subseteq V^n$ iff f is p.r. in χ_A.

(ii) X is *p.r. closed* (*p.r. closed in* A) iff X is closed under the p.r. functions (the p.r. in A functions, respectively).

(iii) α is *p.r. closed* if L_α is p.r. closed.

2.2 Lemma.

(i) *ω is p.r. closed.*

(ii) *Let $\alpha > \omega$. Then α is p.r. closed iff α is closed under the functions f_i $(i < \omega)$ where:*

$$f_0(\nu) = \omega^\nu; \qquad f_{i+1}(\nu) = \text{the } \nu\text{th fixed point of } f_i.$$

If the reader desires then the above can be taken as a definition of a p.r. closed ordinal. It follows relatively easily that:

(1) $\alpha > \omega \wedge \alpha$ p.r. closed $\implies \omega\alpha = \alpha$ and hence $J_\alpha = L_\alpha$.

(2) $\omega\alpha$ p.r. closed $\iff J_\alpha$ p.r. closed.

(3) $\{\alpha \mid \alpha \text{ is p.r. closed}\}$ is closed and unbounded in On.

(4) If α is p.r. closed but not a limit point of the p.r. closed ordinals, then $\alpha = \sup_i\{f_i(\beta) \mid \beta < \alpha\}$ and hence $\mathrm{cf}(\alpha) = \omega$.

Noting that p.r. closure is essentially a Π_2 property of a structure we obtain the relevant property for our subsequent definitions:

2.3 Lemma. *Let $\omega\alpha$ be p.r. closed, and let $f : J_{\overline{\alpha}} \longrightarrow_{\Sigma_1} J_\alpha$. Then $\omega\overline{\alpha}$ is p.r. closed.*

(As we commented above, the property of α being a limit of admissibles, would have satisfied the consequent of the above lemma too.)

2.4 Theorem ($V = L$). *Let $S \subseteq \mathrm{Sing}$ be as above. There is a uniformly definable class $\langle C_\nu \mid \nu \in S \rangle$ so that:*

(i) *$C_\nu \subseteq \nu$ is a set of ordinals unbounded in ν and closed beneath it;*

(ii) *$\mathrm{ot}(C_\nu) < \nu$;*

(iii) *$\overline{\nu} \in (C_\nu)^* \implies \overline{\nu} \in S \wedge C_{\overline{\nu}} = \overline{\nu} \cap C_\nu$;*

(iv) *If $f : \langle J_{\overline{\nu}}, \overline{C} \rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu \rangle$ then $\overline{\nu} \in S$ and $\overline{C} = C_{\overline{\nu}}$.*

We think of C_ν as a canonical singularising sequence for ν, i.e. it provides a sequence cofinal in ν, and of order type less than ν. It will be defined using the least level of the J-hierarchy, over which we can define a map from a bounded subset of ν cofinally into ν. Our desire for short sequences is expressed by (ii). The important clause is (iii) the *coherence property*: the limit points $\bar{\nu}$ of C_ν, are themselves singular, and the initial segment $C_\nu \cap \bar{\nu}$ gives the canonical singularising sequence for $\bar{\nu}$ itself. (i)–(iii) are often referred to as (one form of) Global $\Box$. Clause (iv) is also a strong one, providing as it does information on condensation-like coherency. We shall first give a construction of a global sequence satisfying most of the above: it simply will not always be unbounded in ν—this may happen when $\mathrm{cf}(\nu) = \omega$. The theorem below contains the heart of the argument. Obtaining the theorem above from it, is relatively speaking, at least over L, a minor adjustment.

2.5 Theorem ($V = L$). *There is a uniformly definable class $\langle C_\nu \mid \nu \in S \rangle$ so that:*

(i) *C_ν is a set of ordinals ν and closed beneath it; if $\mathrm{cf}(\nu) > \omega$ then C_ν is unbounded in ν;*

(ii) *$\mathrm{ot}(C_\nu) < \nu$;*

(iii) *$\overline{\nu} \in (C_\nu) \implies \overline{\nu} \in S \wedge C_{\overline{\nu}} = \overline{\nu} \cap C_\nu$;*

(iv) *If $f : \langle J_{\overline{\nu}}, \overline{C} \rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu \rangle$ then $\overline{\nu} \in S$ and $\overline{C} = C_{\overline{\nu}}$.*

As indicated if ν is a singular ordinal, then there will be a least level $J_{\beta(\nu)}$ of the J-hierarchy over which ν is *definably singularised*, i.e. there will be a function (possibly partial) that is $\Sigma_\omega(J_{\beta(\nu)})$-definable (possibly in some parameters) mapping (a subset of) some γ cofinally into ν. However any such function is also good $\Sigma_1^{(n)}(J_{\beta(\nu)})$ for some n. This level $J_{\beta(\nu)}$ will be our *singularising structure* M_ν, and n is the level of complexity at which we shall work. We set out a formal definition of our subject matter. Notice, that given $J_{\beta(\nu)}$ all the objects defined below are Σ_0-definable from it.

2.6 Definition. Let $\nu \in S$. Then we associate the following to ν:

(a) $n_\nu =_{\rm df}$ the least $n \in \omega$ so that there is a good $\boldsymbol{\Sigma}_{\mathbf{1}}^{\boldsymbol{(n)}}(M_\nu)$ function singularising ν, where $M_\nu = J_{\beta(\nu)}$.

(b) $M_\nu^k =_{\rm df} M_\nu^{k,p_{M_\nu}\restriction k}$ for $k \leq n_\nu$.

(c) $h_\nu^k =_{\rm df} h_{M_\nu}^{k,p_{M_\nu}\restriction k}$; $h_\nu =_{\rm df} h_\nu^{n_\nu}$; $\widetilde{h}_\nu =_{\rm df} \widetilde{h}_{M_\nu}^{n_\nu+1}$.

(d) (i) $\omega\rho_\nu =_{\rm df} \mathrm{On} \cap M_\nu^{n_\nu}$; (ii) $\kappa_\nu \simeq$ the largest cardinal of J_ν, if such exists.

(e) $p_\nu =_{\rm df} p_{M_\nu} \backslash \nu$ if ν is a limit cardinal of J_ν; $p_\nu =_{\rm df} p_{M_\nu} \backslash \kappa_\nu$ otherwise; $q_\nu =_{\rm df} p_\nu \cap \omega\rho_{M_\nu}^{n_\nu}$.

(f) $\alpha_\nu =_{\rm df} \max\{\alpha < \nu \mid \nu \cap \widetilde{h}_\nu(\alpha \cup \{p_\nu\}) = \alpha\}$, setting $\max \varnothing = 0$.

(g) $\gamma_\nu =_{\rm df} \min\{\gamma < \nu \mid$ there exists a $\Sigma_1^{(n)M_\nu}(\{p_\nu\})$ partial function
$$F : a \longrightarrow \nu, \text{ cofinal, with } a \subseteq \gamma\}.$$

Thus if $\nu = \kappa_\nu^+$, note then we may possibly have κ_ν in p_ν (if κ_ν happens to be in p_ν). We also note that $p_\nu \backslash \nu$ ($= p_\nu$ unless κ_ν is defined) is the least parameter p so that $\widetilde{h}_\nu(\nu \cup p) = J_{\beta(\nu)}$. It is easy to see that α_ν in (f) is always defined, as the set of α's specified is closed; note also that α_ν is perforce strictly less than the first ordinal γ partially mapped by $\widetilde{h}_\nu$(with parameter p_ν) cofinally into ν. We set that first ordinal γ at (g) to be γ_ν. Such a γ_ν exists, since by assumption there is some good $\boldsymbol{\Sigma}_{\mathbf{1}}^{\boldsymbol{(n)}}(M_\nu)$ function singularising ν, using some parameter r. In fact as we are in L we may take $r = p_{M_\nu}$. If $\gamma' = \max(p_{M_\nu} \cap \nu)+1$ ($\max(p_{M_\nu} \cap \kappa_\nu)+1$ if κ_ν is defined), then clearly $\widetilde{h}_\nu(\gamma' \cup p_\nu)$ is cofinal in ν (as we have enough parameters in the domain of this hull to define our cofinalising map); hence $\gamma_\nu \leq \gamma'$ exists.

2.7 Lemma. $\omega\rho_{M_\nu}^{n_\nu} \geq \nu \geq \omega\rho_{M_\nu}^{n_\nu+1}$.

Proof. Let $n = n_\nu$. If the second inequality failed, then the partial function $\boldsymbol{\Sigma}_{\mathbf{1}}^{\boldsymbol{(n)}}(J_{\beta(\nu)})$-singularising ν would be a subset of ν and thus a bounded subset of $\omega\rho_{M_\nu}^{n+1}$ belonging to $J_{\beta(\nu)}$. Suppose the first inequality failed. Then $n > 0$, and $\widetilde{h}_{M_\nu}^n$ maps $H_{M_\nu}^n = J_{\rho_{M_\nu}^n}$ onto $J_{\beta(\nu)} = M_\nu$. However $\widetilde{h}_{M_\nu}^n$ has a $\Sigma_1^{(n-1)}$ definition, which can be easily amended to provide a $\Sigma_1^{(n-1)}$ map from $\omega\rho_{M_\nu}^n$ onto ν—thus contradicting our definition of n. $\dashv$

2.8 Definition. For $\nu, \overline{\nu} \in S$:

(i) We set $f : \bar{\nu} \Longrightarrow \nu$ if $|f| : J_{\overline{\nu}} \longrightarrow_{\Sigma_1} J_\nu$, and $|f|$ is the restriction of $f^* : J_{\beta(\overline{\nu})} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n = n_\nu$, $\nu = f^*(\overline{\nu})$ if $\nu < \beta(\nu)$; $\kappa_\nu \in \mathrm{ran}(f)$ (if κ_ν is defined); α_ν, p_ν are both in $\mathrm{ran}(f^*)$.

(ii) $\mathbb{F} =_{\mathrm{df}} \{\langle \bar{\nu}, |f|, \nu \rangle \mid f : \bar{\nu} \Longrightarrow \nu\}$. We sometimes write if $f : \bar{\nu} \Longrightarrow \nu$ that $\bar{\nu} = d(f)$ and $\nu = r(f)$.

(iii) If $\nu < \beta$, we set $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu, \nu, \kappa_\nu\}$ (the latter if κ_ν is defined); if $\nu = \beta$ then $p(\nu) =_{\mathrm{df}} p_\nu \cup \{\alpha_\nu, \kappa_\nu\}$ (again the latter only if it is defined).

(iv) $f^*_{(\delta,v,\nu)}$ is the inverse of the transitive collapse of the hull $\widetilde{h}_\nu(\delta \cup \{p(\nu), v\})$ in $J_{\beta(\nu)}$.

2.9 Lemma. *If $\exists \bar{\nu}(f : \bar{\nu} \Longrightarrow \nu)$ then $|f|$ and f^* are uniquely determined by* $\mathrm{ran}(|f|) \cap \nu$.

Proof. As $J_{\beta(\nu)}$ is sound, we have by our definitions, and Fact (i) before Lemma 1.19, that $\widetilde{h}_\nu(\omega\nu \cup \{p_\nu\}) = J_{\beta(\nu)}$ (and a similar statement with $\bar{\nu}$ replacing ν throughout). We have a $\Delta_1(J_\nu)$ onto map $g : \omega\nu \twoheadrightarrow J_\nu$. Thus, if $Y = \widetilde{h}_\nu((\omega\nu \cap \mathrm{ran}(|f|)) \cup \{p_\nu\})$, then $Y = \widetilde{h}_\nu(\mathrm{ran}(|f|) \cup \{p_\nu\}) = \mathrm{ran}(f^*)$: if $x \in \mathrm{ran}(f^*)$, then by soundness of $J_{\bar{\beta}}$ above $\bar{\nu}$, $x = f^*(h_{\bar{\nu}}(\bar{\xi}, p_{\bar{\nu}}))$ for some $\bar{\xi} < \omega\bar{\nu}$; thus $x \in Y$. However $f^*(h_{\bar{\nu}}(\bar{\xi}, p_{\bar{\nu}})) = h_\nu(|f|(\bar{\xi}), p_\nu)$ since f^* is $\Sigma_1^{(n)}$-preserving, we have by Lemma 1.22 that $f^*(p_{\bar{\nu}}) = p_\nu$. The converse inclusion is immediate. ⊣

2.10 Remark. The next lemma will show that in clause (iv) there is some μ and a restriction map $|f_{(\delta,q,\nu)}|$ of $f^*_{(\delta,q,\nu)}$ so that $\langle \mu, |f_{(\delta,q,\nu)}|, \nu \rangle \in \mathbb{F}$.

In the sequel we shall not carefully distinguish $|f|$ (or f), from its *canonical extension* f^* (observe Lemma 2.9) and sometimes write $f : J_{\overline{\nu}} \longrightarrow_{\Sigma_1} J_\nu$ where more correctly we should write $f^* \restriction J_{\overline{\nu}}$ or $|f| : J_{\overline{\nu}} \longrightarrow_{\Sigma_1} J_\nu$ (as in the conclusion of the next lemma). Or conversely if $f : \bar{\nu} \Longrightarrow \nu$ then we should properly write "$f^* : J_{\beta(\overline{\nu})} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$" but sometimes we are slip-shod and again simply substitute f for f^* by virtue of the last lemma.

We shall first prove a pair of lemmata concerning relationships between singularising structures, and associated maps between them. These will then facilitate the definition of our C_ν sequences.

2.11 Lemma. *Let $f : J_{\bar{\beta}} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n = n_\nu$; suppose $f(\bar{\alpha}, \bar{p}) = \alpha_\nu, p_\nu$, and (where appropriate) $f(\bar{\kappa}, \bar{\nu}) = \kappa_\nu, \nu$. (The latter if $\nu < \beta(\nu)$; if $\nu = \beta(\nu)$ then we take $\bar{\nu} = \bar{\beta}$.) Then $\bar{\nu} \in S$, $\bar{\beta} = \beta(\bar{\nu})$ (thus $M_{\bar{\nu}} = J_{\bar{\beta}}$), $f : \bar{\nu} \Longrightarrow \nu$; moreover $n, \bar{\alpha}, \bar{p}, \bar{\kappa}$ (the latter when defined) are $n_{\bar{\nu}}, \alpha_{\bar{\nu}}, p_{\bar{\nu}}, \kappa_{\bar{\nu}}$.*

(Remark: we shall drop the qualification "when defined" from now on and will also let the reader include or exclude the ordinal ν as appropriate.)

Proof. We verify that $J_{\bar{\beta}}$ is the singularising structure for $\bar{\nu}$ and the other parameters are as shown, and thus are moved correctly by f. That $\bar{\nu} \in S$ is trivial if $\nu \in \operatorname{ran}(f)$ and follows from Lemma 2.3 otherwise.

(1) $\bar{p} = p_{J_{\bar{\beta}}} \backslash \bar{\nu}$.

Proof of (1). By the Condensation Lemma 1.22. $\dashv$ (1)

Setting $\beta = \beta(\nu)$, we let $\overline{h}$ have the same functionally absolute definition over $J_{\bar{\beta}}$ as $\widetilde{h}_\nu$ does over J_β. $\overline{h}$ is thus $\mathbf{\Sigma_1^{(n)}}(J_{\bar{\beta}})$.

(2) $\overline{\alpha}$ *is defined from* $J_{\bar{\beta}}$ *as* α *was defined from* $J_{\beta(\nu)}$.

Proof of (2). Set

$$\begin{array}{lcl} H(\xi^n, \zeta^n) & \iff & \widetilde{h}_\nu(\omega\xi^n \cup \{p_\nu\}) \cap \nu \subseteq \zeta^n, \\ \overline{H}(\xi^n, \zeta^n) & \iff & \bar{h}(\omega\xi^n \cup \{\bar{p}\}) \cap \bar{\nu} \subseteq \zeta^n. \end{array}$$

The former H is then $\Pi_1^{(n)J_\beta}(\{p_\nu, \nu\})$, ($\nu$ is omitted if $\nu = \omega\rho^n_{J_\beta}$); the latter $\overline{H}$ is $\Pi_1^{(n)J_{\bar{\beta}}}(\{\bar{p}, \bar{\nu}\})$, by the same definition (again, we include $\bar{\nu}$ when $\bar{\nu} < \omega\rho^n_{J_{\bar{\beta}}}$, which occurs by the $\Sigma_1^{(n)}$-elementarity of f^* when $\nu = \omega\rho^n_{J_\beta}$). As $J_\beta \models H(\alpha, \alpha)$ we have $J_{\bar{\beta}} \models \overline{H}(\bar{\alpha}, \bar{\alpha})$. However for any ξ with $\bar{\alpha} < \xi^n < \bar{\nu}$ we must have $J_{\bar{\beta}} \models \neg\overline{H}(\xi^n, \xi^n)$, because $J_\beta \models \neg H(f(\xi^n), f(\xi^n))$ since $\alpha < f(\xi^n) < \nu$. These properties describe α and $\bar{\alpha}$. $\dashv$ (2)

(3) $\exists\overline{\xi}^n < \bar{\nu}(\widetilde{h}_\nu(f(\bar{\xi}^n) \cup \{p_\nu\})$ is unbounded in ν).

If (3) holds for some $\overline{\xi}$ then, in fact $\bar{h}(\xi \cup \{\bar{p}\})$ is cofinal in $\bar{\nu}$, since

$$J_\beta = M_\nu \models (\forall\zeta^n < \nu)(\exists\delta^n < f(\bar{\xi}^n))(\exists i < \omega)(\zeta^n < \widetilde{h}_\nu(i, \langle\delta^n, p_\nu\rangle) < \nu)$$

is a $\Pi_2^{(n)}$ expression and hence goes down to $J_{\bar{\beta}}$ in the objects $\bar{p}, \bar{h}$, $\bar{\xi}$ and $\bar{\nu}$ (the latter if $\nu = f(\bar{\nu}) < \omega\rho^n_{J_\beta}$) It thus witnesses that $\bar{\beta} = \beta(\bar{\nu})$ and $n \geq n_{\bar{\nu}}$. Thus also $\bar{h}$ is a suitable singularising function for $\bar{\nu}$. So suppose (3) fails. Note that then we must have $\tau =_{\mathrm{df}} \sup f``\bar{\nu} \leq \gamma_\nu < \nu$. However, then we have:

(4) $\tau \neq \nu \cap \widetilde{h}_\nu(\tau \cup \{p_\nu\})$.

We cannot have equality here as that would imply that $\tau \leq \alpha_\nu$. However the latter is in $\operatorname{ran}(f{\restriction}\bar{\nu})$!

Hence the following true in J_β:

$$\exists i \in \omega \exists \xi^n < \tau(\nu > \widetilde{h}_\nu(i, \langle \xi^n, p_\nu \rangle) \geq \tau).$$

Let i, ξ^n witness this, and pick $\bar{\delta} < \bar{\nu}$ so that $f(\bar{\delta}) > \xi^n$. Then for any $\bar{\mu} < \bar{\nu}$ as $f(\bar{\mu}) < \tau$:

$$J_\beta \models (\exists \zeta^n < f(\bar{\delta}))(\nu > \widetilde{h}_\nu(i, \langle \zeta^n, p_\nu \rangle) \geq f(\bar{\mu})).$$

This is $\Sigma_1^{(n)}$ and hence, for all $\bar{\mu} < \bar{\nu}$, goes down to $J_{\bar{\beta}}$. Now:

$$J_{\bar{\beta}} \models \forall \bar{\mu} < \bar{\nu} \exists \zeta^n < \bar{\delta}(\bar{\nu} > \bar{h}(i, \langle \zeta^n, \bar{p} \rangle \geq \bar{\mu})).$$

Again, this says that $\bar{h}$ is a suitable singularising function for $\bar{\nu}$. The only possibility left is that (4) could hold. However that would imply that $\tau \leq \alpha_\nu$ which we have already ruled out.

(5) $\bar{\beta} = \beta(\bar{\nu})$ and $n = n_{\bar{\nu}}$.

We have just seen that $\bar{\beta} = \beta(\bar{\nu})$ and $n \geq n_{\bar{\nu}}$. Suppose $m < n$ and that $\bar{g}$ is a $\Sigma_1^{(m)}(J_{\bar{\beta}})$ good function in the parameter $\bar{r}$. Let g be $\Sigma_1^{(m)}(J_\beta)$ using the same functionally absolute definition and the parameter $f(\bar{r})$. Suppose $\bar{\delta} < \bar{\nu}$. By the $\Sigma_1^{(n)}$-elementarity of f we have the following $\Sigma_1^{(n)}$ statement holds in J_β (as $\operatorname{ran}(g \restriction f(\bar{\delta}))$ is bounded in ν):

$$(\exists \xi^n < \nu)(\forall \zeta^m < f(\bar{\delta}))(\forall \eta^m < \nu)(g(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n)$$

(assuming $\nu < \beta$; otherwise drop the bound ν). As f is actually $\Sigma_1^{(n)}$-preserving, we have in $J_{\bar{\beta}}$:

$$(\exists \xi^n < \bar{\nu})(\forall \zeta^m < \bar{\delta})(\forall \eta^m < \bar{\nu})(\bar{g}(\zeta^m) = \eta^m \longrightarrow \eta^m < \xi^n).$$

As $\bar{\delta}$ was arbitrary, we conclude that $\operatorname{ran}(\bar{g} \restriction \xi)$ is bounded on any $\xi < \bar{\nu}$. Hence $n \leq n_{\bar{\nu}}$. $\dashv$

2.12 Definition. If $f : \bar{\nu} \implies \nu$, let $\lambda(f) =_{\mathrm{df}} \sup f``\overline{\nu}$; $\rho(f) =_{\mathrm{df}} \sup f``\rho_{\bar{\nu}}$.

2.13 Lemma. *Suppose $f : \bar{\nu} \implies \nu$, and let $\lambda = \lambda(f)$. Then $\lambda \in S$ and there exists a unique $f_0 : \bar{\nu} \implies \lambda$ with $f \restriction \overline{\nu} = f_0 \restriction \overline{\nu}$.*

Proof. With $J_{\beta(\bar{\nu})} = M_{\bar{\nu}}$ and $J_{\beta(\nu)} = M_\nu$ a direct application of the Interpolation Lemma 1.51 with λ as $\widetilde{\nu}$, $M_{\bar{\nu}}$, M_ν as $\overline{M}, M$ respectively, and using $f^* : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$ (where f^* is the canonical extension of f) we have the structure $\widetilde{M} = J_{\widetilde{\beta}}$ and maps $\widetilde{f}$, f' as specified.

(1) $\lambda \in S, n = n_\lambda$.

Proof of (1). As $\widetilde{h}_{\bar{\nu}}(\gamma_{\bar{\nu}} \cup \{p_{M_{\overline{\nu}}}, r\})$ is cofinal in $\overline{\nu}$ for some parameter r then $\lambda \cap \widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{f}(\gamma_{\bar{\nu}}) \cup \{p', \widetilde{f}(r)\})$ is cofinal in λ (setting $p' = \widetilde{f}(p_{\overline{\nu}}) = f'^{-1}(p_\nu)$).

$$\begin{tikzcd}
\overline{M} = J_{\beta(\bar\nu)} \arrow[r, "\widetilde{f}"'] \arrow[rr, bend left, "f^*"] & \widetilde{M} = J_{\widetilde{\beta}} \arrow[r, "f'"'] & J_{\beta(\nu)} = M \\
J_{\bar\nu} \arrow[u, hook] \arrow[r, "f^* \restriction J_{\bar\nu}"] & J_\lambda \arrow[u, hook] \arrow[r, "f'^* \restriction J_\lambda", "\mathrm{id}"'] & J_\nu \arrow[u, hook]
\end{tikzcd}$$

Figure 10.2: Lemma 2.13

Thus λ is p.r. closed, but not $\Sigma_1^{(n)}$-regular in $J_{\widetilde{\beta}}$. Hence $n \geq n_\lambda$. We need to show that λ is $\Sigma_1^{(n-1)}$-regular in $J_{\widetilde{\beta}}$. Suppose this fails and there is a good $\boldsymbol{\Sigma_1^{(n-1)}}(\widetilde{M})$ function $\widetilde{g}$, mapping a subset of some $\widetilde{\delta} < \lambda$ cofinally into λ. $\widetilde{g}(\xi)$ can be taken to be of the form $\widetilde{G}(\xi, s)$ in some parameter s where the latter is, by the construction of the pseudo-ultrapower, of the form $\widetilde{f}(\overline{g}_0)(\eta)$ for some good $\boldsymbol{\Sigma}_1^{(n-1)}(M_{\overline{\nu}})$ function $\overline{g}_0$ and some $\eta < \lambda$. We now carry back via $\widetilde{f}$ the supposed singularisation of λ to one of $\overline{\nu}$ which is a contradiction. Let $\overline{G}$ be the good $\Sigma_1^{(n-1)}(M_{\overline{\nu}})$ function by the same functionally absolute definition as $\widetilde{G}$. Define a good $\boldsymbol{\Sigma}_1^{(n-1)}(M_{\overline{\nu}})$ partial function by $\overline{g}(\xi) \simeq \overline{G}((\xi)_0, \overline{g}_0((\xi)_1))$. Then $\overline{g}$ will singularise $\overline{\nu}$: pick a $\overline{\tau} < \overline{\nu}$ sufficiently large so that $\widetilde{f}(\overline{\tau}) > \max\{\widetilde{\delta}, \eta\}$; we show that $\bar{g}$ takes a subset of $\bar{\tau}$ cofinally into $\bar{\nu}$. Let $\zeta < \overline{\nu}$ be arbitrary. We find a $\xi < \bar{\tau}$ with $\overline{g}(\xi) > \zeta$, and this will be our contradiction. As $\widetilde{g}$ is assumed cofinalising, there is an $\iota < \widetilde{\delta}$ so that $\widetilde{f}(\zeta) < \widetilde{g}(\iota) = \widetilde{G}(\iota, \widetilde{f}(\bar{g}_0)(\eta)) < \lambda$.

Thus the following is $\Sigma_1^{(n)}$:

$$\exists u^n [u^n < \widetilde{f}(\overline{\tau}) \wedge \widetilde{f}(\zeta) < \widetilde{G}((u^n)_0, \widetilde{f}(\overline{g}_0)((u^n)_1)) < \lambda].$$

Hence in $M_{\overline{\nu}}$ we have as required:

$$\exists u^n [u^n < \overline{\tau} \wedge \zeta < \overline{G}((u^n)_0, \overline{g}_0((u^n)_1)) = \overline{g}(u^n) < \bar{\nu}].$$

$\dashv$ (1)

(2) $p' = p_\lambda$.

Proof of (2). By the pseudo-ultrapower construction we have $\widetilde{M} = \widetilde{h}^{n+1}_{\widetilde{M}}(\lambda \cup \{p'\})$ (and equals $\widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{\kappa} \cup \{p'\})$ where $\widetilde{\kappa} = \widetilde{f}(\kappa_{\overline{\nu}})$ if $\kappa_{\overline{\nu}}$ is defined). Because of this we have that $p' \in R^n_{\widetilde{M}}$ and can thus be lengthened to a $p'' \in P^{n+1}_{\widetilde{M}} (= R^{n+1}_{\widetilde{M}}$ by the soundness of the L-hierarchy). By the minimality of the standard parameter and the definition of p_λ we thus have $p_\lambda \leq^* p'$. However if $p_\lambda <^* p'$ held, we should have for some $i \in \omega$, $\vec{\xi} < \lambda$ that $p' = \widetilde{h}^{n+1}_{\widetilde{M}}(i, \langle \vec{\xi}, p_\lambda \rangle)$, and thus $p_\nu = \widetilde{h}^{n+1}_\nu(i, \langle f'(\vec{\xi}), f'(p_\lambda)\rangle)$ whence $M_\nu = \widetilde{h}^{n+1}_\nu(\nu \cup f'(p_\lambda))$; this is a contradiction as $f'(p_\lambda) <^* p_\nu$. ⊣ (2)

We may set $f^*_0 = \widetilde{f}$ once we have shown:

(3) *If* $\widetilde{\alpha} =_{\mathrm{df}} \widetilde{f}(\alpha_{\overline{\nu}})$ *then* $\widetilde{\alpha} = \alpha_\lambda$.

We note that $\alpha_\nu = \widetilde{\alpha}$.

That $h_\lambda(\widetilde{\alpha} \cup \{p_\lambda\}) \cap \lambda \subseteq \widetilde{\alpha}$ is proven using:

$$\overline{H}(\xi^n, \zeta^n) \iff h_{\bar{\nu}}(\omega\xi^n \cup \{p_{\bar{\nu}}\}) \cap \bar{\nu} \subseteq \zeta^n.$$

Then $\overline{H}(\alpha_{\bar{\nu}}, \alpha_{\bar{\nu}})$ is a $\Pi^{(n)}_1$ fact about $\alpha_{\bar{\nu}}$ that is preserved up by $\widetilde{f}$ to yield $h_\lambda(\widetilde{\alpha} \cup \{p_\lambda\}) \cap \lambda \subseteq \widetilde{\alpha}$.

This shows that $\widetilde{\alpha}$ is sufficiently closed. To show that no larger ordinal γ with $\widetilde{\alpha} < \gamma < \lambda$ satisfies $h_\lambda(\gamma \cup \{p_\lambda\}) \cap \lambda \subseteq \gamma$; as $\widetilde{f}{\restriction}\bar{\nu}$ is cofinal into λ we may set $\bar{\gamma} = \widetilde{f}^{-1}{}``\gamma$ and then have $\alpha_{\bar{\nu}} < \bar{\gamma}$ and $\widetilde{f}(\bar{\gamma}) \geq \gamma$. So for some $i \in \omega$ and $\bar{\zeta} < \bar{\gamma}$ we have $\bar{\gamma} \leq h_{\bar{\nu}}(i, \langle \bar{\zeta}, p_{\bar{\nu}}\rangle) < \bar{\nu}$. Applying $\widetilde{f}$ we have that $\gamma \leq h_\lambda(i, \langle \widetilde{f}(\bar{\zeta}), p_\lambda\rangle) < \lambda$. As $\widetilde{f}(\bar{\zeta}) < \gamma$ we see that γ is not a closed ordinal. ⊣ (2.13)

2.14 Lemma. *Suppose* $f : \bar{\nu} \implies \nu$. *Then* $\lambda(f) < \nu \longleftrightarrow \rho(f) < \rho_\nu$.

Proof. Let $\lambda = \lambda(f)$. We prove both directions by contraposition. So in the forward direction, suppose $\rho(f) = \rho_\nu$. Then, in the diagram of the previous Lemma the map f' is not only $\Sigma^{(n)}_0$ but is cofinal at the nth level, and thus $\Sigma^{(n)}_1$-preserving. This together with $f'(\lambda, p_\lambda) = \nu, p_\nu$, implies that $\nu \cap f'{}``h_\lambda(\lambda \cup \{p_\lambda\}) \subseteq \nu \cap h_\nu(\lambda \cup \{p_\nu\}) = \lambda$. Were $\lambda < \nu$ this would contradict the fact that $\lambda > \alpha_\nu$ as the latter is in $\mathrm{ran}(f)$.

Conversely, suppose $\lambda = \nu$. Let $\bar{k}$ be the good $\Sigma^{(n)}_1(J_{\beta(\nu)})$ partial function in a parameter $\bar{q}$, cofinalising $\bar{\nu}$. Applying the $\Sigma^{(n)}_1$-preserving f^* we have a good function, k, with the same definition as $\bar{k}$ (in the parameter $f^*(\bar{q})$). Now suppose $\rho' =_{\mathrm{df}} \rho(f) < \rho_\nu$ for a contradiction. Then q would be in $J_{\rho(f)}$, but moreover all the witnesses to the existential quantifiers of type n needed to see that k is cofinal in $\lambda = \nu$ would be in $J_{\rho(f)}$, and those quantifiers could thus be replaced by bounded ones: if, say, $\bar{k}(\xi) = \zeta \leftrightarrow \exists x^n \varphi(x^n, \xi, \zeta, \bar{q})$ with $\varphi \in \Sigma^{(n)}_0$ we then have

$$k(f^*(\xi)) = f^*(\zeta) \iff \exists x^n \in J_{\rho(f)} \varphi(x^n, f^*(\xi), f^*(\zeta), q).$$

Hence over $J_{\beta(\nu)}$ we have a $\boldsymbol{\Sigma^{(n)}_0}$ definition of a singularising function for ν. However as $\nu = \lambda \leq \rho(f) < \rho_\nu$ we have that this function is in $J_{\beta(\nu)}$. This is a contradiction. ⊣

2.1. Defining C_ν

We define outright what the C_ν-sequences will be:

2.15 Definition. Let $\nu \in S$; $C^+_\nu =_{\mathrm{df}} \{\lambda(f) \mid f \Longrightarrow \nu\}$; $C_\nu =_{\mathrm{df}} C^+_\nu \backslash \{\nu\}$.

2.16 Definition. Let $f : \bar{\nu} \Longrightarrow \nu$. Then:

$$\beta(f) =_{\mathrm{df}} \max\{\beta \leq \nu \mid f{\restriction}\beta = \mathrm{id}{\restriction}\beta\}.$$

Simple closure says that $\beta(f)$ is defined. Note that $\beta(f) = \nu$ iff $f = \mathrm{id}_\nu$ iff $f(\beta(f)) \not> \beta(f)$ (if $\nu < \beta(f)$). Further, it is easy to see that $M_{\bar{\nu}} \models$ "$\beta(f)$ is a regular cardinal" (this last is a standard argument: if $\beta(f) < \nu$ were singular in $M_{\bar{\nu}}$ then $\beta(f) > \beta$ and Σ_1-elementarity of f would be a contradiction).

The next lemma lists some properties of $f_{(\gamma,q,\nu)}$ which were defined at Definition 2.8 (and see Remark 2.10). The first of these is a *minimality property* of $f_{(\gamma,q,\nu)}$.

2.17 Lemma.

(i) *If* $\gamma \leq \nu$ *then* $f_{(\gamma,q,\nu)}$ *is the least* $f \Longrightarrow \nu$ *such that* $f{\restriction}\gamma = \mathrm{id}{\restriction}\gamma$ *with* $q, p(\nu) \in \mathrm{ran}(f^*)$, *in that if* g *is any other such with these two properties* (*meaning that if* $g \Longrightarrow \nu$ *with extension* g^* *is so that* $\gamma \cup \{q, p(\nu)\} \subseteq \mathrm{ran}(g^*)$), *then* $g^{-1} f_{(\gamma,q,\nu)} \in \mathbb{F}$.

(ii) $f_{(\gamma,q,\nu)} = f_{(\beta,q,\nu)}$ *where* $\beta = \beta(f_{(\gamma,q,\nu)})$.

(iii) $f_{(\nu,0,\nu)} = \mathrm{id}_\nu$.

(iv) *Let* $f : \bar{\nu} \Longrightarrow \nu$ *with* $\bar{\gamma} \leq \bar{\nu}, f``\bar{\gamma} \subseteq \gamma \leq \nu, \bar{q} \in J_{\bar{\nu}}, f^*(\bar{q}) = q$, *then* $\mathrm{ran}(f^* f^*_{(\bar{\gamma},\bar{q},\bar{\nu})}) \subseteq \mathrm{ran}(f^*_{(\gamma,q,\nu)})$.

With (i) *this implies: if* $\beta(f) \geq \gamma$ *then* $f f_{(\bar{\gamma},\bar{q},\bar{\nu})} = f_{(\gamma,q,\nu)}$.

(v) *Set* $g = f_{(\gamma,q,\nu)}$; $\lambda = \lambda(g)$ *and* $g_0 = \mathrm{red}(g)$. *Then* $q \in J_\lambda$ *and* $g_0 = f_{(\gamma,q,\lambda)}$.

Proof. For (i) note this makes sense since we have specified in effect that $\mathrm{ran}(g^*) \supseteq \mathrm{ran}(f^*_{(\gamma,q,\nu)})$. (i)–(iv) are easy consequences of the definitions. We verify (v). We know that $g_0 \Longrightarrow \lambda$. Set $g'_0 = f_{(\gamma,q,\lambda)}$ and we shall argue that $g_0 = g'_0$. Let $k = g_0^{-1} g'_0$. The argument of Lemma 2.13 shows that $d(g_0) = d(g)$; as $g_0{\restriction}\gamma = \mathrm{id}{\restriction}\gamma$, and $q \in \mathrm{ran}(g_0)$ by (i) the minimality of $g'_0 \Longrightarrow \lambda$ implies we have such a k defined. Thus $k \in \mathbb{F}$. But $k \Longrightarrow d(g_0)$ so we conclude, as $d(g_0) = d(g)$, that $gk \in \mathbb{F}$. But $\mathrm{ran}((gk)^*) \cap \lambda = \mathrm{ran}(g^*) \cap \lambda$. So, using that $gk{\restriction}\gamma = \mathrm{id}{\restriction}\gamma$, and $q, p(\nu) \in \mathrm{ran}(gk)$, and then (i) again, we have $(gk)^{-1} g = k^{-1} \in \mathbb{F}$. Hence $k = \mathrm{id}_{d(g'_0)}$ and thus $g_0 = g'_0$. ⊣

When we have a *cofinal* map f, meaning that $|f|$ is cofinal into $r(f)$, as in the next lemma, then our definitions are preserved through $\Longrightarrow$:

2.18 Lemma. *Let* $f : \bar\nu \Longrightarrow \nu$ *with* $\lambda(f) = \nu$. *Let* $\bar\gamma < \bar\nu$, *and* $\overline{q} \in J_{\bar\nu}$, *with* $\gamma = f(\bar\gamma)$, $f(\bar q) = q$. *Set* $\bar g = f_{(\bar\gamma,\bar q,\bar\nu)}$; $g = f_{(\gamma,q,\nu)}$. *Then*

(i) $\lambda(\bar g) < \bar\nu \Longleftrightarrow \lambda(g) < \nu$;

(ii) *If* $\lambda(\bar g) < \bar\nu$ *then* $f(\lambda(\bar g)) = \lambda(g)$ *and* $f(\beta(\bar g)) = \beta(g)$.

Proof. We first assume $\lambda(\bar g) < \bar\nu$. Set $\lambda' = f(\lambda(\bar g))$. The following is $\Pi_1^{(n)M_{\bar\nu}}(\{\lambda(\bar g), \bar\gamma, p(\bar\nu)\})$:

$$\forall x^n \forall \xi^n < \bar\gamma \forall i < \omega (x^n = \tilde h_{\bar\nu}(i, \langle \xi^n, \overline{q}, p(\bar\nu)\rangle) \wedge x^n < \bar\nu \longrightarrow x^n < \lambda(\bar g)).$$

(if $\bar\nu = \mathrm{On} \cap M_{\bar\nu}$ the conjunct $x^n < \bar\nu$ is omitted). Hence

$$\forall x^n \forall \xi^n < \gamma \forall i < \omega (x^n = \tilde h_{\nu}(i, \langle \xi^n, q, p(\nu)\rangle) \wedge x^n < \nu \longrightarrow x^n < \lambda')$$

as f is $\Pi_1^{(n)}$-preserving. Hence $\lambda' \geq \lambda(g)$.

Claim 1. $\lambda' \leq \lambda(g)$.

As $\lambda(\bar g) < \bar\nu$ we have $\omega\rho(\bar g) < \omega\rho_{\bar\nu}$ by Lemma 2.14. Hence if we set $\bar N = \langle J_{\rho(\bar g)}, A^{n,p_{\bar\nu}\restriction n} \cap J_{\rho(\bar g)}\rangle$ we have that $\overline{N} \in M_{\bar\nu}$ and is an amenable structure, with $\lambda(\bar g) = \sup(\bar\nu \cap h_{\overline{N}}(\bar\gamma \cup \{\bar q, p(\bar\nu) \cap \omega\rho_{\bar\nu}\})$.

Applying f^*, and with $N = f(\overline{N})$, we have (noting that $f(\{\bar q, p(\bar\nu) \cap \omega\rho_{\bar\nu}\}) = \{q, p(\nu) \cap \omega\rho_\nu\})$

$$\lambda' = \sup(\nu \cap h_N(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\}).$$

However for amenable structures (including N) we have a uniform definition of the canonical $\Sigma_1(N)$-Skolem function h_N. As $\langle N, A_N\rangle$ is a Σ_0 substructure of $\langle M^n_\nu, A^n_\nu\rangle$, we have that $h_N \subseteq h_\nu$. Hence

$$\lambda' = \sup(\nu \cap h_\nu(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\})) = \sup(\nu \cap \tilde h_\nu(\gamma \cup \{q, p(\nu)\})).$$

Thus $\lambda' \leq \lambda(g)$. This finishes *Claim 1*.

Claim 2. $f(\beta(\bar g)) = \beta(g)$.

Let $\beta = f(\beta(\bar g))$. Note that $\bar g = f_{(\beta(\bar g), \bar q, \bar\nu)}$. Consequently $\beta(\bar g) \notin \mathrm{ran}(\bar g)$.

$$\begin{aligned}\beta = f(\beta(\bar g)) &= f(\sup\{\bar\delta < \bar\nu \mid \bar\delta \subseteq \mathrm{ran}(\bar g)\}) \\ &= f(\sup\{\bar\delta < \bar\nu \mid \bar\delta \subseteq h_{\overline{N}}(\bar\gamma \cup \{\bar q, p(\bar\nu) \cap \omega\rho_{\bar\nu}\})\}) \\ &= \sup\{\delta < \nu \mid \delta \subseteq h_{\bar N}(\gamma \cup \{q, p(\nu) \cap \omega\rho_\nu\})\}.\end{aligned}$$

And by the argument above is less than or equal to $\sup\{\delta < \nu \mid \delta \subseteq h_\nu(\delta \cup \{q, p(\nu) \cap \omega\rho_\nu\}) = \beta(g)$. Suppose however $\beta < \beta(g)$. Then in M_ν we have:

$$\exists \xi^n < \gamma \exists i < \omega (\beta = \tilde h_\nu(i, \langle \xi, q, p(\nu)\rangle).$$

However f is $\Sigma_1^{(n)}$-preserving, so this goes down to $M_{\bar\nu}$ as:

$$\exists \bar\xi^n < \bar\gamma \exists i < \omega(\beta(\bar g) = \tilde h_{\bar\nu}(i, \langle \bar\xi^n, \bar q, p(\bar\nu)\rangle).$$

But this implies $\beta(\bar g) \in \mathrm{ran}(\bar g)$ after all which is a contradiction! This finishes *Claim 2* and (ii). Finally, just note for ($\Longleftarrow$) of (i) as $\rho(f) = \rho_\nu$, if $\lambda(g) < \nu$ then by Lemma 2.14 there is an $\eta = f(\bar\eta) < \rho(f)$ with $\tilde h_\nu(\gamma \cup \{q, p(\nu)\}) \cap \omega\rho_\nu \subseteq \eta$. As before this is $\Pi_1^{(n)}$ and goes down to $M_{\bar\nu}$ as $\tilde h_{\bar\nu}(\bar\gamma \cup \{\bar q, p(\bar\nu)\}) \cap \omega\rho_{\bar\nu} \subseteq \bar\eta$. Hence $\lambda(\bar g) < \lambda$. $\dashv$

2.19 Definition. Let $\nu \in S$, $q \in J_\nu$. $B(q,\nu) =_{\mathrm{df}} B^+(q,\nu)\backslash\{\nu\}$ where $B^+(q,\nu) =_{\mathrm{df}} \{\beta(f_{(\gamma,q,\nu)}) \mid \gamma \leq \nu\}$.

Then $B(q,\nu)$ is the set of those $\beta < \nu$ so that $\beta = \beta(f)$ where $f = f_{(\beta,q,\nu)}$. Recall that B^* is always the class of limit points of B for any set $B \subseteq \mathrm{On}$.

2.20 Lemma. *Let $f = f_{(\gamma,q,\nu)}$ where $q \in J_\nu$.*

(i) *Suppose $\gamma \in B(q,\nu)^*$. Then* $\mathrm{ran}(f) = \bigcup_{\beta\in B(q,\nu)\cap\gamma} \mathrm{ran}(f_{(\beta,q,\nu)})$.

(ii) *Let $\gamma \leq \nu$. Suppose $\bar\nu$ is such that $f : \bar\nu \Longrightarrow \nu$ with $f(\bar q) = q$. Then $\gamma \cap B(q,\nu) = B(\bar q, \bar\nu)$.*

(iii) *Let $\lambda = \lambda(f)$; $f_0 = \mathrm{red}(f)$. Then $\gamma \cap B(q,\lambda) = \gamma \cap B(q,\nu)$.*

Proof. (i) is clear; (ii) follows from Lemma 2.17(iv), and (iii) from (ii) and Lemma 2.17(v). $\dashv$

2.21 Definition. Let $\nu \in S$, $q \in J_\nu$.

$$\Lambda^+(q,\nu) =_{\mathrm{df}} \{\lambda(f_{(\gamma,q,\nu)}) \mid \gamma \leq \nu\},$$
$$\Lambda(q,\nu) =_{\mathrm{df}} \Lambda^+(q,\nu)\backslash\{\nu\}.$$

Notice that $\Lambda(q,\nu) \subseteq C_\nu$ and we can think of these as first approximations to C_ν as q varies. We proceed to analyse these sets.

2.22 Lemma. *Let $\nu \in S, q \in J_\nu$.*

(i) *$\Lambda(q,\nu)$ is closed below ν;*

(ii) *$\mathrm{ot}(\Lambda(q,\nu)) \leq \nu$;*

(iii) *if $\lambda \in \Lambda(q,\nu)$ then $q \in J_\lambda$ and $\Lambda(q,\lambda) = \lambda \cap \Lambda(q,\nu)$.*

Proof. Set $\Lambda = \Lambda(q,\nu)$.

(i) Let $\eta \in \Lambda^*$. We are claiming that $\eta \in \Lambda^+(q,\nu)$. For each $\lambda \in \Lambda(q,\nu) \cap \eta$ pick $\beta_\lambda \in B(q,\nu)$ with $\lambda(f_{(\beta,q,\nu)}) = \lambda$. Clearly $\lambda \leq \lambda' \longrightarrow \beta_{\lambda'} \leq \beta_\lambda$. Let γ be the supremum of these β_λ, and then using the closure of $B(q,\nu)$ from (i) of the last lemma, we have $\lambda(f_{(\gamma,q,\nu)}) = \sup_\lambda \lambda(f_{(\beta_\lambda,q,\nu)}) = \eta$.

(ii) Clear.

(iii) Let $\lambda \in \Lambda$, and $g = \lambda(f_{(\beta,q,\nu)})$, where we take $\beta = \beta(g)$. Suppose $g : \bar{\nu} \Longrightarrow \nu$. Let $g(\overline{q}) = q$ and set $g_0 = \mathrm{red}(g)$. Then by Lemma 2.17(v) $g_0 = \lambda(f_{(\beta,q,\nu)})$). If $\gamma \geq \beta$ then $\lambda = \lambda(f_{(\gamma,q,\lambda)}) \leq \lambda(f_{(\gamma,q,\nu)})$. If $\gamma \leq \beta$ then

$$|f_{(\gamma,q,\lambda)})| = |g_0||f_{(\gamma,\bar{q},\bar{\nu})}| = |g||f_{(\gamma,\bar{q},\bar{\nu})}) = |f_{(\gamma,q,\nu)}|$$

where the first equality is justified by Lemma 2.17(v). ⊣

2.23 Lemma. *If $f : \bar{\nu} \Longrightarrow \nu$, $\mu = \lambda(f)$, $\overline{q} \in J_{\overline{\overline{\nu}}}$, $f(\overline{q}) = q$, then:*

(i) $\Lambda(\overline{q}, \bar{\nu}) = \varnothing \Longrightarrow \mu \cap \Lambda(q, \nu) = \varnothing$*;*

(ii) f *"*$\Lambda(\overline{q}, \bar{\nu}) \subseteq \Lambda(q, \mu)$*;*

(iii) *if* $\overline{\lambda} = \max(\Lambda(\overline{q}, \bar{\nu}))$ *and* $\lambda = f(\bar{\lambda})$ *then* $\lambda = \max(\mu \cap \Lambda(q, \nu))$.

Proof. (i) By its definition, if $\Lambda(\overline{q}, \bar{\nu}) = \varnothing$ then $f_{(0,\overline{q},\bar{\nu})}$ is cofinal into $\bar{\nu}$. Hence $\mathrm{ran}(f f_{(0,\overline{q},\bar{\nu})})$ is both cofinal in μ, and contained in $\mathrm{ran}(f_{(0,q,\nu)})$ by Lemma 2.17(iv), thus $\mu \cap \Lambda(q, \nu) = \varnothing$. This finishes (i). Note that by Lemma 2.22(iii) $\Lambda(q, \mu) = \mu \cap \Lambda(q, \nu)$. Let $f_0 = \mathrm{red}(f)$.

(ii) Let $\overline{\lambda} = \lambda(f_{(\bar{\beta},\overline{q},\bar{\nu})}) \in \Lambda(\overline{q}, \bar{\nu})$, and let $f(\overline{\beta}, \overline{\lambda}) = \beta, \lambda = f_0(\overline{\beta}, \overline{\lambda})$. Then $f_0(\lambda(f_{(\bar{\beta},\bar{q},\bar{\nu})})) = \lambda(f_{(\beta,q,\mu)}) \in \Lambda(q, \mu)$.

(iii) Let $\overline{\beta} = \sup\{\gamma \mid \lambda(f_{(\gamma,\bar{q},\bar{\nu})}) \leq \overline{\lambda}\}$. Then $\lambda(f_{(\bar{\beta},\bar{q},\bar{s})}) = \overline{\lambda}$, and by the assumed maximality of $\overline{\beta}$ we have $\lambda(f_{(\bar{\beta}+1,\bar{q},\bar{\nu})})) = \bar{\nu}$. Set $\beta = f(\bar{\beta}) = f_0(\bar{\beta})$, then $\lambda = f_0(\bar{\lambda}) = \lambda(f_{(\beta,q,\mu)})$ using Lemma 2.18 for second equality. However $\lambda(f_{(\beta+1,q,\mu)}) \geq \mu$, since, again by Lemma 2.17(iv), $\mathrm{ran}(f_0 f_{(\bar{\beta}+1,}\bar{q}, \bar{\nu})) \subseteq \mathrm{ran}(f_{(\beta+1,q,\mu)})$. Hence $\lambda = \max(\Lambda(q, \mu)) = \max(\mu \cap \Lambda(q, \nu))$. ⊣

We now note that our definitions of $\lambda(f)$, $B(q, \nu)$, $\Lambda(q, \nu)$, etc. are extremely uniform (and indeed are primitive recursively defined) in the appropriate parameters. In particular if $\mu \in S$, then we can define $F_\mu = \{f_{(\gamma,q,\nu)} \mid \nu \in S \cap \mu, q \in J_\nu, \gamma \leq \nu\}$, $E_\mu = \{\langle \nu, M_\nu, p(\nu), \tilde{h}_\nu \rangle \mid \nu \in S \cap \mu\}$, and $G_\mu = \{\langle \langle \nu, q \rangle, \Lambda(q, \nu) \rangle \mid q \in J_\nu, \nu \in S \cap \mu\}$.

2.24 Remark. E_μ, F_μ, G_μ are uniformly $\Delta_1(J_\mu)$ for $\mu \in S$, with, for $\mu' < \mu, E_{\mu'}, F_{\mu'}, G_{\mu'} \in J_\mu$.

2.25 Lemma. *Let $f : \bar{\nu} \Longrightarrow \nu$ with $\overline{q} \in J_{\overline{\overline{\nu}}}, f(\overline{q}) = q$. Then*

(i) *If $\lambda(f) = \nu$ then $|f| : \langle J_{\bar{\nu}}, \Lambda(\bar{q}, \bar{\nu}) \rangle \longrightarrow_{\Sigma_1} \langle J_\nu, \Lambda(q, \nu) \rangle$;*

(ii) *Otherwise: $|f| : \langle J_{\bar{\nu}}, \Lambda(\bar{q}, \bar{\nu}) \rangle \longrightarrow_{\Sigma_0} \langle J_\nu, \Lambda(q, \nu) \rangle$.*

Proof. (i) It suffices to show that $|f|(\Lambda(\bar{q}, \bar{\nu}) \cap \bar{\tau}) = \Lambda(q, \nu) \cap |f|(\tau)$ for arbitrarily large $\overline{\tau} < \bar{\nu}$. This will follow easily from the last lemma.

If $\bar{\lambda} \in \Lambda(\bar{q}, \bar{\nu})$, then $\Lambda(\bar{q}, \bar{\nu}) \cap \bar{\lambda} = \Lambda(\bar{q}, \bar{\lambda})$ by Lemma 2.22, and by the Remark, if $f(\bar{\lambda}) = \lambda$, we have $f(\Lambda(\bar{q}, \bar{\lambda})) = \Lambda(q, \lambda) = \lambda \cap \Lambda(q, \nu)$ (with the latter equality by Lemma 2.22 again). If $\Lambda(\bar{q}, \bar{\nu})$ is unbounded in $\bar{\nu}$, this suffices; if it is empty or bounded, then the last lemma takes care of these cases.

For non-cofinal maps (ii) we still have, if $\lambda(f) = \mu$, that

$$|f_0| : \langle J_{\bar\nu}, \Lambda(\bar q, \bar\nu)\rangle \longrightarrow_{\Sigma_1} \langle J_\mu, \Lambda(q, \mu)\rangle$$

where $f_0 = \mathrm{red}(f)$. However $\Lambda(q, \mu) = \mu \cap \Lambda(q, \nu)$, and $|f_0| = |f|$. ⊣

We turn to a decomposition of the C_ν sets into a finite sequence of sets of the form $\Lambda(l^i_\nu, \nu)$. This will enable us to extend the results of the last few lemmas from the $\Lambda(q, \nu)$ sets to C_ν sets, whilst at the same time giving us a handle on other data about the C_ν.

2.26 Definition. Let $s \in S$, $\eta \le \nu$. $l^i_{\eta\nu} < \nu$ is defined for $i < m_{\eta\nu} \le \omega$ by induction on i:

$$l^0_{\eta\nu} = 0; \qquad l^{i+1}_{\eta\nu} \simeq \max(\eta \cap \Lambda(l^i_{\eta\nu}, \nu)).$$

We also write l^i for $l^i_{\eta\nu}$ if the context is clear; also we set $l^i_\nu \simeq l^i_{\nu\nu}$; $m_\nu = m_{\nu\nu}$.

We note some easily checked facts about this definition (which we shall use without comment; the last here is by induction on i):

Facts.

- $l^i_{\eta\nu} \le l^{i+1}_{\eta\nu}$ $(i < m_{\eta\nu})$.
- $i > 0$ implies $l^i_{\eta\nu} \in \eta \cap C_\nu$.
- Let $l^i_{\eta\nu}$ be defined, and suppose $l^i_{\eta\nu} < \mu \le \eta$. Then $l^i_{\eta\nu} = l^i_{\mu\nu}$.

2.27 Lemma. *Let $f : \bar\nu \implies \nu$.*

(i) *If $\lambda = \lambda(f)$ then $l^i_{\lambda\nu} \simeq f(l^i_{\bar\nu})$;*

(ii) *let $\overline{\eta} < \bar\nu$, $f(\overline{\eta}) = \eta$; then $l^i_{\eta\nu} \simeq f(l^i_{\overline{\eta}\bar\nu})$.*

Proof. (i) By induction on i. If $i = 0$ this is trivial. Suppose $i = j + 1$. Then, as inductive hypothesis $l^j_{\lambda\nu} = f(l^j_{\bar\nu})$, and thus $|f| : \langle J_{\overline{\nu}}, \Lambda(l^j_{\bar\nu}, \bar\nu)\rangle \longrightarrow_{\Sigma_1} \langle J_\lambda, \Lambda(l^j_{\lambda\nu}, \lambda)\rangle$, by the last lemma, as $|\mathrm{red}(f)| = |f|$. However $\Lambda(l^j_{\lambda\nu}, \lambda) = \lambda \cap \Lambda(l^j_{\lambda\nu}, \nu)$, by Lemma 2.22. Hence, $f(l^i_{\bar\nu}) \simeq f(\max \Lambda(l^j_{\bar\nu}, \bar\nu)) \simeq \max(\lambda \cap \Lambda(l^j_{\lambda\nu}, \nu)) \simeq l^i_{\lambda\nu}$ with the middle equality holding by the inductive hypothesis and Lemma 2.23(iii).

(ii) is proved similarly. ⊣

2.28 Corollary.

(i) *Let $f : \bar\nu \implies \nu$ cofinally. Then $l^i_\nu \simeq f(l^i_{\bar\nu})$.*

(ii) *Let $\lambda \in C_\nu$. Then $l^i_{\lambda\nu} \simeq l^i_\lambda$.*

Proof. (i) is immediate. For (ii) choose $f : \bar\nu \implies \nu$ with $\lambda = \lambda(f)$, and set $f_0 = \mathrm{red}(f)$. Then $l^i_{\lambda\nu} \simeq f(l^i_{\bar\nu}) \simeq f_0(l^i_{\bar\nu}) \simeq l^i_\lambda$ with the last equality holding from (i). $\dashv$

2.29 Lemma. *Let $\eta \le \nu$, $\lambda = \min(C^+_\nu \setminus \eta)$. Then $l^i_\nu \simeq l^i_{\lambda\nu} \simeq l^i_{\eta\nu}$ (for any $i < \omega$ for which any term is defined).*

Proof. Induction on i, again $i = 0$ is trivial. Suppose $l^j_\nu = l^j_{\eta\nu} = l^j_{\lambda\nu}$ and $i = j+1$. Set $l = l^j_{\eta\nu}$; then we have: $\Lambda(l,\nu) \cap \eta = \Lambda(l,\nu) \cap \lambda$, since $\Lambda(l,\nu) \subseteq C_\nu$ and $C_\nu \cap [\eta,\lambda) = \varnothing$. Suppose, without loss of generality that $l^i_{\eta\nu}$ is defined. Then $l^i_{\eta\nu} = \max(\eta \cap \Lambda(l,\nu)) = \max(\lambda \cap \Lambda(l,\nu)) = l^i_{\lambda\nu} = l^i_\lambda$. $\dashv$

2.30 Lemma. *Let $j \le i < m_\nu$. Set $l = l^i_\nu$. Then $l^j_\nu \in \mathrm{ran}(f_{(0,l,\nu)})$.*

Proof. Set $f = f_{(0,l,\nu)}$. Suppose $f : \bar\nu \implies \nu$, and $\lambda = \lambda(f)$. Then $l^j_{\lambda\nu} \simeq f(l^j_{\bar\nu})$ by Lemma 2.27*(i)*. But l^j_ν exists, and $l^j_\nu < \lambda \le \nu$. Hence $l^j_\nu = l^j_{\lambda\nu} = f(l^j_{\bar\nu})$. $\dashv$

We now prove a key lemma showing that the $\langle l^j_{\lambda\nu}\rangle$ sequences are finite.

2.31 Lemma. *Let $\nu \in S$, $\eta \le \nu$. Then $m_{\eta\nu} < \omega$.*

Proof. Suppose this fails. Then for some $\eta \le \nu$ we have that $l^i_{\eta\nu}$ is defined for $i < \omega$. Let $\lambda = \min(C^+_\nu \setminus \eta)$. Then $l^i_{\lambda\nu} = l^i_{\eta\nu}$ by Lemma 2.29. Choose $f : \bar\nu \implies \nu$ with $\lambda = \lambda(f)$. Then $l^i_{\lambda\nu} = l^i_\lambda = f(l^i_{\bar\nu})$ for $i < \omega$ by Corollary 2.28(ii) and Lemma 2.27(i). Hence, taking ν as λ, we may assume, without loss of generality, that l^i_ν is defined for $i < \omega$ for some $\nu \in S$. This will yield a contradiction. We obtain an infinite descending chain of ordinals by showing that as i increases, and thus l^i_ν does so strictly too, the maximal β^i that must be contained in the range of any $f :\implies \nu$ together with l^i_ν in order for $\mathrm{ran}(f)$ to be unbounded in ν strictly *decreases.* This is absurd.

Set $l = l^i_\nu$. Define: $\beta^i = \beta^i_\nu =_{\mathrm{df}} \max\{\beta \mid \lambda(f_{(\beta,l,\nu)}) < \nu\}$. By the definition of l^{i+1}_ν we have that $\lambda(f_{(\beta,l,\nu)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta,l,\nu)}) \le l^{i+1}_\nu$. Furthermore, by the definition of β^i:

(1) $\lambda(f_{\beta^i,l,\nu}) \le l^{i+1}_\nu$;

(2) $\lambda(f_{\beta^i+1,l,\nu}) = \nu$.

Claim. $\beta^{i+1} < \beta^i$ for $i < \omega$.

Proof of Claim. Set $f = f_{(\beta^{i+1},l^{i+1},\nu)}$. Then $\lambda(f) = l^{i+2}$. (We omit the subscript ν from now on.) Let $f : \bar\nu \implies \nu$. Then $l^j_{\bar\nu}$ exists and $f(l^j_{\bar\nu}) = l^j_{l^{i+1},\nu} = l^j_\nu$ for $j < i+1$ since $l^j < l^{i+1} < \nu$. (The first equality comes from Lemma 2.27(i) and (1), the second from Lemma 2.29.) $\dashv$ *Claim*

(3) $\beta^i \geq \beta^{i+1}$.

Proof of (3). Suppose not. Then $(\beta^i + 1) \cup \{l^i\} \subseteq \mathrm{ran}(f)$. Hence we have $\mathrm{ran}(f_{(\beta^i+1,l^i,\nu)}) \subseteq \mathrm{ran}(f)$. Consequently by (2), $\lambda(f) = \nu > l^{i+2}$. Contradiction! $\dashv$ (3)

(4) $\beta^i \neq \beta^{i+1}$.

Proof of (4). Suppose not. As β^{i+1} is the first ordinal moved by f we conclude that $f(\beta^i) > \beta^i$. Set $g = f_{(\beta^i,l,\nu)}, \bar{g} = f_{(\beta^i,\bar{l},\bar{\nu})}$ where $\bar{l} = l^i_{\bar{\nu}}$. Then $g = f\bar{g}$, since $f{\restriction}\beta^i = \mathrm{id}, f(\bar{l}) = l(= l^i_\nu)$. Hence $l^{i+1} = \lambda(g) = \lambda(f\bar{g}) < l^{i+2} = \lambda(f)$. Hence $\lambda(\bar{g}) < \bar{\nu}$. Now we set: $g_0 = f_{(\beta^i,l,l^{i+2})}$. If further $f_0 = \mathrm{red}(f)$, then we have also $g_0 = f_0\bar{g}$ by Lemma 2.17(iv). As $l^{i+1} = \lambda(g) < l^{i+2}$, Lemma 2.18(ii) applies and:

$$f(\beta(\bar{g})) = f_0(\beta(\bar{g})) = \beta(g_0) = \beta(g) = \beta^i.$$

Hence $\beta^i \in \mathrm{ran}(f)$ which is a contradiction. This proves the *Claim* and hence the lemma. $\dashv$

We now set $l_{\eta\nu} = l^{m-1}_{\eta\nu}$, where $m = m_{\eta\nu}$. Again we write l_ν for $l_{\nu\nu}$. We note that then $\Lambda(l_{\eta\nu}, \nu) \cap \eta$ is either unbounded in η or is empty. We analyse the latter case further.

2.32 Lemma. *Suppose* $\Lambda(l_{\eta\nu}, \nu) \cap \eta = \varnothing$. *Set* $l = l_{\eta\nu}$. *Then:*

(i) $l = 0 \Longrightarrow C_\nu \cap \eta = \varnothing$;

(ii) $l > 0 \Longrightarrow l = \max(C_\nu \cap \eta)$;

(iii) $\eta \in C^+_\nu \Longrightarrow \eta = \lambda(f_{(0,l,\nu)})$.

Proof. Set $\rho = \min(C^+_\nu \setminus (l+1)$.

(1) $l = l_{\rho\nu}$.

Proof of (1). Set $n = m_{\eta\nu} - 1$. Then $l = l^n_{\eta\nu} < l+1 < \eta$. Hence (by Fact after Definition 2.1) $l = l^n_{l+1,\nu}$. But $\Lambda(l,\nu) \cap (l+1) = \varnothing$. Hence $l^{n+1}_{l+1,\nu}$ is undefined and $l = l_{l+1,\nu}$. Hence $l = l_{\rho,\nu}$ by Lemma 2.29. $\dashv$ (1)

(2) $\lambda(f_{(0,l,\nu)}) = \rho$.

Proof of (2). Choose $f : \bar{\nu} \implies \nu$, with $\lambda(f) = \rho$ witnessing that $\rho \in C_\nu$. Then, by Lemma 2.27(i), $f(l_{\bar{\nu}}) = l_{\rho\nu} = l$. Set $\bar{l} = l_{\bar{\nu}}$. Now note that we must have that $\lambda(f_{(0,\bar{l},\bar{\nu})}) = \bar{\nu}$. For, if this failed then $f(\lambda(f_{(0,\bar{l},\bar{\nu})})) = \lambda(f_{(0,l,\nu)}) < \rho$ by Lemma 2.18, and so the latter is in $C^+_\nu \cap (l,\rho)$, which is absurd! Then $\lambda(f_{(0,l,\nu)}) = \lambda(ff_{(0,\bar{l},\bar{\nu})}) = \lambda(f) = \rho$. $\dashv$ (2)

From (2) it follows that $\rho \geq \eta$ (for otherwise this contradicts the definition of l as $l_{\eta\nu}$). Hence we have the two alternatives:

If $l = 0$ then (i) holds: $\rho = \min(C^+_\nu \backslash 1) = \min(C^+_\nu) \geq \eta$. If $l > 0$ then $l = \max(C_\nu \cap \eta)$ since $(C_\nu \cap \eta)\backslash(l+1) \subseteq (C_\nu \cap \rho)\backslash(l+1) = \varnothing$ and thus we have (ii).

Finally for (iii) if $\eta \in C^+_\nu$ then $\eta = \max(C^+_\nu \backslash (l+1) = \rho = \lambda(f_{(0,l,\nu)})$. $\dashv$

We now get the closure of the sets C^+_ν as well as a precise characterisation of the points they contain.

2.33 Lemma. *Let λ be an element or a limit point of C^+_ν. Let $l = l_{\lambda\nu}$. Then there is a β such that $\lambda = \lambda(f_{(\beta,l,\nu)})$. Hence C_ν is closed in ν, and*

$$C^+_\nu = \{\lambda(f_{(\beta,l,\nu)}) \mid \beta \leq \nu, l < \nu\}.$$

Proof. The last sentence is immediate from the penultimate one.

Case 1. $\lambda \cap \Lambda(l, \nu) = \varnothing$.

Then $C_\nu \cap \lambda = \varnothing$ or $l = \max(C_\nu \cap \lambda)$ by the last lemma. Hence λ is not a limit point of C^+_ν. Hence $\lambda \in C^+_\nu$, and thus $\lambda = \lambda(f_{(0,l,\nu)})$ by (iii) of that lemma.

Case 2. $\lambda \cap \Lambda(l, \nu)$ is unbounded in λ.

Given $\mu \in \Lambda(l, \nu) \cap \lambda$, let β_μ be such that $\lambda(f_{(\beta_\mu,l,\nu)}) = \mu$. Then $\lambda(f_{(\beta,l,\nu)}) = \lambda$ where $\beta = \sup_\mu \beta_\mu$. $\dashv$

The following is (iii) of Theorem 2.5.

2.34 Lemma. $\lambda \in C_\nu \Longrightarrow \lambda \cap C_\nu = C_\lambda$.

Proof. Assume inductively the lemma proven for all ν' with $\nu' < \nu$, and we prove the result for ν by induction on λ. Let $l = l_{\lambda\nu}$. Hence by Corollary 2.28 $l = l_\lambda$. By Lemma 2.33 $\lambda \in \Lambda(l, \nu)$. Set $\Lambda = \lambda \cap \Lambda(l, \nu)$. Then by Lemma 2.22(ii) $\Lambda = \Lambda(l, \lambda)$.

Case 1. $\Lambda = \varnothing$.

If $l = 0$, then $C_\lambda \subseteq \lambda \cap C_\nu = \varnothing$ (the latter by Lemma 2.32). If $l > 0$, then $l = l_\lambda = \max(C_\lambda \cap \lambda) = \max(C_\lambda) = l_{\lambda\nu} = \max(\lambda \cap C_\nu)$ with the last equality also by the same lemma. As $l < \lambda$, we use the inductive hypothesis on λ: $l \cap C_\nu = C_l$. However, $C_l = l \cap C_\lambda$ by the overall inductive hypothesis taking λ as a $\nu' < \nu$. Hence $C_\lambda = \lambda \cap C_\nu = C_l \cup \{l\}$.

Case 2. Λ is unbounded in λ.

Then $\mu \in \Lambda \longrightarrow \mu \in C_\nu \cap C_\lambda$. Hence by the overall inductive hypothesis $C_\mu = \mu \cap C_\lambda$ and (as $\mu < \lambda$) $C_\mu = \mu \cap C_\nu$. Hence $C_\lambda = \lambda \cap C_\nu = \bigcup_{\mu \in \Lambda} C_\mu$. $\dashv$

The following completes (i) of Theorem 2.5:

2.35 Lemma. $\sup(C_\nu) < \nu \Longrightarrow \mathrm{cf}(\nu) = \omega$.

Proof. Let $l = \sup(C_\nu) = l_\nu$. Then $X = \mathrm{On} \cap \mathrm{ran}(f_{(0,l,\nu)})$ is countable, and cofinal in ν. $\dashv$

We now observe that we readily have (ii) of Theorem 2.5, since $\mathrm{ot}(C_\nu) \leq \sum_{i=0}^{m_\nu - 1} \beta_\nu^i < \nu$ where the β_ν^i are from Lemma 2.31.

2.36 Lemma. *Let $f : \bar\nu \Longrightarrow \nu$. Then $|f| : \langle J_{\bar\nu}, C_{\bar\nu}\rangle \longrightarrow_{\Sigma_0} \langle J_\nu, C_\nu\rangle$.*

Proof. It suffices to show that for arbitrarily large $\tau < \bar\nu$ that $|f|(C_{\bar\nu} \cap \tau) = C_\nu \cap |f|(\tau)$. (As usual we shall write "f" for "$|f|$".)

Case 1. $\Lambda(l_{\bar\nu}, \bar\nu)$ is unbounded in $C_{\bar\nu}$.

If $\bar\lambda \in C_{\bar\nu}$ and $\lambda = f(\bar\lambda)$ then by Lemma 2.23 (and Lemma 2.22) $\lambda \in \Lambda(f(l_{\bar\nu}), \nu) \subseteq C_\nu$. By Remark 2.24 we have $E_{\bar\lambda} \in J_{\bar\nu}$ and $f(E_{\overline\lambda}) = E_\lambda$. By Lemma 2.32 $C_{\bar\lambda} = \{\lambda(f_{(0,l,\nu)}) < \bar\lambda \mid l < \bar\lambda\} \in J_{\bar\nu}$ and is uniformly Σ_0 definable from $E_{\bar\lambda}$ over $J_{\bar\nu}$. Consequently $|f|(C_{\overline\lambda}) = C_\lambda$ by Σ_1-elementarity of $|f|$. But $C_{\overline\lambda} = \overline\lambda \cap C_{\bar\nu}$ and $C_\lambda = \lambda \cap C_\nu$.

Case 2. $\Lambda(l_{\bar\nu}, \bar\nu) = \varnothing$.

Set $l_{\bar\nu} = \bar l$ and let $f(l_{\bar\nu}) = l$. Then $l = l_{\lambda\nu}$ where $\lambda = \lambda(f)$. However $\lambda(f_{0,\bar l,\bar\nu}) = \bar\nu$ by our case hypothesis. Thus $\lambda(f_{(0,l,\nu)}) = \lambda(f f_{(0,\bar l,\bar\nu)}) = \lambda$. Hence $\Lambda(l,\nu) \cap \lambda = \varnothing$. So we have one of the following two cases holding, by Lemma 2.32:

Case 2.1. $\bar l = l = 0$. Then, $C_{\bar\nu} = C_\nu \cap \lambda = \varnothing$, and the result is trivial.

Case 2.2. $\bar l = \max C_{\bar\nu}$. Then $l > 0$ and thus $l = \max(C_\nu \cap \lambda)$. Hence for sufficiently large $\bar\tau > \bar l$:

$$f(\bar\tau \cap C_{\bar\nu}) = f(C_{\bar\nu}) = f(C_{\bar\nu} \cap \bar l \cup \{\bar l\}) = (C_\nu \cap l) \cup \{l\} = C_\nu \cap \lambda = f(\bar\tau) \cap C_\nu.$$

$\dashv$

The following completes (iv) of Theorem 2.4 on the condition that C_ν is cofinal in ν. We shall see afterwards how to tweak a C'_ν to get the result for the non-cofinal C_ν.

2.37 Lemma. *If $f : \langle J_{\overline\nu}, \overline{C}\rangle \longrightarrow_{\Sigma_1} \langle J_\nu, C_\nu\rangle$ and $\sup(C_\nu) = \nu$ then $\overline\nu \in S$ and $\overline{C} = C_{\overline\nu}$.*

Proof. Suppose $f, \overline{C}, \overline\nu$ are as in the antecedent. For $\tau \in C_\nu$, let $f_\tau : \overline\nu_\tau \Longrightarrow \nu$ with $\lambda(f_\tau) = \tau$. By the Interpolation Lemma we have (a) $f_{\overline\nu_\tau \nu_\tau} : M_{\overline\nu_\tau} \longrightarrow_{\Sigma_1^{(n)}} M_\tau$, (b) $\pi_\tau : M_\tau \longrightarrow_{\Sigma_0^{(n)}} M_\nu$; (c) $\pi_\tau \restriction \tau = \mathrm{id} \restriction \tau$, and (d) $\pi_\tau(p_\tau) = p_\nu$. Notice that the map $f_{\overline\nu_\tau \nu_\tau}$ can be defined from $M_{\overline\nu_\tau}$ and M_τ inside J_ν, and both of them in turn are there definable just from τ, since $\beta(\tau) < \nu$.

(1) π_τ, M_τ depend only on τ, and π_τ is the unique $\Sigma_0^{(n)}$-preserving map satisfying (b)–(d).

For such τ let $X = \bigcup_{\tau \in C_\nu} \mathrm{ran}(\pi_\tau)$.

(2) $M_\nu = X$.

Proof of (2). Let $x \in M_\nu$. Clearly X is a $\Sigma_0^{(n)}$ substructure of M_ν. Suppose $x = \widetilde{h}_\nu(i, \xi, p_\nu)$ for a $\xi < \nu$. Suppose we take for "$y = \widetilde{h}_\nu(u, v)$" a functionally absolute $\Sigma_1^{(n)}$ definition of the form $\exists z^n H(z^n, y, u, v)$ with $H \in \Sigma_0^{(n)}$. It will suffice to find $\tau > \xi$ and some $x_\tau \in M_\tau$ with $x_\tau = \widetilde{h}_\tau(i, \xi, p_\tau)$. We then would have $\pi_\tau(x_\tau) = x$. For this we need to be able to find such a suitable witnessing $z^n \in \mathrm{ran}(\pi_\tau)$ for some sufficiently large $\tau \in C_\nu$. For this it will suffice to know:

(3) $\sup_{\tau \in C_\nu}(\mathrm{ran}(\pi_\tau) \cap \omega\rho_\nu) = \omega\rho_\nu$.

Proof of (3). Let $l = l_\nu$. Then $\Lambda(l, \nu)$ is unbounded in ν. Let $\gamma = \sup\{\beta \mid \lambda(f_{(\beta,l,\nu)}) < \nu\}$. Then $\lambda(f_{(\gamma,l,\nu)}) = \nu$ and $\sup(\mathrm{ran}(f_{(\gamma,l,\nu)}) \cap \omega\rho_\nu) = \omega\rho_\nu$ by Lemma 2.14. For $\beta < \gamma$ let $\lambda_\beta = \lambda(f_{(\beta,l,\nu)})$. Then $\sup\{\lambda_\beta \mid \beta < \gamma\} = \nu$. However by considering $k_\beta =_{\mathrm{df}} \mathrm{red}(f_{(\beta,l,\nu)})$ we see that $\mathrm{ran}(k_\beta) \subseteq M_{\lambda_\beta}$ for $\beta < \gamma$. Hence $\mathrm{ran}(f_{(\beta,l,\nu)}) \subseteq \mathrm{ran}(\pi_{\lambda_\beta})$. Thus $\mathrm{ran}(f_{(\beta,l,\nu)}) \subseteq X$ and so $\sup(X \cap \omega\rho_\nu) = \omega\rho_\nu$. ⊣ (3)

If $Y = \mathrm{ran}(f)$ let $\widetilde{Y} = \widetilde{h}_\nu(Y \cup \{p_\nu\})$.

(4) If $y \in \widetilde{Y} \cap J_\nu$ then $\exists \tau \in C_\nu(y \in \widetilde{h}_\tau((Y \cap J_\tau) \cup \{p_\tau\})$, and hence $y \in Y$.

Proof of (4). This is because if "$y = \widetilde{h}_\nu(i, \xi, p_\nu)$" then by (2) this holds relativised to $\mathrm{ran}(\pi_\tau)$ for some $\tau \in C_\nu$. Hence $y = \widetilde{h}_\tau(i, \xi, p_\tau)$. But $M_\tau, p_\tau, \widetilde{h}_\tau$ etc. are Σ_1-definable over J_ν from τ, for any $\tau \in S \cap \nu$; hence "$\exists \tau \in C_\nu(i, \xi, p_\tau \in \mathrm{dom}(\widetilde{h}_\tau))$" is a $\Sigma_1^{\langle J_\nu, C_\nu \rangle}$ statement. Hence there is such a $\tau = f(\overline{\tau})$ in $\mathrm{ran}(f)$. Hence

$$\pi_\tau \widetilde{h}_\tau(i, \xi, p_\tau) = \widetilde{h}_\nu(i, \xi, p_\nu) = y.$$

Thus $y \in Y$. ⊣ (4)

We may now transitivise $\widetilde{Y}$ and obtain $\widetilde{f} \supseteq f$ with $\widetilde{f} : \widetilde{M} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$, and our usual arguments show that $\overline{\nu} \in S$, $\widetilde{M} = M_{\overline{\nu}}$, and $\langle \overline{\nu}, f, \nu \rangle \in \mathbb{F}$, etc. We have yet to argue that $\overline{C} = C_{\overline{\nu}}$. Clearly $\overline{C}$ is unbounded in $\overline{\nu}$. Let $\overline{l} = l_{\overline{\nu}}$. Then $\Lambda(\overline{l}, \overline{\nu})$ is unbounded in $\overline{\nu}$ (otherwise if $l =_{\mathrm{df}} f(\overline{l})$, then $l = l_\nu$ and $\Lambda(l, \nu)$—and C_ν—is bounded in ν).

Claim $\overline{C} \subseteq C_{\overline{\nu}}$.

Proof of Claim. If not let $\overline{\lambda}$ be least in $\overline{C}\backslash C_{\overline{\nu}}$. Suppose (for example) $\overline{\eta}_0 < \overline{\lambda}, \overline{\eta}_0 \in C_{\overline{\nu}}$ and $\overline{\eta} > \overline{\lambda}$ is $\min(\Lambda(\overline{l}, \overline{\nu})\backslash\overline{\lambda})$. As $\overline{\eta} \in \Lambda(\overline{l}, \overline{\nu}), \eta \in \Lambda(l, \nu)$ (where $\eta =_{\mathrm{df}} f(\overline{\eta})$ and as above $l = f(\overline{l}) = l_\nu$). So $\eta \in C_\nu$. Hence as $\overline{\lambda} \in \overline{C}\backslash(\overline{\eta}_0 + 1), f(\overline{\lambda}) \in f(\overline{C} \cap \overline{\eta}) = C_\nu \cap \eta = C_\eta$. However then $\overline{\lambda} \in C_{\overline{\eta}} \subseteq C_{\overline{\nu}}$! ⊣ *Claim*

However now, for $\overline{\lambda} \in \overline{C}$, we have

$$f(\overline{C} \cap \overline{\lambda}) = C_\nu \cap \lambda = C_\lambda = f(C_{\overline{\lambda}}) = f(C_{\overline{\nu}} \cap \overline{\lambda}).$$

So, $\overline{C} \cap \overline{\lambda} = C_{\overline{\nu}} \cap \overline{\lambda}$ for unboundedly many $\overline{\lambda} < \overline{\nu}$, and we are done. ⊣

We now indicate how to modify C_ν in case it is bounded in ν. As we know, if this happens then $\mathrm{cf}(\nu) = \omega$. To deal with this situation we substitute a new C'_ν. (In all other cases we shall alter nothing and set $C'_\nu = C_\nu$.) Clearly, if we set $l = l_\nu$ we have that $\widetilde{h}_\nu$"$\omega \times \{\langle l, p_\nu\rangle\}$ is unbounded in ν, and hence by Lemma 2.14 in $\omega\rho_\nu$.

Let $u_n = \nu \cap \widetilde{h}_\nu$"$n \times \{\langle l, p_\nu\rangle\}$. Then $u_n \in J_\nu$ and is finite, with $\bigcup_n u_n$ cofinal in ν. We take $\xi_{n,\nu} =_{\mathrm{df}} \prec n, l, \alpha_\nu, u_0, \ldots, u_n \succ$ (for some suitable iterate of the Gödel pairing function $\prec - \succ$.) Then $l < \xi_{n,\nu}$ and the latter is monotone as a function of n in ν.

2.38 Lemma. *Let* $\sup(C_\nu) < \nu$. *Let* $f : J_{\nu'} \longrightarrow_{\Sigma_0} J_\nu$ *with* $\xi_{n,\nu} \in \mathrm{ran}(f)$ *for every* $n < \omega$. *Then* (i) $\nu' \in S$; (ii) $\langle \nu', f, \nu\rangle \in \mathbb{F}$; *and* (iii) *if* $f(\xi'_n) = \xi_{n,\nu}$ *then* $\xi'_n = \xi_{n,\nu'}$ *and thus* $C_{\nu'} = \{\xi'_n \mid n < \omega\}$.

Proof. As f is cofinal into ν, it is Σ_1-preserving. (i) is immediate. Let $l = l_\nu$. Let $g = f_{(0,l,\nu)}$. Then $\forall n(\xi_{n,\nu} \in \mathrm{ran}(g))$ and $\lambda(g) = \nu$. Let $g : \overline{\nu} \implies \nu$, and $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M_\nu$.

Hence by the Generalised Interpolation Lemma 1.52 there is an M' (obtained by taking $f, f^{-1} \circ g, g$ here as σ_1, σ_0, g respectively there) and there is a $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} M'$ and $g_1 : M' \longrightarrow_{\Sigma_1^{(n)}} M_\nu$ (both being cofinal at the nth level). Just as in the proof of Lemma 2.11 $M' = M_{\nu'}$, $n = n(\nu')$ etc. Then as $g_1 \supseteq f$, (ii) follows. If $g(\overline{l}) = l$ then $\overline{l} = l_{\overline{\nu}}$ by Corollary 2.28; taking $l' = f^{-1} \circ g(\overline{l})$ this too shows that $l' = l_{\nu'}$. Then if $f(\xi'_n) = \xi_{n,\nu}$ we have $\xi'_n = \xi_{n,\nu'}$ by the $\Sigma_1^{(n)}$-elementarity of g_1. Hence $C'_{\nu'} = \{\xi_{n,\nu'} \mid n < \omega\}$, and we are done. ⊣

We now mention how a more careful computation of the order types of the C_ν-sequences can be obtained. This is needed for some applications and was done using ordinal addition in [2], but the following is essentially the same calculation and is from [29]. We first generalise the definition of $\beta^i = \beta^i_\nu$.

2.39 Definition. For $\eta \leq \nu$ set : $\beta^i_{\eta\nu} =_{\mathrm{df}} \max\{\beta \mid \lambda(f_{(\beta, l^i_{\eta\nu}, \nu)}) < \eta\}$.

Many of the previous properties of the $\beta^i = \beta^i_\nu$ carry over to these $\beta^i_{\eta\nu}$. Namely:

1. $\lambda(f_{(\beta,l^i_{\eta\nu},\nu)}) < \nu \iff \lambda(f_{(\beta,l^i_{\eta\nu},\nu)}) \leq l^{i+1}_{\eta\nu}$.
2. $\beta^i_{\eta\nu}$ is defined if and only if $l^{i+1}_{\eta\nu}$ is defined, i.e. when $i+1 < m_{\eta\nu}$.
3. $\beta^i_{\eta\nu} \simeq \beta^i_{\lambda\nu}$ if $\lambda = \min(C^+_\nu \backslash \eta)$. (Again as before, $\lambda(f_{(\beta,l^i_{\eta\nu},\nu)}) < \eta \iff \lambda(f_{(\beta,l^i_{\eta\nu},\nu)}) < \lambda$.)
4. $\beta^{i+1}_{\eta\nu} < \beta^i_{\eta\nu}$ when defined. (By the same argument as for $\beta^{i+1} < \beta^i$.)

Now we set $b_\eta = b_{\eta\nu} =_{\mathrm{df}} \{\beta^i_{\eta\nu} \mid i+1 < m_{\eta\nu}\}$. For $\eta \in C_\nu$ we then set $d_\eta = d_{\eta\nu} =_{\mathrm{df}} b_{\eta^+\nu}$ where $\eta^+ = \min(C^+_\nu \backslash (\eta+1))$. (For the rest of the proof we shall drop the subscript ν on ordinals, which remains unaltered throughout.) Then we shall have:

5. Let $\eta \in C_\nu$, with $l^i_{\eta^+} < \eta$. Then $l^i_{\eta^+} = l^i_\eta$ (proof by induction on i). Moreover:
6. Let $\eta \in C_\nu$, with $l^i_{\eta^+} < \eta$ then:

$$l^{i+1}_{\eta^+} = \eta \text{ if } \eta \in \Lambda(l^i_\eta, \nu), \text{ and equals } l^{i+1}_\nu \text{ if not.}$$

Proof of 6: $l^i_{\eta^+} = l^i_\eta$ by 5. If $\eta \in \Lambda(l^i_\eta, \nu)$ then η is maximal in this set below η^+. So the first alternative holds. Note that $i \neq m_{\eta\nu} - 1$ (otherwise by Lemma 2.33 for some β, $\eta = \lambda(f_{(\beta,l^i_{\eta\nu},\nu)}) \in \Lambda(l^i_\eta, \nu)$). Thus l^{i+1}_η is defined and $l^{i+1}_{\eta^+}$ must equal this.

2.40 Lemma. *Let $\eta, \mu \in C_\nu$, with $\eta < \mu$. Then $d_\eta <^* d_\mu$.*

Proof. Let $\eta^+ = \min(C^+_\nu \backslash (\eta+1))$, $\mu^+ = \min(C^+_\nu \backslash (\mu+1))$. Let i be maximal so that $l^i_{\mu^+} = l^i_{\eta^+}$. Then $\beta^j_{\mu^+} = \beta^j_{\eta^+}$ for $j < i$. As $l^i_{\mu^+} \leq \eta < \mu$, we have by 6. above that $l^{i+1}_{\mu^+}$ is defined and $l^{i+1}_{\mu^+} = \mu$ or l^{i+1}_μ. Moreover then $\beta^i_{\mu^+}$ is defined, and by maximality of i, $l^{i+1}_{\eta^+} \neq l^{i+1}_{\mu^+}$.

Claim $l^{i+1}_{\eta^+} < l^{i+1}_{\mu^+}$.

That $l^{i+1}_{\mu^+} < \eta^+$ is ruled out: otherwise $l^{i+1}_{\eta^+} = l^{i+1}_{\mu^+}$ again. So $l^{i+1}_{\eta^+} < \eta^+ \leq l^{i+1}_{\mu^+}$, establishing the Claim.

As $\beta^i_{\mu^+}$ is defined, if $\beta^i_{\eta^+}$is undefined, then we would be finished. Set $l = l^i_{\mu^+} = l^i_{\eta^+}$. Then $\lambda(f_{(\beta^i_{\eta^+},l,\nu)}) = l^{i+1}_{\eta^+}$ and $\lambda(f_{(\beta^i_{\mu^+},l,\nu)}) = l^{i+1}_{\mu^+}$. Hence $\beta^i_{\eta^+} < \beta^i_{\mu^+}$ and thus $d_\eta <^* d_\mu$ as required. ⊣

2.41 Lemma. *Let α be p.r. closed so that for some $\alpha_0 < \alpha$, $\lambda(f_{(\alpha_0,0,\nu)}) = \nu$. Then* $\mathrm{ot}(C_\nu) < \alpha$.

Proof. First note that $\mathrm{ot}(\langle [\alpha]^{<\omega}, <^* \rangle) = \alpha$. Let $\alpha_0 < \alpha$ have the property that $\lambda(f_{(\alpha_0,0,\nu)}) = \nu$. Then $\{\beta^i_{\eta\nu} \mid \eta \leq \nu, i+1 < m_{\eta\nu}\} \subseteq \alpha_0$. Thus $\mathrm{ot}\langle \{d_\eta \mid \eta \in C_\nu\}, <^* \rangle \leq \mathrm{ot}(\langle [\alpha]^{<\omega}, <^* \rangle) < \alpha$. Thus $\mathrm{ot}(C_\nu) < \alpha$. ⊣

We make some remarks on different versions of this proof. We have obtained a rich global $\Box$ sequence by taking as C_ν the set of all possible $\lambda(f)$ where $f \implies \nu$. There are other ways: the C_ν sequences from [18] are defined by a finite recursion piecing together enumerations of initial segments C^i_ν. [30] does not construct the C_ν sets directly, but defines the "smooth category" of functions $f : \bar{\nu} \implies \nu$ and the class of triples $\mathbb{F}$. They establish that $\mathbb{F}$ satisfies a short list of axioms of a smooth category (much as a morass can specified by a list of axioms, as is done below). From the category one can derive Global $\Box$ without any reference to the model whence the category came. The argument is purely combinatorial. This is implicit in [2], and explicit in [29]. These category like objects were used by Jensen to axiomatise various structures such as premorasses, morasses, and the like, and were further used by Stanley [49] to construct morasses *via* forcing. The proof of Global $\Box$ in [19] involves tying the construction to the notion of *semi-singularisation*, and is based on the account in [1] of Silver's proof of $\Box$ using Silver machines.

In larger fine structural models Global $\Box$ can also be proved. The extent of this is discussed below. We sketch here what has to be done (and overcome) in [30] in order to establish the principle in the core model built assuming that it does not have any level M with an ordinal κ with $M \models$ "the Mitchell order of κ, $o(\kappa)$, is κ^{++})". One problem that has to be addressed from the outset is the failure of condensation in the pure form of Lemma 1.22. (Even at the level of a single measurable cardinal this will occur.) A key part of that paper is proving a form of a suitably enhanced Condensation Lemma. In such models there will be many structures that can putatively singularise an ordinal ν. One defines the class S as now a set of pairs $s = \langle \nu_s, M_s \rangle$ where $\nu_s \in \mathrm{Sing}$, and M_s is a mouse $\Sigma^{(n)}_1$ over which (for some n) ν_s is definably singularised. One requires that M_s be a mouse which is *sound above* ν_s. If two mice M_s and M'_s satisfy this, with both hierarchies of M_s and M'_s agreeing up to ν, then in fact a comparison argument shows they are equal. One then defines much as above category maps $f : \bar{s} \implies s$ between $J_{\bar{s}}$ and J_s, where $J_s =_{\mathrm{df}} J^{E^{M_s}}_{\nu_s}$. f now canonically extends to an $f^* : M_{\bar{s}} \longrightarrow M_s$ at the appropriate level of $\Sigma^{(n_s)}_1$-definability. These maps f^* must carry in their range a potentially extended parameter set for the version of the Condensation Lemma to be proven (we discuss these parameters and condensation lemmas a little more below) under the smallness assumption of $\neg\exists\kappa(o(\kappa) = \kappa^{++})$. Having got a smooth category of maps between pairs, we assume a unique choice of mouse M_s and so unique $s = \langle \nu_s, M_s \rangle$, for ν_s is made, in some closed and unbounded class of ordinals D containing all singular cardinals. This should be done in some sensible way so that if $\lambda < \nu_s$ and $\lambda \in D$ then $s{\restriction}\lambda$ can be defined as an ordered pair of λ together with an initial segment of M_s which is a mouse singularising λ, and is also our canonical choice at λ. Then one can get an appropriate form of Global $\Box$.

2.2. Variants and Generalities on $\square$

Variants on $\square$: The Principles $\square_\kappa$

In [24] the principles $\square_\kappa$ for $\kappa > \omega$ a cardinal were first defined.

2.42 Definition. Let $\kappa > \omega$ be a cardinal. The principle $\square_\kappa$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Sing} \cap \kappa^+\rangle$ so that:

(i) C_α is closed unbounded in α;

(ii) $\mathrm{cf}(\alpha) < \kappa \implies |C_\alpha| < \kappa$;

(iii) $\beta \in (C_\alpha)^* \implies \beta \cap C_\alpha = C_\beta$.

Note that this implies that if $\mathrm{cf}(\alpha) = \kappa$ then $\mathrm{ot}(C_\alpha) = \kappa$. An equivalent variant on this is that with (ii) replaced by the apparently weaker:

(ii)$'$ $\mathrm{ot}(C_\alpha) \leq \kappa$.

Note first that this is immediate for regular κ. Otherwise to see this fix $D \subseteq \kappa+1$ closed and unbounded with $\mathrm{ot}(D) = \mathrm{cf}(\kappa)+1$, $0 \in D$, and $\max(D) = \kappa$. Assuming we have a sequence of C'_α's satisfying (i), (iii), and the new clause (ii)$'$ then, if $\mathrm{ot}(C'_\alpha) \in (D)^*$, replace C'_α with $C_\alpha = \{\beta \in C'_\alpha \mid \mathrm{ot}(C'_\alpha \cap \beta) \in D\}$; otherwise there is a maximal $\delta \in D$ with $\delta < \mathrm{ot}(C'_\alpha)$; in this case set $C_\alpha = \{\beta \in C'_\alpha \mid \mathrm{ot}(C_\alpha \cap \beta) \geq \delta\}$.

A further equivalent variant adds to (ii)$'$ a strengthened

(iii)$'$ $\beta \in C_\alpha \Longrightarrow \beta \cap C_\alpha = C_\beta$,

but accordingly a weakened

(i)$'$ C_α is closed below $\alpha \wedge (\mathrm{cf}(\alpha) > \omega \implies C_\alpha$ is unbounded in $\alpha)$.

It is plausible from the definition of $\square$ that it implies $\forall \kappa > \omega(\kappa \in \mathrm{Card} \longrightarrow \square_\kappa)$. The proof of this is a combinatorial argument (see [8, VI 6.2]) which is not fine-structural, so we do not give it. In fact Jensen showed:

2.43 Theorem (Jensen [24]). *Assume $V = L$; then for any infinite cardinal κ, $\square_\kappa$ holds with the addition that there is a stationary $E \subseteq \kappa^+$ so that for any α, $\beta \in (C_\alpha)^* \longrightarrow \beta \notin E$.*

In the proof of the latter theorem a particular $E \subseteq \mathrm{Cof}(\omega) =_{\mathrm{df}} \{\beta \in \mathrm{On} \mid \mathrm{cf}(\beta) = \omega\}$ was designated. In fact $\square_\kappa$ implies that for any stationary $S \subseteq \kappa^+$ there is some stationary $T \subseteq S$ which does not reflect in this sense: for any $\alpha < \kappa^+$(with $\mathrm{cf}(\alpha) > \omega$ of course) $T \cap \alpha$ is not stationary. (Given any such S, by Fodor's Lemma, find $T \subseteq S$ with $\alpha \in T \longrightarrow \mathrm{ot}(C_\alpha) = \delta$ for some fixed δ. Now if $\mathrm{cf}(\beta) > \omega$, $|(C_\beta)^* \cap T| \leq 1$ and thus $T \cap \beta$ cannot be stationary.)

It is consistent relative to the existence of a Mahlo cardinal that $\square_\kappa$ fails at a regular cardinal by a result of Solovay. On the other hand:

2.44 Theorem (Jensen [24]). *Assume $V = L[A]$ where $A \subseteq \kappa^+$ and that for any ν less than κ^+, we have: $L[A \cap \nu] \models$"ν is singular". Then $\Box_\kappa$ holds.*

In L the existence of $\Box_\kappa$-sequences can be proved by localised methods similar to that of Global $\Box$. For the $L[A]$ result above one defines singularising structures of the form $J^a_{\beta(\nu)} \models$"$a \subseteq \nu$" for the relativised J^a-hierarchies in general. The assumption on the $A \cap \nu$ is to ensure that sufficiently often there are such singularising structures for $a = A \cap \nu$ in $L[A]$. One can prove the local $\Box_\kappa$ by reworking the local proof of $\Box_\kappa$; in [2] this is done in a global fashion for all possible singularising structures for arbitrary a (see Theorem 6.21 op. cit.) and then specialised results such as $\Box_\kappa$, or combinatorial principles necessary for the Coding Theorem, are obtained. Thus Solovay's result is exact: if κ^+ is not Mahlo in L we may take A as a closed and unbounded set in κ^+ of L-singular cardinals and construct a $\Box_\kappa(A)$ sequence which implies the existence of $\Box_\kappa$ (see the next definition and discussion below).

2.45 Definition. Let $S \subseteq$ Sing be a class. Then denote by $\Box(S)$ the assertion that there exists a $\Box$-sequence $\langle C_\alpha \mid \alpha \in S\rangle$, satisfying:

(i) C_ν is a closed subset of $\nu \cap S$; if $\mathrm{cf}(\nu) > \omega$ then it is unbounded in ν;

(ii) $\mathrm{ot}(C_\nu) < \nu$;

(iii) $\bar{\nu} \in (C_\nu)* \implies C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$.

Let $T \subseteq \kappa^+ \cap \mathrm{Sing}$ be closed and unbounded. $\Box_\kappa(T)$ is the assertion that there exists a $\Box$-sequence $\langle C_\alpha \mid \alpha \in T\rangle$, satisfying:

(i) C_ν is a closed subset of $\nu \cap T$; if $\mathrm{cf}(\nu) > \omega$ then it is unbounded in ν;

(ii) $\mathrm{ot}(C_\nu) \leq \kappa$;

(iii) $\bar{\nu} \in (C_\nu)^* \implies C_{\bar{\nu}} = \bar{\nu} \cap C_\nu$.

It is a straightforward combinatorial argument in ZFC once one has a $\Box_\kappa(T)$ sequence for a T closed and unbounded in κ^+ to fill in the gaps between successive members of T with suitable C_β's to enlarge the $\Box_\kappa(T)$ to a $\Box_\kappa(\kappa^+) = \Box_\kappa$ sequence. For S equal to all of SingCard $=_{\mathrm{df}}$ Sing $\cap$ Card, if we have $\Box(S)$ and $\forall\kappa(\Box_\kappa)$ then we have Global $\Box$ (and conversely (Jensen)). Thus if $S \supseteq$ SingCard and moreover for all $\kappa > \omega$ $S \cap (\kappa, \kappa^+)$ is unbounded, then $\Box(S) \Longrightarrow \Box$ again by the "filling in of gaps" just referred to.

Variants on $\Box$: Weakenings and Extensions, $\Box$ with Scales

2.46 Definition. Let $0 < \lambda \leq \kappa$ be cardinals, with κ uncountable. The principle $\Box_\kappa^{<\lambda}$ asserts the existence of a sequence $\langle \mathcal{F}_\alpha \mid \mathrm{Lim}(\alpha), \alpha \in (\kappa, \kappa^+)\rangle$ with, for every limit α, $0 < |\mathcal{F}_\alpha| < \lambda$, and further so that $C \in \mathcal{F}_\alpha$ implies

(i) C is closed unbounded in α;

(ii) $\mathrm{ot}(C) \leq \kappa$;

(iii) $\beta \in C^* \implies \beta \cap C \in \mathcal{F}_\beta$.

$\Box_\kappa^{<\lambda^+}$ is abbreviated as $\Box_\kappa^\lambda$.

In [24] $\Box_\kappa^\kappa$ was formulated in an equivalent formulation called $\Box_\kappa^*$, and clearly $\Box_\kappa^1$ is $\Box_\kappa$. These intermediate so-called *weak square principles* were introduced by Schimmerling in [42]. We defer discussion of these principles until later, but Jensen had already remarked in [24] that if $2^\kappa = \kappa^+$ then $\Box_\kappa^*$ is equivalent to the existence of a special Aronszajn tree on κ^+ (also due to him is the straight equivalence of these two principles, without the assumption that $2^\kappa = \kappa^+$); in fact if $\kappa^{<\kappa} = \kappa$ then one can construct a $\Box_\kappa^*$ sequence; hence that if κ is regular $\Box_\kappa^*$ follows from ZFC + GCH; further notice that the case of κ singular is of particular interest for the question of the existence of $\Box_\kappa^*$ sequences. Whereas from $\Box_\kappa$ we can construct non-reflecting stationary subsets of κ^+ as above, this cannot be done from $\Box_\kappa^*$ alone (although $\Box_\kappa^{<\omega}$ does suffice; see [6]). Mitchell has shown that $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa$ is a Mahlo cardinal$)) \implies \mathrm{Con}(\mathrm{ZFC} + \neg\Box_{\omega_1}^*)$ by a suitable forcing collapsing κ to $\aleph_2$.

Improved-$\Box_\kappa^{<\lambda}$ sequences add a requirement (iv) to the above definition by insisting that at least one element $C \in \mathcal{F}_\alpha$ has $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$. In general $\Box_\kappa$ does not imply Improved-$\Box_\kappa^{<\lambda}$: in the Mitchell model mentioned above, if one effects a further forcing to add back a $\Box_{\omega_2}$-sequence without adding any ω_2-sequences, one ends up with a model where Improved- $\Box_{\omega_2}^{<\omega_2}$ still fails (see [6, Sect. 5]). However [6, Theorem 10] Global $\Box$ implies Improved-$\Box_\kappa^{<\omega}$ for every uncountable cardinal κ.

Enhancements of $\Box$ have been developed. We detail just one here. The following is a natural property allying $\Box_{\aleph_n}$-sequences for $n < \omega$ with a scale on $\aleph_\omega$. It was defined in [7] at 3.1. Let $I_n = \mathrm{Lim} \cap (\omega_n, \omega_{n+1})$ for $n \leq \omega$. CS below will essentially be asserting the existence of $\Box_{\aleph_n}$-sequences, albeit with domains I_n rather than all of $\mathrm{Sing} \cap \omega_{n+1}$, plus a scale for $\aleph_\omega$.

2.47 Definition. CS asserts the existence of $\langle C_\alpha^n \mid 0 < n \leq \omega, \alpha \in I_n\rangle$ and $\langle f_\alpha \,|\, \alpha \in I_\omega\rangle$ so that:

(1) $\forall n \forall \alpha \in I_n$

(a) $C_\alpha^n \subset \alpha \cap I_n$ and is closed and unbounded in α;

(b) $\mathrm{cf}(\alpha) < \omega_n \implies \mathrm{ot}(C_\alpha^n) < \omega_n$.

(2) $\forall\alpha \in I_\omega$ f_α is a function so that:

  (a) $\mathrm{dom}(f_\alpha) = (k, \omega]$ for some $k < \omega$ such that $\mathrm{ot}(C^\omega_\alpha) < \omega_k$;

  (b) $\forall n \in \mathrm{dom}(f_\alpha)$, $f_\alpha(n) \in I_n$;

  (c) $\beta \in (C^\omega_\alpha)^* \Longrightarrow \mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(f_\beta)$;

  (d) $\forall n \in \mathrm{dom}(f_\alpha)$, $(C^n_{f_\alpha(n)})^* = \{f_\beta(n) \mid \beta \in (C^\omega_\alpha)^*\}$.

(3) $\langle f_\alpha \mid \alpha \in I_\omega\rangle$ forms a scale in $\Pi_n \aleph_{n+1}$; *i.e.* it is increasing and cofinal in the eventual domination ordering.

In [7] the consistency of CS is established by forcing. (With some work the methods here of the proof of Global square restricted to $\aleph_{\omega+1}$ together with a few "premorass-like" considerations from the next section, establish the existence of the scale sequence $\langle f_\alpha \mid \alpha \in I_\omega\rangle$ and hence that CS holds in L.) They use CS to demonstrate the existence of a mutually stationary sequence on $\aleph_\omega$ that is not tightly stationary. CS is a special case of the *condensation coherent* global $\Box$-sequences of Donder et al. [13], also shown to hold in L, and can be derived from them. The authors used condensation coherent sequences to demonstrate the existence of *squared scales* of Abraham and Shelah. Donder et al. [13] derive their principle again from a "category" like object. A similar structure is the *Fine Scale Principle* of Friedman [17] used in his proof of the Coding Theorem.

Applications

We discuss only very briefly some positive applications of $\Box$-sequences. The reader should see the chapter by Todorčević in this Handbook. We do not intend here to go extensively into applications of $\Box$ and weak $\Box^\lambda_\kappa$-sequences to *stationary reflection* properties but refer the reader to the extensive [6] for many results in this area, and to the chapter by Eisworth in this Handbook.

In [24, Sect. 6] $\Box$ arguments were originally used to prove two characterisations of weakly compact cardinals in L:

2.48 Theorem (Jensen [24]). *Assume $V = L$; let $\kappa \in \mathrm{Reg}$ but not weakly compact. Then there is a κ-Suslin tree.*

The proof proceeds by establishing:

2.49 Theorem (Jensen [24]). *Assume $V = L$; then for any non-weakly compact $\kappa \in \mathrm{Reg}$, there is a stationary $E \subseteq \kappa$ and a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap \kappa\rangle$ such that* (i) *C_α is closed and unbounded in α, and* (ii) *$\alpha, \beta \in (C_\alpha)^* \Longrightarrow (\beta \notin E \wedge \beta \cap C_\alpha = C_\beta)$.*

From this, one characterisation of weakly compact cardinals in L already follows. Stationary reflection is a consequence of Π^1_1-indescribability: if $E \subseteq \kappa$

is stationary, then for some $\beta < \kappa$ we have that $E \cap \beta$ is stationary. The theorem above thus shows that in L, for non-weakly compact κ this must fail. The use of the $\langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap \kappa \rangle$ sequences from the above facilitates the inductive construction of a κ-Suslin tree. Even outside of L we may obtain (see [8]):

- CH $+ \Box_{\omega_1} \implies$ there exists an ω_2-Suslin tree (Gregory [22]);
- GCH $+ \Box_\kappa \implies$ there exists a κ^+-Suslin tree.

The role of the C_α-sequences in all these (and other) proofs is typically to ensure that the inductive definition of the required structure can be continued at limit stages of uncountable cofinality.

A final result from Jensen's paper is a cardinal transfer theorem due to Silver (see Definition 3.8 for this notation).

2.50 Theorem (Silver [24]). *If* GCH *holds, κ is a singular cardinal, and $\Box_\kappa$ holds, then* $(\omega_1, \omega) \longrightarrow (\kappa^+, \kappa)$.

One of the major applications of $\Box$ and $\Box$-like principles is in *class forcing*: the construction of many coding conditions and coding-style conditions for forcing over L or other models, involves an extensive use of $\Box$-like (and morass-like) machinery. Jensen's original Coding Theorem (see [2]) heavily exploited such principles, especially when formulating conditions for coding information down past singular cardinals. Work of Sy Friedman has transformed notions of class forcing for a variety of constructions over L that similarly involve fine-structural arguments for defining conditions (see [18] and the chapter by Friedman in this Handbook). Just as an example, his Fine Scale Principle alluded to above is employed in his proof of coding.

A recent use of the (proof of) Global $\Box$ in $L[E]$ models is that of Koepke-Welch [33] deriving large cardinal strength from the assumptions of a mutual stationarity principle of Foreman-Magidor (see again [7]). Fine structural methods are also employed by Ishiu [23] to demonstrate the existence of *strong club guessing sequences* in L. Let $S \subseteq \mu$ be stationary with $\mu \in \mathrm{Reg}$. An S-strong club guessing sequence $\langle C_\alpha \mid, \alpha \in S \rangle$ is a sequence where each $C_\alpha \subseteq \alpha$ is closed and unbounded, and so that for any closed and unbounded $D \subseteq \mu$, there is a further closed and unbounded $E \subseteq \mu$ so that $\beta \in E \cap S \implies \exists \delta (C_\beta \backslash \delta \subseteq D)$. Ishiu shows that in L the existence of a μ-strong club guessing sequence is equivalent to the non-ineffability of μ. The heart of the proof is to show that there is a $\mathrm{Sing} \cap \mu$-strong club guessing sequence. He directly constructs the C_α's using fine structural arguments reminiscent partly of $\diamondsuit^*$, and partly of $\Box$ methods, by looking at singularising structures. As he does not require coherence of the guessing sequence this is simpler than proving the existence of Global $\Box$ up to μ.

The Failure of $\Box_\kappa$

It was an early (1974) result of Solovay that showed that the κ^+-supercompactness of a cardinal λ implied the failure of $\Box_\kappa$ [47] (this was later reduced to κ^+-compactness by Gregory; see also [48] for a discussion of this). With the advent of Jensen's Covering Lemma for L it was clear that if any cardinal κ satisfied $\kappa^+ = (\kappa^+)^L$ then the absolute nature of the defining clauses of $\Box_\kappa$ implied that a $\Box_\kappa$ sequence in L is a $\Box_\kappa$ sequence in V. As the Covering Lemma (under the assumption of $\neg 0^\#$) implied that $\kappa^+ = (\kappa^+)^L$ for any singular, or weakly compact cardinal κ, this means that then V would have such $\Box_\kappa$-sequences. In larger extender models K the Weak Covering Lemma (and thus the correct computation of cardinal successors of singulars) can be achieved for K up to a strong cardinal [27] and up to a Woodin cardinal [37]. Hence the failure of, say, $\Box_{\aleph_\omega}$ expressed directly the failure of the Covering Lemma over an inner model (or at least the correct cardinal successor computation) at the cardinal $\aleph_\omega$. Indeed this failure came to be seen as a test question concerning inner models, and implies (at least if $\aleph_\omega$ is a strong limit cardinal) the existence of $\mathrm{AD}^{L[\mathbb{R}]}$; see Theorem 2.59 below.

The principle $\Box_\kappa$ was seen to hold at other classes of cardinals, assuming the rigidity of a suitable canonical inner model. For Jensen's Core Model K^{MOZ} built with measures of order zero, Vickers and Welch [55] showed that successors of Jónsson cardinals were correctly computed in K^{MOZ}, hence $\Box_\kappa$ held at Jónsson κ (the point here is that regular Jónsson cardinals, although weakly inaccessible need not be weakly compact, nevertheless we may show correct cardinal successor computation at κ). Welch [59] showed the same conclusion for the core model K^{Steel}, assuming no inner model with a Woodin cardinal below the measurable cardinal Ω needed for K^{Steel}'s construction. (That $\kappa^+ = (\kappa^+)^{K^{\mathrm{Steel}}}$ under these assumptions, and hence that $\Box_\kappa$ holds, for κ weakly compact is due to Schimmerling-Steel [43].) The Jónsson property in fact yields some strong reflection properties on $\Box$ sequences, assuming the Weak Covering Lemma over the inner model: if there is no inner model with a Woodin cardinal, if Ω is measurable, $\kappa < \Omega$ any regular Jónsson, then the set of regular cardinals $\mu < \kappa$ such that $\Box_\mu$ holds in V, is stationary below κ ([59]). Chang's Conjecture implies the failure of $\Box_{\omega_1}$ as shown by Todorčević.

There are extensive results in the area of forcing and $\Box$ which we shall only lightly touch on here, but we do mention that the proper forcing axiom PFA causes a dramatic failure of square: Todorčević proved [52] that $\mathrm{ZFC} + \mathrm{PFA} \vdash \forall\kappa\neg\Box_\kappa$. Hence PFA implies the existence of an inner model with a measurable cardinal, for example. Magidor noted that Todorčević's proof actually shows that $\mathrm{ZFC} + \mathrm{PFA} \vdash \forall\kappa\neg\Box_\kappa^{\aleph_1}$; however this can be contrasted with his further result that showed (unpublished 1995) that $\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa \text{ a supercompact cardinal})) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{PFA} + \forall\kappa\Box_\kappa^{\aleph_2})$. Hence the possibility arises to use the various weak square principles as a means of calibrating consistency strength. The impact of Martin's Maximum (MM) is stronger than that of PFA: Foreman and Magidor formulated a "very weak square" principle [15] VWS_κ, weaker than $\square^*_\kappa$, and the latter showed that MM implies $\neg\,\mathrm{VWS}_\kappa$ for κ with $\mathrm{cf}(\kappa) = \omega$. By Solovay's result mentioned above, a supercompact destroys the possibility of any $\square_\lambda$ holding for any $\lambda > \kappa$; Shelah showed from the same assumption that any $\square^*_\lambda$ will also fail for any singular λ satisfying $\mathrm{cf}(\lambda) < \kappa < \lambda$ (see [15]). However it is consistent with a supercompact κ that for a cardinal λ satisfying: $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ that $\square_\lambda^{\mathrm{cf}(\lambda)}$ holds [6, Theorem 17]. This complements the observation of Burke and Kanamori that Solovay's methods can show that if κ is λ^+ strongly compact, and $\mathrm{cf}(\lambda) < \kappa$ then $\square_\lambda^{<\mathrm{cf}(\lambda)}$ fails.

If $\mathbb{P}$ is Prikry forcing at a measurable cardinal, then it was shown in [3] that $\square^*_\kappa$ holds in $V^{\mathbb{P}}$. This was improved in [5] to that of $\square^\omega_\kappa$ holding; results of [6] however show that if κ is supercompact then in $V^{\mathbb{P}}$ one has the failure of $\square^{<\omega}_\kappa$.

Jensen [29] showed that for regular κ the principles $\square^\lambda_\kappa$ decreased in strength as λ increases from 1 to κ. In [6] it is shown how this can be done for singulars: from a supercompact a forcing extension can be constructed in which some previously chosen cardinals $0 < \nu < \mu$ (which can be finite) satisfy $\square^\mu_{\aleph_\omega}$ but $\neg\square^\nu_{\aleph_\omega}$.

2.3. $\square$ in Fine-Structural Inner Models

We come now to the interesting question of the status of $\square$-principles in canonical inner models. As for L, Global $\square$, and for all κ, $\square_\kappa$ were shown by Welch [57] to hold in the first such model to go beyond L : the Dodd-Jensen Core Model K^{DJ} built assuming the non-existence of an inner model with a measurable cardinal. Wylie [60] proved the same for Jensen's Core Model for measures of order zero. Whereas [57] had used the older fine structure that was available at that time, Wylie's thesis used the modern arrangement of the hierarchies and the corresponding fine structure.

Several issues surface at this point: Firstly, the *indexing* of extenders plays an important role. Secondly, it was also apparent that *extender fragments* which had been introduced by Mitchell and Steel in their fine structure for $L[E]$ models [36] arise also naturally when considering proofs of $\square$ for models with many measures. Thirdly, additional further condensation results would be needed to be proven both for them, and in general. Concerning this third issue: for example, whereas Skolem hulls of an initial segment of the L-hierarchy collapse to levels of L, for $L[E]$ hierarchies (even with E coding only sequences of measures) certain hulls may collapse not to an initial segment of the J^E_α hierarchy in which they were taken, but to initial segments of an ultrapower of this J^E_α hierarchy, thus leading to multi-claused condensation lemmas. (This has already been mentioned above for Jensen-Zeman [30] and is a substantial mathematical preface to their proof of Global $\square$ in the model below $o(\kappa) = \kappa^{++}$.) Wylie's thesis contains the first kind of condensation results of a type needed to prove $\square$ in the modern setting.

Concerning the first two issues, the indexing of extenders: several possibilities arise for how one might build extender E-hierarchies. Mitchell and Steel built in [36] an extender model $L[\vec{E}\,]$ with a Woodin cardinal using an indexing rule for building E-sequences which indexed with ordinal α an extender E if, when taking the ultrapower $j : M \longrightarrow N = \mathrm{Ult}(M, E)$ then $\alpha = (\nu(E)^+)^N$ where ν was the supremum of the *generators* of the extender E (or simply $(\mathrm{crit}(E)^+)^N$ if $\mathrm{crit}(E)$ was the sole generator). (This ordinal ν can be characterised as the least ordinal $\nu' \geq (\mathrm{crit}(E)^+)^N$ so that the extender of length α derived from j gives rise to the same ultrapower as E.) The *fragments* of the extender E_α which are of the form $E_\alpha \cap ([\eta]^{<\omega} \times J^{\vec{E}}_\xi\,)$ (where $\xi < (\kappa^+)^{J^{\vec{E}}_\alpha}$) are members of $J^{\vec{E}}_\alpha$ (see the discussion at [50, 2.9]). Moreover the fragments just defined appear cofinally in α as ξ rises to $(\kappa^+)^{J^{\vec{E}}_\alpha}$. The extender E_α may now be coded as an *amenable* predicate which is essential for fine structure to work.

Alternative indexing schemes have been proposed. Notably, S. Friedman realised that reindexing could affect proofs of fine-structural principles and suggested the scheme whereby an index α for an extender would be $(j(\kappa))^{+N}$, with j and N as above. This suggestion has been fully worked out by Jensen in [25], and shows *inter alia* how the results of Mitchell and Steel can be performed using this indexing. Each scheme has its advantages, but for the proofs of combinatorial principles such as $\Box_\kappa$ the Friedman-Jensen scheme allows certain difficulties to be postponed to later in the large cardinal hierarchy, beyond superstrong cardinals as we shall see later.

The analysis of fragments and condensation principles for the proof of $\Box$ there, at that time by Wylie and S. Friedman, would be continued independently by Friedman, Jensen, Schimmerling, and Zeman.

Notwithstanding the above analyses, in $L[E]$ models beyond measures the question of $\Box_\kappa$ became increasingly complex. Schimmerling [42] proved that in the Steel Core model built below a Woodin cardinal that (a) $\forall\kappa\, \Box^{<\kappa}_\kappa$ held; (b) that in the core model below n strong cardinals $\Box^{n+1}_\kappa$ held for all κ. Below one strong cardinal Schimmerling divided a certain $S \subseteq \kappa^+$ into two parts $S = S^0 \cup S^1$ and defined a $\Box_\kappa$-sequence $\langle C^i_\alpha \mid \alpha \in S^i \rangle$ for $i < 2$. However this was not a disjoint partition; hence setting $\mathcal{F}_\alpha = \{C^i_\alpha \mid \alpha \in S^i, i < 2\}$ yielded only a $\Box^2_\kappa$ sequence. A modification of the latter proof by Jensen in [29] showed that $\forall\kappa\, \Box_\kappa$ holds in the core model below 1 strong cardinal. Zeman [61] showed Global $\Box$ in the same model.

Jensen and Zeman's paper [30] proves Global $\Box$ in an $L[E]$ model built using Jensen's fine structure and utilising the indexing of extenders of [27]. The model is relatively small, as the assumption is that there are no inner models with a cardinal κ with Mitchell order $o(\kappa) = \kappa^{++}$. The proof establishes the existence of a "smooth category" like formulation of a set of axioms, that in turn automatically, in an entirely separate and purely combinatorial argument, can be shown to yield up Global $\Box$. The proof reveals some of the difficulties of extending condensation arguments to larger models. They are able to prove the necessary sequence of lemmata by adding to the requirements on the range of the maps $f : \bar{\nu} \implies \nu$ that we specified above, that it contained certain further parameters. In short if certain canonical witness maps σ sending W_M^{ν,p_M} into M were not cofinal at the nth stage where $\omega\rho_M^n > \nu \geq \omega\rho_M^{n+1}$ (where $\nu \in p_M$), then it is necessary to require the maps f to contain the ordinal $\sup(\mathrm{ran}(\sigma)) \cap \omega\rho_M^n$ witnessing that non-cofinality. (As there are only finitely many $\nu \in p_M$) this could be at most finitely many requirements.) With this strengthening a form of the Condensation Lemma (under the no $o(\kappa) = \kappa^{++}$ assumption) is provable and the proof can proceed. For larger models it is not clear how one can in general preserve standard parameters: one could put the witnesses themselves into the ranges of the maps, and thus preserve the standard parameters "downwards" into suitable substructures. However the smooth category approach also requires sufficient preservation in an upward direction, so it is unclear whether it is possible to perform this in a larger model context (and is one reason we did not demonstrate this approach in our account here).

There are serious difficulties in generalising the $\square_\kappa$ arguments to larger $L[E]$ models. The problems have to do with condensation and dealing with the extenders fragments that Schimmerling had already encountered, and which had been already seen earlier as a hurdle as described above. This was definitively solved in the last few years by work of Jensen on the former properties of condensation, and of Schimmerling and Zeman on the analysis of extender fragments. We now have a complete picture of how $\square_\kappa$ can be proven in Jensen style $L[E]$ models and at precisely which place this first breaks down, which we now describe.

Firstly a theorem due to Burke [4].

2.51 Theorem. *If* $\{a \subseteq \kappa^+ \mid \mathrm{ot}(a) \in \mathrm{Card}\}$ *is stationary then* $\square_\kappa$ *fails.*

Secondly, we make the following definition due to (but not christened by) Jensen. It was obtained by extracting the kernel of Solovay's proof of the failure of $\square_\kappa$ from a strongly compact.

2.52 Definition. A cardinal κ is *subcompact* iff for every $B \subseteq \kappa^+$, there are $\mu < \kappa$, $A \subseteq \mu^+$, and an elementary embedding $j : \langle H(\mu^+), \in, A\rangle \longrightarrow \langle H(\kappa^+), \in, B\rangle$ with $\mathrm{crit}(j) = \mu$.

It is easy to check here that μ's implicit weak inaccessibility implies the regularity of κ as $j(\mu) = \kappa$. The following is easily derived from Theorem 2.51; the argument is very similar to Solovay's, and seems to have been rediscovered by Jensen. As it is direct we give it:

2.53 Theorem (Jensen). *If* κ *is subcompact, then* $\square_\kappa$ *fails.*

Proof. Suppose, for a contradiction, that $B \subseteq \kappa^+$ codes up a $\square_\kappa$-sequence $\langle D_\alpha \mid \alpha < \kappa^+\rangle$. Let μ, A, j be suitable witnesses to the definition of subcompactness of κ for this B. By elementarity we have $\langle C_\alpha \mid \alpha < \mu^+\rangle$, a $\square_\mu$ sequence, coded by A. Let $S = \{\nu < \mu^+ \mid \mathrm{cf}(\nu) < \mu\}$. Then easily $j``S$ is $<\mu$-closed; set $\gamma = \sup(j``\mu^+)$. Then $(D_\gamma)^* \cap j``S$ is unbounded in γ and $<\mu$-closed. Let $T = \{\nu < \mu^+ \mid j(\nu) \in (D_\gamma)^*\}$. Then T is unbounded in μ^+. Finally let $E = \bigcup\{C_\nu \mid \nu \in T\}$; then E is a union of cohering sets and hence is closed and unbounded in μ^+ and $E \cap \nu = C_\nu$ for all $\nu \in T$. However this would imply that for all sufficiently large $\nu \in T$, $\text{ot}(C_\nu) > \mu$ which contradicts the definition of $\Box_\mu$ sequence. ⊣

A modification of the argument above yields also Burke's theorem 2.51. Schimmerling and Zeman's main theorem is:

2.54 Theorem (Schimmerling and Zeman [45]). *In any Jensen-style $L[E]$ model the following are equivalent:*

(i) $\Box_\kappa$;

(ii) $\Box_\kappa^{<\kappa}$;

(iii) κ *is not subcompact;*

(iv) $\{\nu < \kappa^+ \mid E_\nu \neq \varnothing\}$ *is not stationary in* κ^+.

(Here (i) implies (iii) is the Burke/Jensen result mentioned above, and (ii) implies (iii) is an easy modification of that; Schimmerling and Zeman proved the main result here that (iv) implies (i), and Jensen proved (i) implies (iv) around the time that the notion of Definition 2.52 was formulated. Later Zeman noticed that this argument can be modified to obtain that (iii) implies (iv).)

Thus in such $L[E]$ models $\Box_\kappa$ holds wherever $\Box_\kappa^{<\kappa}$ does. Recall that in this style of $L[E]$ model, if $J_\nu^E \models$ "κ is the largest cardinal", then if $\lambda = \text{crit}(E_\nu)$, then $E_\nu(\lambda) = \kappa$.

As $D =_{\text{df}} \{\nu \,|\, J_\nu^E \models$ "κ is the largest cardinal"$\}$ is closed and unbounded in κ^+, we have, if (iv) fails, that for stationarily many ν that $\nu \in D \wedge E_\nu \neq \varnothing$. Then for some cardinal $\lambda < \kappa$ there are unboundedly many (in fact stationary many) $\nu < \kappa^+$ with $E_\nu \neq \varnothing \wedge \text{crit}(E_\nu) = \lambda \wedge j_{E_\nu}(\lambda) = \kappa$. Thus λ is superstrong in $L[E]$, as κ is a cardinal of $L[E]$ and $V_\kappa^{L[E]} \subseteq \text{Ult}(L[E], E_\nu)$.

The proof of $\Box_\kappa$ for a non-subcompact κ involves splitting κ^+ into *disjoint* pieces S^0, S^1. On one of these the E-sequence is relatively benign and an L-like construction of a $\Box_\kappa$-sequence is possible. The harder, and new, part is on the set S^1 where the work is dealing with extender fragments rather than levels of $L[E]$, and in both cases it has to be shown that there is no conflict between the sequences of the two sides, e.g. that $\alpha \in S^i \Longrightarrow C_\alpha \subseteq S^i$. However the important feature remains that in either case the sequence C_ν is first order definable over the least level $J_{\beta(\nu)}^E$ of the $L[E]$ hierarchy where ν becomes singular. Finally:

2.55 Theorem (Zeman). *In any Jensen style $L[E]$ model,* $\Box(\text{SingCard})$ *holds.*

The use of methods from Woodin's core model induction enabled strong conclusions to be obtained from the failure of, first $\Box_\kappa^{<\omega}$ and now with the above results, that of $\Box_\kappa$. We give some samples of the applications of Schimmerling and Zeman's theorem 2.54, but refer the reader to their [44] for a full survey.

2.56 Theorem. *Let κ be a singular cardinal satisfying $\mu^{\aleph_0} < \mu$ for all $\mu < \kappa$. Then if $\square_\kappa$ fails Projective Determinacy holds.*

2.57 Theorem. *Let κ be a weakly compact cardinal and suppose $\square_\kappa$ fails; then every set of reals in $L(\mathbb{R})$ is determined.*

The last two were first proven by Schimmerling and Steel essentially in [43] assuming the failure of $\square_\kappa^{<\omega}$. The latter theorem concluded with PD, but was then improved by Woodin using the full Core Model Induction to get the result quoted, still from the failure of $\square_\kappa^{<\omega}$.

2.58 Theorem. *Let κ be a measurable cardinal and suppose $\square_\kappa$ fails; then there is a transitive proper class model of* $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$.

We lastly remark that:

2.59 Theorem (Steel [51]). *If κ is a singular strong limit cardinal and $\square_\kappa$ fails then every set in $L(\mathbb{R})$ is determined.*

It is an open question whether the requirement on being a strong limit can be lifted here, for example at $\aleph_\omega$, or whether the result can be proven from the failure of $\square_\kappa$ at a Jónsson cardinal κ. In these last four theorems the use of the failure of $\square_\kappa$ is not as a simple quotation of the failure of correct cardinal successor computation in an inner model: there may be no canonical inner model at hand. In essence fine structural segments are pieced together to approximate some form of a hierarchy coded by subsets $A_\alpha \subseteq \kappa$ for $\alpha < \kappa^+$. Typically some first subset A_0 codes V_κ and then the construction puts together mouse-like segments over A_α's. In these segments parts of a $\square_\kappa$ sequence are pieced together (or of $\square_\kappa^{<\omega}$ in fine structural hierarchies using the earlier work of Schimmerling). The definitions of the various $\langle C_\nu \mid \nu < \lambda_\alpha \rangle$'s are purely local (for λ_α increasing unboundedly below κ^+; as we remarked just before Theorem 2.55, the sequence C_ν is definable in some sort of similar fashion to our proof above in L: it is first-order definable over $J^E_{\beta(\nu)}$ where $\beta(\nu)$ names the first place in the $L[E]$-hierarchy where ν is definably singularised). Hence the result can be derived without knowing that there is some inner model (which might have required some further assumption concerning a larger measurable cardinal, for example) over which the Covering Lemma held.

The reader should consult [41, Sect. 5] for further applications in this area.

3. Morasses

The notion of morass is a somewhat complex one. In many ways morasses seem to encapsulate the totality of the fine structure available in a model, although the $\square$ concept has been more influential. The notion can be motivated through its original application to the Gap-2 Cardinal Transfer Theorems of Jensen (cf. Theorem 3.9 below). However they can be construed as pictures of the extremely regular behaviour of, e.g. the inner model L, in that one sees how structures, or levels of the L_α-hierarchy, say L_{κ^+}, can be approximated by directed systems of levels L_β for β of smaller cardinality then κ. A morass at a cardinal κ is characterised by a gap parameter. A morass at κ of gap 1 (a "$(\kappa, 1)$ morass") then is a system approximating the levels L_α for $\alpha \in (\kappa, \kappa^+]$ by means of levels L_β for $\beta < \kappa$ and maps between them: $f_{\beta\beta'}$. A gap-2 morass at κ is a "higher gap morass", and approximates $L_{\kappa^{++}}$ by means again of structures L_β for $\beta < \kappa$ and a more complex system of maps, which can be construed as a "morass of morasses" building up a double gap approximating process through systems of maps. There is no need to restrict to gaps of length two, and indeed for regular κ one can define (κ, γ) morasses for any $\gamma < \kappa$, once one has seen how to do it for 2.

We shall first give a definition of a gap-1 morass at κ in a manner somewhat reminiscent of a category. This is a formal definition of maps and structures which encapsulate the notion. The notation is intentionally similar to what we have done for Global $\square$: this will bring out the similarities, and also make plausible the existence of such a structure in L.

3.1 Definition. Let $\kappa > \omega$ be regular. A *gap-1 morass at* κ is a pentuple $\langle S, S^1, \prec, \langle \mathfrak{A}_\nu \mid \nu \in S^1\rangle, \langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu\rangle\rangle$ satisfying the following:

M0(i) S is a set of p.r. closed pairs of ordinals $\langle \alpha, \nu\rangle$ with $\alpha < \nu < \kappa^+$ such that: $\langle \alpha, \nu\rangle, \langle \alpha', \nu'\rangle \in S \wedge \alpha' < \alpha \longrightarrow \nu' < \alpha$.

M0(ii) $S^1 =_{\rm df} \{\nu \mid \exists\alpha\langle\alpha, \nu\rangle \in S\}$; for $\nu \in S^1$ we let α_ν denote the unique α such that $\langle \alpha, \nu\rangle \in S$; $S^0 =_{\rm df} \{\alpha_\nu \mid \nu \in S^1\}$; $S_\alpha =_{\rm df} \{\nu \mid \alpha_\nu = \alpha\}$. Then: For $\alpha \in S^0$, S_α is closed in $\sup(S_\alpha)$; $\kappa = \sup(S^0 \cap \kappa) = \max(S^0)$; S_κ is unbounded in κ^+.

M0(iii) $\prec$ is a tree on S^1; $\bar\nu \prec \nu \longrightarrow \alpha_{\bar\nu} < \alpha_\nu$.

M0(iv) (a) For $\nu \in S^1$, $\mathfrak{A}_\nu$ is a transitive amenable structure, $J_\nu \subseteq |\mathfrak{A}_\nu|$; if $\tau \in S_{\alpha_\nu} \cap \nu$ then $|\mathfrak{A}_\tau| \subseteq |\mathfrak{A}_\nu|$.

(b) If h is a Σ_1-Skolem function for $\mathfrak{A}_\nu$ then $|\mathfrak{A}_\nu| = h``(\omega \times J^{<\omega}_{\alpha_\nu})$.

M0(v) $\langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu\rangle$ is a commuting system of maps; $\widehat{f}_{\bar\nu\nu} : \mathfrak{A}_{\bar\nu} \longrightarrow_{\Sigma_1} \mathfrak{A}_\nu$; $\widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\nu} : \langle J_{\bar\nu}, S_{\alpha_{\bar\nu}} \cap \bar\nu\rangle \longrightarrow_Q \langle J_\nu, S_{\alpha_\nu} \cap \nu\rangle$; $\widehat{f}_{\bar\nu\nu}(\alpha_{\bar\nu}) = \alpha_\nu$; $\widehat{f}_{\bar\nu\nu}{\restriction}\alpha_{\bar\nu} = \mathrm{id}{\restriction}\alpha_{\bar\nu}$.

CP1 If $B_\nu =_{\rm df} \{\alpha_{\bar\nu} \mid \bar\nu \prec \nu\}$ and $B^+_\nu =_{\rm df} B_\nu \cup \{\alpha_\nu\}$ then B^+_ν is closed.

M1 If $\tau \in S_{\alpha_\nu} \cap \nu$ then B_τ is unbounded in α_ν.

M2 If $\bar\nu \prec \nu$, $\bar\tau \in S_{\alpha_{\bar\nu}} \cap \bar\nu$ and $\tau = \widehat{f}_{\bar\nu\nu}(\bar\tau)$, then $\bar\tau \prec \tau \wedge \widehat{f}_{\bar\tau\tau}{\restriction}J_{\bar\tau} = \widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\tau}$.

M3 B_ν unbounded in α_ν implies $|\mathfrak{A}_\nu| = \bigcup_{\bar\nu \prec \nu} \mathrm{ran}(\widehat{f}_{\bar\nu\nu})$.

CP2 If $\bar\nu \in (S_{\alpha_{\bar\nu}})^*$ then:

(a) $\bar\nu \prec \nu \wedge \sup(\widehat{f}_{\bar\nu\nu}``\bar\nu) = \lambda < \nu \longrightarrow \bar\nu \prec \lambda \wedge \widehat{f}_{\bar\nu\lambda}{\restriction}J_{\bar\nu} = \widehat{f}_{\bar\nu\nu}{\restriction}J_{\bar\nu}$;

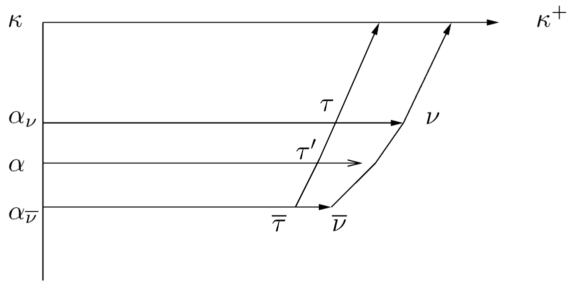

Figure 10.3: Continuity property CP2b

(b) $\bar{\nu} \prec \nu \wedge \sup(\widehat{f}_{\bar{\nu}\nu}``\bar{\nu}) = \nu \wedge \alpha > \alpha_{\bar{\nu}} \longrightarrow [(\forall \tau \in S_{\alpha_\nu} \cap \mathrm{ran}(\widehat{f}_{\bar{\nu}\nu})\ \alpha \in B_\tau) \longrightarrow \alpha \in B_\nu]$.

A morass is *universal* if $H(\kappa^+) = \bigcup_{\nu \in S^1} |\mathfrak{A}_\nu|$.

Note (i). Other forms of closure property can be used in M0(i) for example that of being a limit of admissibles. By M0(i) the definition of α_ν (in M0(ii)) makes sense. In M0(v) recall [46, 1.18] the definition of *Q-embedding*: A *Q-formula* is one of the form $\forall u \exists v \supseteq \varphi(v)$, where φ is Σ_1 and must not contain u. Notice that this can suitably express the notion of cofinality. Then $\pi : M \longrightarrow_Q N$ if π preserves all Q-formulae.

Note (ii). If we are not interested in universality, then it is possible to give a similar "axiomatic" treatment of a gap-1 morass which does not mention L or J structures, but simple ordinal structures $\langle \nu, \in \rangle$: one omits the references to structures, thus M3 becomes "B_ν is unbounded α_ν implies $\nu + 1 = \bigcup_{\bar{\nu} \prec \nu} \mathrm{ran}(f_{\bar{\nu}\nu})$" where now we only consider maps$f_{\bar{\nu}\nu} : \bar{\nu} + 1 \longrightarrow \nu + 1$. M0(v) now must be formulated in a way that expresses the similarity of the structure $\langle \bar{\nu} + 1, S_{\alpha_{\bar{\nu}}} \cap \bar{\nu} \rangle$ to that of $\langle \nu + 1, S_{\alpha_\nu} \cap \nu \rangle$ (see, for example, [8] for details). M0–M3 are known as the *morass axioms* and CP1, CP2 as the first and second *continuity properties* respectively. It is in particular the latter that gives the morass its strength. CP2(b) can be equivalently stated as:

CP2(b)$'$
$$[\bar{\nu} \prec \nu \wedge \sup(\hat{f}_{\bar{\nu}\nu}``\bar{\nu}) = \nu \wedge \alpha \in \bigcap_{\overline{\tau} \in S_{\alpha_{\overline{\nu}}} \cap \overline{\nu}} \{\alpha_{\tau'} \mid \overline{\tau} \prec \tau' \prec \widehat{f}_{\bar{\nu}\nu}(\overline{\tau})\}]$$
$$\longrightarrow (\exists \nu' \prec \nu)(\alpha_{\nu'} = \alpha).$$

3.1. Construction of Gap-1 Morasses in L

Let κ be a regular cardinal. We give the construction of a universal $(\kappa, 1)$-morass in the constructible universe. We assume then $V = L$. In this we can make use of some of the notions and lemmata of the section on Global $\Box$.

Let $S = \{\langle \alpha, \nu \rangle \mid \alpha < \nu,\ \nu$ is a limit of admissibles, $\nu \in \mathrm{Sing} \cap \kappa^+$, $J_\nu \models$ "α is regular and is the largest cardinal"$\}$.

Note that for $\langle\alpha,\nu\rangle\in S$, as α is a cardinal in the sense of J_ν then it is *a fortiori* a limit of admissibles. We remark that our taking of ν as a limit of admissibles is overkill: much less would suffice: we just need to be able to take the transitivisation of certain simply defined hulls of smaller J_η inside J_ν. (As, for example, the intersection of On with such a hull has order type less than or equal to η, very mild recursions of length at most η can define the transitive collapse maps.) Using the p.r. closed ordinals here would be possible. M0(i) is then clear (by appealing to the acceptability of the J-hierarchy!). Likewise M0(ii) is true of the sets S^0, S^1, S_α defined there. As $\nu\in\mathrm{Sing}$ the concepts of the least level $\beta(\nu)$ at which ν is singularised, by some $\boldsymbol{\Sigma_1^{(n)}}(J_{\beta(\nu)})$ function etc., as used in the proof of Global $\Box$ make sense, and we adopt them here too.

3.2 Definition. Let $\nu\in S^1$. Then we associate the following objects to ν:

(a) The same n_ν, M^k_ν, h^k_ν, h_ν, $\widetilde{h}_\nu$, ρ_ν from Definition 2.6;

(b) $\alpha_\nu =_{\mathrm{df}}$ the largest cardinal of J_ν;

(c) $p_\nu =_{\mathrm{df}} p_{M_\nu}\backslash\alpha_\nu$; $q_\nu =_{\mathrm{df}} p_\nu\cap\omega\rho^{n_\nu}_{M_\nu}$.

Note that p_ν is thus the $<^*$-least parameter so that $\widetilde{h}_\nu(\alpha_\nu\cup\{p_\nu\}) = J_{\beta(\nu)}$. It is really also the same p_ν from before; it is only that we have renamed κ_ν there as α_ν. It has become customary to use α_ν for the largest cardinal in the sense of J_ν. Although "κ_ν" would have been more consistent with the Global $\Box$ proof, this is only apparent: even for the $\Box$-proof, it is the ordinal α_ν that is important, rather than the κ_ν; here the important ordinal is again this α_ν. q_ν also has the same definition.

We set $\mathfrak{A}_\nu = M^{n_\nu}_\nu = \langle J_{\rho_\nu}, A^{n_\nu}_{\beta(\nu)}\rangle$. M0(iv) is then immediate, as h_ν is indeed a suitable Σ_1-Skolem function.

3.3 Definition. For $\nu,\overline{\nu}\in S^1$:

(i) We set $f:\overline{\nu}\implies\nu$ if $f: J_{\overline{\nu}}\longrightarrow_{\Sigma_1} J_\nu$, and f is the restriction of $f^*: J_{\beta(\overline{\nu})}\longrightarrow_{\Sigma_1^{(n)}} J_{\beta(\nu)}$ where $n=n_\nu, \nu=f^*(\overline{\nu})$ if $\nu<\beta(\nu); \alpha_\nu\in\mathrm{ran}(f)$; $p_\nu\in\mathrm{ran}(f^*)$.

(ii) $\mathbb{F}=\{\langle\overline{\nu},f,\nu\rangle \mid f:\overline{\nu}\implies\nu\}$.

(iii) If $\nu<\beta$, we set $p(\nu)=_{\mathrm{df}} p_\nu\cup\{\alpha_\nu,\nu\}$; otherwise $p(\nu)=_{\mathrm{df}} p_\nu\cup\{\alpha_\nu\}$.

(iv) $f^*_{(\delta,q,\nu)}$ is the inverse of the transitive collapse of the hull $\widetilde{h}_\nu(\delta,\{p(\nu)\cup q\})$ in $J_{\beta(\nu)}$.

Now we have a new version of Lemma 2.9:

3.4 Lemma. *If* $\exists\bar{\nu}(f:\overline{\nu}\implies\nu)$ *then* f *and* f^* *are uniquely determined by* $\mathrm{ran}(f)\cap\alpha_\nu$.

This justifies the following definition:

3.5 Definition. If $f : \bar{\nu} \implies \nu$ then we set:

(i) $f^*_{\bar{\nu}\nu}$ to be that unique extension f^* given by the last lemma;

(ii) $\widehat{f}_{\bar{\nu}\nu} = f^*_{\bar{\nu}\nu} \restriction \mathfrak{A}_{\bar{\nu}} \longrightarrow \mathfrak{A}_\nu$.

Lemma 2.11 still applies in this context, where now we amend the notation to read "$\nu \in S^1$" or "$\bar{\nu} \in S^1$" rather than just plain "S", and drop all references to κ. If $f : \bar{\nu} \implies \nu$ we see that $f(\alpha_{\bar{\nu}}) = \alpha_\nu$. This is clear if $\nu < \beta(\nu)$ (or equivalently $\bar{\nu} < \beta(\bar{\nu})$) since then $f : J_{\bar{\nu}} \longrightarrow J_\nu$ is in fact elementary. However even if $\beta(\nu) = \nu$, $\alpha_\nu \in \mathrm{ran}(f)$, hence $J_\nu \models$ "$f(\alpha_{\bar{\nu}})$ is a cardinal"; but clearly $f(\alpha_{\bar{\nu}})$ cannot be strictly greater than α_ν (as such are not J_ν-cardinals) nor strictly less (since $f^{-1}(\alpha_\nu)$ is a $J_{\bar{\nu}}$-cardinal $\geq \alpha_{\bar{\nu}}$).

3.6 Definition.

(i) For $f = \langle \bar{\nu}, f, \nu \rangle \in \mathbb{F}$ we call f^* *good* if

$$f^* \restriction J_{\bar{\nu}} : \langle J_{\bar{\nu}}, S_{\alpha_{\bar{\nu}}} \cap \bar{\nu} \rangle \longrightarrow_Q \langle J_\nu, S_{\alpha_\nu} \cap \nu \rangle.$$

(ii) $\bar{\nu} \prec \nu$ iff there is an $f = \langle \bar{\nu}, f, \nu \rangle \in \mathbb{F}$ with f^* good, and $f^* \restriction \alpha_{\bar{\nu}} = \mathrm{id} \restriction \alpha_{\bar{\nu}}$. We shall write in this case $f : \bar{\nu} \prec \nu$. We set $\bar{\nu} \preccurlyeq \nu \longleftrightarrow \bar{\nu} \prec \nu \vee \bar{\nu} = \nu$.

One should bear in mind that again, just as for similar remarks about E_ν in the proof of Global $\Box$, "$y = S_\gamma \cap \eta$" for $\eta < \nu$ is a simple $\Sigma_0^{J_\nu}$ relation of y, γ and J_η. Thus, in particular, if $f = \langle \bar{\nu}, f, \nu \rangle \in \mathbb{F}$, then $f(S_{\alpha_{\bar{\nu}}} \cap \eta) = S_{f(\alpha_{\bar{\nu}})} \cap f(\eta) = S_{\alpha_\nu} \cap f(\eta)$. What is the Q-preservation property for? For the simple reason that if $f = \langle \bar{\nu}, f, \nu \rangle \in \mathbb{F}$, then without Q-preservation of $f^* \restriction J_{\bar{\nu}}$ we may have $S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$ unbounded in $\bar{\nu}$ (a Q-property) whilst $S_{\alpha_\nu} \cap \nu$ is bounded in ν. If we insist on Q-preservation then we shall have, if $f : \bar{\nu} \implies \nu$:

- $S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$ unbounded in $\bar{\nu}$ iff $S_{\alpha_\nu} \cap \nu$ is unbounded in ν.

The next three properties are Σ_1:

- $\eta = \min(S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}) \longrightarrow f(\eta) = \min(S_{\alpha_\nu} \cap \nu)$;
- η the successor element in $S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$ of η' implies $f(\eta)$ is the successor element of $f(\eta')$ in $S_{\alpha_\nu} \cap \nu$;
- $\eta \in (S_{\alpha_{\bar{\nu}}} \cap \bar{\nu})^* \longrightarrow f(\eta) \in (S_{\alpha_\nu} \cap \nu)^*$.

The above can be easily checked. If we simply define gap-1 morasses without "structural properties" i.e. simply as maps between ordinals, one specifies these last three bullet points as an axiom on the S_α's (see [8, p. 341]).

For almost all triples $f = \langle \bar{\nu}, f, \nu \rangle \in \mathbb{F}$ the Q-preservation property holds anyway. Just as remarked before this definition, this is clear if $\nu < \beta(\nu)$ since then $f : J_{\bar{\nu}} \longrightarrow J_\nu$ is even elementary. It is still clear if $n_\nu > 1$ since then $f : J_{\bar{\nu}} \longrightarrow J_\nu$ is Σ_2-preserving. It is only if $n_\nu = 1$ and $\beta(\nu) = \nu$ that we may not get Q-preservation automatically.

It is now reasonably clear that $\prec$ will be a tree: this is because if $f : \bar{\nu} \prec \nu$ and $g : \nu' \prec \nu$ then, if $\alpha_{\bar{\nu}} < \alpha_{\nu'}$ we shall have $\mathrm{ran}(f^*) \subsetneq \mathrm{ran}(g^*)$ and hence $\langle \bar{\nu}, g^{-1} \circ f, \nu' \rangle \in \mathbb{F}$. It is also easy to check that the composed map $g^{-1} \circ f : J_{\bar{\nu}} \longrightarrow J_{\nu'}$ is Q-preserving, and this map witnesses that $\bar{\nu} \prec \nu'$. Equally clearly if $\alpha_{\bar{\nu}} = \alpha_{\nu'}$, then we must have $\bar{\nu} = \nu'$. Thus we have $\prec$ is a tree, and the second part of M0(iii) follows too.

The only remaining part of M0 is part (v). That the system of maps commutes follows from the same for the system for $\Longrightarrow$. The Q-preservation property of $f : J_{\bar{\nu}} \longrightarrow_Q J_\nu$ has been built into the definition of $\prec$.

For the first continuity property CP1, if $D \subseteq B^+_\nu$ then for each $\alpha_{\bar{\nu}} \in D$ we may consider the hull sets $H_{\bar{\nu}} =_{\mathrm{df}} \mathrm{ran}(f^*_{\bar{\nu}\nu})$ in $J_{\beta(\nu)}$. Let $Y = \bigcup_{\alpha_{\bar{\nu}} \in D} H_{\bar{\nu}}$, and then $\alpha =_{\mathrm{df}} \sup(D) \subseteq Y$. Then one may check that if the inverse of the transitive collapse of Y is $g^* : J_{\bar{\beta}} \longrightarrow_{\Sigma_1^{(n)}} J_\beta$ where $\mathrm{ran}(g^*) = Y$, then setting $g = g^* \restriction J_\tau$, $g^*(\tau) = \nu$ (if $\nu < \beta(\nu)$) and $\alpha_\tau = \alpha$, we have by Lemma 2.11 $\bar{\beta} = \beta(\tau)$, and $\langle \tau, g, \nu \rangle \in \mathbb{F}$. That g is a Q-preserving embedding between J_τ and J_ν is also easily checked: as before this only comes into question if $n = 1$ and $\nu = \beta(\nu)$. But here we can see that a union of "Q-hulls" of the form $H_{\bar{\nu}}$ forms a Q-elementary hull Y. Thus $g = g^* \restriction \tau$ is Q-preserving. Thus $\tau \preccurlyeq \nu$ and $\alpha_\tau \in B^+_\nu$.

For M1: let $\tau \in S_{\alpha_\nu} \cap \nu$. As $J_\nu \models$ "α_ν is the largest cardinal" we must have that $\beta(\tau) < \nu$. Now by recursion form a chain of $\Sigma_1^{(n_\tau)}$ hulls $H_\gamma = \widetilde{h}_\tau(\gamma \cup \{p(\tau)\})$ for $\gamma < \alpha_\nu$. This recursion can be effected inside J_ν as the latter is an admissible set (or is the union of such); moreover we can (i) pick out a closed and unbounded $D \subseteq \alpha_\nu$ where $\gamma \in D \longrightarrow H_\gamma \cap \alpha_\nu = \gamma$; and (ii) form the transitive collapses of such H_γ. This is because α_ν is a regular cardinal of J_ν. Then $\gamma \in D \longrightarrow \gamma = \alpha_{\bar{\tau}}$ for some $\alpha_{\bar{\tau}} \in B_\tau$.

Note that we have just shown for such τ that B_τ is unbounded in τ; but we have also shown that $B_\tau \in J_\nu$. In fact the relation "$\alpha_{\bar{\tau}} \in B_\tau$" is a uniform $\Sigma_0^{J_\nu}$ relation of $\bar{\tau}, \tau$ and $\beta(\tau)$. Consequently J_ν knows all about the morass relations for $\langle \alpha, \tau \rangle \in S$ with $\langle \alpha, \tau \rangle <_{\mathrm{lex}} \langle \alpha_\nu, \nu \rangle$.

For M2: note that, with the notation of the hypothesis, $f^*_{\bar{\nu}\nu}(J_{\beta(\bar{\tau})}) = J_{\beta(\tau)}$ and thus $f^*_{\bar{\nu}\nu} \restriction J_{\beta(\bar{\tau})} : J_{\beta(\bar{\tau})} \longrightarrow_{\Sigma_\omega} J_{\beta(\tau)}$. However $\mathrm{ran}(f^*_{\bar{\tau}\tau})$ is determined by $\mathrm{ran}(f^*_{\bar{\tau}\tau}) \cap \alpha_\nu = \alpha_{\bar{\tau}} = \alpha_{\bar{\nu}} = \mathrm{ran}(f^*_{\bar{\nu}\nu}) \cap \alpha_\nu$. Hence $f^*_{\bar{\nu}\nu} \restriction J_{\bar{\tau}} = \widehat{f}_{\bar{\nu}\nu} \restriction J_{\bar{\tau}} = \widehat{f}_{\bar{\tau}\tau} \restriction J_{\bar{\tau}}$. As $f^*_{\bar{\nu}\nu}(J_{\bar{\tau}}) = J_\tau$ we certainly have $\widehat{f}_{\bar{\nu}\nu} \restriction J_{\bar{\tau}} : \langle J_{\bar{\tau}}, S_{\alpha_{\bar{\tau}}} \cap \bar{\tau} \rangle \longrightarrow_Q \langle J_\tau, S_{\alpha_\tau} \cap \tau \rangle$. Hence we have all the conditions for $\bar{\tau} \prec \tau$.

For M3: Suppose $x \in \mathfrak{A}_\nu = M_\nu^{n_\nu} = \langle J_{\rho_\nu}, A^{n_\nu}_{\beta(\nu)} \rangle$. Let $h = h_{\mathfrak{A}_\nu} = h_\nu$. Then $x = h_\nu(i, \vec{\alpha}, \alpha_\nu, q_\nu \backslash \alpha_\nu)$ for some $\vec{\alpha} \in [\alpha_\nu]^{<\omega}$. As B_ν is unbounded in α_ν find $\alpha' > \max(\vec{\alpha})$, with $\alpha' = \alpha_\tau \in B_\nu$. Then $x = \widehat{f}_{\tau\nu}(h_\tau(i, \vec{\alpha}, \alpha_\tau, q_\tau \backslash \alpha_\tau))$.

Now for the distinguishing axioms of the morass: CP2. For (a) we use the Interpolation Lemma in the form of a direct application of Lemma 2.13 to get that there is an $f_0 : \bar{\nu} \Longrightarrow \lambda$ with $f \restriction \overline{\nu} = f_0 \restriction \overline{\nu}$ (where $f = f^*_{\bar{\nu}\nu}$). However f_0 is nothing other than the required map $f^*_{\bar{\nu}\lambda}$. (That $f^*_{\bar{\nu}\lambda} \restriction J_{\bar{\nu}}$ is a Q-embedding is also immediate, as by assumption it is cofinal into λ!.) Hence $\bar{\nu} \prec \lambda \wedge f^*_{\bar{\nu}\lambda} \restriction J_{\bar{\nu}} = f^*_{\bar{\nu}\nu} \restriction J_{\bar{\nu}}$.

For CP2(b): We work in the terms of the equivalent statement CP2(b)$'$.

Given the hypotheses there, let $\tau = \widehat{f}_{\bar{\nu}\nu}(\bar{\tau})$ for $\bar{\tau} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$. Let $X = \bigcup_{\bar{\tau} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}} X_{\bar{\tau}}$ where $X_{\bar{\tau}} = \mathrm{ran}(\widehat{f}_{\tau'\tau} \restriction J_{\tau'})$ and τ' is that unique η satisfying $\bar{\tau} \prec \eta \prec \tau$ and with $\alpha_\eta = \alpha$. Note that $X_{\bar{\tau}} \cap \alpha_\nu = \alpha$ and so $X \cap \alpha_\nu = \alpha$. $X \prec_{\Sigma_0} J_\nu$. Let the transitive collapse map be $g_1^{-1} : J_\nu \restriction X \cong J_{\nu'}$. Define $g_0 = g_1^{-1} \circ (\widehat{f}_{\bar{\nu}\nu} \restriction J_{\bar{\nu}})$. Note moreover, that as $\bar{\nu} \in (S_{\alpha_{\bar{\nu}}})^*$:

$$w \in J_{\bar{\nu}} \quad \Longrightarrow \quad \exists \bar{\tau} \in S_{\alpha_{\bar{\nu}}}[w \in J_{\bar{\tau}} \wedge \widehat{f}_{\bar{\tau}\tau}(w) = \widehat{f}_{\tau'\tau}(\widehat{f}_{\bar{\tau}\tau'}(w)) = \widehat{f}_{\bar{\nu}\nu}(w)]$$

by M2. Thus g_0 is total on $J_{\overline{\nu}}$. By assumption $f^*_{\bar{\nu}\nu}$ is cofinal into ν. We then have that both g_0 and g_1 are cofinal. Now we may apply the Generalised Interpolation Lemma, setting g, σ and σ_1 there as $f^*_{\bar{\nu}\nu}$, g_0, g_1 here respectively. We obtain $\widetilde{g} : M_{\overline{\nu}} \longrightarrow_{\Sigma_1^{(n)}} J_{\beta'}$ and $g' : J_{\beta'} \longrightarrow_{\Sigma_0^{(n)}} M_\nu$ with $\widetilde{g} \restriction J_{\overline{\nu}} = g_0$, $g' \restriction J_{\nu'} = g_1$. However g_1 is cofinal into ν and $f^*_{\bar{\nu}\nu}$ is cofinal at the $n$level (as $f_{\bar{\nu}\nu} : \bar{\nu} \Longrightarrow \nu$ cofinally, we can obtain this latter fact from Lemma 2.14); the proof of the Generalised Interpolation Lemma shows that in this case g' is cofinal at the nth level, and so is in fact also $\Sigma_1^{(n)}$-preserving. As $g_1(\alpha) = \alpha_\nu$ and $p_\nu \backslash \alpha_\nu \in \mathrm{ran}(f^*_{\overline{\nu}\nu})$ then $p_\nu \backslash \alpha_\nu \in \mathrm{ran}(g')$. We have, setting $g'(p') = p_\nu \backslash \alpha_\nu$, by an application of Lemma 2.11, that $p' = p_{\nu'} \backslash \alpha$, $M_{\nu'} = J_{\beta'}$ and $g' = f^*_{\nu'\nu}$, whence $\langle \nu', g_1, \nu \rangle \in \mathbb{F}$. However then we have that $\langle \overline{\nu}, g_0, \nu' \rangle \in \mathbb{F}$ with $\widetilde{g}_0 = f^*_{\overline{\nu},\nu'}$ by the same Lemma again.

The Σ_0-preserving yet cofinality of the maps g_0 and g_1 guarantees that they are Q-preserving (note that $S_{\alpha_{\overline{\nu}}} \cap \overline{\nu}, S_{\alpha_{\nu'}} \cap \nu', S_{\alpha_\nu} \cap \nu$ are all cofinal in their respective ordinals). Thus $f^*_{\overline{\nu}\nu'}$, and $f' = f^*_{\nu'\nu}$ are good (in the sense of Definition 3.6), and $\overline{\nu} \prec \nu' \prec \nu$ with $\alpha_{\nu'} = \alpha \in B_\nu$ as required. ⊣

3.2. Variants

Higher Gap Morasses

In the construction of the gap-1 morass at ω_1 say, we could have defined S^1 as the class of pseudo-successor cardinals, ordinals ν so that there exists $\alpha = \alpha_\nu < \nu$ so that $J_\nu \models$ "$\alpha = \omega_1$ and is the largest cardinal". A gap-n morass at ω_1 ($n < \omega$) would ask for classes S^i ($1 \leq i \leq n$) with $\nu \in S^i \longrightarrow \exists \alpha < \nu J_\nu \models$ "$\alpha = \omega_1 \wedge \mathrm{ot}\{\gamma > \omega \mid \gamma \in \mathrm{Card}\} = i$". Here we are allowing any $\nu < \omega_{n+1}$. In the gap-2 case, we would think of a system that approximates to ω_3 via a system of maps built out of countable objects approximating objects of size ω_1 reaching up to ω_2 much as a gap-1 morass does, but then additional maps piece these together in turn to obtain objects of size ω_2 that will reach up to ω_3. An enhanced set of axioms extending those of Definition 3.1 regulate this overall structure. Additional axioms (the "logical axioms") are also needed to control continuity properties and ensure that maps correctly transfer pieces of the morass from one structure to another. These higher gap morasses were introduced by Jensen [26].

There is no particular need to consider only finite gaps: countable length gaps (for morasses at ω_1); and generally gap-η morasses at κ can be constructed in L for $\eta < \kappa$, $\kappa \in \mathrm{Reg}$. Venturing beyond that, Irrgang has proposed definitions for gap-β morasses at ω_1 for $\beta > \omega_1$ and demonstrated their construction.

Hyperfine structural constructions exist of gap-1 morasses (Friedman et al. [21]), and for the gap-2 case (Friedman and Piwinger [20]); the latter is ostensibly different from the Jensen morass as they come with enhanced preservation properties. This "perfect preservation" comes about from the extremely fine gradations of locations, that enables even more stages in the hierarchy than for the J-hierarchy—thus developing a structure not *prima facie* available for the usual J-hierarchy.

Coarse Morasses

On occasions the full structure of the morass is simply not used and a coarse morass suffices. As its name implies, it is obtained by dropping some clauses from that of a full morass: namely those of the second continuity property CP2 and M0(v). As we don't need such fine elementarity between the structures we can simplify M0(iv) as well. (Indeed as we shall see, this principle requires no fine structure for its construction at all.)

3.7 Definition. Let $\kappa > \omega$ be regular. A *coarse gap-1 morass* at κ is a quadruple $\langle S, S^1, \prec, \langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu\rangle\rangle$ satisfying M0(i)–(iii), M0(iv)(a), CP1, M1-3 of Definition 3.1, together with

M0(v)$'$ $\langle \widehat{f}_{\bar\nu\nu} \mid \bar\nu \prec \nu\rangle$ is a commuting system of maps; $\widehat{f}_{\bar\nu\nu} : \mathfrak{A}_{\bar\nu} \longrightarrow_{\Sigma_\omega} \mathfrak{A}_\nu$;
$\widehat{f}_{\bar\nu\nu}(\alpha_{\bar\nu}) = \alpha_\nu; \widehat{f}_{\bar\nu\nu}\restriction\alpha_{\bar\nu} = \mathrm{id}\restriction\alpha_{\bar\nu}$.

In L coarse morasses are easily constructed: take the class S of pairs $\langle\alpha,\nu\rangle$ such that:

- Both J_α, J_ν are limits of ZF^- models, and $J_\nu \models$ "$\alpha = \alpha_\nu$ is the largest cardinal".

If $\langle\alpha,\nu\rangle \in S$ set $\mathfrak{A}_\nu =_{\mathrm{df}} J_\nu$. We define ν^* to be the least $\nu' \geq \nu$ so that (i) $J_{\nu'} \models \mathrm{ZF}^-$; (ii) for some $p \in J_{\nu'}$, every $x \in J_{\nu'}$ is $J_{\nu'}$-definable from parameters in $\alpha_\nu \cup \{p\}$. An elementary argument shows that such a ν^* must exist, and we set p_ν to be the $<_L$-least p satisfying (ii). Then set $q_\nu = p_\nu$ if $\nu^* = \nu$, or to equal $\langle p_\nu, \nu\rangle$ otherwise. The standard kinds of argument we have been using show quickly that if $f^* : J_{\nu'} \longrightarrow_{\Sigma_\omega} J_{\nu^*}$ with $q_\nu \in \mathrm{ran}(f^*)$, then $f^*(\bar\nu) = \nu$ implies $\nu' = \bar\nu^*$; we may define $\bar\nu \prec \nu$ if there exists an $f^* : J_{\bar\nu^*} \longrightarrow_{\Sigma_\omega} J_{\nu^*}$ such that $f^*\restriction\alpha_{\bar\nu} = \mathrm{id}\restriction\alpha_{\bar\nu}$, $f^*(\alpha_{\bar\nu}) > \alpha_{\bar\nu}$ and $q_\nu \in \mathrm{ran}(f^*)$. We set $\widehat{f}_{\bar\nu\nu} =_{\mathrm{df}} f^*\restriction J_{\bar\nu}$. It can be verified without much new work that if we now fix a regular cardinal κ and restrict our pairs $\langle\alpha,\nu\rangle$ so that $\nu < \kappa^+$ with $J_\nu \models$ "$\alpha_\nu \in \mathrm{Reg}$", then (setting S^1 to be the obvious class of such ν) our relation $\prec\restriction\kappa^+ \times \kappa^+$, together with all possible maps $\widehat{f}_{\bar\nu\nu}$, satisfies the coarse morass definitions. For more detail the reader may consult [11], where it is further shown how to extend the gap-1 coarse notion easily to a generalised gap *global coarse morass* of arbitrarily large gaps.

Morasses with Extra Structure

Various extra structure has been imposed on the morass concept in order to get strengthened results. This has occasioned the so-called *morasses with linear limits* [12], *morasses with built-in* $\diamondsuit$, and *morasses with built-in* $\square$ of Friedman [17].

Simplified Morasses

A substantial simplification of the morass notion is that of the *simplified morass* of Velleman [53]. Working to try and obtain a Martin's axiom-like postulate equivalent to the existence of gap-1 morasses, he derived a very simple and short list of axioms concerning sets of maps between ordinals. He showed that given a morass one could show (by forcing) the existence of a structure satisfying this axiom list: the simplified morass. This work derived from earlier work of Shelah and Stanley who were trying also to obtain similar forcing axioms. The advantage here is indeed that of simplicity: it delivers an easily comprehended structure ready for possible application.

Velleman left open the question of whether such morasses could be constructed directly in L. He later went on to develop higher gap simplified morasses [54] and morasses with linear limits. That simplified morasses existed in L was proven first by Donder [12] in the gap-1 case, and by Jensen (unpublished), building on Donder's work, for the higher gaps. For a construction of gap-1 and -2 morasses from simplified morasses see [38]. A Silver machine construction of a gap-1 morass was given in [40].

Applications

Jensen's original applications were to cardinal transfer theorems.

3.8 Definition. $(\kappa, \lambda) \longrightarrow (\theta, \eta)$ holds if for every structure whose universe has size κ with a distinguished unary predicate with extension in the structure of size λ, also has a model of size θ whose predicate has size η.

3.9 Theorem (Jensen) (The Gap-$n+1$ Cardinal Transfer Theorem). *If there is a gap-n morass at ω_1 then $(\kappa^{+(n+1)}, \kappa) \longrightarrow (\omega_{n+1}, \omega)$.*

Jensen also derived the following form of *Prikry's principle* from gap-1 morasses, symbolically this is:

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \not\longrightarrow \begin{bmatrix} \aleph_0 \\ \aleph_1 \end{bmatrix}_{\aleph_1}.$$

Interpreted this says that there is a partition of $\omega_2 \times \omega_1$ into $\aleph_1$ many sets $\{I_\gamma \mid \gamma < \omega_1\}$ such that $\forall A \subseteq [\omega_2]^{\aleph_0} \forall B \subseteq [\omega_1]^{\aleph_1} \forall \gamma < \omega_1 (A \times B) \cap I_\gamma \neq \varnothing$. (See [39] for more on this and some further generalisations.)

We state as a further example here, versions of the non-existence of free subsets from gap-2 morasses. The notion of "free" here is as a kind of maximal independent set. We typically ask that for every means of associating ordinals with tuples we can find some large set which is free of associations. Let $\mathrm{Fr}(\omega_3, 4, \omega_1)$ denote: "For every $F : [\omega_3]^4 \longrightarrow \omega_3$ such that for every quadruple a from ω_3 satisfies $F(a) \notin a$, there exists X a subset of ω_3 of cardinality ω_1 with $X \cap F``[X]^4 = \varnothing$".

3.10 Theorem (Jensen, unpublished). *Suppose* CH + *"there exists a gap-2 morass". Then* $\neg\,\mathrm{Fr}(\omega_3, 4, \omega_1)$.

Set-theoretical applications of morasses are not legion. The reader may consult [39] and [31] as a survey of early principles derivable from gap-1 morasses. A Morass with built in $\square$ forms a building block of the Strong Coding forcing constructed in [16]. For applications of gap-1 and global coarse morasses to derive in L one and two cardinal versions of $\diamond$ sequences, Kurepa trees without Aronszajn subtrees, and some negative partition relations, see Donder [11].

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11. Elementary Embeddings and Algebra

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Contents

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It has been observed for many years that computations with elementary embeddings entail some purely algebraic features—as opposed to the logical nature of the embeddings themselves. The key point is that the operation of applying an embedding to another satisfies, when defined, the self-distributivity law $x(yz) = (xy)(xz)$. Using specific properties of the elementary embeddings and their critical ordinals, hence assuming some large cardinal hypotheses, Richard Laver established two purely algebraic results about sets equipped with a self-distributive operation (LD-systems), namely the decidability of the associated word problem in 1989, and the unboundedness of the periods in some finite LD-systems in 1993. The large cardinal assumption was eliminated from the first result by the author in 1992 using an argument that led to unexpected results about Artin braid groups; as for the second of Laver's results, no proof in ZF has been discovered so far, and the only result known to date is that it cannot be proved in Primitive Recursive Arithmetic.

1. Iterations of an Elementary Embedding

Our aim is to study the algebraic operation obtained by *applying* an elementary embedding to another one. For $j, k : V \prec M$, we can apply j to any set-restriction of k, and, in good cases, the images of these restrictions cohere so as to form a new elementary embedding that we shall denote by $j[k]$. It is then easy to see that the application operation so defined satisfies various algebraic laws.

Convention: All elementary embeddings we consider here are supposed to be distinct from the identity. An easy rank argument shows that every such embedding moves some ordinal; in particular, the least ordinal moved by j is called the *critical ordinal* of j, and denoted $\mathrm{crit}(j)$.

1.1. Kunen's Bound and Axiom (I3)

If j is an elementary embedding of V into a proper subclass M, then $j[j]$, whenever it is defined, is an elementary embedding of M into a proper subclass M' of M, and it is not clear that $j[j]$ can in turn be applied to j, whose set-restrictions need not belong to M in general. So, if we wish the application operation on elementary embeddings to be everywhere defined, we should consider embeddings where the source and the target models coincide. Here comes an obstruction.

1.1 Proposition (AC; Kunen [15]). *There is no $j : V \prec V$.*

Proof. Assume $j : V \prec V$. Let $\kappa_0 = \mathrm{crit}(j)$, and, recursively, $\kappa_{n+1} = j(\kappa_n)$. Let $\lambda = \sup_n \kappa_n$. By standard arguments, each κ_n is an inaccessible cardinal, so λ is a strong limit cardinal. Fix an injection i_n of $\mathcal{P}(\kappa_n)$ into λ. Then the mapping $X \mapsto (i_n(X \cap \kappa_n))_{n\in\omega}$ defines an injection of $\mathcal{P}(\lambda)$ into λ^ω. Using

AC, we fix an enumeration $(\gamma_\xi, X_\xi)_{\xi<\nu}$ of $\lambda \times [\lambda]^\lambda$, and then recursively construct an injective sequence $(s_\xi)_{\xi<\nu}$ in λ^ω such that s_ξ belongs to $[X_\xi]^\omega$: this is possible because the cardinality of $\lambda \times [\lambda]^\lambda$ equals that of λ^ω. Let $f : \lambda^\omega \to \lambda$ be defined by $f(s) = \gamma_\xi$ for $s = s_\xi$, and $f(s) = 0$ for s not of the form s_ξ. Let $X \in [\lambda]^\lambda$. Then, for each $\gamma < \lambda$, there exists a $\xi < \nu$ satisfying $(\gamma, X) = (\gamma_\xi, X_\xi)$. For this ξ, we have $s_\xi \in [X]^\omega$ by hypothesis, and $f(s_\xi) = \gamma_\xi$. Hence the function f, which lies in $V_{\lambda+2}$, has the property that the range of $f{\restriction}X^\omega$ is λ for every X in $[\lambda]^\lambda$.

Let us consider $j(f)$. We have $j(\lambda) = \sup_n \kappa_{n+1} = \lambda$, hence $j(f)$ is a function of λ into itself, and, as j is elementary, $j(f)$ has the property that, for every X in $[\lambda]^\lambda$, the range of $j(f){\restriction}X^\omega$ is λ. Now, let X be the set $\{\theta < \lambda \mid \theta \in \mathrm{ran}(j)\}$. For every s in X^ω, we have $s(n) \in \mathrm{ran}(j)$ for every n, hence $s = j(s')$ for some s', and $j(f)(s) = j(f)(j(s')) = j(f(s')) \in X$. As X is a proper subset of λ, the range of $j(f){\restriction}X^\omega$ is not λ, and we have got a contradiction. ⊣

We are thus led to considering weaker assumptions, involving embeddings that are defined on ranks rather than on the whole universe.

1.2 Definition (Gaifman, Solovay-Reinhardt-Kanamori [21]).

$$\text{Axiom (I3): For some } \delta\text{, there exists a } j : V_\delta \prec V_\delta.$$

Assume $j : V_\delta \prec V_\delta$. Let $\kappa_0 = \mathrm{crit}(j)$, and $\kappa_n = j^n(\kappa_0)$. The proof of Proposition 1.1 shows that, letting $\lambda = \sup_n \kappa_n$, it is impossible (assuming AC) that the function called f there belongs to the target model of j. The function f belongs to $V_{\lambda+2}$, so $\delta \geqslant \lambda+2$ is impossible, and the only remaining possibilities for Axiom (I3) are $\delta = \lambda$, and $\delta = \lambda + 1$. The second possibility subsumes the first:

1.3 Lemma. *Assume $j : V_{\delta+1} \prec V_{\delta+1}$. Then we have $j{\restriction}V_\delta : V_\delta \prec V_\delta$.*

Proof. First, $j(\delta) < \delta$ is impossible, so we necessarily have $j(\delta) = \delta$, and, therefore, $j{\restriction}V_\delta$ maps V_δ to itself. As for elementarity, an easy induction shows that, for $\vec{a}$ in V_δ and Φ a first-order formula, $V_\delta \models \Phi(\vec{a})$ is equivalent to $V_{\delta+1} \models \Phi^{V_\delta}(\vec{a})$, and, therefore, $V_\delta \models \Phi(\vec{a})$ is equivalent to $V_{\delta+1} \models \Phi^{V_\delta}(\vec{a})$, hence to $V_{\delta+1} \models \Phi^{V_{j(\delta)}}(j(\vec{a}))$, and finally to $V_\delta \models \Phi(j(\vec{a}))$. ⊣

Thus, without loss of generality, we can restrict to the case $j : V_\lambda \prec V_\lambda$ in the sequel, i.e., when using Axiom (I3), we can add the assumption that δ is the supremum of the cardinals $j^n(\mathrm{crit}(j))$.

Before turning to the core of our study, let us observe that Axiom (I3) lies very high in the hierarchy of large cardinals.

1.4 Proposition. *Assume $j : V_\delta \prec V_\delta$, with $\kappa = \mathrm{crit}(j)$. Then there exists a normal ultrafilter on κ concentrating on cardinals that are n-huge for every n.*

Proof. As above, let $\kappa_n = j^n(\kappa)$. Let $U_n = \{X \subseteq \mathcal{P}(\kappa_n) \mid j``\kappa_n \in j(X)\}$. Then U_n is a κ-complete ultrafilter U_n on $\mathcal{P}(\kappa_n)$, and, for every $i < n$, the set $\{x \in \mathcal{P}(\kappa_n) \mid \mathrm{ot}(x \cap \kappa_{i+1}) = \kappa_i\}$ belongs to U_n, since its image under j is $\{x \in \mathcal{P}(\kappa_{n+1}) \mid \mathrm{ot}(x \cap \kappa_{i+2}) = \kappa_{i+1}\}$, which contains $j``\kappa_n$ as we have $j``\kappa_n \cap \kappa_{i+2} = j``\kappa_{i+1}$. By [14, 24.8], this means that κ is n-huge.

Then we use a classical reflection argument, especially easy here. Let $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Then U is a normal ultrafilter over κ. Let X_0 be the set of all cardinals below κ that are n-huge for every n. Then $j(X_0)$ is the set of all cardinals below $j(\kappa)$ that are n-huge for every n, which contains κ as was seen above. So X_0 belongs to U. ⊣

1.2. Operations on Elementary Embeddings

For λ a limit ordinal, we denote by $\mathcal{E}_\lambda$ the family of all $j : V_\lambda \prec V_\lambda$. In most cases $\mathcal{E}_\lambda$ is empty, and Axiom (I3) asserts that at least one $\mathcal{E}_\lambda$ is nonempty.

No function $f : V_\lambda \to V_\lambda$ is an element of V_λ. However, we can approximate f by its restrictions $f{\restriction}V_\gamma$ with $\gamma < \lambda$, each of which belongs to V_λ. If g is (another) function defined on V_λ, then g can be applied to each restriction $f{\restriction}V_\gamma$. If g happens to be an elementary embedding, the images $g(f{\restriction}V_\gamma)$ form a coherent system, and, in this way, we can *apply* g to f.

1.5 Definition. For $j, k : V_\lambda \to V_\lambda$, the *application of* j *to* k is defined by

$$j[k] = \textstyle\bigcup_{\gamma<\lambda} j(k{\restriction}V_\gamma).$$

This definition makes sense, as, by construction, $k{\restriction}V_\gamma$ belongs to $V_{k(\gamma)+3}$, and therefore to V_λ.

1.6 Lemma. *Assume* $j, k \in \mathcal{E}_\lambda$. *Then* $j[k]$ *belongs to* $\mathcal{E}_\lambda$, *and we have* $\mathrm{crit}(j[k]) = j(\mathrm{crit}(k))$.

Proof. When γ ranges over λ, the various mappings $k{\restriction}V_\gamma$ are compatible. As j is elementary, $j(k{\restriction}V_\gamma)$ is a partial mapping defined on $V_{j(\gamma)}$, and the partial mappings $j(k{\restriction}V_\gamma)$ and $j(k{\restriction}V_{\gamma'})$ associated with different ordinals γ, γ' agree on $V_{j(\gamma)} \cap V_{j(\gamma')}$. Hence $j[k]$ is a mapping of V_λ into itself.

Let $\Phi(\vec{x})$ be a first-order formula. For each γ in λ, we have

$$\forall \vec{x} \in V_\gamma (\Phi(\vec{x}) \iff \Phi((k{\restriction}V_\gamma)(\vec{x}))),$$

hence, applying j,

$$\forall \vec{x} \in V_{j(\gamma)} (\Phi(\vec{x}) \iff \Phi(j(k{\restriction}V_\gamma)(\vec{x}))),$$

so $j[k]$ is an elementary embedding of V_λ into itself.

The equality $\mathrm{crit}(j[k]) = j(\mathrm{crit}(k))$ follows from the fact that $k(\mathrm{crit}(k)) > \mathrm{crit}(k)$ implies $j[k](j(\mathrm{crit}(k)) > j(\mathrm{crit}(k))$, while $\forall \gamma < \mathrm{crit}(k)\, (k(\gamma) = \gamma)$ implies $\forall \gamma < j(\mathrm{crit}(k))\, (j[k](\gamma) = \gamma)$. ⊣

Note that, for j, k in $\mathcal{E}_\lambda$ and $\gamma < \lambda$, the equality

$$j[k]\restriction V_{j(\gamma)} = j(k\restriction V_\gamma) \tag{11.1}$$

is true by construction, as well as the formula

$$j[k](x) = jkj^{-1}(x) \tag{11.2}$$

whenever x belongs to the image of j.

Besides the application operation, composition is another binary operation on $\mathcal{E}_\lambda$. We insist that application is *not* composition. As (11.2) shows, application is a sort of conjugation with respect to composition.

Let us turn to the algebraic study of the application and composition operations. The former is neither commutative nor associative, but the following algebraic relations are satisfied. As usual, id_X denotes the identity map on set X.

1.7 Lemma (Folklore). *For $j, k, \ell \in \mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$, we have*

$$j[k[\ell]] = j[k][j[\ell]],\ j{\circ}k = j[k]{\circ}j,\ (j{\circ}k)[\ell] = j[k[\ell]],\ j[k{\circ}\ell] = j[k]{\circ}j[\ell]. \tag{11.3}$$

Proof. Let $\gamma < \lambda$. Then $\ell\restriction V_\gamma$ belongs to V_β for some $\beta < \lambda$. From the definition, we have $k[\ell]\restriction V_{k(\gamma)} = (k\restriction V_\beta)(\ell\restriction V_\gamma)$. Applying j we get

$$j(k[\ell]\restriction V_{k(\gamma)}) = j(k\restriction V_\beta)[j(\ell\restriction V_\gamma)].$$

By (11.1), the left factor is $j[k[\ell]]\restriction V_{j(k(\gamma))}$, and $j(k(\gamma)) = j[k](j(\gamma))$ implies that the right factor is $j[k][j[\ell]]\restriction V_{j(k(\gamma))}$. As γ is arbitrary, we deduce $j[k[\ell]] = j[k][j[\ell]]$.

Let $x \in V_\lambda$. For γ sufficiently large, we have $x \in \mathrm{dom}(k\restriction V_\gamma)$, hence

$$j(k(x)) = j((k\restriction V_\gamma)(x)) \ = j(k\restriction V_\gamma)(j(x)) = j[k](j(x)),$$

which establishes the equality $j{\circ}k = j[k]{\circ}j$. Applying the latter to $x = \ell\restriction V_\gamma$, one easily deduces $(j{\circ}k)[\ell] = j[k[\ell]]$.

Finally, using the fact that j preserves composition, we obtain

$$\begin{aligned} j[k{\circ}\ell]\restriction V_{j(\gamma)} &= j((k{\circ}\ell)\restriction V_\gamma) = j((k\restriction V_{\ell(\gamma)}){\circ}(\ell\restriction V_\gamma)) \\ &= (j[k]\restriction V_{j\ell(\gamma)}){\circ}(j[\ell]\restriction V_{j(\gamma)}) = (j[k]{\circ}j[\ell])\restriction V_{j(\gamma)}, \end{aligned}$$

for every γ, and hence $j[k{\circ}\ell] = j[k]{\circ}j[\ell]$. ⊣

Also $j[\mathrm{id}_{V_\lambda}] = \mathrm{id}_{V_\lambda}$ and $\mathrm{id}_{V_\lambda}[j] = j$ hold for every j in $\mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$. In order to fix the vocabulary for the sequel, we put the following definitions:

1.8 Definition.

(i) $(S, *)$ is a *left self-distributive system*, or LD-*system*, if $*$ is a binary operation on S satisfying

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

(ii) $(M, *, \cdot, 1)$ is a *left self-distributive monoid*, or LD-*monoid*, if $(M, \cdot, 1)$ is a monoid and $*$ is a binary operation on M satisfying

$$x \cdot y = (x * y) \cdot x, \qquad (x \cdot y) * z = x * (y * z),$$
$$x * (y \cdot z) = (x * y) \cdot (x * z), \qquad x * 1 = 1. \tag{11.4}$$

Observe that an LD-monoid is an LD-system and $1 * x = x$ always holds, as (11.4) implies $x * (y * z) = (x \cdot y) * z = ((x * y) \cdot x) * z = (x * y) * (x * z)$, and, similarly, $1 * x = (1 * x) \cdot 1 = 1 \cdot x = x$. With these definitions (various other names have been used in literature), we can restate Lemma 1.7 as

1.9 Proposition. *Let λ be a limit ordinal. Then $\mathcal{E}_\lambda$ equipped with application is an* LD-*system, and $\mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$ equipped with application and composition is an* LD-*monoid.*

Before developing our study further, we conclude this section with an independent result which we shall see in Sect. 3 leads to interesting consequences.

1.10 Proposition. *Assume $j : V_\lambda \prec V_\lambda$. Then we have $j[j](\alpha) \leqslant j(\alpha)$ for every ordinal $\alpha < \lambda$.*

Proof. Let β satisfy $j(\beta) > \alpha$ and $\forall \xi < \beta\, (j(\xi) \leqslant \alpha)$. As j is elementary, we deduce $j[j](j(\beta)) > j(\alpha)$ and $\forall \xi < j(\beta)\, (j[j](\xi) \leqslant j(\alpha))$—we can make things rigorous by replacing the parameter j with some approximation of the form $j \restriction V_\gamma$ with γ sufficiently large. As $\alpha < j(\beta)$ holds, we can take $\xi = \alpha$ in the second formula, which gives $j[j](\alpha) \leqslant j(\alpha)$. $\dashv$

1.3. Iterations of an Elementary Embedding

We now turn to the specific study of the iterations of a fixed elementary embedding $j : V_\lambda \prec V_\lambda$, as developed by Laver. So we concentrate on the countable subfamily of $\mathcal{E}_\lambda$ consisting of those embeddings that can be obtained from j using application (or both application and composition).

1.11 Definition. For $j \in \mathcal{E}_\lambda$, we denote by $\mathrm{Iter}(j)$ the sub-LD-system of $\mathcal{E}_\lambda$ generated by j, and by $\mathrm{Iter}^*(j)$ the sub-LD-monoid of $\mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$ generated by j. The elements of $\mathrm{Iter}^*(j)$ are called the *iterates* of j, while the elements of $\mathrm{Iter}(j)$ are called the *pure iterates* of j.

By definition, the pure iterates of j are those elementary embeddings that can be obtained from j using the application operation repeatedly, so they comprise $j, j[j], j[j[j]], j[j][j]$, etc. As application is a non-associative operation, the iterates of j do not reduce to powers of j; however, even the notion of a power has to be made precise. We shall use the following notation:

1.12 Definition. For j in $\mathcal{E}_\lambda$—or, more generally, in any binary system—we recursively define the nth *right power* $j^{[n]}$ of j and the nth *left power* $j_{[n]}$ of j by $j^{[1]} = j_{[1]} = j$, $j^{[n+1]} = j[j^{[n]}]$, and $j_{[n+1]} = j_{[n]}[j]$.

For future use, let us mention some relations between the powers in an arbitrary LD-system:

1.13 Lemma. *The following relations are satisfied in every* LD-*system:*

$$x^{[p+1]} = x^{[q]}[x^{[p]}] \quad \text{for } 1 \leqslant q \leqslant p, \qquad (x^{[p]})^{[q]} = x^{[p+q-1]} \quad \text{for } 1 \leqslant p, q. \tag{11.5}$$

In the sequel, we investigate the possible quotients of the algebraic structures $\mathrm{Iter}(j)$ and $\mathrm{Iter}^*(j)$, i.e., we look for equivalence relations that are compatible with the involved algebraic operation(s). A simple idea would be to concentrate on critical ordinals, or, more generally, on the values at particular fixed ordinals, but this naïve approach is not relevant beyond the first levels. Another idea would be to consider the restrictions of the embeddings to a fixed rank, i.e., to consider equivalence relations of the form $j{\restriction}V_\gamma = j'{\restriction}V_\gamma$. However, such relations are not compatible with the application operation in general, and we are led to slightly different relations.

1.14 Definition (Laver). Assume j, $j' \in \mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$. For γ a limit below λ, we say that j and j' are *γ-equivalent*, denoted $j \stackrel{\gamma}{\equiv} j'$, if, for every x in V_γ, we have $j(x) \cap V_\gamma = j'(x) \cap V_\gamma$.

By definition, $\stackrel{\gamma}{\equiv}$ is an equivalence relation on $\mathcal{E}_\lambda \cup \{\mathrm{id}_{V_\lambda}\}$. Note that $j \stackrel{\gamma}{\equiv} j'$ implies $j(x) \cap V_\gamma = j'(x) \cap V_\gamma$ for every x in V_λ—not only in V_γ—since, for $y \in V_\beta$ with $\beta < \gamma$, the relation $y \in j(x) \cap V_\gamma$ is equivalent to $y \in j(x \cap V_\beta) \cap V_\gamma$, and $x \cap V_\beta$ belongs to $V_{\beta+1}$, hence to V_γ since γ is limit.

We begin with easy observations.

1.15 Lemma. *Assume $j \stackrel{\gamma}{\equiv} j'$ and $\alpha < \gamma$. Then we have either $j(\alpha) < \gamma$, whence $j'(\alpha) = j(\alpha)$, or $j(\alpha) \geqslant \gamma$, whence $j'(\alpha) \geqslant \gamma$. So, in particular, we have either* $\mathrm{crit}(j) = \mathrm{crit}(j') < \gamma$, *or both* $\mathrm{crit}(j) \geqslant \gamma$ *and* $\mathrm{crit}(j') \geqslant \gamma$.

Proof. Assume $j' \stackrel{\gamma}{\equiv} j$ and $\alpha, \beta < \gamma$. Then, by definition, $j(\alpha) > \beta$ is equivalent to $j'(\alpha) > \beta$. ⊣

1.16 Lemma. *Assume $j, k \in \mathcal{E}_\lambda$. Then $j[k]$ and k are* $\mathrm{crit}(j)$*-equivalent.*

Proof. Let $\gamma = \mathrm{crit}(j)$. An induction on rank shows that $j{\restriction}V_\gamma$ is the identity mapping. Then $y \in k(x)$ is equivalent to $j(y) \in j[k](j(x))$, hence to $y \in j[k](x)$ for x, y in V_γ. ⊣

1.17 Proposition. *For limit $\gamma < \lambda$, γ-equivalence is compatible with composition.*

Proof. Assume $j \stackrel{\gamma}{\equiv} j'$ and $k \stackrel{\gamma}{\equiv} k'$. Let $x, y \in V_\gamma$, and $y \in j(k(x))$. As γ is limit, we have $x, y \in V_\beta$ for some $\beta < \gamma$, so $y \in j(k(x))$ implies $y \in j(k(x) \cap V_\beta) \cap V_\gamma$. By hypothesis, we have $k(x) \cap V_\beta = k'(x) \cap V_\beta \in V_{\beta+1} \subseteq V_\gamma$, hence

$$j'(k'(x) \cap V_\beta) \cap V_\gamma = j(k'(x) \cap V_\beta) \cap V_\gamma = j(k(x) \cap V_\beta) \cap V_\gamma.$$

We deduce $y \in j'(k'(x))$, hence $j(k(x)) \cap V_\gamma \subseteq j'(k'(x)) \cap V_\gamma$. By symmetry, we obtain $j(k(x)) \cap V_\gamma = j'(k'(x)) \cap V_\gamma$, so $j \circ k$ and $j' \circ k'$ are γ-equivalent. $\dashv$

1.18 Lemma. *Let $j : V_\lambda \prec V_\lambda$. Then, for each γ satisfying $\mathrm{crit}(j) < \gamma < \lambda$, there exists a δ satisfying $\delta < \gamma \leqslant j(\delta) < j(\gamma)$.*

Proof. Let $\kappa = \mathrm{crit}(j)$. Let δ be the least ordinal satisfying $\gamma \leqslant j(\delta)$: since $\gamma \leqslant j(\gamma)$ is always true, δ exists, and we have $\delta \leqslant \gamma$. Assume $\delta = \gamma$. This means that $\xi < \gamma$ implies $j(\xi) < \gamma$, hence $j^n(\xi) < \gamma$ for each n. This contradicts $\gamma < \lambda$ and (by the remark after Lemma 1.3) $\lambda = \sup_n j^n(\kappa)$. $\dashv$

1.19 Proposition. *Assume $j \overset{\gamma}{\equiv} j'$ and $k \overset{\delta}{\equiv} k'$ with $j(\delta) \geqslant \gamma$. Then we have $j[k] \overset{\gamma}{\equiv} j'[k']$.*

Proof. Assume first $\mathrm{crit}(j) \geqslant \gamma$. By Lemma 1.15, we also have $\mathrm{crit}(j') \geqslant \gamma$. Moreover, $\delta \geqslant \gamma$ holds, for $\delta < \gamma$ would imply $j(\delta) = \delta < \gamma$. Hence, $k \overset{\delta}{\equiv} k'$ implies $k \overset{\gamma}{\equiv} k'$. Then, by Lemma 1.16, we find $j[k] \overset{\gamma}{\equiv} k \overset{\gamma}{\equiv} k' \overset{\gamma}{\equiv} j'[k']$.

Assume now $\mathrm{crit}(j) < \gamma$, and, therefore, $\mathrm{crit}(j') = \mathrm{crit}(j)$. Since $k \overset{\delta}{\equiv} k'$ implies $k \overset{\delta'}{\equiv} k'$ for $\delta' \leqslant \delta$, we may assume without loss of generality that δ is minimal satisfying $j(\delta) \geqslant \gamma$, which, by Lemma 1.18, implies $\gamma > \delta$. Let $j \overset{*}{\cap} V_\alpha$ denote the set $\{(x, y) \in V_\alpha^2 \mid y \in j(x)\}$. By definition, $j \overset{\alpha}{\equiv} j'$ is equivalent to $j \overset{*}{\cap} V_\alpha = j' \overset{*}{\cap} V_\alpha$. We have

$$j[k] \overset{*}{\cap} V_\gamma = (j[k] \overset{*}{\cap} V_{j(\delta)}) \cap V_\gamma^2 = j(k \overset{*}{\cap} V_\delta) \cap V_\gamma^2.$$

By construction, $k \overset{*}{\cap} V_\delta$ is a set of ordered pairs of elements of V_δ, hence an element of V_γ. The hypotheses $k \overset{*}{\cap} V_\delta = k' \overset{*}{\cap} V_\delta$ and $j(x) \cap V_\gamma = j'(x) \cap V_\gamma$ for $x \in V_\gamma$ imply

$$j[k] \overset{*}{\cap} V_\gamma = j(k \overset{*}{\cap} V_\delta) \cap V_\gamma = j'(k \overset{*}{\cap} V_\delta) \cap V_\gamma = j'(k' \overset{*}{\cap} V_\delta) \cap V_\gamma = j'[k'] \overset{*}{\cap} V_\gamma,$$

so $j[k]$ and $j'[k']$ are γ-equivalent. $\dashv$

Let $j, k, \ell \in \mathcal{E}_\lambda$. Left self-distributivity gives $j[k[\ell]] = j[k][j[\ell]]$, but these embeddings need not be equal to $j[k][\ell]$, unless $j[\ell] = \ell$ holds. Now, by Lemma 1.16, $j[\ell]$ and ℓ are $\mathrm{crit}(j)$-equivalent, which implies that $j[k[\ell]]$ and $j[k][\ell]$ are $j[k](\mathrm{crit}(j))$-equivalent. Generalizing the argument, we obtain the following technical lemma. The convention is that $j[k][\ldots]$ means $(j[k])[\ldots]$.

1.20 Lemma. *Assume $j, j_1, \ldots, j_p \in \mathcal{E}_\lambda$, and let $\gamma = \mathrm{crit}(j)$.*

(i) *Assume $j[j_1[j_2]\ldots[j_\ell]](\gamma) \geqslant \gamma'$ for $1 \leqslant \ell \leqslant p-1$. Then we have*

$$j[j_1][j_2]\ldots[j_p] \overset{\gamma'}{\equiv} j[j_1[j_2]\ldots[j_p]]. \tag{11.6}$$

(ii) *Assume* $\mathrm{crit}(j_1[j_2]\ldots[j_\ell]) < \gamma$ *for* $1 \leqslant \ell \leqslant p-1$ *and* $\mathrm{crit}(j_1[j_2]\ldots[j_p]) \leqslant \gamma$. *Then we have*

$$\mathrm{crit}(j[j_1][j_2]\ldots[j_p]) = j(\mathrm{crit}(j_1[j_2]\ldots[j_p])). \tag{11.7}$$

Proof. (i) Use induction on p. For $p=1$, (11.6) is an equality. Otherwise, we have, by induction hypothesis, $j[j_1][j_2]\ldots[j_{p-1}] \overset{\gamma'}{\equiv} j[j_1[j_2]\ldots[j_{p-1}]]$, and, therefore,

$$j[j_1][j_2]\ldots[j_{p-1}][j_p] \overset{\gamma'}{\equiv} j[j_1[j_2]\ldots[j_{p-1}]][j_p]. \tag{11.8}$$

Lemma 1.16 gives $j_p \overset{\gamma}{\equiv} j[j_p]$, which implies

$$j[j_1[j_2]\ldots[j_{p-1}]][j_p] \overset{\gamma'}{\equiv} j[j_1[j_2]\ldots[j_{p-1}]][j[j_p]], \tag{11.9}$$

since $j[j_1[j_2]\ldots[j_{p-1}]](\gamma) \geqslant \gamma'$ holds by hypothesis. The right factor of (11.9) is also $j[j_1[j_2]\ldots[j_p]]$ by left self-distributivity, and combining (11.8) and (11.9) gives (11.6).

(ii) The case $p=1$ is trivial. Assume $p \geqslant 2$, and let γ' be the smallest of $j[j_1](\gamma)$, $j[j_1[j_2]](\gamma)$, $\ldots$, $j[j_1[j_2]\ldots[j_{p-1}]](\gamma)$. Applying (i), we find

$$j[j_1][j_2]\ldots[j_p] \overset{\gamma'}{\equiv} j[j_1[j_2]\ldots[j_p]]. \tag{11.10}$$

Let q be minimal satisfying $\gamma' = j[j_1[j_2]\ldots[j_q]](\gamma)$, and $j' = j_1[j_2]\ldots[j_q]$. Then we have $\gamma' = j[j'](\gamma)$. By hypothesis, we have $\mathrm{crit}(j') < \gamma$, so there exists a δ satisfying $\delta < \gamma \leqslant j'(\delta)$. From (11.10) we deduce

$$j(\gamma) \leqslant j(j'(\delta)) = j[j'](j(\delta)) = j[j'](\delta) < j[j'](\gamma) = \gamma'.$$

Hence $\mathrm{crit}(j_1[j_2]\ldots[j_p]) \leqslant \gamma$ implies $\mathrm{crit}(j[j_1[j_2]\ldots[j_p]]) \leqslant j(\gamma) < \gamma'$. Therefore the right embedding in (11.10) has its critical ordinal below γ', and, by Lemma 1.15, so has the left-hand embedding, and the two critical ordinals are equal. ⊣

1.4. Finite Quotients

By Proposition 1.19, γ-equivalence is compatible with the application operation, so that taking quotients under $\overset{\gamma}{\equiv}$ leads to a well-defined LD-system. We shall describe this quotient LD-system completely when γ happens to be the critical ordinal of some iteration of the investigated embedding.

By construction, for $j : V_\lambda \prec V_\lambda$, the sets $\mathrm{Iter}(j)$ and $\mathrm{Iter}^*(j)$ consist of countably many elementary embeddings, each of which except the identity has a critical ordinal. So, we can associate with j the countable family of all critical ordinals of iterates of j.

1.21 Definition. The ordinal $\mathrm{crit}_n(j)$ is defined to be the $(n+1)$th element in the increasing enumeration of the critical ordinals of iterations of j.

The formulas $\mathrm{crit}(j[k]) = j(\mathrm{crit}(k))$, $\mathrm{crit}(j\circ k) = \inf(\mathrm{crit}(j), \mathrm{crit}(k))$ and an obvious induction show that $\mathrm{crit}(i) \geqslant \mathrm{crit}(j)$ holds for every iterate i of j. Hence $\mathrm{crit}_0(j)$ is always $\mathrm{crit}(j)$. We shall prove below the values $\mathrm{crit}_1(j) = j(\mathrm{crit}(j))$ and $\mathrm{crit}_2(j) = j^2(\mathrm{crit}(j))$. Things become complicated subsequently. At this point, we do not know (yet) that the sequence of the ordinals $\mathrm{crit}_n(j)$ exhaust all critical ordinals in $\mathrm{Iter}^*(j)$: it could happen that some nontrivial iterate i of j has its critical ordinal beyond all $\mathrm{crit}_n(j)$'s.

1.22 Theorem (Laver). *Assume $j : V_\lambda \prec V_\lambda$. Then $\mathrm{crit}_n(j)$-equivalence is a congruence on the* LD*-monoid* $\mathrm{Iter}^*(j)$*, and the quotient* LD*-monoid has 2^n elements, namely the classes of j, $j_{[2]}$, ..., $j_{[2^n]}$, the latter also being the class of the identity.*

The proof requires several preliminary results.

1.23 Lemma. *Assume that i_1, i_2, ..., i_{2^n} are iterates of j. Then we have $\mathrm{crit}(i_1[i_2]\dots[i_p]) \geqslant \mathrm{crit}_n(j)$ for some p with $p \leqslant 2^n$.*

Proof. We use induction on n. For $n = 0$, the result is the inequality $\mathrm{crit}(i_1) \geqslant \mathrm{crit}(j)$, which we have seen holds for every iterate i_1 of j. Otherwise, we apply the induction hypothesis twice. First, we find $q \leqslant 2^{n-1}$ satisfying

$$\mathrm{crit}(i_1[i_2]\dots[i_q]) \geqslant \mathrm{crit}_{n-1}(j). \tag{11.11}$$

If the inequality is strict, we have $\mathrm{crit}(i_1[i_2]\dots[i_q]) \geqslant \mathrm{crit}_n(j)$, and we are done. So, we can assume from now on that (11.11) is an equality. By applying the induction hypothesis again, we find $r \leqslant 2^{n-1}$ satisfying

$$\mathrm{crit}(i_{q+1}[i_{q+2}]\dots[i_{q+r}]) \geqslant \mathrm{crit}_{n-1}(j).$$

If r is taken to be minimal, we can apply Lemma 1.20(i) with $p = r$, $j = i_1[i_2]\dots[i_q]$, $j_1 = i_{q+1}, \dots, j_p = i_{q+r}$, $\gamma = \mathrm{crit}_{n-1}(j)$, and $\gamma' = \mathrm{crit}_n(j)$. Indeed, with these notations, we have $\mathrm{crit}(j_1[j_2]\dots[j_s]) < \gamma$ for $s < r$, hence

$$\mathrm{crit}(j[j_1[j_2]\dots[j_s]]) = j(\mathrm{crit}(j_1[j_2]\dots[j_s])) = \mathrm{crit}(j_1[j_2]\dots[j_s]) < \gamma,$$

and, therefore, $j[j_1[j_2]\dots[j_s]](\gamma) > \gamma$, which gives $j[j_1[j_2]\dots[j_s]](\gamma) \geqslant \gamma'$ by definition. So we have

$$j[j_1][j_2]\dots[j_p] \overset{\gamma'}{\equiv} j[j_1[j_2]\dots[j_p]].$$

We have $\mathrm{crit}(j[j_1[j_2]\dots[j_p]]) \geqslant j(\gamma) \geqslant \gamma' = \mathrm{crit}_n(j)$, so, by Lemma 1.15, we deduce

$$\mathrm{crit}(j[j_1][j_2]\dots[j_p]) \geqslant \mathrm{crit}_n(j)$$

i.e., $\mathrm{crit}(i_1[i_2]\dots[i_q]\dots[i_{q+r}]) \geqslant \mathrm{crit}_n(j)$, as was expected. ⊣

The main task is now to show that all the iterates of j can be approximated by left powers of j up to $\mathrm{crit}_n(j)$-equivalence. Firstly, we approximate arbitrary iterates by pure iterates.

1.24 Lemma. *Assume that n is a fixed integer, and i is an iterate of j. Then there exists a pure iterate i' of j that is* $\mathrm{crit}_n(j)$*-equivalent to i.*

Proof. Let $\gamma = \mathrm{crit}_n(j)$, and let A be the set of those iterates of j that are γ-equivalent to some pure iterate of j. The set A contains j, and it is obviously closed under application. So, in order to show that A is all of $\mathrm{Iter}^*(j)$, it suffices to show that A is closed under composition, and, because γ-equivalence is compatible with composition, it suffices to show that, if i_1, i_2 are pure iterates of j, then some pure iterate of j is γ-equivalent to $i_2{\circ}i_1$. To this end, we define recursively a sequence of pure iterates of j, say i_3, i_4, ... by the clause $i_{p+2} = i_{p+1}[i_p]$. Then we have

$$i_3{\circ}i_2 = i_2[i_1]{\circ}i_2 = i_2{\circ}i_1,$$

and, recursively, $i_{p+1}{\circ}i_p = i_2{\circ}i_1$ for every p. We claim that $\mathrm{crit}(i_p) \geqslant \gamma$ holds for at least one of the values $p = 2n$ or $p = 2n+1$. If this is known, we find $i_2{\circ}i_1 = i_p{\circ}i_{p-1} = i_p[i_{p-1}]{\circ}i_p \overset{\gamma}{\equiv} i_{p-1}$, and we are done.

In order to prove the claim, we separate the cases $\mathrm{crit}(i_2) > \mathrm{crit}(i_1)$ and $\mathrm{crit}(i_2) \leqslant \mathrm{crit}(i_1)$. In this first case, we have

$$\mathrm{crit}(i_3) = i_2(\mathrm{crit}(i_1)) = \mathrm{crit}(i_1) \quad \text{and} \quad \mathrm{crit}(i_4) = i_3(\mathrm{crit}(i_2)) > \mathrm{crit}(i_2).$$

An immediate induction gives

$$\mathrm{crit}(i_1) = \mathrm{crit}(i_3) = \mathrm{crit}(i_5) = \cdots, \qquad \mathrm{crit}(i_2) < \mathrm{crit}(i_4) < \mathrm{crit}(i_6) < \cdots.$$

By definition, we have $\mathrm{crit}(i_1) \geqslant \mathrm{crit}_0(j)$, and, therefore, $\mathrm{crit}(i_2) \geqslant \mathrm{crit}_1(j)$, and, inductively, $\mathrm{crit}(i_{2n}) \geqslant \gamma$, as was claimed.

Assume now $\mathrm{crit}(i_2) \leqslant \mathrm{crit}(i_1)$. Similar computations give

$$\mathrm{crit}(i_1) < \mathrm{crit}(i_3) < \mathrm{crit}(i_5) < \cdots, \qquad \mathrm{crit}(i_2) = \mathrm{crit}(i_4) = \mathrm{crit}(i_6) = \cdots,$$

and we find now $\mathrm{crit}(i_{2n+1}) \geqslant \gamma$. So the claim is established, and the proof is complete. ⊣

Let us e.g. consider $i = j{\circ}j$. We are in the case "$\mathrm{crit}(i_2) \leqslant \mathrm{crit}(i_1)$", and we know that the pure iterate i_{2n+1} as above is a $\mathrm{crit}_n(j)$-approximation of i. For instance, we find $i_3 = j_{[2]}$, $i_4 = j_{[3]}$, $i_5 = j_{[3]}[j_{[2]}] = {j_{[4]}}^{[2]}$, so $j{\circ}j$ and $(j_{[4]})^{[2]}$ are $\mathrm{crit}_2(j)$-equivalent. It can be seen that the critical ordinal of $(j_{[4]})^{[2]}$, i.e., $j_{[4]}(\mathrm{crit}(j_{[4]}))$, is larger than $\mathrm{crit}_2(j)$, namely it is $\mathrm{crit}_3(j)$, so the previous equivalence is actually a $\mathrm{crit}_3(j)$-equivalence.

1.25 Proposition. *Assume $j : V_\lambda \prec V_\lambda$, $i \in \mathrm{Iter}^*(j)$, and $n \geqslant 0$. Then i is* $\mathrm{crit}_n(j)$*-equivalent to $j_{[p]}$ for some p with $p \leqslant 2^n$.*

Proof. By Lemma 1.24, we may assume that i is a pure iterate of j. The idea is to iteratively divide by j on the right, i.e., we construct pure iterates of j, say $i_0, i_1, \ldots$ such that i_0 is i, and i_p is $\mathrm{crit}_n(j)$-equivalent to $i_{p+1}[j]$

for every p. So, i is $\mathrm{crit}_n(j)$-equivalent to $i_p[j]\dots[j]$ (p times j) for every p. We stop the process when we have either $i_p = j$, in which case i is $\mathrm{crit}_n(j)$-equivalent to $j_{[p+1]}$, or $p = 2^n$: in this case, we have obtained a sequence of 2^n iterates of j, and Lemma 1.23 completes the proof.

Let us go into detail. In order to see that the construction is possible, let us assume that i_p has been obtained. If $i_p = j$ holds, we are done. Otherwise, i_p has the form $i'_1[i'_2[\dots[i'_r[j]]\dots]]$, where $i'_1, \dots, i'_r$ are some uniquely defined pure iterates of j. Applying the identity $j[k[\ell]] = (j{\circ}k)[\ell]$ $r-2$ times, we find $i_p = (i'_1{\circ}\cdots{\circ}i'_r)[j]$, and we define i_{p+1} to be a pure iterate of j that is $\mathrm{crit}_n(j)$-equivalent to $i'_1{\circ}\cdots{\circ}i'_r$.

Assume that the construction continues for at least 2^n steps, and let us consider the 2^n embeddings $i_{2^n}[j]$, $i_{2^n}[j][j], \dots, i_{2^n}[j][j]\dots[j]$ (2^n times j). By 1.23, there must be a $p \leqslant 2^n$ so that the critical ordinal of $i_{2^n}[j][j]\dots[j]$ (p times j) is at least $\mathrm{crit}_n(j)$. Let i' be the latter elementary embedding. Then i is $\mathrm{crit}_n(j)$-equivalent to $i_{2^n}[j][j]\dots[j]$ (2^n times j), which is also $i'[j][j]\dots[j]$ ($2^n - p$ times j), and, therefore, i is $\mathrm{crit}_n(j)$-equivalent to $\mathrm{id}[j][j]\dots[j]$ ($2^n - p$ times j), i.e., to $j_{[2^n-p]}$, and we are done as well. ⊣

The previous argument is effective. Starting with an arbitrary iteration i of j and a fixed level of approximation $\mathrm{crit}_n(j)$, we can find some left power of j that is $\mathrm{crit}_n[j]$-equivalent to i in a finite number of steps. However, the computation becomes quickly intricate, and there is no uniform way to predict how many steps are needed. For instance, let $i = j^{[3]}$, the simplest iterate of j that is not a left power. We write $i = (j{\circ}j)[j]$, and have to find an approximation of $j{\circ}j$. Now, $j{\circ}j$ is $\mathrm{crit}_3(j)$-equivalent to $j_{[3]}$, and, in this particular case, we obtain directly that $j^{[3]}$ is $\mathrm{crit}_3(j)$-equivalent to $j_{[3]}[j]$, i.e., to $j_{[4]}$. If we look for $\mathrm{crit}_4(j)$-equivalence, the computation is much more complicated. The results below will show that, if i is $\mathrm{crit}_3(j)$-equivalent to $j_{[4]}$, then it is $\mathrm{crit}_4(j)$-equivalent either to $j_{[4]}$ or to $j_{[12]}$. By determining the critical ordinal of $i[j][j][j][j]$, we could finally prove that $j^{[3]}$ is $\mathrm{crit}_4(j)$-equivalent to $j_{[12]}$. We shall see an easier alternative way for proving such statements in Sect. 3.2 below.

1.26 Proposition. *Assume $j : V_\lambda \prec V_\lambda$. Then, for every p, we have $\mathrm{crit}(j_{[p]}) = \mathrm{crit}_m(j)$, where m is the largest integer such that 2^m divides p.*

Proof. Let $\mathcal{H}_n$ be the conjunction of the following relations:

(i) $\mathrm{crit}(j_{[2^n]}) \geqslant \mathrm{crit}_n(j)$,

(ii) $p < 2^n$ implies $\mathrm{crit}(j_{[p]}) = \mathrm{crit}(j_{[2^m]})$ with m maximal such that 2^m divides p,

(iii) $m < n$ implies $\mathrm{crit}(j_{[2^m]}) < \mathrm{crit}(j_{[2^n]})$.

We prove $\mathcal{H}_n$ using induction on n. First, $\mathcal{H}_0$ reduces to $\mathrm{crit}(j) = \mathrm{crit}_0(j)$. Then, assume $\mathcal{H}_n$ and consider the embeddings $j_{[2^n+p]}$ for $1 \leqslant p \leqslant 2^n$. By definition, we have $j_{[2^n+p]} = j_{[2^n]}[j]\ldots[j]$ (p times j). Then $\mathcal{H}_n$(ii) and (iii) imply $\mathrm{crit}(j_{[s]}) < \mathrm{crit}(j_{[2^n]})$ for $s < 2^n$, while $\mathcal{H}_n$(i) implies $\mathrm{crit}(j_{[2^n]}) \geqslant \mathrm{crit}_n(j)$. By Lemma 1.20(ii) applied with $j = j_{[2^n]}$ and $j_1 = \cdots = j_p = j$, we have

$$\mathrm{crit}(j_{[2^n+p]}) = j_{[2^n]}(\mathrm{crit}(j_{[p]})).$$

For $p < 2^n$, we deduce $\mathrm{crit}(j_{[2^n+p]}) = \mathrm{crit}(j_{[p]}) = \mathrm{crit}_m(j)$ where m is the largest integer such that 2^m divides p, which is also the largest integer such that 2^m divides $2^n + p$, which is (ii) of $\mathcal{H}_{n+1}$. For $p = 2^n$, we obtain

$$\mathrm{crit}(j_{[2^{n+1}]}) = j_{[2^n]}(\mathrm{crit}(j_{[2^n]}) > \mathrm{crit}(j_{[2^n]}) = \mathrm{crit}_n(j),$$

and we deduce $\mathrm{crit}(j_{[2^{n+1}]}) \geqslant \mathrm{crit}_{n+1}(j)$, which is (i) of $\mathcal{H}_{n+1}$, and $\mathrm{crit}(j_{[2^{n+1}]}) > \mathrm{crit}(j_{[2^n]})$, which is (iii) of $\mathcal{H}_{n+1}$. Hence $\mathcal{H}_n$ is satisfied for each n.

Now, it follows from Proposition 1.25 that the critical ordinal of any iterate of j is either equal to the critical ordinal of some left power of j, or is larger than all ordinals $\mathrm{crit}_m(j)$. Since the sequence of all ordinals $\mathrm{crit}(j_{[2^n]})$ is increasing, the only possibility is $\mathrm{crit}(j_{[2^n]}) = \mathrm{crit}_n(j)$. ⊣

1.27 Lemma. *The left powers $j_{[p]}$ and $j_{[p']}$ are $\mathrm{crit}_n(j)$-equivalent if and only if $p = p' \bmod 2^n$ holds.*

Proof. We have $\mathrm{crit}(j_{[2^n]}) = \mathrm{crit}_n(j)$, so $j_{[2^n]}$ is $\mathrm{crit}_n(j)$-equivalent to the identity mapping, which by Proposition 1.19 inductively implies that $j_{[p]}$ and $j_{[2^n+p]}$ are $\mathrm{crit}_n(j)$-equivalent for every p. Hence the condition of the lemma is sufficient. On the other hand, we prove using induction on $n \geqslant 0$ that $1 \leqslant p < p' \leqslant 2^n$ implies that $j_{[p]}$ and $j_{[p']}$ are not $\mathrm{crit}_n(j)$-equivalent. The result is vacuously true for $n = 0$. Otherwise, for $p' \neq 2^{n-1} + p$, the induction hypothesis implies that $j_{[p]}$ and $j_{[p']}$ are not $\mathrm{crit}_{n-1}(j)$-equivalent, and *a fortiori* they are not $\mathrm{crit}_n(j)$-equivalent. Now, assume $p' = 2^{n-1} + p$ and $j_{[p]}$ and $j_{[p']}$ are $\mathrm{crit}_n(j)$-equivalent. By applying Proposition 1.19 $2^{n-1}-p$ times, we deduce that $j_{[2^{n-1}]}$ and $j_{[2^n]}$ are $\mathrm{crit}_n(j)$-equivalent, which is impossible as we have $\mathrm{crit}(j_{[2^{n-1}]}) < \mathrm{crit}_n(j)$ and $\mathrm{crit}(j_{[2^n]}) \geqslant \mathrm{crit}_n(j)$. ⊣

We are now ready to complete the proof of Theorem 1.22.

Proof. The result is clear from Proposition 1.25 and Lemma 1.27. That $j_{[2^n]}$ and the identity mapping are $\mathrm{crit}_n(j)$-equivalent follows from $\mathrm{crit}_n(j)$ being the critical ordinal of $j_{[2^n]}$. ⊣

1.5. The Laver-Steel Theorem

Assume $j : V_\lambda \prec V_\lambda$. By Lemma 1.6, $j^n(\mathrm{crit}(j))$ is the critical ordinal of $j^{[n+1]}$, which is also, by Lemma 1.13, $j^{[n]}[j^{[n]}]$: so, in the sequence of right powers $j, j^{[2]}, j^{[3]}, \ldots$, every term is a left divisor of the next one. Kunen's bound asserts that the supremum of the critical ordinals in the previous sequence is λ. Actually, this property has nothing to do with the particular choice of the embeddings $j^{[n]}$, and it is an instance of a much stronger statement, called the Laver-Steel theorem in the sequel, which is itself a special case of a general result of John Steel [22] about the Mitchell ordering:

1.28 Theorem (Steel). *Assume that $j_1, j_2, \ldots$ is a sequence in $\mathcal{E}_\lambda$ that is increasing with respect to divisibility,* i.e., *for every n, we have $j_{n+1} = j_n[k_n]$ for some k_n in $\mathcal{E}_\lambda$. Then we have $\sup_n \mathrm{crit}(j_n) = \lambda$.*

Here we shall give a simple proof of this result due to Randall Dougherty.

1.29 Definition. Assume $j \in \mathcal{E}_\lambda$, and $\gamma < \lambda$. We say that the ordinal α is *γ-representable by j* if it can be expressed as $j(f)(x)$ where f and x belong to V_γ and f is a mapping with ordinal values. The set of all ordinals that are γ-representable by j is denoted $S_\gamma(j)$.

1.30 Lemma. *Assume $j' = j[k]$ in $\mathcal{E}_\lambda$, and let γ be an inaccessible cardinal satisfying $\mathrm{crit}(j) < \gamma < \lambda$. Then the order type of $S_\gamma(j)$ is larger than the order type of $S_\gamma(j')$.*

Proof. The point is to construct an increasing mapping of $S_\gamma(j')$ into some proper initial segment of $S_\gamma(j)$. The idea is that $S_\gamma(j')$ is (more or less) the image under j of some set $S_\delta(k)$ with $\delta < \gamma$, which we can expect to be smaller than $S_\gamma(j)$ because $\delta < \gamma$ holds and γ is inaccessible.

By Lemma 1.18, there exists an ordinal δ satisfying $\delta < \gamma \leqslant j(\delta)$. Let G be the function that maps every pair (f, x) in V_δ^2 such that f is a function with ordinal values and x lies in the domain of $k(f)$ to $k(f)(x)$. By construction, the image of G is the set $S_\delta(k)$. The cardinality of this set is at most that of V_δ^2, hence it is strictly less than γ since γ is inaccessible. So the order type of the set $S_\delta(k)$ is less than γ, and, by ordinal recursion, we construct an order-preserving mapping H of $S_\delta(k)$ onto some ordinal β below γ. Let us apply now j: the mapping $j(H)$ is also order-preserving, and it maps $j(S_\delta(k))$, which is $S_{j(\delta)}(j')$, onto $j(\beta)$. By hypothesis, $j(\delta) \geqslant \gamma$ holds, so $S_{j(\delta)}(j')$ includes $S_\gamma(j')$. Let α be an ordinal in the latter set: by definition, there exist f and x in V_γ with f a mapping with ordinal values and x an element in the domain of $j'(f)$ satisfying $\alpha = j'(f)(x)$, and we have

$$j(H)(\alpha) = j(H)(j'(f)(x)) = j(H)(j(G)((f,x))) = j(H{\circ}G)((f,x)). \quad (11.12)$$

Now both $H{\circ}G$ and (f, x) are elements of V_γ. Thus (11.12) shows that the ordinal $j(H)(\alpha)$ is γ-representable by j, and the mapping $j(H)$ is an order-preserving mapping of $S_\gamma(j')$ into $S_\gamma(j)$. Moreover, the image of the mapping H is, by definition, the ordinal β, so the image of $j(H)$ is the ordinal $j(\beta)$, and, therefore, $j(H)$ is an order-preserving mapping of $S_\gamma(j')$ into $\{\xi \in S_\gamma(j) \mid \xi < j(\beta)\}$. Now we have $j(\beta) = j(f)(0)$, where f is the mapping $\{(0, \beta)\}$. Since $\beta < \gamma$ holds, we deduce that $j(\beta)$ is itself γ-representable by j, and that the above set $\{\xi \in S_\gamma(j) \mid \xi < j(\beta)\}$ is a proper subset of $S_\gamma(j)$. So the order type of $S_\gamma(j')$, which is that of $\{\xi \in S_\gamma(j) \mid \xi < j(\beta)\}$, is strictly smaller than the order type of $S_\gamma(j)$. $\dashv$

We can now prove the Laver-Steel theorem easily.

Proof. Assume for a contradiction that there exists an ordinal γ satisfying $\gamma < \lambda$ and $\gamma > \mathrm{crit}(j_n)$ for every n. We may assume that γ is an inaccessible cardinal: indeed, by Kunen's bound, there exists an integer m such that $j_1^m(\mathrm{crit}(j_1)) \geqslant \gamma$ holds, and we know that $j_1^m(\mathrm{crit}(j_1))$ is inaccessible. Now Lemma 1.30 applies to each pair (j_n, j_{n+1}), showing that the order types of the sets $S_\gamma(j_n)$ make a decreasing sequence, which is impossible. ⊣

1.31 Theorem (Laver). *Assume $j : V_\lambda \prec V_\lambda$.*

(i) *The ordinals* $\mathrm{crit}_n(j)$ *are cofinal in* λ, i.e., *there exists no θ with $\theta < \lambda$ such that* $\mathrm{crit}_n(j) < \theta$ *holds for every n.*

(ii) *For every iterate i of j, we have* $\mathrm{crit}(i) = \mathrm{crit}_m(j)$ *for some integer m, and, therefore, i is* $\mathrm{crit}_m(j)$*-equivalent to the identity.*

Proof. (i) By definition, every entry in the sequence j, $j_{[2]}$, $j_{[3]}$, $\ldots$ is a left divisor of the next one; hence Theorem 1.28 implies that the critical ordinals of j, $j_{[2]}$, $\ldots$ are cofinal in λ. By definition, these critical ordinals are exactly the ordinals $\mathrm{crit}_n(j)$.

(ii) Proposition 1.25 implies that either $\mathrm{crit}(i) > \mathrm{crit}_m(j)$ holds for every m, or there exists m satisfying $\mathrm{crit}(i) = \mathrm{crit}_m(j)$. By (i), the first case is impossible. ⊣

Observe that the point in the previous argument is really the Laver-Steel theorem, because Proposition 1.25 or Lemma 1.23 alone do not preclude the critical ordinal of some iterate i lying above all $\mathrm{crit}_m(j)$'s.

It follows from the previous result that, for every m, the image under j of the critical ordinal $\mathrm{crit}_m(j)$ is again an ordinal of the form $\mathrm{crit}_n(j)$. Indeed, $\mathrm{crit}_m(j)$ is the critical ordinal of $j_{[2^m]}$, and, therefore, $j(\mathrm{crit}_m(j))$ is the critical ordinal of $j[j_{[2^m]}]$, hence the critical ordinal of some iterate of j and, therefore, an ordinal of the form $\mathrm{crit}_n(j)$ for some finite n.

1.6. Counting the Critical Ordinals

As we already observed, the definition of an elementary embedding implies that the critical ordinal of $j[k]$ is the image under j of the critical ordinal of k. Hence every embedding in $\mathcal{E}_\lambda$ induces an increasing injection on the critical ordinals of $\mathcal{E}_\lambda$. In particular, every iterate of an embedding j acts on the critical ordinals of the iterates of j, which we have seen in the previous section consists of an ω-indexed sequence $(\mathrm{crit}_n(j))_{n<\omega}$. Let us introduce, for $j : V_\lambda \prec V_\lambda$, two mappings $\hat{j}, \tilde{j} : \omega \to \omega$ by

$$\hat{j}(m) = p \quad \text{if and only if} \quad j(\mathrm{crit}_m(j)) = \mathrm{crit}_p(j),$$

and $\tilde{j}(n) = \hat{j}^n(0)$. By definition, $\mathrm{crit}_{\tilde{j}(n)}$ is $j^n(\mathrm{crit}_0(j))$, so, if we use κ for $\mathrm{crit}(j)$ and κ_n for $j^n(\kappa)$, we simply have $\mathrm{crit}_{\tilde{j}(n)} = \kappa_n$: thus $\tilde{j}(n)$ is the number of critical ordinals of iterates of j below κ_n.

The aim of this section is to prove the following result:

1.32 Theorem (Dougherty [7]). *For $j : V_\lambda \prec V_\lambda$, the function $\tilde{\jmath}$ grows faster than any primitive recursive function.*

For the rest of the section, we fix $j : V_\lambda \prec V_\lambda$, and write γ_m for $\mathrm{crit}_m(j)$. Thus $\hat{\jmath}$ is determined by $\gamma_{\hat{\jmath}(m)} = j(\gamma_m)$ and $\tilde{\jmath}$ by $\gamma_{\tilde{\jmath}(n)} = j^n(\gamma_0)$. We are going to establish lower bounds for the values of the function $\tilde{\jmath}$. The first values of the function $\tilde{\jmath}$ can be computed exactly by determining sequences of iterated values for $j_{[p]}$. We use the notation

$$i : \Vdash\!\!\!\rightarrow \theta_0 \mapsto \theta_1 \mapsto \cdots$$

to mean that we have $\theta_0 = \mathrm{crit}(i)$, $\theta_1 = i(\theta_0)$ $(= \mathrm{crit}(i^{[2]}))$, etc. For instance, by definition of $\tilde{\jmath}$, we have

$$j : \gamma_0 \mapsto \gamma_{\tilde{\jmath}(1)} \mapsto \gamma_{\tilde{\jmath}(2)} \mapsto \gamma_{\tilde{\jmath}(3)} \mapsto \cdots .$$

Now, for each sequence of the form

$$i : \Vdash\!\!\!\rightarrow \theta_0 \mapsto \theta_1 \mapsto \theta_2 \mapsto \cdots ,$$

we deduce for each elementary embedding j_0 a new sequence

$$j_0[i] : \Vdash\!\!\!\rightarrow j_0(\theta_0) \mapsto j_0(\theta_1) \mapsto j_0(\theta_2) \mapsto \cdots .$$

Applying the previous principle to the above sequence with $j_0 = j$, and using $\tilde{\jmath}(1) = 1$, we obtain the sequence

$$j_{[2]} : \Vdash\!\!\!\rightarrow \gamma_1 \mapsto \gamma_{\tilde{\jmath}(2)} \mapsto \gamma_{\tilde{\jmath}(3)} \mapsto \cdots .$$

Applying the same principle with $j_0 = j_{[2]}$, we obtain

$$j_{[3]} : \Vdash\!\!\!\rightarrow \gamma_0 \mapsto \gamma_{\tilde{\jmath}(2)} \mapsto \gamma_{\tilde{\jmath}(3)} \mapsto \cdots .$$

Then $\gamma_2 = \mathrm{crit}(j_{[4]})$ implies $\gamma_2 = j_{[3]}(\gamma_0)$, so the previous sequence shows that the latter ordinal is $\gamma_{\tilde{\jmath}(2)}$, i.e., we have proved $\gamma_{\tilde{\jmath}(2)} = \gamma_2$, and, therefore we have $\hat{\jmath}(1) = 2$. Similar (but more tricky) arguments give $\hat{\jmath}(2) = 4$. Equivalently, we have $\tilde{\jmath}(1) = 1$, $\tilde{\jmath}(2) = 2$, $\tilde{\jmath}(3) = 4$, which means that the critical ordinals of the right powers j, $j^{[2]}$, and $j^{[3]}$ are γ_1, γ_2, and γ_4 respectively.

We turn now to the proof of Theorem 1.32. The basic argument is the following simple observation.

1.33 Lemma. *Assume that some iterate i of j satisfies $i : \gamma_p \mapsto \gamma_q \mapsto \gamma_r$. Then we have $r - q \geqslant q - p$.*

Proof. As the restriction of i to ordinals is increasing, $\gamma_p < \alpha < \alpha' < \gamma_q$ implies $\gamma_q < i(\alpha) < i(\alpha') < \gamma_r$. Moreover, if α is the critical ordinal of i_1, $i(\alpha)$ is that of $i[i_1]$, and, if i_1 is an iterate of j, so is $i[i_1]$. Hence the number of critical ordinals of iterates of j between γ_q and γ_r, which is $r - q - 1$, is at least the number of critical ordinals of iterates of j between γ_p and γ_q, which is $q - p - 1$. $\dashv$

1.34 Definition. A sequence of ordinals $(\alpha_0, \ldots, \alpha_p)$ is said to be *realizable* (with respect to j) if we have $i : \Vdash\!\!\to \alpha_0 \mapsto \cdots \mapsto \alpha_p$ for some iterate i of j. We say that the sequence $(\alpha_0, \ldots, \alpha_p)$ is a *base* for the sequence $\vec{\theta} = (\theta_0, \ldots, \theta_n)$ if, for each $m < n$, the sequence $(\alpha_0, \ldots, \alpha_p, \theta_m, \theta_{m+1})$ is realizable.

Observe that the existence of a base for a sequence $\vec{\theta}$ implies that $\vec{\theta}$ is increasing, and that, if $(a_0, \ldots, a_p)$ is a base for $\vec{\theta}$, so is every final subsequence of the form $(a_m, \ldots, \alpha_p)$: if i admits the critical sequence $\Vdash\!\!\to \alpha_0 \mapsto \cdots \mapsto \theta_m \mapsto \theta_{m+1}$, then $i^{[2]}$ admits the critical sequence $\Vdash\!\!\to \alpha_1 \mapsto \cdots \mapsto \theta_m \mapsto \theta_{m+1}$.

1.35 Lemma. *Assume that the sequence* $(\theta_0, \theta_1, \ldots)$ *admits a base. Then* $\theta_n \geqslant \gamma_{2^n}$ *holds for every* n.

Proof. Assume that (γ_p) is a base for $(\theta_0, \theta_1, \ldots)$. Define f by $\theta_n = \gamma_{f(n)}$. Lemma 1.33 gives $f(n+1) - f(n) \geqslant f(n) - p$ for every n. As $f(0) > p$ holds by definition, we deduce $f(n) \geqslant 2^n + p$ inductively. $\dashv$

For instance, the embedding $j_{[2]}$ leaves γ_0 fixed maps $\gamma_{\tilde{\jmath}(r)}$ to $\gamma_{\tilde{\jmath}(r+1)}$ for $r \geqslant 1$. So its $(r-1)$-th power with respect to composition satisfies

$$(j_{[2]})^{r-1} : \Vdash\!\!\to \gamma_1 \mapsto \gamma_{\tilde{\jmath}(r)}, \qquad \gamma_2 \mapsto \gamma_{\tilde{\jmath}(r+1)}.$$

Applying these values to the critical sequence of j, we obtain

$$(j_{[2]})^{r-1}[j] : \Vdash\!\!\to \gamma_0 \mapsto \gamma_{\tilde{\jmath}(r)} \mapsto \gamma_{\tilde{\jmath}(r+1)}.$$

Hence (γ_0) is a base for the sequence $(\gamma_{\tilde{\jmath}(1)}, \gamma_{\tilde{\jmath}(2)}, \ldots)$. Lemma 1.35 gives $\tilde{\jmath}(n) \geqslant 2^{n-1}$. In particular, we find $\tilde{\jmath}(4) \geqslant 8$. This bound destroys any hope of computing an exact value by applying the scheme used for the first values: indeed this would entail computing values until at least $j_{[255]}$. We shall see below that the value of $\tilde{\jmath}(4)$ is actually much larger than 8.

In order to improve the previous results, we use the following trick to expand the sequences admitting a base by inserting many intermediate new critical ordinals.

1.36 Lemma. *Assume* $(\alpha_0, \ldots, \alpha_p, \beta, \gamma)$ *is realizable,* $\vec{\theta}$ *is based on* (β) *and it goes from* γ *to* δ *in* n *steps. Then there exists a sequence based on* $(\alpha_0, \ldots, \alpha_p)$ *that goes from* β *to* δ *in* 2^n *steps.*

Proof. We use induction on $n \geqslant 0$. For $n = 0$, the sequence (β, γ) works, since $(\alpha_0, \ldots, \alpha_p, \beta, \gamma)$ being realizable means that (β, γ) is based on $(\alpha_0, \ldots, \alpha_p)$. For $n > 0$, let δ' be the next to last term of $\vec{\theta}$. The induction hypothesis gives a sequence $\vec{\tau}\,'$ based on $(\alpha_0, \ldots, \alpha_p)$ that goes from γ to δ' in 2^{n-1} steps. As (δ', δ) is based on (β), there exists an embedding i satisfying

$$i : \Vdash\!\!\to \beta \mapsto \delta' \mapsto \delta.$$

We define the sequence $\vec{\tau}$ by extending $\vec{\tau}\,'$ with 2^{n-1} additional terms

$$\tau_{2^{n-1}+m} = i(\tau'_m) \quad \text{for } 1 \leqslant m \leqslant 2^{n-1}.$$

By hypothesis, we have $\tau'_{2^{n-1}} = \delta'$, hence $\tau_{2^n} = i(\delta') = \delta$. So $\vec{\tau}$ goes from β to δ in 2^n steps. Moreover, $(\alpha_0, \dots, \alpha_p)$ is a base for $\vec{\tau}\,'$, so, for $0 \leqslant m < 2^{n-1}$, there exists an i'_m satisfying

$$i'_m : \Vdash\!\!\rightarrow \alpha_0 \mapsto \cdots \mapsto \alpha_p \mapsto \tau'_m \mapsto \tau'_{m+1}.$$

As β is the critical ordinal of i and $\alpha_p < \beta$ holds, this implies

$$i[i'_m] : \Vdash\!\!\rightarrow \alpha_0 \mapsto \cdots \mapsto \alpha_p \mapsto i(\tau'_m) \mapsto i(\tau'_{m+1}),$$

which shows that $(\alpha_0, \dots, \alpha_p)$ is a base for $\vec{\tau}$. Note that the case $m = 0$ works because $\tau'_0 = \beta$ implies $i(\tau'_0) = i(\beta) = \delta' = \tau_{2^{n-1}}$, as is needed. ⊣

By playing with the above construction one more time, we can obtain still longer sequences. In order to specify them, we use an *ad hoc* iteration of the exponential function, namely g_p recursively defined by $g_0(n) = n$, $g_{p+1}(0) = 0$, and $g_{p+1}(n) = g_{p+1}(n-1) + g_p(2^{g_{p+1}(n-1)})$. Thus, g_1 is an iterated exponential. Observe that $g_p(1) = 1$ holds for every p.

1.37 Lemma. *Assume $(\beta_0, \dots, \beta_{p+1}, \gamma)$ is realizable, $\vec{\theta}$ is based on (β_p, β_{p+1}) and it goes from γ to δ in n steps. Then there exists a sequence based on (β_{p+1}) that goes from γ to δ in $g_{p+1}(n)$ steps.*

Proof. We use induction on $p \geqslant 0$, and, for each p, on $n \geqslant 1$. For $n = 1$, the sequence (γ, δ) works, since, if i satisfies $\Vdash\!\!\rightarrow \beta_p \mapsto \beta_{p+1} \mapsto \gamma \mapsto \delta$, then $i^{[2]}$ satisfies $\Vdash\!\!\rightarrow \beta_{p+1} \mapsto \gamma \mapsto \delta$. Assume $n \geqslant 2$. Let δ' be the next to last term of $\vec{\theta}$. By induction hypothesis, there exists a sequence $\vec{\tau}\,'$ based on (β_{p+1}) that goes from γ to δ' in $g_{p+1}(n-1)$ steps. As in Lemma 1.36, we complete the sequence by appending new terms, but, before translating it, we still fatten it one or two more times. First, we apply Lemma 1.36 to construct a new sequence $\vec{\tau}\,''$ based on (β_p, β_{p+1}) that goes from β_{p+1} to δ' in $2^{g_{p+1}(n-1)}$ steps and is based on (β_{p-1}, β_p) for $p \neq 0$ (resp. on (β_p) for $p = 0$). For $p \neq 0$, we are in position for applying the current lemma with $p-1$ to the sequence of $\vec{\tau}\,''$. So we obtain a new sequence $\vec{\tau}\,'''$ based on (α_p), and going from β_{p+1} to δ' in $g_p(2^{g_{p+1}(n-1)})$ steps. For $p = 0$, we simply take $\vec{\tau}\,''' = \vec{\tau}\,''$: as $g_0(N) = N$ holds, this remains consistent with our notations. Now we make the translated copy: we choose i satisfying $\Vdash\!\!\rightarrow \beta_p \mapsto \beta_{p+1} \mapsto \delta' \mapsto \delta$, and complete $\vec{\tau}\,'$ with the new terms

$$\tau_{g_{p+1}(n-1)+m} = i(\tau'''_m) \quad \text{for } 0 < m \leqslant g_p(2^{g_{p+1}(n-1)}).$$

The sequence $\vec{\tau}$ has length $g_{p+1}(n-1) + g_p(2^{g_{p+1}(n-1)}) = g_{p+1}(n)$, and it goes from γ to $i(\delta')$, which is δ. It remains to verify the base condition for the new terms. Now assume that i'''_m satisfies $\Vdash\!\!\rightarrow \beta_p \mapsto \tau'''_m \mapsto \tau'''_{m+1}$. As in the proof of Lemma 1.36, we see that $i[i'''_m]$ satisfies $\Vdash\!\!\rightarrow \beta_{p+1} \mapsto i(\tau'''_m) \mapsto i(\tau'''_{m+1})$, which completes the proof, as $i(\tau'''_0) = \delta'$ guarantees continuity. ⊣

By combining Lemmas 1.36 and 1.37, we obtain:

1.38 Lemma. *Assume $(\beta_0, \dots, \beta_{p+1}, \gamma)$ is realizable, $\vec{\theta}$ is based on (β_p, β_{p+1}) and it goes from γ to δ in n steps. Then there exists a sequence based on (β_0) that goes from β_1 to δ in $h_1(h_2(\dots(h_{p+1}(n))\dots))$ steps, where $h_q(m)$ is defined to be $2^{g_q(m)}$.*

Proof. We use induction on $p \geqslant 0$. In every case, Lemma 1.37 constructs from $\vec{\theta}$ a new sequence $\vec{\theta}'$ based on (β_{p+1}) going from β_{p+1} to δ in $g_{p+1}(n)+1$ steps. Then, Lemma 1.36 constructs from $\vec{\theta}'$ a new sequence $\vec{\theta}''$ that goes from (β_{p+1}) to δ in $2^{g_{p+1}(n)}+1 = h_{p+1}(n)+1$ steps, a sequence based on (α_{p-1}, α_p) for $p \neq 0$, and on (α_p) for $p = 0$. For $p = 0$, the sequence $\vec{\theta}''$ works. Otherwise, we are in position for applying the induction hypothesis to $\vec{\theta}''$. ⊣

We deduce the following lower bound for the function $\tilde{\jmath}$.

1.39 Proposition. *Assume $j : V_\lambda \prec V_\lambda$. Then, for $n \geqslant 3$, we have*

$$\tilde{\jmath}(r) \geqslant 2^{h_1(h_2(\dots(h_{n-2}(1))\dots))}. \tag{11.13}$$

Proof. By definition, $(\gamma_{\tilde{\jmath}(n-1)}, \gamma_{\tilde{\jmath}(n)})$ is based on $(\gamma_{\tilde{\jmath}(n-3)}, \gamma_{\tilde{\jmath}(n-2)})$, and the auxiliary sequence $(\gamma_0, \dots, \gamma_{\tilde{\jmath}(n-2)})$ is realizable. Indeed, j satisfies

$$j : \Vdash\!\!\!\rightarrow \gamma_{\tilde{\jmath}(0)} \mapsto \gamma_{\tilde{\jmath}(1)} \mapsto \gamma_{\tilde{\jmath}(2)} \mapsto \gamma_{\tilde{\jmath}(3)},$$

and, therefore, we have

$$j^{[n+1]} : \Vdash\!\!\!\rightarrow \gamma_{\tilde{\jmath}(n)} \mapsto \gamma_{\tilde{\jmath}(n+1)} \mapsto \gamma_{\tilde{\jmath}(n+2)} \mapsto \gamma_{\tilde{\jmath}(n+3)}$$

for every n. By applying Lemma 1.38, we find a new sequence based on (γ_0) that goes from γ_1 to $\gamma_{\tilde{\jmath}(n)}$ in $h_1(h_2(\dots(h_{n-2}(1))\dots))$ steps. We conclude using Lemma 1.38. ⊣

We thus proved $\tilde{\jmath}(4) \geqslant 2^8 = 256$, and $\tilde{\jmath}(5) \geqslant 2^{h_1(h_2(h_3(1)))} = 2^{2^{g_1(16)}}$. It follows that $\tilde{\jmath}(5)$ is more than a tower of base 2 exponentials of height 17.

Let us recall that the Ackermann function f_p^{Ack} is defined recursively by $f_0^{\mathrm{Ack}}(n) = n+1$, $f_{p+1}^{\mathrm{Ack}}(0) = f_p^{\mathrm{Ack}}(1)$, and $f_{p+1}^{\mathrm{Ack}}(n+1) = f_p^{\mathrm{Ack}}(f_{p+1}^{\mathrm{Ack}}(n))$. We put $f_\omega^{\mathrm{Ack}}(n) = f_n^{\mathrm{Ack}}(n)$. Using the similarity between the definitions of f_p^{Ack} and g_p, it is easy to complete the proof of Theorem 1.32.

Proof. The function f_ω^{Ack} is known to grow faster than every primitive recursive function, so it is enough to show $2^{h_1(h_2(\dots(h_{n-2}(1))\dots))} \geqslant f_\omega^{\mathrm{Ack}}(n-1)$ for $n \geqslant 5$. First, we have $g_p(n+3) > f_p^{\mathrm{Ack}}(n)$ for all p, n. This is obvious for $p = 0$. Otherwise, for $n = 0$, using $g_p(2) \geqslant 3$, we find

$$g_p(3) > g_{p-1}(2^{g_p(2)}) > f_{p-1}^{\mathrm{Ack}}(6) > f_{p-1}^{\mathrm{Ack}}(1) = f_p^{\mathrm{Ack}}(0).$$

Then, for $n > 0$, we obtain

$$g_p(n+3) > g_{p-1}(2^{g_p(n+2)}) > g_{p-1}(f_p^{\mathrm{Ack}}(n-1)+3) > f_{p-1}^{\mathrm{Ack}}(f_p^{\mathrm{Ack}}(n-1)) = f_p^{\mathrm{Ack}}(n).$$

Finally, we have $g_2(n) = n+2$ for every n, and therefore

$$2^{h_1(h_2(\ldots(h_{n+2}(1))\ldots))} = 2^{h_1(h_2(\ldots(h_{n+1}(2))\ldots))} = 2^{h_1(h_2(\ldots(h_n(2^{n+3}))\ldots))} > g_n(2^{n+3})) \geqslant g_n(n+3) > f_n^{\mathrm{Ack}}(n),$$

hence $2^{h_1(h_2(\ldots(h_{n+2}(1))\ldots))} \geqslant f_\omega^{\mathrm{Ack}}(n-1)$. ⊣

Let us finally mention without proof the following strengthening of the lower bound for $\tilde{\jmath}(4)$:

1.40 Proposition (Dougherty). *For $j : V_\lambda \prec V_\lambda$, we have*

$$\tilde{\jmath}(4) \geqslant f_9^{\mathrm{Ack}}(f_8^{\mathrm{Ack}}(f_8^{\mathrm{Ack}}(254))).$$

In other words, there are at least the above huge number of critical ordinals below κ_4 in $\mathrm{Iter}(j)$.

2. The Word Problem for Self-Distributivity

The previous results about iterations of elementary embeddings have led to several applications outside set theory. The first application deals with free LD-systems and the word problem for the self-distributivity law (LD): $x(yz) = (xy)(xz)$. In 1989, Laver deduced from Lemma 1.20 that the LD-system $\mathrm{Iter}(j)$ has a specific algebraic property, namely that left division has no cycle in this LD-system, and he derived a solution for the word problem for (LD). Here we shall describe these results, following the independent and technically more simple approach of [4].

2.1. Iterated Left Division in LD-systems

For $(S, *)$ a (non-associative) algebraic system, and x, y in S, we say that x is a *left divisor* of y if $y = x * z$ holds for some z in S; we say that x is an *iterated left divisor* of y, and stipulate $x \sqsubset y$ if, for some positive k, there exist $z_1, \ldots, z_k$ satisfying $y = (\ldots((x * z_1) * z_2)\ldots) * z_k$. So $\sqsubset$ is the transitive closure of left divisibility. In the sequel, we shall be interested in LD-systems where left division (or, equivalently, iterated left division) has no cycle.

We let T_n be the set of all terms constructed using the variables $x_1, \ldots, x_n$ and a binary operator $*$, and T_∞ for the union of all T_n's. We denote by $=_{LD}$ the congruence on T_∞ generated by all pairs of the form $(t_1 * (t_2 * t_3)), (t_1 * t_2) * (t_1 * t_3))$. Then, by standard arguments, $T_n/{=_{LD}}$ is a free LD-system with n generators, which we shall denote by F_n. The word problem for (LD) is the question of algorithmically deciding the relation $=_{LD}$.

2.1 Theorem (Dehornoy [4]; also Laver [18] for an independent approach). *Assume that there exists at least one* LD*-system where left division has no cycle.*

(i) *Iterated left division in a free* LD*-system with one generator is a linear ordering.*

(ii) *The word problem for* (LD) *is decidable.*

The rest of this subsection is an outline of the proof of this statement, which can be skipped by a reader exclusively interested in set theory.

2.2 Definition. For t, t' terms in T_∞, we say that t' is an LD-*expansion* of t if we can go from t to t' by applying finitely many transformations consisting of replacing a subterm of the form $t_1 * (t_2 * t_3)$ with the corresponding term $(t_1 * t_2) * (t_1 * t_3)$.

By definition, t' being LD-equivalent to t means that we can transform t to t' by applying the law (LD) in either direction, i.e., from $x * (y * z)$ to $(x * y) * (x * z)$ or *vice versa*, while t' being an LD-expansion of t means that we transform t to t' by applying (LD), but only in the expanding direction, i.e., from $x * (y * z)$ to $(x * y) * (x * z)$, but not in the converse, contracting direction.

2.3 Definition. For t a term and k small enough, we denote by $\mathrm{left}^k(t)$ the kth iterated left subterm of t: we have $\mathrm{left}^0(t) = t$ for every t, and $\mathrm{left}^k(t) = \mathrm{left}^{k-1}(t_1)$ for $t = t_1 * t_2$ and $k \geqslant 1$. For t_1, t_2 in T_∞, we say that $t_1 \sqsubset_{LD} t_2$ is true if we have $t'_1 = \mathrm{left}^k(t'_2)$ for some k, t'_1, t'_2 satisfying $k \geqslant 1$, $t'_1 =_{LD} t_1$, and $t'_2 =_{LD} t_2$.

By construction, saying that $t_1 \sqsubset_{LD} t_2$ is true in T_1 is equivalent to saying that the class of t_1 in the free LD-system F_1 is an iterated left divisor of the class of t_2. The core of the argument is:

2.4 Proposition. *Let t_1, t_2 be one-variable terms in T_1. Then at least one of $t_1 \sqsubset_{LD} t_2$, $t_1 =_{LD} t_2$, $t_2 \sqsubset_{LD} t_1$ holds.*

2.5 Corollary. *If $(S, *)$ is an* LD*-system with one generator, then any two elements of S are comparable with respect to iterated left division.*

Proving Proposition 2.4 relies on three specific properties of left self-distributivity. As in Sect. 1, we use the notation $x^{[n]}$ for the nth right power of x.

2.6 Lemma.

(i) *For every term t in T_1, we have $x^{[n+1]} =_{LD} t * x^{[n]}$ for n sufficiently large.*

(ii) *Assume that $\mathrm{left}^n(t)$ is defined, and t' is an* LD*-expansion of t. Then $\mathrm{left}^{n'}(t')$ is an* LD*-expansion of $\mathrm{left}^n(t)$ for some $n' \geqslant n$.*

(iii) *Any two* LD-*equivalent terms admit a common* LD-*expansion.*

We skip the purely algebraic proofs of these properties, and just explain how to derive Proposition 2.4.

Proof. Let t_1, t_2 be arbitrary terms in T_1. By Lemma 2.6(i), we have $t_1 * x^{[n]} =_{LD} x^{[n+1]} =_{LD} t_2 * x^{[n]}$ for n sufficiently large. Fix such a n. By Lemma 2.6(iii), the terms $t_1 * x^{[n]}$ and $t_2 * x^{[n]}$ admit a common LD-expansion, say t. By Lemma 2.6(ii), there exist nonnegative integers n_1, n_2 such that, for $i = 1, 2$, the term $\mathrm{left}^{n_i}(t)$ is an LD-expansion of $\mathrm{left}(t_i * x^{[n]})$, i.e., of t_i. Thus we have $t_1 =_{LD} \mathrm{left}^{n_1}(t)$, and $t_2 =_{LD} \mathrm{left}^{n_2}(t)$. Three cases may occur: for $n_1 > n_2$, $\mathrm{left}^{n_1}(t)$ is an iterated left subterm of $\mathrm{left}^{n_2}(t)$, and, therefore, $t_1 \sqsubset_{LD} t_2$ holds; for $n_1 = n_2$, t_1 and t_2 both are LD-equivalent to $\mathrm{left}^{n_1}(t)$, and $t_1 =_{LD} t_2$ is true; finally, for $n_1 < n_2$, $\mathrm{left}^{n_2}(t)$ is an iterated left subterm of $\mathrm{left}^{n_1}(t)$, and, therefore, $t_2 \sqsubset_{LD} t_1$ holds. ⊣

Finally, we can complete the proof of Theorem 2.1.

Proof. (i) Proposition 2.4 tells us that any two elements of the free LD-system F_1 are comparable with respect to the iterated left divisibility relation. Assume that S is any LD-system. The universal property of free LD-systems gives a homomorphism π of F_1 into S. If $(a_1, \ldots, a_n)$ is a cycle for left division in F_1, then $(\pi(a_1), \ldots, \pi(a_n))$ is a cycle for left division in S. So, if there exists at least one LD-system S where left division has no cycle, the same is true for F_1, i.e., the iterated left divisibility relation of F_1 is irreflexive. As it is always transitive, it is a (strict) linear ordering.

(ii) Let us consider the case of one variable terms first. For t_1, t_2 in T_1, we can decide whether $t_1 =_{LD} t_2$ is true as follows: we systematically enumerate all pairs (t_1', t_2') such that t_1' is LD-equivalent to t_1 and t_2' is LD-equivalent to t_2. By Proposition 2.4, there will eventually appear some pair (t_1', t_2') such that either t_1' and t_2' are equal, or t_1' is a proper iterated left subterm of t_2', or t_2' is a proper iterated left subterm of t_1'. In the first case, we conclude that $t_1 =_{LD} t_2$ is true, in the other cases, we can conclude that $t_1 =_{LD} t_2$ is false whenever we know that $t \sqsubset_{LD} t'$ excludes $t =_{LD} t'$, i.e., whenever we know that left division has no cycle in F_1.

The case of terms with several variables is not more difficult. For t in T_∞, let $t^\dagger$ denote the term obtained from t by replacing all variables with x_1. For t_1, t_2 in T_n, we can decide whether $t_1 =_{LD} t_2$ is true as follows. First we compare $t_1^\dagger$ and $t_2^\dagger$ as above. If the latter terms are not LD-equivalent, then t_1 and t_2 are not LD-equivalent either. Otherwise, we can find a common LD-expansion t of $t_1^\dagger$ and $t_2^\dagger$. Then we consider the LD-expansion t_1' of t_1 obtained in the same way as t is obtained from $t_1^\dagger$, i.e., by applying (LD) at the same successive positions, and, similarly, we consider t_2' obtained from t_2 as t is obtained from $t_2^\dagger$. Then, either t_1' and t_2' are equal, in which case we conclude that $t_1 =_{LD} t_2$ is true, or t_1' and t_2' have some variable clash, in which case, using the techniques of Lemma 2.6(iii), we can conclude that $t_1 =_{LD} t_2$ is false. ⊣

2.2. Using Elementary Embeddings

In the mid-1980's, Laver showed the following:

2.7 Proposition (Laver). *Left division in the* LD*-system* $\mathcal{E}_\lambda$ *has no cycle.*

Proof. Assume that $j_1, \dots, j_n$ is a cycle for left division in $\mathcal{E}_\lambda$. Consider the infinite periodic sequence $j_1, \dots, j_n, j_1, \dots, j_n, j_1, \dots$. Theorem 1.28 applies, and it asserts that the supremum of the critical ordinals in this sequence is λ. But, on the other hand, there are only n different embeddings in the sequence, and the supremum of finitely many ordinals below λ cannot be λ, a contradiction. ⊣

The original proof of the previous result in [18] did not use the Laver-Steel theorem, but instead a direct computation based on Lemma 1.20.

Using the results of Sect. 2.1, we immediately deduce:

2.8 Theorem (Laver, 1989). *Assume Axiom* (I3). *Then:*

(i) *Iterated left division in a free* LD*-system with one generator is a linear ordering.*

(ii) *The word problem for* (LD) *is decidable.*

Another application of Proposition 2.7 is a complete algebraic characterization of the LD-system made by the iterations of an elementary embedding.

2.9 Lemma ("Laver's criterion"). *A sufficient condition for an* LD*-system* S *with one generator to be free is that left division in* S *has no cycle.*

Proof. Assume that left division in S has no cycle. Let π be a surjective homomorphism of F_1 onto S, which exists by the universal property of F_1. Let x, y be distinct elements of F_1. By Corollary 2.5, at least one of $x \sqsubset y$, $y \sqsubset x$ is true in F_1, which implies that at least one of $\pi(x) \sqsubset \pi(y)$, $\pi(y) \sqsubset \pi(x)$ is true in S. The hypothesis that left division has no cycle in S implies that, in S, the relation $a \sqsubset b$ excludes $a = b$. So, here, we deduce that $\pi(x) \neq \pi(y)$ is true in every case, which means that π is injective, and, therefore, it is an isomorphism, i.e., S is free. ⊣

We deduce the first part of the following result

2.10 Theorem (Laver). *Assume* $j : V_\lambda \prec V_\lambda$. *Then* $\mathrm{Iter}(j)$ *equipped with the application operation is a free* LD*-system, and* $\mathrm{Iter}^*(j)$ *equipped with application and composition is a free* LD*-monoid.*

We skip the details for the LD-monoid structure, which are easy. The general philosophy is that, in an LD-monoid, most of the nontrivial information is concentrated in the self-distributive operation. In particular, if X is any set and F_X is the free LD-system based on X, then the free LD-monoid based on X is the free monoid generated by F_X, quotiented under the congruence generated by the pairs $(x \cdot y, (x * y) \cdot x)$. It easily follows that there exists a realization of the free monoid based on X inside the free LD-system based on X. So, in particular, every solution for the word problem for (LD) gives a solution for the word problem of the laws that define LD-monoids.

2.3. Avoiding Elementary Embeddings

The situation created by Theorem 2.8 was strange, as one would expect no link between large cardinals and such a simple combinatorial property as the word problem for (LD). Therefore, finding an alternative proof not relying on a large cardinal axiom—or proving that some set-theoretic axiom is needed here—was a natural challenge.

2.11 Theorem (Dehornoy [5]). *That left division in the free* LD*-system with one generator has no cycle is a theorem of* ZFC*.*

Outline of Proof. The argument of [5] consists in studying the law (LD) by introducing a certain monoid $\mathcal{G}_{LD}$ that captures its specific geometry. Viewing terms as binary trees, one considers, for each possible address α of a subterm, the partial operator Ω_α on terms corresponding to applying (LD) at position α in the expanding direction, i.e., expanding the subterm rooted at the vertex specified by α. If $\mathcal{G}_{LD}$ is the monoid generated by all operators $\Omega_\alpha^{\pm 1}$ using composition, then two terms t, t' are LD-equivalent if and only if some element of $\mathcal{G}_{LD}$ maps t to t'. Because the operators Ω_α are partial in an essential way, the monoid $\mathcal{G}_{LD}$ is not a group. However, one can guess a presentation of $\mathcal{G}_{LD}$ and work with the group G_{LD} admitting that presentation. Then the key step is to construct a realization of the free LD-system with one generator in some quotient of G_{LD}, a construction that is reminiscent of Henkin's proof of the completeness theorem. The problem is to associate with each term t in T_1 a distinguished operator in $\mathcal{G}_{LD}$ (or its copy in the group G_{LD}) in such a way that the obstruction to satisfying (LD) can be controlled. The solution is given by Lemma 2.6(i): the latter asserts that, for each term t, the term $x^{[n+1]}$ is LD-equivalent to $t * x^{[n]}$ for n sufficiently large, so some operator χ_t in $\mathcal{G}_{LD}$ must map $x^{[n+1]}$ to $t * x^{[n]}$, i.e., in some sense, construct the term t. Moreover Lemma 2.6(i) gives an explicit recursive definition of χ_t in terms of χ_{t_1} and χ_{t_2} when t is $t_1 * t_2$. Translating this definition into G_{LD} yields a self-distributive operation on some quotient of G_{LD}, and proving that left division has no cycle in the LD-system so obtained is then easy—but requires a number of verifications. $\dashv$

2.12 Remark. A relevant geometry group can be constructed for every algebraic law (or family of algebraic laws). When the self-distributivity law is replaced with the associativity law, the corresponding group is Richard Thompson's group F [2]. So G_{LD} is an analog of F.

Theorem 2.11 allows one to eliminate any set-theoretic assumption from the statements of Theorem 2.8. Actually, it gives more. Indeed, the quotient of G_{LD} appearing in the above proof turns out to be Artin's braid group B_∞, and the results about G_{LD} led to unexpected braid applications.

Artin's braid group B_n admits many equivalent definitions. Usually, B_n is introduced for $2 \leqslant n \leqslant \infty$ as the group generated by elements σ_i, $1 \leqslant i < n$, subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| \geqslant 2, \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{for } |i-j| = 1. \tag{11.14}$$

The connection with braid diagrams comes when σ_i is associated with an n-strand diagram where the $(i+1)$st strand crosses over the ith strand; then the relations in (11.14) correspond to ambient isotopy.

2.13 Theorem (Dehornoy [5]). *For x, y in B_∞, say that $x < y$ holds if, among all possible expressions of $x^{-1}y$ in terms of the $\sigma_i^{\pm 1}$, there is at least one where the generator σ_i of minimal index i occurs only positively (i.e., no σ_i^{-1}). Then the relation $<$ is a left-invariant linear ordering on B_∞.*

The result is a consequence of Theorem 2.11. Indeed, there exists a (partial) action of the group B_n on the nth power of every left cancellative LD-system, and one obtains a linear ordering on B_n by defining, for x, y in B_n and $\vec{a}$ in F_1^n, the relation $x <_{\vec{a}} y$ to mean that $\vec{a} \cdot x$ is lexicographically smaller than $\vec{a} \cdot y$. One then checks that $<_{\vec{a}}$ does not depend on the choice of $\vec{a}$ and it coincides with the relation $<$ of Theorem 2.13. In this way, one obtains the previously unknown result that braid groups are orderable. A number of alternative definitions of the braid order have been found subsequently, in particular in terms of homeomorphisms of a punctured disk, and of hyperbolic geometry [6]. Various results have been derived, in particular new efficient solutions for the word problem of B_n with possible cryptographic applications.

The following result, first discovered by Laver (well-foundedness), was then made more explicit by Serge Burckel (computation of the order type):

2.14 Theorem (Laver [20], Burckel [1]). *For each n, the restriction of the braid ordering to the braids that can be expressed without any σ_i^{-1} is a well-ordering of type $\omega^{\omega^{n-2}}$.*

Returning to self-distributivity, we can mention as a last application a simple solution to the word problem for (LD) involving the braid group B_∞. Indeed, translating the inductive proof of Lemma 2.6(i) to B_∞ leads to the explicit operation

$$x * y = x\,\mathrm{sh}(y)\sigma_1\mathrm{sh}(x)^{-1}, \tag{11.15}$$

where sh is the endomorphism that maps σ_i to σ_{i+1} for every i. Laver's criterion Lemma 2.9 implies that every sub-LD-system of $(B_\infty, *)$ with one generator is free, i.e., that $(B_\infty, *)$ is a torsion-free LD-system. Then, in order to decide whether two terms on one variable are LD-equivalent, it suffices to compare their evaluations in B_∞ when x is mapped to 1 and (11.15) is used. Note that, once (11.15) has been guessed, checking that it defines a self-distributive operation on B_∞ is easy, and, therefore, any argument proving that left division in $(B_\infty, *)$ has no cycle is sufficient for fulfilling the assumptions of Theorem 2.1 without resorting to the rather convoluted construction of G_{LD}. Several such arguments have been given, in particular by David Larue using automorphisms of a free group [16] and by Ivan Dynnikov using laminations [6].

The developments sketched above have no connection with set theory. As large cardinal axioms turned out to be unnecessary, one could argue that set theory is not involved here, and deny that any of these developments can be called an application of set theory. The author disagrees with such an opinion. Had not set theory given the first hint that the algebraic properties of LD-systems are a deep subject [17, 3], then it is not clear that anyone would have tried to really understand the law (LD). The production of an LD-system with acyclic division using large cardinals gave evidence that some other example might be found in ZFC, and hastened its discovery. Without set theory, it is likely that the braid order would not have been discovered, at least as soon:[1] could not this be accepted as a definition for an application of set theory? It is tempting to compare the role of set theory here with the role of physics when it gives evidence for some formulas that remain then to be proved in a standard mathematical framework.

3. Periods in the Laver Tables

Here we describe another combinatorial application of the set theoretic results of Sect. 1. This application involves some finite LD-systems discovered by Laver in his study of iterations of elementary embeddings [19]. In contrast to the results mentioned in Sect. 2, these results have not yet received any ZF proof.

3.1. Finite LD-Systems

The results of Sect. 1.4 give, for each $j : V_\lambda \prec V_\lambda$, an infinite family of finite quotients of $\mathrm{Iter}(j)$, namely one with 2^n elements for each n. The finite LD-systems so obtained will be called the Laver tables. In this section, we show how to construct the Laver tables directly, and list some of their properties.

Let us address the question of constructing a finite LD-system with one generator. We start with an incomplete table on the elements $1, \dots, N$, and try to complete it by using the self-distributivity law. Here, we consider the case when the first column is assumed to be cyclic, i.e., we have

$$a * 1 = a + 1, \quad \text{for } a = 1, \dots, N-1, \; N * 1 = 1. \tag{11.16}$$

[1] *A posteriori*, it became clear that the orderability of braid groups could have been deduced from old work by Nielsen, but this was not noted until recently.

3.1 Lemma.

(i) *For every N, there exists a unique operation $*$ on $\{1, \ldots, N\}$ satisfying* (11.16) *and, for all a, b,*

$$a * (b * 1) = (a * b) * (a * 1).$$

(ii) *The following relations hold in the resulting system:*

$$a * b \begin{cases} = b & \textit{for } a = N, \\ = a + 1 & \textit{for } b = 1, \textit{ and for } a * (b-1) = N, \\ > a * (b-1) & \textit{otherwise.} \end{cases}$$

For $a < N$, there exists $p \leqslant N - a$ and $c_1 = a + 1 < c_2 < \cdots < c_p = N$ such that, for every b, we have $a * b = c_i$ with $i = b \pmod p$, hence, in particular, $a * b > a$.

We denote by S_N the system given by Lemma 3.1. At this point, the question is whether S_N is actually an LD-system: by construction, certain occurrences of (LD) hold in the table, but this does not guarantee that the law holds for all triples. Actually, it need not: for instance, the reader can check that, in S_5, one has $2 * (2 * 2) = 3 \neq (2 * 2) * (2 * 2) = 5$.

3.2 Proposition.

(i) *If N is not a power of 2, there exists no* LD-*system satisfying* (11.16).

(ii) *For each n, there exists a unique* LD-*system with domain $\{1, \ldots, 2^n\}$ that satisfies* (11.16)*, namely the system S_{2^n} of Lemma 3.1.*

The combinatorial proof relies on an intermediate result, namely that S_N is an LD-system if and only if the equality $a * N = N$ is true for every a. It is not hard to see that this is impossible when N is not a power of 2. On the other hand, the verification of the property when N is a power of 2 relies on the following connection between S_N and $S_{N'}$ when N' is a multiple of N:

3.3 Lemma.

(i) *Assume that S is an* LD-*system and $g_{[N'+1]} = g$ holds in S. Then mapping a to $g_{[a]}$ defines a homomorphism of $S_{N'}$ into S.*

(ii) *In particular, if S_N is an* LD-*system and N divides N', then mapping a to $a \bmod N$ defines a homomorphism of $S_{N'}$ onto S_N.*

(Here $a \bmod N$ denotes the unique integer equal to a *modulo* N lying in the interval $\{1, \ldots, N\}$.)

3.4 Definition. For $n \geqslant 0$, the nth Laver table, denoted A_n, is defined to be the LD-system S_{2^n}, i.e., the unique LD-system with domain $\{1, 2, \ldots, 2^n\}$ that satisfies (11.16).

The first Laver tables are

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

The reader can compute that the first row of A_4 is $2, 12, 14, 16, 2, \ldots$, while that of A_5 is $2, 12, 14, 16, 28, 30, 32, 2, \ldots$.

By Lemma 3.1, every row in A_n is periodic and it comes in the proof of Proposition 3.2 that the corresponding period is a power of 2. In the sequel, we write $o_n(a)$ for the number such that $2^{o_n(a)}$ is the period of a in A_n, i.e., the number of distinct values in the row of a. The examples above show that the periods of 1 in $A_0, \ldots, A_5$ are $1, 1, 2, 4, 4$, and 8 respectively, corresponding to the equalities $o_0(1) = 0$, $o_1(1) = 0$, $o_2(1) = 1$, $o_3(1) = 2$, $o_4(1) = 2$, $o_5(1) = 3$. Observe that the above values are non-decreasing.

It is not hard to prove that, for each n, the unique generator of A_n is 1, its unique idempotent is 2^n, and we have $2^n *_n a = a$ and $a *_n 2^n = 2^n$ for every a.

An important point is the existence of a close connection between the tables A_n and A_{n+1} for every n (we write $*_n$ for the multiplication in A_n):

3.5 Lemma.

(i) *For each n, the mapping $a \mapsto a \bmod 2^n$ is a surjective morphism of A_{n+1} onto A_n.*

(ii) *For every n, and every a with $1 \leqslant a \leqslant 2^n$, there exists a number $\theta_{n+1}(a)$ with $0 \leqslant \theta_{n+1}(a) \leqslant 2^{o_n(a)}$ and $\theta_{n+1}(2^n) = 0$ such that, for every b with $1 \leqslant b \leqslant 2^n$, we have*

$$a *_{n+1} b = a *_{n+1} (2^n + b) = \begin{cases} a *_n b & \text{for } b \leqslant \theta_{n+1}(a), \\ a *_n b + 2^n & \text{for } b > \theta_{n+1}(a), \end{cases}$$

$$(2^n + a) *_{n+1} b = (2^n + a) *_{n+1} (2^n + b) = a *_n b + 2^n.$$

For instance, the values of the mapping θ_4 are

a	1	2	3	4	5	6	7	8
$\theta_4(a)$	1	1	2	4	2	2	1	0

We obtain in this way a short description of A_n: the above 8 values contain all information needed for constructing the table of A_4 (16×16 elements) from that of A_3.

The LD-systems A_n play a fundamental role among finite LD-systems. In particular, it is shown in [12] how every LD-system with one generator can be obtained by various explicit operations (analogous to products) from a well-defined unique table A_n. Let us mention that as an LD-system A_n admits the presentation $\langle g \mid g_{[m+1]} = g\rangle$ for every number m of the form $2^n(2p+1)$, and that the structure $(A_n, *)$ can be enriched with a second binary operation so as to become an LD-monoid:

3.6 Proposition. *There exists a unique associative product on A_n that turns $(A_n, *, \cdot)$ into an* LD*-monoid, namely the operation defined by*

$$a \cdot b = (a * (b+1)) - 1 \quad \text{for } b < 2^n, \qquad a \cdot b = a \quad \text{for } b = 2^n. \tag{11.17}$$

3.2. Using Elementary Embeddings

In order to establish a connection between the tables A_n of the previous section and the finite quotients of $\mathrm{Iter}(j)$ described in Sect. 1.4, we shall use the following characterization:

3.7 Lemma. *Assume that S is an* LD*-system admitting a single generator g satisfying $g_{[2^n+1]} = g$ and $g_{[a]} \neq g$ for $a \leqslant 2^n$. Then S is isomorphic to A_n.*

Proof. Assume that S is an LD-system generated by an element g satisfying the above conditions. A double induction gives, for $a, b \leqslant 2^n$, the equality $g_{[a]} * g_{[b]} = g_{[a*b]}$, where $a * b$ refers to the product in A_n. So the set of all left powers of g is closed under product, and S, which has exactly 2^n elements, is isomorphic to A_n. ⊣

We immediately deduce from Theorem 1.22:

3.8 Proposition (Laver [19]). *For $j : V_\lambda \prec V_\lambda$, the quotient of* $\mathrm{Iter}(j)$ *under* $\mathrm{crit}_n(j)$*-equivalence is isomorphic to A_n.*

Under the previous isomorphism, the element a of A_n is the image of the class of the embedding $j_{[a]}$, and, in particular, 2^n is the image of the class of $j_{[2^n]}$, which is also the class of the identity map.

By construction, if S is an LD-system, and a is an element of S, there exists a well-defined evaluation for every term t in T_1 when the variable x is given the value a. We shall use $t(1)^{A_n}$, or simply $t(1)$, for the evaluation in A_n of a term $t(x)$ of T_1 at $x = 1$, and $t(j)$ for the evaluation of $t(x)$ in $\mathrm{Iter}(j)$ at $x = j$. With this notation, it should be clear that, for every term $t(x)$, the image of the $\mathrm{crit}_n(j)$-equivalence class of $t(j)$ in A_n under the isomorphism of Proposition 3.8 is $t(1)^{A_n}$.

The previous isomorphism can be used to obtain results about the iterations of an elementary embedding. For instance, let us consider the question of determining which left powers of j are $\mathrm{crit}_4(j)$-approximations of $j \circ j$ and of $j^{[3]}$. By looking at the table of the LD-monoid A_4, we obtain

$A_4 \models 1 \circ 1 = 11$, and $A_4 \models 1^{[3]} = 12$. We deduce that $j \circ j$ is $\mathrm{crit}_4(j)$-equivalent to $j_{[11]}$ and $j^{[3]}$ is $\mathrm{crit}_4(j)$-equivalent to $j_{[12]}$.

The key to further results is the possibility of translating into the language of the finite tables A_n the values of the critical ordinals associated with the iterations of an elementary embedding.

3.9 Proposition. *Assume $j : V_\lambda \prec V_\lambda$. Then, for every term t and for $n \geqslant m \geqslant 0$ and $n \geqslant a \geqslant 1$,*

(i) $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ *is equivalent to* $A_n \models t(1) = 2^n$.

(ii) $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ *is equivalent to* $A_{n+1} \models t(1) = 2^n$.

(iii) $t(j)(\mathrm{crit}_m(j)) = \mathrm{crit}_n(j)$ *is equivalent to* $A_{n+1} \models t(1) * 2^m = 2^n$.

(iv) $j_{[a]}(\mathrm{crit}_m(j)) = \mathrm{crit}_n(j)$ *is equivalent having the period of a jump from* 2^m *to* 2^{m+1} *between* A_n *and* A_{n+1}.

Proof. (i) By definition, $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ is equivalent to $t(j)$ being $\mathrm{crit}_n(j)$-equivalent to the identity mapping, hence to the image of $t(j)$ in A_n being the image of the identity, which is 2^n.

(ii) Assume $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$. Then we have $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, so, by (i), $A_n \models t(1) = 2^n$ and $A_{n+1} \not\models t(1) = 2^{n+1}$. Now $A_n \models t(1) = 2^n$ implies $A_{n+1} \models t(1) = 2^n$ or 2^{n+1}, so 2^n is the only possible value here. Conversely, $A_{n+1} \models t(1) = 2^n$ implies $A_n \models t(1) = 2^n$ and $A_{n+1} \not\models t(1) = 2^{n+1}$, so, by (i), $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$.

(iii) As $\mathrm{crit}_m(j)$ is the critical ordinal of $j_{[2^m]}$, we have $t(j)(\mathrm{crit}_m(j)) = \mathrm{crit}(t(j)[j_{[2^m]}])$. By (ii), $\mathrm{crit}(t(j)[j_{[2^m]}]) = \mathrm{crit}_n(j)$ is equivalent to $A_{n+1} \models t(1) * 1_{[2^m]} = 2^n$. Now we have $A_{n+1} \models 1_{[2^m]} = 2^m$ for $n \geqslant m$.

(iv) The image of $j_{[a]}$ is a both in A_n and A_{n+1}, hence (iii) tells us that $j_{[a]}(\mathrm{crit}_m(j)) = \mathrm{crit}_n(j)$ is equivalent to $A_{n+1} \models a * 2^m = 2^n$. If the latter holds, the period p of a in A_{n+1} is 2^{m+1}: indeed, $A_{n+1} \models a * 2^m < 2^{n+1}$ implies $p > 2^m$, while $2 \times 2^n = 2^{n+1}$ implies $p \leqslant 2 \times 2^m$. Conversely, assume that the period of a is 2^m in A_n and 2^{m+1} in A_{n+1}. We deduce $A_n \models a * 2^m = 2^n$ and $A_{n+1} \not\models a * 2^m = 2^{n+1}$, so the only possibility is $A_{n+1} \models a * 2^m = 2^n$. ⊣

For instance, we can check $A_3 \models 1^{[3]} = 4$, and $A_5 \models 1^{[4]} = 16$. Using the dictionary, we deduce that the critical ordinal of $j^{[3]}$ is $\mathrm{crit}_2(j)$, while the critical ordinal of $j^{[4]}$ is $\mathrm{crit}_4(j)$. Also, we find $A_4 \models 4 * 4 = 8$, which implies that $j_{[4]}$ maps $\mathrm{crit}_2(j)$ to $\mathrm{crit}_3(j)$—as can be established directly. Similarly, we have $A_5 \models 1 * 4 = 16$, corresponding to $j(\mathrm{crit}_2(j)) = \mathrm{crit}_4(j)$. As for (iv), we see that the period of 1 jumps from 1 to 2 between A_1 and A_2, that it jumps from 2 to 4 between A_2 and A_3, and that it jumps from 4 to 8 between A_4 and A_5. We deduce that, if j is an elementary embedding of V_λ into itself, then j maps $\mathrm{crit}_0(j)$ to $\mathrm{crit}_1(j)$, $\mathrm{crit}_1(j)$ to $\mathrm{crit}_2(j)$, and $\mathrm{crit}_2(j)$ to $\mathrm{crit}_4(j)$, i.e., we have $\kappa_2 = \gamma_4$ with the notations of Sect. 1.6. Similarly, the period of 3 jumps from 8 to 16 between A_5 and A_6: we deduce that $j_{[3]}$ maps $\mathrm{crit}_3(j)$ to $\mathrm{crit}_5(j)$.

By Proposition 3.9(iii): $\hat{\jmath}(m) = n$ is equivalent to $A_{n+1} \models 1 * 2^m = 2^n$. As the latter condition does not involve j, we deduce

3.10 Corollary. *For $j : V_\lambda \prec V_\lambda$, the mappings $\hat{\jmath}$ and $\tilde{\jmath}$ do not depend on j.*

In the previous examples, we used the connection between the iterates of an elementary embedding and the tables A_n to deduce information about elementary embeddings from explicit values in A_n. We can also use the correspondence in the other direction, and deduce results about the tables A_n from properties of the elementary embeddings.

Now, the existence of the function $\hat{\jmath}$ and, therefore, of its iterate $\tilde{\jmath}$, which we have seen is a direct consequence of the Laver-Steel theorem, translates into the following asymptotic result about the periods in the tables A_n. We recall that $o_n(a)$ denotes the integer such that the period of a in A_n is $2^{o_n(a)}$.

3.11 Proposition (Laver). *Assume Axiom* (I3). *Then, for every a, the period of a in A_n tends to infinity with n. More precisely, for $j : V_\lambda \prec V_\lambda$,*

$$o_n(a) \leqslant \tilde{\jmath}(r) \quad \textit{if and only if} \quad n \leqslant \tilde{\jmath}(r+1) \tag{11.18}$$

holds for $r \geqslant a$. In particular, (11.18) *holds for every r in the case $a = 1$.*

Proof. Assume first $a = 1$. Then j maps $\mathrm{crit}_{\tilde{\jmath}(r)}(j)$ to $\mathrm{crit}_{\tilde{\jmath}(r+1)}(j)$ for every r. Hence, by Proposition 3.9(iv), the period of 1 doubles from $2^{\tilde{\jmath}(r)}$ to $2^{\tilde{\jmath}(r)+1}$ between $A_{\tilde{\jmath}(r+1)}$ and $A_{\tilde{\jmath}(r+1)+1}$. So we have

$$o_{\tilde{\jmath}(r+1)}(1) = \tilde{\jmath}(r) \quad \text{and} \quad o_{\tilde{\jmath}(r+1)+1}(1) = \tilde{\jmath}(r) + 1,$$

which gives (11.18). Assume now $a \geqslant 2$. By Lemma 1.13, we have $(j_{[a]})^{[r]} = j^{[r+1]}$ for $r \geqslant a$, so the critical ordinal of $(j_{[a]})^{[r]}$ is $\mathrm{crit}_{\tilde{\jmath}(r)}(j)$. Hence, for $r \geqslant a$, the embedding $j_{[a]}$ maps $\mathrm{crit}_{\tilde{\jmath}(r)}(j)$ to $\mathrm{crit}_{\tilde{\jmath}(r+1)}(j)$, and the argument is as for $a = 1$. ⊣

We conclude with another result about the periods in the tables A_n.

3.12 Proposition (Laver). *Assume Axiom* (I3). *Then, for every n, the period of* 2 *in A_n is at least the period of* 1.

Proof. Assume that the period of 1 in A_n is 2^m. Let n' be the largest integer such that the period of 1 in $A_{n'}$ is 2^{m-1}. By construction, the period of 1 jumps from 2^{m-1} to 2^m between $A_{n'}$ and $A_{n'+1}$. Assume that j is a nontrivial elementary embedding of a rank into itself. By Proposition 3.9(iv), j maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{n'}(j)$. Now, by Proposition 1.10, $j[j]$ maps $\mathrm{crit}_m(j)$ to some ordinal of the form $\mathrm{crit}_{n''}(j)$ with $n'' \leqslant n'$. This implies that the period of 2 jumps from 2^{m-1} to 2^m between $A_{n''}$ and $A_{n''+1}$. By construction, we have $n'' \leqslant n' < n$, hence the period of 2 in A_n is at least 2^m. ⊣

3.3. Avoiding Elementary Embeddings

Once again, the situation of Propositions 3.11 and 3.12 is strange, as it is not clear why any large cardinal hypothesis should be involved in the asymptotic behavior of the periods in the finite LD-systems A_n. So we would either get rid of the large cardinal hypothesis, or prove that it is necessary.

We shall mention partial results in both directions. In the direction of eliminating the large cardinal assumption, i.e., of getting arithmetic proofs, Dougherty and Aleš Drápal have proposed a scheme that essentially consists in computing the rows of (sufficiently many) elements $2^p - a$ in A_n using induction on a, which amounts to constructing convenient families of homomorphisms between the A_n's. Here we shall mention statements corresponding to the first two levels of the induction:

3.13 Theorem (Drápal [11]).

(i) *For every d, and for $0 \leqslant m \leqslant 2^d + 1$, $b \mapsto 2^{2^d} b$ defines an injective homomorphism of A_m into A_{m+2^d}; it follows that, for $2^d \leqslant n \leqslant 2^{d+1} + 1$, the row of $2^{2^d} - 1$ in A_n is given by*

$$(2^{2^d} - 1) *_n b = 2^{2^d} b.$$

(ii) *For every d, and for $0 \leqslant m \leqslant 2^{2^{d+1}}$, the mapping f_d defined by $f_d : 2^i \mapsto 2^{(i+1)2^d} - 2^{i2^d}$ and $f_d(\sum b_i 2^i) = \sum b_i f_d(2^i)$ defines an injective homomorphism of A_m into A_{m2^d}; it follows that, for $0 \leqslant n \leqslant 2^{2^{d+1}+d}$ such that 2^d divides n, the row of $2^{2^d} - 2$ in A_n is given by*

$$(2^{2^d} - 2) *_n b = f_d(b).$$

So far, the steps $a \leqslant 4$ have been completed, but the complexity quickly increases, and whether the full proof can be completed remains open.

3.4. Not Avoiding Elementary Embeddings?

We conclude with a result in the opposite direction:

3.14 Theorem (Dougherty-Jech [9]). *It cannot proved in* PRA *(Primitive Recursive Arithmetic) that the period of* 1 *in the table A_n goes to infinity with n.*

The idea is that enough of the computations of Sect. 1.6 can be performed in PRA to guarantee that, if the period of 1 in A_n tends to infinity with n, then some function growing faster than the Ackermann function provably exists.

Assume $j : V_\lambda \prec V_\lambda$. For every term t in T_1, the elementary embedding $t(j)$ acts on the family $\{\mathrm{crit}_n(j) \mid n \in \omega\}$, and, as was done for j, we can associate with $t(j)$ an increasing injection $\widetilde{t(j)} : \omega \to \omega$ by

$$\widetilde{t(j)}(m) = n \quad \text{if and only if} \quad t(j)(\mathrm{crit}_m(j)) = \mathrm{crit}_n(j).$$

If t and t' are LD-equivalent terms, we have $t(j) = t'(j)$, hence $\widetilde{t(j)} = \widetilde{t'(j)}$, so, for a in the free LD-system F_1, we can define f^j_a to be the common value of $\widetilde{t(j)}$ for t representing a. We obtain in this way an F_1-indexed family of increasing injections of ω to itself, distinct from identity, and, by construction, the equality

$$\mathrm{crit}(f^j_{a*b}) = f^j_a(\mathrm{crit}(f^j_b)) \tag{11.19}$$

is satisfied for all a, b in F_1, where we define $\mathrm{crit}(f)$ to be the least m satisfying $f(m) > m$. The sequence $(f^j_a \mid a \in F_1)$ is the trace of the action of j on critical ordinals, and we shall see it captures enough of the combinatorics of elementary embeddings to deduce the results of Sect. 1.6.

Let us try to construct directly, without elementary embedding, some similar family of injections on ω satisfying (11.19). To this end, we can resort to the Laver tables. Indeed, by Proposition 3.9, the condition $t(j)(\mathrm{crit}_m(j)) = \mathrm{crit}_n(j)$ in the definition of $\widetilde{t(j)}$ is equivalent to $A_{n+1} \models t(1) * 2^m = 2^n$. So we are led to

3.15 Definition (PRA). For a in F_1, we define f_a to be the partial mapping on ω such that $f_a(m) = n$ holds if, for some term t representing a, we have $A_{n+1} \models t(1) * 2^m = 2^n$.

As A_{n+1} is an LD-system, the value of $t(1)*2^m$ computed in A_{n+1} depends on the LD-class of t only, so the previous definition is non-ambiguous. If there exists a $j : V_\lambda \prec V_\lambda$, then, for each a in F_1, the mapping f_a coincides with f^j_a, and, therefore, each f_a is a total increasing injection of ω to ω, distinct from identity, and the f_a's satisfy the counterpart of (11.19). In particular, we can state

3.16 Proposition (ZFC + I3). *For each a in F_1, the function f_a is total.*

Some of the previous results about the f_a's can be proved directly. Let us define a *partial increasing injection* on ω to be an increasing function of ω into itself whose domain is either ω, or a finite initial segment of ω. We shall say that a partial increasing injection f is *nontrivial* if $f(m) > m$ holds for at least one m, and that m is the *critical integer* of f, denoted $m = \mathrm{crit}(f)$, if we have $f(n) = n$ for $n < m$, and $f(m) \neq m$, i.e., either $f(m) > m$ holds or $f(m)$ is not defined.

For f a partial increasing injection on ω, and m, n in ω, we write $f(m) \tilde{\geqslant} n$ if either $f(m)$ is defined and $f(m) \geqslant n$ holds, or $f(m)$ is not defined; we write $\mathrm{crit}(f) \tilde{\geqslant} m$ for $\forall n < m\,(f(n) = n)$. Then $f(m) = n$ is equivalent to the conjunction of $f(m) \tilde{\geqslant} n$ and $f(m) \tilde{\ngeqslant} n+1$, and $\mathrm{crit}(f) = m$ is equivalent to the conjunction of $\mathrm{crit}(f) \tilde{\geqslant} m$ and $\mathrm{crit}(f) \tilde{\ngeqslant} m+1$.

3.17 Lemma (PRA).

(i) *For every p, we have* $\mathrm{crit}(f_{x_{[2^p]}}) = p$.

(ii) *For t representing a, and for $n \geqslant m$, $f_a(m) \tilde{\geqslant} n$ is equivalent to $A_n \models t(1) * 2^m = 2^n$.*

(iii) *The mapping* f_a *is a partial increasing injection.*

(iv) *The relation* $\mathrm{crit}(f_a) \tilde{\geqslant} n$ *is equivalent to* $A_n \models t(1) = 2^n$*.*

(v) *If* $\mathrm{crit}(f_b)$ *and* $f_a(\mathrm{crit}(f_b))$ *are defined, so is* $\mathrm{crit}(f_{a*b})$ *and we have* $\mathrm{crit}(f_{a*b}) = f_a(\mathrm{crit}(f_b))$*.*

Proof. (i) First, $f_{x_{[2^p]}}(m) = m$ is equivalent to $A_{m+1} \models 1_{[2^p]} * 2^m = 2^m$ by definition. This holds for $m < p$, as we have $A_{m+1} \models 1_{[2^p]} = 2^{m+1}$, and $A_{m+1} \models 2^{m+1} * x = x$ for every x. On the other hand, $A_{m+1} \models 1_{[2^p]} = 2^m$ holds, hence so does $A_{m+1} \models 1_{[2^p]} * 2^m = 2^{m+1} \neq 2^m$. So $\mathrm{crit}(f_{x_{[2^p]}})$ exists, and it is p.

(ii) If $f_a(m) = p$ holds for some $p \geqslant n$, $A_{p+1} \models t(1) * 2^m = 2^{p+1}$, hence $A_n \models t(1) * 2^m = 2^n$ by projecting. And $f_a(m)$ not being defined means that there exists no p satisfying $A_{p+1} \models t(1) * 2^m < 2^{p+1}$: in other words $A_{p+1} \models t(1) * 2^m = 2^{p+1}$ for $p+1 \geqslant m$, and, in particular, for $p+1 = n$.

(iii) Assume $f_a(m+1) = n+1$. Then $A_{n+2} \models t(1)*2^{m+1} = 2^{n+1}$ holds, i.e., $t(1)$ has period 2^{m+2} at least in A_{n+2}. By projecting from A_{n+2} to A_{n+1}, we deduce that $t(1)$ has period 2^{m+1} at least in A_{n+1}, hence $A_{n+1} \models t(1)*2^m \leqslant 2^n$. If the latter relation is an equality, we deduce $f_a(m) = n$. Otherwise, by projecting, we find some integer $p < n$ for which $A_{p+1} \models t(1) * 2^m = 2^p$, and we deduce $f_a(m) = p$. In both cases, $f_a(m)$ exists, and its value is at most n. This shows that the domain of f_a is an initial segment of ω, and that f_a is increasing.

(iv) Assume $\mathrm{crit}(f_a) \tilde{\geqslant} n$, i.e., $f_a(m) = m$ holds for $m < n$. We have $f_a(n-1) \tilde{\ngeqslant} n$, hence $A_n \models t(1) * 2^{n-1} \leqslant 2^{n-1}$, whence $A_n \models t(1) = 2^n$, as $A_n \models a * 2^{n-1} = 2^n$ holds for $a < 2^n$. Conversely, assume $A_n \models t(1) = 2^n$, and $m < n$. By projecting from A_n to A_{m+1}, we obtain $A_{m+1} \models t(1) = 2^{m+1}$, hence $A_{m+1} \models t(1) * 2^m = 2^m < 2^{m+1}$, which gives $f_a(m) \tilde{\ngeqslant} m+1$ by (ii). As $f_a(m) \tilde{\geqslant} m$ holds by (ii), we deduce $f_a(m) = m$.

(v) Let $a, b \in F_1$ be represented by t_1 and t_2 respectively. Assume first $f_a(p) \tilde{\geqslant} n$ and $\mathrm{crit}(f_b) \tilde{\geqslant} p$. By (iv), the hypotheses are $A_n \models t_1(1)*2^p = 2^n$, and $A_p \models t_2(1) = 2^p$. By projecting from A_n to A_p, we deduce that $t_2(1)^{A_n}$ is a multiple of 2^p. Hence, the hypothesis $A_n \models t_1(1) * 2^p = 2^n$ implies $A_n \models (t_1 * t_2)(1) = t_1(1) * t_2(1) = 2^n$, hence, by (iv), $\mathrm{crit}(f_{a*n}) \tilde{\geqslant} n$.

Assume now $f_a(p) \tilde{\ngeqslant} n+1$ and $\mathrm{crit}(f_b) \tilde{\ngeqslant} p+1$. The hypotheses are $A_{n+1} \models t_1(1) * 2^p \neq 2^{n+1}$, i.e., the period of $t_1(1)$ in A_{n+1} is 2^{p+1} at least, and $A_{p+1} \models t_2(1) \neq 2^{p+1}$, hence $A_{p+1} \models t_2(1) \leqslant 2^p$. We cannot have $A_{n+1} \models t_2(1) \geqslant 2^{p+1}$ because, by projecting from A_{n+1} to A_{p+1}, we would deduce $A_{p+1} \models t_2(1) = 2^{p+1}$, contradicting our hypothesis. Hence we have $A_{n+1} \models t_2(1) \leqslant 2^p$, and the hypothesis that the period of $t_1(1)$ in A_{n+1} is 2^{p+1} at least implies $A_{n+1} \models t_1(1) * t_2(1) \leqslant 2^n$, hence $\mathrm{crit}(f_{a*b}) \tilde{\ngeqslant} n+1$. So the conjunction of $f_a(p) = n$ and $\mathrm{crit}(f_b) = p$ implies $\mathrm{crit}(f_{a*b}) = n$. ⊣

The only point we have not proved so far is that the function f_a be total. Before going further, let us observe that the latter property is connected with the asymptotic behavior of the periods in the tables A_n, as well as with several equivalent statements:

3.18 Proposition (PRA). *The following statements are equivalent:*

(i) *For each a in F_1, the function f_a is total.*

(ii) *For every term t, the period of $t(1)$ in A_n goes to infinity with n—so, in particular, the period of every fixed a in A_n goes to infinity with n.*

(iii) *The period of 1 in A_n goes to infinity with n.*

(iv) *For every r, there exists an n satisfying $A_n \models 1^{[r]} < 2^n$.*

(v) *The subsystem of the inverse limit of all A_n's generated by $(1, 1, \ldots)$ is free.*

Proof. Let t be an arbitrary term in T_1, and a be its class in F_1. Saying that the period of $t(1)$ in A_n goes to ∞ with n means that, for every m, there exists n with $A_n \models t(1) * 2^m < 2^n$, i.e., $f_a(m) \tilde{\ngeqslant} n$. If the function f_a is total, such an n certainly exists, so (i) implies (ii). Conversely, if (ii) is satisfied, the existence of n satisfying $f_a(m) \tilde{\ngeqslant} n$ implies that $f_a(m)$ is defined, so (i) and (ii) are equivalent, and they imply (iii), which is the special case $t = x$ of (ii).

Assume now (iii). By the previous argument, the mapping f_x is total. If f_a and f_b are total, then, by Lemma 3.17(v), $\mathrm{crit}(f_{b^{[n]}})$ exists for every n, and so does $f_a(\mathrm{crit}(f_{b^{[n]}}))$, which is $\mathrm{crit}(f_{(a*b)^{[n]}})$. This proves that $f_{a*b}(m)$ exists for arbitrary large values of m, and this is enough to conclude that f_{a*b} is total. So, inductively, we deduce that f_a is total for every a, which is (i).

Then, we prove that (ii) implies (iv) using induction on $r \geqslant 1$. The result is obvious for $r = 1$. Let p be maximal satisfying $A_p \models 1^{[r-1]} = 2^p$, which exists by induction hypothesis. By (ii), we have $A_n \models 1 * 2^p < 2^n$ for some $n > p$, so the period of 1 in A_n is a multiple of 2^{p+1}. By hypothesis, we have $A_{p+1} \models 1^{[r-1]} = 2^p$, hence $A_n \models 1^{[r-1]} = 2^p \bmod 2^{p+1}$, so 2^p is the largest power of 2 that divides $1^{[r-1]}$ computed in A_n. As the period of 1 in A_n is a multiple of 2^{p+1}, we obtain $A_n \not\models 1 * 1^{[r-1]} = 2^n$, so $A_n \models 1^{[r]} = 1 * 1^{[r-1]} < 2^n$.

Assume now (iv), and let t be an arbitrary term. By Lemma 2.6(i), there exist q, r satisfying $t^{[r]} =_{LD} x^{[q]}$. By (iv), $A_n \models 1^{[q]} = t(1)^{[r]} < 2^n$ for some n, hence $A_n \models t(1) < 2^n$, since every right power of 2^n in A_n is 2^n. Hence (iv) implies (iii).

Assume (i), and let $t, t_1, \ldots, t_p$ be arbitrary terms. By (ii), we can find n such that none of the terms $t, t*t_1, (t*t_1)*t_2, \ldots, (\ldots(t*t_1)\ldots)*t_p$ evaluated at 1 in A_n is 2^n: this is possible since $A_n \models t(1) \neq 2^n$ implies $A_m \models t(1) \neq 2^m$ for $m \geqslant n$. So we have

$$A_n \models t(1) < (t * t_1)(1) < ((t * t_1) * t_2)(1) < \cdots$$

and, in particular, $A_n \models t(1) \neq (\cdots(t * t_1)*) \cdots * t_p)(1)$. This implies that left division in the sub-LD-system of the inverse limit of all A_n's generated by $(1, 1, \ldots)$ has no cycle, and, therefore, by Laver's criterion, this LD-system is free. Conversely, assume that (i) fails, i.e., there exists a $p \geqslant 1$ such that $A_n \models 1 * 2^p = 2^n$ for every n. Let α denote the sequence $(1, 1, \ldots)$ in the inverse limit. Then we have $\alpha_{[2^p]} = (1, 2, \ldots, 2^p, 2^p, \ldots)$ and

$$\alpha * \alpha_{[2^p+1]} = (\alpha * \alpha_{[2^p]}) * (\alpha * \alpha) = \alpha * \alpha.$$

The sub-LD-system generated by α cannot be free, since $g * g = g * g_{[2^p+1]}$ does not hold in the free LD-system generated by g. So (v) is equivalent to (i)–(iv). ⊣

The status of the equivalent statements of Proposition 3.18 remains currently open. However, the results of Sect. 1.6 enable us to say more. We have seen that the function $\tilde{\jmath}$ associated with an elementary embedding j grows faster than any primitive recursive function. In terms of the functions f_a^j, we have $\tilde{\jmath}(n) = (f_x^j)^n(0)$. As the functions f_a^j and f_a coincide when the former exist, it is natural to look at the values $f_x^n(0)$. The point is that we can obtain for this function the same lower bound as for its counterpart f_x^j without using any set theoretic hypothesis:

3.19 Proposition (PA). *Assume that, for each a, the function f_a is total. Then the function $n \mapsto f_a^n(0)$ grows faster than any primitive recursive function.*

Proof. We consider the proof of Proposition 1.32, and try to mimic it using f_a and critical integers instead of f_a^j and critical ordinals. This is possible, because the only properties used in Sect. 1.6 are the left self-distributivity law and Relation (11.19) about critical ordinals. First, the counterpart of Lemma 1.33 is true since every value of f_a is an increasing injection and its domain is an initial interval of ω. Then the definitions of a base and of a realizable sequence can be translated without any change. Let us consider Lemma 1.36. With our current notation, the point is to be able to deduce from the hypothesis

$$f_b : \Vdash\!\!\!\to m_0 \mapsto m_1 \mapsto \cdots \mapsto m_p \tag{11.20}$$

the conclusion

$$f_{a*b} : \Vdash\!\!\!\to f_a(m_0) \mapsto f_a(m_1) \mapsto \cdots \mapsto f_a(m_p). \tag{11.21}$$

An easy induction on r gives the equality $(f_a)^n(\mathrm{crit}(f_a)) = \mathrm{crit}(f_{a^{[n+1]}})$. Now (11.20) can be restated as

$$\mathrm{crit}(f_b) = m_0, \quad \mathrm{crit}(f_{b^{[2]}}) = m_1, \quad \ldots, \quad \mathrm{crit}(f_{b^{[n+1]}}) = m_n.$$

By applying f_a and using Lemma 3.17(v), we obtain

$$\mathrm{crit}(f_{a*b}) = f_a(m_0), \quad \ldots, \quad \mathrm{crit}(f_{a*b^{[n+1]}}) = f_a(m_n).$$

By (LD), we have $f_{a*b^{[n]}} = f_{(a*b)^{[n]}}$, and therefore (11.20) implies (11.21).

So the proof of Lemma 1.36 goes through in the framework of the f_a's, and so do those of the other results of Sect. 1.6. We deduce that, for $n > 3$, there are at least $2^{h_1(h_2(\dots(h_{n-2}(1))\dots))}$ critical integers below the number $f_x^n(0)$, where h_p are the fast growing function of Sect. 1.6, and, finally, we conclude that the function $n \mapsto f_x^n(0)$ grows at least as fast as the Ackermann function. $\dashv$

It is then easy to complete the proof of Theorem 3.14:

Proof. By Proposition 3.18, proving that the period of 1 in A_n goes to infinity with n is equivalent (in PRA) to proving that the functions f_a are total. By Proposition 3.19, such a proof would also give a proof of the existence of a function growing faster than the Ackermann function. The latter function is not primitive recursive, and, therefore, such a proof cannot exist in PRA. $\dashv$

As the gap between PRA and Axiom (I3) is large, there remains space for many developments here.

To conclude, let us observe that, in the proof of Proposition 3.19, the hypothesis that the injections are total is not really used. Indeed, we establish lower bounds for the values, and the precise result is an alternative: for each r, either the value of $f_x^n(0)$ is not defined, or this value is at least some explicit value. In particular, the result is local, and the lower bounds remain valid for small values of r even if $f_x^n(0)$ is not defined for some large n. So, for instance, we have seen in Sect. 1.6 that, for $j : V_\lambda \prec V_\lambda$, we have $\tilde{\jmath}(4) \geqslant 256$, which, when translated into the language of A_n, means that the period of 1 in A_n is 16 for every n between 9 and 256 at least. The above argument shows that this lower bound remains valid even if Axiom (I3) is not assumed. The same result is true with the stronger inequality of Proposition 1.40, so we obtain

3.20 Theorem (Dougherty). *If it exists, the first integer n such that the period of 1 in A_n reaches 32 is at least $f_9^{\mathrm{Ack}}(f_8^{\mathrm{Ack}}(f_8^{\mathrm{Ack}}(254)))$.*

We refer to [8, 10, 13] (and to unpublished work by Laver) for many more computations about the critical ordinals of iterated elementary embeddings.

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12. Iterated Forcing and Elementary Embeddings

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Contents

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1. Introduction

In this chapter we present a survey of the area of set theory in which iterated forcing interacts with elementary embeddings. The original plan was to concentrate on forcing constructions which preserve large cardinal axioms, particularly *reverse Easton* iterations. However this plan proved rather restrictive, so we have also treated constructions such as Baumgartner's consistency proof for the Proper Forcing Axiom. The common theme of the constructions which we present is that they involve extending elementary embeddings. We have not treated the preservation of large cardinal axioms by "Prikry-type" forcing, for example by Radin forcing or iterated Prikry forcing. For this we refer the reader to Gitik's chapter in this Handbook.

After some preliminaries, the bulk of this chapter consists of fairly short sections, in each of which we introduce one or two technical ideas and give one or more examples of the ideas in action. The constructions are generally of increasing complexity as we proceed and have more techniques at our disposal. Especially at the beginning, we have adopted a fairly leisurely and discursive approach to the material. The impatient reader is encouraged to jump ahead and refer back as necessary. At the end of this introduction there is a brief description of the contents of each section.

Here is a brief review of our notation and conventions. We defer the discussion of forcing to Sect. 5.

- $P(X)$ is the power set of X. If X is a subset of a well-ordered set then $\mathrm{ot}(X)$ is the order-type of X. V_α is the set of sets with rank less than α. $\mathrm{tc}(X)$ is the transitive closure of X. H_θ is the set of x such that $\mathrm{tc}(\{x\})$ has cardinality less than θ.

- For τ a term and M a model, τ_M or τ^M denotes the result of interpreting the set-theoretic term τ in the model M, for example V_α^M or 2^ω_M. When $\tau_M = \tau \cap M$ we sometimes write "$\tau \cap M$" instead of "τ_M", especially when τ is a term of the form "$P(X)$" or "V_α".

- f is a *partial function f from X to Y* ($f : X \rightsquigarrow Y$) if and only if $f \subseteq X \times Y$ and for every $a \in X$ there is at most one $b \in Y$ with $(a, b) \in f$. f is a *total function from X to Y* ($f : X \to Y$) if and only if for every $a \in X$ there is exactly one $b \in Y$ with $(a, b) \in f$. As usual we write "$f(a) = b$" for "$(a, b) \in f$". id_X is the identity function on X.

- We use On for the class of ordinals, Card for the class of cardinals, Lim for the class of limit ordinals, Reg for the class of regular cardinals and Sing for the class of singular ordinals.

- If α is a limit ordinal then $\mathrm{cf}(\alpha)$ is the cofinality of α. If δ is a regular cardinal then $\mathrm{Cof}(\delta)$ is the class of limit ordinals α such that $\mathrm{cf}(\alpha) = \delta$. Expressions like "$\mathrm{Cof}(<\kappa)$" have the expected meaning.

- $|X|$ is the cardinality of X.

- ${}^X Y$ is the set of all functions from X to Y. If κ and λ are cardinals then $\kappa^\lambda = |{}^\lambda \kappa|$.

- We will make the following abuse of notation. When M and N are transitive models with $M \subseteq N$ we will write "$N \models {}^\beta M \subseteq M$" to mean that every β-sequence from M which lies in N actually lies in M, even in situations where possibly M is not definable in N. A similar convention applies when we write "$N \models {}^\beta \mathrm{On} \subseteq M$".

- $[X]^\lambda$ is the set of subsets of X of cardinality λ. Expressions like $[X]^{\leq\lambda}$ have the obvious meaning. If κ is regular and $\kappa \leq \lambda$ then $P_\kappa \lambda = \{a \in [\lambda]^{<\kappa} : a \cap \kappa \in \kappa\}$; this is a departure from the more standard notation in which the terms "$P_\kappa \lambda$" and "$[\lambda]^{<\kappa}$" are synonymous.

- A *tree* is a structure $(T, <_T)$ where $<_T$ is a well-founded strict ordering on T, and each element of T has a linearly ordered set of predecessors. T_α is the set of elements of height α, $T \restriction \alpha$ is the set of elements of height less than α.

- A tree is *normal* if and only if it is nonempty, has a unique minimal element, and has the properties that every element has two immediate successors and that every element of limit height is determined uniquely by the set of its predecessors in the tree. For κ regular a *κ-tree* is a normal tree of height κ, in which every level has size less than κ.

- ω_α is the αth infinite cardinal.

- Throughout we use "inaccessible" to mean "strongly inaccessible" and "Mahlo" to mean "strongly Mahlo".

- An *ideal on X* is a non-empty family of subsets of X which is downwards closed and closed under finite unions; a *filter on X* is a non-empty family of subsets of X which is upwards closed and closed under finite intersections. An ideal I is *proper* if $X \notin I$, and a filter F is *proper* if $\emptyset \notin F$; most of the ideals and filters appearing in this chapter will be proper. If I is an ideal on X then $\{X \setminus A : A \in I\}$ is a filter on X, which is called the *dual filter* and will often be denoted by I^*; similarly if F is a filter then $F^* = \{X \setminus A : A \in F\}$ is an ideal.

Ideals often arise in measure theory, where the class of measure zero sets for a (complete) measure on X is an ideal. If I is an ideal on X then we say that $A \subseteq X$ is *positive for I* or *I-positive* iff $A \notin I$, and we often write I^+ for the class of positive sets; we also sometimes say that A is *measure one for I* if $A \in I^*$. Similarly if F is a filter we say A is *F-positive* iff $A \notin F^*$, and is *F-measure one* iff $A \in F$.

- An *ultrafilter* on X is a maximal proper filter on X, or equivalently a filter U such that for all $A \subseteq X$ exactly one of the sets $A, X \setminus A$ is in U. An ultrafilter is *principal* if and only if it is of the form $\{A \subseteq X : a \in A\}$ for some $a \in X$.

- If I is an ideal and λ is a cardinal, then I is *λ-complete* if and only if I is closed under unions of length less than λ; similarly a filter F is *λ-complete* if and only if F is closed under intersections of length less than λ.

  If κ is a regular cardinal then a *measure on κ* is a κ-complete non-principal ultrafilter on κ. The measure U is *normal* if and only if it is closed under *diagonal intersections*, that is for every sequence $\langle X_i : i < \kappa \rangle$ with $X_i \in U$ for all $i < \kappa$, the *diagonal intersection* $\{\beta : \forall \alpha < \beta \; \beta \in X_\alpha\}$ of the sequence lies in U.

The prerequisites for reading this chapter are some familiarity with iterated forcing and the formulation of large cardinal axioms in terms of elementary embeddings. Knowledge of the material in Baumgartner's survey paper on iterated forcing [6, Sects. 0, 1, 2 and 5] and Kanamori's book on large cardinals [43, Sects. 5, 22, 23, 24 and 26] should be more than sufficient.

I learned much of what I know about elementary embeddings and forcing from Hugh Woodin, and would like to thank him for many patient explanations. I have also profited greatly from conversations with Uri Abraham, Arthur Apter, Jim Baumgartner, Matt Foreman, Sy Friedman, Moti Gitik, Aki Kanamori, Menachem Magidor, Adrian Mathias and Saharon Shelah.

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We conclude this introduction with the promised road map of the chapter.

- Section 2 discusses basic facts about elementary embeddings.

- Section 3 describes how we approximate elementary embeddings by ultrapowers and more generally by *extenders*, a special kind of limit ultrapower.

- Section 4 reviews some basic large cardinal axioms and their formulation in terms of elementary embeddings.

- Section 5 contains a discussion of the basics of forcing. Our convention (following Kunen [46]) is that a *notion of forcing* is a preordering with a designated largest element; we discuss the relationship with the other standard approaches to forcing. We review the basic closure, distributivity and chain condition properties and introduce some variants (the Knaster property and strategic closure) which are important later. We also introduce some basic forcing posets, Cohen forcing and the standard cardinal collapsing posets.

- Section 6 defines four forcing posets which enable us to distinguish different closure properties and will all play various roles later in the chapter. These are the posets to add a Kurepa tree, a non-reflecting stationary set, a square sequence and finally a club set disjoint from a prescribed co-stationary set in ω_1.

- Section 7 reviews iterated forcing, essentially following the approach of Baumgartner's survey [6]. We discuss the preservation of various closure and chain conditions and the idea of a factor iteration.

- Section 8 describes how to build generic objects over sufficiently closed inner models for sufficiently closed forcing posets. We apply this to construct a variant form of Prikry forcing first isolated by Foreman and Woodin in their work on the global failure of GCH [20].

- Section 9 proves a key lemma of Silver's on lifting elementary embeddings to generic extensions, discusses the properties of the lifted embeddings and gives some easy applications.

- Section 10 discusses the key idea of a *generic elementary embedding*, constructs some examples and applies them to a discussion of stationary reflection at small cardinals.

- Section 11 describes Silver's idea of iterating forcing with Easton supports. As a first application we sketch a simpler proof of a theorem by Kunen and Paris [47], that under GCH a measurable cardinal κ may carry κ^{++} normal measures.

- Section 12 introduces another key idea of Silver's, that of a *master condition*. As a first example of a master condition argument we give something close to Silver's original consistency proof for the failure of GCH at a measurable cardinal, starting from the hypothesis that there is a model of GCH in which some κ is κ^{++}-supercompact.

- Section 13 describes a technique, which is due to Magidor, for doing without a master condition under some circumstances. As an example we redo the failure of GCH at a measurable cardinal from the hypothesis that there is a model of GCH in which some κ is κ^+-supercompact.

- Section 14 describes how we may absorb κ-closed forcing posets into a large enough κ-closed collapsing poset, so that the quotient is also κ-closed. We then apply this to prove a theorem of Kunen [45] about saturated ideals, a theorem of the author from joint work with Džamonja and Shelah [13] about *strong non-reflection*, and Magidor's theorem [55] that consistently every stationary set in $\omega_{\omega+1}$ reflects.

- Section 15 discusses how to transfer generic filters between models of set theory, and sketches an application to constructing generalized versions of Prikry forcing.

- Section 16 shows that we may apply the ideas in this chapter in the context of weak large cardinal axioms such as weak compactness, and sketches a proof that GCH may first fail at a weakly compact cardinal.

- Section 17 proves two theorems of Jech, Magidor, Mitchell and Prikry [41]. The first result is that ω_1 may carry a precipitous ideal, the second is that in fact the non-stationary ideal on ω_1 may be precipitous. The argument for the second result uses the absorption idea from Sect. 14, and also involves iterating a natural forcing for *shooting club sets* through stationary sets.

- Section 18 sketches the proof of Gitik's result [23] that the precipitousness of NS_{ω_2} is equi-consistent with a cardinal of Mitchell order two.

- Section 19 gives two more applications of iterated club shooting, Jech and Woodin's result [40] that $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ can be κ^+-saturated for a Mahlo cardinal κ and Magidor's result [55] that consistently every stationary set of cofinality ω ordinals in ω_2 may reflect at almost all points of cofinality ω_1.

- Section 20 discusses some variant collapsing posets which are often useful, Kunen's *universal collapse* [45] and the *Silver collapse*. We sketch Kunen's proof [45] that ω_1 can carry an ω_2-saturated ideal, starting from the hypothesis that there is a huge cardinal.

- Section 21 sketches some results primarily due to Hamkins which put limits on what we can achieve by reverse Easton forcing. As a sample application we sketch an easy case of Hamkins' *superdestructibility theorem* [32].

- Section 22 describes an idea of Laver's for introducing a kind of universal generic object by forcing with a poset of terms. As an application we sketch an unpublished proof by Magidor [51] of his celebrated theorem [52] that the least measurable cardinal can be strongly compact.

- Section 23 introduces the idea of analyzing iterations by term forcing. As an example we introduce yet another collapsing poset and give a version of Mitchell's proof [57] that ω_2 may have the tree property.

- Section 24 discusses how to build universal iterations using prediction principles. We prove Laver's theorem that supercompact cardinals carry *Laver diamonds*, and use this to give Baumgartner's proof for the consistency of the Proper Forcing Axiom [15] and Laver's proof that a supercompact cardinal κ can be made indestructible under κ-directed closed forcing [49].

- Section 25 introduces an idea due to Woodin for altering generic objects, and then applies this to give Woodin's consistency proof for the failure of GCH at a measurable from an optimal assumption.

2. Elementary Embeddings

We will be concerned with elementary embeddings $k : M \longrightarrow N$ where M and N are transitive models of ZFC and k, M, and N are all classes of some universe of set theory. It will not in general be the case that k or N are classes of M or that $N \subseteq M$. In particular we will be interested in the situation of a "generic embedding" where $j : V \longrightarrow M \subseteq V[G]$ for $V[G]$ a generic extension of V, and j, M are defined in $V[G]$.

This notion is straightforward if M and N are sets but one needs to be a little careful when M and N are proper classes. We refer the reader to Kanamori's book [43, Sects. 5 and 19] for a careful discussion of the metamathematical issues. From now on we will freely treat elementary embeddings between proper classes as if those classes were sets, a procedure which can be justified by the methods of [43]. We reserve the term "inner model" for a transitive class model of ZFC which contains all the ordinals.

We start by recalling a few basic facts about elementary embeddings.

2.1 Proposition. *Let M and N be transitive models of* ZFC *and let the map $k : M \longrightarrow N$ be elementary. Then*

1. *The pointwise image $k``M$ is an elementary substructure of $N$, the Mostowski collapse of the structure $(k``M, \in)$ is M, and k is the inverse of the collapsing isomorphism from $k``M$ to M.*

2. *$k(\alpha) \geq \alpha$ for all $\alpha \in M \cap \mathrm{On}$.*

3. *If $k\restriction(\beta + 1) = \mathrm{id}_{\beta+1}$ and $A \in M$ with $A \subseteq \beta$, then $k(A) = A$.*

Proof. Easy. $\dashv$

2.2 Proposition. *Let M be a transitive model of* ZFC*, let $x \in M$ and let $M \models$ "$x \in H_{\lambda^+}$" where λ is an infinite M-cardinal. Then there is a set $A \subseteq \lambda$ such that $A \in M$ and for any transitive model N of* ZF*, $A \in N$ implies that $x \in N$.*

Proof. Let $f \in M$ be an injection from $\mathrm{tc}(\{x\})$ to λ, let G be Gödel's pairing function and let

$$A = \{G(f(a), f(b)) : a, b \in \mathrm{tc}(\{x\}) \text{ and } a \in b\}.$$

If $A \in N$ then N can compute x by forming the Mostowski collapse of the well-founded extensional relation $\{(\alpha, \beta) : G(\alpha, \beta) \in A\}$, and then finding the element of maximal rank in this set. $\dashv$

We abbreviate the rather cumbersome assertion "A is a set of ordinals such that $\{(\gamma, \delta) : G(\gamma, \delta) \in A\}$ is a well-founded relation whose transitive collapse is $\mathrm{tc}(\{x\})$" by "A codes x". The assertions "A codes x" and "A codes something" are both Δ_1^{ZFC} and are thus absolute between transitive models of ZFC.

2.3 Proposition. *Let M and N be transitive models of* ZFC *and let the map $k : M \longrightarrow N$ be elementary. If $k``(M \cap \mathrm{On})$ is cofinal in $N \cap \mathrm{On}$ then exactly one of the following is true:*

1. *$k = \mathrm{id}_M$ and $M = N$.*
2. *There exists a $\delta \in M \cap \mathrm{On}$ such that $k(\delta) > \delta$.*

Proof. Suppose the second alternative fails, so that $k \restriction (M \cap \mathrm{On})$ is the identity. Let $x \in M$ and find a set of ordinals $A \in M$ such that A codes x. Then $A = k(A)$ by Proposition 2.1, $k(A)$ codes $k(x)$ by elementarity, and so $k(x) = x$. Since x was arbitrary, $k = \mathrm{id}_M$.

Since $k = \mathrm{id}_M$, $M \cap \mathrm{On} = N \cap \mathrm{On}$ and $V_\beta^N = V_{k(\beta)}^N = k(V_\beta^M) = V_\beta^M$ for all $\beta \in M \cap \mathrm{On}$. So $M = N$. $\dashv$

From now on we will say that $k : M \longrightarrow N$ is *nontrivial* if $k \neq \mathrm{id}_M$.

2.4 Remark. It was crucial in Proposition 2.3 that k should map $M \cap \mathrm{On}$ cofinally into $N \cap \mathrm{On}$. For example the theory of sharps [43, Sect. 9] shows that if $0^\#$ exists then L_{ω_1} and L_{ω_2} are models of ZFC and $L_{\omega_1} \prec L_{\omega_2}$.

2.5 Remark. Let $k : M \longrightarrow N$ be elementary, where M is an inner model and N is transitive. Then N is an inner model, and the hypotheses of Proposition 2.3 are satisfied.

If $k : M \longrightarrow N$ is elementary then the least δ such that $k(\delta) > \delta$ (if it exists) is called the *critical point* of k and is denoted by $\mathrm{crit}(k)$. It is not hard to see that $\mathrm{crit}(k)$ is a regular uncountable cardinal in M.

It is natural to ask how much agreement there must be between the models M and N. The following proposition puts a lower bound on the level of agreement.

2.6 Proposition. *If $k : M \longrightarrow N$ is an elementary embedding between transitive models of* ZFC *and* $\mathrm{crit}(k) = \delta$*, then* $H^M_{\delta^+} \subseteq N$.

Proof. Let $x \in H^M_{\delta^+}$ and let $A \in M$ code x with $A \subseteq \delta$. Then for $\alpha < \delta$ we have

$$\alpha \in A \quad \Longleftrightarrow \quad k(\alpha) \in k(A) \quad \Longleftrightarrow \quad \alpha \in k(A),$$

so $A = k(A) \cap \delta \in N$. Therefore $x \in N$. ⊣

In general we cannot say much more, as illustrated by the following two examples. In Example 2.7 $M = N$, while in Example 2.8 M and N agree only to the extent indicated by Proposition 2.6.

2.7 Example. Suppose that $0^\#$ exists. Then there is a nontrivial elementary embedding $k : L \longrightarrow L$ [43, Sect. 9].

2.8 Example. It is consistent (from large cardinals) that there exist inner models M and N and an embedding $k : M \longrightarrow N$ such that $\mathrm{crit}(k) = \omega_1^M$ and $V_{\omega+1} \cap M \subsetneq V_{\omega+1} \cap N$. We will construct such an example in Theorem 10.2.

If the critical point is inaccessible in M we can say more:

2.9 Proposition. *If $k : M \longrightarrow N$ is an elementary embedding between transitive models of* ZFC*, and* $\mathrm{crit}(k) = \delta$ *where δ is inaccessible in M, then* $V_\delta \cap M = V_\delta \cap N$.

Proof. For $\alpha < \delta$, the set $V_\alpha \cap M$ is coded by a bounded subset of δ lying in M. In particular it is fixed by k, so as α is also fixed by elementarity $V_\alpha \cap M = V_\alpha \cap N$. ⊣

In the theory of large cardinals we are most interested in embeddings of the following type, where usually M will be an inner model.

2.10 Definition. An embedding $k : M \longrightarrow N$ is *definable* if and only if k and N are definable in M.

The analysis of these embeddings is due to Scott [61] and is summarized in the following proposition.

2.11 Proposition. *Let M and N be inner models and let $k : M \longrightarrow N$ be a nontrivial definable elementary embedding with* $\mathrm{crit}(k) = \delta$*. Let*

$$U = \{X \subseteq \delta : X \in M, \delta \in k(X)\}.$$

Then

1. *$U \in M$ and $M \models$ "U is a normal measure on δ".*
2. *$V^M_{\delta+1} = V^N_{\delta+1}$.*
3. *$k \restriction V^M_\delta = \mathrm{id}_{V^M_\delta}$.*

4. *For all* $A \in V^M_{\delta+1}$, $A = k(A) \cap V^M_\delta$.

Proof. See [43, Sect. 5]. ⊣

2.12 Remark. Neither of the embeddings from Examples 2.7 and 2.8 is definable.

3. Ultrapowers and Extenders

It will be important for us to be able to describe embeddings between models by ultrapowers and limit ultrapowers. We give a sketchy outline here and refer the reader to [43, Sects. 19 and 26] for the details.

Let M be a transitive model of ZFC, let $X \in M$ and let U be an ultrafilter on $P(X) \cap M$. Then we may form $\mathrm{Ult}(M, U)$, the collection of U-equivalence classes of functions $f \in M$ with $\mathrm{dom}(f) = X$. As usual we let $[f]_U$ denote the class of f, and for $x \in M$ we let $j_U(x) = [f_x]_U$ where f_x is the function with domain X and constant value x. $\mathrm{Ult}(M, U)$ is made into a structure for the language of set theory by defining

$$[f]_U E [g]_U \quad \iff \quad \{x : f(x) \in g(x)\} \in U,$$

and we make a mild abuse of notation by writing "$\mathrm{Ult}(M, U)$" for the structure $(\mathrm{Ult}(M, U), E)$.

3.1 Remark. When M is an inner model $[f]_U$ is typically a proper class, which makes the definition of $\mathrm{Ult}(M, U)$ appear problematic. This can be fixed by *Scott's trick* in which $[f]_U$ is redefined as the set of functions with minimal rank which are equivalent to f modulo U. Similar remarks apply to ultrapowers throughout this chapter.

Since M is a model of ZFC Łoś's theorem holds, that is to say that for any formula $\phi(x_1, \ldots, x_n)$ and any functions $F_1, \ldots, F_n \in M$ with domain X,

$$\mathrm{Ult}(M, U) \models \phi([F_1]_U, \ldots, [F_n]_U)$$

if and only if

$$\{x : M \models \phi(F_1(x), \ldots, F_n(x))\} \in U.$$

In particular j_U is an elementary embedding from M to $\mathrm{Ult}(M, U)$. When $\mathrm{Ult}(M, U)$ is well-founded we will identify it with its transitive collapse. The following propositions are standard.

3.2 Proposition. *Let* $k : M \longrightarrow N$ *be an elementary embedding between transitive models of* ZFC, *let* $a \in N$ *and let* $B \in M$ *with* $a \in k(B)$. *Let* $E_a = \{A \subseteq B : A \in M, a \in k(A)\}$. *Then*

1. E_a *is an ultrafilter on* $P(B) \cap M$. *For notational convenience we define* $M_a = \mathrm{Ult}(M, E_a)$ *and* $j_a = j_{E_a}$.

2. *If we define* $k_a : M_a \longrightarrow N$ *by* $k_a([f]_{E_a}) = k(f)(a)$ *then* k_a *is a well-defined elementary embedding and* $k_a \circ j_a = k$. k_a *and* M_a *do not depend on the choice of* B.

3. M_a *is isomorphic via* k_a *to* X_a, *where*

$$X_a = \{k(F)(a) : F \in M, \mathrm{dom}(F) = B\}.$$

4. M_a *is well-founded and, when we identify it with its transitive collapse* k_a, *is the inverse of the transitive collapsing map on* X_a.

5. *If* k *is definable then* $E_a \in M$ *and* j_a *is definable.*

3.3 Proposition. *Let* $k : M \longrightarrow N$ *be an elementary embedding between transitive models of* ZFC. *Let* $a_1 \in k(B_1)$, $a_2 \in k(B_2)$ *and let* E_1, E_2 *be the associated ultrafilters. Suppose that* $F : B_2 \longrightarrow B_1$ *is such that* $k(F)(a_2) = a_1$. *Then* F *induces an elementary embedding*

$$F^* : \mathrm{Ult}(M, E_1) \longrightarrow \mathrm{Ult}(M, E_2),$$

where $F^*([g]_{E_1}) = [g \circ F]_{E_2}$. *Moreover* $j_{E_2} = F^* \circ j_{E_1}$.

3.4 Proposition. *Let* $\lambda \in N \cap \mathrm{On}$ *be such that* $\lambda \leq \sup(k``(M \cap \mathrm{On}))$. *For each* $a \in [\lambda]^{<\omega}$ *let* μ_a *be the least ordinal such that* $a \subseteq j(\mu_a)$ *and let*

$$E_a = \{A \subseteq [\mu_a]^{|a|} : A \in M, a \in k(A)\}.$$

Let M_a, j_a, k_a, *and* X_a *be as in Proposition 3.2. If* $a, b \in [\lambda]^{<\omega}$ *and* $a \subseteq b$ *then define*

$$F_{ab}(x) = \{\gamma \in x : \exists \gamma^* \in a \ \mathrm{ot}(x \cap \gamma) = \mathrm{ot}(b \cap \gamma^*)\}$$

for $x \in [\mu_b]^{|b|}$. *Then*

1. $F_{ab} : [\mu_b]^{|b|} \longrightarrow [\mu_a]^{|a|}$ *and* $k(F_{ab})(b) = a$. *We let* j_{ab} *denote the embedding from* M_a *to* M_b *induced by* F_{ab}.

2. $M_0 = M$, $k_0 = k$, $j_{0a} = j_a$.

3. *The system of structures* M_a *and embeddings* j_{ab} *is a directed system, so has a direct limit* M_∞. *There are elementary embeddings* $j_{a\infty} : M_a \longrightarrow M_\infty$ *such that* $M_\infty = \bigcup_a j_{a\infty}[M_a]$ *and* $j_{b\infty} \circ j_{ab} = j_{a\infty}$.

4. *There is an elementary embedding* $l : M_\infty \longrightarrow M$ *such that* $l \circ j_{a\infty} = k_a$ *for all* a.

5. M_∞ *is isomorphic via* l *to* $X_\infty = \bigcup_a X_a$, *and* l *is the inverse of the Mostowski collapsing map on* X_∞. *In particular* M_∞ *is well-founded.*

6. *If* k *is definable and* M *is an inner model then* $j_{0\infty}$ *is definable.*

$$\begin{tikzcd}
M \arrow[rr, "k"] \arrow[rrd, "j_{0\infty}"] \arrow[d, "j_a"'] \arrow[ddr, "j_b"] & & N \\
M_a \arrow[rru, "k_a"] \arrow[rr, "j_{a\infty}"] \arrow[dr, "j_{ab}"'] & & M_\infty \arrow[u, "l"'] \\
& M_b \arrow[ruu, "k_b"] \arrow[ru, "j_{b\infty}"'] &
\end{tikzcd}$$

If $k : M \longrightarrow N$ is elementary where M and N are inner models then we may make X_∞ contain arbitrarily large initial segments of N by choosing λ sufficiently large. M_∞ is the transitive collapse of X_∞, l is the inverse of the collapsing map and $l \circ j_{0\infty} = k$. It follows that we may make $j_{0\infty}$ approximate k to any required degree of precision by a suitable choice of λ.

3.5 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC with $\mathrm{crit}(k) = \delta$, and let $\lambda \le \sup(k``(M \cap \mathrm{On}))$. If

$$E = \{E_a : a \in [\lambda]^{<\omega}\}$$

where E_a is defined as above, then we call E the *M-(δ, λ)-extender derived from k*.

It is possible [43, Sect. 26] to give an axiomatization of the properties enjoyed by E as in Definition 3.5, thus arriving at the concept of an "M-(δ, λ)-extender". Given an M-(δ, λ)-extender E we can compute the limit ultrapower of M by E to get a well-founded structure $\mathrm{Ult}(M, E)$ and an embedding $j_E : M \longrightarrow \mathrm{Ult}(M, E)$.

If E is the extender derived from $k : M \longrightarrow N$ as in Proposition 3.4 then in the notation of that proposition, $\mathrm{Ult}(M, E) = M_\infty$ and $j_E = j_{0\infty}$. If E is an M-(δ, λ)-extender and E' is the M-(δ, λ)-extender derived from the ultrapower map $j_E : M \longrightarrow \mathrm{Ult}(M, E)$ then $E = E'$.

When E is a V-(δ, λ)-extender lying in V we will just refer to E as a "(δ, λ)-extender".

3.6 Definition. An M-(δ, λ)-extender E is called *short* if all the measures E_a concentrate on $[\delta]^{<\omega}$, or equivalently if $\lambda \le j_E(\delta)$.

We now make a couple of (non-standard) definitions which will give us a convenient way of phrasing some results later. See for example Propositions 3.9 and 15.1.

3.7 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC, and let μ be an ordinal. The embedding k *has width* $\le \mu$ if and only if every element of N is of the form $k(F)(a)$ for some $F \in M$, $a \in N$ where $M \models |\mathrm{dom}(F)| \le \mu$.

3.8 Definition. Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC, and let $A \subseteq N$. The embedding k is *supported on* A if and only if every element of N is of the form $k(F)(a)$ for some $F \in M$ and $a \in A \cap \mathrm{dom}(k(F))$.

The following easy proposition will be useful later.

3.9 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC *with* $\mathrm{crit}(k) = \kappa$*, and let*

$$U = \{X \subseteq \kappa : X \in M, \kappa \in k(X)\}.$$

Then k is the ultrapower map computed from M and U if and only if k is supported on $\{\kappa\}$.

4. Large Cardinal Axioms

We briefly review some standard large cardinal axioms and their formulation in terms of elementary embeddings and ultrapowers. Once again we refer the reader to Kanamori's book [43] for the details.

We start with the characterizations in terms of elementary embeddings.

- κ is *measurable* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$.
- κ is λ-*strong* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. κ is *strong* if and only if it is λ-strong for all λ.
- κ is λ-*supercompact* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$. κ is *supercompact* if and only if it is λ-supercompact for all λ.
- κ is λ-*strongly compact* if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and there is a set $X \in M$ such that $M \models |X| < j(\kappa)$ and $j``\lambda \subseteq X$. κ is strongly compact if and only if it is λ-strongly compact for all λ.
- κ is *huge with target* λ if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and ${}^\lambda M \subseteq M$. κ is *almost huge with target* λ if and only if there is a definable $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and ${}^{<\lambda} M \subseteq M$.

Each of these concepts can also be characterized using ultrafilters or extenders.

- κ is measurable if and only if there is a *measure on* κ (that is, a normal κ-complete non-principal ultrafilter on κ).

- Assuming GCH, κ is $(\kappa + \beta)$-strong if and only if there is a short $(\kappa, \kappa^{+\beta})$-extender E such that $V_{\kappa+\beta} \subseteq \mathrm{Ult}(V, E)$.
- For $\lambda \geq \kappa$, κ is λ-supercompact if and only if there is a normal, fine and κ-complete ultrafilter on $P_\kappa \lambda$. We will generally refer to such an object as a *supercompactness measure on* $P_\kappa \lambda$.
- For $\lambda \geq \kappa$, κ is λ-strongly compact if and only if there is a fine and κ-complete ultrafilter on $P_\kappa \lambda$. We will generally refer to such an object as a *strong compactness measure on* $P_\kappa \lambda$.
- For $\lambda \geq \kappa$, κ is huge with target λ if and only if there is a normal, fine and κ-complete ultrafilter on $P^\kappa \lambda$, where $P^\kappa \lambda$ is the set of $X \subseteq \lambda$ with order type κ. Almost-hugeness has a rather technical characterization in terms of a direct limit system of supercompactness measures on $P_\kappa \mu$ for $\mu < \lambda$.

4.1 Remark. If $j : V \longrightarrow M$ is a definable embedding such that $\mathrm{crit}(j) = \kappa$ and $j``\lambda \in M$, then $\{X \in P_\kappa \lambda : j``\lambda \in j(X)\}$ is a supercompactness measure.

4.2 Remark. Weak compactness may also be characterized in terms of elementary embeddings, we discuss this in Sect. 16.

For use later we record the definition of the *Mitchell ordering* $\lhd$ and a few basic facts about it.

4.3 Definition. Let κ be a measurable cardinal and let U_0 and U_1 be measures on κ. Then $U_0 \lhd U_1$ if and only if $U_0 \in \mathrm{Ult}(V, U_1)$.

The theory of the Mitchell ordering is developed in Mitchell's first chapter in this Handbook. The relation $\lhd$ is a strict well-founded partial ordering. If U is a measure then $o(U)$ is defined to be the height of U in $\lhd$, and the *Mitchell order* $o(\kappa)$ of κ is defined to be the height of $\lhd$. In the usual canonical inner models for large cardinals, $\lhd$ is a linear ordering.

The following propositions collect some easy but useful facts about the behavior of elementary embeddings.

4.4 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC*, and let k have width $\leq \mu$. If $M \models \mathrm{cf}(\alpha) > \mu$ then $\sup(k``\alpha) = k(\alpha)$.*

If $\sup(k``\alpha) = k(\alpha)$ we will say that k is *continuous at α.*

4.5 Proposition. *If $U \in V$ is a countably complete ultrafilter on X, X has cardinality κ and $j : V \longrightarrow M$ is the associated ultrapower map then $|j(\mu)| < (|\mu|^\kappa)^+$ for all ordinals μ.*

4.6 Proposition. *If $E \in V$ is a short (κ, λ)-extender and $j : V \longrightarrow M$ is the associated ultrapower map then $|j(\mu)| < (\lambda \times |\mu|^\kappa)^+$ for all ordinals μ.*

4.7 Proposition. *Let M be an inner model of V. If ${}^{\lambda}M \subseteq M$ then the cardinals of V and M agree up to and including λ^+. If* GCH *holds, κ is inaccessible and $V_{\kappa+\beta} \subseteq M$ then the cardinals of V and M agree up to $\beth_\beta(\kappa)$.*

The following example illustrates how these ideas can be used. There are many similar calculations in later sections, where we will generally suppress the details.

4.8 Example. Let GCH hold and let U be a supercompactness measure on $P_\kappa \kappa^+$, with $j : V \longrightarrow M$ the associated ultrapower map. Then

1. j is continuous at κ^{++} and κ^{+++}.

2. $\kappa^{++} < j(\kappa)$.

3. $j(\kappa^{+++}) = \kappa^{+++}$.

Proof. $|P_\kappa \kappa^+| = \kappa^+$, so by Proposition 4.4 j is continuous at κ^{++} and κ^{+++}. By the definition of a supercompactness measure ${}^{\kappa^+}M \subseteq M$, and so by Proposition 4.7 $\kappa^{++} = \kappa^{++}_M$. By elementarity $j(\kappa)$ is an M-inaccessible cardinal greater than κ, and so $\kappa^{++} < j(\kappa)$.

For every $\eta < \kappa^{+++}$, Proposition 4.5 and GCH imply that $j(\eta) < \kappa^{+++}$. Since j is continuous at κ^{+++} we have $j(\kappa^{+++}) = \kappa^{+++}$ as required. ⊣

5. Forcing

We assume that the reader is familiar with forcing; in this section we establish our forcing conventions and review some of the basic definitions and facts. We will essentially follow the treatment of forcing in Kunen's text [46]. Proofs of all the facts that we mention in this section can be found in at least one of the texts by Kunen [46] or Jech [39].

Our approach to forcing is based on posets with a largest element. We justify this by the sociological observation that when a set theorist writes down a new set of forcing conditions it is almost always of this form.

For technical reasons we sometimes work with preordered sets rather than partially ordered sets; recall that a *preordering* is a transitive and reflexive relation, and that if $\leq$ is a preordering of $\mathbb{P}$ we may form the quotient by the equivalence relation

$$p \, E \, q \quad \iff \quad p \leq q \leq p$$

to get a partially ordered set. We refer to this as the *quotient poset*.

A *largest element* in a preordered set $\mathbb{P}$ is an element b such that $a \leq b$ for all a. A preordering may have many largest elements, which will all be identified when we form the quotient poset.

A *notion of forcing* is officially a triple $(\mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ where $\leq_{\mathbb{P}}$ is a preordering of $\mathbb{P}$ and $1_{\mathbb{P}}$ is a largest element. A *forcing poset* is a notion of forcing where $\leq_{\mathbb{P}}$ is a partial ordering; if $\mathbb{P}$ is a notion of forcing then the quotient poset is a forcing poset. If p, q are conditions in a notion of forcing $\mathbb{P}$ then $p \leq q$ means that p is stronger than q.

5.1 Remark. It might seem more natural just to use forcing posets in our discussion of forcing. However this would cause irritating problems when we come to discuss iterated forcing; for example in a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ we may have $p \Vdash \dot{q}_1 = \dot{q}_2$, in which case the conditions $(p, \dot{q}_1)$ and $(p, \dot{q}_2)$ are equivalent but not identical.

For $p \in \mathbb{P}$ we denote by $\mathbb{P}/p$ the subset $\{q \in \mathbb{P} : q \leq p\}$ with the inherited ordering. It is a standard fact that there is a bijection between $\mathbb{P}$-generic filters G with $p \in G$, and $(\mathbb{P}/p)$-generic filters, in which G corresponds to $G \cap (\mathbb{P}/p)$. If $p \in G$ then $V[G] = V[G \cap (\mathbb{P}/p)]$.

We say $\mathbb{P} \subseteq \mathbb{Q}$ is *dense* if every condition in $\mathbb{Q}$ has an extension in $\mathbb{P}$. There is a bijection between $\mathbb{Q}$-generic filters G and $\mathbb{P}$-generic filters, in which G corresponds to $G \cap \mathbb{P}$ and $V[G] = V[G \cap \mathbb{P}]$.

If $\mathbb{P}$ is a notion of forcing then the class $V^{\mathbb{P}}$ of $\mathbb{P}$-names is defined recursively so that σ is a $\mathbb{P}$-name if and only if every element of σ has the form (τ, p) for some $\mathbb{P}$-name τ and condition $p \in \mathbb{P}$.

We denote by $i_G(\sigma)$ the result of interpreting the name σ with respect to the filter G, that is

$$i_G(\sigma) = \{i_G(\tau) : \exists p \in G \ (\tau, p) \in \sigma\}.$$

We let $\check{x}$ denote the standard forcing name for the ground model object x, that is $\check{x} = \{(\check{y}, 1_{\mathbb{P}}) : y \in x\}$. $\dot{G} = \{(\check{p}, p) : p \in \mathbb{P}\}$ is the standard name for the generic filter.

A notion of forcing is *non-trivial* if and only if it is forced by every condition that $V[G] \neq V$, or equivalently that $G \notin V$. *The trivial forcing* is the forcing poset with just one element; we usually denote the trivial forcing by "$\{1\}$".

It is easy to see that $p \Vdash \check{q} \in \dot{G}$ if and only if every extension of p is compatible with q; we will say that a notion of forcing is *separative* when $p \Vdash \check{q} \in \dot{G} \iff p \leq q$. It is routine to check that if $\mathbb{P}$ is a separative notion of forcing then the quotient forcing poset is also separative.

It is a standard fact that for any notion of forcing $\mathbb{P}$ there is a separative forcing poset $\mathbb{Q}$ and an order and incompatibility preserving surjection $h : \mathbb{P} \to \mathbb{Q}$. The map h and forcing poset $\mathbb{Q}$ are unique up to isomorphism, $\mathbb{Q}$ is called the *separative quotient* of $\mathbb{P}$ and forcing with $\mathbb{Q}$ is equivalent to forcing with $\mathbb{P}$.

If $\mathbb{P}$ is a separative forcing poset then the Boolean algebra $\mathrm{ro}(\mathbb{P})$ of regular open subsets of $\mathbb{P}$ is complete, and $\mathbb{P}$ is isomorphic to a dense set in $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$. It follows that there is a bijection between $\mathbb{P}$-generic filters and $\mathrm{ro}(\mathbb{P})$-generic ultrafilters, so that forcing with the poset $\mathbb{P}$ is equivalent to forcing with the complete Boolean algebra $\mathrm{ro}(\mathbb{P})$. We sometimes abuse notation and write $\mathrm{ro}(\mathbb{P})$ for the regular open algebra of the separative quotient of a notion of forcing $\mathbb{P}$.

In general when $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing we will say that they are *equivalent* if and only if for every $\mathbb{P}$-generic filter G there is a $\mathbb{Q}$-generic filter H with $V[G] = V[H]$, and symmetrically for every $\mathbb{Q}$-generic filter H there is a $\mathbb{P}$-generic filter G with $V[H] = V[G]$. It is routine to see that this can be formulated in a first-order way which does not mention generic filters.

Complete Boolean algebras have the advantage that they allow a straightforward discussion of the relationship between different forcing extensions. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{Q}$ if and only if $\mathrm{ro}(\mathbb{P})$ is isomorphic to $\mathrm{ro}(\mathbb{Q})$. For $\mathbb{C}$ a complete Boolean algebra and G a $\mathbb{C}$-generic ultrafilter over V, the models of ZFC intermediate between V and $V[G]$ are precisely the models of form $V[G \cap \mathbb{B}]$ for $\mathbb{B}$ a complete subalgebra of $\mathbb{C}$.

In particular when $\mathbb{B}$ is a complete subalgebra of $\mathbb{C}$ then $\dot{G}_{\mathbb{C}} \cap \mathbb{B}$ is a $\mathbb{C}$-name for a $\mathbb{B}$-generic ultrafilter. Conversely for any complete $\mathbb{B}$ and $\mathbb{C}$, a $\mathbb{C}$-name for a $\mathbb{B}$-generic ultrafilter gives a complete embedding of $\mathbb{B}$ into $\mathbb{C}$.

Since we are wedded to an approach to forcing via posets, it is helpful to have some sufficient conditions which guarantee that a $\mathbb{Q}$-generic extension contains a $\mathbb{P}$-generic one without mentioning the regular open algebras.

5.2 Definition. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then a *projection* from $\mathbb{Q}$ to $\mathbb{P}$ is a map $\pi : \mathbb{Q} \to \mathbb{P}$ such that π is order-preserving, $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$, and for all $q \in \mathbb{Q}$ and all $p \leq \pi(q)$ there is a $\bar{q} \leq q$ such that $\pi(\bar{q}) \leq p$.

The following facts are standard; see e.g. Abraham's chapter in this Handbook.

1. If H is $\mathbb{Q}$-generic over V then π"H generates a $\mathbb{P}$-generic filter G.

2. Conversely if G is $\mathbb{P}$-generic over V and we set

$$\mathbb{Q}/G = \{q \in \mathbb{Q} : \pi(q) \in G\},$$

with the partial ordering inherited from $\mathbb{Q}$, then any $H \subseteq \mathbb{Q}/G$ which is $\mathbb{Q}/G$-generic over $V[G]$ is $\mathbb{Q}$-generic over V.

5.3 Remark. In general if $\mathbb{Q}$ and $\mathbb{P}$ are forcing posets such that forcing with $\mathbb{Q}$ adds a generic object for $\mathbb{P}$, then there is a projection from $\mathbb{Q}$ to the poset of nonzero elements of $\mathrm{ro}(\mathbb{P})$.

5.4 Definition. If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing then a *complete embedding* from $\mathbb{P}$ to $\mathbb{Q}$ is a function $i : \mathbb{P} \to \mathbb{Q}$ such that $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$, and

$$p_1 \leq p_2 \quad \iff \quad i(p_1) \leq i(p_2)$$

for all p_1 and p_2 in $\mathbb{P}$, and for every $q \in \mathbb{Q}$ there is a condition $p \in \mathbb{P}$ such that $i(\bar{p})$ is compatible with q for all $\bar{p} \leq p$.

The following facts are standard [46]:

1. If H is $\mathbb{Q}$-generic over V then $G = i^{-1}``H$ is a $\mathbb{P}$-generic filter.
2. Conversely if G is $\mathbb{Q}$-generic over V and we set $\mathbb{Q}/G$ to be the set of $q \in \mathbb{Q}$ which are compatible with all elements of $i``G$, any H which is $\mathbb{Q}/G$-generic over $V[G]$ is $\mathbb{P}$-generic over V.

5.5 Remark. In the context of projections or complete embeddings as above $\mathbb{Q}/G$ may not be separative, even if $\mathbb{P}$ and $\mathbb{Q}$ both are.

5.6 Remark. We have overloaded the notation "$\mathbb{Q}/G$", defining it both in the setting of a projection from $\mathbb{Q}$ to $\mathbb{P}$ and of a complete embedding from $\mathbb{P}$ to $\mathbb{Q}$. This is (we assert) harmless in the sense that if we have both a projection $\pi : \mathbb{Q} \to \mathbb{P}$ and a complete embedding $i : \mathbb{P} \to \mathbb{Q}$, and $i \circ \pi = \mathrm{id}_{\mathbb{P}}$, then the two definitions of $\mathbb{Q}/G$ give equivalent notions of forcing.

We will make some use of the *Maximum Principle*: if $\mathbb{P}$ is a notion of forcing and $p \in \mathbb{P}$ forces $\exists x\ \phi(x)$, then there is a term $\dot{\tau} \in V^{\mathbb{P}}$ such that $p \Vdash \phi(\dot{\tau})$. This needs the Axiom of Choice, but that presents no obstacle for us.

When we say that $\mathbb{P}$ *adds* some kind of object or *forces* some statement to hold, we mean that this is forced by every condition in $\mathbb{P}$, or equivalently it is forced by $1_{\mathbb{P}}$. This is important because some natural notions of forcing are highly inhomogeneous.

We will frequently use the standard forcing posets for adding subsets to a regular cardinal κ, and for collapsing cardinals to have cardinality κ. Each forcing poset consists of a family of partial functions ordered by reverse inclusion.

5.7 Definition. Let κ be a regular cardinal, and let λ be any ordinal.

1. (Cohen forcing) $\mathrm{Add}(\kappa, \lambda)$ is the set of all partial functions from $\kappa \times \lambda$ to 2 of cardinality less than κ.
2. $\mathrm{Col}(\kappa, \lambda)$ is the set of all partial functions from κ to λ of cardinality less than κ.
3. (The Levy collapse) $\mathrm{Col}(\kappa, {<}\lambda)$ is the set of all partial functions p from $\kappa \times \lambda$ to λ such that
   (a) $|p| < \kappa$.
   (b) $p(\alpha, \beta) < \beta$ for all $(\alpha, \beta) \in \mathrm{dom}(p)$.

5.8 Definition. Let $\mathbb{P}$ be a notion of forcing and let κ be an uncountable cardinal. Then

1. $\mathbb{P}$ is *κ-chain condition (κ-c.c.)* if and only if $\mathbb{P}$ has no antichain of size κ.

2. $\mathbb{P}$ is *κ-closed* if and only if every decreasing sequence of conditions in $\mathbb{P}$ with length less than κ has a lower bound.

3. $\mathbb{P}$ is *(κ, ∞)-distributive* if and only if forcing with $\mathbb{P}$ adds no new $<\kappa$-sequence of ordinals.

4. $\mathbb{P}$ is *κ-directed closed* if and only if every directed set of size less than κ of conditions in $\mathbb{P}$ has a lower bound.

5.9 Remark. If $\mathbb{P}$ is separative, then $\mathbb{P}$ is (κ, ∞)-distributive if and only if every $<\kappa$-sequence of dense open subsets of $\mathbb{P}$ has a nonempty intersection.

The following fact is easy but crucial. See [39, 20.5] for a proof.

5.10 Fact (Easton's Lemma). Let κ be a regular uncountable cardinal. Let $\mathbb{P}$ be κ-c.c. and let $\mathbb{Q}$ be κ-closed. Then

1. $\Vdash_{\mathbb{P}\times\mathbb{Q}}$ "$\check{\kappa}$ is a regular uncountable cardinal".

2. $\Vdash_{\mathbb{Q}}$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c.".

3. $\Vdash_{\mathbb{P}}$ "$\check{\mathbb{Q}}$ is $(\check{\kappa}, \infty)$-distributive".

It is sometimes useful to consider a stronger form of the κ-c.c. See Kunen and Tall's paper [48] for more information about the following property.

5.11 Definition. Let κ be an uncountable regular cardinal. A poset $\mathbb{P}$ is *κ-Knaster* if and only if for every κ-sequence of conditions $\langle p_\alpha : \alpha < \kappa\rangle$ there is a set $X \subseteq \kappa$ unbounded such that $\langle p_\alpha : \alpha \in X\rangle$ consists of pairwise compatible conditions.

For example the standard Δ-system proof [46, Theorem 1.6] that the Cohen poset $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-c.c. actually shows that $\mathrm{Add}(\kappa, \lambda)$ is $(2^{<\kappa})^+$-Knaster. The following easy fact shows that the Knaster property is in some ways better behaved than the property of being κ-c.c. It is not in general the case that the product of two κ-c.c. posets is κ-c.c.

5.12 Fact. Let κ be regular and let $\mathbb{P}$, $\mathbb{Q}$ be two notions of forcing. Then

1. If $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster then $\mathbb{P}\times\mathbb{Q}$ is κ-Knaster.

2. If $\mathbb{P}$ is κ-c.c. and $\mathbb{Q}$ is κ-Knaster then $\mathbb{P}\times\mathbb{Q}$ is κ-c.c.

5.13 Remark. In general the property of being κ-Knaster is stronger than that of being κ-c.c. For example if T is an ω_1-Suslin tree then $(T, \geq)$ is ω_1-c.c. but is not ω_1-Knaster, by Fact 5.12 and the easy remark that $T \times T$ is not ω_1-c.c.

We will also need some properties intermediate between κ-closure and (κ, ∞)-distributivity, involving the idea of a game on a poset. This concept was introduced by Jech [37] and studied by Foreman [18] and Gray [30] among others.

5.14 Definition. Let $\mathbb{P}$ be a notion of forcing and let α be an ordinal. We define $G_\alpha(\mathbb{P})$, a two-player game of perfect information. Two players Odd and Even take turns to play conditions from $\mathbb{P}$ for α many moves, with Odd playing at odd stages and Even at even stages (including all limit stages). Even must play $1_\mathbb{P}$ at move zero. Let p_β be the condition played at move β; the player who played p_β loses immediately unless $p_\beta \leq p_\gamma$ for all $\gamma < \beta$. If neither player loses at any stage $\beta < \alpha$, then player Even wins.

5.15 Definition. Let $\mathbb{P}$ be a notion of forcing and let κ be a regular cardinal.

1. $\mathbb{P}$ is *$<\kappa$-strategically closed* if and only if for all $\alpha < \kappa$, player Even has a winning strategy for $G_\alpha(\mathbb{P})$.
2. $\mathbb{P}$ is *κ-strategically closed* if and only if player Even has a winning strategy for $G_\kappa(\mathbb{P})$.
3. $\mathbb{P}$ is *$(\kappa+1)$-strategically closed* if and only if player Even has a winning strategy for $G_{\kappa+1}(\mathbb{P})$, where we note that it is player Even who must make the final move.

5.16 Remark. More general forms of strategic closure have been studied [18] and are sometimes useful, but this one is sufficient for us.

5.17 Remark. It is not difficult to see that the conclusions of Lemma 5.10 remain true when we weaken the hypothesis of κ-closure to κ-strategic closure. This *strategic Easton lemma* is part of the folklore.

6. Some Forcing Posets

It is easy to see that every κ-directed closed poset is κ-closed, every κ-closed poset is κ-strategically closed, every κ-strategically closed poset is $<\kappa$-strategically closed and every $<\kappa$-strategically closed poset is (κ, ∞)-distributive. The following examples illustrate that these concepts are distinct, and will all find some use later in this chapter.

The first example shows that κ-closure does not in general imply κ-directed closure.

6.1 Example (Adding a Kurepa tree at an inaccessible cardinal). Recall that if κ is inaccessible then a *κ-Kurepa tree* is a normal tree of height κ such that

- $|T_\alpha| \leq |\alpha| + \omega$ for $\alpha < \kappa$.
- T has at least κ^+ cofinal branches.

Devlin's book about constructibility [16] contains more information about Kurepa trees, including a discussion of when such trees exist in L. We note that if κ is *ineffable* (ineffability is a large cardinal axiom intermediate between weak compactness and measurability) then there is no κ-Kurepa tree, and that in L there is such a tree for every non-ineffable inaccessible κ.

6.2 Remark. It might have seemed more natural generalize the definition of a Kurepa tree to inaccessible κ by dropping the first condition and requiring only that it be a κ-tree with more than κ cofinal branches. But this would be uninteresting because the complete binary tree of height κ is always such a tree.

6.3 Remark. It is very easy to see that there is no κ-Kurepa tree for κ measurable. For if T is such a tree and $j : V \longrightarrow M$ is elementary with critical point κ, then the map which takes each cofinal branch b to the unique point of $j(b)$ on level κ is one-to-one, so in M level κ of $j(T)$ has more than κ points.

Given κ inaccessible we define a forcing poset $\mathbb{P}$ to add a κ-Kurepa tree. Conditions are pairs (t, f) where

1. t is a normal tree of height $\beta + 1$ for some $\beta < \kappa$.

2. $|t_\alpha| \leq |\alpha| + \omega$ for all $\alpha \leq \beta$.

3. f is a function with $\mathrm{dom}(f) \subseteq \kappa^+$, $\mathrm{ran}(f) = t_\beta$ and $|\mathrm{dom}(f)| \leq |\beta| + \omega$.

Intuitively $f(\delta)$ is supposed to be the point in which branch δ meets t_β. Accordingly we say that $(u, g) \leq (t, f)$ if and only if

1. t is an initial segment of u.

2. $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$.

3. For all $\delta \in \mathrm{dom}(f)$, $f(\delta) \leq_u g(\delta)$.

It is easy to see that $\mathbb{P}$ is κ-closed and κ^+-c.c. and that $\mathbb{P}$ adds a κ-Kurepa tree. We claim that $\mathbb{P}$ is not κ-directed closed. To see this, let $\{x_\alpha : \alpha < 2^\omega\}$ enumerate ${}^\omega 2$ and let S be the family of conditions (t, f) such that $t = {}^n 2$ for some finite n, f has domain a countable subset of 2^ω and $f(\delta) = x_\delta \restriction n$ for all $\delta \in \mathrm{dom}(f)$. S is directed and $|S| = 2^\omega < \kappa$. However S cannot have a lower bound because, if (t, g) is a lower bound for S then t must have 2^ω points on level ω.

6.4 Remark. Similar arguments show that $\mathbb{P}$ has no dense κ-directed closed dense subset, and is not κ-directed closed below any condition. We will see in Theorem 24.12 that it is consistent for there to exist a measurable cardinal κ whose measurability is preserved by any κ-directed closed forcing, while by contrast forcing with the κ-closed poset $\mathbb{P}$ always destroys the measurability of κ.

Our next example shows that in general κ-strategic closure is a weaker property than κ-closure.

6.5 Example (Adding a non-reflecting stationary set). Let $\kappa = \mathrm{cf}(\kappa) \geq \omega_2$. We define a forcing poset $\mathbb{P}$ which aims to add a *non-reflecting stationary set of cofinality ω ordinals in κ*, that is to say a stationary $S \subseteq \kappa \cap \mathrm{Cof}(\omega)$ such that $S \cap \alpha$ is non-stationary for all $\alpha \in \kappa \cap \mathrm{Cof}(> \omega)$. $p \in \mathbb{P}$ if and only if p is a function such that

1. $\mathrm{dom}(p) < \kappa$, $\mathrm{ran}(p) \subseteq 2$.
2. If $p(\alpha) = 1$, $\mathrm{cf}(\alpha) = \omega$.
3. If $\beta \leq \mathrm{dom}(p)$ and $\mathrm{cf}(\beta) > \omega$ then there exists a set $c \subseteq \beta$ club in β such that $\forall \alpha \in c\; p(\alpha) = 0$.

It is easy to see that $\mathbb{P}$ is countably closed, and that it adds the characteristic function of a stationary subset of κ. It is also easy to see that if we let S be any stationary set of limit ordinals in ω_1, let $\chi_S : \omega_1 \to 2$ be the characteristic function of S, and define $p_\alpha = \chi_S \restriction \alpha$ for $\alpha < \omega_1$, then $\langle p_\alpha : \alpha < \omega_1 \rangle$ is a decreasing sequence of conditions in $\mathbb{P}$ with no lower bound and so $\mathbb{P}$ fails to be ω_2-closed.

We now claim that $\mathbb{P}$ is κ-strategically closed, which we will prove by exhibiting a winning strategy for Even. At stage α Even will compute $\gamma_\alpha = \mathrm{dom}(\bigcup_{\beta<\alpha} p_\beta)$, and will then define p_α by setting $\mathrm{dom}(p_\alpha) = \gamma_\alpha + 1$, $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta<\alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = 0$. This strategy succeeds because at every limit stage β of uncountable cofinality the set $\{\gamma_\alpha : \alpha < \beta\}$ is club in γ_β, and Even has ensured that p_β is 0 at every point of this club set.

The following example shows that in general the property of $<\kappa$-strategic closure is weaker than that of κ-strategic closure. The forcing is due to Jensen.

6.6 Example (Adding a square sequence). Let λ be an uncountable cardinal. Recall that a *$\square_\lambda$-sequence* is a sequence $\langle C_\alpha : \alpha \in \lambda^+ \cap \mathrm{Lim} \rangle$ such that for all α

1. C_α is club in α.
2. $\mathrm{ot}(C_\alpha) \leq \lambda$.
3. $\forall \beta \in \lim(C_\alpha)\; C_\alpha \cap \beta = C_\beta$.

We define a forcing poset $\mathbb{P}$ to add such a sequence. Conditions are initial segments of successor length of such a sequence and the ordering is extension. More formally $p \in \mathbb{P}$ iff

- $\mathrm{dom}(p) = (\beta + 1) \cap \mathrm{Lim}$ for some $\beta \in \lambda^+ \cap \mathrm{Lim}$.
- $p(\alpha)$ is club in α, $\mathrm{ot}(p(\alpha)) \leq \lambda$ for all $\alpha \in \mathrm{dom}(p)$.
- If $\alpha \in \mathrm{dom}(p)$ then $\forall \beta \in \lim p(\alpha)\; p(\alpha) \cap \beta = p(\beta)$.

If $p, q \in \mathbb{P}$ then $q \le p$ if and only if $p = q \restriction \mathrm{dom}(p)$.

It can be checked that $\mathbb{P}$ is $<\lambda^+$-strategically closed, so that $\mathbb{P}$ preserves cardinals up to λ^+ and adds a $\square_\lambda$-sequence. The author's joint paper with Foreman and Magidor [14] has a detailed discussion of the poset $\mathbb{P}$ and several variations.

We claim that $\mathbb{P}$ is not in general λ^+-strategically closed. To see this we observe that if player Even can win $G_{\lambda^+}(\mathbb{P})$, then the union of the sequence of the moves in a winning play is actually a $\square_\lambda$-sequence. So if $\square_\lambda$ fails then $\mathbb{P}$ is not λ^+-strategically closed. Ishiu and Yoshinobu [36] have observed that the principle $\square_\lambda$ is in fact equivalent to the λ^+-strategic closure of $\mathbb{P}$.

6.7 Remark. The difference between the last two examples is essentially that "S is a stationary subset of κ" is a second-order statement in the structure (H_κ, S) while "$\vec{C}$ is a $\square_\lambda$-sequence" is a first-order statement in the structure $(H_{\lambda^+}, \vec{C})$. This difference was exploited in [9].

Our final example shows that in general (κ, ∞)-distributivity is weaker than $<\kappa$-strategic closure. This forcing is due to Baumgartner, Harrington and Kleinberg [7].

6.8 Example (Killing a stationary subset of ω_1). Let $S \subseteq \omega_1$ be stationary and co-stationary. We define a forcing $\mathbb{P}$ to destroy the stationarity of S. The conditions in $\mathbb{P}$ are the closed bounded subsets c of ω_1 ordered by end-extension such that $c \cap S = \emptyset$.

We claim that $\mathbb{P}$ is (ω_1, ∞)-distributive. To see this let $\vec{D} = \langle D_n : n < \omega \rangle$ be an ω-sequence of dense open sets and let $c \in \mathbb{P}$. Fix θ some large regular cardinal and $<_\theta$ a well-ordering of H_θ. Find an elementary substructure $N \prec (H_\theta, \in, <_\theta)$ such that

1. $c, \mathbb{P}, S, \vec{D} \in N$.
2. N is countable.
3. $N \cap \omega_1 \notin S$ (this is possible because S is co-stationary).

Let $\delta = N \cap \omega_1$ and fix $\langle \delta_n : n < \omega \rangle$ an increasing and cofinal sequence in δ. Now build a chain of conditions $\langle c_n : n < \omega \rangle$ as follows: $c_0 = c$ and c_{n+1} is the $<_\theta$-least condition d such that $d \le c_n$, $d \in D_n$ and $\max(d) \ge \delta_n$. An easy induction shows that $c_n \in N$, so in particular $\max(c_n) \in N \cap \omega_1 = \delta$. It follows that if $c_\infty = \bigcup_n c_n \cup \{\delta\}$ then $c_\infty \in \mathbb{P}$, and by construction $c_\infty \in D_n$ for all n.

On the other hand $\mathbb{P}$ is not $<\omega_1$-strategically closed. To see this we show that for any $\mathbb{Q}$, if Even wins $G_{\omega+1}(\mathbb{Q})$ then $\mathbb{Q}$ preserves stationary subsets of ω_1. Let σ be a winning strategy for Even in $G_{\omega+1}(\mathbb{Q})$. Let $T \subseteq \omega_1$ be stationary. Let $q \Vdash_{\mathbb{Q}}$ "$\dot{C}$ is club in ω_1" and let $q, \mathbb{Q}, T, \dot{C}, \sigma \in N \prec (H_\theta, \in, <_\theta)$, where N is countable with $\delta = N \cap \omega_1 \in T$. Let $\langle E_n : n < \omega \rangle$ enumerate the dense subsets of $\mathbb{Q}$ which lie in N.

Now consider a run $\langle q_n : n \le \omega \rangle$ of $G_{\omega+1}(\mathbb{Q})$ such that

1. $q_0 = 1_{\mathbb{P}}$ and $q_1 = q$.
2. Even plays according to σ.
3. For $n > 0$, q_{2n+1} is the $<_\theta$-least condition r such that $r \leq q_{2n}$ and $r \in E_{n-1}$.

It is easy to see that $q_n \in \mathbb{Q} \cap N$ for $n < \omega$. The condition q_ω forces that $\delta \in \lim(\dot{C})$, so $q_\omega \Vdash \delta \in \dot{C} \cap \check{T}$ and we have shown that the stationarity of T is preserved.

6.9 Remark. The question of preservation of stationarity by forcing is one to which we will return several times in this chapter. The argument of Example 6.8 shows that for any ordinal λ of uncountable cofinality, any stationary $S \subseteq \lambda \cap \mathrm{Cof}(\omega)$ and any $(\omega+1)$-strategically closed $\mathbb{Q}$, forcing with $\mathbb{Q}$ preserves the stationarity of S. The situation is more complex for uncountable cofinalities, because if we build a structure N as in the last part of Example 6.8 and then try to build a suitable chain of conditions in N, we may in general wander out of N after ω steps. We will return to this topic in Lemma 10.6 and the proof of Theorem 14.10.

It will be convenient to fix some notation for the kind of forcing poset constructed in Example 6.8.

6.10 Definition. Let κ be a regular cardinal and let T be a stationary subset of κ. Then $\mathrm{CU}(\kappa, T)$ is the forcing poset whose conditions are closed bounded subsets of T, ordered by end-extension.

The poset of Example 6.8 is $\mathrm{CU}(\omega_1, \omega_1 \setminus S)$. For $\kappa > \omega_1$ the poset $\mathrm{CU}(\kappa, T)$ may not be well-behaved, in particular it may collapse cardinals; consider for example the situation where $\kappa = \omega_2$ and $T = \omega_2 \cap \mathrm{Cof}(\omega_1)$. See Sect. 18 for a detailed discussion of this issue.

7. Iterated Forcing

In this section we review the definition of iterated forcing and some basic facts about iterated forcing constructions. We will basically follow Baumgartner's survey paper [6] in our treatment of iterated forcing. Many readers may have learned iterated forcing from the excellent account in Kunen's book [46], and for their benefit we point out that there is one rather significant difference between the Baumgartner and Kunen treatments.

This involves the precise definition of a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P}$ is a notion of forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing. In [46] the elements of $\mathbb{P} * \dot{\mathbb{Q}}$ are all pairs $(p, \dot{q})$ such that $p \in \mathbb{P}$ and $\dot{\mathbb{Q}}$ contains some pair of the form $(\dot{q}, r)$; Baumgartner [6] adopts a more liberal definition in which $\dot{q}$ is chosen from some set X of $\mathbb{P}$-names such that every $\mathbb{P}$-name for a member of $\dot{\mathbb{Q}}$ is forced to be equal to some name in X. This distinction makes for some (essentially trivial) differences in the theory, for example it is possible with the definition from [46] that $\mathbb{P}$ is countably closed and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is countably closed" but $\mathbb{P} * \dot{\mathbb{Q}}$ is not countably closed.

In the interests of precision we make the following definition, which really amounts to specifying the set of names X from the last paragraph.

7.1 Definition. Let $\mathbb{P}$ be a notion of forcing.

- A $\mathbb{P}$-name $\dot{x}$ is *canonical* iff there is no $\dot{y}$ such that $|\operatorname{tc}(\dot{y})| < |\operatorname{tc}(\dot{x})|$ and $\Vdash_{\mathbb{P}} \dot{x} = \dot{y}$.
- If $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing then $\mathbb{P} * \dot{\mathbb{Q}}$ is the set of all pairs such that $p \in \mathbb{P}$, $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$ and $\dot{q}$ is canonical.

The advantage of this convention will be that we get equality rather than just isomorphism in statements like Lemma 12.10 below.

We recall the standard facts about two step iterations:

1. $\mathbb{P} * \dot{\mathbb{Q}}$ is ordered as follows: $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ if and only if $p_0 \leq p_1$ and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$.
2. There is a bijection between V-generic filters for $\mathbb{P} * \dot{\mathbb{Q}}$ and pairs (G, H) where G is V-generic for $\mathbb{P}$, and H is $V[G]$-generic for $i_G(\dot{Q})$.

As we mentioned above we will follow the treatment of iterated forcing from Baumgartner's survey paper [6]. We give a brief review. We make the convention that whenever we have a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a notion of forcing, $\dot{1}_{\mathbb{Q}}$ names the specified largest element of $\dot{\mathbb{Q}}$.

An iteration of length α is officially an object of the form

$$(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle)$$

where for every $\beta \leq \alpha$

- $\mathbb{P}_\beta$ is a notion of forcing whose elements are β-sequences.
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$ then $p \restriction \gamma \in \mathbb{P}_\gamma$.
- If $\beta < \alpha$ then $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathbb{Q}}_\beta$ is a notion of forcing".
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, then $p(\gamma)$ is a $\mathbb{P}_\gamma$-name for an element of $\dot{\mathbb{Q}}_\gamma$.
- If $\beta < \alpha$ then $\mathbb{P}_{\beta+1} \simeq \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, via the map which takes $h \in \mathbb{P}_{\beta+1}$ to $(h \restriction \beta, h(\beta))$.
- If $p, q \in \mathbb{P}_\beta$ then $p \leq_{\mathbb{P}_\beta} q$ iff $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \leq_{\dot{\mathbb{Q}}_\gamma} q(\gamma)$ for all $\gamma < \beta$.
- $1_{\mathbb{P}_\beta}(\gamma) = \dot{1}_{\mathbb{Q}_\gamma}$ for all $\gamma < \beta$.
- If $p \in \mathbb{P}_\beta$, $\gamma < \beta$ and $q \leq_{\mathbb{P}_\gamma} p \restriction \gamma$ then $q ^\frown p \restriction [\gamma, \beta) \in \mathbb{P}_\beta$.

In a standard abuse of notation we will sometimes use "$\mathbb{P}_\alpha$" as a shorthand for the iteration $(\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle)$. We usually write "$\Vdash_\beta$" for "$\Vdash_{\mathbb{P}_\beta}$".

The key points in the definition of iteration are that if G_α is $\mathbb{P}_\alpha$-generic over V and $\beta < \alpha$ then

- $G_\beta =_{\mathrm{def}} \{p \restriction \beta : p \in G_\alpha\}$ is $\mathbb{P}_\beta$-generic over V.
- $g_\beta =_{\mathrm{def}} \{i_{G_\beta}(p(\beta)) : p \in G_\alpha\}$ is $i_{G_\beta}(\dot{\mathbb{Q}}_\beta)$-generic over $V[G_\beta]$.

7.2 Remark. It is sometimes useful to weaken the conditions in the definition of iteration and to admit as a forcing iteration any pair $(\vec{\mathbb{P}}, \vec{\mathbb{Q}})$ where

1. $\mathbb{P}_\beta$ is a forcing poset whose conditions are β-sequences.
2. $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing poset.
3. $\mathbb{P}_{\beta+1} \simeq \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, via the map which takes $h \in \mathbb{P}_{\beta+1}$ to $(h \restriction \beta, h(\beta))$.
4. The restriction map from $\mathbb{P}_\beta$ to $\mathbb{P}_\gamma$ for $\gamma < \beta$ is a projection.

Note that the "key properties" from the last paragraph will still be true in this setting. An important example is Prikry iteration with Easton support (see [22]) which are iterations in this more general sense.

7.3 Remark. Some arguments which we need to do subsequently work most smoothly with forcing posets which are separative partial orderings. The poset absorption argument of Sect. 14 is an example. In our definition of iterated forcing $\mathbb{P}_\alpha$ is just a notion of forcing. However it is routine to check that if we form an iteration such that each factor $\dot{\mathbb{Q}}_i$ is forced to be a separative partial ordering, then the quotient poset of $\mathbb{P}_\alpha$ is separative. We will sometimes blur the distinction between the preordering $\mathbb{P}_\alpha$ and its associated quotient partial ordering.

If $p \in \mathbb{P}_\alpha$ then the *support of* p ($\mathrm{supp}(p)$) is $\{\beta < \alpha : p(\beta) \neq \dot{1}_{\mathbb{Q}_\beta}\}$.

Let λ be a limit ordinal and let an iteration of length λ be given. We define the *inverse limit* $\varprojlim \vec{\mathbb{P}} \restriction \lambda$ to be the set of sequences p of length λ such that $\forall \alpha < \lambda \; p \restriction \alpha \in \mathbb{P}_\alpha$. The *direct limit* $\varinjlim \vec{\mathbb{P}} \restriction \lambda$ is the subset of the inverse limit consisting of those p such that $p(\alpha) = \dot{1}_{\mathbb{Q}_\alpha}$ for all sufficiently large α. The definition of a forcing iteration implies that if we have an iteration of length greater than λ then

$$\varinjlim \vec{\mathbb{P}} \restriction \lambda \subseteq \mathbb{P}_\lambda \subseteq \varprojlim \vec{\mathbb{P}} \restriction \lambda.$$

To specify a forcing iteration it will suffice to describe the names $\dot{\mathbb{Q}}_\beta$ and to give a procedure for computing $\mathbb{P}_\lambda$ for λ limit. In many iterations the only kinds of limit which are used are direct and inverse ones.

7.4 Remark. Let κ be inaccessible, and suppose that we have an iteration of length κ where $\dot{\mathbb{Q}}_\beta \in V_\kappa$ for all $\beta < \kappa$ and a direct limit is taken at stage κ. Then

- $\mathbb{P}_\beta \subseteq V_\kappa$ for all $\beta < \kappa$.
- While it is not literally true that $\mathbb{P}_\kappa \subseteq V_\kappa$, for every $p \in \mathbb{P}_\kappa$ there exist $\beta < \kappa$ and $q \in \mathbb{P}_\beta$ such that $p(\alpha) = q(\alpha)$ for $\alpha < \beta$, $p(\alpha) = \dot{1}_{\mathbb{Q}_\alpha}$ for $\beta \leq \alpha < \kappa$. We will often blur the distinction between $\mathbb{P}_\kappa$ and $\bigcup_\alpha \mathbb{P}_\alpha$, which actually is a subset of V_κ.

7.5 Definition. If κ is regular then an *iteration with $<\kappa$-support* is an iteration in which direct limits are taken at limit stages of cofinality greater than or equal to κ, and inverse limits are taken at limit stages of cofinality less than κ. An *iteration with Easton support* is an iteration in which direct limits are taken at regular limit stages and inverse limits are taken elsewhere.

As this terminology would suggest, the support of a condition in an iteration with $<\kappa$-support has size less than κ. The support of a condition in an Easton iteration is an *Easton set*, that is to say a set of ordinals which is bounded in every regular cardinal.

The following are a few key facts about two-step iterations. Proofs are given in [6, Sect. 2] for 1 and 2, while the proofs for 3 are easy variations of the proof for 2.

7.6 Proposition. *Let $\kappa = \mathrm{cf}(\kappa) > \omega$, let $\mathbb{P}$ be a notion of forcing and let $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a notion of forcing".*

1. *$\mathbb{P} * \dot{\mathbb{Q}}$ is κ-c.c. iff $\mathbb{P}$ is κ-c.c. and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-c.c.".*
2. *If $\mathbb{P}$ is κ-closed and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-closed" then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-closed.*
3. *Let X be any of the properties "κ-directed closed", "$<\kappa$-strategically closed" or "κ-strategically closed". If $\mathbb{P}$ is X and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is X" then $\mathbb{P} * \dot{\mathbb{Q}}$ is X.*

7.7 Proposition. *Let κ be inaccessible and $\mathbb{P} \in V_\kappa$. If $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \dot{V}_\kappa$ then $\mathbb{P} * \dot{\mathbb{Q}} \in V_\kappa$, and if $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \subseteq V_\kappa$ then $\mathbb{P} * \dot{\mathbb{Q}} \subseteq V_\kappa$.*

In general the preservation of chain condition in iterations is a very delicate question. See [6, Sect. 4] and [62] to get an idea of the difficulties surrounding preservation of the ω_2-c.c. by countable support iterations. Fortunately we can generally get away with some comparatively crude arguments.

The following fact is proved in [6, Sect. 2].

7.8 Proposition. *Let α be limit and let $\mathbb{P}_\alpha = \varinjlim \vec{\mathbb{P}} \restriction \alpha$. Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Suppose that*

- *For every $\beta < \alpha$, $\mathbb{P}_\beta$ is κ-c.c.*
- *If* $\mathrm{cf}(\alpha) = \kappa$ *then* $\{\gamma < \alpha : \mathbb{P}_\gamma = \varinjlim \vec{\mathbb{P}} \restriction \gamma\}$ *is stationary in α.*

Then $\mathbb{P}_\alpha$ is κ-c.c.

The following fact is proved in [6, Sect. 2] for the case when X equals "κ-closed". The proofs for the other closure properties are similar.

7.9 Proposition. *Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Let X be any of the properties "κ-closed", "κ-directed closed", "$<\kappa$-strategically closed" or "κ-strategically closed". Suppose that*

- $\Vdash_\beta$ *"$\dot{\mathbb{Q}}_\beta$ is X" for $\beta < \alpha$.*
- *All limits are direct or inverse, and inverse limits are taken at every limit stage with cofinality less than κ.*

Then $\mathbb{P}_\alpha$ is X.

7.10 Remark. The moral of Proposition 7.8 is that one should take many direct limits to preserve chain condition properties; the moral of Proposition 7.9 is that one should take many inverse limits to preserve closure properties.

We will also need to analyze the quotient of an iteration by some initial segment. Once again we quote from [6, Sect. 5].

7.11 Proposition. *If $\beta < \alpha$ then there exists a term $\dot{\mathbb{R}}_{\beta,\alpha} \in V^{\mathbb{P}_\beta}$ such that*

1. $\Vdash_\beta$ *"$\mathbb{R}_{\beta,\alpha}$ is an iteration of length $\alpha - \beta$".*
2. *There is a dense subset of $\mathbb{P}_\beta * \dot{\mathbb{R}}_{\beta,\alpha}$ which is isomorphic to $\mathbb{P}_\alpha$.*

The definition of the iteration $\mathbb{R}_{\beta,\alpha}$ is simple at successor stages; we translate $\dot{\mathbb{Q}}_\gamma$ in the canonical way to a $\mathbb{P}_\beta$-name for an $\mathbb{R}_{\beta,\gamma}$-name for a notion of forcing and force with that poset at stage $\gamma - \alpha$. Limits are trickier because while a direct limit in V still looks like a direct limit in $V^{\mathbb{P}_\beta}$ the same may not be true in general of an inverse limit. We will usually write $\mathbb{P}_\alpha / G_\beta$ for $i_{G_\beta}(\dot{\mathbb{R}}_{\beta,\alpha})$.

The following proposition is proved in [6, Sect. 5] for the case when X equals "κ-closed", and once again can be proved in a very similar way for the other closure properties.

7.12 Proposition. *Let $\kappa = \mathrm{cf}(\kappa) > \omega$. Let X be any of the properties "κ-closed", "κ-directed closed", "$<\kappa$-strategically closed" or "κ-strategically closed". Let $\mathbb{P}_\alpha$ be an iteration of length α in which all limits are inverse or direct. Let $\beta < \alpha$ and suppose that*

1. *$\mathbb{P}_\beta$ is such that every set of ordinals of size less than κ in $V[G_\beta]$ is covered by a set of size less than κ in V.*

2. *For* $\beta \leq \gamma < \alpha$, $\Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma$ *is* X".

3. *Inverse limits are taken at all limit* γ *such that* $\beta \leq \gamma < \alpha$ *and* $\mathrm{cf}(\gamma) < \kappa$.

Then $\Vdash_\beta$ "$\dot{\mathbb{R}}_{\beta,\alpha}$ *is* X".

The following result is easily proved by the methods of [6, Sect. 2].

7.13 Proposition. *Let* κ *be inaccessible and let* $\mathbb{P}_\kappa$ *be an iteration of length* κ *such that*

1. $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \in V_\kappa$ *for all* $\alpha < \kappa$.

2. *A direct limit is taken at* κ *and on a stationary set of limit stages below* κ.

Then

- $\mathbb{P}_\kappa$ *is* κ*-Knaster and has cardinality* κ.
- *If* $\delta < \kappa$ *then in* $V[G_\delta]$ *the quotient forcing* $\mathbb{R}_{\delta,\kappa}$ *is a* κ*-Knaster and has cardinality* κ.

7.14 Remark. The hypothesis 2 of Proposition 7.13 will be satisfied if the iteration is done with $<\lambda$-support for some $\lambda < \kappa$, and also if the iteration is done with Easton support and κ is Mahlo.

8. Building Generic Objects

A crucial fact about forcing is that if M is a countable transitive model of set theory and $\mathbb{P} \in M$ is a notion of forcing, then there exist filters which are $\mathbb{P}$-generic over M. We give an easy generalization of this fact, which will be used very frequently in the constructions to follow.

8.1 Proposition. *Let* M *and* N *be two inner models with* $M \subseteq N$ *and let* $\mathbb{P} \in M$ *be a non-trivial notion of forcing. Let* $\mathcal{A}$ *be the set of* $A \in M$ *such that* A *is an antichain of* $\mathbb{P}$, *and note that* $\mathcal{A} \in M$.

Let $p \in \mathbb{P}$ *and let* λ *be an* N*-cardinal. If*

$$N \models \text{“}\mathbb{P} \textit{ is } \lambda\textit{-strategically closed and } |\mathcal{A}| \leq \lambda\text{”}$$

then there is a set in N *of* N*-cardinality* 2^λ *of filters on* $\mathbb{P}$, *each one of which contains* p *and is generic over* M.

Proof. We work in N. Let $\langle A_\alpha : \alpha < \lambda\rangle$ enumerate $\mathcal{A}$. Let σ be a winning strategy for player Even in the game $G_\lambda(\mathbb{P}/p)$.

We now build a binary tree $\langle p_s : s \in {}^{<\lambda}2\rangle$ of conditions in $\mathbb{P}/p$ such that

1. $p_{\langle\rangle} = p$.

2. If $\mathrm{lh}(s)$ is even, say $\mathrm{lh}(s) = 2\alpha$, then $p_{s\frown 0}$ and $p_{s\frown 1}$ are incompatible and each of them refines some element of A_α.

3. If $\mathrm{lh}(s) = 2(1+\alpha)$, then p_s is the response dictated by σ at move 2α in the run of the game $G_\lambda(\mathbb{P}/p)$ where $p_{s\restriction(2+i)}$ is played at move i for $i < 2\alpha$.

Then every branch generates a generic filter, and any two branches contain incompatible elements so generate distinct filters. ⊣

The following easy propositions will be useful in applications of Proposition 8.1.

8.2 Proposition. *Let M and N be inner models of* ZFC *such that $M \subseteq N$. Let $N \models$ "κ is a regular uncountable cardinal". Then $N \models {}^{<\kappa}M \subseteq M$ if and only if $N \models {}^{<\kappa}\mathrm{On} \subseteq M$.*

8.3 Proposition. *Let M and N be inner models of* ZFC *such that $M \subseteq N$. Let $N \models$ "κ is a regular uncountable cardinal" and let $N \models {}^{<\kappa}M \subseteq M$. Let $\mathbb{P} \in M$ be a notion of forcing and let X be any of the properties "κ-directed closed", "κ-closed", "κ-strategically closed" and "$<\kappa$-strategically closed".*

If $M \models$ "$\mathbb{P}$ is X" then $N \models$ "$\mathbb{P}$ is X".

8.4 Proposition. *Let M and N be inner models of* ZFC *with $M \subseteq N$ and let $\mathbb{P} \in M$ be a notion of forcing.*

1. *If $N \models {}^{<\lambda}M \subseteq M$, $N \models$ "$\mathbb{P}$ is λ-c.c." and G is $\mathbb{P}$-generic over N then $N[G] \models {}^{<\lambda}M[G] \subseteq M[G]$.*

2. *If $V_\lambda \cap M = V_\lambda \cap N$ and*

$$N \models \text{“Every canonical } \mathbb{P}\text{-name for a member of } V_\lambda^{N^{\mathbb{P}}} \text{ is in } V_\lambda\text{”}$$

then $V_\lambda \cap M[G] = V_\lambda \cap N[G]$.

We digress from our main theme to give a sample application of Proposition 8.1, namely, building a generalized version of Prikry forcing.

8.5 Lemma. *Let κ be measurable with $2^\kappa = \kappa^+$, and let U be a normal measure on κ. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map constructed from U, and let $\mathbb{Q} = \mathrm{Col}(\kappa^{++}, {<}j(\kappa))_M$. Then there is a filter $g \in V$ which is $\mathbb{Q}$-generic over M.*

Proof. In M, $\mathbb{Q}$ is a forcing of size $j(\kappa)$ which is $j(\kappa)$-c.c. Since $j(\kappa)$ is measurable in M it is surely inaccessible in M, and so

$$M \models \text{“}\mathbb{Q} \text{ has } j(\kappa) \text{ maximal antichains”}.$$

By Proposition 4.5 $V \models |j(\kappa)| = 2^\kappa = \kappa^+$, so

$$V \models \text{“}\mathbb{Q} \text{ has } \kappa^+ \text{ maximal antichains lying in } M\text{”}.$$

Clearly $M \models$ "$\mathbb{Q}$ is κ^{++}-closed", and $V \models {}^{\kappa}M \subseteq M$. By Proposition 8.3 it follows that $V \models$ "$\mathbb{Q}$ is κ^{+}-closed". Applying Proposition 8.1 we may therefore construct $g \in V$ which is $\mathbb{Q}$-generic over M. ⊣

The forcing we are about to describe is essentially a special case of the forcing $\mathbb{P}^{\pi}$ from Foreman and Woodin's paper on failure of GCH everywhere [20], and is also implicitly present in Magidor's work on failure of SCH [53]. We learned this presentation from Woodin.

8.6 Example. Let κ be measurable with $2^{\kappa} = \kappa^{+}$. Then there is a κ^{+}-c.c. poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \check{\kappa} = \dot{\omega}_{\omega}$.

Sketch of Proof. Let U, $\mathbb{Q}$ and g be as in Lemma 8.5. Conditions in $\mathbb{P}$ have the form $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ where

- The κ_i are inaccessible with $\kappa_1 < \cdots < \kappa_n < \kappa$.
-
  - $p_0 \in \mathrm{Col}(\omega_2, {<}\kappa_1)$.
  - $p_i \in \mathrm{Col}(\kappa_i^{++}, {<}\kappa_{i+1})$ for $0 < i < n$.
  - $p_n \in \mathrm{Col}(\kappa_n, {<}\kappa)$.
- H is a function such that $\mathrm{dom}(H) \in U$, $H(\alpha) \in \mathrm{Col}(\alpha^{++}, {<}\kappa)$ for $\alpha \in \mathrm{dom}(H)$ and $[H]_U \in g$.

We refer to n as the *length* of this condition.

Intuitively H constrains the possibilities for adding in new objects in the same way as the measure one set constrains new points in Prikry forcing. Formally $(q_0, \lambda_1, q_1, \ldots, \lambda_m, q_m, I)$ extends $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ iff

- $m \geq n$.
- For every $i \leq n$, $\lambda_i = \kappa_i$ and q_i extends p_i.
- For every i with $n < i \leq m$, $\lambda_i \in \mathrm{dom}(H)$ and $q_i \leq H(\lambda_i)$.
- $\mathrm{dom}(I) \subseteq \mathrm{dom}(H)$ and $I(\lambda)$ extends $H(\lambda)$ for every $\lambda \in \mathrm{dom}(I)$.

The second condition will be called a *direct* extension of the first if and only if $m = n$.

It is easy to see that $\mathbb{P}$ is κ^{+}-c.c. because any two elements in g are compatible. The poset $\mathbb{P}$ adds an increasing ω-sequence $\langle \kappa_i : i < \omega \rangle$ cofinal in κ (which is actually a Prikry-generic sequence for the measure U) and a sequence $\langle g_i : i < \omega \rangle$ where g_i is $\mathrm{Col}(\kappa_i^{++}, {<}\kappa_{i+1})$-generic over V.

The key lemma about $\mathbb{P}$ is that any statement in the forcing language can be decided by a direct extension. This is proved by an argument very similar to that for Prikry forcing. It can then be argued as in Magidor's paper [53] that below κ only the cardinals in the intervals $(\kappa_i^{++}, \kappa_{i+1})$ have collapsed. Thus $\mathbb{P}$ is a κ^{+}-c.c. forcing poset which makes κ into the ω_ω of the extension. ⊣

9. Lifting Elementary Embeddings

A key idea in this chapter is that it is sometimes possible to take an elementary embedding of a model of set theory and extend it to an embedding of some generic extension of that model. This idea goes back to Silver's consistency proof for the failure of GCH at a measurable, a proof which we will outline in Sect. 12.

9.1 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC*. Let $\mathbb{P} \in M$ be a notion of forcing, let G be $\mathbb{P}$-generic over M and let H be $k(\mathbb{P})$-generic over N. The following are equivalent:*

1. *$\forall p \in G\ k(p) \in H$.*
2. *There exists an elementary embedding $k^+ : M[G] \longrightarrow N[H]$, such that $k^+(G) = H$ and $k^+ \restriction M = k$.*

Proof. Clearly the second statement implies the first one. For the converse let $k``G \subseteq H$ and attempt to define k^+ by

$$k^+(i_G(\dot{\tau})) = i_H(k(\dot{\tau})).$$

To check that k^+ is well-defined, let $i_G(\dot{\sigma}) = i_G(\dot{\tau})$ and fix $p \in G$ such that $p \Vdash^M_{\mathbb{P}} \dot{\sigma} = \dot{\tau}$. Now by elementarity $k(p) \Vdash^N_{k(\mathbb{P})} k(\dot{\sigma}) = k(\dot{\tau})$, and since $k(p) \in H$ we have $i_H(k(\dot{\sigma})) = i_H(k(\dot{\tau}))$.

A similar proof shows that k^+ is elementary. If $x \in M$ and $\check{x}$ is the standard $\mathbb{P}$-name for x then $k(\check{x})$ is the standard $k(\mathbb{P})$-name for $k(x)$ and so $k^+(x) = k^+(i_G(\check{x})) = i_H(k(\check{x})) = k(x)$. Similarly if $\dot{G}$ is the standard $\mathbb{P}$-name for the $\mathbb{P}$-generic filter then $k(\dot{G})$ is the standard $k(\mathbb{P})$-name for the $k(\mathbb{P})$-generic filter, and so $k^+(G) = H$. ⊣

The following propositions give some useful structural information about the lifted embedding k^+. Recall that we defined the *width* and *support* of an embedding in Definitions 3.7 and 3.8.

9.2 Proposition. *Let $k : M \longrightarrow N$ be an elementary embedding between transitive models of* ZFC *and let G, H, $k^+ : M[G] \longrightarrow N[H]$ be as in Proposition 9.1. Then $N \cap \operatorname{ran}(k^+) = \operatorname{ran}(k)$.*

Proof. Let $y \in N$ with $y = k^+(x)$ for some $x \in M[G]$. If $\alpha = 1 + \operatorname{rk}(x)$ then by elementarity $y \in V^N_{k(\alpha)}$. Since k^+ extends k and k is an elementary embedding, $k^+(V^M_\alpha) = k(V^M_\alpha) = V^N_{k(\alpha)}$. So $k^+(x) \in k^+(V^M_\alpha)$, and since k^+ is elementary $x \in V^M_\alpha$. So $x \in M$ and $y = k^+(x) = k(x)$, thus $y \in \operatorname{ran}(k)$. ⊣

9.3 Proposition. *Let $k : M \longrightarrow N$, G, H, $k^+ : M[G] \longrightarrow N[H]$ be as in Proposition 9.1. If k has width $\leq \mu$ then k^+ has width $\leq \mu$. If k is supported on A then k^+ is supported on A.*

Proof. Suppose first that k has width $\leq \mu$. Let $y \in N[H]$, so that $y = i_H(\dot{\tau})$ for some $k(\mathbb{P})$-name $\dot{\tau} \in N$. By our assumptions about k, $\dot{\tau} = k(F)(a)$ where $F \in M$, $a \in N$ and $M \models |\operatorname{dom}(F)| \leq \mu$. Without loss of generality we may assume that $F(x)$ is a $\mathbb{P}$-name for all $x \in \operatorname{dom}(F)$.

Now we define a function $F^* \in M[G]$ by setting $\operatorname{dom}(F^*) = \operatorname{dom}(F)$ and $F^*(x) = i_G(F(x))$ for all $x \in \operatorname{dom}(F)$. By elementarity

$$k^+(F^*)(a) = i_{k^+(G)}(k^+(F)(a)) = i_H(\dot{\tau}) = y.$$

Therefore k^+ has width $\leq \mu$. The argument for the property "supported on A" is very similar. ⊣

In Sect. 4 we gave characterizations of various large cardinal axioms in terms of *definable* elementary embeddings. When we apply Proposition 9.1 to a definable embedding it is likely that definability will be lost; the next section gives an example of this phenomenon.

One of our major themes is forcing iterations which preserve large cardinal axiom, so we would like to preserve definability when applying Proposition 9.1. This motivates the following proposition, where the key hypothesis for getting a definable embedding is that we are choosing $H \in V[G]$.

9.4 Proposition. *Let $\kappa < \lambda$ and let $j : V \longrightarrow M$ be an elementary embedding with critical point κ. Let $\mathbb{P} \in V$ be a notion of forcing, and let G be $\mathbb{P}$-generic over V. Let H be $j(\mathbb{P})$-generic over M with $j``G \subseteq H$, and let $j^+ : V[G] \longrightarrow M[H]$ be the unique embedding with $j^+ \restriction V = j$ and $j^+(G) = H$. Let $H \in V[G]$.*

1. *If there is in V a short V-(κ, λ)-extender E such that j is the ultrapower of V by E, then there is in $V[G]$ a short $V[G]$-(κ, λ)-extender E^* such that j^+ is the ultrapower of $V[G]$ by E^*. Moreover $E_a = E^*_a \cap V$ for all $a \in [\lambda]^{<\omega}$.*

2. *If there is in V a supercompactness measure U on $P_\kappa\lambda$ such that j is the ultrapower of V by U, then there is in $V[G]$ a supercompactness measure U^* on $P_\kappa\lambda$ such that j^+ is the ultrapower of $V[G]$ by U^*. Moreover $U = U^* \cap V$.*

In both cases j^+ is definable.

Proof. Assume first that j is the ultrapower by some (κ, λ)-extender E. For each $a \in [\lambda]^{<\omega}$ let μ_a be minimal with $a \subseteq j(\mu_a)$. Arguing exactly as in the proof of Proposition 9.3,

$$M[H] = \{j^+(F)(a) : a \in [\lambda]^{<\omega}, F \in V[G], \operatorname{dom}(F) = [\mu_a]^{|a|}\}.$$

If we now let E^* be the (κ, λ)-extender derived from j^+ then it follows easily from the equation above and Proposition 3.4 that $\operatorname{Ult}(V[G], E^*) = M[H]$ and

$j_{E^*} = j^+$. Since $H \in V[G]$ we see that $E^* \in V[G]$ and so j^+ is definable. Finally if $X \in V$ and $X \subseteq [\kappa]^{|a|}$ then $j(X) = j^+(X)$, so

$$X \in E_a \iff a \in j(X) \iff a \in j^+(X) \iff X \in E^*_a,$$

that is to say $E_a = E^*_a \cap V$.

The argument for j arising from a supercompactness measure is similar. ⊣

9.5 Remark. Either clause of Proposition 9.4 can be used to argue that κ is measurable in $V[G]$. By Remark 4.1 we may use the second clause to conclude without further work that κ is λ-supercompact in $V[G]$; preservation of some strength witnessed by E will need some argument about the resemblance between $V[G]$ and $M[H]$.

In what follows we will see a number of ways of arranging that k"$G \subseteq H$. We start by proving a classical result by Levy and Solovay (which implies in particular that standard large cardinal hypotheses cannot resolve the Continuum Hypothesis).

9.6 Theorem (Levy and Solovay [50]). *Let κ be measurable. Let $|\mathbb{P}| < \kappa$ and let G be $\mathbb{P}$-generic. Then κ is measurable in $V[G]$.*

Proof. Let U be a measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(M, U)$ be the ultrapower map. Without loss of generality $\mathbb{P} \in V_\kappa$, so that $j \restriction \mathbb{P} = \mathrm{id}_{\mathbb{P}}$ and $j(\mathbb{P}) = \mathbb{P}$. In particular j"$G = G$, and since $M \subseteq V$ and $\mathbb{P} \in M$ we have that G is $\mathbb{P}$-generic over M. Now by Proposition 9.1 we may lift j to get a new map $j^+ : V[G] \longrightarrow M[G]$. By Proposition 9.4 j^+ is definable in $V[G]$. j^+ extends j and so $\mathrm{crit}(j^+) = \mathrm{crit}(j) = \kappa$, and thus κ is measurable in $V[G]$. ⊣

9.7 Remark. Usually when we lift an embedding we will denote the lifted embedding by the same letter as the original one.

The Levy-Solovay result actually applies to most other popular large cardinal axioms, for example "κ is λ-strong" or "κ is λ-supercompact".

9.8 Theorem. *Let $|\mathbb{P}| < \kappa$ and let $\lambda > \kappa$. Then forcing with $\mathbb{P}$ preserves the statements "κ is λ-strong" and "κ is λ-supercompact".*

Proof. Without loss of generality $\mathbb{P} \in V_\kappa$. Let G be $\mathbb{P}$-generic over V.

Suppose that κ is λ-strong. Let $j : V \longrightarrow M$ be such that $\mathrm{crit}(j) = \kappa$ and $V_\lambda \subseteq M$. Build $j^+ : V[G] \longrightarrow M[G]$ as in Theorem 9.6. By Proposition 8.4 we see that $V_\lambda^{V[G]} \subseteq M[G]$. By Proposition 9.4 j^+ is definable in $V[G]$. Since $\mathrm{crit}(j^+) = \kappa$, κ is λ-strong in $V[G]$.

The argument for λ-supercompactness is analogous. ⊣

9.9 Remark. Levy and Solovay also showed that small forcing cannot create instances of measurability. We prove a generalization of this in Theorem 21.1. In Theorem 14.6 a forcing poset of size κ makes a non-weakly compact cardinal κ measurable. See Sect. 21 for more discussion of these matters.

10. Generic Embeddings

It is a common situation that in some generic extension $V[G]$ we are able to define an elementary embedding $j : V \longrightarrow M \subseteq V[G]$. Such embeddings are usually known as *generic embeddings*. Foreman's chapter in this Handbook contains a wealth of information about generic embeddings, e.g. about the following situation.

10.1 Example. If I is an ω_2-saturated ideal on ω_1 and U is generic for the poset of I-positive sets, then in $V[U]$ the ultrapower $\mathrm{Ult}(V, U)$ is well-founded and we get a map $j : V \longrightarrow M \subseteq V[U]$ with $\mathrm{crit}(j) = \omega_1$ and $j(\omega_1) = \omega_2$.

We now honor a promise made in Sect. 2. The embedding that we describe is a generic embedding with critical point ω_1 and is added by a very simple poset. See Theorems 14.6, 23.2 and 24.11 for some applications of generic embeddings added by more elaborate posets.

10.2 Theorem. *Let κ be measurable, let U be a normal measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$ and let G be $\mathbb{P}$-generic. There is a forcing poset $\mathbb{Q} \in M$ such that*

1. *For any H a $\mathbb{Q}$-generic filter over $V[G]$, j can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*
2. *If*
$$U_G = \{X \in P(\kappa) \cap V[G] : \kappa \in j_G(X)\},$$
then U_G is a $V[G]$-κ-complete $V[G]$-normal $V[G]$-ultrafilter on κ. Also $M[G * H] = \mathrm{Ult}(V[G], U_G)$ and j_G is the ultrapower map.

Proof. By elementarity $j(\mathbb{P}) = \mathrm{Col}(\omega, {<}j(\kappa))$. Let $\mathbb{Q}$ be the set of finite partial functions q from $\omega \times (j(\kappa) \setminus \kappa)$ to $j(\kappa)$ such that $q(n, \alpha) < \alpha$ for all $(n, \alpha) \in \mathrm{dom}(p)$, ordered by reverse inclusion. Clearly the map which sends p to $(p\restriction(\omega \times \kappa), p\restriction(\omega \times (j(\kappa) \setminus \kappa)))$ sets up an isomorphism in M between $j(\mathbb{P})$ and $\mathbb{P} \times \mathbb{Q}$.

Now let H be $\mathbb{Q}$-generic over $V[G]$, so that by the Product Lemma $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V. Let H^* be the $j(\mathbb{P})$-generic object which is isomorphic to $G \times H$ via the isomorphism from the last paragraph, that is

$$H^* = \{r \in j(\mathbb{P}) : r\restriction(\omega \times \kappa) \in G, r\restriction(\omega \times (j(\kappa) \setminus \kappa)) \in H\}.$$

If $p \in G$ then $\mathrm{dom}(p)$ is a finite subset of $\omega \times \kappa$ and $p(n, \alpha) < \alpha < \kappa$ for all $(n, \alpha) \in \mathrm{dom}(p)$. It follows that $\mathrm{dom}(j(p)) = j(\mathrm{dom}(p)) = \mathrm{dom}(p)$, and what is more, if $p(n, \alpha) = \beta$ then $j(p)(n, \alpha) = j(p(n, \alpha)) = j(\beta) = \beta$. So $p = j(p) \in H^*$.

We now work in $V[G * H]$. By Proposition 9.1 we can lift j to get $j_G : V[G] \longrightarrow M[H^*] = M[G * H]$. It then follows from Proposition 3.9 $M[G * H] = \mathrm{Ult}(V[G], U_G)$ and j_G is precisely the ultrapower embedding. Of course U_G does not exist in $V[G]$; it is only definable in the generic extension $V[G * H]$. $\dashv$

10.3 Remark. This theorem provides an example of an elementary embedding $k : M_1 \longrightarrow M_2$ with critical point $\omega_1^{M_1}$ and $P(\omega) \cap M_1 \subsetneq P(\omega) \cap M_2$. This shows that Proposition 2.6 is sharp.

10.4 Remark. An important feature of the last proof was the product analysis of $j(\mathbb{P})$. In that proof we were careful to stress that $G \times H$ and H^* are isomorphic rather than identical.

In what follows we will follow the standard practice and be more cavalier about these issues. The cavalier way of writing the main point in the last proof is to say "$p \in G$ implies that $j(p) = p \in G \times H$".

Theorem 10.2 can be generalized in a way that is important for several later results.

10.5 Theorem. *Let κ be measurable, let U be a normal measure on κ and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Let δ be an uncountable regular cardinal less than κ. Let $\mathbb{P} = \mathrm{Col}(\delta, {<}\kappa)$ and let G be $\mathbb{P}$-generic. There is a δ-closed forcing poset $\mathbb{Q} \in M$ such that for any H a $\mathbb{Q}$-generic filter, j can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*

The proof is just like that of Theorem 10.2. It will be useful later to know that some reflection properties of the original measurable cardinal κ survive in $V^{\mathbb{P}}$. We need a technical lemma on the preservation of stationary sets by forcing.

10.6 Lemma. *Let δ be regular with $\delta^{<\eta} = \delta$ for all regular $\eta < \delta$, and let $S \subseteq \delta^+ \cap \mathrm{Cof}({<}\delta)$ be stationary. Then the stationarity of S is preserved by δ-closed forcing.*

Proof. Let $\mathbb{P}$ be δ-closed and let $p \in \mathbb{P}$ force that $\dot{C}$ is a club set in δ^+. Fix $\eta < \delta$ such that $S \cap \mathrm{Cof}(\eta)$ is stationary in δ^+ and a large regular θ, then build $M \prec (H_\theta, \in)$ such that $p, \mathbb{P}, \dot{C} \in M$, $|M| = \delta$, M is closed under ${<}\eta$-sequences and $\gamma = M \cap \delta^+ \in S \cap \mathrm{Cof}(\eta)$. Now build a decreasing sequence p_i for $i < \eta$ of conditions in $\mathbb{P} \cap M$, and a sequence γ_i of ordinals increasing and cofinal in γ, such that $p_{i+1} \Vdash \gamma_i \in \dot{C}$. The construction is easy, using elementarity at successor stages and the closure of M at limit stages. Since $\mathbb{P}$ is δ-closed we may choose q a lower bound for the p_i and then $q \Vdash \gamma \in \dot{C}$. $\dashv$

10.7 Remark. In general it is *not* true for $\delta > \omega_1$ and $\gamma > \delta$ that every stationary subset of $\gamma^+ \cap \mathrm{Cof}({<}\delta)$ is preserved by δ-closed forcing, even if we assume GCH. Shelah has given an incisive analysis of when we may expect stationarity to be preserved; the author's survey paper [10] contains an exposition of the resulting "$I[\lambda]$ theory".

10.8 Theorem (Baumgartner [5]). *In the model $V[G]$ of Theorem 10.5, where $\kappa = \delta^+$, every stationary $S \subseteq \kappa \cap \mathrm{Cof}({<}\delta)$ reflects to a point of cofinality δ.*

Proof. Consider the generic embedding $j_G : V[G] \longrightarrow M[G * H]$ where H is generic for δ-closed forcing. We know that $j_G(S) \cap \kappa = S$, and since $M[G] \subseteq V[G]$ and $V[G] \models {}^{\kappa}M[G] \subseteq M[G]$ the set S is a stationary subset of $\kappa \cap \mathrm{Cof}(<\delta)$ in $M[G]$. The conditions of Lemma 10.6 apply (in fact $\delta^{<\delta} = \delta$) so S is stationary in $M[G*H]$, and so by the elementarity of j_G the set S has a stationary initial segment. Finally $\mathrm{cf}(\kappa) = \delta$ in $M[G * H]$ and $j_G(\delta) = \delta$, so stationarity reflects to an ordinal of cofinality δ. ⊣

10.9 Remark. Actually the conclusion of Theorem 10.8 holds if κ is only weakly compact, and this was the hypothesis used by Baumgartner. If κ is supercompact and we force with $\mathrm{Col}(\omega_1, <\kappa)$, then Shelah [5] observed that we get a model where for every regular $\lambda > \omega_1$ every stationary subset of $\lambda \cap \mathrm{Cof}(\omega)$ reflects.

Shelah [63] has also shown that it is consistent that (roughly speaking) "all stationary sets that can reflect do reflect". This is tricky because of the preservation problems alluded to in Remark 10.7.

10.10 Remark. The fact that we needed Lemma 10.6 to complete the proof of Theorem 10.8 is an example of a very typical phenomenon in the theory of generic embeddings, where we often need to know that the forcing which adds the embedding is in some sense "mild". See Theorem 23.2 for an example where the needed preservation lemmas involve not adding cofinal branches to trees.

11. Iteration with Easton Support

When defining an iterated forcing one of the key parameters is the type of support which is to be used. Silver realized that iteration with Easton support (see Definition 7.5) is a very useful technique in doing iterations which preserve large cardinal axioms. Easton [17] had already used Easton sets as the supports in products of forcings defined in V; the method of iteration with Easton supports has often been called "reverse Easton", "backwards Easton" or "upwards Easton" to distinguish it from Easton's product construction.

We give an example of forcing with Easton support which is due in a slightly different form to Kunen and Paris [47]. The goal is to produce a measurable cardinal κ with the maximum possible number of normal measures; if we assume GCH for simplicity, then the maximal possible number of normal measures is $2^{2^\kappa} = \kappa^{++}$. Kunen's work on iterated ultrapowers [44] shows that if κ is measurable then in the canonical minimal model $L[U]$ in which κ is measurable, κ carries exactly one normal measure.

The arguments of Levy and Solovay [50] show that if κ is measurable a forcing of size less than κ cannot increase the number of normal measures on κ. It follows that we need to force with a forcing poset of size at least κ. The simplest such poset which does not obviously destroy the measurability of κ is $\mathrm{Add}(\kappa, 1)$, however it is not hard to see that if we force over $L[U]$ this poset destroys the measurability of κ.

We will build an iterated forcing of size κ which adds subsets to many cardinals less than κ. As we will see shortly, we need the initial segments of the iteration to have a reasonable chain condition, and the final segments to have a reasonable degree of closure. Silver realized that the right balance between closure and chain condition could be achieved by doing an iteration with Easton support. We will assume that GCH holds in V. Assuming GCH is no burden because GCH is true in $L[U]$.

11.1 Theorem. *Let κ be measurable and let* GCH *hold. Then there exists $\mathbb{P}$ such that*

1. $|\mathbb{P}| = \kappa$.
2. $\mathbb{P}$ *is κ-c.c.*
3. GCH *holds in $V^{\mathbb{P}}$.*
4. *κ is measurable in $V^{\mathbb{P}}$.*
5. *κ carries κ^{++} normal measures in $V^{\mathbb{P}}$.*

The proof will occupy the rest of this subsection.

Let $A \subseteq \kappa$ be the set of those $\alpha < \kappa$ such that α is the successor of a singular cardinal. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map. Since $M \subseteq V$ we see that κ is inaccessible in M, so that $\kappa \notin j(A)$ or equivalently $A \notin U$. In the iteration we will add subsets to cardinals lying in A. The exact choice of A is irrelevant so long as it is a set of regular cardinals less than κ and is a set of measure zero for U.

We will now let $\mathbb{P} = \mathbb{P}_\kappa$ be an iteration of length κ with Easton support. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha$ be a name for the trivial forcing unless $\alpha \in A$, in which case $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, 1)_{V^{\mathbb{P}_\alpha}}$. By Proposition 7.13, if $\delta \leq \kappa$ and δ is Mahlo then $\mathbb{P}_\delta$ is δ-c.c.

11.2 Lemma. *Let $\delta \leq \kappa$ be Mahlo and let $\lambda = \delta^{+\omega+1}$. As in Proposition 7.11 let $\dot{\mathbb{R}}_{\delta,\kappa}$ name the canonical iteration of length $\kappa - \delta$ such that $\mathbb{P}_\kappa \simeq \mathbb{P}_\delta * \dot{\mathbb{R}}_{\delta,\kappa}$. Then $V[G_\delta] \models$ "$\mathbb{R}_{\delta,\kappa}$ is λ-closed".*

Proof. If we fix $\delta \leq \kappa$ a Mahlo cardinal then it follows from the chain condition of $\mathbb{P}_\delta$ that every set of ordinals of size less than δ in $V[G_\delta]$ is covered by a set of size less than δ in V. $\lambda = \min(A \setminus \delta)$ and the iteration is only non-trivial at points of A, and so for all γ with $\delta \leq \gamma < \kappa$ we see that $\Vdash_\gamma$ "$\dot{\mathbb{Q}}_\gamma$ is λ-closed". By Proposition 7.12, $\mathbb{R}_{\delta,\kappa}$ is λ-closed in $V[G_\delta]$. ⊣

11.3 Remark. We already have enough information to see that all Mahlo cardinals $\delta \leq \kappa$ are preserved by $\mathbb{P}$. A more delicate analysis as in Hamkins' paper [31] shows that in fact this iteration preserves all cardinals and cofinalities.

11.4 Lemma. GCH *holds in* $V[G_\kappa]$ *above* κ.

Proof. If $\lambda \geq \kappa$ and $\Vdash_{\mathbb{P}} \dot{\tau} \subseteq \lambda$ then the interpretation of $\dot{\tau}$ is determined by $\{(p, \alpha) : p \Vdash \check{\alpha} \in \dot{\tau}\}$. There are only $2^{\kappa \times \lambda} = \lambda^+$ possibilities for this set. ⊣

11.5 Remark. With more care we can show that GCH holds everywhere.

We now need to compare $j(\mathbb{P})$ with $\mathbb{P}$. Elementarity implies that from the point of view of M, $j(\mathbb{P})$ is an Easton iteration of length $j(\kappa)$, with Easton support, in which we add a Cohen subset to each $\alpha \in j(A)$.

11.6 Lemma. $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$ *and* $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \{1\}$.

Proof. If $\alpha < \kappa$ then $\mathbb{P}_\alpha \in V_\kappa$ and so $j(\mathbb{P})_\alpha = j(\mathbb{P})_{j(\alpha)} = j(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$. κ is inaccessible in M so a direct limit is taken at stage κ in the construction of $j(\mathbb{P})$. The direct limit construction is absolute so $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. Finally $\kappa \notin j(A)$ and so $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \{1\}$. ⊣

Let G be $\mathbb{P}$-generic over V. Since $M \subseteq V$ and $\mathbb{P} \in M$, G is $\mathbb{P}$-generic over M and $M[G] \subseteq V[G]$.

11.7 Lemma. *Let* $\mathbb{R} = i_G(\dot{\mathbb{R}}_{\kappa, j(\kappa)})$.

$V[G] \models$ "$\mathbb{R}$ *is* κ^+*-closed and has* κ^+ *maximal antichains lying in* $M[G]$".

Proof. By Lemma 11.2 applied in M to $j(\mathbb{P})$, if $\lambda = \kappa^{+\omega+1}_{M[G]}$ then

$$M[G] \models \text{“}\mathbb{R} \text{ is } \lambda\text{-closed”}.$$

Since $\mathbb{P}$ is κ-c.c. it follows from Proposition 8.4 that $V[G] \models {}^\kappa M[G] \subseteq M[G]$. So $V[G] \models$ "$\mathbb{R}$ is κ^+-closed".

$\mathbb{P}$ is κ-c.c. forcing poset with size κ, and in M we have $j(\mathbb{P}) \simeq \mathbb{P} * \dot{\mathbb{R}}$. It follows from Proposition 7.13 that in $M[G]$, $\mathbb{R}$ is $j(\kappa)$-c.c. forcing with size $j(\kappa)$, so if Z is the set of maximal antichains of $\mathbb{R}$ which lie in $M[G]$ then $M[G] \models |Z| = j(\kappa)$.

V is a model of GCH and so $V \models |j(\kappa)| = 2^\kappa = \kappa^+$. Therefore

$$V[G] \models \text{“}\mathbb{R} \text{ has } \kappa^+ \text{ maximal antichains lying in } M[G]\text{”}.$$

⊣

From now on we work in $V[G]$. Applying Proposition 8.1 we construct a sequence $\langle H_\alpha : \alpha < \kappa^{++} \rangle$ of κ^{++} distinct $\mathbb{R}$-generic filters over $M[G]$. For each α the set $G * H_\alpha$ is $\mathbb{P} * \dot{\mathbb{R}}$-generic over M, and since $\mathbb{P} * \dot{\mathbb{R}}$ is canonically isomorphic to $j(\mathbb{P})$ in M we will regard $G * H_\alpha$ as a $j(\mathbb{P})$-generic filter over M.

11.8 Lemma. *For all* $p \in G$ *and all* $\alpha < \kappa^{++}$, $j(p) \in G * H_\alpha$.

Proof. $\mathbb{P}_\kappa = \varinjlim \vec{\mathbb{P}} \restriction \kappa$. Fix $\beta < \kappa$ such that $p(\gamma) = 1$ for $\beta \le \gamma < \kappa$, and observe that $j(\beta) = \beta$ and so by elementarity $j(p)(\gamma) = 1$ for $\beta \le \gamma < j(\kappa)$. What is more $p \restriction \beta \in V_\kappa$ and so $j(p) \restriction \beta = j(p \restriction \beta) = p \restriction \beta$. It follows that $j(p) \in G * H_\alpha$. ⊣

Accordingly we can find κ^{++} extensions $j_\alpha : V[G] \longrightarrow M[G * H_\alpha]$ with $j_\alpha \restriction V = j$ and $j_\alpha(G) = G * H_\alpha$. They are distinct because the filters H_α are distinct. $H_\alpha \in V[G]$ and so by Proposition 9.4 j_α is definable in $V[G]$. We will be done if we can show that each j_α is an ultrapower map computed from some normal measure on κ in $V[G]$.

11.9 Lemma. *For every α, j_α is the ultrapower of $V[G]$ by U_α where*

$$U_\alpha = \{X \subseteq \kappa : X \in V[G], \kappa \in j_\alpha(X)\}.$$

Proof. j is the ultrapower of V by the normal measure U, so that by Proposition 3.9 j is supported on $\{\kappa\}$. By Proposition 9.3 j_α is also supported on $\{\kappa\}$. By Proposition 3.9 again j_α is the ultrapower of $V[G]$ by U_α. ⊣

12. Master Conditions

We are now in a position to give Silver's proof that GCH can fail at a measurable cardinal. We will need Silver's idea of the *master condition*, which is a technique for arranging the compatibility between generic filters required to apply Proposition 9.1.

12.1 Definition. Let $k : M \longrightarrow N$ be elementary and let $\mathbb{P} \in M$. A *master condition for k and* $\mathbb{P}$ is a condition $q \in k(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ with $D \in M$, there is a condition $p \in D$ such that q is compatible with $k(p)$.

Suppose that q is a master condition for k, and H is any N-generic filter on $\mathbb{Q}$ with $q \in H$. It is routine to check that k^{-1}“H generates an M-generic filter G such that k“$G \subseteq H$, and so again Proposition 9.1 can be applied to lift k. In general different choices of H will give different filters G.

12.2 Definition. Let $k : M \longrightarrow N$ be elementary and let $\mathbb{P} \in M$. A *strong master condition for k and* $\mathbb{P}$ is a condition $q \in k(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ with $D \in M$, there is a condition $p \in D$ such that $q \le k(p)$.

If q is a strong master condition then let $G = \{p \in \mathbb{P} : q \le k(p)\}$. It is routine to check that G is an M-generic filter, and that k^{-1}“$H = G$ for any N-generic filter H on $\mathbb{Q}$ with $q \in H$. Under these circumstances we will often say that q is a *strong master condition for k and G*.

12.3 Remark. A similar distinction occurs in the theory of proper forcing. See Remarks 24.4 and 24.5 for more on this.

12.4 Remark. Most of the master conditions which we build will be of the strong persuasion.

For use later we record a remark on the connection between existence of strong master conditions and distributivity.

12.5 Theorem. *Let $\pi : M \longrightarrow N$ be elementary, let $\mathbb{P} \in M$, let G be $\mathbb{P}$-generic, and let $q \in j(\mathbb{P})$ be such that $q \leq j``G$. Then for every $\delta < \mathrm{crit}(\pi)$, M and $M[G]$ have the same δ-sequences of ordinals.*

Proof. Suppose not, and fix $p \in G$ and $\dot{\tau} \in M$ such that p forces $\dot{\tau}$ to be a new δ-sequence of ordinals. For each $i < \delta$ there is a condition $p_i \in G$ such that p_i determines $\dot{\tau}(i)$. By elementarity $\pi(p_i)$ determines $\pi(\dot{\tau})(i)$ for each $i < \pi(\delta) = \delta$, and so since $q \leq \pi(p_i)$ we have that q determines $\pi(\dot{\tau})(i)$ for all $i < \delta$, that is, q forces that $\pi(\dot{\tau})$ is equal to some element of N; but $q \leq \pi(p)$ and by elementarity $\pi(p)$ forces that $\pi(\dot{\tau})$ is a new sequence of ordinals, contradiction. $\dashv$

It is easy to see that if U is a normal measure on κ and $2^\kappa \geq \kappa^{+n}$ then $\{\alpha < \kappa : 2^\alpha \geq \alpha^{+n}\} \in U$. In the light of this remark and the result of the last section, a natural strategy for producing a failure of GCH at a measurable is to start with a model of GCH with a measurable κ, and to do an iteration of length $\kappa + 1$ violating GCH on $A \cup \{\kappa\}$ for some suitably large A.

This strategy can be made to work but it is necessary to use a fairly strong large cardinal assumption. We will give here a version of Silver's original proof, using the hypothesis that GCH holds and there is a cardinal κ which is κ^{++}-supercompact. In Sects. 13 and 25 we will see how to weaken this large cardinal assumption.

12.6 Theorem. *Let* GCH *hold and let κ be κ^{++}-supercompact. Then there is a forcing poset $\mathbb{P}$ such that*

1. *$|\mathbb{P}| = \kappa^{++}$.*
2. *$\mathbb{P}$ is κ^+-c.c.*
3. *κ is measurable in $V^{\mathbb{P}}$.*
4. *$2^\kappa = \kappa^{++}$ in $V^{\mathbb{P}}$.*

Proof. Let U be a κ-complete normal fine ultrafilter on $P_\kappa\kappa^{++}$, and define $j : V \longrightarrow M$ to be the associated ultrapower map. Arguing exactly as in Example 4.8, we have

1. $j(\kappa) > \kappa^{+++}$.
2. $j(\kappa^{+4}) = \kappa^{+4}$.

Let A be the set of inaccessible cardinals less than κ. As in the last section the exact choice of A is more or less irrelevant, so long as A is a set of inaccessible cardinals and $A \in U_0$, where $U_0 = \{X \subseteq \kappa : \kappa \in j(X)\}$.

We now let $\mathbb{P} = \mathbb{P}_{\kappa+1}$ be the iteration of length $\kappa+1$ with Easton supports in which $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, \alpha^{++})_{V^{\mathbb{P}_\alpha}}$ if $\alpha \in A \cup \{\kappa\}$, and names the trivial forcing otherwise. Let G_κ be $\mathbb{P}_\kappa$-generic over V, let g_κ be $\mathbb{Q}_\kappa$-generic over $V[G]$ and let $G_{\kappa+1} = G_\kappa * g_\kappa$.

The next lemma is similar to Lemma 11.2 from the last section, the crucial difference being that this time $\delta \in A$ and so the iteration $\mathbb{P}$ acts at stage δ.

12.7 Lemma. *Let $\delta < \kappa$ be Mahlo. Then*

1. *$\mathbb{P}_\delta$ is δ-c.c.*
2. *$\mathbb{P}_{\delta+1}$ is δ^+-c.c.*
3. *If λ is the least inaccessible greater than δ then*

$$V[G_{\delta+1}] \models \text{“}\mathbb{R}_{\delta+1,\kappa} \textit{ is } \lambda\textit{-closed”}.$$

Proof. By Proposition 7.13 $\mathbb{P}_\delta$ is δ-c.c. and has size δ. Then $V[G_\delta] \models \delta^{<\delta} = \delta$, and so $\Vdash_\delta$ “$\mathbb{Q}_\delta$ is δ^+-c.c.”. By Proposition 7.6 $\mathbb{P}_{\delta+1}$ is δ^+-c.c.

Since A is a set of inaccessible cardinals we are guaranteed that $\dot{\mathbb{Q}}_\alpha$ names the trivial forcing for $\delta < \alpha < \lambda$. Every set of ordinals of size less than λ in $V[G_{\delta+1}]$ is covered by a such a set in V, so by Proposition 7.12 $\mathbb{R}_{\delta+1,\kappa}$ is λ-closed in $V[G_{\delta+1}]$. ⊣

The next lemma follows by exactly the same argument as that for Lemma 11.4 in the last section.

12.8 Lemma. *$\mathbb{P}_\kappa$ is κ-c.c. with size κ, and* GCH *holds above κ in $V^{\mathbb{P}_\kappa}$.*

The standard arguments counting names also give us

12.9 Lemma. *$\mathbb{P}$ is κ^+-c.c. with size κ^{++}, and* GCH *holds above κ^+ in $V^{\mathbb{P}}$.*

We now need to analyze the iteration $j(\mathbb{P})$.

12.10 Lemma. $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$.

Proof. We can argue exactly as in Lemma 11.6 that $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. By Proposition 8.4 we see that $V[G] \models {}^{\kappa^{++}}M[G] \subseteq M[G]$, so that

$$\mathrm{Add}(\kappa, \kappa^{++})_{V[G]} = \mathrm{Add}(\kappa, \kappa^{++})_{M[G]}.$$

Every condition in $\mathrm{Add}(\kappa, \kappa^{++})_{V^{\mathbb{P}_\kappa}}$ has a name which lies in $H_{\kappa^{+++}}$, and $H_{\kappa^{+++}} \subseteq M$ so that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$. ⊣

12.11 Lemma. *Let* $\mathbb{R} = i_{G_{\kappa+1}}(\dot{\mathbb{R}}_{\kappa+1,j(\kappa)})$. *Then*

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \textit{ is } \kappa^{+++}\textit{-closed”}$$

and

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \textit{ has } \kappa^{+++} \textit{ maximal antichains lying in } M[G_{\kappa+1}]\text{”}.$$

Proof. By Lemma 12.7 applied in M to $j(\mathbb{P}_\kappa)$, if λ is the least M-inaccessible greater than κ then $M[G_{\kappa+1}] \models$ “$\mathbb{R}$ is λ-closed”. Since $\mathbb{P}$ is κ^+-c.c. it follows from Proposition 8.4 that $V[G_{\kappa+1}] \models {}^{\kappa^{++}}M[G_{\kappa+1}] \subseteq M[G_{\kappa+1}]$. So $V[G_{\kappa+1}] \models$ “$\mathbb{R}$ is κ^{+++}-closed”.

$\mathbb{P}_\kappa$ is κ-c.c. with size κ, and in M we have $j(\mathbb{P}_\kappa) \simeq \mathbb{P}_{\kappa+1} * \dot{\mathbb{R}}$. It follows from Proposition 7.13 that in $M[G_{\kappa+1}]$, $\mathbb{R}$ is $j(\kappa)$-c.c. with size $j(\kappa)$, so if Z is the set of maximal antichains of $\mathbb{R}$ which lie in $M[G_{\kappa+1}]$ then $M[G_{\kappa+1}] \models |Z| = j(\kappa)$.

By Proposition 4.5, $V \models |j(\kappa)| = \kappa^{+++}$. So

$$V[G_{\kappa+1}] \models \text{“}\mathbb{R} \text{ has } \kappa^{+++} \text{ maximal antichains in } M[G_{\kappa+1}]\text{”}.$$

$\dashv$

Applying Proposition 8.1 we may find a filter $H \in V[G_{\kappa+1}]$ such that H is $\mathbb{R}$-generic over $M[G_{\kappa+1}]$. Let $G_{j(\kappa)} = G_{\kappa+1} * H$, so that $G_{j(\kappa)}$ is $j(\mathbb{P}_\kappa)$-generic over M. The argument of Lemma 11.8 shows that $j``G_\kappa \subseteq G_{j(\kappa)}$, so that by Proposition 9.1 we may lift $j : V \longrightarrow M$ and obtain an elementary embedding $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$.

To finish the proof we need to construct a filter $h \in V[G_{\kappa+1}]$ such that

1. h is $\mathrm{Add}(j(\kappa), j(\kappa^{++}))_{M[G_{j(\kappa)}]}$-generic over $M[G_{j(\kappa)}]$.
2. $j``g_\kappa \subseteq h$.

The first of these conditions can be met using methods we have seen already, once we have done some counting arguments.

12.12 Lemma. $V[G_{\kappa+1}] \models {}^{\kappa^{++}}M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$.

Proof. $\mathbb{P}_{\kappa+1}$ is κ^+-c.c. and so $V[G_{\kappa+1}] \models {}^{\kappa^{++}}\mathrm{On} \subseteq M[G_{\kappa+1}]$ by Proposition 8.4. $M[G_{\kappa+1}] \subseteq M[G_{j(\kappa)}]$ so $V[G_{\kappa+1}] \models {}^{\kappa^{++}}\mathrm{On} \subseteq M[G_{j(\kappa)}]$. The result follows by Proposition 8.2. $\dashv$

12.13 Lemma. *Let* $\mathbb{Q} = \mathrm{Add}(j(\kappa), j(\kappa^{++}))_{M[G_{j(\kappa)}]}$. *Then*

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \textit{ is } \kappa^{+++}\textit{-closed”}$$

and

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \textit{ has } \kappa^{+++} \textit{ maximal antichains in } M[G_{j(\kappa)}]\text{”}.$$

Proof. $M[G_{j(\kappa)}] \models$ "$\mathbb{Q}$ is $j(\kappa)$-closed", so by Proposition 8.3 it follows that $V[G_{\kappa+1}] \models$ "$\mathbb{Q}$ is κ^{+++}-closed".

Lemma 12.8 and an easy counting argument give that

$$V[G_\kappa] \models \text{“}\mathrm{Add}(\kappa, \kappa^{++}) \text{ has } \kappa^{+++} \text{ maximal antichains”}.$$

$j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ is elementary and so

$$M[G_{j(\kappa)}] \models \text{“}\mathbb{Q} \text{ has } j(\kappa^{+++}) \text{ maximal antichains”}.$$

Since $V \models |j(\kappa^{+++})| = \kappa^{+++}$,

$$V[G_{\kappa+1}] \models \text{“}\mathbb{Q} \text{ has } \kappa^{+++} \text{ maximal antichains in } M[G_{j(\kappa)}]\text{”}$$

and we are done. ⊣

We can now build $h \in V[G_{\kappa+1}]$ which is suitably generic. To ensure that $j``g_\kappa \subseteq h$ we use the "strong master condition" idea from Definition 12.2.

12.14 Lemma. *There is a strong master condition for the elementary embedding $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ and the generic object g_κ.*

Proof. If $p \in g_\kappa$ then p is a partial function from $\kappa \times \kappa^{++}$ to 2 with size less than κ, so in particular $j(p) = j``p$. $g_\kappa \in M[G_{j(\kappa)}]$ and $j \restriction (\kappa \times \kappa^{++}) \in M$, so that $j``g_\kappa \in M[G_{j(\kappa)}]$. Working in $M[G_{j(\kappa)}]$ the cardinality of $j``g_\kappa$ is $\kappa^{++}$, $j``g_\kappa$ is a directed subset of $\mathbb{Q}$ and $\mathbb{Q}$ is $j(\kappa)$-directed closed; it follows that we may find a condition $r \in \mathbb{Q}$ such that $r \leq j(p)$ for all $p \in g_\kappa$. ⊣

12.15 Remark. We can give an explicit description of an r with this property; let $\mathrm{dom}(r) = \kappa \times j``\kappa^{++}$ and $r(\alpha, j(\beta)) = j(F(\alpha, \beta)) = F(\alpha, \beta)$ where $F : \kappa \times \kappa^{++} \longrightarrow 2$ is given by $F = \bigcup g_\kappa$.

We now use Proposition 8.1 to build h which is $\mathbb{Q}$-generic over $M[G_{j(\kappa)}]$ with $r \in h$. Let $G_{j(\kappa)+1} = G_{j(\kappa)} * h$. Then by construction we have $j``g \subseteq h$, so that we may lift $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ to $j : V[G_{\kappa+1}] \longrightarrow M[G_{j(\kappa)+1}]$. $G_{j(\kappa)+1} \in V[G_{\kappa+1}]$ and so by Proposition 9.4 the elementary embedding $j : V[G_{\kappa+1}] \longrightarrow M[G_{j(\kappa)+1}]$ is definable in $V[G_{\kappa+1}]$, that is to say it is a definable embedding in the sense of Definition 2.10. It follows that κ is measurable in $V[G_{\kappa+1}]$. ⊣

12.16 Remark. By Proposition 9.4 and Remark 4.1, κ is actually κ^{++}-supercompact in $V[G_{\kappa+1}]$.

12.17 Remark. If we had forced with $\mathrm{Add}(\alpha, 1)$ instead of $\mathrm{Add}(\alpha, \alpha^{++})$ at each stage in $A \cup \{\kappa\}$, then we could have proved that the measurability of κ was preserved assuming only that GCH holds and κ is measurable in the ground model. Of course we would not have violated GCH this way, and indeed it is known [59, 24] that to violate GCH at a measurable cardinal requires the strength of a cardinal κ with Mitchell order κ^{++}.

13. A Technique of Magidor

In this section we describe a technique due to Magidor [54] for lifting elementary embeddings in situations where we do not have enough closure to build a strong master condition. The trick will be to build an "increasingly masterful" sequence of conditions into our final generic filter. As an example we will redo the result from the last section from a weaker large cardinal hypothesis.

We assume that GCH holds and that κ is κ^+-supercompact, and we let $j : V \longrightarrow M$ be the ultrapower map arising from some κ^+-supercompactness measure on $P_\kappa \kappa^+$. As in Example 4.8 we see that

- $\kappa^{++} = \kappa^{++}_M < j(\kappa) < j(\kappa^+) < j(\kappa^{++}) < j(\kappa^{+++}) = \kappa^{+++}$.
- j is continuous at κ^{++} and κ^{+++}.
- j is discontinuous at every limit ordinal of cofinality κ^+.

We now perform exactly the same forcing construction as in the last section, namely, we perform an Easton support iteration of length $\kappa + 1$ in which we add α^{++} Cohen subsets to every inaccessible $\alpha \leq \kappa$. We let G_κ be $\mathbb{P}_\kappa$-generic over V and g_κ be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$.

As in Lemma 12.10 from the last section we see that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$. As in the last section we let $\mathbb{R} = i_{G_{\kappa+1}}(\dot{\mathbb{R}}_{\kappa+1,j(\kappa)})$, that is to say $\mathbb{R}$ is the forcing that one would do over $M[G_{\kappa+1}]$ to produce a $j(\mathbb{P}_\kappa)$-generic object extending $G_{\kappa+1}$.

Modifying the proof of Lemma 12.11 we see that

$$V[G_{\kappa+1}] \models \text{"}\mathbb{R} \text{ is } \kappa^{++}\text{-closed"}$$

and

$$V[G_{\kappa+1}] \models \text{"}\mathbb{R} \text{ has } \kappa^{++} \text{ maximal antichains lying in } M[G_{\kappa+1}]\text{"}.$$

By Proposition 8.1 we build a filter $H \in V[G_{\kappa+1}]$ which is $\mathbb{R}$-generic over $M[G_{\kappa+1}]$, and lift $j : V \longrightarrow M$ to get $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$ where $G_{j(\kappa)} = G_\kappa * g_\kappa * H$. We observe that $V[G_{\kappa+1}] \models {}^{\kappa^+}M[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]$.

As in the last section we may apply Proposition 8.1 to build $h \in V[G_{\kappa+1}]$ which is $j(\mathbb{Q}_\kappa)$-generic over $M[G_{j(\kappa)}]$, and the remaining problem is to build h is such a way that $j``g_\kappa \subseteq h$. At this point we can no longer imitate the proof of the last section because we no longer have enough closure.

We do some analysis of an antichain A of $j(\mathbb{Q}_\kappa)$ with $A \in M[G_{j(\kappa)}]$. Let $A = j(F)(j``\kappa^+)$ where $F \in V[G_\kappa]$, $\mathrm{dom}(F) = P_\kappa \kappa^+$, and without loss of generality $F(x)$ is a maximal antichain in $\mathbb{Q}_\kappa$ for all x. Working in $V[G_\kappa]$, for each $\zeta < \kappa^{++}$ we let $X_\zeta = \mathrm{Add}(\kappa, \zeta)$, so that $X_\zeta \subseteq \mathbb{Q}_\kappa$, $|X_\zeta| = \kappa^+$, and $\mathbb{Q}_\kappa = \bigcup_\zeta X_\zeta$. A routine argument in the style of the Löwenheim-Skolem theorem shows that for each x there is a club subset C_x of κ^{++} such that for all $\alpha \in C_x \cap \mathrm{Cof}(\kappa^+)$ the antichain $F(x) \cap X_\alpha$ is maximal in X_α. Let C be the intersection of the C_x for $x \in P_\kappa \kappa^+$, then C is club and for every $\alpha \in C \cap \mathrm{Cof}(\kappa^+)$ the antichain A is maximal in $j(X_\alpha) = \mathrm{Add}(j(\kappa), j(\alpha))_{M[G_{j(\kappa)}]}$.

Now we work in $V[G_{\kappa+1}]$ to build a suitable filter h. Define Q a partial function from $j(\kappa) \times j``\kappa^{++}$ by setting $Q$ to be the union of $j(p)$ for $p \in g_\kappa$. It is routine to check that $\mathrm{dom}(Q) = \kappa \times j``\kappa^{++}$, and while Q is not even in $M[G_{j(\kappa)}]$, for all $\zeta < \kappa^{++}$ the partial function $q_\zeta = Q \restriction (j(\kappa) \times j(\zeta))$ is in $j(X_\zeta)$ and is a strong master condition for j and $g_\kappa \cap X_\zeta$.

Working in $V[G_{\kappa+1}]$, we may enumerate all the maximal antichains of $j(\mathbb{Q}_\kappa)$ as $\langle A_i : i < \kappa^{++} \rangle$. Using the analysis of such antichains given above we choose an increasing sequence $\alpha_i \in \kappa^{++} \cap \mathrm{Cof}(\kappa^+)$ such that $A_i \cap j(X_{\alpha_i})$ is maximal in $j(X_{\alpha_i})$ for all $i < \kappa^{++}$. Now we build a decreasing sequence of conditions $r_i \in j(\mathbb{Q}_\kappa)$ such that for each $i < \kappa^{++}$

1. $r_i \in j(X_{\alpha_i})$.
2. $r_i \le q_{\alpha_i}$.
3. r_i extends some member of A_i.

The construction of r_i goes as follows. We start by forming $r = \bigcup_{j<i} r_j$, where we note that the support of r is contained in $j(\kappa) \times \sup_{j<i} j(\alpha_j)$. We claim that r is compatible with q_{α_i}. To see this let $(\delta, j(\gamma))$ be an arbitrary point in the domain of q_{α_i}, that is, $\gamma < \alpha_i$ and $\delta < \kappa$. If $\gamma < \alpha_j$ for some j then since $r \le r_j \le q_j$ we have

$$q_{\alpha_i}(\delta, j(\gamma)) = Q(\delta, j(\gamma)) = r(\delta, j(\gamma)),$$

while if $\gamma \ge \alpha_j$ for all $j < i$ then $(\delta, j(\gamma)) \notin \mathrm{dom}(r)$.

So we may take the union $r \cup q_{\alpha_i}$ to get a condition in $j(X_{\alpha_i})$ and since $A_i \cap j(X_{\alpha_i})$ is maximal in $j(X_{\alpha_i})$ we may choose $r_i \le r \cup q_{\alpha_i}$ so that $r_i \in j(X_{\alpha_i})$ and r_i extends some condition in A_i.

It is now easy to see that the sequence of r_i generates a generic filter h with $h \supseteq j``g$. We may then proceed as in the previous section to lift the embedding to $V[G_{\kappa+1}]$.

13.1 Remark. In fact κ is still κ^+-supercompact in $V[G_{\kappa+1}]$.

13.2 Remark. The forcing technique described here has many applications in the theory of precipitous and saturated ideals. See Sects. 17 and 18, and also Foreman's chapter in this Handbook.

14. Absorption

In this section we discuss an idea which is used in many forcing constructions (for example in building Solovay's model in which every set is Lebesgue measurable [65]) and is particularly useful for our purposes, namely, the idea of embedding a complex poset into a simple one. This is one area of the subject where the presentation of forcing in terms of complete Boolean algebras is very helpful.

The "simple posets" into which we typically absorb more complex ones are the Cohen forcing $\mathrm{Add}(\kappa, \lambda)$ and the collapsing poset $\mathrm{Col}(\kappa, \lambda)$. We note that for any regular κ the forcing $\mathrm{Col}(\kappa, \kappa)$ is equivalent to $\mathrm{Add}(\kappa, 1)$ so we phrase our whole discussion in terms of the collapsing poset.

The following universal property of the collapsing poset is key:

14.1 Theorem. *Let κ be regular. Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a separative forcing poset such that $\mathbb{P}$ is κ-closed, $|\mathbb{P}| = \lambda$, every condition in $\mathbb{P}$ has λ incompatible extensions and $\mathbb{P}$ adds a surjection from κ to λ.*

Then $\mathbb{P}$ is equivalent to the collapsing poset $\mathrm{Col}(\kappa, \lambda)$.

Notice that if $\lambda > \kappa$ and λ is regular, then the demand that $\mathbb{P}$ adds a surjection from κ to λ implies that for no $p \in \mathbb{P}$ can $\mathbb{P}/p$ be λ-c.c., and so the demand that every condition should have λ incompatible extensions follows from the other conditions.

Proof. Let $\dot{f}$ name a surjective map from κ to $\dot{G}$, where $\dot{G}$ names the generic filter on $\mathbb{P}$. We will build a dense subset of $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$ which is isomorphic to $\mathrm{Col}(\kappa, \lambda)$. Let $\mathbb{P}^*$ be the canonical isomorphic copy of $\mathbb{P}$ in $\mathrm{ro}(\mathbb{P})$, so that $\mathbb{P}^*$ is a dense κ-closed subset of $\mathrm{ro}(\mathbb{P}) \setminus \{0\}$.

We will build a family b_s indexed by $s \in \mathrm{Col}(\kappa, \lambda)$ with the following properties:

1. $b_0 = 1$, and $b_s \in \mathrm{ro}(\mathbb{P}) \setminus \{0\}$ for all s.
2. For all s and t, $t \leq s$ implies that $b_t \leq b_s$.
3. For all $\alpha < \kappa$, $\{b_s : \mathrm{dom}(s) = \alpha\}$ is a maximal antichain.
4. For all s with $\mathrm{dom}(s)$ a successor ordinal, $b_s \in \mathbb{P}^*$.
5. For all $\alpha < \kappa$ and s with domain α, b_s determines $\dot{f} \restriction \alpha$.
6. For all s with $\mathrm{dom}(s)$ a limit ordinal μ, $b_s = \bigwedge_{i<\mu} b_{s \restriction i}$.

We will construct $\{b_s : \mathrm{dom}(s) = \alpha\}$ by recursion on α. At successor stages we construct $\{b_{s ^\frown j} : j < \lambda\}$ to be a maximal antichain below b_s, consisting of conditions that lie in $\mathbb{P}^*$ and determine $\dot{f} \restriction (\mathrm{dom}(s) + 1)$.

For limit μ we define (as we are compelled to) b_s as the infimum of $\{b_{s \restriction i} : i < \mu\}$ for all s with $\mathrm{dom}(s) = \mu$. By κ-closure of $\mathbb{P}$, $b_s \neq 0$. Clearly $\{b_s : \mathrm{dom}(s) = \mu\}$ is an antichain. To show it is maximal, it will suffice to show that it meets every generic G. Let G be generic, and let $s : \mu \to \lambda$ be the unique function with $b_{s \restriction i} \in G$ for all $i < \mu$; by closure $s \in V$, so s is a condition in $\mathrm{Col}(\kappa, \lambda)$ and by genericity $b_s \in G$.

This completes the construction, and it remains to see that the set of all b_s is dense. Let $p \in \mathbb{P}$, so that p forces $p \in G$, and find a condition $q \leq p$ and an ordinal $i < \kappa$ such that $q \Vdash \dot{f}(i) = p$. The condition q is compatible with b_s for s such that $\mathrm{dom}(s) = i + 1$. Now b_s determines $\dot{f}(i)$, so b_s forces that $\dot{f}(i) = p$, in particular $b_s \Vdash p \in G$ and so by separativity $b_s \leq p$. ⊣

In particular a separative κ-closed forcing poset of cardinality κ is equivalent to $\mathrm{Add}(\kappa, 1)$ and a separative forcing poset of size λ which makes λ countable is equivalent to $\mathrm{Col}(\omega, \lambda)$. Moreover if $\mathbb{P}$ is κ-closed then for a sufficiently large μ we see that $\mathbb{P} \times \mathrm{Col}(\kappa, \mu)$ is equivalent to $\mathrm{Col}(\kappa, \mu)$; this is the key point in Theorems 14.2 and 14.3.

Theorem 14.1 has the following corollaries. We separate the cases of $\mathrm{Col}(\omega, {<}\kappa)$ and $\mathrm{Col}(\delta, {<}\kappa)$ for uncountable δ because (as detailed below) we may say significantly more in the former case.

14.2 Theorem. *Let κ be an inaccessible cardinal and let $\mathbb{C} = \mathrm{Col}(\omega, {<}\kappa)$. Let $\mathbb{P}$ be a separative forcing poset with $|\mathbb{P}| < \kappa$ and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a separative forcing poset of size less than κ. Then*

1. *There is a complete embedding $i : \mathrm{ro}(\mathbb{P}) \to \mathrm{ro}(\mathbb{C})$.*
2. *For any such i and any $\mathbb{P}$-generic g, $\mathbb{C}/i(g)$ is equivalent in $V[g]$ to $\mathrm{Col}(\omega, {<}\kappa)$.*
3. *Any such i may be extended to a complete $j : \mathrm{ro}(\mathbb{P} * \dot{\mathbb{Q}}) \to \mathrm{ro}(\mathbb{C})$.*

In the general case we have:

14.3 Theorem. *Let κ be an inaccessible cardinal, let $\delta < \kappa$ be regular and let $\mathbb{D} = \mathrm{Col}(\delta, {<}\kappa)$. Let $|\mathbb{P}| < \kappa$ where $\mathbb{P}$ is a δ-closed separative forcing poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a δ-closed separative forcing poset of size less than κ. Then*

1. *There is a complete embedding $i : \mathrm{ro}(\mathbb{P}) \to \mathrm{ro}(\mathbb{D})$ such that $\mathbb{D}/i(g)$ is equivalent in $V[g]$ to $\mathrm{Col}(\delta, {<}\kappa)$.*
2. *Any such i may be extended to a complete $j : \mathrm{ro}(\mathbb{P} * \dot{\mathbb{Q}}) \to \mathrm{ro}(\mathbb{D})$ such that $\mathbb{D}/i(g * h)$ is equivalent in $V[g * h]$ to $\mathrm{Col}(\delta, {<}\kappa)$.*

Notice that Theorem 14.2 asserts that *however* we embed the small poset $\mathbb{P}$ in the collapse $\mathrm{Col}(\omega, {<}\kappa)$, the quotient forcing is $\mathrm{Col}(\omega, {<}\kappa)$. Theorem 14.3 just asserts that there is *some* way of embedding the small δ-closed poset $\mathbb{P}$ in the collapse $\mathrm{Col}(\delta, {<}\kappa)$, so that the quotient forcing is $\mathrm{Col}(\delta, {<}\kappa)$.

To underline the difference between these theorems we consider the following example, which implies that we can find an embedding of $\mathrm{Add}(\omega_1, 1)$ into $\mathrm{Add}(\omega_1, 1)$ where the quotient is not countably closed or even proper.

14.4 Example. Let CH hold, let $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$. If $F : \omega_1 \to 2$ is the function added by $\mathbb{P}$ and $S = \{\alpha : F(\alpha) = 1\}$ then we claim that S is stationary in $V[F]$. To see this let p force that $\dot{C}$ is club and build a chain of extensions $p_0 = p \geq p_1 \geq p_2 \geq \cdots$ so that p_{n+1} forces some ordinal

$\alpha_n > \mathrm{dom}(p_n)$ into $\dot{C}$. Then if $p^* = \bigcup p_n$ and $\alpha^* = \sup_n \alpha_n$ we have that $\mathrm{dom}(p^*) = \alpha^*$ and $p^* \Vdash \alpha^* \in \dot{C}$, and may then extend to force that $\alpha^* \in S$. A similar argument shows that S^c is also stationary.

Working in $V[F]$, we now let $\mathbb{Q}$ be the forcing from Example 6.8 to add a club set $D \subseteq S^c$. We claim that $\mathbb{P} * \dot{\mathbb{Q}}$ has a countably closed dense subset of size ω_1, namely, the set of pairs $(p, \check{c})$ where $\mathrm{dom}(p) = \max(c) + 1$ and $p(\alpha) = 0$ for all $\alpha \in c$: the proof is just like the proof that S is stationary from the last paragraph. So now $\mathbb{P} * \mathbb{Q}$ is equivalent to $\mathrm{Add}(\omega_1, 1)$, while $\mathbb{Q}$ destroys the stationarity of S so is not countably closed (or even proper).

Now we give some examples of the absorption idea in action. The first one is due to Kunen [45]. Since there is a detailed account in Foreman's chapter in this Handbook we shall only sketch the result.

As motivation we recall some facts about saturated ideals, weakly compact cardinals, and stationary reflection.

1. If κ is strongly inaccessible and carries a λ-saturated ideal for some $\lambda < \kappa$ then κ is measurable [45].

2. If κ is weakly compact and carries a κ-saturated ideal then κ is measurable [45].

3. If κ is weakly compact then every stationary subset of κ reflects.

4. If $V = L$ then for every regular κ, κ is weakly compact if and only if every stationary subset of κ reflects [42].

5. If $\mathbb{P} \times \mathbb{P}$ is κ-c.c. and κ is measurable in $V^{\mathbb{P}}$ then κ is measurable in V (see Theorem 21.1).

In the model which we present there is an inaccessible cardinal κ which carries a κ-saturated ideal and reflects stationary sets, and there is also a κ-Suslin tree T (so in particular κ is not weakly compact). It follows that 1 and 2 above are close to optimal, and that in general the conclusion of 4 fails. The key property of the model will be that adding a branch through the κ-Suslin tree T resurrects the measurability of κ so that 5 is also close to optimal.

We will use a standard device due to Kunen [45] for manufacturing saturated ideals.

14.5 Lemma. *Let $\mathbb{P}$ be λ-c.c. and let $\dot{U}$ be a $\mathbb{P}$ name for a V-ultrafilter on κ. Let I be the set of those $X \in P(\kappa) \cap V$ such that $\Vdash_{\mathbb{P}} \kappa \setminus X \in \dot{U}$. Then*

1. *I is a λ-saturated ideal.*

2. *If $\dot{U}$ is forced to be V-κ-complete then I is κ-complete, and if $\dot{U}$ is forced to be V-normal then I is normal.*

Often we have a $\mathbb{P}$-name for a generic embedding $j : V \longrightarrow M \subseteq V[G_{\mathbb{P}}]$, and U will be $\{X \in P(\kappa) \cap V : \kappa \in j(X)\}$. This kind of *induced ideal* is discussed at length in Foreman's chapter in this Handbook.

We use a result by Kunen [45].

14.6 Fact. Let α be inaccessible. Then there is a forcing poset $\mathbb{P}_0(\alpha)$ such that

1. $\mathbb{P}_0(\alpha)$ has cardinality α and adds an α-Suslin tree T_α.
2. If $\mathbb{P}_1(\alpha) \in V^{\mathbb{P}_0(\alpha)}$ is the forcing poset $(T_\alpha, \geq_{T_\alpha})$, then $\mathbb{P}_0(\alpha) * \dot{\mathbb{P}}_1(\alpha)$ has a dense subset isomorphic to the Cohen poset $\mathrm{Add}(\alpha, 1)$.

We assume that κ is measurable and GCH holds. We do an iteration with Easton support of length $\kappa+1$. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha$ name the trivial forcing unless α is inaccessible, in which case $\dot{\mathbb{Q}}_\alpha$ names $\mathrm{Add}(\alpha, 1)_{V^{\mathbb{P}_\alpha}}$. We let $\dot{\mathbb{Q}}_\kappa$ name $\mathbb{P}_0(\kappa)_{V^{\mathbb{P}_\kappa}}$ where $\mathbb{P}_0(\kappa)$ is the forcing from Fact 14.6.

As usual, let G_κ be $\mathbb{P}_\kappa$-generic over V and let g_κ be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$.

14.7 Claim. *In $V[G_{\kappa+1}]$:*

- *κ is not weakly compact.*
- *κ carries a normal κ-saturated ideal.*
- *κ reflects stationary sets.*

Proof. Let T be the tree added by the $\mathbb{P}_0(\kappa)$-generic filter g_κ. Since T is a κ-Suslin tree in $V[G_{\kappa+1}]$, κ is not weakly compact in $V[G_{\kappa+1}]$.

Let H be generic over $V[G_{\kappa+1}]$ for $\mathbb{P}_1(\kappa)$, that is to say $(T, \geq_T)$. Since T is a κ-Suslin tree, H is generic for κ-c.c. (κ, ∞)-distributive forcing over $V[G_{\kappa+1}]$.

Since $\mathbb{P}_0(\kappa) * \mathbb{P}_1(\kappa)$ is isomorphic to $\mathrm{Add}(\kappa, 1)$, $V[G_{\kappa+1} * H]$ is a model obtained by forcing with $\mathrm{Add}(\alpha, 1)_{V[G_\alpha]}$ at every inaccessible $\alpha \leq \kappa$. By Remark 12.17 κ is measurable in $V[G_{\kappa+1} * H]$, and we may fix $U \in V[G_{\kappa+1} * H]$ which is a normal measure on κ. Let $\dot{U}$ be a $\mathbb{P}_1(\kappa)$-name for U.

Working in $V[G_{\kappa+1}]$ we now define an ideal I on κ by

$$X \in I \iff \Vdash_{\mathbb{P}_1(\kappa)} \kappa \setminus X \in U.$$

By Lemma 14.5 this is a normal κ-saturated ideal.

Finally let $V[G_{\kappa+1}] \models$ "S is a stationary subset of κ". κ-c.c. forcing preserves the stationarity of stationary subsets of κ and so S is stationary in $V[G_{\kappa+1} * H]$. Measurable cardinals reflect stationary sets and so there is an ordinal $\alpha < \kappa$ such that $V[G_{\kappa+1} * H] \models$ "$S \cap \alpha$ is a stationary subset of α". It follows easily that

$$V[G_{\kappa+1}] \models \text{“}S \cap \alpha \text{ is a stationary subset of } \alpha\text{”}.$$

$\dashv$

As our second example we sketch a result from the author's program of joint work [13, 12] with Džamonja and Shelah on *strong non-reflection*. The argument has the interesting feature that we are creating a strong master condition by forcing.

14.8 Definition. Let $\kappa < \lambda < \mu$ be regular cardinals. Then the *Strong Non Reflection* principle $\mathrm{SNR}(\kappa, \lambda, \mu)$ is the assertion that there is a function F from $\mu \cap \mathrm{Cof}(\kappa)$ to λ, such that for every $\delta \in \mu \cap \mathrm{Cof}(\lambda)$ there is a set C club in δ with $F \restriction (C \cap \mathrm{Cof}(\kappa))$ strictly increasing.

It is easy to see that if F witnesses $\mathrm{SNR}(\kappa, \lambda, \mu)$, $S \subseteq \mu \cap \mathrm{Cof}(\kappa)$ is stationary and we use Fodor's Lemma to find stationary $T \subseteq S$ with F constant on T, then T reflects at no point of cofinality λ. The next theorem shows that this idea can be used to make fine distinctions between stationary reflection principles. The hypothesis can be improved to the existence of a weakly compact cardinal with a little more work.

14.9 Theorem. *Suppose that it is consistent that there exists a measurable cardinal. Then it is consistent that every stationary subset of* $\omega_3 \cap \mathrm{Cof}(\omega)$ *reflects to a point of cofinality* ω_2*, while at the same time every stationary subset of* $\omega_3 \cap \mathrm{Cof}(\omega_1)$ *contains a stationary set which reflects at no point of cofinality* ω_2*.*

Proof. We start with κ a measurable cardinal. Fix U a normal measure on κ and let $j : V \longrightarrow M$ be the ultrapower map. We let $\mathbb{P} = \mathrm{Col}(\omega_2, {<}\kappa)$. As we saw in Theorem 10.8 in $V^{\mathbb{P}}$ every stationary subset of $\omega_3 \cap (\mathrm{Cof}(\omega) \cup \mathrm{Cof}(\omega_1))$ reflects to a point of cofinality ω_2.

Let $\mathbb{Q}$ be the natural poset to add a witness to $\mathrm{SNR}(\omega_1, \omega_2, \omega_3)$ by initial segments. More precisely the elements of $\mathbb{Q}$ are partial functions f with domain an initial segment of $\omega_3 \cap \mathrm{Cof}(\omega_1)$ and the property that if $\alpha \leq \mathrm{dom}(f)$ and $\alpha \in \mathrm{Cof}(\omega_2)$ then there is a set C club in α with $f \restriction (C \cap \mathrm{Cof}(\omega_1))$ strictly increasing. The ordering is end-extension.

It is easy to see that $\mathbb{Q}$ is ω_2-closed and that player II wins the strategic closure game of length $\omega_2 + 1$; to see the second claim consider a strategy where player II moves as follows: at every stage $\alpha \in \omega_2 \cap \mathrm{Cof}(\omega_1)$, player II extends the existing function f_α to $f_{\alpha+1} = f_\alpha \cup \{\mathrm{dom}(f_\alpha), \alpha)\}$. In particular $\mathbb{Q}$ adds no ω_2-sequences and so preserves cardinals up to and including ω_3.

We now make a suggestive false start. As usual we factor $j(\mathbb{P}) = \mathbb{P} \times \mathbb{R}$ where $\mathbb{R}$ is an ω_2-closed forcing poset collapsing cardinals in the interval $[\kappa, j(\kappa))$. If G is $\mathbb{P}$-generic and g is $\mathbb{Q}$-generic then by Theorem 14.3 we may absorb $\mathbb{P} * \dot{\mathbb{Q}}$ into $j(\mathbb{P})$ so that the quotient is ω_2-closed, and so build an embedding $j : V[G] \longrightarrow M[G * g * h]$ where h is generic for ω_2-closed forcing.

If F is the function added by g then $F = \bigcup g = \bigcup j``g$. It is natural to try and use F as a strong master condition. Since $\mathrm{cf}(\kappa) = \omega_2$ in $M[G * g * h]$ we need to know that F is increasing on a club set to see that $F \in j(\mathbb{Q})$, but this is not immediately clear.

To resolve this problem we work in $V[G * g]$ and define a poset $\mathbb{S}$ as follows: conditions in $\mathbb{S}$ are closed bounded subsets c of κ such that $|c| \le \omega_1$ and $F \restriction (c \cap \mathrm{Cof}(\omega_1))$ is strictly increasing. It is easy to see that $\mathbb{S}$ is countably closed in $V[G * g]$. We claim that in $V[G]$ there is a dense ω_2-closed set of conditions in $\mathbb{Q} * \dot{\mathbb{S}}$, consisting of those conditions $(f, \check{c})$ such that $\mathrm{dom}(f) = (\max(c) + 1)) \cap \mathrm{Cof}(\omega_1)$ and f is strictly increasing on $c \cap \mathrm{Cof}(\omega_1)$. The proof is routine.

We now force over $V[G * g]$ with $\mathbb{S}$ to obtain a club set $C \subseteq \kappa$ such that C has order type ω_2 and $F \restriction (C \cap \mathrm{Cof}(\omega_1))$ is increasing. Since $\mathbb{Q} * \dot{\mathbb{S}}$ has an ω_2-closed dense set we may absorb $G * g * C$ into $j(\mathbb{P})$ with an ω_2-closed quotient and then lift to obtain $j : V[G] \longrightarrow M[G * g * C * h]$ where h is generic for ω_2-closed forcing. C serves as witness that $F \in j(\mathbb{Q})$ so we may force with $j(\mathbb{Q})/F$ to obtain a generic g^+ and then lift to get $j : V[G * g] \longrightarrow M[G * g * C * h * g^+]$.

This elementary embedding exists in a generic extension of $V[G * g]$ by countably closed forcing, so exactly as in Theorem 10.8 in $V[G * g]$ every stationary set in $\kappa \cap \mathrm{Cof}(\omega)$ reflects to a point of cofinality ω_2. By construction we also have $\mathrm{SNR}(\omega_1, \omega_2, \omega_3)$ in $V[G * g]$ so we are done. ⊣

As a third example we sketch Magidor's proof that consistently every stationary subset of $\omega_{\omega+1}$ reflects.

14.10 Theorem. *If it is consistent that there exist ω supercompact cardinals then it is consistent that every stationary subset of $\omega_{\omega+1}$ reflects.*

Proof. We start by fixing an increasing sequence $\langle \kappa_n : 0 < n < \omega \rangle$ of supercompact cardinals. We also fix $j_n : V \longrightarrow M_n$ witnessing that κ_n is λ^+-supercompact where $\lambda = \sup_n \kappa_n$. We then define a full support iteration of length ω by setting $\mathbb{P}_1 = \mathbb{Q}_0 = \mathrm{Col}(\omega, {<}\kappa_1)$, $\mathbb{Q}_n = \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})_{V^{\mathbb{P}_n}}$ for all $n > 0$, $\mathbb{P}_{n+1} = \mathbb{P}_n * \mathbb{Q}_n$, $\mathbb{P}_\omega = \varprojlim \mathbb{P}_n$.

Let G_ω be $\mathbb{P}_\omega$ generic, let G_n be the $\mathbb{P}_n$-generic filter induced by G_ω and let g_n be the corresponding $\mathbb{Q}_n$-generic filter over $V[G_n]$. The following claims are easy:

- $\kappa_n = \omega_n$, $\lambda = \omega_\omega$, and $\lambda^+ = \omega_{\omega+1}$ in $V[G_\omega]$.
- For every n, $\mathbb{P}_\omega / G_n$ is κ_n-directed-closed in $V[G_n]$.
- For every $n > 0$ let us factor G_ω as $G_{n-1} * g_{n-1} * H_n$. Then j_n can be lifted to an elementary embedding $j_n : V[G_{n-1}] \longrightarrow M_n[G_{n-1}]$, and in $V[G_{n-1}]$ we may embed $\mathbb{P}_\omega / G_{n-1}$ into $j_n(\mathbb{Q}_n)$ so that the quotient forcing is κ_{n-1}-closed.

It follows from this discussion that we may lift j_n to an embedding with domain $V[G_\omega]$ in three steps:

1. Lift to $j_n : V[G_{n-1}] \longrightarrow M_n[G_{n-1}]$.

2. Lift to $j_n : V[G_{n-1} * g_{n-1}] \longrightarrow M_n[G_{n-1} * g_{n-1} * H_n * I_n] = M_n[j_n(G_n)]$ where I_n is generic over $V[G_\omega]$ for κ_{n-1}-closed forcing.

3. Use the closure of M_n to show that j_n“$H_n \in M_n[j_n(G_n)]$, and then use the fact that $j_n(\kappa_n) > |H_n|$ and directedness to find a suitable strong master condition r. Then force with $j_n(\mathbb{P}_\omega/G_n)/r$ and lift once more to $j_n : V[G_n * H_n] \longrightarrow M_n[j_n(G_n) * j_n(H_n)]$.

The key points are that

1. By forcing over $V[G_\omega]$ with κ_{n-1}-closed forcing we have added a generic embedding $j_n : V[G_\omega] \longrightarrow M_n[j_n(G_\omega)]$ with critical point κ_n.

2. j_n“$\lambda^+ \in M_n$.

It remains to argue that in $V[G_\omega]$ every stationary subset of λ^+ reflects. By the completeness of the club filter, every stationary set in λ^+ has a stationary subset of ordinals with a constant cofinality, so it will suffice to show that for all n any stationary subset S of $\lambda^+ \cap \mathrm{Cof}(\omega_n)$ reflects.

We consider the generic embedding $j_{n+2} : V[G_\omega] \longrightarrow M_n[j_n(G_\omega)]$ constructed above. It is easy to see that if $\gamma = \sup j_n$“λ^+ then $\gamma < j_n(\lambda^+)$ and j“$S \cap \gamma$ is stationary in $M_n[G_\omega]$, because the map j_{n+2} is continuous at points of cofinality κ_n. The only problem is to see that S (and hence j“$S \cap \gamma$) is still stationary in $M_n[j_n(G_\omega)]$, so it will certainly suffice to see that the stationarity of S is preserved by any ω_{n+1}-closed forcing.

Unfortunately it is not true in general [10] that κ^+-closed forcing preserves stationary subsets of $\mu \cap \mathrm{Cof}(\kappa)$ when μ is the successor of a singular cardinal. We address this problem using an idea of Shelah to show that in our model $V[G_\omega]$ every stationary subset of $\omega_{\omega+1} \cap \mathrm{Cof}(\omega_n)$ is preserved by ω_{n+1}-closed forcing.

We start by fixing in V for every $\beta < \lambda^+$ a decomposition $\beta = \bigcup_{i<\omega} b_i^\beta$ where the b_i^β are disjoint and $|b_i^\beta| \leq \kappa_i$. We define $F(\alpha, \beta)$ to be the unique $i < \omega$ with $\alpha \in b_i^\beta$. The key technical claim is that in $V[G_\omega]$ any ordinal $\rho < \lambda^+$ with uncountable cofinality contains an unbounded homogeneous set for F.

We fix such a ρ and let n be the unique integer such that in V we have $\kappa_n \leq \mathrm{cf}(\rho) < \kappa_{n+1}$. We note that if $\sigma =_{\mathrm{def}} \sup(j_n$“$\rho)$ then $\sigma < j(\rho)$, and so we may define in V an ultrafilter $U =_{\mathrm{def}} \{A \subseteq \rho : \sigma \in j(A)\}$. Clearly $\rho \setminus \alpha \in U$ for all $\alpha < \rho$, and U is κ_n-complete in V.

We now fix for each $\alpha \in \rho$ a U-large set A_α on which $F(\alpha, -)$ is constant. In $V[G_\omega]$ we have that $\mathrm{cf}(\rho) = \kappa_n$, and we will build by recursion an increasing and cofinal sequence $\langle \alpha_i : i < \kappa_n \rangle$ in ρ such that $\alpha_j \in A_{\alpha_i}$ for $i < j$. This is possible because $\langle A_{\alpha_i} : i < j \rangle$ is in $V[G_n]$ so is covered by a subset Y_j of U which lies in V and has size less than κ_n; the intersection of Y_j is in U, and any element of this intersection will do as α_j. It is then easy to thin out the “tail homogeneous” sequence of α_i to a cofinal homogeneous set.

To finish we show the needed stationary preservation fact. We work in $V[G_\omega]$. Let $T \subseteq \lambda^+ \cap \mathrm{Cof}(\kappa_n)$ be stationary, let $\mathbb{Q}$ be κ_{n+1}-closed, let $\dot{C}$ be $\mathbb{Q}$-name for a club subset of λ^+. We build $N \prec H_\theta$ for some large θ such that N contains all the relevant parameters, $|N| = \lambda$, all bounded subsets of λ are in N and $\delta = N \cap \lambda^+ \in T$. Fix $A \subseteq \delta$ a cofinal set of order type κ_n and $i \in \omega$ so that A is i-homogeneous for F. We claim that all proper initial segments of A lie in N: for if $\beta \in A$ then $A \cap \beta \subseteq b_i^\beta$, and since $b_i^\beta \in N$ with $|b_i^\beta| \leq \kappa_i$ and also $P(\kappa_i) \subseteq N$ we see easily that $A \cap \beta \in N$.

The endgame of the argument is now very similar to the proof of Lemma 10.6. We enumerate the elements of A in increasing order as α_i for $i < \kappa_n$. We then build a decreasing sequence $\langle q_j : j < \kappa_n \rangle$ of conditions in $\mathbb{Q} \cap N$, where q_j is the least condition which both determines $\min(\dot{C} \setminus \alpha_j)$ and is below q_i for all $i < j$. We need to see that $q_j \in N$ for all $j < \kappa_n$; the key point is that $\langle q_i : i < j \rangle$ is definable from $A \cap \alpha_j$, and so can be computed in N. To finish we choose q a lower bound for $\langle q_i : i < \kappa_n \rangle$, and observe that $q \Vdash \alpha \in \dot{C} \cap T$. $\dashv$

15. Transfer and Pullback

It is sometimes possible to transfer a generic filter over one model to another model along an elementary embedding, and then to lift that elementary embedding. The following proposition makes this precise

15.1 Proposition. *Let $k : M \longrightarrow N$ have width $\leq \mu$, and let $\mathbb{P} \in M$ be a separative notion of forcing such that*

$$M \models \text{“}\mathbb{P} \textit{ is } (\mu^+, \infty)\textit{-distributive”}.$$

Let G be $\mathbb{P}$-generic over M and let H be the filter on $k(\mathbb{P})$ which is generated by k“G. Then H is $k(\mathbb{P})$-generic over N.

Proof. Let $D \in N$ be a dense open subset of $k(\mathbb{P})$. Let $D = k(F)(a)$ for some $a \in N$ and some $F \in M$ such that $M \models |\mathrm{dom}(F)| \leq \mu$; we may as well assume that for every $x \in \mathrm{dom}(F)$, $F(x)$ is a dense open subset of $\mathbb{P}$.

Now let $E = \bigcap_{x \in \mathrm{dom}(F)} F(x)$. By the distributivity assumption E is a dense subset of $\mathbb{P}$, and clearly $E \in M$, so that $E \cap G \neq \emptyset$. If $p \in G \cap E$ then $k(p) \in k(F)(a) = D$, so that $H \cap D \neq \emptyset$ and so H is generic as claimed. $\dashv$

15.2 Remark. Given the conclusion of the last proposition, it follows from Proposition 9.1 that k can be lifted to get $k^+ : M[G] \longrightarrow N[H]$.

As an example of this proposition in action, we prove a result reminiscent of Lemma 8.5.

15.3 Lemma. *Let* GCH *hold. Let E be a (κ, κ^{++}) extender and let the map $j : V \longrightarrow M = \mathrm{Ult}(V, E)$ be the ultrapower. Let $\mathbb{Q} = \mathrm{Col}(\kappa^{+++}, {<}j(\kappa))_M$. Then there is a filter $g \in V$ which is $\mathbb{Q}$-generic over M.*

Proof. Let $U = \{X \subseteq \kappa : \kappa \in j(X)\}$ and let $i : V \longrightarrow N = \mathrm{Ult}(V, U)$. As in Proposition 3.2 we may define an elementary embedding $k : N \longrightarrow M$ by $k([F]_U) = j(F)(\kappa)$, and $j = k \circ i$.

Let $\lambda = \kappa_N^{++}$. It is easy to see that

$$\begin{aligned} M &= \{j(F)(a) : a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(F) = [\kappa]^{|a|}\} \\ &= \{k(H)(a) : a \in [\kappa^{++}]^{<\omega}, \mathrm{dom}(H) = [\lambda]^{|a|}\}. \end{aligned}$$

It follows that k is an embedding of width at most λ.

Now let $\mathbb{Q}_0 = \mathrm{Col}(\lambda^+, {<}i(\kappa))_N$, and notice that $k(\mathbb{Q}_0) = \mathbb{Q}$. By exactly the same argument as in Lemma 8.5 there is $g_0 \in V$ which is $\mathbb{Q}_0$-generic over N. By Proposition 15.1 $k``g_0$ generates a filter g which is $\mathbb{Q}$-generic over M. ⊣

15.4 Remark. This lemma can be used to construct posets along the lines of the generalized Prikry forcing from Example 8.6, collapsing κ to become for example ω_{ω_1}. See [8] and [27] for details.

15.5 Remark. See Sects. 22 and 25 for applications of Proposition 15.1 in reverse Easton constructions.

Proposition 15.1 admits a kind of dual in which the traffic goes the other way:

15.6 Proposition. *Let $k : M \longrightarrow N$ have critical point δ, let $\mathbb{P} \in M$ be a notion of forcing such that*

$$M \models \text{“}\mathbb{P} \text{ is } \delta\text{-c.c.”}.$$

Let H be $k(\mathbb{P})$-generic over N and let $G = k^{-1}``H$. Then G is $\mathbb{P}$-generic over M.

Proof. Let $A \in M$ be a maximal antichain of $\mathbb{P}$. Then $k(A) = k``A$ and it is maximal in $k(\mathbb{P})$, so $k``A$ meets H and hence A meets G. It is routine to check that G is a filter. ⊣

16. Small Large Cardinals

One of the main themes of this chapter has been preservation of large cardinal axioms in forcing extensions, using the characterization of those large cardinal axioms in terms of elementary embeddings. It might seem that this method can only work for large cardinal hypotheses at least as strong as the existence of a measurable cardinal, because after all the critical point of any definable $j : V \longrightarrow M$ is always measurable (and even the existence of a generic embedding $j : V \longrightarrow M \subseteq V[G]$ implies the existence of an inner model with a measurable cardinal).

However it turns out that we can get down to the level of weakly compact cardinals by working with elementary embeddings whose domains are sets which do not contain the full power set of the critical point. We record a number of equivalent characterizations of weak compactness. The last one (which is due to Hauser [34]) has the surprising feature that the target model of the embedding contains the embedding itself, a fact which can be used to good effect in master condition arguments [35, 34].

16.1 Theorem. *The following are equivalent for an inaccessible cardinal κ:*

1. *κ is weakly compact.*
2. *κ is Π^1_1-indescribable.*
3. *κ has the tree property.*
4. *For every transitive set M with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \longrightarrow N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$ and $\mathrm{crit}(j) = \kappa$.*
5. *For every transitive set M with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \longrightarrow N$ where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$, $\mathrm{crit}(j) = \kappa$ and in addition j and M are both elements of N.*

Proof. The equivalence of the first four statements is standard [43]. So we only show that the last one follows from weak compactness. Given M a transitive set with $|M| = \kappa \in M$ and ${}^{<\kappa}M \subseteq M$ we find a transitive $\bar{M}$ with the same properties so that $M \in \bar{M}$. We fix in $\bar{M}$ a well founded relation E on κ so that (κ, E) collapses to $(M, \in)$.

By weak compactness we may find an embedding $j : \bar{M} \longrightarrow \bar{N}$ with critical point κ such that $|\bar{N}| = \kappa$ and ${}^{<\kappa}\bar{N} \subseteq \bar{N}$. Let $N = j(M)$ and $i = j\restriction M$ so that $i : M \longrightarrow N$ is elementary. Since $j(E) \in \bar{N}$ it is easy to see that M and i are both in $\bar{N}$; but by elementarity N is closed under κ-sequences in $\bar{N}$ so that M and i are in N. ⊣

16.2 Example. We show that it is consistent for the first failure of GCH to occur at a weakly compact cardinal. This needs a little work. For example if $V = L$ and we add κ^{++} Cohen subsets to a weakly compact cardinal κ then this destroys the weak compactness of κ. The point is that for $X \subseteq \kappa$ the statement "$X \notin L$" is Π^1_1 in $(V_\kappa, \in, X)$, so that if κ is weakly compact and $X \notin L$ then by Π^1_1-indescribability some initial segment of X is not in L.

We will assume that GCH holds in V and that κ is weakly compact. We will force with Easton support to add α many Cohen subsets to each inaccessible $\alpha < \kappa$, and will then add κ^{++} many Cohen subsets to κ. Let $\mathbb{P}_\kappa$ be the iteration up to κ and let $\mathbb{Q} = \mathrm{Add}(\kappa, \kappa^{++})_{V^{\mathbb{P}_\kappa}}$. Let G be $\mathbb{P}_\kappa$-generic over V and let g be $\mathbb{Q}$-generic over $V[G]$.

For the sake of variety we show that κ has the tree property in $V[G * g]$. Let $T \in V[G*g]$ be a κ-tree. T is essentially a subset of κ, and so by the κ^+-c.c. there is in V a set $X \subseteq \kappa^{++}$ with $|X| = \kappa$ such that $T \in V[G * g_0]$ where $g_0 = g \restriction (\kappa \times X)$. Without loss of generality we may as well assume that $X = \kappa$. So now $T \in V[G * g_0]$, where $g_0 = g \restriction (\kappa \times \kappa)$ and g_0 is $\mathbb{Q}_0 = \mathrm{Add}(\kappa, \kappa)$-generic.

Working in V we fix a suitable transitive model M such that $\dot{T} \in M$, and then choose $j : M \longrightarrow N$ as in clause 5 of Theorem 16.1. We now proceed to lift j. We need to be slightly careful about issues of closure. Our models are less closed than in the context of measurable cardinals, but since they are themselves small sets this is not a problem.

Since $\mathbb{P}_\kappa$ is κ-c.c. and $V \models {}^{<\kappa}N \subseteq N$, we have by Proposition 8.4 that $V[G] \models {}^{<\kappa}N[G] \subseteq N[G]$. $\mathbb{Q}_0$ adds no $<\kappa$-sequences so by Proposition 8.2 $V[G * g_0] \models {}^{<\kappa}N[G * g_0] \subseteq N[G * g_0]$. Since $|N[G * g_0]| = \kappa$ and the factor iteration $j(\mathbb{P}_\kappa)/G * g_0$ is $<\kappa$-closed in $V[G * g_0]$, we may as usual build $H \in V[G * g_0]$ suitably generic and lift to get $j : M[G] \longrightarrow N[G * g_0 * H]$. As usual $V[G * g_0] \models {}^{<\kappa}N[G * g_0 * H] \subseteq N[G * g_0 * H]$. Finally since $j \restriction g_0 = \mathrm{id}_{g_0}$ we may use $r = \bigcup g_0$ as a strong master condition, construct a suitable generic filter for $j(\mathbb{Q}_0)/R$ and lift the embedding onto $M[G * g_0]$. Since $j(T) \restriction \kappa = T$, we may use any point on level κ of $j(T)$ to generate a cofinal branch of T lying in $V[G * g_0]$.

17. Precipitous Ideals I

In this section we prove some theorems about precipitous ideals due to Jech, Magidor, Mitchell and Prikry [41]. As a warm-up we show it is consistent that there exists a precipitous ideal (precipitousness is defined below) on ω_1, then we show that the non-stationary ideal on ω_1 can be precipitous. The hypothesis used is the existence of a measurable cardinal, which is known [41] to be optimal.

The proof has several very interesting technical features including:

- The use of the universal properties of the Levy collapsing poset, an idea which goes back to Solovay's proof that every set of reals can be measurable [65].
- The use of forcing to add simultaneously filters G and H such that an embedding $M \longrightarrow N$ lifts to an embedding $M[G] \longrightarrow N[H]$.
- The use of an iterated club-shooting forcing to make the club filter exhibit properties that are characteristic of filters derived from elementary embeddings.

17.1. A Precipitous Ideal on ω_1

We refer readers to Foreman's chapter in this Handbook for the basic theory of precipitous ideals. We recall that if I is an ideal on κ then we may force with the forcing poset $P(\kappa)/I \setminus \{0\}$ (equivalence classes of I-positive sets modulo I) to add a V-ultrafilter U such that $U \cap I = \emptyset$. Working in $V[U]$ we may then form $\mathrm{Ult}(V, U)$ using functions in V ordered modulo U. The ideal I is said to be *precipitous* if and only if $\mathrm{Ult}(V, U)$ is forced to be well-founded. We will follow a common practice and abuse notation by saying that the ultrafilter U is "$P(\kappa)/I$-generic".

The following fact is key for us: to show that an ideal I on a cardinal κ is precipitous, it suffices to produce (typically by forcing) for every $A \notin I$ a V-ultrafilter U on κ such that $A \in U$, U is $P(\kappa)/I$-generic, and $\mathrm{Ult}(V, U)$ is well-founded. The point is that if I fails to be precipitous there is $A \notin I$ which forces this, and for such an A no U as above can exist.

We will reuse an example from earlier in this chapter. Assume that κ is measurable, and let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map from a normal measure U on κ. Let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$. Then as we saw in Theorem 10.2:

1. $j(\mathbb{P})$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$ where $\mathbb{Q}$ is the poset which adds a surjection from ω onto each ordinal in $[\kappa, j(\kappa))$ with finite conditions. We will usually be careless and identify the posets $j(\mathbb{P})$ and $\mathbb{P} \times \mathbb{Q}$.

2. If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over $V[G]$ then $G * H$ is $j(\mathbb{P})$-generic over V, and $j``G \subseteq G * H$, so in $V[G * H]$ we can lift our original j to $j_G : V[G] \longrightarrow M[G * H]$ with $j_G(G) = G * H$. So from the point of view of $V[G]$ the embedding j_G is a generic embedding added by forcing with $\mathbb{Q}$.

3. Since $M = \{j(f)(\kappa) : f \in V\}$ we have that
$$M[G * H] = \{j_G(f)(\kappa) : f \in V[G]\},$$
so that $M[G * H]$ is the ultrapower $\mathrm{Ult}(V[G], U_G)$ where
$$U_G = \{X \in P(\kappa) \cap V[G] : \kappa \in j_G(X)\}.$$
Here U_G is a $V[G]$-normal $V[G]$-κ-complete $V[G]$-ultrafilter and j_G is the associated ultrapower map.

Foreman's chapter in this Handbook gives a rather general framework for defining precipitous ideals by way of generic elementary embeddings. In the interests of being self-contained, we describe how this plays out in the setting of the embedding from Theorem 10.2.

We caution the reader that the following arguments involve viewing the universe $V[G * H]$ both as an extension of V by $j(\mathbb{P})$ and an extension of $V[G]$ by $\mathbb{Q}$. We are quietly identifying $j(\mathbb{P})$-names in V with $\mathbb{Q}$-names in $V[G]$, resolving any possible confusion by making explicit which model we are forcing over and with which poset.

Working in $V[G]$ we define an ideal I on $\omega_1^{V[G]} (= \kappa)$ by
$$I = \{X \subseteq \kappa : \Vdash_{\mathbb{Q}}^{V[G]} \kappa \notin j_G(X)\}.$$

Equivalently I consists of those sets which are forced by $\mathbb{Q}$ not to be in the ultrafilter U_G.

Working in $V[G]$ we define a Boolean algebra homomorphism from $P(\kappa)$ to $\mathrm{ro}(\mathbb{Q})$ which maps X to the truth value $[X \in \dot{U}_G]_{\mathrm{ro}(\mathbb{Q})}$. The kernel of this map is exactly I so we may induce a map ι from $P(\kappa)/I$ to $\mathrm{ro}(\mathbb{Q})$.

The key point is that, as we see in a moment, the range of ι is dense. From this it follows that for any H which is $\mathbb{Q}$-generic over $V[G]$, the ultrafilter U_G is $P(\kappa)/I$-generic over $V[G]$; in fact it follows from the truth lemma that $X \in U_G \iff \iota(X) \in H$, so that in a very explicit way forcing with $\mathbb{Q}$ is equivalent to forcing with $P(\kappa)/I$.

To establish that the range of ι is dense recall that $\mathbb{Q}$ is densely embedded in $\mathrm{ro}(\mathbb{Q})$. Let $q \in \mathbb{Q}$, so that $q = j(F)(\kappa)$ where $F \in V$ is a function such that $F(\alpha) \in \mathrm{Col}(\omega, [\alpha, \kappa))$ for all $\alpha < \kappa$. Working in $V[G]$, define $X = \{\alpha : F(\alpha) \in G\}$. Since j_G extends j, $q = j_G(F)(\kappa)$ and so for any H we have that

$$\iota(X) \in H \quad \iff \quad \kappa \in j_G(X) \quad \iff \quad q \in j_G(G) \quad \iff \quad q \in H.$$

It is an immediate conclusion that I must be precipitous. For if $A \notin I$ then we may choose H inducing j_G such that $\kappa \in j_G(A)$. Arguing as above we get that U_G is $P(\kappa)/I$-generic with $A \in U_G$, so we are done.

It is interesting to note that the ideal I is precisely the ideal generated in $V[G]$ by the ideal dual to the ultrafilter U. It is immediate that I contains this ideal, so we only need to prove that I is contained in this ideal.

Let $p \Vdash^V_{\mathbb{P}} \dot{X} \in \dot{I}$. We claim that if we define $A = \{\alpha : p \Vdash \alpha \notin \dot{X}\}$ then $A \in U$. For if not then we may define a function F on A^c such that $F(\alpha) \leq p$ and $F(\alpha) \Vdash \alpha \in \dot{X}$. But then if we let $q = j(F)(\kappa)$ and force to get $G * H$ containing q, we obtain a situation in which $p \in G$ and yet $\kappa \in j_G(X)$, so that $X \in U_G$ and we have a contradiction.

17.1 Remark. Really we have just worked through a very special case of Foreman's Duality Theorem. See Foreman's chapter in this Handbook for more on this subject.

17.2. Iterated Club Shooting

In Sect. 17.1 we produced a precipitous ideal I on ω_1. It is not hard to see that this ideal is not the non-stationary ideal. For example if in V we define $S = \kappa \cap \mathrm{Cof}(\omega)$, then S is stationary in $V[G]$ by the κ-c.c. of the collapsing poset $\mathbb{P}$. Since $\kappa \notin j(S)$ we see that $\Vdash_{\mathbb{Q}} \check{S} \notin \dot{U}_G$, so that $S \in I$.

To make the non-stationary ideal precipitous, we will iteratively shoot clubs so as to destroy the stationarity of inconvenient sets such as the set S from the last paragraph. The argument is somewhat technical so we give an overview before launching into the details.

Overview

Working in $V[G]$ we build a countable support iteration $\mathbb{R}$ of length κ^+ (which is the ω_2 of $V[G]$). At each stage we shoot a club set through some stationary subset of ω_1. A key point will be that this iteration adds no ω-sequences of ordinals; from this it will follow that

1. ω_1 is preserved.

2. At each stage of the iteration $\mathbb{R}$, the conditions in the club shooting forcing used at that stage (which are closed and bounded subsets of ω_1) actually lie in the model $V[G]$.

Recall from Sect. 17.1 that $\mathbb{P} = \mathrm{Col}(\omega, <\kappa)$, $j(\mathbb{P}) = \mathbb{P} \times \mathbb{Q}$, and for any H which is $\mathbb{Q}$-generic over $V[G]$ we may lift $j : V \to M$ and get $j_G : V[G] \to M[G*H]$. It is the existence of these generic embeddings which is responsible for the precipitousness of I in $V[G]$.

For each α we will find an embedding of $\mathbb{P} * \mathbb{R}_\alpha$ into $j(\mathbb{P})$, and we will use this to produce generic embeddings $j_\alpha : V[G * g_\alpha] \to M[G * H * h_\alpha]$ where g_α is $\mathbb{R}_\alpha$-generic and $G * g_\alpha$ is embedded into $G * H$. From these embeddings j_α we will define normal ideals $I_\alpha \in V[G * g_\alpha]$ which are analogous to the ideal I; the construction will be organized so that

1. I_α increases with α.

2. In the final model $\bigcup_\alpha I_\alpha$ is the non-stationary ideal.

To finish, we will use the embeddings j_α to show that the non-stationary ideal is precipitous. This argument, which is similar to but more complicated than that of Sect. 17.1, appears as Lemma 17.3 below.

Details

Working in $V[G]$ we construct by induction on $\alpha < \kappa^+$

1. A countable support iteration $\mathbb{R}_\alpha$.

2. An embedding $i_\alpha : \mathbb{P} * \mathbb{R}_\alpha \to j(\mathbb{P})$ extending the identity embedding of $\mathbb{P}$ into $j(\mathbb{P})$. Our convention in what follows is that H is always some $\mathbb{Q}$-generic filter over $V[G]$ and g_α is always the $\mathbb{R}_\alpha$-generic filter induced by H via the embedding i_α.

3. A $\mathbb{Q}$-name $\dot{D}_\alpha$ for a strong master condition appropriate for the embedding $j_G : V[G] \to M[G * H]$ and generic object g_α; that is to say $\dot{D}_\alpha$ names a condition in $j(\mathbb{R}_\alpha)$ which is a lower bound for $j``g_\alpha$.

4. A $\mathbb{Q} * j(\mathbb{R}_\alpha)/D_\alpha$-name for $j_\alpha : V[G * g_\alpha] \to M[G * H * h_\alpha]$ extending $j_G : V[G] \to M[G * H]$.

5. An $\mathbb{R}_\alpha$-name $\dot{I}_\alpha$ for a normal ideal on κ; this is defined as a *master condition ideal* of the sort discussed in Foreman's chapter of this Handbook; to be more precise, it is defined to be the set of those $X \subseteq \kappa$ in $V[G * g_\alpha]$ such that it is forced over $V[G * g_\alpha]$ by $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$ that $\kappa \notin j_\alpha(X)$.

6. An $\mathbb{R}_\alpha$-name $\dot{S}_\alpha$ for a set in the filter dual to the ideal I_α.

We will maintain the hypotheses that

1. $\mathbb{R}_\alpha$ adds no ω-sequences of ordinals and has the κ^+-c.c.

2. For $\beta < \gamma \leq \alpha$

   (a) i_γ extends i_β (from which it follows that g_γ extends g_β).

   (b) It is forced over $V[G]$ (by the appropriate forcing posets) that $D_\gamma \restriction j(\beta) = D_\beta$, $j_\gamma \restriction V[G * g_\beta] = j_\beta$, and $I_\gamma \cap V[G_\beta] = I_\beta$.

3. The set of *flat* conditions is dense in $\mathbb{R}_\alpha$, where a condition r in $\mathbb{R}_\alpha$ is *flat* if

   (a) For every η in the support of r, $r(\eta)$ is a canonical $\mathbb{R}_\eta$-name $\check{d}_\eta$ for some $d_\eta \in V[G]$, where d_η is a closed and bounded subset of κ.

   (b) There is an ordinal $\gamma < \kappa$ such that $\gamma = \max(d_\eta)$ for every η in the support of p.

We will explain why some of the hypotheses are maintained and then give the details of the construction. Since D_α is a strong master condition for $\mathbb{R}_\alpha$ it follows from Theorem 12.5 that forcing with $\mathbb{R}_\alpha$ adds no ω-sequences of ordinals. As we see shortly D_α is flat, and since $D_\alpha \leq j``g_\alpha$ it follows from elementarity that the set of flat conditions is dense. Standard Δ-system arguments show that the set of flat conditions has the κ^+-c.c. and so since this set is dense $\mathbb{R}_\alpha$ has the κ^+-c.c. The remaining "coherence" hypotheses will be satisfied by construction.

- At successor stages we take $\mathbb{R}_{\alpha+1} \simeq \mathbb{R}_\alpha * \mathrm{CU}(\kappa, S_\alpha)$. For α limit, $\mathbb{R}_\alpha$ is constructed as the direct limit of $\langle \mathbb{R}_\beta : \beta < \alpha \rangle$ if α has uncountable cofinality and the inverse limit if α has countable cofinality.

- i_α is defined using the universal properties of the Levy collapse as in Theorem 14.2.

- Recall that j_α exists in the extension of $V[G * g_\alpha]$ by the forcing poset $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$. Working in $V[G * g_\alpha]$ we define I_α to be the ideal of those $X \subseteq \kappa$ such that it is forced over $V[G * g_\alpha]$ by $j(\mathbb{P})/(G * g_\alpha) * j(\mathbb{R}_\alpha)/D_\alpha$ that $\kappa \notin j_\alpha(X)$.

- Let $\langle C_\eta : \eta < \alpha \rangle$ be the sequence of club sets added by g_α. We construct D_α as follows: the support of D_α is $j``\alpha$, and $D_\alpha(j(\eta))$ is the canonical $j(\mathbb{R}_\eta)$-name for $C_\eta \cup \{\kappa\}$.

  Of course we need to check that D_α is a strong master condition. The salient points are that

  - Since $\alpha < \kappa^+$, $j``\alpha \in M$.
  - Since $\kappa^+ < j(\kappa)$, $j``\alpha$ is countable in $M[G * H]$. In particular D_α has countable support.
  - If $f \in g_\alpha$ then $f \in V[G]$ and the support of f has size less than $\mathrm{crit}(j_G)$. Hence the support of $j_G(f)$ is $j``\mathrm{dom}(f)$.
  - For every $\beta < \alpha$, $D_\beta = D_\alpha \restriction j(\beta)$ is a lower bound for $j``g_\beta$. In particular it is immediate for α limit that D_α is a strong master condition, so we may concentrate on the case when $\alpha = \beta + 1$.

    Recall that by induction $\mathbb{P}_\beta$ adds no ω-sequences of ordinals. Let $r \in g_\alpha$, then it follows from the distributivity of $\mathbb{P}_\beta$ that we may write $r = r_0 {}^\frown r_1$ where $r_0 \in g_\beta$ and (without loss of generality) r_1 is the canonical name for $C_\beta \cap (\eta + 1)$ where $\eta \in C_\beta$. By induction $D_\beta \leq j(r_0)$. By the distributivity of $\mathbb{P}_\beta$ again, every initial segment of C_β is in $V[G]$, and is fixed by j_G. So $D_\beta(= D_\alpha \restriction j(\beta))$ forces that $D_\alpha(j(\beta))$ is an end-extension of $j(r_1)$.

    It only remains to check that D_β forces that $C_\beta \cup \{\kappa\}$ is a legitimate condition in $j_\beta(\mathrm{CU}(\kappa, S_\beta))$. But this is immediate because S_β was chosen to lie in the filter dual to I_β.

- D_α is flat. This is straightforward: we have already checked that it is a condition, and by construction each entry is a canonical name for a closed set of ordinals with maximum element κ.

- j_α is defined in the standard way as a lifting of $j_G : V[G] \to M[G * H]$. The fact that D_α is a strong master condition ensures the necessary compatibility of generic filters.

- The sets S_α are chosen according to a suitable book-keeping scheme so that after κ^+ steps it is forced that every element of $\bigcup_\alpha I_\alpha$ has become non-stationary.

17.2 Remark. The trick of using a dense subset of conditions which are "flat" (in a suitable sense) is very often useful in situations when we are iteratively shooting clubs through stationary sets. It would have been tempting to define $\mathbb{P}_\alpha$ as the set of flat conditions with a suitable ordering, but this raises problems of its own; in particular we would have needed to verify that the flat poset is an iteration of club shooting forcing, which amounts to showing that flat conditions are dense.

In several subsequent arguments that involve iterated club shooting we have cheated (in a harmless way) by just defining the set of flat conditions, and leaving it to the reader to check that this set is dense in the corresponding iteration. See in particular Lemma 17.3 and the results of Sects. 18 and 19.

Precipitousness

It remains to see that the non-stationary ideal is precipitous in $V[G * g_{\kappa^+}]$. The argument runs parallel to that for the precipitous ideal in the preceding section, but is harder because we now need a generic elementary embedding with domain $V[G * g_{\kappa^+}]$. The main technical difficulties are that

1. $j``\kappa^+ \notin M$, indeed it is cofinal in $j(\kappa^+)$, so that we cannot hope to cover it by any countable set in $M[G * H]$. So there is no chance of building a strong master condition.

2. The method for doing without a strong master condition which we described in Sect. 13 uses a reasonably large amount of closure but $\mathbb{P}_{\kappa^+}$ is not even countably closed.

3. $\mathbb{P}_{\kappa^+}$ is not sufficiently distributive to transfer a generic object as in Sect. 15, nor does it obey a strong enough chain condition to pull back a generic object as in Proposition 15.6.

Since we will use the same set of ideas again in Sect. 18 when we build a model in which NS_{ω_2} is precipitous, we state a rather general lemma about constructing precipitous ideals by iterated club-shooting. This is really just an abstraction of an argument from [41]. In the applications which we are making of this lemma the preparation forcing $\mathbb{P}$ will make κ into the successor of some regular $\delta < \kappa$.

17.3 Lemma. *Suppose that κ is measurable and $2^\kappa = \kappa^+$. Let U be a normal measure on κ and let $j : V \longrightarrow M$ be the associated ultrapower map. Let $\mathbb{P}$ be a κ-c.c. poset with $\mathbb{P} \subseteq V_\kappa$. As usual $\mathbb{P}$ is completely embedded in $j(\mathbb{P})$, so that if G is $\mathbb{P}$-generic and H is $j(\mathbb{P})/G$-generic then $j : V \longrightarrow M$ can be lifted to an elementary embedding $j_G : V[G] \longrightarrow M[G * H]$.*

Let $\langle \dot{\mathbb{Q}}_\alpha : \alpha \leq \kappa^+ \rangle$ be an $\mathbb{P}$-name for a sequence of forcing posets such that in $V[G]$

1. $\mathbb{Q}_\alpha$ is a complete subposet of $\mathbb{Q}_\beta$ for $\alpha \leq \beta \leq \kappa^+$.

2. Forcing with $\mathbb{Q}_\alpha$ adds no $<\kappa$-sequences of ordinals.

3. $\mathbb{Q}_{\kappa^+}$ is κ^+-c.c.

4. Every condition in $\mathbb{Q}_\alpha$ is a partial function q such that $\mathrm{dom}(q) \subseteq \alpha$, $|\mathrm{dom}(q)| < \kappa$ and $q(\eta)$ is a closed and bounded subset of κ for all $\eta \in \mathrm{dom}(q)$.

5. If g is a $\mathbb{Q}_\alpha$-generic filter then $\bigcup g$ is a sequence $\langle C_\beta : \beta < \alpha \rangle$ of club subsets of κ.

Suppose further that there are sequences $\langle i_\alpha : \alpha < \kappa^+ \rangle$ and $\langle \dot{D}_\alpha : \alpha < \kappa^+ \rangle$ such that

1. i_α is a complete embedding of $\mathbb{P} * \dot{\mathbb{Q}}_\alpha$ into $j(\mathbb{P})$, with $i_0 = \mathrm{id}$.

2. i_β extends i_α for $\alpha \leq \beta \leq \kappa^+$.

3. $\dot{D}_\alpha$ is a $j(\mathbb{P})/\mathbb{P} * \dot{\mathbb{Q}}_\alpha$-name for a condition $Q \in j_G(\mathbb{Q}_\alpha)$ such that $\mathrm{dom}(Q) = j``\alpha$, and for every $\eta \in j``\alpha$ $Q(\eta) = C_\eta \cup \{\kappa\}$, where $\langle C_\beta : \beta < \alpha \rangle$ is the sequence of club subsets of κ added by $\mathbb{Q}_\alpha$.

4. It is forced that $\dot{D}_\beta$ extends $\dot{D}_\alpha$ for $\alpha \leq \beta \leq \kappa^+$.

Let G be $\mathbb{P}$-generic over V, and let H be $j(\mathbb{P})/G$-generic over $V[G]$. For each $\alpha < \kappa^+$, let g_α be the filter on $\mathbb{Q}_\alpha$ induced by i_α and H (so that g_α is $V[G]$-generic).

For each $\nu < \kappa^+$ the hypotheses above imply that it is forced over $V[G * g_\nu]$ by $j(\mathbb{P})/G * g_\nu * j_G(\mathbb{Q}_\nu)/D_\nu$ that j_G can be extended to a generic embedding j_ν with domain $V[G * g_\nu]$. Let $J_\nu \in V[G * g_\nu]$ be the ideal of those $X \subseteq \kappa$ such that it is forced that $\kappa \notin j_\nu(X)$. Let $g = \bigcup_\nu g_\nu$ and $J = \bigcup_\nu J_\nu$.

Then

1. g is $\mathbb{Q}_{\kappa^+}$-generic over $V[G]$.

2. J is precipitous in $V[G * g]$.

17.4 Remark. In the intended applications, J will end up being the non-stationary ideal. However it may be (as will be the case in Sect. 18) that this is accomplished by a more sophisticated strategy than arranging that every $A \in J$ is disjoint from some C_i.

Proof. The induced filters g_ν are compatible in the sense that if $\rho \leq \sigma$ then $g_\rho = g_\sigma \cap \mathbb{Q}_\rho$. It follows easily from the κ^+-c.c that if we set $g = \bigcup_\rho g_\rho$ then g is $\mathbb{Q}_{\kappa^+}$-generic. Let D_ρ be the strong master condition computed from g_ρ.

We will construct K which is $j_G(\mathbb{Q}_{\kappa^+})$-generic over $M[G * H]$, and is compatible with g in the sense that $j_G``g \subseteq K$. We do this as follows: define $\mathbb{Q}^*$ to be the subset of $j_G(\mathbb{Q}_{\kappa^+})$ consisting of those F such that for some $\mu < \kappa^+$ we have $F \in j_G(\mathbb{Q}_\mu)/D_\mu$. Of course $\mathbb{Q}^* \subseteq M[G * H]$, but $\mathbb{Q}^* \notin M[G * H]$ since its definition requires a knowledge of $j \restriction \kappa^+$.

We force to get K_0 which is $\mathbb{Q}^*$-generic over $V[G * H]$. Let K be the upwards closure of K_0 in $j_G(\mathbb{Q}_{\kappa^+})$, then we claim that K is $j_G(\mathbb{Q}_{\kappa^+})$-generic over $M[G * H]$. Let $A \in M[G * H]$ be a maximal antichain in $j_G(\mathbb{Q}_{\kappa^+})$. We will show that conditions extending some element of A are dense in $\mathbb{Q}^*$. Let $F \in \mathbb{Q}^*$, and fix μ such that $F \in j_G(\mathbb{Q}_\mu)/D_\mu$. By a familiar chain condition argument we may fix a $\rho > \mu$ such that A is a maximal antichain in $j_G(\mathbb{Q}_\rho)$. Working in $M[G * H]$ we first extend F to $F' = F \cup D_\rho \in j_G(\mathbb{Q}_\rho)/D_\rho$, and then extend F' to a condition $F'' \in j(\mathbb{Q}_\rho)$ which also extends some member of A.

In the usual way we may now extend $j_G : V[G] \longrightarrow M[G * H]$ to an embedding $j^* : V[G * g] \longrightarrow M[G * H * K]$. From the point of view of the model $V[G * g]$ this is a generic embedding added by forcing with $j(\mathbb{P})/(G * g) * \mathbb{Q}^*$. It is routine to check that J is the ideal on κ induced by this generic embedding; more explicitly for every $X \in V[G * g]$

$$X \in J \quad \Longleftrightarrow \quad \Vdash^{V[G*g]}_{j(\mathbb{P})/(G*g)*\mathbb{Q}^*} \kappa \notin j^*(X).$$

For any $X \notin J$ we may therefore force to obtain some embedding j^* such that $\kappa \in j(X)$. To finish it will suffice to show that for any generic embedding j^* as above, if we define a $V[G * g]$-ultrafilter U^* by

$$U^* = \{Y : \kappa \in j^*(Y)\}$$

then U^* is $P(\kappa)/J$-generic over $V[G * g]$ and gives a well-founded ultrapower. Since as in the last section we have $\mathrm{Ult}(V[G * g], U^*) = M[G * H * K]$, the well-foundedness is immediate.

The argument that U^* is $P(\kappa)/J$-generic is similar to that from the last section but there are some extra subtleties. We note in particular that the embedding j^* is defined in a generic extension of V by $j(\mathbb{P}) * \mathbb{Q}^*$, but is a lifting of j to a map from the extension of V by $\mathbb{P} * \mathbb{Q}_{\kappa^+}$ to the extension of M by $j(\mathbb{P}) * j(\mathbb{Q}_{\kappa^+})$.

Let $\dot{\mathcal{A}}$ be a $\mathbb{P} * \mathbb{Q}_{\kappa^+}$-name for a maximal antichain of J-positive sets and suppose towards a contradiction that some condition $(P, \dot{F}) \in j(\mathbb{P}) * \mathbb{Q}^*$ forces that for every $B \in \mathcal{A}$, $\kappa \notin j^*(B)$. We will find a J-positive set T which has J-small intersection with every $B \in \mathcal{A}$, contradicting the maximality of $\mathcal{A}$.

Since $\mathbb{Q}^* \subseteq j(\mathbb{P}_{\kappa^+})$ we see that $(P, \dot{F}) \in j(\mathbb{P} * \mathbb{Q}_{\kappa^+})$, and so we may choose in V a function $R : \kappa \to \mathbb{P} * \mathbb{Q}_{\kappa^+}$ such that $j(R)(\kappa) = (P, \dot{F})$. Let $\dot{T}$ name the set $\{\alpha : R(\alpha) \in G * g\}$; then it is easy to see that it is forced by $j(\mathbb{P}) * \mathbb{Q}^*$ that for every $B \in \mathcal{A}$, $\kappa \notin j^*(B \cap T)$; the key point is that by construction $\kappa \in j^*(T) \iff (P, F) \in G * H * K$.

We now force to get $G * H$ which is $j(\mathbb{P})$-generic over V with $P \in G * H$, and from this we obtain as usual a g which is $\mathbb{Q}_{\kappa^+}$-generic over $V[G]$. Moving to $V[G * g]$ and using the fact that J is the ideal induced by j^*, we see that $B \cap T \in J$ for all $B \in \mathcal{A}$.

To finish the argument we show that $T \notin J$. By forcing over $V[G * H]$ with $\mathbb{Q}^*/F$ we obtain an elementary embedding $j^* : V[G * g] \longrightarrow M[G * H * K]$ where $(P, F) \in G * H * K$, so that $\kappa \in j^*(T)$ by the construction of R and T. Since J is the ideal induced by j^*, $T \notin J$ and we are done. $\dashv$

17.5 Remark. The technique used in this lemma is discussed in a more general and abstract setting in Foreman's chapter of this Handbook.

18. Precipitous Ideals II

In this section we discuss some work of Moti Gitik in which he obtains various results of the form "$\mathrm{NS}_\kappa \restriction \mathrm{Cof}(\mu)$ can be precipitous" from hypotheses which are optimal or close to optimal. We will describe in some detail the proof of

18.1 Theorem (Gitik [23]). *The precipitousness of* NS_{ω_2} *is equi-consistent with the existence of a cardinal of Mitchell order two.*

We will then give a much less detailed discussion of some of Gitik's consistency and equi-consistency results for cardinals greater than ω_2, which use many of the same ideas. Throughout this section, we will be using the general machinery of Lemma 17.3 to construct precipitous ideals. We will focus on the technical problems that need to be overcome to invoke this machinery, and on their solutions.

18.2 Remark. As discussed in Foreman's chapter in this Handbook, the simplest known model [21] for the precipitousness of NS_κ is obtained by taking a Woodin cardinal $\delta > \kappa$ and forcing with $\mathrm{Col}(\kappa, {<}\delta)$. The point here is to use the optimal hypotheses, which turn out to be much weaker.

18.3 Remark. We note that NS_{ω_2} is precipitous if and only if both of the restrictions $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$ and $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ are precipitous.

18.1. A Lower Bound

We start by sketching a proof of a lower bound for the strength of "NS_{ω_2} is precipitous". Suppose for a contradiction that NS_{ω_2} is precipitous and there is no inner model with a cardinal κ such that $o(\kappa) = 2$, and let K be the core model for sequences of measures constructed by Mitchell [59]. Let $\lambda = \omega_2^V$ and let $\mathcal{F}$ be the measure sequence of K. We recall the key facts that K is definable and invariant under set forcing, and that any elementary $i : K \longrightarrow N \subseteq V$ is an iterated ultrapower of K by $\mathcal{F}$.

By precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$, we may force to get a V-ultrafilter U which concentrates on ordinals of cofinality ω_1 and has $M = \mathrm{Ult}(V, U)$ well-founded. Let $j : V \longrightarrow M \subseteq V[U]$ be the ultrapower map. By the usual arguments $\mathrm{crit}(j) = \lambda = [\mathrm{id}]_U$, $P(\lambda)^V \subseteq M$, and $\mathrm{cf}^M(\lambda) = \omega_1^M = \omega_1^V$. Note also that if A is an ω-club subset of λ in V, then the same is true of A in M.

Let $i = j \restriction K$, then by the properties of K mentioned above we know that $i : K \longrightarrow K' = K^M$ and i is an iterated ultrapower of K with critical point λ. In particular λ is measurable in K, and so $\mathcal{F}(\lambda, 0)$ exists. By our initial hypotheses λ is not measurable in K', and so in K the only measure on λ is $\mathcal{F}(\lambda, 0)$. Note also that $P(\lambda)^K = P(\lambda)^{K'}$.

Let C be the ω-club filter on λ as computed in V, let $A \in \mathcal{F}(\lambda, 0)$ and let W be an arbitrary V-generic ultrafilter added by forcing with C-positive sets. Then by precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$, we get an elementary embedding $j_W : V \longrightarrow \mathrm{Ult}(V, W)$, and if $i_W = j_W \restriction K$ then i_W is an iterated ultrapower of K with critical point λ; since in K the only measure on λ is $\mathcal{F}(\lambda,0)$, we see that $\kappa \in j_W(A)$, that is, $A \in W$. Since it is forced that $A \in W$, we have that $A \in C$. So $\mathcal{F}(\lambda,0) \subseteq C \cap K$, and since the left hand side is a K-ultrafilter in fact $\mathcal{F}(\lambda,0) = C \cap K$.

Now let D be the ω-club filter on λ as computed in M. This makes sense because $\mathrm{cf}^M(\lambda) = \omega_1$. We know that $C \subseteq D$ and $P(\lambda)^K = P(\lambda)^{K'}$, so easily $\mathcal{F}(\lambda,0) = D \cap K'$. Since D is a countably complete filter in M we see that $M' = \mathrm{Ult}(K', \mathcal{F}(\lambda,0))$ is wellfounded and we get in M an elementary embedding $j' : K' \longrightarrow M'$ with critical point λ; since $K' = K^M$ this is an iteration of K', but that is impossible because λ is not measurable in K'.

18.2. Precipitousness for $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$

We have established that if NS_{ω_2} is precipitous then there is an inner model with a cardinal κ such that $o(\kappa) = 2$. We will prove that this is an equiconsistency, but before we do that we warm up with a sketch of the easier argument that starting from a measurable cardinal $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous [41].

We have already introduced in Sect. 17 most of the ideas needed to show that $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous. What is still missing is a discussion of how we should shoot club sets through stationary subsets of ω_2. The arguments of Lemmas 18.5 and 18.6 are due to Stavi (see [3]).

As we saw in Sect. 6 if S is a stationary subset of ω_1 then it is possible to add a club set C with $C \subseteq S$, using a forcing poset which does not add any ω-sequences of ordinals. Suppose now that instead S is a stationary subset of ω_2. In general we may not be able to shoot a club set through S without collapsing cardinals, for example if $S = \omega_2 \cap \mathrm{Cof}(\omega_1)$.

In a way the rather trivial example from the last paragraph is misleading. If we aim to make $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ precipitous then we need to take a stationary $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ and add, without collapsing ω_1 or ω_2, a club subset C of ω_2 such that $C \cap \mathrm{Cof}(\omega_1) \subseteq S$. This is fairly easy.

We recall that $\mathrm{CU}(\delta, A)$ is the forcing poset whose conditions are closed bounded subsets of δ which are contained in A, ordered by end-extension. We will need a technical lemma on the existence of countably closed structures.

18.4 Lemma. *Let* CH *hold and let $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ be stationary. Let θ be a large regular cardinal and let $x \in H_\theta$. Then there exists $N \prec H_\theta$ such that $\omega_1 \cup \{x\} \subseteq N$, $|N| = \omega_1$, ${}^\omega N \subseteq N$ and $N \cap \omega_2 \in S$.*

Proof. We build an increasing and continuous chain $\langle N_j : j < \omega_2 \rangle$ such that $N_j \prec H_\theta$, $\omega_1 \cup \{x\} \subseteq N_0$, $|N_j| = \omega_1$ and ${}^\omega N_j \subseteq N_{j+1}$. Since $\omega_1 \subseteq N_j$ we see that $N_j \cap \omega_2 \in \omega_2$, and so by continuity and the stationarity of S we may choose j such that $\mathrm{cf}(j) = \omega_1$ and $N_j \cap \omega_2 \in S$; it is easy to see that ${}^\omega N_j \subseteq N_j$. ⊣

18.5 Lemma. *Let* CH *hold, and let $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega_1)$ be stationary. Let $\mathbb{P} = \mathrm{CU}(\omega_2, (\omega_2 \cap \mathrm{Cof}(\omega)) \cup S)$. Then $\mathbb{P}$ is countably closed and adds no ω_1-sequences of ordinals.*

Proof. Countable closure is immediate, so suppose that c forces that $\dot{\tau}$ is a function from ω_1 to On. By Lemma 18.4 we may find $N \prec H_\theta$ for some large θ so that N contains everything relevant, $|N| = \omega_1$, ${}^\omega N \subseteq N$, and $\delta =_{\mathrm{def}} N \cap \omega_2$ lies in S. Now we build a decreasing chain of conditions $\langle c_i : i < \omega_1 \rangle$ so that $c_i \in N$, c_{i+1} decides $\tau(i)$ and the sequence $\langle \delta_i : i < \omega_1 \rangle$ where $\delta_i =_{\mathrm{def}} \max(c_i)$ is cofinal in δ. If $\lambda < \omega_1$ is a limit stage there is no problem because ${}^\omega N \subseteq N$, and we may safely choose $\delta_\lambda = \sup_{i<\lambda} \delta_i$, $c_\lambda = \bigcup_{i<\lambda} c_i \cup \{\delta_\lambda\}$. To finish we choose $d = \bigcup_{i<\omega_1} c_i \cup \{\delta\}$, which is legal since $\delta \in S$, and then d is a condition which refines c and determines $\dot{\tau}$. $\dashv$

An equivalent formulation would be that we are shooting an ω_1-club set through S by forcing with bounded ω_1-closed subsets of S. Using the forcing of Lemma 18.5 and the ideas of Sect. 17, it is now fairly straightforward to show that starting with a measurable cardinal κ we may produce a model where $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ is precipitous. We force first with $\mathrm{Col}(\omega_1, <\kappa)$ and then iterate club shooting, absorb forcing posets and construct strong master conditions more or less exactly as in Sect. 17.

If we are interested in the full ideal NS_{ω_2} then we need to shoot club sets rather than ω_1-club sets. This is more subtle; a little thought shows that if $S \subseteq \omega_2$ and we wish to shoot a club set through S without adding ω_1-sequences, then there must be stationarily many $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ such that $S \cap \alpha$ contains a closed cofinal set of order type ω_1. The next result shows that (at least under CH) this is the only obstacle.

18.6 Lemma. *Let* CH *hold, and let $S \subseteq \omega_2$ be such that for stationarily many $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ there exists a set $C \subseteq S \cap \alpha$ with C club in α. Let $\mathbb{P} = \mathrm{CU}(\omega_2, S)$. Then $\mathbb{P}$ adds no ω_1-sequences of ordinals.*

Proof. The proof is similar to that of Lemma 18.5. Let T be the stationary set of $\alpha \in S \cap \mathrm{Cof}(\omega_1)$ such that there exists a set $C \subseteq S \cap \alpha$ with C club in α. Suppose that c forces that $\dot{\tau}$ is a function from ω_1 to On. Build an elementary $N \prec H_\theta$ for some large θ so that N contains everything relevant, $|N| = \omega_1$, ${}^\omega N \subseteq N$, and $\alpha =_{\mathrm{def}} N \cap \omega_2 \in T$. By hypothesis there is a set $C \subseteq S \cap \alpha$ with C club in α. We build a strictly decreasing chain of conditions $\langle c_i : i < \omega_1 \rangle$ so that $c_i \in N$, c_{i+1} decides $\tau(i)$ and $\delta_i \in C$ where $\delta_i =_{\mathrm{def}} \max(c_i)$. To finish we choose $d = \bigcup_{i<\omega_1} c_i \cup \{\alpha\}$, so that d is a condition which refines c and determines $\dot{\tau}$. $\dashv$

18.3. Outline of the Proof and Main Technical Issues

We will use measures $U_0 \lhd U_1$ of Mitchell orders zero and one respectively. We let B be the set of $\alpha < \kappa$ with $o(\alpha) = 1$, so that $B \in U_1$ and $B \notin U_0$. We fix measures W_i for $i \in B$ so that W_i is a measure of order zero on i (so in particular $B \cap i \notin W_i$) and $\langle W_i : i \in B\rangle$ represents U_0 in $\mathrm{Ult}(V, U_1)$, or more concretely for $X \subseteq \kappa$

$$X \in U_0 \iff \{i : X \cap i \in W_i\} \in U_1.$$

The rough idea is this: we start with some preparation forcing which adds no reals, makes κ into ω_2, makes all inaccessible α lying in B into ordinals of cofinality ω_1, and makes all inaccessible α not lying in B into ordinals of cofinality ω. We then iterate shooting club sets so that U_0 extends to the ω-club filter and U_1 extends to the ω_1-club filter. Roughly speaking U_0 will be responsible for the precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega)$ and U_1 will be responsible for the precipitousness of $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$.

There are several technical obstacles to be overcome.

- In the proof sketched above that $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$ can be precipitous, the forcing which is being iterated to shoot ω_1-club subsets of ω_2 is countably closed. In particular it can be absorbed into any sufficiently large countably closed collapsing poset. This means that the "preparation stage" of the preceding construction can be the simple forcing $\mathrm{Col}(\omega_1, {<}\kappa)$. In the construction to follow we will be shooting club subsets of ω_2 in a way which destroys stationary subsets of $\omega_2 \cap \mathrm{Cof}(\omega)$, so that the forcing can not be embedded into any countably closed forcing (or even any proper forcing). This is one reason why the preparation stage for the construction to follow has to be more complicated.

- The measure U_0 will be extended to become the ω-club filter. So we need to shoot ω-club sets through (at least) all $A \in U_0$, and we will therefore need to shoot closed sets of order type ω_1 through many initial segments of A, in order to appeal to a suitable version of Lemma 18.6. We need some way of organizing the construction so that all $A \in U_0$ are anticipated.

  Recall that if $A \in U_0$ then there are many $i \in B$ such that $A \cap i \in W_i$. At many $i \in B$ we will add a club subset of i which has order type ω_1, and is eventually contained in every member of W_i.

- To build the preparation forcing, we need some way of iterating forcings which change cofinality without adding reals. This will require an appeal to Shelah's machinery of revised countable support iteration.

- In the arguments for the precipitousness of NS_{ω_1} and $\mathrm{NS}_{\omega_2} \restriction \mathrm{Cof}(\omega_1)$, we iterated to shoot club sets through stationary sets which were measure one for certain "master condition ideals" (in the sense of Foreman's chapter) arising along the way. In the current setting it is not clear that we can do this in a distributive way, so we finesse the question and shoot club sets through some more tractable sets, then argue that this is enough.

To be a bit more precise, suppose that $V[G_\kappa]$ is the result of the preparation stage. We will build a κ^+-c.c. iteration $\mathbb{Q}_{\kappa^+}$, shooting club sets through subsets of κ. At successor stages we will shoot club sets through certain sets of inaccessibles from the ground model of the form $X \cup Y$ where $X \in U_0$, $Y \in U_1$, $X \subseteq \kappa \setminus B$ and $Y \subseteq B$.

As the construction proceeds we will show, by induction on ν, that the embeddings j_i can be lifted onto the extension of $V[G_\kappa]$ by $\mathbb{Q}_\nu$. For limit ν we will write $\mathbb{Q}_\nu$ as the union $\bigcup_{\alpha<\kappa} \mathbb{Q}^\alpha_\nu$ of a continuous sequence of subsets each of size less than κ. The existence of the lifted embeddings implies that if H_ν is $\mathbb{Q}_\nu$-generic over $V[G_\kappa]$, then there are many $\alpha < \kappa$ such that $H_\nu \cap \mathbb{Q}^\alpha_\nu$ is $\mathbb{Q}^\alpha_\nu$-generic over $V[G_\alpha]$; we will ensure that a club set is shot through each such set of "generic points".

We then argue that using the club sets which are added in this process, for each set in one of the relevant master condition ideals we may define a club set which is disjoint from it. Below, at the end of Sect. 18.6, we will work through a toy example which illustrates this central idea.

- In order to realize the idea of the last item, we need that the closed sets of order type ω_1 added to points of B during the preparation stage have an additional property. Namely, if $i \in B$ and c is the club set in i which is added at stage i during the preparation, then we require that for every $\beta \in \lim(c)$ the set $c \cap \beta$ must intersect every club subset of β which lies in $V[G_\beta]$.
- Let $j_i : V \longrightarrow M_i$ be the ultrapower by U_i for $i = 0, 1$. We will need to embed $\mathbb{P}_\kappa * \mathbb{Q}_\nu$ into both $j_0(\mathbb{P}_\kappa)$ and $j_1(\mathbb{P}_\kappa)$. Naturally the iterations $j_0(\mathbb{P}_\kappa)$ and $j_1(\mathbb{P}_\kappa)$ differ at stage κ; the forcing at stage κ will change the cofinality of κ to the values ω and ω_1 respectively.

18.4. Namba Forcing, RCS Iteration and the S and $\mathbb{I}$ Conditions

Several important ingredients in the proof come from Shelah's work [64] on iterated forcing. The technical issue is that in the preparation stage we need to iterate forcing posets which change cofinalities to ω and add no reals, in such a way that the whole iteration adds no reals. A detailed discussion of Shelah's techniques would take us too far afield, so we content ourselves with a very brief overview.

The preparation iteration will be done using Shelah's *Revised Countable Support* (RCS) technology. This is a version of countable support iteration in which (very roughly speaking) we allow the supports of conditions used in the iteration to be countable sets which are not in the ground model, but arise in the course of the iteration: the point of doing this is to cope gracefully with iteration stages δ such that $\mathrm{cf}(\delta) > \omega$ in V but the cofinality of δ is changed to ω in the course of the iteration.

For motivation, consider the case of Namba forcing. The conditions are trees $T \subseteq {}^{<\omega}\omega_2$ with a unique *stem element* stem(T), such that every element of T is comparable with stem(T) and every element extending stem(T) has ω_2 immediate successors in T. Forcing with these conditions adds an ω-sequence cofinal in ω_2.

Namba [60] showed that under CH this forcing poset adds no reals; we sketch an argument for this which is due to Shelah. Let $\dot{r}$ be a name for a real and let S be a condition. We first find a refinement $T \leq S$ such that stem(T) = stem(S), and T forces that $\dot{r}(n)$ is determined by the first n points of the generic branch. We then appeal to a partition theorem for trees (proved from CH, by applying Borel determinacy to each of a family of ω_1 "cut and choose" games played on T) to find a refinement $U \subseteq S$ such that stem(U) = stem(S) and every branch through U determines the same real r, so that $U \Vdash \dot{r} = \check{r}$.

The decisive points for the arguments of the last paragraph were that Namba forcing satisfies a version of the fusion lemma and that (under CH) the ideal of bounded subsets of ω_2 is $(2^\omega)^+$-complete. Motivated by these ideas Shelah formulated a technical condition on forcing posets known as the *S-condition*, where S is some set of regular cardinals; this is an abstract form of fusion, saying *very* roughly that a tree of conditions which has cofinally many λ-branching points for each $\lambda \in S$ can be fused. Shelah also showed that under the right circumstances an RCS iteration of S-condition forcing does not add reals. The variant of Namba forcing in which conditions are subtrees of ${}^{<\omega}\omega_2$ such that cofinally many points have ω_2 successors satisfies the S-condition for $S = \{\omega_2\}$.

An important ingredient in the proof we are describing that NS_{ω_2} can be precipitous is a variant Nm' of Namba forcing. Conditions in Nm' are subtrees T of ${}^{<\omega}\omega_3$, such that for $i \in \{2, 3\}$ there are cofinally many points $t \in T$ with $\{\alpha : t^\frown\alpha \in T\}$ an unbounded subset of ω_i.

The salient facts about Nm' are encapsulated in the following result. The first fact in this list is quite hard, but the remaining ones follow easily.

18.7 Lemma. *Let* CH *hold. Then*

1. Nm' *satisfies Shelah's S-condition for $S = \{\omega_2, \omega_3\}$, in particular it adds no reals (and so preserves ω_1).*
2. Nm' *adds cofinal ω-sequences in ω_2^V and ω_3^V.*
3. *In the generic extension ω_3^V can be written as the union of ω many sets which each lie in V and have V-cardinality ω_1.*
4. *Assuming that $2^{\omega_2} = \omega_3$ in V, in the generic extension by* Nm' *there is an ω-sequence $\langle E_n \rangle$ such that*
   - (a) *$E_n \in V$ and $V \models$ "E_n is a club subset of ω_2" for each $n < \omega$.*
   - (b) *For every $E \in V$ such that $V \models$ "E is a club subset of ω_2" there is an integer n such that $E_n \subseteq E$.*

5. *Assuming that* $2^{\omega_2} = \omega_3$ *in* V, *if* $\mathbb{R}$ *is any forcing poset of size* ω_2 *which adds no* ω_1*-sequences then forcing with* Nm' *adds a generic filter for the poset* $\mathbb{R}$.

For use later we note that Gitik and Shelah defined a generalized version of the S-condition known as the $\mathbb{I}$-*condition*, where $\mathbb{I}$ is a family of ideals on some set S of regular cardinals. The $\mathbb{I}$-condition is just like the S-condition except that the branching in the fusion trees now has to be positive cofinally often with respect to every ideal in $\mathbb{I}$. Gitik and Shelah extended the iteration theorems for RCS iteration which we mentioned above to cover the $\mathbb{I}$-condition, subject to additional technical conditions.

18.5. The Preparation Iteration

We will start by forcing with an RCS iteration $\mathbb{P}_\kappa$. Among the important features of this forcing poset will be that

1. $\mathbb{P}_\kappa$ adds no reals.

2. For every inaccessible $\alpha \leq \kappa$,

   (a) $\mathbb{P}_\alpha$ is isomorphic to the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha \}$.

   (b) $\mathbb{P}_\alpha \subseteq V_\alpha$.

   (c) $\mathbb{P}_\alpha$ is α-c.c.

   (d) $\mathbb{P}_\alpha$ collapses α to become $\omega_2^{V[G_\alpha]}$.

   (e) After forcing with $\mathbb{P}_{\alpha+1}$, α is an ordinal of cardinality ω_1, which has cofinality ω for $\alpha \notin B$ and cofinality ω_1 for $\alpha \in B$.

   (f) For $\alpha \in B$, $V[G_\alpha]$ and $V[G_{\alpha+1}]$ have the same ω-sequences of ordinals.

18.8 Remark. It follows from the properties of $\mathbb{P}_\kappa$ we just listed that

1. All bounded subsets of κ in $V[G_\kappa]$ appear in $V[G_\beta]$ for some $\beta < \kappa$.

2. All elements of ${}^\omega\alpha$ which are in $V[G_\kappa]$ already appear in $V[G_{\alpha+1}]$, and if $\alpha \in B$ then such ω-sequences are actually in $V[G_\alpha]$.

As usual it suffices to define the poset which is used at each stage i of the iteration.

- Case 1: If i is not inaccessible we force with $\mathrm{Col}(\omega_1, 2^{\omega_1})^{V[G_i]}$.

- Case 2: If i is inaccessible and $i \notin B$ then we force with $(\mathrm{Nm}')^{V[G_i]}$ (where we note that in $V[G_i]$ we will have $i = \omega_2$ and $i^+ = \omega_3$).

- Case 3: If i is inaccessible and $i \in B$ then we force with $\mathbb{P}^*[W_i]$ defined as follows: conditions are pairs (c, A) such that c is a countable closed subset of $(\kappa \setminus B) \cap i$ consisting of V-inaccessibles, $A \subseteq (\kappa \setminus B) \cap i$ with $A \in W_i$, and for every $\beta \in \lim(c)$ the set $c \cap \beta$ meets every club subset of β lying in the model $V[G_\beta]$. The condition (c', A') extends (c, A) if and only if c' end-extends c, $A' \subseteq A$ and $c' - c \subseteq A$.

18.9 Remark. Let i fall under case 3, let g be a $V[G_i]$-generic subset of $\mathbb{P}^*[W_i]$ and let $e = \bigcup\{c : \exists A\ (c, A) \in g\}$. Then e is a club subset of i with order type ω_1, e is eventually contained in every element of W_i, and every element of e falls under case 2.

18.10 Remark. The definition of $\mathbb{P}^*[W_i]$ can be simplified by the observation that (by the β-c.c.) every club subset of β in $V[G_\beta]$ contains a club subset of β in V.

A key technical point (which we are glossing over here) is that the poset $\mathbb{P}^*[W_i]$ satisfies a suitable version of Gitik and Shelah's $\mathbb{I}$-condition [28]. In fact the argument we are describing was one of the main motivations for the development of the $\mathbb{I}$-condition. Once it is has been checked that Nm' has the S-condition and $\mathbb{P}^*[W_i]$ satisfies the $\mathbb{I}$-condition for suitable S and $\mathbb{I}$, an appeal to standard facts about RCS iterations lets us conclude that $\mathbb{P}_\kappa$ has the properties listed above.

18.11 Lemma. *If $i \in B$ then forcing with $\mathbb{P}^*[W_i]$ adds no ω-sequences of ordinals to $V[G_i]$.*

Sketch of Proof. Take a $\mathbb{P}_i$-name for a sequence $\langle D_n : n < \omega\rangle \in V[G_i]$ of dense open subsets of $\mathbb{P}^*[W_i]$. Working in V we fix an elementary chain of models M_β for $\beta \in i$ such that $M_\beta \prec (H_\theta, \dots)$, M_0 contains everything relevant and $M_\beta \cap i \in i$. Now we choose an inaccessible $\beta \notin B$ such that $M_\beta \cap i = \beta$ and $\beta \in A$ for every $A \in M_\beta \cap W_i$. Since $\mathbb{P}_\beta$ is β-c.c. and $\mathbb{P}_\beta \subseteq M_\beta$, routine arguments as in the theory of proper forcing show that $M_\beta[G_\beta] \prec H_\theta[G_\kappa]$ and $M_\beta[G_\beta] \cap V = M_\beta$.

As we observed already β must fall under case 2 in the definition of the preparation iteration $\mathbb{P}_\kappa$, so that by Lemma 18.7 there is in $V[G_i]$ an ω-sequence $\langle E_m : m < \omega\rangle$ which "diagonalizes" the club subsets of β lying in $V[G_\beta]$. We may now construct a sequence $\langle (c_n, A_n) : n < \omega\rangle$ of conditions in $\mathbb{P}^*[W_i] \cap M_\beta[G_\beta]$ such that $c_{2n+1} \in D_n$ and $\max(c_{2n+2}) \in E_n$. Let $d =_{\mathrm{def}} \bigcup_n c_n \cup \{\beta\}$ and $A^* = \bigcap(M_\beta \cap W_i)$, then (d, A^*) is a condition in $\mathbb{P}^*[W_i]$ which lies in the intersection of the D_n. ⊣

18.6. A Warm-up for the Main Iteration

Throughout the discussion that follows we are working in $V[G_\kappa]$, in particular $\kappa = \omega_2$ and $\kappa^+ = \omega_3$. We will eventually describe an iteration of length κ^+ in which we shoot club sets through subsets of κ without adding bounded subsets of κ. Before we do that, for purposes of motivation we will describe a much simpler three step iteration $\mathbb{R}_0 * \dot{\mathbb{R}}_1 * \dot{\mathbb{R}}_2$ of club-shooting forcing, and sketch proofs of its salient properties which contain most of the ideas needed for the full iteration.

To describe $\mathbb{R}_0$ we fix sets of inaccessibles $X \in U_0$ and $Y \in U_1$ such that $X \subseteq \kappa \setminus B$ and $Y \subseteq B$. Let $A = X \cup Y$ and define $\mathbb{R}_0 = \mathrm{CU}(\kappa, A)$, the poset of closed and bounded subsets of A ordered by end-extension.

18.12 Lemma. *Forcing with $\mathbb{R}_0$ over $V[G_\kappa]$ adds no ω_1-sequences of ordinals.*

Proof. Working in V, let $T = \{\beta \in Y : X \cap \beta \in W_\beta\}$. Then $T \in U_1$, because $X \in U_0$ and U_0 is represented by $\langle W_i : i \in B\rangle$ in $\mathrm{Ult}(V, U_1)$. In particular T is stationary in κ. The poset $\mathbb{P}_\kappa$ is κ-c.c. and so T is stationary in $V[G_\kappa]$. For each $\beta \in T$, the preparation forcing added a closed set of order type ω_1 which is contained in $X \cap \beta$, and we are done by Lemma 18.6. ⊣

One of the key ideas in Gitik's arguments is that of a "local master condition". We give a more precise formulation in a moment, but the rough idea is to look at conditions which induce generic filters over a submodel of the universe for subposets of a forcing poset. The idea is similar to that of a strongly generic condition in proper forcing (see Remark 24.5) but the relevant submodels here are the classes $V[G_\beta]$ for $\beta < \kappa$. We will construct our iterations so that there are many local master conditions; as we see at the end of this section, this is vital when it comes to lifting the elementary embeddings j_0 and j_1 in the required way.

The set T defined in the proof of Lemma 18.12 is stationary, so by the usual reflection arguments the set of points where T reflects is a measure one set for any normal measure. We let $A' = X' \cup Y'$, where

$$X' = \{\beta \in X : T \cap \beta \text{ is stationary in } \beta\},$$
$$Y' = \{\beta \in Y : T \cap \beta \text{ is stationary in } \beta\}.$$

For $\beta < \kappa$ we define $\mathbb{R}_{0,\beta}$ to be the set of $d \in \mathbb{R}_0$ such that $\max(d) < \beta$ and $d \in V[G_\beta]$. It is easy to see that $\mathbb{R}_0 = \bigcup_{\beta<\kappa} \mathbb{R}_{0,\beta}$, and that

$$\mathbb{R}_{0,\gamma} = \textstyle\bigcup_{\beta<\gamma} \mathbb{R}_{0,\beta} = \mathbb{R}_0 \cap V_\gamma[G_\gamma]$$

when γ is V-inaccessible.

18.13 Remark. By the usual conventions, for λ an uncountable regular cardinal and X a set with $|X| = \lambda$, a *filtration* of X is an increasing and continuous sequence $\langle X_i : i < \lambda\rangle$ such that $X_i \subseteq X$, $X = \bigcup_i X_i$, and $|X_i| < \lambda$. The key property is that given filtrations X_i, X'_i we have $X_j = X'_j$ for a club set of j.

Technically the sequence of posets $\mathbb{R}_{0,\beta}$ is not a filtration of $\mathbb{R}_0$ because it is only continuous at V-inaccessible points. Until the end of this section we will abuse notation and refer to such sequences as filtrations.

A local version of the argument of Lemma 18.12 shows immediately that

18.14 Lemma. *For every $\beta \in A'$, forcing with $\mathbb{R}_{0,\beta}$ over $V[G_\beta]$ adds no ω_1-sequences of ordinals.*

The next lemma can be seen as a more refined version of this result. Let $\beta \in A'$. We will say that $c \in \mathbb{R}_0$ is a *β-master condition for* $\mathbb{R}_0$ if $\max(c) = \beta$, and $\{c \cap (\alpha+1) : \alpha \in \beta \cap \lim(c)\}$ is a $V[G_\beta]$-generic subset of $\mathbb{R}_{0,\beta}$.

18.15 Lemma. *For every $\beta \in A'$ and every $d \in \mathbb{R}_{0,\beta}$ there is a β-master condition $c \leq d$ with $c \in V[G_{\beta+2}]$.*

Proof. In $V[G_\beta]$ we have $\beta = \omega_2$ and $(\beta^+)^V = \omega_3$. By the previous lemma, $\mathbb{R}_{0,\beta}$ is (ω_1, ∞)-distributive in $V[G_\beta]$. We distinguish the cases $\beta \in X'$ and $\beta \in Y'$.

$\beta \in X'$: At stage β in the preparation forcing we forced with Nm'. So we are done by an appeal to clause 5 of Lemma 18.7, and in fact we can build a suitable c in $V[G_{\beta+1}]$.

$\beta \in Y'$: Again $\mathbb{R}_{0,\beta}$ is (ω_1, ∞)-distributive in $V[G_\beta]$. In $V[G_{\beta+2}]$ we have $\mathrm{cf}(\beta) = \mathrm{cf}(\beta^+) = \omega_1$; so if $\mathcal{D}$ is the set of dense open subsets of $\mathbb{R}_{0,\beta}$ which lie in $V[G_\beta]$, working in $V[G_{\beta+2}]$ we may write $\mathcal{D} = \bigcup_{i<\omega_1} \mathcal{D}_i$ where $\mathcal{D}_i \in V[G_\beta]$ and $V[G_\beta] \models |\mathcal{D}_i| = \omega_1$.

We fix $D \in V[G_{\beta+2}]$ such that D is a club subset of β of order type ω_1 and $D \subseteq X \cap \beta$. Now we build a chain of conditions $c_i \in \mathbb{R}_{0,\beta}$ such that $\max(c_i) \in D$ and $c_{i+1} \in \bigcap \mathcal{D}_i$ for all i. Since $V[G_{\beta+2}]$ and $V[G_\beta]$ have the same ${}^\omega\beta$, there is no problem at limit stages. As usual we may now set $c = \bigcup_i c_i \cup \{\beta\}$ to finish. $\dashv$

We now define $\mathbb{R}_1$. Let E be the generic club subset of κ added by $\mathbb{R}_0$. Then $\mathbb{R}_1$ is the set of those closed bounded sets d such that $d \subseteq E \cap A'$, and $E \cap (\beta+1)$ is a β-master condition for every $\beta \in d$.

18.16 Remark. We remind the reader of the discussion of the "flat condition trick" in Remark 17.2. We will be using that trick heavily in what follows. In particular when we get to the main construction in Sect. 18.7 we will just define the set of flat conditions and leave all the details to the reader.

We define a suitable concept of flatness for conditions in the two-step iteration $\mathbb{R} =_{\mathrm{def}} \mathbb{R}_0 * \dot{\mathbb{R}}_1$. The flat conditions are pairs $(c, \check{d})$ where $c \in \mathbb{R}_0$, $c \Vdash \check{d} \in \mathbb{R}_1$ and $\max(c) = \max(d)$. We define $\mathbb{R}_\gamma = \mathbb{R} \cap V_\gamma[G_\gamma]$ for inaccessible $\gamma < \kappa$.

18.17 Lemma. *Forcing with $\mathbb{R}$ over $V[G_\kappa]$ adds no ω_1-sequences of ordinals, and the set of flat conditions is dense in $\mathbb{R}$.*

Proof. Working in V we fix a $\mathbb{P}_\kappa$-name $\dot{D}$ for an ω_1-sequence of dense subsets of $\mathbb{R}$, where we may as well assume that $\dot{D} \subseteq V_\kappa$. By routine arguments there is a club set $F \subseteq \kappa$ in V such that for every inaccessible $\gamma \in F$, $\dot{D} \cap V_\gamma$ is a $\mathbb{P}_\gamma$-name for a sequence of dense sets in $\mathbb{R}_\gamma$.

Now we choose $\gamma \in Y'$ such that $F \cap X' \cap \gamma \in W_\gamma$. By the definition of the preparation forcing there is a club set $e \subseteq \gamma$ in $V[G_{\gamma+1}]$ such that $\mathrm{ot}(e) = \omega_1$, $e \subseteq F \cap X'$, and for every $\beta \in \lim(e)$ the set $e \cap \beta$ meets every club subset of β lying in $V[G_\beta]$.

We will now work in $V[G_\kappa]$. Let r be an arbitrary condition in $\mathbb{R}$; we will show that r can be extended to a flat condition which lies in the intersection of the dense sets D_i for $i < \omega_1$, establishing both of our claims about $\mathbb{R}$. We will build a decreasing sequence of conditions $(c_i, \dot{d}_i)$ for $i \leq \omega_1$, such that

1. $r = (c_0, \dot{d}_0)$.
2. For every $i < \lambda$,
   (a) $(c_i, \dot{d}_i) \in \mathbb{R}_\gamma$.
   (b) The condition c_{i+1} determines $\dot{d}_i$, that is, $c_{i+1} \Vdash \dot{d}_i = \check{d}_i$ for some $d_i \in V[G_\kappa]$.
   (c) $(c_{i+1}, \dot{d}_{i+1}) \in D_i$.
   (d) The condition c_{i+1} forces that $\max(\dot{d}_{i+1}) > \max(c_i)$.
   (e) The ordinal $\beta_i =_{\mathrm{def}} \max(c_i)$ lies in the set e, and c_i is a β_i-master condition.
3. The sequence $\langle \beta_i : i \leq \omega_1 \rangle$ is increasing and continuous.
4. For every limit $\lambda \leq \omega_1$, $(c_\lambda, \dot{d}_\lambda)$ is a flat condition.

The successor steps in this construction are easy by an appeal to Lemmas 18.14 and 18.15, and the fact we reflected the density of the dense sets down to each β_j.

The subtle point is that for a limit ordinal $\lambda \leq \omega_1$ we are safe to set $\beta_\lambda = \sup_{i<\lambda} \beta_i$, $c_\lambda = \bigcup_{i<\lambda} c_i \cup \{\beta_\lambda\}$ and $\dot{d}_\lambda$ equal to the canonical name for $d_\lambda = \bigcup_{i<\lambda} d_i \cup \{\beta_\lambda\}$. The issue is to check that c_λ is a β_λ-master condition, so we set $\beta = \beta_\lambda$ and fix a $\mathbb{P}_\beta$ name $Z \subseteq V_\beta$ for a dense subset of $\mathbb{R}_{0,\beta}$. We then find a club set $C_Z \subseteq \beta$ such that if $\alpha \in C_Z$ is inaccessible then $Z \cap V_\alpha$ names a dense subset of $\mathbb{R}_{0,\alpha}$. Now the key point is that $\beta \in \lim(e)$ so $e \cap \beta$ meets C_Z, and we have $i < \lambda$ such that $\beta_i \in C_Z$. Let $\alpha = \beta_i$, then we are done since c_i is an α-master condition and it generates a filter which meets the dense set named by $Z \cap \mathbb{R}_{0,\alpha}$ (which is an initial segment of the dense set named by Z itself). ⊣

We may now define the notion of a β-master condition for $\mathbb{R}$ and prove analogues of Lemmas 18.15 and 18.17. To be a bit more explicit, we say that (c, d) is a *β-master condition* for $\mathbb{R}$ if and only if it is flat, $\max(c) = \max(d) = \beta$, and $\{(c \cap (\alpha+1), d \cap (\alpha+1)) : \alpha \in d\}$ is $\mathbb{R}_\beta$-generic over $V[G_\beta]$. We define T', X'', Y'', A'' from X' and Y' in just the same way that

T, X', Y', A' were defined from X and Y. Then the analogue of Lemma 18.15 says that if $\beta \in A''$ any condition in $\mathbb{R}_\beta$ extends to a β-master condition, and there is a similar generalization of Lemma 18.17.

We now sketch the main ideas in the argument that we can make the restriction of NS_{ω_2} to $\mathrm{Cof}(\omega)$ precipitous. Similar arguments apply to the restriction to $\mathrm{Cof}(\omega_1)$.

Applying the elementary embedding j_0 to the result of Lemma 18.15, we obtain the result that every condition in $\mathbb{R}_0$ can be extended in M_0 to a κ-master condition in $j_0(\mathbb{R}_0)$. Implicitly this defines an embedding of $\mathbb{P}_\kappa * \mathbb{R}_0$ into $j_0(\mathbb{P}_\kappa)$, and a strong master condition suitable for lifting the elementary embedding j_0 to the extension by $\mathbb{P}_\kappa * \mathbb{R}_0$. A similar argument applies to the iteration $\mathbb{R}_0 * \mathbb{R}_1$.

We now return to a point which we already mentioned in Sect. 18.3, namely, that can achieve the same kind of effect as in the construction of Sect. 17.2 by performing an iteration where every step is either like $\mathbb{R}_0$ or like $\mathbb{R}_1$. To fix ideas let H_0 be $\mathbb{R}_0$-generic over $V[G_\kappa]$, and let $a \in V[G_\kappa * H_0]$ be in the master condition ideal for j_0. Explicitly this means that it is forced that $\kappa \notin j_0^+(a)$ where j_0^+ is the lifting of j_0 onto $V[G_\kappa * H_0]$ described in the preceding paragraph. We will show how to add an ω-club set disjoint from a.

Let $\dot{a}$ be a $\mathbb{P}_\kappa * \dot{\mathbb{R}}_0$-name for a and let $(p, q) \in G_\kappa * H_0$ force that $\dot{a}$ is in the master condition ideal. That is to say, (p, q) forces that "it is forced that $\kappa \notin j_0^+(a)$". Analyzing the lifting construction and viewing p now as a condition in $j_0(\mathbb{P}_\kappa)$, p forces over M_0 that for every κ-master condition $Q \leq j_0(q)$, Q forces that $\kappa \notin j_0(\dot{a})$.

Now let C be the set of $\alpha < \kappa$ such that

1. $q \in \mathbb{R}_{0,\alpha}$.
2. p forces (over V for the forcing poset $\mathbb{P}_\kappa$) that for every $Q \leq q$ which is an α-master condition for $\mathbb{R}_0$, Q forces (over $V[G_\kappa]$ for the forcing poset $\mathbb{R}_0$) that $\alpha \notin \dot{a}$.

By Łoś's theorem we see that $C \in U_0$.

Define $\mathbb{R}_2$ to be similar to $\mathbb{R}_0$, adding a club contained in $C \cup D$ for some $D \in U_1$. One can do an analysis of $\mathbb{R}_0 * \mathbb{R}_1 * \mathbb{R}_2$ which is similar to the analyses of $\mathbb{R}_0$ and $\mathbb{R}_0 * \mathbb{R}_1$ given above. Let E_i be the club set added by $\mathbb{R}_i$. Then

- By the construction of $\mathbb{R}_1$, $E_0 \cap (\alpha + 1)$ is an α-master condition for every $\alpha \in E_1$.
- By the construction of $\mathbb{R}_2$, for every $\alpha \in E_2 \cap \mathrm{Cof}(\omega)$ we have $\alpha \in C$.
- So for every $\alpha \in E_1 \cap E_2 \cap \mathrm{Cof}(\omega)$, it follows from the definition of C that $\alpha \notin a$.

We have argued that in the extension by $\mathbb{R}_0 * \mathbb{R}_1 * \mathbb{R}_2$ there is an ω-club set disjoint from a. In the next subsection we will show how to iterate and achieve the same effect for every set which appears in some master condition ideal during the course of the iteration.

18.7. The Main Iteration

Recall from the last section that we defined $\mathbb{R}_0$ from a set $A = X \cup Y$ and then $\mathbb{R}_1$ from a set $A' \subseteq A$, where $\beta \in A'$ if $\beta \in A$ and there are stationarily many $\gamma < \beta$ such that $X \cap \gamma \in W_\gamma$. The poset $\mathbb{R}_0$ shot a club set E through A, and the poset $\mathbb{R}_1$ shot a club set through the set of points $\beta \in E \cap A'$ such that $E \cap \beta$ was $\mathbb{R}_{0,\beta}$-generic over $V[G_\beta]$. The main iteration, which we will only describe in outline, can be viewed as iterating this kind of construction many times for every possible A simultaneously. The main difficulty in defining the iteration is that when we have iterated ν times and have obtained an iteration $\mathbb{Q}_\nu$, we need to define a suitable notion of *β-master condition for* $\mathbb{Q}_\nu$; this requires choosing a filtration of $\mathbb{Q}_\nu$, and the filtrations for different values of ν must fit together nicely.

The main iteration is defined from some parameters $\langle A_\nu, i_\nu, C_\nu : \nu < \kappa^+\rangle$, which are chosen in V. They must satisfy a long list of technical conditions, most of which we are omitting. In particular

1. A_ν is the union of sets of inaccessibles $X_\nu \subseteq \kappa \setminus B$ and $Y_\nu \subseteq B$, with $X_\nu \in U_0$ and $Y_\nu \in U_1$.

2. Every set of inaccessibles $X \subseteq \kappa \setminus B$ with $X \in U_0$ is enumerated as X_ν for some successor ν, and similarly every set of inaccessibles $Y \subseteq B$ with $Y \in U_1$ is enumerated as Y_ν for some successor ν.

3. i_ν is a surjection from κ to ν, which is also injective for $\nu \geq \kappa$. Note that for any normal measure on κ, the map which takes $\beta < \kappa$ to the order-type of $i_\nu``\beta$ represents ν in the ultrapower.

4. C_ν is club in κ.

5. If $\kappa \leq \nu_1 < \nu_2$, $\beta \in C_{\nu_2}$ and $\nu_1 \in i_{\nu_2}``\beta$, then $\beta \in C_{\nu_1}$.

We define $X_{\nu,\beta} = i_\nu``\beta$ for $\beta < \kappa$, so that the $X_{\nu,\beta}$'s form a filtration of ν.

We define by recursion posets $\mathbb{Q}_\nu$ for $\nu < \kappa^+$, and for each $\mathbb{Q}_\nu$ also a filtration in which $\mathbb{Q}_\nu$ is written as the union of subsets $\mathbb{Q}_{\nu,\beta}$ for $\beta < \kappa$.

18.18 Remark. Once again the remarks about the "flat condition trick" in Remark 17.2 are somewhat applicable. We are defining a sequence of posets, whose conditions are comprised of closed bounded sets from the ground model, and claiming that they can be considered as an iteration. However in this instance it would be hard to write down a genuine iteration and then identify our conditions as a dense subset. To give a complete account of the proof we would have to check that the sequence of posets $\mathbb{Q}_\nu$ can be considered as an iteration, but this is only one of many details that we are omitting.

Conditions in $\mathbb{Q}_\nu$ are sequences of the form $q = \langle q_\alpha : \alpha \in X_{\nu,\beta} \rangle$ where (omitting one condition for the moment)

I. $\beta \in C_\nu$ (we will denote this ordinal β by β_q in what follows).

II. For successor α in the support of q, $q_\alpha \in \mathrm{CU}(\kappa, A_\alpha)$.

III. For limit α in the support of q, $q_\alpha \in \mathrm{CU}(\kappa, A_\alpha \cap C_\alpha)$.

IV. For limit α in the support of q, for every $\eta \in q_\alpha$,

   (a) $\eta \leq \beta_q$.

   (b) $X_{\alpha,\eta} \subseteq X_{\nu,\beta_q}$.

   (c) $\eta \in q_\tau$ for every $\tau \in X_{\alpha,\eta}$.

To qualify as a member of $\mathbb{Q}_\nu$ a sequence q as above must satisfy a fifth property (property V), whose description we defer until we have made a few definitions.

Once we have defined $\mathbb{Q}_\nu$, we define $\mathbb{Q}_{\nu,\beta}$ for $\beta < \kappa$ to be the set of those $p \in \mathbb{Q}_\nu$ such that

1. $p \in V[G_\beta]$.

2. $\beta_p < \beta$.

3. For every τ in the support X_{ν,β_p} of p, p_τ is bounded in β.

If $q \in \mathbb{Q}_\nu$, α is a limit ordinal in the support X_{ν,β_q} of q and $\beta \in q_\alpha$ then we define $q\restriction(\alpha, \beta) = \langle q_\tau \cap (\beta + 1) : \tau \in X_{\alpha,\beta} \rangle$. Notice that by the conditions we imposed on q we have that the support $X_{\alpha,\beta}$ of $q\restriction(\alpha, \beta)$ is contained in the support X_{ν,β_q} of q; also $\beta \in q_\tau$ for all $\tau \in X_{\alpha,\beta}$.

The intuition here is that $q\restriction(\alpha, \beta)$ is of the right general shape to be a β-*master condition* for $\mathbb{Q}_\alpha$. To be a bit more formal we say that r is a β-master condition for $\mathbb{Q}_\alpha$ if

1. The support of r is $X_{\alpha,\beta}$.

2. For every $\tau \in X_{\alpha,\beta}$, $\beta = \max(r_\tau)$.

3. The set of conditions $p \in \mathbb{Q}_\alpha\restriction\beta$ such that p_τ is an initial segment of $r_\tau \cap \beta$ for all τ is a $V[G_\beta]$-generic filter on $\mathbb{Q}_\alpha\restriction\beta$.

Now we can complete the description of $\mathbb{Q}_\nu$. Intuitively the following condition says that at limit stages we are shooting clubs through certain sets of “generic points”.

V. For every limit α in the support X_{ν,β_q} of q and every $\beta \in q_\alpha$, $q\restriction(\alpha, \beta)$ is a β-master condition for $\mathbb{Q}_\alpha$.

If $p, q \in \mathbb{Q}_\nu$ then $p \leq_\nu q$ iff $\beta_p \geq \beta_q$ and p_α end-extends q_α for all $\alpha \in X_{\nu,\beta_q}$.

The key lemmas are proved by similar means to those used in the last section.

18.19 Lemma. *For $\nu < \mu < \kappa^+$, $\mathbb{Q}_\nu$ is a complete subordering of $\mathbb{Q}_\mu$. Defining $\mathbb{Q}_{\kappa^+} = \bigcup_\nu \mathbb{Q}_\nu$, $\mathbb{Q}_{\kappa^+}$ has the κ^+-c.c.*

The following lemma is the technical heart of the whole construction. The proof (which we omit) is by a very intricate double induction on the pairs (μ, β) with $\beta \in A_\mu \cap C_\mu$, ordered lexicographically.

18.20 Lemma. *If ν is limit, $\alpha \in A_\nu \cap C_\nu$, $p \in \mathbb{Q}_{\nu,\alpha}$ then there exists a condition $q = \langle q_\tau : \tau \in X_{\nu,\alpha} \rangle \leq_\nu p$ in $V[G_{\alpha+2}]$ such that q is an α-master condition for $\mathbb{Q}_\nu$.*

The following is an easy corollary:

18.21 Lemma. *Let $\nu < \kappa^+$ be limit and let $\alpha \in A_\nu \cap C_\nu$. Forcing over $V[G_\alpha]$ with $\mathbb{Q}_{\nu,\alpha}$ adds no ω_1-sequence of ordinals.*

Let $j_i : V \longrightarrow M$ be the ultrapower by the normal measure U_i, and observe that since $V \models {}^\kappa M \subseteq M$ and $\mathbb{P}_\kappa$, $V[G_\kappa] \models {}^\kappa M_i[G_\kappa] \subseteq M_i[G_\kappa]$. Observe also that by normality $\kappa \in j(A_\nu \cap C_\nu)$ for all limit $\nu < \kappa^+$. Accordingly we see that

18.22 Lemma. *For every limit $\nu < \kappa^+$, in $M_i[G_{\kappa+1}]$ there is a condition $q \in \mathbb{Q}_{j(\nu)}$ such that $q = \langle q_\tau : \tau \in j``\nu \rangle$, $q$ induces a $j(\mathbb{Q})_{\nu,\kappa}$-generic filter over $V[G_\kappa]$, and $\max(q_\tau) = \kappa$ for every $\tau \in j``\nu$.*

Each condition in $\mathbb{Q}_\nu$ is an object of size less than κ. It follows easily that

18.23 Lemma. *For every limit $\nu < \kappa^+$ and every $\alpha \in A_\nu \cap C_\nu$, there is an isomorphism between $\mathbb{Q}_{\nu,\alpha}$ and $j(\mathbb{Q})_{\nu,\alpha}$ in $M_i[G_\kappa]$.*

18.24 Lemma. *There exists an isomorphism between $\mathbb{Q}_\nu$ and $j(\mathbb{Q})_{\nu,\kappa}$ in $M_i[G_\kappa]$.*

Putting these various pieces of information together, we get

18.25 Lemma. *For every limit $\nu < \kappa^+$, there is a $\mathbb{Q}_\nu$-generic filter over $V[G_\kappa]$ in $M_i[G_{\kappa+1}]$, which is induced by a condition as in Lemma 18.22.*

18.8. Precipitousness of the Non-Stationary Ideal

We are now in precisely the situation of Lemma 17.3, so we have produced two precipitous ideals I_0 and I_1, where I_a concentrates on points of cofinality ω_a. It remains to be seen that these are in fact restrictions of the non-stationary ideal. We will show that the ideal I_0 induced by the construction with j_0 is the ω-nonstationary ideal, the argument for I_1 is exactly the same. We worked through a simple case of the argument at the end of Sect. 18.6, the idea here is very similar.

Let H be generic over $V[G_\kappa]$ for $\mathbb{Q}_{\kappa^+}$. We work in $V[G_\kappa * H]$. We denote by $H \restriction \nu$ the $\mathbb{Q}_\nu$-generic object induced by H. Let t_j be the club set added by H at stage j.

Since I_0 is a normal ideal concentrating on points of cofinality ω, I_0 must contain the ω-nonstationary ideal. The other direction is trickier, since we did not explicitly shoot ω-club sets through every I_0-large set.

18.26 Claim. *I_0 is contained in the ω-nonstationary ideal.*

Proof. Suppose that a is in I_0. Unwrapping the definition, this means that at some stage ν we have that $a \in V[G_\kappa * H_\nu]$ and it is forced that $\kappa \notin j_{0,\nu}(a)$ where $j_{0,\nu}$ is the lifting of j_0 to $V[G_\kappa * H_\nu]$.

We now fix $\dot{a}$ a $\mathbb{P}_\kappa * \mathbb{Q}_\nu$-name for a and a condition $(p, q) \in G_\kappa * H_\nu$ forcing (over M_0 for $\mathbb{P}_\kappa * \mathbb{Q}_\nu$) that "it is forced (over $M_0[\dot{G}_\kappa][\dot{H}_\nu]$ by the forcing poset $(j_{0,\nu}(\mathbb{P}_\kappa)/\dot{G}_\kappa * \dot{H}_\nu) * j_{0,\nu}(\mathbb{Q}_\nu)/\dot{m}_\nu$, where m_ν is the master condition) that $\kappa \notin j_{0,\nu}(\dot{a})$". Regarding p as a condition in $j_0(\mathbb{P}_\kappa)$, p forces (over M_0 for $j_0(\mathbb{P}_\kappa)$) that for every κ-master condition $Q \leq j_0(q)$ for $j_0(\mathbb{Q})_\nu$, Q forces (over $M_0[\dot{G}_{j_0(\kappa)}]$ for $j_0(\mathbb{Q})_\nu$) that $\kappa \notin j_0(\dot{a})$.

Now we apply Łoś's theorem to see that $R \in U_0$, where R is the set of α such that $q \in \mathbb{Q}_{\nu,\alpha}$ and p forces (over V for $\mathbb{P}_\kappa$) that for every $Q \leq q$ with Q an α-master condition for $\mathbb{Q}_\nu$, Q forces (over $V[\dot{G}_\kappa]$ for $\mathbb{Q}_\nu$) that $\alpha \notin \dot{a}$.

Let $\eta > \nu$ be some limit stage. The construction of the forcing poset implies that for all sufficiently large $\alpha \in t_\eta$, there is a condition $Q \leq q$ in H which is an α-master condition for $\mathbb{Q}_\nu$. So for all sufficiently large $\alpha \in t_\eta \cap R$, $\alpha \notin a$.

In the construction we enumerated R as $X_{\bar{\eta}}$ for some $\bar{\eta}$. By definition $A_{\bar{\eta}} = X_{\bar{\eta}} \cup Y_{\bar{\eta}}$, and in $V[G_\kappa]$ the preparation forcing arranged that all points of $X_{\bar{\eta}}$ have cofinality ω while all points of $Y_{\bar{\eta}}$ have cofinality ω_1. At stage $\bar{\eta}$ in the main iteration we added a club set $t_{\bar{\eta}} \subseteq A_{\bar{\eta}}$, so $t_{\bar{\eta}} \cap \mathrm{Cof}(\omega) \subseteq R$.

Combining these results, all sufficiently large $\alpha \in t_\eta \cap t_{\bar{\eta}} \cap \mathrm{Cof}(\omega)$ fail to be in a. We conclude that a is ω-nonstationary in $V[G_\kappa * H]$, as required. $\dashv$

18.9. Successors of Larger Cardinals

Gitik [25, 26] has also obtained rather similar equi-consistency results for regular cardinals $\kappa > \omega_2$. The idea is broadly the same, but the preparation forcing is an iteration of Prikry-style forcing with Easton supports followed by an iteration of Cohen forcing (for κ inaccessible) or a Lévy collapse (for κ a successor cardinal). The main iteration is essentially the same.

We content ourselves with quoting some of the main results. When stating the lower bounds we assume throughout that there is no inner model with a cardinal λ such that $o(\lambda) = \lambda^{++}$, and we let K be the Mitchell core model for sequences of measures and $\mathcal{F}$ its measure sequence. Let $\vec{U}$ be a coherent sequence of measures. An ordinal α is an (ω, δ) *repeat point over* κ if and only if $\mathrm{cf}(\alpha) = \omega$ and for every $A \in \bigcap\{U(\kappa, \zeta) : \alpha \leq \zeta < \alpha + \delta\}$ there are unboundedly many $\gamma < \alpha$ such that $A \in \bigcap\{U(\kappa, \zeta') : \gamma \leq \zeta' < \gamma + \delta\}$.

The result for successors of regular cardinals greater than ω_1 is exact.

18.27 Theorem (Gitik [26]). *Let $\lambda = \mathrm{cf}(\lambda) < \kappa$ and suppose that* GCH *holds and there is a measure sequence with an $(\omega, \lambda + 1)$-repeat point over κ. Then there is a generic extension in which* GCH *holds, cardinals up to and including λ are preserved, $\kappa = \lambda^+$ and NS_κ is precipitous.*

18.28 Theorem (Gitik [25]). *Suppose that $\mu = \mathrm{cf}(\mu) > \omega_1$,* GCH *holds and NS_κ is precipitous where $\kappa = \mu^+$. Then in K there is an $(\omega, \mu + 1)$-repeat point over κ.*

Interestingly enough, the proof uses only the precipitousness of the restrictions of NS_κ to cofinality ω and cofinality μ. When κ is inaccessible the strength of "NS_κ is inaccessible" is bounded from above by an $(\omega, \kappa + 1)$-repeat and from below by an $(\omega, {<}\kappa)$-repeat.

19. More on Iterated Club Shooting

In this section we give sketches of two more theorems obtained by iterated club shooting. The first theorem is due to Jech and Woodin [40] and shows that it is consistent for $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ to be a κ^+-saturated ideal. The second is due to Magidor [55] and shows that it is consistent for every stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$ to reflect at almost every point of $\omega_2 \cap \mathrm{Cof}(\omega_1)$. Apart from their intrinsic interest we have included them because they illustrate some new ideas: the theorem by Jech and Woodin involves embedding one iteration in another "universal" iteration, while the theorem by Magidor gives another example of shooting clubs to make a natural filter (defined in this case via stationary reflection) become the club filter.

As some motivation for Theorem 19.1 we sketch a proof that if κ is weakly compact then $\mathrm{NS}_\kappa \restriction \mathrm{Reg}$ is not κ^+-saturated. We start by recalling a classical result of Solovay: if κ is a regular uncountable cardinal and $S \subseteq \kappa$ is stationary then $T = \{\alpha \in S : S \cap \alpha \text{ is non-stationary in } \alpha\}$ is stationary (given a club C look at the first place where $\lim(C)$ meets S). In particular $T \cap \alpha$ is non-stationary in α for every $\alpha \in T$, in what follows we refer to stationary sets which reflect at no point of themselves as *thin.*

We now consider an ordering on stationary subsets of inaccessible cardinals investigated by Jech [38]. Given an inaccessible cardinal κ and stationary subsets $S, T \subseteq \kappa$ we write $S < T$ when $S \cap \alpha$ is stationary for almost every $\alpha \in T$ (modulo the club filter). It is easy to check that $<$ is well-founded, and by the result of Solovay from the last paragraph $<$ is irreflexive. If $S < T$ with S and T both thin, then clearly $S \cap T$ is non-stationary.

Assume now that κ is weakly compact. We will produce a $<$-increasing sequence $\langle S_\alpha : \alpha < \kappa^+ \rangle$ of thin stationary sets of regular cardinals. Let $S_0 =$

$\kappa \cap \text{Reg}$. At stage α fix a surjection f from κ to α, and use Π^1_1-indescribability to show that

$$S = \{\delta : \forall \gamma < \delta \ S_{f(\gamma)} \cap \delta \text{ is stationary in } \delta\}$$

is stationary. Then choose S_α to be a thin stationary subset of this set S. If $\beta < \alpha$ then $S_\beta \cap \delta$ is stationary for all large $\delta \in S_\alpha$, so $S_\beta < S_\alpha$. Since the S_α for $\alpha < \kappa^+$ have pairwise non-stationary intersections, $\text{NS}_\kappa \restriction \text{Reg}$ is not κ^+-saturated.

The proof we just gave shows essentially that if κ is κ^+-Mahlo then $\text{NS}_\kappa \restriction \text{Reg}$ is not κ^+-saturated. Jech and Woodin showed [40] that for any $\alpha < \kappa^+$ we may have κ which is α-Mahlo with $\text{NS}_\kappa \restriction \text{Reg}$ κ^+-saturated, starting from a measurable cardinal of Mitchell order α. This is known [38] to be optimal.

19.1 Theorem. *Let κ be measurable and let* GCH *hold. Then in a suitable generic extension* $\text{NS}_\kappa \restriction \text{Reg}$ *is κ^+-saturated.*

Proof. Let δ be inaccessible and let $S \subseteq \text{Reg} \cap \delta$. We define a forcing poset $\text{CU}_{\text{Reg}}(\delta, S) = \text{CU}(\delta, (\text{Sing} \cap \delta) \cup S)$; to be more explicit conditions are closed bounded subsets c of δ such that $c \cap \text{Reg} \subseteq S$, ordered by end-extension.

It is easy to see that for every $\gamma < \delta$ the set of conditions c with $\max(c) > \gamma$ is dense and γ-closed, so that $\text{CU}_{\text{Reg}}(\delta, S)$ forces that almost every regular cardinal is in S while adding no $<\delta$-sequences.

We now describe a kind of "universal" iteration of this forcing. To be more precise we define by recursion $\mathbb{Q}_\alpha$ for $\alpha \leq \delta^+$ and $\mathbb{Q}_\alpha$-names $\dot{S}_\alpha$ for $\alpha < \delta^+$ so that

1. $f \in \mathbb{Q}_\alpha$ if and only if
   (a) f is a partial function on α.
   (b) $\text{dom}(f)$ has size less than δ, and $f(\beta)$ is a closed bounded subset of δ for all $\beta \in \text{dom}(f)$.
   (c) For all $\alpha \in \text{dom}(f)$, $f \restriction \alpha \Vdash_{\mathbb{Q}_\alpha} f(\alpha) \cap \text{Reg} \subseteq \dot{S}_\alpha$.
2. For conditions $f, g \in \mathbb{Q}_\alpha$, $f \leq g$ if and only if $\text{dom}(g) \subseteq \text{dom}(f)$ and $f(\beta)$ end-extends $g(\beta)$ for all $\beta \in \text{dom}(g)$.
3. (Universality) Every $\mathbb{Q}_{\delta^+}$-name for a subset of δ is equivalent to $\dot{S}_\alpha$ for unboundedly many $\alpha < \delta^+$.

For every α, $\mathbb{Q}_\alpha$ is δ^+-c.c by an easy Δ-system argument. Also for all γ and α the set of $f \in \mathbb{Q}_\alpha$ such that $\max(f(\beta)) > \gamma$ for all $\beta \in \text{dom}(f)$ is dense and γ-closed. The GCH assumption and the δ^+-c.c make it possible to satisfy universality.

19.2 Remark. We are cheating slightly, in the sense that we should really verify that $\mathbb{Q}_\alpha$ is equivalent to an iteration of club-shooting forcing. See the remarks on the "flat condition trick" in Sect. 17.

19.3 Lemma. *Let $\mathbb{Q}^*_{\delta^+}$ be built in a similar way from a sequence of names $\dot{S}^*_\alpha$ satisfying clauses 1 and 2 above. Then there is a complete embedding of $\mathbb{Q}^*_{\delta^+}$ into $\mathbb{Q}_{\delta^+}$.*

Sketch of Proof. This is almost immediate if we use the flat conditions trick to regard $\mathbb{Q}_{\delta^+}$ and $\mathbb{Q}^*_{\delta^+}$ as dense sets in iterations of club shooting forcing. We may also proceed quite explicitly by constructing for each α a complete embedding i_α of $\mathbb{Q}_\alpha$ into $\mathbb{Q}^*_{\beta_\alpha}$ for a suitable $\alpha < \delta^+$. At successor stages we use i_α to identify the $\mathbb{Q}^*_\alpha$-name $\dot{S}^*_\alpha$ with a $\mathbb{Q}_{\beta_\alpha}$-name, use universality to find $\gamma > \beta_\alpha$ such that this name is $\dot{S}_\gamma$, and then set $\beta_{\alpha+1} = \gamma + 1$ and extend to $i_{\alpha+1} : \mathbb{Q}^*_{\alpha+1} \to \mathbb{Q}_{\gamma+1}$ in the obvious way; at limits we just take a suitable limit of the embeddings i_α and check that everything works. ⊣

We are now ready to build the model. We will do a reverse Easton iteration of length $\kappa + 1$. For $\alpha < \kappa$ we let $\dot{\mathbb{Q}}_\alpha = \{0\}$ unless α is inaccessible, in which case we let $\dot{\mathbb{Q}}_\alpha$ name some universal iteration as above for α.

We fix some normal measure U and let $j : V \longrightarrow M$ be the associated ultrapower map. Let $\dot{\mathbb{Q}}$ be the member of M represented by $\langle \mathbb{Q}_\alpha : \alpha < \kappa \rangle$. Since ${}^\kappa M[G_\kappa] \subseteq M[G_\kappa]$, it is routine to check that $\mathbb{Q}$ is a universal iteration in $V[G_\kappa]$; we let $\dot{S}_j$ be the set which is used at stage j.

The last step $\mathbb{Q}_\kappa$ in our iteration will be a certain sub-iteration of $\mathbb{Q}$. The idea is to build a submodel $V[G * g_0]$ of $V[G * g]$ (where g is $\mathbb{Q}$-generic) and an embedding j which is defined in $V[G * g]$ and has domain $V[G * g_0]$, in such a way that if $S \in V[G * g_0]$ then $\Vdash \kappa \in j(S)$ if and only if S contains a club. A slightly subtle point is that as the construction proceeds we can anticipate in $V[G_\kappa]$ which of the names $\dot{S}_i$ are naming sets S of this type, and pick out the sub-iteration $\mathbb{Q}_\kappa$ so that we shoot a club through each one.

By the usual arguments $\mathbb{P}_\kappa * \mathbb{Q}$ is an initial segment of $j(\mathbb{P}_\kappa)$. If $G_\kappa * g$ is $\mathbb{P}_\kappa * \mathbb{Q}$-generic then as usual we may build in $V[G_\kappa * g]$ a $M[G_\kappa * g]$-generic filter H for the factor iteration $j(\mathbb{P}_\kappa)/G * g$, and then extend to get $j : V[G_\kappa] \longrightarrow M[G_\kappa * g * H]$.

Working in $V[G_\kappa]$, we construct an increasing sequence $\langle \alpha_i : i < \kappa^+ \rangle$ of ordinals, subiterations $\mathbb{Q}^*_i$ of $\mathbb{Q}$ and names for conditions $r_i \in j(\mathbb{Q}^*_i)$ as follows:

1. $\mathbb{Q}^*_i$ is the subiteration of $\mathbb{Q}$ which adds a club subset $C_j \subseteq \kappa$ with $C_j \cap \mathrm{Reg} \subseteq S_{\alpha_j}$ for each $j < i$.

2. r_i is a $j(\mathbb{P}_\kappa)$-name for a condition in $j(\mathbb{Q}^*_i)$ which is a strong master condition for j and the $\mathbb{Q}^*_i$-generic object g_i.

3. α_i is chosen least so that S_{α_i} is a $\mathbb{Q}^*_i$-name and
$$\Vdash_{(j(\mathbb{P}_\kappa)/(\mathbb{P}_\kappa * \mathbb{Q}^*_i)) * (j(\mathbb{Q}^*_i)/r_i)} \kappa \in j(S_{\alpha_i}).$$

4. The domain of r_i is $j``i$, and if $\langle C_k : k < i \rangle$ is the sequence of club sets added by $\mathbb{Q}^*_i$ then $r(j(k)) = C_k \cup \{\kappa\}$.

The construction is very similar to that of Sect. 17.2 and we omit all details.

Let $\mathbb{Q}_\kappa = \mathbb{Q}^*_{\kappa^+}$ and let g_0 be $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$. By forcing over $V[G_\kappa * g_0]$ with $\mathbb{Q}/g_0$ we may obtain g which is $\mathbb{Q}$-generic over $V[G]$, and working in $V[G * g]$ we may lift to get $j : V[G] \longrightarrow M[G * g * H]$ as above. Using Magidor's method from Sect. 13 and the sequence of partial strong master conditions r_i, we may build in $V[G*g]$ an $M[G*g*H]$-generic filter I on $j(\mathbb{Q}_\kappa)$ with $j``g_0 \subseteq I$ and then lift to get $j : V[G*g_0] \longrightarrow M[G*g*H*I]$.

The construction guarantees that for any $T \in V[G*g_0]$ with $T \subseteq \mathrm{Reg} \cap \kappa$, T is non-stationary if and only if $\Vdash_{\mathbb{Q}/g_0} \kappa \notin j(T)$. Since $\mathbb{Q}/g_0$ has the κ^+-c.c. it follows by Lemma 14.5 that $\mathrm{NS}\restriction\mathrm{Reg}$ is κ^+-saturated. ⊣

We now sketch Magidor's result that consistently every stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$ reflects almost everywhere in $\omega_2 \cap \mathrm{Cof}(\omega_1)$. The construction is quite similar to that for the precipitousness of NS_{ω_1}; we use this as the pretext for omitting many details.

19.4 Remark. Magidor used the optimal hypothesis of weak compactness; to simplify the exposition we use a measurable cardinal.

19.5 Theorem. *If κ is measurable, then in some generic extension $\kappa = \omega_2$ and for every $S \subseteq \omega_2 \cap \mathrm{Cof}(\omega)$ there is a club set C such that $S \cap \alpha$ is stationary for all $\alpha \in C \cap \mathrm{Cof}(\omega_1)$.*

Proof. Let $\mathbb{P} = \mathrm{Col}(\omega_1, {<}\kappa)$ and let $j : V \longrightarrow M$ be the ultrapower map arising from some normal measure U on κ. The idea of the proof is that after forcing with $\mathbb{P}$ every stationary set reflects stationarily often, and we may then shoot club sets to arrange the desired result. Of course new stationary sets will arise as we iterate so some care is required.

Much as in Sect. 17.2 we will work in $V[G]$ where G is $\mathbb{P}$-generic over V, and define $\mathbb{Q}$ which has the effect of iterating club-shooting with supports of size ω_1. We will be constructing certain strong master conditions as we go, whose existence will imply by Theorem 12.5 that no ω_1-sequences of ordinals are added to $V[G]$ by $\mathbb{Q}$. This is why we can set things up so that the conditions in $\mathbb{Q}$ are just functions in $V[G]$.

Explicitly in $V[G]$ we define by recursion $\mathbb{Q}_\alpha$ and $\mathbb{Q}_\alpha$-names $\dot{S}_\alpha$ such that

1. $\dot{S}_\alpha$ is a $\mathbb{Q}_\alpha$-name for a stationary subset of $\omega_2 \cap \mathrm{Cof}(\omega)$.
2. $f \in \mathbb{Q}_\alpha$ if and only if
   (a) f is a partial function on α with $|\mathrm{dom}(f)| \leq \omega_1$.
   (b) For all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a closed bounded subset of ω_2 and $f\restriction\beta$ forces that
$$f(\beta) \subseteq \mathrm{Cof}(\omega) \cup \{\gamma \in \mathrm{Cof}(\omega_1) : S_\beta \cap \gamma \text{ is stationary in } \gamma\}.$$

Clearly $\mathbb{Q}_\alpha$ is countably closed and an easy Δ-system argument shows that it is κ^+-c.c.

19.6 Remark. Once again we are cheating slightly in the definition of the forcing by using only "flat" conditions. See the remarks on the "flat condition trick" in Sect. 17.

Exactly as in Sect. 17.2 we will build embeddings i_α of $\mathbb{P} * \mathbb{Q}_\alpha$ into $j(\mathbb{P})$, with the added wrinkle that we use Theorem 14.3 to ensure that the quotient forcing for prolonging a $\mathbb{P} * \mathbb{Q}_\alpha$-generic to a $j(\mathbb{P})$-generic is countably closed. As we see soon this is crucial for the success of the master condition argument.

At a stage $\alpha < \kappa^+$, if $\langle C_\beta : \beta < \alpha \rangle$ is the sequence of club sets added by $\mathbb{Q}_\alpha$, then we define r_α as follows: $\mathrm{dom}(r_\alpha) = j``\alpha$, and $r_\alpha(j(\beta)) = C_\beta \cup \{\kappa\}$ for every $\beta < \alpha$. We verify that r_α is a strong master condition just as in Sect. 17.2, the only sticky point is that since $\mathrm{cf}(\kappa) = \omega_1$ after forcing with $j(\mathbb{P})$ we need to know that r_β forces that $j(S_\beta) \cap \kappa$ is stationary. This is easy because (by virtue of being a master condition) r_β forces that $j(S_\beta) \cap \kappa = S_\beta$, and since we are in a countably closed extension of $V[G * g_\beta]$ we see that the stationarity of S_β is preserved.

It is now easy to see that forcing with $\mathbb{Q}_{\kappa^+}$ adds no ω_1-sequences of ordinals to $V[G]$, so that κ is preserved. By the usual book-keeping we may arrange that every $\mathbb{Q}_{\kappa^+}$-name for a stationary subset of $\kappa \cap \mathrm{Cof}(\omega)$ appears as S_α for some $\alpha < \kappa^+$. If H is $\mathbb{Q}_{\kappa^+}$-generic over $V[G]$ then $V[G * H]$ is as required. $\dashv$

20. More on Collapses

We have seen many applications of the Levy collapse. In this section we discuss two situations where the Levy collapse cannot be used, one involving master conditions and the other involving absorbing "large" forcing posets into a collapsing poset. We shall describe some more exotic collapsing posets which can sometimes be used in these situations, namely, the Silver collapse and Kunen's universal collapse. We then show how these can be applied by sketching Kunen's consistency proof [45] for an ω_2-saturated ideal on ω_1.

We have seen many situations where we are given $\mathbb{P} = \mathrm{Col}(\delta, {<}\kappa)$ and an elementary embedding j with critical point κ, and wish to lift j to the extension by $\mathbb{P}$. Here there is no master condition issue because $j \restriction \mathbb{P} = \mathrm{id}_{\mathbb{P}}$ and $\mathbb{P}$ is just an initial segment of $j(\mathbb{P})$.

But now consider the following situation: $k : M \longrightarrow N$ has critical point κ, $\mathbb{P} = \mathrm{Col}(\kappa, {<}\lambda)_M$, G is $\mathbb{P}$-generic over M and both G and $k \restriction \lambda$ are in N. Certainly we may form in N a partial function $Q = \bigcup k``G$, where $\mathrm{dom}(Q) = \kappa \times k``\lambda$; but if $\lambda \geq k(\kappa)$ then Q has the wrong shape to be a condition in $k(\mathbb{P})$.

To fix this we consider a cardinal collapsing poset due to Silver, which was first used by him in the consistency proof for Chang's Conjecture.

20.1 Definition. Let κ be inaccessible and let $\delta = \mathrm{cf}(\delta) < \kappa$. The *Silver collapse* $\mathbb{S}(\delta, {<}\kappa)$ is the set of those partial functions f on $\delta \times \kappa$ such that $\mathrm{dom}(f) = \alpha \times X$ for some $\alpha < \delta$ and some $X \in [\kappa]^\delta$, and $f(\beta, \gamma) < \gamma$ for all $\beta < \alpha$ and $\gamma \in X$. The ordering is extension.

It is easy to see that $\mathbb{S}(\delta, {<}\kappa)$ is δ-closed and κ-c.c.

Returning for a moment to the discussion preceding Definition 20.1, if we let $\mathbb{P} = \mathbb{S}(\kappa, {<}\lambda)$ where $\lambda = k(\kappa)$ then it *is* possible to build a strong master condition. We will use this shortly, but first we discuss another problem with the Levy collapse.

Suppose that $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$ and that $\mathbb{B}$ is a complete subalgebra of $\mathrm{ro}(\mathbb{P})$. Then as we saw in Theorem 14.2 we can embed $\mathbb{B} * \dot{\mathbb{C}}$ into $\mathbb{P}$ when $\dot{\mathbb{C}}$ names an algebra of size less than κ. However there is no guarantee that this is possible when $\dot{\mathbb{C}}$ has size κ, even if $\mathbb{C}$ is forced to have the κ-c.c.

Kunen [45] showed that it is possible to construct a poset with stronger universal properties. We sketch a version of his construction. Let κ be an inaccessible cardinal, and let U be a function which returns for each complete Boolean algebra $\mathbb{B}$ of size less than κ a $\mathbb{B}$-name $U(\mathbb{B})$ for a κ-Knaster poset of size κ. We aim to build a κ-c.c. poset $\mathbb{P}$ of size κ such that for every complete subalgebra $\mathbb{B}$ of $\mathrm{ro}(\mathbb{P})$ with size less than κ, the inclusion embedding of $\mathbb{B}$ into $\mathrm{ro}(\mathbb{P})$ extends to a complete embedding of $\mathbb{B} * \mathrm{ro}(U(\mathbb{B}))$ into $\mathrm{ro}(\mathbb{P})$.

To construct the universal collapse we build a finite support κ-c.c. iterated forcing poset $\mathbb{P}_\kappa$ of length and cardinality κ, where each step $\mathbb{P}_\alpha$ is κ-c.c. with cardinality κ. At stage α we choose by some book-keeping scheme some $\mathbb{B}_\alpha$ which is a complete subalgebra of $\mathrm{ro}(\mathbb{P}_\alpha)$ with $|\mathbb{B}_\alpha| < \kappa$. Given an $\mathbb{B}_\alpha$-generic filter g we may form in $V[g]$ the product $\mathbb{P}_\alpha/g \times U(\mathbb{B}_\alpha)$, which is κ-c.c. by Theorem 5.12. Back in V we see that $\mathbb{B}_\alpha * (\mathbb{P}_\alpha/\dot{g} \times U(\mathbb{B}_\alpha))$ is κ-c.c. and embeds both $\mathbb{P}_\alpha$ and $\mathbb{B}_\alpha * U(\mathbb{B}_\alpha)$, and choose $\mathbb{P}_{\alpha+1}$ accordingly. With appropriate book-keeping we may arrange that every small subalgebra of $\mathrm{ro}(\mathbb{P}_\kappa)$ has appeared as $\mathbb{B}_\alpha$ for some α, giving the desired universal property for $\mathbb{P}_\kappa$. Preservation of κ-c.c. is easy since we are iterating with finite support.

20.2 Remark. The construction of the universal collapse is an example of "iteration with amalgamation", a technique which is frequently used in forcing constructions to build saturated ideals. Note that in the construction we amalgamated $\mathbb{P}_\alpha$ and $U(\mathbb{B}_\alpha)$ over $\mathbb{B}_\alpha$. The point in applications will typically be that we can absorb an iteration $\mathbb{P} * \dot{\mathbb{Q}}$ into $j(\mathbb{P})$ in a context where $\mathbb{P} * \dot{\mathbb{Q}}$ is "large".

20.3 Remark. Laver showed that it is sometimes possible to build λ-closed collapsing posets with similar universal properties. Naturally one needs to be a little more careful about the chain condition.

We are now ready to sketch Kunen's consistency proof for an ω_2-saturated ideal on ω_1. More details will be found in Foreman's chapter in this Handbook.

20.4 Theorem. *Let κ be a huge cardinal with target λ. Then in some generic extension κ is ω_1, $\lambda = \omega_2$ and there is an ω_2-saturated ideal on ω_1.*

Proof. We fix an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, and ${}^{\lambda}M \subseteq M$. We start by constructing as above a κ-c.c. poset $\mathbb{P}$

of size κ such that for every subalgebra $\mathbb{B}$ of $\mathrm{ro}(\mathbb{P})$ with $|\mathbb{B}| < \kappa$, the inclusion embedding extends to an embedding of $\mathbb{B} * \mathrm{ro}(\mathbb{S}(\omega_1^{V^{\mathbb{B}}}, {<}\kappa))$. For convenience we assume (as we clearly may) that $\mathbb{P} \subseteq V_\kappa$.

It is easy to see that after forcing with $\mathbb{P}$, κ is the new ω_1. Let G be generic for $\mathbb{P}$, let $\mathbb{Q} = \mathbb{S}(\kappa, {<}\lambda)_{V[G]}$ and let H be $\mathbb{Q}$-generic over $V[G]$.

We will show that there is a λ-saturated ideal on κ in $V[G * H]$. Working in M we fix an embedding of $\mathrm{ro}(\mathbb{P} * \mathbb{Q})$ into $\mathrm{ro}(j(\mathbb{P}))$ extending the identity embedding of $\mathrm{ro}(\mathbb{P})$. Since $V[G * H] \models {}^\lambda M[G * H] \subseteq M[G * H]$, we see that $j(\mathbb{P})/G * H$ is λ-c.c. in $V[G * H]$. Forcing with this poset over $V[G * H]$, we obtain an embedding $j^+ : V[G] \longrightarrow M[G * H * I]$ in $V[G * H * I]$. Since $\mathbb{Q}$ is a Silver collapse, and H and $j \restriction \lambda$ are both in M, we may construct a strong master condition $r = \bigcup j``H \in j(\mathbb{Q})$ and force with $j(\mathbb{Q})/r$ to obtain a compatible generic object J and an embedding $j^{++} : V[G * H] \longrightarrow M[G * H * I * J]$.

Unfortunately this is not quite enough because the $V[G * H]$-ultrafilter $U = \{X \in P(\kappa) \cap V[G * H] : \kappa \in j^{++}(X)\}$ lives in the extension by $j(\mathbb{P})/G * H * j(\mathbb{Q})/r$, which is not λ-c.c. in $V[G * H]$. To fix this we note that in $V[G * H * I]$ the $V[G * H]$-powerset of κ has size λ, and $j^+(\mathbb{Q})$ is λ-closed; so we may build a decreasing sequence $\langle r_i : i < \lambda \rangle$ with $r_0 = r$ deciding whether $\kappa \in j^{++}(X)$ for all $X \in P(\kappa) \cap V[G * H]$, and then let $U_0 = \{X : \exists i \; r_i \Vdash \kappa \in j^{++}(X)\}$. Then U_0 is a $V[G * H]$-ultrafilter which lives in $V[G * H * I]$, so that we may derive a λ-saturated ideal by Lemma 14.5. $\dashv$

20.5 Remark. A Woodin cardinal is all that is required to get the consistency of an ω_2-saturated ideal on ω_1. The argument given here was generalized by Laver to get saturated ideals on larger cardinals. Magidor [54] showed that the kind of argument given here can be done from an almost huge cardinal.

21. Limiting Results

In this section we sketch some results which put limits on the effects which we can achieve in reverse Easton constructions. We are not sure to whom the following result should be attributed; it has a family resemblance to some results by Kunen [45] on the question of whether an inaccessible cardinal λ can carry a λ-saturated ideal.

21.1 Theorem. *If $\mathbb{P} \times \mathbb{P}$ is κ-c.c. and $\mathbb{P}$ forces that κ is measurable then κ is measurable.*

Proof. Clearly κ is inaccessible in V. Let $\dot{U}$ name a normal measure and suppose that κ is not measurable in V. If A is a potential member of $\dot{U}$ then it can be split into two disjoint potential members of $\dot{U}$, otherwise we could read off a measure on κ in V. Using this we build a binary tree of height κ with root node κ such that the levels form increasingly fine partitions of κ into fewer than κ many pieces. At successor steps every node is partitioned by its two immediate successors, and if a node is a potential member of $\dot{U}$ then so are both of its immediate successors; at limit steps λ, every branch through the binary tree of height λ which has been constructed so far is continued by putting at level λ the intersection of the nodes on that branch.

Now let G be $\mathbb{P}$-generic and realize $\dot{U}$ as U_G; then there is a unique branch through the tree consisting of members of U_G. Choosing for each A on the branch a condition which forces the successor of A which is not in U_G into $\dot{U}$, we build an antichain of size κ in $\mathbb{P}$, contradicting our assumption that $\mathbb{P} \times \mathbb{P}$ has the κ-c.c. $\dashv$

We now sketch some results of Hamkins [33]. The key technical result is Theorem 21.3 which involves two notions of resemblance between inner models of ZFC.

21.2 Definition. Let M and N be inner models of ZFC with $M \subseteq N$. Let δ be a regular uncountable cardinal in N. Then

1. *δ-covering holds between M and N* if and only if for every set $A \subseteq \mathrm{On}$ such that $A \in N$ and $N \models |A| < \delta$, there exists a set $B \subseteq \mathrm{On}$ such that $B \in M$, $A \subseteq B$ and $M \models |B| < \delta$.

2. *δ-approximation holds between M and N* if and only if for every $A \subseteq \mathrm{On}$ with $A \in N$, if $A \cap a \in M$ for all $a \in M$ with $M \models |a| < \delta$, then $A \in M$.

21.3 Theorem. *Let V and $\bar{V}$ be inner models with $V \subseteq \bar{V}$. Let $j : \bar{V} \longrightarrow \bar{M}$ be a definable elementary embedding with* $\mathrm{crit}(j) = \kappa$, *and let $M = \bigcup j``V$ so that $j \restriction V$ is an elementary embedding from V to M.*

If there is a cardinal $\delta < \kappa$ regular in $\bar{V}$ such that $\bar{V} \models {}^{\delta}\bar{M} \subseteq \bar{M}$, and the δ-covering and δ-approximation properties hold between V and $\bar{V}$, then

1. *$M = \bar{M} \cap V$, in particular $V \models {}^{\delta}M \subseteq M$.*

2. *$j \restriction A \in V$ for all $A \in V$.*

Proof. Throughout the proof we work in $\bar{V}$. In particular all cardinalities are computed in $\bar{V}$ unless otherwise specified. By elementarity and the fact that $\delta < \kappa$, the δ-covering and δ-approximation properties hold between M and $\bar{M}$.

We claim that every set of ordinals A with $|A| < \delta$ is contained in a set of ordinals $B \in V \cap M$ such that $|B| \leq \delta$. To see this we build (starting with A) an increasing and continuous chain of length δ consisting of sets of size less than δ, with even successor elements in V and odd successor elements in M. If B is the union then by the approximation property $B \in V \cap M$.

Next we claim that for every set of ordinals A with $|A| < \delta$, $A \in V$ if and only if $A \in M$. To see this find a set $B \in V \cap M$ with $A \subseteq B$ and $\gamma = \mathrm{ot}(B) < \delta^+$. Since $\gamma < \kappa$ and κ is inaccessible in V, it follows from Proposition 2.9 that $P(\gamma) \cap M = P(\gamma) \cap V$.

Now we claim that $M = \bar{M} \cap V$. Let $A \in M$ where by Proposition 2.2 we may assume that A is a set of ordinals. Clearly $A \in \bar{M}$. Let $a \in V$ with $|a| < \delta$. Applying the preceding claim $a \in M$, hence $A \cap a \in M$, hence by another application of the preceding claim $A \cap a \in V$. By the approximation property $A \in V$. Conversely let $A \in \bar{M} \cap V$ be a set of ordinals; arguing just as before $A \cap a \in M$ for all $a \in M$ with $|a| < \delta$, so that $A \cap a \in M$.

To finish we show that $j \restriction A \in V$ for all sets of ordinals $A \in V$. By approximation it will suffice to show that $j \restriction a \in V$ for all $a \in V$ with $a \subseteq A$ and $|a| < \delta$. Since $\mathrm{ot}(a) < \kappa$ we see that $j``a = j(a) \in M \subseteq V$, and since $j \restriction a$ is the order-isomorphism between $a$ and $j``a$ we have $j \restriction a \in V$. $\dashv$

21.4 Corollary. *Under the hypotheses of Theorem 21.3, if $\bar{V}$ is a set-generic extension of V then $j \restriction V$ is definable in V. It is also easy to see that if j witnesses the λ-supercompactness or λ-strongness of κ in $\bar{V}$ then $j \restriction V$ will do the same in V.*

Of course the interest of Theorem 21.3 hinges on there being some examples of extensions with the covering and approximation properties. The following result [33] shows that many extensions by reverse Easton iterations have these properties.

21.5 Theorem. *Let δ be a cardinal. Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a forcing iteration where $|\mathbb{P}| \leq \delta$, $\mathbb{P}$ is non-trivial and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is $(\delta+1)$-strategically closed. Then the δ^+-covering and δ^+-approximation properties hold between V and the extension by $\mathbb{P} * \dot{\mathbb{Q}}$.*

Proof. The covering is easy so we concentrate on the approximation. Let $G * H$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter and let $S : \theta \to 2$ be such that $S \in V[G * H]$ and $S \restriction a \in V$ for all $a \in V$ with $|a| \leq \delta$. By induction we may assume that $S \restriction \lambda \in V$ for all $\lambda < \theta$. Let $\dot{S}$ name S.

If $\mathrm{cf}(\theta) \leq \delta$ then $S \in V[G]$, so without loss of generality $\dot{S}$ is a $\mathbb{P}$-name. Consider the tree T of potential proper initial segments of $\dot{S}$; it is easy to see that there are at most δ many sequences t such that both $t^\frown 0$ and $t^\frown 1$ are in T. So by specifying δ many bits in S we determine S, hence $S \in V$.

If $\mathrm{cf}(\theta) > \delta$ in V, we note that this remains true in $V[G * H]$. So since $|G| \leq \delta$, there is a condition $p \in G$ such that for all $i < \theta$ there is a condition $q \in H$ so that (p, q) determines $\dot{S} \restriction i$. We may thus find a condition $(p, \dot{q}_0) \in G * H$ forcing that $\dot{S} \notin \check{V}$ and that p has this property; so easily for all i and all $(p, \dot{q}_1) \leq (p, \dot{q}_0)$ there is a condition $(p, \dot{q}_2) \leq (p, \dot{q}_1)$ determining $\dot{S} \restriction i$.

Using the non-triviality of $\mathbb{P}$ we can find a function $h \in V[G] \setminus V$ such that $h : \beta \to 2$ for some $\beta \leq \delta$, where (by choosing β to be minimal) we may also assume that $h \restriction j \in V$ for all $j < \beta$. Using the strategic closure of $\dot{Q}$, the choice of p and the fact that $\dot{S}$ is forced to be new we build $\langle \dot{q}_t : t \in {}^{<\beta}2 \rangle$ and $\langle \dot{\beta}_t : t \in {}^{<\beta}2 \rangle$ such that

1. For each t, $\dot{q}_t$ is a $\mathbb{P}$-name for a condition in $\mathbb{Q}$, and $\dot{\beta}_t$ is a $\mathbb{P}$-name for an element of ${}^{<\theta}2 \cap V$.

2. The sequences $\langle \dot{q}_t : t \in {}^{<\beta}2 \rangle$ and $\langle \dot{\beta}_t : t \in {}^{<\beta}2 \rangle$ lie in V.

3. It is forced by p that for any branch x of the tree ${}^{<\beta}2 \cap V$, the sequence $\langle q_{x \restriction j} : j < \beta \rangle$ has a lower bound.

4. $(p, q_{t^\frown i})$ forces that $\beta_t^\frown i$ is an initial segment of S.

Now working in $V[G]$ we choose a lower bound q for $\langle q_{h \restriction j} : j < \beta \rangle$. If we force so that H contains q we obtain a situation in which h can be computed from a proper initial segment of S, contradiction! $\dashv$

As an example of these ideas in action we sketch an easy case of the *superdestructibility* theorem of Hamkins [32]. A supercompact cardinal κ is said to be *Laver indestructible* if it is supercompact in every extension by κ-directed closed forcing; we show in Sect. 24 that any supercompact cardinal can be made indestructible.

21.6 Corollary. *Let κ be supercompact and let $\mathbb{P} = \mathrm{Add}(\omega, 1)$. Then κ is not Laver indestructible after forcing with $\mathbb{P}$.*

Proof. Let g be $\mathbb{P}$-generic, and let $\mathbb{Q} = \mathrm{Add}(\kappa, 1)_{V[g]}$. Let G be $\mathbb{Q}$-generic over $V[g * G]$. we show that κ is not measurable in $V[g * G]$.

Let $\bar{V} = V[g*G]$ and suppose that $j : \bar{V} :\longrightarrow \bar{M}$ is the ultrapower by some normal measure in $\bar{V}$. By Theorems 21.5 and 21.3 we have $j \restriction V : V \longrightarrow M$ where $M \subseteq V$. Now easily $\bar{M} = M[g * j(G)]$, by the closure of ultrapowers $G \in \bar{M}$, and by the closure of $j(\mathbb{Q})$ we have $G \in M[g]$. This is impossible as $M \subseteq V$ and $G \notin V[g]$. $\dashv$

22. Termspace Forcing

In this section we introduce a very useful idea due to Laver, that of the *term forcing* or *termspace forcing*. The idea is roughly that given a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ we can add by forcing over V a sort of "universal generic object", from which given any G which is $\mathbb{P}$-generic over V we may compute in a uniform way an H which is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$.

Magidor [52] showed by iterated Prikry forcing that the least measurable cardinal can be strongly compact. In unpublished work Magidor [51] gave an alternative proof, using termspace forcing and an Easton iteration of the forcing from Example 6.5. We outline the proof here, a more detailed account is given in a joint paper by Apter and the author [4] which further exploits these ideas.

22.1 Definition. Let $\mathbb{P}$ be a notion of forcing and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a notion of forcing. Then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is the notion of forcing whose underlying set is the set of canonical $\mathbb{P}$-terms for members of $\dot{\mathbb{Q}}$, with the ordering being given by

$$\dot{\sigma} \leq_{\mathrm{A}(\mathbb{P},\dot{\mathbb{Q}})} \dot{\tau} \quad \iff \quad \Vdash_{\mathbb{P}} \dot{\sigma} \leq_{\dot{\mathbb{Q}}} \dot{\tau}.$$

22.2 Remark. Several notations for the termspace forcing are in use, for example $\mathbb{Q}^*$ and $\mathbb{Q}^{\mathbb{P}}$. We follow Foreman's paper [19] in using $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$, emphasizing the importance of $\mathbb{P}$.

The following proposition is the key to the applications of term forcing.

22.3 Proposition. *Let G be $\mathbb{P}$-generic over V and let H be $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$-generic over V. Define $I = \{i_G(\dot{\tau}) : \dot{\tau} \in H\}$. Then I is an $i_G(\dot{\mathbb{Q}})$-generic filter over $V[G]$.*

Proof. We begin by checking that I is a filter. If $\dot{\sigma}$ and $\dot{\tau}$ are in H then there is a term $\dot{\rho} \in H$ such that $\Vdash_{\mathbb{P}} \dot{\rho} \leq \dot{\sigma}, \dot{\tau}$. It follows that $i_G(\dot{\rho}) \leq i_G(\dot{\sigma}), i_G(\dot{\tau})$ so that I is a directed set.

If $i_G(\dot{\sigma}) \leq i_G(\dot{\tau})$ with $\dot{\sigma} \in H$ then we fix $p \in G$ such that $p \Vdash_F P\dot{\sigma} \leq \dot{\tau}$. Let $\dot{\rho}$ be a name which is interpreted as $\dot{\tau}$ if p is in the generic filter and as the trivial condition otherwise, so that $p \Vdash_F P\dot{\rho} = \dot{\tau}$ and $\Vdash_{\mathbb{P}} \dot{\sigma} \leq \dot{\rho}$. Then $\dot{\tau} \in H$ and so $i_G(\dot{\rho}) = i_G(\dot{\tau}) \in I$. It follows that I is upwards closed, and so is a filter.

Finally let $D = i_G(\dot{D})$ where $\dot{D}$ is forced to be a dense subset of $\dot{\mathbb{Q}}$. If $E = \{\dot{\sigma} : \Vdash_{\mathbb{P}} \dot{\sigma} \in \dot{D}\}$ then by the Maximum Principle E is a dense subset of $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$. We find a term $\dot{\sigma} \in E \cap H$, and observe that $i_G(\dot{\sigma}) \in D \cap I$. It follows that I is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$ as required. $\dashv$

The next result is an easy application of the Maximum Principle.

22.4 Proposition. *If it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is κ-strategically closed then $\mathrm{A}(\mathbb{P}, \mathbb{Q})$ is κ-strategically closed.*

Foreman's paper "More saturated ideals" [19] contains a wealth of other structural results about $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$. We quote some here.

22.5 Proposition. *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset.*

1. *If $\mathbb{P}$ is non-trivial and it is not forced that $\dot{\mathbb{Q}}$ is κ-c.c. then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is not 2^κ-c.c.*
2. *If κ is inaccessible, $\mathbb{P}$ is κ-c.c. and it is forced that $\mathbb{Q}$ is κ-c.c. then $\mathrm{A}(\mathbb{P}, \dot{\mathbb{Q}})$ is κ-c.c.*
3. *If $\langle P_i, \dot{\mathbb{Q}}_i \rangle$ is a forcing iteration with supports in an ideal I, then the limit of the iteration can be completely embedded in the product of the termspace posets $\mathrm{A}(\mathbb{P}_i, \dot{\mathbb{Q}}_i)$ taken with supports in I.*

We will now use termspace forcing to give a proof (due to Magidor) that the least measurable cardinal can be strongly compact. The idea of the proof is to shoot a non-reflecting stationary set through each measurable cardinal below a supercompact cardinal κ, and then argue that the strong compactness of κ is preserved and no new measurable cardinals are created.

To get an embedding witnessing strong compactness we use the following easy result.

22.6 Proposition. *Let $j : V \longrightarrow M$ be an embedding with critical point κ, and let $\lambda \geq \kappa$ be such that $j``\lambda \in M$ and $\lambda < j(\kappa)$. Let $k : M \longrightarrow N$ be any embedding with* $\mathrm{crit}(k) \geq \kappa$ *and let $X = k(j``\lambda)$. Then* $\mathrm{crit}(k \circ j) = \kappa$, *$X \in N$, $(k \circ j)``\lambda \subseteq X$ and $N \models \mathrm{ot}(X) < k \circ j(\kappa)$.*

In particular, if $k \circ j$ is definable then $k \circ j$ witnesses that κ is λ-strongly compact. If $V[G]$ is a generic extension of V, and $i : V[G] \longrightarrow M[H]$ is an embedding definable in $V[G]$ extending $k \circ j$, then i witnesses that κ is λ-strongly compact in $V[G]$.

We now fix a ground model in which GCH holds, κ is supercompact, and there is no measurable cardinal greater than κ. This last hypothesis is a technical one which simplifies some later arguments; it entails no loss of generality because we can truncate the universe at the least measurable greater than κ if such a cardinal exists. Notice that since κ is supercompact there are unboundedly many measurable cardinals less than κ.

Let A be the set of $\alpha < \kappa$ which are measurable in V. We will define an iteration $\mathbb{P}_\kappa$ of length κ with Easton support, in which $\dot{\mathbb{Q}}_\alpha$ names the trivial forcing unless $\alpha \in A$. If $\alpha \in A$ then $\dot{\mathbb{Q}}_\alpha$ names the poset from Example 6.5 to add a non-reflecting stationary set to α, as defined in $V[G_\alpha]$. It is clear that this iteration will destroy the measurability of every α in A. We will show that no new measurable cardinals are created.

Let G_κ be $\mathbb{P}_\kappa$-generic over V and suppose for a contradiction that $\alpha < \kappa$ and α is measurable in $V[G_\kappa]$. By construction $\alpha \notin A$, and if γ is the least measurable greater than α then arguments as in Lemma 11.2 show that $V[G_\kappa]$ is an extension of $V[G_\alpha]$ by γ-strategically closed forcing. In particular α is measurable in $V[G_\alpha]$, from which it easily follows that α must be a Mahlo cardinal in V. Since α is Mahlo, by Proposition 7.13 $\mathbb{P}_\alpha$ is α-Knaster. It follows that $\mathbb{P}_\alpha \times \mathbb{P}_\alpha$ is α-c.c. By Theorem 21.1 α must be measurable in V, which is a contradiction as $\alpha \notin A$.

To finish, we show that κ is still strongly compact in $V[G_\kappa]$. We fix a regular cardinal $\lambda > \kappa$ and let $j : V \longrightarrow M$ be the ultrapower of V by a supercompactness measure on $P_\kappa\lambda$. The argument of Example 4.8 shows that $V \models |j(\kappa)| = \lambda^+$.

By GCH $\lambda \geq 2^\kappa$ and so κ is measurable in M, and we may find a measure U on κ such that $U \in M$ and U is minimal in the Mitchell ordering [58]; we let $k : M \longrightarrow N$ be the ultrapower of M by U, so that in particular $N \models$ "κ is not measurable". It is easy to see that $\kappa \notin k \circ j(A)$.

Consider the iteration $j(\mathbb{P}_\kappa)$, which is an iteration defined in M in which a non-reflecting stationary set is added to each $\alpha \in j(A)$. The cardinal κ is measurable in M, so $\kappa \in j(A)$ and $j(\mathbb{P})_\kappa$ adds a set at κ. There are no measurable cardinals above κ in V and $P(\lambda) \subseteq M$, so if γ is the least M-measurable cardinal greater than κ then $\gamma > \lambda$.

Notice that since we are aiming to show that κ is strongly compact (and so a fortiori measurable) in $V[G_\kappa]$ we cannot hope to find a $\mathbb{Q}_\kappa$-generic filter over $M[G_\kappa]$ in $V[G_\kappa]$. It is at this point that we use termspace forcing.

Working in M we may factor $j(\mathbb{P}_\kappa)$ as $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ adds a non-reflecting stationary subset of κ. Working in $M[G_\kappa]$ we get a factorization of the rest of the iteration as $\mathbb{Q} * \dot{\mathbb{R}}$.

22.7 Lemma. *$\dot{\mathbb{R}}$ is $j(\kappa)$-c.c. and γ-strategically closed in $M[G_\kappa]^{\mathbb{Q}}$.*

Proof. It follows from Proposition 7.13 that $\mathbb{R}$ is $j(\kappa)$-c.c. The closure follows from Proposition 7.12. ⊣

22.8 Lemma. *In $M[G_\kappa]$, $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$ is $j(\kappa)$-c.c. and λ^+-strategically closed.*

Proof. We work in the model $M[G_\kappa]$. The strategic closure follows by Proposition 22.4.

For the chain condition, assume for a contradiction that $\langle \dot{r}_\alpha : \alpha < j(\kappa)\rangle$ is an antichain in $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$. If $\alpha < \beta$ then $\dot{r}_\alpha$ and $\dot{r}_\beta$ are incompatible, which means that there is no term for a condition forced to refine both of them; by the Maximum Principle this is equivalent to saying that $\dot{r}_\alpha$ and $\dot{r}_\beta$ are not forced to be compatible in $\dot{\mathbb{R}}$.

For $\alpha < \beta$ we choose $q_{\alpha\beta} \in \mathbb{Q}$ such that $q_{\alpha\beta} \Vdash^{M[G_\kappa]}_{\mathbb{Q}} \dot{r}_\alpha \perp \dot{r}_\beta$. $j(\kappa)$ is measurable in M and so by the Levy-Solovay Theorem [50] $j(\kappa)$ is measurable in $M[G_\kappa]$. By Rowbottom's theorem we may therefore find a fixed $q \in \mathbb{Q}$ and $X \subseteq j(\kappa)$ unbounded such that $q_{\alpha\beta} = q$ for all $\alpha, \beta \in X$. q forces that $\{\dot{r}_\alpha : \alpha \in X\}$ is an antichain of size $j(\kappa)$ in $\mathbb{R}$, contradicting Lemma 22.7. So $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$ is $j(\kappa)$-c.c. in $M[G_\kappa]$. ⊣

Appealing to Proposition 8.1 we may now build $H \in V[G_\kappa]$ which is $\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}})$-generic over $M[G_\kappa]$.

We now consider the embedding k and the iteration $k(\mathbb{P}_\kappa)$. Since κ is not a point at which this iteration adds a set, we may argue exactly as in Sect. 11 to build $g \in M[G_\kappa]$ such that $G_\kappa * g$ is $k(\mathbb{P}_\kappa)$-generic over $N[G_\kappa]$, and may lift to get $k : M[G_\kappa] \longrightarrow N[G_\kappa * g]$. By similar arguments we may also build $h \in M[G_\kappa]$ which is $k(\mathbb{Q})$-generic over $N[G_\kappa * g]$.

By Proposition 9.3 this lifted embedding has width $\leq \kappa$, so by Proposition 15.1 we may transfer H along k to get H^+ which is $k(\mathrm{A}(\mathbb{Q}, \dot{\mathbb{R}}))$-generic over $N[G_\kappa * g]$. If we let $I = \{i_h(\dot{\sigma}) : \dot{\sigma} \in H^+\}$ then I is $k(\mathbb{R})$-generic over $N[G_\kappa * g * h]$.

Putting everything together we get $G_\kappa * g * h * I$ which is $k \circ j(\mathbb{P}_\kappa)$-generic over N, and then as in Sect. 11 we may lift $k \circ j$ to get a map from $V[G_\kappa]$ to $N[G_\kappa * g * H * I]$. This map is definable by Proposition 9.4, and so by Proposition 22.6 we see that κ is λ-strongly compact in $V[G_\kappa]$.

23. More on Termspace Forcing and Collapsing

In this section we show that the termspace forcing ideas of Sect. 22 may be used to analyze iterations. We also introduce yet another cardinal collapsing poset, this time one due to Mitchell [57].

We give an outline of Mitchell's model [57] in which there are no ω_2-Aronszajn trees. Our treatment of this material owes much to Abraham [2]. For simplicity we build the model using a measurable cardinal. Mitchell actually used a weakly compact cardinal and this is known to be optimal [57].

Throughout this section we assume that κ is measurable. We recall the easy proof that κ has the tree property; let T be a κ-tree, let $j : V \longrightarrow M$ have critical point κ, then $j(T)\restriction\kappa$ is isomorphic to T and any point on level κ of $j(T)$ gives us a branch through T.

We start by making an instructive false start. Let $\mathbb{P} = \mathrm{Col}(\omega_1, {<}\kappa)$ and as in Theorem 10.5 factor $j(\mathbb{P})$ as $\mathbb{P} \times \mathbb{Q}$. If $G * H$ is $j(\mathbb{P})$-generic then we may build as usual an embedding $j : V[G] \longrightarrow M[G * H]$. If $T \in V[G]$ is a κ-tree then as above $j(T)\restriction\kappa$ is isomorphic to κ, so by choosing any point on level κ we may determine a branch b of T.

It is well-known that CH implies there is a special ω_2-Aronszajn tree, and since $V[G]$ is a model of CH and $\kappa = \omega_2$ there is a κ-Aronszajn tree in $V[G]$. This is not a contradiction to the argument of the previous paragraph; the point is that $j(T)$ only exists in $M[G*H]$, so the branch b that we constructed is a member of $V[G * H]$ but not in general a member of $V[G]$.

To put the problem more abstractly, we need to create a situation in which a generic embedding with critical point ω_2 is added by a poset which does not add any branches through any ω_2-Aronszajn tree. By the remarks made above we also need the continuum to be at least ω_2.

Before the main argument we need a technical fact about trees.

23.1 Lemma. *Let $2^\omega = \omega_2$ and let T be an ω_2-Aronszajn tree. Let $\mathbb{S}$ and $\mathbb{T}$ be forcing posets such that*

1. *$\mathbb{T}$ is countably closed forcing and collapses ω_2.*
2. *$\mathbb{S}$ is ω_1-Knaster in $V^{\mathbb{T}}$.*

Then forcing with $\mathbb{S} \times \mathbb{T}$ does not add a cofinal branch of T.

Proof. Let $G_{\mathbb{S}} \times G_{\mathbb{T}}$ be $\mathbb{S} \times \mathbb{T}$-generic. We claim first that T has no cofinal branch in $V[G_{\mathbb{T}}]$. To see this suppose $p \in \mathbb{T}$ forces that $\dot{b}$ is a cofinal branch, and use the fact that $b \notin V$ to build a binary tree $\langle p_s : s \in {}^{<\omega}2\rangle$ and increasing $\langle \alpha_n : n < \omega\rangle$ such that $p_0 = p$ and for each n the conditions $\{p_s : s \in {}^n2\}$ decide where $\dot{b}$ meets level α_n in 2^n different ways. Then let $\alpha = \sup_n \alpha_n$ and observe that level α must have at least 2^ω elements, contradicting our assumptions that T is an ω_2-tree and $2^\omega = \omega_2$.

Choose in $V[G_{\mathbb{T}}]$ a sequence β_j for $j < \omega_1$ which is cofinal in ω_2^V. Suppose for a contradiction that some $q \in \mathbb{S}$ forces over $V[G_{\mathbb{T}}]$ that $\dot{c}$ is cofinal in T, and then choose for each j a condition $q_j \leq q$ deciding where the branch $\dot{c}$ meets level β_j. In $V[G_{\mathbb{T}}]$ a subfamily of size ω_1 of $\{q_j\}$ must be pairwise compatible, but this implies that there is a cofinal branch in $V[G_{\mathbb{T}}]$. ⊣

23.2 Theorem. *Let κ be measurable. Then in some ω_1-preserving generic extension, $2^\omega = \omega_2 = \kappa$ and κ has the tree property.*

Proof. Let

$$\mathbb{P} = \mathrm{Add}(\omega, \kappa), \qquad \mathbb{P}_\alpha = \mathrm{Add}(\omega, \alpha), \qquad \mathbb{R}_\alpha = \mathrm{Add}(\omega_1, 1)_{V^{\mathbb{P}_\alpha}}.$$

We define $\mathbb{Q}$ as follows; a condition is a pair (p, f) where $p \in \mathbb{P}$, f is a partial function on κ with countable support, and $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $\mathbb{R}_\alpha$. $(p_2, f_2) \leq (p_1, f_1)$ iff $p_2 \leq p_1$ in $\mathbb{P}$, $\mathrm{supp}(f_1) \subseteq \mathrm{supp}(f_2)$, and $p_2 \restriction (\omega \times \alpha) \Vdash f_2(\alpha) \leq f_1(\alpha)$ for all $\alpha \in \mathrm{supp}(f_1)$.

It is easy to see that $\mathbb{Q}$ is κ-c.c. Since adding a Cohen subset of ω_1 collapses the continuum to ω_1, it is also easy to see that $\mathbb{Q}$ collapses every α between ω_1 and κ. It may not be immediately clear that $\mathbb{Q}$ preserves ω_1. This will fall out from the product analysis of $\mathbb{Q}$ which we give below.

For any inaccessible $\delta < \kappa$ we may truncate the forcing at δ in the obvious way, to get $\mathbb{Q} \restriction \delta$ which forces $2^\omega = \omega_2 = \delta$. We note that if G_δ is $\mathbb{Q} \restriction \delta$-generic then $\mathbb{Q}/G_\delta$ is very similar to $\mathbb{Q}$.

To analyze $\mathbb{Q}$ we define a variation of the sort of termspace forcing we studied in Sect. 22. Let $\mathbb{R}$ be the set of g such that g is a function on κ with countable support, and $g(\alpha)$ is $\mathbb{P}_\alpha$-name for an element of $\mathbb{R}_\alpha$. Order $\mathbb{R}$ by setting $r_2 \leq r_1$ if and only if $\mathrm{supp}(r_1) \subseteq \mathrm{supp}(r_2)$, and $\Vdash r_2(\alpha) \leq r_1(\alpha)$ for all $\alpha \in \mathrm{supp}(r_1)$. It is routine to check that the identity is a projection map from $\mathbb{P} \times \mathbb{R}$ to $\mathbb{Q}$. It follows that if G is $\mathbb{Q}$-generic with projection g on the first coordinate then we may view $V[G]$ as a submodel of $V[g \times h]$ where $g \times h$ is $\mathbb{P} \times \mathbb{R}$-generic. By Easton's Lemma all countable sequences from $V[G]$ are in $V[g]$, so in particular ω_1 is preserved.

We now finish the argument by showing there are no ω_2-Aronszajn trees in $V[G]$. To do this we start by noting that (morally speaking) $\mathbb{Q} \subseteq V_\kappa$, so that we may regard $\mathbb{Q}$ as an initial segment of $j(\mathbb{Q})$ where $j : V \longrightarrow M$ is the ultrapower by some normal measure on κ. As usual we may then build a generic embedding $j : V[G] \longrightarrow M[G * H]$ where H is $j(\mathbb{Q})/G$-generic.

Suppose for contradiction that $T \in V[G]$ is a κ-tree. By the usual chain condition arguments $T \in M[G]$, and since $j(T) \in M[G * H]$ we see that forcing over $M[G]$ with $j(\mathbb{Q})/G$ has added a branch to the κ-Aronszajn tree T. We observe that in $M[G]$ we have that $2^\omega = \kappa = \omega_2$. It is not hard to see that $j(\mathbb{Q})/G$ is susceptible exactly to the same kind of product analysis as $\mathbb{Q}$ or $j(\mathbb{Q})$, so that by Lemma 23.1 it is not possible for $j(\mathbb{Q})/G$ to add a branch through T. This concludes the proof. $\dashv$

23.3 Remark. Abraham [2] showed that it is consistent for both ω_2 and ω_3 to simultaneously have the tree property. Foreman and the author [11] built a model where ω_n has the tree property for $1 < n < \omega$. Magidor and Shelah [56] showed that $\omega_{\omega+1}$ may have the tree property. Foreman and the author [11] constructed a model where ω_ω is strong limit and $\omega_{\omega+2}$ has the tree property.

Mitchell also showed that if κ is Mahlo and we force with the poset $\mathbb{Q}$ of Theorem 23.2, then in the extension there is no special ω_2-Aronszajn tree. By work of Jensen there is a special ω_2-Aronszajn tree if and only if the weak square principle $\square^*_{\omega_1}$ holds, so in Mitchell's model $\square^*_{\omega_1}$ fails. We sketch a proof (due to Mitchell) that an even weaker version of square fails in the model; for more on the ideal $I[\lambda]$ see [10].

Recall that $I[\omega_2]$ is the (possibly improper) ideal of $A \subseteq \omega_2$ such that there exist $\langle x_\alpha : \alpha < \omega_2 \rangle$ and a club set $C \subseteq \omega_2$, such that for every $\alpha \in C \cap A \cap \mathrm{Cof}(\omega_1)$ there is a set $d \subseteq \alpha$ with d club in α, $\mathrm{ot}(d) = \omega_1$, and every proper initial segment of d appearing as x_β for some $\beta < \alpha$. It is easy to see that if $\square^*_{\omega_1}$ then $\omega_2 \in I[\omega_2]$.

23.4 Theorem. *If κ is Mahlo and we force with $\mathbb{Q}$ as in Theorem 23.2 then in the extension $\omega_2 \notin I[\omega_2]$.*

Proof. Let G be $\mathbb{Q}$-generic. An argument similar to that of Theorem 21.5 shows that if $\alpha < \kappa$ is inaccessible and $X \in P(\alpha) \cap V[G]$ with $X \cap \beta \in V[G_\alpha]$ for all $\beta < \alpha$, then $X \in V[G_\alpha]$. Suppose for contradiction that $\langle x_\alpha : \alpha < \kappa \rangle$ and C witness in $V[G]$ that $\omega_2 \notin I[\omega_2]$.

Then since β is Mahlo and $\mathbb{Q}$ is β-c.c. there is a V-inaccessible cardinal $\beta \in C$ such that $\langle x_\alpha : \alpha < \beta \rangle \in V[G_\beta]$, and so there is in $V[G]$ a club subset $d \subseteq \beta$ such that $\mathrm{ot}(d) = \omega_1$ and every initial segment of d is in $V[G_\beta]$. By the remarks of the last paragraph we have $d \in V[G_\beta]$, which is impossible because $\beta = \omega_2^{V[G_\beta]}$. ⊣

24. Iterations with Prediction

In this section we look at some theorems proved using the powerful reflection properties of supercompact cardinals. Both of the results we prove depend on the following theorem of Laver [49] which may be viewed as a kind of diamond principle.

24.1 Theorem. *Let κ be a supercompact cardinal. Then there exists a function $f : \kappa \to V_\kappa$ such that for all $\lambda \geq \kappa$ and all $x \in H_{\lambda^+}$ there is a supercompactness measure U on $P_\kappa \lambda$ such that $j_U(f)(\kappa) = x$.*

Proof. Fix a well-ordering $\prec$ of V_κ. We define $f(\alpha)$ by recursion on α. We set $f(\alpha) = 0$ unless there exists a cardinal λ with $\alpha \leq \lambda < \kappa$ and $x \in H_{\lambda^+}$, such that for no supercompactness measure U on $P_\alpha \lambda$ does $j_U(f \restriction \alpha)(\alpha) = x$. In this case we choose the minimal such λ and then the $\prec$-minimal such $x \in H_{\lambda^+}$, and set $f(\alpha) = x$.

Suppose for a contradiction that there exist $\lambda \geq \kappa$ and $x \in H_{\lambda^+}$ such that for no supercompactness measure U on $P_\kappa \lambda$ does $j_U(f)(\kappa) = x$. Let $\rho = 2^{2^\lambda}$, let W be a supercompactness measure on $P_\kappa \rho$, and let the ultrapower by W be $j : V \longrightarrow N = \mathrm{Ult}(V, W)$. Observe that $H_{\lambda^+} \subseteq (V_{j(\kappa)})_N$.

All supercompactness measures on $P_\kappa\lambda$ and all functions from $P_\kappa\lambda$ to V_κ lie in N. It follows easily that

$$N \models \text{"for no supercompactness measure } U \text{ on } P_\kappa\lambda \text{ does } j_U(f)(\kappa) = x\text{"}.$$

Let μ be minimal such that for some $y \in H_{\mu^+}$ there is no supercompactness measure U on $P_\kappa\mu$ with $j_U(f)(\kappa) = y$; clearly $\mu \leq \lambda$, so in particular $y \in (V_{j(\kappa)})_N$. Let y be $j(\prec)$-minimal such $y \in H_{\mu^+}$. By elementarity, the definition of f, and the agreement between V and N we may conclude that $j(f)(\kappa) = y$.

Now we define $U = \{X \subseteq P_\kappa\mu : j``\mu \in j(X)\}$ so that $U$ is a supercompactness measure on $P_\kappa\mu$. Let $i : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map, and observe that by Proposition 3.2 there is an elementary embedding $k : M \longrightarrow N$ given by $k : [F]_U = j(F)(j``\mu)$. We also have that $k \circ i = j$.

We now analyze the embedding k. The definition of k gives easily that $j``V \subseteq \mathrm{ran}(k)$ and $j``\mu \in \mathrm{ran}(k)$. If $X \subseteq \mu$ then

$$X = \{\mathrm{ot}(\gamma \cap j``\mu) : \gamma \in j``\mu \cap j(X)\},$$

so that $X \in \mathrm{ran}(k)$. It follows that $H_{\mu^+} \subseteq \mathrm{ran}(k)$ and so in particular $k \restriction H_{\mu^+} = \mathrm{id}_{H_{\mu^+}}$.

Since $y \in H_{\mu^+}$, $k(y) = y$. We also know that $k(\kappa) = \kappa$ and $k \circ i = j$, so $k(i(f)(\kappa)) = j(f)(\kappa) = y$. Contradiction!

It follows that for all $\lambda \geq \kappa$ and all $x \in H_{\lambda^+}$ there is a supercompactness measure U on $P_\kappa\lambda$ with $j_U(f)(\kappa) = x$. $\dashv$

24.2 Remark. Using extenders in the place of supercompactness measures it is possible to prove a similar result for strong cardinals. See Gitik and Shelah's paper [29] for this result and some applications.

In this section we prove the consistency of the Proper Forcing Axiom (defined below) and of the statement "the supercompactness of κ is indestructible under κ-directed closed forcing". These statements have in common that they involve a universal quantification over a proper class; they will both be proved by doing a set forcing and using some reflection arguments.

In each of the two consistency proofs we will begin with a supercompact cardinal κ. We fix a function f as in Theorem 24.1 (a *Laver function*) and use this function as a guide in building an iteration of length κ which anticipates a proper class of possibilities for what may happen at stage κ.

The details of the constructions are of course somewhat different, but they each involve taking a generic object for some forcing we may do stage κ, and copying it via some supercompactness embedding j to a filter on the image of that forcing under j. In the argument for the Proper Forcing Axiom the existence of this filter is reflected back to give a witness for the truth of the axiom, while in the indestructibility theorem the filter is used to construct a strong master condition and lift the embedding j.

We now give Baumgartner's consistency proof [15] for the Proper Forcing Axiom. We begin with a brief review of proper forcing.

24.3 Definition. Let θ be regular with $\mathbb{P} \in H_\theta$. Let $<_\theta$ be a well-ordering of H_θ and let $\mathbb{P} \in N \prec (H_\theta, \in, <_\theta)$ where N is countable. $p \in \mathbb{P}$ is $(N, \mathbb{P})$-*generic* if and only if for every maximal antichain A of $\mathbb{P}$ with $A \in N$, $A \cap N$ is predense below p.

24.4 Remark. This notion is closely related to the ideas about lifting embeddings from Proposition 9.1. Let $\bar{N}$ be the Mostowski collapse of N, let $\pi : \bar{N} \longrightarrow N$ be the inverse of the Mostowski collapse and let $\bar{x}$ be the collapse of x for $x \in N$.

Then it is easy to see that p is $(N, \mathbb{P})$-generic if and only if p forces that $\bar{G} =_{\text{def}} \{\bar{p} : p \in G \cap N\}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$; that is, p is a master condition for π in the sense of Definition 12.1. The definition of $\bar{G}$ implies that $\pi``\bar{G} \subseteq G$. Therefore if p is $(N, \mathbb{P})$-generic and $p \in G$ for some G which is $\mathbb{P}$-generic over V, then π can be lifted to a map $\pi^+ : \bar{N}[\bar{G}] \longrightarrow N[G]$ which is the inverse of the Mostowski collapse map for $N[G]$.

We note that for example in the Martin's Maximum paper [21] $(N, \mathbb{P})$-generic conditions are referred to as "$(N, \mathbb{P})$-master conditions". We have chosen to follow the conventions of Shelah's book on proper forcing [64].

24.5 Remark. In the study of proper forcing it is often interesting to look at conditions which are *strongly* $(N, \mathbb{P})$-*generic*, where (adopting the notation of the last remark) a condition $p \in \mathbb{P}$ is strongly $(N, \mathbb{P})$-generic if and only if $g_p =_{\text{def}} \{\bar{q} : q \in N \cap \mathbb{P}, p \leq q\}$ is $\bar{\mathbb{P}}$-generic over $\bar{N}$. Such a condition is precisely a strong master condition for g_p and π in the sense of Definition 12.2.

24.6 Definition. $\mathbb{P}$ is *proper* if and only if for all large θ, all countable N with $\mathbb{P} \in N \prec (H_\theta, \in, <_\theta)$, and all $p \in \mathbb{P} \cap N$ there exists a condition $q \leq p$ which is $(N, \mathbb{P})$-generic.

See Abraham's chapter in this Handbook for an exposition of proper forcing. The only fact about properness we will need is that a countable support iteration of proper forcing is proper.

24.7 Definition. The Proper Forcing Axiom (PFA) is the statement: for every proper $\mathbb{P}$ and every sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$ there exists a filter F on $\mathbb{P}$ such that $F \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

Before we prove the consistency of PFA we make a few remarks.

24.8 Remark. It would be hopeless to ask to meet ω_2 sets in the statement of PFA, because we could then apply the axiom to the proper forcing poset $\text{Col}(\omega_1, \omega_2)$ and produce a surjection from ω_1 onto ω_2.

24.9 Remark. The axiom PFA is known [66] to have a very high consistency strength. One way of seeing this is that by work of Todorčević [67] PFA implies the failure of $\square_\kappa$ for κ singular, which implies in turn that the weak covering lemma fails over any reasonable core model.

24.10 Remark. In the consistency proof for MA_{ω_1} [6] the first step is to observe that we only need to deal with forcing posets of size ω_1. The point is that the property of being c.c.c. is inherited by any completely embedded subposet, and so to deal with ω_1 dense subsets of an arbitrary $\mathbb{P}$ we may as well work in some subposet Q of size ω_1 in which all those dense sets remain dense.

This is not so for proper forcing. Notice that by Example 6.6 we may force $\square_\kappa$ without changing H_{κ^+}, so that PFA cannot in general be "localised" to a statement in H_{κ^+} for any κ.

24.11 Theorem. *Let κ be supercompact. Then there is a forcing iteration of length κ such that in $V^{\mathbb{P}_\kappa}$*

1. PFA *holds.*
2. $2^\omega = \kappa = \omega_2$.

Proof. Let $f : \kappa \longrightarrow V_\kappa$ be a function as in Theorem 24.1. The poset $\mathbb{P}_\kappa$ will be an inductively defined iteration of length κ with countable support, with each $\mathbb{Q}_\alpha$ forced to be proper in $V^{\mathbb{P}_\alpha}$. The name $\dot{\mathbb{Q}}_\alpha$ will name the trivial forcing unless $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for a proper forcing poset, in which case $\mathbb{Q}_\alpha = f(\alpha)$.

By the Properness Iteration Theorem [1] the poset $\mathbb{P}_\kappa$ is proper, so preserves ω_1. By Proposition 7.13 $\mathbb{P}_\kappa$ is κ-c.c. with cardinality κ, so in particular κ is preserved.

Now let $\dot{Q}$ be the standard $\mathbb{P}_\kappa$-name for $\mathrm{Add}(\omega, 1)$. Let $\lambda = 2^{2^\kappa}$ and find a supercompactness measure U on $P_\kappa\lambda$ such that $j_U(f)(\kappa) = \dot{Q}$. Arguing as in Lemma 11.6, $j_U(\mathbb{P}_\kappa)$ is an iteration of length $j(\kappa)$ in $\mathrm{Ult}(V, U)$ whose first κ stages are exactly those of $\mathbb{P}_\kappa$.

$\mathrm{Ult}(V, U)$ agrees that $\dot{Q}$ is a $\mathbb{P}_\kappa$-name for $\mathrm{Add}(\omega, 1)$, so by the usual reflection argument there are unboundedly many $\alpha < \kappa$ such that $f(\alpha)$ is a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\omega, 1)$. Since $\mathrm{Add}(\omega, 1)$ is proper, there are unboundedly many α where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\omega, 1)$, so that in the course of the iteration $\mathbb{P}_\kappa$ we add κ many subsets of ω. A very similar argument shows that $\mathbb{Q}_\alpha = \mathrm{Col}(\omega_1, \alpha)_{V^{\mathbb{P}_\alpha}}$ for many $\alpha < \kappa$, so that $2^\omega = \kappa = \omega_2$ in $V^{\mathbb{P}_\kappa}$.

To finish the proof we need to show that PFA holds in $V^{\mathbb{P}_\kappa}$. Let G be $\mathbb{P}_\kappa$-generic over V, and let $\mathbb{Q} = i_G(\dot{\mathbb{Q}})$ where $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$-name for a proper forcing poset. Find a cardinal λ such that $\dot{\mathbb{Q}} \in H_\lambda$, let $\mu = 2^{2^\lambda}$, and let U be a supercompactness measure on $P_\kappa\mu$ such that $j_U(f)(\kappa) = \dot{\mathbb{Q}}$. Let $j : V \longrightarrow M = \mathrm{Ult}(V, U)$ be the ultrapower map.

By Proposition 8.4 we see that $V[G] \models {}^\mu M[G] \subseteq M[G]$, so in particular $\Vdash^M_{\mathbb{P}_\kappa}$ "$\dot{\mathbb{Q}}$ is proper". It follows that in the iteration $j(\mathbb{P}_\kappa)$, the forcing which is used at stage κ is precisely $\dot{\mathbb{Q}}$.

Now let g be $\mathbb{Q}$-generic over $V[G]$. Working in $M^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ let $\dot{\mathbb{R}}$ name the canonical forcing such that $j(\mathbb{P}_\kappa) \simeq \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, and let $\mathbb{R} = i_{G*g}(\dot{\mathbb{R}})$. Let H be $\mathbb{R}$-generic over $V[G * g]$.

Since $\mathbb{P}_\kappa$ is an iteration with countable support, the support of every condition in G is bounded in κ. This implies that $j``G \subseteq G * g * H$, and so we may lift j to get a map $j_G : V[G] \longrightarrow M[G * g * H]$.

Now let $\vec{D} = \langle D_\alpha : \alpha < \omega_1 \rangle$ be an ω_1-sequence of dense subsets of $\mathbb{Q}$, with $\vec{D} \in V[G]$. Since g is generic over $V[G]$, $g \cap D_\alpha \neq \emptyset$ for each α. By the choice of μ and U we know that $j \restriction \dot{\mathbb{Q}} \in M$, from which it follows by the definition of j_G in Proposition 9.1 that $j_G \restriction \mathbb{Q} \in M[G * g * H]$.

Now let F be the filter on $j_G(\mathbb{Q})$ generated by $j_G``g$. By the arguments of the last paragraph we see that $F \in M[G{*}g{*}H]$. Since $\mathrm{crit}(j_G) = \mathrm{crit}(j) = \kappa$, we see that $j_G(\vec{D}) = \langle j_G(D_\alpha) : \alpha < \omega_1 \rangle$. By genericity $g \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$, and so by elementarity $F \cap j_G(D_\alpha) \neq \emptyset$ for all $\alpha < \omega_1$.

It follows that

$$M[G * g * H] \models \text{“}F \text{ meets every set in } j_G(\vec{D})\text{”}.$$

By the elementarity of j_G,

$$V[G] \models \text{“}\exists f\ f \text{ meets every set in } \vec{D}\text{”}.$$

It follows that $V[G]$ is a model of PFA. ⊣

We will prove the following theorem of Laver on making any supercompact cardinal Laver indestructible.

24.12 Theorem (Laver [49]). *Let κ be supercompact and let $\delta < \kappa$. There is a forcing iteration $\mathbb{P}_\kappa$ of length κ such that*

1. *κ is supercompact in $V^{\mathbb{P}_\kappa}$, and in any extension of $V^{\mathbb{P}_\kappa}$ by κ-directed closed forcing.*
2. *$\mathbb{P}_\kappa$ has cardinality κ, is κ-c.c. and is δ-directed closed.*

24.13 Remark. The proof of Theorem 24.12 is similar in its outline to the consistency proof for PFA. One significant difference is that we will leave long gaps in the iteration in which nothing happens. This is natural when we consider that

- κ is supposed to be supercompact in $V^{\mathbb{P}_\kappa}$.
- $\mathbb{P}_\kappa$ will have no effect above κ on cardinals, cofinalities and the continuum function.

An easy reflection argument shows that there should be arbitrarily long intervals (α, β) below κ in which $V^{\mathbb{P}_\kappa}$ should resemble V.

Proof. Let $f : \kappa \longrightarrow V_\kappa$ be a function as in Theorem 24.1. The poset $\mathbb{P}_\kappa$ will be an iteration of length κ with Easton support, such that for each α

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ is } \alpha\text{-directed closed”}.$$

$\dot{\mathbb{Q}}_\alpha$ will name the trivial forcing unless

1. $\alpha \geq \delta$.
2. $f(\alpha)$ is a pair $(\lambda, \mathbb{Q})$ where $\mathbb{Q}$ is a $\mathbb{P}_\alpha$-name for an α-directed closed forcing poset and λ is an ordinal.
3. For all $\beta < \alpha$, if $f(\beta)$ is an ordered pair whose first entry $f(\beta)_0$ is an ordinal, then $f(\beta)_0 < \alpha$.

In this case we let $\mathbb{Q}_\alpha = \mathbb{Q}$.

By Proposition 7.13 $\mathbb{P}_\kappa$ is κ-c.c. with cardinality κ, and by Proposition 7.9 $\mathbb{P}_\kappa$ is δ-directed closed.

Let G be $\mathbb{P}_\kappa$-generic over V. We need to show that the supercompactness of κ is indestructible; accordingly we fix $\mathbb{Q} \in V[G]$ such that

$$V[G] \models \text{“}\mathbb{Q} \text{ is a } \kappa\text{-directed closed forcing poset”},$$

and a cardinal λ with $\lambda \geq \kappa$, and we prove that κ is λ-supercompact in $V[G]^{\mathbb{Q}}$. Notice that the trivial forcing is (trivially) κ-directed closed, so that our proof will show in particular that κ is supercompact in $V[G]$.

Let $\mathbb{Q} = i_G(\dot{\mathbb{Q}})$, where (increasing λ if necessary) we may as well assume that $\dot{\mathbb{Q}} \in H_\lambda$. Let $\mu = 2^{2^\lambda}$. Let W be a supercompactness measure on $P_\kappa\mu$ such that $j_W(f)(\kappa) = (\mu, \mathbb{Q})$. Let $j = j_W$ and $N = \mathrm{Ult}(V, W)$.

Let g be $\mathbb{Q}$-generic over V. Working in N let $\dot{\mathbb{R}}$ be the standard name for the iteration such that $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{R}} \simeq j(\mathbb{P}_\kappa)$, let $\mathbb{R} = i_{G*g}(\dot{\mathbb{R}})$ and let H be $\mathbb{R}$-generic over $V[G*g]$.

Arguing exactly as in the consistency proof for PFA, we may lift j to get $j_G : V[G] \longrightarrow N[G*g*H]$. We may also argue exactly as before that $M[G] \models |Q| < \lambda$, $V[G*g] \models {}^\mu M[G*g] \subseteq M[G*g]$ and $j_G \restriction \mathbb{Q} \in M[G*g*H]$.

We now need to lift j_G to an embedding of $V[G*g]$, which we will do using Silver's master condition argument as in the proof of Theorem 12.6. By the last paragraph j_G“$g \in M[G*g*H]$, and clearly j_G“g is a directed set of conditions in $j_G(\mathbb{Q})$. We recall that $j(\kappa) > \lambda$, $M[G] \models |\mathbb{Q}| < \lambda$ and by the elementarity of j_G

$$M[G*g*H] \models \text{“}j(\mathbb{Q}) \text{ is } j(\kappa)\text{-directed closed”}.$$

It follows that there is a condition $q \in j_G(\mathbb{Q})$ such that $\forall p \in g\ q \leq j(p)$, that is to say q is a strong master condition for g and j_G. Let h be $j_G(\mathbb{Q})$-generic over $V[G*g*G]$ with $q \in h$. We may lift j_G as usual to get an elementary embedding $j^{++} : V[G*g] \longrightarrow M[G*g*H*h]$.

The argument is not finished at this point because j^{++} can only be defined in $V[G*g*H*h]$. The final stage of the proof is to find an approximation to j^{++} which can be defined in $V[G*g]$. For notational simplicity let $V^* = V[G*g]$ and $M^* = M[G*g*H*h]$. Let

$$U = \{X \subseteq P_\kappa\lambda : X \in V^*, j\text{“}\lambda \in j^{++}(X)\}.$$

As in Proposition 3.2 we may factor j^{++} as $k \circ j_U$ where j_U is the ultrapower of V^* by U.

Recall that $|\mathbb{P}_\kappa * \dot{\mathbb{Q}}| < \lambda$. The definition of the iteration $j(\mathbb{P}_\kappa)$ and the fact that $j(f)(\kappa) = (\mu, \dot{\mathbb{Q}})$ together imply that in the iteration $j(\mathbb{P}_\kappa)$ we do trivial forcing at every stage between κ and μ. It follows that

$$M[G * g] \models \text{“}\mathbb{R} * j(\dot{\mathbb{Q}}) \text{ is } \mu\text{-closed”}.$$

Since $V^* \models {}^{\mu}M[G * g] \subseteq M[G * g]$, V^* agrees that $\mathbb{R} * j(\dot{\mathbb{Q}})$ is μ-closed.

Now since $V^* \models \lambda^{<\kappa} < \mu$, the arguments of the last paragraph imply that $U \in V^*$. It is easy to check that

$$V^* \models \text{“}U \text{ is a supercompactness measure on } P_\kappa\lambda\text{”}.$$

This concludes the proof that κ is λ-supercompact in $V[G * g]$. $\dashv$

24.14 Remark. It is easy to see that if κ is measurable then there is no κ-Kurepa tree. Since Example 6.1 shows that a κ-Kurepa tree can be added by a κ-closed forcing poset for any inaccessible κ, it is not possible to improve Laver's result to cover all κ-closed forcing posets.

25. Altering Generic Objects

In this final section we introduce an idea due to Woodin, namely, that it is sometimes possible to alter generic objects so as to enforce the compatibility requirements of Proposition 9.1. Returning to the theme of failure of GCH at a measurable cardinal, we will prove a result of Woodin which gets GCH to fail at a measurable cardinal from the optimal large cardinal hypothesis.

25.1 Theorem. *Let* GCH *hold and let $j : V \longrightarrow M$ be a definable embedding such that $\mathrm{crit}(j) = \kappa$, ${}^{\kappa}M \subseteq M$ and $\kappa^{++} = \kappa^{++}_M$. Then there is a generic extension in which κ is measurable and* GCH *fails.*

25.2 Remark. The hypotheses of Theorem 25.1 can easily be had from a cardinal κ which is $(\kappa + 2)$-strong. Work of Gitik [24] shows that they can be forced starting with a model of $o(\kappa) = \kappa^{++}$, and by work of Mitchell [59] this is optimal.

Proof of Theorem 25.1. By the arguments of Sect. 3, we may assume that $j = j^V_E$ for some (κ, κ^{++})-extender E. We define $U = \{X : \kappa \in j(X)\}$ and form the ultrapower map $i : V \longrightarrow N \simeq \mathrm{Ult}(V, U)$. We write $j = k \circ i$ where $k : N \longrightarrow M$ is given by $k([F]) = j(F)(\kappa)$.

Let $\lambda = \kappa^{++}_N$. Then $\lambda < i(\kappa)$, since $i(\kappa)$ is inaccessible in N. Since GCH holds $i(\kappa) < \kappa^{++}$. On the other hand $k(\lambda) = \kappa^{++}_M = \kappa^{++}$, so that $\mathrm{crit}(k) = \lambda$. It is also easy to see that k is an embedding of width $\leq \lambda$.

As before we will let $\mathbb{P} = \mathbb{P}_{\kappa+1}$ be an iteration with Easton support, where for $\alpha \leq \kappa$ we let $\mathbb{Q}_\alpha = \mathrm{Add}(\alpha, \alpha^{++})_{V[G_\alpha]}$ for α inaccessible and let it be the trivial forcing otherwise.

Let G be $\mathbb{P}_\kappa$-generic over V and let g be $\mathbb{Q}_\kappa$-generic over $V[G]$. The iterations $\mathbb{P}$, $i(\mathbb{P})$ and $j(\mathbb{P})$ agree up to stage κ.

The following lemmas are easy.

25.3 Lemma. $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$.

25.4 Lemma. $i(\mathbb{P})_{\kappa+1} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}^*_\kappa$, *where* $\mathbb{Q}^*_\kappa = \mathrm{Add}(\kappa, \lambda)_{V[G_\kappa]}$.

Let $g_0 = g \cap \mathbb{Q}^*_\kappa$, then g_0 is $\mathbb{Q}^*_\kappa$-generic over $V[G]$ and also over $N[G]$. It follows from Proposition 8.4 that $V[G * g_0] \models {}^\kappa N[G * g_0] \subseteq N[G * g_0]$.

Let $\mathbb{R}_0 = \mathbb{R}^N_{\kappa+1, i(\kappa)}$ be the factor forcing to prolong $G * g_0$ to a generic object for $i(\mathbb{P}_\kappa)$. By Proposition 8.1 we may build $H_0 \in V[G * g_0]$ which is $\mathbb{R}_0$-generic over $N[G * g_0]$.

Since $\mathrm{crit}(k) = \lambda$ it is easy to see that $k``G = G$, and we may lift $k : N \longrightarrow M$ to get $k : N[G] \longrightarrow M[G]$. Since $\lambda = \kappa^{++}_N$, if $q \in \mathbb{Q}^*_\kappa$ then the support of $q$ is contained in $\kappa \times \mu$ for some $\mu < \lambda$, and so $k(q) = q$. It follows that $k``g_0 = g_0 \subseteq g$, and so we may lift again to get $k : N[G * g_0] \longrightarrow M[G * g]$.

Since $N[G * g_0] \models$ "$\mathbb{R}_0$ is λ^+-closed" and we bounded the width of k, we may now appeal to Proposition 15.1 and transfer H_0 along k. We obtain H which is $\mathbb{R}$-generic over $M[G * g]$, where $\mathbb{R} = \mathbb{R}^M_{\kappa+1 j(\kappa)}$. We may then build a commutative triangle

$$\begin{tikzcd} V[G] \arrow[r, "j"] \arrow[dr, "i"'] & M[G * g * H] \\ & N[G * g_0 * H_0] \arrow[u, "k"'] \end{tikzcd}$$

Let $\mathbb{S}_0 = i(\mathbb{Q}_\kappa)$, that is, $\mathbb{S}_0 = \mathrm{Add}(i(\kappa), i(\kappa^{++}))_{N[G * g_0 * H_0]}$.

25.5 Lemma. $\mathbb{S}_0$ *is* κ^+*-closed and* κ^{++}*-Knaster in the model* $V[G * g_0]$.

Proof. The closure is easy because $V[G * g_0] \models {}^\kappa N[G * g_0] \subseteq N[G * g_0]$. Let $\langle p_\alpha : \alpha < \kappa^{++} \rangle$ be a sequence of conditions, and let $p_\alpha = i(f_\alpha)(\kappa)$ where $f_\alpha : \kappa \longrightarrow \mathbb{Q}_\kappa$, $f_\alpha \in V[G]$. An easy Δ-system argument shows that κ^{++} of the functions f_α are pointwise compatible, from which it follows that κ^{++} of the conditions p_α are compatible. ⊣

25.6 Remark. A more delicate analysis shows that $\mathbb{S}_0$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})_{V[G * g_0]}$.

25.7 Lemma. $\mathbb{S}_0$ *is* (κ^+, ∞)*-distributive and* κ^{++}*-c.c. in* $V[G * g]$.

Proof. $V[G * g]$ is a generic extension of $V[G * g_0]$ by a forcing isomorphic to $\mathbb{Q}_\kappa$. The poset $\mathbb{Q}_\kappa$ is κ^+-c.c. in $V[G * g_0]$ and so by Easton's Lemma $\mathbb{S}_0$ is (κ^+, ∞)-distributive in $V[G * g]$. By Proposition 5.12 $\mathbb{S}_0 \times \mathbb{Q}_\kappa$ is κ^{++}-c.c. in $V[G * g_0]$ and so by Easton's Lemma again $\mathbb{S}_0$ is κ^{++}-c.c. in $V[G * g]$. ⊣

Now we force over $V[G * g]$ with $\mathbb{S}_0$, and denote the generic object by f_0. By the last lemma forcing with $\mathbb{S}_0$ preserves cardinals. Since k has width $\leq \lambda$, we may use Proposition 15.1 and transfer f_0 to get f which is $\mathbb{S}$-generic over $M[G * g * H]$, where $\mathbb{S} = \mathrm{Add}(j(\kappa), j(\kappa^{++}))$. The problem is now that we wish to lift j, but it may not be the case that $j``g \subseteq f$.

There is no hope of using any of our previous methods for doing without a master condition, since f_0 is built by forcing (and even if we could build f_0 in a suitably compatible way, this would not guarantee compatibility for f). We adopt a different approach based on the observation that we only need to adjust any given condition in f on a small set to make it agree with $j``g$. We will work in $V[G * g * f_0]$ and construct a suitable f^*, by altering each element of f to conform with j and g.

To be precise let $Q = \bigcup j``g$, so that $Q$ is a partial function from $\kappa \times j``\kappa^{++}$ to 2. Let $p \in f$, so that $p = j(P)(a)$ for some $a \in [\kappa^{++}]^{<\omega}$ and some function $P : [\kappa]^{|a|} \to \mathbb{Q}_\kappa$ with $P \in V[G]$. If $(\gamma, j(\delta)) \in \mathrm{dom}(p)$ then by elementarity $(\gamma, \delta) \in \mathrm{dom}\, P(x)$ for at least one $x \in [\kappa]^{|a|}$, so that there are at most κ many points in the intersection of $\mathrm{dom}(Q)$ and $\mathrm{dom}(p)$. If we then define p^* to be the result of altering p on $\mathrm{dom}(p) \cap \mathrm{dom}(Q)$ to agree with Q, then since $V[G * g] \models {}^{\kappa}M[G * g * H] \subseteq M[G * g * H]$ we see that $p^* \in M[G * g]$ and hence $p^* \in j(\mathbb{Q}_\kappa)$.

It remains to see that $f^* = \{p^* : p \in f\}$ is $j(\mathbb{Q}_\kappa)$-generic over $M[G * g * H]$. To see this we work temporarily in $V[G]$. Let $\delta < \kappa$ and let D be dense in $\mathbb{Q}_\kappa$. Let E be the set of those $p \in D$ such that every q obtained by altering p on a set of size δ is in D. An easy argument shows that E is also dense. Returning to $M[G * g * H]$ and applying this remark with κ in place of δ, we see that f^* meets every dense set in $M[G * g * H]$.

We may now lift to get $j : V[G * g] \longrightarrow M[G * g * H * f^*]$. We are not quite done because f^* only exists in the extension $V[G * g * f_0]$. However since f_0 is generic for (κ, ∞)-distributive forcing and j has width $\leq \kappa$, we may transfer f_0 to get a suitable generic object h and finish by lifting to get $j : V[G * g * f_0] \longrightarrow M[G * g * H * f^* * h]$. $\dashv$

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13. Ideals and Generic Elementary Embeddings

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Contents

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1. Introduction

This chapter covers the technique of generic elementary embeddings. These embeddings are closely analogous to conventional large cardinal embeddings, the difference being that they are definable in forcing extensions of V rather than in V itself. The advantage of allowing the embeddings to be generic is that the critical points of the embeddings can be quite small, even as small as ω_1. For this reason they have many consequences for accessible cardinals, settling many classical questions of set theory.

In analogy to conventional large cardinals, two parameters help describe the strength of a generic elementary embedding $j : V \to M$: where j sends ordinals and the closure properties of M. For generic embeddings we have a third parameter: the nature of the forcing required to define j. This chapter focuses on how the "three parameters" interact to determine the impact j has on the universe.

An Overview of the Chapter

In Sect. 2, we introduce the basic techniques of generic embeddings and generic ultrapowers and describe briefly the relationship to ideals. Topics include criteria for precipitousness, the disjointing property, normality, limitations on closure, canonical functions, selectivity and use of generic embeddings for reflection. Starting with Sect. 2.3, unless otherwise specified, we will assume that our ideals are countably complete.

Section 3 gives examples of relevant ideals and describes a distinction between *natural* and *induced* ideals; the latter bear close resemblance to the duals of conventional large cardinal ultrafilters that have been collapsed by forcing. The nonstationary ideal (in its modern sense) is described along with several lesser known natural ideals. The analogies between induced ideals and proper forcing are explored via master condition ideals and the idea of *goodness*. The relationship between *self-genericity* and saturation is first introduced here and is exploited in later sections on the nonstationary ideal on ω_1 and towers.

In Sect. 4, we look more closely at the theory of generic ultrapowers. Topics include presaturation, layered ideals, Rudin-Keisler projections, computations of where ordinals go under generic embeddings, and the sizes of sets of measure one. It is shown how to iterate embeddings coming from ideals, and generic elementary embeddings arising from towers of ideals are introduced.

In Sect. 5, we consider the consequences of "generic large cardinals". Generic large cardinals, taken together with their prototypical cousins, conventional large cardinals, tend to give a coherent collection of answers to many of the classical questions of set theory. There is one extremely important counterexample: the assumption that the nonstationary ideal on ω_1 is $\aleph_2$-saturated. Topics include CH, GCH and SCH, graph and partition theory, Chang's Conjecture and Jónsson cardinals, square, stationary set reflection, Suslin and Kurepa trees, descriptive set theory, and non-regular ultrafilters.

In Sect. 6, we discuss limitations on generic large cardinal assumptions. These results play a role analogous to the Kunen limitation on conventional large cardinals. Because generic large cardinals have more parameters, the limitations involve such issues as the nature of the forcing required for the generic embeddings. Given the ubiquitous consequences of generic large cardinals, it is not surprising that this topic is more involved than with conventional large cardinals.

Sections 7 and 8 deal with consistency results for generic large cardinals. These are split as far as possible into consistency results for induced ideals and consistency results for natural ideals. Forcing constructions are given for ideals with strong properties in all three parameters. Special attention is paid to results that show the consistency of properties that are either used as hypotheses in Sect. 5 or that show that results in Sect. 6 are sharp. There is a general theorem (the Duality Theorem) that describes how to compute the forcing necessary to construct an elementary embedding coming from an induced ideal. This result allows one to control the third parameter in consistency results. Various corollaries are drawn such as the indestructibility of the generic supercompactness of ω_1 under proper forcing. An attempt is made to unify the treatment of various forcing techniques constructing precipitous and saturated ideals. The section ends with a brief discussion of methods for destroying precipitous and saturated ideals.

Section 8 deals with consistency results for natural ideals. Unfortunately, due to time and space limitations, consistency results that are proved by adding choice to determinacy models are not included, although some references are given. Starting with large cardinals, consistency results for natural ideals are shown in two ways. Both methods are explored in this section. The first is to take an induced ideal and by some technique, such as shooting closed unbounded sets, turning it into a natural ideal while preserving its generic embedding. The second technique arose from [47] and uses inherent properties of the natural ideal. The notions of *goodness* and *self-genericity* are important for taking an existing natural ideal and making it have nice properties. The section focuses on the two natural ideals for which good consistency results are known: the nonstationary ideals and the club guessing ideals. The section includes results from Martin's Maximum and about $\square_\kappa$ and c.c.c.-destructible and indestructible ideals.

Section 9 gives a brief introduction to tower forcing. It discusses induced towers, giving an example of a saturated tower. Then it goes on to natural towers. The notion of catching an antichain is discussed and connections with the forcing for making the nonstationary ideal on ω_1 saturated are made, as is the notion of self-genericity in this context. Woodin's towers are shown to be presaturated, and generalizations first considered by Douglas Burke are discussed. Some applications are given and an example due to Burke of a non-precipitous tower is given.

Section 10 deals with the consistency strength of generic ideal assumptions. The most cogent results of this section are covered in the chapters of this Handbook devoted to inner model theory. They exposit results that show fine structural inner models with large cardinals exist, under various ideal assumptions. For this reason, this section focuses on results that are of a different flavor. These results state that the existence of generic embeddings j with the property that the images of a very small finite number of sets are determined independently of the generic object imply inner models with very large (supercompact or huge) cardinals. Results are also quoted that show that there is a strict hierarchy of consistency strength among the ideal axioms involving the ω_n's.

Section 11 is a speculative discussion of the possibility of adopting generalized large cardinals along with conventional large cardinals as additional axioms for mathematics.

Apology and Acknowledgments

This chapter was first conceived as a 40-page introduction to the theory of generic elementary embeddings. As I began to outline the chapter and look at references it became clear that there was no general survey of the area in the literature. At that moment the chapter began to grow.

There are several decisions that had to be made early on. My belief is that long lists of theorems without indications of the ideas of the proofs are not very useful. I have never been able to read such a "survey". The first decision was to try to include at least sketches of arguments for almost all of the results.

Secondly, the existing literature included a lot of overlap and duplication of ideas under different names, ad hoc arguments for special cases and the use of generic embeddings in the middle of arguments for other theorems. Rather than repeat many variations of the same argument, I decided to try to isolate common lemmas that could be used repeatedly.

A corollary of the attempt to unify a lot of disparate literature was that the logical structure of the chapter became tree-like. Most of the latter sections depend on elements introduced in the earlier sections. Without the background information in the first four sections, it is hard to motivate or illuminate the deeper results in the later sections. For this reason alone, more detail is given to the earlier sections. Moreover, reading the later sections may require glancing back over definitions and results in the earlier parts of the chapter.

The main aim of the chapter is to illustrate that there is a coherent *theory* here, that there are unifying fundamental ideas that occur frequently in many different contexts. These include master condition ideals, natural and induced ideals, disjointing, self-genericity, the role of diagonal unions for representing Boolean sums, good elementary substructures—the list is long.

The third decision was to try to organize the theory around a few basic principles. These include the categorization of ideals as natural and induced and the "three parameters" that determine a generic elementary embedding $j : V \to M$, which again are: where j sends ordinals, the closure properties of M, and the nature of the forcing required to define j.

The three parameters provide a backbone for the chapter. There are sections and examples devoted to each of them, and a range of ideal-theoretic properties, like the various saturation and normality properties, are explored in terms of the three parameters.

Despite the attempt to give arguments in as much detail as space allows, there are parts of the chapter that do little more than adumbrate results. This reflects my ignorance and lack of energy. I have attempted to give accurate references for results that the reader may want to learn on his or her own. I apologize also to authors whose results were left out due to my ignorance or sloth.

Given the problematic length of the chapter, several people have suggested that it be expanded to a stand-alone book. As this would require that near complete arguments be given, I estimate such a book might need 800 to 900 pages. I will leave this for others.

Many people have been incredibly generous with their time and criticism. I especially want to thank Aki Kanamori for his continuing support and encouragement. James Cummings, Joel Hamkins and Tetsuya Ishiu provided ongoing corrections that were very useful. Ishiu was involved with the chapter since its inception. Many other people, including David Aspero, Andrés Caicedo, Neus Castells, Moti Gitik, Paul Larson, Rich Laver, Luis Pereira, John Rapalino, Hugh Woodin and Martin Zeman gave me substantial commentary and feedback.

I would also like to thank Menachem Magidor, Saharon Shelah and Hugh Woodin for their help in understanding some of the topics in this chapter. Magidor's views on generic elementary embeddings influenced me a great deal.

Some Conventions Used in the Chapter

In describing generic elementary embeddings, Boolean algebras of the form $P(Z)/I$ occur naturally as forcing notions. Moreover, via Łoś's theorem, the Boolean value of a forcing statement is frequently very easily computed directly as a subset of Z. In this context the "Boolean valued" approach to forcing has its advantages. In several places, an elementary familiarity with Boolean algebras may be useful to the reader. To convert the language of Boolean algebras to other approaches to forcing, one can very roughly say that Σ translates to an infinite disjunction and $\prod$ translates into an infinite conjunction. Rigorously, if $\mathbb{B}$ is a Boolean algebra, and $A \subseteq \mathbb{B}$, then ΣA is the least upper bound of A and $\prod A$ is the greatest lower bound. The obvious De Morgan's laws apply. If $\mathbb{B}$ is not complete, then ΣA and $\prod A$ need not exist.

An effort was made to keep notation as conventional as possible. Although $\omega_\alpha = \aleph_\alpha$, we try to use the former when regarding it as a set of ordinals and the latter when appealing to its cardinality aspect. Some notation that may not be standard include the use of "$\mathrm{Cof}(\kappa)$" and "$\mathrm{Cof}({<}\kappa)$" for the classes of ordinals of cofinality κ and less than κ respectively. To denote the cofinality of a particular ordering $(I, <)$ we write $\mathrm{cf}(I)$. If $g : Z \to A$, we denote the corresponding element of the ultrapower A^Z/U as $[g]$ or $[g]^U$. If $V^Z/U \cong M$ and the isomorphism with M is understood from the context then we will write $[g]^M$ for the image of $[g]$ in M. If $\mathfrak{A}$ is a structure with domain A that has canonical Skolem functions and $z \subseteq A$ we will write $\mathrm{Sk}^{\mathfrak{A}}(z)$ for the elementary substructure of $\mathfrak{A}$ generated by z. We will follow current practice by taking $H(\theta)$ to stand for a structure in a countable language expanding $\langle H(\theta), \in, \Delta\rangle$ with domain the collection of sets of hereditary cardinality less than θ and where Δ is a fixed-in-advance well-ordering of the domain. Unless otherwise stated, the convention is that θ is a very large regular cardinal reflecting all relevant statements from the universe V. We use the abbreviation $\mathrm{ot}(X)$ for the order type of X. We lapse into jargon from time to time; for example we will write $\kappa \ll \lambda$ to mean that λ is substantially larger than κ. "Substantially larger" will depend on context, but usually means that λ is a regular cardinal at least $(2^\kappa)^+$. We will be sloppy in our notation in the following way: if $\dot\tau$ is a $\mathbb{P}$-term for an element of a generic extension $V[G]$ we will write $\dot\tau^{V[G]}$ for its realization by G. This is strictly speaking incorrect as the interpretation of τ depends on G, rather than the model $V[G]$. Similarly we will often omit the traditional ˇ's and ˙'s whenever the context makes them clear.

Throughout this chapter we will assume, without loss of generality, that all of our partial orderings have the property that for all compatible conditions p, q there is a greatest lower bound, which we will denote $p \wedge q$. We will say that $p \wedge q = 0$ if p and q are not compatible.

If $e : \mathbb{P} \to \mathbb{Q}$ is an order-preserving map and $G \subseteq \mathbb{P}$ is generic, then we can define a partial ordering $\mathbb{Q}/G$ by setting the domain of $\mathbb{Q}/G$ to be $\{q \in \mathbb{Q} : \text{for all } p \in G, q \wedge e(p) \neq 0\}$ and the ordering of $\mathbb{Q}/G$ to be $\leq_{\mathbb{Q}}$. An order-preserving map e will be said to be a *regular embedding* iff e preserves incompatibility and takes maximal antichains to maximal antichains.

Despite the fact that $\mathbb{Q}/G$ can be a strange non-separative partial ordering, if e is a regular embedding it is always the case that:

$$\mathbb{Q} \sim \mathbb{P} * \mathbb{Q}/G.$$

We assume general familiarity with the combinatorial principles $\diamondsuit$ and $\square$, much discussed in the various Handbook chapters. We refer to the early sections of Cummings' chapter for basic definitions and notations for forcing and (conventional) large cardinals. We will use the following partial orderings: $\mathrm{Add}(\kappa, \lambda)$ is the set of partial functions from $\kappa \times \lambda$ to 2 of cardinality less than κ. This is the usual notion of forcing for adding λ Cohen subsets of κ. We write $\mathrm{Add}(\kappa)$ for $\mathrm{Add}(\kappa, 1)$. $\mathrm{Col}(\kappa, \lambda)$ is the set of partial functions from κ to λ of cardinality less than κ, the usual notion of forcing for collapsing λ to κ. $\mathrm{Col}(\kappa, <\lambda)$ is the set of partial functions p from $\kappa \times \lambda$ to λ of cardinality less that κ such that $p(\alpha, \beta)$ when defined is less than β. This is the usual notion of forcing, the Levy collapse, for collapsing all cardinals below λ to κ. A property is *c.c.c.-destructible* if it can be destroyed by a c.c.c. forcing. We often regard large cardinal axioms as providing elementary embeddings, with phrases like "supercompact embedding" and "almost huge embedding" meaning the elementary embedding with the closure properties associated with supercompact or almost huge cardinals.

2. Basic Facts

We begin with some classical algebraic definitions about Boolean algebras. See, for example, Sikorski's book [108] for more details. We recall that a *Boolean algebra* is a structure $\mathbb{B} = \langle F, \wedge, \vee, \neg, 0, 1\rangle$ that is isomorphic to a field of sets with the operations of $\cup, \cap$ and complement, and with 0 denoting $\emptyset$ and 1 denoting the complement of the empty set. In such a structure we can define an ordering by setting $b \le c$ iff $b \wedge c = b$.

The class of Boolean algebras is an elementary class. A Boolean algebra $\mathbb{B}$ will be called *κ-complete* iff for all sets $A \subseteq F$ of size less than κ there is an element of $\mathbb{B}$, denoted ΣA, that is a least upper bound for all of the elements of A. This is equivalent to the requirement that for all $A \subseteq F$ of cardinality less than κ there is a greatest lower bound ΠA for all elements of A. Of course, ΣA and ΠA correspond to infinite unions and intersections respectively. The significance of this for forcing is that $\Sigma A \Vdash \dot{G} \cap A \neq \emptyset$.

A *homomorphism* between Boolean algebras $\mathbb{B}$ and $\mathbb{C}$ is a function from the domain of $\mathbb{B}$ to the domain of $\mathbb{C}$ that preserves the operations $\wedge, \vee, \neg$. A homomorphism will be called *κ-complete* iff it preserves the infinitary operation of ΣA for $A \subseteq \mathbb{B}$ having cardinality less than κ. A κ-complete homomorphism preserves ΠA as well, for $A \subseteq \mathbb{B}$ of cardinality less than κ.

The Boolean algebra will be *complete* iff it is κ-complete for all κ. It is a standard fact (see [63]) that if $\mathbb{P}$ is a separative partial ordering then there is a complete Boolean algebra, which we denote by $\mathcal{B}(\mathbb{P})$, that has a dense subset $D \subseteq \mathcal{B}(\mathbb{P})$ such that $(D, \le_{\mathcal{B}(\mathbb{P})}) \cong \mathbb{P}$. Moreover, for any partial ordering $\mathbb{P}$ there is an equivalence relation $\sim$ such that the map $p \to [p]_\sim$ is order preserving, forcing with $\mathbb{P}$ is equivalent to forcing with $\mathbb{P}/\sim$ and, $\mathbb{P}/\sim$ is separative. In an abuse of notation we will write $\mathcal{B}(\mathbb{P})$ for $\mathcal{B}(\mathbb{P}/\sim)$.

2.1 Definition. Let $\mathbb{B} = \langle B, \wedge, \vee, \le, \neg, 0, 1\rangle$ be a Boolean algebra. A non-empty set $I \subseteq \mathbb{B}$ is an *ideal* on $\mathbb{B}$ if it is closed under finite joins and $\le$. The dual of an ideal I is $\breve{I} = \{\neg A : A \in I\}$. It is called a *filter*. If $S \subseteq \mathbb{B}$ we will denote the ideal generated by S as $\bar{S}$. An ideal is *proper* iff $1 \notin I$.

In most contexts the Boolean algebra $\mathbb{B}$ will be of the form $P(Z)$ for some set Z. In this case we will say that I is an ideal *on* Z.[1] Clearly I is an ideal on Z iff I is closed under finite unions and subsets. I is proper iff $Z \notin I$.

A maximal proper ideal is called *prime* and its dual is an *ultrafilter*. We will usually assume without further comment that our ideals and filters are proper and, if $\mathbb{B} \subseteq P(Z)$, that the ideals contain all singletons of elements from Z.

The ideal I is called *κ-complete* provided that it is closed under joins (unions) of size less than κ. Ideals that are ω_1-complete are also called *countably complete* or *σ-additive*. For a proper ideal I the *completeness* of I is the least cardinal κ such that there is a collection $\{X_\alpha : \alpha < \kappa\} \subseteq I$ such that $\bigcup_{\alpha<\kappa} X_\alpha \notin I$. The completeness of an ideal is easily seen to be a regular cardinal. We will denote the completeness of the ideal I by $\mathrm{comp}(I)$.

If I is an ideal on Z and $Y \subseteq Z$ we say that the ideal I (or its dual filter) *concentrates on* Y iff $Y \in \breve{I}$.

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[1] This is clearly an abuse of terminology, but follows standard practice. It might be better to follow [69] and say that I is an ideal *over* Z.

An ideal I on Z is *uniform* iff every subset of Z of cardinality smaller than the cardinality of Z is in I.

Given an ideal I on Z we will be concerned with the Boolean algebra $P(Z)/I$. Elements of this algebra are equivalence classes of subsets of Z under the relation $S \sim_I T$ iff $S\Delta T \in I$ and are denoted $[S]_I$, the subscript deleted when clear from the context. The usual operations of union and intersection are well-defined modulo this equivalence relation and the result is a Boolean algebra. It is a standard fact that if the ideal I is κ-complete then the resulting Boolean algebra is κ-complete.

The collection of subsets of Z not in I are called the *I-positive* sets and are denoted I^+. If $S \subseteq Z$ is in I^+ we define $I\restriction S$ to be the ideal $I \cap P(S)$. Depending on context, we occasionally want to view $I\restriction S$ as an ideal on Z. In this context we define $I\restriction S$ to be the ideal $\overline{I \cup \{Z \setminus S\}}$ generated by I and the single set $Z \setminus S$. The rationalization for this ambiguity is the canonical Boolean algebra isomorphism $P(Z)/(\overline{I \cup \{Z \setminus S\}}) \cong P(S)/(I \cap P(S))$. Note that the completeness of $I\restriction S$ is at least as large as the completeness of I.

An ideal I is *non-principal* iff every singleton from Z is a member of I and *atomless* iff $P(Z)/I$ has no atoms. Being atomless is equivalent to the property that every $S \subseteq Z$ that is not in I contains at least two disjoint subsets not in I. Clearly, if I is atomless it is non-principal. Further, if $[S] \in P(Z)/I$ is an atom, then I induces an ultrafilter on S in a natural way.

For the rest of this chapter, take "proper and non-principal" as part of the definition of ideal or filter, unless otherwise indicated.

We will use the locution "I is nowhere φ" to mean that no I-positive subset of Z has property φ. So, for example, an ideal is "nowhere prime" means that there is no positive set S such that $I\restriction S$ is prime.

An important property of any Boolean algebra is its chain condition, or equivalently, in a complete Boolean algebra, its antichain condition. Recall that an *antichain* in a Boolean algebra is a collection of elements A such that the meet of any pair of distinct elements of A is empty.

2.2 Definition. The *saturation* of an ideal I on Z, $\mathrm{sat}(I)$, is the least cardinal κ such that every antichain $\mathcal{A} \subseteq P(Z)/I$ has cardinality less than κ. The ideal I is said to be *λ-saturated* iff $\mathrm{sat}(I) \leq \lambda$.

It is a result of Tarski [117] that the saturation of a Boolean algebra is always a regular cardinal.

2.1. The Generic Ultrapower

An important modern application of the theory of ideals is that of *generic ultrapowers*. This technique was developed by Solovay [111] in order to show that the existence of a real-valued measurable cardinal implies the existence of an inner model with a measurable cardinal. The essence of this technique was isolated in the definition of a *precipitous ideal* due to Jech and Prikry [62, 64].

The technique uses forcing to build an ultrafilter G over V and then take the V-ultrapower by G. If we remove the zero element from $P(Z)/I$, then the usual Boolean algebra partial ordering on $P(Z)/I$ gives a separative partial ordering that is appropriate for forcing. This can be defined explicitly by setting $[X]_I \leq [Y]_I$ iff $X \setminus Y \in I$. Call the resulting partial ordering $\mathbb{B}$.

In general, in the context of forcing, we will be sloppy about distinguishing between $P(Z)/I$ and $\mathbb{B}$. The reader also has the option of viewing this as Boolean-valued forcing. An equivalent, pre-partial ordering has as elements I^+ under the ordering $A \leq B$ iff $A \setminus B \in I$.

Forcing with $\mathbb{B}$ gives an ultrafilter $G \subseteq \mathbb{B}$ that is V-complete in the following sense: if $\langle X_j : j \in J \rangle \subseteq P(Z)/I$ is a maximal antichain in V below some element $Y \in G$, then there is a j with $X_j \in G$. If $P(Z)/I$ is complete, this is equivalent to the statement that whenever $\langle Y_j : j \in J \rangle \in V$ and for all j, $Y_j \in G$ then $\prod Y_j \in G$.

We can interpret G as an ultrafilter G' on $P(Z)^V$ by taking all subsets of Z that belong to an I equivalence class that belongs to G. The forcing to produce G' is equivalent to the forcing to produce G and we will not, in general, distinguish between G and G'.

In $V[G]$, we can use this ultrafilter to form the ultrapower of V by G using functions from V. More formally: Let $f, g : Z \to V$ be elements of V. Then $\{z \in Z : f(z) = g(z)\}$ and $\{z \in Z : f(z) \in g(z)\}$ lie in V and hence are "G-measurable". Define an equivalence relation of the collection of such functions by setting $f \sim g$ iff $\{z \in Z : f(z) = g(z)\} \in G$. For equivalence classes $[f], [g]$, we set $[f]\, E\, [g]$ iff $\{z \in Z : f(z) \in g(z)\} \in G$; E is "the ultrapower of the $\in$ relation". It is easy to check that this is well-defined.

Let $(V^Z/G, E)$ denote the resulting class ultrapower. The "usual tricks" are employed to deal with the fact that each equivalence class is a proper class, exactly as in the case of measurable cardinals.

The appropriate version of Łoś's Theorem is easily verified and as a consequence, there is a canonical elementary embedding

$$j : V \to V^Z/G$$

that lies in $V[G]$.

The next lemma is an important tool for computing generic ultrapowers:

2.3 Lemma. *Suppose that I is an ideal on a set Z. Let $G \subseteq P(Z)/I$ be generic and $j : V \to V^Z/G$ be the generic elementary embedding. Let* $\mathrm{id} : Z \to Z$ *be the identity function, given by* $\mathrm{id}(z) = z$ *for $z \in Z$. Then*

1. *for $A \subseteq Z$, $A \in G$ iff $[\mathrm{id}]^{V^Z/G}\, E\, j(A)$, where E is the ultrapower of the $\in$ relation and*
2. *for all $g : Z \to V$ with $g \in V$, we have $[g]^{V^Z/G} = j(g)([\mathrm{id}]^{V^Z/G})$.*

The identity function thus plays a pivotal role. As we consider various Z's, id will continue to denote the corresponding identity function with the Z implicit.

In some lucky cases V^Z/G is well-founded:

2.4 Definition (Jech-Prikry [64]). An ideal I on Z is *precipitous* iff for all generic $G \subseteq P(Z)/I$, the ultrapower V^Z/G is well-founded.

In the case where V^Z/G is well-founded, we replace it by its transitive collapse, M. Note that $M \subseteq V[G]$. In a slight abuse of notation we consider j as an elementary embedding from V to M, $j : V \to M$. Moreover, if $g : Z \to V$ with $g \in V$ we will denote the unique element of M corresponding to g by $[g]^M$.

Recall that if $j : V \to M \subseteq V[G]$ is elementary then the critical point of j, denoted $\mathrm{crit}(j)$, is the least ordinal moved by j. A standard fact is that if M and N are two well-founded models of ZFC that have the same ordinals and $j : M \to N$ is an elementary embedding that does not move ordinals, then $M = N$ and j is the identity map.[2]

We now compute the critical point of j in terms of the completeness of the ideal.

2.5 Proposition. *Let I be a precipitous ideal on Z. Let $G \subseteq P(Z)/I$ be generic and $j : V \to M \subseteq V[G]$ the generic elementary embedding. Then j is not the identity map and the critical point of j is the largest κ such that there is an $S \in G$ with the completeness of $I{\restriction}S$ equal to κ.*

Proof. As we observed above, if $S \subseteq T$ then $\mathrm{comp}(I{\restriction}S) \geq \mathrm{comp}(I{\restriction}T)$.

Let G be generic. A simple density argument shows that there is a $Y \in G$, and a sequence $\langle A_\alpha : \alpha < \gamma\rangle \subseteq I$, where γ is the completeness of $I{\restriction}Y$, such that $\bigcup_{\alpha<\gamma} A_\alpha \in G$. By the observation, this γ is the largest κ such that for some $S \in G$, $\mathrm{comp}(I{\restriction}S) = \kappa$.

Let $W = \bigcup_{\alpha<\gamma} A_\alpha$. Let $F : Z \to \gamma$ be defined by:

$$F(z) = \begin{cases} 0 & \text{if } z \notin W, \\ \alpha & \text{if } \alpha \text{ is least such that } z \in A_\alpha. \end{cases}$$

Then, for each $\alpha < \gamma$, $\{z \in Z : F(z) \leq \alpha\} \subseteq (\bigcup_{\beta\leq\alpha} A_\beta \cup Z \setminus W) \in I{\restriction}W$. Hence, by Łoś's Theorem, $[F]^M > j(\alpha) \geq \alpha$. On the other hand for every $z \in Z$, $F(z) \in \gamma$, and hence $[F] < j(\gamma)$. Thus the critical point of j exists and is less than or equal to γ. We have shown that there is a set $Y \in G$ such that $\mathrm{crit}(j) \leq \mathrm{comp}(I{\restriction}Y)$. In particular, $\mathrm{crit}(j) \leq \sup\{\mathrm{comp}(I{\restriction}Y) : Y \in G\}$.

To finish, for all generic G there is an $S \in G$ such that for some κ, $[S] \Vdash \mathrm{crit}(j) = \kappa$ and for some function $F : Z \to V$, with $F \in V$ we have $[S] \Vdash [F]^M = \kappa$. It suffices to show that $\mathrm{comp}(I{\restriction}S) \leq \mathrm{crit}(j)$. For each $\alpha < \kappa$, consider $A_\alpha = \{z \in S : F(z) < \alpha\}$. Then $A_\alpha \in I{\restriction}S$, but $\bigcup_{\alpha<\kappa} A_\alpha = S$. Hence the completeness of $I{\restriction}S$ is less than or equal to κ. Using the observation and the fact that S was arbitrary deciding F and κ, we have shown that the crit(j) is at least as big as the completeness of $I{\restriction}S$ for any $S \in G$. ⊣

[2] We include the case where M and N are proper classes.

2.6 Corollary. *Every precipitous ideal is countably complete.*

We now describe a combinatorial criterion for an ideal to be precipitous due to Jech and Prikry [64].

Let $\mathbb{P}$ be a partial ordering. A *tree of maximal antichains* is a sequence $\langle \mathcal{A}_n : n \in \omega \rangle$ of maximal antichains of $\mathbb{P}$ such that $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$. A *branch* through a tree of maximal antichains is a decreasing sequence of conditions $\langle p_n : n \in \omega \rangle$ such that $p_n \in \mathcal{A}_n$.

In the case that $\mathbb{P} = (P(Z)/I) \setminus \{[\emptyset]\}$ and $\mathcal{A}_n \subseteq P(Z)$ we will say that $\langle \mathcal{A}_n : n \in \omega \rangle$ is a *tree of maximal antichains* iff $\langle \mathcal{A}'_n : n \in \omega \rangle$ is a tree of maximal antichains in $\mathbb{P}$ where $\mathcal{A}'_n = \{[a] : a \in \mathcal{A}_n\}$.

2.7 Proposition (Jech-Prikry [64]). *Let I be an ideal. Then I is precipitous iff for any I-positive set S and any sequence $\langle \mathcal{A}_n : n \in \omega \rangle$ with $\mathcal{A}_n \subseteq P(Z)$ that forms a tree of maximal antichains below $[S]$ there is a sequence $\langle a_n : n \in \omega \rangle$ such that:*

1. *$a_n \in \mathcal{A}_n$,*
2. *$\langle [a_n]_I : n \in \omega \rangle$ is a branch through the tree,*
3. *$\bigcap_n a_n \neq \emptyset$.*

Proof. Suppose that I is precipitous. Consider a tree of maximal antichains $\langle \mathcal{A}_n : n \in \omega \rangle$ below a condition $[S]$. Let $j : V \to M \subseteq V[G]$ be a generic elementary embedding where M is transitive, $M \cong V^Z/G$ and $[S] \in G$. In M, consider the tree $\mathcal{T}$ of sequences

$$\{(a_0, \ldots, a_n) : a_i \in j(\mathcal{A}_i),\ [\mathrm{id}]^M \in a_i,\ [a_{i+1}]_{j(I)} \leq [a_i]_{j(I)}\}.$$

For each n, there is a unique $a^*_n \in \mathcal{A}_n$ such that $[\mathrm{id}]^M \in j(a^*_n)$. Hence the sequence $\langle j(a^*_n) : n \in \omega \rangle$ is an infinite branch through $j(\mathcal{T})$ lying in $V[G]$. Since M is well-founded there is a branch through this tree in M. Hence $M \models$ "there is a sequence $\langle a_n : n \in \omega \rangle$ with $a_n \in j(\mathcal{A}_n)$ and $\bigcap_{n\in\omega} a_n \neq \emptyset$". By the elementarity of j we get such a sequence in V.

Suppose that I is not precipitous. Let $[S] \Vdash$ "V^I/G is ill-founded". Choose terms $\dot{F}_n$ such that $[S] \Vdash \dot{F}_n : \check{Z} \to V$, $[S] \Vdash \dot{F}_n \in V$ and $[S] \Vdash [\dot{F}_{n+1}]\, E\, [\dot{F}_n]$, where E is the ultrapower of the $\in$ relation.

Inductively build a tree of antichains $\mathcal{A}_n \subseteq P(Z)$ (with $\mathcal{A}_{-1} = \{S\}$) such that:

1. the collection $\{[a]_I : a \in \mathcal{A}_{n+1}\}$ is a maximal antichain below $[S]$ that refines $\{[a]_I : a \in \mathcal{A}_n\}$,
2. for all $a \in \mathcal{A}_n$, there is an $f^n_a \in V$ such that $[a]_I \Vdash \check{f}^n_a \restriction a = \dot{F}_n \restriction a$, and
3. if $a \in \mathcal{A}_{n+1}$, $b \in \mathcal{A}_n$ and $[a]_I \leq_I [b]_I$ then $a \subseteq b$ and for all $z \in a$, $f^{n+1}_a(z) \in f^n_b(z)$.

Then the sequence $\langle \mathcal{A}_n : n \in \omega \rangle$ has no branch with non-empty intersection; for if b were a branch through the $\langle \mathcal{A}_n : n \in \omega \rangle$ and $z \in \bigcap b$ then $\langle f^n_a(z) : a \in b \cap \mathcal{A}_n \rangle$ is a descending $\in$-sequence in V. ⊣

2.2. Game Characterization of Precipitousness

Let I be a proper countably complete ideal on a set Z. Consider the following game: Players W and B alternate playing subsets:

$$W_0 \supseteq B_0 \supseteq W_1 \supseteq B_1 \cdots$$

of Z that form a decreasing sequence of I-positive sets. Player B wins the game iff $\bigcap_{i\in\omega} B_i \neq \emptyset$.

Galvin, Jech and Magidor proved the following useful characterization of precipitousness:

2.8 Theorem (Galvin et al. [50]). *I is a precipitous ideal iff W does not have a winning strategy.*

2.3. Disjointing Property and Closure of Ultrapowers

An important tool in investigating the precipitousness of an ideal as well as the closure of the generic ultrapower is the *disjointing property* and its weaker relatives. We begin with an easy proposition:

2.9 Proposition. *Let κ be a cardinal. Suppose that I is κ^+-saturated and κ-complete. Then every antichain in $P(Z)/I$ has a pairwise disjoint system of representatives.*

Proof. Let $\mathcal{A} = \langle A_\alpha : \alpha < \kappa\rangle$ be an antichain. For all α, let $A'_\alpha = A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. Then $A'_\alpha \sim_I A_\alpha$ and the sequence $\langle A'_\alpha : \alpha < \kappa\rangle$ is pairwise disjoint. ⊣

2.10 Definition. An ideal I has the *disjointing property* iff every antichain $\mathcal{A}$ in $P(Z)/I$ has a pairwise disjoint system of representatives, i.e. $\{S_a : a \in \mathcal{A}\}$ such that each $S_a \in a$ and $S_a \cap S_b = \emptyset$ for $a \neq b$.

2.11 Example. If $\mu : P(\kappa) \to [0, 1]$ is a measure witnessing that κ is real-valued measurable and I is the ideal of null sets of μ, then I is $\aleph_1$-saturated. Hence I has the disjointing property. In this case we have $\aleph_1$-completeness and $\aleph_1$-saturation, which is stronger than necessary to apply Proposition 2.9.

One of the main consequences of the disjointing property is embodied in the following proposition, due to Solovay [111].

2.12 Proposition. *Suppose that I has the disjointing property. Let $\mathbb{B} = P(Z)/I$ and $\dot{f}$ be a $\mathbb{B}$-term whose realization in the generic extension is a function from Z to V that lies in V. Then there is a $g : Z \to V$ with $g \in V$ such that for all generic $G \subseteq \mathbb{B}$, $[\dot{f}]^{V^Z/G} = [\check{g}]^{V^Z/G}$.*

The conclusion can be restated equivalently by saying that for all generic G, $V[G] \models \{z : \dot{f}(z) = \check{g}(z)\} \in G$.

Proof. Choose a maximal antichain $\mathcal{A}$ such that for each $a \in \mathcal{A}$ there is a function g_a such that $a \Vdash \dot{f} = \check{g_a}$. By the disjointing property we can find a pairwise disjoint system of representatives $\{S_a : a \in \mathcal{A}\}$ of elements of $\mathcal{A}$. Define a function

$$g(z) = \begin{cases} g_a(z) & \text{if } z \in S_a \text{ for some } a, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is well-defined since the S_a's are pairwise disjoint, and for each $a \in \mathcal{A}$, $a \Vdash [\dot{f}]^{V^Z/G} = [\check{g}]^{V^Z/G}$. Since $\mathcal{A}$ is a maximal antichain, we are done. ⊣

2.13 Remark. The disjointing property is equivalent to the conclusion of Proposition 2.12. For if we are given an antichain $\mathcal{A} = \langle A_\alpha : \alpha \in \lambda \rangle$, we can take $\dot{f}$ to be the term for the function with constant value α where $A_\alpha \in G$. If we take $g : Z \to V$ such that $1 \Vdash [\dot{f}]^{V^Z/G} = [\check{g}]^{V^Z/G}$, then $A_\alpha =_I \{z \in Z : g(z) = \alpha\}$.

The importance of the disjointing property is expressed by the following proposition:

2.14 Proposition. *Suppose that I is a countably complete ideal with the disjointing property. Then*

1. *I is precipitous, and*
2. *if $G \subseteq P(Z)/I$ is generic, $j : V \to M \subseteq V[G]$ is the generic ultrapower and $\mathrm{crit}(j) = \kappa$, then $M^\kappa \cap V[G] \subseteq M$.*

Proof. To see that I is precipitous, we use Proposition 2.7. Let $\{\mathcal{A}_n : n \in \omega\}$ be a tree of maximal antichains below some $S \in I^+$.

Using the disjointing property we can proceed by induction on n to refine each element B of $\mathcal{A}_n$ to a B' such that

1. $B' \subseteq B$,
2. $B' \sim_I B$,
3. for each $A \in \mathcal{A}_{n-1}$, B' is either disjoint from A' or included in A', and
4. $\{B' : B \in \mathcal{A}_n\}$ is pairwise disjoint.

We thus assume our original sequence of $\mathcal{A}_n$'s have this property.

Since each $\mathcal{A}_n$ is a maximal antichain, every $\bigcup \mathcal{A}_n$ is in the dual of $I \restriction S$. Hence, by the countable completeness of I, $\bigcap_n \bigcup \mathcal{A}_n$ is in the dual of $I \restriction S$. Let $z \in \bigcap_n \bigcup \mathcal{A}_n$. Then for each n there is a unique B'_n in $\mathcal{A}_n$ such that $z \in B'_n$. Then $\langle B'_n : n \in \omega \rangle$ is a branch through the tree with $z \in \bigcap B'_n$.

To establish the second part of the proposition, it suffices to see that if $S = \langle x_\alpha : \alpha < \kappa \rangle \subseteq M$ is a subset of M that lies in $V[G]$, then $S \in M$. Let $\dot{S} = \langle \dot{x}_\alpha : \alpha < \kappa \rangle$ be a $\mathbb{B}$-term for such an S. Let $[Y] \in G$ with

$[Y] \Vdash \mathrm{crit}(j) = \kappa$. By Proposition 2.12, there is a $k : Z \to V$ such that $k \in V$ and $[Y] \Vdash [k]^M = \kappa$.

Again, by Proposition 2.12, we can find $\langle g_\alpha : \alpha < \kappa\rangle$ with $g_\alpha : Z \to V$ such that for all generic G, $[g_\alpha]^M = \dot{x}_\alpha^{V[G]}$. Define $g : Z \to V$ by setting $g(z) = \langle g_\alpha(z) : \alpha < k(z)\rangle$. Then $[g]^M = \dot{S}^{V[G]}$. ⊣

As an illustration of a technique, we now prove the following result due to Tarski [117].

2.15 Theorem. *Suppose that I is a κ-complete, κ^+-saturated ideal on κ. Then $P(\kappa)/I$ is a complete Boolean algebra.*

Proof. To establish completeness it suffices to show that for every antichain $\mathcal{A}$ there is a least upper bound $\Sigma\mathcal{A}$ for $\mathcal{A}$.

By the disjointing property, there is a function $f : \kappa \to \kappa$ such that for all generic $G \subseteq P(\kappa)/I$, if $j : V \to M \subseteq V[G]$ is the generic embedding then $[f]^M = \kappa$. If $\mathcal{A} = \{a_\alpha : \alpha < \kappa\}$ is an antichain, we choose representatives A_α of a_α. We can assume that $A_\alpha \cap \{\beta : f(\beta) \leq \alpha\} = \emptyset$. Define the "$f$-diagonal union" to be $A = \{\beta : \text{for some } \alpha < f(\beta),\ \beta \in A_\alpha\}$.

We claim that the f-diagonal union is the Boolean sum of $\mathcal{A}$. It is clear that $A \geq_I A_\alpha$, for all $\alpha < \kappa$. Suppose that $B \subseteq \kappa$ is above every A_α, but $A \setminus B \notin I$. Let G be generic with $A \setminus B \in G$. Let $j(\{A_\alpha : \alpha < \kappa\}) = \{A^j_\alpha : \alpha < j(\kappa)\}$. Then $[\mathrm{id}]^M \in j(A \setminus B)$. Hence $M \models$ "there is an $\alpha < j(f)([\mathrm{id}]^M)$ so that $[\mathrm{id}]^M \in A^j_\alpha$". Since $j(f)([\mathrm{id}]^M) = \kappa$, there is an $\alpha < \kappa$, $[\mathrm{id}]^M \in A^j_\alpha$. Hence there is an $\alpha < \kappa$ such that $A_\alpha \in G$. But $B \notin G$ and hence $B \not\geq A_\alpha$. ⊣

The more general fact has an easier proof:

2.16 Theorem. *Suppose that $I \subseteq P(Z)$ is an ideal that has the disjointing property. Then $P(Z)/I$ is a complete Boolean algebra.*

Proof. Let κ be the minimal cardinality of a set $\mathcal{B} \subseteq P(Z)/I$ for which ΣB does not exist. Enumerating $\mathcal{B}$ as $\langle b_\alpha : \alpha < \kappa\rangle$ and applying induction we can replace each b_α by $a_\alpha = b_\alpha \setminus \Sigma_{\beta<\alpha} b_\beta$. Then $\mathcal{A} = \{a_\alpha : \alpha < \kappa \text{ and } a_\alpha = [A]$ for an $A \notin I\}$ is an antichain with $\Sigma\mathcal{A} = \Sigma\mathcal{B}$, if either sum exists.

Choose a $\mathcal{C} \subseteq P(Z)/I$ so that $\mathcal{A} \cup \mathcal{C}$ forms a maximal antichain. Choose a pairwise disjoint system of representatives for elements of $\mathcal{A} \cup \mathcal{C}$, $\{A_\alpha : \alpha < \kappa\} \cup \{C_\delta : \delta < \gamma\}$. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. We claim that $[A]_I = \Sigma\mathcal{A}$.

If not, there is a set U such that $A_\alpha \subseteq_I U$ for all $\alpha < \kappa$ and such that $A \setminus U \notin I$. Since $A \setminus U \notin I$ there is either an α such that $(A \setminus U) \cap A_\alpha \notin I$ or a δ such that $(A \setminus U) \cap C_\delta \notin I$. The former is impossible because $A_\alpha \subseteq_I U$ and the latter is impossible since $A \cap C_\delta = \emptyset$ for all $\delta < \gamma$. ⊣

2.4. Normal Ideals

If we add more structure to Z, we can achieve the disjointing property by a generalization of Proposition 2.9. If we assume that $Z \subseteq P(X)$ for some set X, we can define the notion of a *normal* ideal on Z.

2.17 Definition. If $A \subseteq P(X)$ for some set X and $f : A \to X$, then f is *regressive* iff for all $a \in A$, $f(a) \in a$. Let I be an ideal on Z, where $Z \subseteq P(X)$ for some set X.[3] Then I is *normal* iff for every set $A \in I^+$ and every regressive function $f : A \to X$, there is an I-positive set $B \subseteq A$ and an $x \in X$ such that for all $a \in B$, $f(a) = x$. An ideal is *fine* iff for every $x \in X$, $\{z \in Z : x \in z\} \in \breve{I}$.

We will give several examples of such ideals in the next section. "Most" natural ideals are normal and fine. We will often relativize the notion of normality to an inner model of set theory. For example, for such a model W if $X, Z \in W$ and I is an ideal on Z, then we will say that I is W*-normal* iff for all $A \in W \cap P(Z)$ not in I and all regressive functions $f : A \to X$ lying in W, there is an I-positive set $B \subseteq A$ on which f is constant. Note that if such a B exists we can assume $B \in W$. Similarly, I will be said to be κ*-complete for* W iff whenever $\beta < \kappa$ and $\{A_\alpha : \alpha < \beta\} \subseteq I$ and $\{A_\alpha : \alpha < \beta\} \in W$, we have $\bigcup\{A_\alpha : \alpha < \beta\} \in I$.

There is a closely related definition:

2.18 Definition. Let $\mathcal{A} = \langle A_x : x \in X\rangle$ be a sequence of subsets of $Z \subseteq P(X)$. Then the *diagonal union* of $\mathcal{A}$, denoted $\triangledown\mathcal{A}$, is defined to be $\{z \in Z :$ for some $x \in z,\ z \in A_x\}$. The *diagonal intersection* of $\mathcal{A}$, denoted $\Delta\mathcal{A}$, is $\{z \in Z :$ for all $x \in z,\ z \in A_x\}$.

The next proposition is standard:

2.19 Proposition. *Let X be a set and I an ideal on $Z \subseteq P(X)$. Then:*

1. *I is normal iff I is closed under diagonal unions iff the filter dual to I is closed under diagonal intersections.*

2. *If $\kappa \subseteq X$ and I is a normal, fine ideal, then I is κ-complete iff for each $\alpha < \kappa$, $\{z : \alpha \not\subseteq z\} \in I$. Moreover, in this case $\{z : z \cap \kappa \notin \kappa + 1\} \in I$.*

If I is a normal, fine, countably complete ideal on $Z \subseteq P(X)$ and $f : X \to X$ is a function then almost every $z \in Z$ is closed under f. For if there were a positive set $A \subseteq Z$ of z that are not closed under f, then we could define a regressive function g on A by setting $g(z) = x$ for some $x \in z$ for which $f(x) \notin z$. This function would have to be constant on an I-positive subset $B \subseteq A$, say with value x_0. Let $y = f(x_0)$. By fineness, $\hat{y} = \{z : y \in z\} \in \breve{I}$. Hence $\hat{y} \cap B \neq \emptyset$, a contradiction.

Using the countable completeness of the ideal one can extend this to functions of several variables and to countable collections of functions. Thus we get:

2.20 Proposition. *Suppose that I is a normal, fine, countably complete ideal on $Z \subseteq P(X)$ and $\{f_i : i \in \omega\}$ is a countable sequence of functions where $f_i : X^{n_i} \to X$. Then $\{z \in Z :$ for all $i,\ z$ is closed under $f_i\} \in \breve{I}$.*

[3] So $I \subseteq P(Z) \subseteq P(P(X))$.

At first glance it may appear as though $\triangledown\{a_x : x \in X\}$ depends on the indexing of $\langle a_x : x \in X\rangle$. However, modulo a normal ideal I any two indexings give the same element of $P(Z)/I$. By Proposition 2.20 if $f : X \to X$ is a bijection, then $\{z \in Z : z$ is closed under $f, f^{-1}\} \in \breve{I}$. Hence:

2.21 Proposition. *Let I be a normal, fine ideal on $Z \subseteq P(X)$ and $\mathcal{A} = \langle a_x : x \in X\rangle$. Suppose that $f : X \to X$ is a bijection. Let $D = \{z :$ for some $x \in z,\ z \in a_{f(x)}\}$. Then $[D]_I = [\triangledown\mathcal{A}]_I$.*

Under the same assumptions on I, let $\langle v_x : x \in X\rangle$ be a one-to-one enumeration of the finite sequences of elements of X. By Proposition 2.20 $\{z \in Z : \{v_x : x \in z\}$ is an enumeration of the finite sequences of elements of $z\}$ belongs to $\breve{I}$. If we have a collection of sets $\mathcal{A} = \langle A_{\vec{x}} : \vec{x} \in X^{<\omega}\rangle \subseteq I$, we can use the enumeration $\langle v_x : x \in X\rangle$ to index $\mathcal{A}$ by elements of X. Since for typical z the enumeration restricted to z gives all finite sequences from z, we see that $\{z :$ for some $\vec{x} \in z^{<\omega},\ z \in A_{\vec{x}}\} \in I$.

Thus, for example, if $\mathcal{A} = \langle A_{x,y} : x, y \in X\rangle \subseteq I$ then the "diagonal union", $\{z :$ for some $x, y \in z,\ z \in A_{x,y}\}$, belongs to I. Dually, the "diagonal intersection" of a collection of sets $\{C_{x,y} : x, y \in X\} \subseteq \breve{I}$ defined as $\{z :$ for all $x, y \in z$, $z \in C_{x,y}\} \in \breve{I}$.

In the interplay between the forcing and combinatorial properties of the Boolean algebra it is very convenient to be able to compute Boolean sums. For example, the Boolean value of an infinite disjunction is the sum of the Boolean values of the disjuncts. For quotients by normal ideals, this has an elegant formulation.

2.22 Proposition. *Suppose that $Z \subseteq P(X)$ and I is a normal, fine ideal on Z. Suppose that $A = \{[a_x] : x \in X\} \subseteq P(Z)/I$. Then*

$$\Sigma A = [\triangledown\{a_x : x \in X\}]_I$$

in the Boolean algebra $P(Z)/I$.

Proof. Let $[a_x] \in A$, then $\hat{x} =_{\text{def}} \{z : x \in z\} \in \breve{I}$, and $a_x \cap \hat{x} \subseteq \triangledown\{a_x : x \in X\}$ so $a_x \leq_I \triangledown\{a_x : x \in X\}$. For the other direction, it suffices to show that if $b \leq_I \triangledown\{a_x : x \in X\}$ and b is not in I, then there is an $x \in X$ such that $b \cap a_x \notin I$. Without loss of generality we can assume that $b \subseteq \triangledown\{a_x : x \in X\}$. For each $z \in b$ there is an $x(z) \in z$ such that $z \in a_{x(z)}$. So the map from b to X defined by $z \mapsto x(z)$ is regressive. Hence there is a positive set $b' \subseteq b$ and a fixed x such that for all $z \in b'$, $x(z) = x$. But then $b' \subseteq a_x$ as desired. $\dashv$

Normal ideals frequently have the disjointing property.

2.23 Proposition. *Suppose that $Z \subseteq P(X)$ and I is a normal, fine ideal on Z. Then:*

1. *I is $|X|^+$-saturated implies that,*
2. *I has the disjointing property, which in turn implies that,*

3. I is $|Z|^+$-saturated.

Proof. Assume that I is $|X|^+$-saturated. Suppose that $\mathcal{A} \subseteq P(Z)/I$ is an antichain. Then we can index a collection of representatives of elements of $\mathcal{A}$ by elements of X, i.e. for some $X' \subseteq X$, $\mathcal{A} = \{[A_x] : x \in X'\}$. Since I is fine, we can assume that for all $x \in X'$ and all $z \in A_x$, $x \in z$.

For each pair x, y of distinct elements of X' there is a set $C_{x,y}$ in $\breve{I}$, such that $A_x \cap A_y \cap C_{x,y} = \emptyset$. Let $C = \Delta C_{x,y}$. Then for all $x, y \in X'$, $A_x \cap A_y \cap C = \emptyset$. We get a sequence of disjoint representatives by taking $A'_x = A_x \cap C$.

The second implication is immediate. ⊣

Thus we see:

2.24 Theorem. *Suppose that I is a normal, fine $|X|^+$-saturated ideal on $Z \subseteq P(X)$. Then $P(Z)/I$ is a complete Boolean algebra. Given any set $\mathcal{A} \subseteq P(Z)/I$ there is a subset $\mathcal{B} \subseteq \mathcal{A}$ having cardinality at most $|X|$ such that $\Sigma\mathcal{B} = \Sigma\mathcal{A}$. The join of a collection of elements of $P(Z)/I$ of size at most $|X|$ is given by its diagonal union and the meet is given by diagonal intersection.*

We remark that the completeness of the Boolean algebra $P(Z)/I$ is quite convenient. It implies that for each term τ in the forcing language and each formula ϕ there is a set $b \subseteq Z$ such that for all generic $G \subseteq P(Z)/I$ we have $[b] \in G$ iff $V[G] \models \phi(\tau)$. We will refer to the set b as the *Boolean value* of $\phi(\tau)$.

Normal, fine ideals that have the disjointing property have generic ultrapowers with strong closure properties.

2.25 Theorem. *Suppose that I is a normal, fine, precipitous*[4] *ideal on $Z \subseteq P(X)$, where $|X| = \lambda$. Let $G \subseteq P(Z)/I$ be generic, and M the generic ultrapower of V by G. Then $P(\lambda) \cap V \subseteq M$. Further, if I has the disjointing property, then $M^\lambda \cap V[G] \subseteq M$.*

Proof. Note that without loss of generality $X = \lambda$. Consider the generic embedding $j : V \to M$.

2.26 Claim. *Let I be a normal, fine ideal on $Z \subseteq P(X)$.*[5] *Let G be generic for $P(Z)/I$ and $M = V^Z/G$. Then the identity function* $\mathrm{id} : Z \to Z$ *represents the j-image of λ, which we denote by $j``\lambda$.*

Proof. By fineness, for all $\alpha \in \lambda$, $\{z : \alpha \in z\} \in \breve{I}$. Thus $j``\lambda \subseteq^M [\mathrm{id}]^M$. By normality, together with a density argument, if $f \in V$ and $R = \{z : f(z) \in z\} \in I^+$ then the collection of $W \subseteq Z$ such that f is constant on W is dense below R. Hence if $f \in V$ and $[f]^M \in^M [\mathrm{id}]^M$, there is a $W \in G$ on which f has some constant value α. Hence $[f]^M = j(\alpha)$. ⊣

[4] Using Proposition 2.34, we can weaken the assumption that I is precipitous to I being countably complete.

[5] In this claim we do not assume that I is precipitous.

Suppose that $A \subseteq \lambda$. Since j is an elementary embedding, the function $f_A(z) = A \cap z$ represents $j(A) \cap j``\lambda$. Hence, $M$ contains both $j``\lambda$ and $j(A) \cap j``\lambda$, from which $M$ can easily decode $A$ as the image of $j(A) \cap j``\lambda$ under the canonical isomorphism between $j``\lambda$ and λ, i.e. j^{-1}.

Now suppose that I has the disjointing property. Let $\dot{A} = \langle \dot{a}_\alpha : \alpha < \lambda \rangle$ be a term for a λ-sequence of elements of M. By the disjointing property, there is a sequence of functions $\mathcal{G} = \langle g_\alpha : \alpha < \lambda \rangle$ such that for all generic $G \subseteq \mathbb{B}$, $[g_\alpha]^M = \dot{a}_\alpha^G$. Denote $j(\mathcal{G})$ by $\langle j(g)_\alpha : \alpha < j(\lambda) \rangle$.

Define a function $g : Z \to V$ by setting $g(z) = \langle g_\alpha(z) : \alpha \in z \rangle$. Then by Lemma 2.3, $[g]^M = j(g)(j``\lambda) = \langle j(g)_\beta(j``\lambda) : \beta \in j``\lambda \rangle$. Hence the function that sends $\alpha$ to $j(g_\alpha)(j``\lambda)$ (for $\alpha < \lambda$) lies in M and gives the realization of the sequence $\dot{A}^G$. ⊣

2.27 Remark. From Claim 2.26, it is easy to see that for $\alpha < \lambda$, the function $f(z) = \mathrm{ot}(z \cap \alpha)$ represents α in every transitive $M \cong V^Z/G$.

Weak Normality

We now discuss a variant of normality that is useful in considering non-regular ultrafilters among other topics.

2.28 Definition. Suppose that I is an ideal on $Z \subseteq P(\lambda)$. Then I is *weakly normal* iff for any regressive function $f : Z \to \lambda$ there is an $\alpha < \lambda$ such that $\{z \in Z : f(z) < \alpha\} \in \breve{I}$.

2.29 Proposition. *Let λ be a regular cardinal. Suppose that I is a normal, λ-saturated ideal on $Z \subseteq P(\lambda)$. Then I is weakly normal.*

Proof. Let f be a regressive function defined on Z. Then there is a maximal antichain $\mathcal{A} \subseteq P(Z)/I$ such that for all $a \in \mathcal{A}$, f is constant on a. By saturation $|\mathcal{A}| < \lambda$, and hence there is a $\beta < \lambda$ such that f is bounded by β on a set in $\breve{I}$. ⊣

2.5. More General Facts

We now establish a limitation on the closure of M, which is a standard fact in the case of ordinary ultrapowers:

2.30 Proposition. *Suppose that I is a proper non-principal ideal on $Z \subseteq P(X)$. Let $j : (V, \in) \to (V^Z/G, E)$ be the elementary embedding induced by a generic $G \subseteq P(Z)/I$. Then there is no $A \in V^Z/G$ such that for all $a \in V^Z/G$, $a \, E \, A$ iff $a = j(\beta)$ for some $\beta < (|Z|^+)^V$.*

Note that in the case that V^Z/G is well-founded and M is the transitive collapse this is saying that $j``|Z|^+ \notin M$.

Proof. Suppose not. Let $a \Vdash [f]^{V^Z/G} = A$. Let $L = \{z \in a : |f(z)| > |Z|\}$ and $S = \{z \in a : |f(z)| \leq |Z|\}$. Then $|\bigcup_{z \in S} f(z)| \leq |Z|$, so we can find an ordinal $\alpha \in |Z|^+$ such that $\alpha \notin \bigcup_{z\in S} f(z)$. Let $g : a \to |Z|^+$ be defined so that:

1. $g(z) = \alpha$ when $z \in S$, and
2. g is an injective function on L with $g(z) \in f(z)$ for all $z \in L$.

This is possible since $|f(z)| > |Z|$ for $z \in L$.

Suppose that $S \in G$. Then $j(\alpha)$ is not in the E relation with $[f]^{V^Z/G}$. If $L \in G$, then $[g]^{V^Z/G}\, E\, [f]^{V^Z/G}$, but $[g]^{V^Z/G} \neq j(\beta)$ for any $\beta \in |Z|^+$. $\dashv$

The properties of the forcing used to create a generic elementary embedding interact with the behavior of that embedding on the ordinals. We illustrate this by discussing continuity points of embeddings, where an ordinal δ is called a *continuity point* of j iff $\sup(j``\delta) = j(\delta)$.

We borrow a fact from the study of Chang's Conjecture:

2.31 Proposition. *Suppose that $j : W \to W'$ is an elementary embedding with critical point κ and that W and W' are models of a sufficiently large fragment of* ZFC*. Then:*

1. *An ordinal δ is a continuity point of j iff $\mathrm{cf}(\delta)^W$ is a continuity point of j,*
2. *every ordinal δ with W-cofinality less than κ is a continuity point of j,*
3. *if $\eta < \kappa$ is regular, $\mathrm{Cof}(\eta)^W \cap \lambda = \mathrm{Cof}(\eta)^{W'} \cap \lambda$, and $j``\delta \in W'$ for all $\delta < \lambda$ then $j``\lambda$ is η-closed, in particular,*
4. *if*
   - *(a) $\eta < \kappa$ is regular,*
   - *(b) $Z \subseteq P(\lambda)$ and $I \in W$ is a normal ideal on Z,*
   - *(c) $W' \cong W^Z/G$ for a W-generic ultrafilter $G \subseteq P(Z)/I$ and j is the ultrapower embedding,*
   - *(d) $\mathrm{Cof}(\eta)^W \cap \lambda = \mathrm{Cof}(\eta)^{W'} \cap \lambda$, and*
   - *(e) $(W')^\eta \cap W[G] \subseteq W'$,*

   then $j``\lambda$ is η-closed in $W[G]$.

Proof. Suppose that δ has cofinality η in W. Let $\langle \delta_i : i \in \eta\rangle \in W$ be an increasing cofinal sequence in δ. Since $j(\langle \delta_i : i \in \eta\rangle)$ is an increasing cofinal sequence in $j(\delta)$ of length $j(\eta)$ the following are equivalent:

a. δ is a continuity point of j,

b. $\langle j(\delta)_{j(i)} : i \in \eta\rangle$ is cofinal in $j(\delta)$,

c. $\langle j(i) : i \in \eta\rangle$ is cofinal in $j(\eta)$, and

d. η is a continuity point of j.

If $W \models \mathrm{cf}(\delta) = \eta$ and $\eta < \kappa$, then $j \restriction \eta$ is the identity map and hence $j(\langle \delta_i : i \in \eta \rangle) = \langle j(\delta)_i : i \in \eta \rangle$. Hence $\langle j(\delta)_i : i \in \eta \rangle$ is cofinal in $j(\delta)$ and δ is a continuity point.

Supposing the hypotheses of clause 3 we show that $j``\lambda$ is $\eta$-closed. If not, a counterexample is an ordinal $\alpha$ that is a limit point of $j``\lambda$, $\alpha \notin j``\lambda$ and $W' \models \mathrm{cf}(\alpha) = \eta$. Let $\delta$ be the least element of $\lambda$ such that $j(\delta) > \alpha$. Then $j``\delta \cup P(\delta)^W \subseteq W'$ and, moreover, $\{j``\delta\}$ is cofinal in α. Hence if $\mu = \mathrm{cf}(\delta)^W$, then $\mathrm{cf}(\mu)^{W'} = \eta$. Since W and W' agree about which ordinals have cofinality η, we must have $\mu = \eta$. But then δ must be a continuity point by clause 2, a contradiction.

Clause 4 is immediate from the other clauses, since it implies that $\mathrm{cf}(\eta)$ is absolute between W' and $W[G]$. ⊣

We have a converse to Proposition 2.31:

2.32 Proposition. *Suppose that $j : V \to M \subseteq V[G]$ is a generic elementary embedding with critical point κ, $j``\lambda$ is η-closed (as a class of ordinals) in $V[G]$ and $M^\eta \cap V[G] \subseteq M$. Then for all ordinals $\alpha < \lambda$, $V \models \mathrm{cf}(\alpha) = \eta$ iff $V[G] \models \mathrm{cf}(\alpha) = \eta$.*

Proof. Let μ be the least counterexample. Then μ must be a regular cardinal, since otherwise the cofinality of $\mathrm{cf}(\mu)^V$ is a smaller counterexample. Thus $M \models$ "$j(\mu)$ is regular". But μ is a continuity point of j by the hypothesis that $j``\lambda$ is η-closed. Thus $\mathrm{cf}(j(\mu))^{V[G]} = \mathrm{cf}(\mu)^{V[G]} = \eta$. Hence $M \models$ "$\mathrm{cf}(j(\mu)) = \eta$" and so $j(\mu) = \eta$. But $\eta < \mu$, a contradiction. ⊣

2.6. Canonical Functions

An important technical tool in studying normal ideals is the sequence of canonical functions: a sequence of canonical representatives for the ordinals less than λ^+ in the generic ultrapower with respect to any countably complete normal ideal I on $Z \subseteq P(\lambda)$.

We define $\langle f_\alpha : \alpha < \lambda^+ \rangle$ by induction on α. For $\alpha < \lambda$, we let

$$f_\alpha(z) = \mathrm{ot}(z \cap \alpha).$$

Suppose that we have defined $f_{\alpha'}$ for $\alpha' < \alpha$. Let $g : \lambda \to \alpha$ be a bijection. Define

$$f_\alpha(z) = \sup\{f_{g(\eta)}(z) + 1 : \eta \in z\}.$$

If g_1, g_2 are two bijections between λ and α, then the collection of z such that $\{g_1(\eta) : \eta \in z\} = \{g_2(\eta) : \eta \in z\}$[6] belongs to the dual of any normal ideal. As a consequence, modulo every normal ideal, the definition of f_α is independent of the choice of g.

[6] Using Proposition 2.20 one can prove that this set is closed and unbounded, in the sense of Example 3.2.

2.33 Definition. The functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ are called the *canonical functions* for normal ideals on Z.

This definition is slightly misleading as the canonical functions are only well-defined modulo the nonstationary ideal.

2.34 Proposition. *Suppose that $\lambda = |X|$. Let I be a normal, fine, countably complete ideal on $Z \subseteq P(X)$. Let G be generic for $P(Z)/I$. Then the well-founded part of the ordinals in V^Z/G includes $(\lambda^+)^V$ and for each $\alpha \in \lambda^+$, $[f_\alpha]^G = \alpha$.*

Proof. Without loss of generality we can assume that $X = \lambda$. As in Claim 2.26, we see that if $i : V \to V^Z/G$ is the generic embedding, then $[\mathrm{id}]^G = i``\lambda$. Hence V^Z/G is a model of ZFC with a well-founded set of order type λ. This implies that the ordinals of V^Z/G are well-founded up to λ. To see that the well-founded part of V^Z/G includes λ^+ we need to look more closely.

Let $\langle f_\alpha : \alpha < \lambda^+ \rangle$ be a sequence of canonical functions. Then:

1. for $\alpha < \alpha', \{z \in P(\lambda) : f_\alpha(z) < f_{\alpha'}(z)\} \in \breve{I}$,
2. for any normal, countably complete ideal I and $\alpha < \lambda^+$, if $h(z) < f_\alpha(z)$ for all z in a set $A \in I^+$, then there is a dense collection of $B \subseteq_I A$ such that there is an $\alpha(B) < \alpha$ with $h(z) = f_{\alpha(B)}(z)$ for all $z \in B$.

Hence for a generic G, $\{[f_\alpha]^G : \alpha < \lambda^+\}$ form a well-ordered initial segment of the V^Z/G ordinals that has order type λ^+. ⊣

A fact slightly stronger than Proposition 2.34 is true: if $N \subseteq V$ and I is a fine ideal on $P(\lambda)^N$ that lies in V that is normal and countably complete for sequences of sets that lie in N, then for all generic $G \subseteq P(\lambda)^N/I$ and all $\alpha < (\lambda^+)^N$, $[f_\alpha]^G = \alpha$.

The next example is well-known.

2.35 Example. Let M be a well-founded model of $V = L$ and suppose that $G \subseteq (P(\omega_1)/\mathrm{NS}_{\omega_1})^M$ is generic over M. Let N be the generic ultrapower of M by G. By Proposition 2.34, N is well-founded to ω_2^M. Moreover, if $\alpha = \omega_1^M$, $N \models$ "α is countable". Hence there is an $x \in N$ such that $N \models$ "x is the least countable ordinal such that there is a subset of ω constructed at stage x that codes a bijection between α and ω". Clearly x must be bigger than the first ω_2^M many ordinals of N.

Define $f : \omega_1 \to \omega_1$ in M, by setting $f(\alpha)$ to be the least β such that there is a subset of ω in $L^M_{\beta+1}$ that codes a bijection between α and ω. Then $x = [f]_G > [f_\delta]_G$ for any canonical function f_δ, where $\delta < \omega_2^M$. Since G is arbitrary we see that for all $\delta < \omega_2^M, \{\alpha : f(\alpha) > f_\delta(\alpha)\}$ contains a closed unbounded set.

The consistency strength of having the canonical functions from ω_1 to ω_1 bound every function from ω_1 to ω_1 is exactly an inaccessible limit of measurable cardinals: Larson and Shelah [81] showed that from a model with an inaccessible limit of measurable cardinals one can force a model of CH such that every function from ω_1 to ω_1 is bounded by a canonical function. Deiser and Donder [20] showed that if every function is bounded by a canonical function, then ω_2 is an inaccessible limit of measurable cardinals in an inner model of set theory.

2.7. Selectivity

The following definition comes from Baumgartner, Taylor and Wagon [9], where the notions of *selective, P-* and *Q*-ideals are explored in detail. In particular they showed that if $2^{\omega_1} = \omega_2$ and the nonstationary ideal on ω_1 is $\aleph_2$-saturated, then an ideal on a cardinal κ is selective iff it is isomorphic to a normal ideal on κ. One direction of this is shown below.

2.36 Definition. Let I be an ideal on κ. Then I is *selective* iff whenever $f : \kappa \to V$ is a function that is not constant on any I-positive set, then there is a set $A' \in \breve{I}$ such that f is one-to-one on A'.

2.37 Lemma. *Suppose that $j : N \to M$ is an elementary embedding with critical point δ, where M is a model of* ZF$^-$ *(*ZF *minus Power Set) that is well-founded to $\delta + 1$. Suppose that $f \in N$ is a function with domain $A \subseteq \delta$ such that $\delta \in j(A)$. Then either:*

1. *there is an $c \in N$ such that $\delta \in j(\{\alpha : f(\alpha) = c\})$, or*
2. *there is an $A' \subseteq A$ with $A' \in N$ such that $\delta \in j(A')$ and f is one-to-one on A'.*

Moreover, if I is a normal ideal on a cardinal κ, then I is selective.

Proof. Define $g : A \to A$ by setting $g(\alpha) = \min\{\beta : f(\beta) = f(\alpha)\}$. Then $g \in N$. Let $A' = \{\alpha : g(\alpha) = \alpha\}$. If $\delta \notin j(A')$, let $\beta = j(g)(\delta)$. Then $\beta < \delta$ and $c = f(\beta)$ belongs to N. Moreover $j(f)(\delta) = j(f)(\beta) = j(c)$, and so $\delta \in j(\{\alpha : f(\alpha) = c\})$.

Suppose that I is a normal ideal on κ. Given an $f : \kappa \to V$, define g and A' as in the previous paragraph. If $A' \notin \breve{I}$, then g is a regressive function on the I-positive set $\kappa \setminus A'$. By normality, g is constant on an I-positive set. ⊣

The notions in [9] have not been explored on normal, fine ideals on $Z \subseteq P(X)$ other than κ-complete ideals on $Z = \kappa$. In particular, the analogue of Lemma 2.37 is not immediately clear. We ask the general question: what one can say about P-, Q- and selective ideals on $[\lambda]^{<\kappa}$ or $[\lambda]^{\kappa}$?

2.8. Ideals and Reflection

The utility of ideals lies largely in their ability to capture many of the reflection properties of large cardinals, with the additional advantage that they can "live" on relatively small sets such as ω_1 or $\mathbb{R}$. This will be a recurring theme in this chapter; we mention here only the basic idea for deducing such reflection.

Let I be an ideal and $j : V \to V^Z/G$ be the canonical elementary embedding. We can view the generic ultrapower V^Z/G as having an *ideal element*, namely the element $i = [\mathrm{id}]^G$, where again $\mathrm{id} : Z \to Z$ is the identity function. This element is handy in relating the elementary embedding j to the ultrapower. For example, Lemma 2.3 shows that $[f] = j(f)(i)$ and that for every $Y \subseteq Z$, $Y \in G$ iff $i\, E\, j(Y)$, where again E is the ultrapower of the $\in$ relation in the possibly ill-founded generic ultrapower.

As a consequence of Łoś's Theorem, the properties of the ideal element i are reflected to the sets that the ideal I concentrates on. Explicitly, suppose that $\phi(x, y)$ is a formula in the language of set theory, and $X \subseteq Z$ is such that $[X]_I \Vdash$ "$V^Z/G \models \phi(i, j(\check{a}))$"; then for almost every $z \in X$, $V \models \phi(z, a)$.

Hence, for example, if $A \in P(Z)/I$ forces that $i \in [j(\lambda)]^{<j(\kappa)}$, the ideal $I{\restriction}A$ can be taken to be an ideal on $[\lambda]^{<\kappa}$. Similarly if $A \Vdash i \in [j(\lambda)]^{j(\kappa)}$, then $I{\restriction}A$ can be taken to be an ideal on $[\lambda]^{\kappa}$.

To put this in better focus, suppose that I is a normal, fine, κ-complete precipitous ideal on a cardinal κ. Then the ideal element represents κ in the generic ultrapower. If the ideal concentrates on ordinals α that are regular cardinals, then κ is regular in M. Otherwise, by normality, there will be a set $Y \in G$ consisting of ordinals of a fixed cofinality $\gamma < \kappa$. Then $j(\gamma) = \gamma$ and hence $M \models \mathrm{cf}(\kappa) = \gamma$.

As another example of this technique, suppose that $Z = [\lambda]^{<\kappa}$ and I is a normal, fine, precipitous ideal on Z. We know that i represents $j``\lambda$ in $M$. If $\eta < \lambda$, $\mu < \kappa$ and $I$ concentrates on $\{z : \mathrm{ot}(z \cap \eta) = \mu\}$, then we know that $M \models \mathrm{ot}(i \cap j(\eta)) = j(\mu)$. Since $i = j``\lambda$, the order type of $i \cap j(\eta)$ is η. Hence $j(\mu) = \eta$. Similar remarks can be made using inequalities. For example, if I concentrates on the collection of z where the order type of $z \cap \eta$ is less than μ, then $j(\mu) > \eta$.

We now illustrate this by giving a reflection argument that we will use later:

2.38 Lemma. *Suppose that I is a normal, fine, precipitous ideal on $[\lambda]^{<\kappa}$. Let $G \subseteq P([\lambda]^{<\kappa})/I$ be generic.*

1. *Let μ and ν be less than κ and A be the collection of $z \in [\lambda]^{<\kappa}$ such that*

   (a) $z \cap \kappa \in \kappa$,

   (b) $|z| = |z \cap \kappa|$,

   (c) $\mathrm{cf}(z \cap \kappa) = \mu$, *and*

   (d) $\mathrm{cf}(\sup(z)) = \nu$.

   If $A \in G$ then in M, μ and ν are regular cardinals, $|\lambda| = |\kappa|$, $\mathrm{cf}(\kappa) = \mu$ and $\mathrm{cf}(\lambda) = \nu$.

2. Let $\kappa = \rho^+$ and B be the collection of $z \in [\lambda]^{<\kappa}$ such that

(a) $z \cap \kappa \in \kappa$,

(b) $|z| = |z \cap \kappa|$, and

(c) $\mathrm{cf}(z \cap \kappa) = \mathrm{cf}(\sup(z)) \neq \mathrm{cf}(\rho)$.

If $B \in G$, then in M, ρ and $\mathrm{cf}(\rho)$ are preserved and $|\lambda| = |\kappa| = \rho$, $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\rho)$.

Moreover, were I to be λ^+-saturated, then we can draw the same conclusions about $V[G]$ as we did about M.

Proof. Let $j : V \to M$ be the generic ultrapower. By Theorem 2.25, $j``\lambda \in M$, $P(\lambda)^V \subseteq M$ and if $I$ is $\lambda^+$-saturated, $M^\lambda \cap V[G] \subseteq M$. Hence all cardinals and cofinalities below $\kappa$ are preserved between $V$ and $M$. Moreover, $[\mathrm{id}]^M = j``\lambda$.

We can now see that the critical point of j is κ. To see that it is at least κ we note that for each $\beta < \kappa, \{z : \beta \in z\} \in G$. Hence, by assumption, $\{z : \beta + 1 \subseteq z\} \in G$. By the elementarity of j, $j(\beta + 1) \subseteq j``\lambda$. But then the critical point of $j$ must be at least $j(\beta + 1)$ which is greater than or equal to $\beta + 1$. If the critical point of $j$ is bigger than $\kappa$, then $\kappa + 1 \subseteq j``\lambda$. Since $j\restriction(\kappa + 1)$ is the identity, $\{z : \kappa + 1 \subseteq z\} \in G$. But this contradicts the assumption that $\{z : z \cap \kappa \in \kappa\} \in G$.

If $A \in G$, using Łoś's Theorem we see that in M, $\mathrm{cf}(j``\lambda \cap j(\kappa)) = \mu$. Since $j``\lambda \cap j(\kappa) = \kappa$, $\mathrm{cf}(\kappa) = \mu$. Finally, another application of Łoś's Theorem implies that $\mathrm{cf}(\sup(j``\lambda)) = \nu$. Since $\mathrm{cf}(\sup(j``\lambda)) = \mathrm{cf}(\lambda)$, $\mathrm{cf}(\lambda) = \nu$.

If $B \in G$, $|j``\lambda| = |j``\lambda \cap j(\kappa)| = |\kappa|$. Since $\kappa = \rho^+$ in V and is moved by j, we must have $|\kappa|^M = \rho$. On the other hand $\mathrm{cf}(\lambda)^M = \mathrm{cf}(j``\lambda)^M \neq \mathrm{cf}(\rho)$.

If I is λ^+-saturated, then the closure of M implies that these statements also hold in $V[G]$. $\dashv$

From examining in detail the order properties of the sets in a typical element of the generic ultrapower, one can generally get a complete picture of where j sends cardinals. The variations on this technique are myriad, and will be explored further in Sects. 5 and 6.

3. Examples

In this section we give examples of ideals. These ideals fall into roughly two categories: the ideals that have an intrinsic definition (such as the nonstationary ideal, or the ideal of null sets for Lebesgue measure) and those ideals that are defined extrinsically in terms of an elementary embedding. We will refer to the former as "natural" ideals, the latter as "induced" ideals.

Our main concern in this chapter is with ideals that give rise to generic elementary embeddings into transitive models. These ideals will always be the induced ideals from their own embeddings. Thus, if a "natural" ideal gives rise to a well-founded ultrapower, it is also an "induced" ideal. Another way induced ideals frequently arise is as remnants of large cardinal embeddings that can be generically extended even after the associated large cardinal has been modified by forcing.

Various induced ideals' existence is often posited as non-constructive existence principles, similar in spirit to the supposition of the existence of large cardinal ultrafilters. These ideals can function in two ways, as existence principles in their own right and as intermediaries in relative consistency results.

3.1. Natural Ideals

We begin by describing several natural ideals.

3.1 Example. Let κ be a regular cardinal. The collection of subsets of κ that have cardinality less than κ forms a κ-complete ideal on κ called the *bounded ideal* or the *ideal of bounded sets*. Jech and Prikry [64] showed that this ideal is never precipitous.

Similarly, if $\lambda \geq \kappa$ then there is a smallest κ-complete, fine ideal on $[\lambda]^{<\kappa}$, namely $I_{\kappa\lambda} = \{X \subseteq [\lambda]^{<\kappa} : \text{for some } a \in [\lambda]^{<\kappa},\ X \cap \{b : a \subseteq b\} = \emptyset\}$. Matsubara and Shioya showed that $I_{\kappa\lambda}$ is never precipitous. We show here that the bounded ideal on κ is not precipitous, but postpone the second assertion to Sect. 6, Corollary 6.30.

Let I be the bounded ideal, and let $G \subseteq P(\kappa)/I$ be generic. A density argument shows that in $V[G]$ there is a sequence $\langle Y_n : n \in \omega\rangle \subseteq G$ such that if $f_n : Y_n \to \kappa$ is the unique order-preserving bijection between Y_n and κ, then for all $\alpha \in Y_{n+1}$, $f_{n+1}(\alpha) < f_n(\alpha)$. But then, the sequence $[f_n]^{V^Z/G}$ is an infinite decreasing sequence of ordinals in V^Z/G.

The Closed Unbounded Filter and the Nonstationary Ideal

We now discuss the nonstationary ideal. This ideal is covered in some depth in Jech's chapter in this Handbook. The definition given here was used implicitly by Shelah since the early 1980's, appeared in [47] and was exploited dramatically by Woodin in 1989 in his work on stationary towers.

3.2 Example. We fix a base set X, and define an ideal on $Z = P(X)$. By an *algebra* on X we will mean a structure $\mathfrak{A} = \langle X, f_n\rangle_{n\in\omega}$ where each $f_n : X^k \to X$, for some k.[7] For notational purposes, and without loss of generality, we will usually assume that f_n is an n-ary function. For such an algebra $\mathfrak{A}$, we let $C_{\mathfrak{A}}$ be the collection of all $z \subseteq X$ that are closed under all of the functions f_n. Following [47] such $C_{\mathfrak{A}}$ will be called *strongly closed unbounded* and the filter generated by the collection of all $C_{\mathfrak{A}}$ will be called the *strongly closed unbounded filter* on $P(X)$.

[7] In the sequel, we will refer to structures $\mathfrak{B}$ in an arbitrary countable language as *algebras*, when strictly speaking, we are referring to an algebra of Skolem functions for $\mathfrak{B}$.

Using Skolem functions this filter can also be generated by taking as a typical generating set the collection of all elementary substructures of an arbitrary structure $\mathfrak{A}$ in a countable language with universe X. This is sometimes a more convenient definition.

The *strongly nonstationary* ideal on $P(X)$ is the ideal dual to the strongly closed unbounded filter. Positive sets for this ideal are called *weakly stationary* sets.

We now show that this filter is quite well-behaved and provides a nice generalization of the classical closed unbounded filter. Moreover its restriction to various weakly stationary sets yields exactly the classical filter.

If $F : [\lambda]^{<\omega} \to \lambda$ we let $C_F = \{x \in P(\lambda) : x \text{ is closed under } F\}$. We note that for every algebra $\mathfrak{A}$ on λ there is a function F such that $C_F \subseteq C_{\mathfrak{A}}$ and vice versa.

3.3 Lemma. *The filter $\mathcal{F}$ of strongly closed unbounded sets is normal and fine.*

Proof. The proof of this theorem is an illustration of a standard trick.

First, the collection of generating sets is clearly closed under countable intersection: a countable collection of algebras can be merged into a single algebra. If $\mathcal{F}$ were not normal then there would be a weakly stationary set A and a regressive function $g : A \to X$ such that for all $x \in X$, $\{z \in A : g(z) = x\}$ is not weakly stationary. This means that for all $x \in X$ we can associate an algebra $\mathfrak{A}_x = \langle X, f_n^x \rangle_{n\in\omega}$ such that no $z \in A$ that is closed under the functions of $\mathfrak{A}_x$ has $g(z) = x$. Define an algebra $\mathfrak{A} = \langle X, f_n \rangle_{n\in\omega}$, by setting $f_n(x, x_0, \ldots, x_{n-2}) = f_{n-1}^x(x_0, \ldots, x_{n-2})$. Now suppose that $z \in A$ is closed under all of the functions f_n. Then for all $x \in z$, z is closed under all of the f_n^x. In particular, if $g(z) = x$ we have that z is closed under the functions of $\mathfrak{A}_x$. This is a contradiction.

Fineness is trivial, for if $x \in X$, then any algebra containing the constant function with value x gives a set in the filter, all of whose elements contain x. $\dashv$

We now discuss the relationship between the definition of the strongly closed unbounded filter and the more customary definitions of the filters of closed unbounded sets. First, it is clear that this notion is distinct: by the downward Löwenheim-Skolem theorem, every strongly closed unbounded set contains countable sets. However when we restrict this filter to standard sets we recover the various definitions of closed unbounded set. We now look at some examples where this is true.

The most common objects called the closed unbounded filter and the nonstationary ideal are defined on regular cardinals κ and are κ-complete. We want to see that these are given by the strongly closed unbounded filter restricted to a weakly stationary set. To do this we consider a regular cardinal κ, not as a collection of ordinals, but as a collection of subsets of κ.

3.4 Example. Let $X = \kappa$ where κ is a regular cardinal. Then $K = \{z \subseteq \kappa : z \cap \kappa \in \kappa\}$ is a weakly stationary set that is canonically isomorphic with κ via the identity map. Moreover, an elementary substructure argument shows that the filter of strongly closed sets restricted to K gives the usual closed unbounded filter on κ.

3.5 Example. Jech [61] generalized the notion of closed unbounded filter from a filter on a regular cardinal κ to a filter on the base set $[\lambda]^{<\kappa}$ for regular κ.[8] A generating set for this filter is a set C that is:

1. *closed* in the sense that C is closed under unions of chains of length less than κ, and

2. *unbounded* in the sense that for all $y \subseteq \lambda$ having cardinality less than κ, there is an $x \in C$ such that $y \subseteq x$.

The first clause is equivalent to the statement that C is closed under directed unions of size less than κ. The second clause states that $C \cap [\lambda]^{<\kappa}$ is cofinal in the structure $\langle [\lambda]^{<\kappa}, \subseteq \rangle$.

Before we prove the next theorem, we discuss a fundamental Skolemization trick used in the proof of that theorem and in many other contexts such as the study of projections of ideals. The trick is used where we have sets $X \subseteq Y$ and a structure $\mathfrak{A}$ with domain Y in a countable language. By adding countably many functions to the type of $\mathfrak{A}$ we can assume that $\mathfrak{A}$ is fully Skolemized, and that the functions mentioned in the type of $\mathfrak{A}$ are closed under all possible compositions.

List the functions mentioned in $\mathfrak{A}$ as $\langle f_n : n \in \omega \rangle$ where $f_n : Y^{k_n} \to Y$. Fix an element $x_0 \in X$. Define $g_n : X^{k_n} \to X$ by

$$g_n(\vec{x}) = \begin{cases} f_n(\vec{x}\,) & \text{if } f_n(\vec{x}\,) \in X, \\ x_0 & \text{otherwise.} \end{cases}$$

Using the fact that the f_n's are closed under compositions, if we take a set $B \subseteq X$ that is closed under all of the g_n's and has $x_0 \in B$, then $\mathrm{Sk}^{\mathfrak{A}}(B) \cap X = B$.

The following theorem of Kueker [76] is fundamental:

3.6 Theorem. *Suppose that $D \subseteq [\lambda]^{<\kappa}$ is closed unbounded in the sense of Jech. Then there is a function $F_D : \lambda^{<\omega} \to \lambda$ such that $\{z : z \cap \kappa \in \kappa$ and z is closed under the function $F_D\} \subseteq D$.*

Proof (Sketch). Fix $\rho \gg \lambda$. Let $\mathfrak{B} = \langle H(\rho), \in, \Delta_\rho, D \rangle$, where Δ_ρ is a well-ordering of $H(\rho)$. Suppose that $N \prec \mathfrak{B}$ has cardinality less than κ and $N \cap \kappa \in \kappa$. We observe that:

[8] For more on this filter see Jech's chapter in this Handbook.

1. $N \cap D$ is directed,
2. for all $\alpha \in N \cap \lambda$ there is a $y \in D \cap N$ with $\alpha \in y$, and
3. since $N \cap \kappa \in \kappa$, if $x \in N \cap [\lambda]^{<\kappa}$, then $x \subseteq N \cap \lambda$.

Hence $N \cap \lambda = \bigcup(N \cap D)$. Since D is closed under directed unions, $N \cap \lambda$ belongs to D. By the Skolemization trick we can find an $F : \lambda^{<\omega} \to \lambda$ such that if $A \subseteq \lambda$ is closed under F and A' is the Skolem hull of A in $\mathfrak{B}$, then $A = A' \cap \lambda$. Then F satisfies the conclusion of the theorem. ⊣

The next lemma appears in [47].

3.7 Lemma. *The filter of closed unbounded sets in the sense of Jech is the filter on $[\lambda]^{<\kappa}$ generated by the strongly closed unbounded filter and $\{x : x \cap \kappa \in \kappa\}$.*

Proof. Clearly $\{z \subseteq \lambda : z \cap \kappa \in \kappa\}$ is closed and unbounded in Jech's sense. Moreover, given a function $F : \lambda^{<\omega} \to \lambda$, C_F is closed and unbounded in the sense of Jech. Hence every set in the filter generated by the strongly closed unbounded filter and $\{x : x \cap \kappa \in \kappa\}$ is in the closed unbounded filter in the sense of Jech.

For the other direction, if D is closed unbounded in the sense of Jech we consider F_D as in Kueker's theorem. Then $C_F \subseteq D$ and belongs to the filter generated by the strongly closed unbounded filter and $\{N : N \cap \kappa \in \kappa\}$. ⊣

From this lemma we see that the strongly closed unbounded filter on $[\lambda]^{<\kappa}$ is distinct from the Jech closed unbounded filter exactly when $\{z \in [\lambda]^{<\kappa} : z \cap \kappa \notin \kappa\}$ is weakly stationary. Thus if $\kappa = \omega_1$ the two filters coincide.

As noted earlier if $\kappa > \omega_1$, the filters differ for trivial cardinality reasons, most prominently because the collection of countable subsets of λ is weakly stationary. To rule out such cardinality reasons for the difference we can take $\kappa = \rho^+ > \omega_1$ and look at those elements of $[\lambda]^{<\kappa}$ of cardinality ρ. This is a closed unbounded set in the sense of Jech, and it makes sense to discuss the strongly closed unbounded filter restricted to this set, which we call T.

The two filters differ on T iff for all algebras $\mathfrak{A}$ on λ, there is an elementary substructure $z \prec \mathfrak{A}$ such that z has cardinality ρ, but $\rho \not\subseteq z$. By the Skolemization trick, this can be equivalently rephrased as stating that $\{z \in [\rho^+]^{<\rho^+} : \rho \not\subseteq z\}$ is weakly stationary.

We now digress to define Chang's Conjecture properties. We will use the following notation: $(\kappa_n, \ldots, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_0)$ is the statement that every structure $\mathfrak{A} = \langle \kappa_n; f_i, R_j, c_k \rangle_{i,j,k \in \omega}$ in a countable language has an elementary substructure $\mathfrak{B}$ of cardinality λ_n such that for all i, $|\mathfrak{B} \cap \kappa_i| = \lambda_i$. The same notation with "λ_i" replaced by "$<\lambda_i$" for some of the i's changes the cardinality requirement to $|\mathfrak{B} \cap \kappa_i| < \lambda_i$. Statements of this form are called *Chang's Conjectures.*

The classical Chang's Conjecture is the statement $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$. It is easily seen to be equivalent to the statement that

$$\{z \in [\omega_2]^{<\omega_2} : |z| = \omega_1 \text{ and } z \cap \omega_2 \notin \omega_2\}$$

is weakly stationary. In particular Chang's Conjecture is equivalent to the statement that the strongly closed unbounded filter on $[\omega_2]^{<\omega_2}$ differs from the Jech closed unbounded filter. It was pointed out in [47] that by fixing the parameters κ, λ, one can describe every instance of Chang's Conjecture by asking that the appropriate set be weakly stationary.

Closely related to Chang's Conjecture are Jónsson cardinals. A cardinal κ is said to be *Jónsson* iff every structure $\mathfrak{A}$ with domain κ in a countable language has a proper elementary substructure of cardinality κ. A counterexample to κ being Jónsson is called a *Jónsson algebra*.[9]

Jónsson cardinals are also relevant to the discussion of the difference between weak stationarity and Jech stationarity. For simplicity we take $\rho = \aleph_\omega$. Since $\aleph_\omega$ is a limit cardinal, $\{z : z \in [\aleph_{\omega+1}]^{<\aleph_{\omega+1}} : |z| = \aleph_\omega \text{ and } \aleph_\omega \not\subseteq z\}$ is weakly stationary iff $\{z : z \in [\aleph_\omega]^{\aleph_\omega}, z \neq \aleph_\omega\}$ is weakly stationary. The latter statement is just a rephrasing of the statement that $\aleph_\omega$ is Jónsson.

Summarizing: each difference between the strongly closed unbounded filter restricted to T and the Jech closed unbounded filter is an instance of a form of Chang's Conjecture or of ρ being Jónsson. All instances of Chang's Conjecture imply the existence of an inner model with a Rowbottom cardinal (and many instances are considerably stronger[10]). Thus, if there is a difference between the two filters restricted to T, then at least mild large cardinals exist.

The next example of a natural filter expands on this:

3.8 Example. Suppose that $(\kappa_n, \dots, \kappa_0) \twoheadrightarrow (\lambda_n, \dots, \lambda_0)$ holds. Then the strongly closed unbounded filter restricted to $\{z \subseteq \kappa_n : \text{for all } i,\ |z \cap \kappa_i| = \lambda_i\}$ is called the *Chang filter* and the dual ideal, the *Chang ideal.*

In many cases we can assume that the Chang ideal has the maximal degree of completeness. We now discuss this situation. The following proposition appears in [32] and is an application of ideas of Shelah.

3.9 Proposition. *Let $\kappa_n > \kappa_{n-1} > \cdots > \kappa_0$ and $\lambda_n > \lambda_{n-1} > \cdots > \lambda_0$ be regular cardinals. Suppose that either:*

1. GCH *holds, or*

2. *there are at most countably many cardinals between λ_0 and κ_n,*

and that $(\kappa_n, \dots, \kappa_1, \kappa_0) \twoheadrightarrow (\lambda_n, \dots, \lambda_1, {<}\lambda_0)$. Let $X \supseteq \kappa_n$ be a set. Then the nonstationary ideal restricted to $Z =_{\text{def}} \{z \in P(X) : |z \cap \kappa_i| = \lambda_i \text{ for } i > 0, |z \cap \kappa_0| < \lambda_0 \text{ and } z \cap \lambda_0 \in \lambda_0\}$ is a proper, λ_0-complete, normal, fine ideal.

For any regular λ and set $X \supseteq \lambda$ it is not difficult to check that the strongly nonstationary ideal restricted to $\{z \subseteq X : z \cap \lambda \in \lambda\}$ is λ-complete. Hence the issue in Proposition 3.9 is whether or not the ideal is proper. To show that the ideal is proper, one must show that if $\mathfrak{B}$ is a structure with

[9] No $\aleph_n$ can be Jónsson for finite n. It is a prominent open problem whether $\aleph_\omega$ can be Jónsson.

[10] See Sect. 10 for more details.

domain X, then there is an elementary substructure z of $\mathfrak{B}$ with the property that $|z \cap \kappa_i| = \lambda_i$ for $i > 0$, $|z \cap \kappa_0| < \lambda_0$ and $z \cap \lambda_0 \in \lambda_0$.

We content ourselves here to showing a lemma that easily implies Proposition 3.9 in the case that there are only finitely many cardinals between λ_0 and κ_n.

3.10 Lemma. *Let $\lambda \leq \kappa \ll \theta$ be cardinals with λ and θ regular and with $\mathrm{cf}(\kappa) \geq \lambda$. Let $\mathfrak{A}$ be a structure expanding $\langle H(\theta), \in, \Delta, \{\kappa, \lambda\}\rangle$ and $N_0 \prec \mathfrak{A}$. Let $N_1 = \mathrm{Sk}^{\mathfrak{A}}(N_0 \cup \sup(N_0 \cap \lambda))$. Then*

$$\sup(N_1 \cap \kappa) = \sup(N_0 \cap \kappa).$$

Proof. Let τ be a Skolem function for $\mathfrak{A}$. Since we are considering the intersection of N_1 with κ, without loss of generality we can assume that

$$\tau : H(\theta) \times \lambda \to \kappa.$$

We must show that for each $a \in N_0$ and $\delta \in \sup(N_0 \cap \lambda)$ there is a $\beta \in N_0$ with $\beta \geq \tau(a, \delta)$.

Fix such an $a \in N_0$ and δ. Choose a $\gamma \in N_0 \cap \lambda$ with $\delta < \gamma$. Let

$$\beta = \sup\{\tau(a, \alpha) : \alpha < \gamma\}.$$

Then β is definable in N_0 and is clearly at least $\tau(a, \delta)$, as required. ⊣

Proof of Proposition 3.9. We now explain how to prove Proposition 3.9 using Lemma 3.10 in the case that $\kappa_n = \lambda_0^{+l}$ for some $l \in \omega$. We are given a structure $\mathfrak{B}$ with domain X and we assume that Chang's Conjecture holds. We must find a suitable $z \prec \mathfrak{B}$. Let $\theta \gg \sup X$, and take $\mathfrak{A}$ to expand $\langle H(\theta), \in, \Delta, \{\kappa_n, \kappa_{n-1}, \ldots, \kappa_0, \lambda_n, \lambda_{n-1}, \ldots, \lambda_0\}, \mathfrak{B}\rangle$. By the Chang's Conjecture assumption we can find an $N_0 \prec \mathfrak{A}$ such that $|N_0| = |N_0 \cap \kappa_n| = \lambda_n$; for all $1 \leq i \leq n$, $|N_0 \cap \kappa_i| = \lambda_i$; and $|N_0 \cap \kappa_0| < \lambda_0$.

Let $N_1 = \mathrm{Sk}^{\mathfrak{A}}(N_0 \cup \sup(N_0 \cap \lambda_0))$. By Lemma 3.10, $N_1 \cap \lambda_0 = \sup(N_0 \cap \lambda_0)$ and for all $0 \leq j < \omega$, $\sup(N_1 \cap \lambda_0^{+j}) = \sup(N_0 \cap \lambda_0^{+j})$. From this one sees inductively that for $0 \leq j < \omega$, $|N_1 \cap \lambda_0^{+j}| = |N_0 \cap \lambda_0^{+j}|$. In particular, $|N_1 \cap \kappa_i| = |N_0 \cap \kappa_i|$ for $0 \leq i \leq n$.

Let $z = N_1 \cap X$. Since $\mathfrak{B}$ is definable in $\mathfrak{A}$, we must have $z \prec \mathfrak{B}$. Since $z \cap \kappa_i = N_1 \cap \kappa_i$, the proposition follows. ⊣

There is a converse to Proposition 3.9:

3.11 Proposition. *Let $\kappa_n > \kappa_{n-1} > \cdots > \kappa_0$ and $\lambda_n > \lambda_{n-1} > \cdots > \lambda_0$ be regular cardinals and $X \supseteq \kappa_n$. If there is a proper, normal, fine, countably complete ideal concentrating on $Z =_{\mathrm{def}} \{z \in P(X) : |z \cap \kappa_i| = \lambda_i$ for $i > 0$, $|z \cap \kappa_0| < \lambda_0$ and $z \cap \lambda_0 \in \lambda_0\}$, then $(\kappa_n, \ldots, \kappa_1, \kappa_0) \twoheadrightarrow (\lambda_n, \ldots, \lambda_1, {<}\lambda_0)$.*

Proof. This follows easily from the fact that if $F : [X]^{<\omega} \to X$ and I is normal, fine and countably complete, then $\{z : z$ is closed under $F\}$ belongs to $\breve{I}$. ⊣

We will denote the nonstationary ideal concentrating on z's with $|z \cap \kappa_i| = \lambda_i$ by $\mathrm{CC}(\vec{\kappa}, \vec{\lambda})$. Note that the ideal $\mathrm{CC}((\kappa, \lambda, \mu), (\kappa', \lambda', \mu'))$ projects[11] to the ideal $\mathrm{CC}((\lambda, \mu), (\lambda', \mu'))$ via the map $z \mapsto z \cap \lambda$. We can also define an ideal $\mathrm{CC}((\kappa, \lambda), (\kappa', {<}\lambda'))$ by restricting the nonstationary ideal to those N where $|N \cap \kappa| = \kappa'$ and $|N \cap \lambda| < \lambda'$ and $N \cap \lambda' \in \lambda'$. By Proposition 3.9, we see that this ideal is λ'-complete. If $\lambda' = \mu^+$ then $\mathrm{CC}((\kappa, \lambda), (\kappa', {<}\lambda'))$ is equal to $\mathrm{CC}((\kappa, \lambda), (\kappa', \mu))$ restricted to those z such that $\mu \subseteq z$.

Stipulation. These examples help illustrate the idea that the ideal of strongly nonstationary sets is a generalization of several particular notions of the "nonstationary ideal". By restricting the strongly nonstationary ideal to particular weakly stationary sets we recover the particular ideals. **We will use this as a rationalization to drop the adjective "strongly", and refer to this ideal as "the ideal of nonstationary sets" and its dual filter as the "filter of closed unbounded sets".** When the underlying set X is clear from the context, we will denote the nonstationary ideal by simply NS and this ideal restricted to a (weakly) stationary set $Y \subseteq P(X)$ by $\mathrm{NS}\restriction Y$ or NS_Y. In particular, for regular uncountable cardinals κ, NS_κ will denote the usual ideal of nonstationary subsets of κ.

The closed unbounded filter is the minimal normal and fine filter:

3.12 Lemma. *Suppose that I is a normal, fine, countably complete ideal on a set $Z \subseteq P(X)$. If $A \subseteq Z$ is nonstationary, then $A \in I$.*

Proof. Suppose that $\mathfrak{A} = \langle X, f_n \rangle_{n \in \omega}$ is an algebra witnessing the nonstationarity of A, and suppose that $A \notin I$. Then no element of A is closed under every function from $\mathfrak{A}$. Hence for each $z \in A$, there is a function f_n and $\{x_0, \ldots, x_{n-1}\} \subseteq z$ such that the point $f_n(x_0, \ldots, x_{n-1}) \notin z$. For each $z \in A$ choose such a $(x_0, \ldots, x_{n-1})$. By the normality and countable completeness of I, there is an I-positive set $B \subseteq A$ such that n and $(x_0, \ldots, x_{n-1})$ are constant for all $z \in B$. Let $x = f_n(x_0, \ldots, x_{n-1})$. Then by fineness, there is a $z \in B$ such that $x \in z$, a contradiction. $\dashv$

In Sect. 4.4 we show how to condition the nonstationary ideal on an ideal on a smaller set, while preserving properties similar to Lemma 3.12.[12]

Natural Ideals on $P(\mathbb{R})$

The next two examples of ideals are quite well-known:

3.13 Example. The ideal of null sets for a countably additive measure on the unit interval. This ideal becomes precipitous if the measure is defined on all subsets of the unit interval. In this case the ideal is $\aleph_1$-saturated and countably complete, hence has the disjointing property.

[11] The definition of a *projection* is officially given by Definition 4.17.

[12] The conditional nonstationary ideals are analogous to the ideals arising from conditional probability functions analysis.

This example is not totally misrepresentative; a consequence of the Hahn-Banach theorem is that if I is an ideal on Z then there is a finitely additive probability measure $\mu : P(Z) \to [0,1]$, such that I is the ideal of sets of μ-measure zero. Inspired by this remark, in the sequel we will use phrases such as "almost all" and "almost every", and "in a set of measure one" to refer to sets in $\breve{I}$. So for example, "almost every element of $X \ldots$" means that "there is a set $C \in \breve{I}$ such that for all $z \in C \cap X \ldots$".

3.14 Example. The ideal of meager subsets of the unit interval.

It is not known if either the ideal of Lebesgue null sets or the ideal of meager sets can be precipitous. However Komjáth [73] has shown that it is consistent, relative to a measurable cardinal, that there is a non-meager set $A \subseteq \mathbb{R}$ such that $P(A)/\{\text{meager sets}\}$ is c.c.c. It follows that the ideal of meager subsets of A forms a precipitous ideal on $P(A)$. We give a sketch of Komjáth's argument in Sect. 8.

$I[\lambda]$ and Related Ideals

We now describe several interesting examples due to Shelah. The first is the ideal $I[\lambda]$. This example is particularly interesting on successors of singular cardinals. Doing justice to the significance of this ideal is beyond the scope of this chapter. The author notes that the exposition here is not as thorough as the treatment of these ideals given by Shelah.[13]

3.15 Example. Let λ be a regular cardinal. Let θ be a regular cardinal "much larger" than λ and Δ a well-ordering of $H(\theta)$. A generating set for the ideal $I[\lambda]$ is determined by a structure $\mathfrak{A}$ in a countable language that expands the structure $\langle H(\theta), \in, \Delta\rangle$.

An ordinal α is *approachable* with respect to $\mathfrak{A}$ iff $\mathrm{Sk}^{\mathfrak{A}}(\alpha) \cap \lambda = \alpha$ and there is a sequence $\langle \alpha_i : i \in \mathrm{cf}(\alpha)\rangle$ that is cofinal in α and is such that for all $j < \mathrm{cf}(\alpha)$, $\langle \alpha_i : i < j\rangle \in \mathrm{Sk}^{\mathfrak{A}}(\alpha)$.

The set $S_{\mathfrak{A}}$ determined by $\mathfrak{A}$ is the collection of ordinals α such that either $\mathrm{Sk}^{\mathfrak{A}}(\alpha) \cap \lambda \neq \alpha$ or α is approachable with respect to $\mathfrak{A}$. We let $I[\lambda]$ be the ideal generated by all of the $S_{\mathfrak{A}}$'s.

Clearly this ideal includes the ideal of bounded subsets of λ. Moreover we get an equivalent definition if we allow the structures $\mathfrak{A}$ to vary over expansions of $H(\theta)$ that have languages of size less than λ.

Since λ is regular, we can merge any small collection $\langle \mathfrak{A}_\delta : \delta < \eta < \lambda\rangle$ of structures in various languages of cardinality less than λ into a single structure $\mathfrak{A}$, in a language of cardinality less than λ. An ordinal $\alpha > \eta$ that is approachable with respect to some $\mathfrak{A}_\delta$ remains approachable with respect to $\mathfrak{A}$. Thus we can argue that the ideal $I[\lambda]$ is λ-complete.

[13] See Shelah's book [105], and his many related papers, for extensive information. An exposition of these results is given in Eisworth's chapter in this Handbook.

Arguments similar to those given in Lemma 3.3 can be extended to show that the ideal $I[\lambda]$ is normal and extends the nonstationary ideal on λ.

This ideal may fail to be proper however. For example, if there is a square sequence $\langle C_\alpha : \alpha < \lambda\rangle$ of length λ then the collection of ordinals that are approachable with respect to the algebra $\mathfrak{A} = \langle H(\theta), \in, \Delta, \langle C_\alpha : \alpha < \lambda\rangle\rangle$ is closed unbounded. Hence the ideal $I[\lambda]$ is not a proper ideal.

To see this, suppose that we have a typical α and we assume that C_α is closed and unbounded in α and has order type less than α. Using the coherence of the square sequence, we see that every initial segment of C_α is in $\mathrm{Sk}^{\mathfrak{A}}(\alpha)$. If $\gamma = \mathrm{ot}(C_\alpha)$ then $\mathrm{Sk}^{\mathfrak{A}}(\alpha)$ contains a closed unbounded subsequence $D \subseteq \gamma$ that has order type the cofinality of α. Copying D over inside C_α, we get a sequence cofinal in α and of order type the cofinality of α such that every initial segment is in $\mathrm{Sk}^{\mathfrak{A}}(\alpha)$.

Similar arguments show for example, that if GCH holds and κ is regular then $I[\kappa^+]$ is not proper.

Shelah has shown that if $\mu^+ < \lambda$ and μ is regular, then there is always a stationary set of approachable ordinals of cofinality μ. Moreover, if $\lambda^{<\lambda} = \lambda$, $I[\lambda]$ is the nonstationary ideal restricted to a single stationary set that contains ordinals of all cofinalities less than λ. As a consequence, we can view the approachable ordinals as being a canonical stationary set. This stationary set is closed unbounded just in case the ideal is not a proper ideal.

It is consistent that the approachable ordinals constitute a co-stationary set. To make this happen at μ^+, where μ is regular, one collapses a weakly compact cardinal greater than μ to be the successor of μ in the manner of Mitchell's model [96] for no Aronszajn trees on μ^+. This result is presented in Cummings' chapter in this Handbook. At the successor of a singular cardinal larger than a supercompact, the approachable ordinals are always co-stationary. To arrange a model where the approachable ordinals are co-stationary in $\aleph_{\omega+1}$, one Levy collapses a supercompact cardinal to be $\aleph_2$.[14]

An important property of $I[\lambda]$ is the following:

3.16 Theorem (Foreman and Magidor [42]). *Let $\kappa < \lambda$ be regular cardinals such that $\kappa^{<\kappa} = \kappa$. Let $S \subseteq \lambda \cap \mathrm{Cof}(\kappa)$ be a stationary set. Then the following are equivalent:*

1. *For all $<\kappa^+$-closed partial orderings $\mathbb{P}$ and for all generic $G \subseteq \mathbb{P}$, $V[G] \models S \subseteq \lambda$ is stationary.*
2. *There is a set $T \in I[\lambda]$ such that $S \cap T$ is stationary.*

We remind the reader that a set N of regular cardinality κ is *internally approachable of length* κ iff there is a sequence $\langle N_\alpha : \alpha < \kappa\rangle$ such that:

1. $|N_\alpha| < \kappa$,
2. for all $\alpha < \beta < \kappa$, $N_\alpha \subseteq N_\beta \subseteq N$,

[14] See [43] for further information and references.

3. for all $\beta < \kappa$, $\langle N_\alpha : \alpha < \beta \rangle \in N$, and
4. $N = \bigcup_{\alpha<\kappa} N_\alpha$.

Another natural ideal discovered by Shelah is closely related to $I[\lambda]$.

3.17 Example. Let $\lambda > \kappa$ be regular cardinals. We define the ideal $I_d(\lambda, \kappa)$. Fix a large regular $\theta \gg \lambda$. $I_d(\lambda, \kappa)$ will be generated by sets $S_{\mathfrak{A}}$ as $\mathfrak{A}$ ranges over expansions of $\langle H(\theta), \in, \Delta \rangle$ in a countable language. Given such a structure, we define $S_{\mathfrak{A}}$ to be the collection of δ such that there is no internally approachable $N \prec \mathfrak{A}$ of cardinality and length κ with $\sup(N \cap \lambda) = \delta$.

Familiar arguments show that $I_d(\lambda, \kappa)$ is a proper, normal ideal on λ. The ideal $I_d(\lambda, \kappa)$ is orthogonal to the ideal $I[\lambda]\restriction\mathrm{Cof}(\kappa)$ in the following sense:

3.18 Lemma (Shelah). *If $S_1 \in I[\lambda]\restriction\mathrm{Cof}(\kappa)$ and $S_2 \in I_d(\lambda, \kappa)$ then $S_1 \cap S_2$ is nonstationary.*

See [43] for a proof of this lemma.

Club Guessing Ideals

We now turn to the class of *guessing* ideals. We exhibit some samples, the *club guessing ideals*. Here, as elsewhere, we use $\subseteq^*$ to mean eventual inclusion.

3.19 Example. Let $\kappa > \mu$ be regular cardinals and $S \subseteq \kappa \cap \mathrm{Cof}(\mu)$. Let $\langle C_\alpha : \alpha \in S \rangle$ be a sequence such that C_α is unbounded in α. We define two filters, the *club guessing filter* and the *tail club guessing filter* on S.

For $D \subseteq \kappa$ closed unbounded, let $G(D) = \{\alpha \in S : C_\alpha \subseteq D\}$ and $E(D) = \{\alpha \in S : C_\alpha \subseteq^* D\}$.

1. The club guessing filter on S is the filter generated by the sets $\{G(D) : D \text{ is closed unbounded}\}$ together with the closed unbounded sets.
2. The tail club guessing filter on S is the filter generated by the sets $\{E(D) : D \text{ is closed unbounded}\}$ together with the closed unbounded sets.

The sequence $\langle C_\alpha : \alpha \in S \rangle$ is *club guessing* iff the club guessing filter on S is a proper filter, and *tail club guessing* iff the tail club guessing filter on S is a proper filter.

The next lemma was observed by Ishiu; the third clause was proved earlier in Shelah's work [105]:

3.20 Lemma. *Let $\langle C_\alpha : \alpha \in S \rangle$ be a club guessing sequence. Then:*

1. *The smallest normal filter containing the club guessing filter is not proper.*

2. *The club guessing filter is incompressible.*[15]

3. *The tail club guessing filter is normal.*

It was shown by Ishiu [59] that if one collapses a Woodin cardinal to be the successor of a regular cardinal μ, then every tail club guessing filter is precipitous. Foreman and Komjáth [40] showed that it is consistent relative to an almost huge cardinal to have a club guessing filter on any given regular cardinal μ be μ^+-saturated.

Example 3.19 can easily be generalized: If $\mathcal{F}$ is a filter on a regular cardinal κ and $\langle C_\alpha : \alpha \in \kappa\rangle$ is any sequence, we can define a new filter $\mathcal{G}$ generated by sets of the form:

$$G(A) = \{\alpha < \kappa : C_\alpha \subseteq^* A \cap \alpha\}$$

for $A \in \mathcal{F}$. If $\mathcal{F}$ is normal, then $\mathcal{G}$ is normal. If $\mathcal{F}$ is a normal filter and $\langle C_\alpha\rangle$ is a diamond sequence that guesses on positive sets for $\mathcal{F}$, then the corresponding filter $\mathcal{G}$ is proper.

Ideals of Sets Without Guessing Sequences

A different kind of ideal is given by the non-diamond ideal.

3.21 Example. Let κ be a regular cardinal. Recall that for $A \subseteq \kappa$, $\langle S_\alpha : \alpha \in A\rangle$ is a $\Diamond(A)$-*sequence* iff for any $X \subseteq \kappa$, $\{\alpha \in A : X \cap \alpha = S_\alpha\}$ is stationary in κ, and $\Diamond_\kappa$ is the assertion that there is a $\Diamond(\kappa)$ sequence. Let I be the collection of sets $A \subseteq \kappa$ such that there is no diamond sequence defined on A. Explicitly, $A \in I$ iff there is no $\langle S_\alpha : \alpha \in A\rangle$ that is a $\Diamond(A)$ sequence. I is called the *non-diamond ideal.*

Let $\langle\ ,\ \rangle : \kappa \times \kappa \to \kappa$ be a bijective pairing function and $\langle S_\alpha : \alpha \in B\rangle$ be a $\Diamond(B)$ sequence. Suppose that $B = \bigtriangledown\langle B_\gamma : \gamma < \kappa\rangle$. Define $\langle S^\beta_\alpha : \alpha \in B_\beta\rangle$ by $S^\beta_\alpha = \{\gamma < \alpha : \langle\beta, \gamma\rangle \in S_\alpha\}$. Then it is easy to see that there is a β such that $\langle S^\beta_\alpha : \alpha \in B_\beta\rangle$ is a $\Diamond(B_\beta)$ sequence. This establishes that this ideal is normal and κ-complete.

For the next proposition we need $\Diamond_\kappa$ to see that the non-diamond ideal is proper.

3.22 Proposition (Abraham-Magidor). *Suppose that κ is a regular cardinal and $\Diamond_\kappa$. Then the non-diamond ideal on κ is nowhere κ^+-saturated.*

Proof. We show that if there is an almost disjoint family of subsets of κ that has size λ, then below any positive set there is an antichain of size λ. Fix a bijective pairing function $\langle\ ,\ \rangle : \kappa \times \kappa \to \kappa$.

[15] This means that there is a minimal non-constant function $f : \kappa \to \kappa$ with respect to the club guessing filter.

If A is a positive set, then $\diamondsuit(A)$ holds. Let $\langle S_\alpha : \alpha \in A\rangle$ be a $\diamondsuit(A)$ sequence. For $Y \subseteq \kappa$ define $S^Y_\alpha = z$ iff $z \subseteq \alpha$ is such that S_α is the image of $(Y \cap \alpha) \times z$ under the map $\langle\ ,\ \rangle$. Let $A_Y = \{\alpha : S^Y_\alpha$ is defined$\}$. We claim that for all $Y \subseteq \kappa$, $\langle S^Y_\alpha : \alpha \in A_Y\rangle$ is a $\diamondsuit(A_\gamma)$ sequence.

Let $X \subseteq \kappa$. Let X' be the image of $Y \times X$ under $\langle\ ,\ \rangle$. Then the set of α such that the $\langle\ ,\ \rangle$-image of $(Y \cap \alpha) \times (X \cap \alpha)$ is $X' \cap \alpha$ is closed unbounded. Choose an α where this holds and where $S_\alpha = X' \cap \alpha$. Then S^Y_α is defined and $S^Y_\alpha = X \cap \alpha$.

To finish the non-saturation argument note that if $Y \cap Y'$ has cardinality less than κ, then $A_Y \cap A_{Y'}$ is nonstationary. Hence taking a large almost disjoint family of Y's yields a large antichain in the non-diamond ideal. ⊣

There is another example of a natural normal ideal, the non-weak diamond ideal.

Recall that weak diamond at λ is the statement that for all $F : 2^{<\lambda} \to 2$ there is a $g : \lambda \to 2$ such that for all $f : \lambda \to 2$, the collection $\{\alpha : g(\alpha) = F(f\restriction\alpha)\}$ is stationary in λ. The idea is that g is guessing information about F. Devlin and Shelah [22] showed that weak diamond holds at a successor cardinal $\lambda = \kappa^+$ iff $2^\kappa < 2^\lambda$.

3.23 Example. Let λ be a regular cardinal. Let $F : 2^{<\lambda} \to 2$. We can define an ideal I_F by putting $S \in I_F$ iff for all $g : \lambda \to 2$ there is an $f : \lambda \to 2$ such that $\{\alpha \in S : g(\alpha) = F(f\restriction\alpha)\}$ is nonstationary. The union of the ideals I_F can be verified to be a λ-complete ideal, which we call the *non-weak diamond ideal*. The ideal consists of those sets on which weak diamond fails. Thus it is a proper ideal just in case weak diamond holds at λ.

3.24 Theorem (Devlin-Shelah [22]). *Let λ be a regular uncountable cardinal. Then the non-weak diamond ideal is a normal λ-complete ideal.*

Proof. We will verify the normality of the ideal. The proof of λ-completeness is similar, but easier. Suppose that $\langle S_\alpha : \alpha < \lambda\rangle$ is a sequence of sets in the non-weak diamond ideal. We need to see that $S =_{\mathrm{def}} \triangledown_{\alpha<\lambda} S_\alpha$ is in the non-weak diamond ideal. Fix $\langle F_\alpha : \alpha < \lambda\rangle$ such that F_α witnesses that S_α is small.

Let $\langle\ ,\ \rangle : \lambda \times \lambda \to \lambda$ be a pairing function. We will work on the closed unbounded collection C of ξ such that $\langle\ ,\ \rangle : \xi \times \xi \to \xi$ is a bijection. For $\gamma : \xi \to 2$ we let $\gamma^\alpha(\beta) = \gamma(\langle\alpha, \beta\rangle)$. We define a function F as follows: for $\xi \in \triangledown S_\alpha \cap C$, let $\alpha < \xi$ be the least ordinal such that $\xi \in S_\alpha$. Define $F(\gamma) = F_\alpha(\gamma^\alpha)$.

Suppose now that there is a g such that for all $f : \lambda \to 2$, there are stationarily many $\xi \in S$ such that $g(\xi) = F(f\restriction\xi)$. Fix such a g. Then for each $\alpha < \lambda$ there is an f_α such that the collection of $\xi \in S_\alpha$ with $g(\xi) = F_\alpha(f_\alpha\restriction\xi)$ is nonstationary. Define $f : \lambda \to \lambda$ by setting $f(\langle\alpha, \beta\rangle) = f_\alpha(\beta)$. Note that for this $f, (f\restriction\xi)^\alpha = f_\alpha\restriction\xi$.

Since there are stationarily many $\xi \in S$ with $g(\xi) = F(f\restriction\xi)$, there is an α such that for stationarily many $\xi \in S_\alpha \setminus \bigcup_{\alpha'<\alpha} S_{\alpha'}$ we have $g(\xi) = F(f\restriction\xi)$.

But then for these ξ,

$$g(\xi) = F(f{\restriction}\xi) = F_\alpha((f{\restriction}\xi)^\alpha) = F_\alpha(f_\alpha{\restriction}\xi)$$

a contradiction. ⊣

Shelah has generalized the non-weak diamond ideal to the analogous ideal for colorings into n colors for $n \in \omega$. Surprisingly, it is not true in general that the ideals are identical for different finite numbers of colors. We refer the reader to [26] and [103] for further information.

Uniformization Ideals

Barney considered closely related ideals, the *uniformization ideals*. We describe his work on such ideals on ω_1. Let $S \subseteq \omega_1$ be a set of limit ordinals. We consider a *ladder system on* S to be a sequence of functions η_δ for $\delta \in S$ where η_δ is an increasing and cofinal map from ω into δ. An α*-coloring* of the ladder system is a sequence of functions $\langle c_\delta : \delta \in S\rangle$ where $c_\delta : \omega \to \alpha$. If $f : \omega \to \omega$, then an f-coloring is an ω-coloring such that for all $\delta \in S$ and $n \in \omega$, $c_\delta(n) \in f(n)$. A *monochromatic* α*-coloring* is an α-coloring with the property that each c_δ is a constant function.

An α-coloring is *uniformized* if there is an $h : \omega_1 \to \alpha$ such that for all $\delta \in S$ and all but finitely many n, $h(\eta_\delta(n)) = c_\delta(n)$.

We let Unif_α be the collection of sets S such that there is a ladder system on S such that every α-coloring $\langle c_\delta : \delta \in S\rangle$ of the ladder system can be uniformized. The sets $\mathrm{mUnif}_\alpha, \mathrm{Unif}_f$ are defined similarly for monochromatic colorings and f-colorings.

Shelah showed that weak diamond on a set S implies that for every ladder system on S, there is a monochromatic 2-coloring that is not uniformizable. Hence mUnif_2 is a subset of the weak diamond ideal, which is denoted WD_2 in what follows to emphasize 2-colorings.

Barney proved:

3.25 Proposition (Barney). *For $\alpha \in \omega + 1$ and $f : \omega \to \omega$, the sets* Unif_α, mUnif_α, *and* Unif_f *are normal, countably complete ideals. Moreover:*

$$\mathrm{Unif}_\omega \subseteq \mathrm{Unif}_f \subseteq \mathrm{Unif}_n \subseteq \mathrm{Unif}_2 \subseteq \mathrm{mUnif}_2 \subseteq \mathrm{WD}_2.$$

Unlike the case for the weak diamond ideal, it is not possible to separate the uniformization ideals for finitely many colors or even f-colorings:

3.26 Proposition (Barney). *Let $f : \omega \to \omega \setminus \{0, 1\}$. Then* $\mathrm{Unif}_2 = \mathrm{Unif}_f$.

However this is all one can prove:

3.27 Theorem (Barney). *There is a partial ordering $\mathbb{P}$ such that in $V^{\mathbb{P}}$,*

$$\mathrm{W}D_2 \neq \mathrm{mUnif}_2 \neq \mathrm{Unif}_2 \neq \mathrm{Unif}_\omega,$$

i.e. all of the inclusions are proper.

Weakly Compact and Ineffable Ideals

There is a family of natural ideals definable on large cardinals, such as ineffable cardinals. We give two examples here, one is the weakly compact filter.

Recall that κ is weakly compact iff for all Π^1_1 formulas, $\phi(x,y)$, $a \in V_\kappa$, and $A \subseteq V_\kappa$, if $V_\kappa \models \phi(A,a)$, then there is a stationary set of $\alpha < \kappa$ such that $V_\alpha \models \phi(A \cap V_\alpha, a)$.

3.28 Example. Suppose that κ is weakly compact. The *weakly compact filter* on κ is generated by sets of the form $R = \{\alpha : V_\alpha \models \phi(A \cap V_\alpha, a)\}$, where $A \subseteq V_\kappa$, $a \in V_\kappa$, ϕ is a Π^1_1 formula and $V_\kappa \models \phi(A,a)$. This filter is proper, normal and κ-complete. The *weakly compact ideal* is the dual ideal to the weakly compact filter.

We show in Proposition 6.4 that the weakly compact filter on a weakly compact cardinal κ is not κ-saturated. A. Hellsten, in unpublished work, has shown that it is consistent that there be weakly compact cardinal κ such that the weakly compact ideal on κ is κ^+-saturated.

A variant on the weakly compact ideal is the ineffable ideal.

3.29 Example. Let κ be an ineffable cardinal. Then for all sequences $\vec{A} = \langle A_\alpha : \alpha < \kappa \rangle$ with $A_\alpha \subseteq \alpha$, there is a stationary set $S_{\vec{A}}$ such that for $\alpha < \beta$ both in $S_{\vec{A}}$, $A_\alpha = A_\beta \cap \alpha$. Then the collection of $S_{\vec{A}}$ generate a normal κ-complete filter on κ.

As far as the author knows the properties of the generic ultrapowers by this filter have not been investigated.

3.2. Induced Ideals

Most induced ideals arise as special cases of the following observation:

3.30 Example. Let X, Z be sets with $Z \subseteq P(X)$. Let $\mathbb{Q}$ be a partial ordering such that for all $H \subseteq \mathbb{Q}$ that are generic, the $\mathbb{Q}$-term $\dot{U}$ denotes a V-normal, fine, V-κ-complete filter on $P(Z)^V$. Then in V, we can define an ideal I by setting $A \in I$ iff every condition in $\mathbb{Q}$ forces that A is in the dual of $\dot{U}$. This ideal is easily seen to be normal, fine and κ-complete in V. Moreover for all generic $H \subseteq \mathbb{Q}$, $\dot{U}^{V[H]} \supseteq \check{I}$. If $\mathbb{Q}$ is κ-c.c. then the Boolean algebra $P(Z)/I$ is κ-c.c.

The example is a special case of a more general fact: If $\mathcal{B}$ is a Boolean algebra and $\dot{U}$ is a term for an ultrafilter on $\mathcal{B}$ in a κ-c.c. forcing extension $\mathbb{P}$ and we let I be the ideal of elements of b such that $1 \Vdash_{\mathbb{P}} b \notin \dot{U}$, then $\mathcal{B}/I$ is κ-c.c.

In slightly more generality we can see:

3.31 Lemma. *Let X, Z be sets with $Z \subseteq P(X)$. Suppose that $\mathbb{P}$ is a κ-c.c. partial ordering and $\dot{J}$ is a $\mathbb{P}$ term for a κ-saturated ideal on $P(Z)^{V^{\mathbb{P}}}$.*

Define an ideal I in V by setting $A \in I$ iff $\|A \in \dot{J}^+\|_{\mathbb{P}} = 0$. Then I is a κ-saturated ideal on $P(Z)$. Moreover if $V^{\mathbb{P}} \models$ "$\dot{J}$ is normal and fine", then I is a normal and fine ideal.

We note that the statement $\|A \in \dot{J}^+\|_{\mathbb{P}} = 0$ is equivalent to $\|A \in \dot{J}\|_{\mathbb{P}} = 1$. In certain circumstances this version is easier to work with.

In many situations the filter U is an ultrafilter associated with an elementary embedding that is defined in the extension $V[H]$:

3.32 Definition. Let M and N be models of a sufficient amount of set theory[16] with $i \in N$ and $Z \in M$. If $j : M \to N$ is an elementary embedding and $i \in j(Z)$ then i generates an ultrafilter $U(j, i)$ on $P(Z)^M$. Namely, for $A \subseteq Z$

$$A \in U(j, i) \quad \text{iff} \quad i \in j(A).$$

In our previous language, the element i is functioning as an "ideal element". This ultrafilter can be used to induce ideals in various ways. We begin with a typical example that follows immediately in one direction from Lemma 3.31 applied to $U(j, \kappa)$ and in the other by taking a generic ultrapower.

3.33 Example. The following are equivalent:

1. There is an elementary embedding $j : V \to M \subseteq V[G]$ defined in $V[G]$ such that:
   (a) $\mathrm{crit}(j) = \kappa$, and
   (b) $G \subseteq \mathbb{P}$ is V-generic where $\mathbb{P}$ is κ^+-c.c.
2. There is a normal, fine, κ-complete, κ^+-saturated ideal on $P(\kappa)$.

The next example is very general.

3.34 Example. Suppose that N expands $\langle H(\lambda), \in, \Delta\rangle$ where Δ is a well-ordering of $H(\lambda)$. Suppose that M is the transitive collapse of an elementary substructure of N and j is the inverse of the collapsing map. Then the M-ultrafilter $U(j, i)$ induces a filter (and hence an ideal) on $P(Z)^V$. If the critical point of j is α, then this filter is M-α-complete. If $\kappa < \alpha$ and $M^{<\kappa} \subseteq M$, then this filter is κ-complete.

This example was used very fruitfully in the case $i = \mathrm{crit}(j)$ by Baumgartner, Hajnal and Todorčević [7].

Continuing with an elementary embedding $j : M \to N$:

3.35 Definition. Suppose that $\mathbb{P}$ is a partial ordering in M. A condition $m \in j(\mathbb{P})$ is called a *master condition* (or an *M-generic condition*) iff for all dense subsets D of $\mathbb{P}$ with $D \in M$ and all $q \leq m$ belonging to $j(\mathbb{P})$ there is a $p \in D$ such that $j(p)$ is compatible with q.

[16] ZF$^-$, ZF minus Power Set, is more than enough.

If there is a master condition for j and $H \subseteq j(\mathbb{P})$ is a filter, we define a filter $G \subseteq \mathbb{P}$ by setting $p \in G$ iff $j(p) \in H$.[17] If H is sufficiently generic then G is generic over M and the elementary embedding j can be extended to an elementary embedding

$$\hat{j} : M[G] \to N[H].$$

This idea comes up in several contexts: If $\mathbb{Q}$ is a partial ordering and λ is a regular cardinal much bigger than $|\mathbb{Q}|$, let $N = \langle H(\lambda), \in, \Delta, \mathbb{Q}\rangle$ and M be the transitive collapse of a countable elementary substructure and $\mathbb{P}$ be the collapse of $\mathbb{Q}$. Let j be the inverse of the collapse map. Then the notion of a master condition coincides exactly with the notion of a generic condition in the sense of proper forcing. The properness of $\mathbb{Q}$ is equivalent to the statement:

> For all such M and all $p \in \mathbb{P}$ there is an $m \leq j(p)$ such that for all generic $H \subseteq \mathbb{Q}$, with $m \in H$, if $G = j^{-1}(H)$ then G is M generic and j extends to an elementary embedding
>
> $$\hat{j} : M[G] \to H(\lambda)[H].$$

Semiproperness can be formulated similarly. The link between properness, semiproperness and ideals is not a coincidence [47].

The next example is the usual way that strong ideals are manufactured from the remnants of a large cardinal using master conditions and a forcing construction.

3.36 Example. Suppose that $j : V \to M$ is a large cardinal embedding such that $j``X \in M$. Let $Z \subseteq P(X)$ be such that $j``X \in j(Z)$. Suppose that $\mathbb{P} \in V$ is a partial ordering and $m \in j(\mathbb{P})$ is a master condition. Then forcing over V with $j(\mathbb{P})$ below m we get a generic $H \subseteq j(\mathbb{P})$ such that $G = j^{-1}(H) \subseteq \mathbb{P}$ is generic over V. Then j can be extended to an elementary embedding $\hat{j} : V[G] \to M[H]$.

In $V[G]$ we define an ideal on Z by setting $A \in I$ iff whenever $H \subseteq j(\mathbb{P})$ is generic and extends $j``G \cup \{m\}$ we have $A \notin U(\hat{j}, j``X)$. This ideal is called the *master condition ideal*. If we let $\mathbb{Q}$ be the partial ordering consisting of those conditions in $j(\mathbb{P})/j``G$ that are below m, we have:

$$A \in I \quad \text{iff} \quad \|A \in U(\hat{j}, j``X)\|_{\mathbb{Q}} = 0.$$

Thus we are in the context of Example 3.30.

In Example 7.7 we will see that if we collapse a supercompact cardinal to be ω_1 then every proper forcing in $V[G]$ of cardinality κ gives rise to a precipitous master condition ideal on $[\lambda]^{<\omega_1}$ for every $\lambda > 2^\kappa$.

[17] We will abuse notation and write $j^{-1}(H)$ for this G.

General Induced Ideals

We now summarize this discussion by describing the general situation in which induced ideals arise.

3.37 Example. Let $H \subseteq \mathbb{Q}$ be generic and suppose that in $V[H]$ there is an elementary embedding $j : V \to M \subseteq V[H]$, where M is transitive. Let i be a $\mathbb{Q}$-name for an element of M and Z a set such that $\|i \in j(Z)\|_{\mathbb{Q}} = 1$. Then in V we can define an ideal $I \subseteq P(Z)$ by setting $A \in I$ iff every condition in $\mathbb{Q}$ forces $A \notin U(j, i)$. For all generic $H \subseteq \mathbb{Q}$ the ultrafilter $U(j, i)$ extends $\breve{I}$.

3.38 Proposition. *Assume the hypotheses of Example 3.37. Suppose that for all V-generic $G \subseteq \mathbb{Q}$, $V[G] \models i = j``X$. Then the ideal I is a normal, fine, countably complete ideal on $Z = P(X)$.*

3.39 Definition. We will call any elementary embedding j that is definable in a forcing extension of V a *generic* elementary embedding. We will call the ideal defined in Example 3.37 the ideal *induced by j and i.*[18] If I is an ideal induced by a generic elementary embedding we will call I *pre-precipitous.*

The dual of every pre-precipitous ideal can be extended to an ultrafilter lying in a generic extension that has a well-founded ultrapower.

There are examples of partial orderings $\mathbb{Q}$ and generic $G \subseteq \mathbb{Q}$ where $U(j, i)$ is not generic for the partial ordering $P(Z)/I$. In these examples, the pre-precipitous ideal I is not precipitous. This is discussed in Sect. 7.3.

Burke [12] showed that if I is *any* countably complete ideal on a set Z and there is a Woodin cardinal $\delta > |Z|$,[19] then there is a partial ordering $\mathbb{P}$ which produces an ultrafilter U on $P(Z)$ such that V^Z/U is well-founded, $\breve{I} \subseteq U$ and if $j : V \to N \cong V^Z/U$ is the canonical embedding, then $\mathrm{crit}(j) = \mathrm{comp}(I)$. A small improvement of his argument actually shows that every countably complete ideal is pre-precipitous. We show this in Proposition 9.44.

3.40 Proposition. *Suppose that I is an ideal on a set Z. Then the following are equivalent:*

1. *I is precipitous.*
2. *There is a partial ordering $\mathbb{P}$ such that for all generic $G \subseteq \mathbb{P}$ there is a $j : V \to M \subseteq V[G]$ and a $\mathbb{P}$-term i such that:*
   - *(a) I is the ideal induced by j and i, and*
   - *(b) $U(j, i)$ is generic for $P(Z)/I$.*

Proof. Suppose that I is precipitous. Let $\mathbb{P} = P(Z)/I$, $j : V \to M$ the generic ultrapower embedding and i a term for $[\mathrm{id}]^M$.

[18] Usually, $\mathbb{Q}$ is implicit in the definition of j.

[19] Woodin cardinals occur prominently in several Handbook chapters. We defer the actual definition to Definition 9.22 where we use it for the first time.

For the other direction, note that the map $\iota : P(Z) \to \mathcal{B}(\mathbb{P})$ defined by setting

$$\iota(A) = \|A \in U(j,i)\|$$

has kernel I. Hence we can view $\iota : P(Z)/I \to \mathcal{B}(\mathbb{P})$. The hypothesis (b) implies that ι is a regular embedding. By factoring $\mathcal{B}(\mathbb{P})$ as $P(Z)/I * \mathcal{B}(\mathbb{P})/\iota``U$ we see that whenever $U \subseteq P(Z)/I$ is generic there is a generic $G \subseteq \mathbb{P}$ such that $U(j,i)^{V[G]} = U$.

Let U be generic for $P(Z)/I$. We need to see that the ultrapower V^Z/U is well-founded. Let $G \subseteq \mathbb{P}$ be such that $U(j,i) = U$. Then there is a commuting diagram:

$$\begin{tikzcd} & V \arrow[dl, "j'"'] \arrow[dr, "j"] & \\ V^Z/U(j,i) \arrow[rr, "k"'] & & M \end{tikzcd}$$

where j' is the ultrapower map and k is defined by setting

$$k(f) = j(f)(i).$$

Since M is well-founded we see that $V^Z/U(j,i)$ is well-founded. $\dashv$

3.41 Remark. Any large cardinal embedding is a generic elementary embedding, as we can take $\mathbb{Q}$ to be the trivial partial ordering. The ultrafilters used to define standard large cardinal axioms are of the form $U(j,i)$ and hence their duals are "induced ideals" from the trivial partial ordering.

Goodness and Self-Genericity

A very powerful situation can arise when a natural ideal is simultaneously an induced ideal. For this to happen we need a criterion for genericity, which we give in the "reflected" form. Attempts to make natural ideals be induced[20] use the idea of self-genericity, to be defined below. The next example sets the stage for this important definition.

3.42 Example. Suppose that J is a countably complete ideal on a set $Z^* \subseteq P(X)$. Let θ be a regular cardinal much larger than the cardinality of Z^*, and suppose that $M' \prec \langle H(\theta), \in, \Delta, J, Z^*, \ldots \rangle$. Let M be the transitive collapse of M' and $j : M \to H(\theta)$ be the inverse of the collapse map. Suppose that $Z = j^{-1}(Z^*)$ and $I = j^{-1}(J)$. Letting $i = M' \cap X$, if $i \in Z^*$ we can build the M-ultrafilter $U(j,i) \subseteq P(Z)$.

We want to consider the ultrafilter $U(j,i)$ and its associated ideal both in the situation of extension via an elementary embedding $j : V \to M$ and in the situation of reflection via taking elementary submodels. For this reason we give the definition of "goodness" in a way that highlights the role of j.

[20] E.g. by making them precipitous.

3.43 Definition. The model M' is *good* iff

$$M' \cap X \in \bigcap\{j(A) : A \subseteq Z \text{ and } A \in \breve{I}\}.$$

We can say this another way: taking $i = M' \cap X$ the goodness of M' is equivalent to $U(j,i) \supseteq \breve{I}$. Proposition 3.44 shows that goodness is also equivalent to the condition that $M' \cap X \in \bigcap\{B : B \in \breve{J} \text{ and } B \in M'\}$. This latter condition is sometimes easier to verify in practice.

The point of the definition is that it describes what happens in every generic ultrapower. Let I be a normal, fine ideal I on $Z \subseteq P(\mu)$ for some $\mu \ll \theta$. Let $j : V \to N \cong V^Z/G \subseteq V[G]$ be the ultrapower map coming from a generic $G \subset P(Z)/I$. Then $M' =_{\text{def}} j``H(\theta)$ is a good elementary substructure of $j(H(\theta))$ for the ideal $J = j(I)$.

The next proposition shows that most $z \in Z^*$ generate good elementary substructures of $H(\theta)$.

3.44 Proposition. *Let $\mathfrak{A}$ be an algebra expanding $\langle H(\theta), \in, \Delta, J, Z^*, \ldots\rangle$ and suppose that J is a normal, fine, countably complete ideal. Then:*

1. *There is a set $C \in \breve{J}$ such that for all $z \in C$ if $M' = \mathrm{Sk}^{\mathfrak{A}}(z)$, then M' is good.*
2. *If M' is good, then $\{j(A) : A \in U(j,i)\} \supseteq \breve{J} \cap M'$, where $i = M' \cap X$.*

Proof. We can assume that $\mathfrak{A}$ has a complete set of Skolem functions. By normality there is a set $D \in \breve{J}$ such that for all $z \in D$, $\mathrm{Sk}^{\mathfrak{A}}(z) \cap X = z$. Suppose that there is a J-positive set B of counterexamples z. Using the countable completeness of J, we can assume that there is a particular Skolem function f such that for all $z \in B$ there is an $a \in [z]^{<\omega}$ such that $f(a)$ is a set in $\breve{J}$ and $z \notin f(a)$. Applying normality again, we can assume that there is a fixed a such that for all $z \in B$, $a \subseteq z$ and $z \notin f(a)$. But this contradicts $B \in J^+$.

For the second assertion, suppose that M' is good. By definition, $\breve{J} \cap M' = \{j(A) : A \in \breve{I}\}$ and since M' is good, $M' \cap X \in \bigcap\{j(A) : A \in \breve{I}\} = \bigcap\{B : B \in \breve{J} \cap M'\}$. Thus $M' \cap X$ belongs to every set in $\breve{J} \cap M'$ and hence $\breve{J} \cap M' \subseteq \{j(A) : A \in U(j,i)\}$. ⊣

3.45 Definition. If $U(j, M' \cap X)$ is generic over M for $P(Z)/I$, we say that M' is *self-generic*.

This idea first appeared in [47]. Note that a good M' is self-generic iff for every $A \subseteq P(Z^*)$ that is a maximal antichain in $P(Z^*)/J$ and lies in M', there is an $a \in A \cap M'$ such that $M' \cap X \in a$.[21] Equivalently, if $A \subseteq P(Z)$ belongs to M and is an M-maximal antichain in $P(Z)/I$ there is an $a \in A$ such that $M' \cap X \in j(a)$.

[21] See Definition 8.19 and Proposition 8.20 in Sect. 8.

A guiding idea for making various nonstationary ideals have some degree of saturation, presaturation or precipitousness is to use devices such as semiproperness to create some degree of self-genericity. The next lemma shows that goodness and self-genericity give a condition equivalent to saturation.

3.46 Lemma. *Let $Z^* \subseteq P(X)$ for some set X and J be a normal, fine, countably complete ideal on $P(Z^*)$. Suppose that $\mathfrak{A}$ is an algebra expanding $\langle H(\theta), \in, \Delta, J, Z^*, \ldots\rangle$.*

1. *If the ideal J is the nonstationary ideal on $P(X)$ restricted to a stationary set Z^*, then every $M' \prec \mathfrak{A}$ with $M' \cap X \in Z^*$ is good.*
2. *If the ideal J is normal, fine and $|X|^+$-saturated, then every good $M' \prec \mathfrak{A}$ is self-generic.*
3. *Suppose that every good $M' \prec \mathfrak{A}$ is self-generic and $|Z^*| \leq |X|$. Then J is $|X|^+$-saturated.*

Note that in most interesting cases $|Z^*| = |X|$ and in these cases, the third assertion is an exact converse to the second.

Proof. If $C \in \breve{I}$, then $C \in M$ and C is closed and unbounded. Let $\mathfrak{B} \in M$ be an algebra on $j^{-1}(X)$ such that every elementary substructure of $\mathfrak{B}$ is in C. Since $M' \cap X$ is an elementary substructure of $j(\mathfrak{B})$, we see that $M' \cap X \in j(C)$.

To establish the second assertion, suppose that $\mathcal{A} \subseteq P(Z)$ is a maximal antichain relative to I. Since I is $|X|^+$-saturated, $\mathcal{A}$ has cardinality at most $|X|$ and we can let $C = \triangledown\mathcal{A}$. Then C is in $\breve{I}$. Hence, by elementarity, $j(\mathcal{A})$ is a maximal antichain in $P(Z^*)/J$ and $j(C) \in \breve{J}$. Since M' is good, $M' \cap X \in j(C)$. By the definition of diagonal union, this implies that there is an $a' \in j(\mathcal{A}) \cap M'$ such that $M' \cap X \in a'$. Letting $a = j^{-1}(a')$, we see that $a \in U(j, M' \cap X) \cap \mathcal{A}$. Thus $U(j, M' \cap X)$ has non-empty intersection with every maximal antichain in $P(Z)/I$ that lies in M and hence $U(j, M' \cap X)$ is generic for $P(Z)/I$.

To establish the third assertion, indirectly assume that $\mathcal{A}$ is a maximal antichain in $P(Z^*)/J$ of size at least $|X|^+$. We can assume without loss of generality that $\mathcal{A}$ is definable in $\mathfrak{A}$. Let $C \in \breve{J}$ be such that for all $z \in C$, $\mathrm{Sk}^{\mathfrak{A}}(z)$ is good and $\mathrm{Sk}^{\mathfrak{A}}(z) \cap X = z$. Let $\mathcal{B} = \{a \in \mathcal{A} :$ for some $z \in C,\ a \in \mathrm{Sk}^{\mathfrak{A}}(z)\}$. Note that $\mathcal{B}$ has cardinality at most $|X| \cdot |Z^*| = |X|$. Hence we will be done if we can show that $\mathcal{B} \supseteq \mathcal{A}$.

If not, let $a \in \mathcal{A} \backslash \mathcal{B}$. For each $b \in \mathcal{B}$ choose a $D_b \in \breve{J}$ such that $D_b \cap b \cap a = \emptyset$. Let $D = \Delta_{b \in \mathcal{B}} D_b$, and $z \in (a \cap C \cap D)$. Since $\mathrm{Sk}^{\mathfrak{A}}(z)$ is self-generic there is a $c \in \mathcal{A} \cap \mathrm{Sk}^{\mathfrak{A}}(z)$ such that $z \in c$. Since $z \in C$ we have that $c \in \mathcal{B}$. Since $c \in z$, $z \in D_c$. But then $z \in D_c \cap c \cap a$, a contradiction. ⊣

A special case of the example above may be illustrative.

3.47 Example. Suppose that the nonstationary ideal on ω_1 is $\aleph_2$-saturated. Let θ be a large regular cardinal, and $M' \prec \langle H(\theta), \in, \Delta, \ldots \rangle$ be a countable elementary substructure. Let $\delta = M' \cap \omega_1$. Then $U(j, \delta)$, the *critical point ultrafilter*, is generic over M, the transitive collapse of M', for the partial ordering $P(\omega_1)/\mathrm{NS}_{\omega_1}$ as computed in M. Moreover, the ultrapower of M by $U(j, \delta)$ is isomorphic to $\mathrm{Sk}^{H(\theta)}(M' \cup \{\delta\})$.

Conversely, if $\lambda \geq (2^{\omega_1})^+$ and there is a closed unbounded set $C \subseteq [H(\lambda)]^{<\omega_1}$ such that every $M \in C$ is self-generic, then NS_{ω_1} is $\aleph_2$-saturated.

Example 3.47 foreshadows techniques from [47] that played an important role in work of Woodin.

4. A Closer Look

In this section we explore further the various properties an ideal can have. These include various saturation properties, both strengthening and weakening conventional saturation. Some of these are preserved under *projection*, an operation that is closely related to factoring the generic ultrapower through smaller generic ultrapowers. We will touch on *towers* of ideals, mostly in relation to ideals on fixed sets.

4.1. A Structural Property of Saturated Ideals

We begin with some theorems that give some insight into the structure of saturated ideals. These theorems are due to Baumgartner, Taylor and Wagon [8] in the case that the ideal is κ-complete. We present the first theorem in slightly greater generality.

4.1 Theorem. *Let κ be a regular cardinal. Suppose that $I \subseteq P(P(\kappa))$ is a normal, fine, countably complete, κ^+-saturated ideal. If I is contained in $J \subseteq P(P(\kappa))$ where J is a normal countably complete ideal, then there is an I-positive set A such that:*

$$J = I \restriction A.$$

In particular, J is saturated.[22]

Proof. Let $\gamma \leq \kappa$ and $\langle B_\alpha : \alpha < \gamma \rangle$ be a maximal I-antichain of elements of J. Let $B = \triangledown B_\alpha$. We claim that J is the ideal generated by I and B.

Since J is normal, $B \in J$. Hence $\overline{I \cup \{B\}} \subseteq J$. Suppose that there is a set $C \in J$ not in the ideal generated by $I \cup \{B\}$. Then $C \setminus B \notin I$. Replacing C by $C \setminus B$ we can assume that $C \cap B_\alpha \in I$ for all $\alpha < \gamma$. This shows that $\langle B_\alpha : \alpha < \gamma \rangle$ is not a maximal I-antichain of elements of J, a contradiction. $\dashv$

[22] We remind the reader that $I \restriction A$ can be viewed both as $I \cap P(A)$ and as the ideal generated by $I \cup \{P(\kappa) \setminus A\}$. See Sect. 2 for a discussion of this.

In the rarer situation where the ideal I is κ-complete and κ-saturated, normality is not needed:

4.2 Theorem. *Let κ be a regular cardinal. Suppose that I is a κ-complete, κ-saturated ideal on a set Z. If $J \supseteq I$ is a κ-complete ideal extending I, then there is a set $A \in I^+$ such that $J = I \restriction A$.*

Proof. Let $\langle A_\alpha : \alpha < \beta \rangle$ be a maximal antichain in I^+ of elements of J. Then $\beta < \kappa$, so $[\bigcup_{\alpha<\beta} A_\alpha]_I = \Sigma_{\alpha<\beta}[A_\alpha]$ in $P(X)/I$. Hence $J = I \restriction (Z \setminus A)$, where $A = \bigcup_{\alpha<\beta} A_\alpha$. ⊣

4.2. Saturation Properties

We now enumerate a hierarchy of saturation properties of ideals. Many of these properties were explored in papers by Baumgartner and Taylor [5, 6], although we have modified the terminology very slightly. The saturation properties of ideals are in direct correspondence to the chain condition properties of arbitrary partial orderings, for obvious reasons. Let $\mathbb{B} = P(Z)/I$.[23] We define some properties that $\mathbb{B}$ may have; we will say that an ideal I or a tower of ideals $\mathcal{T}$ has a property iff the corresponding partial ordering $\mathbb{B}$ has the property.

The partial ordering $\mathbb{B}$ is:

- κ-*dense* iff $\mathbb{B}$ has a dense subset of cardinality κ,
- (κ, λ)-*centered* iff $\mathbb{B} = \bigcup_{\alpha<\kappa} F_\alpha$, where each F_α is a λ-complete filter on $\mathbb{B}$. I is κ-*centered* iff I is $(\kappa, \aleph_0)$-centered,
- (κ, η, λ)-*saturated* iff every subset A of $\mathbb{B}$ having cardinality κ contains a subset B of cardinality η such that the meet (intersection) of any λ elements of B is non-zero, and
- κ-*linked* iff $\mathbb{B} = \bigcup_{\alpha<\kappa} F_\alpha$, where the intersection of any two elements of F_α is not in I.

These properties are listed in roughly descending order of strength, provided that the parameters are set correctly. For example, if I is κ-dense then I is (κ, λ)-centered for all λ. Hence $P(Z)$ can be written as a union of κ many filters that are comp(I)-complete. Similarly, if I is (κ, λ)-centered, then I is $(\kappa^+, \kappa^+, \gamma)$-saturated for all $\gamma < \lambda$. We leave it to the reader to explore the possibilities. These ideal properties were defined or used variously in [123, 83, 35, 126].

Taylor showed:

4.3 Theorem (Taylor [119, 120]). *The following are equivalent:*

1. *There is a countably complete, $\aleph_1$-dense ideal on ω_1.*

[23] As usual we will be sloppy in not carefully distinguishing between the Boolean algebra $P(Z)/I$ and the partial ordering $\langle P(Z)/I, \subseteq_I \rangle$, with the 0 element removed.

2. *(Ulam's Problem for normal ideals) There is a collection $\mathcal{I}$ of countably complete, normal ideals on ω_1 with $|\mathcal{I}| = \aleph_1$ such that for all $A \subseteq \omega_1$, there is an $I \in \mathcal{I}, A \in I \cup \breve{I}$.*

Moreover, both properties fail if MA_{ω_1} *holds.*

We now give a characterization of κ-dense ideals due to Shelah [102] in the particular case that an ideal on κ is normal and κ-complete. The equivalence of properties 1 and 2 is also implicit in Taylor [119]. Note the analogy between the third part of the characterization and (μ^+, μ^+, μ)-saturation.

4.4 Theorem. *Let I be a normal, fine, countably complete ideal on $Z \subseteq P(X)$ and suppose that $\kappa \leq |X|$ is a regular cardinal. Then the following are equivalent:*

1. *$P(Z)/I$ has a dense set of size $\leq \kappa$.*

2. *There are normal, fine, countably complete ideals $\langle J_i : i < \kappa \rangle$ such that $I \subseteq J_i$ for each i and every set in I^+ belongs to $\bigcup_i \breve{J}_i$.*

If in addition $2^\kappa = \kappa^+$, these are also equivalent to:

3. *Whenever $\{A_l : l \in \kappa^+\}$ is a collection of I-positive sets there is a set $L \subseteq \kappa^+$ of cardinality κ^+ such that the diagonal intersection of any $L' \subseteq L$ of size κ is I-positive.*

Proof. That property 1 implies property 2 is immediate.

Assume property 2, towards showing property 1. Then $P(Z)/I$ is κ^+-c.c. For each i, let $\{A_\beta : \beta \in \gamma\} \subseteq \breve{J}_i$ be maximal and strictly I-decreasing. Then $|\gamma| \leq \kappa$. Hence $A_i =_{\mathrm{def}} \Delta_{\beta \in \gamma} A_\beta \in \breve{J}_i$ and is the I-minimal element of $\breve{J}_i$, i.e. $J_i = I \restriction A_i$. Since every element of I^+ belongs to some $\breve{J}_i$, $\{[A_i]_I : i < \kappa\}$ is a dense subset of $P(Z)/I$.

Assume property 2, towards showing property 3. If we are given a sequence $\{A_l : l \in \kappa^+\}$ of I-positive sets, then there must be some j such that $L =_{\mathrm{def}} \{l : A_l \in \breve{J}_j\}$ has cardinality κ^+. Since J_j is normal, property 3 follows.

Assume property 3 and $2^\kappa = \kappa^+$, and towards showing property 1, that I does not have a dense set of size $\leq \kappa$. Then we can build a sequence of sets $\langle A_\alpha : \alpha < \kappa^+ \rangle$ such that:

1. if $D \in [\kappa^+]^\kappa$, and $\Delta_{\alpha \in D} A_\alpha \neq \emptyset$, then there is a $\beta < \kappa^+$ such that $A_\beta = \Delta_{\alpha \in D} A_\alpha$, and

2. for $\alpha < \beta$, $A_\alpha \nsubseteq_I A_{\beta+1}$.

Using property 3, we can find an unbounded set $L \subseteq \kappa^+$ of successor ordinals such that for all $D \in [L]^\kappa$, $\Delta_{\alpha \in D} A_\alpha \neq_I \emptyset$. For $\beta < \kappa^+$, let $B_\beta = \Delta_{L \cap \beta} A_\alpha$. By the κ^+-saturation of I there is a β such that for all $\beta' > \beta$, $B_\beta =_I B_{\beta'}$. Suppose that $B_\beta = A_\gamma$. Choose an $\alpha \in L \setminus (\gamma + 1)$. Then $B_\beta \subseteq_I A_\alpha$, since $B_\beta =_I B_{\alpha+1}$, but $A_\gamma \nsubseteq_I A_\alpha$, a contradiction. ⊣

The next few properties are weaker than saturation.

4.5 Definition. An ideal I, with corresponding $\mathbb{B} = P(Z)/I$, is:

- *κ-preserving* iff the forcing $\mathbb{B}$ preserves the cardinal κ,
- *cardinal preserving* iff $\mathbb{B}$ preserves all cardinals greater than or equal to $|Z|^+$ and below the completeness of I,
- *κ-presaturated* iff I is precipitous and κ-preserving,
- *presaturated* iff $|Z| = \kappa$ and $\mathbb{B}$ is κ^+-presaturated, and
- *weakly (λ, κ)-saturated* iff for any $\gamma < \lambda$ and any collection of antichains $\{\mathcal{A}_\alpha : \alpha < \gamma\}$ in $\mathbb{B}$ there is a dense collection of $Y \in \mathbb{B}$ such that for all $\alpha < \gamma$, $|\{b \in \mathcal{A}_\alpha : Y \wedge b \neq 0\}| < \kappa$.

The first and last properties are abstract properties of the partial ordering $\mathbb{B}$, and do not depend on $\mathbb{B}$ being of the form $P(Z)/I$. Indeed, the following is a forcing exercise:

4.6 Proposition. *Suppose that $\gamma < \kappa$ are regular cardinals and $\mathbb{P}$ is a partial ordering of size less than or equal to κ. Then the following are equivalent:*

1. *$\mathbb{P}$ is weakly (γ^+, κ)-saturated.*
2. *If $G \subseteq \mathbb{P}$ is generic, then $\mathrm{cf}(\kappa) > \gamma$ in $V[G]$.*

Proof. To see that weak saturation implies that $\mathrm{cf}(\kappa) > \gamma$, suppose that $b \Vdash$ "$\dot{f}$ is a term for a cofinal function from γ into κ". Let A_α be a maximal antichain below b deciding $\dot{f}(\alpha)$. Let $c \leq b$ be such that for all $\alpha < \gamma$, $|\{a \in A_\alpha : c \wedge a \neq 0\}| < \kappa$. Let $\xi = \sup_{\alpha<\gamma}\{\beta :$ there is an $a \in A_\alpha$ such that $c \wedge a \neq 0$ and $a \Vdash \dot{f}(\alpha) = \beta\}$. Then $c \Vdash$ "for all α, $f(\alpha) < \xi$", a contradiction.

Now suppose that $\langle A_\alpha : \alpha < \gamma\rangle$ is a sequence of maximal antichains and $b \in \mathbb{P}$. Enumerate A_α as $\{a_\alpha^\beta : \beta < |A_\alpha|\}$. Define a term $\dot{f}$ for a function by setting $\|\dot{f}(\alpha) = \beta\| = a_\alpha^\beta$. Let $c \leq b$ be such that for some $\xi < \kappa$, $c \Vdash$ "$\dot{f}$ is bounded by ξ". Then for all α, $|\{\beta : c \wedge a_\alpha^\beta \neq 0\}| \leq |\xi| < \kappa$. ⊣

4.7 Lemma. *Suppose that $\mathbb{P}$ is a partial ordering and κ is regular. Let $\rho < \kappa$ be the least cardinal such that there are maximal antichains $\langle \mathcal{A}_\alpha : \alpha < \rho\rangle$ such that the collection of Y such that for all $\alpha < \rho$, $|\{b \in \mathcal{A}_\alpha : Y \wedge b \neq 0\}| < \kappa$ is not dense. Then ρ is a regular cardinal.*

Proof. Suppose not. Let $\eta = \mathrm{cf}(\rho)$ and $\langle \rho_i : i \in \eta\rangle$ be a cofinal sequence in ρ. For each $i \in \eta$ we can define a maximal antichain $\mathcal{B}_i$ such that for all $Y \in \mathcal{B}_i$ and all $\alpha < \rho_i$, $|\{b \in \mathcal{A}_\alpha : Y \wedge b \neq 0\}| < \kappa$. Since $\eta < \rho$, we can find a dense collection D of Y such that for each $i \in \eta$, $|\{c \in \mathcal{B}_i : Y \wedge c \neq 0\}| < \kappa$.

Let $Y \in D$, and $\alpha < \rho$. Let $C_i = \{c \in \mathcal{B}_i : Y \wedge c \neq 0\}$. Let $\mu = \sup_{i\in\eta} |C_i|$. For each $c \in C_i$ let $\mathcal{A}_c = \{b \in \mathcal{A}_\alpha : b \wedge c \neq 0\}$. Let $\gamma = \sup_{c\in C_i, i<\eta} |\mathcal{A}_c|$. Then $\{b \in \mathcal{A}_\alpha : Y \wedge b \neq 0\}$ has cardinality at most $\mu \times \gamma < \kappa$, a contradiction. ⊣

With some favorable cardinal arithmetic, Baumgartner and Taylor [6] showed that these properties are equivalent. Most of the results in this subsection are an elaboration of theorems from that paper.

4.8 Proposition. *Assume* GCH *and that I is a normal, fine, countably complete ideal on $Z \subseteq P(\kappa)$ with $|Z| \leq \kappa$. Let λ be regular cardinal less than or equal to κ. Then the following are equivalent:*

1. *For all generic $G \subseteq P(Z)/I$, $V[G] \models \mathrm{cf}(\kappa^+) > \lambda$.*
2. *I is weakly (λ^+, κ^+)-saturated.*
3. *If $\{\mathcal{A}_\alpha : \alpha < \lambda\}$ is a collection of maximal antichains in $P(Z)/I$, then there is a dense collection of Y in $P(Z)/I$, such that for each α there is a pairwise disjoint collection of representatives for $\mathcal{A}_\alpha \restriction Y = \{b \in \mathcal{A}_\alpha : Y \cap b \notin I\}$.*
4. *I is precipitous and for all generic $G \subseteq P(Z)/I$ if $j : V \to M \subseteq V[G]$ is the generic elementary embedding, then $M \models \mathrm{cf}(\kappa^+) > \lambda$ and $M^\lambda \cap V[G] \subseteq M$.*

We note that the only part of the following proof that uses GCH is the implication that property 1 implies property 2. Weaker assumptions than GCH suffice for proving 2, for example the assumption that $2^\kappa < \kappa^{+\omega}$ and that for all $n \in \omega$, $\mathrm{cf}(\kappa^{+n})^{V[G]} > \lambda$. We leave this to the reader. See [18] for some sufficient conditions involving pcf theory. We do point out that the argument given below can be easily generalized without any GCH assumption under the hypothesis that I is a normal, fine, countably complete, $\kappa^{+\omega}$-saturated ideal on $Z \subseteq P(\kappa)$ with $|Z| \leq \kappa$ that preserves every cardinal κ^{+n} for $n \geq 1$.

Proof. To see the proposition, assume property 1 and fix maximal antichains $\langle \mathcal{A}_\alpha : \alpha < \lambda \rangle$. Then each $\mathcal{A}_\alpha$ has cardinality less than or equal to κ^+. Enumerate $\mathcal{A}_\alpha$ as $\langle a^\alpha_\beta : \beta < \gamma_\alpha \rangle$ in a one-to-one way so that $\gamma_\alpha \leq \kappa^+$. Define a function $f : \lambda \to \kappa^+$ lying in $V[G]$ for generic $G \subseteq P(Z)/I$, by setting $f(\alpha) = \beta$ iff $G \cap \mathcal{A}_\alpha = \{a^\alpha_\beta\}$. Then, since κ^+ has cofinality greater than λ in $V[G]$, there is a set $Y \subseteq Z$ such that $[Y] \Vdash \sup(f``\lambda) = \eta$. Then for all α, $\sup\{\beta : Y \cap a^\alpha_\beta \notin I\} \leq \eta$. Hence we have verified that I is weakly (λ^+, κ^+)-saturated.

Now suppose that I is weakly (λ^+, κ^+)-saturated, i.e. property 2. Then arguments similar to Proposition 2.23 show that if $|\{b \in \mathcal{A}_\alpha : b \cap Y \notin I\}| \leq \kappa$, then we can choose a system of disjoint representatives for $\{b \cap Y : b \in \mathcal{A}_\alpha \text{ and } b \cap Y \notin I\}$. Hence given a collection of antichains $\langle \mathcal{A}_\alpha : \alpha < \lambda \rangle$, there is a dense collection of $Y \in P(Z)/I$ such that for all $\alpha < \lambda$ there is a disjoint system of representatives for $\{b \cap Y : b \in \mathcal{A}_\alpha \text{ and } b \cap Y \notin I\}$.

As in the proof that the disjointing property implies precipitousness (see Proposition 2.14), this immediately gives that I is precipitous. Moreover, it shows that the analogue of Proposition 2.12 holds: given $P(Z)/I$ terms $\{\dot{f}_\alpha : \alpha < \lambda\}$ for functions in V with domain Z, there is a dense collection $D \subseteq P(Z)/I$ such that for each $Y \in D$ there is a sequence of functions $\{g_\alpha : \alpha < \lambda\}$ that lies in V such that $[Y] \Vdash [\dot{f}_\alpha]^M = [\check{g}_\alpha]^M$. Using normality, this suffices to argue the closure of the ultrapower as in Proposition 2.14.

Finally, it is easy to check that property 4 implies property 1. ⊣

To summarize we get the following corollaries:

4.9 Corollary. *Assume* GCH *and that I is a normal, fine, countably complete ideal on $Z \subseteq P(\kappa)$ with $|Z| = \kappa$ such that $P(Z)/I$ preserves κ^+. Then I is precipitous. If $G \subseteq P(Z)/I$ is generic and $j : V \to M$ is the generic embedding then $M^\kappa \cap V[G] \subseteq M$.*[24]

4.10 Corollary. *In addition to the hypotheses of Proposition 4.8, assume that I is λ^{++}-complete and weakly (λ^+, κ^+)-saturated. Then every cardinal and cofinality less than λ^+ is preserved.*

We now show a small generalization of a theorem Solovay proved about c.c.c. ideals:

4.11 Proposition. *If I is a countably complete ideal on $P(\kappa)$ such that $P(\kappa)/I$ is a proper partial ordering, then I is weakly $(\aleph_1, \aleph_1)$-saturated. Thus, I is precipitous and the corresponding generic ultrapower is closed under ω-sequences.*

Proof. Suppose that $P(\kappa)/I$ is a proper partial ordering. Let $\{\mathcal{A}_n : n \in \omega\}$ be a collection of maximal antichains and $S \in I^+$. Let $N \prec H(\theta)$ be countable with $I, \kappa, \{\mathcal{A}_n : n \in \omega\} \in N$ and T a generic condition for N. Then $\{a \in \mathcal{A}_n : a \cap T \notin I\} \subseteq N$ and hence is countable. Since I is countably complete we can disjointify the $\mathcal{A}_n$'s below T, and proceed as in Proposition 4.8. ⊣

The saturation properties of the ideals often greatly restrict the possible partial orderings that can arise as the quotient $P(Z)/I$. An extreme example is:

4.12 Example. Suppose that I is a countably complete, $\aleph_1$-dense ideal. Then $P(Z)/I$ is a complete Boolean algebra that collapses ω_1 and has a dense set of size $\aleph_1$. Hence it is isomorphic to the complete Boolean algebra generated by $\mathrm{Col}(\omega, \omega_1)$.

We note that Balcar and Franek showed that if I is an ideal on ω_1 that is nowhere precipitous and $\mathcal{B}(P(\omega_1)/I)$ has a dense set of size ω_2, then $\mathcal{B}(P(\omega_1)/I) \cong \mathcal{B}(\mathrm{Col}(\omega, \omega_2))$.

[24] Note that we do not assume that the critical point of j is κ.

4.3. Layered Ideals

The quotient algebras of certain saturated ideals are organized enough to allow powerful inductive constructions of ultrafilters. (See e.g. [48, 36] and Sects. 5.9 and 7.4.) The main notion is that of a layered ideal.

4.13 Definition. Let I be a normal, fine, κ-complete ideal on κ. Then I is *layered* iff $\mathcal{B}(P(\kappa)/I) = \bigcup_{\alpha<\kappa^+} \mathcal{B}_\alpha$ where:

1. each $\mathcal{B}_\alpha$ is a Boolean subalgebra of $\mathcal{B}(P(\kappa)/I)$ with $|\mathcal{B}_\alpha| = \kappa$,
2. if $\alpha < \beta$ then $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$,
3. for limit β, $\mathcal{B}_\beta = \bigcup_{\alpha<\beta} \mathcal{B}_\alpha$, and
4. for stationarily many $\alpha \in \kappa^+ \cap \operatorname{Cof}(\kappa)$, $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}(P(\kappa)/I)$.

A sequence $\langle \mathcal{B}_\alpha : \alpha < \kappa^+ \rangle$ is called a *filtration* if it satisfies clauses 1 to 3 in the definition. A filtration that also satisfies clause 4 is a *layering sequence*.

4.14 Remark (Shelah). If I is a layered ideal on κ then I is κ^+-saturated.

Proof. Suppose that $\mathcal{A} \subseteq \mathcal{B}(P(\kappa)/I)$ is a maximal antichain. Then for a closed unbounded set of $\alpha < \kappa^+$, $\mathcal{A} \cap \mathcal{B}_\alpha$ is a maximal antichain in $\mathcal{B}_\alpha$. For such an α where $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}$, we must have $\mathcal{A} \cap \mathcal{B}_\alpha$ maximal in $\mathcal{B}$. In particular, $\mathcal{A} \cap \mathcal{B}_\alpha = \mathcal{A}$ and hence $|\mathcal{A}| \leq \kappa$. ⊣

4.15 Definition. A layered ideal I is *strongly layered* iff there is a layering sequence $\langle \mathcal{B}_\alpha : \alpha < \kappa^+ \rangle$ such that there is a closed unbounded set $C \subseteq \kappa^+$ such that for all $\alpha \in C \cap \operatorname{Cof}(\kappa)$, $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}$.

If $\langle \mathcal{B}_\alpha : \alpha < \kappa^+ \rangle$ witnesses strong layering then we can pass to a subsequence with the property that for *all* $\alpha \in \operatorname{Cof}(\kappa)$, $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}$.

It is well-known that if $A \subseteq \kappa^+ \cap \operatorname{Cof}(\kappa)$ is stationary, then there is a partial ordering $\mathbb{P}$ that adds a closed unbounded set $C \subseteq \kappa^+$ to V with $C \cap \operatorname{Cof}(\kappa) \subseteq A$ in such a way that $\mathbb{P}$ adds no new κ-sequences to V. As a consequence, if I is a layered ideal on κ and A is the stationary subset witnessing clause 4 of the definition of "layered", we can add a closed unbounded subset of A without changing $P(\kappa)$. In the resulting extension I is strongly layered. Thus it is a consequence of the following theorem that if we have $\kappa^{<\kappa} = \kappa$, and a layered ideal I on κ, there is a (κ^+, ∞)-distributive forcing extension in which I is κ-centered. The following is a result of Shelah.

4.16 Theorem (Shelah [102]). *If I is a strongly layered ideal on κ and $\kappa^{<\kappa} = \kappa$ then I is κ-centered.*

We prove this theorem for the case CH and $\kappa = \omega_1$. After the proof we briefly discuss how to extend the theorem to arbitrary κ.

Proof. We begin with a review of the elementary theory of Boolean algebras. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are complete Boolean algebras, $\mathbb{P}$ is dense in $\mathcal{B}$ and $\mathbb{Q}$ is dense in $\mathcal{C}$. If there is a regular embedding i of $\mathbb{P}$ into $\mathbb{Q}$ then there is a complete embedding $\iota : \mathcal{B} \to \mathcal{C}$ extending i. In this case we can define a map $\pi_\iota : \mathcal{C} \to \mathcal{B}$ by setting $\pi(c) = \prod\{b : \iota(b) \geq c\}$. Then $\pi \circ \iota = \mathrm{id}$ and $\iota \circ \pi \geq \mathrm{id}$. Moreover, if $b \in \mathcal{B}$ and $b \leq \pi(c)$ then $\iota(b) \wedge c \neq 0$. The map π will be called a projection of $\mathcal{C}$ to $\mathcal{B}$.

Suppose now that we have complete embeddings $\iota_0 : \mathcal{B} \to \mathcal{C}$ and $\iota_1 : \mathcal{C} \to \mathcal{D}$ where $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are all complete Boolean algebras. Then it is easy to check that if π_0 and π_1 are the projections associated with ι_0 and ι_1, then $\pi_0 \circ \pi_1$ is the projection associated with $\iota_1 \circ \iota_0$.

If $i : \mathbb{P} \to \mathbb{Q}$ is a regular embedding, then for $q \in \mathbb{Q}$ we define the "preprojection"

$$\mathrm{pp}_{\mathbb{P}}(q) = \{p \in \mathbb{P} : p \leq \pi(q)\}.$$

For $p \in \mathbb{P}$, $\mathrm{pp}_{\mathbb{P}}(i(p)) = \{r \in \mathbb{P} : r \leq p\}$ and for all $q \in \mathbb{Q}$,

$$\sum\nolimits^{\mathcal{B}(\mathbb{P})} \mathrm{pp}_{\mathbb{P}}(q) = \pi(q).$$

If we have a further regular embedding $j : \mathbb{Q} \to \mathbb{R}$, then for all $r \in \mathbb{R}$:

$$\sum\nolimits^{\mathcal{B}(\mathbb{P})} \bigcup\{\mathrm{pp}_{\mathbb{P}}(q) : q \in \mathrm{pp}_{\mathbb{Q}}(r)\} = \pi_i \circ \pi_j(r).$$

Inductively, given a sequence of Boolean algebras $\{\mathcal{B}_\alpha : \alpha \in \{\alpha_0, \ldots, \alpha_n\}\}$ indexed by an increasing sequence of ordinals α_i and a commuting system of regular embeddings $\iota_{\alpha,\alpha'} : \mathcal{B}_\alpha \to \mathcal{B}_{\alpha'}$ for $\alpha < \alpha'$ and $b \in \mathcal{B}_{\alpha_n}$, we can define $\mathrm{pp}_{\vec{\alpha}}(b)$ to be

$$\begin{aligned}&\bigcup\{\mathrm{pp}_{\alpha_0}(q_1) : q_1 \in \bigcup\{\mathrm{pp}_{\alpha_1}(q_2) : \ldots \\ &\qquad q_{n-2} \in \bigcup\{\mathrm{pp}_{\alpha_{n-2}}(q_{n-1}) : q_{n-1} \in \mathrm{pp}_{\alpha_{n-1}}(b)\}\} \ldots\}.\end{aligned}$$

Then $\sum \mathrm{pp}_{\vec{\alpha}}(b) = \pi_{\alpha_0}(b)$ and if $\vec{\alpha}'$ is a subsequence of $\vec{\alpha}$ then $\mathrm{pp}_{\vec{\alpha}}(b) \subseteq \mathrm{pp}_{\vec{\alpha}'}(b)$.

Let $\mathcal{B} = P(\omega_1)/I$ and let $\langle \mathcal{B}_\alpha : \alpha \in \omega_2\rangle$ be a filtration such that for all $\alpha \in \mathrm{Cof}(\omega_1)$, $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}$. Fix a large regular cardinal θ and $b \in \mathcal{B}$, let

$$\mathfrak{A}_b = \langle H(\theta), \in \Delta, \mathcal{B}, \langle \mathcal{B}_\alpha : \alpha \in \omega_2\rangle, \dot{b}\rangle,$$

where $\dot{b}^{\mathfrak{A}_b} = b$.

Claim. For each well-founded countable model $N \equiv \mathfrak{A}_b$ there is a filter U_N on $\mathcal{B}^N$, with $\dot{b}^N \in U_N$ such that for all decreasing sequences $\vec{\alpha} \in [(\mathrm{Cof}(\omega_1) \cap \omega_2)^N]^{<\omega}$ and all $d \in U_N$, there is a $c \in \mathrm{pp}_{\vec{\alpha}}(d) \cap U_N$.

To see this, we build U_N by finite approximations. We can assume that N is transitive. Enumerate $(\mathrm{Cof}(\omega_1) \cap \omega_2)^N$ as $\langle \beta_n : n \in \omega\rangle$. Using the remarks above about preprojections, inductively build an increasing sequence of finitely generated filters so that at stage n, the filter is generated by a set $\{d_i : i < i(n)\}$ containing $\dot{b}^N$ with the property that for all decreasing sequences $\vec{\alpha}$ chosen from $\{\beta_k : k < n\}$ and all i, there is a j such that $d_j \in \mathrm{pp}_{\vec{\alpha}}(d_i)$.

We now describe the centering of $\mathcal{B}$. Let $\langle N_\gamma : \gamma \in \omega_1\rangle$ enumerate the transitive countable N that for some $b \in \mathcal{B}$, N is elementarily equivalent to $\mathfrak{A}_b$. These are the transitive countable models satisfying the hypothesis of the claim. Each N_γ and its associated U_{N_γ} will act as a template for a filter $\mathcal{F}_\gamma$ on $\mathcal{B}$. The collection of filters witnessing the centering is $\langle \mathcal{F}_\gamma : \gamma \in \omega_1\rangle$.

For $b \in \mathcal{B}$, let $N_b = \mathrm{Sk}^{\mathfrak{A}_b}(\emptyset)$. Clearly N_b satisfies the hypothesis of the claim. We let $\mathcal{F}_\gamma$ be the filter generated by those b for which $N_b \cong N_\gamma$. Since the transitive collapse of each N_b is among the N_γ's, each b belongs to some $\mathcal{F}_\gamma$. We must see that $\mathcal{F}_\gamma$ is a proper filter.

Fix a particular γ. We will be done if we can show that if $b_0, \dots, b_{n-1} \in \mathcal{F}_\gamma$ then $\prod_{i<n} b_i \neq 0$. Fix such a collection $\{b_0, \dots, b_{n-1}\}$. For notational simplicity, write N_{b_i} as N_i.

We look at how the traces of the N_i's on ω_2 fit together. Note that any two N_i's have the same intersection with ω_1, namely $\omega_1^{N_\gamma}$. For $i, j < n$, $N_i \cap N_j \cap \omega_2$ is an initial segment of $N_i \cap \omega_2$. After this common initial segment they are disjoint below ω_2 and thus N_j divides N_i into intervals that contain no elements of N_j.

If $\beta \in N_j \cap \omega_2 \setminus N_i \cap \omega_2$ and there is an $\alpha \in N_i \cap \omega_2$ that is bigger than β then the least such α is a limit ordinal of uncountable cofinality and there are unboundedly many ordinals of cofinality ω_1 in $N_i \cap \alpha$. If $d \in N_i \cap \mathcal{B}_\alpha$ then there is a $\delta \in N_i \cap \mathrm{Cof}(\omega_1)$ such that $d \in N_i \cap \mathcal{B}_\delta$. Furthermore, if $d \in \mathrm{pp}_\alpha(c)$ for some $c \in \mathcal{B} \cap N_i$, then $d \in \mathrm{pp}_\beta(c)$ for all $\delta \leq \beta \leq \alpha$.

When we have three or more N_i's the picture is a bit more complicated as the initial segments of $N_i \cap \omega_2$ coming from the intersections with N_j and N_k can differ. With this in mind we define the *stem* of N_i to be those ordinals α in $N_i \cap \omega_2$ with the property that for all $j \neq i$, every $\beta \in N_j \cap \alpha$ belongs to N_i. We will take the *stem* of a collection of N_i's to be the longest stem among the stems of the N_i's in the collection.

Let $\pi_i : N_i \cong N_\gamma$ be the transitive collapse. Let U_i be the π_i inverse image of U_{N_γ}. Then $U_i \subseteq N_i$ and satisfies the conclusion of the claim. Because $N_i \cap N_j \cap \omega_2$ is an initial segment of $N_i \cap \omega_2$ and $N_j \cap \omega_2$, the transitive collapses of N_i and N_j agree on $\mathcal{B}_\alpha$ for all α in $N_i \cap N_j \cap \omega_2$. Hence, if $d \in N_i \cap N_j \cap \mathcal{B}$, then $d \in U_i$ iff $d \in U_j$.

For $i \leq n$, define decreasing sequences of ordinals $\vec{\alpha}_i = \alpha_0^i, \alpha_1^i, \dots$ and elements $c_m^i \in \mathcal{B}_{\alpha_m^i}$ such that:

1. $c_o^i = b_i$ and α_0^i is equal to the least $\alpha \in \mathrm{Cof}(\omega_1)$ such that $b_i \in \mathcal{B}_\alpha$,

2. both α_j^i and c_j^i belong to N_i and α_j^i is the least ordinal α of uncountable cofinality such that $c_j^i \in \mathcal{B}_\alpha$,

3. the last ordinal on each sequence belongs to the stem of a collection of N_i's,

4. $c^i_{m+1} \in \mathrm{pp}_{\alpha^i_{m+1}}(c^i_m) \cap U_i$ (and hence inductively, we see that $c^i_m \in \mathrm{pp}_{\alpha^i_0,\alpha^i_1,\dots,\alpha^i_m}(b_i) \cap U_i$ for all α^i_m), and

5. if $i \neq j$ and $\alpha^i_k > \alpha^j_{k'}$ with $\alpha^j_{k'} \notin N_i$, then there is an l such that $c^i_l \in \mathrm{pp}_\beta(c^i_k)$ where β is the least element of $N_i \cap \omega_2$ above $\alpha^j_{k'}$.

We note that $c^i_l \in \mathrm{pp}_\delta(c^i_k)$ for some $\delta \in \omega_2 \cap \beta$. This $\delta < \alpha^j_{k'}$ and hence $c^i_l \in \mathrm{pp}_{\alpha^j_{k'}}(c^i_k)$.

These sequences are built by induction simultaneously for all i with exactly one $\vec{\alpha}_i$ being extended at each step of the induction. At the inductive step, one considers the greatest $\alpha^j_{k'}$ for which there is an α^i_k for which the clause 5 does not hold (if such a bad $\alpha^j_{k'}$ exists). We can assume inductively that k is the length of the sequence $\vec{\alpha}_i$ defined so far. Let β be the least ordinal in N_i above $\alpha^j_{k'}$. Let c^i_{k+1} be an element of $\mathrm{pp}_\beta(c^i_k) \cap U_i$. Let α^i_{k+1} be the least $\rho \in \mathrm{Cof}(\omega_1) \cap \omega_2$ such that $c^i_{k+1} \in \mathcal{B}_\rho$.

If there is no bad $\alpha^j_{k'}$ at some step in the induction, we claim that among the least elements of the sequences $\vec{\alpha}_j$ there is at most one that is not in the stem. Otherwise we would have $\alpha^i_k > \alpha^j_{k'}$ such that α^i_k and $\alpha^j_{k'}$ are the least elements of $\vec{\alpha}_i$ and $\vec{\alpha}_j$ respectively and neither one is in the stem of the system. Then $\alpha^j_{k'} \notin N_i$, and hence there is an α^i_k for which the clause 5 does not hold for $\alpha^j_{k'}$; i.e. $\alpha^j_{k'}$ is bad.

The induction continues until either all of the least elements of the sequences $\vec{\alpha}_i$ belong to the stem, or until there are no counterexamples to item 5. If the latter case holds and α^i_k is the only last element not in the stem, we let β be the least element of N_i above the stem, $c^i_{k+1} \in \mathrm{pp}_\beta(c^i_k) \cap U_i$, and α^i_{k+1} the least ordinal ρ of cofinality ω_1 such that $c^i_{k+1} \in \mathcal{B}_\rho$.

Claim. For all ordinals ρ in $\bigcup_i \vec{\alpha}^i$, if $\{d_0, \dots, d_l\}$ are the elements $\{c^i_m\}_{i,m}$ of $\mathcal{B}$ that belong to $\mathcal{B}_\alpha$ for α in $\rho + 1 \cap \bigcup_i \vec{\alpha}^i$, then $\prod d_j \neq 0$.

The claim suffices, since we can take $\rho = \max \bigcup \vec{\alpha}$, and see that $\prod b_i \neq 0$, finishing the proof.

We establish the claim by induction on $\rho \in \bigcup_i \vec{\alpha}^i$. Let ρ be the greatest ordinal in $\bigcup_i \vec{\alpha}^i$ that lies in the stem of the system of N_i's and suppose that $\rho \in N_k$. Then π_k agrees with each of the other π_i's on the stem of N_i. In particular, if $\{d_0, \dots, d_l\}$ are the elements of $\mathcal{B}$ associated with ordinals in the stems of all of the N_i's then $\{d_0, \dots, d_l\} \subseteq U_k$. Thus $\prod_{i \le l} d_i \neq 0$.

Suppose that we succeed with the induction to ρ and $\{d_0, \dots, d_l\}$ the elements of $\mathcal{B}$ associated with the ordinals in $\rho + 1 \cap \bigcup_i \vec{\alpha}^i$. Let α be the least element of $\bigcup_i \vec{\alpha}^i$ greater than ρ and let c be the element of $\mathcal{B}$ associated with α. Suppose that $\alpha = \alpha^i_k \in N_i$. Then α^i_{k+1} and c^i_{k+1} are built in one of two ways corresponding to whether they are the result of a bad $\alpha^j_{k'}$ or not. The first possibility is that α^i_k is the least element of $\bigcup_i \vec{\alpha}_i$ not in the stem. If β is the least element of N_i above the stem, then $c^i_{k+1} \in \mathcal{B}_\beta$. But $\prod_{i \le l} d_i \le c^i_{k+1} \in \mathrm{pp}_\beta(c^i_k)$. Hence $\prod_{i \le l} d_i \cap c^i_k \neq 0$.

The other way that α^i_{k+1} can be constructed is as a result of a bad $\alpha^j_{k'} \in \{d_0, \dots, d_l\}$. In this case if β is the least element of N_i above $\alpha^j_{k'}$ then $c^i_{k+1} \in \mathrm{pp}_\beta(c^i_k)$. Thus, as in the other case $\prod_{i\le l} d_i \le c^i_{k+1} \in \mathrm{pp}_\beta(c^i_k)$ and so $\prod_{i\le l} d_i \cap c^i_k \ne 0$. ⊣

We now describe the modifications necessary for $\kappa > \omega_1$. By replacing countable structures N with structures N' of cardinality less than κ with $N' \cap \kappa \in \kappa$, most of the proof above goes through without change. The main problem is the construction of the filters U_N. To remedy this problem, in the Claim we use structures N with the property that $N = \bigcup_{n\in\omega} N_n$ where $N_n \in N_{n+1}$ and $|N_n| < \kappa$ and $N_n \cap \kappa$ an initial segment of $N_{n+1} \cap \kappa$.

We may assume without loss of generality that for each $\alpha \in \mathrm{Cof}(\kappa)$, $\mathcal{B}_\alpha$ is closed under sums of size less than κ. In particular, if $N \prec \mathfrak{A}_b$ is the union of an increasing sequence N_n with $N_n \in N_{n+1}$, $N_n \cap \kappa \in \kappa$, and $\vec{\alpha}$ is a decreasing sequence in $\mathrm{Cof}(\kappa) \cap N_l$ with $c \in \mathcal{B} \cap N_l$, then $c_{\vec{\alpha},n} = \sum(\mathrm{pp}_{\vec{\alpha}}(c) \cap N_n)$ exists in N_l for all $n < l$. One can then check that for $N \equiv \mathfrak{A}_b$, the set $\{b_{\vec{\alpha},n} : \vec{\alpha} \in N_l \cap \mathrm{Cof}(\kappa)^N, n < l\}$ generates a filter U_N that makes the rest of the argument work.

4.4. Projections

We now investigate ideals under the Rudin-Keisler ordering. We establish an important result of Burke stating that any ideal is the projection of the nonstationary ideal restricted to a stationary set. Our priority is to see which of the properties listed above are preserved under projections, rather than the structure of the Rudin-Keisler ordering itself. The latter topic has been explored extensively in [9].

4.17 Definition. Let $\pi : Z \to Z'$ and I be an ideal on Z. Then the *projection* of I to Z' is the ideal I' defined by setting $B \in I'$ iff $\pi^{-1}(B) \in I$. It is easy to check that if I is κ-complete then its projection I' is also κ-complete.

Let $\langle A, <_A\rangle$ be a linearly ordered set. Given a collection of sets $\langle Z_a : a \in A\rangle$ and a commuting sequence of functions $\pi_{a,a'} : Z_a \to Z_{a'}$ for $a' <_A a$, the sequence of ideals $\langle I_a : a \in A\rangle$ is a *tower* iff for all $a' < a$, $I_{a'}$ is the projection of I_a by the function $\pi_{a,a'}$.

We consider some standard examples:

4.18 Example. Let $\kappa < \lambda' < \lambda$ be cardinals, with κ regular. Define the function $\pi : [\lambda]^{<\kappa} \to [\lambda']^{<\kappa}$ by setting $\pi(z) = z \cap \lambda'$. Then a normal, fine, γ-complete ideal I on $[\lambda]^{<\kappa}$ projects to a normal, fine, γ-complete ideal on $[\lambda']^{<\kappa}$.

The ideal of nonstationary sets on $[\lambda]^{<\kappa}$ projects to the ideal of nonstationary sets on $[\lambda']^{<\kappa}$ (see [47]). Hence the nonstationary ideals form a tower of ideals. More can be said (see [42]): the nonstationary ideals restricted to $\{x : x \cap \mathrm{On}$ is ω-closed$\}$ form a tower, as do the nonstationary ideals restricted to the internally approachable sets. It is not true in general that if $S \subseteq [\lambda]^{<\kappa}$ is a stationary set and $\lambda' < \lambda$ then the projection of the nonstationary ideal to $[\lambda']^{<\kappa}$ is the nonstationary ideal on the projection of S to λ'. Corollary 4.21 below shows that this is false in the most dramatic way.

Example 4.18 is not special to $[\lambda]^{<\kappa}$. If $\lambda > \lambda'$ define $\pi : P(\lambda) \to P(\lambda')$ by setting $\pi(z) = z \cap \lambda'$. Then π projects the nonstationary ideal on $P(\lambda)$ to the nonstationary ideal on $P(\lambda')$.

Going in the other direction, ideals on sets $Z \subseteq P(X)$ naturally give ideals on sets $P(X')$ for $X' \supseteq X$:

4.19 Definition. Suppose that J is a normal, fine, κ-complete ideal on $Z \subseteq P(X)$ and suppose that $X' \supseteq X$. Then the *conditional closed unbounded filter on $P(X')$ relative to J* is defined to be the smallest normal, fine, κ-complete filter on $P(X')$ projecting to $\breve{J}$ by the map $\pi(N) = N \cap X$. The nonstationary ideal conditioned on J is the dual of the conditional closed unbounded filter.

The terminology "conditional" is taken in analogy with probability theory, where one conditions one σ-algebra on another and takes the ideals of null sets.

For sufficiently large θ relative to $|Z|$, we can characterize conditional closed unbounded filters[25] which take a particularly simple form.

4.20 Proposition. *Suppose that J is a normal, fine, κ-complete filter on $Z \subseteq P(X)$ and that θ is a regular cardinal with $J, P(Z) \in H(\theta)$. Then the conditional closed unbounded filter on $H(\theta)$ relative to J is the closed unbounded filter on $H(\theta)$ restricted to*

$$\{N \prec H(\theta) : N \cap X \in Z \textit{ and } N \textit{ is good for } J\}.$$

Proof. Without loss of generality we can assume that $\kappa \subseteq X$. Since J is normal, fine and κ-complete, we can assume that for all $z \in Z$, $z \cap \kappa \in \kappa + 1$.

Let I be the closed unbounded filter restricted to $\{N \prec H(\theta) : N \cap X \in Z$ and N is good for $J\}$. By Proposition 3.44, I is a proper ideal. Clearly I is normal and fine. Since it concentrates on those N such that $N \cap X \in Z$ it is κ-complete. We need to see that it projects to J.

Suppose that $A \subseteq H(\theta)$ is positive with respect to I. Let $D \in \breve{J}$. Then $\{N \subseteq H(\theta) : D \in N\} \in \breve{I}$ and for every good $N \in A$ with $D \in N$ we have $N \cap X \in D$. Hence $\pi``A \in J^+$.

For the other direction, let $B \in J^+$ and $\mathfrak{A}$ an algebra on $H(\theta)$. Then by Proposition 3.44, for J-almost all $z \in B$, $\mathrm{Sk}^{\mathfrak{A}}(z)$ is good. In particular, there is a $z \in B$ such that $\mathrm{Sk}^{\mathfrak{A}}(z) \cap X = z$ and $\mathrm{Sk}^{\mathfrak{A}}(z)$ is good. Hence $\pi^{-1}(B) \in I^+$. This shows that I is a normal ideal that projects to J.

Suppose that I' is a normal ideal projecting to J. Then a normality argument easily shows for all $\mathfrak{A}$, the collection of those N that are good and are elementary substructures of $\mathfrak{A}$ belongs to the dual of I'. Hence $I \subseteq I'$. ⊣

[25] See Lemma 3.12 for the basic result on unconditional club filters.

We draw the following corollary due to Burke [11], using somewhat different methods.

4.21 Corollary. *Suppose that I is a normal, fine ideal on a stationary set $Z \subseteq P(X)$. Then for all $Y \supseteq X$ with $|Y| \geq 2^Z$, there is a stationary set $Z' \subseteq P(Y)$ such that I is the projection of the nonstationary ideal on Y restricted to Z'.*

Note that Proposition 4.20 allows us to characterize the conditional closed unbounded filter on arbitrary sets $P(X')$ for $X' \supseteq X$ as the projection to $P(X')$ of the conditional closed unbounded filter on some $H(\theta)$ with θ large and regular. Thus the conditional closed unbounded filter exists on all $X' \supseteq X$.

We now explore projections in terms of generic elementary embeddings. The following proposition is an exercise in standard large cardinal techniques.

4.22 Proposition. *Suppose that I is an ideal on a set Z and $\pi : Z \to Z'$ yields a projection of I to an ideal I'. If $j : V \to M = V^Z/G$ is the ultrapower embedding given by a generic $G \subseteq P(Z)/I$,*[26] *i is the ideal element $[\mathrm{id}]^M$ and $i' = j(\pi)(i)$ then I' is the ideal induced by j and the ideal element i'. Moreover if M' is the ultrapower $V^{Z'}/U(j, i')$, then the map $k : M' \to M$ defined by $k(f) = j(f)(i')$ is a well-defined elementary embedding making the following diagram commute:*

$$\begin{tikzcd} & V \arrow[dl, "j'"'] \arrow[dr, "j"] & \\ M' \cong V^{Z'}/U(j,i') \arrow[rr, "k"'] & & M \end{tikzcd}$$

In particular, if M is well-founded then M' is.

We note that Proposition 4.22 should not be interpreted as claiming that I' is precipitous. This is false in general, as shown by Theorem 7.8. The next example shows that we can often find normal ideals as projections of precipitous ideals.

4.23 Example. Suppose that I is a precipitous ideal on $Z \subseteq P(X)$. Let $X' \subseteq X$. Suppose that for all generic $G \subseteq P(Z)/I$, if $j : V \to M$ is the associated generic embedding, then $j``X' \in M$. Then there is a dense collection of $Y \subseteq Z$ such that $I \restriction Y$ projects to a normal ideal on $P(X')$. In particular, if κ is the critical point of j for all generic G, then densely often I projects to a normal, κ-complete ideal on κ.

Proof. To see this, suppose that $[Y] \Vdash$ "$[f]$ represents $j``X'$". Then f is *incompressible* in the sense of [111], namely if g is any function with domain Y such that for all $z \in Y$, $g(z) \in f(z)$ then g is constant on an I-positive subset

[26] For this proposition we do not assume that V^Z/G is well-founded.

of Y. Define a projection map from $I\restriction Y$ to a normal ideal on $P(X')$ by setting $\pi(z) = f(z)$ for $z \in Y$. Let J be the projection of I to $Z' = P(X')$. It is easy to check that $W \in J$ iff there is no $Y' \subseteq Y$ such that $[Y'] \Vdash j``X' \in j(W)$. Since f is incompressible, the ideal J is normal. $\dashv$

Note that this example shows that if I is a κ-complete precipitous ideal on κ, then for a dense set of $[Y] \in P(\kappa)/I$, there is a projection of $I\restriction Y$ to a normal κ-complete ideal on κ.

Given a projection map $\pi : Z \to Z'$ and ideals I and I' such that I' is the projection of I, we can consider the Boolean algebras $\mathcal{B} = P(Z)/I$ and $\mathcal{B}' = P(Z')/I'$. The map $\iota : \mathcal{B}' \to \mathcal{B}$ given by $\iota([Y']) = [\pi^{-1}(Y')]$ is a well-defined Boolean algebra homomorphism. In particular, ι sends antichains to antichains. We thus can make the following observation:

4.24 Remark. Suppose that $\pi : Z \to Z'$, I is an ideal on Z that is λ-saturated and I' is the projection of I to an ideal on Z'. Then I' is λ-saturated.

The function ι is not necessarily a regular embedding. However the next proposition gives a criterion for when it is.

4.25 Proposition. *Suppose that $X' \subseteq X$, $Z \subseteq P(X)$, $Z' = \{z \cap X' : z \in Z\}$ and $\pi : Z \to Z'$ is defined by $\pi(z) = z \cap X'$. Let I be a normal, fine, countably complete ideal on Z and I' be the projection of I to Z'. If I' is $|X'|^+$-saturated then ι is a regular embedding.*

Proof. Let $\{a_x : x \in X\}$ be a maximal antichain in $P(Z')/I'$. Then $\bigtriangledown_{x\in X} a_x \in \breve{I}'$. Since $\iota(\bigtriangledown_{x\in X} a_x) = \bigtriangledown_{x\in X}\iota(a_x)$ we see that $\bigtriangledown_{x\in X}\iota(a_x) \in \breve{I}$. Hence $\{\iota(a_x) : x \in X\}$ is a maximal antichain in $P(Z)/I$. $\dashv$

It is easy to check that the properties of (κ, λ)-centeredness, (κ, η, λ)-saturation, κ-linkedness and κ-saturation are all preserved under projections. (See [111, 6].) As a consequence, the existence of an ideal I with one of these properties implies that the projection of I to a normal ideal on the completeness of I inherits that property. So, for example, the existence of a κ-complete, κ^+-saturated ideal on κ implies the existence of a normal, κ-complete, κ^+-saturated ideal on κ.

Following [35] we give a sample of these types of arguments:

4.26 Lemma. *Suppose that $X' \subseteq X$, $Z \subseteq P(X)$, $Z' = \{z \cap X' : z \in Z\}$ and $\pi : Z \to Z'$ is defined by $\pi(z) = z \cap X'$. Let I be a normal, fine, countably complete ideal on Z and I' be the projection of I to Z'. Let $\mu \le |X'|$. Then if I is μ-dense, so is I'.*

Proof. Define $\Pi : P(Z) \to P(Z')$ by setting $\Pi(A) = \{\pi(z) : z \in A\}$. Let $D \subseteq P(Z)/I$ be a dense set of size μ. Choose representatives $\langle A_d : d \in D\rangle$ with $A_d \subseteq Z$ and $[A_d]_I = d$. For each $C \in \breve{I}$, let $b^d_C = Z' \setminus \Pi(C \cap A_d)$.

By the μ^+-saturation of I', we can find $\langle b^d_{C_\alpha} : \alpha < \mu\rangle$ such that

$$\nabla_{\alpha<\mu} b^d_{C_\alpha} \supseteq_{I'} b^d_C$$

for all $C \in \check{I}$. Let $C_d = \Delta_{\alpha<\mu} C_\alpha$. Then for all $C \in \check{I}$, $\Pi(C_d \cap A_d) \cap b^d_C \in I'$.

Let

$$E = \{[\Pi(C_d \cap A_d)]_{I'} : d \in D\}.$$

We claim that E is dense in $P(Z')/I'$. Choose an arbitrary $B \in P(Z')/I'$. Then there is a d such that $A_d \setminus \{z : \pi(z) \in B\} \in I$. Choose a C such that $\Pi(A_d \cap C) \subseteq B$. Then $\Pi(C_d \cap A_d) \subseteq_{I'} \Pi(C \cap A_d) \subseteq B$. ⊣

We give a sample corollary:

4.27 Corollary. *Suppose that there is a countably complete, uniform, $\aleph_1$-dense ideal on ω_n. Then there is a weakly normal, countably complete, uniform, $\aleph_1$-dense ideal on ω_n.*

Not all of the desirable properties of ideals are preserved under projections. Laver [85] showed that if the existence of a supercompact cardinal is consistent then it is consistent to have a precipitous ideal on $[\omega_2]^{<\omega_1}$ such that its canonical projection to a normal ideal on ω_1 is not precipitous. This is Theorem 7.8 in Sect. 7.

Gitik, in recent unpublished work, gave an example of a precipitous ideal on ω_1 such that the canonical projection to a normal ideal on ω_1 is not precipitous.

By Laver's example, we know that precipitous ideals are not closed under projection. As far as the author knows it is open whether presaturated ideals are closed under projection, even in the presence of GCH.

4.5. Where the Ordinals Go

In this section we give some examples of techniques for determining where the ordinals are sent by generic elementary embeddings. These examples are chosen to be representative and are easy to generalize to handle any particular situation.

4.28 Proposition. *Let κ be a successor cardinal, $n \in \omega$ and I be a uniform κ-complete, κ^+-saturated ideal on a set $Z \subseteq P(\kappa^{+n})$ that has cardinality at least κ^{+n}. Suppose that either:*

1. *Z has cardinality κ^{+n}, or*

2. *I is normal and fine.*

Let $j : V \to M$ be a generic embedding induced by the ultrapower of V by a generic $G \subseteq P(Z)/I$. Then the critical point of j is κ and for all $i \leq n$, $j(\kappa^{+i}) = \kappa^{+i+1}$.

Proof. Let $\lambda = \mathrm{crit}(j)$.

Suppose that $|Z| = \kappa^{+n}$. Then there are functions $\langle f_\alpha : \alpha < \kappa^{+n+1}\rangle$ such that $f_\alpha : Z \to Z$ and for all $\alpha < \beta$, $|\{z \in Z : f_\alpha(z) \geq f_\beta(z)\}| < \kappa^{+n}$. Hence $j(\kappa^{+n}) \geq (\kappa^{+n+1})^V$ so $\lambda \leq \kappa^{+n}$.

Suppose now that I is normal and fine. Define $f(z)$ to be the least element of $\kappa^{+n} \setminus z$ and let $\mathrm{id} : Z \to Z$ be the identity map. Then $j(\kappa^{+n}) > [f] \notin [\mathrm{id}]^M = j``\kappa^{+n}$. Hence $\lambda \leq \kappa^{+n} < j(\kappa^{+n})$. Moreover, $P(\kappa^{+n})^V \subseteq M$, so $j(\kappa^{+n}) \geq \kappa^{+n+1}$.

In either case, $\lambda = \kappa^{+i}$ for some $i \leq n$ and $j(\kappa^{+n}) \geq \kappa^{+n+1}$. If $\lambda = \kappa^{+i}$ for some $i > 0$, then we have $j(\lambda) = (\kappa^{+i})^M$, but $j(\lambda) > \lambda$ and this is impossible without collapsing some cardinal greater than or equal to κ^+. Hence $\lambda = \kappa$ and $j(\lambda) = (\kappa^+)^V$. Counting cardinals shows that $j(\kappa^{+i}) = \kappa^{+i+1}$. ⊣

4.29 Example. Suppose that I is a normal, κ-complete, κ^+-saturated ideal on a successor cardinal κ. Let $j : V \to M$ be a generic embedding induced by the ultrapower of V by a generic $G \subseteq P(\kappa)/I$. Then:

1. $j(\kappa) = \kappa^+$,
2. $j(\kappa^+) \in ((\kappa^+)^V, (\kappa^{+2})^V)$, $j``\kappa^+$ is cofinal in $j(\kappa)^+$,
3. for all $i \in [2, \omega]$, $j(\kappa^{+i}) = (\kappa^{+i})^V$, and
4. there is a $<\kappa$-closed and unbounded subset of κ^{+i} lying in V consisting of ordinals α such that for all generic G, $j(\alpha) = \alpha$.

Proof. Since κ is the critical point of j and κ is a successor cardinal, if $j(\kappa) > \kappa^+$, the forcing would collapse κ^+, a contradiction. Since $P(\kappa)^V \subseteq M$, $j(\kappa) \geq \kappa^+$ and hence $j(\kappa) = \kappa^+$.

Let $f : \kappa \to \kappa^+$. Then the range of f is bounded by some ordinal $\alpha < \kappa^+$, and hence $[f]^M < j(\alpha)$. Thus $j``\kappa^+$ is cofinal in $j(\kappa^+)$. Since κ^{+2} is preserved by the forcing, we see that $j(\kappa^+) \leq (\kappa^{+2})^V$. But in $V[G]$ the cofinality of $j(\kappa^+)$ is equal to the cofinality of $(\kappa^+)^V$, and hence $j(\kappa^+)$ must lie between κ^+ and κ^{+2}. Since κ^{+2} is preserved, $j(\kappa^+) < (\kappa^{+2})^V$.

Since $j(\kappa^{+2}) \geq (\kappa^{+2})^V$ we must have $j(\kappa^{+2}) = (\kappa^{+2})^V$. Similarly we must have $j(\kappa^{+n}) = \kappa^{+n}$.

For each $n > 2$ and each ordinal $\alpha < \kappa^{+n}$ there is a $\gamma(\alpha) < \kappa^{+n}$ such that $j(\alpha) < \gamma(\alpha)$ for all generic G. Let α^* be a closure point of the function $\alpha \mapsto \gamma(\alpha)$ having cofinality less than κ. Let $\langle \alpha_\xi : \xi \in \delta\rangle$ be increasing and cofinal in α^*, where $\delta < \kappa$. Then

$$\alpha^* \leq j(\alpha^*) = \sup\{j(\alpha_\xi) : \xi \in \delta\} = \alpha^*.$$

Thus the set of closure points that have cofinality less than κ in V is a $<\kappa$-closed unbounded set $F \subseteq \kappa^{+n}$ with $F \in V$ consisting of ordinals that are fixed by every embedding j. ⊣

4.30 Example. Let κ be a successor cardinal. Suppose that I is a normal, fine, κ-complete, κ^+-saturated ideal on $Z \subseteq P(\kappa^{+n})$. Let $j : V \to M$ be a generic embedding induced by taking the ultrapower of V by a generic $G \subseteq P(Z)/I$. Then:

1. for all $i \leq n$, $j(\kappa^{+i}) = \kappa^{+i+1}$, and
2. I concentrates on the collection of $x \in P(\kappa^{+n})$ such that for all $i \leq n-1$, $\mathrm{ot}(x \cap \kappa^{+i+1}) = \kappa^{+i}$.

If $|Z| = \kappa^{+n}$, then

3. $j``\kappa^{+n+1}$ is cofinal in $j(\kappa^{+n+1})$.

For all $i > n+1$,

4. $j(\kappa^{+i}) = \kappa^{+i}$, and
5. there is a $<\kappa$-closed unbounded subset of κ^{+i} lying in V consisting of ordinals α such that for all generic G, $j(\alpha) = \alpha$.

Remark. The hypothesis that $|Z| = \kappa^{+n}$ is redundant as we shall see later in Corollary 4.43, as such an I always concentrates on a set Z with $|Z| = \kappa^{+n}$.

Proof. Note that the critical point of j is κ by Proposition 4.28. Since $M^{\kappa^{+n}} \cap V[G] \subseteq M$, we know that $j(\kappa) \geq \kappa^+$ and can show inductively that for all $i \leq n$, $j(\kappa^{+i}) \geq \kappa^{+i+1}$. Since κ^{+i+1} is preserved by the forcing, we must have $j(\kappa^{+i}) = \kappa^{+i+1}$.

By the remarks in Sect. 2.8, we see that I concentrates on the collection of $x \in P(\kappa^{+n})$ such that for all $i \leq n-1$, $\mathrm{ot}(x \cap \kappa^{+i+1}) = \kappa^{+i}$.

Suppose that $|Z| = \kappa^{+n}$ and argue as before: Suppose that $f : Z \to \kappa^{+n+1}$. Then f is bounded by a constant function. Hence $j``\kappa^{+n+1}$ is cofinal in $j(\kappa^{+n+1})$. Since $j(\kappa^{+n+1}) > \kappa^{+n+1}$ we see that $j(\kappa^{+n+1})$ is singular in $V[G]$. Since cardinals are not collapsed, $j(\kappa^{+n+1}) < \kappa^{+n+2}$.

The rest of the example is exactly parallel to the previous one. ⊣

The next example is slightly different but uses similar ideas:

4.31 Example. Suppose that I is a normal, fine, κ-complete ideal on $Z \subseteq [\kappa^{+n}]^{<\kappa}$ that is κ^{+n+1}-saturated. Suppose that $2^{|Z|} = \kappa^{+n+1}$. Let $j : V \to M$ be the generic embedding induced by the ultrapower of V by a generic $G \subseteq P(Z)/I$, then:

1. κ is the critical point of j,
2. $j(\kappa) = \kappa^{+n+1}$, and
3. for all $i \geq n+2$, κ^{+i} is a fixed point of j.

Without normality we can still get results:

4.32 Example. Suppose that κ is a successor cardinal and I is a uniform, κ-complete, κ^+-saturated ideal on κ^{+n}. Let $j : V \to M$ be a generic embedding induced by the ultrapower of V by a generic $G \subseteq P(Z)/I$, then for all $0 \le i \le n-1$,

1. $j(\kappa^{+i}) = \kappa^{+i+1}$,
2. $j(\kappa^{+n}) \le \kappa^{+n+1}$, and
3. for all $i \ge 2$, $j(\kappa^{+n+i}) = (\kappa^{+n+i})^V$.

4.6. A Discussion of Large Sets

In earlier lemmas we have frequently had the hypothesis that an ideal I concentrates on some set $Z \subseteq P(X)$ with $|Z| \le |X|$. With one important exception, in every case we consider one can simply prove that $|Z| \le |X|$. In many cases there are closed unbounded sets of small cardinality, as shown by Baumgartner [4].

The exceptional case is when we are dealing with closed unbounded sets relative to the countable subsets of a cardinal $\kappa \ge \omega_2$. In Corollary 6.16 we see that every closed unbounded subset of $[\kappa]^\omega$ has cardinality $\kappa^{\aleph_0}$. The more typical situations are covered in this section.

We begin by discussing the situation with precipitous ideals before proceeding to saturated ideals where the situation is much simpler. First some prerequisites:

4.33 Definition. For K a collection of regular cardinals, define the *characteristic function* of a set X on K, $\chi_X^K : K \to \mathrm{On}$, by setting:

$$\chi_X^K(\kappa) = \sup(X \cap \kappa).$$

We will say that X is *weakly ω_1-uniform on K* if for all $\kappa \in K$, $\mathrm{cf}(X \cap \kappa) > \omega$.

The following lemma is standard and easy to prove (see e.g. [18]).

4.34 Lemma. *Suppose that K is a collection of regular cardinals between μ and ρ for some cardinals μ, ρ with μ regular. Then there is a closed unbounded set of $C \subseteq \{z \in P(\rho) : z \cap \mu \in \mu\}$ such that for all $z_1, z_2 \in C$ if:*

1. *z_1, z_2 both are weakly ω_1-uniform on K, and*
2. *$\chi_{z_1}^K = \chi_{z_2}^K$,*

then $z_1 = z_2$. Thus, if $\rho = \mu^{+n}$ for some $n \in \omega$ and I is a normal, fine, countably complete ideal concentrating on $\{z : z \cap \mu \in \mu$ and for all $0 \le i \le n$, $\mathrm{cf}(z \cap \mu^{+i}) > \omega\}$, then there is a set $A \in \breve{I}$ that is canonically well-ordered by the $n+1$-tuple $(z \cap \mu, z \cap \mu^+, \ldots, z \cap \mu^{+n})$. In particular, $|A| = \rho$.

We will use the following result from [42] (Theorem 2.15):

4.35 Theorem. *Suppose that $\kappa < \lambda < \mu$ are three consecutive cardinals. Then there is a closed unbounded set $C \subseteq P(\mu)$ such that for all $z \in C$ if $|z| = \lambda$ and $|z \cap \lambda| = \kappa$, then $\mathrm{cf}(z \cap \lambda) = \mathrm{cf}(\kappa)$.*

We see as a corollary:

4.36 Lemma. *If κ is regular, uncountable, and $(\kappa^{+n}, \kappa^{+}) \twoheadrightarrow (\kappa^{+n-1}, \kappa)$, then:*

1. *$(\kappa^{+n}, \kappa^{+n-1}, \ldots, \kappa^{+}) \twoheadrightarrow (\kappa^{+n-1}, \kappa^{+n-2}, \ldots, \kappa)$,*
2. *there is a closed unbounded set of $z \in [\kappa^{+n}]^{\kappa^{+n-1}}$ such that if $|z \cap \kappa^{+}| = \kappa$ then z is weakly ω_1-uniform on the interval $K = [\kappa^{+}, \kappa^{+n}]$, and*
3. *there is a closed unbounded $C \subseteq \kappa^{+n}$ such that for all $z_1, z_2 \in C \cap [\kappa^{+n}]^{\kappa^{+n-1}}$ if:*
   (a) $\kappa \subseteq z_1 \cap z_2$,
   (b) $|z_1 \cap \kappa^{+}| = |z_2 \cap \kappa^{+}| = \kappa$, and
   (c) $\chi^K_{z_1} = \chi^K_{z_2}$,

   then $z_1 = z_2$.

In particular, each $z \in C \cap [\kappa^{+n}]^{\kappa^{+n-1}}$ with $\kappa \subseteq z$ is determined by the finite sequence of ordinals $\sup(z \cap \kappa^{+i}), 1 \leq i \leq n$.

As a corollary we immediately see:

4.37 Theorem. *Suppose that κ is a regular uncountable cardinal and I is a normal, fine, κ^{+}-complete ideal on $[\kappa^{+n}]^{\kappa^{+n-1}}$. Then there is a set $Z \subseteq [\kappa^{+n}]^{\kappa^{+n-1}}$ such that $Z \in \breve{I}$ and $|Z| = \kappa^{+n}$.*

Proof. We note that if I is a proper ideal then $(\kappa^{+n}, \kappa^{+}) \twoheadrightarrow (\kappa^{+n-1}, \kappa)$, since any normal, fine ideal extends the nonstationary ideal. For $1 \leq i \leq n$, let $A_i = \{z : z \cap \kappa^{+i} \in \kappa^{+i}\}$. Then by Proposition 2.19, $\bigcup_{1 \leq i \leq n} A_i \in \breve{I}$ and for all $B \subseteq A_i$ with $B \in I^{+}$, $\mathrm{comp}(I \restriction B) = \kappa^{+i}$. Applying Lemma 4.36, we see that on each A_i almost every z is determined by the finite sequence of ordinals $\{\sup(z \cap \kappa^{+i}), \sup(z \cap \kappa^{+i+1}), \sup(z \cap \kappa^{+i+2}) \ldots\}$. Hence almost every element $z \in [\kappa^{+n}]^{\kappa^{+n-1}}$ is determined by a finite sequence of ordinals less than κ^{+n}. $\dashv$

4.38 Remark. Suppose that $n > 0$ is an integer and I is a normal, fine, countably complete ideal on $Z \subseteq P(\kappa^{+n})$ where κ is a regular cardinal. If $\{z : z \not\supseteq \kappa\} \in \breve{I}$ then for all $i \geq 1$, $\{z : |z \cap \kappa^{+i}| < \kappa^{+i}\} \in \breve{I}$. For otherwise, κ^{+i} would be a Jónsson cardinal, but no successor of a regular cardinal is Jónsson [122, 124]. Similar remarks are true for κ a singular cardinal, not the limit of measurable cardinals [105].

It is often useful to know when an ideal concentrates on sets of ordinals that are μ-closed.

4.39 Proposition. *Let $\mu \leq \lambda$. Suppose that $\mathbb{P}$ is a partial ordering such that for all generic $G \subseteq \mathbb{P}$, $j : V \to M \subseteq V[G]$ is a generic elementary embedding with critical point κ with $M^{j(\mu)} \cap V[G] \subseteq M$. Suppose that $j``\lambda \in M$. Then the following are equivalent:*

1. *For all generic $G \subseteq \mathbb{P}$, $j``\lambda$ is $j(\mu)$-closed in $V[G]$.*
2. *The ideal induced by $j, \mathbb{P}$, and $j``\lambda$ concentrates on $\{z : z \subseteq \lambda$ and z is μ-closed$\}$.*

Moreover, if $\mu < \kappa$ then these are also equivalent to

3. *For all cardinals $\rho < \lambda$, $\mathrm{cf}(\rho)^V = \mu$ iff for all generic $G \subseteq \mathbb{P}$, $\mathrm{cf}(\rho)^{V[G]} = \mu$.*

Proof. That the first two clauses are equivalent is an easy consequence of remarks in Sect. 2.8 and the fact that $M^{j(\mu)} \cap V[G] \subseteq M$.

The equivalence between clause 1 and clause 3 follows from Propositions 2.32 and 2.31. ⊣

4.40 Corollary. *Let $\mu \leq \kappa \leq \lambda$. Suppose that $\mathbb{P}$ is a partial ordering such that for all generic $G \subseteq \mathbb{P}$, $j : V \to M \subseteq V[G]$ is a generic elementary embedding with critical point κ with $M^{<\mu} \cap V[G] \subseteq M$. Suppose that $j``\lambda \in M$. Then the following are equivalent:*

1. *For all generic $G \subseteq \mathbb{P}$, $j``\lambda$ is $<\mu$-closed in $V[G]$.*
2. *The ideal induced by $j, \mathbb{P}, j``\lambda$ concentrates on $\{z : z \subseteq \lambda$ and z is $<\mu$-closed$\}$.*
3. *For all cardinals $\rho < \lambda$, $\mathrm{cf}(\rho)^V \geq \mu$ iff for all generic $G \subseteq \mathbb{P}$, $\mathrm{cf}(\rho)^{V[G]} \geq \mu$.*

4.41 Proposition (Baumgartner [4]). *Suppose that*

$$N \prec \langle H(\kappa^{+n}), \in, \Delta, f_i, \ldots \rangle$$

with $N \cap \kappa \in \kappa$. Suppose that $\eta \notin \{\mathrm{cf}(N \cap \kappa), \mathrm{cf}(N \cap \kappa^+), \ldots, \mathrm{cf}(N \cap \kappa^{+n})\}$. Then $N \cap \mathrm{On}$ is η-closed.

Proof. Let $\langle \xi_i : i < \eta \rangle$ be an increasing sequence of ordinals in N. Let $\psi \in N$ be the least element above each ξ_i. Such a ψ exists by the cofinality assumptions. Let $\nu = \mathrm{cf}(\psi)$. Then $\nu < \psi$, $\nu \in N$, and $N \models \nu = \mathrm{cf}(\psi)$. Hence, $\eta = \mathrm{cf}(N \cap \psi) = \mathrm{cf}(N \cap \nu)$. Since ν is regular, we must have either $\nu < N \cap \kappa$ or $\nu = \kappa^{+i}$ some i. The latter cannot happen by the cofinality assumptions. Hence $\nu < N \cap \kappa$, and so $\eta = \nu$. Thus $\psi = \sup\{\xi_i : i < \eta\}$ as desired. ⊣

From this we can deduce the following result:

4.42 Theorem. *Suppose that κ is a regular cardinal and I is a normal, fine, κ-complete, κ^{+n}-saturated ideal on $Z \subseteq P(\kappa^{+n})$ and there is an infinite cardinal $\lambda < \kappa$ such that for all $0 \le i \le n$, $\{z : \mathrm{cf}(z \cap \kappa^{+i}) \ne \lambda\} \in \breve{I}$. Then there is an $A \in \breve{I}$ such that the function $z \mapsto \sup(z)$ is one-to-one on A.*

Proof. By normality we can assume that for all $z \in Z$, if N is the Skolem hull of z in $H(\kappa^{+n})$, then $N \cap \kappa^{+n} = z$. Hence by Proposition 4.41, we can assume that every z in Z is λ-closed.

Partition the ordinals in κ^{+n} of cofinality λ into κ^{+n} disjoint stationary sets, $\{S_\alpha : \alpha \in \kappa^{+n}\}$.

Let $G \subseteq P(Z)/(I{\restriction}Z)$ be generic and $j : V \to M \subseteq V[G]$ be the generic elementary embedding. Then $\mathrm{crit}(j) \ge \kappa$. Let S^j_δ be the δth member of $j(\{S_\alpha : \alpha \in \kappa^{+n}\})$. We claim that in M for all $\alpha < j(\kappa^{+n})$:

$$S^j_\alpha \cap \sup(j``\kappa^{+n}) \text{ is stationary iff } \alpha \in j``\kappa^{+n}.$$

By the κ^{+n}-c.c., the forcing $P(Z)/(I{\restriction}Z)$ preserves stationary subsets of κ^{+n}. By Proposition 4.39, we see that $j``\kappa^{+n}$ is $j(\lambda)$-closed. Since $\lambda < \kappa$, $j(\lambda) = \lambda$. Thus in $V[G]$, $\lambda$ is a cardinal and $j``\kappa^{+n}$ is λ-closed.

For $\beta \in \kappa^{+n}$, $S_\beta \subseteq (\kappa^{+n})^V$ is still stationary in $V[G]$, so $j``S_\beta$ is stationary in $\sup(j``\kappa^{+n})$. Since $j``S_\beta \subseteq S^j_{j(\beta)}$ we have shown one direction of the result.

Suppose that S^j_δ is stationary in $\sup(j``\kappa^{+n})$. Since $j``\kappa^{+n}$ is λ-closed, there is a γ such that $j(\gamma) \in S^j_\delta \cap j``\kappa^{+n}$. But $\gamma \in S_\alpha$ for a unique α, so $j(\alpha) = \delta$, as required.

By reflection, $\{N \in Z : \alpha \in N \text{ iff } S_\alpha \cap \sup(N) \text{ is stationary}\} \in \breve{I}$. Hence there is a measure one subset of Z on which the supremum function is one-to-one. ⊣

4.43 Corollary. *Under the assumptions of the theorem, there is a set $Z \in \breve{I}$ such that $|Z| = \kappa^{+n}$.*

4.44 Corollary. *Suppose that κ is a regular cardinal and I is a normal, fine, κ-complete, κ^{+n}-saturated ideal on $Z \subseteq P(\kappa^{+n})$, and either:*

1. *$\kappa > \omega_1$ is a successor cardinal and I is κ^+-saturated, or*

2. *$\kappa > \omega_{n+1}$.*

Then there is a finite partition P of an $A \in \breve{I}$ such that the supremum function is one-to-one on each element of P.

Proof. If the first hypothesis holds then by Proposition 4.28, we see that for all $0 < i \le n$, κ^{+i} is preserved, and $\mathrm{ot}(z \cap \kappa^{+i}) = \kappa^{+i-1}$ for almost every $z \in Z$. Since $\kappa > \omega_1$, we see that at least one of $\lambda = \omega$ or $\lambda = \omega_1$ satisfies the hypothesis of Theorem 4.42. Hence there is a set $A \in \breve{I}$ on which the supremum function is one-to-one.

If $\kappa > \omega_n$, then we can divide Z into $\{A_i : 0 \leq i \leq n\}$ where $\bigcup A_i \in \breve{I}$ and for all $z \in A_i$ we have $\aleph_i \notin \{\mathrm{cf}(z \cap \kappa), \mathrm{cf}(z \cap \kappa^+), \ldots, \mathrm{cf}(z \cap \kappa^{+n-1}), \mathrm{cf}(z)\}$. We can then apply Theorem 4.42 to each $I \restriction A_i$ separately. ⊣

4.45 Proposition. *Suppose that κ is an uncountable regular cardinal and $\delta \geq \kappa$ is a cardinal. Let I be a normal, fine, κ-complete ideal on $Z \subseteq P(\delta)$ such that $P(Z)/I$ contains a dense countably closed set. Then there is a set $Z \in \breve{I}$ such that the function $z \mapsto \sup(z)$ is one-to-one.*

Proof. If $P(Z)/I$ contains a countably closed dense set, then

1. $M^\omega \cap V[G] \subseteq M$,
2. $\mathrm{cf}(\alpha)^V = \omega$ iff $\mathrm{cf}(\alpha)^{V[G]} = \omega$, and
3. $j``\delta$ is ω-closed.

Taking $\lambda = \omega$ we can follow the proof Theorem 4.42. ⊣

The hypotheses of the previous theorem typically hold for induced ideals with critical point at least ω_2 that arise from collapsing large cardinals.

So far we have not addressed the issue of ideals whose associated elementary embedding has critical point ω_1. By Corollary 6.16, the analogue to the previous proposition cannot hold for ideals that are not $\aleph_2$-complete. We must content ourselves to note that by Theorem 5.9 if there is a countably complete $\aleph_1$-dense ideal on ω_2 then CH holds. In particular:

4.46 Proposition. *Suppose that I is a normal, fine, countably complete, $\aleph_1$-dense ideal on $P(\omega_n)$, then there is a set $Z \in \breve{I}$ that has cardinality $\aleph_n$.*

Proof. By Example 4.30, I concentrates on $[\omega_n]^{\omega_{n-1}}$. By Theorems 5.9 and 5.10, GCH holds below ω_n. In particular, $[\omega_n]^{\omega_{n-1}}$ has cardinality ω_n. ⊣

4.7. Iterating Ideals

We now give a brief discussion of the theory of iterations of embeddings induced by ideals.[27] The theory is highly analogous to that of iterating large cardinal embeddings discovered by Gaifman [49] and Kunen [77]; however, one has to take into account the forcing. We will discuss the case where we are iterating the embeddings coming from a single ideal. The reader can generalize this to more complicated situations, such as generic iteration trees.

For the rest of this discussion, let I be a precipitous ideal on a set Z lying in a transitive model M of a sufficiently strong subtheory of ZFC. We do not assume that M contains all of the ordinals. A *generic iteration* of M by I is a triple

$$\{\langle M_\alpha : \alpha \leq \mu\rangle, \langle j_{\alpha,\alpha'} : \alpha < \alpha' \leq \mu\rangle, \langle G_\alpha : \alpha < \mu\rangle\}$$

with the following properties:

[27] Iterations of generic embeddings were first used in [35].

1. $j_{\alpha,\alpha'} : M_\alpha \to M_{\alpha'}$ is an elementary embedding for all $\alpha < \alpha' \leq \mu$ and the family of embeddings commutes,
2. $M_0 = M$,
3. G_α is generic for $j_{0,\alpha}(P(Z)/I)$ over M_α,
4. $M_{\alpha+1}$ is the generic ultrapower of M_α by G_α and $j_{\alpha,\alpha+1}$ is the canonical embedding from the ultrapower, and
5. for α' a limit ordinal, $M_{\alpha'}$ is the direct limit of the M_α for $\alpha < \alpha'$.

An obvious question is whether generic iterations yield well-founded models. Since the ideal I is precipitous, its images are precipitous in the appropriate models. Hence $M_{\alpha+1}$ is well-founded provided that M_α is.

The following theorem of Woodin extends work of Gaifman [49] and Solovay (see [35]):

4.47 Theorem (Woodin [126]). *Suppose that*

$$\{\langle M_\alpha : \alpha \leq \mu\rangle, \langle j_{\alpha,\alpha'} : \alpha < \alpha' \leq \mu\rangle, \langle G_\alpha : \alpha < \mu\rangle\}$$

is a generic iteration and $\mu < \mathrm{On} \cap M$. Then M_μ is well-founded.

Proof. We outline the proof of the theorem. The first step is to show that if there is an ill-founded generic iteration of M then there is one in a generic extension of M. To see this we note that if there is an ill-founded iteration of M of length μ, then in $V^{\mathrm{Col}(\omega,2^{2^\mu})}$ all of the data required to define this information is countable. Hence in $M^{\mathrm{Col}(\omega,2^{2^\mu})}$ there is a tree whose ill-foundedness is equivalent to the existence of an ill-founded generic iteration of M. Hence the existence of such an iteration is absolute between V and $M^{\mathrm{Col}(\omega,2^{2^\mu})}$, and the first step is accomplished.[28]

The second step is to use a variation of Gaifman's arguments about iterated ultrapowers: In M, let μ be the smallest ordinal such that there is a generic iteration of length μ lying in a forcing extension of M that is ill-founded. Let η be the least ordinal such that there is a some generic iteration of length μ such that $j_{0,\mu}(\eta)$ is above a decreasing infinite sequence of "ordinals" in M_μ.

Arguments similar to the argument of the first step show that η, μ have absolute definitions; i.e. M correctly defines η and μ.

Consider an ill-founded generic iteration of length μ witnessing the definitions of η and μ. Then μ is a limit ordinal, so for some $\alpha < \mu$, there is an ordinal $\eta' < j_{0,\alpha}(\eta)$ such that $j_{\alpha,\mu}(\eta')$ is above an infinite decreasing sequence of M_μ-ordinals. Let η' be the least such.

But again, we can argue that M_α correctly identifies η' as the least ordinal above an infinite decreasing sequence of "ordinals" in the shortest ill-founded generic iteration beginning at M_α. However, $\eta' < \eta$, contradicting the elementarity of $j_{0,\alpha}$. ⊣

[28] Note that we have used the fact that the length of the iteration is less than the supremum of the ordinals of M.

4.8. Generic Ultrapowers by Towers

The concept of a *tower* of ideals was given in Definition 4.17. We now explore generic ultrapowers by towers of ideals.

Suppose that $\langle U, <_U\rangle$ is a linearly ordered set and we have a tower of ideals $\mathcal{T} = \langle I_a : a \in U\rangle$ on sets $\langle Z_a : a \in U\rangle$ via projections $\langle \pi_{a,a'} : a' <_U a\rangle$. Let $\mathcal{B}_a = P(Z_a)/I_a$. For $a' <_U a$ we get a well-defined embedding $i_{a',a} : \mathcal{B}_{a'} \to \mathcal{B}_a$ given by the formula, $i_{a',a}([A']_{I_{a'}}) = [\{z \in Z_a : \pi_{a,a'}(z) \in A'\}]_{I_a}$. We can form the direct limit of the system of Boolean algebras $\langle \mathcal{B}_a, i_{a',a}\rangle$ and call it $\mathcal{B}_\infty(\mathcal{T})$.

By a slight modification of Proposition 4.22, we see that a generic $G \subseteq \mathcal{B}_\infty(\mathcal{T})$ yields ultrafilters $G_a \subseteq \mathcal{B}_a$ and if N_a is the ultrapower V^{Z_a}/G_a and $j_a : V \to N_a$ is the canonical embedding then there are embeddings $k_{a',a} : N_{a'} \to N_a$ ($a' <_U a$) such that the following commutes:

$$\begin{tikzcd} & V \arrow[dl, "j_{a'}"'] \arrow[dr, "j_a"] & \\ N_{a'} \arrow[rr, "k_{a',a}"'] & & N_a \end{tikzcd}$$

We will call $N_\infty = \lim\limits_{\to}\langle N_a, k_{a',a} : a' <_U a\rangle$ the *generic ultrapower of* V *by the tower*.

In concrete terms, we usually have $Z_\alpha \subseteq P(\alpha)$ for some collection U of ordinals α and $\pi_{\alpha,\alpha'}(z) = z \cap \alpha'$. We can define a partial ordering $\mathcal{P}_\mathcal{T}$ whose domain is a quotient of $\bigcup_{\alpha\in U} P(Z_\alpha)$ by an equivalence relation. For $\alpha' < \alpha$ and $A' \in P(Z_{\alpha'})$ and $A \in P(Z_\alpha)$ we set $A' \sim A$ iff $\iota_{\alpha',\alpha}([A']_{I_{\alpha'}}) = [A]_{I_\alpha}$ and $[A'] \le [A]$ iff $\iota_{\alpha',\alpha}(A') \subseteq_{I_\alpha} A$.

Viewing $\mathcal{P}_\mathcal{T}$ as the direct limit of the partial orderings $P(Z_\alpha)/I_\alpha$, the non-zero elements of $\mathcal{B}_\infty(\mathcal{T})$ can be identified with elements of $P_\mathcal{T}$. With $\delta = \sup(U)$, $\mathcal{B}_\infty(\mathcal{T})$ can also be described as equivalence classes of those subsets $a \subseteq P(\delta)$ such that for some $\alpha \in U$ and all $z, z' \in P(\delta)$, if $z \cap \alpha = z' \cap \alpha$, then $z \in a$ iff $z' \in a$. The ordinal α can be viewed as a *support* of a. If a is a subset of $P(\delta)$ with support α then for any $\beta > \alpha$, we can identify a with $[\{z \cap \beta : z \in a\}]_{I_\beta}$ and consider its class in $P(Z_\beta)/I_\beta$. If $a, b \subseteq P(\delta)$ have supports less than δ we can consider any β that is a support of both a and b and set $a \le_\infty b$ iff $[a]_{I_\beta} \le [b]_{I_\beta}$. The ordering $\le_\infty$ is well-defined since the ideals form a tower, and $\mathcal{B}_\infty(\mathcal{T})$ is isomorphic to the separative quotient of the resulting partial ordering.

4.48 Remark. We note that we will often write formulas like "$N \in S$", where $S \subseteq P(\alpha)$ but N is not a subset of α. This will be interpreted as "$N \cap \alpha \in S$". Thus in the context of towers, we view sets $S \subseteq P(\alpha)$ as subsets of $P(\beta)$ for $\beta > \alpha$.

We can check the following lemma using the proof of Lemma 2.22:

4.49 Lemma. *Suppose that δ is a limit cardinal, $U \subseteq \delta$ is unbounded and $\langle I_\alpha \subseteq P(P(\alpha)) : \alpha \in U\rangle$ is a tower of normal ideals. If $A \subseteq P(P(\alpha))$ for some $\alpha \in U$ then $\Sigma A = \bigtriangledown A$ in $\mathcal{B}_\infty(\mathcal{T})$.*

For notational simplicity we will use the partial ordering $\mathcal{P}_{\mathcal{T}}$ (defined above) instead of $\mathcal{B}_\infty(\mathcal{T})$. For most towers the index set U will be a collection of ordinals and for $\alpha \in U$, I_α will be an ideal on $P(\alpha)$. In this case we will call $\sup(U)$ the *height* of the tower $\mathcal{T}$ and it will usually be denoted δ. Burke [11] proved the following result as a generalization of Corollary 4.21:

4.50 Proposition. *Suppose that $\mathcal{T}$ is a tower of normal ideals of height δ, which is an inaccessible cardinal. Then there is a stationary set $S \subseteq P(\delta)$ such that for all $\alpha \in U$, I_α is the projection to $P(\alpha)$ of the nonstationary ideal on $P(\delta)$ restricted to S.*

4.51 Definition. The tower $\mathcal{T}$ is said to be *precipitous* iff for all generic $G \subseteq \mathcal{P}_{\mathcal{T}}$, the direct limit N_∞ of the ultrapowers N_α is well-founded.

The combinatorial criterion for precipitousness of towers takes a slightly different form than that as would be given by forcing with individual ideals. The analogue of Proposition 2.7 in the context of towers of ideals is:

4.52 Proposition. *Let $\mathcal{T}$ be a tower of ideals. Then $\mathcal{T}$ is precipitous iff whenever $[X] \in \mathcal{P}_{\mathcal{T}}$ and $\mathcal{A}_n \subseteq \bigcup_{a\in U} P(Z_a)$ for $n \in \omega$ form a tree of maximal antichains in $\mathcal{P}_{\mathcal{T}}$ below $[X]$, there are $\langle a_n : n \in \omega\rangle$ and $s : \omega \to U$ such that:*

1. *$a_n \in \mathcal{A}_n \cap P(Z_{s(n)})$,*
2. *$\langle [a_n]_{I_{s(n)}} : n \in \omega\rangle$ is a branch through the tree, and*
3. *there is a sequence $z_n \in Z_{s(n)}$ with $\pi_{s(n+1),s(n)}(z_{n+1}) = z_n$ and $z_n \in a_n$.*

In the concrete case where U is a set of ordinals, $Z_\alpha \subseteq P(\alpha)$ and $\pi_{\alpha,\alpha'}(z) = z \cap \alpha'$, the conditions of Proposition 4.52 can be stated by taking $s(n)$ to be the support of a_n and demanding that there is a set N such that for all n, $N \cap P(s(n)) \in a_n$.

Burke proved the game theoretic version of precipitousness for towers corresponding to Theorem 2.8:

4.53 Theorem (Burke [11]). *Let $\mathcal{T}$ be a tower of ideals on $P(\alpha)$ for $\alpha \in U$ such that for $\alpha' < \alpha$, $\pi_{\alpha,\alpha'}(z) = z \cap \alpha'$. Then the following are equivalent:*

1. *$\mathcal{T}$ is precipitous.*
2. *Player I does not have a winning strategy in the following game: I and II alternately play sets $p_n \in \bigcup_{\alpha\in U} P(\alpha)$ such that:*
   (a) *if $p_n \in P(\alpha)$ then $[p_n]_{I_\alpha} \in \mathcal{P}_{\mathcal{T}}$, and*
   (b) *$p_0 \geq_{\mathcal{P}_{\mathcal{T}}} p_1 \geq_{\mathcal{P}_{\mathcal{T}}} p_2 \geq_{\mathcal{P}_{\mathcal{T}}} \dots, p_n, \dots$.*

II wins iff there is an $a \subseteq \sup(U)$ *such that* $a \cap \mathrm{supp}(p_n) \in p_n$ *for every* n*.*

Somewhat surprisingly, in fairly general circumstances, towers of ideals automatically have the disjointing property:

4.54 Theorem (Burke [11]). *Suppose that* $\mathcal{T}$ *is a tower of ideals of height* δ *where* δ *is inaccessible. Then for all antichains* $\mathcal{A} \subseteq \mathcal{P}_\mathcal{T}$ *there is a pairwise disjoint collection of sets* $B \subseteq \bigcup_{\alpha\in U} P(\alpha)$ *such that each* $b \in B$ *has support less than* δ *and for all* $a \in \mathcal{A}$ *there is a* $b \in B$ *with* $[b]_{I_{\mathrm{supp}(b)}} = a$ *in* $\mathcal{P}_\mathcal{T}$*.*

Proof (Sketch). Enumerate $\mathcal{A}$ as $\langle a_i : i < \gamma \leq \delta\rangle$. Build $B = \langle b_i : i < \gamma\rangle$ by induction so that $[b_i]_{\mathcal{P}_\mathcal{T}} = a_i$, $b_i \subseteq P(s_i)$ for some $s_i \in U$. If we have built $\langle b_j : j < i\rangle$, we construct b_i by choosing an $s_i \in U$ larger than $\sup\{s_j : j < i\}$ and larger than $\mathrm{supp}(a_i)$. Let a be a representative of $[a_i]$ in $P(\mathrm{supp}(a_i))$ and $b_i = \{z \subseteq P(s_i) : b \cap \mathrm{supp}(a_i) \in a \text{ and } \sup\{s_j : j < i\} \in b_i\}$. ⊣

Unfortunately the disjointing property for towers of ideals is not as useful as for individual ideals: In Sect. 9 we give examples due to Burke of towers of ideals of inaccessible height that are not precipitous.

The property analogous to the disjointing property for individual ideals is the *bounded disjointing property*, i.e. that given any antichain $\mathcal{A} \subseteq \mathcal{P}_\mathcal{T}$ there is a pairwise disjoint collection B of representatives for members of $\mathcal{A}$ such that $\sup\{\mathrm{supp}(b) : b \in B\} < \delta$. This is equivalent to the tower $\mathcal{T}$ being δ-saturated, a fairly rare occurrence in our current state of knowledge. However, being able to locally disjointify a collection of antichains with representatives of bounded support follows from weak saturation properties and yields closure of the generic ultrapower. This is the usual situation for presaturated towers. The following is the analogue of Proposition 4.8 in the context of towers, and can be proved similarly:

4.55 Proposition. *Suppose that* $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ *is a tower of normal, fine ideals with height* δ*,* δ *is inaccessible and* $\mathcal{P}_T$ *is weakly* $(\aleph_1, \delta)$*-saturated. Then* $\mathcal{T}$ *is precipitous. Further, if* $\mathcal{P}_\mathcal{T}$ *is weakly* (η^+, δ)*-saturated* $(\omega \leq \eta < \delta)$*, and* $G \subseteq \mathcal{P}_T$ *is generic then* $N_\infty^\eta \cap V[G] \subseteq N_\infty$*.*

Proof (Sketch). Suppose that $\mathcal{T}$ is weakly $(\aleph_1, \delta)$-saturated. Let $\langle \mathcal{A}_n : n \in \omega\rangle$ be a tree of maximal antichains and $[S] \in \mathcal{T}$ a condition. By the weak saturation we can find a $[T] \leq_{\mathcal{P}_\mathcal{T}} [S]$ such that for each n, $\kappa_n =_{\mathrm{def}} |\{a \in \mathcal{A}_n : [a \cap T] \in \mathcal{P}_\mathcal{T}\}| < \delta$. Since δ is inaccessible we can find an $\alpha < \delta$ such that for all n and all $a \in \mathcal{A}_n$, if a is compatible with $[T]$ then the support of a is an ordinal less than α. Moreover, by increasing α slightly we can assume that α is regular and $\kappa_n < \alpha$.

By the normality of I_α, we can disjointify $a \in \mathcal{A}_n$ by choosing subsets of each a that have support α. Call the resulting antichains $\mathcal{A}'_n$. By shrinking still further we can assume that the $\mathcal{A}'_n$ still form a tree of antichains. Since $\mathcal{A}'_n$ is an I_α maximal antichain below $[T]$,

$$T =_{I_\alpha} \bigcap_n \bigcup \mathcal{A}'_n.$$

If $z \in T \cap \bigcap_n \bigcup \mathcal{A}'_n$ then z determines a branch through $\mathcal{A}'$ and hence a branch through $\mathcal{A}$. Let $a_n \in \mathcal{A}_n$ be the element of $\mathcal{A}_n$ given by z. Let $s(n)$ be the support of a_n and $z_n = z \cap s(n)$. Applying Proposition 4.52, we see that $\mathcal{T}$ is precipitous.

If $\mathcal{P}_\mathcal{T}$ is weakly (η^+, δ)-saturated, and $\langle \mathcal{A}_\alpha : \alpha \in \eta \rangle$ is a sequence of maximal antichains, then densely often in $\mathcal{P}_\mathcal{T}$ we can find a $[T]$ such that for all α, $\mathcal{A}_\alpha$ can be disjointified below $[T]$. It follows that if we are given a set $\{\dot{F}_\alpha : \alpha < \eta\}$ of terms for elements of N_∞ then there is a dense collection of $[T]$ for which we can find a sequence of functions $\{f_\alpha : \alpha < \eta\}$ such that the domain of each f_α is $Z_{\mathrm{supp}(T)}$ and

$$[T] \Vdash [\dot{F}_\alpha]_{N_\infty} = [f_\alpha]_{N_\infty}.$$

Let $\beta \in U$ be bigger than $\max(\mathrm{supp}(T), \eta)$ and let $: Z_\beta \to V$ be defined by setting

$$g(z) = \{f_\alpha(z \cap \mathrm{supp}(T)) : \alpha \in z \cap \eta\}.$$

Then $[g]_{N_\infty} = \{[\dot{F}_\alpha]_{N_\infty} : \alpha \in \eta\}$. ⊣

Tracking where ordinals go in ultrapowers by towers is highly analogous to the situation when forcing with individual ideals. One can easily verify:

4.56 Proposition. *Let $\mathcal{T}$ be a precipitous tower of normal and fine ideals of height δ. Let ρ and $\lambda < \delta$, $G \subseteq \mathcal{P}_\mathcal{T}$ be generic and $j : V \to M$ be the canonical generic embedding determined by G. Then:*

1. *if $\{z : z \cap \rho \in \rho\} \in G$, then $\mathrm{crit}(j) = \rho$,*
2. *if each I_α is λ-complete, then $\mathrm{crit}(j) \geq \lambda$,*
3. *if $\{z : \mathrm{ot}(z \cap \lambda) = \rho\} \in G$, then $j(\rho) = \lambda$, and*
4. *if $\{z : \mathrm{ot}(z \cap \lambda) < \rho\} \in G$, then $j(\rho) > \lambda$.*

From Propositions 4.56 and 4.55, we can get most of the information about the "three parameters" determining the strength of a generic embedding. For example, if $\mathcal{T}$ is a tower of height δ and:

1. for all α, I_α concentrates on $[\alpha]^{<\rho}$ and is ρ-complete, and
2. the tower $\mathcal{P}_\mathcal{T}$ is weakly (ρ, δ)-saturated,

then the critical point of j is ρ, $j(\rho) = \delta$ and M is closed under $<\rho$-sequences from $V[G]$.

If in addition, $\rho = \mu^+$ in V, then all of the ordinals less than δ have cardinality μ in $V[G]$. Hence closure under $<\rho$ sequences from $V[G]$ implies closure under $<\delta$ sequences from $V[G]$.

There is an intimate relationship between towers of ideals and ideals on a single set in a generic extension of the universe. For example, suppose that $\mathcal{T}$ is a tower of μ-complete ideals on $\langle P(\lambda_\alpha) : \alpha \in U \rangle$ and $\sup\{\lambda_\alpha \mid \alpha \in U\} = \delta$, where δ is an inaccessible cardinal. Then in $V^{\mathrm{Col}(\mu,<\delta)}$, each ideal I_α becomes isomorphic to an ideal on $P(\mu)$ and these ideals cohere in the obvious way. Unfortunately, the normal closure of the union of the ideals may fail to remain proper.

However there are circumstances where the limit ideal on μ can be shown to be proper. This occurs, for example, in the stationary tower of ideals restricted to the internally approachable sets. The resulting limit ideal is the nonstationary ideal on μ in the extension. An example of this is given in Proposition 9.4.

These topics are discussed at length in Sect. 9.

5. Consequences of Generic Large Cardinals

In this section we discuss various consequences of the existence of strong ideals. Indeed these consequences are so ubiquitous and striking that ideal axioms have been proposed as candidates for axioms for set theory. We discuss this in Sect. 11.

In our current state of knowledge, ideal assumptions divide into two mutually contradictory collections. One can loosely be termed "generic large cardinals".[29] These are the axioms that can be stated in terms of the "three parameters" that determine a generic elementary embedding $j : V \to M$, which again are: where j sends ordinals, the closure properties of M, and the nature of the forcing required to define j. The consequences of generic large cardinals affect virtually all of set theory and settle many of the classical problems of set theory. These ideal properties are easily seen to be rather straightforward generalizations of conventional large cardinal axioms, suitably modified so that they can "live" on relatively small cardinalities such as ω_1.

The second collection of ideal assumptions has just one essential element that we know of: the assumption that the nonstationary ideal on ω_1 is $\aleph_2$-saturated. In the presence of a measurable cardinal, this assumption implies that the Continuum Hypothesis fails in a particularly dramatic way, and that $L(\mathbb{R})$ is quite close to the universe, in a precise way. This assumption fits well with the $\mathbb{P}_{\max}$ theory of Woodin.[30]

At the moment the author has no strongly compelling reason to prefer the former collection of results over the latter. One could hope perhaps, that there is some happy reconciliation such as was devised between the Axiom of Choice and the Axiom of Determinacy.

5.1. Using Reflection

We illustrate the use of reflection in the context of saturated ideals with two simple examples.

[29] As this chapter went to the publishers, Woodin showed that there were more incompatible pairs of generic large cardinal axioms.
[30] See P. Larson's chapter in this Handbook for more information.

5.1 Theorem. *Suppose that there is a partial ordering $\mathbb{P}$ such that for all generic $G \subseteq \mathbb{P}$ there is an elementary embedding $j : V \to M$ such that for some $n \in \omega$:*

1. $j(\omega_{n-1}) = \omega_n$,
2. $j``\omega_n \in M$, *and*
3. $\mathbb{P}$ *is* $\aleph_1$*-centered.*

Then:

1. *If G is a graph on ω_k with $1 < k \leq n$ and G has chromatic number $\omega_{k'}$ with $1 < k' \leq k$, then G has an induced subgraph of cardinality ω_{k-1} and chromatic number $\omega_{k'-1}$, and*
2. *if G is a group (resp. Abelian group) of cardinality ω_k with $1 \leq k \leq n$ and every subgroup of G of cardinality less than ω_k is free*[31] *(resp. free Abelian), then $G = \bigcup_{\alpha<\omega_1} G_\alpha$ where G_α is free (free Abelian).*

For a given graph or group G, we can draw the same conclusions as Theorem 5.1 if we weaken the assumptions to asking that the critical point of j is ω_1 and the forcing for producing j is of the form $\mathbb{P} * \mathbb{Q}$ where $\mathbb{P}$ is $\aleph_1$-centered and $\mathbb{Q}$ is $|G|^V$-closed in $V^{\mathbb{P}}$. An example of this type of result is the next theorem whose hypotheses are shown consistent in Theorem 7.70.

5.2 Theorem. *Suppose that there is a partial ordering of the form $\mathbb{P} * \mathbb{Q}$ such that if $G * H \subseteq \mathbb{P} * \mathbb{Q}$ is generic, then in $V[G * H]$ there is a generic elementary embedding $j : V \to M$ such that:*

1. $\mathrm{crit}(j) = \omega_1$,
2. $j``\omega_2 \in M$, *and*
3. $\mathbb{P}$ *is* $\aleph_1$*-centered and* $\mathbb{Q}$ *is* $<(\omega_2)^V$*-closed in* $V[G]$.

Then

1. *If G is a graph on ω_2 with chromatic number ω_2 then G has a subgraph of size $\aleph_1$ and chromatic number ω_1, and*
2. *if G is a group of cardinality $\aleph_2$ and every subgroup of cardinality $\aleph_1$ is free, then $G = \bigcup_{\alpha<\omega_1} G_\alpha$, where each G_α is a free group.*

Proof of Theorem 5.1. Let $j : V \to M \subseteq V[H]$ be the elementary embedding associated with a generic $H \subseteq \mathbb{P}$. Since $\mathbb{P}$ is $\aleph_1$-centered, $\mathbb{P}$ preserves all cardinals above ω_1, and hence the critical point of j must be ω_1. It follows that for all $0 < k < n$, $j(\omega_k) = \omega_{k+1}^V$. The assumption that $j``\omega_n \in M$ implies that $P(\omega_n)^V \subseteq M$.

[31] I.e. G an almost free group.

Let $\mathbb{P} = \bigcup_{\alpha<\omega_1} F_\alpha$ where F_α is a filter on $\mathbb{P}$. Suppose that G is a graph on ω_k of chromatic number $\omega_{k'}$. Then G is isomorphic to the graph on $j``G$ induced by $j(G)$ on $j``\omega_k$ and $j``G$ has cardinality $\omega^M_{k-1}$. By elementarity, it suffices to show that $j``G$ has chromatic number $\omega_{k'-1}$ in M.

If this fails then there is a coloring $C : j``G \to \kappa$ lying in M where $\kappa < \omega^M_{k'-1} = \omega^V_{k'}$. For each $g \in G$ choose a $p_g \in \mathbb{P}$ such that $p_g \Vdash \dot{C}(j(g)) = \alpha$ for some $\alpha < \kappa$. In V, define $C' : G \to \kappa \times \omega_1$ by setting $C'(g) = (\alpha, \beta)$ where $p_g \in F_\beta$ and $p_g \Vdash \dot{C}(j(g)) = \alpha$. Then this is a coloring of G into $|\kappa \times \omega_1| < \omega_{k'}$ colors, a contradiction.

Now suppose that G is a group of cardinality ω_k such that every subgroup of cardinality ω_{k-1} is free. Then $j``G$ is a subgroup of G of cardinality ω^M_{k-1}, and hence G is free in M. Let $\dot{A}$ be a term for a free generating set for G. In V, for $\alpha < \omega_1$, let G_α be the subgroup of G generated by $\{g :$ for some $p \in F_\alpha, p \Vdash g \in A\}$. ⊣

Results of the type of 2 are clearly general for many algebraic objects where there is a suitable notion of freeness.

5.2. Chang's Conjectures, Jónsson Cardinals and Square

We begin with a simple remark [35]:

5.3 Proposition. *Let $j : V \to M$ be a generic elementary embedding.*

1. *If $\mathfrak{A}$ is a structure on a cardinal λ, then $j``\lambda \prec j(\mathfrak{A})$.*
2. *If $\lambda' < \lambda$ is a cardinal, and*
   - *(a) $j``\lambda \in M$,*
   - *(b) $j(\kappa) = \mathrm{ot}(j``\lambda)$, and*
   - *(c) $j(\kappa') = \mathrm{ot}(j``\lambda')$,*

   then $(\lambda, \lambda') \twoheadrightarrow (\kappa, \kappa')$.

Proof. Let $\mathfrak{A}$ be an algebra on a cardinal λ. We can assume that $\mathfrak{A}$ is fully Skolemized. Let f be a Skolem function and $\vec{\alpha} \in [j``\lambda]^{<\omega}$. Then there is a $\vec{\beta}$ such that $j(\vec{\beta}) = \vec{\alpha}$. Let $\beta^* = f(\vec{\beta})$ and $\alpha^* = j(\beta^*)$. Then $\alpha^* = j(f)(\vec{\alpha})$ and $\alpha^* \in j``\lambda$. Thus $j``\lambda$ is closed under the Skolem functions of $j(\mathfrak{A})$.

For the second assertion, we see that $M \models \mathrm{ot}(j``\lambda) = j(\kappa)$, and $\mathrm{ot}(j``\lambda \cap j(\lambda')) = j(\kappa')$. Thus we know that $M \models$ "there is an elementary substructure of $j(\mathfrak{A})$ of type $(j(\kappa), j(\kappa'))$". The result now follows from the elementarity of j. ⊣

We saw in Proposition 3.9, that $(\lambda, \lambda') \twoheadrightarrow (\kappa, \kappa')$ is equivalent to the existence of a proper, normal, fine, countably complete ideal on $P(\lambda)$ concentrating on $\{z : \mathrm{ot}(z) = \kappa$ and $\mathrm{ot}(z \cap \lambda') = \kappa'\}$, namely the Chang ideal.

A sort of converse of Proposition 5.3 is true: If there is a Woodin cardinal μ above λ and $(\lambda,\lambda') \twoheadrightarrow (\kappa,\kappa')$, then in $V^{\mathrm{Col}(\lambda,<\mu)}$, the Chang ideal is precipitous (Theorem 8.37). This was shown in [47] when μ is a supercompact cardinal. The version of the theorem with the optimal hypothesis stated here was published in [58].

Since Chang's Conjecture principles and square principles are antithetical combinatorial properties, we get the next corollary. This result and more general versions appear in [37].

5.4 Corollary. *Let $\mu < \kappa$ be cardinals and $\kappa < \aleph_{\mu^+}$. Suppose that there is a proper, normal, fine ideal on $[\kappa^+]^{\mu^+}$ that concentrates on $\{x : \mathrm{ot}(x \cap \kappa) = \mu\}$. Then $\square_\kappa(\mathrm{Cof}(\leq\mu))$ fails. Hence:*

If there is a generic elementary embedding $j : V \to M$ such that

1. *$j``\kappa^+ \in M$, and*
2. *$j(\mu) = \mathrm{ot}(j``\kappa)$,*

then $\square_\kappa(\mathrm{Cof}(\leq\mu))$ fails.

By adjusting the cardinals involved in Chang's Conjecture we get many consequences, as we shall prove later. For example, if κ is regular and $(\kappa^+,\kappa) \twoheadrightarrow (\kappa,<\kappa)$, then there are no Kurepa trees on κ. Hence, for example, if there is a generic elementary embedding $j : V \to M$ such that $j(\omega_1) = \omega_2$ and $j``\omega_2 \in M$, then there are no Kurepa trees on ω_1.

Chang's Conjecture also has GCH consequences: If $(\lambda^{+2},\lambda) \twoheadrightarrow (\kappa^{+2},\kappa)$ and $2^\kappa = \kappa^+$, then $2^\lambda = \lambda^+$.

It is also possible to see that Jónsson cardinals exist assuming the appropriate generic elementary embeddings:

5.5 Proposition. *Suppose that there is a generic elementary embedding $j : V \to M$ such that:*

1. *$\mathrm{crit}(j) < \kappa$,*
2. *$j``\kappa \in M$, and*
3. *$j(\kappa) = \kappa$,*

then κ is Jónsson.

Again, this is almost an equivalence: If $\mu > \kappa$ is Woodin then κ is Jónsson iff $V^{\mathrm{Col}(\kappa^+,<\mu)} \models$ "there is a proper, normal, fine, precipitous ideal on $[\kappa^+]^{<\kappa^+}$ concentrating on $\{z : |z \cap \kappa| = \kappa$ and $\kappa \not\subseteq z\}$". This follows from Theorem 8.37.

If we have an elementary embedding $j : V \to M$ we define the sequence of cardinals by setting $\kappa_0 = \mathrm{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$, and finally $\kappa_\omega = \sup\{\kappa_n : n \in \omega\}$. To see that κ_ω is Jónsson, formally less than the previous proposition is required:

5.6 Theorem. *Suppose that $\langle \kappa_i : i \in \omega \rangle$ is an increasing sequence of uncountable cardinals with supremum κ_ω. Suppose that there is a sequence of normal, fine, countably complete ideals I_n on $P([\kappa_{n+1}]^{\kappa_n})$ such that the I_n's form a precipitous tower $\mathcal{T}$ of ideals.*[32] *Then κ_ω is Jónsson.*

Proof. Suppose that $\mathfrak{A}$ is a structure with domain κ_ω. By Skolemizing we can assume that if $\mathfrak{A}_n$ is the structure with domain κ_n determined by restricting the functions of $\mathfrak{A}$ to κ_n, and $B \subseteq \kappa_n$ is the domain of an elementary substructure of $\mathfrak{A}_n$, then $\mathrm{Sk}^{\mathfrak{A}}(B) \cap \kappa_n = B$.

Let S be the tree of finite sequences $(z_0, z_1, \ldots, z_n)$ such that:

1. $z_i \in [\kappa_{i+1}]^{\kappa_i}$,
2. $z_{i+1} \cap \kappa_{i+1} = z_i$, and
3. $z_i \prec \mathfrak{A}_{i+1}$

ordered by extension. It suffices to see that S is an ill-founded tree.

Let $G \subseteq \mathcal{P}_{\mathcal{T}}$ be generic for the tower of ideals and $j : V \to N_\infty$ be the generic ultrapower of V by the tower. We claim that $j(S)$ is ill-founded in N_∞. Since N_∞ is well-founded, by absoluteness, it suffices to show that $j(S)$ is ill-founded in $V[G]$. But this is immediate since the sequence $\langle j``\kappa_{i+1} : i \in \omega \rangle$ is a branch through $j(S)$. ⊣

5.3. Ideals and GCH

In this section we consider the consequences strong ideals can have on the Generalized Continuum Hypothesis, deferring the case of the nonstationary ideal on ω_1 to its own subsection.

We begin with an easy example of an axiom that implies instances of GCH:

5.7 Proposition. *Suppose that there is a generic elementary embedding $j : V \to M$ with critical point κ^+ in a (κ, ∞)-distributive forcing extension of V. Then $\kappa^{<\kappa} = \kappa$ holds.*

Since it is easy to see the consistency of the statement there is a normal $\aleph_2$-complete ideal I on ω_2 such that $P(\omega_2)/I$ has a dense countably closed subset, it is interesting to note the following corollary:

5.8 Corollary. *Suppose that there is an ideal I on a set Z that has completeness ω_2 such that $P(Z)/I$ has a dense countably closed subset. Then* CH *holds.*

Proof of Proposition 5.7. Let $\vec{r} = \langle r_\alpha : \alpha < \kappa^{<\kappa} \rangle$ be a one-to-one enumeration of $[\kappa]^{<\kappa}$. If $\kappa^{<\kappa} > \kappa$, then $j(\vec{r}) \neq \vec{r}$ and hence there is an element of $[\kappa]^{<\kappa}$ in M that is not in V. This contradicts the distributivity of the forcing. ⊣

We note that the axiom stating that "ω_n is generically supercompact by $<\omega_n$-closed forcing" implies the hypothesis of Example 5.7 and hence that $\omega_n^{<\omega_n} = \omega_n$.

[32] See Definition 4.51.

Woodin's Theorem Showing CH Holds

Perhaps the first clue that $\aleph_1$-dense ideals can have an affect on the value of $2^{\aleph_0}$ was Taylor's Theorem 4.3, which showed that if there is a countably complete, $\aleph_1$-dense ideal on ω_1 then MA_{ω_1} fails.

In [35], the author showed that strong ideal axioms implied GCH. In later work the author showed that if there is a generic elementary embedding $j : V \to M \subseteq V[G]$ where $G \subseteq \mathbb{P}$ is generic and:

1. $j(\omega_1) = \omega_2$,
2. $j``\omega_2 \in M$, and
3. $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$,

then the Continuum Hypothesis holds. Woodin improved both of these by showing that the existence of an $\aleph_1$-dense ideal on ω_2 implies the Continuum Hypothesis.

5.9 Theorem (Woodin). *Suppose that there is a uniform, countably complete $\aleph_1$-dense ideal on ω_2. Then the Continuum Hypothesis holds.*

Proof (Sketch). We will use the forthcoming Lemma 6.23, essentially due to Gitik and Shelah. That lemma is shown in its appropriate context in Sect. 6.6, where it serves as the main ingredient for the proof of Theorem 6.22 there. Woodin made the appropriate modifications in the lemma so that it now serves also to prove the present theorem. The thrust of the proof is that Lemma 6.23 shows that four hypotheses I–IV cannot jointly hold, yet the existence of the dense ideal together with the failure of CH implies that these hypotheses simultaneously hold after all.

Since I has the disjointing property, it is precipitous and if $G \subseteq P(\omega_2)/I$ is generic and $j : V \to M \subseteq V[G]$ is the generic ultrapower then $M^\omega \cap V[G] \subseteq M$.

In V, we can define an ideal I' by setting $X \in I'$ iff $1 \Vdash_{P(\omega_2)/I} \omega_1 \notin j(X)$. Then I' is the induced ideal from the ultrafilter $U(j, \omega_1^V)$. By the results in Sect. 4, I' is a projection of I and there is a natural embedding ι from $P(\omega_1)/I'$ to $P(\omega_2)/I$ sending $[X]$ to the Boolean value $\|\omega_1 \in j(X)\|$ in the complete Boolean algebra $P(\omega_2)/I$. Moreover, I' is $\aleph_1$-dense and ι is a complete embedding. For generic $G \subseteq P(\omega_2)/I$ the ultrafilter $U = U(j, \omega_1^V)$ is generic for $P(\omega_1)/I'$.

Form the ultrapower V^{ω_1}/U, and let N be its transitive collapse, and $i : V \to N$ be the canonical embedding. Then $N^\omega \cap V[U] \subseteq N$. Let k be the natural factor map from this ultrapower to M given by $k([f]) = j(f)(\omega_1)$. Then $j = k \circ i$:

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "j"] & \\ N \arrow[rr, "k"'] & & M \end{tikzcd}$$

As in Examples 4.28 and 4.29:

1. $\omega_1^N = \omega_2^V = \omega_1^M$,
2. $(\omega_2)^{V[G]} = (\omega_2)^{V[U]} = \omega_3^V$,
3. $\sup(i``\omega_2^V)$ is cofinal in $i(\omega_2^V)$,
4. $i(\omega_2^V) < \omega_3^V$,
5. $\omega_3^N = i(\omega_3^V) = \omega_3^V$ and there is a set $C \subseteq \omega_3$ in V that is ω-closed in V and for all $\alpha \in C$, $i(\alpha) = \alpha$,
6. $j(\omega_2^V) = \omega_3^V$, and
7. $\mathrm{crit}(k) = \omega_2^N$.

In particular, the cardinal successor of the ordinal ω_2^N computed in N is the same as the cardinal successor of ω_2^N computed in $V[U]$.

We now assume the Continuum Hypothesis fails, and derive a contradiction. Let $\vec{x} = \langle x_\alpha : \alpha < 2^{\aleph_0}\rangle$ be a one-to-one enumeration of the real numbers in V. Then $i(\vec{x})$ is an enumeration of all of the real numbers in $V[U]$ and has length at least $\omega_2^N = \mathrm{crit}(k)$. Hence $k(i(\vec{x})) \neq i(\vec{x})$ and the range of $k(i(\vec{x}))$ differs from the range of $i(\vec{x})$. Since the diagram commutes, $k(i(\vec{x})) = j(\vec{x})$ is an enumeration of the reals in M. Since M has all of the reals of $V[G]$, we see that the collection of real numbers in $V[U]$ is different from the collection of real numbers in $V[G]$. Hence the embedding $\iota : P(\omega_1)/I' \to P(\omega_2)/I$ does not map onto a dense set: $V[U]$ is a proper subclass of $V[G]$.

The relationship between forcing with $P(\omega_1)/I'$ and $P(\omega_2)/I$ is easy to describe. Since both are $\aleph_1$-dense, each forcing is equivalent to forcing with $\mathrm{Col}(\omega, \omega_1)$. If we first force with $P(\omega_1)/I'$, then $P(\omega_2)/I$ has a countable dense set. Hence the quotient forcing from the embedding ι is equivalent to Cohen forcing.[33] If $U \subseteq P(\omega_1)/I'$ is generic, then one can force over $V[U]$ with Cohen forcing to produce a generic real s. In $V[U][s]$ we can recover G, M and k.

This is the main idea of the proof: by adding a Cohen real to $V[U]$ we can force the existence of a non-trivial generic embedding with domain N. Were $N = V[U]$ this would give an immediate contradiction to Theorem 6.22. The heart of this theorem is arguing that N is close enough to $V[U]$ to derive a similar contradiction.

We now verify the hypotheses I–IV that Lemma 6.23 shows are impossible. We will follow the notation given there. We take the model $W = V[U]$, and the N in the lemma to be the N that is isomorphic to V^{ω_1}/U. That N is closed under ω-sequences from $V[U]$ is immediate from Proposition 2.14. Let $\kappa = \omega_2^N$. Then $(\kappa^+)^N = (\kappa^+)^W = \omega_3^V$.

[33] See the forcing fact mentioned in Sect. 1.

Let $J \in V[U]$ be the ideal on $P(\kappa)^N$ induced by κ and k. Then J is normal and κ-complete for functions that lie in N. Since the forcing producing s is c.c.c., Example 3.30 shows that $P(\kappa)^N/J$ is c.c.c. in $V[U]$.

If $\mathcal{A} \subseteq P(\kappa)^N/J$ is a maximal antichain lying in $V[U]$, then $\mathcal{A}$ is countable and hence there is a disjoint collection $\mathcal{R} \subseteq P(\kappa)^N$ of representatives of $\mathcal{A}$ that lies in $V[U]$. Since N is ω-closed $\mathcal{R}$ belongs to N and $\bigcup \mathcal{R} \in \breve{J}$. Hence there is a unique element A of $\mathcal{R}$ such that $\kappa \in k(A)$. From this we conclude that $H = \{[A] : \kappa \in k(A)\} \subseteq P(\kappa)^N/J$ is generic over W.

Thus the forcing $P(\kappa)^N/J$ is a regular subalgebra of the Boolean algebra for adding a single Cohen real. Since every subalgebra of the Cohen algebra is again a Cohen algebra, we see that $P(\kappa)^N/J$ has a countable dense subset in W.

Let $H \subseteq P(\kappa)^N/J$ be generic over W. Then we can do further forcing to produce s, G, M and k so that $H = \{[A] : \kappa \in k(A)\}$. Hence there is an elementary embedding of N^κ/H into M. In particular, N^κ/H is well-founded. Let N' be the transitive collapse of N^κ/H, i' be the ultrapower embedding of N into N' and $k' : N' \to M$ by $k'([f]^{N'}) = k(f)(\kappa)$. Then we get a commutative diagram of elementary embeddings:

$$\begin{tikzcd}
 & V \arrow[dl, "i"'] \arrow[dr, "j"] & \\
N \arrow[rr, "k"'] \arrow[dr, "i'"] & & M \\
 & N' \arrow[ur, "k'"'] &
\end{tikzcd}$$

The next claim finishes our reduction to Lemma 6.23, by establishing Hypothesis IV. To each generic $H \subseteq P(\kappa)^N/J$ we can canonically associate a Cohen real $r = r(H)$ such that $W[H] = W[r]$, and vice versa. In the notation of the hypotheses for Lemma 6.23, $\langle r_\xi : \xi < \kappa\rangle \in N$ is a one-to-one sequence such that for all generic $H \subseteq P(\kappa)/J$, $k'(\langle r_\xi : \xi < \kappa\rangle)(\kappa) = r$. We let $k'(\langle r_\xi : \xi < \kappa\rangle) = \langle r_\xi^{k'} : \xi < k'(\kappa)\rangle$. A sequence of Cohen terms $\langle s_\alpha : \alpha < \kappa^+\rangle \in W$ is chosen so that $1 \Vdash^W s_\xi = r_\xi^{k'}$ for all $\xi < \kappa^+$. To each Cohen term t we canonically associate a Borel function $T : \omega^\omega \to \omega^\omega$ in the usual way. Since s_α and s_β represent distinct reals, $\{x : S_\alpha(x) = S_\beta(x)\}$ is meager.

Let $X \in V[U]$ be a set of Borel functions with $|X| = \omega_1^{V[U]}$. Then for any unbounded set $Y \subseteq \omega_3$ in $V[U]$, there is an $\alpha \in Y$ such that for all $T \in X$, $\{x : T(x) = S_\alpha(x)\}$ is meager. For otherwise, there would be $T \in X$ and a $B \subseteq Y$ of cardinality ω_3^V such that for all $\alpha \in B$, $\{x : T(x) = S_\alpha(x)\}$ is non-meager. Hence there would be a single $p \in \omega^{<\omega}$ and $\alpha < \beta \in B$ such that $\{x \in [p] : S_\beta(x) = S_\alpha(x) = T(x)\}$ is comeager in $[p]$ a contradiction.

Claim. There is a sequence of Cohen terms $\langle t_\alpha : \alpha < \omega_3^V\rangle \in N$ such that:

(a) for a cofinal set of $\alpha < \omega_3^N$, $1 \Vdash t_\alpha = s_\alpha$, and

(b) for all $\alpha < \beta < \omega_3^N$, $\{x : T_\alpha(x) = T_\beta(x)\}$ is meager.

We are viewing Cohen terms for real numbers as hereditarily countable objects, so that each $s_\alpha \in N$. Working in V, for each $\alpha < \omega_3$ let $f_\alpha : \omega_1 \to H(\omega_1)$ be such that $1 \Vdash_{P(\omega_1)/I'} [f_\alpha]^N = s_\alpha$. Define

$$F : \omega_3 \times \omega_1 \to H(\omega_1)$$

by setting $F(\alpha, \xi) = f_\alpha(\xi)$.

In N, define a sequence of Cohen terms $\langle t_\alpha : \alpha < \omega_3^V \rangle$ by induction. Let $t_0 = s_0$. Suppose that we have defined $\langle t_\alpha : \alpha < \alpha_0 \rangle$.

Let Y be the set of fixed points of i below ω_3. By the previous paragraphs there is a $\beta \in Y \setminus \alpha_0$ such that S_β disagrees with each T_α on a comeager set. Since β is a fixed point of i, $i(F)(\beta, \omega_1^V) = s_\beta$. Hence the Borel function associated with $i(F)(\beta, \omega_1^V)$ disagrees with each of the T_α on a comeager set. Let β_0 be the least ordinal $\beta \geq \alpha_0$ such that $i(F)(\beta, \omega_1^V)$ has this property and $t_{\alpha_0} = i(F)(\beta_0, \omega_1^V)$. Clearly for $\alpha < \beta < \omega_3^V, \{x : T_\alpha(x) = T_\beta(x)\}$ is meager.

It remains to see that $\{\alpha : t_\alpha = s_\alpha\}$ is cofinal in ω_3^V. Suppose that this fails. Let η be a fixed point of i. If s_η is not on the sequence $\langle t_\alpha : \alpha < \omega_3 \rangle$, then there is a $\beta < \eta$ such that S_η agrees with T_β on a non-meager set. Since the set Y of fixed points of i is $V[U]$-stationary, we can find a stationary set $Z \subseteq Y$ and a fixed β such that for $\alpha \in Z$, S_α agrees with T_β on a non-meager set. In particular, we can find a fixed $p \in \omega^{<\omega}$ and $\alpha_0 < \alpha_1 \in Z$ such that both S_{α_0} and S_{α_1} agree with T_β on a comeager subset of $[p]$. But this means that S_{α_0} and S_{α_1} agree on a non-meager set, a contradiction.

We have now verified the hypotheses I–IV that Lemma 6.23 (proved in Sect. 6) shows are impossible. $\dashv$

We now describe how to use ideal assumptions to transfer CH to GCH. The first theorem of this sort was due to Jech and Prikry [64], who showed that if CH holds and there is an $\aleph_2$-saturated ideal on ω_1, then $2^{\aleph_1} = \aleph_2$. The more general version of this theorem appears in Foreman [35]:

5.10 Theorem. *Assume that $2^\kappa = \kappa^+$. Suppose that either:*

1. $(\lambda^{+2}, \lambda) \twoheadrightarrow (\kappa^{+2}, \kappa)$, *or*
2. *there is an elementary embedding $j : V \to M \subseteq V[G]$ where $G \subseteq \mathbb{P}$ is generic such that*
   - *(a)* $j(\kappa^+) = \lambda$,
   - *(b)* $j``\lambda \in M$, *and*
   - *(c)* $\mathbb{P}$ *is* λ^+*-c.c.*

Then $2^\lambda = \lambda^+$.

Proof. Assume the first hypothesis and suppose that $\langle x_\alpha : \alpha < \lambda^{+2}\rangle$ is a sequence of distinct subsets of λ. Then there is an elementary substructure $\mathfrak{A}$ of $\langle H(\theta), \in, \langle x_\alpha : \alpha < \lambda^{+2}\rangle\rangle$ such that the order type of $\mathfrak{A}$ intersected with λ^{+2} is κ^{+2} and the intersection of $\mathfrak{A}$ with λ has cardinality κ. Then $\langle x_\alpha : \alpha \in \mathfrak{A} \cap \lambda^{+2}\rangle$ is a sequence of κ^{+2} distinct subsets of $\lambda \cap \mathfrak{A}$. Hence $2^\kappa \geq \kappa^{+2}$, a contradiction.

Let j be the embedding from the second hypothesis. If I is the ideal induced by j and $j``\lambda$, then I is a normal, fine, countably complete, λ^+-saturated ideal on $[\lambda]^{\kappa^+}$. We can assume that j comes from an ultrapower of V by a generic $G \subseteq P([\lambda]^{\kappa^+})/I$. From this assumption, we can conclude that $M^\lambda \cap V[G] \subseteq M$. Since $M \models |P(\lambda)| = \lambda^+$ we must have $V[G] \models |P(\lambda)| = \lambda^+$. Since all cardinals greater than or equal to λ^+ are preserved we see that $2^\lambda = \lambda^+$ in V. $\dashv$

Moreover, since all Chang's Conjectures are consequences of the appropriate ideal hypothesis, we can formulate other axioms that "transfer" instances of GCH. For example, $(\lambda^{+2}, \lambda) \twoheadrightarrow (\kappa^{+2}, \kappa)$ and $2^\kappa = \kappa^+$ implies $2^\lambda = \lambda^+$. Hence:

5.11 Proposition (Foreman [35]). *Suppose that there is a pre-precipitous ideal on $[\lambda^+]^{\kappa^+}$ concentrating on $[\lambda]^\kappa$. Then $2^\kappa = \kappa^+$ implies $2^\lambda = \lambda^+$.*

Abe's Results on SCH

Abe [1] found another way of deducing instances of GCH and the Singular Cardinals Hypothesis (SCH) from ideal assumptions.[34] These arguments follow Solovay's proof of SCH above a supercompact cardinal, weakening Solovay's assumption of the existence of a normal ultrafilter on $[\lambda]^{<\kappa}$ to the existence of a weakly normal ideal on $[\lambda]^{<\kappa}$. The first theorem deals with regular cardinals and implicitly SCH by considering the successor of a singular cardinal.

5.12 Theorem. *If λ is regular and there is a weakly normal, fine, κ-complete ideal I on $[\lambda]^{<\kappa}$. Then $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$.*

5.13 Corollary. *Suppose that λ is regular and there is a normal, fine, κ-complete, λ-saturated ideal on $[\lambda]^{<\kappa}$. Then $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$.*

Proof of Theorem 5.12. Let U be the dual of I and D be the projection of U to a filter on λ via the map $z \mapsto \sup(z)$. Then D is a uniform, weakly normal, κ-complete filter on λ concentrating on ordinals of cofinality less than κ.

For ordinals $\alpha \in \lambda \cap \mathrm{Cof}({<}\kappa)$, let $A_\alpha \subseteq \alpha$ be a cofinal subset of α of order type the cofinality of α. Define an increasing continuous sequence of ordinals $\langle \eta_\xi : \xi < \lambda\rangle$ by induction by letting $\eta_0 = 0$, and $\eta_{\xi+1}$ the least

[34] Informally speaking, SCH asserts that 2^λ for a singular λ is the least possibility given by cardinal arithmetic and the powers of smaller regular cardinals. It is much discussed in various chapters of this Handbook.

ordinal such that $\{\alpha : A_\alpha \cap [\eta_\xi, \eta_{\xi+1}) \neq \emptyset\} \in D$. (Such an $\eta_{\xi+1}$ exists by the weak normality of D.)

Let $M_\alpha = \{\xi < \lambda : A_\alpha \cap [\eta_\xi, \eta_{\xi+1}) \neq \emptyset\}$. Then $M_\alpha \in [\lambda]^{<\kappa}$ and for each ξ, $\{\alpha : \xi \in M_\alpha\} \in D$. Hence, by the κ-completeness for all $z \in [\lambda]^{<\kappa}$ there is an α with $z \subseteq M_\alpha$. In particular, $[\lambda]^{<\kappa} = \bigcup_{\alpha<\lambda} P(M_\alpha)$. $\dashv$

Abe gets two more results directly addressing the case of singular cardinals:

5.14 Theorem. *Suppose that λ is a singular cardinal and there is a weakly normal, fine, κ-complete ideal on $[\lambda]^{<\kappa}$. Then*

1. *if* $\mathrm{cf}(\lambda) < \kappa$, *then* $\lambda^{<\kappa} = \lambda^+ \cdot 2^{<\kappa}$, *and*
2. *if* $\mathrm{cf}(\lambda) = \kappa$, *then* $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$.

The Value of Θ

We now turn to more subtle ideas about the continuum. Let M be a model of ZF but not necessarily the Axiom of Choice. We define Θ^M to be the supremum of all ordinals α for which there is a surjection $f : \mathbb{R}^M \to \alpha$ with $f \in M$. If M is a model of the Axiom of Choice, then $\Theta^M = (\mathfrak{c}^+)^M$, where $\mathfrak{c} = 2^{\aleph_0}$. In [42] the following analogue of the Continuum Hypothesis was considered:

5.15 Definition. The *Constructive Continuum Hypothesis* is the assertion that $\Theta^{L(\mathbb{R})} \leq \omega_2$.

Thus the Constructive Continuum Hypothesis asserts that it is not possible to effectively construct a counterexample to the Continuum Hypothesis in $L(\mathbb{R})$. If there are measurable cardinals, then $\Theta^{L(\mathbb{R})}$ has cofinality ω and consequently is either strictly larger than ω_2 or strictly smaller than ω_2.

In Sect. 5.11, we will prove Woodin's theorem that it is a consequence of the statement "the nonstationary ideal on ω_1 is $\aleph_2$-saturated and there is a measurable cardinal", that $\underset{\sim}{\delta}^1_2 = \omega_2$. Since $\underset{\sim}{\delta}^1_2$ is much smaller than $\Theta^{L(\mathbb{R})}$ this shows that there is a constructive/effective counterexample to the Continuum Hypothesis. However, we work in the other direction for the moment.

5.16 Definition. A partial ordering $\mathbb{P}$ is *reasonable* iff for all ordinals α and all V-generic $G \subseteq \mathbb{P}$, $([\alpha]^{<\omega_1})^V$ is stationary in $V[G]$.

It is easy to verify many standard classes of partial orderings are reasonable. For example, any proper partial ordering is reasonable and any $\aleph_\omega$-c.c. partial ordering that preserves all cardinals is reasonable [42].

5.17 Lemma. *Suppose that I is an $\aleph_n$-complete, $\aleph_\omega$-saturated, cardinal preserving ideal on ω_n. If there is a set $A \in I[\omega_n]$ with $A \subseteq \mathrm{Cof}(\omega_{n-1})$ and $A \in \breve{I}$, then $P(\omega_n)/I$ is reasonable.*

Proof. We must see that if α is an ordinal, $G \subseteq P(\omega_n)/I$ is generic, and $\mathfrak{A} \in V[G]$ is a structure in a countable language with domain α, then there is a countable set $z \in V$ such that $z \prec \mathfrak{A}$.[35] We can assume that α is cardinal in V.

Using the downward Löwenheim-Skolem theorem, it is easy to verify the following general result due to Abraham:

Fact. Suppose that $V \subseteq W$ are models of set theory and $\alpha \in \mathrm{On}$ with $(\alpha^+)^V$ a cardinal in W. If $V \cap [\alpha]^{<\omega_1}$ is stationary in W then so is $V \cap [\alpha^+]^{<\omega_1}$.

Abraham's fact together with the hypothesis that I is cardinal preserving implies that for all $\beta \leq \omega_{n-1}$, $([\beta]^{<\omega})^V$ remains stationary after forcing with $P(\omega_n)/I$.

By a theorem of Tarski [117] the saturation of a Boolean algebra is always a regular cardinal. Hence $P(\omega_n)/I$ is $\aleph_k$-c.c. for some $k \in \omega$. This implies that if $G \subseteq P(\omega_n)/I$ is generic and $\mathfrak{A} \in V[G]$ is a structure with domain some $\lambda \geq \omega_k$, then there is a set $N \in V$ with $|N|^V = \aleph_k$ such that $N \prec \mathfrak{A}$. Since ω_k is ω_n^{+j} for some finite j, the lemma is reduced to seeing that if ρ is ω_n^V, then $([\rho]^{<\omega_1})^V$ is stationary in $V[G]$.

Let $\langle f_\gamma : \gamma \in [\omega_{n-1}, \omega_n]^V \rangle \in V$ such that for each $\gamma \in [\omega_{n-1}, \omega_n]^V$, $f_\gamma : \omega_{n-1}^V \to \gamma$ is a bijection. Let $\mathfrak{A}$ be an expansion of a large $H(\theta)$ witnessing that $A \in I[\omega_n]$.

Let $G \subseteq P(A)/I$ be generic and $j : V \to M \subseteq V[G]$ be the generic elementary embedding. Let $\mathfrak{B} \in V[G]$ be a structure in a countable language with domain ρ. We must find a countable $z \subseteq \rho$ lying in V such that $z \prec \mathfrak{B}$.

For all $\gamma < \rho$, $\mathrm{Sk}^{j(\mathfrak{A})}(\gamma) \cap \rho^{<\rho} = \mathrm{Sk}^{\mathfrak{A}}(\gamma) \cap \rho^{<\rho}$. Hence, $\mathrm{Sk}^{j(\mathfrak{A})}(\rho) \cap \rho^{<\rho} = \mathrm{Sk}^{\mathfrak{A}}(\rho) \cap \rho^{<\rho} \subseteq V$. Since $A \in G$, there is a sequence $\langle \rho_i : i \in \omega_{n-1} \rangle \in V[G]$ cofinal in ρ such that every initial segment $\langle \rho_i : i \in j \rangle \in \mathrm{Sk}^{j(\mathfrak{A})}(\rho)$. In particular, we see that for $j \in \omega_{n-1}$, $\langle \rho_i : i \in j \rangle$ is in V.[36]

Let $X_j = \bigcup_{i<j} f_{\rho_i}``j$. Then each $X_j \in V$ and in $V[G]$ the sequence $\langle X_j : j < \omega_{n-1} \rangle$ is an increasing, continuous sequence of sets of cardinality less than ω_{n-1} whose union is ρ. In particular, there is a j such that X_j is an elementary substructure of $\mathfrak{B}$.

Since $([X_j]^{<\omega_1})^V$ is stationary in $P(X_j)^{V[G]}$, there is a $z \in ([X_j]^{<\omega_1})^V$ such that $z \prec \mathfrak{B}$. $\dashv$

A somewhat easier argument shows:

5.18 Lemma. *Suppose that $P(\omega_n)/I$ is $\aleph_{n+1}$-saturated, where I is the non-stationary ideal on ω_n restricted to the ordinals of cofinality ω_{n-1}. Then $P(\omega_n)/I$ is reasonable.*

Foreman and Magidor established the following result [42, Theorem 3.4]:

[35] More precisely, z is the domain of an elementary substructure of $\mathfrak{A}$.
[36] This is where we use the hypothesis that I concentrates on a set in $I[\omega_n]$.

5.19 Theorem. *Suppose that $\sim$ is a κ-weakly homogeneously Suslin equivalence relation, and $\mathbb{P}$ is a reasonable partial ordering of cardinality less than κ. Suppose that there is a $\tau \in V[G] \cap \omega^\omega$ such that for all $f \in V \cap \omega^\omega$, $\tau \not\sim f$. Then in V, there is a perfect set of $\sim$ inequivalent elements of ω^ω.*

Suppose that $f : \omega^\omega \to \rho$ is a function in $L(\mathbb{R})$, where $\rho \in \mathrm{On}$. Then f induces an equivalence relation on ω^ω defined by setting $x \sim_f y$ iff $f(x) = f(y)$. Given a sufficient number of conventional large cardinals, this equivalence relation is κ-weakly homogeneously Suslin for some measurable cardinal κ.

Suppose that $\sim$ is a κ-weakly homogeneously Suslin equivalence relation on ω^ω with at least $\aleph_n$ classes. Enumerate the classes as $\langle [x_\alpha] : \alpha < \gamma \rangle$. Let I be a precipitous ideal on ω_n. Let $G \subseteq P(\omega_n)/I$ be generic, and $j : V \to M \subseteq V[G]$ be the generic elementary embedding. Then one can show that the ω_nth equivalence class on the sequence $j(\langle [x_\alpha] : \alpha < \gamma \rangle)$ is a new class for $\sim$.

As a corollary we get the following results:

5.20 Theorem. *Suppose that any of the following hypotheses (together with conventional large cardinal hypotheses):*

1. *there is an $\aleph_n$-complete, cardinal preserving ideal I on ω_n that is $\aleph_\omega$-saturated with an element $A \in I[\omega_n] \cap I^+$, or*
2. *the nonstationary ideal on ω_n restricted to cofinality ω_{n-1} is $\aleph_{n+1}$-saturated, or*
3. *there is an $\aleph_1$-complete ideal I on ω_n such that the forcing $P(\omega_n)/I$ is proper.*

Then $\Theta^{L(\mathbb{R})} < \omega_n$.

There are a host of other possible hypotheses that work as well as 1–3 in Theorem 5.20.

Woodin used this result to prove the following:

5.21 Theorem. *Suppose that there is an $\aleph_2$-saturated, uniform ideal I on ω_2 and sufficiently many conventional large cardinals. Then $\Theta^{L(\mathbb{R})} < \omega_2$.*

Proof. Let $\sim$ be a weakly homogeneous equivalence relation with at least ω_2 classes. We show that there is a perfect set of $\sim$ inequivalent reals.

As in the proof of Theorem 5.9, there is a normal, $\aleph_2$-saturated ideal I' on ω_1 defined by setting $X \in I'$ iff $1 \Vdash_{P(\omega_2)/I} \omega_1 \notin j(X)$, where $j : V \to M$ is the elementary embedding induced by a generic ultrafilter for $P(\omega_2)/I$. Moreover, I' is a projection of I and $P(\omega_1)/I'$ is regularly embedded in $P(\omega_2)/I$. Let $G \subseteq P(\omega_2)/I$ be generic and U be the projection to a generic object for $P(\omega_1)/I$. We note that G is in a c.c.c. extension of $V[U]$, since $\omega_2^V = \omega_1^{V[U]}$. In particular, the forcing creating G over $V[U]$ is reasonable. We argue that it adds a new $\sim^{V[U]}$ equivalence class.

If $i : V \to N$ is the generic elementary embedding induced by I' and the ultrapower by U then there is an elementary embedding $k : N \to M$ making the following diagram commute:

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "j"] & \\ N \arrow[rr, "k"'] & & M \end{tikzcd}$$

Again, as in Theorem 5.9, the critical point of k is ω_2^N. Let $\langle [x_\alpha] : \alpha < \gamma\rangle \in N$ be an enumeration of the $\sim^N$ equivalence classes. Since $N \cap \mathbb{R} = V[U] \cap \mathbb{R}$, and $\sim$ is weakly homogeneous, we see that this is also an enumeration of the $\sim^{V[U]}$ equivalence classes.

Moreover, the ω_2^N element of $k(\langle [x_\alpha] : \alpha < \gamma\rangle)$ is a new $\sim$ equivalence class. Hence we can apply Theorem 5.19, to see that there is a perfect set of inequivalent reals. ⊣

5.4. Stationary Set Reflection

Generic embeddings imply the same stationary set reflection as the corresponding large cardinal embedding, provided that the forcing preserves stationarity. One example of this phenomenon is the following proposition:

5.22 Proposition. *Suppose that there is a generic elementary embedding with critical point κ^+ in a $<\kappa$-closed forcing extension of V. Then if $S \subseteq \kappa^+ \cap \mathrm{Cof}(<\kappa)$ is stationary, there is a $\gamma < \kappa^+$ such that $S \cap \gamma$ is stationary.*

Proof. Let $j : V \to M \subseteq V[G]$ have critical point κ^+. Since $\kappa^{<\kappa} \cap V = \kappa^{<\kappa} \cap V[G]$ we must have $V \models \kappa^{<\kappa} = \kappa$. It follows that $<\kappa$-closed forcing preserves stationary subsets $S \subseteq \kappa^+ \cap \mathrm{Cof}(<\kappa)$. Consequently, if S is a stationary subset of κ^+ consisting of ordinals of cofinality less than κ, S remains stationary in $V[G]$.

Since, $j(S) \cap \kappa^+ = S$, we know that $M \models$ "there is a $\gamma < j(\kappa)$ such that $j(S) \cap \gamma$ is stationary". By the elementarity of j, this holds in V. ⊣

Note that the hypothesis of Proposition 5.22 holds if there is a normal, κ^+-complete ideal I on κ^+ such that $P(\kappa^+)/I$ has a dense $<\kappa$-closed subset.

Since proper forcing preserves stationary sets of ordinals of cofinality ω, the proof of Proposition 5.22 gives the following observation:

5.23 Proposition. *Suppose that $j : V \to M$ is an elementary embedding with critical point κ in a generic extension of V by proper forcing. Suppose that $P \subseteq \kappa$ with $\|\kappa \in j(P)\| \neq 0$ and $S \subseteq \kappa \cap \mathrm{Cof}(\omega)$ is stationary, then there is an $\alpha \in P$ such that $S \cap \alpha$ is stationary.*

It follows from Proposition 5.23 that if I is a normal κ-complete ideal on κ such that $P(\kappa)/I$ is proper, then for every stationary set $S \subseteq \kappa \cap \mathrm{Cof}(\omega)$ and every $P \in I^+$, there is an $\alpha \in P$ such that $S \cap \alpha$ is stationary.

The interested reader can easily formulate other generic embedding assumptions that yield every ordinal reflection property at successor of regular cardinals.

An interesting assumption at a successor of a singular cardinal is:

5.24 Theorem. *Suppose that there are partial orderings $\langle \mathbb{P}_n : n \in \omega \rangle$ such that $\mathbb{P}_n$ is $<\omega_n$-closed and whenever $G \subseteq \mathbb{P}_n$ there is a generic elementary embedding $j : V \to M$ with critical point ω_n such that $j(\omega_n) > \aleph_{\omega+1}$ and $j``\aleph_{\omega+1} \in M$. Suppose further that $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$. Then every stationary subset of $\aleph_{\omega+1}$ reflects to an ordinal α.*

5.25 Corollary. *Suppose that for all n there is a normal, fine, countably complete ideal $I_n \subseteq [\aleph_{\omega+1}]^{<\omega_n}$ such that $P([\aleph_{\omega+1}]^{<\omega_n})/I$ is $<\omega_n$-closed and $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$. Then every stationary subset of $\aleph_{\omega+1}$ reflects.*

Magidor showed that the hypothesis of this theorem is consistent relative to large cardinal assumptions in [91]; see Cummings' or Eisworth's chapter in this Handbook for a proof of this fact.

Proof. Suppose that $S \subseteq \aleph_{\omega+1}$ is stationary. Then there is a stationary subset of S consisting of ordinals of a fixed cofinality ω_n. We assume that S has this property. Let $m > n$ and suppose that j_m is the generic embedding with critical point ω_m, arising from a generic object $G \subseteq \mathbb{P}_m$.

By Theorem 3.16, S is still stationary in $V[G]$. Let $\gamma = \sup(j``\aleph_{\omega+1})$. Since $j``\aleph_{\omega+1}$ is $\aleph_n$-closed unbounded in γ, we see that $j``S$ is stationary in $\gamma$. Since $j``S \subseteq j(S)$,

$$M \models j(S) \cap \gamma \text{ is stationary in } \gamma.$$

By elementarity, $V \models$ there is a $\delta, S \cap \delta$ is stationary in δ. $\dashv$

A. Sharon has proved that if every stationary subset of $\omega_n \cap \mathrm{Cof}(\omega)$ reflects, then $I[\aleph_{\omega+1}] \restriction \mathrm{Cof}(\omega_n)$ contains a closed unbounded set. The proof extends to show that if there is a normal, $\aleph_n$-complete ideal I on ω_n such that $P(\omega_n)/I$ contains a dense set that is $<\omega_{n-1}$-closed, then $I[\aleph_{\omega+1}] \restriction \mathrm{Cof}(\omega_n)$ contains a club. In particular, the hypothesis of Corollary 5.25 can be weakened to the assumption that the ideals I_n exist and that $I[\aleph_{\omega+1}] \restriction \mathrm{Cof}(\omega_1)$ contains a relative club.

One can also prove the following with analogous techniques:

5.26 Theorem. *Suppose that there is a normal, fine, $\aleph_2$-complete ideal I on $Z = [\lambda]^{<\omega_2}$ such that $P(Z)/I$ is a proper forcing. Then for every countable collection of stationary subsets $\{S_i : i \in \omega\}$ of $P_{\omega_1}(\lambda)$, there is an X of cardinality ω_1 with $\omega_1 \subseteq X$ such that for all i, $S_i \cap P_{\omega_1}(X)$ is stationary and for all regular γ, $\mathrm{cf}(X \cap \gamma) = \omega_1$.*

Note that the hypotheses of the theorem hold if $P(Z)/I$ has a dense countably closed subset. Foreman and Todorčević [45] showed that the type of reflection in the conclusion of this theorem implies the Singular Cardinals Hypothesis.

Burke and Matsubara have shown that hypothesis on the closure of the quotient algebra can be replaced with chain conditions, even for non-normal ideals. Strengthening standard results about strongly compact cardinals they showed:

5.27 Theorem (Burke-Matsubara [13]). *Suppose that $\kappa < \lambda$ are uncountable regular cardinals and there is a fine, κ-complete, λ-saturated ideal on $[\lambda]^{<\kappa}$. Then every stationary subset of $\lambda \cap \mathrm{Cof}(<\kappa)$ reflects.*

Proof. Let I be a λ-saturated, fine, κ-complete ideal on $[\lambda]^{<\kappa}$. Define an ideal I' on λ by putting $A \in I'$ iff $\{z \in [\lambda]^{<\kappa} : \sup(z) \in A\} \in I$. Then I' is a uniform λ-saturated, κ-complete ideal on λ. We show that the existence of such an I' implies the desired reflection.

Let $A \subseteq \lambda \cap \mathrm{Cof}(<\kappa)$ be stationary. Suppose that A does not reflect. Without loss of generality we can assume that there is a fixed $\eta < \kappa$ such that $A \subseteq \lambda \cap \mathrm{Cof}(\eta)$. For each $\alpha \in A$ choose an increasing sequence $\langle \alpha_i : i \in \eta \rangle$ cofinal in α.[37]

Using the fact that for each limit point $\beta \in \lambda$ there is a closed unbounded set $C_\beta \subseteq \beta$ disjoint from A, one can show by induction that for all $\beta \in \lambda$ there is an $f_\beta : \beta \cap A \to \eta$ such that for $\delta, \rho \in A \cap \beta$

$$\{\rho_i : i > f_\beta(\rho)\} \cap \{\delta_i : i > f_\beta(\delta)\} = \emptyset.$$

Force with $P(\lambda)/I'$ to get a generic G. Then for each $\delta \in A$, there is an $X \in G$ and an $i(\delta)$ such that for all $\beta \in X$, $f_\beta(\delta) = i(\delta)$. But then for all $\rho, \delta \in A$

$$\{\rho_i : i > i(\rho)\} \cap \{\delta_i : i > i(\delta)\} = \emptyset.$$

This contradicts the fact that A is stationary in $V[G]$. ⊣

5.5. Suslin and Kurepa Trees

The author noticed that if there is a normal, fine, countably complete $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$ then there is a Suslin tree on ω_1. To see this note that Shelah showed that if c is Cohen generic over V then there is a Suslin tree on ω_1 in $V[c]$. View $P([\omega_2]^{\omega_1})/I$ as the Boolean completion of $\mathrm{Col}(\omega, \omega_1)$. Let $G \subseteq \mathrm{Col}(\omega, \omega_1)$ be generic. Then we can write $G \sim G_0 * G_1$, where G_0 is V-generic for $\mathrm{Col}(\omega, \omega_1)$ and G_1 is Cohen generic over $V[G_0]$. Hence by Shelah's theorem there is a Suslin tree T in $V[G]$. Let $j : V \to M \subseteq V[G]$ be the generic embedding induced by G. By the closure of M, we must have that $T \in M$ and is a Suslin tree in M. Since $\omega_1^{V[G]} = \omega_2^V = j(\omega_1^V)$ we can apply elementarity to see that there is a Suslin tree on ω_1 in V.

Woodin [126] proved:

[37] I.e. a ladder system.

5.28 Theorem. *Suppose that there is a countably complete, $\aleph_1$-dense ideal on ω_1. Then there is a Suslin tree on ω_1.*

Proof. Since the canonical projection of an $\aleph_1$-dense ideal to a normal ideal is also $\aleph_1$-dense, we can assume we start with a normal $\aleph_1$-dense ideal I. Let $G \subseteq P(\omega_1)/I$ be generic. By Example 4.12 we can consider G to be a generic subset of $\mathrm{Col}(\omega, \omega_1)$, i.e. a surjective function from $\omega \to \omega_1$.

Suppose now that $Z = \omega_1$, and $j : V \to M$ is the generic elementary embedding. Then $G \in M$ and we can choose a function $f : \omega_1 \to \omega_1^\omega$ such that the empty condition forces that $[f]^M = G$. We can assume that for all limit ordinals α, $f(\alpha) : \omega \to \alpha$ is a surjection.

We construct a tree T and show that it is Suslin. The elements of T at level α will be the ordinals $[\alpha\omega, (\alpha+1)\omega)$. We will denote level α by $(T)_\alpha$.

At successor stages $\alpha = \beta + 1$ we arbitrarily assign successors to each γ at level β so that each γ has countably many successors.

At a limit stage α we will have defined a countable tree $(T)_{<\alpha}$ and we must decide which branches through $(T)_{<\alpha}$ we extend to level α. Given a $\gamma \in (T)_{<\alpha}$ and $f(\alpha)$ we have an attempt at a branch b_γ through $(T)_{<\alpha}$ defined as follows:

1. Let $b_0(\gamma) = \gamma$.

2. Suppose that $b_n(\gamma)$ is defined and k is the least natural number such that $f(\alpha)(k) = b_n(\gamma)$. Let l be the least element of ω greater than k such that $b_n(\gamma) <_T f(\alpha)(l)$, and $b_{n+1}(\gamma) = f(\alpha)(l)$.

Let $b_\gamma = \langle b_n(\gamma) : n \in \omega \rangle$.

Case 1. For all $\gamma \in (T)_{<\alpha}$, b_γ is a branch through $(T)_{<\alpha}$ that is cofinal in α.

In this case we put a $\delta \in (\alpha+1)\omega$ above each b_γ.

Case 2. Otherwise.

In this case we arbitrarily choose a countable collection of branches through $(T)_{<\alpha}$ that are cofinal in α and such that every $\gamma \in (T)_{<\alpha}$ belongs to one of them.

We claim that this defines a Suslin tree on ω_1. The following suffices:

Claim. Let A be a maximal antichain in T. Then there is an $\alpha < \omega_1$ such that Case 1 holds, and for every $\gamma \in (T)_{<\alpha}$ there is an element $a \in A$ and an n such that $a < b_n(\gamma)$.

To see the claim, fix an antichain A and let $j : V \to M$ be the generic elementary embedding induced by a generic $G \subseteq \mathrm{Col}(\omega, \omega_1)$. Note that $j(A) \cap \omega_1 = A$, $j(T) \cap \omega_1 = T$ and $j(f)(\omega_1^V) = G$. An easy density argument yields that from the point of view of M, ω_1^V is in Case 1 and that for all $\gamma \in (\omega_1)^V$ there is an n and an $a \in A$ such that $a <_{j(T)} b_n(\gamma)$. Reflecting this to V finishes the proof of the claim. ⊣

This theorem easily generalizes to show:

5.29 Theorem. *Suppose that there is a κ^+-complete, normal ideal I on $P(\kappa^+)$ so that there is a dense subset of $P(\kappa^+)/I$ isomorphic to* $\mathrm{Col}(\kappa,\kappa^+)$. *Then there is a Suslin tree on κ^+.*

Recall that a Kurepa tree on κ is a normal tree T on κ such that each level has cardinality less than κ but T has at least κ^+ branches.

We have a couple of ways of showing that there are no Kurepa trees:

5.30 Theorem. *Suppose that $(\kappa^+,\kappa) \twoheadrightarrow (\kappa,{<}\kappa)$. Then there is no Kurepa tree on κ.*

Proof. Let $\theta \gg \kappa$ and $\mathcal{T}$ be a Kurepa tree on κ. Let $\langle b_\alpha : \alpha < \kappa^+\rangle$ enumerate distinct branches through $\mathcal{T}$ of length κ. Let $\mathfrak{A} \prec \langle H(\theta), \in, \Delta, \mathcal{T}\rangle$ be such that $|\mathfrak{A} \cap \kappa^+| = \kappa$ and $|\mathfrak{A} \cap \kappa| < \kappa$.

Let $\delta = \sup(\mathfrak{A}\cap\kappa)$. For $\alpha \neq \beta$ with α and $\beta \in \mathfrak{A}\cap\kappa^+$, there is a $\gamma \in \mathfrak{A}\cap\kappa$ such that b_α and b_β differ at level γ. Hence b_α and b_β differ at level δ. But this means that level δ must have cardinality at least κ, a contradiction. $\dashv$

5.31 Theorem. *Suppose that $\kappa = \mu^+$ is a successor cardinal and there is an elementary embedding $j : V \to M \subseteq V[G]$ where $G \subseteq \mathbb{P}$ is generic where:*

1. $\mathrm{crit}(j) = \kappa$, *and*

2. $\mathbb{P}$ *is κ^+-c.c.*

Then there is no Kurepa tree on κ.

Proof. Let $j : V \to M \subseteq V[G]$ be the generic elementary embedding of V to M. Since κ is a successor cardinal, $j(\kappa) = \kappa^+$. Suppose that $\mathcal{T}$ is a Kurepa tree on κ. Let $\langle b_\alpha : \alpha < \kappa^+\rangle$ enumerate distinct branches of length κ through $\mathcal{T}$.

Then for all α, $j(b_\alpha) \cap \kappa = b_\alpha$ and there is a point $\tau_\alpha \in j(\mathcal{T})$ at level κ that belongs to the branch $j(b_\alpha)$. This τ_α lies above every element of b_α. In particular, for $\alpha \neq \beta, \tau_\alpha \neq \tau_\beta$. Hence there are at least $(\kappa^+)^V$ points at level κ in $j(\mathcal{T})$. Since $M \models j(\mathcal{T})$ is Kurepa, $V[G] \models |(\kappa^+)^V| = \mu$, a contradiction. $\dashv$

5.6. Partition Properties

In this section we discuss the use of ideals for proving partition properties.[38] We remind the readers of some definitions:

Let κ be a cardinal, γ and $\langle \kappa_\nu : \nu < \gamma\rangle$ be ordinals. Then $\kappa \to (\kappa_\nu)^r_\gamma$ iff for every function $f : [\kappa]^r \to \gamma$ there is a $\nu < \gamma$ and a set $X \subseteq \kappa$ of order type κ_ν

[38] We refer the reader to the chapter by Hajnal and Larson in this Handbook for an extensive discussion of partition relations.

such that f has constant value ν on $[X]^r$. The set X is called *homogeneous* for f. If all of the κ_ν's are constantly λ then $\kappa \to (\lambda)^r_\gamma$.

Baumgartner, Hajnal and Todorčević in [7] showed how to use ideal theory effectively to simplify classical proofs of partition theorems and to prove new strong partition theorems. We will content ourselves to mention two applications of the theory and relations to generic elementary embeddings.

As noted in Sect. 4, for a κ-complete ideal $I \subseteq P(\kappa)$, the following properties are decreasing in strength:

1. I is prime.[39]
2. I is κ-dense.
3. I is $(\kappa^+, \kappa^+, \kappa)$-saturated.

These various properties yield decreasing amounts of partition strength.

5.32 Definition. Let κ be a measurable cardinal. Then $\Omega(\kappa)$ is the least ordinal greater than κ that is a uniform indiscernible for bounded subsets of κ.[40]

We note that $\Omega(\kappa)$ is a very large ordinal; for example, $L_{\Omega(\kappa)} \models \text{ZFC}$. Thus $\Omega(\kappa)$ is closed under primitive recursive set functions, and much more.

The following theorems use the theory of ideals developed by Baumgartner, Hajnal and Todorčević [7] along with assumptions about ideals that use generic embeddings:

5.33 Theorem (Foreman-Hajnal [39]). *Suppose that κ carries a κ-complete prime ideal. Then for all $\rho < \Omega(\kappa)$ and $m \in \omega$,*

$$\kappa^+ \to (\rho)^2_m.$$

Weaker hypotheses give weaker conclusions:

5.34 Theorem (Foreman-Hajnal [39]). *Suppose that there is a κ-complete κ-dense ideal on κ and $\kappa^{<\kappa} = \kappa$. Then:*

$$\kappa^+ \to (\kappa^2 + 1, \alpha)^2_2 \quad \textit{for all } \alpha < \kappa^+.$$

Laver and later Kanamori independently showed:

5.35 Theorem (Kanamori [68]). *Suppose that $\kappa^{<\kappa} = \kappa$ and there is a $(\kappa^+, \kappa^+, \kappa)$-saturated ideal on κ. Then:*

$$\kappa^+ \to (\kappa \times 2 + 1, \alpha)^2_2 \quad \textit{for all } \alpha < \kappa^+.$$

[39] I.e. κ is measurable.

[40] For each $\alpha < \kappa$ and $a \subseteq \alpha$, the measurability of κ implies that there is a closed unbounded class of indiscernibles C_a for the structure $\langle L[a], \in, a, \xi\rangle_{\xi \leq \alpha}$. (One proceeds here in direct analogy to the development of the canonical indiscernibles for L and the theory $0^\#$.) Then $\Omega(\kappa)$ is the least member of $\bigcap\{C_a : a \subseteq \alpha \text{ for some } \alpha < \kappa\}$. For a broader, combinatorial definition of $\Omega(\kappa)$ see Definition 5.6 of the Hajnal-Larson chapter in this Handbook.

As usual with strong ideal axioms, the advantage of Theorems 5.34 and 5.35 is that the existence of these strong ideals is consistent at small cardinals such as ω_1. For example, a corollary of Theorem 5.33 is that it is consistent that $\omega_2 \to (\omega_1^2 + 1, \alpha)^2$ for all $\alpha < \omega_2$.

To illustrate how strong ideal assumptions could be relevant we outline the proof of the following theorem of Laver, dealing with the partition property:

$$\binom{\omega_2}{\omega_1} \to \binom{\omega_1}{\omega_1}_\omega$$

which says that whenever $f : \omega_2 \times \omega_1 \to \omega$ there are uncountable sets $A \subseteq \omega_1$ and $B \subseteq \omega_2$ such that f is constant on $A \times B$.[41]

5.36 Theorem (Laver [83]). *Assume* CH *and that there is a normal, countably complete* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated ideal* I *on* ω_1*. Then:*

$$\binom{\omega_2}{\omega_1} \to \binom{\omega_1}{\omega_1}_\omega .$$

Proof. Fix an $f : \omega_1 \times \omega_2 \to \omega$. For each $k < \omega$ and $\beta < \omega_2$, let $f_k(\beta) = \{\alpha \in \omega_1 : f(\alpha, \beta) = k\}$. Then there is a set O of size $\aleph_2$ and a $k \in \omega$ such that for all $\beta \in O, f_k(\beta) \in I^+$. By the saturation we can find a set $S \subseteq O$ of size $\aleph_2$ such that for all countable sets $S' \subseteq S$, $|\bigcap_{\beta \in S'} f_k(\beta)| = \omega_1$. Let $\langle \rho_\eta : \eta < \omega_2 \rangle$ enumerate S in order.

Let θ be a large regular cardinal, $N \prec \langle H(\theta), \in, \Delta, f, I, S, \ldots \rangle$ be an elementary substructure of cardinality $\aleph_1$ with $N^\omega \subseteq N$ and $\xi = N \cap \omega_2$.

We build sequences of ordinals $\langle \alpha_i : i \in \omega_1 \rangle$ and $\langle \beta_i : i \in \omega_1 \rangle$ with the α_i's countable and the β_i's from $S \cap N$ by induction on i. At stage j we will assume the following:

1. for all $i < j$, $\alpha_i \in f_k(\rho_\xi)$, and
2. for all i and i' less than j, $\alpha_{i'} \in f_k(\beta_i)$.

Having defined $\langle \alpha_i : i < j \rangle$ and $\langle \beta_i : i < j \rangle$ satisfying these inductive hypotheses, we need to define β_j and α_j. By our assumptions on S, $|\bigcap_{i<j} f_k(\beta_i) \cap f_k(\rho_\xi)| = \omega_1$. Choose $\alpha_j > \sup_{i<j} \alpha_i$ in this intersection. We can then reflect the statement "$\{\alpha_i : i \leq j\} \subseteq f_k(\rho_\xi)$" to find a $\rho_{\xi'} > \sup_{i<j} \beta_i$ with $\xi' \in N$ satisfying "$\{\alpha_i : i \leq j\} \subseteq f_k(\rho_{\xi'})$". Let $\beta_j = \rho_{\xi'}$. ⊣

The next theorem has nothing directly to do with ideals:

5.37 Theorem (Hajnal-Juhasz). *Suppose that* $G \subseteq \mathrm{Add}(\omega, \omega_1)$ *is generic for the partial ordering adding* ω_1 *Cohen reals. Then:*

$$V[G] \models \binom{\omega_2}{\omega_1} \not\to \binom{\omega}{\omega_1}_2 .$$

[41] Variants of this partition property appearing elsewhere in this chapter have the expected analogous meaning.

Proof. View $G : \omega_1 \to 2$. Let $\langle A_\alpha : \alpha < \omega_2\rangle$ be a sequence of elements of $[\omega_1]^{\omega_1}$ such that for all $\alpha \neq \beta$, $|A_\alpha \cap A_\beta| \leq \aleph_0$. For each $\alpha \in \omega_2$ let $\langle \rho^\alpha_i : i \in \omega_1\rangle$ enumerate A_α in increasing order.

Define a function $f : \omega_1 \times \omega_2 \to 2$ by $f(\gamma, \alpha) = j$ iff $G(\rho^\alpha_\gamma) = j$. We claim that for all countably infinite sets $S \subseteq \omega_2$ there is a tail of $\gamma < \omega_1$ such that there are $\alpha, \beta \in S$, $f(\gamma, \alpha) \neq f(\gamma, \beta)$. Indeed, suppose that S is a countably infinite set. Choose a γ_0 such that for all $\gamma > \gamma_0$ and all $\alpha, \beta \in S$ we have $\rho^\alpha_\gamma \neq \rho^\beta_\gamma$. Since G is generic we can find $\alpha, \beta \in S$ such that $G(\rho^\alpha_\gamma) \neq G(\rho^\beta_\gamma)$, as desired. ⊣

As Woodin pointed out, it follows from these results that the following three properties are collectively inconsistent:

1. CH,
2. there is a normal, fine, countably complete, $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$, and
3. there is a normal, fine, countably complete ideal on $[\omega_2]^{\omega_1}$ with quotient algebra $\mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V)$.

Since Woodin's argument only used the positive partition property to deduce his inconsistency, property 2 can be replaced by the assumption that there is an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1.

5.38 Corollary. *The following are inconsistent with each other:*

1. *There is a normal* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated ideal on* ω_1.
2. *There is a normal, fine, countably complete ideal on* $[\omega_2]^{\omega_1}$ *such that* $P([\omega_2]^{\omega_1})/I$ *has a dense subset isomorphic to* $\mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V)$.

Proof. The first ideal assumption implies CH and that there is an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1. Hence $\binom{\omega_2}{\omega_1} \not\to \binom{\omega_1}{\omega_1}_\omega$. Assume the second ideal exists. Let $G \subseteq \mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V)$ be generic and $j : V \to M$ be the generic embedding associated with G. Then $G \in M$. Write $G = G_0 * G_1$ where $G_0 \subseteq \mathrm{Col}(\omega, \omega_1)$ and $G_1 \subseteq \mathrm{Add}(\omega, \omega_2^V)$. Then G_0 and G_1 are both in M. Let f be the example built from G_1 in $V[G_0 * G_1]$ that shows $\binom{\omega_2}{\omega_1} \not\to \binom{\omega}{\omega_1}_{<\omega}$. Since the construction of f is sufficiently absolute relative to a subset of ω_1^M coding an almost disjoint sequence of sets we see that $f \in M$. By the elementarity of j, $V \models \binom{\omega_2}{\omega_1} \not\to \binom{\omega}{\omega_1}_2$. ⊣

5.39 Remark. It is easy to see from Theorem 7.14 that if I is a normal, fine, $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$ in V and $G \subseteq \mathrm{Add}(\omega, \omega_1)$ is V-generic then in $V[G]$, $P([\omega_2]^{\omega_1})/I$ has a dense subset isomorphic to $\mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V)$. Hence if it is consistent that there is an $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$, then there are two consistent ideal properties that are mutually inconsistent.

In Sect. 7, Example 7.25 shows that it is consistent that CH holds and there is an inaccessible cardinal λ and a normal, fine, countably complete ideal $I \subseteq [\lambda]^{\omega_1}$ such that

$$P([\lambda]^{\omega_1})/I \cong \mathcal{B}(\mathrm{Col}(\omega, {<}\lambda)).$$

The argument for Corollary 5.38, works equally well to show that such an ideal implies

$$\binom{\omega_2}{\omega_1} \not\to \binom{\omega}{\omega_1}_2.$$

By Theorem 7.40, we see that it is consistent that CH holds and there is an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal. Hence, assuming the consistency of a huge cardinal, we see that there are two individually consistent, but mutually inconsistent ideal assumptions.

5.7. The Normal Moore Space Conjecture and Variants

In this section we describe how to prove the Normal Moore Space Conjecture and related properties from generic large cardinals.[42] The generic embeddings posited here stand in stark contrast to the other axioms in that they imply that the continuum is larger than the first weakly Mahlo cardinal. We note that their statements can be distinguished from other generic large cardinal postulations in that the “three parameters” directly refer to the value of $2^{\aleph_0}$. They thus “settle” CH in a somewhat circular way.

5.40 Definition. We give the basic definitions:

1. A *development* of a topological space (X, τ) is a collection $\{\mathcal{U}_n : n \in \omega\}$ of open coverings such that for every $x \in X$ and every open neighborhood O of x, there is an n such that:

$$\bigcup\{U \in \mathcal{U}_n : x \in U\} \subseteq O.$$

2. A *Moore Space* is a regular space with a development.

We note that if (X, τ) is metrizable, then it has a development. If a space (X, τ) has a development then every point in X has a countable neighborhood base. “Developments” are an attempt to capture the difference between metrizability and first countability.

The Normal Moore Space Conjecture [97] is the following statement:

Every normal Moore space is metrizable.

[42] The author would like to thank Alan Dow for helpful correspondence and certification of the results in this section.

The question of metrizability was reduced to a set-theoretic context by work of Bing [10] who showed that collectionwise normal Moore spaces are metrizable. Thus the conjecture follows if normal implies collectionwise normal. This was just the beginning: the apparently innocuous statement of the Normal Moore Space Conjecture involves a considerable amount of set theory. For example, assuming MA_{ω_1}, Silver [116] showed that the Normal Moore Space Conjecture is false. Fleissner [29] showed that it failed assuming the Continuum Hypothesis and proved [28] that the Normal Moore Space Conjecture implies that there are inner models with measurable cardinals. In the positive direction, work of Kunen; Nyikos [98]; and Dow, Tall, and Weiss [25] and others have shown it is true after forcing over of models of ZFC that have supercompact cardinals.[43]

Given that the proofs rely on computing the quotient algebra of a generic elementary embedding, it is not surprising that it is possible to make an axiomatic statement that suffices to prove the Normal Moore Space Conjecture and its more sophisticated variants.

The following result is a simple codification of the reflection results necessary to prove the Normal Moore Space Conjecture in a generic extension of V by Cohen or random reals.

5.41 Theorem. *Suppose that for all $\lambda \geq 2^{\aleph_0}$ there is a $2^{\aleph_0}$-complete, normal, fine ideal $I \subseteq [\lambda]^{<2^{\aleph_0}}$ such that $P([\lambda]^{<2^{\aleph_0}})/I$ is either the Boolean algebra for adding Cohen reals or random reals. Then:*

1. *Every normal space of character less than $2^{\aleph_0}$ is collectionwise normal. In particular the Normal Moore Space Conjecture is true.*
2. *(Balogh) Every normal, locally compact space is collectionwise normal.*

5.8. Consequences in Descriptive Set Theory

As we have seen, assumptions about the existence of generic elementary embeddings prove many combinatorial properties that are relevant to fine structure and the core model theory. For example, the existence of generic embeddings constructed in sufficiently closed forcing extensions imply that every stationary subset of a successor of a singular cardinal reflects. (Theorems 5.22 and 5.24.) Thus, using core model theory Projective Determinacy holds.

Woodin has shown that the existence of a countably complete, $\aleph_1$-dense ideal on ω_1 implies that the Axiom of Determinacy (AD) holds in $L(\mathbb{R})$. Ketchersid showed that stronger ideal axioms imply AD^+, a stronger form of the axiom due to Woodin, holds in an inner model containing the real numbers with Θ relatively large. We are content here to show a very easy result from [35], one that foreshadowed later, much more impressive results.[44]

[43] Theorem 2(1) of [25].

[44] Even before this result, Magidor [90] also had results showing that ideal axioms gave the Lebesgue measurability of Σ^1_4-sets.

5.42 Theorem. *If there is a normal, fine, $2^{\aleph_0}$-dense ideal I on $[(2^{\aleph_0})^+]^{\omega_1}$ or on $[2^{\aleph_0}]^{<\omega_1}$, then the following are true in $L(\mathbb{R})$:*

1. *Every set of reals is Lebesgue measurable, completely Ramsey and has the property of Baire.*

2. *The partition relation $\omega \to (\omega)^\omega$.*

Woodin showed, somewhat earlier:

5.43 Theorem. *If* CH *holds and there is a countably complete, $\aleph_1$-dense ideal on ω_1, then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable, has the property of Baire and $L(\mathbb{R}) \models \omega \to (\omega)^\omega$.*

Proof. Both theorems have essentially the same proof: In either case, by forcing with $\mathrm{Col}(\omega, 2^{\aleph_0})$, we get a generic object G and an elementary embedding $j : V \to M \subseteq V[G]$, where $M^\omega \cap V[G] \subseteq M$ and $|\mathbb{R}|^V = \omega$. In each case $j : L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$. Let $\phi(x, y_1, \dots, y_n, z_1, \dots, z_m)$ define a set of real numbers in $L(\mathbb{R})^V$ by

$$A = \{r : \phi^{L(\mathbb{R})}(r, a_1, \dots, a_n, \alpha_1, \dots, \alpha_m)\}$$

where $a_1, \dots, a_n \in \mathbb{R}$ and $\alpha_1, \dots, \alpha_m \in \mathrm{On}$. Then $j(A)$ is the set of reals defined in $L(\mathbb{R})^{V[G]}$ by ϕ, the a's and $j(\alpha_1), \dots, j(\alpha_m)$. Hence $j(A)$ is a set that is definable in $L(\mathbb{R})^{V[G]}$ using reals in V. Since G is the result of forcing with a Levy collapse, $j(A)$ must be Lebesgue measurable (see Solovay [110]). By elementarity, A must be measurable.

The results for the property of Baire and the partition property follow the same outline. ⊣

5.9. Connections with Non-Regular Ultrafilters

Early investigations into ultraproducts [15] were concerned with the coarsest possible property: their cardinality. In this section we focus on ultraproducts of structures whose domains have cardinality $\aleph_0$ and $\aleph_1$. The generalizations to structures with larger domains are straightforward. The main result of the section are Theorems 5.47 and 5.51, which for $\kappa \in \{\omega_1, \omega_2\}$ give the existence of an ultrafilter F on κ such that $|\omega^\kappa/F| = \aleph_1$.

If U is a countably complete ultrafilter, then ω^κ/U has cardinality $\aleph_0$. On the other hand, for countably incomplete ultrafilters the "obvious" cardinality of an ultrapower ω^κ/U is 2^κ. It became a prominent question whether the "obvious" cardinality was always obtained [72, 15].

We begin with a simple result:

5.44 Proposition. *Suppose that U is an ultrafilter on a cardinal κ that is not countably complete. Then $|\omega^\kappa/U| \geq 2^{\aleph_0}$.*

Proof. Let $\langle X_n : n \in \omega\rangle$ be a partition of κ into sets that are not in U. For $\alpha \in \kappa$ let $n(\alpha)$ be such that $\alpha \in X_{n(\alpha)}$. For $A \subseteq \omega$, let $\langle m_i : i \in \omega\rangle$ enumerate A in increasing order, and $\langle p_i : i \in \omega\rangle$ enumerate the prime numbers. Define a function $f_A : \kappa \to \omega$ by setting $f_A(\alpha) = \prod_{m_i < n(\alpha)} p_i^{m_i}$.

We claim that if $A \neq B$, then $[f_A] \neq [f_B]$. For if $A \neq B$, we can find an $n \in A\Delta B$. For all $\alpha \in \bigcup_{n' \geq n} X_{n'}$, $f_A(\alpha) \neq f_B(\alpha)$. ⊣

The following is the negation of a standard condition that implies maximal cardinalities for ultrapowers.

5.45 Definition. An ultrafilter U on Z is (μ, γ)-*non-regular* iff whenever $\langle X_\alpha : \alpha < \gamma\rangle \subseteq U$, there is an $S \subseteq \gamma$ with $|S| = \mu$ and $\bigcap_{\alpha \in S} X_\alpha \neq \emptyset$.

Any fine ultrafilter on $[\kappa]^{<\omega}$ is easily seen to be (ω, κ)-regular. Hence for each cardinal κ, there is a regular ultrafilter on $P(\kappa)$. If U is an (ω, κ)-regular ultrafilter on a set Z of size κ, then ω^Z/U has cardinality 2^κ [15].

Duals of ideals with nice quotient algebras turn out to be filters that can be extended to highly non-regular ultrafilters. The first example of a theorem of this type is due to Magidor. We shall see in Theorem 7.43 that the hypotheses of the next theorem are consistent.

5.46 Theorem (Magidor [89]). *Assume* GCH. *Suppose that I is a normal, fine, countably complete, $\aleph_3$-saturated, $\aleph_3$-dense ideal on $[\omega_3]^{\omega_1}$, and let F be any ultrafilter on $[\omega_3]^{\omega_1}$ extending $\breve{I}$. Then $|\omega^{[\omega_3]^{\omega_1}}/F| = \aleph_3$, and even $|\omega_1^{[\omega_3]^{\omega_1}}/F| = \aleph_3$.*

Note that since GCH implies that $|[\omega_3]^{\omega_1}| = \aleph_3$, the result yields an ultrafilter $F' \subseteq P(\omega_3)$ such that $\omega_1^{\omega_3}/F'$ has cardinality $\aleph_3$.

Proof. Let $Z = [\omega_3]^{\omega_1}$. We show the stronger fact that the reduced product ω^Z/I has cardinality $\aleph_3$.

Since there is a canonical bijection between functions $f : [\omega_3]^{\omega_1} \to \omega$ (modulo I) and partitions $\langle A_n : n \in \omega\rangle$ of $P(Z)/I$, it suffices to count partitions. Let $D = \{d_\alpha : \alpha < \omega_3\} \subseteq P(Z)/I$ be a dense set of size $\aleph_3$. By the $\aleph_3$-c.c., $P(Z)/I$ is a complete Boolean algebra, and for each $A \in P(Z)/I$ there is a $\beta < \omega_3$ and a set $B \subseteq \beta$ such that $A = \bigtriangledown_{\alpha \in B} d_\alpha$. Hence for all partitions $\langle A_n : n \in \omega\rangle$, there is a $\beta < \omega_3$ and sets $B_n \subseteq \beta$ such that for all n, $A_n = \bigtriangledown_{B_n} d_\alpha$. Hence the cardinality of the set of partitions is $\aleph_3 \times (2^{\aleph_2})^{\aleph_0} = \aleph_3$.

To see the stronger claim that ω_1^Z/I has cardinality $\aleph_3$ we show that there is a cofinal subset of ω_1^Z/I that has cardinality $\aleph_3$. This suffices, since if $f \in \omega_1^Z$, the previous results show $|\prod_{z \in Z} f(z)/I| = \aleph_3$.

For $\alpha \in \omega_3$ let

$$f_\alpha : \{z : \alpha \in z\} \to \omega_1$$

by setting $f_\alpha(z) = \gamma$ iff α is the γth element of z. Clearly if $\alpha < \beta$ then $\{z : f_\alpha(z) < f_\beta(z)\} \in \breve{I}$. Let $g : Z \to \omega_1$ be arbitrary. Define h by setting $h(z) = \gamma$ where γ is the $g(z)$th element of z. Then h is a regressive function and hence we can find a maximal antichain of I-positive sets on which h is constant. By the $\aleph_3$-saturation of I, there is an α bigger than all of these constant values. The function f_α dominates g on each element of the antichain and hence on a set in $\breve{I}$. $\dashv$

Magidor proved a quite delicate result that if I is a normal, fine, countably complete, $\aleph_3$-dense, $\aleph_3$-saturated ideal on $[\omega_3]^{\omega_1}$ in V, and $G \subseteq \mathrm{Col}(\omega, \omega_1)$ is generic, then in $V[G]$ any ultrafilter F extending $\breve{I}$ satisfies $|\omega^{\omega_2}/F| = \aleph_2$. This consistency result is improved by Theorem 5.51.

The first result getting an ultrapower of ω to have cardinality ω_1 is due to Laver [84]:

5.47 Theorem. *Suppose that $\diamondsuit_{\omega_1}$ holds and there is a countably complete, $\aleph_1$-dense ideal on ω_1. Then there is an (ω, ω_1)-non-regular ultrafilter D such that $|\omega^{\omega_1}/D| = \aleph_1$.*

Laver's construction yielded the stronger property that there is an $\aleph_1$-generated ultrafilter over the $\aleph_1$-dense ideal. Woodin later eliminated the diamond assumption for getting an $\aleph_1$-generated ultrafilter although CH is needed to get the small ultrapower. Laver's result was improved by Kanamori to:

5.48 Theorem (Kanamori [67]). *Assume that there is an $\aleph_1$-dense ideal I on ω_1 and $\diamondsuit_{\omega_1}$. Suppose that D is any non-principal ultrafilter on ω, and $f : \omega_1 \to \omega$ is a map such that $f^{-1}(\{n\}) \in I^+$ for all $n \in \omega$. Then there is an $\aleph_1$-generated ultrafilter U over ω_1 extending I^* such that $f_*(U) = D$.*

Kanamori showed that for regular λ, there is a weakly normal ultrafilter U on λ^+ concentrating on $\mathrm{Cof}(\lambda)$ iff there is a (λ, λ^+) non-regular ultrafilter on λ^+.

In [48], it was shown that:

5.49 Theorem. *Suppose that there is a layered ideal I on $\kappa = \lambda^+$ and $\diamondsuit_\kappa$. Then there is a (κ^+, ∞)-distributive partial ordering $\mathbb{P}$ that adds a weakly normal ultrafilter on κ extending $\breve{I}$.*

Since a saturated ideal on κ concentrates on $\mathrm{Cof}(\lambda)$ this result yields a fully non-regular ultrafilter. For ultrafilters with small ultrapower, a strongly layered ideal can be used:

5.50 Theorem (Foreman et al. [48]). *Suppose that $\diamondsuit_{\omega_1}$ and there is a strongly layered ideal I on ω_1. Then there is an (ω_2, ∞)-distributive forcing adding an ultrafilter $D \supseteq \breve{I}$ such that $|\omega^{\omega_1}/D| = \aleph_1$.*

As remarked before Theorem 4.16, it is easy to force a strongly layered ideal from a layered ideal.

Fully Non-Regular Ultrafilters on ω_2

In this section we consider some ideal properties shown to be consistent in Sect. 7.

Corollary 7.66 says: if κ is regular, $\kappa^{<\kappa} = \kappa$ and there is a very strongly layered ideal I on κ^+, $\square_{\kappa^+}$ and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$, then for all uniform ideals J on κ, there is a uniform ideal K on κ^+ such that:

$$P(\kappa^+)/K \;\cong\; P(\kappa)/J.$$

Furthermore, the degree of completeness of K equals the degree of completeness of J, and if J is κ^+-saturated then K is weakly normal.

By Theorem 7.63, the hypotheses of Corollary 7.66 are consistent with $\kappa = \omega_1$ and the existence of an $\aleph_1$-dense ideal on ω_1. Moreover, in unpublished work, the author showed that the $\square$ assumption is not necessary.

Thus the hypotheses of the following theorem are consistent:

5.51 Theorem. *Assume* GCH. *Suppose that ω_1 carries a countably complete uniform, $\aleph_1$-dense ideal on ω_1, and there is a very strongly layered ideal on ω_2. Then there is an ultrafilter F on ω_2 such that $|\omega^{\omega_2}/F| = \aleph_1$ and $|\omega_1^{\omega_2}/F| = \aleph_2$.*

Proof (Sketch). By Corollary 7.66, we can start with a uniform, countably complete, $\aleph_1$-dense ideal I on ω_2. Woodin's proof of Laver's $\diamondsuit_{\omega_1}$ result gives in ZFC that if $\mathcal{B}$ is a Boolean algebra with a dense subset $\mathcal{D}$ of size $\aleph_1$, then there is an ultrafilter $\tilde{F} \subseteq \mathcal{B}$ such that for any $A \subseteq \mathcal{D}$ with $\bigvee A = 1$ then there is a countable $B \subseteq A$ such that $\bigvee B \in \tilde{F}$. Suppose that $\mathcal{B} = P(\omega_2)/I$ has a dense set $\mathcal{D}$ of size $\aleph_1$ and $\tilde{F}$ is such an ultrafilter. Let F be the ultrafilter on ω_2 induced by $\tilde{F}$ over $\breve{I}$. Suppose that $f : \omega_2 \to \omega$ is an arbitrary function. Let A_n be those elements of $\mathcal{D}$ below $f^{-1}(\{n\})$. Then there is a countable collection $B \subseteq \bigcup_n A_n$ such that $\bigvee B \in \tilde{F}$. Choose disjoint representatives $\{b_m : m \in \omega\}$ for the elements of B. Define $g : \bigcup B \to \omega$, by setting $g(\alpha) = n$ iff $\alpha \in b_m$ and $b_m \subseteq A_n$. Then $g \equiv f \pmod{\tilde{F}}$. Since there are only $\aleph_1$ many such g's $(\mathrm{mod}\ \tilde{F})$ we see that ω^{ω_2}/F has cardinality $\aleph_1$.

It is a general fact that if $|\lambda^\kappa/U| = \delta$ then $|(\lambda^+)^\kappa/U| \leq \delta^+$. We know that $|\omega_1^{\omega_2}| \geq \aleph_2$ and hence $\omega_2 \leq |\omega_1^{\omega_2}/F| \leq \aleph_2$. ⊣

Taking $\mu = \omega_1$ in the model of Theorem 7.63, Corollary 7.66 shows that there is a uniform, weakly normal, countably complete ideal K on ω_2 such that $P(\omega_2)/K$ has a dense subset of size $\aleph_1$. Thus we get the following corollary:

5.52 Corollary. *Suppose that there are almost huge embeddings j_0, j_1 with critical points κ_0, κ_1 respectively such that $j_0(\kappa_0) = \kappa_1$. Then there is a forcing extension in which* GCH *holds and there is an ultrafilter F on ω_2 such that $|\omega^{\omega_2}/F| = \aleph_1$ and $|\omega_1^{\omega_2}/F| = \aleph_2$.*

Layered ideals on ω_2 also give the following generalization of Theorem 5.48:

5.53 Theorem. *Suppose* $\diamondsuit_{\omega_2}(\mathrm{Cof}(\omega_1))$ *and that there is a very strongly layered ideal* I *on* ω_2*. Then for all functions* $f : \omega_2 \to \omega_1$ *which are not bounded in* ω_1 *on a set in* $\breve{I}$ *there is a uniform, weakly normal, countably complete ideal* K *on* ω_2 *such that:*

1. $P(\omega_2)/K \cong P(\omega_1)/\{\text{countable sets}\}$*, and*

2. *for all* $g : \omega_2 \to \omega_1$ *there is a* $h : \omega_1 \to \omega_1$ *with* $g \equiv_K h \circ f$*.*

Kanamori [67] calls the function f in the theorem a *finest partition* relative to K. We note the following corollary:

5.54 Corollary. *Under the hypotheses of Theorem 5.53 together with the assumption that* $2^{\omega_1} = \omega_2$*, there is a uniform, countably complete ideal* K *on* ω_2 *such that if* F *is any ultrafilter extending* $\breve{K}$ *then*

$$|\omega_1^{\omega_2}/F| = \aleph_2.$$

For cardinals $\kappa > \omega_2$, the cardinality of ultrapowers ω^κ/F is remains a mystery for the most part. For more information on non-regular ultrafilters we refer the reader to [119, 71, 67].

5.10. Graphs and Chromatic Numbers

Erdős and Hajnal defined the following graph in the early 1960's:

$$\mathfrak{G}(\kappa, \lambda) = \langle \{f | f : \kappa \to \lambda\}, \perp \rangle,$$

where $f \perp g$ iff $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$. This graph is of interest partly because Erdős and Hajnal showed that if G is a graph of cardinality ω_2 such that each subgraph has countable chromatic number, then G can be embedded into $\mathfrak{G}(\omega_2, \omega)$ as a subgraph.[45]

In particular, they showed that CH implies that $\mathfrak{G}(\omega_2, \omega)$ has uncountable chromatic number and asked whether this graph could have chromatic number $\aleph_1$. It is immediate that if there is a uniform ultrafilter F on ω_2 such that ω^{ω_2}/F has cardinality ω_1, then $\mathfrak{G}(\omega_2, \omega)$ has chromatic number $\aleph_1$. To see this one colors an element of $\mathfrak{G}(\omega_2, \omega)$ by the member of the ultrapower it determines. Connected elements of $\mathfrak{G}(\omega_2, \omega)$ are functions that eventually differ and hence are in different classes of the ultrapower by any uniform ultrafilter.

In particular, we get:

5.55 Theorem (Foreman [36]). *Assume* GCH *and that there is a uniform, countably complete,* $\aleph_1$*-dense ideal on* ω_2*. Then the Erdős-Hajnal graph has chromatic number* ω_1*.*

5.56 Corollary. *Assume* GCH *and that there is a uniform countably complete,* $\aleph_1$*-dense ideal on* ω_2*. If* G *is a graph of cardinality* ω_2 *and chromatic number* ω_2*, then* G *has an induced subgraph of cardinality* ω_1 *and chromatic number* ω_1*.*

[45] The image of G is not necessarily an *induced* subgraph.

5.11. The Nonstationary Ideal on ω_1

We now will consider the family of assumptions around the statement "the nonstationary ideal on ω_1 is $\aleph_2$-saturated". This statement seems singular in that there are no known analogues at other cardinals. For example, the assertion that the nonstationary ideal on ω_2 is $\aleph_3$-saturated is inconsistent.[46] Weakening this a bit, one can imagine that the nonstationary ideal on ω_2 restricted to the collection of ordinals of cofinality ω_1 is $\aleph_3$-saturated. However this assumption implies that $\Theta^{L(\mathbb{R})} < \omega_2$ and this is inconsistent with the assumption that the nonstationary ideal on ω_1 is $\aleph_2$-saturated.

We discuss this further in Sect. 8.2 (after Theorem 8.8) and again in Sect. 11.

Our first result is due to Shelah. Recall that Devlin and Shelah [22] showed that $2^{\aleph_0} < 2^{\aleph_1}$ is equivalent to the following *weak diamond* property:[47]

> For all $F : 2^{<\omega_1} \to 2$ there is a function $g : \omega_1 \to 2$ such that for all $f : \omega_1 \to 2$ the set $\{\alpha : g(\alpha) = F(f\restriction\alpha)\}$ is stationary.

5.57 Theorem (Shelah [102]). *Suppose that weak diamond holds. Then the nonstationary ideal on ω_1 is not $\aleph_1$-dense.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a collection of stationary sets such that $\{[A_\alpha]_I : \alpha < \omega_1\}$ is dense in $P(\omega_1)/I$ where I is the nonstationary ideal on ω_1. Given a function $f : \delta \to 2$ define $F(f)$ as follows:

$$F(f) = \begin{cases} 1 & \text{if } \delta \notin \bigcup\{A_\alpha : f(\alpha) = 1\}, \\ 0 & \text{if } \delta \in \bigcup\{A_\alpha : f(\alpha) = 1\}. \end{cases}$$

Suppose that g is as in the definition of weak diamond. Let $B = \{\alpha : g(\alpha) = 1\}$. Then B must be stationary. Let $f : \omega_1 \to 2$ be defined by setting $f(\alpha) = 1$ iff $A_\alpha \subseteq_I B$. Then $[\triangledown\{A_\alpha : f(\alpha) = 1\}]_I = [B]_I$. Let C be a closed and unbounded set such that $C \cap \triangledown\{A_\alpha : f(\alpha) = 1\} = C \cap \{\beta : \text{for some } \alpha < \beta, \beta \in A_\alpha \text{ and } f(\alpha) = 1\} = C \cap B$.

Suppose now that $\delta \in C$ and $F(f\restriction\delta) = g(\delta)$. Then $F(f\restriction\delta) = 1$ iff $\delta \notin C \cap \triangledown\{A_\alpha : f(\alpha) = 1\}$ iff $\delta \notin B$ iff $g(\delta) = 0$, a contradiction. ⊣

5.58 Corollary. *Suppose that the nonstationary ideal on ω_1 is $\aleph_1$-dense. Then $2^{\aleph_0} = 2^{\aleph_1}$; in particular,* CH *fails.*

We now show Woodin's remarkable theorem that $\underset{\sim}{\delta}^1_2 = \omega_2$ if the nonstationary ideal on ω_1 is saturated and there is a measurable cardinal.

We need some facts about indiscernibles. For general information we direct the reader to [63]. Suppose that N is a countable, transitive structure in a countable language that satisfies ZFC and has Skolem functions. Suppose further that $\langle i_n : i \in \omega\rangle \subseteq \mathrm{On}^N$ is a sequence of indiscernibles for N such that

[46] See Corollary 6.11.
[47] See Example 3.23.

$N = \mathrm{Sk}^N(\langle i_n : n \in \omega\rangle)$. Let $\Gamma(\langle i_n : n \in \omega\rangle)$ be the theory of $\langle N, \langle i_n : n \in \omega\rangle\rangle$. Suppose that $\gamma \geq \omega$ is an ordinal and define N' to be the *stretch* of N by adding γ indiscernibles to N. This is the Ehrenfeucht-Mostowski model constructed by adding new constant symbols $\langle c_\alpha : \alpha < \gamma\rangle$ to our language and using Γ as a "blueprint". If Γ is "remarkable", then every such N' is well-founded and we identify it with its transitive collapse. In this case N is an initial segment of N'.

The following lemma is standard:

5.59 Lemma. *Suppose that $\mathfrak{A}$ is a transitive structure of cardinality κ, where κ is a measurable cardinal. Let U be a normal ultrafilter on κ and suppose that $X \in U$ is a set of indiscernibles for $\mathfrak{A}$. Let $\{j_n : n \in \omega\} \subseteq X$ and N be the transitive model isomorphic to $\mathrm{Sk}^{\mathfrak{A}}(\{j_n : n \in \omega\})$. Then the theory $\langle N, \langle j_n : n \in \omega\rangle\rangle$ is remarkable.*

We say that N is *remarkable* if it is generated by indiscernibles over a remarkable blueprint. We will use remarkable structures for iterations:

5.60 Lemma. *Suppose that N is a remarkable structure and $I \in N$ is a precipitous ideal on a set $Z \in N$. Let μ be any ordinal.*[48] *Then any generic iteration of N by I of length μ is well-founded.*

Proof. Let $\gamma > \mu$ be an ordinal. Let N' be the stretch of N by adding γ indiscernibles. Let

$$\{\langle N_\alpha : \alpha \leq \mu\rangle, \langle j_{\alpha,\alpha'} : \alpha < \alpha' \leq \mu\rangle, \langle G_\alpha : \alpha < \mu\rangle\}$$

be a generic iteration of N. Then one shows by induction on δ that there is an iteration

$$\left\{\langle N'_\alpha : \alpha \leq \delta\rangle, \langle j'_{\alpha,\alpha'} : \alpha < \alpha' \leq \delta\rangle, \langle G'_\alpha : \alpha < \delta\rangle\right\}$$

with the properties that for all $\alpha < \alpha' \leq \delta$:

1. $N'_0 = N'$,
2. N_α is an initial segment of N'_α,
3. $j'_{\alpha,\alpha'} \restriction N_\alpha = j_{\alpha,\alpha'}$,
4. for all $w \in N$, $j'_{0,\delta}(w) = j_{0,\delta}(w)$, and
5. $G'_\alpha = G_\alpha$ for all $\alpha < \delta$.

By Theorem 4.47, we see that N'_μ is well-founded. Since N_μ is an initial segment of N'_μ we see that N_μ is well-founded. ⊣

[48] We note that $\mu > \mathrm{On}^N$ is allowed.

We now fake some descriptive set theory:

5.61 Lemma. *Let δ be a countable ordinal. Suppose that N is countable and $N \models \mathrm{ZFC}$ + "I is a precipitous ideal". Then $R = \{x : x$ is a linear ordering coding On^M where M is the result of a generic iteration of N by I of length $\delta\}$ is a $\mathbf{\Sigma}^1_1$ set.*

Proof. We first remark that the statement "M_0 is a code for a generic ultrapower of M_1 by G" is Σ^1_1 in reals coding M_0 and M_1 and G. Fix real numbers $\tilde{N}$, $\tilde{\delta}$ coding N and δ.

We then see that $x \in R$ iff $\exists \langle N_i : i < \tilde{\delta}\rangle, \langle j_{i,i'} : i < i' \in \tilde{\delta}\rangle, \langle G_i : i \in \tilde{\delta}\rangle$, M such that:

1. $N_0 = N$,
2. for all i, N_{i+1} is the generic ultrapower of N_i by G_i and $j_{i,i+1}$ is the canonical ultrapower embedding of N_i to N_{i+1},
3. for all $i < i'$, $j_{i,i'} : N_i \to N_{i'}$ and the $j_{i,i'}$'s commute,
4. for limit i', $N_i = \varinjlim \langle N_i : i < i'\rangle$ by the embeddings $\langle j_{i,k} : i < k < i'\rangle$,
5. the maps $j_{i,i'}$ are the canonical maps into the direct limit,
6. $M = \varinjlim \langle N_i : i \in \delta\rangle$ by the maps $\langle j_{i,i'} : i < i' < \delta\rangle$, and
7. x is isomorphic to On^M.

⊣

We get the following immediate corollaries:

5.62 Corollary. *Let δ be a countable ordinal. Suppose that N is countable and $N \models \mathrm{ZFC}$ + "I is a precipitous ideal" and every countable generic iteration of N by I is well-founded. Then for every $\delta < \omega_1$ there is a $\beta(\delta) < \omega_1$ such that the order type of the ordinals in a generic iteration of N of length δ is less than $\beta(\delta)$.*

Proof. Since N is iterable, the set R above is a $\mathbf{\Sigma}^1_1$ subset of $\mathcal{WO}$, the set of reals coding well-orderings, and hence bounded. ⊣

5.63 Corollary. *Suppose that $V \subseteq W$ are transitive models of* ZFC*, and $N \in V$ is countable and $N \models \mathrm{ZFC}$ + "I is a precipitous ideal". Suppose that $V \models \delta, \gamma$ are countable ordinals. Then the statement "there is a generic iteration M of N of length δ where the order type of $\mathrm{On}^M = \gamma$" is absolute between V and W.*

We now apply these ideas to prove:

5.64 Theorem (Woodin [126]). *Suppose that the nonstationary ideal on ω_1 is $\aleph_2$-saturated and there is a measurable cardinal. Then $\underset{\sim}{\delta}^1_2 = \omega_2$.*

As a coarse consequence of the hypotheses of Theorem 5.64, $\Theta^{L(\mathbb{R})} > \omega_2$ and CH fails.

Proof. We will use the characterization:

$$\underset{\sim}{\delta}^1_2 = \sup\{(\kappa^+)^{L(x)} : x \in \mathbb{R}\}$$

where $\kappa = \omega_1^V$.

Let $\alpha < \omega_2$ be an ordinal. We will find a real number we will call $\tilde{N}$ such that $(\kappa^+)^{L(\tilde{N})} \geq \alpha$. Let $\mathfrak{A} = \langle H(\mu), \in, \Delta, \alpha\rangle$, where μ is a measurable cardinal. Let I be a remarkable set of indiscernibles for $\mathfrak{A}$, and N be the model $\mathrm{Sk}^{\mathfrak{A}}(\{i_n : n \in \omega\})$, where $\{i_n : n \in \omega\}$ are the first ω elements of I. By Lemma 5.60, we see that any generic iteration of N by the nonstationary ideal on ω_1 is well-founded.

Let N_0 be the transitive collapse of N and $j_0 : N_0 \to \mathfrak{A}$ be the inverse of the collapse map.

Claim. There is an iteration $\langle N_\gamma : \gamma \leq \omega_1\rangle$ of N_0 of length ω_1 such that $\mathrm{On}^{N_{\omega_1}} > \alpha$.

Proof of Claim. Inductively define models N_γ and embeddings $j_\gamma : N_\gamma \to \mathfrak{A}$. Suppose that we are given N_γ, j_γ. By Example 3.47, if M_γ is the image of N_γ under j_γ, and $\delta_\gamma = M_\gamma \cap \omega_1$, then $U(j_\gamma, \delta_\gamma)$ is generic over N_γ. Let $N_{\gamma+1}$ be the transitive collapse of the ultrapower of N_γ by $G = U(j_\gamma, \delta_\gamma)$. Define $j_{\gamma+1} : N_{\gamma+1} \to \mathfrak{A}$ by $j_{\gamma+1}([f]_G) = j_\gamma(f)(\delta_\gamma)$. At limit stages γ we take direct limits of the N_η with the j_η for $\eta < \gamma$.

It is easy to check inductively that $M_{\gamma+1} = \mathrm{Sk}^{\mathfrak{A}}(M_\gamma \cup \{\delta_\gamma\})$. In particular, $M_{\omega_1} \supseteq \omega_1$. Since $M_{\omega_1} \prec \mathfrak{A}$, $\alpha \subseteq M_{\omega_1}$ and hence the order type of $\mathrm{On}^{M_{\omega_1}} > \alpha$, and hence the order type of $\mathrm{On}^{N_{\omega_1}} > \alpha$. This establishes the claim. $\dashv$

Let $\tilde{N}$ be a real number coding N. We show that $(\kappa^+)^{L(\tilde{N})} > \alpha$, where $\kappa = \omega_1^V$. Let G be generic for $\mathrm{Col}(\omega, \kappa)$ over $L(\tilde{N})$. Apply Corollary 5.62 in $L(\tilde{N})[G]$ with $\delta = \kappa$, to see that there is a $\beta(\kappa) < \omega_1^{L(\tilde{N})[G]}$ such that all iterations of N of length κ in $L(\tilde{N})[G]$ have ordinals bounded by $\beta(\kappa)$. By Corollary 5.63, and the previous claim we see that $\alpha < \beta(\kappa) < \omega_1^{L(\tilde{N})[G]}$. Since G is generic for $\mathrm{Col}(\omega, \kappa)$, $\omega_1^{L(\tilde{N})[G]} = (\kappa^+)^{L(\tilde{N})}$, so $\alpha < (\kappa^+)^{L(\tilde{N})}$, as desired. $\dashv$

6. Some Limitations

In this section we discuss some limitations on the types of ideals that can exist. These limitations affect all three of the parameters determining the nature of the generic embeddings:

- where the cardinals go,

- the closure of the ultrapower (in particular, the saturation of the ideal), and
- the nature of the quotient algebra of the ideal.

We note that Burke found a limitation on the closure of the generic ultrapower when forcing with a tower of ideals that appears in a different section as Proposition 9.48.

6.1. Soft Limitations

For this section we let $j : V \to M$ be a generic elementary embedding and let κ be the critical point of j. The following facts are straightforward:

6.1 Proposition. *κ is regular in V. If $2^\lambda < \kappa$ and $M^\lambda \cap V[G] \subseteq M$ then $P(\lambda) \cap V[G] = P(\lambda) \cap V$.*

Note however the last conclusion cannot be strengthened to say that $V^\lambda \cap V[G] \subseteq V$. In Sect. 9, there are examples of stationary tower forcings that induce elementary embeddings with critical point $\aleph_{\omega+1}$ (and GCH holds). Hence the forcing adds new ω-sequences to $\aleph_\omega$ even though the critical point of the generic embedding is above $2^{\aleph_0}$.

6.2 Proposition. *Suppose that $\lambda < \delta$ are cardinals such that:*

1. *$\eta^{<\lambda} < \kappa$ for all cardinals $\eta < \kappa$,*
2. *$\delta^{<\lambda} > \delta$, and*
3. *$P(\delta^+)^V \subseteq M$.*

Then $j(\kappa) \neq \delta^+$.

From Proposition 6.2 we see, for example, that CH implies that there is no normal, fine, precipitous ideal on $[\aleph_{\omega+1}]^{\omega_2}$. We see this by setting $\kappa = \omega_2$, $\lambda = \omega_1$ and $\delta = \aleph_\omega$.

A weak version of the next result was proved by Ulam [123] using Ulam matrices and improved by Baumgartner, Taylor and Wagon [8].

6.3 Proposition. *Suppose that κ is not a weakly Mahlo cardinal of high degree. Then there is no κ-saturated ideal I on any set Z that has completeness κ.*

Proof. To see this note that if $j : V \to M$ is the generic embedding from a κ-complete, κ-saturated ideal on Z, then:

1. $\mathrm{crit}(j) = \kappa$,
2. $M^\kappa \cap V[G] \subseteq M$, and
3. κ is regular in M.

If $\kappa = \delta^+$ in V then $M \models \delta^+ = j(\kappa) > \kappa$ and hence κ is not a cardinal in M, contradicting κ-saturation. Thus κ must be a regular limit cardinal. Let C be a closed unbounded set in κ. Then $j(C)$ is unbounded in κ and hence $M \models \kappa \in j(C)$. Thus by reflection, C contains a regular cardinal and so κ is Mahlo. Clearly this argument can be used to show that κ is also highly Mahlo. ⊣

We note that one ingredient in the proof is the fact that if $j : V \to M$ is a generic elementary embedding in $V[G]$ with critical point κ where κ is a successor cardinal in V, then κ is not a cardinal in M.

Recall the definition of the weakly compact filter from Example 3.28. From Proposition 6.1 we can easily see:

6.4 Proposition. *Let κ be a weakly compact cardinal. Then the weakly compact filter on κ is not κ-saturated.*

Proof. Toward a contradiction, assume that κ is the least weakly compact cardinal where the weakly compact ideal is κ-saturated.

Let F be the weakly compact filter, and I be the dual ideal. Let $G \subseteq P(\kappa)/I$ be generic and $j : V \to M$ be the generic embedding. Since I is κ-saturated and κ-complete, $M^\kappa \cap V[G] \subseteq M$.

We first claim that the forcing $P(\kappa)/I$ is (κ, ∞)-distributive. Since the critical point of j is κ, $j \restriction V_\kappa$ is the identity. Since κ is inaccessible and M is closed under κ-sequences from $V[G]$, this implies that $P(\kappa)/I$ adds no new bounded subsets of κ.

If $P(\kappa)/I$ is not (κ, ∞) distributive, let τ be a term for a new function from γ into V for some $\gamma < \kappa$. Let $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$ be a sequence of maximal antichains such that each $p \in \mathcal{A}_\alpha$ decides the value of $\tau(\alpha)$. Then each $\mathcal{A}_\alpha$ has cardinality less than κ. Let $\mu = \sup\{|\mathcal{A}_\alpha| : \alpha < \gamma\}$. In V choose a sequence of injections $i_\alpha : \mathcal{A}_\alpha \to \mu$. If G is generic, then the sequence $\langle G \cap \mathcal{A}_\alpha : \alpha < \gamma \rangle$ determines τ. However the sequence $\langle i_\alpha(G \cap A_\alpha) : \alpha < \gamma \rangle$ is a bounded sequence in κ, hence it lies in V. But then the sequence $\langle G \cap \mathcal{A}_\alpha : \alpha < \gamma \rangle$, and hence τ also lie in V, a contradiction.

We now claim that there is a new subset of κ in $V[G]$. If not, then $P(\kappa)^{V[G]} = P(\kappa)^V = P(\kappa)^M$. Hence, κ is weakly compact in M and has the same weakly compact filter.

Since κ is weakly compact the product of two κ-c.c. partial orderings is still a κ-c.c. partial ordering. So $P(\kappa)/I \times P(\kappa)/I$ is κ-c.c. Hence I is still κ-saturated in M. But $M \models$ "$j(\kappa)$ is the least weakly compact cardinal where the weakly compact filter is saturated", a contradiction.

Thus $P(\kappa)/I$ is κ-c.c., (κ, ∞)-distributive and adds a new subset to κ. Let $\dot{\tau}$ be a term for a new subset of κ. Let $\mathcal{B}$ be the complete Boolean subalgebra generated by $\{\|\alpha \in \tau\| : \alpha < \kappa\}$. Then $\mathcal{B}$ is a Suslin algebra, contradicting the weak compactness of κ. ⊣

In recent unpublished work Hellsten has shown that it is consistent for the weakly compact ideal on a weakly compact cardinal κ to be κ^+-saturated.

6.2. The "Kunen Argument"

Let $j : V \to M$ be an elementary embedding and let $\kappa_0 = \mathrm{crit}(j)$, $\kappa_{n+1} = j(\kappa_n)$ and $\kappa_\omega = \sup_n \kappa_n$. Then $j(\kappa_\omega) = \kappa_\omega$ and for all $\alpha < \mathrm{crit}(j)$, $j(\kappa_\omega^{+\alpha}) = (\kappa_\omega^{+\alpha})^M$.

Kunen [78] showed that it is inconsistent for there to be a V-definable elementary embedding $j : V \to M$ such that $V_{\kappa_\omega+1} \subseteq M$ or even an elementary embedding $j : V_{\kappa_\omega+2} \to V_{\kappa_\omega+2}$. Kunen's proof uses the construction of an "ω-Jónsson algebra" on κ_ω due to Hajnal and Erdős [27].

As such, Kunen's argument does not immediately generalize to the case where j is not definable in V. For example, if $\mathbb{P}$ is the partial ordering producing a generic embedding j, then $\mathbb{P}$ might destroy ω-Jónsson algebras. Nonetheless, his general technique is quite relevant.

We begin by reminding the reader of a basic definition. An algebra on a cardinal λ with no proper elementary substructure of cardinality λ is called a *Jónsson algebra*. A cardinal λ is called *Jónsson* if there is no Jónsson algebra on λ. Equivalently, every algebra on λ has a proper elementary substructure of cardinality λ.

At the heart of the Kunen technique is the result shown in Sect. 5 that if $\mathfrak{A}$ is an algebra on a cardinal λ then $j``\lambda \prec j(\mathfrak{A})$. In particular, if we have an elementary embedding $j$ and a cardinal $\lambda > \mathrm{crit}(j)$ such that $j(\lambda) = \lambda$ and $j``\lambda \in M$, then λ must be a Jónsson cardinal in V. Since results of Woodin, Tryba [122] and others have shown that the successors of regular cardinals are *not* Jónsson we immediately see the following results:

6.5 Theorem. *Let κ be a successor of a regular cardinal. Then there is no normal, fine, precipitous ideal on $[\kappa]^\kappa$.*

Proof. Note that if I is a non-trivial normal, fine, precipitous ideal on $[\kappa]^\kappa$, then for I-almost every $z \in [\kappa]^\kappa$, there is an $\alpha \in \kappa \setminus z$. By normality, for each I-positive set $A \subseteq [\kappa]^\kappa$ there is a $\beta < \kappa$ and an I-positive $B \subseteq A$ such that for all $z \in B$ there is a $\gamma \in \beta \setminus z$.

Hence if $j : V \to M$ is the generic embedding from the ideal I then the critical point of j must be less than κ. Since I concentrates on $[\kappa]^\kappa$, κ is a fixed point of j. Moreover, by normality and fineness, $j``\kappa \in M$. Thus κ is Jónsson, a contradiction. ⊣

The next corollary follows immediately from the existence of a Jónsson algebra on $j``\kappa_\omega^{+2}$.

6.6 Corollary. *There is no generic elementary embedding $j : V \to M$ such that $j``\kappa_\omega^{+2} \in M$ and $(\kappa_\omega^{+2})^M = (\kappa_\omega^{+2})^V$.*

Shelah [105] has shown that many successors of singular cardinals carry Jónsson algebras. In particular this is true for all successors of singulars below the first inaccessible cardinal.[49] As a result, Kunen's theorem follows for all

[49] Actually, much further.

elementary embeddings whose critical point lies below the first inaccessible cardinal.

6.7 Proposition. *Suppose that κ_ω is below the first inaccessible cardinal. Then there is no elementary embedding $j : V \to M$ such that:*

- $(\kappa_\omega^+)^V = (\kappa_\omega^+)^M$, *and*
- $j``(\kappa_\omega^+)^V \in M$.

In particular, there is no normal, fine, precipitous ideal on $[\aleph_{\omega+1}]^{\aleph_{\omega+1}}$.

Of course there are finer versions of this theorem ruling out elementary embeddings of $V_{\kappa_\omega+2}$ into $M_{\kappa_\omega+2}$ when M contains $j``(\kappa_\omega^+)^V$.

Zapletal [130] pointed out that the techniques used in Shelah's theorem about Jónsson algebras on successors of singular cardinals (pcf theory) can be used to directly show Kunen's theorem that there can be no $j : V \to M$ with $V_{\kappa_\omega+1} \subseteq M$. We briefly review the definitions:

6.8 Definition. Let a be a set of regular cardinals, and I an ideal on a. We consider the reduced product $\prod a/I$ as a partial ordering, by giving each $\kappa \in a$ the ordering of "$\in$". A $<_I$-increasing sequence $\langle f_\alpha : \alpha < \nu\rangle$ is a *scale* in $\prod a/I$ iff for all $g \in \prod a$ there is an α such that $g <_I f_\alpha$.

Shelah [105] showed that for all singular cardinals λ there is a cofinal set $a \subseteq \lambda$ of regular cardinals such that there is a scale $\langle f_\alpha : \alpha < \lambda^+\rangle$ in $\prod a/\{\text{bounded sets in } \lambda\}$.

Using this fact we can give Zapletal's proof of Kunen's theorem:

Let $a \subseteq \kappa_\omega$ be a cofinal set of regular cardinals on which there is a scale $\langle f_\alpha : \alpha < \kappa_\omega^+\rangle$. By thinning out if necessary, we can assume that there is at most one element of a between each κ_n and κ_{n+1} and the least element of a is above κ_0. In particular, $j(a) = j``a$. Suppose that $j : V \to M$ is such that $j``\kappa_\omega \in M$. Then we can verify the following claims:

1. $j(\kappa_\omega^+) = \kappa_\omega^+$ and $j``\kappa_\omega^+$ is cofinal in κ_ω^+.
2. $M \models$ "$j(\langle f_\alpha : \alpha < \kappa_\omega^+\rangle)$ is a scale in $\prod j(a)$" and hence for all $g \in \prod j(a) \cap M$ there is an $\alpha < \kappa_\omega^+$ such that $g <_{\text{bounded}} j(f_\alpha)$.
3. For each $\mu \in a$, we have $j``\mu$ is bounded in $j(\mu)$.
4. For each $\alpha, \mu \in a$ we have $j(f_\alpha)(j(\mu)) \in j``\mu$.
5. Let $g \in \prod j(a)$ be defined as $g(j(\mu)) = \sup(j``\mu)$. Then every function in $\langle j(f)_\alpha : \alpha \in \kappa_\omega^+\rangle$ is bounded everywhere by g. Note that $g \in M$ by the assumption on M.
6. This contradicts clause 2.

Several people, including the author, Burke and Matsubara noted that Zapletal's proof gives information about generic elementary embeddings too:

6.9 Theorem. *Let λ be a singular cardinal. Then there is no λ-saturated normal, fine ideal on $[\lambda]^\lambda$. In particular, there is no normal, fine, countably complete, $\aleph_n$-saturated ideal on $[\aleph_\omega]^{\aleph_\omega}$.*

Proof. We follow the same steps as Zapletal's argument: the first two items follow for identical reasons. The third item must be slightly modified. By the closure of M, we know that every $j(\mu)$ for $\mu \in a$ is a cardinal in V. Since the ideal is λ-saturated there is a tail of $\mu \in j(a)$ that remain regular cardinals in $V[G]$. Since $\mu < j(\mu)$, $j``\mu$ must be bounded in $j(\mu)$. The rest of the proof is the same. ⊣

Burke and Matsubara improved this result to eliminate the assumption of normality. The author noted that the proof above works assuming only λ^+-saturation: If I is λ^+-saturated, then it has the disjointing property. Hence, if $j : V \to M$ is the generic embedding, $M^\lambda \cap V[G] \subseteq M$. In particular, for all $\mu \in a$, $j(\mu)$ is regular in $V[G]$, so $j``\mu$ is bounded in μ and the proof works as before.

We end this subsection by remarking that the requirement on M that it share some cardinal structure with V is necessary. The various stationary tower forcings that yield elementary embeddings with many fixed points give examples of elementary embeddings with $j : V \to M$ where M contains $j``\alpha$ for many α much bigger than κ_ω.

6.3. Saturated Ideals and Cofinalities

We now state two theorems that are of use in showing that various ideals cannot be saturated.

6.10 Theorem (Shelah [103]). *Let $V \subseteq W$ be two models of set theory with the same ordinals and suppose that*

$$V \models \kappa \textit{ is a regular cardinal and } \lambda = \kappa^+.$$

If λ is a cardinal in W then

$$W \models \mathrm{cf}(\kappa) = \mathrm{cf}(|\kappa|).$$

We note that the hypotheses of this theorem are satisfied if W is a λ-c.c. forcing extension of V.

For the reader's convenience (and because the proof is so nice) we give the proof of Shelah's theorem.

Proof of Theorem 6.10. Working in V, let $\langle A_\alpha : \alpha < \kappa^+\rangle$ be a sequence of subsets of κ such that the intersection of any pair has cardinality less than κ. Then for all $\gamma < \kappa^+$, there is a function $f_\gamma : \gamma \to \kappa$ such that $\langle A_\alpha \setminus f_\gamma(\alpha) : \alpha < \gamma\rangle$ is a pairwise disjoint sequence of sets.

If κ is a cardinal in W then there is nothing to show. Otherwise suppose that $W \models$"$\mu = |\kappa|$ and $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\mu)$". Let $f \in W$ be a bijection between μ

and κ. Then for all $\alpha < \lambda$ there is a $\delta_\alpha < \mu$ such that $f``\delta_\alpha$ is unbounded in A_α. Since λ is a cardinal in W there is a set $Y \subseteq \lambda$ lying in W having cardinality λ such that for all $\alpha, \beta \in Y$ we have $\delta_\alpha = \delta_\beta$. Let the common value be δ. Choose a $\gamma \in \lambda$ such that $Y \cap \gamma$ has ordertype at least μ. Then $W \models$"$\langle f^{-1}(A_\alpha \setminus f_\gamma(\alpha)) \cap \delta : \alpha \in Y \cap \gamma\rangle$ is a sequence of μ disjoint non-empty subsets of δ". This contradicts the fact that μ is a cardinal in W. $\dashv$

Shelah's theorem gives an immediate corollary for saturated ideals:

6.11 Corollary. *Suppose that I is a κ-complete, κ^+-presaturated ideal on $\kappa = \mu^+$. Then $\{\delta : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\} \in \breve{I}$.*

Proof. Consider the generic embedding $j : V \to M$. The critical point of j is κ and $M^\kappa \cap V[G] \subseteq M$. Hence $V[G] \models |\kappa| = \mu$. From Theorem 6.10 it follows that $V[G] \models \mathrm{cf}(\kappa) = \mathrm{cf}(\mu)$. The corollary now follows from the results of Sect. 1.5. $\dashv$

In particular, this shows that the nonstationary ideal can never be κ^+-saturated or even presaturated on any successor cardinal larger than ω_1. Theorem 6.10 leaves open the possibility that the nonstationary ideal on ω_2 can be saturated when restricted to the ordinals of cofinality ω_1. As of this writing this is open, although there are closely related results in Sect. 7.

Shelah and Matsubara established a formally similar result for ideals whose quotient is proper in the sense of "proper forcing":

6.12 Proposition (Matsubara and Shelah [94]). *Suppose that $Z \subseteq P(X)$ and I is a normal, fine, κ-complete ideal on Z such that $P(Z)/I$ is proper. If $\rho < \kappa$ is regular, then $\{z : z \cap \kappa \in \mathrm{Cof}({<}\rho) \cap \kappa\} \in I$.*

Proof. If this fails, then there is a regular δ with $\delta^+ < \kappa$ such that $T = \{z : z \cap \kappa \in \mathrm{Cof}(\delta)\} \in I^+$. Partition $\kappa \cap \mathrm{Cof}(\omega)$ into stationary sets $\langle S_\alpha : \alpha < \delta^+\rangle$. Each α of the form $z \cap \kappa$ for $z \in T$ has cofinality δ. Hence there is a closed unbounded set $C \subseteq \alpha$ of order type δ. In particular, there is a $\gamma(\alpha)$ such that $S_{\gamma(\alpha)} \cap \alpha$ is not stationary.

By the κ-completeness of the ideal there is a $\gamma^* < \delta^+$ and a positive set T' such that for all $z \in T'$, $\gamma(z \cap \kappa) = \gamma^*$. Letting $P = \{z \cap \kappa : z \in T'\}$ we see that for all $\alpha \in P$, $S_{\gamma^*} \cap \alpha$ is nonstationary. This contradicts Proposition 5.23. $\dashv$

Cummings generalized Theorem 6.10 to singular cardinals:

6.13 Theorem (Cummings [16]). *Let $V \subseteq W$ be two transitive models of* ZFC *with the same ordinals and suppose that $V \models \lambda = \kappa^+$. Suppose further that for all V-stationary sets $S \subseteq \lambda$, $W \models$ "S is stationary". Then:*

$$W \models \mathrm{cf}(\kappa) = \mathrm{cf}(|\kappa|).$$

Again the hypotheses of this theorem are satisfied if W is a λ-c.c. forcing extension of V.

For general regular κ, Gitik and Shelah were able to prove:

6.14 Theorem (Gitik-Shelah [56]). *If κ is a regular cardinal then for all regular cardinals δ with $\delta^+ < \kappa$, the nonstationary ideal on κ restricted to points of cofinality δ is not κ^+-saturated.*

If κ is the successor of a regular cardinal, this already follows from Corollary 6.11. Hence the substance of this theorem is when:

1. $\kappa = \mu^+$ where μ is singular, or
2. κ is weakly inaccessible.

Proof (Sketch). Towards a contradiction, assume that the nonstationary ideal on κ restricted to points of cofinality δ is κ^+-saturated.

- We take as a fact that for all stationary sets $S \subseteq \kappa \cap \mathrm{Cof}(\delta)$ there is a club guessing sequence consisting of points of high cofinality. Precisely: there is a sequence of sets $\langle S_\alpha : \alpha \in S \rangle$ for all $\alpha \in S$:

  1. $S_\alpha \subseteq \alpha \cap \mathrm{Cof}(>\delta)$, and
  2. S_α unbounded in α

  such that for all closed unbounded sets $C \subseteq \kappa$ the set $\{\alpha : S_\alpha \subseteq C\}$ is stationary in κ.

- We claim that the saturation of the ideal implies that for all S and club guessing sequences $\langle S_\alpha : \alpha \in S \rangle$, there is a stationary $S^* \subseteq S$ such that $\langle S_\alpha : \alpha \in S^* \rangle$ is a *strong club guessing sequence* on S^*. This means that:

  1. S_α is unbounded in α, and
  2. for all closed unbounded sets $C \subseteq \kappa$,

  $$\{\alpha : \text{a tail of } S_\alpha \text{ is a subset of } C\}$$

  is closed unbounded relative to S^*.

  For if this failed, for each $T \subseteq S$ there would be a closed unbounded set $C(T)$ such that $T^* =_{\mathrm{def}} \{\alpha : \text{a tail of } S_\alpha \text{ is a contained in } C(T)\}$ would be stationary and co-stationary in T. We inductively define a sequence $\langle \langle T_\beta, C_\beta \rangle : \beta < \kappa^+ \rangle$. Set $T_0 = S$ and $C_0 = C(S)$. Suppose that we have defined C_β. Let $T_\beta = \{\alpha \in S : S_\alpha \text{ is eventually included in } C_\beta\}$. Given T_β, let $C_{\beta+1} \subseteq C_\beta \cap C(T_\beta)$ be any closed and unbounded set. At limit β choose C_β so that it is eventually included in $\langle C_{\beta'} : \beta' < \beta \rangle$. Then:

  1. $\langle C_\beta : \beta < \kappa^+ \rangle$ is a sequence of closed unbounded sets so that $\beta' < \beta$ implies C_β is eventually included in $C_{\beta'}$,
  2. $T_{\beta+1}$ is a stationary and co-stationary subset of T_β, and
  3. $\langle T_\beta : \beta < \kappa^+ \rangle$ is a decreasing sequence of stationary sets.

But then $\{T_\beta \setminus T_{\beta+1} : \beta < \kappa^+\rangle$ is an antichain with respect to NS_κ, contradicting the κ^+ saturation assumption.

- Again by the saturation of the ideal, we can find a pairwise disjoint collection of such S^* whose union contains a closed unbounded set in κ. Gluing together the guessing sets yields a strong club guessing sequence $\langle S_\alpha : \alpha \in \kappa \cap \mathrm{Cof}(\delta)\rangle$ on $\kappa \cap \mathrm{Cof}(\delta)$.

  Since our original S_α's consisted of points of large cofinality, the sequence continues to have the additional property that for all α,

$$S_\alpha \subseteq \alpha \cap \mathrm{Cof}(>\delta).$$

  Shelah and Gitik called a strong club guessing sequence with this property $\diamondsuit^*_{\mathrm{club}}(\kappa, \delta)$.

- $\diamondsuit^*_{\mathrm{club}}(\kappa, \delta)$ is inconsistent with ZFC. To see this in the case where $\delta > \omega$, build a decreasing sequence of closed unbounded subsets of κ, $\langle E_n : n < \omega\rangle$. Let E_0 be the set of ordinals below κ which are limits of ordinals of cofinality bigger than δ. Suppose that we have defined E_n. Let E_{n+1} be a closed unbounded set witnessing $\diamondsuit^*_{\mathrm{club}}(\kappa, \theta)$ for the collection of limit points of E_n. Without loss of generality assume that E_{n+1} is a subset of the limit points of E_n.

  Let $E = \bigcap E_n$. Let η be the least point of E of cofinality δ. Then for all $n \in \omega$ the limit points of E_n contain a tail of S_η, and δ has uncountable cofinality. Hence E contains a tail of S_η. Let $\beta \in E \cap S_\eta$. Then β has cofinality greater then δ and is a limit point of every E_n. Hence $E \cap \beta$ is closed unbounded in β. But then there is a $\gamma \in E \cap \beta$ of cofinality δ, contradicting the minimality of η.

  The case where $\delta = \omega$ is similar, except that one builds a sequence of E_n's with length ω_1.

$\dashv$

In contrast to this, Foreman and Komjáth [40] jointly established a general result that when $\kappa = \omega_2$ or $\kappa = \aleph_{\omega+1}$ shows that it is consistent for:

1. $\mathrm{NS}_\kappa \restriction S$ is κ^+ saturated for a stationary $S \subseteq \kappa$, and

2. strong club guessing at κ.

This is Theorem 8.14, which is outlined in Sect. 8.2. The resolution of the apparent contradiction between Theorem 8.14 and the proof of Theorem 6.14, is that the cofinality of the ordinals in the strong club guessing sequence in the model for Theorem 8.14 is small. Hence $\diamondsuit^*_{\mathrm{club}}(\kappa, \delta)$ fails.

6.4. Closed Unbounded Subsets of $[\kappa]^\omega$

In this section we present a theorem from [5] that says that every closed unbounded relative to $[\kappa]^\omega$ is large. This stands in marked contrast to the results of Sect. 4.6, where we saw that for exponents other than ω and normal ideals there were usually sets of measure one of small cardinality.[50]

As a corollary of the theorem, we see that $[\kappa]^\omega$ can be split into a large collection of disjoint stationary sets. In particular, the nonstationary ideal on $[\kappa]^\omega$ cannot be $2^{\aleph_0}$-saturated for any regular $\kappa \geq \omega_2$.

Since the partial ordering for adding Cohen reals is c.c.c., it does not make a stationary subset of $[\kappa]^\omega$ into a nonstationary set. Thus, we see that it is consistent that there be stationary subsets of $[\omega_2]^\omega$ of size $\aleph_2$ even with $2^{\aleph_0}$ arbitrarily large.

6.15 Theorem (Baumgartner-Taylor [5]). *Let $\kappa \geq \omega_2$ be regular. Suppose that C is a closed unbounded subset of $P(\kappa)$. Then there is a countable subset $R \subseteq \kappa$ such that $|C \cap [R]^\omega| = 2^{\aleph_0}$.*

Proof. For $\alpha \in \kappa \cap \mathrm{Cof}(\omega)$ choose an increasing cofinal sequence $\vec{\alpha} = \{\alpha(n) : n \in \omega\}$.

Without loss of generality we can assume that there is a function $f : \kappa^{<\omega} \to \kappa$ such that C is the collection of $z \subseteq \kappa$ closed under f. For a set $z \subseteq \kappa$, let $\mathrm{Sk}^f(z)$ be the closure of z under f. We can assume that if $\alpha \in \mathrm{Sk}^f(z)$ then $\vec{\alpha} \subseteq \mathrm{Sk}^f(z)$.

Let Z be the collection of $\alpha < \kappa$ of cofinality ω that are closed under f. For $\vec{\xi} \in [\kappa]^{<\omega}$, let $P_{\vec{\xi}} = \{\alpha \in Z : \vec{\xi} \subseteq \vec{\alpha}\}$.

We note that if $T \subseteq P_{\vec{\xi}}$ is stationary, then there are unboundedly many $\eta < \kappa$ such that $T \cap P_{\vec{\xi}^\frown\eta}$ is stationary. For if there were a bound, then each $\alpha \in T$ above the bound has some n where $\alpha(n)$ is above the bound. For stationarily many α in T, we have the same n, and by a regressive function argument, the same $\alpha(n)$. But this shows that $P_{\vec{\xi}^\frown\alpha(n)} \cap T$ is stationary, a contradiction.

Define by induction a sequence $\langle\langle Z_s, \xi_s\rangle : s \in 2^{<\omega}\rangle$ such that for all $s \in 2^{<\omega}$ and $i \in \{0,1\}$:

1. Z_s is a stationary subset of $P_{\langle \xi_{s\restriction j} : j \leq \mathrm{length}(s)\rangle}$,
2. $Z_{s^\frown i} \subseteq Z_s$,
3. For all $\alpha \in Z_{s^\frown i}$, the ordinal $\xi_{s^\frown(1-i)} \notin \mathrm{Sk}^f(\vec{\alpha})$.

We first show that these conditions imply that if h and g are distinct elements of 2^ω, then $\mathrm{Sk}^f(\{\xi_{h\restriction j} : j \in \omega\})$ is different than $\mathrm{Sk}^f(\{\xi_{g\restriction j} : j \in \omega\})$. Letting $R = \mathrm{Sk}^f(\{\xi_s : s \in 2^{<\omega}\})$, this establishes the theorem.

[50] Assuming the consistency of large cardinals, Baumgartner [4] showed that it is consistent that there is a closed unbounded set whose intersection with $[\kappa]^{\omega_1}$ has cardinality less than κ^{ω_1}.

Let h and g be different elements of 2^ω. Consider the least n such that $h(n) \neq g(n)$. Then $\xi_{h\restriction n+1} \in \mathrm{Sk}^f(\{\xi_{h\restriction j} : j \in \omega\})$. On the other hand, if $\xi_{h\restriction n+1} \in \mathrm{Sk}^f(\{\xi_{g\restriction j} : j \in \omega\})$, then there would be a finite $m \geq n+1$ such that $\xi_{h\restriction n+1} \in \mathrm{Sk}^f(\{\xi_{g\restriction j} : j \in m\})$. If we let $\alpha \in Z_{\langle \xi_{g\restriction j} : j \in m\rangle}$, then $\alpha \in Z_{\langle \xi_{g\restriction j} : j \in n+1\rangle}$ and $\xi_{h\restriction n+1} \in \mathrm{Sk}^f(\vec{\alpha})$, contradicting clause 3.

We are reduced to defining the sequence $\langle\langle Z_s, \xi_s\rangle : s \in 2^{<\omega}\rangle$ satisfying clauses 1–3. Suppose that we have defined Z_s and $\xi_{s\restriction j}$ for $j \leq \mathrm{length}(s)$. Let $\vec{\xi}$ be the sequence $\langle \xi_{s\restriction j} : j \leq \mathrm{length}(s)\rangle$. Let $K \subseteq \kappa$ be unbounded so that for all $\eta \in K$, $P_{\vec{\xi}^\frown \eta} \cap Z_s$ is stationary. Let K' be the first ω_1 many elements of K.

For each $\alpha \in Z_s$, there is an $\eta(\alpha) \in K'$ such that $\eta(\alpha) \notin \mathrm{Sk}^f(\vec{\alpha})$. Hence we can find an unbounded set $K_1 \subseteq K$ and a fixed η_0 such that for all $\zeta \in K_1$ the set of α in $P_{\vec{\xi}^\frown \zeta} \cap Z_s$ with $\eta(\alpha) = \eta_0$ is stationary.

Repeating this argument we can find an $\eta_1 \in K_1$ such that the collection Z_0 of $\alpha \in Z_s \cap P_{\vec{\xi}^\frown \eta_0}$ such that $\eta_1 \notin \mathrm{Sk}^f(\vec{\alpha})$ is stationary.

Let $Z_{s^\frown 0} = Z_0$, $Z_{s^\frown 1} = \{\alpha \in Z_s \cap P_{\vec{\xi}^\frown \eta_1} : \eta_0 \notin \mathrm{Sk}^f(\vec{\alpha})\}$, $\xi_{s^\frown 0} = \eta_0$, and $\xi_{s^\frown 1} = \eta_1$. ⊣

6.16 Corollary. *Let $\kappa \geq \omega_2$ be a cardinal. Every closed unbounded subset relative to $[\kappa]^\omega$ has cardinality $\kappa^{\aleph_0}$.*

Proof. Let C be a closed unbounded subset of $P(\kappa)$. Define a map $f : [\kappa]^\omega \to C$, by sending $x \in [\kappa]^\omega$ to a countable $z \in C$ such that $x \subseteq z$. The map f is at most $2^{\aleph_0}$-to-one. Hence $\kappa^{\aleph_0} = 2^{\aleph_0} \times |C \cap [\kappa]^\omega|$. Since $|C \cap [\kappa]^\omega| \geq 2^{\aleph_0}$, we see that $|C \cap [\kappa]^\omega| = \kappa^{\aleph_0}$. ⊣

We can now see

6.17 Corollary. *Suppose that $\omega_2 \leq \kappa \leq 2^{\aleph_0}$. Then there is a collection of $2^{\aleph_0}$ disjoint stationary subsets of $[\kappa]^\omega$.*

Proof. Let $\langle (A_\alpha, f_\alpha) : \alpha < 2^\omega\rangle$ be an enumeration of the pairs $\langle (A, f)\rangle$ such that $A \in [\kappa]^\omega$, $f : A^{<\omega} \to A$, and there are $2^{\aleph_0}$ distinct subsets of A closed under f. We inductively build continuum many pairwise disjoint sets $\langle S_\beta : \beta < 2^{\aleph_0}\rangle$ such for all β and α, S_β contains some subset of A_α closed under f_α. By Theorem 6.15, this suffices. ⊣

6.5. Uniform Ideals on Ordinals

Our attention has been focused on normal ideals on $Z \subseteq P(X)$, where we can always take X to be a cardinal λ. We frequently use the ordering on λ for reflection arguments using Łoś's Theorem. In many situations it might be convenient to give up normality in order that X have a well-ordering whose length is not a cardinal and have I uniform in the sense of order type rather than cardinality; e.g. all subsets of X of small order type belong to I. The next result says that this is not possible for κ-complete, κ^+-saturated ideals.

6.18 Theorem (Foreman-Hajnal [39]). *Let κ be a successor cardinal. Suppose that I is a κ-complete, κ^+-saturated ideal on an ordinal γ having cardinality κ such that I is uniform in the sense that if $a \subseteq \gamma$ has order type less than γ, then $a \in I$. Then $\gamma = \kappa$.*

Proof. The Milner-Rado Paradox says that every ordinal less than κ^+ can be written as a countable union of subsets of order type less than κ^ω.[51] Hence if γ carries a countably complete, uniform ideal, then $\gamma < \kappa^\omega$. Since γ must be indecomposable, we know $\gamma = \kappa^n$ for some n. Let I be an ideal satisfying the hypothesis of the theorem.

We view the ordinal κ^n as a product of n copies of κ and show that there is a function of the first coordinate in the product which bounds the other coordinates on a set of positive measure for the ideal I. This will contradict the uniformity of I. To reduce to one coordinate we must take some Rudin-Keisler reductions of the usual ultrapower.

Let $\phi : (\gamma, \in) \to \langle \kappa \times \kappa \times \cdots \times \kappa, <_{\text{lex}} \rangle$ be an isomorphism between γ and the product of n copies of κ, where $<_{\text{lex}}$ is the left-to-right lexicographical ordering. The fact that I is uniform implies that if $A \subseteq \gamma$ and $A \notin I$ then $\phi[A]$ contains a κ-splitting tree isomorphic to the n-fold product of κ's. Via the isomorphism ϕ we can regard I as a κ-complete, κ^+ saturated ideal on $\{(\alpha_0, \ldots, \alpha_{n-1}) : \alpha_i \in \kappa\}$ containing all sets that do not have such a tree. For the rest of this proof we will write κ^n for this n-fold product of κ.

Suppose that $\kappa = \mu^+$. Let $G \subseteq P(\kappa^n)/I$ be generic, and $j : V \to M \subseteq V[G]$ be the generic ultrapower. As usual $M^\kappa \cap V[G] \subseteq M$ and $[a]_I \in G$ iff $[\text{id}]^M \in j(a)$, where $\text{id} : \kappa^n \to V$ is the identity function. Let $[\text{id}]_M = (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1})$.

We define:

$$H_1 = \{A \subseteq \kappa : A \times \kappa^{n-1} \in G\}.$$

It is easy to verify that $H_1 = \{A \subseteq \kappa : \tilde{\alpha}_0 \in j(A)\}$. Thus H_1 is an ultrafilter and we can construct an ultrapower V^κ/H_1. Define:

$$k_1 : V^\kappa/H_1 \to M$$

by setting $k_1([f]) = j(f)(\tilde{\alpha}_0)$. Then k_1 is well-defined and elementary. Hence V^κ/H_1 is well-founded and we replace it by its transitive collapse N_1 and reconstrue k_1 to have domain N_1.

Denote the ultrapower embedding from V to N_1 by j_1. One can check easily that $j = k_1 \circ j_1$, and that the critical point of j_1 is κ. Moreover, $j_1(\kappa) = (\mu^+)^{N_1}$ and $P(\kappa)^V \subseteq N_1$, so $j_1(\kappa) = (\kappa^+)^V = j(\kappa)$. Thus the critical point of k_1 must be greater than the κ^+ of V.

Let $i : \kappa \to V$ be such that $[i]_{N_1} = \kappa$. Define

$$H_0 = \{A \subseteq \kappa : i^{-1}(A) \in H_1\}.$$

[51] Ordinal exponentiation.

Then $H_0 = \{A \subseteq \kappa : \kappa \in j_1(A)\}$. Repeating the analysis we did for H_1 we see that the ultrapower V^κ/H_0 is well-founded and if N_0 is the transitive collapse, there is an elementary embedding $k_0 : N_0 \to N_1$ given by $k_0([f]_{N_0}) = j_1(f)(\kappa)$. Moreover if j_0 is the ultrapower embedding $j_0 : V \to N_0$, then the critical point of j_0 is κ and the critical point of k_0 is bigger than $(\kappa^+)^V$, and finally $j_1 = k_0 \circ j_0$.

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_0"'] \arrow[dr, "j_1"] & M \\ N_0 \arrow[r, "k_0"'] & N_1 \arrow[u, "k_1"'] \end{tikzcd}$$

Since $j(\kappa) = \kappa^+$, we know that $\max\{\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-1}\} < \kappa^+$. By the κ^+ saturation of I, there is an $\eta < \kappa^+$ such that

$$\| \max\{\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-1}\} < \eta \|_{P(\kappa^n)/I} = 1.$$

Let $f : \kappa \to \kappa$ be in V such that $[f]_{N_0} = \eta$. Since the critical points of k_0 and k_1 are both greater than the κ^+ of V, $k_1 \circ k_0(\eta) = k_0(\eta) = \eta$. Hence, $j_0(f)(\kappa) = \eta$ and thus $j(f)(j(i)(\tilde{\alpha}_0)) = \eta$.

Let $B = \{(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) : \max\{\alpha_0, \dots, \alpha_{n-1}\} < f(i(\alpha_0))\}$. Then $B \in G$, since $\max\{\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{n-1}\} < \eta = j(f)(j(i)(\tilde{\alpha}_0))$. But this is a contradiction, since B does not contain a tree isomorphic to κ^n and hence $B \in I$. ⊣

6.6. Restrictions on the Quotient Algebra

In this section we discuss theorems of Gitik and Shelah about the nature of the quotient algebra of an ideal.

If κ is a successor cardinal then a precipitous κ-complete ideal on κ yields an embedding that has critical point κ. As remarked after Proposition 6.3 this implies that the forcing arising from I must collapse κ. On the other hand, as we shall see in Sect. 7, if κ is a measurable cardinal, $\eta < \kappa$ and $\mathbb{P}$ is an η-c.c. forcing, then in $V^{\mathbb{P}}$ there is a κ-complete, η-c.c. ideal on κ. In particular, adding at least κ Cohen or random reals to a model where κ is measurable gives a highly saturated ideal whose quotient algebra is isomorphic to the algebra for adding more Cohen or random reals.

The question arises, can this algebra actually have small cardinality rather than just small chain condition? More succinctly: What are the possible densities of $P(\kappa)/I$ if I is a κ-complete ideal on κ?

This question was asked classically by Ulam [123] in the context of determining the smallest size of a collection of countably complete measures such that every subset of the real numbers is measurable with respect to one of the measures. The question is clearly natural for cardinals such as $\aleph_1, \aleph_2$, as well as $2^{\aleph_0}$.

Ulam showed that there is no measure that measures every subset of ω_1; i.e. ω_1 is not measurable. Alaoglu and Erdős showed that for every countable collection of measures on $\aleph_1$ there is a set that is not measurable by one of the measures. Ulam asked whether there could be a collection of ω_1 countably complete measures such that every subset of ω_1 is measurable with respect to one of them. If the ideals corresponding to the measures are normal, then Taylor showed (Theorem 4.3) that this is equivalent to the existence of a countably complete, $\aleph_1$-dense ideal.

The next theorem gives a characterization of ideals with very small density in terms of a version of Ulam's property:

6.19 Theorem (Taylor [119]). *Let $\delta < \kappa$ be regular cardinals. Suppose that $\langle \mu_\alpha : \alpha < \delta \rangle$ is a collection of $<\kappa$-additive 0-1 measures over Z such that every subset of Z that is not measure zero with respect to all of them is measure one with respect to one of them. Let I be the κ-complete ideal of subsets of Z that have measure zero with respect to each μ_α. Then $P(Z)/I$ has a dense set of size δ.*

Proof. Note that the second assertion easily implies the first. For if $\{d_\alpha : \alpha \in \delta\}$ is a collection of subsets of Z such that every subset of Z not in I contains some d_α modulo I, then the measures associated to $I \restriction d_\alpha$ measure every subset of Z in the prescribed manner.

For the other direction, suppose that $\langle \mu_\alpha : \alpha \in \delta \rangle$ is a sequence of measures such that every subset of Z not measure zero with respect to all of them is measure one with respect to one of them. Let I_α be the ideal of sets of measure zero for μ_α and $I = \bigcap I_\alpha$. We claim that $P(Z)/I$ has a dense set of size δ.

Clearly $P(Z)/I$ has the δ^+-c.c. For if $\langle A_\gamma : \gamma \in \delta^+ \rangle$ were an antichain, then we would have to have a γ, a β and an α such that A_γ and A_β both are measure one for μ_α. But then $A_\gamma \cap A_\beta \notin I_\alpha \supseteq I$.

Thus for a fixed α, any strictly I-decreasing chain of μ_α-measure one sets has length less than δ^+. Using the additivity of μ_α, for each α we can find a d_α that is measure one for μ_α such that for any a that is measure one for μ_α, $d_\alpha \leq_I a$. Then if $a \in I^+$ is arbitrary, there is an α such that a has measure one for μ_α. Hence $d_\alpha \leq_I a$. Thus $\{d_\alpha : \alpha \in \delta\}$ is a dense set in $P(Z)/I$.[52] $\dashv$

We note that the converse of Theorem 6.19 is easy: if there is a dense set of size δ, then there is a collection of measures satisfying the hypothesis. Thus we have an equivalence to the existence of an ideal with quotient containing a countable dense set.

Since any complete Boolean algebra with a countable dense set is isomorphic to the Boolean algebra for adding a single Cohen real, the question becomes whether there is a κ-complete, uniform ideal on a set Z such that adding a Cohen real adds a generic object for $P(Z)/I$.

[52] Alternatively, we could apply Theorem 4.2, to find sets d_α such that each $I_\alpha = I \restriction d_\alpha$ and the $\{[d_\alpha]_I : \alpha \in \delta\}$ would be dense.

If $|Z|$ is a measurable cardinal and $\breve{I}$ is the dual of a $|Z|$-complete ultrafilter on $|Z|$ then $|P(Z)/I| = 2$. To rule out this trivial case we require our ideal to be *nowhere prime*, i.e. there is no $A \in I^+$ such that $\breve{I}\restriction A$ is an ultrafilter on A.

Tarski showed in a 1962 paper [118] that:

6.20 Theorem. *There is no cardinal $\lambda > 2^{\aleph_0}$ that carries a nowhere prime, countably complete, uniform, $\aleph_0$-dense ideal.*

Krawczyk and Pelc [74] found an extension of this theorem to sets of size continuum:

6.21 Theorem. *The continuum does not carry a countably complete uniform $\aleph_0$-dense ideal.*

Finally Gitik and Shelah [55] finished the problem by showing:

6.22 Theorem. *There is no countably complete, $\aleph_0$-dense, nowhere prime ideal on any set Z.*

Proof. The proof will use two standard facts about Cohen forcing. Let r be V-generic for the Cohen forcing $\mathrm{Add}(\omega)$. Then:

1. $r \notin \bigcup\{M : M$ is a meager set belonging to $V\}$.
2. If $A \subseteq \omega^\omega$ is not meager in V, then A is not meager in $V[r]$.

We now describe a complicated scenario that allows us to state a lemma useful for proving both Theorem 6.22 and Theorem 5.9. The argument proving the lemma is essentially due to Gitik and Shelah, although Woodin saw its relevance to Theorem 5.9 and modified it appropriately. We adapt Woodin's approach to formulate a lemma that works for both theorems. We will give the assumptions forming the hypotheses of the lemma Roman numerals.

I. There are class models $N \subseteq W$ of ZFC with N closed under ω-sequences from W.

II. κ is an N-regular cardinal and $(\kappa^+)^N = (\kappa^+)^W$.

III. There is an ideal $J \in W$ on $P(\kappa)^N$ such that

   (a) J is normal and κ-complete with respect to sequences lying in N.

   (b) $P(\kappa)^N/J$ has a countable dense set in W and hence is forcing equivalent to $\mathrm{Add}(\omega)$, the partial ordering for adding a single Cohen real. We will view $\mathrm{Add}(\omega)$ as the tree $\langle \omega^{<\omega}, \supseteq\rangle$.

   (c) If $H \subseteq P(\kappa)^N/J$ is W-generic, then N^κ/H is well-founded.

Assume for the moment that I–III hold. Let $D \subseteq P(\kappa)^N/J$ be countable and dense with $D \in W$. Since N is closed under ω-sequences, we can find a collection $\langle A_p : p \in \omega^{<\omega}\rangle \in N$ such that:

1. $A_{p\frown i} \subseteq A_p \subseteq \kappa$,
2. $A_{p\frown i} \cap A_{p\frown j} = \emptyset$ for $i \neq j$, and
3. $\{[A_p]_J : p \in \omega^{<\omega}\}$ is dense in $P(\kappa)^N/J$.

We redefine D to be equal to $\{[A_p]_J : p \in \omega^{<\omega}\}$.

The collection of sets $\langle A_p : p \in \omega^{<\omega}\rangle$ determines a canonical bijection between W-generic ultrafilters $H \subseteq P(\kappa)^N/J$ and W-generic reals $r \subseteq \mathrm{Add}(\omega, 1)$. We will write $r(H)$ for the Cohen real associated with H, and if r is a Cohen real we will write $H(r)$ for the associated ultrafilter on $P(\kappa)^N/J$.

Let $H \subseteq P(\kappa)^N/J$ be generic over W. Let $i' : N \to N' \cong N^\kappa/H$ be the ultrapower map, where N' is a transitive model. By the N-normality of J, if $\mathrm{id} : \kappa \to \kappa$ is the identity map then $[\mathrm{id}]^{N'} = \kappa$.

If τ is a $P(\kappa)^N/J$ term in W for an element of N', then we can find a maximal antichain $\mathcal{A} \subseteq D$ and a collection of functions $\{f_a : a \in \mathcal{A}\} \subseteq N$ such that $a \Vdash [f_a]^{N'} = \tau$. Since $\mathcal{A} \subseteq D$, if $a = [A_p]_J$ and $b = [A_q]_J$ are distinct elements of $\mathcal{A}$, then $A_p \cap A_q = \emptyset$. Since N is ω-closed, $\{A_p : [A_p]_J \in \mathcal{A}\}$ and $\{f_{[A_p]} : [A_p]_J \in \mathcal{A}\}$ both belong to N. Define $f = \bigcup_{[A_p]\in\mathcal{A}} f_{[A_p]} \restriction A_p$. Then $1 \Vdash [f]_H = \tau$. Hence to each W-term τ for an element of N' we can associate a function f in N that represents it in N', no matter which generic object is chosen. A similar argument shows that $(N')^\omega \cap W[H] \subseteq N'$.

Since J is κ-complete in N and N is closed under ω-sequences

$$B = \bigcap_{n\in\omega} \bigcup_{|p|=n} A_p \in \breve{J}.$$

For $\xi \in B$, let $r_\xi : \omega \to \omega$ be the unique function such that $\xi \in \bigcap_{n\in\omega} A_{r_\xi \restriction n}$. Then the sequence $\langle r_\xi : \xi \in B\rangle \in N$ and for all W-generic H,

$$i'(\langle r_\xi : \xi \in B\rangle)(\kappa) = r(H).$$

Each $r_\xi \in N$ but $r(H) \notin N$. Hence $\{\xi : \text{there is an } \eta < \xi, r_\xi = r_\eta\} \in J$. By thinning slightly we can assume that $\langle r_\xi : \xi \in B\rangle$ is one-to-one. Modifying the sequence again on a set of measure zero we can assume that

1. $\langle r_\xi : \xi < \kappa\rangle \in N$ is one-to-one,
2. $A_p = \{\xi : r_\xi \in [p]\}$, and
3. $1 \Vdash i'(\langle r_\xi : \xi < \kappa\rangle)(\kappa) = r(H)$.

Suppose that $i'(\langle r_\xi : \xi < \kappa\rangle) = \langle r_\xi^{i'} : \xi < i'(\kappa)\rangle$. Then each $r_\xi^{i'} \in W[r(H)]$ and hence there is an $\mathrm{Add}(\omega)$-term s_ξ in W such that $1 \Vdash s_\xi = r_\xi^{i'}$. Since a Cohen term is a countable object, $s_\xi \in N$. Moreover, any Cohen term $t \in N$ for an element of ω^ω can be canonically viewed as a Borel function $T : \omega^\omega \to \omega^\omega$ that is coded in N. Our final assumption is:

IV. There is a sequence of Cohen terms $\langle t_\alpha : \alpha < \kappa^+\rangle \in N$ such that:

(a) $\{\alpha : 1 \Vdash t_\alpha = s_\alpha\}$ is unbounded in κ^+, and

(b) for all $\alpha < \beta < \kappa^+$, $\{x : T_\alpha(x) = T_\beta(x)\}$ is meager.

6.23 Lemma. *Hypotheses* I, II, III, *and* IV *are jointly inconsistent.*

We begin by reducing Theorem 6.22 to Lemma 6.23. Let $N = W = V$. Let I be a countably complete $\aleph_0$-dense, nowhere prime ideal on any set Z. Force with $P(Z)/I$ and consider the resulting generic elementary embedding $j : V \to N'$. Let κ be the critical point of j. If J is the ideal induced by using κ as the "ideal point" (as in Example 3.37), then J is a uniform, normal, κ-complete ideal, and the countable dense set in $P(Z)/I$ projects to a countable dense set in $P(\kappa)/J$. Thus we can assume without loss of generality that we have a J that is a uniform, normal, κ-complete ideal on a regular cardinal κ and that $P(\kappa)/J$ has a countable dense set.

Let $\langle t_\alpha : \alpha < \kappa^+\rangle = \langle s_\alpha : \alpha < \kappa^+\rangle$. We know that $1 \Vdash t_\alpha \neq t_\beta$ for $\alpha \neq \beta$. Hence $\{x : T_\alpha(x) \neq T_\beta(x)\}$ is meager and $\langle t_\alpha : \alpha < \kappa^+\rangle$ satisfy the hypotheses of the lemma.

We now sketch a proof of the lemma.

Proof (Sketch). Let H be generic for $P(\kappa)^N/J$, $r = r(H)$ and $i' : N \to N'$ the generic ultrapower embedding with N' transitive.

Here are the main points of the proof.

1. For sets of reals that belong to N, meagerness is absolute between N and W.

2. We will make the harmless simplifying assumption that if $1 \Vdash_{\mathrm{Add}(\omega)} t_\alpha^{V[r]} = s_\alpha^{V[r]}$ then $s_\alpha = t_\alpha$.

3. Every Borel set in W has a code that lies in N. If c is a Borel code then $i'(c) = c$. We claim that for each $A \in \breve{J}$ the set $\{r_\xi : \xi \in A\}$ is non-meager inside each basic open interval $[p]$. For otherwise, there would be a meager set X coded in N with $\{r_\xi : \xi \in A_p \cap A\} \subseteq X$. If we take H with $[A_p]_J \in H$ we would see that $r \in i'(X)$, a contradiction.

4. Since Cohen forcing preserves non-meagerness, for $A \in \breve{J}$ and $p \in \omega^{<\omega}$, $\{r_\xi : \xi \in A\} \cap [p]$ is non-meager in N'. Since $i'(A) \cap \kappa = A$ we can use i' and reflection to see that there is a $\xi_0 < \kappa$ such that $\{r_\xi : \xi \in \xi_0 \cap A\}$ is non-meager inside each basic open interval $[p]$.

5. If $\vec{f} = \langle f_\alpha : \alpha < \kappa^+\rangle$ is a sequence of canonical functions representing the ordinals less than κ^+ in the ultrapower[53] then $\vec{f} \in N$. Let $\langle r_\xi^{i'} : \xi < i'(\kappa)\rangle = i'(\langle r_\xi : \xi < \kappa\rangle)$. For $\alpha < \kappa^+$ the function $g_\alpha(\xi) = r_{f_\alpha(\xi)}$ represents $r_\alpha^{i'}$.

[53] As in Proposition 2.34.

6. For $\alpha < \kappa^+$, let $A_\alpha = \{\xi : T_\alpha(r_\xi) = g_\alpha(\xi)\}$. Then the sequence $\langle A_\alpha : \alpha < \kappa^+\rangle$ is definable from the sequences $\langle T_\alpha : \alpha < \kappa^+\rangle$, $\langle r_\xi : \xi < \kappa\rangle$ and $\langle f_\alpha : \alpha < \kappa^+\rangle$. Hence $\langle A_\alpha : \alpha < \kappa^+\rangle \in N$.

7. If $\alpha < \kappa^+$ with $t_\alpha = s_\alpha$ then $r_\alpha^{i'} = T_\alpha(r_\kappa^{i'}) = i'(g_\alpha)(\kappa)$. Hence, $A_\alpha \in \breve{J}$.

8. Working in N, let $\xi(\alpha)$ be the least $\xi < \kappa$ such that

   (a) ξ is closed under f_α, and

   (b) $\{r_\xi : \xi \in A_\alpha \cap \xi(\alpha)\}$ is not meager in any interval $[p]$,

   if such a ξ exists and 0 otherwise. By clauses 4 and 7 if $t_\alpha = s_\alpha$ then $\xi(\alpha)$ is non-zero.

9. In W, there is a set $E \subseteq \{\alpha : t_\alpha = s_\alpha\}$ with size κ^+ and a non-zero $\xi^* < \kappa$ such that for all $\alpha \in E$, $\xi(\alpha) = \xi^*$. Let F be the collection of $\alpha < \kappa^+$ such that $\xi(\alpha) = \xi^*$. Then $F \in N$ and $E \subseteq F$ so $|F| = \kappa^+$.

10. For $\alpha \in F$, $\{r_\xi : \xi < \xi^*$ and $T_\alpha(r_\xi) = r_\eta$ for some $\eta < \xi^*\}$ is non-meager in any interval $[p]$. Note that i' fixes the sequence $\langle r_\xi : \xi < \xi^*\rangle$.

11. Let $\alpha^* \in i'(F)$ with $\alpha^* \notin (i')``\kappa^+$. Let $\langle t_\alpha^{i'} : \alpha < i'(\kappa^+)\rangle$ be the sequence $i'(\langle t_\alpha : \alpha < \kappa^+\rangle)$. Consider the function $T_{\alpha^*}^{i'}$ in N' corresponding to $t_{\alpha^*}^{i'}$. Then $N' \models \{r_\xi : \xi < \xi^*$ and $T_{\alpha^*}^{i'}(r_\xi) = r_\eta$ for some $\eta < \xi^*\}$ is non-meager in any interval $[p]$. Since N' is closed under ω-sequences from $W[r]$, this is true in $W[r]$ as well.

12. Since Cohen forcing is countable, there is a condition p, a non-meager set $B \subseteq \{r_\xi : \xi < \xi^*\}$ and a function $w : B \to \{r_\xi : \xi < \xi^*\}$, such that $p \Vdash^W w = T_{\alpha^*}^{i'} \restriction B$.

13. Let $\tau \in N$ be a Cohen term for the function $T_{\alpha^*}^{i'}$. Working in W, we can view τ as a Borel function $T : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that for a generic r and all $x \in \omega^\omega \cap N[r]$, $T(r, x) = T_{\alpha^*}^{i'}(x)$. Then $p \Vdash^W T_{\alpha^*}^{i'}(r_\eta) = r_\xi$ iff $W \models \{y : T(y, r_\eta) = r_\xi\}$ is comeager in the interval $[p]$. Since T is coded in N and comeagerness is absolute between N and W we can find a non-meager set $B \subseteq \{r_\xi : \xi < \xi^*\}$ and a function w that lies in N such that $p \Vdash^W w = T_{\alpha^*}^{i'} \restriction B$.

14. Note that $i'(B) = B$ and $i'(w) = w$. Consider $\phi(\eta, \gamma, B, w)$ saying:

$$\exists \alpha (\eta < \alpha < \gamma \wedge w = T_\alpha \restriction B).$$

   Let γ be the least ordinal such that $i(\gamma) > \alpha^*$ and $\eta < \gamma$ be arbitrary. Then $N' \models \phi(i'(\eta), i'(\gamma), i'(B), i'(w))$. Using α^* we can reflect twice to find $\alpha < \beta < \gamma$ such that $T_\alpha \restriction B = T_\beta \restriction B = w$. Since B is non-meager this contradicts condition IV.

⊣

Gitik and Shelah's results extend to show that if I is a nowhere prime ideal on a set X then the completeness of I is less than or equal to the density of I. Moreover, there are many forcing notions that are not possible as quotients of κ-complete ideals. These include:

- Any forcing P that has a filtration $\langle P_\alpha \rangle$ into forcings of size less than κ that are neatly embedded in P,[54]
- random real forcing,
- Cohen $*$ random forcing,
- Hechler forcing,
- Miller forcing,
- Sacks forcing.

6.7. Yet Another Result of Kunen

From the Gitik-Shelah limitations on $\aleph_0$-dense ideals, one sees that the strongest consistent ideal property on a cardinal below the first measurable cardinal is that an ideal be $\aleph_1$-dense. Kunen showed the following result which put a limitation on when countably complete ideals can even be $\aleph_2$-saturated.

6.24 Proposition. *There is no uniform, countably complete, $\aleph_2$-saturated ideal on any cardinal between $\aleph_\omega$ and $\aleph_{\omega_1}$.*

We can generalize Kunen's result as follows (see [31] for a complete discussion):

6.25 Definition. Let κ be a regular cardinal. Define $\mathcal{C}_\kappa$ to be the smallest class of ordinals such that:[55]

1. $\kappa \subseteq \mathcal{C}_\kappa$,
2. if $\alpha, \beta \in \mathcal{C}_\kappa$ and $\beta^{+\alpha} \geq \kappa^{+\omega}$ then $\beta^{+\alpha} \in \mathcal{C}_\kappa$, and
3. if $\beta \in \mathcal{C}_\kappa$ and $\beta \geq \kappa$, then every cardinal in the interval $[\beta, \beta^{+\kappa})$ belongs to $\mathcal{C}_\kappa$.

An easy induction shows that for all κ, $[\kappa, \kappa^{+\omega}) \cap \mathcal{C}_\kappa = \emptyset$. The next result generalizes Kunen's proposition:

6.26 Proposition. *Let κ be a successor cardinal. Suppose that λ is a regular cardinal in $\mathcal{C}_\kappa$. Then there is no κ-complete, κ^+-saturated uniform ideal on λ.*

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[54] An example of such a forcing is the partial ordering for adding κ Cohen reals.
[55] Here we will use $\beta^{+\alpha}$ to mean the αth cardinal successor of β.

In the section on consistency results we will see that this theorem is sharp at the upper end of the interval, and that there is some evidence that it is sharp at the lower end.

Proof. As in the examples in Sect. 4.5, we see that if I is a uniform ideal on λ and $j : V \to M \subseteq V[G]$ is the generic elementary embedding coming from $G \subseteq P(\lambda)/I$, then $j(\lambda) > \lambda$.

6.27 Claim. *Suppose that α is an ordinal in $\mathcal{C}_\kappa$. Then for all κ-complete, κ^+-saturated ideals I on a set Z and all generic $G \subseteq P(Z)/I$, α is fixed by the generic elementary embedding determined by G.*

Proof. We show that the collection of fixed points of such an embedding j are closed under clauses 1–3 of Definition 6.25 of $\mathcal{C}_\kappa$. Let $G \subseteq P(Z)/I$ be generic and $j : V \to M \cong V^Z/G$ be the associated embedding, with M transitive.

Clause 1 is immediate since the ideal is κ-complete.

For clause 2: we remark that every cardinal of $V[G]$ is a cardinal of M. In particular, since the only cardinal collapsed by forcing with $P(Z)/I$ is $(\kappa^+)^V$ we know that for all α, β with $\beta^{+\alpha} \geq \kappa^{+\omega}$,

$$(\beta^{+\alpha})^M \leq (\beta^{+\alpha})^V.$$

Suppose that α, β are fixed points of j and $\beta^{+\alpha} \geq \kappa^{+\omega}$. Then,

$$(\beta^{+\alpha})^V \leq j(\beta^{+\alpha}) = (j(\beta)^{+j(\alpha)})^M = (\beta^{+\alpha})^M \leq (\beta^{+\alpha})^V.$$

For clause 3, note that if $\beta \geq \kappa^{+\omega}$ is a fixed point of j then every cardinal in the interval $[\beta, \kappa^{+\beta})$ is of the form $\beta^{+\alpha}$ for a fixed point α. Clause 3 then follows from the fact that $[\kappa, \kappa^{+\omega}) \cap \mathcal{C}_\kappa = \emptyset$. ⊣

Proposition 6.26 now follows from the fact that $j(\lambda) > \lambda$ and Claim 6.27. ⊣

6.8. The Matsubara-Shioya Theorem

Example 3.1 showed that the ideal of bounded sets on a regular cardinal κ is never precipitous. Matsubara and Shioya's theorem shows the analogous result for the minimal fine ideal on $[\lambda]^{<\kappa}$ where κ is a regular uncountable cardinal. The result is a corollary of an elegant general theorem that was discovered amazingly late in the study of ideals. The contents of this section are very similar to their paper [95].

6.28 Definition. Let I be a countably complete ideal on a set Z such that $P(Z)/I$ is non-atomic. Let

- $\pi(I)$ be the cardinality of the smallest I-positive set, and
- $\gamma(I)$ be the smallest cardinality of a set that generates I by taking finite unions and subsets.

It is an unpublished observation of Johnson that:

1. $\omega \leq \pi(I) \leq \gamma(I)$, and
2. if $\pi(I) = \gamma(I)$, then every I-positive set contains an I-positive set of cardinality $\pi(I)$.

To see the first clause, choose a set $\mathcal{G} \subseteq I$ of cardinality $\gamma(I)$ closed under finite unions that generates I. Then we can find a set $T = \{z_A : A \in \mathcal{G}\}$ such that for all $A \in \mathcal{G}$, $z \in Z \setminus A$. Then $T \notin I$, since otherwise there would be an $A \in \mathcal{G}$ such that $T \subseteq A$.

To see the second clause, fix an I-positive set S. Repeat the previous argument with respect to a generating family $\mathcal{G}'$ for $I \restriction S$ that has cardinality less than or equal to $\gamma(I)$. We get an I-positive $T \subseteq S$ of cardinality less than or equal to $|\mathcal{G}'|$. Since $\pi(I) \leq |T| \leq \gamma(I)$ and $\pi(I) = \gamma(I)$, we see $|T| = \pi(I)$.

6.29 Theorem (Matsubara-Shioya [95]). *If $\pi(I) = \gamma(I)$ then I is nowhere precipitous.*

We defer the proof of Theorem 6.29 until we state and prove some significant corollaries.

Recall from Example 3.1 the bounded ideal on $[\lambda]^{<\kappa}$, $I_{\kappa\lambda}$. We can also describe $I_{\kappa\lambda}$ as the smallest κ-complete, fine ideal on $P(\lambda)$.

6.30 Corollary. *Let $\kappa \leq \lambda$ and κ regular and uncountable. Then $I_{\kappa\lambda}$ is not precipitous.*

The corollary follows from Theorem 6.29 by observing that $\pi(I_{\kappa\lambda}) = \gamma(I_{\kappa\lambda})$ because they are both equal to the smallest cardinality of a subset $\mathcal{A}$ of $[\lambda]^{<\kappa}$ such that every element of $[\lambda]^{<\kappa}$ is covered by a set in $\mathcal{A}$.

Note as well that $\pi(I_{\kappa\lambda})$ has cofinality at least κ. Otherwise, let $\mathcal{A}$ be a covering subset of $[\lambda]^{<\kappa}$ of cardinality $\pi(I_{\kappa\lambda})$. Write $\mathcal{A} = \bigcup_{\alpha<\nu} A_\alpha$ where:

1. $|A_\alpha| < |A_{\alpha'}| < \pi(I_{\kappa\lambda})$ for $\alpha < \alpha'$,
2. each A_α is not a covering set, and
3. $\nu < \kappa$.

Then there is an $a_\alpha \in [\lambda]^{<\kappa}$ such that there is no element $b \in A_\alpha$ with $a_\alpha \subseteq b$. Since $\nu < \kappa$, the cardinality of $a = \bigcup_{\alpha<\nu} a_\alpha$ is less than κ and a is not covered by any element of $\mathcal{A}$, a contradiction.

Let the nonstationary ideal on λ restricted to $[\lambda]^{<\kappa}$ be denoted $\mathrm{NS}_{\kappa\lambda}$. Using the definition from Example 3.2 it is easy to verify that $\gamma(\mathrm{NS}_{\kappa\lambda}) \leq 2^\lambda$.

Moreover, any positive set for $\mathrm{NS}_{\kappa\lambda}$ is positive for $I_{\kappa\lambda}$, and hence $\pi(I_{\kappa\lambda}) \leq \pi(\mathrm{NS}_{\kappa\lambda})$. Thus if λ is singular of cofinality less than κ and $2^\lambda = \lambda^+$ the last three of the following inequalities are actually equalities:

$$\lambda < \pi(I_{\kappa\lambda}) \leq \pi(\mathrm{NS}_{\kappa\lambda}) \leq \gamma(\mathrm{NS}_{\kappa\lambda}) \leq 2^\lambda.$$

Thus we have:

6.31 Corollary. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and* $2^\lambda = \lambda^+$. *Then* $\mathrm{NS}_{\kappa\lambda}$ *is nowhere precipitous. In particular, if* GCH *holds then for all* n, $\mathrm{NS}_{\aleph_n\aleph_\omega}$ *is nowhere precipitous.*

Matsubara and Shioya draw several other corollaries including:

6.32 Corollary.

1. *If* $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$ *then* $\mathrm{NS}_{\kappa\lambda}$ *is nowhere precipitous.*
2. *If* $\sup\{\mathrm{cf}(\prod a/U) : a \in [\lambda]^{<\kappa}$ *is a collection of regular cardinals greater than* κ, *and* U *is an ultrafilter on* $a\} = 2^\lambda$, *then* $\mathrm{NS}_{\kappa\lambda}$ *is nowhere precipitous.*
3. *If* $2^{\aleph_0} < \aleph_\omega^{\aleph_0}$ *then* $\mathrm{NS}_{\aleph_n\aleph_\omega}$ *is nowhere precipitous.*

Proof of Theorem 6.29. First note that it suffices to prove the following:

Claim. Let $\kappa = \pi(I) = \gamma(I)$. For all I-positive sets X and one-to-one functions $f : X \to \kappa$ there is an I-positive set $Y \subseteq X$ and a one-to-one function $g : Y \to \kappa$ such that for all $y \in Y$, $g(y) < f(y)$.

To see that the claim suffices, we suppose that $I\restriction A$ is precipitous. We use the claim to build maximal antichains $A_n \subseteq P(Z)/(I\restriction A)$ such that:

- A_{n+1} refines A_n,
- for each $a \in A_n$ we have a one-to-one function $f_a : a \to \kappa$, and
- if $a \in A_{n+1}$, $b \in A_n$ and $a \subseteq b$, then for all $y \in a$, $f_a(y) < f_b(y)$.

Then each sequence of functions $\langle f_a : a \in A_n\rangle$ determines a term for a function $F_n \in V^Z$ such that for all n, $1 \Vdash [F_{n+1}]^{\dot{M}} < [F_n]^{\dot{M}}$, where $\dot{M}$ is a term for the transitive model isomorphic to V^Z/G.

To see the claim: enumerate a generating set for I as $\langle J_\alpha : \alpha < \kappa\rangle$ and inductively choose a $y_\alpha \in X \setminus (J_\alpha \cup f^{-1}(\alpha+1) \cup \{y_\beta : \beta < \alpha\})$. Let $Y = \{y_\alpha : \alpha < \kappa\}$ and $g(y_\alpha) = \alpha$. ⊣

Matsubara and Shelah were able to extend Theorem 6.29 to show:

6.33 Theorem (Matsubara-Shelah [94]). *If* λ *is a strong singular limit cardinal and* $\kappa < \lambda$ *is regular, then* $\mathrm{NS}_{\kappa\lambda}$ *is nowhere precipitous.*

The proof fixes a continuous cofinal sequence $\langle \lambda_\alpha : \alpha \in \mathrm{cf}(\lambda)\rangle$ and uses pcf theory to reduce the issue to those λ_α with $\mathrm{cf}(\lambda_\alpha) < \kappa$. It then uses the proof of Theorem 6.29 to deal with these cardinals individually.

6.9. The Nonstationary Ideal on $[\lambda]^{<\kappa}$

In the previous section we saw that GCH implies that the nonstationary ideal on $[\lambda]^{<\kappa}$ is not precipitous in many cases, such as $[\aleph_\omega]^{<\omega_n}$. In this section we describe a result saying that the nonstationary ideal is never saturated unless $\kappa = \lambda = \omega_1$.

This problem was studied by Burke and Matsubara [14] who made substantial progress on it. The cases they left open were finished by Foreman and Magidor in [44].

6.34 Theorem. *Let κ be a regular cardinal. The nonstationary ideal on $[\lambda]^{<\kappa}$ is not λ^+-saturated unless $\kappa = \lambda = \omega_1$.*

We note that the statement that the nonstationary ideal on ω_1[56] is $\aleph_2$-saturated has a complicated history; see Sect. 8.

Theorem 6.34 is not a local statement. Gitik proved:

6.35 Theorem (Gitik [52]). *Suppose that there is a supercompact cardinal κ. Then there is a generic extension in which κ remains inaccessible and there is a set $S \subseteq \{z \in [\kappa^+]^{<\kappa} : \mathrm{ot}(z) = \mathrm{ot}(z \cap \kappa)^+\}$ such that the nonstationary ideal restricted to S is κ^+-saturated.*[57]

This was later improved by Krueger [75] to show that it is possible to have $\mathrm{NS}\restriction\{z \in [\kappa^+]^{<\kappa} : \mathrm{ot}(z) = \mathrm{ot}(z \cap \kappa)^+\}$ be κ^+-saturated.

Proof of Theorem 6.34. The proof splits into several cases depending on the properties of κ and λ. We discuss the proof in some of the cases and refer the reader to [44] for a complete proof.

Note that if $\kappa = \lambda$ then there is a stationary subset S of $[\lambda]^{<\kappa}$ such that $\mathrm{NS}\restriction S = \mathrm{NS}_\lambda$. For $\kappa = \lambda \neq \omega_1$, Theorem 6.14 implies that $\mathrm{NS}_{\kappa\lambda}$ is not λ^+-saturated. Hence we can assume that $\lambda > \kappa$.

Case 1. λ is a regular cardinal, and κ is a successor cardinal.

Suppose that $\kappa = \mu^+$ for some cardinal μ. Let $\eta = \mathrm{cf}(\mu)$. We claim that the nonstationary ideal on λ restricted to points of cofinality η is not λ^+-saturated.

We can see this from various theorems depending on what type of cardinal λ is. If λ is either the successor of a singular cardinal or weakly inaccessible we use Theorem 6.14. If λ is a successor of a regular cardinal then we use Corollary 6.11.

Let $\{T_\alpha : \alpha < \lambda^+\}$ be an antichain in $P(\lambda)/(\mathrm{NS}\restriction \mathrm{Cof}(\eta))$. Let $S_\alpha = \{N \in [\lambda]^{<\kappa} : \sup(N) \in T_\alpha\}$. Then each S_α is stationary and $\{S_\alpha : \alpha < \lambda^+\}$ forms an antichain in $P([\lambda]^{<\kappa})/\mathrm{NS}_{\kappa\lambda}$.

Case 2. $\kappa = \omega_1$ and $\lambda > \omega_1$.

[56] Or equivalently, the nonstationary ideal on $[\omega_1]^{<\omega_1}$.

[57] An alternate proof was given by Shioya [107].

Significant progress was made in this case by Donder and Matet as well as Shelah who showed the following $\diamondsuit$-principle under various cardinal arithmetic assumptions. The diamond principle was later shown to hold in ZFC by Shelah [106] using techniques similar to those in [44].

6.36 Definition. If $\lambda > \omega_1$ is a cardinal and $S \subseteq [\lambda]^{<\omega_1}$ is stationary, then $\diamondsuit_{\omega_1,\lambda}(S)$ asserts the existence of a sequence $\langle s_a \colon a \in [\lambda]^{<\omega_1}\rangle$ such that for all $A \subseteq \lambda$ the set $\{a \in S \colon s_a = A \cap a\}$ is stationary in $[\lambda]^{<\omega_1}$.

This principle immediately implies that the nonstationary ideal on $[\lambda]^{<\omega_1}$ is not 2^λ-saturated.

The original proof for the case of $\kappa = \omega_1$ is due to Foreman and Magidor [44] who developed the theory of mutually stationary sets:

6.37 Definition. Let K be a set of regular cardinals, and $\mathcal{S} = \langle S_\kappa : \kappa \in K\rangle$ be a sequence of sets such that $S_\kappa \subseteq \kappa$. Then the sequence $\mathcal{S}$ is called *mutually stationary* iff for all algebras $\mathfrak{A}$ on $\sup(K)$, there is an elementary substructure $N \prec \mathfrak{A}$ such that for all $\kappa \in K \cap N$, $\sup(N \cap \kappa) \in S_\kappa$.

Note that if the sequence $\mathcal{S}$ is mutually stationary, then in particular, each S_κ is stationary.

See [44] for the following facts:

1. Mutually stationary sequences remain mutually stationary under finite variations.[58]

2. If K is a collection of measurable cardinals then every sequence of stationary sets is mutually stationary.

3. (Welch) If $f : \omega \to \omega$ is a function that is not eventually constant and $S_n = \omega_n \cap \mathrm{Cof}(\omega_{f(n)})$ defines a mutually stationary sequence, then there is an inner model with a measurable cardinal. A weak converse has been proved by Liu and Shelah [87].

4. In L for all $k > 0$, there is a sequence of stationary sets $\mathcal{S} = \langle S_n : n > k\rangle$ such that $S_n \subseteq \omega_n \cap \mathrm{Cof}(\omega_k)$ is stationary but the sequence $\mathcal{S}$ is *not* mutually stationary.

5. In ZFC, if $\mathcal{S} = \langle S_\kappa : \kappa \in K\rangle$ is a sequence of stationary sets such that each $S_\kappa \subseteq \kappa \cap \mathrm{Cof}(\omega)$, then $\mathcal{S}$ is mutually stationary.

We use property 5, to settle case 2. If λ is regular, then case 2 follows from the first case.

If λ is singular, then we choose a cofinal sequence $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ such that each λ_α is regular. By a theorem of Solovay (see [63]) we can partition $\lambda_\alpha \cap \mathrm{Cof}(\omega)$ into λ_α disjoint stationary sets $\{S^\alpha_\beta : \beta < \lambda_\alpha\}$.

By property 5, for each function $f \in \prod \lambda_\alpha$, the sequence $\mathcal{S}_f = \langle S^\alpha_{f(\alpha)} : \alpha < \mathrm{cf}(\lambda)\rangle$ is mutually stationary.

[58] Subject to a mild cofinality restriction.

In particular, for each $f \in \prod \lambda_\alpha$ if we let

$$T_f = \{N : \text{ for all } \alpha \in N, \ \sup(N \cap \lambda_\alpha) \in S^\alpha_{f(\alpha)}\},$$

then each T_f is stationary and for $f \neq g, T_f \cap T_g$ is not stationary. Since there are at least λ^+ such f, we get an antichain of cardinality λ^+.

Case 3. $\mathrm{cf}(\lambda) \geq \kappa$ and κ is a successor cardinal, or κ is weakly inaccessible and $\mathrm{cf}(\lambda) > \kappa$.

In this case Burke and Matsubara used Cummings' theorem (Theorem 6.13) to show that the nonstationary ideal on $[\lambda]^{<\kappa}$ is not saturated. For if it were, then forcing with $\mathbb{P} = P([\lambda]^{<\kappa})/\mathrm{NS}_{\kappa\lambda}$ would preserve stationary subsets of λ^+. Hence by Cummings' theorem, if $G \subseteq \mathbb{P}$ is generic then $V[G]$ satisfies "$\mathrm{cf}(\lambda) = \mathrm{cf}(|\lambda|)$". Hence the generic ultrapower M must satisfy "$\mathrm{cf}(\lambda) = \mathrm{cf}(|\lambda|)$" as well.

The following lemma is standard (see [4]):

6.38 Lemma. *Let κ and λ be cardinals with κ regular.*

1. *Suppose that $\mathrm{cf}(\lambda) > \kappa$, and μ, ν are regular cardinals less than κ. Let S_1 be the collection of x in $[\lambda]^{<\kappa}$ such that:*

   (a) $x \cap \kappa \in \kappa$,

   (b) $|x| = |x \cap \kappa|$,

   (c) $\mathrm{cf}(x \cap \kappa) = \mu$, *and*

   (d) $\mathrm{cf}(\sup(x)) = \nu$.

   Then S_1 is stationary.

2. *Suppose that $\kappa = \rho^+ \geq \omega_2$ and $\mathrm{cf}(\lambda) \geq \mathrm{cf}(\rho)$. Let S_2 be the set of x in $[\lambda]^{<\kappa}$ such that:*

   (a) $x \cap \kappa \in \kappa$,

   (b) $|x| = |x \cap \kappa|$, *and*

   (c) $\mathrm{cf}(x \cap \kappa) = \mathrm{cf}(\sup(x)) \neq \mathrm{cf}(\rho)$.

   Then S_2 is stationary.

Assuming the lemma we get an immediate contradiction to Cummings' theorem. If κ is a successor cardinal, we force with $P([\lambda]^{<\kappa})/\mathrm{NS}_{\kappa\lambda}$ below S_2. By Lemma 2.38, we see that in $V[G]$, $|\lambda| = \rho$, but $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\rho)$, a contradiction.

If κ is weakly inaccessible and $\mathrm{cf}(\lambda) > \kappa$, then forcing below S_1 yields a model $V[G]$ satisfying "$|\lambda| = \kappa$" but $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\kappa)$, again contradicting Cummings' theorem.

Case 4. $\mathrm{cf}(\lambda) < \kappa$.

In this case Burke and Matsubara showed that the nonstationary ideal on $[\lambda]^{<\kappa}$ is not λ^+-saturated, but it is shown in [44] that even more is true: the nonstationary ideal on $[\lambda]^{<\kappa}$ is not even λ^{++}-saturated.

By appealing to the Gitik-Shelah theorem one sees that there is a large antichain of stationary sets of approachable ordinals in λ^+ (see [43]). Via Shelah's pcf theory, such a set in the antichain can be used to code a stationary set in $[\lambda]^{<\kappa}$ through comparisons of characteristic functions and elements of a scale.

Case 5. κ is weakly inaccessible and $\mathrm{cf}(\lambda) = \kappa$.

This case is handled with an argument combining Shelah's "trichotomy" theorem in pcf theory with generic ultrapowers. $\dashv$

7. Consistency Results

In this section we discuss some consistency results for ideals. It is a trivial, but significant observation that classical large cardinals are special cases of ideal axioms; namely they are typically equivalent to the existence of normal, fine, prime ideals, or systems of such. One of the main techniques for proving consistency results is to take prime ideals on large cardinals and transform them, via forcing, into ideals on more accessible sets.

Most of these results rely on rather technical arguments involving iterated forcing and master conditions. As a result we will do no more than outline the deeper results. The reader is referred to Cummings' chapter in this Handbook for information about backwards Easton forcing. It is beyond the scope of this chapter to attempt to describe the intricate lore of constructing master conditions.

Trivial Master Conditions

We will use the following lemma which describes the situation where no master condition is needed:

7.1 Lemma. *Suppose that $j : V \to M \subseteq V[G]$ is a generic elementary embedding with critical point κ, where $G \subseteq \mathbb{Q}$ is generic. Let $\mathbb{P} \in V$ be a κ-c.c. partial ordering. Then for all M-generic $\hat{H} \subseteq j(\mathbb{P})$ the filter $H = \{p : j(p) \in \hat{H}\} \subseteq \mathbb{P}$ is V-generic.*

If $H \subseteq \mathbb{P}$ is V-generic then one can force over $V[H]$ to produce a $V[G]$-generic $\hat{H} \subseteq j(\mathbb{P})$ such that j extends to an elementary embedding

$$\hat{j} : V[H] \to M[\hat{H}] \subseteq V[G * \hat{H}].$$

Proof. Let $\hat{H}$ be generic over M. For all maximal antichains $\mathcal{A} \subseteq \mathbb{P}$ lying in V, we know that $|\mathcal{A}| < \kappa$. Hence $j(\mathcal{A}) = j``\mathcal{A}$. Since $\hat{H} \cap j(\mathcal{A}) \neq \emptyset$ there is a $p \in \mathcal{A}$ with $j(p) \in \hat{H}$. Hence $H \cap \mathcal{A} \neq \emptyset$, and so H is generic.

The proof shows that in $V[G]$ the map $j\restriction\mathbb{P}$ sends V-maximal antichains $\mathcal{A}\subseteq\mathbb{P}$ to M-maximal antichains $j(\mathcal{A})\subseteq j(\mathbb{P})$. Hence if we force over $V[G]$ to get a generic H, then there is a further forcing $\mathbb{R}$ to get a $V[G]$ generic $\hat{H}\subseteq j(\mathbb{P})$ with $j``H\subseteq\hat{H}$. This can be summarized by the formula:

$$\mathbb{Q}*j(\mathbb{P})\sim(\mathbb{Q}*\mathbb{P})*\mathbb{R}.$$

Since both $\mathbb{P}$ and $\mathbb{Q}$ lie in V, the product lemma tells us that $\mathbb{Q}*\mathbb{P}\sim\mathbb{Q}\times\mathbb{P}\sim\mathbb{P}*\mathbb{Q}$. Rearranging the terms, we see that if $H\subseteq\mathbb{P}$ is generic, then forcing over $V[H]$ with $\mathbb{Q}*\mathbb{R}$ produces both G and $\hat{H}$ with the property that for all $p\in\mathbb{P}$, $p\in H$ iff $j(p)\in\hat{H}$.

Define a map $\hat{j}$ by setting $\hat{j}(\tau^{V[H]})=j(\tau)^{M[\hat{H}]}$. That $\hat{j}$ is well-defined and elementary follows from the fact that $j``H\subseteq\hat{H}$. ⊣

The Basic Idea

We here avail ourselves of the notation and discussion in Sect. 3, referring to ideal elements, master conditions, $U(j,i)$ and induced ideals, and so forth. For most of the results in this section we will be in the situation described by Example 3.37. We suggest the reader reread that example before proceeding.

Let $\mathbb{Q}$ be a partial order. We extend our notation slightly to write $\mathbb{Q}/H$ for an arbitrary subset $H\subseteq\mathbb{Q}$ to mean the structure with domain $\{q\in\mathbb{Q}:$ for all $h\in H$, q is compatible with $h\}$, and the relation $\leq_{\mathbb{Q}}$. For $p\in\mathbb{Q}$ we write $\mathbb{Q}/p$ for $\mathbb{Q}/\{p\}$.

Hence if $j:V\to M$ is elementary, $\mathbb{P}\in V$ and $G\subseteq\mathbb{P}$, then the conditions in $j(\mathbb{P})/j``G$ that are compatible with a condition $m$ will be written $j(\mathbb{P})/(j``G\cup\{m\})$. If $H\subseteq\mathbb{Q}$ is a finite set, then forcing with the ordering $\mathbb{Q}/H$ is equivalent to forcing with $\{q\in\mathbb{Q}:$ for all $h\in H$, $q\leq_{\mathbb{Q}}h\}$.

If $m\in j(\mathbb{P})$ is a master condition and $\hat{G}\subseteq j(\mathbb{P})/m$ is V-generic, then $\{p:j(p)\in\hat{G}\}\subseteq\mathbb{P}$ is V-generic. Hence there is a $p\in\mathbb{P}$ such that the restriction of j to $\mathbb{P}/p$ is a regular embedding from $\mathbb{P}/p$ to $j(\mathbb{P})/\{p,m\}$.

A typical construction in this section will start with an elementary embedding $j:V\to M$ that has critical point κ and choose an ideal object $i\in M$. We will force with a partial ordering $\mathbb{P}$ and find a master condition $m\in j(\mathbb{P})/j``G$ where $G\subseteq\mathbb{P}$ is generic. If $\hat{G}\subseteq j(\mathbb{P})/(j``G\cup\{m\})$ is generic then we can extend j to $\hat{j}:V[G]\to M[\hat{G}]$. If Z is such that $m\Vdash_{j(\mathbb{P})/j``G} i\in j(Z)$ then, as in Example 3.37 applied to the partial ordering consisting of the collection of conditions in $j(\mathbb{P})/j``G$ that lie below m, we get an ideal $I\subseteq P(Z)$ that depends on $\hat{j},i$ and m for its definition.

Let $\mathcal{B}$ be the completion of $j(\mathbb{P})/(j``G\cup\{m\})$. Then we get an order and antichain preserving embedding:

$$\iota:P(Z)/I\to\mathcal{B}$$

defined by setting $\iota([X])=\|i\in j(X)\|$.

As a consequence, if $\mathcal{B}$ has the γ-c.c. then I is γ-saturated. Moreover if ι maps $P(Z)/I$ onto a dense subset of $\mathcal{B}$ or simply takes maximal antichains to maximal antichains, then the canonical ultrafilter $U(\hat{j}, i)$ defined by setting $X \in U(\hat{j}, i)$ iff $i \in \hat{j}(X)$ is generic for the partial ordering $P(Z)/I$.

Let $k : V^Z/U(\hat{j}, i) \to M$ be defined by setting $k([g]) = j(g)(i)$. Since the diagram:

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr, "\hat{j}"] & \\ V^Z/U(\hat{j}, i) \arrow[rr, "k"'] & & M \end{tikzcd}$$

commutes, we see that $V^Z/U(\hat{j}, i)$ is well-founded. If ι takes maximal antichains to maximal antichains, the ultrafilter $U(\hat{j}, i)$ is generic for $P(Z)/I$. Hence there is a condition $A \in P(Z)/I$ that forces "if $U \subseteq P(Z)/I$ is generic then V^Z/U is well-founded". Thus $I \restriction A$ is precipitous.

If we know that the range of ι is dense in $\mathcal{B}$ then forcing with $P(Z)/I$ is the same as forcing with $j(\mathbb{P})/(j``G \cup \{m\})$. In this case we have an exact description of the quotient algebra of the ideal. In Sect. 7.4 we give a general method for computing the quotient algebras $P(Z)/I$.

7.2 Remark. We note the close connection with proper forcing ideas. The notion of a "generic" condition in proper forcing is exactly the same as what we are calling a "master condition". The difference in context is that proper forcing deals with the specific case that $\kappa = \omega_1$ and the elementary embedding involved comes from reflection, rather than extension.[59] Corollary 7.18 also illustrates the connection.

The next example gives the simplest mechanism for getting saturated ideals.

7.3 Example. Let κ be measurable and $j : V \to M$ be the elementary embedding induced by an ultrapower by a κ-complete ultrafilter U on κ. If $\mathbb{P}$ is the partial ordering for adding Cohen or random reals, then $\mathbb{P}$ has the c.c.c., and hence the empty condition is a master condition. By the remarks above the ideal induced by j using κ as the ideal element is precipitous and has the c.c.c. In particular, it has the disjointing property.

Solovay [111] proved that adding κ random reals to a measurable cardinal makes κ real-valued measurable, i.e. there is a countably complete probability measure defined on all subsets of κ. In this model $\kappa = 2^{\aleph_0}$, and the ideal of null sets for the measure is c.c.c. and hence precipitous.

Note that this example generalizes quite easily to get generically supercompact, and huge elementary embeddings. Namely, if j is λ-supercompact (or huge, with $j(\kappa) = \lambda$) then we choose the ideal element to be $j``\lambda$ and Z to be either $[\lambda]^{<\kappa}$ (or $[\lambda]^{\kappa}$) and argue in exactly the same way to get a normal, fine, countably complete c.c.c. ideal on $[\lambda]^{<\kappa}$ (or $[\lambda]^{\kappa}$).

59 See the discussion around Definition 3.43.

7.1. Precipitous Ideals on Accessible Cardinals

Let κ be a measurable cardinal. Let $j : V \to M$ be the ultrapower of V by a κ-complete ultrafilter U on κ. Let $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa)$. Let $G \subseteq \mathbb{P}$ be generic. Since $\mathbb{P}$ is κ-c.c., if $H \subseteq j(\mathbb{P})/G$ is an arbitrary generic filter, then j can be extended to $\hat{j} : V[G] \to M[H]$. Hence we can take the empty condition in $j(\mathbb{P})/j``G$ to be a master condition and, working in $V[G]$, define an ideal I as above using $\hat{j}$ and $i = \kappa$ as the ideal element. Then in $V[G]$, I is a precipitous ideal on ω_1. More generally:

7.4 Theorem. *Suppose that κ is a measurable cardinal. If $\nu < \kappa$ is a regular cardinal and $G \subseteq \mathrm{Col}(\nu, {<}\kappa)$ is generic, then there is a precipitous ideal on ν^+ in $V[G]$.*

As a corollary we get the following theorem:

7.5 Corollary (Jech et al. [65]). *Suppose that it is consistent that there is a measurable cardinal. Then it is consistent that there is a precipitous ideal on ω_1.*

Proof of Theorem 7.4. We show that the map ι maps $(P(\kappa)/I)^{V[G]}$ to a dense subset of $j(\mathbb{P})/G$. This suffices, since it shows that the ultrafilter $U(\hat{j}, \kappa)$ is generic for the forcing $P(\kappa)/I$ and as we argued above, the ultrapower $V^\kappa/U(\hat{j}, \kappa)$ embeds into M and hence is well-founded.

Let $p \in j(\mathbb{P})/G$. Then there is a function $f : \kappa \to \mathbb{P}$ lying in V such that $\hat{j}(f)(\kappa) = p$.

Working in $V[G]$, let $X = \{\alpha : f(\alpha) \in G\}$. Then $X \in U(\hat{j}, i)$ iff $\kappa \in \hat{j}(X)$ iff $\hat{j}(f)(\kappa) \in H$. But $j(f)(\kappa) = p$. Hence $\iota(X) = p$, and we have established the theorem. ⊣

With some extra work one can show that the ideal I defined in the proof of Theorem 7.4 is exactly the ideal generated in $V[G]$ by the dual of U.

The argument above is an example of a more general phenomenon explored in the Duality Theorem and a special case, Proposition 7.13. The latter immediately implies:

7.6 Theorem (Laver[60]). *Let κ be a measurable cardinal, and $\nu < \kappa$ be regular. If $G \subseteq \mathrm{Col}(\nu, {<}\kappa)$ is generic, then in $V[G]$ there is a κ-complete, normal, precipitous ideal I such that $P(\kappa)/I$ has a ${<}\nu$-closed dense subset.*

The main point of the argument above is that the set Z and the ideal element i generating the ideal are chosen to be the same as those for the original large cardinal embedding. Thus with minor variations, one can check that if I is a master condition ideal generated by an ideal element $j``\lambda$ over an ultrapower of $V$ by a ultrafilter on $[\lambda]^{<\kappa}$, then $I$ is precipitous and $P([\lambda]^{<\kappa})/I$ is isomorphic to the quotient forcing for $j(\mathbb{P})/(j``G \cup \{m\})$.

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[60] It is remarked in [50] that this was proved independently by Galvin, Jech and Magidor.

The following example, discovered independently by Laver and the author, and used fruitfully by Laver in [85], partially illustrates the close connection between ideals and proper forcing.

7.7 Example. Let κ be supercompact, and let $G \subseteq \mathrm{Col}(\omega, {<}\kappa)$ be generic. Let $\lambda > \kappa$ be a regular cardinal and $j : V \to M$ be a λ-supercompact embedding. Then for all generic $H \subseteq \mathrm{Col}(\omega, j(\kappa) \setminus \kappa)$ there is an elementary embedding $\hat{j} : V[G] \to M[G, H]$.

Let $\mathbb{P}$ be a proper partial ordering in $V[G]$, and suppose that we take $\lambda \geq (2^{|\mathbb{P}|})^+$. Then in $M[G, H]$:

1. $j(\mathbb{P})$ is proper, and
2. $j``H(\lambda)$ is countable.

So by properness, in $M[G, H]$, there is a "generic" condition m for $j``H(\lambda)$. From the point of view of $V[G]$ the condition m is a master condition for the forcing $\mathbb{P}$ with respect to the embedding j. If $G^* \subseteq \mathbb{P}$ is generic and compatible with m, then in $V[G * G^*]$ there is a precipitous ideal I on $[\lambda]^{<\kappa} = Z$ such that the quotient algebra $P(Z)/I$ is isomorphic to $\mathrm{Col}(\omega, j(\kappa) \setminus \kappa) * j(\mathbb{P})/(j``G^* \cup \{m\})$.

In particular, if there is a single condition m in $j(\mathbb{P})$ below every element of $j``G^*$ then the quotient algebra is isomorphic to the Levy collapse followed by forcing with $j(\mathbb{P})$ below m.

7.2. Strong Master Conditions

In this section we briefly describe the construction of non-trivial master conditions. This topic is covered in depth in Cummings' chapter in the Handbook. The first use of this technique was due to Silver who showed that it was consistent for GCH to fail at a measurable cardinal, provided the existence of a supercompact cardinal is consistent.

In a typical situation, we are given an elementary embedding $j : V \to M$ that is closed under λ-sequences for some cardinal $\lambda \geq \kappa$. We will force with a partial ordering of the form $\mathbb{P} * \mathbb{Q}$ where $\mathbb{P}$ is κ-c.c. and $|\mathbb{Q}| \leq \lambda$. The result will be a generic filter of the form $G * H \subseteq \mathbb{P} * \mathbb{Q}$. To extend the embedding j we must find a generic $\hat{G} * \hat{H} \subseteq j(\mathbb{P} * \mathbb{Q})$ such that for all $(p, \dot{q}) \in \mathbb{P} * \mathbb{Q}$, $(p, \dot{q}) \in G * H$ iff $(j(p), j(\dot{q})) \in \hat{G} * \hat{H}$.

Since $\mathbb{P}$ has the κ-c.c. this is automatic for $p \in \mathbb{P}$ and $G, \hat{G}$. If one can arrange that the generic object $H \subseteq \mathbb{Q}$ lies in $M[\hat{G}]$, then it is frequently possible to construct a condition m such that for all $\dot{q} \in H$, $M[\hat{G}] \models m \leq j(\dot{q})$. Such an m is called a *strong master condition*. Forcing with $j(\hat{\mathbb{Q}})$ below m then gives an $\hat{H}$ with the requisite properties.

H can often be constructed in $V^{j(\mathbb{P})}$ because $\mathbb{P}$ is a sufficiently strong Levy collapse that $\mathbb{Q}$ is embedded in $j(\mathbb{P})/G$. If $j(\mathbb{Q})$ is ${<}\lambda^+$-directed-closed then in $M[G]$ the desired condition m is defined by taking a condition below $\bigcup j``H$.

There are examples when $\bigcup j``H$ is a condition in $j(\mathbb{Q})$ even without $<\lambda^+$-closure due to the geometry of the conditions in $\mathbb{Q}$. This is the reason for forcing with the Silver collapse, rather than the Levy collapse in many models given below.

7.3. Precipitousness is not Preserved Under Projections

In this subsection we outline a result of Laver that illustrates two interesting points: that not every master condition ideal is precipitous and that projection maps do not preserve the property of being precipitous. Gitik has very recently given an example of a precipitous ideal on a regular cardinal whose canonical projection to a normal ideal is not precipitous.

7.8 Theorem (Laver). *Suppose that it is consistent that there to be a supercompact cardinal. Then it is consistent that:*

1. *There is a master condition ideal on ω_1 that is not precipitous.*
2. *There is a precipitous ideal on $[\omega_2]^{<\omega_1}$ whose projection to ω_1 is not precipitous.*

Proof (Sketch). A basic building block for Laver's partial construction is a version of Heckler forcing. It is a partial ordering D designed to add a generic function $f : \omega_1 \to \omega_1$ that dominates every ground model function. Conditions in D are ordered pairs (g, s) where $g : \omega_1 \to \omega_1$ and s is a countable approximation to f. The ordering on D is given by taking (g, s) to be stronger than (h, t) iff g eventually dominates h and $s \supseteq t$. Assuming the CH, this partial ordering is $\aleph_2$-c.c. and countably closed.

Let $j : V \to M$ be a λ-supercompact embedding where $\kappa = \mathrm{crit}(j)$ and $\lambda = \kappa^+$. Let

$$\mathbb{P} = \mathrm{Col}(\omega, <\kappa) * \prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha,$$

where each D_α is a copy of the partial ordering D defined in $V^{\mathrm{Col}(\omega,<\kappa)}$ and the product is taken with countable supports. It is routine to check that $\mathbb{P}$ has cardinality λ, is λ-c.c. and belongs to M.

- There is a regular embedding $e : \mathbb{P} \to \mathrm{Col}(\omega, <j(\kappa))$ that lies in M such that the restriction of e to $\mathrm{Col}(\omega, <\kappa)$ is the identity map. If $\hat{G} \subseteq \mathrm{Col}(\omega, <j(\kappa))$ is generic over V then in M, we can build a V-generic object $G*H \subseteq \mathrm{Col}(\omega, <\kappa)*\prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha$. In the model $V[G*H]$ we note that $\kappa = \omega_1$ and $\lambda = \omega_2$.
- By the κ-c.c. of $\mathrm{Col}(\omega, <\kappa)$ there is an extension of j to $j_1 : V[G] \to M[\hat{G}]$, and by the supercompactness of j, $j_1``H \in M[\hat{G}]$. Since $j_1``H$ is countable, we can form the master condition $m = \bigcup j_1``H \in j_1(\prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha)$. Forcing with $j_1(\prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha)$ below m we get an $\hat{H}$ and an elementary embedding $\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}]$.

- Arguing as in the discussion before Remark 7.2[61] the ideal J on $[\omega_2]^{<\omega_1}$ induced by $\hat{j}$ and j“λ is precipitous in the model $V[G * H]$. We now show that the projection of this ideal to ω_1 is *not* precipitous. Note that the projected ideal I on ω_1 is the ideal induced by j and $i = \omega_1$.

- Let f_α be the generic function determined by D_α. To show that this ideal is not precipitous it suffices to show that for all $\alpha < \omega_2$ and all I-positive sets X, there is an I-positive set $Y \subseteq X$ and a $\beta < \omega_2$ such that $f_\beta(\delta)$ is less than $f_\alpha(\delta)$ for all $\delta \in Y$. This allows one to construct terms for a decreasing sequence of functions in the generic ultrapower.

- For this latter statement, it suffices to show that for all $X \in I^+$, and all f_α, there is a condition $q \in j(\prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha)$ below m and a β such that q forces that $\kappa \in \hat{j}(X)$ and $\hat{j}(f_\alpha)(\kappa) > \hat{j}(f_\beta)(\kappa)$.

- Note that each $X \subseteq \omega_1$ in $V[G * H]$ is in $V[G * (H \cap \prod_A^{\text{ctbl sppt}} D_\alpha)]$ for some $A \subseteq (\omega_2)^{V[G*H]}$ of cardinality $\omega_1^{V[G*H]}$. Without loss of generality we can assume that $A = \omega_1^{V[G*H]} = \kappa$.

- All of the conditions involved in deciding $\|\kappa \in j(X)\|$ are in the product of the first $j(\kappa)$ coordinates. Hence, if $\beta \notin \omega_1$, then we can extend the condition m to an m_1 that decides $j(f_\beta)(\kappa)$ to be any value that is above $\sup\{j(g)(\kappa) : g \in \kappa^\kappa \cap V[G]\}$, without affecting the truth value of $\|\kappa \in j(X)\|$.

- Thus we will be done if we can show that for all sets $X \in I^+$ and all $\alpha < \kappa^+$ and all $\theta < j(\kappa)$ there is an extension q of m belonging to $j(\prod_{\alpha<\lambda}^{\text{ctbl sppt}} D_\alpha)$ such that q forces $\kappa \in j(X)$ and $j(f_\alpha)(\kappa) \geq \theta$.

The latter point uses a delicate reflection argument to show that any such bound θ can be taken to be of the form $j(g)(\kappa)$ for some function $g \in \kappa^\kappa \cap V$, and “works” relative to an auxiliary embedding j_κ also determined by a function with domain κ that lies in the ground model. The two embeddings j and j_κ then can be simultaneously extended, but the function $j(f_\alpha)$ eventually dominates g, yielding a contradiction. The reader is referred to [85] for details. $\dashv$

7.4. Computing Quotient Algebras and Preserving Strong Ideals under Generic Extensions

Many generic embeddings are constructed by taking standard large cardinal embeddings and modifying them by forcing. Since the nature of the forcing is one of the three parameters of the strength of a generic embedding, characterizing it is a fundamental problem. Indeed most consistency results about

[61] Or applying Proposition 7.13.

ideals explicitly refer to properties, such as saturation, of the quotient algebras $P(Z)/I$. Thus it is desirable to have a general method for computing exactly what the quotient algebra is.

Such a general method must necessarily be abstract. Formulating it requires a balance between retaining its generality and keeping it concrete enough for actual use in consistency arguments. Three results of this type are given here. Proposition 7.13 is the most concrete and easy to use. It covers many common cases, but does not directly apply in several important arguments. In the author's view, the Duality Theorem (Theorem 7.14) gives the best balance. It is completely general in the case that master conditions exist, is easy to use and has many standard theorems as consequences. Theorem 7.30 and the discussion that follows it give completely general results. At the moment, they have only been applied in fairly exotic situations.

A related topic of current interest is the preservation of various axioms under generic extensions. In the context of generalized large cardinals this can be viewed as calculating the quotient algebra of a generic embedding after forcing. Hence the Duality Theorem has corollaries about the preservation properties of ideals under fairly general circumstances. As corollaries we deduce theorems of Kakuda [66], Baumgartner-Taylor [6], and Laver [83].

We only give a brief tour of the known results. The proofs appear in [30].

Preservation of Normality and a Warm-Up Theorem

We start with some basic remarks about ideals in generic extensions. Let $\mathbb{P}$ be a partial ordering and J an ideal on Z for some $Z \subseteq P(X)$. If $H \subseteq \mathbb{P}$ is generic we will denote the ideal generated by J in $V[H]$ by $\bar{J}$. Thus for $W \subseteq Z$ and $W \in V[H]$ we have $W \in \bar{J}$ iff there is a $Y \in J$ such that $W \subseteq Y$. Note that with this definition it is a triviality that $\bar{J} \cap P(Z)^V = J$. For shorthand, if $\mathbb{P}$ and H are clear from context we will denote $P(Z)^{V[H]}$ by $\overline{P(Z)}$.

The next observation is standard:

7.9 Proposition. *Suppose that J is a κ-complete ideal on Z and $\mathbb{P}$ is a κ-c.c. partial ordering. Then for all generic $H \subseteq \mathbb{P}$, $\bar{J}$ is a κ-complete ideal in $V[H]$. Moreover, if J is also normal then $\bar{J}$ is normal in $V[H]$.*

Prior to a discussion of calculating quotients of the closure of an ideal after forcing, there is the more basic question about whether the closure remains normal. The next result is a general criterion for the preservation of normality of an ideal J under a generic extension. Note the close resemblance to definitions used in proper forcing.

7.10 Lemma. *Suppose that J is a normal, fine, κ-complete ideal on $Z \subseteq P(X)$, $\mathbb{P}$ is a $|X|^+$-c.c. partial ordering and θ is a regular cardinal sufficiently large so that $X, Z, \mathbb{P} \in H(\theta)$. Let $H \subseteq \mathbb{P}$ be generic. Then for $L \in P(Z)^{V[H]}$, the following are equivalent:*

1. *$\bar{J}\restriction L$ is normal, fine and κ-complete.*

2. *For all algebras $\mathfrak{A} \in V[H]$ on $H(\theta)^V$, $\{z \in Z : H$ is generic over $\mathrm{Sk}^{\mathfrak{A}}(z)\}$ is in the dual of $\bar{J}\restriction L$.*[62]

3. *For all algebras $\mathfrak{A} \in V$ on $H(\theta)^V$, $\{z \in Z : H$ is generic over $\mathrm{Sk}^{\mathfrak{A}}(z)\}$ is in the dual of $\bar{J}\restriction L$.*

In the case that we have an ideal J on a set $Z \subseteq P(X)$ and a partial ordering $\mathbb{P}$ that is not $|X|^+$-c.c. Lemma 7.10 can be applied to the nonstationary ideal I on $H(\gamma)$ conditioned[63] on J for some γ such that $\mathbb{P}$ is γ^+-c.c. After forcing with $\mathbb{P}$, the projection of $\bar{I}$ is $\bar{J}$. Thus the question of preservation of the normality of J can be reduced to the question of the preservation of the normality of the conditional nonstationary ideal. In this way, Lemma 7.10 gives an almost complete answer to the question of preservation of normality under forcing.

The following example shows that Lemma 7.10 has content.

7.11 Example. Let U be a supercompact ultrafilter on $[\kappa^+]^{<\kappa}$. Let $J = \breve{U}$ Then J is a 2-saturated, normal, fine ideal on $[\kappa^+]^{<\kappa} = Z \subseteq P(X)$, where $X = \kappa^+$. Let $\mathbb{P} = \mathrm{Col}(\kappa, \kappa^+)$. Then $|\mathbb{P}| \leq |X|$. Let $H \subseteq \mathbb{P}$ be generic. Then in $V[H]$, $\bar{J}$ is κ-complete. However, $\{x : |x \cap \kappa| < |x|\} \in U$, but $\{x : |x \cap \kappa| = |x|\} \in \breve{I}$ for all normal, fine, κ-complete ideals I on Z in $V[H]$. Hence $\bar{J}$ is not normal in $V[H]$.

The next proposition partially describes when forcing with $\mathbb{P}$ preserves maximal antichains in $P(Z)/J$, an approximation to characterizing the forcing $P(Z)^{V^{\mathbb{P}}}/\bar{J}$.

7.12 Proposition. *Suppose that J is a normal, fine, countably complete ideal on $Z \subseteq P(X)$ and $\mathbb{P}$ is a partial ordering such that $1 \Vdash \bar{J}$ is normal. Suppose that either:*

1. *J is $|X|^+$-saturated in V, or*

2. *$|\mathbb{P}| \leq |X|$.*

Then if $\mathcal{A} \subseteq P(Z)/J$ is a maximal antichain in V, then $\mathcal{A} \subseteq P(Z)^{V[H]}/\bar{J}$ is a maximal antichain in $V[H]$. In particular, if $\bar{J}$ is precipitous in $V[H]$, then J is precipitous in V.

Note that this proposition does *not* say that in $V^{\mathbb{P}}$, the identity map from $P(Z)/J$ to $P(Z)^{V^{\mathbb{P}}}/\bar{J}$ is a regular embedding (see Corollary 7.26).

Here is a warm-up for the full Duality Theorem. The warm-up is often easier to apply, though much less general:

[62] Recall that H is generic over $N \prec H(\theta)$ iff for all maximal antichains $\mathcal{A} \in N, H \cap \mathcal{A} \cap N \neq \emptyset$.

[63] See Definition 4.19.

7.13 Proposition. *Suppose that $\mathbb{Q}$ is a partial ordering, and $Z \subseteq P(X)$ is such that for all generic $H \subseteq \mathbb{Q}$ there is a V-normal, V-κ-complete ultrafilter U on $P(Z)^V$ belonging to $V[H]$ such that:*

1. *V^Z/U is well-founded, and*
2. *there are functions $h, Q, \{f_q : q \in \mathbb{Q}\}$ in V such that for all generic $H \subseteq \mathbb{Q}$, if M is the transitive collapse of V^Z/U, then $[Q]^M = \mathbb{Q}$, $[h]^M = H$ and for all $q \in \mathbb{Q}$, $[f_q]^M = q$.*

In V, let $J = \{A \subseteq Z : \|A \in \dot{U}\|_{\mathbb{Q}} = 0\}$. Then J is normal, fine, κ-complete and precipitous and $\mathcal{B}(P(Z)/J) \cong \mathcal{B}(\mathbb{Q})$.

The isomorphism defined in Proposition 7.13 maps $[X] \in P(Z)/J$ to $\|X \in \dot{U}\|_{\mathbb{Q}}$. To see that this maps onto a dense set, let $q \in \mathbb{Q}$. Then $A = \{z : f_q(z) \in h(z)\}$ gets sent to q.

When Master Conditions Exist

For the rest of the subsection we will have a precipitous ideal J on Z for some set Z and a partial ordering $\mathbb{P}$. As usual we will consider a generic $G \subseteq P(Z)/J$ and the associated elementary embedding $j : V \to V^Z/G \cong M \subseteq V[G]$, where M is transitive. We explicitly *include* the case where J is a prime ideal.[64]

We are interested in calculating the quotient algebra of the ideal $\bar{J}$ after forcing with $\mathbb{P}$ provided that it still yields a generic elementary embedding. In most situations this is equivalent to the existence of a master condition m. We now make that equivalence explicit. For the general situation we refer the reader to Theorem 7.30 and the remarks after it.

We will consider the partial ordering $j(\mathbb{P})$ in M and force with it over $V[G]$ to get an $\hat{H}$. If $m \Vdash^{V[G]}_{j(\mathbb{P})} j^{-1}(\hat{H}) \subseteq \mathbb{P}$ is generic, then there is a condition $q \in \mathbb{P}$ such that $q \Vdash^V_{\mathbb{P}}$ "j"$H \cup \{m\}$ can be generically extended to a generic $\hat{H} \subseteq j(\mathbb{P})$". Then j is a regular embedding from $\mathbb{P}/q$ to $j(\mathbb{P})/(m \wedge j(q))$. Replacing the original $\mathbb{P}$ by the conditions in $\mathbb{P}$ below q we can see that $j : \mathbb{P} \to j(\mathbb{P})/m$ is a regular embedding in $V[G]$.

Summarizing: if $\mathbb{P}$ is a partial ordering in V such that for all generic $G \times H \subseteq P(Z)/J \times \mathbb{P}$ there is a $V[G]$-generic $\hat{H}$ such that the embedding $j : V \to M \subseteq V[G]$ induced by the ultrapower can be extended to an embedding:

$$\hat{j} : V[H] \to M[\hat{H}]$$

[64] This is the situation where j is a conventional large cardinal embedding. In this case the "generic" G is trivial.

then there is a condition $\dot{m} \in j(\mathbb{P})$ such that

$$\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m,$$

is a regular embedding.[65]

This is exactly the main hypothesis of the next theorem, which is due to the author, and says that forcing with $\mathbb{P} * P(Z)/\bar{J}$ is equivalent to forcing with $P(Z)/J * j(\mathbb{P})$.

The Duality Theorem and Its Consequences

7.14 Theorem (The Duality Theorem; Foreman [30]). *Suppose that there is a condition $\dot{m} \in j(\mathbb{P})$ such that the embedding*

$$\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m$$

is a regular embedding. Then there are conditions $p \in P(Z)/J * j(\mathbb{P})$ and $q \in \mathbb{P} * P(Z)/\bar{J}$ such that:

$$(\mathbb{P} * P(Z)/\bar{J})/q \sim (P(Z)/J * j(\mathbb{P}))/p.$$

7.15 Remark. We can make the Duality Theorem more explicit. Under the hypotheses of the Duality Theorem, if $[A] \in P(Z)/J$ is such that $[A] \Vdash_{P(Z)/J} m = [f]$ and $\mathcal{M} \in V^{\mathbb{P}}$ is defined as $\{z \in A : f(z) \in H\}$ (where $H \subseteq \mathbb{P}$ is the generic object), then there is a canonical isomorphism ι witnessing:

$$\mathcal{B}(\mathbb{P} * P(Z)/(\bar{J}\restriction\mathcal{M})) \cong \mathcal{B}(P(Z)/(J\restriction A) * j(\mathbb{P})/m).$$

We now reduce several other theorems to the duality theorem:

7.16 Corollary. *Suppose that J is a precipitous ideal on $Z \subseteq P(X)$ and $\mathbb{P}$ is a partial ordering satisfying the hypotheses of Theorem 7.14. Then for all generic $H \subseteq \mathbb{P}$ there is a set $\mathcal{M}$ such that $\bar{J}\restriction\mathcal{M}$ is precipitous.*

Proof. For $V[H]$-generic $\bar{G} \subseteq P(Z)/(\bar{J}\restriction\mathcal{M})$, we get a generic object $G * \hat{H}$ for $P(Z)/J * j(\mathbb{P})$ such that $j``H \subseteq \hat{H}$. Hence there is an elementary embedding $\hat{j} : V[H] \to M[\hat{H}]$. Moreover $\bar{G} = \{A \subseteq Z : A \in V[H]$ and $[\mathrm{id}]^M \in \hat{j}(A)\}$. In particular, the ultrapower $V^Z/\bar{G}$ (taken in $V[H]$, using functions from $V[H]$) is well-founded for all generic $\bar{G}$. Hence $\bar{J}$ is precipitous. ⊣

A special case of this is the following result that was discovered independently by Magidor:

7.17 Corollary (Kakuda's Theorem [66]). *Suppose that $\mathbb{P}$ is a κ-c.c. partial ordering on κ and that J is a κ-complete precipitous ideal. Then $\bar{J}$ is precipitous in $V^{\mathbb{P}}$.*

---

[65] It is important that $\hat{H}$ is $V[G]$-generic. Without this requirement we cannot argue that m exists.

Proof. This follows immediately, since the condition $\mathbf{1} \in j(\mathbb{P})$ works for m. Hence in the theorem, we need only consider $A = Z$. Moreover, from the definition of $\mathcal{M}$ we see that in this case $\mathcal{M} = Z$. ⊣

From Kakuda's Theorem one easily sees that collapsing a large cardinal κ to be a successor cardinal using κ-c.c. forcing yields precipitous ideals; in particular, Theorem 7.4 follows. Moreover, collapsing a λ-supercompact cardinal to μ^+ yields a precipitous ideal on $[\lambda]^{<\mu^+}$, collapsing a huge cardinal κ to μ^+ yields the existence of a precipitous ideal on $[\lambda]^{\mu^+}$ for some λ and so on. The next corollary implies that the property:

For all sufficiently large λ there is a precipitous ideal on $[\lambda]^{<\omega_1}$

is indestructible by proper forcing (compare with Example 7.7).

7.18 Corollary. *Suppose that I is a normal, fine, precipitous ideal on $[\lambda]^{<\omega_1}$ for some λ, and $\mathbb{P}$ is a proper partial ordering with $2^{|\mathbb{P}|} \leq \lambda$. Then there is a dense collection of sets $A \in P([\lambda]^{<\omega_1})/I$ such that $I{\restriction}A$ precipitous in $V^{\mathbb{P}}$.*

Proof. We claim that $I{\restriction}A$ and $\mathbb{P}$ satisfy the hypotheses of the theorem for a dense collection of A. Without loss of generality we can take I to be on $[X]^{<\omega_1}$, where X is a transitive set with $P(\mathbb{P}) \subseteq X$. Let $G \subseteq P([X]^{<\omega_1})/I$ be generic, and $j : V \to M$ be the generic elementary embedding.

Let θ be a large regular cardinal. We can assume that for each $z \in [X]^{<\omega_1}$ and all conditions $p \in \mathrm{Sk}^{H(\theta)}(z)$ there is a condition m stronger than p that is $\mathrm{Sk}^{H(\theta)}(z)$-generic. Thus in M, there is a condition $m \in j(\mathbb{P})$ that is $\mathrm{Sk}^{H(j(\theta))}(j``X)$-generic. Then $m \Vdash^M$ "for all dense sets $D \in \mathrm{Sk}^{H(j(\theta))}(j``X)$, there is a $p \in \mathrm{Sk}^{H(j(\theta))}(j``X) \cap D \cap \hat{H}$", where $\hat{H}$ is the canonical term for the generic object for $j(\mathbb{P})$ in M. Hence $m \Vdash^{V[G]}$ "$\{p \in \mathbb{P} : j(p) \in \hat{H}\}$ is V-generic".

Let $A \in P([X]^{<\omega_1})/I, f : [X]^{<\omega_1} \to \mathbb{P}$ be such that $A \Vdash [f]^M = m$. Then A and $\mathbb{P}$ satisfy the hypotheses of the theorem. ⊣

The next example shows that it is not a theorem of ZFC that precipitous ideals are preserved under $<\kappa$-directed closed forcing.

7.19 Example. Let U be a κ-complete, normal ultrafilter on κ. Force over the model $L[U]$ with the partial ordering $\mathrm{Add}(\kappa)$ to get a generic G. Then in $V = L[U][G]$, the ideal generated by the dual of U is no longer precipitous. For if it were and we took the generic elementary embedding $j : V \to M \subseteq V[H]$ then $M \models P(\kappa) \subseteq L[j(U)]$. But by the standard theory of $L[U]$, $L[j(U)]$ is an iterate of $L[U]$, and hence $P(\kappa)^{L[j(U)]} = P(\kappa)^{L[U]}$. But $G \in M$, and $G \notin L[U]$ a contradiction.

Another immediate consequence of Theorem 7.14 is:

7.20 Corollary. *Suppose that J is an $|X|^+$-saturated ideal on $Z \subseteq P(X)$ and $\mathbb{P}$ is a partial ordering satisfying the hypotheses of Theorem 7.14. Let $A, f, \mathcal{M}$ be as in Remark 7.15. Then $\bar{J}{\restriction}\mathcal{M}$ is $|X|^+$-saturated in $V^{\mathbb{P}}$ iff $j(\mathbb{P})/m$ is $|X|^+$-c.c. in $V[G]$ for all generic $G \subseteq P(Z)/J$.*

The next corollary is immediate from the previous one, since we do not need m (hence A or $\mathcal{M}$).

7.21 Corollary (Baumgartner-Taylor [6] and Laver independently). *Suppose that J is a κ-complete, κ^+-saturated ideal on κ and $\mathbb{P}$ is κ-c.c. Then $\bar{J}$ is $|\kappa|^+$-c.c. in $V^{\mathbb{P}}$ iff $j(\mathbb{P})$ is κ^+-c.c. in $V[G]$ for all generic $G \subseteq P(\kappa)/J$.*

We give an example of a c.c.c.-destructible saturated ideal on ω_1 in the discussion of Theorem 8.52 and show in Theorem 8.46 that the nonstationary ideal on ω_1 can be a c.c.c.-indestructible $\aleph_2$-saturated ideal.

Another consequence is a generalization of a classical result of Solovay [111] which also follows easily from the Duality Theorem.

7.22 Corollary. *Let $\lambda, \delta < \kappa$ be regular and $\gamma \leq |X|^+$. Suppose that J is a normal, fine, weakly (λ, γ)-saturated, κ-complete ideal on $Z \subseteq P(X)$. Let $\mathbb{P}$ be a partial ordering and $H \subseteq \mathbb{P}$ be generic. If $\mathbb{P}$ is λ-c.c. then in $V[H]$, $\bar{J}$ is weakly (λ, γ)-saturated and $\operatorname{sat}(\bar{J})^{V[H]} \leq \max\{\lambda, \operatorname{sat}(J)^V\}$.*

In particular, if $\mathbb{P}$ is δ-c.c. and J is a η-saturated ideal on κ with $\eta \leq \kappa^+$, J remains η-saturated in $V[H]$.

Proof. Since $\mathbb{P}$ is κ-c.c. the hypotheses of Theorem 7.14 are automatically satisfied. Hence $\mathbb{P} * P(Z)/\bar{J} \sim P(Z)/J * j(\mathbb{P})$. Let $G \subseteq P(Z)/J$ be generic and $j : V \to M$ be the generic ultrapower. Then M is closed under λ-sequences and $j(\lambda) = \lambda$. Hence $j(\mathbb{P})$ is λ-c.c. in $V[G]$. From this we conclude that $\mathbb{P} * P(Z)/\bar{J}$ is weakly (λ, γ)-saturated and has the $\max\{\lambda, \operatorname{sat}(J)^V\}$-c.c. It follows abstractly that $P(Z)/\bar{J}$ has the same properties in $V[H]$ for a generic $H \subseteq \mathbb{P}$. ⊣

The next examples illustrate how the Duality Theorem can be used to calculate quotients of large cardinal ideals after doing some forcing. Many lengthy calculations of quotient algebras such as those that appear in [48] and [41] are also made quite easy by the Duality Theorem.

7.23 Example (Laver). Suppose that κ is a measurable cardinal and U a normal ultrafilter on κ. Let $H \subseteq \operatorname{Col}(\mu, {<}\kappa)$ be generic. Then in $V[H]$, $P(\kappa)/\breve{U}$ contains a dense set isomorphic to $\operatorname{Col}(\mu, {<}j(\kappa))$. In particular, the quotient has a ${<}\mu$-closed dense subset.

Proof. Take $J = \breve{U}$ and $\mathbb{P} = \operatorname{Col}(\mu, {<}\kappa)$ in the theorem. Note that $P(\kappa)/J$ is the trivial, two-element Boolean algebra.

Since $\mathbb{P}$ is κ-c.c. we can take m to be the trivial condition and hence $A = \mathcal{M} = \kappa$. By the theorem, in $V[H]$, $\mathcal{B}(P(Z)/\bar{J}) \cong \mathcal{B}(j(\mathbb{P})/H)$. Since $j(\mathbb{P}) = \operatorname{Col}(\mu, {<}j(\kappa))$ and $j \restriction \operatorname{Col}(\mu, {<}\kappa)$ is the identity, we see that $\mathcal{B}(j(\mathbb{P})/H) \cong \mathcal{B}(\operatorname{Col}(\mu, {<}j(\kappa) \setminus \kappa))$. ⊣

7.24 Example. Let I be a normal, countably complete, $\aleph_1$-dense ideal on ω_1. Let $\mathbb{P}$ be the partial ordering $\operatorname{Add}(\omega, \omega_1)$ for adding ω_1 Cohen subsets of ω with finite conditions. Then in $V^{\mathbb{P}}$, $P(\omega_1)/\bar{I}$ is isomorphic to the partial ordering that collapses ω_1 with finite conditions and then adds ω_2 Cohen subsets of ω with finite conditions.

Proof. To see this we apply duality. Note that $\mathrm{Add}(\omega,\omega_1) * P(\omega_1)/\bar{I}$ is isomorphic to the quotient of $P(\omega_1)/I * \mathrm{Add}(\omega,\omega_2^V) \sim \mathrm{Col}(\omega,\omega_1) * \mathrm{Add}(\omega_1,\omega_2^V)$. If H adds ω_1 Cohen reals then, by duality the ordering $P(\omega_1)/\bar{I}$ is isomorphic to the quotient of $\mathrm{Col}(\omega,\omega_1) * \mathrm{Add}(\omega,\omega_1)/H$ which is isomorphic to $\mathrm{Col}(\omega,\omega_1) * \mathrm{Add}(\omega,\omega_1)$. ⊣

The next example plays an unfortunate role in the discussion of generalized large cardinals as axioms for set theory, as we shall see in Sect. 11.2. It follows immediately since $j(\mathrm{Col}(\omega,{<}\kappa)) = \mathrm{Col}(\omega,{<}\lambda)$ and $\mathrm{Col}(\omega,{<}\lambda)/G \sim \mathrm{Col}(\omega,{<}\lambda)$:

7.25 Example. Suppose that $j : V \to M$ is a huge embedding with critical point κ and $j(\kappa) = \lambda$. Let I be the dual of the huge ultrafilter on $[\lambda]^\kappa$ and $G \subseteq \mathrm{Col}(\omega,{<}\kappa)$ be generic. Then in $V[G]$, λ is inaccessible and:

$$P([\lambda]^\kappa)/I \cong \mathcal{B}(\mathrm{Col}(\omega,{<}\lambda)).$$

Understanding the Embeddings: Master Conditions Must Exist

Suppose that J is a normal, fine ideal on $Z \subseteq P(X)$, and $\bar{J}$ is precipitous in $V[G]$. One can ask:

> Suppose that $\hat{j}$ is a generic elementary embedding coming from a $V[H]$-generic $\bar{G} \subseteq P(Z)^{V[H]}/\bar{J}$. How close is $\hat{j}$ to a generic elementary embedding j arising from a V-generic $G \subseteq P(Z)/J$?

One answer is given by the following corollary:

7.26 Corollary. *Suppose that*

$$\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m$$

is a regular embedding. Then there is a J-positive set $A \in V$ and a $\bar{J}$-positive set $L \in V^{\mathbb{P}}$ such that

$$\mathrm{id} : \mathbb{P} \times P(Z)/(J \restriction A) \to \mathbb{P} * P(Z)/(\bar{J} \restriction L)$$

sending $(p, A) \mapsto (p, \check{A})$ is a regular embedding. Moreover, if $\bar{G} \subseteq P(Z)/\bar{J}$ is $V[H]$-generic and G is the V-generic ultrafilter induced by id*, and $\hat{j} : V[H] \to M[\hat{H}]$ and $j : V \to M$ are the two generic ultrapower embeddings then j and $\hat{j}$ agree on V.*

Since there are many examples of consistency results that rely on extending generic elementary embeddings, it is interesting to understand if this can be done using methods besides constructing master conditions. Unfortunately the full force of the Duality Theorem requires master conditions.

Indeed Corollary 7.26 is a near equivalence: the conclusions almost imply the existence of a master condition. From the proof of the duality theorem we can derive more information that yields an exact converse.

7.27 Remark. Under the hypotheses of the Duality Theorem, if $H * \bar{G} \subseteq \mathcal{B}(\mathbb{P} * P(Z)/(\bar{J}\restriction\mathcal{M}))$ is generic, then:

1. $\mathrm{comp}(J\restriction A)^V = \mathrm{comp}(\bar{J}\restriction\mathcal{M})^{V[H]}$,
2. for all $f : Z \to V$ with $f \in V[H]$ there is a $g \in V$ such that $[f]_{\bar{G}} = [g]_{\bar{G}}$, and
3. the isomorphism
$$\iota : \mathcal{B}(\mathbb{P} * P(Z)/(\bar{J}\restriction\mathcal{M})) \cong \mathcal{B}(P(Z)/(J\restriction A) * j(\mathbb{P})/m)$$
is such that if $G \subseteq P(Z)/(J\restriction A)$ is generic, $G^* = \iota^{-1}(G)$ and $\bar{\imath}$ is the induced isomorphism from
$$\{\mathcal{B}(\mathbb{P} * P(Z)/(\bar{J}\restriction\mathcal{M}))\}/G^* \cong \{\mathcal{B}(P(Z)/(J\restriction A) * j(\mathbb{P})/m)\}/G,$$
then $\bar{\iota}\restriction\mathbb{P} = j\restriction\mathbb{P}$ and is a regular embedding from $\mathbb{P}$ to $j(\mathbb{P})/m$ in $V[G]$.

Master conditions must exist if we assume properties 1–3:

7.28 Proposition. *Suppose that $\mathbb{P}$ is a partial ordering, J is a precipitous ideal on Z and there are $A \in P(Z)$, $f : Z \to \mathbb{P}$ and $\mathcal{M} \in P(Z)^{V^{\mathbb{P}}}$ such that 1–3 of Remark 7.27 hold with $m = [f]^M$.*
Then
$$\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m$$
is a regular embedding.

When the forcing $\mathbb{P}$ preserves the saturation of J, we can succinctly state a converse to the Duality Theorem.

7.29 Theorem. *Let J be a normal, fine, precipitous ideal on $Z \subseteq P(X)$ and $\mathbb{P}$ a partial ordering. Suppose that:*

1. *$\bar{J}\restriction\mathcal{M}$ is normal and $|X|^+$-saturated in $V^{\mathbb{P}}$,*
2. *for all generic $H * \bar{G} \subseteq \mathbb{P} * P(Z)/(\bar{J}\restriction\mathcal{M})$ and all $f : Z \to V$ in $V[H]$ there is a $g : Z \to \mathbb{P}$ in V such that $[h]_{\bar{G}} = [g]_{\bar{G}}$,*
3. *the identity mapping:*
$$\mathbb{P} \times P(Z)/J \to \mathbb{P} * \overline{P(Z)}/(\bar{J}\restriction\mathcal{M})$$
is a regular embedding, and
4. *for all generic $G \subseteq P(Z)/J$, $j(\mathbb{P})$ is $|X|^+$-saturated in $V[G]$.*

Then there is a condition $\dot{m} \in j(\mathbb{P})$ such that the mapping:
$$\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/m$$
is a regular embedding.

Generalizations of the Duality Theorem Without Master Conditions

In later sections it will be convenient to use a generalization of the Duality Theorem. We state a version here (and an even more general version in the remarks following it).

7.30 Theorem. *Let J be a precipitous ideal on a set Z and $\mathbb{P}$ be a partial ordering. Let $\dot{F}$ be a $P(Z)/J$-term for a filter on $j(\mathbb{P})$. Suppose that:*

1. $\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/\dot{F}$ *is a regular embedding,*
2. $\dot{F}$ *is generated by* $\langle \dot{m}_\alpha : \alpha < \gamma \rangle$,
3. $A \in J^+$ *is such that for each* α, $A \Vdash [f_\alpha]_M = \dot{m}_\alpha$, *and*
4. $A \Vdash_{P(Z)/J}$ *if* $\hat{H} \subseteq j(\mathbb{P})/\dot{F}$ *is* $V[G]$*-generic, then* $\hat{H} \subseteq j(\mathbb{P})$ *is generic over* M.[66]

Let $H \subseteq \mathbb{P}$ be generic and $\mathcal{J}_F \in V[H]$ be the ideal generated over $\bar{J}$ by the sets $\mathcal{M}_\alpha = \{z \in A : f_\alpha(z) \notin H\}$. Then there is a canonical isomorphism ι witnessing:

$$\mathcal{B}(\mathbb{P} * P(Z)/\mathcal{J}_F) \cong \mathcal{B}(P(Z)/(J{\restriction}A) * j(\mathbb{P})/\mathcal{F}).$$

We can weaken our assumptions of the generalized duality theorem further. Suppose that:

1. $\dot{F}$ is a $P(Z)/J$-term for a filter on $j(\mathbb{P})$ generated by a collection of terms T,
2. $\mathrm{id} \times \dot{j} : P(Z)/J \times \mathbb{P} \to P(Z)/J * j(\mathbb{P})/\dot{F}$ is a regular embedding, and
3. if $\hat{H} \subseteq j(\mathbb{P})/\dot{F}$ is generic then $\hat{H} \subseteq j(\mathbb{P})$ is generic over M.

Then if $H \subseteq \mathbb{P}$ is generic, there is a minimal ideal $\mathcal{J}_F \in V[H]$ such that for all $A \in J^+$, $\tau \in T$ if $A \Vdash f^\tau_A = \tau$, then $\{z \in A : f^\tau_A(z) \notin H\} \in \mathcal{J}_F$. For this ideal, there is a canonical isomorphism ι witnessing:

$$\mathcal{B}(\mathbb{P} * P(Z)/\mathcal{J}_F) \cong \mathcal{B}(P(Z)/(J{\restriction}A) * j(\mathbb{P})/\mathcal{F}).$$

7.5. Pseudo-Generic Towers

In this section we discuss a technique for showing strong properties of an ideal closely related to a master condition ideal.

In the typical situation where this technique is used, we have a large cardinal embedding $j : V \to M$ and $M^\lambda \subseteq M$. We will force over V to get

[66] Where $G \subseteq P(Z)/J$ is the canonical term for the V-generic filter.

$G * H \subseteq \mathbb{P} * \mathbb{Q}$ and $\hat{G} * \hat{H} \subseteq j(\mathbb{P} * \mathbb{Q})$ and extend the embedding to a

$$\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}].$$

If $j(\mathbb{P})$ is λ-c.c. then $M[\hat{G}]$ is closed under λ-sequences from $V[\hat{G}]$. Suppose now that $j(\mathbb{Q})$ is $<\lambda$-closed over $M[\hat{G}]$. Then, working in $V[\hat{G}]$, for all sets $Z \in V[G * H]$ such that $|P(Z)^{V[G*H]}| \leq \lambda$ and all ultrafilters $U(\hat{j}, i)$ induced by $\hat{j}$ on Z we can build a tower of conditions $\mathcal{T} \subseteq j(\mathbb{Q})$ such that for all $X \in P(Z)^{V[G*H]}$, there is a $p \in \mathcal{T}$ such that $p \parallel$ "$i \in \hat{j}(X)$".

In $V[\hat{G}]$ we define an ultrafilter $\tilde{U}$ on $P(Z)^{V[G*H]}$ by setting $X \in \tilde{U}$ iff there is a $p \in \mathcal{T}$ such that $p \Vdash i \in \hat{j}(X)$. Then $\tilde{U}$ is closed under intersections of $<\kappa$-sequences that lie in $V[G * H]$.

We can phrase this as building the tower $\mathcal{T}$ to be generic for some collection $\mathcal{D}$ of dense sets in $j(\mathbb{Q})$. Such a tower is called *pseudo-generic.*

This is particularly interesting in the case where $i = \eta \in \mathrm{On}$ or $i = j``\mu$ for some ordinal μ. In either of these cases, we can arrange that the tower meets a larger collection of dense sets so that $\tilde{U}$ is closed under diagonal intersections of sequences of sets that lie in $V[G * H]$.

Pseudo-generic tower arguments can be formalized with the following result:

7.31 Lemma (Kunen [79]). *Assume that $\kappa^{<\kappa} = \kappa$. Suppose that $\mathbb{P}$ is a $<\kappa$-closed partial ordering and ϕ is formula in the language of set theory supplemented by an n-ary predicate symbol X. Suppose that $a_1, \ldots, a_n \in H(\kappa)$ and there is a condition $p \in \mathbb{P}$ such that:*

$$p \Vdash \exists X \subseteq H(\kappa)^n, \langle H(\kappa), \in, X\rangle \models \phi(X, a_1, \ldots, a_n).$$

Then $\exists X \subseteq H(\kappa), \langle H(\kappa), \in, X\rangle \models \phi(X, a_1, \ldots, a_n)$.

The proof of this lemma builds a pseudo-generic tower deciding longer and longer initial segments of X for enough dense sets. The cardinal arithmetic assumption guarantees that the list of dense sets to be met is not too large.

7.6. A κ-Saturated Ideal on an Inaccessible Cardinal κ

In this section we prove the following theorem of Kunen:

7.32 Theorem (Kunen [79]). *Let κ be a measurable cardinal. Then there is a partial ordering $\mathbb{P}$ such that for all generic $G \subseteq \mathbb{P}$, $V[G] \models \kappa$ is inaccessible but not weakly compact and there is a κ-complete, κ-saturated ideal on κ.*

Proof. We begin with some preliminaries:

Let λ be a regular cardinal and $\mathcal{T}$ be the collections of sets $(t, <_t)$ with the following properties:

1. $t \subseteq \lambda \times \lambda$, $|t| < \lambda$ and $<_t$ is a tree ordering on t such that the αth level of $<_t$ is $t \cap (\lambda \times \{\alpha\})$,

2. the tree has a "top level", i.e. there is a maximal $\alpha \in \lambda$ such that $t \cap (\lambda \times \{\alpha\}) \neq \emptyset$,
3. every node in the tree at level ν is below at least two nodes at level $\nu + 1$ and some node at the top level of the tree, and
4. for all $\sigma, \tau \in t$ in the same level of $<_t$ there is an automorphism of $(t, <_t)$ sending σ to τ.

For conditions $s, t \in \mathcal{T}$ we define $t \leq_{\mathcal{T}} s$ iff $t \supseteq s$ and $<_t$ is an end extension of $<_s$ and every automorphism of s extends to an automorphism of t. Then $\mathcal{T}$ is ω-closed and $<\lambda$-strategically closed. Further, forcing with $\mathcal{T}$ adds a λ-Suslin tree $\mathbb{T}$ to V.

The following lemma is crucial for Kunen's theorem and is of independent interest:

7.33 Lemma. *Suppose that* $\lambda^{<\lambda} = \lambda$ *and let* $\mathbb{R} = \mathcal{T} * \mathbb{T}$. *Then forcing with* $\mathbb{R}$ *is equivalent to forcing with* $\mathrm{Add}(\lambda)$.

Proof. The Boolean algebra $\mathcal{B}(\mathrm{Add}(\lambda))$ can be characterized as the unique complete Boolean algebra with a dense $<\lambda$-closed subset of cardinality λ. Hence it suffices to show that $\mathcal{T} * \mathbb{T}$ is $<\lambda$-closed and λ-dense.

Let $\mathbb{Q}$ be the partial ordering consisting of pairs (p, b) where $p \in \mathcal{T}$ and b is a branch through the tree of p that includes a node on the top level of p. We let $(q, c) \leq (p, b)$ iff $q \leq_{\mathcal{T}} p$ and c extends b.

A few moments' thought shows that $\mathbb{Q}$ is $<\lambda$-closed and λ-dense, so it suffices to see that forcing with $\mathbb{Q}$ is equivalent to forcing with $\mathcal{T} * \mathbb{T}$.

Let $G \subseteq \mathbb{Q}$ be generic. Then the sequence of first coordinates of conditions in G is a generic filter H for $\mathcal{T}$, and the sequence of second coordinates is a branch through $\mathcal{T}$ of length λ. Since any λ-branch through a λ-Suslin tree is generic, forcing with $\mathbb{Q}$ yields a generic object for $\mathcal{T} * \mathbb{T}$.

The set of conditions D of the form $(p, \check{b}) \in \mathcal{T} * \mathbb{T}$, where $b \in V$ and b has an element on the top level of the tree given by p, is dense in $\mathcal{T} * \mathbb{T}$ and we have seen that the identity embedding from D into $\mathbb{Q}$ is a regular embedding. Since these conditions are dense in $\mathbb{Q}$, the two forcings are equivalent. $\dashv$

Let κ be a measurable cardinal and $j : V \to M$ be the elementary embedding from a normal ultrafilter on κ. Let $\mathbb{P}$ be an iteration of length κ with Easton supports that adds a generic object for $\mathrm{Add}(\alpha)$ at stage α for every inaccessible α and does nothing at other stages. Then $\mathbb{P}$ is κ-c.c. Let G be generic for $\mathbb{P}$ and $H \subseteq \mathrm{Add}(\kappa)^{V[G]}$ be generic.

We claim that κ is measurable in $V[G*H]$. Note that $j(\mathbb{P}) = \mathbb{P}*\mathrm{Add}(\kappa)*\mathbb{R}$, where $\mathbb{R}$ is $<(2^\kappa)^+$-closed. In particular, it is possible to extend $G * H$ to a V-generic object $G * H * K \subseteq j(\mathbb{P})$. Thus in M we can define the master condition $m = \bigcup j``H$ and extend j to a $\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}]$, where $\hat{H} \subseteq \mathrm{Add}(j(\kappa))$ is generic.

By the closure of $\mathbb{R}$, the ultrafilter $U(\hat{j}, \kappa) \in V[G * H]$, and hence κ is measurable in $V[G * H]$.

By Lemma 7.33, we can split $H \sim H_0 * H_1$, where $V[G * H_0]$ has a Suslin tree $\mathbb{T}$ on κ and H_1 is generic for this tree.

The final model is $V' = V[G * H_0]$. In this model we can force with the Suslin tree $\mathbb{T}$ to make κ measurable. If $\dot{\mu}$ is a term for a normal ultrafilter on κ, then $I = \{X \subseteq \kappa : \|X \notin \dot{\mu}\| = 1\}$ is a normal, κ-saturated ideal on κ. This establishes Kunen's theorem. ⊣

7.34 Remark. An application of Proposition 7.13 shows that if I is the ideal defined at the end of the previous proof, then forcing with $P(\kappa)/I$ is equivalent to forcing with the Suslin tree $\mathbb{T}$.

Kunen's result extends to ideals on $[\lambda]^{<\kappa}$ as follows: Let κ be a supercompact cardinal and let $f : \kappa \to \kappa$ be a function such that for all $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ that is λ-supercompact and $j(f)(\kappa) > \lambda$.[67] To get a κ-saturated ideal on $[\lambda]^{<\kappa}$, start with a 2^λ-supercompact elementary embedding j with $j(f)(\kappa) > 2^\lambda$. Modify the previous iteration by only forcing at inaccessibles α that are closed under the function f.

Using the notation above, the partial ordering $\mathbb{R}$ is $<(2^\lambda)^+$-closed, and hence the induced ultrafilter $U(\hat{j}, j``\lambda)$ on $Z = ([\lambda]^{<\kappa})^{V[G*H]}$ lies in $V[G*H]$. The rest of the argument is identical.

7.7. Basic Kunen Technique: κ^+-Saturated Ideals

We saw in Sect. 6 that it was impossible to have a κ-saturated, κ-complete ideal on a successor cardinal κ. In this subsection we describe Kunen's technique for producing an $\aleph_2$-saturated ideal on ω_1.

This technique and its many variations has proved to be the main tool for producing saturated ideals on accessible cardinals, with one notable exception detailed in Sect. 8.

The basic idea behind Kunen's argument is to start with a huge embedding $j : V \to M$ with critical point κ and $j(\kappa) = \lambda$ and collapse κ to be $\aleph_1$ and λ to be $\aleph_2$ by a two stage forcing $\mathbb{P} * \mathbb{S}$. In order to find an induced ideal, we must be able to extend the embedding j in the resulting model. A basic obstacle arises in that the generic object for $\mathbb{S}$ has cardinality λ which is $\aleph_1$ is $M^{j(\mathbb{P})}$. But $\mathbb{S}$ can be at most countably closed and so the master conditions do not need to exist by virtue of the closure alone.[68] In addition to closure, Kunen used the "shape" of the conditions in a partial ordering invented by Silver to show that a master condition exists.

7.35 Definition. Let $\kappa < \lambda$ be regular cardinals and λ inaccessible. The *Silver Collapse* of λ to be κ^+ is the collection of functions p with domain included in $\kappa \times \lambda$ that map into λ and satisfy:

[67] Such a function always exists, see e.g. [82].

[68] In Sect. 7.11 we will see a method of Magidor for circumventing this problem.

1. $|\operatorname{dom}(p)| \leq \kappa$ and there is an $\eta < \kappa$ such that the domain of p is a subset of $\eta \times \lambda$, and
2. for all $(\alpha, \beta) \in \operatorname{dom}(p)$, $p(\alpha, \beta) < \beta$.

The ordering on the conditions in the Silver Collapse is reverse inclusion.

For inaccessible λ, we will denote the Silver collapse of λ to be κ^+ by $\mathbb{S}(\kappa, \lambda)$. It is easily checked that it is λ-c.c., $<\kappa$-closed and makes every ordinal between κ and λ have cardinality κ.

7.36 Theorem (Kunen [79]). *Suppose that $j : V \to M$ is a huge embedding with critical point κ and $j(\kappa) = \lambda$. Then there is a κ-c.c. partial ordering $\mathbb{P}$ such that if $G * H \subseteq \mathbb{P} * \mathbb{S}(\kappa, \lambda)$ is generic then $V[G * H]$ satisfies the statements $\kappa = \omega_1$, $\lambda = \omega_2$ and there is an $\aleph_2$-saturated ideal on ω_1.*

Proof. Kunen's proof starts by building a "highly saturated" κ-c.c. partial ordering $\mathbb{P} \subseteq V_\kappa$.[69] Since there are many variations on Kunen's actual construction, we begin by describing the cogent properties that allow it to be generalized. At the end of the discussion we will give a specific example of this type of construction.

The saturation property desired of $\mathbb{P}$ is that if $\mathbb{Q}$ is a regular subordering of $\mathbb{P}$ of inaccessible cardinality $\alpha < \kappa$, then there is a regular embedding $i : \mathbb{Q} * \mathbb{S}(\alpha, \kappa) \to \mathbb{P}$ extending the identity mapping of $\mathbb{Q}$ into $\mathbb{P}$.

Suppose that we have succeeded in constructing such a $\mathbb{P}$. We let $G \subseteq \mathbb{P}$ be generic over V. By the κ-c.c. of $\mathbb{P}$ we can find a $\hat{G} \subseteq j(\mathbb{P})$ that is $V[G]$-generic and an extension of j to a $j_1 : V[G] \to M[\hat{G}]$. Moreover, $M[\hat{G}]^\lambda \cap V[\hat{G}] \subseteq M[\hat{G}]$.

Note that $j \restriction \mathbb{P}$ is the identity mapping, since $\mathbb{P} \subseteq V_\kappa$. By the κ-c.c. of $\mathbb{P}$, j is a regular embedding, and hence $\mathbb{P}$ is a regular subordering of $j(\mathbb{P})$. Using the definition of $j(\mathbb{P})$, there is a regular embedding i extending j of $\mathbb{P} * \mathbb{S}(\kappa, \lambda)$ into $j(\mathbb{P})$. In $M[\hat{G}]$ we have a generic object $H \subseteq \mathbb{S}(\kappa, \lambda)$ over $V[G]$. Our final model is $V[G * H]$.

We want to extend the embedding j. Let $m = \bigcup j``H$. We claim that $m \in \mathbb{S}(\lambda, j(\lambda))^{M[\hat{G}]}$. Note that m has the right cardinality, namely λ. Moreover, if $p \in H$, then there is an ordinal $\eta < \kappa$ such that the domain of p is a subset of $\eta \times \lambda$. Hence the domain of $j_1(p)$ is a subset of $\eta \times j(\lambda)$. Thus we see that the domain of m is a subset of $\kappa \times j(\lambda)$ and so m has the right shape. It is now easy to verify that m is a condition in $\mathbb{S}(\lambda, j(\lambda))^{M[\hat{G}]}$. Forcing below m in the partial ordering $\mathbb{S}(\lambda, j(\lambda))$ we get a generic $\hat{H}$ and an extension of j to a $\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}]$.

We are now ready to apply the technique of pseudo-generic towers to find an ultrafilter $\tilde{U}$ on $P(\kappa)^{V[G*H]}$ in $V[\hat{G}]$ mimicking the properties of $U(\hat{j}, \kappa)$. In particular the ultrafilter $\tilde{U}$ is normal and κ-complete for sequences of sets that lie in $V[G * H]$.

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[69] This is a different sense of *saturation* than "chain condition", related to the model-theoretic idea of saturation.

Working in $V[G * H]$, let I be the collection of $X \subseteq \kappa$ such that $\|X \in \tilde{U}\|_{j(\mathbb{P})/(G*H)} = 0$. Then I is normal and κ-complete and since $j(\mathbb{P})/G * H$ is $\lambda = \kappa^+$-c.c. the ideal is κ^+-saturated.

We now give an explicit example of such a construction. $\mathbb{P}$ will be an iteration-with-amalgamation of length κ with finite supports. To get the construction of $\mathbb{P}$ started take $\mathbb{P}_0 = \mathrm{Col}(\omega, <\kappa)$. At a typical α, $\mathbb{P}_\alpha$ will have been defined. There will be a a regular subordering $\mathbb{Q}_\alpha$ of $\mathbb{P}_\alpha$ having cardinality less than or equal to α. Then $\mathbb{P}_{\alpha+1}$ is:

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{S}^{\mathbb{Q}_\alpha}(\alpha, \kappa)$$

where this is defined to consist of pairs (p, τ) where $p \in \mathbb{P}_\alpha$ and τ is a $\mathbb{Q}_\alpha$ term for an element of $\mathbb{S}(\alpha, \kappa)^{V^{\mathbb{Q}_\alpha}}$. Since $\mathbb{Q}_\alpha$ is a regular subordering of $\mathbb{P}_\alpha$, the partial ordering $\mathbb{S}(\alpha, \kappa)^{V^{\mathbb{Q}_\alpha}}$ has a canonical realization in $V^{\mathbb{P}_\alpha}$ and the ordering on $\mathbb{P}_{\alpha+1}$ is defined accordingly.[70]

To finish the description, we take a sequence $\langle \mathbb{Q}_\alpha : \alpha < \kappa \rangle$ to be a sequence of partial orderings dovetailed so that every regular subordering of any $\mathbb{P}_\beta$ for $\beta < \kappa$ occurs as $\mathbb{Q}_\alpha$ for cofinally many $\alpha < \kappa$. $\dashv$

Laver remarked that instead of finite supports, Kunen's construction works with countable or Easton supports. In particular, if you start Kunen's construction with $\mathrm{Col}(\omega_1, <\kappa)$ and iterate with countable supports you can get an $\aleph_3$-saturated ideal on ω_2.

Hence, as a corollary of Kunen's techniques, we get:

7.37 Corollary (Laver using [79]). *Suppose that μ is a regular cardinal less than a huge cardinal κ. Then there is a μ-directed closed partial ordering $\mathbb{P}$ such that for all generic $G \subseteq \mathbb{P}$,*

$$V[G] \models \textit{ there is a } \mu^{+2}\textit{-saturated ideal on } \mu^+.$$

As remarked by the author in [48], this proof combined with Proposition 7.13 gives more:

7.38 Corollary. *Suppose that μ is a regular cardinal less than a huge cardinal κ. Then there is a partial ordering $\mathbb{P}$ such that for all generic $G \subseteq \mathbb{P}$,*

$$V[G] \models \textit{ there is a layered ideal on } \mu^+.$$

Proof. The partial ordering $\mathbb{P} = \mathbb{P}_\kappa$ in Kunen's proof is κ-c.c. and $\mathbb{P} \subseteq V_\kappa$. Hence $j : \mathbb{P} \to j(\mathbb{P})$ is a regular embedding and $M \models j(\mathbb{P}) \cap V_\kappa$ is a regular subordering of $j(\mathbb{P})$.

By elementarity we see that there is a stationary $R \subseteq \kappa$ such that for all $\alpha \in R$, $\mathbb{P} \cap V_\alpha$ is a regular subordering of V_κ and $\mathbb{P} \cap V_\alpha$ has the α-c.c. Since M is closed under $<\lambda$-sequences we see that for all $\alpha \in j(R)$, $j(\mathbb{P}) \cap V_\alpha$ is a regular subordering of $j(\mathbb{P})$ and has the α-c.c.

[70] An alternate description of $\mathbb{P}_{\alpha+1}$ is $\mathbb{Q}_\alpha * (\mathbb{P}_\alpha/\mathbb{Q}_\alpha \times S(\alpha, \kappa))$.

Standard arguments then show that if i is the canonical embedding from $\mathbb{P} * \mathbb{S}(\kappa, \lambda)$ into $j(\mathbb{P})$ given in the Kunen construction and $G * H \subseteq \mathbb{P} * \mathbb{S}(\kappa, \lambda)$ is generic, then $V[G * H] \models$

$$\text{for } \alpha \in j(R),\ (j(\mathbb{P})/G * H) \cap V_\alpha \text{ is a regular subordering of } j(\mathbb{P})/G * H.$$

Let $\langle \mathcal{B}_\alpha : \alpha < \lambda \rangle$ be an increasing, continuous sequence of Boolean subalgebras of $\mathcal{B}(j(\mathbb{P})/G * H)$. Then there is a closed unbounded set C of α, for all $\alpha \in C$, $(j(\mathbb{P})/G * H) \cap V_\alpha$ is dense in $\mathcal{B}_\alpha$. In particular, for $\alpha \in C \cap j(R)$, $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}$.

Applying Proposition 7.13 to Kunen's ideal I, we see that $P(\kappa)/I$ is isomorphic to $\mathcal{B}$, and hence I is a layered ideal. ⊣

Kunen pointed out that these models also satisfy various Chang's Conjectures: Using the notation of the proof of Kunen's theorem and assuming we are doing Laver's variation, let

$$\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}]$$

be the generic embedding. Suppose further that $V[G * H] \models \kappa = \mu^+$. Let $\mathfrak{A}$ be a structure in a countable language whose domain is $\kappa^+ = \lambda$. Then $j``\lambda$ is the domain of an elementary substructure of $j(\mathfrak{A})$, and $M[\hat{G} * \hat{H}] \models |j``\lambda| = \mu^+$ and $|j``\lambda \cap j(\kappa)| = \mu$. By the elementarity of j,

$$V[G * H] \models \text{there is an elementary substructure of } \mathfrak{A} \text{ of type } (\mu^+, \mu).$$

To summarize, we see that in a model built by the Kunen technique $(\mu^{+2}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ holds:

7.39 Corollary. *Suppose that μ is a regular cardinal less than a huge cardinal κ. Then for all finite $n > 0$ there is a partial ordering $\mathbb{P}_n$ such that for all generic $G \subseteq \mathbb{P}_n$,*

$$V[G] \models (\aleph_{n+1}, \aleph_n) \twoheadrightarrow (\aleph_n, \aleph_{n-1}).$$

7.8. $(\aleph_2, \aleph_2, \aleph_0)$-Saturated Ideals

In this section we outline Laver's result giving the consistency of the existence of $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideals. Though Woodin constructed an ideal with stronger saturation properties from an almost huge cardinal (Theorem 7.60), Laver's construction retains certain advantages. The primary one is that it does not require passing to an inner model to find an ideal satisfying the desired conditions.

Laver modified the Silver collapse still further to give the "Eastonized" version. Define a set $X \subseteq \sigma \in \mathrm{On}$ to be δ-*Easton* iff for all regular ρ between δ and σ, $|X \cap \rho| < \rho$. Let $\mathbb{L}(\delta, \sigma)$ be the subset of $\prod_{\alpha<\sigma} \mathrm{Col}(\delta, \alpha)$ consisting of those p such that the support of p is a δ-Easton subset of σ and there is a $\xi < \delta$ such that for all $\alpha \in \mathrm{supp}(p)$, $p(\alpha) \subseteq \xi \times \alpha$. Explicitly, $p \in \mathbb{L}(\delta, \sigma)$ iff p is a partial function from $\sigma \times \delta$ to σ such that for all α, ζ we have

$p(\alpha,\zeta) < \alpha$, $\{\alpha : \text{there is a } \zeta, (\alpha,\zeta) \in \mathrm{dom}(p)\}$ is δ-Easton and for some $\xi < \delta$, $\mathrm{dom}(p) \subseteq \sigma \times \xi$.[71]

The ordering of $\mathbb{L}(\delta,\sigma)$ is inclusion. It is routine to check that this partial ordering is $<\delta$-closed, is σ-c.c. for σ Mahlo and collapses σ to be δ^+. If $p \in \mathbb{L}(\delta,\sigma)$ and $\alpha < \sigma$ we let $p\restriction\alpha$ denote the condition $p \cap (\alpha \times \delta \times \alpha)$.

7.40 Theorem (Laver [83]). *Suppose that there is a huge cardinal κ and $\mu < \kappa$ is regular. Then there is a forcing extension in which $\kappa = \mu^+$ and there is a normal κ-complete ideal I on κ that is (κ^+,κ^+,μ)-saturated.*

Proof. Let $j : V \to M$ be the embedding with critical point κ and $j(\kappa) = \lambda$. Following the general Kunen outline we build an iteration-with-amalgamation $\mathbb{P}$ of length κ with μ-Easton supports.[72] Let $\mathbb{P}_0 = \mathbb{L}(\mu,\kappa)$. At stage α, if $V_\alpha \cap \mathbb{P}_\alpha$ is a regular subordering of $\mathbb{P}_\alpha$, then we let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{L}^{\mathbb{P}_\alpha \cap V_\alpha}(\alpha,\kappa)$. As in the Kunen construction, the αth coordinate in an element of $\mathbb{P}_{\alpha+1}$ is a $\mathbb{P}_\alpha \cap V_\alpha$-term for an element of $\mathbb{L}(\alpha,\kappa)$. Otherwise $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * 1$, where 1 is the trivial partial ordering. We note that at each stage in the iteration we are forcing with terms that lie in some $V^{\mathbb{Q}}$, where $|\mathbb{Q}| < \kappa$.[73]

The main point of this version of the construction is that $\mathbb{P}$ has a particularly strong chain condition property. At each stage of the forcing we are using terms from an extension of V by a small forcing. Hence we see that for all $X \in [\mathbb{P}]^\kappa$ there are arbitrarily large $\beta < \kappa$ and $Y \in [X]^\kappa$ such that the following strong chain condition property holds:

1. The collection $\{\mathrm{supp}(p) : p \in Y\}$ forms a Δ-system with kernel contained in β,
2. if $\alpha \in \beta$ and $\mathbb{P}_{\alpha+1} \neq \mathbb{P}_\alpha * 1$, then $1 \Vdash_{\mathbb{P}_\alpha \cap V_\alpha} \{p(\alpha) : p \in Y\}$ forms a Δ-system with kernel contained in $\beta \times \beta \times \beta$, and
3. if $p, q \in Y$ and $\alpha < \beta$ then $1 \Vdash_{\mathbb{P}_\alpha \cap V_\alpha} p(\alpha)\restriction\beta = q(\alpha)\restriction\beta$.

If β is regular and Y is a collection of conditions that satisfy 1–3, then for any $Z \in [Y]^{<\beta}$ there is a $q \in \mathbb{P}$ such that for all $p \in Z$:

- $q \leq p$,
- for $\alpha < \beta$, $q(\alpha)\restriction\beta = p(\alpha)\restriction\beta$, and
- for $\alpha \geq \beta$ in the support of q there is a unique $p \in Z, q(\alpha) = p(\alpha)$.

Since $\mathbb{P}$ is κ-c.c. and $j\restriction\mathbb{P}$ is the identity map, $\mathbb{P}$ is a regular subordering of $j(\mathbb{P})$. By the interpretation of the construction in M, $\mathbb{P} * \mathbb{L}(\kappa,\lambda)$ is regularly embedded into $j(\mathbb{P})$ via a canonical map i that is the identity on $\mathbb{P}$ and sends

[71] This description is analogous to the description of the Silver collapse with the domains of the conditions "turned sideways" from the domains given in Definition 7.35.

[72] These are supports S with the property that for all regular $\alpha > \mu$, $|S \cap \alpha| < \alpha$.

[73] We are viewing conditions in an iteration to be partial functions defined on their supports. Thus $\mathbb{P} \subseteq V_\kappa$ and $j(\mathbb{P}) \cap V_\kappa = \mathbb{P}$.

terms in $\mathbb{L}(\kappa,\lambda)$ to the κth-coordinate of $j(\mathbb{P})$. If $\hat{G}\subseteq j(\mathbb{P})$ is generic, i gives a V-generic $G*H\subseteq\mathbb{P}*\mathbb{L}(\kappa,\lambda)$. Then $m=\bigcup j``H\in L^{j(\mathbb{P})}(\lambda,j(\lambda))$, and if $\hat{H}$ is generic with $m\in\hat{H}$ we can extend j to $\hat{j}:V[G*H]\to M[\hat{G}*\hat{H}]$. Moreover in this model $\kappa=\mu^+$, and $\lambda=\kappa^+$.

Using a pseudo-generic tower argument, in $V[\hat{G}]$, we can build a $V[G*H]$-normal and κ-complete ultrafilter U on $P(\kappa)^{V[G*H]}$. As usual we can define a normal κ-complete ideal I in $V[G*H]$ by setting $A\in I$ iff $\|A\in U\|_{j(\mathbb{P})/G*H}=0$. It is this ideal I that we claim is (κ^+,κ^+,μ)-saturated.

If not, let $(r,l)\in\mathbb{P}*\mathbb{L}(\kappa,\lambda)$ and $\langle\tau_\alpha:\alpha<\lambda\rangle$ be a sequence of $\mathbb{P}*\mathbb{L}(\kappa,\lambda)$-terms such that $(r,l)\Vdash$ "$\langle\tau_\alpha:\alpha<\lambda\rangle$ is a counterexample to (λ,λ,μ) saturation". Then we can find a sequence of conditions $\langle p_\alpha:\alpha<\lambda\rangle\in j(\mathbb{P})$ such that $p_\alpha\leq i(r,l)$ and $p_\alpha\Vdash_{j(\mathbb{P})}\tau_\alpha\in U$.

Since j is a huge embedding, $j(\mathbb{P})$ has the strong chain condition property mentioned above in V. Applying the strong chain condition we get a $Y\subseteq[\lambda]^\lambda$ and a $\beta\geq\kappa$ satisfying 1–3. We can find an $r_0\in\mathbb{P}$ such that for all $\gamma<\kappa$, $\alpha\in Y$, $p_\alpha(\gamma)\restriction\kappa=r_0(\gamma)$. Moreover, $r_0\Vdash_{\mathbb{L}(\kappa,\lambda)}|\{\alpha\in Y:p_\alpha(\kappa)\in H\}|=\lambda$. Let $Y'=\{\alpha\in Y:p_\alpha(\kappa)\in H\}$. We claim that $\{\tau_\alpha^{V[G*H]}:\alpha\in Y'\}$ is the witness to (λ,λ,μ)-saturation.

Let $(r_1,l_1)\leq(r_0,l)$ be such that $(r_1,l_1)\Vdash_{\mathbb{P}*\mathbb{L}(\kappa,\lambda)}Z\in[Y']^\mu$. Then $Z\in V[G]$. In V, we can find a collection $Z'\in[Y]^{<\kappa}$ such that $r_1\Vdash Z\subseteq Z'$ and for all $\alpha\in Z'$, $r_1\Vdash_{\mathbb{P}}$ "l_1 and $p_\alpha(\kappa)$ are compatible". By 1–3 there is a $q\in j(\mathbb{P})$ such that:

- for $\gamma<\kappa$, $q(\gamma)\restriction\kappa=r_1(\gamma)$,
- $q(\kappa)\leq l_1$, and
- for all $\alpha\in Z'$, $q\leq p_\alpha$.

Then $r_1\Vdash_{j(\mathbb{P})}$ "$q(\kappa)\in\mathbb{L}(\kappa,\lambda)$ and $q(\kappa)\leq l_1$". Thus, $(r_1,q(\kappa))\Vdash_{\mathbb{P}*\mathbb{L}(\kappa,\lambda)}q\in j(\mathbb{P})/G*H$. But for all $\alpha\in Z'$, $q\Vdash_{j(\mathbb{P})}\tau_\alpha\in U$ and thus $q\Vdash_{j(\mathbb{P})}\bigcap_{\alpha\in Z'}\tau_\alpha\in U$. Hence, $V[G*H]\models\bigcap_{\alpha\in Z}\tau_\alpha\notin I$. ⊣

7.9. Chang Ideals with Simple Quotients I

The Kunen technique also allows "skipping cardinals". This useful variant is a component of many proofs. Fix a regular cardinal $\mu<\kappa$ and a successor ordinal $\gamma<\kappa$. We follow the outline of the Kunen proof. We begin by constructing a partial ordering $\mathbb{P}$. This partial ordering will be a κ-stage iteration with $<\mu$-supports. The first stage of $\mathbb{P}$ is $\mathrm{Col}(\mu,<\kappa)$. For regular suborderings $\mathbb{Q}$ of $\mathbb{P}$ that have cardinality $\alpha<\kappa$ we arrange that there is a regular embedding $i:\mathbb{Q}*\mathbb{S}(\alpha^{+\gamma},\kappa)\to\mathbb{P}$ extending the identity mapping of $\mathbb{Q}$ into $\mathbb{P}$. The result is that $\mathbb{P}$ is a κ-c.c. partial ordering that collapses κ to μ^+.

The final forcing is $\mathbb{P}*\mathbb{S}(\kappa^{+\gamma},\lambda)$. This forcing makes $\kappa=\mu^+$ and $\lambda=\kappa^{+\gamma+1}$. As in the original Kunen construction, if $G*H\subseteq\mathbb{P}*\mathbb{S}(\kappa^{+\gamma},\lambda)$ is generic then there is a generic $\hat{G}*\hat{H}$ such that the huge embedding j can be extended to $\hat{j}:V[G*H]\to M[\hat{G}*\hat{H}]$.

If we assume that the original embedding $j : V \to M$ is somewhat stronger than a huge cardinal, namely that $M^{\lambda^{+\gamma}} \subseteq M$, then the Kunen argument for Chang's Conjecture gives more; it shows that for all $\delta \leq \gamma$,

$$(\mu^{+1+\gamma+1+\gamma}, \mu^{1+\gamma+1+\delta}, \mu^{+1+\gamma+1}, \mu^{+1+\gamma}) \twoheadrightarrow (\mu^{+1+\gamma}, \mu^{+1+\delta}, \mu^{+}, \mu)$$

holds in this model. Taking $\gamma = n$ and $\mu = \omega_k$, one gets the following:

7.41 Corollary. *Suppose that there is an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)^{+n}} \subseteq M$. Then for each $k \in \omega$ there is a forcing extension in which*

$$(\aleph_{2n+k+2}, \aleph_{n+k+1}) \twoheadrightarrow (\aleph_{n+k+1}, \aleph_k).$$

Taking $\gamma = \omega + 1, \delta = \omega$ and $\mu = \omega$ one sees:

7.42 Corollary. *Suppose that there is an elementary embedding $j : V \to M$ with critical point κ such that $M^{j(\kappa)^{+\omega+1}} \subseteq M$. Then there is a forcing extension in which*

$$(\aleph_{\omega+\omega+1}, \aleph_{\omega+\omega}) \twoheadrightarrow (\aleph_{\omega+1}, \aleph_\omega).$$

Using arguments very similar to Proposition 3.9, we see that in a model W where $(\aleph_{\omega+\omega+1}, \aleph_{\omega+\omega}) \twoheadrightarrow (\aleph_{\omega+1}, \aleph_\omega)$, every algebra $\mathfrak{A}$ on $\aleph_{\omega+\omega+1}$ has an elementary substructure $\mathfrak{B}$ of size $\aleph_{\omega+1}$ with $\aleph_\omega \subseteq \mathfrak{B}$. Starting from this property, Levinski, Magidor and Shelah in [86] showed that if $G \subseteq \mathrm{Col}(\omega, \aleph_\omega)$ is generic over W, then in the resulting model $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

We note that the "skipping cardinals" technique is very flexible. It is no longer necessary to force with the Silver collapse, as the geometry of the conditions is no longer relevant. Indeed any $\kappa^{+\gamma}$-closed forcing, such as the Levy collapse, works.

Also, there is no need to fix γ. We can use a function $f : \kappa \to \kappa$ and force with $\mathbb{S}(\alpha^{+f(\alpha)}, \kappa)$ in the construction of $\mathbb{P}$. For the final model we force with $\mathbb{P} * \mathbb{S}(\kappa^{j(f)(\kappa)}, \lambda)$. The partial ordering $\mathbb{P}$ need not have fixed supports, but can have Easton or many other types of support.

$\aleph_3$-Dense Ideals on ω_3

Magidor [89] showed how to use the skipping cardinals technique to get ideals with simple quotients on sets of the form $[\lambda]^\kappa$ where there is a gap between λ and κ.

We start with a model of GCH and a huge embedding j and skip γ cardinals, so that our forcing is $\mathbb{P} * \mathbb{S}(\kappa^{+\gamma}, \lambda)$. Let $G * H \subseteq \mathbb{P} * \mathbb{S}(\kappa^{+\gamma}, \lambda)$ be generic and $\hat{G} \subseteq j(\mathbb{P})$ be generic extending i"$G * H$, where i is the regular embedding of $\mathbb{P} * \mathbb{S}(\kappa^{+\gamma}, \lambda)$ into $j(\mathbb{P})$.

By GCH, $|P([\lambda]^\kappa)^{V[G*H]}| = \lambda^+$ in $V[\hat{G}]$. Since $j(\mathbb{S}(\kappa^{+\gamma}, \lambda))$ is λ^+-closed, in $V[\hat{G}]$ we can build a pseudo-generic tower $\mathcal{T} \subseteq j(\mathbb{S}(\kappa^{+\gamma}, \lambda))$ such that for all $A \in P([\lambda]^\kappa)^{V[G*H]}$ there is a $q \in \mathcal{T}$ deciding the statement j"$\lambda \in j(A)$, and for each regressive function $f : [\lambda]^\kappa \to \lambda$ that lies in V there is a $q \in \mathcal{T}$ deciding the value of $j(f)(j$"$\lambda)$.

Let $\tilde{U} \in V[\hat{G}]$ be the ultrafilter on $P([\lambda]^\kappa)^{V[G*H]}$ defined by setting $A \in \tilde{U}$ iff there is a $q \in \mathcal{T}$ such that $q \Vdash j``\lambda \in j(A)$. Then $\tilde{U}$ is $V[G*H]$ κ-complete, fine and normal for functions that lie in $V[G*H]$. Working in $V[G*H]$, let I be the collection of $X \subseteq [\lambda]^\kappa$ such that $\|X \in \tilde{U}\|_{j(\mathbb{P})/(G*H)} = 0$. Applying Proposition 7.13, we conclude that in $V[G*H]$,

$$\mathbb{B}(P([\lambda]^\kappa)/I) \cong \mathbb{B}(\mathbb{P}).$$

In particular, $P([\lambda]^\kappa)/I$ has a dense set of size λ.

From this we conclude:

7.43 Theorem (Magidor [89]). *Suppose that $\mu < \kappa$ are regular, κ is huge, and* GCH *holds. Let $\gamma < \kappa$ be a successor ordinal. Then there is a forcing extension in which $\kappa = \mu^+$, $\lambda = \kappa^{+\gamma+1}$ and there is a normal, fine, κ-complete, λ-dense, λ-saturated ideal on $[\lambda]^\kappa$.*

Magidor's original paper only claimed that the ideal is λ-centered; however the slight additional strength stated is immediate from Proposition 7.13.

This theorem is most striking when μ and γ are small:

7.44 Corollary. *Suppose that there is a huge cardinal and* GCH *holds. Then there is a forcing extension in which there is a normal, fine, countably complete, $\aleph_3$-dense, $\aleph_3$-saturated ideal on $[\omega_3]^{\omega_1}$.*

In particular, by projecting to ω_3 one sees that it is consistent that there is a countably complete, uniform $\aleph_3$-dense ideal on ω_3.

The author notes that a similar projection argument also shows that in Magidor's model, if we make $\kappa = \omega_1$, $\lambda = \omega_3$ as in the previous corollary, then in the resulting model we get a normal, fine, countably complete, $\aleph_3$-saturated ideal on $[\omega_2]^{<\omega_1}$. Similar results hold for general κ and γ.

7.10. Higher Chang's Conjectures and $\aleph_\omega$ Jónsson

In this section we continue the discussion begun in Sect. 5.2. The Kunen technique of extending generic elementary embeddings comes tantalizingly close to showing the consistency of $\aleph_\omega$ being Jónsson. Recall the definition in Sect. 5.2 of the critical sequence $\langle \kappa_n : n \in \omega \rangle$. By the result of Sect. 5.2, if there is a normal, fine, countably complete ideal on $[\kappa_\omega]^{\kappa_\omega}$, then κ_ω is Jónsson. It is natural to attempt to try to force the existence of such an ideal on $\aleph_\omega$ with the κ_n's being a subsequence of the ω_n's.

Indeed it suffices to start with an elementary embedding $j : V \to M$ such that $V_{\kappa_\omega} \subseteq M$, and force to make the κ_n's a subsequence of the ω_n's in such a way that the embedding j can be extended generically. To date this program has limited success:

7.45 Theorem (Foreman [33]). *Suppose that $j : V \to M$ is a 2-huge embedding and $1 \le n_0 < n_1 < n_2$ are elements of ω such that $n_2 - n_1 \ge n_1 - n_0$. Then there is a forcing extension $\mathbb{P} * \mathbb{R} * \mathbb{S}$ such that:*

1. *$\mathbb{P} * \mathbb{R} * \mathbb{S}$ is $<\omega_{n_0-1}$-closed and κ_2-c.c.,*
2. *$\mathbb{S}$ is $<\kappa_1^{+n_2-n_1-1}$-closed in $V^{\mathbb{P}*\mathbb{R}}$,*
3. *In $V^{\mathbb{P}*\mathbb{R}*\mathbb{S}}$, $\kappa_i = \omega_{n_i}$, and*
4. *If $G * H * K \subseteq \mathbb{P} * \mathbb{R} * \mathbb{S}$ is generic then there are $\hat{G} * \hat{H} * \hat{K} \subseteq j(\mathbb{P} * \mathbb{R} * \mathbb{S})$ such that j can be extended to a*

$$\hat{j} : V[G * H * K] \to M[\hat{G} * \hat{H} * \hat{K}].$$

From this we get the following corollaries:

7.46 Corollary. *Suppose that having a 2-huge cardinal is consistent. Then for $1 \leq n_0 < n_1 < n_2$ such that $n_2 - n_1 \geq n_1 - n_0$ it is consistent that*

$$(\aleph_{n_2}, \aleph_{n_1}, \aleph_{n_0}) \twoheadrightarrow (\aleph_{n_1}, \aleph_{n_0}, \aleph_{n_0-1}).$$

Using a standard pseudo-generic tower argument:

7.47 Corollary. *Suppose that having a 2-huge cardinal is consistent. Then for $1 \leq n$ it is consistent that $[\omega_{n+1}]^{\omega_n}$ carries a normal, fine, $\aleph_n$-complete, $\aleph_{n+2}$-saturated ideal.*

The proof of Theorem 7.45 is a quite complicated adaptation of the Kunen technique and beyond the scope of this chapter. An attempt to extend Theorem 7.45 to 3-huge cardinals and beyond runs into serious technical difficulties known as "ghost coordinates". This approach has not been successful to date.

7.11. The Magidor Variation

In this section we present a development of Kunen's technique invented by Magidor. The method allows one to use the Kunen partial ordering at an *almost* huge cardinal κ to produce saturated ideals. Indeed Magidor's method of avoiding the use of master conditions allows the Kunen partial ordering to be simplified. The Levy collapse can be substituted for the Silver collapse, as the function of the Silver collapse was to allow the construction of a master condition. What is lost, however, is the fact that Chang's Conjecture holds in the resulting extensions.

We will use elaborations of Magidor's techniques in Sect. 8.2 where we present forcing constructions making various natural ideals saturated. We present here a version from [40] that we will use for the more demanding situations in Sect. 8.2.

Frequently, given $j : V \to M$, a partial ordering $\mathbb{P}$ and a V-generic $G \subseteq \mathbb{P}$, we will want to extend j to an elementary embedding $\hat{j} : V[G] \to M[H]$. Magidor realized that it is only necessary that H be M-generic, rather than fully V-generic. In some situations this allows us to extend j without having a master condition.

The embedding $j : \mathcal{B}(\mathbb{P}) \to \mathcal{B}^M(j(\mathbb{P}))$ is a Boolean homomorphism. The embedding j can be extended iff there is an M-generic H such that $j``G \subseteq H$. Translating this into the language of Boolean algebras, this is saying that $H$ is disjoint from the ideal $\mathcal{I} = \{q \in \mathcal{B}^M(j(\mathbb{P})) :$ there is a $p \in j``G$ such that $p \wedge q = 0\}$. We can force over V with the partial ordering $\mathcal{B}^M(j(\mathbb{P}))/\mathcal{I}$, and hope that it yields an $H \subseteq \mathcal{B}^M(j(\mathbb{P}))$ that is M-generic. The next lemma is Remark 20 from [40] and gives a sufficient criterion for this to happen.

7.48 Lemma. *Let λ be a regular cardinal. Let $M \subseteq V$ be a model of set theory such that $M^{<\lambda} \subseteq M$. Suppose that $\mathbb{P} \in M$ is a $<\lambda$-closed partial ordering and that $\mathcal{F} \subseteq \mathbb{P}$ is a $<\lambda$-closed filter with dual $\mathcal{I}$ such that every dense set $D \subseteq \mathbb{P}$ lying in M is dense in $\mathbb{P}/\mathcal{I}$. Then $\mathbb{P}/\mathcal{I}$ is a $<\lambda$-closed partial ordering such that forcing with $\mathbb{P}/\mathcal{I}$ adds an M-generic filter H for $\mathbb{P}$ such that $\mathcal{F} \subseteq H$.*

A corollary of this is the following:

7.49 Corollary. *Let λ be a regular cardinal with $\lambda^{<\lambda} = \lambda$. Let $M \subseteq V$ be a model of set theory such that $M^{<\lambda} \subseteq M$. Suppose that $|\eta| = \lambda$ but $M \models$ "η is an inaccessible cardinal". Suppose that $\mathcal{F} \in V$ is a filter on* $\mathrm{Col}(\lambda, <\eta)$ *generated by a decreasing sequence of conditions $\langle m_\alpha : \alpha < \lambda\rangle$ such that for all $\beta < \eta$ there is an m_α such that*

$$\mathcal{F} \restriction \mathrm{Col}(\lambda, <\beta) = \{p \in \mathrm{Col}(\lambda, <\beta) : m_\alpha \leq_{\mathrm{Col}(\lambda,<\eta)} p\}.$$

Let $\mathcal{I}$ be the ideal dual to $\mathcal{F}$. Then forcing with $\mathrm{Col}(\lambda, <\eta)/\mathcal{I}$ *adds an M-generic object H to* $\mathrm{Col}(\lambda, <\eta)$ *with $\mathcal{F} \subseteq H$. Moreover, in V,* $\mathrm{Col}(\lambda, <\eta)/\mathcal{I}$ *is isomorphic to* $\mathrm{Add}(\lambda)$.

This corollary works for any partial ordering $\mathbb{Q} \in M$ of cardinality λ in V such that in V, there is a filtration $\langle \mathbb{Q}_\alpha : \alpha < \lambda\rangle$ of $\mathbb{Q}$ and a V-stationary collection of α such that $\mathbb{Q}_\alpha$ is a regular subordering of $\mathbb{Q}$. In particular, it also applies to the Silver collapse $\mathbb{S}(\lambda, \eta)$.

The following fact about almost huge cardinals is standard. It is proved in [40] among other places:

7.50 Lemma. *Suppose that κ is an almost huge cardinal. Then there is an almost huge embedding $j : V \to M$ with critical point κ, $j(\kappa) = \lambda$, $j``\lambda$ is cofinal in $j(\lambda)$ and $|j(\lambda)| = \lambda$. Moreover, if λ is the least such, then λ is not Mahlo.*

We now can outline:

7.51 Theorem (Magidor). *Suppose that κ is an almost huge cardinal and $\mu < \kappa$ is regular. Then there is a forcing extension in which $\kappa = \mu^+$ and there is a κ-complete, κ^+-saturated ideal on κ.*

Proof. We assume that j satisfies the conclusion of Lemma 7.50. Let $\mathbb{P}$ be the partial ordering constructed by Kunen, and $\mathbb{Q}$ be the Silver collapse $\mathbb{S}(\kappa, \lambda)$

to make λ into κ^+, as constructed in $V^{\mathbb{P}}$. Let $G*H \subseteq \mathbb{P}*\mathbb{Q}$ be generic. As in the Kunen construction, $\mathbb{P}*\mathbb{Q}$ is regularly embedded into $j(\mathbb{P})$ by a canonical map i that is the identity on $\mathbb{P}$ and there is a generic $\hat{G} \subseteq j(\mathbb{P})$ extending $i``(G*H)$. By the κ-c.c., we see that j can be extended to $\hat{j} : V[G] \to M[\hat{G}]$. Moreover, $M[\hat{G}]$ is closed under $<\lambda$-sequences from $V[\hat{G}]$.

For $\alpha < \lambda$, let $m_\alpha = \bigcup\{j(q) : q \in H \cap \mathbb{S}(\kappa,\alpha)\}$ and $\mathcal{F}$ be the filter generated by $\{m_\alpha : \alpha \in \lambda\}$. Applying Corollary 7.49 (for the Silver collapse, as in the remarks following the corollary), we see that we can force with $\mathbb{S}(\lambda, j(\lambda))^M/\mathcal{I}$ over V to get an M-generic filter $\hat{H}$ extending $\mathcal{F}$.

Since $\hat{H} \supseteq \mathcal{F}$, we can extend j to $\hat{j} : V[G*H] \to M[\hat{G}*\hat{H}]$. The rest of the argument goes as before: $\mathbb{S}(\lambda, j(\lambda))^M/\mathcal{I}$ is $<\lambda$-closed, and hence a pseudo-generic tower argument gives us a $V[G*H]$-ultrafilter on κ that is κ-complete, and normal for sequences that lie in $V[G*H]$. This ultrafilter lies in $V[\hat{G}]$, which is a κ-c.c. extension of $V[G*H]$. By Example 3.30, we are done. $\dashv$

7.52 Remark. As in the Kunen construction, the ideal in Magidor's model can be seen to be layered; this is an immediate corollary of Proposition 7.13.

There are several advantages to Magidor's variation. In the Kunen construction, the use of the Silver collapse was important: the conditions had to have the right "shape" for the union of $j``H$ to be an element of $j(\mathbb{Q})$. The Magidor variation does not use $\bigcup j``H$ as a condition; it extends the filter generated by $j``H$. For this one only needs the $<\lambda$-closure of the conditions. Any $<\kappa$-closed partial ordering that collapses λ to κ^+ and has cardinality λ works. In particular, one can do a similar construction where $\mathbb{Q} = \mathrm{Col}(\kappa, <\lambda)$.

If we work a bit harder, we can see that in the Kunen/Magidor situation, it is not necessary to force beyond $j(\mathbb{P})$ to extend the elementary embedding j. Since $|j(\lambda)| = \lambda$, the cardinality of $\mathbb{S}^{M[\hat{G}]}(\lambda, j(\lambda))$ is λ in $V[\hat{G}]$ and $M[\hat{G}]$ has λ-many dense subsets of $S^{M[\hat{G}]}(\lambda, j(\lambda))$. Since $S^{M[\hat{G}]}(\lambda, j(\lambda))$ is $<\lambda$-closed in $V[\hat{G}]$ there is an $M[\hat{G}]$-generic filter $\hat{H} \subseteq S^{M[\hat{G}]}(\lambda, j(\lambda))$ lying in $V[\hat{G}]$. Using Lemma 7.48, this filter can be built to extend $j``H$. Hence in $V[\hat{G}]$ there is a generic elementary embedding $\hat{j} : V[G*H] \to M[\hat{G}*\hat{H}]$.

7.12. More Saturated Ideals

Is it consistent to have a κ^+-saturated ideal if κ is the successor of a singular cardinal? Can every regular cardinal κ carry a κ^+-saturated ideal? Can there be a model where

$$\text{for all } n > m > 0,\ (\aleph_n, \aleph_{n-1}) \twoheadrightarrow (\aleph_m, \aleph_{m-1})?$$

These questions were answered in the paper [34]. We briefly outline the construction that answers these questions.

Let's now analyze key points of Kunen's construction, with the intention of isolating the essential elements. The main goal of the original construction is to have $\mathbb{P} * \mathbb{S}(\kappa, \lambda)$ sit as a regular subordering of $j(\mathbb{P})$. In the construction, the family of Silver collapses could be replaced by any family $\mathcal{Q} = \{\mathbb{Q}(\alpha, \beta)\}$ of uniformly definable partial orderings indexed by ordinals α, β such that $\mathbb{Q}(\alpha, \beta) \subseteq V_\beta$. Then the construction could be arrange so that $\mathbb{P} * \mathbb{Q}(\kappa, \lambda)$ is regularly embedded into $j(\mathbb{P})$.

In our original description of the construction, we diagonalized over *all* small regular suborderings $\mathbb{Q}$ of $\mathbb{P}$ to make sure that $\mathbb{Q} * S(\alpha, \kappa)$ is regularly embedded into $\mathbb{P}$. As we saw in the Laver construction, this is not necessary. Since $\mathbb{P}$ is $j(\mathbb{P}) \cap V_\kappa$, we need only consider $\mathbb{Q}$'s of the form $\mathbb{P} \cap V_\alpha$. Thus at a typical stage α in the iteration, we force with $\mathbb{Q}^{V^{\mathbb{P} \cap V_\alpha}}(\alpha, \kappa)$ just in case $\mathbb{P}_\alpha \cap V_\alpha$ is a regular subordering of $\mathbb{P}_\alpha$. This gives us a partial ordering $\mathbb{P}$ that is definable in the parameter κ. Reflecting the definition, we get another family of partial orderings $\mathbb{P}(\alpha)$ for many $\alpha < \kappa$. Provided that partial orderings in $\mathcal{Q}$ are not too exotic, it is easy to verify that $\mathbb{P}(\alpha) = \mathbb{P} \cap V_\alpha$ for most α.

Summarizing, if one is given a family $\mathcal{Q}$, the construction produces another family of partial orderings $\mathbb{P}(\alpha)$ such that for most $\alpha < \beta$,

- $\mathbb{P}(\alpha)$ is α-c.c. and makes α into a cardinal such as ω_1 or ω_2,
- $\mathbb{P}(\alpha)$ is naturally included in $\mathbb{P}(\beta)$ and is a regular subordering, and
- there is a regular embedding of $\mathbb{P}(\alpha) * \mathbb{Q}(\alpha, \beta)$ into $\mathbb{P}(\beta)$ extending the inclusion of $\mathbb{P}(\alpha)$ into $\mathbb{P}(\beta)$.

Since the construction is "top down" there is no problem iterating it for finitely many cardinals. For example, to do a construction like this for three cardinals one would start with two huge embeddings j_0 and j_1 with the property that $\kappa_0 = \mathrm{crit}(j_0)$ and $j_0(\kappa_0) = \kappa_1$, where $\kappa_1 = \mathrm{crit}(j_1)$. We let $\kappa_2 = j_1(\kappa_1)$.

We can define a family of partial orderings $\mathbb{Q}(\alpha, \beta)$ by doing the Kunen construction with α-closed partial orderings as in the construction showing Corollary 7.37. The inductive construction yields a definable family of partial orderings $\mathbb{Q}(\alpha, \beta)$ defined such that $\mathbb{Q}(\kappa_0, \kappa_2)$ has $\mathbb{Q}(\kappa_0, \kappa_1) * S(\kappa_1, \kappa_2)$ as a regular subordering. Let $\{\mathbb{P}(\alpha)\}$ be the partial orderings anticipating the family of $\mathbb{Q}(\alpha, \beta)$'s.

If we force with $\mathbb{P}(\kappa_0) * \mathbb{Q}(\kappa_0, \kappa_1) * S(\kappa_1, \kappa_2)$, then we get a model where $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$, $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ and there are saturated ideals on both ω_1 and ω_2.

For a fixed m, similar methods using the Silver collapse at the top and working downwards give the consistency of the statement that:

- for all positive $n < m$ there is an $\aleph_{n+1}$-saturated ideal on ω_n, and
- $(\aleph_{n+1}, \aleph_n) \twoheadrightarrow (\aleph_n, \aleph_{n-1})$.

It requires a new idea to prove the analogous results for an infinite interval of cardinals. This was done in [34], where the author showed how to uniformly construct a family of partial orderings $\mathbb{R}(\vec{\alpha})$ for finite increasing sequences of Mahlo cardinals[74] $\vec{\alpha}$ such that:

- $\mathbb{R}(\alpha,\beta) \subseteq V_\beta$, $\mathbb{R}(\alpha,\beta)$ is $<\alpha$-closed, β-c.c. and collapses β to be the successor of α,
- if $\vec{\alpha} = (\alpha_0, \ldots, \alpha_n)$ and $\beta > \alpha_n$, then $\mathbb{R}(\vec{\alpha}^\frown\beta) = \mathbb{R}(\vec{\alpha}) * \mathbb{R}(\alpha_n, \beta)$,
- if $\vec{\alpha}_0$ and $\vec{\alpha}_1$ are increasing sequences of Mahlo cardinals and β is a Mahlo cardinal where $\max \vec{\alpha}_0 < \beta < \min \vec{\alpha}_1$, then there is a natural embedding of $\mathbb{R}(\vec{\alpha}_0^\frown\beta^\frown\vec{\alpha}_1)$ into $\mathbb{R}(\vec{\alpha}_0^\frown\vec{\alpha}_1)$ and these embeddings commute in the obvious sense, and
- $\mathbb{R}$ has the type of "geometry" that allows master conditions to exist.

If we are given a huge embedding $j_n : V \to M_n$ with critical point κ_n and $j_n(\kappa_n) = \kappa_{n+1}$ and a finite increasing sequence of Mahlo cardinals $\vec{\alpha}$ below κ_n, then there is a natural embedding:

$$i_n : \mathbb{R}(\vec{\alpha}, \kappa_n) * \mathbb{R}(\kappa_n, \kappa_{n+1}) \to \mathbb{R}(\vec{\alpha}, \kappa_{n+1}) = j(\mathbb{R}(\vec{\alpha}, \kappa_n)).$$

Thus if $\hat{G} \subseteq \mathbb{R}(\vec{\alpha}, \kappa_{n+1})$ is generic then it induces a generic $G * H \subseteq \mathbb{R}(\vec{\alpha}, \kappa_n) * \mathbb{R}(\kappa_n, \kappa_{n+1})$, and there is an $m \in \mathbb{R}(\kappa_{n+1}, j_n(\kappa_{n+1}))$ such that

$$M[\hat{G}] \models \text{for all } p \in H,\ m \leq j_n(p).$$

In particular, it is possible to extend the elementary embedding j_n to a

$$\hat{j_n} : V[G * H] \to M[\hat{G} * \hat{H}].$$

This allows us to apply the pseudo-generic tower arguments to conclude that in $V[G*H]$, there is a κ_n-complete, κ_n^+-saturated ideal on κ_n and if the largest element of $\vec{\alpha}$ is α_n, then $(\kappa_{n+1}, \kappa_n) \twoheadrightarrow (\kappa_n, \alpha_n)$.

We need to see that the various $\mathbb{R}(\kappa_n, \kappa_{n+1})$ do not destroy the effects of $\mathbb{R}(\kappa_m, \kappa_{m+1})$ for $m < n$. For this the following lemma was first proved in [34]. It is proved using a pseudo-generic tower argument.

7.53 Lemma. *Suppose that $\mathbb{P}$ is a $<\kappa$-closed partial ordering and $\kappa > \lambda \geq \kappa' > \lambda'$. Let $G \subseteq \mathbb{P}$ be generic. Then*

- *if $(\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$ then $V[G] \models (\kappa, \lambda) \twoheadrightarrow (\kappa', \lambda')$, and*
- *if there is a λ-complete, λ^+-saturated ideal on λ then $V[G] \models$ "there is a λ-complete, λ^+-saturated ideal on λ".*

[74] For these purposes, we include ω as a Mahlo cardinal.

Suppose now that we are given a sequence of huge embeddings $\langle j_n : n \in \omega \rangle$ where the critical point of j_n is κ_n and $j_n(\kappa_n) = \kappa_{n+1}$. We have argued that if we take a generic $G_n \subseteq \mathbb{R}(\omega, \kappa_0, \kappa_1, \ldots, \kappa_{n+1})$ then in $V[G_n]$ there is a saturated ideal on ω_{n+1} and $(\aleph_{n+2}, \aleph_{n+1}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$. If we force with the inverse limit over $n \in \omega$ of the $\mathbb{R}(\omega, \kappa_0, \kappa_1, \ldots, \kappa_{n+1})$ we will have the conclusion that for all $n \in \omega$ there is a saturated ideal on ω_{n+1} and $(\aleph_{n+2}, \aleph_{n+1}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$.

What remains is to outline the construction of the $\mathbb{R}$'s. An important tool for this is the *termspace* partial ordering. This stratagem is due to Laver and was exploited most fruitfully by Abraham [2].

7.54 Definition. Let $\mathbb{P}$ be a partial ordering and $\mathbb{Q}$ be a $\mathbb{P}$-term for a partial ordering. The *termspace* partial ordering $\mathbb{Q}^*$ is defined to be the partial ordering whose domain consists of all $\mathbb{P}$-terms for elements of $\mathbb{Q}$, with the ordering that $\sigma \leq_{\mathbb{Q}^*} \tau$ iff $1 \Vdash_{\mathbb{P}} \sigma \leq \tau$.

We can take the domain of $\mathbb{Q}^*$ to be a set, since there are only a set of equivalence classes of elements of $\mathbb{Q}$ with respect to the relation $\sigma \sim \tau$ iff $1 \Vdash_{\mathbb{P}} \sigma = \tau$. Note also that the termspace partial ordering depends on both $\mathbb{P}$ and $\mathbb{Q}$. It is denoted here by $A(\mathbb{P}, \mathbb{Q})$.

One must prove some basic facts about the termspace partial ordering such as:

1. (Laver) The identity map defines a projection from $\mathbb{P} \times \mathbb{Q}^*$ to $\mathbb{P} * \mathbb{Q}$.

2. (Abraham) If $1 \Vdash$ "$\mathbb{Q}$ is $<\lambda$-closed", then $\mathbb{Q}^*$ is $<\lambda$-closed.

3. (Foreman) $A(\mathbb{P}, A(\mathbb{Q}, \mathbb{R})) \cong A(\mathbb{P} * \mathbb{Q}, \mathbb{R})$ canonically.

4. (Foreman) $A(\mathbb{P}, \prod_{n\in\omega} \mathbb{Q}_n) \cong \prod_{n\in\omega} A(\mathbb{P}, \mathbb{Q}_n)$ canonically.

For regular $\alpha < \beta$ we define the partial orderings $\mathbb{R}(\alpha, \beta)$ by induction on inaccessible β simultaneously in all generic extensions of V. Suppose that we have defined $\mathbb{R}(\alpha, \beta)$ for all pairs $\alpha < \beta$ which lie below λ. We now define $\mathbb{R}(\alpha, \lambda)$.

- If λ is the first inaccessible above α, let $\mathbb{R}(\alpha, \lambda) = \mathbb{S}(\alpha, \lambda)$.

- Otherwise, by induction on $n \in \omega$ we define partial orderings $S^n(\alpha, \lambda)$. We will define this in the ground model. The same definition, relativized appropriately, will allow us to define a partial ordering $S^n(\alpha, \lambda)^{\mathbb{R}(\gamma,\alpha)}$ in a generic extension of V by a partial ordering of the form $\mathbb{R}(\gamma, \alpha)$.

  For $n = 0$, $S^0(\alpha, \lambda) = \mathbb{S}(\alpha, \lambda)$.

  Assume that we have defined $S^n(\delta, \lambda)$ in every extension of V of the form $\mathbb{R}(\gamma, \delta)$ for $\alpha < \delta < \lambda$ and δ inaccessible. Then we let $S^{n+1}(\alpha, \lambda)$ be the product with supports of size α of all the partial orderings $A(\mathbb{R}(\alpha, \delta), S^n(\delta, \lambda))$ as δ ranges over Mahlo cardinals in the interval between α and λ.

We let $\mathbb{R}(\alpha,\lambda)=\prod_{n\in\omega} S^n(\alpha,\lambda)$.

The main point of this construction is the following: if the identity mapping of $\mathbb{R}(\alpha,\delta)$ into $\mathbb{R}(\alpha,\lambda)$ is a regular embedding, then for each n, $A(\mathbb{R}(\alpha,\delta), S^n(\delta,\lambda))$ is a factor in $S^{n+1}(\alpha,\lambda)$. Hence, there is an embedding of $\mathbb{R}(\alpha,\delta)\times\prod_{n\in\omega} A(\mathbb{R}(\alpha,\delta), S^n(\delta,\lambda))$ into $\mathbb{R}(\alpha,\lambda)=\prod_{n\in\omega} S^n(\alpha,\lambda)$ that extends the identity mapping on $\mathbb{R}(\alpha,\delta)$. By property 4 above, we see that $\mathbb{R}(\alpha,\delta)\times A(\mathbb{R}(\alpha,\delta),\prod_{n\in\omega} S^n(\delta,\lambda))$ is canonically embedded in $\mathbb{R}(\alpha,\lambda)$, and hence by property 1 above we see that $\mathbb{R}(\alpha,\delta)*\mathbb{R}(\delta,\lambda)$ is canonically embedded in $\mathbb{R}(\alpha,\lambda)$.

We now discuss the chain conditions satisfied by the $\mathbb{R}(\alpha,\beta)$'s. Let $\alpha<\beta$ be Mahlo cardinals, and $\mathbb{P}$ be an α-c.c. partial ordering. Then one can check that $A(\mathbb{P},\mathbb{S}(\alpha,\beta))$ is isomorphic to a subcollection of the order-preserving maps from $\mathbb{P}$ to $\mathbb{S}(\alpha,\beta)$ ordered by setting $f\le g$ iff there is a dense set $D\subseteq\mathbb{P}$ such that for all $p\in D$, $f(p)\le g(p)$. In particular, the chain condition of $A(\mathbb{P},\mathbb{S}(\alpha,\beta))$ is less than or equal to the chain condition of the product of $|\mathbb{P}|$-copies of $\mathbb{S}(\alpha,\beta)$ ordered coordinatewise. Though the details are quite technical this is the basis for the argument showing:

7.55 Lemma. *For all regular β and all Mahlo λ, $\mathbb{R}(\beta,\lambda)$ is λ-c.c. and has cardinality λ.*

We have outlined the proof of the following theorem:

7.56 Theorem. *Suppose that $\langle j_n : n\in\omega\rangle$ is a sequence of huge embeddings and the critical point of j_n is κ_n and $j_n(\kappa_n)=\kappa_{n+1}$. Then there is a partial ordering $\mathbb{P}$ such that for all generic $G\subseteq\mathbb{P}$, $V[G]$ satisfies:*

1. *for all $n\in\omega$ there is an $\aleph_{n+1}$-complete, $\aleph_{n+2}$-saturated ideal on ω_{n+1}, and*
2. *for all $n\in\omega$, $(\aleph_{n+2},\aleph_{n+1})\twoheadrightarrow(\aleph_{n+1},\aleph_n)$.*

What about the consistency of a κ^+-saturated ideal if κ is the successor of a singular cardinal? This is also answered in [34]. From the machinery we have developed this is not difficult.

Suppose that κ is an indestructibly supercompact cardinal[75] and that there is a huge cardinal μ above κ. We can do the Kunen construction to collapse μ to be κ^+ and arrange that there is a μ^+-saturated ideal I on μ. Then κ is still supercompact so we can follow this by forcing to make κ singular, or even $\aleph_\omega$.

As long as the forcing making κ singular is κ-centered, Corollary 7.21 implies that the ideal I remains μ^+-saturated. Typically, Prikry-type forcings are κ centered even when mixed with collapsing to make κ into $\aleph_\omega$.[76]

[75] That is, κ is supercompact and remains so after any forcing extension via a $<\kappa$-directed closed forcing. Laver [82] showed how to make a supercompact cardinal indestructible, and his construction is given in Cummings's chapter in this Handbook. See also Definition 11.4.
[76] For example, one can use the "projected" version of Magidor forcing collapsing κ to be $\aleph_\omega$ is κ-centered (see [88, 46]).

We have outlined the argument that if we first force the existence of a supercompact κ such that κ^+ carries a κ^{+2}-saturated ideal, and then collapse κ to be $\aleph_\omega$ in the usual way, the resulting model W satisfies the sentence:

$$\aleph_{\omega+1} \text{ carries an } \aleph_{\omega+2}\text{-saturated ideal.}$$

This outline is very general. For example, if κ is supercompact and the nonstationary ideal on $\mu = \kappa^+$ restricted to a particular stationary set is saturated,[77] then after making $\kappa = \aleph_\omega$ in the standard way, the nonstationary ideal on μ restricted to that stationary set is still saturated, and $\mu = \aleph_{\omega+1}$.

Life is not quite so simple if one wants to have all cardinals less than or equal to $\aleph_{\omega+1}$ carry a saturated ideal. We now outline the proof that this is possible and even:

7.57 Theorem (Foreman [34]). *Suppose that there is a huge cardinal. Then there is a model of* ZFC + *"for all regular cardinals κ there is a κ-complete, κ^+-saturated ideal on κ".*

Here is a brief outline of the proof of this theorem. We start with a model of GCH with a λ-supercompact cardinal κ that has some additional properties. If $j : V \to M$ is the λ-supercompact embedding, we will assume that there is a set $X \subseteq \kappa$ such that:

1. $\kappa \in j(X)$,
2. for $\alpha < \beta$ belonging to X there is an almost huge embedding $j_{\alpha,\beta}$ that has critical point α and $j_{\alpha,\beta}(\alpha) = \beta$, and
3. λ is bigger than the first 5 elements of $j(X)$ above κ.

The existence of such a pair κ and λ follows from the assumption of a huge cardinal.

To get the necessary anticipation properties we modify the partial orderings $\mathbb{R}(\alpha, \beta)$ defined in the proof of Theorem 7.56 so that if $\alpha < \alpha'$ are limit elements of X, then $\mathbb{R}(\alpha, \alpha_1) * \mathbb{R}(\alpha_1, \alpha_2) * \mathbb{R}(\alpha_2, \alpha_3) * \mathbb{R}(\alpha_3, \alpha_4) * (\mathrm{Col}(\alpha_4, \alpha') \times \mathbb{R}(\alpha', \alpha'_1))$ is canonically embedded into $\mathbb{R}(\alpha, \alpha_1) * \mathbb{R}(\alpha_1, \alpha_2) * \mathbb{R}(\alpha_2, \alpha_3) * \mathbb{R}(\alpha_3, \alpha'_1)$. This involves essentially the same technique as before.

As is common in "singular cardinals" type constructions, we do a preparatory forcing before we change any cofinalities. For $\alpha \in X$, denote the next four elements of X as $\alpha_1, \alpha_2, \alpha_3$ and α_4. The preparatory forcing in this proof is an Easton iteration of length $\kappa + 1$ where, at a limit stage α of X that lies in X, we will collapse $\alpha_1, \ldots, \alpha_4$ using the partial ordering of the form $\mathbb{R}(\alpha, \alpha_1) * \mathbb{R}(\alpha_1, \alpha_2) * \mathbb{R}(\alpha_2, \alpha_3) * \mathbb{R}(\alpha_3, \alpha_4)$. So, if I is the iteration, then I has Easton supports and $I_{\alpha+1} = I_\alpha * (\mathbb{R}(\alpha, \alpha_1) * \mathbb{R}(\alpha_1, \alpha_2) * \mathbb{R}(\alpha_2, \alpha_3) * \mathbb{R}(\alpha_3, \alpha_4))$. At κ we force with $\mathbb{R}(\kappa, \kappa_1) * \mathbb{R}(\kappa_1, \kappa_2) * \mathbb{R}(\kappa_2, \kappa_3) * \mathbb{R}(\kappa_3, \kappa_4)$. Standard arguments show that after forcing with I, κ remains $2^{2^{2^\kappa}}$-supercompact.

[77] Section 8 establishes the consistency of this.

We use the variant definition of the $\mathbb{R}$'s for the preparatory forcing. This is used to establish the following:

7.58 Lemma. *If V_1 is the resulting model after the preparatory forcing, and $\alpha < \alpha'$ are limit elements of X and $G \subseteq \mathrm{Col}(\alpha_4, \alpha')$ is generic over V_1, then in $V_1[G]$, α'_1 is the successor of α_4 and α_4 carries an α_4-complete, α_4^+-saturated ideal.*

This takes care of the local problem of producing saturated ideals on successor of regular cardinals. In V_1, all limit points α of X have the property that for $i \in \{1, 2, 3\}$, $\alpha_i = \alpha^{+i}$ carries a saturated ideal.

A forcing extension of V_1 yields our next model V_2, which has the properties that:

1. all cardinals below κ and above some fixed κ_0 are elements of X, and

2. all successor cardinals between κ_0 and κ are of the form α_i for some $1 \leq i \leq 4$.

The final model is $V_2[g]$ where $g \subseteq \mathrm{Col}(\omega, \kappa_0)$ is generic. The generic collapse g makes all of the remaining regular cardinals below the first inaccessible be of the form α_i for some $\alpha \in X$ and $1 \leq i \leq 4$, while preserving the saturation of the ideals on all of the cardinals above κ_0. If δ is the first inaccessible cardinal in the final model $V_2[g]$ and W is the collection of sets in $V_2[g]$ of rank less than δ, then W is a model of "every regular cardinal carries a saturated ideal".

The crux of the issue is the forcing to produce V_2 from V_1. This follows the general outline of [46], which in turn, adapted Magidor's techniques of [88] to work with Radin forcing.

There are two relevant partial orderings $\mathbb{P}$ and $\mathbb{P}^\pi$ for producing V_2 from V_1. The model V_2 is the result of forcing with $\mathbb{P}^\pi$ over V_1. The first partial ordering $\mathbb{P}$ uses a supercompact Radin forcing to add a closed unbounded set through $\{z \in [\kappa^{+3}]^{<\kappa} : z \cap \kappa \in X\}$. It also uses Levy collapses to make the successor cardinals of V_2 be successors of elements of X. This forcing preserves the fact that κ is a highly Mahlo cardinal.

Since $\mathbb{P}$ adds a closed unbounded set C in $[\kappa^{+3}]^{<\kappa}$, if $\alpha \in X$ is a limit point of $C^\pi = \{z \cap \kappa : z \in C\}$, then $\mathbb{P}$ adds a closed unbounded set through $(\alpha^+)^V$ of order type the same as the order type of $C^\pi \cap \alpha$.

There will be a projection map from $\pi : \mathbb{P} \to \mathbb{P}^\pi$. The forcing $\mathbb{P}^\pi$ adds the Radin-generic closed unbounded subset $C^\pi \subseteq X \cap \kappa$, while collapsing cardinals. The following are some of the properties of $\mathbb{P}^\pi$:

1. If κ_0 is the first point in C^π, then between κ_0 and κ, every limit cardinal α is an element of C and $C \cap \alpha$ is Radin generic for a Radin forcing involving α.

2. If $\alpha < \beta$ are successive points of C^π, then the forcing $\mathbb{P}^\pi$ factors as $\mathbb{P}^\pi_\alpha \times \mathrm{Col}(\alpha_4, \beta) \times \mathbb{P}^\pi_{>\beta}$, where $\mathbb{P}^\pi_\alpha$ is α-centered, and $\mathbb{P}^\pi_{>\beta}$ adds no new β-sequences.

From the factoring we see that for all $\alpha \in X$ that lie on the Radin sequence, we have preserved the saturated ideals on α_1, α_2 and α_3. Each successor element of X on the Radin sequence is collapsed. So to complete the picture we must see that α_4 carries a saturated ideal.

This follows from our remarks about the preparatory forcing. The factoring properties of $\mathbb{P}^\pi$ imply that the subsets of β_1 in the final model are those in the model after forcing over V_1 to with $\mathbb{P}^\pi_\alpha \times \mathrm{Col}(\alpha_4, \beta)$. The forcing $\mathbb{P}^\pi_\alpha$ is α-centered, so it suffices to see that there is a saturated ideal on α_4 in the model produced over V' by forcing with $\mathrm{Col}(\alpha_4, \beta)$. This is the content of the lemma mentioned in the beginning of the proof.

3. $\mathbb{P}$ and $\mathbb{P}^\pi$ have the relevant Prikry type properties.

4. GCH holds in $V^{\mathbb{P}^\pi}$.

These properties ensure that every regular cardinal α less than the first inaccessible cardinal carries an α^+-saturated ideal. We remark that using the techniques of [41] it is easy to verify that the ideals produced in [34] are more than just α^+-saturated—they are α-centered.

We note that a novel element in the proof of Theorem 7.57 is that the successor points on the Radin sequence are collapsed by the forcing. This makes no difference in the proofs of the relevant Prikry properties, as was pointed out by Woodin.

7.13. Forbidden Intervals

Proposition 6.26 showed that for each successor cardinal κ there is a proper class $\mathcal{C}_\kappa$ consisting of intervals I of cardinals such that no regular cardinal λ in I can carry a uniform κ-complete, κ^+-saturated ideal on λ. Each of these intervals was of the form $[\gamma, \delta)$ where δ had cofinality κ. We now state a consistency result appearing in [31] that shows that the limitations given by these intervals is sharp at the upper ends of each interval.

7.59 Theorem. *Suppose that there is a huge cardinal and* GCH *holds. Then there is a partial ordering $\mathbb{P}$ and an α such that in $V^{\mathbb{P}}_\alpha$:*

1. ZFC *holds,*

2. every regular uncountable cardinal ξ carries a normal, ξ-centered ideal on ξ, and

3. if λ is a singular cardinal of cofinality ξ, then for all ideals I on ξ there is a uniform ideal J on λ^+ such that

(a) $P(\xi)/I \cong P(\lambda^+)/J$,

(b) the completeness of I is equal to the completeness of J, and

(c) if I is normal, then J is weakly normal.

After this section was written, Magidor showed that $\aleph_{\omega_1}$ can carry a uniform, countably complete, $\aleph_2$-saturated ideal, a case not covered by Theorem 7.59.

The conclusions of Theorem 7.59 hold in the model constructed in the proof of Theorem 7.57. We indicate some of the elements one needs to see this.

Let $\kappa_0 < \alpha < \delta$ be a singular cardinal of cofinality ξ in $V[G^\pi]$ and I a uniform ideal ξ. To construct an ideal J on the successor of α with quotient isomorphic $P(\xi)/I$, we force with a sufficiently large part of $\mathbb{P}/\mathbb{P}^\pi$ to get a G which projects to G^π, in which $P(\xi^+)^{V[G^\pi]} = P(\xi^+)^{V[G]}$ and which makes $\mathrm{cf}(\alpha^+) = \xi$. In $V[G]$, we choose a closed unbounded subset $\langle \delta_i : i < \xi \rangle$ of $(\alpha^+)^V =_{\mathrm{def}} \eta$ of order type ξ and "copy" the ideal onto η. The resulting ideal $\bar{J}$ in $V[G]$ has the same completeness as I, and the map $\iota : P(\delta)/\bar{J} \to P(\xi)/I$ defined by $\iota(A) = \{i : \delta_i \in A\}$ defines an isomorphism.

Looking carefully at the forcing used to produce G in $V[G^\pi]$ we see that it is sufficiently homogeneous so that $J =_{\mathrm{def}} (P(\delta)^{V[G^\pi]} \cap \bar{J})$ and the isomorphism ι both lie in $V[G^\pi]$. In particular, we see that J has the same degree of completeness as I does and is weakly normal, if I is normal.

7.14. Dense Ideals on ω_1

In the late 1970's Woodin showed that it is consistent to have a countably complete, $\aleph_1$-dense ideal on ω_1, assuming the consistency of ZF + $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular". Later Woodin [124] improved this result showing, assuming the consistency of an almost huge cardinal, that the following is consistent: For all $\aleph_2$-c.c. partial orderings $\mathbb{P}$ that collapse $\aleph_1$ and have cardinality at most ω_2, there is a countably complete ideal I on ω_1 such that $P(\omega_1)/I$ has a dense set isomorphic to $\mathbb{P}$.

We present here a special case of this result: that it is consistent to have an $\aleph_1$-dense ideal on ω_1.

7.60 Theorem. *Let j be an almost huge embedding with critical point κ_0 and $j(\kappa_0) = \kappa_1$. Let $C_0 \subseteq \mathrm{Col}(\omega, {<}\kappa_0)$ be generic over V. Let $\mathbb{R}_0 = P(\omega) \cap V[C_0]$ and $V_1 = V(\mathbb{R}_0) \subseteq V[C_0]$. Let $\mathbb{Q} = \mathrm{Add}(\omega_1) * \mathbb{Q}'$ be a κ_1-c.c. partial ordering in V_1 that is countably closed, has cardinality κ_1, and collapses κ_1 to be ω_2. Then for all V_1-generic $G \subseteq \mathbb{Q}$, $V_1[G] \models \mathrm{ZFC} + \diamondsuit_{\omega_1}$ + "There is a normal $\aleph_1$-dense ideal J on ω_1".*

We will use a forcing fact:

7.61 Lemma. *Let $V \subseteq W$ be models of* ZFC. *Suppose that:*

1. *κ is an inaccessible cardinal in V,*
2. *$\kappa = \omega_1^W$, and*
3. *if $r \in \mathbb{R}^W$, then there is a partial ordering $\mathbb{Q} \in V$ with $|\mathbb{Q}| < \kappa$ and a V-generic $H \subseteq \mathbb{Q}$ belonging to W such that $r \in V[H]$.*

Then there is a partial ordering $\mathbb{C}$ in W such that if $U \subseteq \mathbb{C}$ is W-generic there is a V-generic object $H \subseteq \mathrm{Col}(\omega, {<}\kappa)$ lying in $W[U]$ such that $\mathbb{R}^{V[H]} = \mathbb{R}^W$. Moreover, if $H_\alpha \subseteq \mathrm{Col}(\omega, {<}\alpha)$ is generic over V and belongs to W, then H can be taken to extend H_α.

Proof (Sketch). Let $\mathbb{C}$ be the partial ordering in W whose domain consists of V-generic filters $H_\alpha \subseteq \mathrm{Col}(\omega, {<}\alpha)$ for some $\alpha < \kappa$. If H_α and H_β are elements of $\mathbb{C}$, then H_α is a stronger condition than H_β iff $\alpha > \beta$ and $H_\alpha \supseteq H_\beta$.

If H^* is generic for $\mathbb{C}$ over W, then $H = \bigcup H^*$ is generic over V for $\mathrm{Col}(\omega, {<}\kappa)$ and every real in W belongs to $V[H]$. ⊣

Proof of Theorem 7.60. Let $\mathbb{R}_0 = P(\omega) \cap V[C_0]$. Since $\mathrm{Add}(\omega_1)$ adds a well-ordering of $\mathbb{R}$ in a canonical way, we see that any generic $G \subseteq \mathbb{Q}$ can be decomposed into $G_0 * G_1$, and $V_1[G_0] \models \mathrm{ZFC}$.

Let $G \subseteq \mathbb{Q}$ be $V[C_0]$-generic. Let $C \subseteq \mathrm{Col}(\omega, \omega_1)$ be $V[C_0, G]$-generic and $\mathbb{R}_1 = P(\omega) \cap V_1[G * C]$.

We let W be the model $V_1[G_0]$ and construct the partial ordering $\mathbb{C} \in W$ as in Lemma 7.61. In $V_1[G * C]$, the cardinality of $P(\mathbb{C})^W$ is countable, so we can build a W-generic object for $\mathbb{C}$. This in turn yields a V-generic $C_0' \subseteq \mathrm{Col}(\omega, {<}\kappa_0)$ belonging to $V_1[G * C]$ such that $\mathbb{R}^{V[C_0]} = \mathbb{R}^{V[C_0']}$. We can now apply Lemma 7.61 in $V_1[G * C]$ to force a V-generic $C_1 \subseteq \mathrm{Col}(\omega, {<}\kappa_1)$ such that $\mathbb{R}^{V[C_1]} = \mathbb{R}^{V_1[G,C]}$ and C_1 extends C_0'.

Since $\mathrm{Col}(\omega, {<}\kappa_0)$ is κ_0-c.c., j can be extended to a

$$\hat{j} : V[C_0'] \to M[C_1].$$

The restriction of $\hat{j}$ to $V(\mathbb{R}_0)$ is an elementary embedding from $V(\mathbb{R}_0)$ to $M(\mathbb{R}_1)$. If τ is a V-term for an element of $V(\mathbb{R}_0)$ then $\hat{j}(\tau^{V(\mathbb{R}_0)}) = \tau^{M(\mathbb{R}_1)}$. Hence the restriction of $\hat{j}$ to $V(\mathbb{R}_0)$ can be defined in $V_1[G * C]$ independently of C_0' and C_1.[78]

We work in $V_1[G * C]$. For $\kappa_0 < \alpha < \kappa_1$, let $m_\alpha = \hat{j}``(G \cap V_\alpha)$. Then each m_α is in $M(\mathbb{R}_1)$. Let $\langle x_\alpha : \alpha < \kappa_1 \rangle$ be an enumeration of the V_1-terms for elements of $P(\kappa_0) \cap V_1[G]$ such that for a closed unbounded set of α and all $\beta < \alpha \in \kappa_1$, $x_\beta \in V_1[G \cap V_\alpha]$. Note that for $\beta < \kappa_1$, $\langle x_\alpha : \alpha < \beta \rangle$ is in $M(\mathbb{R}_1)$. Let $\blacktriangle$ be a well-ordering of $\hat{j}(\mathbb{Q}')$ in $M(\mathbb{R}_1)^{j(\mathrm{Add}(\omega_1))}$. Define a descending sequence $\langle p_\alpha : \alpha < \kappa_1 \rangle \subseteq \hat{j}(\mathbb{Q})$ such that:

1. for each α and γ, if $x_\alpha \in V_1[G \cap V_\gamma]$, then $p_{\alpha+1} \cap M_{\hat{j}(\gamma)}$ decides $\|\kappa_0 \in \hat{j}(x_\alpha)\|$,
2. if $p_\alpha \in V_{\hat{j}(\gamma)}$, then p_α is compatible with m_γ, and
3. if $p_{\alpha+1} = q_0^\alpha * q_1^\alpha \in \mathrm{Add}(\omega_1) * \mathbb{Q}'$ then $q_0^\alpha \Vdash$ "q_1^α is the $\blacktriangle$-least element of $\mathbb{Q}'$ so that $p_{\alpha+1} < p_\alpha$ and $p_{\alpha+1}$ has q_0^α as its first coordinate and satisfies 1 and 2".

[78] The generic C_0' and C_1 are used to show that the embedding is elementary, not to define the embedding.

Using 3, and the fact that $\langle q_0^\alpha : \alpha < \beta\rangle \in M(\mathbb{R}_1)$, one can check that for all $\beta < \kappa_1$ the sequence $\langle p_\alpha : \alpha < \beta\rangle \in M(\mathbb{R}_1)$.

The sequence $\langle p_\alpha : \alpha < \kappa_1\rangle$ induces an ultrafilter U on $P(\kappa_0) \cap V_1[G]$ that is κ_0-complete for sequences that lie in $V_1[G]$ and belongs to $V_1[G*C]$. Define an ideal J in $V_1[G]$ by putting $x \in J$ iff $\|x \in U\| = 0$, where the Boolean value is taken in $\mathcal{B}(\mathrm{Col}(\omega, \omega_1))$. Equivalently, $x \in \breve{J}$ iff $\|$there is an α, $p_\alpha \Vdash \kappa_0 \in \hat{j}(x)\| = 1$.

To see that J is a normal ideal, let $\langle x_\beta : \beta < \kappa_0\rangle \in V_1[G]$ be a sequence of elements of $\breve{J}$. Then for all β, $\|$for some α, $p_\alpha \Vdash \kappa_0 \in \hat{j}(x_\beta)\| = 1$. Since in $V_1[G * C]$,

1. $\mathrm{cf}(\kappa_1) > \kappa_0$,

2. $\langle p_\alpha : \alpha \in \kappa_1\rangle$ is a descending sequence, and

3. the forcing yielding $V_1[G * C]$ is κ_1-c.c.,

we see that $\|$for some α, $p_\alpha \Vdash \kappa_0 \in \bigcap\{\hat{j}(x_\beta) : \beta < \kappa_0\}\| = 1$. Hence $\|$for some α, $p_\alpha \Vdash \kappa_0 \in \hat{j}(\Delta\{x_\beta\})\| = 1$. Thus $\breve{J}$ is closed under diagonal intersections, so J is normal.

A similar argument shows that the map $x \mapsto \|x \in U\|$ induces a Boolean algebra monomorphism from $P(\kappa_0)/J$ to $B(\mathrm{Col}(\omega, \omega_1))$. Hence, J is $\aleph_2$-saturated. Since J is normal, this map is a regular embedding and thus $P(\kappa_0)/J$ is isomorphic to a regular subalgebra of $B(\mathrm{Col}(\omega, \omega_1))$ that collapses ω_1. Thus J is an $\aleph_1$-dense ideal. ⊣

7.15. The Lower End

Proposition 6.26 showed that for successor cardinal κ there is a proper class of forbidden intervals that contain no cardinals λ with κ-complete, κ^+-saturated ideals. This generalized a result of Kunen for $\kappa = \omega_1$. Theorem 7.59 showed that the cardinal bounds at the upper ends of the forbidden intervals are sharp. At the other ends of the intervals, the situation is much less clear.

We state a theorem which implies the consistency of an $\aleph_1$-dense, countably complete, uniform ideal on ω_2. This is the best result known at this time. We outline the proof of this consistency result. The whole proof is in Foreman [36]. The proof has two parts: the first part is an unexciting, but original, application of the Woodin and Kunen techniques to get ideals on consecutive cardinals. The second part is a method for transferring ideals from κ to κ^+.[79]

We begin with a definition, one that gives a slight strengthening of the notion of "strongly layered ideal".

[79] On the other hand, the transfer result in the second part is completely general. The obstacle to getting $\aleph_1$-dense ideals on ω_n for $n > 2$ is in generalizing the "unexciting" first part.

7.62 Definition. A normal κ^+-complete ideal I on κ^+ is *very strongly layered* iff

$$P(\kappa^+)/I = \bigcup\{B_\alpha : \alpha < \kappa^{+2}\}$$

where:

1. the sequence $\langle B_\alpha : \alpha < \kappa^{+2}\rangle$ is increasing and continuous, and for all α, $|B_\alpha| = \kappa^+$. In other words $\langle B_\alpha : \alpha < \kappa^{+2}\rangle$ is a filtration,

2. there is a dense set $D \subseteq P(\kappa^+)/I$ that is closed under descending $<\kappa$-sequences and finite non-zero meets (i.e. if $\{d_1, \dots, d_n\}$ are in D and $\bigwedge d_i \neq 0$ then $\bigwedge d_i \in D$),

3. if $\alpha \in \kappa^{+2}$ and $\mathrm{cf}(\alpha) = \kappa^+$ or α is a successor ordinal, then B_α is a regular subalgebra of $P(\kappa^+)/I$. Further, there is a commuting family of projection maps $\{\pi_\alpha : \alpha \in \kappa^{+2} \cap (\mathrm{Cof}(\kappa^+) \cup \mathrm{Succ})\}$ such that $\pi_\alpha : D \to (D \cap B_\alpha)$, $\pi_\alpha \restriction (D \cap B_\alpha)$ is the identity, and for $\alpha < \beta$ we have $\pi_\alpha \circ \pi_\beta = \pi_\alpha$, and

4. there is a dense set $D'_0 \subseteq D \cap B_0$ such that $(D'_0, \leq_I)$ is isomorphic to $\mathrm{Col}(\kappa, \kappa^+)$.

Essentially any model with a layered ideal can be turned into a model with a very strongly layered ideal by forcing a closed unbounded set through the appropriate stationary set. The "very" part of the definition holds in all models with strongly layered ideals known to the author.

The first step in the consistency result is to prove the following:

7.63 Theorem. *Let j_0 and j_1 be almost huge embeddings with critical points κ_0 and κ_1 respectively. Suppose that $j_0(\kappa_0) = \kappa_1$ and that $\kappa_2 = j_1(\kappa_1)$ is Mahlo. Let $\mu < \kappa_0$ be regular. Then there is a partial ordering $\mathbb{P}$ such that there is a definable subclass W of $V^{\mathbb{P}}$ satisfying:*

1. $\kappa_0 = \mu^+$,

2. ZFC + GCH + $\diamondsuit_{\mu^+}$ + $\diamondsuit_{\mu^{+2}}(\mathrm{Cof}(\mu^+))$ + $\square_{\mu^{+2}}$*,*

3. there is an μ^+-dense ideal J on μ^+, and

4. there is a very strongly layered ideal I on μ^{+2}.

7.64 Remark. It is not known how to get a successor cardinal κ with a κ-dense ideal on κ and very strongly layered ideals on κ^+ and κ^{+2}. The weaker property that there are three successor consecutive cardinals with strongly layered ideals is also open.

The next theorem is the heart of the consistency result:

7.65 Theorem (Foreman [36]). *Let κ be a regular cardinal with $\kappa^{<\kappa} = \kappa$. Suppose there is a very strongly layered ideal I on κ^+, $\square_{\kappa^+}$, and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$. Then there is a κ-complete uniform ideal $K \supseteq I$ on κ^+ such that*

$$P(\kappa^+)/K \cong P(\kappa)/\{\text{bounded sets}\}.$$

From this one sees:

7.66 Corollary. *Suppose that κ is regular, $\kappa^{<\kappa} = \kappa$ and there is a very strongly layered ideal I on κ^+, $\square_{\kappa^+}$ and $\diamondsuit_{\kappa^+}(\mathrm{Cof}(\kappa))$. Then for all uniform ideals J on κ, there is a uniform ideal K on κ^+ such that:*

$$P(\kappa^+)/K \;\cong\; P(\kappa)/J.$$

Furthermore, the degree of completeness of K equals the degree of completeness of J, and if J is κ^+-saturated then K is weakly normal.

This corollary shows that in the model built in Theorem 7.63 with $\mu = \omega$, there is a weakly normal, countably complete, uniform $\aleph_1$-dense ideal on ω_2. Since the hypotheses of Theorem 7.65 hold in a model where every $\aleph_2$-c.c., ω_1-collapsing Boolean algebra of cardinality at most ω_2 is realized as a quotient of an ideal on ω_1, we see that every such Boolean algebra is realized as a quotient of a weakly normal, countably complete, uniform ideal on ω_2.

7.67 Remark. There can be no analogue of Corollary 7.66 for normal fine ideals on $[\kappa^+]^\kappa$, because of Remark 5.39. If there were, there would simultaneously be an $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$ and an ideal on $[\omega_2]^{\omega_1}$ with quotient isomorphic to $\mathcal{B}(\mathrm{Col}(\omega, {<}\omega_2))$—an impossibility.

We now outline the first step in the proof of Theorem 7.65. This step is easier to describe than the main construction in [36].

The main idea of Theorem 7.65 is to build a surjective homomorphism $h : P(\kappa^+)/I \to P(\kappa)/\{\text{bounded sets}\}$. The ideal K will be the kernel of h, which we must verify is κ-complete. It follows that

$$P(\kappa^+)/K \cong P(\kappa)/\{\text{bounded sets}\}.$$

Fix a strongly layered ideal I, and witnesses $B_\alpha, D, \pi_\alpha, \ldots$ to the strong layering. Let $D_\alpha = D \cap B_\alpha$.

Given a subset of κ^+ we need to "measure" it by a subset of κ. Any function $f : \kappa \to D$ measures each set $x \subseteq \kappa^+$ by yielding the set $A_x = \{i : f(i) \subseteq_I x\}$. Unfortunately this measurement may be ambiguous in that typically $\kappa^+ \setminus A_x \neq A_{\kappa^+ \setminus x}$ (modulo bounded sets). This failure is equivalent to the statement that it is not the case that for all sufficiently large i, either $f(i) \subseteq_I x$ or $f(i) \subseteq_I \kappa^+ \setminus x$.

It is hopeless to unambiguously measure every subset of κ^+ with a single function f. Hence we need a family of functions. Furthermore, the measurements these functions make must agree with each other. This is the motivation for the first three clauses of the following definition. The last clause is a coding device to make the homomorphism surjective. In what follows we will use $\leq$ to mean $\subseteq_I$.

We will construct a matrix of functions:[80]

$$\mathcal{F} = \{f^\delta_\gamma : \gamma < \kappa^+,\ \delta \in \kappa^{+2} \cap (\mathrm{Cof}(\kappa^+) \cup \mathrm{Succ})\}$$

such that for each δ, γ,

$$f^\delta_\gamma : \kappa \to D_\delta.$$

The family of functions $\mathcal{F}$ will satisfy the following four properties:

1. Horizontal Coherence: For $\gamma < \gamma'$, for all but less than κ many i, $f^\delta_\gamma(i) \geq f^\delta_{\gamma'}(i)$.

2. Vertical Coherence: For $\delta < \delta'$ there is an unbounded set of $\gamma < \kappa^+$ such that for all but less than κ many i, $f^\delta_\gamma(i) = \pi_\delta(f^{\delta'}_\gamma(i))$.

3. Genericity: For each $x \subseteq \kappa^+$, with $x \in B_\delta$ there is a $\gamma < \kappa^+$ such that for all but less than κ many i, either $f^\delta_\gamma(i) \leq x$ or $f^\delta_\gamma(i) \wedge x =_I 0$.

   Let $D'_0 \subseteq D_0$ be dense with $D'_0 \cong \mathrm{Col}(\kappa, \kappa^+)$. If f^0_γ takes values in D'_0, then by using this isomorphism we can assume that for all i, $f^0_\gamma(i) \in \mathrm{Col}(\kappa, \kappa^+)$.

4. Coding: Fix an enumeration $P(\kappa) = \{y_\eta : \eta < \kappa^+\}$. For each $\eta < \kappa^+$, there is a γ_η such that for all i, $f^0_{\gamma_\eta}(i) \in D'_0$, and $\gamma_\eta \in \mathrm{ran}(f^0_{\gamma_\eta}(i))$. Further, letting $\beta_\eta(i)$ be the least β such that $f^0_{\gamma_\eta}(i)(\beta) = \gamma_\eta$, then $y_\eta = \{i : \beta_\eta(i) \text{ is a limit ordinal}\}$.

7.68 Remark. The purpose of the vertical coherence condition is to show that the homomorphism h defined in Claim 7.69 below is well-defined. It can be weakened to the following statement:

Weak Vertical Coherence: For all $\delta < \delta'$ and all γ' there is a $\gamma > \gamma'$ such that for sufficiently large i, $f^\delta_\gamma(i) \leq \pi_\delta(f^{\delta'}_{\gamma'}(i))$.

7.69 Claim. *If there is a set of functions $\mathcal{F}$ satisfying the conditions 1–4, there is a surjective homomorphism $h : P(\kappa^+)/I \to P(\kappa)/\{\text{bounded sets}\}$ with a κ-complete kernel.*

Proof. Given the set of functions $\mathcal{F}$ and an $x \subseteq \kappa^+$, we look at the least δ such that $x \in B_\delta$. By genericity there is a $\gamma < \kappa^+$ such that for sufficiently large i, $f^\delta_\gamma(i) \subseteq_I x$ or $f^\delta_\gamma(i) \cap x =_I 0$. Let $A_x = \{i : f^\delta_\gamma(i) \subseteq_I x\}$.

We claim that for all $\delta' > \delta$ and all sufficiently large γ' (depending on δ'), $A_x = \{i : f^{\delta'}_{\gamma'}(i) \subseteq_I x\}$ modulo bounded sets. Namely, fix a δ' and choose

[80] Here, Succ denotes the class of successor ordinals.

a $\gamma' > \gamma$ where for sufficiently large i, $f^{\delta}_{\gamma'}(i) = \pi_\delta(f^{\delta'}_{\gamma'}(i))$. Since $x \in B_\delta$ and π_δ is a projection map, $f^{\delta'}_{\gamma'}(i) \subseteq_I x$ iff $\pi_\delta(f^{\delta'}_{\gamma'}(i)) \subseteq_I x$. Since $\gamma' > \gamma$ for all but less than κ many i, $f^{\delta}_{\gamma'}(i) \subseteq_I x$ iff $f^{\delta}_{\gamma}(i) \subseteq_I x$ and similarly for $\kappa^+ \setminus x$. Hence for sufficiently large i, $f^{\delta}_{\gamma'}(i) \subseteq_I x$ or $f^{\delta}_{\gamma'}(i) \subseteq_I \kappa^+ \setminus x$, and $A_x = \{i : f^{\delta}_{\gamma'}(i) \subseteq_I x\} = \{i : f^{\delta'}_{\gamma'}(i) \subseteq_I x\}$.

Define a function $h : P(\kappa^+) \to P(\kappa)/\{\text{bounded sets}\}$ by setting $h(x) = [A_x]$. Then h is well-defined by the remarks in the previous paragraph. To see that h is a homomorphism, it suffices to show that h preserves complements and intersections. Clearly, for all δ, γ and all x, y,

$$\{i : f^{\delta}_{\gamma}(i) \subseteq_I x \cap y\} = \{i : f^{\delta}_{\gamma}(i) \subseteq x\} \cap \{i : f^{\delta}_{\gamma}(i) \subseteq y\}.$$

Hence $h(x \cap y) = h(x) \cap h(y)$. Let $x \subseteq \kappa^+$. Choose sufficiently large δ, γ such that $h(x) = \{i : f^{\delta}_{\gamma}(i) \subseteq x\}$ and for all sufficiently large i, $f^{\delta}_{\gamma}(i) \subseteq x$ or $f^{\delta}_{\gamma}(i) \cap x =_I 0$. Then

$$h(\kappa^+ \setminus x) = \{i : f^{\delta}_{\gamma}(i) \subseteq_I \kappa^+ \setminus x\} = \{i : f^{\delta}_{\gamma}(i) \cap x =_I 0\} = \kappa \setminus h(x).$$

To see that h is surjective, fix some $[y_\eta] \in P(\kappa)$. By the coding condition 4 on $\mathcal{F}$, there is a γ_η such that for all i, $f^{0}_{\gamma_\eta}(i) \in D'_0$, $\gamma_\eta \in \operatorname{ran}(f^{0}_{\gamma_\eta}(i))$ and $y_\eta = \{i : \text{the least } \beta \text{ with } f^{0}_{\gamma_\eta}(i)(\beta) = \gamma_\eta \text{ is a limit ordinal}\}$. Since a generic ultrafilter for $P(\kappa^+)/I$ canonically induces a generic ultrafilter on D'_0, we have a canonical term $\dot{G}_0$ for a generic object for $\mathrm{Col}(\kappa, \kappa^+)$.

Let $x = \|\text{the least } \beta \text{ with } \dot{G}_0(\beta) = \gamma_\eta \text{ is a limit}\|$ where the Boolean value is taken in the forcing $P(\kappa^+)/I$. Then $x \in B_\delta$, for some δ. Hence $h(x) = [\{i : f^{\delta}_{\gamma}(i) \subseteq_I x\}]$ for all sufficiently large γ.

By the coding condition 4, for all i, $f^{0}_{\gamma_\eta}(i) \in \mathrm{Col}(\kappa, \kappa^+)$ and γ_η is in the range of $f^{0}_{\gamma_\eta}(i)$. Hence $f^{0}_{\gamma_\eta}(i) \subseteq_I x$ iff the least β with $f^{0}_{\gamma_\eta}(i)(\beta) = \gamma_\eta$ is a limit. Otherwise, $f^{0}_{\gamma_\eta}(i) \subseteq_I \kappa^+ \setminus x$.

Hence, by the coding condition, for $\gamma \geq \gamma_\eta$ and sufficiently large i, $f^{0}_{\gamma}(i) \subseteq_I x$ or $f^{0}_{\gamma}(i) \subseteq_I \kappa^+ \setminus x$. Hence we see that for all sufficiently large $\gamma \geq \gamma_\eta$, $f^{\delta}_{\gamma}(i) \subseteq_I x$ iff $f^{0}_{\gamma}(i) \subseteq_I x$, and $h(x) = [\{i : f^{0}_{\gamma_\eta}(i) \subseteq_I x\}]$.

But $f^{0}_{\gamma_\eta}(i) \subseteq_I x$ iff $i \in y_\eta$, by the coding condition. Hence, $h(x) = [y_\eta]$, and we have shown that h is surjective.

Let K be the kernel of h. To see that K is κ-complete, let $\eta^* < \kappa$ and $\{X_\eta : \eta \in \eta^*\} \subseteq K$. Then for all δ, γ and all η, $\{i : f^{\delta}_{\gamma}(i) \subseteq_I X_\eta\} =_{\{\text{bounded sets}\}} 0$. Let δ, γ be so large that for all $\eta \in \eta^*$ and sufficiently large i, $f^{\delta}_{\gamma}(i) \subseteq_I X_\eta$ or $f^{\delta}_{\gamma}(i) \cap X_\eta =_I 0$ and that $h(\bigcup X_\eta) = \{i : f^{\delta}_{\gamma}(i) \subseteq_I \bigcup X_\eta\}$. Then $h(\bigcup X_\eta) = \bigcup\{i : f^{\delta}_{\gamma}(i) \subseteq_I X_\eta\}$ and so has size less than κ. Hence, $\bigcup X_\eta \in K$. $\dashv$

The heart of Theorem 7.65 is the construction of the matrix of functions $\mathcal{F}$. This uses the powerful $\diamondsuit$ techniques forged by Shelah in his papers *Models with second-order properties* I–V (see e.g. [101]). Though the published proof [36] uses square, it is not necessary for the construction.

7.16. Chang-Type Ideals with Simple Quotients II

In this section we see how the technique of the previous section can be used to produce an ideal I on $[\omega_2]^{\omega_1}$ that is normal and fine, and is such that forcing with $P([\omega_2]^{\omega_1})/I$ is equivalent to forcing with an $\aleph_1$-centered partial ordering $\mathbb{P}$ followed by an $<\aleph_2^V$-closed partial ordering; moreover, GCH holds in the model with this ideal. A pseudo-generic tower argument shows that this implies the existence of an $\aleph_1$-centered ideal on ω_1.

Donder [23] pointed out that if there is an $\aleph_1$-centered ideal on ω_1 and CH holds, then one can force an example of a c.c.c.-destructible $\aleph_2$-saturated ideal on ω_1. This follows because there is a well-known forcing for adding $\square_{\omega_1}$ without adding new subsets of ω_1. The relationship between square and c.c.c.-destructibility is discussed in Sect. 8.6.

7.70 Theorem. *Suppose that there is a huge cardinal κ and $\mu < \kappa$ is regular. Then there is a partial ordering $\mathbb{Q}$ such that for all generic $G \subseteq \mathbb{Q}, V[G]$ satisfies* GCH *and the statement: There is a normal fine ideal I on $[\mu^{+2}]^{\mu^+}$, a μ^+-centered partial ordering $\mathbb{P}$ and an $\mathbb{R} \in V^{\mathbb{P}}$ that is $<(\mu^{+2})^V$-closed in $V^{\mathbb{P}}$ such that:*

$$\mathcal{B}(P([\mu^{+2}]^{\mu^+})/I) \cong \mathcal{B}(\mathbb{P} * \mathbb{R}).$$

We also refer the reader to Theorem 5.2 for some consequences of the existence of such an ideal in combinatorics and algebra.

Proof. We now briefly outline the argument for Theorem 7.70. We will use the Kunen technique to build partial orderings $\mathbb{P}(\alpha)$ and $\mathbb{R}(\alpha,\beta)$ such that both are definable from Mahlo α, β and:

1. $\mathbb{P}(\alpha) \subseteq V_\alpha$, $\mathbb{R}(\alpha,\beta) \subseteq V_\beta$ and $\mathbb{P}(\alpha)$ is α-c.c. and $<\mu$-closed, and $\mathbb{R}(\alpha,\beta)$ is $<\alpha$-closed and β-c.c.
2. $\mathbb{P}(\alpha)$ collapses α to be μ^+ and $\mathbb{R}(\alpha,\beta)$ collapses β to be α^+.
3. For $\alpha < \beta$, if the identity map from $\mathbb{P}(\alpha)$ maps to $\mathbb{P}(\beta)$ and is a regular embedding, then $\mathbb{P}(\alpha) * \mathbb{R}(\alpha,\beta)$ is regularly embedded in $\mathbb{P}(\beta)$ by a map $i_{\alpha,\beta}$ extending the identity.
4. Moreover if $G \subseteq \mathbb{P}(\alpha) * \mathbb{R}(\alpha,\beta)$ is generic then in $V[G]$, the quotient forcing $\mathbb{P}(\beta)/i_{\alpha,\beta}$"$G$ is α-centered.
5. $\mathbb{R}(\alpha,\beta)$ has the right "shape" for the existence of master conditions.

Most of the properties on the list above are familiar, the new one being 4.

The conditions on the list guarantee that if $j : V \to M$ is a huge embedding with critical point κ, $j(\kappa) = \lambda$ and one forces with $\mathbb{P}(\kappa) * \mathbb{R}(\kappa,\lambda)$ to get a G then one can generically extend the huge embedding to a $\hat{j} : V[G] \to M[\hat{G}]$. The ideal I will be the ideal induced by $\hat{j}$ and the ideal element j"λ. The ideal has the correct quotient, since $\mathbb{P}(\lambda) * \mathbb{R}(\lambda, j(\lambda))/G$ is κ-centered followed by $<\lambda$-closed forcing. $\dashv$

The centering argument from [41] can be seen to work in the model discussed in Sect. 7.12. In particular, the construction in Theorem 7.57 gives the stronger statement:

7.71 Theorem. *Suppose that there is a huge cardinal. Then there is a model of* ZFC + *"For all regular cardinals κ there is a κ-complete, κ-centered ideal on κ".*

7.17. Destroying Precipitous and Saturated Ideals

We now consider the possibility of completely ridding the universe of ideals with nice embedding properties. Since the existence of generic elementary embeddings implies the existence of inner models with large cardinals, if one starts with a small enough model (such as L) there are no ideals with nice properties. Even this is not completely understood, as it is not known if there are precipitous ideals on successor cardinals in $L[E]$ models.

Here is what is known about forcing over an arbitrary model V to get rid of precipitous and saturated ideals.

Baumgartner [3] described a partial ordering $\mathbb{D}$ for adding a closed unbounded subset of ω_1 with finite conditions. Elements of $\mathbb{D}$ can be viewed as finite collections of disjoint intervals of countable ordinals that are open at the lower end and closed at the upper end. Extension is by taking a bigger collection of intervals. These intervals approximate the complement of a closed unbounded set $D \subseteq \omega_1$ in the extension. The following facts were shown by Baumgartner:

1. $\mathbb{D}$ adds a closed unbounded subset of ω_1 that does not include any closed unbounded set in V, and
2. $\mathbb{D}$ is proper and hence preserves ω_1.

The following result appears in [6] as Theorem 3.5:

7.72 Theorem (Baumgartner-Taylor [6]). *Suppose that $G \subseteq \mathbb{D}$ is generic over V. Then in $V[G]$ there are no $\aleph_2$-saturated ideals on ω_1.*

Assuming GCH, there are analogues of Baumgartner's forcing for each successor cardinal μ^+. These involve forcing with approximations to the complement of the generic closed unbounded set that have size $< \mu$ and satisfy some continuity conditions. The Baumgartner-Taylor argument generalizes directly to show that after forcing over a model of GCH with a forcing of this type, there are no μ^{+2}-saturated ideals on μ^+.

The situation for precipitous ideals is less satisfactory. In [47], the following theorem is shown:

7.73 Theorem. *For all κ with $\kappa^{<\kappa} = \kappa$, there is a forcing $\mathbb{Q}$ that is $<\kappa$-closed, κ^+-c.c such that if I is a normal ideal in V then its normal closure in $V^{\mathbb{Q}}$ is not precipitous.*

By Theorem 7.73 it is consistent with a supercompact cardinal that the nonstationary ideal on ω_1 is not precipitous, but it is not known how to build a model with a supercompact cardinal where there is no precipitous ideal on ω_1. The results of the next section show that the nonstationary ideal on ω_1 is always a pre-precipitous ideal if one assumes the existence of Woodin cardinals.

8. Consistency Results for Natural Ideals

In this section we discuss methods of creating models where various natural ideals yield well-founded generic ultrapowers. These results fall into three categories. The first type are situations where one starts with an induced ideal with strong properties and forces that ideal to be a natural ideal while maintaining the properties of the generic embedding. One way to do this is to shoot closed unbounded sets through sets in the dual of the ideal.

The second type of result begins with a natural ideal and manipulates its antichain structure in such a way so as to make the generic ultrapower have strong properties. A typical method for making an ideal saturated is to iterate collapsing the size of each maximal antichain, while maintaining its maximality.

The third method for making natural ideals have strong properties was pioneered by Steel and Van Wesep [115]. This technique starts with a model of ZF together with some determinacy hypothesis and forces to add choice to construct a model where the NS_{ω_1} is $\aleph_2$-saturated.

8.1. Forcing over Determinacy Models

The first construction of a model of ZFC where NS_{ω_1} is $\aleph_2$-saturated was done by Steel and Van Wesep. Their technique was extremely original in that it started with an inherently choiceless model of $V = L(\mathbb{R})$ and added a well-ordering of the real numbers. If NS_{ω_1} is dual to an ultrafilter in the original model, then one can hope that the forcing for adding Choice is mild enough to preserve its saturation. Their theorem is:

8.1 Theorem (Steel-Van Wesep [115]). *Suppose that V is a model of* ZF + $\mathrm{AD}_{\mathbb{R}}$ + *"Θ is regular". Then there is a forcing extension of* ZFC *in which NS_{ω_1} is $\aleph_2$-saturated.*

Woodin improved their result by weakening the hypothesis to the assumption AD + $V = L(\mathbb{R})$. Later work of Woodin showed that even the following is possible:

8.2 Theorem (Woodin [126]). *Suppose that V is a model of* AD + $V = L(\mathbb{R})$. *Then there is a forcing extension in which NS_{ω_1} is $\aleph_1$-dense.*

Note that Shelah's Corollary 5.58 implies $2^{\aleph_0} = 2^{\aleph_1}$ in this model. In contrast, Woodin was able to show from similar determinacy hypotheses that it is consistent to have CH together with the statement "there is a dense set $D \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$ such that for all $S \in D$, $P(S)/\mathrm{NS}_{\omega_1}$ is $\aleph_1$-dense".

Using $\mathbb{P}_{\max}$ and $\mathbb{Q}_{\max}$ techniques, Woodin was able to show many related consistency results. For example, he showed the consistency of strong club guessing together with NS_{ω_1} being $\aleph_2$-saturated. Woodin's $\mathbb{P}_{\max}$ techniques are covered in detail in his book [126], as well as in Larson's chapter in this Handbook.

These results have the peculiar feature that it is not clear from the method that *every* model of ZFC can be included in a model in which NS_{ω_1} has strong saturation properties.

8.2. Making Induced Ideals Natural

These constructions are based on various types of forcing that transform sets in an induced ideal into members of a given natural ideal. For example, in the case of the nonstationary ideal, these partial orderings shoot closed unbounded sets through stationary sets S in the dual of the induced ideal. These make the complement of S nonstationary in the generic extension. For other natural ideals the constructions use other mechanisms, such as forcing sets in the ideals to be meager or adding strong club guessing sequences on sets in the dual filter.

The general outline of this type of argument is to start with a generic elementary embedding $j_0 : V_0 \to M_0$ and let I be the induced ideal from $U(j_0, i)$ for some i. A forcing construction is carried out that makes elements of $\breve{I}$ belong to the dual of the natural ideal. Care must be taken to make sure that the generic embedding j can be extended during the forcing construction. The ability to extend the embedding j implies that the critical point of j remains a regular cardinal after the forcing.

A complication to this outline is that if j_α is the generic embedding after α stages of the iteration, then the induced ideal I_α for $U(j_\alpha, i)$ may properly contain the initial ideal I. Thus the construction involves:

1. A "nice" initial ideal $I = I_0$.

2. An iteration $\langle(\mathbb{R}_\alpha, \dot{\mathbb{Q}}_\alpha) : \alpha < \lambda\rangle$ such that in $V_0^{\mathbb{R}_\alpha}$, the original embedding j can be generically extended to an embedding j_α from $V_0^{\mathbb{R}_\alpha}$ to $M_0^{j(\mathbb{R}_\alpha)/m_\alpha}$ where m_α is a master condition. Moreover, for $\alpha < \beta$ we have $m_\beta \leq m_\alpha$ and $j_\alpha \subseteq j_\beta$.

3. An sequence of ideals $\langle I_\alpha : \alpha < \lambda\rangle$ where I_α is the induced ideal for $U(j_\alpha, i)$. Since the j_α's cohere and the m_α's get stronger with α, we have $I_\alpha \subseteq I_\beta$ for $\alpha < \beta$.

4. For every element S that belongs to some I_α for an $\alpha < \lambda$ there is a β so that $\mathbb{Q}_\beta$ puts S into the natural ideal.

5. $I_\infty =_{\mathrm{def}} \bigcup_{\alpha<\lambda} I_\alpha$ is the natural ideal in $V_0^{\mathbb{R}_\lambda}$.

The final model will be $V_0^{\mathbb{R}_\lambda}$. Typically $j_\infty =_{\mathrm{def}} \varinjlim j_\alpha$ gives a generic embedding from $V_0^{\mathbb{R}_\lambda}$ to $M_0^{j(\mathbb{R}_\lambda)}$ and the induced ideal from $U(j_\infty, i)$ is I_∞. By the last clause 5, I_∞ is the natural ideal.

If the original ideal I has nice properties, e.g. that I is saturated or that I is precipitous and the embedding j was the generic ultrapower of I, then I_∞ can be shown to retain some of these properties.

The Null and Meager Ideals

We first deal with two relatively easy examples, the null ideal and the meager ideal. That the ideal of null sets of a measure can be precipitous follows from the next result due to Solovay [111].

8.3 Theorem. *Suppose that κ is measurable. Then there is a forcing extension in which there is a countably additive $\mu : P([0,1]) \to [0,1]$.*

Since the ideal of μ-null subsets of the reals is necessarily c.c.c. and $\aleph_1$-complete, it is an example of a precipitous ideal. Later work of Kunen showed that this ideal can be taken to extend the ideal of Lebesgue null subsets of the real line. Dow remarked that the ideal of Lebesgue null subsets of the unit interval can never be c.c.c. since there is an uncountable pairwise disjoint collection of sets of outer measure one.

As far as the author knows it is an open question whether it is consistent for the ideal of Lebesgue null subsets of $[0,1]$ to be a precipitous ideal on $P([0,1])$.

The analogous result for the meager ideal is open; however, Komjáth showed the following:

8.4 Theorem (Komjáth [73]). *Suppose that there is a measurable cardinal. Then there is a forcing extension in which there is a non-meager set $A \subseteq \mathbb{R}$ such that $P(A)/\{\text{meager sets}\}$ is c.c.c.*

Proof. We sketch the proof. Let κ be measurable and U a normal ultrafilter on κ. The idea is to make $\breve{U}$ into the meager ideal restricted to a non-meager set. More explicitly, one first adds a sequence of Cohen reals $\langle s_\alpha : \alpha < \kappa\rangle$ and then forces that $S \subseteq \{s_\alpha : \alpha < \kappa\}$ is meager iff there is an $Y \in \breve{U}$ such that $S \subseteq \{s_\alpha : \alpha \in Y\}$.

We start by describing a forcing for making a particular set meager. Fix a set $S \subseteq \omega^\omega$. We let $\mathbb{P}_S$ be the collection of conditions

$$p = \langle \{s_0, \ldots, s_n\}, c, \{U^i_j : i, j \in F\}\rangle$$

where:

1. U^i_j are basic open sets in ω^ω and F is a finite subset of ω,
2. $n \in \omega$ and $\{s_0, \ldots, s_n\} \subseteq S$,

3. $c : \{s_0, \dots, s_n\} \to F$, and
4. $s_m \notin \bigcup_{j \in F} U_j^{c(s_m)}$.

For conditions $p, q \in \mathbb{P}_S$ we say that q is stronger than p if q can be built from p by adding elements of S to $\{s_0, \dots, s_n\}$, extending c and expanding the collection of U_j^i to a larger collection $\{U_j^i : i, j \in F' \supseteq F\}$.

$\mathbb{P}_S$ is $\aleph_1$-Knaster[81] and, as is well-known for such partial orderings, arbitrary products of partial orderings of the form $\mathbb{P}_S$ with finite support are c.c.c. Moreover, if H is generic for $\mathbb{P}_S$ and $\{U_j^i : i, j \in \omega\}$ are the basic open sets appearing in the conditions in H, then for each i, $\bigcup_j U_j^i$ is open dense and $S \cap \bigcap_i \bigcup_j U_j^i = \emptyset$

Komjáth's model is built by starting with a κ-complete, normal ultrafilter U on a measurable cardinal κ and adding κ Cohen reals $A = \langle s_\alpha : \alpha < \kappa \rangle$. The second step is to force with the finite support product $\mathbb{P} = \prod_{S \in I} \mathbb{P}_S$, where $S \in I$ iff $S \subseteq A$ and $\{\alpha : s_\alpha \in S\} \in \breve{U}$.

Fix $G * H$ generic for $\mathrm{Add}(\omega, \kappa) * \mathbb{P}$. Since the two step forcing $\mathrm{Add}(\omega, \kappa) * \mathbb{P}$ is c.c.c. it follows from Kakuda's theorem (Corollary 7.17) that $\breve{U}$ remains a precipitous ideal on $P(\kappa)$ in $V[G * H]$.[82] We will be done if we can show that in $V[G * H]$, for all $Y \subseteq A$, Y is meager iff $\{\alpha : s_\alpha \in Y\} \in \breve{U}$.

One direction of this is clear. If $\{\alpha : s_\alpha \in T\} \in \breve{U}$, then there is an $Y \in V \cap \breve{U}$ such that $\{\alpha : s_\alpha \in T\} \subseteq Y$. If $S = \{s_\alpha : \alpha \in Y\}$ then $\mathbb{P}_S$ is a factor of the forcing and hence S is meager.

Suppose now that $T \subseteq A$ is meager. We show that there is a countable collection $C \subseteq I$, such that $T \subseteq \bigcup C$. This suffices, since $\breve{U}$ is countably complete in $V[G * H]$ and $\{\alpha : s_\alpha \in T\} \subseteq \bigcup_{S \in C} \{\alpha : s_\alpha \in S\}$.

In $V[G * H]$ there is a sequence of basic open sets $\langle O_j^i : i, j \in \omega \rangle$ such that in $V[G * H]$, for all i, $\bigcup_j O_j^i$ is dense and $T \cap \bigcap_i \bigcup_j O_j^i = \emptyset$. Since the forcing $\mathrm{Add}(\omega, \kappa) * \mathbb{P}$ is c.c.c. there is an $\alpha_0 < \kappa$ and a countable subset $C \subseteq I$ such that $\langle O_j^i : i, j \in \omega \rangle$ belongs to

$$V\big[\{s_\alpha : \alpha \in \big(\alpha_0 \cup \bigcup C\big)\}\big]\big[H \cap \textstyle\prod_{S \in C} \mathbb{P}_S\big].$$

If T is not a subset of $\bigcup C$ modulo bounded sets, then we can find a $\beta > \alpha_0$ such that $a_\beta \in T \setminus \bigcup C$. But then a_β is generic over the above model and hence belongs to $\bigcap_i \bigcup_j O_j^i$, a contradiction. ⊣

Nonstationary Ideals

The nonstationary ideals on general $Z \subseteq P(\lambda)$ are somewhat mysterious. We show in Theorem 8.37 that they can be made precipitous by collapsing a large cardinal to be λ^+. Not much is known about their saturation properties, with

[81] That is, any uncountable set of conditions has an uncountable subset of pairwise compatible conditions.
[82] We can say more: by Theorem 7.14, the quotient $P(A)/\{\text{meager sets}\}$ is c.c.c.

one notable exception: Gitik was able to give an example of a model with a large cardinal κ such that the nonstationary ideal on a stationary subset of $[\kappa^+]^{<\kappa}$ is κ^+-saturated.[83]

We turn to the nonstationary ideals on cardinals. In these arguments we shoot closed unbounded sets through sets in the dual of the induced ideal. The partial ordering usually used can be described as follows. If S is a stationary subset of a regular cardinal κ, then $\mathbb{P}_S$ consists of closed bounded subsets of S, ordered by end extension.

Forcing with this partial ordering can be disastrous, not only collapsing κ but many other cardinals as well. Equally problematic, if $\mathcal{C}$ is a collection of stationary sets, then the iterated forcing given by the partial orderings $\mathbb{P}_S$ for $S \in \mathcal{C}$ can collapse cardinals even if each individual $\mathbb{P}_S$ is well-behaved. What prevents the situation from being hopeless in the cases we are interested in is the fact that the collection $\mathcal{C}$ forms a filter that has a nice generic ultrapower.

Two examples of arguments of this form are in Cummings' chapter in this Handbook. One is the following theorem:

8.5 Theorem (Magidor; see [65]). *Suppose that κ is a measurable cardinal. Then there is a forcing extension in which $\kappa = \omega_1$ and NS_{ω_1} is precipitous.*

The situation for creating the precipitousness of nonstationary ideals on arbitrary regular cardinals is simple, provided there are sufficiently many large cardinals in the universe. Theorem 8.37 shows that if μ is regular and $\kappa > \mu$ is supercompact, then NS_μ is precipitous in $V^{\mathrm{Col}(\mu,<\kappa)}$.

Though the large cardinal hypothesis required in the proof of Theorem 8.37 can be reduced to a Woodin cardinal, that assumption is far from optimal. The following theorems of Gitik [51, 54, 53] give exact equiconsistency results:

8.6 Theorem. *The statement "NS_{ω_2} is precipitous" is equiconsistent with the existence of a measurable cardinal of Mitchell order 2.*

More generally:

8.7 Theorem. *If $\mu > \omega_1$, then* CH + *"NS_{μ^+} is precipitous" is equiconsistent with μ^+ being an $(\omega, \mu + 1)$-repeat point for the normal ultrafilters on μ^+ in the core model K.*

8.8 Theorem. *The property "κ is an inaccessible cardinal and NS_κ is precipitous" is equiconsistent with κ having an $(\omega, \kappa + 1)$-repeat point for the normal ultrafilters on κ in the core model K.*

See Cummings' chapter in this Handbook particularly for Theorem 8.6.

The saturation of NS_{ω_1} is a very special situation that is dealt with by manipulating the antichain structure of $P(\omega_1)/\mathrm{NS}_{\omega_1}$. This is covered in Sect. 8.3. It is not known how to make NS_{ω_1} saturated by making an induced ideal become the nonstationary ideal. This may be related to the problem of whether CH + "NS_{ω_1} is $\aleph_2$-saturated" is consistent with ZFC.

[83] See Theorem 6.35.

The saturation of nonstationary ideals on cardinals greater than ω_1 is even more mysterious. If κ is the successor of a singular cardinal or is weakly inaccessible then Theorem 6.14 implies that for all regular $\delta < \kappa$, the ideal $\mathrm{NS}_\kappa \restriction \mathrm{Cof}(\delta)$ is not κ^+-saturated.

At successors of regular cardinals above ω_1 essentially nothing is known. Corollary 6.11 shows that if I is a κ^+-saturated ideal on a successor cardinal $\kappa = \mu^+$ then $\{\alpha < \mu^+ : \mathrm{cf}(\alpha) = \mathrm{cf}(\mu)\} \in \breve{I}$, so the best one can hope for is that NS_κ restricted to $\mathrm{Cof}(\mu)$ is κ^+-saturated. A prominent open problem is whether it is consistent with ZFC to have NS_{ω_2} restricted to $\mathrm{Cof}(\omega_1)$ to be $\aleph_3$-saturated.

In the positive direction, Woodin has proved the strongest known consistency result about the saturation of the nonstationary ideals on cardinals at least ω_2:

8.9 Theorem (Woodin). *Suppose that μ is a regular cardinal and $\kappa > \mu$ is almost huge. Then there is a forcing extension by a $<\mu$-closed partial ordering that satisfies "there is a stationary set $S \subseteq \mu^+$ such that $\mathrm{NS}_{\mu^+} \restriction S$ is μ^{+2}-saturated". If* GCH *holds in the ground model, then* GCH *holds in the extension.*

From Woodin's theorem, one can argue as in [34] that it is consistent for the successor κ of a supercompact cardinal μ to have a stationary set $S \subseteq \kappa$ such that $\mathrm{NS}_\kappa \restriction S$ is κ^+-saturated. As described in the discussion of Theorem 7.57 [34], it is then possible to collapse μ to be e.g. $\aleph_\omega$ while preserving the saturation of the nonstationary ideal restricted to S. As a corollary we get:

8.10 Corollary. *Suppose that μ is a supercompact cardinal and $\kappa > \mu$ is an almost huge cardinal. Then there is a forcing extension that satisfies* GCH + *"there is a stationary subset S of $\aleph_{\omega+1}$ such that the nonstationary ideal on $\aleph_{\omega+1}$ restricted to S is $\aleph_{\omega+2}$-saturated".*

8.11 Remark. As proved in Theorem 5.64, if NS_{ω_1} is $\aleph_2$-saturated and there is a measurable cardinal, then CH fails in a concrete way. It remains a prominent open problem whether it is consistent to have CH together with NS_{ω_1} being $\aleph_2$-saturated.

Since measurable cardinals seem to be an accepted extension of ZFC, this problem does not seem relevant to settling CH, but a solution to it would certainly require new and interesting techniques.

We now give a sketch of a proof of Woodin's theorem. Woodin's original proof has not been published. We outline a subroutine that was used in the proof of Theorem 8.14 and was heavily influenced by Woodin's ideas. We assume the reader is familiar with the Kunen proof of the consistency of an $\aleph_2$-saturated ideal on ω_1 and the Magidor variation of that proof. These appear in Sects. 7.7 and 7.11.

Proof. The proof of Woodin's theorem follows the general outline given at the beginning of Sect. 8 for transforming an induced ideal into a natural ideal. One starts with an almost huge embedding $j : V \to M$ with critical point κ. From the general theory of large cardinals, we can assume that $j(\kappa) = \lambda$ where λ is not Mahlo and that $j``\lambda$ is cofinal in $j(\lambda)$. Let $\mathcal{I}$ be the V-inaccessibles below κ. Then $j(\mathcal{I})$ is NS_λ from the point of view of V.

The first forcing uses the Magidor variation on Kunen's theorem described in Sect. 7.11. This produces a partial ordering of the form $\mathbb{P} * \mathrm{Col}(\kappa, {<}\lambda)$ such that for each generic $G * H \subseteq \mathbb{P} * \mathrm{Col}(\kappa, {<}\lambda)$ there are M-generic $\hat{G} * \hat{H}$ such that j can be extended to a generic

$$j_0 : V[G * H] \to M[\hat{G} * \hat{H}].$$

The partial ordering $\mathbb{P}$ is ${<}\mu$-closed, κ-c.c. and collapses κ to be μ^+.[84]

Let I_0 be the induced ideal from $U(j_0, \kappa)$. Then I_0 is κ^+-saturated. The rest of the forcing is an iteration $\mathbb{R}$ of length λ with supports of size less than κ.

For each $\alpha < \lambda$ we will have a master condition $m_\alpha \in \mathbb{R}_\alpha$ guaranteeing that j can be extended to

$$j_\alpha : V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha} \to M^{j(\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha)/m_\alpha}.$$

The ideal I_α will be the induced ideal from $U(j_\alpha, \kappa)$. The partial ordering $\mathbb{R}$ adds closed unbounded sets in such a way that the final ideal I_∞ is the nonstationary ideal restricted to the set $\mathcal{I}$ of V-inaccessibles.

There are two issues in constructing $\mathbb{R}$. The first is making sure that there are master conditions m_α so that the intermediate generic embeddings j_α can be constructed. The second is making sure that the partial orderings $j(\mathbb{R}_\alpha)$ are sufficiently closed that the pseudo-generic tower argument can be used to recover an ultrafilter over $V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}}$ in a λ-c.c. extension of $V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}}$.

The first issue is straightforward. Suppose inductively that we have defined m_α and S is a stationary set lying in $\breve{I}_\alpha \cap V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha}$. We will shoot a closed unbounded set through $S \cup (\kappa \setminus \mathcal{I})$. This makes S closed unbounded *relative* to $\mathcal{I}$. Since $S \in \breve{I}_\alpha$, for all generic extensions of j to j_α, we know that $\kappa \in j_\alpha(S)$. If $C \subseteq \mathbb{P}_{S \cup (\kappa \setminus \mathcal{I})}$ is generic over $V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha}$ and lies in $M^{j(\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha)/m_\alpha}$, then $r = C \cup \{\kappa\}$ is a condition in $j_\alpha(\mathbb{P}_{S \cup (\kappa \setminus \mathcal{I})})$ that lies below $j(c)$ for each $c \in C$. If we let $\mathbb{R}_{\alpha+1} = \mathbb{R}_\alpha * \mathbb{P}_{S \cup (\kappa \setminus \mathcal{I})}$, then we can define a master condition for $\mathbb{R}_{\alpha+1}$ by setting $m_{\alpha+1} = m_\alpha^\frown r$.

To summarize, if we iterate shooting closed unbounded sets $\langle C_\alpha : \alpha < \lambda \rangle$ through sets in the duals of each ideal and each initial segment of $\langle C_\alpha : \alpha < \lambda \rangle$ belongs to $M^{j(\mathbb{P}) * \mathrm{Col}(\lambda, <j(\lambda))}$, we can construct a sequence of master conditions $\langle m_\alpha : \alpha < \lambda \rangle$. The $j(\xi)$th coordinate of m_α will be $C_\xi \cup \{\kappa\}$, where C_ξ is the closed unbounded coming from the ξth forcing in $\mathbb{R}$.[85]

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84 In the notation of the outline given in Sect. 8.2, $V_0 = V[G * H]$ and $M_0 = M[\hat{G} * \hat{H}]$.

85 Very similar arguments are given in Cummings' chapter in this Handbook in the proof of the consistency of "NS_{ω_1} is precipitous".

The second issue is more subtle. Woodin's method relies on the very clever observation that if $\mathbb{P}_S$ is the partial ordering for shooting a closed unbounded subset through S with initial segments and $S \subseteq \kappa$ contains a closed unbounded set then there is a dense subset of $\mathbb{P}_S$ that is $<\kappa$-closed. In particular, assuming $\kappa^{<\kappa} = \kappa$, forcing with $\mathbb{P}_S$ is equivalent to forcing with the partial ordering for adding a Cohen subset of κ.

To illustrate the relevance of this observation, let's consider $\mathbb{R}_0$, the first partial ordering for adding a closed unbounded set. $\mathbb{R}_0$ is defined in $V[G*H]$. A stationary set $S_0 \in \breve{I}$ is chosen. $\mathbb{R}_0$ is defined to be $\mathbb{P}_{S_0 \cup (\kappa \setminus \mathcal{I})}$. Then $j(\mathbb{R}_0)$ is $\mathbb{P}_{j(S_0) \cup (\lambda \setminus j(\mathcal{I}))}$ as defined in $M[\hat{G} * \hat{H}]$. However, from the point of view of V, $\lambda \setminus j(\mathcal{I})$ is closed unbounded. Hence from the point of view of $V[\hat{G}]$ the forcing $(\mathrm{Col}(\lambda, <j(\lambda)) * \mathbb{P}_{j(S_0) \cup (\lambda \setminus j(\mathcal{I}))})^M$ is $<\lambda$-closed. This is sufficient for a pseudo-generic tower argument.

We can now give a summary of Woodin's argument. The partial ordering used will be of the form $\mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}$, where $\mathbb{R}$ is an iteration with $<\kappa$-supports for shooting closed unbounded sets through stationary sets.

The Magidor variation of Kunen's theorem (Sect. 7.11) gives an ideal I_0 that is saturated in $V^{\mathbb{P} * \mathrm{Col}(\kappa, <\lambda)}$ and concentrates on the set $\mathcal{I}$ of ordinals that were inaccessible cardinals in V. Generic elementary embeddings, ideals and master conditions $\langle j_\alpha, I_\alpha, m_\alpha : \alpha < \lambda \rangle$ are defined inductively, with j_0 being the extension of the original almost huge embedding in the Magidor variation.

The αth stage of $\mathbb{R}$ is of the form $\mathbb{P}_{S_\alpha \cup (\kappa \setminus \mathcal{I})}$. Since all of the forcings are of this form, for each $\alpha < \lambda$ the partial ordering $j(\mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha)$ is $<\lambda$-closed over $V[\hat{G}]$ and hence in $V[\hat{G}]$ it is possible to find filters $\hat{H} * \vec{C}^j_\alpha \subseteq j(\mathrm{Col}(\kappa, <\lambda) * \mathbb{R}_\alpha)$ that are sufficiently generic over M to extend the embeddings to j_α, build $U(j_\alpha, i)$, and determine the ideal I_α. Every set in $\bigcup_{\alpha<\lambda} \breve{I}_\alpha$ occurs on the sequence $\langle S_\alpha : \alpha < \lambda \rangle$.

The final model will be $V[G * H * \vec{C}]$ where $G * H * \vec{C} \subseteq \mathbb{P} * \mathrm{Col}(\kappa, <\lambda) * \mathbb{R}$ is V-generic. By the pseudo-genericity arguments, there is an ultrafilter U that is κ-complete and normal for $V[G * H * \vec{C}]$-sequences that lies in $V[\hat{G}]$ where $\hat{G} \subseteq j(\mathbb{P})$. Since $\hat{G}$ lies in a λ-c.c. forcing extension of $V[G * H * \vec{C}]$ and $\lambda = \kappa^+$ we see that the induced ideal from U is κ^+-c.c. On the other hand the induced ideal for U is $\bigcup_{\alpha<\lambda} I_\alpha$ which has been forced to be the ideal $\mathrm{NS} \restriction \mathcal{I}$. ⊣

8.12 Remark. One can apply Theorem 7.30 to see that Woodin's construction and the Foreman-Komjáth construction below actually yield κ-centered ideals on $\kappa = \mu^+$.

Club Guessing Ideals

Recall the following definitions from Example 3.19: Let $\kappa > \mu$ be regular cardinals and $S \subseteq \kappa \cap \mathrm{Cof}(\mu)$. Let $\langle C_\alpha : \alpha \in S \rangle$ be a sequence such that C_α

is unbounded in α. We define two filters, the *club guessing filter* and the *tail club guessing filter* on S.

Let $D \subseteq \kappa$ be closed unbounded. Let $G(D) = \{\alpha \in S : C_\alpha \subseteq D\}$ and $E(D) = \{\alpha \in S : C_\alpha \subseteq^* D\}$.

1. The club guessing filter on S is the filter generated by the sets $\{G(D) : D$ is closed unbounded$\}$ together with the filter of closed unbounded sets.

2. The tail club guessing filter on S is the filter generated by the sets $\{E(D) : D$ is closed unbounded$\}$ together with the filter of closed unbounded sets.

The sequence $\langle C_\alpha : \alpha \in S\rangle$ is *club guessing* iff the club guessing filter on S is a proper filter, and *tail club guessing* iff the tail club guessing filter on S is a proper filter. A sequence $\langle C_\alpha : \alpha \in S\rangle$ is *strong club guessing* iff the tail club guessing filter is the nonstationary ideal restricted to S.[86]

Club guessing sequences and their associated filters play a vital role in singular cardinal combinatorics and other subjects. They are an interesting class of natural filters and the possible generic embeddings associated with club guessing filters are only beginning to be understood.

Woodin was able to show, using a $\mathbb{P}_{\max}$ variation:

8.13 Theorem. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then there is a forcing extension of* $L(\mathbb{R})$ *in which* NS_{ω_1} *is* $\aleph_2$*-saturated and there is a strong club guessing sequence* $\langle C_\alpha : \alpha < \omega_1\rangle$.

Independently of this, Foreman and Komjáth were able to extend the Woodin techniques of Theorem 8.9 to show:

8.14 Theorem (Foreman-Komjáth [40]). *Suppose that* κ *is an almost huge cardinal and* $\mu < \kappa$ *is regular. Then there is a partial ordering* $\mathbb{P}$ *such that in* $V^{\mathbb{P}}$ *the following hold:*

1. $\kappa = \mu^+$,

2. *there is a stationary set* $S \subseteq \kappa$ *such that* $\mathrm{NS}_\kappa \restriction S$ *is* κ^+*-saturated, and*

3. *there is a strong club guessing sequence* $\langle x_\alpha : \alpha < \kappa\rangle$.

8.15 Corollary. *Assuming the hypotheses of Theorem 8.14, there is a forcing extension in which* $\kappa = \mu^+$ *and there is a sequence* $\langle x_\alpha : \alpha \in S\rangle$ *for a stationary* $S \subseteq \kappa$ *such that the tail club guessing filter on* μ^+ *determined by* $\langle x_\alpha : \alpha \in S\rangle$ *is* μ^{+2}*-saturated.*

8.16 Corollary. *Let* α *be an ordinal. Assume that there is an almost huge cardinal* κ *bigger than a supercompact cardinal* $\mu > \alpha$. *Then there is a generic extension of* V *in which* $\kappa = \aleph_{\alpha+1}$, *there is a stationary subset* S *of* κ *on*

[86] The concept of a strong club guessing sequence appears in the proof of Theorem 6.14.

which NS_κ *is* κ^+*-saturated and strong club guessing holds. In particular, there is a club guessing sequence* $\langle x_\beta : \beta \in S\rangle$ *such that the tail club guessing ideal determined by sequence is* κ^+*-saturated.*

Corollary 8.16 stands in strange counterpoint to the use of $\diamondsuit^*_{\text{club}}$ in Theorem 6.14 of Gitik and Shelah. A contradiction appears narrowly avoided by noting that the ordinals in the elements x_α of the guessing sequence have small cofinality.

In fact we can prove a slightly stronger theorem than Theorem 8.14. A slight variant of the proof shows that it is consistent to have $\mathrm{NS}_\kappa\restriction S$ be κ^+-saturated, a strong club guessing sequence defined on S and a $\diamondsuit^+(\kappa\setminus S)$-sequence.

In [40] the following forcing is defined that adds a strong club guessing sequence on a stationary set S:

8.17 Definition. Let κ be a regular cardinal and $S \subseteq \kappa$ be stationary. Let $\mathcal{CG}(S)$ be the iteration of length 2^κ with $<\kappa$-supports of the following components:

1. We let $\mathcal{CG}(S)_0$ be the collection of sequences $\langle x_\alpha : \alpha \in S \cap (\beta+1)\rangle$ for some $\beta < \kappa$, where each x_α is closed and unbounded in α. The ordering of $\mathcal{CG}(S)_0$ is by end extension.

2. We use a bookkeeping system that at stage γ will choose an appropriate closed unbounded set $C_\gamma \subseteq \kappa$ with $C_\gamma \in V^{\mathcal{CG}(S)_\gamma}$ and we will let $S_\gamma = \{\alpha : x_\alpha \subseteq^* C_\gamma \text{ or } \alpha \notin S\}$. Then $\mathcal{CG}(S)_{\gamma+1} = \mathcal{CG}(S)_\gamma * \mathbb{P}_{S_\gamma}$ where $\mathbb{P}_{S_\gamma}$ is the partial ordering for shooting a closed unbounded set through S_γ.

3. We arrange our bookkeeping so that every closed unbounded set in $V^{\mathcal{CG}(S)}$ appears as some C_γ for some $\gamma < 2^\kappa$.

Assuming that $\kappa^{<\kappa} = \kappa$, the partial orderings $\mathcal{CG}(S)_\gamma$ are κ^+-c.c. and hence that it is possible to achieve the third clause in the definition of $\mathcal{CG}(S)$. Moreover, there is a dense set of *flat* conditions that form a $<\kappa$-closed subset of $\mathcal{CG}(S)$.

In the proof of Theorem 8.14, the iteration used in $\mathcal{CG}(S)$ is interdigitated with the iteration used in the Woodin construction and with yet another partial ordering, the partial ordering for adding a "fast club" set:

The following partial ordering appeared in Jensen's proof of the consistency of CH + Suslin's Hypothesis (see [21]):

8.18 Definition. Let $\mu > \omega$ be a regular cardinal. Define $\mathcal{FC}(\mu)$ to be the partial ordering consisting of pairs, (s, C) where s is a closed bounded subset of μ and C is a closed unbounded subset of μ. We will say that a condition (s, C) is stronger than a condition (t, D) iff

1. s is an end extension of t,
2. $C \subseteq D$, and
3. $s \setminus t \subseteq D$.

If we assume that $\mu^{<\mu} = \mu$, then this partial ordering is μ^+-c.c. and $<\mu$-closed. If $G \subseteq \mathcal{FC}(\mu)$ is generic and $E = \bigcup\{s : \text{for some } C, (s, C) \in G\}$ then E is a closed unbounded subset of μ. A simple density argument shows that if $C \in V$ is a closed unbounded subset of μ then there is a condition $(t, D) \in G$ with $D \subseteq C$. In particular, $E \setminus \sup(t) \subseteq D$.

We now have the ingredients to describe the construction in Theorem 8.14 informally:

Proof. We start with an almost huge embedding $j : V \to M$ with critical point κ and $j(\kappa) = \lambda$ where λ is not Mahlo in V. The forcing will have three parts $\mathbb{P}*\mathbb{Q}*\mathbb{R}$. The partial ordering $\mathbb{P}$ will make $\kappa = \mu^+$, $\mathbb{Q}$ will make $\lambda = \kappa^+$ and $\mathbb{R}$ will be a "termspace" forcing, that will ultimately be isomorphic to adding Cohen subsets to λ. Since $\mathbb{R}$ does not add subsets of κ, the ideals built during the construction will be on subalgebras of $P(\kappa)^{V^{\mathbb{P}*\mathbb{Q}}}$.

We will initially use the Magidor variation of the Kunen construction and build a model V_0 where $\kappa = \mu^+$ and there is a saturated ideal I_0 on the set of V-inaccessible cardinals $\mathcal{I} \subseteq \kappa$ induced by a generic elementary embedding $\hat{j}$. We then begin an iteration that shoots closed unbounded sets through the sets of measure one for $\breve{I}_0$ relative to the V-inaccessible cardinals. The result will be an increasing series of saturated ideals I_α for $\alpha < \lambda$. We dovetail adding closed unbounded sets to make sure that these ideals all eventually end up being NS_κ restricted to the old inaccessible cardinals. This iteration will be a subiteration of $\mathbb{Q}$.

We have three tasks:

1. making strong club guessing hold on $\kappa \setminus \mathcal{I}$,
2. shooting the closed unbounded sets to make the induced saturated ideal the nonstationary ideal, and
3. making strong club guessing hold on $\mathcal{I}$.

These are accomplished in three different ways. The first two are the result of the comingling of the forcing for adding a strong club guessing sequence on $\kappa \setminus \mathcal{I}$, and the forcing for making the sets that appear in the ideals I_α nonstationary. This describes a partial ordering $\mathbb{Q}$ that creates a club guessing sequence on $\kappa \setminus \mathcal{I}$ and makes the nonstationary ideal restricted to the V-inaccessible cardinals saturated. As is crucial, Woodin's observation is used to see that $j(\mathbb{Q})$, which iterates shooting closed unbounded sets through M-stationary subsets relative to the V-nonstationary set of M-inaccessible cardinals $j(\mathcal{I})$), is $<\lambda$-closed.

The third task uses a completely different technique: during the construction of the first phase $\mathbb{P}$ of the partial ordering we will frequently be adding fast closed sets c_α to α over the model $V^{\mathbb{P}_\alpha \cap V_\alpha}$. The strong club guessing sequence on $\mathcal{I}$ will be $\langle c_\alpha : \alpha \in \mathcal{I}\rangle$.

This plan assumes that the inductive construction of the ideals I_α works. The ideals I_α are defined inductively *after* $\mathbb{P}$ is defined, and hence the $\mathbb{Q}_\alpha$

are defined after $\mathbb{P}$ is defined. The definition uses the fact that we extend the elementary embeddings to $j_\alpha : V^{\mathbb{P}*\mathbb{Q}_\alpha} \to M^{j(\mathbb{P}*\mathbb{Q}_\alpha)/m_\alpha}$. To extend the embeddings we must have $\mathbb{P}*\mathbb{Q}_\alpha$ regularly embedded into $j(\mathbb{P})$. This presents a problem because we do not have a definable way of anticipating in advance what the inductive construction of $\mathbb{Q}_\alpha$ is.

To build $\mathbb{P}$, we must then do at least two things. First, we need to find a universal way of anticipating a large class of iterations that include many coordinates that shoot closed unbounded sets through α in $V^{\mathbb{P}_\alpha \cap V_\alpha}$. Second, we need to add a closed unbounded subset c_α of α that is "fast" over any of these iterations.

This involves axiomatizing the type of iterations we will anticipate. In the terminology of [40], these are the "α-acceptable, canonically μ-closed iterations with V-limits". These iterations are shown to have the property of *near properness*, an iterable condition on partial orderings. We then define a partial ordering that is universal for this type of iteration:

Let $\mu < \alpha$ be regular cardinals. Let $\mathbb{B}(\mu, \alpha, \gamma)$ be the partial ordering:

$$\prod_{\substack{\gamma\text{-copies} \\ \text{bounded height} \\ <\alpha\text{-supports}}} \mathrm{Col}(\mu, \alpha),$$

the product partial ordering on γ copies of $\mathrm{Col}(\mu, \alpha)$ with $<\alpha$ supports restricted to those elements p of the product such that there is a $\beta < \mu$ for all $\xi \in \mathrm{supp}(p)$, $\mathrm{dom}(p(\xi)) \subseteq \beta$.

This partial ordering is universal in the sense that if $\mu < \alpha$ are regular cardinals and $\mathbb{Q}$ is a V-limit iteration of canonically μ-closed, α-acceptable partial orderings that has length γ, then there is a regular embedding:

$$\mathbb{Q} \to \mathbb{B}(\mu, \alpha, \gamma).$$

The partial ordering $\mathbb{P}$ is then defined inductively as a $<\mu$ support iteration with:

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * (\mathrm{Col}(\alpha, <\kappa) * \mathbb{B}(\mu, \alpha, \kappa) * \mathcal{F}C(\mu))^W$$

where $W = V^{\mathbb{P}_\alpha \cap V_\alpha}$.

Note that since α has cofinality μ in $W^{\mathrm{Col}(\alpha,<\kappa)*\mathbb{B}(\mu,\alpha,\kappa)}$ and W and $W^{\mathrm{Col}(\alpha,<\kappa)*\mathbb{B}(\mu,\alpha,\kappa)}$ have the same $<\mu$-sequences, the partial ordering $\mathcal{F}C(\mu)^W$ adds a closed unbounded subset of α that is eventually included in every closed unbounded subset of α that lies in $W^{\mathrm{Col}(\alpha,<\kappa)*\mathbb{B}(\mu,\alpha,\kappa)}$.

Then at stage κ,

$$j(\mathbb{P})_{\kappa+1} = j(\mathbb{P}_\kappa) * (\mathrm{Col}(\kappa, j(\kappa)) * \mathbb{B}(\mu, \kappa, \lambda), *\mathcal{F}C(\mu))^{V^{\mathbb{P}}}.$$

The fast closed subset of κ added at this stage is eventually inside every closed unbounded subset of κ in the final model $V^{\mathbb{P}*\mathbb{Q}*\mathbb{R}}$. Hence by reflection the fast closed sets added to α at stage α for α in $\mathcal{I}$ will be a club guessing sequence.

It turns out that the iteration $\mathbb{Q}$ described above has a dense set isomorphic to a V-limit of canonically μ-closed, κ-acceptable partial orderings, no matter how the sequence of ideals I_α eventuate. Hence by the universality, we see

$$\mathbb{P} * \mathbb{Q} \to j(\mathbb{P}).$$

We now face the final obstacle. The partial ordering $j(\mathbb{Q})$ is $<\lambda$-closed, so we can attempt pseudo-generic tower arguments in $j(\mathbb{Q})$. However, the inductive definition of the ideals I_α depend on which tower one chooses, and the towers must be collectively generic.

To illustrate why this is a problem, consider a simple case: To define the ideal I_2 we have to build pseudo-generic towers to the first two coordinates of $j(\mathbb{Q})$. To build I_3 we need to build a third tower on the third coordinate of $j(\mathbb{Q})$. However this tower needs to be pseudo-generic over the decisions made by first two towers, which may not be possible if the first two towers were chosen badly. Even more problematic, at a limit stage δ the sequence of pseudo-generic towers built for iterations of length less than α must be pseudo-generic at α.

In this construction the partial ordering $\mathbb{R}$ is a termspace partial ordering that provides *actual* $M^{j(\mathbb{P})}$-generic objects for the iterations $j(\mathbb{Q})_\alpha$, that allow the embeddings $j_\alpha : V^{\mathbb{P}*\mathbb{Q}_\alpha} \to M^{j(\mathbb{P})*j(\mathbb{Q})_{j(\alpha)}/m_\alpha}$ to be extended and cohere sufficiently to give the definitions of the ideals I_α. The interaction of $\mathbb{R}$ with the other partial orderings and the manner in which it provides generic objects for $j(\mathbb{Q})_{j(\alpha)}$ is beyond the scope of this survey. ⊣

8.3. Making Natural Ideals Have Well-Founded Ultrapowers

In this section we briefly summarize some of the results originating from [47]. We start by giving some criteria for a natural ideal to have interesting generic ultrapowers. These involve the notion of "good" structure as given by Definition 3.43.

8.19 Definition. Let I be an ideal on a set $Z \subseteq P(X)$ and $\mathcal{A} = \{a_\alpha : \alpha < \gamma\} \subseteq P(Z)$ be a maximal antichain relative to I.[87] Suppose that θ is a regular cardinal bigger than 2^Z, and $N \prec H(\theta)$ a good structure for I with $I, Z, \mathcal{A} \in N$. Then N *catches an index for* (or just *catches*) $\mathcal{A}$ iff for some $\alpha \in N$, $N \cap X \in a_\alpha$.

Note that the goodness of N implies that it can catch at most one index for $\mathcal{A}$. "Catching an index" is the first step towards being "self-generic" in the sense of Definition 3.45:

8.20 Proposition. *If N is good then the following are equivalent:*

1. *N catches an index for every antichain that belongs to N.*

[87] Precisely, $\{[a_\alpha]_I : \alpha < \gamma\}$ is a maximal antichain in $P(Z)/I$.

2. N is self-generic.

If every good N catches an index for each maximal antichain that belongs to N and $|Z| \leq |X|$, then I is $|X|^+$-saturated.

Proof. Note that N catches an index for $\mathcal{A}$ iff there is an $a \in \mathcal{A} \cap N$ with $N \cap X \in a$. Let $j : \bar{N} \to N$ be the inverse of the transitive collapse map of N. From the definition, N catches an index for every antichain belonging to N iff for all antichains $\mathcal{A} \in N$,

$$\mathcal{A} \cap U(j, N \cap X) \cap N \neq \emptyset.$$

This latter statement is equivalent to the $\bar{N}$-genericity of $U(j, N \cap X)$ for $(P(Z)/I)^{\bar{N}}$.

The final sentence is a restatement of part 3 of Lemma 3.46. ⊣

The method for making natural ideals be precipitous or have saturation properties is to create situations where there is a stationary set of good N that catch many antichains.

The next proposition gives a criterion for precipitousness in these terms.

8.21 Proposition. *Let $Z \subseteq P(X)$ and $I \subseteq P(Z)$ be an ideal. Suppose that for all $S \in I^+$, and all sequences $\langle \mathcal{A}_n : n \in \omega \rangle$ of maximal antichains below $[S]_I$ there is an*

$$N \prec \langle H((2^{|Z|})^+), \in, \Delta, I, Z, X, \langle \mathcal{A}_n : n \in \omega \rangle \rangle,$$

such that:

1. $N \cap X \in S$, and

2. for all n, N catches an index for the antichain $\mathcal{A}_n$.

Then I is precipitous.

Proof. First note that N is good, since this is part of the definition of catching an index for an antichain. We use Proposition 2.7. Let $\langle \mathcal{A}_n : n \in \omega \rangle$ be a tree of maximal antichains below S. Since N is good, there is a a unique element a_n of $\mathcal{A}_n \cap N$ for which $N \cap Z \in a_n$. Since $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, either $a_{n+1} \subseteq_I a_n$ or $a_{n+1} \cap a_n =_I \emptyset$. Since N is good, we must have $a_{n+1} \subseteq_I a_n$. Thus the sequence $\langle a_n : n \in \omega \rangle$ forms a branch through the tree of antichains and $N \cap X \in \bigcap_{n \in \omega} a_n$. ⊣

8.22 Definition. Let $S \subseteq H(\theta)$ be a stationary set. Then S *reflects to a set of size* μ iff there is a set $Y \subseteq H(\theta)$ with $\mu \subseteq Y$ and $|Y| = \mu$ such that $S \cap P(Y)$ is stationary.

If I is a normal, fine, countably complete ideal on $Z \subseteq P(\mu)$, Y is a set of cardinality μ and $S \subseteq P(Y)$, then S corresponds canonically to an element $[\bar{S}]_I$ of $P(Z)/I$. Namely, if one fixes a bijection $f : \mu \to Y$, then $\bar{S} = \{z \in Z : f``z \in S\}$. By the normality of I, $[\bar{S}]_I$ is independent of the choice of f. Hence it is well-defined to say that S is *I-positive* iff $[\bar{S}]_I \neq 0$.

We can generalize Definition 8.22 as follows:

8.23 Definition. Let I be a normal, fine, countably complete ideal on $Z \subseteq P(\mu)$, and $S \subseteq H(\theta)$. Then S *reflects to a set in* I^+ iff there is a set $Y \subseteq H(\theta)$ with $\mu \subseteq Y$ and $|Y| = \mu$ such that $S \cap P(Y)$ is I-positive.

In this language, "S reflects to a set of size μ" is equivalent to saying that S reflects to a positive set with respect to the nonstationary ideal on $P(\mu)$.

In certain circumstances the next proposition yields presaturated ideals.

8.24 Proposition. *Let $Z \subseteq P(\mu)$, $\gamma \leq \mu$ and I be a normal, fine, countably complete uniform ideal on Z. Suppose that for all sequences $\langle \mathcal{A}_\alpha \subseteq P(Z) : \alpha < \gamma \rangle$ of maximal antichains relative to I and $S_0 \in I^+$, if T is the set of $N \prec \langle H((2^{|Z|})^+), \in, \Delta, \{I, S_0\}, \langle \mathcal{A}_\alpha : \alpha \in \gamma \rangle \rangle$ such that:*

1. *$N \cap \mu \in S_0$, and*

2. *for all $\alpha \in N \cap \gamma$, N has an index for the antichain $\mathcal{A}_\alpha$,*

then T reflects to a set in I^+. Then I is weakly (γ^+, μ^+)-saturated.[88]

Proof. Suppose that Y is a set of size μ with $\mu \subseteq Y$ such that $T \cap P(Y)$ is in I^+. Let $f : \mu \to Y$ be a bijection. The set of $M \in P(Y)$ such that M is closed under f and f^{-1} is closed and unbounded. Let T' be the collection of $N \in T \cap P(Y)$ that are closed under f and f^{-1} such that $f^{-1}(N) \in Z$. Then $[\bar{T}]_I = [\bar{T}']_I$. Letting $S = \{N \cap \mu : N \in T'\}$ we see that $S = \bar{T}'$.

Since $S \in I^+$ and $|Y| = \mu$ it suffices to show that for each $\alpha < \gamma$, $\{a \cap S : a \cap S \notin I \text{ and } a \in \mathcal{A}_\alpha \cap Y\}$ is a maximal antichain below S.

Modulo closed unbounded sets, $\bigtriangledown(\mathcal{A}_\alpha \cap Y) \cap S$ is the collection of $z \in S$ such that there is a $\delta \in z$ such that $z \in f(\delta)$ and $f(\delta) \in \mathcal{A}_\alpha$. But this collection is a closed unbounded set relative to S. For if $z \in S$ and $\alpha \in z$, then $N = f``z \in T$. Hence N catches $\mathcal{A}_\alpha$. Thus there is a $\delta \in z$ such that $z \in f(\delta)$ and $f(\delta) \in \mathcal{A}_\alpha$.

Hence, for a closed unbounded set of z, if $z \in S$ then $z \in \bigtriangledown(\mathcal{A}_\alpha \cap Y)$. This implies that $S \leq_I \bigtriangledown(\mathcal{A}_\alpha \cap Y)$. By Theorem 2.24, $\bigtriangledown(\mathcal{A}_\alpha \cap Y)$ is the Boolean sum of $\mathcal{A}_\alpha \cap Y$ in $P(Z)/I$. Hence $\{a \cap S : a \cap S \notin I \text{ and } a \in \mathcal{A}_\alpha \cap Y\}$ is a maximal antichain below S. ⊣

Catching Antichains Using Reflection

We now introduce a technique for modifying an existing natural ideal to make it precipitous. We will follow the original route to these results from

[88] Weak saturation is defined in Definition 4.5.

[47] which use reflection and a supercompact.[89] Woodin [126] discovered how to modify these arguments to work from a Woodin cardinal. His arguments do not use stationary set reflection.

8.25 Lemma. *Suppose that $\theta \gg \mu$ are regular cardinals and I is a normal and fine ideal on $Z \subseteq P(\mu)$. Suppose that $R \subseteq P(H(\theta))$ is a stationary set of good structures such that the projection to $P(\mu)$ of any relatively closed unbounded subset of R belongs to $\breve{I}$ and every stationary subset of R reflects to a set in I^+. Then for each maximal antichain $\mathcal{A} \subseteq P(Z)/I$ and each expansion $\mathfrak{A}$ of $\langle H(\theta), \in, \Delta\rangle$ there is a closed unbounded set C of $N \prec \mathfrak{A}$ such that if:*

(a) $N \in C \cap R$, and

(b) $\mathcal{A} \in N$,

then there is an $N' \prec \mathfrak{A}$ such that:

(A) $N' \in R$,

(B) $N' \cap \mu = N \cap \mu$,

(C) $N \prec N' \prec \mathfrak{A}$, and

(D) N' catches $\mathcal{A}$.

Proof. Suppose not. Then there is a stationary set B of $N \prec \mathfrak{A}$ such that:

(a) $B \subseteq R$,

(b) $\mathcal{A} \in N$ for all $N \in B$,

and such that whenever $N' \prec \mathfrak{A}$ is such that

(A) $N' \in R$,

(B) $N' \cap \mu = N \cap \mu$, and

(C) $N \prec N'$

then N' does not catch $\mathcal{A}$.

Let $Y \subseteq H(\theta)$ with $\mu \subseteq Y$ and $|Y| = \mu$ such that $B \cap P(Y)$ is I-positive. Let $f : \mu \to Y$ be a bijection, and $S = \{z : f``z \cap \mu = z \text{ and } f``z \in T\}$. Then S is I-positive, so there is an $b \in \mathcal{A}$ such that $S \cap b$ is in I^+.

The collection of $M \in R$ such that $M \prec \mathfrak{A}$ and $b, f \in M$ is closed unbounded relative to R. Thus we can find an $N' \prec \mathfrak{A}$ in R with $b, f \in N'$ and $N' \cap \mu \in S \cap b$. Let $N = f``N' \cap \mu$. Then $N \in B$, $N \cap \mu = N' \cap \mu$ and $N \prec N'$, a contradiction. ⊣

[89] Matsubara [93] showed how to interpolate ideal assumptions to get similar results; his arguments use a result similar to Theorem 5.26 that he discovered independently to prove the reflection properties needed to catch antichains.

Reflecting Stationary Sets

The following definition gives a name for a standard idea:

8.26 Definition. Let X be a set, and $Z \subsetneq P(X)$ be stationary. Then Z is *μ-robust* iff for all stationary subsets $S \subseteq Z$ and all $<\mu$-closed partial orderings $\mathbb{P}$, S is stationary in $V^{\mathbb{P}}$.

Abstract forcing arguments easily show that Z is μ-robust iff every stationary subset of Z is preserved under forcing with $\mathrm{Col}(\mu, |Z|)$.

We now give some examples of robust stationary sets.

8.27 Example. Let $X = \mu$ be a cardinal and $Z = \mu$. Since $<\mu$-closed forcing does not destroy stationary subsets of μ, Z is λ-robust for all $\lambda \geq \mu$.

Our next example illustrates the point of robustness. We recall the following definition from [42]. Let $N \subseteq H(\lambda)$. Then N is *internally approachable* (or IA) *of length* β iff there is a sequence $\langle N_\alpha : \alpha < \beta \rangle$ such that:

1. $N = \bigcup_{\alpha<\beta} N_\alpha$,
2. $N_\alpha \subseteq N_\beta$ for $\alpha < \beta$, and
3. for all $\beta' < \beta$, $\langle N_\alpha : \alpha < \beta' \rangle \in N$.

Given an $N \prec \langle H(\lambda), \in, \Delta \rangle$, any two approaching sequences have the same cofinality. If this cofinality is γ we will say that $N \in \mathrm{IA}(\mathrm{Cof}(\gamma))$.

8.28 Example. Let μ be a regular cardinal and let $Z = \{z \in [H(\lambda)]^{<\mu} : z \cap \mu \in \mu$ and z is internally approachable$\}$. Then Z is a maximal μ-robust subset of $\{z \in [H(\lambda)]^{<\mu} : z \cap \mu \in \mu\}$.

This follows from Lemma 2.5 of [42] where it is shown that IA is μ-robust and for all sets $S \subseteq [H(\lambda)]^{<\mu} \cap \{N : N \cap \mu \in \mu\}$ and all V-generic $G \subseteq \mathrm{Col}(\mu, H(\lambda))$,

$$V[G] \models [S]_{\mathrm{NS}} = [S \cap \mathrm{IA}]_{\mathrm{NS}}.$$

We now present a slight generalization of Example 8.28 that covers Chang's Conjecture and clarifies the results in [47] slightly:

8.29 Definition. Suppose that $N \prec H(\theta)$ with $|N| < \mu$. Then N is *μ-weakly approachable* iff there is an increasing continuous sequence $\langle N_\alpha : \alpha < \sup(N \cap \mu) \rangle$ such that:

1. $|N_\alpha| < \mu$ for all α,
2. $N \subseteq \bigcup_{\alpha \in N \cap \mu} N_\alpha$, and
3. for all $\beta \in N \cap \mu$, $\langle N_\alpha : \alpha \in \beta \rangle \in N$.

If $N \in [H(\theta)]^{<\mu}$ and $N \cap \mu \in \mu$, then N is μ-weakly approachable iff N is internally approachable. It is a standard fact (see [47]) that if $\theta > \eta$, $N \prec H(\theta)$ is internally approachable, $\eta \in N$, and $a \in \eta$, then $\mathrm{Sk}^{H(\theta)}(N \cup \{a\})$ is also internally approachable. The following generalization is shown using the same technique:

8.30 Proposition. *Suppose that θ is a regular uncountable cardinal and $\mu < \theta$. Let $N \prec \langle H(\theta), \in, \Delta\rangle$ be a μ-weakly approachable structure and $a \in y$ for some $y \in N$. Then $\mathrm{Sk}^{H(\theta)}(N \cup \{a\})$ is μ-weakly approachable.*

Proof (Sketch). The structure $\langle H(\theta) \in, \Delta\rangle$ has definable Skolem functions. Fix an internally approachable structure $N \prec H(\theta)$ and let $\langle N_\alpha : \alpha \in \gamma\rangle$ be a witness to approachability. For each $x \in N$ the restriction of each of those Skolem functions to x is a member of N. For each α, let N^*_α be the closure of $N_\alpha \cup \{a\}$ under all functions from $H(\theta)$ to $H(\theta)$ that belong to $N_{\alpha+1}$. Then the sequence $\langle N^*_\alpha : \alpha \in \gamma\rangle$ witnesses the fact that $\mathrm{Sk}^{H(\theta)}(N \cup \{a\})$ is weakly approachable. $\dashv$

Weak approachability is intimately tied to robustness. We now describe the relationship.

8.31 Proposition. *Let μ and θ be regular cardinals with $\theta > 2^\mu$. Then any stationary collection $S \subseteq P(H(\theta))$ of μ-weakly approachable structures is μ-robust.*

Proof. Let S be a stationary subset of the μ-weakly approachable subsets of $H(\theta)$ and κ be a cardinal. Suppose that $p \in \mathrm{Col}(\mu, \kappa)$ forces that S is not stationary in $V[G]$, where $G \subseteq \mathrm{Col}(\mu, \kappa)$ is generic. Let $\lambda \gg \max\{|S|, \kappa\}$ be a regular cardinal and $\dot{F}$ be a $\mathrm{Col}(\mu, \kappa)$-term for a function from $H[\theta]^{<\omega}$ to $H(\theta)$ such that $p \Vdash \{N \in [H(\theta)]^{<\mu} : N \text{ is closed under } \dot{F}\} \cap S = \emptyset$. Since S is stationary there is an $N \prec \langle H(\lambda), \in, \Delta, S, \kappa, p, \dot{F}\rangle$ with $|N| < \mu$ such that $N \cap H(\theta) \in S$. Let $\langle N_\alpha : \alpha \in \sup(N \cap \mu)\rangle$ witness the weak approachability of $N \cap H(\theta)$.

Since $\beta \in N$ implies that $\langle N_\alpha : \alpha \in \beta\rangle \in N$, one sees that there is a decreasing sequence of conditions $\langle p_\alpha : \alpha \in \sup(N \cap \mu)\rangle$ below p such that

1. for all $\beta \in N \cap \mu$, $\langle p_\alpha : \alpha < \beta\rangle \in N$, and
2. if $x \in N_\alpha \cap [H(\theta)]^{<\omega}$ and $\alpha < \beta$, then there is a y such that $p_\beta \Vdash \dot{F}(x) = y$.

Since $|N| < \mu$, we see that $q = \bigcup_{\alpha \in N \cap \mu} p_\alpha \in \mathrm{Col}(\mu, \kappa)$. However $q \Vdash$ "$N \cap H(\theta)$ is closed under $\dot{F}$", a contradiction. $\dashv$

We can generalize Lemma 2.5 of [42] with the following remark:

8.32 Remark. Let $S \subseteq [H(\theta)]^{<\mu}$. If $G \subseteq \mathrm{Col}(\mu, |H(\theta)|)$ is generic, then in $V[G]$ there is a closed unbounded set $C \subseteq P(H(\theta)^V)$ such that $S \cap C \subseteq \{N : N \text{ is } \mu\text{-weakly approachable in } V\}$. In particular, in $V[G]$, $[S]_{\mathrm{NS}} = [S \cap \{\text{weakly approachable structures}\}]_{\mathrm{NS}}$.

Proof. View G as a function from μ to $H(\theta)^V$ and let $N_\alpha = G``\alpha$. Let $C = \{N : N \prec \langle H(\theta)^V, \in, \Delta, G\rangle\}$. Then for $N \in C \cap ([H(\theta)]^{<\mu})^V$, the sequence $\langle N_\alpha : \alpha \in \sup(N \cap \mu)\rangle$ witnesses that N is weakly approachable. $\dashv$

We note that if $|H(\mu)| = \mu$ and $\mathfrak{A}$ is an algebra expanding the structure $\langle H(\mu), \in, \Delta\rangle$, then $H(\mu)$ is a union of a continuously increasing sequence of structures $\langle N_\alpha : \alpha < \mu\rangle$ such that

1. $|N_\alpha| < \mu$,
2. for $\beta < \beta'$, $\langle N_\alpha : \alpha < \beta\rangle \in N_{\beta'}$, and
3. $N_\alpha \prec \mathfrak{A}$.

If $M \prec \langle H(\mu), \in, \Delta, \langle N_\alpha : \alpha < \mu\rangle\rangle$ has cardinality less than μ, then M is μ-weakly approachable. Hence, the μ-weakly approachable structures form a closed unbounded set in $[H(\mu)]^{<\mu}$.

Using this and the usual arguments that show that the nonstationary ideals restricted to IA form a tower we see:

8.33 Proposition. *Suppose that $\mu \le \theta < \theta'$ are regular. Then*

1. *The nonstationary ideal on $[H(\theta')]^{<\mu}$ restricted to the μ-weakly approachable structures projects to the nonstationary ideal on $[H(\theta)]^{<\mu}$ restricted to the μ-weakly approachable structures.*
2. *If $|H(\mu)| = \mu$, then the μ-weakly approachable structures form a closed unbounded set in $[H(\mu)]^{<\mu}$.*

8.34 Example (Foreman et al. [47]). Suppose that $(\mu, \rho) \twoheadrightarrow (\mu', \rho')$. Then for all $\theta \ge \mu$, $W = \{N \prec H(\theta) : |N \cap \mu| = \mu'$ and $|N \cap \rho| = \rho'$ and N is μ-weakly approachable$\}$ is stationary. Moreover, if $|H(\mu)| = \mu$ then the projection of $\mathrm{NS}\restriction W$ to an ideal on $P(H(\mu))$ is the Chang ideal on $\{N \prec H(\mu) : |N| = \mu'$ and $|N \cap \rho| = \rho'\}$.

Our use of the notion of robustness is the following lemma.

8.35 Lemma. *Suppose that κ is a supercompact cardinal. Let $\mu < \kappa$ be regular. Let $G \subseteq \mathrm{Col}(\mu, <\kappa)$ be generic. In V[G], if $R \subseteq P(H(\theta))$ is a μ-robust stationary set and $S \subseteq R$ is stationary, then S reflects to a set of size μ.*

Proof. Let $j : V \to M$ be a $|H(\theta)|$-supercompact embedding. Then for all generic $G \subseteq \mathrm{Col}(\mu, <\kappa)$, there is a generic $H \subseteq \mathrm{Col}(\mu, <j(\kappa))$ such that j can be extended to a

$$\hat{j} : V[G] \to M[H].$$

By robustness, $\hat{j}(S) \cap P(\hat{j}``H(\theta))$ is stationary in $V[H]$ and $M[H] \models |\hat{j}``H(\theta)| = \mu$. Hence $M[H] \models$ "there is an $Y \subseteq H(j(\theta))$ with $\mu \subseteq Y$ and $|Y| = \mu$ such that $\hat{j}(S) \cap P(Y)$ is stationary in $P(Y)$". Hence the statement holds in $V[G]$ by elementarity. $\dashv$

Catching Your Tail

We now describe a typical "catch-your-tail" argument. Catching your tail allows one to change "there is a club set of N" in antichain catching lemmas (such as Lemma 8.25) to "for every N". The hypotheses of the next lemma are those of Lemma 8.25, with R being the μ-weakly approachable sets and I being the nonstationary ideal. The improvement on the conclusion of Lemma 8.25 is that instead of saying that there is a closed unbounded set of structures that can be expanded to catch a fixed maximal antichain $\mathcal{A}$, we find a $\mathfrak{B}$ such that for *every* elementary substructure $N \prec \mathfrak{B}$ and *every* maximal antichain $\mathcal{A}$ in N, N can be so extended to catch $\mathcal{A}$.

8.36 Lemma. *Let:*

1. *$\theta \gg \mu$ be regular cardinals,*
2. *R be the set of μ-weakly approachable subsets of $H(\theta)$,*
3. *$\mathfrak{A}$ be a structure expanding $\langle H(\theta), \in, \Delta\rangle$,*
4. *Z be the set of μ-weakly approachable subsets of $H(\mu)$, and*
5. *I be $\mathrm{NS}\restriction Z$.*

Suppose that every stationary subset of R reflects to a stationary set of size μ. Then there is a structure $\mathfrak{B}$ with domain $H(\theta)$ expanding $\langle H(\theta), \in, \Delta, \mu, R\rangle$ and $\mathfrak{A}$ such that for every $N \prec \mathfrak{B}$ with $N \in R$ and for every maximal antichain $\mathcal{A} \subseteq P(Z)/I$ with $\mathcal{A} \in N$ there is an $N' \in R$ with

(A) $N' \cap \mu = N \cap \mu$,

(B) $N \prec N' \prec \mathfrak{B}$, and

(C) N' catches an index for $\mathcal{A}$.

Proof. Let $\theta' \gg \theta$ and $\mathfrak{A}^* = \langle H(\theta'), \in, \Delta, \mathfrak{A}, \{\theta, Z, \mu, R, I\}\rangle$. Let $\langle f_i : i \in \omega\rangle$ be a complete set of Skolem functions for $\mathfrak{A}^*$ closed under composition, where each f_i is definable in $\mathfrak{A}^*$. Let $\mathfrak{B}$ be the expansion of $\mathfrak{A}$ and $\langle H(\theta), \in, \Delta, \mu, R\rangle$ built by adding function symbols for $\langle g_i : i \in \omega\rangle$ where the g_i's are the f_i's restricted to $H(\theta)$. Then any elementary substructure N of $\mathfrak{B}$ is of the form $N^* \cap H(\theta)$ for some $N^* \prec \mathfrak{A}^*$.

Suppose now that $N \prec \mathfrak{B}$ belongs to R and that $\mathcal{A} \in N$ is a maximal antichain. Let N^* be the Skolem hull of N in $\mathfrak{A}^*$. Since $\mathcal{A} \in N^*$ and $\mathfrak{A}$ is definable in $\mathfrak{A}^*$, if C is the closed unbounded set given in the conclusion of Lemma 8.25 applied to $\mathfrak{A}$ and $\mathcal{A}$, then $C \in N^*$. In particular, $N \in C$.

Hence we can find an $M \in R$ with $N \prec M \prec \mathfrak{A}$, $M \cap \mu = N \cap \mu$ and M catches an index for $\mathcal{A}$.

Suppose that a is the index caught by M. Let $N' = \mathrm{Sk}^{\mathfrak{A}}(N \cup \{a\})$. Then $N \prec N' \prec M$ and $N \cap \mu = N' \cap \mu$. Hence N' catches an index for $\mathcal{A}$. By Proposition 8.30, N' is weakly approachable.

It remains to see that $N' \prec \mathfrak{B}$. Every element of $\mathrm{Sk}^{\mathfrak{B}}(N \cup \{a\})$ is of the form $g_i(\vec{x}, a)$ for some $\vec{x} \in [N]^{<\omega}$. Fixing $\vec{x}$, we can view $g_i(\vec{x}, \cdot)$ as a function h from $\mathcal{A}$ to $H(\theta)$. Since this function is definable in $\mathfrak{B}$ from $\vec{x}$ and $N \prec \mathfrak{B}$ it must belong to N. In particular, $h(a) \in \mathrm{Sk}^{\mathfrak{A}}(N \cup \{a\}) = N'$. Thus $g_i(\vec{x}, a) \in N'$, as desired. ⊣

We note that Lemma 8.36 can be reformulated equivalently by having a closed unbounded set D play the role of the structure $\mathfrak{A}$. In this alternate version there is a closed unbounded set $D \subseteq H(\theta)$ replacing $\mathfrak{A}$ in the third clause of the lemma. The conclusion is that there is a structure $\mathfrak{B}$ such that for every weakly approachable structure $N \prec \mathfrak{B}$ in D and every maximal antichain $\mathfrak{A} \in N$ there is a weakly approachable $N' \prec \mathfrak{B}$ belonging to D that catches $\mathcal{A}$ and has the same intersection with μ that N does.

This conclusion can be further reformulated combinatorially by saying for all closed unbounded sets D there is a closed unbounded set $C \subseteq D$ for all weakly approachable $N \in C$ and all maximal antichains $\mathcal{A} \in N$ there is a weakly approachable elementary extension $N' \in C$ catching $\mathcal{A}$ that lies in C. This combinatorial version is close to what is necessary for stationary tower arguments.

The Nonstationary Ideal is Precipitous

Since all $N \prec H(\theta)$ are good for the nonstationary ideal, we immediately deduce:

8.37 Theorem (Foreman et al. [47]). *Suppose that κ is a supercompact cardinal and $\mu < \kappa$ is regular. Suppose that $G \subseteq \mathrm{Col}(\mu, {<}\kappa)$ is generic. Then in $V[G]$ the nonstationary ideal restricted to $[\mu]^{<\mu}$ is precipitous. In particular:*

1. $\mathrm{NS}{\restriction}\mu$ *is precipitous.*

2. $\mathrm{NS}{\restriction}[\mu]^{<\delta}$ *is precipitous for all uncountable regular cardinals δ.*

3. *If $(\mu, \rho) \twoheadrightarrow (\mu', \rho')$ in V for cardinals ρ, μ', ρ' then in $V[G]$, the Chang ideal $\mathrm{CC}((\mu, \rho), (\mu', \rho'))$ is precipitous.*

4. *If $\mu = \eta^+$, where η is a Jónsson cardinal then $\mathrm{NS}{\restriction}\{N \in [H(\mu)]^{<\mu} : |N \cap \eta| = \eta$ and $\eta \nsubseteq N\}$ is precipitous.*

Proof. Let $G \subseteq \mathrm{Col}(\mu, {<}\kappa)$ be generic. We work in $V[G]$. Note that $|H(\mu)| = \mu$ in $V[G]$ so we work with $[H(\mu)]^{<\mu}$.

Let R be the collection of μ-robust subsets of $H((2^\mu)^+)$. Then by Proposition 8.33 the nonstationary ideal on R projects to the nonstationary ideal on $[H(\mu)]^{<\mu}$.

We use Proposition 8.21. Let $\langle \mathcal{A}_n : n \in \omega \rangle$ be a tree of antichains. Choose $N_0 \prec H(\theta)$ such that $N_0 \in R$ and $\langle \mathcal{A}_n : n \in \omega \rangle \in N_0$. Applying Lemma 8.36, we see that we can build a chain $\langle N_n : n \in \omega \rangle$ of elementary substructures of

$H((2^\mu)^+)$ such that N_{n+1} catches $\mathcal{A}_n$ and $N_n \in R$ and $N_n \cap \mu = N_{n+1} \cap \mu$. Let $N = \bigcup_n N_n$. Then $N \cap \mu$ belongs to the projection of R to μ, is good and catches an index for each antichain $\mathcal{A}_n$. ⊣

Saturation of the Nonstationary Ideal

Suppose that κ is supercompact and $G \subseteq \mathrm{Col}(\omega_1, {<}\kappa)$ is generic. We work in $V[G]$, noting that there, for all $\lambda \geq \omega_1$, every stationary subset of $[\lambda]^{<\omega_1}$ reflects to a set of size $\aleph_1$.

Let $\langle \mathcal{A}_n : n \in \omega \rangle$ be a collection of maximal antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ and fix a $\theta \gg \omega_1$. We follow the arguments of Theorem 8.37. Since every countable $N \prec \langle H(\theta), \in, \Delta \rangle$ is internally approachable, if $\mathfrak{A}$ is an expansion of $\langle H(\theta), \in, \Delta, \langle \mathcal{A}_n : n \in \omega \rangle \rangle$ and $N \prec \mathfrak{A}$ then there is a countable $N' \prec \mathfrak{A}$ catching every $\mathcal{A}_n$. Hence the collection S of countable $M \prec \langle H(\theta), \in, \Delta \rangle$ that catch every $\mathcal{A}_n$ is stationary. But then S reflects to a set of size $\aleph_1$. Hence by Proposition 8.24, we see that $P(\omega_1)/\mathrm{NS}_{\omega_1}$ is weakly $(\aleph_1, \aleph_2)$-saturated. In particular, it is presaturated. We have shown:

8.38 Corollary. *Suppose that κ is supercompact and $G \subseteq \mathrm{Col}(\omega_1, {<}\kappa)$ is generic. Then in $V[G]$, CH holds and NS_{ω_1} is presaturated.*

If we want to have NS_{ω_1} $\aleph_2$-saturated, then we need to do more than collapse a large cardinal to be ω_2. We must control the size of maximal antichains.

The most naïve approach would be to take a large maximal antichain $\mathcal{A} = \langle a_\alpha : \alpha < \gamma \rangle$ with $\gamma \geq \omega_2$ in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ and collapse γ to have cardinality ω_1. The problem with this approach is that $\mathcal{A}$ no longer is a maximal antichain after the collapse.

We can make antichains persistently maximal by the following trick. Suppose that $\mathcal{A} \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$ is any antichain of size at most $\aleph_1$, and S is $\bigtriangledown\mathcal{A}$. If we force with the partial ordering $\mathbb{P}_S$ for adding a closed unbounded set inside S to get a generic G, then in $V[G]$, $\mathcal{A}$ is a maximal antichain. Moreover $\mathcal{A}$ remains a maximal antichain in any model $W \supseteq V[G]$ with $\omega_1^W = \omega_1^{V[G]}$.

This suggests a strategy of reducing the size of antichains $\mathcal{A} = \langle a_\alpha : \alpha < \gamma \rangle$ by shooting closed unbounded sets through the diagonal union of the first ω_1 elements of $\mathcal{A}$, $\langle a_\alpha : \alpha < \omega_1 \rangle$, and iterating this forcing for all antichains. The problem with this approach, as is typical for iterations that destroy stationary sets, is that ω_1 is collapsed by the iteration.

The solution is to combine the two approaches into a single forcing:

8.39 Definition (Foreman et al. [47]). Suppose that $\mathcal{A} = \langle a_\alpha : \alpha < \gamma \rangle$ is a maximal antichain in $P(\omega_1)/\mathrm{NS}_{\omega_1}$. The *antichain sealing forcing* for $\mathcal{A}$ is $\mathrm{Col}(\omega_1, \gamma) * \mathbb{P}_T$ where T is the diagonal union of $\mathcal{A}$ in $V^{\mathrm{Col}(\omega_1, \gamma)}$.

Iterating this forcing does not, in general, preserve ω_1. However, it is still relatively gentle to V:

8.40 Lemma. *Suppose that $S \subseteq \omega_1$ is a stationary set. Then for all maximal antichains $\mathcal{A} \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$, the antichain sealing forcing for $\mathcal{A}$ preserves the stationarity of S.*

Proof. Let $\mathbb{P}$ be the antichain sealing forcing. Suppose that a $\mathbb{P}$-term $\dot{C}$ for a closed unbounded subset of ω_1 and $(p,q) \in \mathbb{P}$ are such that $(p,q) \Vdash \dot{C} \cap S = \emptyset$. Let $a \in \mathcal{A}$ be such that $a \cap S$ is stationary. Let $\theta \gg \omega_1$ and $N \prec \langle H(\theta), \in, \Delta, \mathcal{A}, \{p, a, S, \dot{C}\}\rangle$ be a countable set such that $\delta =_{\mathrm{def}} N \cap \omega_1 \in a \cap S$. Choose a descending sequence $\langle (p_i, q_i) : i \in \omega\rangle \subseteq \mathbb{P} \cap N$ below (p,q) such that for all dense sets $D \subseteq \mathbb{P}$ with $D \in N$ there is an i such that $(p_i, q_i) \in D$. Then $m = (\bigcup_{i\in\omega} p_i, \bigcup_{i\in\omega} q_i \cup \{\delta\}) \in \mathbb{P}$ and is stronger than each (p_i, q_i). Hence m is N generic and forces that δ is a limit point of $\dot{C}$, a contradiction. ⊣

There are models where preserving stationary sets is equivalent to being semiproper:

8.41 Theorem (Foreman et al. [47]). *Suppose that $\mathbb{P}$ is a κ-c.c. partial ordering that preserves stationary subsets of ω_1 and collapses κ to be ω_2. Suppose that there is a λ-supercompact embedding $j : V \to M$ with critical point κ such that $j(\mathbb{P}) \sim \mathbb{P} * \mathrm{Col}(\omega_1, \lambda) * \mathbb{R}$, where $\mathbb{R}$ preserves stationary subsets of ω_1 in $V^{\mathbb{P}*\mathrm{Col}(\omega_1,\lambda)}$. Then for all generic $G \subseteq \mathbb{P}$, and partial orderings $\mathbb{Q} \in V[G]$:*

If $\mathbb{Q}$ preserves stationary subsets of ω_1 and $2^{2^{|\mathbb{Q}|}} < \lambda$, then $\mathbb{Q}$ is semiproper.

Proof (Sketch). Suppose that $G \subseteq \mathbb{P}$ is generic and $\mathbb{Q}$ preserves stationary subsets of ω_1. If $\mathbb{Q}$ is not semiproper, then there is a "bad" stationary set $B \subseteq [H((2^{2^{|\mathbb{Q}|}})^+)]^{<\omega_1}$ and a fixed condition $p \in \mathbb{Q}$ such that for all $N \in B$, $p \in N$ and if $H \subseteq \mathbb{Q}$ is generic with $p \in H$, then $N[H] \cap \omega_1 \neq N \cap \omega_1$.

Let $\lambda = |H((2^{2^{|\mathbb{Q}|}})^+)|$. Choose a λ-supercompact embedding $j : V \to M$ such that $j(\mathbb{P}) \sim \mathbb{P} * \mathrm{Col}(\omega_1, \lambda) * \mathbb{R}$, where $\mathbb{R}$ preserves stationary subsets of ω_1. Let $G * H_0 * H_1 \subseteq \mathbb{P} * \mathrm{Col}(\omega_1, \lambda) * \mathbb{R}$ be a generic object extending G and $\hat{j} : V[G] \to M[G * H_0 * H_1]$ be an elementary embedding extending j.

Since $\mathrm{Col}(\omega_1, \lambda)$ is $<\omega_1$-closed, it preserves the stationarity of B. Let $h : \omega_1 \to H((2^{2^{|\mathbb{Q}|}})^+)$ be a bijection lying in $M[G * H_0]$, and $B' \subseteq \omega_1$ be $\{\alpha : h``\alpha \in B,\ h``\alpha \cap \alpha = \alpha\}$. Then B' is stationary in $V[G * H_0]$ and so in $V[G * H_0 * H_1]$.

Let $H_2 \subseteq j(\mathbb{Q})$ be generic over $M[G * H_0 * H_1]$ with $j(p) \in H_2$. Let N' be a countable elementary substructure of

$$\langle H((2^{2^{j(|\mathbb{Q}|)}})^+)^{M[G*H_0*H_1]}, \in, \Delta, j(\mathbb{Q}), \mathbb{Q}, h, j{\restriction}H((2^{2^{|\mathbb{Q}|}})^+), H_2\rangle$$

such that $\delta =_{\mathrm{def}} N' \cap \omega_1 \in B'$. Let $N_0 = h``\delta$. Then $N_0 = N' \cap H((2^{2^{|\mathbb{Q}|}})^+)^V$ and $N_0 \in B$. Let $N = j(N_0)$. Then $N = j``N_0 \subseteq N'$ and $N \in j(B)$. But $N[H_2] \subseteq N'$, so $\delta \subseteq N \cap \omega_1 \subseteq N[H_2] \cap \omega_1 \subseteq N' \cap \omega_1 = \delta$. Hence $N \in j(B)$ and $N \cap \omega_1 = N[H_2] \cap \omega_1$, a contradiction to the definition of $j(B)$ in M. ⊣

If $\mathbb{P}$ satisfies the hypotheses of Theorem 8.41, then in $V^{\mathbb{P}}$ every antichain sealing forcing is semiproper. An elementary chain argument shows that when this is the case, the partial ordering

$$\mathrm{Col}(\omega_1, 2^{\omega_1}) * \prod_{\substack{\mathcal{A}\in V \\ <\omega_1\text{-supports}}} \mathbb{P}_{T_{\mathcal{A}}}$$

is also semiproper, where the product is taken over all V-maximal antichain $\mathcal{A} \subseteq P(\omega_1)^V/\mathrm{NS}_{\omega_1}$ and $\mathbb{P}_{T_{\mathcal{A}}}$ is the partial ordering for shooting a closed unbounded set through $\triangledown\mathcal{A}$.

The Equivalence of "Semiproper" with "Stationary Set Preserving" in the Case of Antichain Sealing Forcing

Since an important special case of Theorem 8.41 is when $\mathbb{Q}$ is the antichain sealing forcing, we discuss the proof in this special case. Assume that $\mathbb{P}$ satisfies the hypotheses of the theorem.

Let $G \subseteq \mathbb{P}$ be generic and $\mathbb{Q} = \mathrm{Col}(\omega_1, \gamma) * \mathbb{P}_T$ be the antichain sealing forcing for $\mathcal{A}$. We show that $\mathbb{Q}$ is semiproper in $V[G]$. Let $\theta = (2^{2^{|\mathbb{Q}|}})^+$.

The argument for Lemma 8.40 shows that if $N' \prec \langle H(\theta), \in, \Delta, \mathcal{A}\rangle$ catches $\mathcal{A}$, and $(p, q) \in N' \cap \mathbb{Q}$ there is a generic condition $m \le (p, q)$ for N'. If $N \prec N'$ and $N \cap \omega_1 = N' \cap \omega_1$ then m is a semigeneric condition for N.

Thus to see that $\mathbb{Q}$ is semiproper it suffices to show that for each $\mathfrak{A}$ there is a closed unbounded set C relative to $[H(\theta)]^{<\omega_1}$ of $N \prec \mathfrak{A}$ with $\mathcal{A} \in N$ such that there is an $N' \prec \mathfrak{A}$:

(A) $N' \cap \omega_1 = N \cap \omega_1$,

(B) $N \prec N'$, and

(C) N' catches $\mathcal{A}$.

If this fails then there is a structure $\mathfrak{A}$ with domain $H(\theta)$ and a bad stationary set $B \subseteq [H(\theta)]^{<\omega_1}$ such that for all $N \in B$, we have $\mathcal{A} \in N$ and whenever $N' \prec \mathfrak{A}$ is such that

(A) $N' \cap \omega_1 = N \cap \omega_1$, and

(B) $N \prec N'$,

then

(C) N' does not catch $\mathcal{A}$.

By the hypotheses of the theorem we can find a $|H(\theta)|$-supercompact embedding $j : V \to M$ such that $j(\mathbb{P}) \sim \mathbb{P} * \mathrm{Col}(\omega_1, \lambda) * \mathbb{R}$, where $\mathbb{R}$ preserves stationary subsets of ω_1. Let $G * H_0 * H_1 \subseteq \mathbb{P} * \mathrm{Col}(\omega_1, \lambda) * \mathbb{R}$ be a generic object extending G and $\hat{j} : V[G] \to M[G * H_0 * H_1]$ be an elementary embedding extending j.

Since $\mathrm{Col}(\omega_1, |H(\theta)|)$ is $<\omega_1$-closed, it preserves the stationarity of B. Let $h : \omega_1 \to H(\theta)$ be a bijection lying in $M[G * H_0]$, and $B' \subseteq \omega_1$ be $\{\alpha : h``\alpha \in B,\ h``\alpha \cap \alpha = \alpha\}$. Then B' is stationary in $V[G * H_0 * H_1]$. Hence there is a $b \in j(\mathcal{A})$ such that B' is compatible with b.

Let $N' \prec j(\mathfrak{A})$ be such that

1. $N' \cap \omega_1 \in b \cap B'$, and
2. $\{j{\restriction}H(\theta), h, b\} \subseteq N'$.

Let $N_0 = h``\delta$, where $\delta = N' \cap \omega_1$. Then $N_0 \in B$ and $N_0 = N' \cap H(\theta)^V$. Let $N = j(N_0)$. Then $N = j``N_0$ and $N \in j(B)$.

Thus in $M[G * H_0 * H_1]$, N belongs to the bad set $j(B)$. But:

(A) $N' \cap \omega_1 = N \cap \omega_1$,

(B) $N \prec N'$, and

(C) N' catches $j(\mathcal{A})$,

which is a contradiction.

We now outline the argument from [47] that one can force NS_{ω_1} to be $\aleph_2$-saturated, provided one starts with a model with a supercompact cardinal. The actual argument given there was more ambitious: it showed that Martin's Maximum[90] is consistent. The iteration given here is specific to the nonstationary ideal.

Define a semiproper iteration $\mathbb{P} = \mathbb{P}_\kappa$ of length κ with revised countable supports that, at a typical limit stage α, seals a maximal antichain $\mathcal{A}_\alpha \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$ lying in $V^{\mathbb{P}_\alpha}$ provided that every antichain sealing forcing lying in $V^{\mathbb{P}_\alpha}$ is semiproper. There are several mechanisms for choosing $\mathcal{A}_\alpha$ that work. We can use a "Laver function" f,[91] and force with $f(\alpha)$ if $f(\alpha)$ is a maximal antichain in $V^{\mathbb{P}_\alpha}$. Equally well, we can generically choose a maximal antichain $\mathcal{A}_\alpha$. Alternately we could simultaneously seal all maximal antichains $\mathcal{A} \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$ lying in $V^{\mathbb{P}_\alpha}$ with the forcing described after Theorem 8.41. In the original proof a Laver function was used.

If there is a maximal antichain which for which the sealing forcing is not semiproper, then at stage α the iteration $\mathbb{P}$ forces with $\mathrm{Col}^{\mathbb{P}_\alpha}(\omega_1, (2^{2^{\omega_2}})^+)$. Then $\mathbb{P}$ is κ-c.c., semiproper and collapses κ to be ω_2.

If $\mathbb{Q}$ is an antichain sealing forcing in $V^{\mathbb{P}}$, then $\mathbb{Q}$ is semiproper. For otherwise, $\mathbb{P}$ would satisfy the hypotheses of Theorem 8.41. But the conclusion of the theorem says that "semiproper" and "stationary set preserving" are equivalent, a contradiction.

By reflection, there are many α such that every antichain sealing forcing at stage α is semiproper. Whichever diagonalization method we used to choose

[90] See Theorem 8.48.

[91] That is, a "universal" function available at supercompacts that anticipates all possibilities; see Laver [82].

the forcing at stage α, this iteration has the property that if $\mathcal{A}$ is a maximal antichain in $V^{\mathbb{P}}$ then for some $\alpha < \kappa$,

$$\mathcal{A}_\alpha = \{z \subseteq \omega_1 : z \in V^{\mathbb{P}_\alpha} \text{ and } z \in \mathcal{A}_\alpha\}$$

lies in $V^{\mathbb{P}_\alpha}$ and is a maximal antichain in $(P(\omega_1)/\mathrm{NS}_{\omega_1})^{\mathbb{P}_\alpha}$ and is sealed at stage α. Thus $\mathcal{A} = \mathcal{A}_\alpha$. Since $|\mathcal{A}_\alpha| = \omega_1$ in $V^{\mathbb{P}}$, we see that the NS_{ω_1} is $\aleph_2$-saturated.

We have outlined a proof of the following theorem:

8.42 Theorem. *Suppose that κ is a supercompact cardinal, then there is a κ-c.c. semiproper forcing that makes NS_{ω_1} $\aleph_2$-saturated.*

Unlike iterations for creating general forcing axioms, the antichain sealing forcing does not require a "guessing function", and hence with some modifications to the proof, the large cardinal hypothesis of Theorem 8.42 can be reduced to a Woodin cardinal (see [126]).

Because of the Woodin result Theorem 5.64, we cannot hope that the forcing for making NS_{ω_1} $\aleph_2$-saturated preserves the Continuum Hypothesis. However, we can do something only slightly weaker:

8.43 Definition (Foreman et al. [47]). Let $S \subseteq \omega_1$ be stationary and $\mathcal{A} \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}\restriction S$ a maximal antichain. The *antichain sealing forcing relative to S* is the partial ordering $\mathrm{Col}(\omega_1, |\mathcal{A}|) * \mathbb{P}_T$ where T is the union of $\omega_1 \setminus S$ and the diagonal union of $\mathcal{A}$ in $V^{\mathrm{Col}(\omega_1,|\mathcal{A}|)}$.

If $G*H \subseteq \mathrm{Col}(\omega_1, |\mathcal{A}|)*\mathbb{P}_T$ is generic over V, then $\mathcal{A}$ is a maximal antichain in $P(\omega_1)/(\mathrm{NS}_{\omega_1}\restriction S)$ in any model $W \supseteq V[G * H]$ such that $\omega_1^V = \omega_1^{V[G*H]}$. Let S_0 be a co-stationary set and $S = \omega_1 \setminus S_0$. Then the antichain sealing forcing relative to S_0 has a property discovered by Shelah [103] known as *S-closure* where $S = \omega_1 \setminus S_0$. If $\mathbb{P}$ is a partial ordering with this property then $\mathbb{P}$ does not add new ω-sequences to V. Moreover, S-closure is preserved under iterations with countable supports.

Hence if we define an iteration with countable supports up to a supercompact cardinal by alternately using the antichain sealing forcing relative to a stationary and co-stationary set S_0 and collapsing ω_2 we construct a model as above in which CH holds and $\mathrm{NS}_{\omega_1}\restriction S_0$ is $\aleph_2$-saturated.

The remarkable thing about this iteration is that it contains $\mathrm{Col}(\omega_1, {<}\kappa)$ as a regular subalgebra. In particular, by Lemma 3.31, we see that if we collapse a supercompact cardinal to be ω_2 using countably closed forcing then there is an $\aleph_2$-saturated ideal on ω_1.

We summarize:

8.44 Theorem (Foreman et al. [47]). *Suppose that κ is a supercompact cardinal. Then:*

1. *If S is a stationary and co-stationary set there is a semiproper forcing that does not add new ω-sequences and makes $\mathrm{NS}_{\omega_1}\restriction S$ $\aleph_2$-saturated. In particular, if CH holds in V then it holds in the extension.*

2. *In $V^{\mathrm{Col}(\omega_1,<\kappa)}$, there is an $\aleph_2$-saturated ideal on ω_1.*

We note that Todorčević [121] later explicitly described the saturated ideal in the second clause above. As with the full nonstationary ideal, modifications of the argument give the same results assuming only that κ is a Woodin cardinal [104].

Recently Ishiu [59, 60] was able to adapt these techniques to show:

8.45 Theorem. *Let κ be a Woodin cardinal. Then:*

1. *If $\mu < \kappa$ is regular and $\mathcal{C} = \langle C_\alpha : \alpha \in S\rangle$ is a club guessing sequence on a stationary set $S \subseteq \mu$, and $G \subseteq \mathrm{Col}(\mu,<\kappa)$ is generic, then in $V[G]$ the club guessing ideal associated with $\mathcal{C}$ is precipitous.*

2. *There is a κ-c.c. partial ordering $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is generic then:*

   (a) V and $V[G]$ have the same ω_1,

   (b) in $V[G]$, κ is ω_2, CH *holds, and*

   (c) there is a club guessing sequence $\mathcal{C} = \langle C_\alpha : \alpha \in \omega_1\rangle$ such that the club guessing ideal associated with $\mathcal{C}$ is $\aleph_2$-saturated. Moreover this ideal is not *the nonstationary ideal restricted to a stationary set.*

For this theorem, Ishiu studied towers of ideals that are analogous to the stationary tower, suitably adapted to club guessing situations.

8.4. Martin's Maximum and Related Topics

The results of this section appear in [47]. We begin with an examination of some properties that NS_{ω_1} has when it is $\aleph_2$-saturated. The first result shows that an appropriate version of Chang's Conjecture implies that the nonstationary ideal is c.c.c.-indestructible. This version of Chang's Conjecture is simpler than one given in [6] for preservation of saturated ideals and implies the other version in the special case of the nonstationary ideal. In Sect. 8.6, we give a method of Donder for producing c.c.c.-destructible saturated ideals on ω_1, and discuss a theorem of Shelah that shows that NS_{ω_1} can be c.c.c.-destructible.

8.46 Theorem. *Suppose that for all structures $\mathfrak{A} = \langle \omega_2, f_i, R_j, c_k\rangle_{i,j,k\in\omega}$ and all stationary sets $T \subseteq \omega_1$ there is a $\mathfrak{B} \prec \mathfrak{A}$ with $|\mathfrak{B}| = \omega_1$ and $\mathfrak{B} \cap \omega_1 \in T$. If NS_{ω_1} is $\aleph_2$-saturated and $\mathbb{P}$ is a c.c.c. partial ordering then NS_{ω_1} is $\aleph_2$-saturated in $V^{\mathbb{P}}$.*

Proof. By Corollary 7.21, it suffices to show that for all generic $G \subseteq P(\omega_1)/\mathrm{NS}_{\omega_1}$, if $j : V \to M$ is the generic ultrapower then $j(\mathbb{P})$ is $\aleph_2^V$-c.c. in $V[G]$. If this fails there is a stationary $T \subseteq \omega_1$ and a collection of functions $\langle f_\alpha : \alpha < \omega_2^V\rangle$ such that $f_\alpha : \omega_1 \to \mathbb{P}$ and for all α, β the set $C_{\alpha,\beta} = \{\delta : \delta \notin T$ or $f_\alpha(\delta)$ is incompatible with $f_\beta(\delta)\}$ is closed unbounded.

Let θ be sufficiently large and $\mathfrak{B} \prec \langle H(\theta), \in, \Delta, T, \mathbb{P}, \langle f_\alpha\rangle\rangle$ be such that $|\mathfrak{B}| = \omega_1$ and $\mathfrak{B} \cap \omega_1 =_{\mathrm{def}} \delta \in T$. Then for all distinct α and β in $\mathfrak{B} \cap \omega_2$,

$C_{\alpha,\beta} \in \mathfrak{B}$, and hence $\delta \in C_{\alpha,\beta}$. But then $\{f_\alpha(\delta) : \alpha \in \mathfrak{B} \cap \omega_2\}$ is an uncountable antichain in $\mathbb{P}$, a contradiction. $\dashv$

Under MA, if NS_{ω_1} is $\aleph_2$-saturated, it yields minimal degrees of reals in forcing extensions.

8.47 Theorem. *Suppose that Martin's Axiom holds and* CH *fails. Let I be an $\aleph_2$-saturated ideal on ω_1 and $G \subseteq P(\omega_1)/I$ be generic. Suppose that r is a real in $V[G]$ that does not belong to V. Then $V[r] = V[G]$.*

Proof. Let $j : V \to M$ be the generic ultrapower. Then $r \in M$, so there is a function $f : \omega_1 \to 2^\omega$ lying in V such that $[f]^M = r$. By Lemma 2.37, we can assume that f is one-to-one.

Fix a recursive enumeration $\langle \sigma_n : n \in \omega \rangle$ of $2^{<\omega}$. For $s \in 2^\omega$, let $\mathrm{Seq}(s) = \{n : \text{for some } k,\ s{\restriction}k = \sigma_n\}$. A standard application of MA shows for all $X \subseteq \omega_1$, there is a set $a_X \subseteq \omega$ such that for all $\alpha \in \omega_1$,

$$\alpha \in X \quad \text{iff} \quad |a_X \cap \mathrm{Seq}(f(\alpha))| < \omega.$$

Then $X \in G$ iff $\mathrm{Seq}(r) \cap a_X$ is finite. Hence from r we can recover G. $\dashv$

We note that we only use MA_{ω_1} and that this method of proof works equally well to show in ZFC + MA_{ω_1} that Namba forcing with stationary branching trees yields a minimal degree.

When the Steel-Van Wesep proof of the consistency of ZFC + "NS_{ω_1} is $\aleph_2$-saturated" first appeared, it was not known that large cardinals implied AD (or $\mathrm{AD}_\mathbb{R}$), and hence it was not known that ZFC + "NS_{ω_1} is $\aleph_2$-saturated" was consistent relative to large cardinals. The original proof of the consistency of ZFC + "NS_{ω_1} is $\aleph_2$-saturated" from large cardinals went by proving the consistency of Martin's Maximum. *Martin's Maximum* is a provably strongest forcing axiom and has the following statement:

> Suppose that $\mathbb{P}$ is a partial ordering with the property that every stationary $S \subseteq \omega_1$ remains stationary after forcing with $\mathbb{P}$. Let $\mathcal{D} = \langle D_\alpha : \alpha \in \omega_1 \rangle$ be a sequence of dense subsets of $\mathbb{P}$. Then there is a filter $F \subseteq \mathbb{P}$ such that for all α, $F \cap D_\alpha \neq \emptyset$.

We now show:

8.48 Theorem. *Suppose that Martin's Maximum holds. Then* NS_{ω_1} *is $\aleph_2$-saturated.*

Proof. Let $\mathcal{A} = \langle a_\alpha : \alpha < \gamma \rangle$ be a maximal antichain in $P(\omega_1)/\mathrm{NS}_{\omega_1}$. Let $\mathbb{P}$ be the antichain sealing forcing for $\mathcal{A}$.[92] Then by Lemma 8.40, $\mathbb{P}$ preserves stationary subsets of ω_1.

Recall that $\mathbb{P}$ is $\mathrm{Col}(\omega_1, \gamma) * \mathbb{P}_T$, where $\mathbb{P}_T$ is the partial ordering for shooting a closed unbounded set through $T = \bigtriangledown_G \mathcal{A}$. A dense collection of the

[92] See Definition 8.39.

conditions are of the form (p, q), where $p \in \mathrm{Col}(\omega_1, \gamma)$ and q is a closed bounded subset of ω_1. Let D_α be the dense collection of conditions with α in the domain of p and $\sup(q) > \alpha$.

Let $F \subseteq \mathbb{P}$ be a filter such that $F \cap D_\alpha \neq \emptyset$ for all α. Set $G = \bigcup\{p :$ for some q, $(p, q) \in F\}$, and $C = \bigcup\{q :$ for some p, $(p, q) \in F\}$. Then C is a closed unbounded subset of ω_1 inside $\triangledown\{a_{G(\alpha)} : \alpha \in \omega_1\}$. Hence $\{a_{G(\alpha)} : \alpha \in \omega_1\}$ is a maximal antichain and so $|\gamma| \leq \omega_1$. ⊣

Since c.c.c. forcing preserves stationary subsets of ω_1, Martin's Maximum implies MA_{ω_1}. Moreover, Martin's Maximum implies Strong Chang's Conjecture in a form even stronger than the hypothesis of Theorem 8.46 (see [47]). Hence we get the following corollary.

8.49 Corollary. *Suppose that Martin's Maximum holds. Then* NS_{ω_1} *is c.c.c. indestructibly* $\aleph_2$*-saturated and any real added by forcing with* $P(\omega_1)/\mathrm{NS}_{\omega_1}$ *is a minimal degree over* V*.*

8.5. Shelah's Results on Ulam's Problem

Shelah showed that many of the properties of the Kunen-style saturated ideals can hold for NS_{ω_1}, from significantly weaker hypotheses. In doing so he gave a consistency proof for Ulam's Problem for ω_1[93] from an assumption much weaker than a huge cardinal. His remarkable theorems from [104] are:

8.50 Theorem. *Suppose that there is a regular cardinal* κ *with stationarily many supercompact cardinals below* κ*. Then there is a forcing extension which preserves* ω_1 *and* κ*, and:*

1. $2^{\aleph_0} = 2^{\aleph_1} = \omega_2 = \kappa$,
2. $P(\omega_1)/\mathrm{NS}_{\omega_1}$ *is* $(\aleph_1, \aleph_1)$*-centered, and*
3. $P(\omega_1)/\mathrm{NS}_{\omega_1} \cong \mathcal{B}(\mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V))$.

In particular, NS_{ω_1} *is strongly layered.*

Note that the statement that $P(\omega_1)/\mathrm{NS}_{\omega_1}$ is $(\aleph_1, \aleph_1)$-centered is equivalent to the property that there is a collection of countably complete filters $\langle F_\alpha : \alpha < \omega_1\rangle$ such that for every non-stationary set $X \subseteq \omega_1$ there is an α with $X \in F_\alpha$. This implies a positive solution to Ulam's problem for ω_1.

Shelah was able to get similar results holding with CH for the NS_{ω_1} restricted to an arbitrary stationary, co-stationary subset of ω_1:

8.51 Theorem. *Suppose that there is a regular cardinal* κ *such that there are stationarily many supercompact cardinals below* κ *and* $S \subseteq \omega_1$ *is stationary and co-stationary. Then there is a partial ordering that does not add real numbers, preserves* κ *and forces:*

[93] Ulam's problem was introduced in Sect. 6.6. At ω_1 it asks whether it is consistent for there to be a collection $\mathcal{I} = \{I_\alpha : \alpha \in \omega_1\}$ of countably complete ideals such that $P(\omega_1) = \bigcup_{\alpha\in\omega_1}(I_\alpha \cup \breve{I}_\alpha)$.

1. $2^{\aleph_0} = \omega_1$, $2^{\aleph_1} = \omega_2$,
2. $P(\omega_1)/\mathrm{NS}_{\omega_1} \restriction S$ *is* $\aleph_1$*-centered, and*
3. $P(\omega_1)/\mathrm{NS}_{\omega_1} \restriction S \cong \mathcal{B}(\mathrm{Col}(\omega, \omega_1) * \mathrm{Add}(\omega, \omega_2^V))$.

8.6. Saturated Ideals and Square

An important tool for distinguishing between elementary embeddings is their tolerance for square and square-like properties. For example, it is a classical result that square is incompatible with supercompact cardinals. It is not difficult to give similar proofs that square is inconsistent with Chang's Conjectures.[94] This leads to the question of the consistency of square with saturated ideals.

The combinatorial properties of square often make it easier to force other properties. For example, Jensen showed that $\square_{\omega_1}$ implies the existence of a c.c.c. forcing for adding a Kurepa tree on ω_1.

The first results showing the consistency of square with a saturated ideal is due to Donder.

8.52 Proposition (Donder). *If there is a countably complete, $\aleph_1$-dense ideal on ω_1, then there is a forcing extension in which there is an $\aleph_1$-dense ideal on ω_1 and $\square_{\omega_1}$ holds. Moreover, if* CH *holds in the ground model then it holds the extension with $\square_{\omega_1}$.*

Donder's proposition follows from the fact that one can add $\square_{\omega_1}$ with a forcing that adds no subsets of ω_1. As remarked earlier in Sect. 7.16, this also shows that if there is an $\aleph_1$-centered ideal on ω_1 then it remains centered after forcing square in this manner.

Donder further pointed out that if I is an $\aleph_2$-saturated ideal on ω_1, and $\square_{\omega_1}$ holds, then, by Jensen's result and Theorem 5.31, I is c.c.c.-destructible. Hence the existence of an $\aleph_1$-dense ideal implies that one can force the existence of a c.c.c.-destructible $\aleph_2$-saturated ideal on ω_1 in a generic extension.

Historically, these remarks were made around the time that Woodin constructed an $\aleph_1$-dense ideal on ω_1 by forcing over models of determinacy. Forcing over a model with large cardinals to get a c.c.c-destructible saturated ideal on ω_1 was first done with Theorem 7.70.

In Theorems 8.50 and 8.51, published in 1987, Shelah showed that it is consistent to have NS_{ω_1} $\aleph_1$-centered. This immediately gives the following result:

8.53 Theorem (Shelah [104]). *Suppose that there is a regular cardinal κ with stationarily many supercompact cardinals below κ. Then:*

1. *There is a forcing extension in which* NS_{ω_1} $\aleph_1$*-centered and* $\square_{\omega_1}$ *holds.*

---

[94] E.g. Corollary 5.4.

2. *There is a forcing extension in which there is a stationary set S such that $\mathrm{NS}_{\omega_1} \restriction S$ is $\aleph_1$-centered and $\square_{\omega_1}$ and* CH *holds.*

In particular, in these models the saturation of NS_{ω_1} *is c.c.c.-destructible.*

Later, several people, including Foreman/Magidor and Velickovic gave constructions from a supercompact cardinal along the lines of the Martin's Maximum construction of models in which NS_{ω_1} is $\aleph_2$-saturated, and $\square_{\omega_1}$ holds. More recently, Woodin has deduced the consistency of the existence of an $\aleph_1$-dense ideal on ω_1 and the failure of CH from the consistency of $\mathrm{AD}^{L(\mathbb{R})}$. This gives a consistency result for a saturated ideal on ω_1 with $\square_{\omega_1}$ from weaker assumptions, but at the cost of the failure of CH.

On larger cardinals, the model for Theorem 7.71, combined with the distributive version of the partial ordering for adding square, answers the analogous questions:

8.54 Theorem. *Suppose that there is an almost huge cardinal κ and $\mu < \kappa$ is regular. Then there is a forcing extension in which there is a μ^+-centered, μ^+-complete ideal on μ^+ and $\square_{\mu^+}$.*

We note that Chang's Conjecture is preserved by c.c.c. forcing and is inconsistent with the existence of a Kurepa tree.

9. Tower Forcing

In Sect. 4.8, we discussed generic ultrapowers of V associated with forcing with towers of ideals.[95] It was discovered in [47] that if one collapses a supercompact cardinal κ to be ω_2, then there is an $\aleph_2$-saturated ideal I on ω_1. If we let $\mathbb{P}$ be $\mathrm{Col}(\omega, {<}\kappa) * P(\omega_1)/I$, then forcing with $\mathbb{P}$ to get $H * G$ yields a generic elementary embedding $j' : V[H] \to M'$ for some transitive M' isomorphic to $V[H]^{\omega_1}/G$. Restricting j' to V we get a transitive M and an elementary embedding $j : V \to M$. Skipping the intermediate step, forcing with $\mathbb{P}$ yields a generic elementary embedding $j : V \to M$ where:

1. M is transitive,
2. $j(\omega_1) = \kappa$, and
3. $M^\omega \cap V[H] \subseteq M$.

Thus we see that large cardinals imply the existence of generic elementary embeddings with small critical points such as ω_1.

Woodin greatly expanded the technology of [47] by showing that large cardinals imply the existence of many precipitous and presaturated towers of ideals. This first appeared in [125]. The tower forcings have proven extremely useful in that they yield generic elementary embeddings with small critical

[95] The definition of a tower of ideals is given in Definition 4.17.

points and their properties can be established directly from large cardinals. They act as a bridge between large cardinals and descriptive set theory which is concerned with definable properties of small sets.

In this section we prove basic facts about certain examples of towers that have nice ultrapowers. The subject of tower forcings and their consequences is well explored in the books of Woodin [126] and Larson [80], so we barely scratch the surface of the theory. As with single ideals the study splits into two closely related types of towers, induced and natural. We give a brief discussion of induced towers and then focus on a particular type of natural tower, the stationary towers. The latter have proved most useful for applications and have a better developed theory.

Recall that if X and Y have the same cardinality and $f : X \to Y$ is a bijection, then f induces a one-to-one correspondence between ideals on X and ideals on Y. Moreover, for normal ideals this correspondence is independent of the choice of f.[96] For this reason, if $\gamma = |H(\alpha)|$, and we are forcing with towers of ideals of height $\delta > \gamma$, it makes no essential difference whether we view the tower as consisting of ideals on sets of the form $P(\gamma)$ or $P(H(\alpha))$. Woodin uses sets of the form V_α as his base sets. Depending on context we will use the most convenient version. We note that for inaccessible α, $V_\alpha = H(\alpha)$ and $|H(\alpha)| = \alpha$.

Throughout this section we will be assuming:

> δ is a strong limit cardinal, $U \subseteq \delta$ is an unbounded set of cardinals and $\mathcal{T} = \langle I_\alpha \subseteq PP(H(\alpha)) : \alpha \in U\rangle$ is a tower of normal, fine, countably complete ideals.

We will call $\delta = \sup(U)$ the *height* of the tower. For $\alpha < \delta$ we let α^* be the least element of U greater than 2^{2^α}. For all of our towers we get an equivalent forcing partial order if we restrict to a cofinal subset of U. For this reason we can assume that α^* is the least element of U above α. We will use the notation of Sect. 4.8. In particular, we will write $\mathcal{P}_\mathcal{T}$ for the forcing associated with a tower $\mathcal{T}$.

If we have conditions $b, c \in \mathcal{P}_\mathcal{T}$ with $\beta = \operatorname{supp}(b) < \gamma = \operatorname{supp}(c)$, then there is a canonical meet of b and c in $\mathcal{P}_\mathcal{T}$. This is given by $\{z : z \cap H(\beta) \in b$ and $z \cap H(\gamma) \in c\}$. In an abuse of notation we will denote this by $b \cap c$.

If $\alpha \leq \sup(U)$, we define $\mathcal{T}_\alpha = \mathcal{T}\restriction\alpha =_{\text{def}} \langle I_\beta : \beta \in U \cap \alpha\rangle$.

9.1 Definition. We will say that a tower $\mathcal{T}$ of inaccessible height δ is *presaturated* iff forcing with $\mathcal{P}_\mathcal{T}$ preserves the statement "δ is a regular cardinal".

9.2 Proposition. *Suppose that $\mathcal{T}$ is a presaturated tower of inaccessible height δ. Then $\mathcal{T}$ is precipitous. If $G \subseteq \mathcal{P}_\mathcal{T}$ is generic and $j : V \to M$ is the generic ultrapower with M transitive, then $M^{<\delta} \cap V[G] \subseteq M$.*

Proof. By Proposition 4.55, it suffices to show that if $\langle \mathcal{A}_\alpha : \alpha < \gamma\rangle$ is a sequence of maximal antichains of length $\gamma < \delta$, then there is a dense set of

[96] In the language of [18], the correspondence is "natural structure".

conditions $S \in \mathcal{P}_\mathcal{T}$ such that for all $\alpha < \gamma$, $|\{a \in \mathcal{A}_\alpha : a$ is compatible with $S\}| < \delta$.

Define a term for a function $\dot{f} : \gamma \to \delta$ by setting $\dot{f}(\alpha)$ to be the least β such that β is the support of a condition $a \in \mathcal{A}_\alpha \cap G$. Since δ is regular in $V[G]$, there is a dense set $D \subseteq \mathcal{P}_\mathcal{T}$ for all $S \in D$ there is a $\rho < \delta$ such that $S \Vdash$ "$\dot{f}$ is bounded by ρ". If $S \in D$ then for all α, $|\{a \in \mathcal{A}_\alpha : S$ is compatible with $S\}| \leq 2^{2^{|H(\rho)|}}$. ⊣

Using this proposition and Proposition 4.56, we can now describe some typical tower behavior:

9.3 Example. Suppose that

1. $\rho \geq \omega_1$ is a successor cardinal,
2. each I_α is ρ-complete and concentrates on $[H(\alpha)]^{<\rho}$, and
3. $\mathcal{P}_\mathcal{T}$ is weakly (ρ, δ)-saturated.[97]

Then $\mathcal{T}$ is presaturated. If $j : V \to M \subseteq V[G]$ is the elementary embedding arising from a generic $G \subseteq \mathcal{P}_\mathcal{T}$ then $\mathrm{crit}(j) = \rho$, $j(\rho) = \delta$ and $M^{<\delta} \cap V[G] \subseteq M$.

9.1. Induced Towers

This section summarizes some of the results of [42].

Let μ be regular and $\delta > \mu$ be an inaccessible cardinal. Suppose that $\mathbb{Q}$ is a partial ordering collapsing δ to be μ^+. Let $\mu < \gamma < \delta$ with γ a cardinal. If $G \subseteq \mathbb{Q}$ is generic and $f : \mu \to H(\gamma)$ is a bijection in $V[G]$, then for each $S \subseteq [H(\gamma)]^{<\mu}$ we can define $\overline{S} \subseteq \mu$ by setting $\overline{S} = \{\beta\colon f``\beta \in S\}$. Modulo $\mathrm{NS}_\mu$, $\overline{S}$ is independent of the choice of the bijection $f$ and $\{\beta\colon f``\beta \cap \mu = \beta\}$ is closed and unbounded. In $V[G]$, the map $S \mapsto \overline{S}$ is $\subseteq$ order-preserving.

Let I be a normal ideal on μ in $V^\mathbb{Q}$. For $\gamma < \delta$, define an ideal $I_\gamma \subseteq P([H(\gamma)]^{<\mu})$ by putting $S \in I_\gamma$ iff $\|\overline{S} \notin I\|_\mathbb{Q} = 0$. Then I_γ is a normal ideal.

Define $\iota'_\gamma : P([H(\gamma)]^{<\mu}) \to \mathcal{B}(\mathbb{Q}) * P(\mu)/I$ by letting $\iota'_\gamma(S) = \langle \|\bar{S} \notin I\|, [\overline{S}]\rangle$. Then ι'_γ has kernel I_γ and hence induces a well-defined order and antichain preserving map $\iota_\gamma : P([H(\gamma)]^{<\mu})/I_\gamma \to \mathcal{B}(\mathbb{Q}) * P(\mu)/I$.[98]

If $\gamma < \gamma'$ then the restriction of $\iota'_{\gamma'}$ to $P([H(\gamma)]^{<\mu})$ is ι'_γ. Hence $\mathcal{T} = \langle I_\gamma\colon \gamma < \delta\rangle$ is a tower of ideals. The direct limit of the ι_γ's gives an order and antichain preserving map from $\mathcal{P}_\mathcal{T}$ into $\mathcal{B}(\mathbb{Q}) * P(\mu)/I$. Hence if I is δ-saturated, then $\mathcal{P}_\mathcal{T}$ has the δ-c.c.

If $\mathbb{Q}$ is $<\mu$-closed then $\mathbb{Q}$ adds a generic object to $\mathrm{Col}(\mu, |H(\gamma)|)$ for each $\gamma < \delta$. If $S \subseteq [H(\gamma)]^{<\mu}$ and $\overline{S}$ is I-positive then $\overline{S}$ is stationary in μ. By our remarks on robustness around Example 8.32 we see that $S \cap \mathrm{IA}$ is positive and S is equivalent to $S \cap \mathrm{IA}$ modulo I_γ. Moreover I_γ extends the nonstationary

[97] The definition is given in the discussion before Proposition 4.6.
[98] A similar situation is discussed in the paragraphs before Proposition 4.25.

ideal on $[H(\gamma)]^{<\mu}\restriction\mathrm{IA}$. If I is the nonstationary ideal on μ (respectively, the nonstationary ideal on μ restricted to cofinality $\rho<\mu$) then I_γ is exactly the nonstationary ideal on $[H(\gamma)]^{<\mu}\cap\mathrm{IA}$ (respectively the nonstationary ideal on $[H(\gamma)]^{<\mu}\cap\mathrm{IA}(\mathrm{Cof}(\rho))$).

We have outlined:

9.4 Proposition. *Let μ be regular and $\delta>\mu$ be inaccessible. Suppose that $\mathbb{Q}$ is $<\mu$-closed, δ-c.c. and $V^{\mathbb{Q}}\models\delta=\mu^+$. If there is a μ-complete, μ^+-saturated ideal on μ in $V^{\mathbb{Q}}$, then there is a tower $\mathcal{T}$ of normal, μ-complete ideals $\langle I_\gamma:\gamma<\delta\rangle$ on $\langle[H(\gamma)]^{<\mu}\cap\mathrm{IA}:\gamma<\delta\rangle$ such that $\mathcal{P}_{\mathcal{T}}$ has the δ-chain condition.*

From this we get the following corollaries:

9.5 Corollary. *Suppose that ρ is a regular cardinal and $\mu>\rho$ is an almost huge cardinal. Then there is a $<\rho$-closed, μ-c.c. partial ordering $\mathbb{P}$ such that in $V^{\mathbb{P}}$:*

1. *$\mu=\rho^+$, and*
2. *there is an inaccessible cardinal δ and a tower of normal, fine, μ-complete ideals $\mathcal{T}=\langle I_\gamma:\gamma<\delta\rangle$ such that I_γ concentrates on $[H(\gamma)]^{<\mu}\cap\mathrm{IA}$ and $\mathcal{P}_{\mathcal{T}}$ has the δ-chain condition.*

Proof. Let $j:V\to M$ be an almost huge embedding with critical point μ and $j(\mu)=\delta$. The partial ordering $\mathbb{P}$ is the first stage of the Magidor variation on the Kunen construction described in Sect. 7.11. That construction builds a partial ordering $\mathbb{P}*\mathbb{Q}$ such that:

1. $\mathbb{P}$ is $<\rho$-closed and μ-c.c.,
2. $\mathbb{Q}$ is $<\mu$-closed and δ-c.c.,[99] and
3. $V^{\mathbb{P}*\mathbb{Q}}\models\delta=\mu^+$ and there is a μ^+-saturated ideal on μ.

Hence we can apply Proposition 9.4 to conclude that in $V^{\mathbb{P}}$ there is a tower $\mathcal{T}$ as desired. ⊣

From Proposition 9.4 and Theorem 8.44, we also see:

9.6 Corollary. *Suppose that δ is a supercompact cardinal. Then there is a δ-saturated tower $\langle I_\gamma:\gamma<\delta\rangle$ of normal, countably complete ideals on $\langle[H(\gamma)]^{<\omega_1}:\gamma<\delta\rangle$.*

Since the hypothesis of Theorem 8.44 can be weakened to the assumption that δ is a Woodin cardinal the conclusions of Corollary 9.6 follow if we simply assume that δ is Woodin.

We can get information about the stationary tower $\mathcal{T}$ on $\langle[H(\gamma)]^{<\omega_1}:\gamma<\delta\rangle$ as well (see Theorem 4.9 of [42]). In this case the maps ι_γ yield a regular embedding from $\mathcal{P}_{\mathcal{T}}$ into $\mathcal{B}(\mathbb{Q}*P(\omega_1)/\mathrm{NS}_{\omega_1})$.

[99] In fact $\mathbb{Q}$ can be taken to be $\mathrm{Col}(\mu,<\delta)$.

9.7 Theorem. *Let δ be an inaccessible cardinal, $\mathbb{Q} = \mathrm{Col}(\omega_1, {<}\delta)$, and suppose that $V^{\mathbb{Q}} \models \mathrm{NS}_{\omega_1}$ is presaturated. Let $\mathcal{T}$ be the tower of nonstationary ideals restricted to $[H(\gamma)]^{<\omega_1}$ for $\gamma < \delta$. Then $\mathcal{P}_{\mathcal{T}}$ is a regular subalgebra of $\mathbb{Q} * P(\omega_1)/\mathrm{NS}_{\omega_1}$.*

In particular, by Example 9.3, forcing with $\mathcal{P}_{\mathcal{T}}$ preserves δ, the generic ultrapower is well-founded and is closed under ω-sequences from $V[G]$.

Woodin proved much more general results as we see in Sect. 9.3.

9.2. General Techniques

Woodin used the ideas of catching antichains to show that towers of natural ideals have nice generic ultrapowers. We now explore this technology but only sketch the proofs. The books of Woodin [126] and Larson [80] are excellent definitive references containing complete proofs and many applications. The notation and terminology adopted in this section is somewhat different than Woodin's in order to place it in the context of the rest of this chapter. Standard terminology is found in the two books.

Good Structures

In this subsection we discuss *good structures*. The antichain catching ideas from [47] are crucial for showing that certain towers have nice properties. For these arguments to work they require good structures. Most of the theory of stationary tower forcing can be developed for arbitrary towers under assumptions about the existence of good structures. If each ideal I_α has the form $\mathrm{NS}\restriction Z_\alpha$ for some stationary set Z_α, the goodness of a structure $N \prec H((2^{2^\delta})^+)$ is equivalent to having $N \cap H(\alpha) \in Z_\alpha$ for all $\alpha \in N$.[100] Those readers who are only interested in stationary towers can simplify some arguments by taking this as the definition of *goodness*.

9.8 Definition. Let N be a set. Then N is *good for* α iff $\alpha \in N$ and for all $C \in \breve{I}_\alpha \cap N$ we have that $N \cap H(\alpha) \in C$. We will say that N is a *good structure* iff for all $\alpha \in N \cap U$, N is good for α.

Note that if $\alpha < \beta$, $N \cap H(2^\beta) \prec H(2^\beta)$, $\alpha \in N$ and N is good for β, then N is good for α. Proposition 3.44 implies that for each α, I_α-almost all z generate structures that are good for α. Under mild assumptions, we can show that good structures exist.

9.9 Proposition. *Let $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ be a tower of normal, fine, countably complete ideals with height δ. Suppose that there is a normal, fine, countably complete ideal J on $P(H(\delta))$ such that for all $\alpha \in U$, the projection of J to an ideal on $P(H(\alpha))$ is I_α. Then for all $\theta \geq \delta^+$ and all Skolemized structures $\mathfrak{A}$ expanding $\langle H(\theta), \in \Delta, J, \mathcal{T}\rangle$ there is a set $C \in \breve{J}$ for all $z \in C$:*

[100] See Lemma 3.46.

1. $\mathrm{Sk}^{\mathfrak{A}}(z) \cap V_\delta = z$, *and*

2. $\mathrm{Sk}^{\mathfrak{A}}(z)$ *is good.*

Proof. Same as Proposition 3.44. $\dashv$

Catching Antichains

The antichain catching arguments used in [47] were informally referred to as "catching an index for an antichain". Woodin formalized this by calling the method "capturing an antichain". We will slightly abuse terminology by frequently using the word "antichain" for a collection of subsets of $P(H(\alpha))$ for $\alpha \in U$ whose I_α equivalence classes represent an antichain in $\mathcal{P}_{\mathcal{T}}$.

9.10 Definition. Let $\mathcal{A}$ be a collection of subsets of $P(H(\alpha))$ for $\alpha \in U$ that form an antichain in $\mathcal{P}_{\mathcal{T}}$. A structure N *captures* $\mathcal{A}$ *below* α iff N is good for α and there is an $a \in N \cap \mathcal{A}$ such that $\mathrm{supp}(a) < \alpha$ and $N \cap H(\mathrm{supp}(a)) \in a$.

The next two results play the role of Proposition 8.24 for towers.

9.11 Lemma. *Let $\mathcal{A}$ be a maximal antichain. Suppose that $\alpha < \delta$ and $[S] \in \mathcal{P}_{\mathcal{T}}$. If $S \subseteq \{z \in H(\delta) : z$ captures $\mathcal{A}$ below $\alpha\}$, then $\{b \in \mathcal{A} : b$ is compatible with $[S]\} \subseteq \{b : \mathrm{supp}(b) < \alpha\}$.*

Proof. By Lemma 4.49,

$$[\triangledown\{b \in \mathcal{A} : \mathrm{supp}(b) < \alpha\}] = \Sigma\{[b] : b \in \mathcal{A} \text{ and } \mathrm{supp}(b) < \alpha\}.$$

Since the right hand side is less than or equal to $\Sigma\{[b] : \mathrm{supp}(b) < \alpha\}$ and $S \subseteq \triangledown\{b \in \mathcal{A} : \mathrm{supp}(b) < \alpha\}$ we see the lemma. $\dashv$

From this we easily deduce:

9.12 Proposition. *Let $\rho \leq \delta$. Suppose that for all $\gamma < \rho$ and all sequences of maximal antichains $\langle \mathcal{A}_\alpha : \alpha < \gamma\rangle$ there is a dense set of $S \in \mathcal{P}_{\mathcal{T}}$ with an η (depending on S) between γ and δ such that if $N \in S$ and $\alpha \in \gamma \cap N$, then N captures $\mathcal{A}_\alpha$ below η. Then $\mathcal{P}_{\mathcal{T}}$ is weakly (ρ, δ)-saturated.*

9.13 Definition. Let $\mathcal{T}$ be a tower of height δ. Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}_{\mathcal{T}}$. Then $\mathcal{T}$ *can capture* $\mathcal{A}$ *at* α iff

1. $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$ is a maximal antichain in $\mathcal{P}_{\mathcal{T}_\alpha}$, and
2. whenever:
   (a) γ is between α and δ, $\sigma < \alpha$ and $\mathfrak{A}$ is a structure in a countable language expanding $\langle H(\gamma^*), \in, \Delta\rangle$,

   there is a closed unbounded set of $N \prec \mathfrak{A}$ such that if:

   (b) N is good for γ,

(c) $\{\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}, \mathcal{T}_\alpha\} \subseteq N$, and

(d) $N^* \prec N$ has cardinality less than α

then there is an $N' \prec \mathfrak{A}$ such that

(A) N' is good for γ,

(B) $N' \cap H(\sigma) = N \cap H(\sigma)$,

(C) $N^* \prec N'$, and

(D) N' captures $\mathcal{A}$ below α.

We will say that $\mathcal{T}$ *captures antichains* iff for all maximal antichains $\mathcal{A}$ there is a stationary set of $\alpha < \delta$ such that $\mathcal{T}$ can capture $\mathcal{A}$ at α.

In all of the arguments we give it suffices to have an unbounded rather than stationary set of α such that $\mathcal{T}$ captures $\mathcal{A}$ at α. Woodin called the property of capturing antichains at α "semiproper at α". The tower $\mathcal{T}$ is typically not semiproper in the sense of the forcing property, so we have shifted terminology in this chapter.

9.14 Remark. If I_α is of the form $\mathrm{NS}\restriction Z_\alpha$, then any $N \in Z_\alpha$ with $N \prec \langle H(\alpha^*), \in, \Delta, I_\alpha \rangle$ is good for α. If $\mathcal{T} = \langle \mathrm{NS}\restriction Z_\alpha : \alpha \in U \rangle$ is a tower, then for $\alpha < \gamma$ in U there is a closed unbounded set C such that for all $z \in C \cap Z_\alpha$ there is an $N \prec \langle H(\gamma^*), \in, \Delta, I_\gamma \rangle$ with $N \cap H(\gamma) \in Z_\gamma$ and $N \cap H(\alpha) = z$. This N is necessarily good for γ.

Thus for towers of this form we can modify Definition 9.13 by taking $\mathfrak{A}$ to be a structure with domain $H(\alpha^*)$ and in clause (b) replace "N is good for γ" by "$N \in Z_{\alpha^*}$".

Catching Your Tail

Next we describe a "catch-your-tail" argument in this context. The point of catching your tail is to turn the quantifier "there is a closed unbounded set of N" in the definition of antichain catching into the quantifier "for every N". This trick is quite familiar in the context of proper forcing.

In a situation where we want to catch antichains, our data will be:

1. a tower $\mathcal{T}$,
2. a maximal antichain $\mathcal{A}$ in $\mathcal{P}_{\mathcal{T}}$ that $\mathcal{T}$ can capture at α,
3. a γ between α and δ,
4. a structure $\mathfrak{A}$ in a countable language expanding $\langle H(\gamma^*), \in, \Delta \rangle$,[101] and
5. the closed unbounded set C posited in clause 2 of Definition 9.13.

[101] Equivalently, we could be given a closed unbounded subset of $H(\gamma^*)$. This discussion parallels the simpler Lemma 8.36.

Let λ be a regular cardinal greater than 2^{γ^*}. Suppose that we are given an arbitrary $M \prec \langle H(\lambda), \in, \Delta\rangle$ such that $\{\alpha, \gamma, \mathfrak{A}, \mathcal{T}_\alpha, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}\} \subseteq M$ and M is good for γ. Let $N = M \cap H(\gamma^*)$. Then $N \prec \mathfrak{A}$ and N is good for γ. We note that M has definable Skolem functions.

Let $M^* \prec M$ have cardinality less than α with $\{\alpha, \gamma, \mathcal{T}_\alpha, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}\} \subseteq M^*$ Letting $N^* = M^* \cap H(\gamma^*)$, we see that if $f(x, y_0, \ldots, y_n)$ is a definable Skolem function, and $a \in M^*$, then the restricted Skolem function $g : [H(\gamma)]^{n+1} \to H(\gamma^*)$ defined by setting:

$$g(a_1, \ldots, a_n) = \begin{cases} f(a, a_1, \ldots, a_n) & \text{if } f(a, a_1, \ldots, a_n) \in H(\gamma^*), \\ 0 & \text{otherwise,} \end{cases}$$

belongs to N^*.

Applying Definition 9.13 we can find an $N' \prec \mathfrak{A}$ that is good for γ, $N^* \prec N'$, N' has the same intersection with $H(\sigma)$ as N does, and N' captures $\mathcal{A}$ below α.

Now let M' be the Skolem hull of $(N' \cap H(\gamma)) \cup M^*$ in $\langle H(\lambda), \in, \Delta\rangle$. Since each restricted Skolem function of $H(\lambda)$ belongs to N', we see that $M' \cap H(\gamma) = N' \cap H(\gamma)$ and $M' \cap H(\gamma^*) \subseteq N'$. In particular:

(A$'$) M' is good for γ,

(B$'$) $M' \cap H(\sigma) = N \cap H(\sigma)$,

(C$'$) $M^* \prec M'$, and

(D$'$) M' captures $\mathcal{A}$ below α.

Hence we see that by passing from γ^* to λ we can change the quantifier "there is a closed unbounded set of $N \prec \mathfrak{A}$" in Definition 9.13 to "for all sufficiently large λ and all $M \prec \langle H(\lambda), \in, \Delta\rangle$ with $\{\alpha, \gamma, \mathfrak{A}, \mathcal{T}_\alpha, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}\} \subseteq M$".

By using standard Skolemization arguments this shows that given the structure $\mathfrak{A}$ in Definition 9.13, we can find an expansion $\mathfrak{A}'$ such that for *every* $N \prec \mathfrak{A}'$ satisfying (b), (c) and (d) there is an N" $\prec \mathfrak{A}'$ satisfying (A)–(D).

Using Antichain Catching

9.15 Theorem. *Suppose that $\mathcal{T}$ is a tower that captures antichains. Then $\mathcal{T}$ is precipitous.*[102]

Proof. We verify the conditions of Proposition 4.52. Suppose that $\langle \mathcal{A}_n : n \in \omega\rangle$ is a tree of maximal antichains below a condition $[X] \in \mathcal{P}_{\mathcal{T}}$. Fix an increasing sequence of ordinals α_n such that $\mathcal{T}$ can capture $\mathcal{A}_n$ at α_n. Let $\gamma \in U \setminus \sup_n \alpha_n$ and $\lambda > 2^{\gamma^*}$. Let $N_0 \prec \langle H(\lambda), \in, \Delta\rangle$ be such that $\{\langle \mathcal{A}_n : n \in \omega\rangle, \langle \alpha_n : n \in \omega\rangle, \mathcal{T}\} \subseteq N_0$ and N_0 is good for γ.

By the remarks on catching your tail, we can build a sequence of structures $\langle N_i : i \in \omega\rangle$ such that:

[102] This is the tower analogue of Theorem 8.37.

1. $N_i \prec \langle H(\lambda), \in, \Delta\rangle$ and is good for γ,
2. $\{\langle \mathcal{A}_n : n \in \omega\rangle, \langle \alpha_n : n \in \omega\rangle, \mathcal{T}\} \in N_i$,
3. N_{i+1} captures $\mathcal{A}_i$ below α_{i+1}, and
4. $N_i \cap H(\alpha_i) = N_{i+1} \cap H(\alpha_i)$.

Then $z_i = N_i \cap H(\alpha_i)$ witnesses the hypothesis of Proposition 4.52. ⊣

We now want to be able to extend this technique to be able to catch antichains in a transfinite sequence. This is not possible in general:

9.16 Example (Foreman and Magidor [42]). Let δ be Woodin. For regular $\alpha < \delta$ let Z_α be the collection of $N \in [H(\alpha)]^{<\omega_2}$ that are internally approachable by a sequence of length ω_1. Then $\langle \mathrm{NS}\restriction Z_\alpha : \alpha \text{ is regular and } \alpha < \delta\rangle$ forms a precipitous tower. Further, if this tower is presaturated then $\Theta^{L(\mathbb{R})} < \omega_2$.

Thus, by Woodin's Theorem 5.64 if NS_{ω_1} is $\aleph_2$-saturated, then this tower is not presaturated.

To build transfinite sequences that catch antichains we need to be able to continue past limit stages. The obstacle in the previous example is that after catching ω many antichains, a model is no longer internally approachable. An extra hypothesis that works is given by the following definition that is idiosyncratic to this chapter.

9.17 Definition. Let $\rho < \delta$ and δ inaccessible. A tower $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ of height δ will be called *ρ-complete* iff for all $\gamma < \rho$ and all increasing sequences $\langle \alpha_i : i < \gamma + 1\rangle$ of elements of U and all regular $\lambda \gg \alpha_\gamma$ and all $u \in H(\lambda)$, if:

1. $\langle N_i : i \in \gamma\rangle$ is a sequence of elementary substructures of $\langle H(\lambda), \in, \Delta, u\rangle$ with
$$\{\langle \alpha_i : i < \gamma + 1\rangle, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\} \subseteq N_j$$
for all $j < \gamma$,
2. N_i good for α_γ, and
3. $N_i \cap H(\alpha_i) = N_j \cap H(\alpha_i)$ for $i < j < \gamma$,

then there is an $N_\gamma \prec \langle H(\lambda), \in, \Delta, u\rangle$ with $\{\langle \alpha_i : i < \gamma + 1\rangle, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\} \subseteq N_\gamma$ that is good for α_γ and for all $i < \gamma$, $N_\gamma \cap H(\alpha_i) = N_{\alpha_i} \cap H(\alpha_i)$.

When each $I_\alpha = \mathrm{NS}\restriction Z_\alpha$, we will say that the sequence of stationary sets $\langle Z_\alpha : \alpha \in U\rangle$ is ρ-complete.

By using the catch-your-tail arguments of the previous section, we see that this is equivalent to the statement that any structure $\mathfrak{A}_0$ with domain $H((2^{\alpha_\gamma})^+)$ can be expanded to a structure $\mathfrak{A}$ so that any sequence of elementary substructures $\langle N_i : i < \gamma\rangle$ with:

1. $\{\langle \alpha_i : i < \gamma + 1\rangle, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\} \subseteq N_j$ for all $j < \gamma$,
2. N_i good for α_γ, and
3. $N_i \cap H(\alpha_i) = N_j \cap H(\alpha_i)$ for $i < j < \gamma$

has a limiting structure $N_\gamma \prec \mathfrak{A}$ containing $\{\langle \alpha_i : i < \gamma + 1\}, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\}$ that is good for α_γ and such that for all $i < \gamma$, $N_\gamma \cap H(\alpha_i) = N_{\alpha_i} \cap H(\alpha_i)$.

9.18 Example. Let δ be inaccessible, $\kappa < \delta$ regular and $Z_\alpha = [H(\alpha)]^{<\kappa}$. Then the tower $\langle \mathrm{NS}\restriction Z_\alpha : \alpha < \delta\rangle$ is κ-complete.

The next example appears in [42] and its significance is discussed in Example 9.36.

9.19 Example. Suppose that δ is inaccessible, and μ and $\kappa > \mu^+$ are regular cardinals less than δ. For regular $\alpha < \delta$ let $Z_\alpha = \{N \in [H(\alpha)]^{<\kappa} : N \cap \alpha$ is $\leq\mu$-closed$\}$. Then $\langle \mathrm{NS}\restriction Z_\alpha : \alpha$ is regular and $\alpha < \delta\rangle$ is a κ-complete tower of ideals.

The main point of the proof of Example 9.19 is to show that if $\alpha < \beta$ are two regular cardinals bigger than 2^κ and $\mathfrak{A}_0$ is a structure in a countable language with domain $H(\beta)$ then there is an expansion of $\mathfrak{A}_0$ to a fully Skolemized structure $\mathfrak{A}$ in a countable language such that if:

1. $z \subseteq H(\alpha)$,
2. $z \cap \alpha$ is $\leq\mu$-closed, and
3. $\mathrm{Sk}^{\mathfrak{A}}(z) \cap H(\alpha) = z$

then $\mathrm{Sk}^{\mathfrak{A}}(z) \cap \beta$ is $\leq\mu$-closed.

9.20 Theorem. *Let $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ be a tower of normal, fine, countably complete ideals of inaccessible height δ. Suppose that:*

1. $\mathcal{T}$ captures antichains, and

2. $\mathcal{T}$ is ρ-complete.

Then $\mathcal{T}$ is weakly (ρ, δ)-saturated.

Proof. We verify the conditions of Proposition 9.12. Let $\gamma < \rho$, $\langle \mathcal{A}_i : i < \gamma\rangle$ be a sequence of maximal antichains in $\mathcal{T}$ and $[X] \in \mathcal{P}_{\mathcal{T}}$. We need to find an $S \in \mathcal{P}_{\mathcal{T}}$ below $[X]$ and an η such that if $N \in S$ and $i \in \gamma \cap N$, then N captures $\mathcal{A}_i$ below η.

Fix an increasing sequence $\langle \alpha_i : i < \gamma\rangle$ drawn from $U \setminus \max\{\gamma, \mathrm{supp}(X)\}$ such that $\mathcal{T}$ can capture $\mathcal{A}_i$ at α_i. Let $\alpha_\gamma \in U$ be much larger than the supremum of the $\langle \alpha_i : i < \gamma\rangle$, and λ a sufficiently large regular cardinal below δ to witness γ^+-completeness.

We let $\eta = \alpha_\gamma$. Indirectly assume that there is no $S \in I_\eta^+$ below $[X]$ as desired. Then there is a $C \in \breve{I}_\eta$ such that if $N \in C \cap X$ then there is an $i \in \gamma \cap N$ such that N does not capture $\mathcal{A}_i$ below α_i. Hence it suffices to show that there is an $N \prec H(\lambda)$ that is good for α_γ such that $C \in N$ and for all $i \in N \cap \gamma$, N captures $\mathcal{A}_i$ below α_i.

Let $N_0 \prec \langle H(\lambda), \in, \Delta\rangle$ be such that

1. $\{\langle \alpha_i : i < \gamma + 1\rangle, \langle \mathcal{A}_i : i < \gamma\rangle, C, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\} \subseteq N_0$,
2. N is good for α_γ.

Using completeness at limit stages and antichain capturing at successor stages, build a sequence of structures $\langle N_i : i \in \gamma + 1\rangle$ such that:

1. $N_i \prec \langle H(\lambda), \in, \Delta\rangle$ and
$$\{\langle \alpha_i : i < \gamma + 1\rangle, \langle \mathcal{A}_i : i < \gamma\rangle, C, \langle I_\alpha : \alpha \in U \cap (\alpha_\gamma + 1)\rangle\} \subseteq N_i,$$
2. N_i is good for α_γ,
3. if $i < j < \gamma$, $N_i \cap H(\alpha_i) = N_j \cap H(\alpha_i)$, and
4. if $i \in N_i$, then N_{i+1} captures $\mathcal{A}_i$ below α_i.

Letting $N = N_\gamma$ we get the desired contradiction. $\dashv$

9.3. Natural Towers

Just as for single ideals there are many examples of natural towers. The only type of natural tower whose generic embeddings have been explored extensively are given by the following definition.

9.21 Definition. The tower $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ will be called a *stationary tower* iff there is a sequence of stationary sets $\langle Z_\alpha \subseteq P(H(\alpha)) : \alpha \in U\rangle$ such that $I_\alpha = \mathrm{NS}{\restriction}Z_\alpha$.

We note that the requirement that $\langle \mathrm{NS}{\restriction}Z_\alpha : \alpha \in U\rangle$ forms a tower is sometimes hard to verify as illustrated in Example 9.19. Moreover, Burke's Corollary 4.21 shows that for sufficiently large α_2 any normal ideal on $P(H(\alpha_1))$ is the projection of the nonstationary ideal restricted to a stationary subset of $P(H(\alpha_2))$. In particular, if α_1 is supercompact we can find Z_2 such that the projection of $\mathrm{NS}{\restriction}Z_2$ to $P(H(\alpha_1))$ is a prime ideal dual to a supercompact ultrafilter!

Methods for Proving Antichain Capturing

We now examine the methods for showing that various towers catch antichains. All of the arguments take the same basic form used to prove Theorem 8.41 in the paper [47]. To verify semiproperness in the context of forcing and antichain catching in the context of an ideal or a tower of ideals, we have a model N and a σ and want to add some ordinals to N without changing $N \cap H(\sigma)$. In the case of semiproperness, $\sigma = \omega_1$. In the case of towers, σ is an ordinal above the support of some elements of antichains N already captures.

In Theorem 9.46, we give Burke's examples of towers that are not precipitous. Burke's methods of proof do not restrict the height of the tower in any way. For example, the non-precipitous towers can have supercompact height. In light of Theorem 9.15, Burke's towers cannot capture antichains. However, with a sufficient large cardinal hypothesis every stationary tower forcing does capture antichains.

All of the antichain catching arguments follow the same pattern. By Remark 9.14, we can simplify the definition of antichain catching by working with structures $\mathfrak{A}$ on $H(\alpha^*)$ and replace "good for γ" with "$N \in Z_{\alpha^*}$". We will assume that $\mathcal{T}$ cannot capture an antichain $\mathcal{A}$ and use our large cardinal hypothesis to create a situation with a cardinal α such that:

1. $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$ is a maximal antichain in $\mathcal{P}_{\mathcal{T}_\alpha}$, and
2. there are:

   (a) a $\sigma < \alpha$ and a structure $\mathfrak{A}$ expanding $\langle H(\alpha^*), \in, \Delta\rangle$[103]

   and a "bad" stationary set B of $N \prec \mathfrak{A}$ such that

   (b) $B \subseteq Z_{\alpha^*}$,

   (c) $\{\mathcal{T}_\alpha, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}\} \in N$ for all $N \in B$, and

   (d) each $N \in B$ has an $N^* \prec N$ of cardinality less than α

   such that whenever $N' \prec \mathfrak{A}$ is such that

   (A) $N' \in Z_{\alpha^*}$,

   (B) $N \cap H(\sigma) = N' \cap H(\sigma)$, and

   (C) $N^* \prec N'$

   then

   (D) N' does not capture $\mathcal{A}$ below α.

The cardinal α is chosen so that there is an appropriately strong $j : V \to M$ with critical point α. Then $j(\mathcal{A})$ is a maximal antichain up to $j(\alpha)$. The set B determines a condition in the forcing $j(\mathcal{P}_{\mathcal{T}})$ and hence is compatible with some $b \in j(\mathcal{A})$ that can be taken to have support bigger than α^*. A structure $N' \prec j(\mathfrak{A})$ is considered with

[103] We are using Remark 9.14.

1. $N' \cap H(\text{supp}(b)) \in b \cap B$,
2. $b \in N'$, and
3. $N' \in j(Z_{\alpha^*})$.

Such an N' captures $j(\mathcal{A})$.

The strength of the embedding is used to see that we can find an N' with properties 1–3 that is sufficiently closed and belongs to M. Depending on Z_{α^*}, this is done in at least two different ways. For simple sequences $\langle Z_\alpha : \alpha \in U\rangle$, as in what we call "Woodin's towers" below, all that is required is that $b \cap B$ is stationary in V. For this, δ being a Woodin cardinal suffices to generate the required embeddings j. When the sequence of Z_α's is more complicated, as in the towers considered by Burke [11], we need the fact that j witnesses some degree of supercompactness.

Next we will have $N_0 = N' \cap H(\alpha^*)$ and $N = j(N_0)$. Since $N_0 \in B$, there are witnesses N^* and σ. By elementarity $j(N^*)$ and $j(\sigma)$ are witnesses to N being in $j(B)$.

Since $j(\sigma) = \sigma$, $N \cap H(\sigma) = N_0 \cap H(\sigma) = N' \cap H(\sigma)$. Since $|N^*| < \alpha$, $j(N^*) = j``N^*$. We use the closure of N' to argue that $j(N^*) \subseteq N'$. This will be done different ways in different contexts.

Arguing in M, we can now reach our desired contradiction by noting that $N' \prec j(\mathfrak{A})$ and:

(A) $N' \in j(Z_{\alpha^*})$,

(B) $N \cap H(\sigma) = N' \cap H(\sigma)$,

(C) $j(N^*) \prec N'$, and

(D) N' captures $j(\mathcal{A})$ below $j(\alpha)$.

We give the explicit arguments in the next two sections.

Woodin's Towers

Woodin's investigation and application of tower forcing focused on cases where the Z_α were of a simple form such as $[H(\alpha)]^{<\kappa}$ or $P(H(\alpha))$. The towers considered in [42], while concentrating on Z_α whose definitions are not as simple, are also amenable to arguments of the style of this section.

Notation. Woodin introduced the following notation, which has become standard for towers:

1. when each $Z_\alpha = P(H(\alpha))$, the tower is called $\mathbb{P}_{<\delta}$, and
2. when each $Z_\alpha = [H(\alpha)]^{<\omega_1}$, the tower is called $\mathbb{Q}_{<\delta}$.

We will use the following definition of a Woodin cardinal:

9.22 Definition. Let δ be a cardinal. Then δ is *Woodin* iff for every $A \subseteq V_\delta$ and all functions $f : \delta \to \delta$ there is an $\alpha < \delta$ closed under f and an elementary embedding $j : V \to M$ with M transitive such that:

1. $\operatorname{crit}(j) = \alpha$ and M is closed under $<\alpha$-sequences,
2. $j(f)(\alpha) = f(\alpha)$, $V_{f(\alpha)} \subseteq M$, and
3. $j(A) \cap V_{f(\alpha)} = A \cap V_{f(\alpha)}$.

9.23 Theorem (Woodin). *Let δ be a Woodin cardinal and $\mathcal{T}$ the stationary tower $\langle \mathrm{NS}{\restriction} Z_\alpha : \alpha \in \delta\rangle$ where either:*

1. *for all α, $Z_\alpha = P(H(\alpha))$, or*
2. *for all α, $Z_\alpha = [H(\alpha)]^{<\kappa}$ for some regular uncountable cardinal $\kappa < \delta$.*

Then $\mathcal{T}$ captures antichains.

Proof. Suppose that the theorem fails. Let $\mathcal{A}$ be a counterexample. Then there is a closed unbounded set $D \subseteq \delta$ such that for all $\alpha \in D$, $\mathcal{T}$ cannot capture $\mathcal{A}$ at α. If our tower is of the form $Z_\alpha = [H(\alpha)]^{<\kappa}$ we can take the least element of D to be above κ. We can assume without loss of generality that for all $\alpha \in D$,

1. $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$ is a maximal antichain in $\mathcal{P}_{\mathcal{T}_\alpha}$, and
2. there are

   (a) a $\sigma < \alpha$ and a structure $\mathfrak{A}_\alpha$ expanding $\langle H(\alpha^*), \in, \Delta\rangle$

   and a "bad" stationary set B_α of $N \prec \mathfrak{A}_\alpha$ such that

   (b) $B_\alpha \subseteq Z_{\alpha^*}$,
   (c) $\{\mathcal{T}_\alpha, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}\} \in N$ for all $N \in B_\alpha$, and
   (d) each $N \in B_\alpha$ has an $N^* \prec N$ of cardinality less than α

   such that if $N' \prec \mathfrak{A}_\alpha$ is such that

   (A) $N' \in Z_{\alpha^*}$,
   (B) $N \cap H(\sigma) = N' \cap H(\sigma)$, and
   (C) $N^* \prec N'$

   then

   (D) N' does not capture $\mathcal{A}$ below α.

Define a function f by setting $f(\alpha)$ to be the least element of D above α if $\alpha \notin D$. For each $\alpha \in D$, let $f(\alpha)$ be some regular β sufficiently large so that:

1. β is above the next inaccessible element of D above α, and
2. there is a $b \in \mathcal{A} \cap V_{f(\alpha)}$ such that $b \cap B_\alpha$ is stationary.

Since δ is Woodin, we can find an $\alpha < \delta$ and an elementary embedding $j : V \to M$ with critical point α such that $\alpha \in D$, $V_{f(\alpha)} \subseteq M$, $j(\mathcal{A}) \cap V_{f(\alpha)} = \mathcal{A} \cap V_{f(\alpha)}$, $j(\mathcal{T}) \cap V_{f(\alpha)} = \mathcal{T} \cap V_{f(\alpha)}$ and M is closed under $<\alpha$-sequences.

Choose a $b \in \mathcal{A} \cap V_{f(\alpha)}$ so that $b \cap B_\alpha$ is stationary in V. We can do this since stationarity is absolute between V and M for bounded subsets of $V_{f(\alpha)}$. We can assume that $\mathrm{supp}(b) > \alpha^*$ and $b \subseteq Z_{\mathrm{supp}(b)}$. Arguing in V, let $\mathfrak{B}$ be a structure in a countable language expanding $j(\mathfrak{A}_\alpha)$ containing constant symbols for $j \restriction H(\alpha^*)$, $\mathfrak{A}_\alpha$ and b. Since $b \cap B_\alpha$ is stationary in V, there is a $z \subseteq H(\mathrm{supp}(b))$ belonging to $b \cap B_\alpha$ such that

$$\mathrm{Sk}^{\mathfrak{B}}(z) \cap H(\mathrm{supp}(b)) = z.$$

Let $N_0 = z \cap H(\alpha^*)$ and $N = j(N_0)$. Since $N_0 \in B_\alpha$, there are witnesses N^* and σ. By elementarity $j(N^*)$ and $j(\sigma)$ are witnesses to N being in $j(B_\alpha)$. Since $|N^*| < \alpha$, $j(N^*) = j``N^* \in M$. Since $\mathrm{Sk}^{\mathfrak{B}}(z)$ is closed under $j \restriction H(\alpha^*)$ and $N^* \subseteq z$ we see that $j(N^*) \subseteq \mathrm{Sk}^{\mathfrak{B}}(z)$. Since $j(\sigma) = \sigma$, $N \cap H(\sigma) = N_0 \cap H(\sigma) = z \cap H(\sigma)$.

Let $N' = \mathrm{Sk}^{j(\mathfrak{A}_\alpha)}(z \cup j(N^*) \cup \{b\})$. Then $N' \subseteq \mathrm{Sk}^{\mathfrak{B}}(z)$. Since z, $j(N^*)$, b and $j(\mathfrak{A}_\alpha)$ belong to M we see that $N' \in M$. Hence

1. $N' \cap \mathrm{supp}(b) = z \in b \cap B_\alpha$,
2. $b \in N'$, and
3. $N' \in j(Z_{\alpha^*})$.[104]

By 1–3, N' captures $j(\mathcal{A})$.

Arguing in M, we can now reach our desired contradiction by noting that $N' \prec j(\mathfrak{A}_\alpha)$ and:

(A) $N' \in j(Z_{\alpha^*})$,

(B) $N \cap H(\sigma) = N' \cap H(\sigma)$,

(C) $j(N^*) \prec N'$, and

(D) N' captures $j(\mathcal{A})$ below $j(\alpha)$.

$\dashv$

9.24 Corollary. *Let δ be a Woodin cardinal. Then the stationary tower forcing is presaturated if either:*

1. $Z_\alpha = P(H(\alpha))$ for all α, or

2. for some successor $\kappa < \delta$, $Z_\alpha = [H(\alpha)]^{<\kappa}$ for all α.

[104] This is automatic in the case of the Woodin's towers, but not for arbitrary stationary towers.

Proof. If $Z_\alpha = P(H(\alpha))$ then the sequence $\langle Z_\alpha\rangle$ is δ-complete. Hence $\mathcal{P}_\mathcal{T}$ is weakly (δ,δ)-saturated.

Suppose that $\kappa = \mu^+$. If $Z_\alpha = [H(\alpha)]^{<\kappa}$, then the sequence $\langle Z_\alpha\rangle$ is $<\kappa$-complete, and hence weakly (κ,δ)-saturated. Let $G \subseteq \mathcal{P}_\mathcal{T}$ be generic and $j : V \to M$ be the generic embedding associated with G. As in Example 9.3, $j(\kappa) \geq \delta$. This implies that $V[G] \models |\delta| \leq \mu^+$. By weak saturation, $V[G] \models |\delta| \geq \mu^+$ and hence δ is a regular a cardinal in $V[G]$. ⊣

Burke's Towers

In [11], Burke considered stationary towers $\mathcal{T} = \langle \mathrm{NS}\restriction Z_\alpha : \alpha \in U \subseteq \delta\rangle$ that were arbitrary save for the restriction that U contained all sufficiently large regular cardinals below δ.

The arguments in his paper yield the somewhat more general result that we prove below. The hypotheses that we impose on the tower to ensure that it captures antichains is that there is a function f that bounds the map sending α to α^* that is sufficiently absolute and that δ is a supercompact cardinal.

9.25 Theorem (Burke [11]). *Let δ be a supercompact cardinal and $\mathcal{T} = \langle \mathrm{NS}\restriction Z_\alpha : \alpha \in U\rangle$ be an arbitrary stationary tower of height δ. Suppose that there is a Σ_2 formula $\phi(x, x', \vec{y})$ and $\vec{p} \in H(\delta)$ such that if we set*

$$f(\alpha) = \alpha' \quad \textit{iff} \quad \phi^V(\alpha, \alpha', \vec{p})$$

then $f : \mathrm{On} \to \mathrm{On}$ and $f\restriction\delta$ bounds the map sending α to α^*, the least element of U above α.[105]

Then $\mathcal{T}$ captures antichains.

Proof. A simple class pigeon-hole argument together with the assumption that ϕ exists yields the following property: There are $\delta^* > 2^\delta$ and a stationary $Z_{\delta^*} \subseteq P(H(\delta^*))$ such that for unboundedly many $\lambda \in \mathrm{On}$ there is a λ-supercompact embedding $j : V \to M$ such that:

1. δ^* is the least element of $j(U)$ above δ, and
2. if we write $j(\langle Z_\alpha : \alpha \in U\rangle) = \langle Z^j_\alpha : \alpha \in j(U)\rangle$, then $Z^j_{\delta^*} = Z_{\delta^*}$.

Let $\mathcal{A}$ be a maximal antichain in $\mathcal{P}_\mathcal{T}$. We show that for every $(2^{\delta^*})^+$-supercompact embedding $j : V \to M$ satisfying 1 and 2, $j(\mathcal{T})$ can catch $j(\mathcal{A})$ at δ. Using a reflection argument we see this implies that there are stationarily many $\alpha < \delta$ such that $\mathcal{T}$ captures $\mathcal{A}$ at α.

Indirectly assume that $\mathcal{A}$ and j are a counterexample. Since $j(\mathcal{A})\restriction\delta = \mathcal{A}$, and $j(\mathcal{T})_\delta = \mathcal{T}$ we see that

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[105] Given an arbitrary stationary tower $\mathcal{T} = \langle \mathrm{NS}\restriction Z_\alpha : \alpha \in U\rangle$, we can interpolate ideals to make another tower $\mathcal{T}'$ with a U' bounded by the function $\alpha \mapsto (2^{2^\alpha})^+$ such that forcing with $\mathcal{P}_\mathcal{T}$ is equivalent to forcing with $\mathcal{P}_{\mathcal{T}'}$. The resulting tower $\mathcal{T}'$ does not necessarily satisfy the hypotheses of Theorem 9.25, since it may not be a stationary tower.

1. $j(\mathcal{A}) \cap j(\mathcal{T})_\delta$ is a maximal antichain in $\mathcal{P}_{j(\mathcal{T})_\delta}$ and
2. there are

   (a) a $\sigma < \alpha$ and a structure $\mathfrak{A}$ expanding $\langle H(\delta^*), \in, \Delta \rangle$

   and a "bad" stationary set B of $N \prec \mathfrak{A}$ such that

   (b) $B \subseteq Z_{\delta^*}$,
   (c) $\{\mathcal{T}_\delta, \mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\delta}\} \in N$ for all $N \in B$, and
   (d) each $N \in B$ has an $N^* \prec N$ of cardinality less than δ

   such that if $N' \prec \mathfrak{A}$ is such that

   (A) $N' \in Z_{\delta^*}$,
   (B) $N \cap H(\sigma) = N' \cap H(\sigma)$, and
   (C) $N^* \prec N'$

   then

   (D) N' does not capture $j(\mathcal{A})$ below δ.

We note that we can rephrase condition (D) equivalently as:

(E) There is no $a \in N' \cap \mathcal{A}$ with $N' \cap H(\mathrm{supp}(a)) \in a$.

Let $b \in j(\mathcal{A})$ be a condition in M such that $M \models b \cap B$ is stationary. We can assume that $\mathrm{supp}(b) > \delta^*$ and $b \subseteq Z_{\mathrm{supp}(b)}$. In M, choose an N' such that

1. $N' \cap \mathrm{supp}(b) \in B \cap b$,
2. $N' \prec j(\mathfrak{A})$ with $\{j \restriction H(\delta^*), B, b, \mathfrak{A}, \mathcal{T}\} \subseteq N'$, and
3. $N' \in j(Z_{\delta^*})$.

Then $b \in N'$ and $N' \cap \mathrm{supp}(b) \in b$.

Let $N_0 = N' \cap H(\delta^*)$, and $N = j(N_0)$. Let $\sigma < \delta$ and N^* witness $N_0 \in B$. Then $j(\sigma)$ and $j(N^*)$ witness $N \in j(B)$.

Since $|N^*| < \delta$ and $j(N^*) = j``N^*$, and moreover $j(\sigma) = \sigma$. Since $N'$ is closed under $j \restriction H(\delta^*)$ and $N^* \subseteq N'$ we see that $j``N^* \subseteq N'$. Moreover, $N \cap H(\sigma) = N_0 \cap H(\sigma) = N' \cap H(\sigma)$.

We have shown that $N' \prec j(\mathfrak{A})$ and:

(A) $N' \in j(Z_{\delta^*})$,

(B) $N \cap H(\sigma) = N' \cap H(\sigma)$,

(C) $j(N^*) \prec N'$, and

(E) There is an $a \in N' \cap j(\mathcal{A})$ with $N' \cap H(\mathrm{supp}(a)) \in a$.

This contradicts $N \in j(B)$ with witnesses $j(N^*)$ and $j(\sigma)$. ⊣

We now compare the proofs of Woodin's theorem and Burke's theorem. In Woodin's proof the strength of the embedding j is used to see that $b \cap B_\alpha$ is stationary in V. This gives the existence of z and hence of an $N' = \mathrm{Sk}^{\mathfrak{A}}(z \cup j(N^*) \cup \{b\})$ with $N' \cap H(\mathrm{supp}(b)) = z$. For simple sequences $\langle Z_\alpha \rangle$, N' is automatically a member of $j(Z_{\alpha^*})$.

In Burke's situation this may not hold. To find an $N' \in j(Z_{\alpha^*})$, one either needs $j(Z_{\alpha^*})$ to be stationary in V or else have $j \restriction H(\alpha^*) \in M$. For arbitrary sequences, the former possibility requires j to have a degree of hugeness. The latter requires only supercompactness. Hence Burke's hypothesis.

We remark on the use of the function f bounding the growth of U. Without the existence of such a function we cannot focus on particular δ^* and Z_{δ^*}. Unless we fix these in advance, among other problems, we cannot necessarily assume that $j \restriction H(\delta^*) \in M$. The assumption that f exists can be avoided at the cost of assuming that δ is a cardinal larger than a supercompact, as the next theorem shows.

In Burke's paper he asks the whether general stationary tower forcing captures antichains. Woodin proved that with sufficient large cardinal strength, it does. He used the following cardinal:

9.26 Definition. Let δ be a cardinal. Then δ is a *Woodinized supercompact* cardinal iff for every $A \subseteq V_\delta$ and all functions $f : \delta \to \delta$ there is an $\alpha < \delta$ closed under f and an elementary embedding $j : V \to M$ with M transitive such that:

1. $\mathrm{crit}(j) = \alpha$ and M is closed under $|V_{j(f)(\alpha)}|$-sequences,
2. $j(f)(\alpha) = f(\alpha)$, and
3. $j(A) \cap V_{f(\alpha)} = A \cap V_{f(\alpha)}$.

9.27 Remark. The definition can be equivalently reformulated to simply demand that for all $f : \delta \to \delta$ there is an $\alpha < \delta$ closed under f and a $j : V \to M$ with critical point α such that M is closed under $|V_{j(f)(\alpha)}|$-sequences.

These cardinals stand in the same relation to supercompact cardinals as Woodin cardinals stand to hypermeasurable cardinals. Standard large cardinal techniques can be used to check that every almost huge cardinal is a Woodinized supercompact cardinal and has many Woodinized supercompact cardinals below it. Moreover, if δ is a Woodinized supercompact cardinal then there is a stationary set of $\kappa < \delta$ such that

$$(V_\delta, \in) \models \kappa \text{ is supercompact.}$$

9.28 Theorem. *Suppose that δ is a Woodinized supercompact cardinal and $\mathcal{T}$ is a stationary tower of height δ. Then $\mathcal{T}$ captures antichains.*

Proof. Suppose that $\mathcal{T} = \langle \mathrm{NS} \restriction Z_\alpha : \alpha \in U \rangle$ is a stationary tower of height δ. Let $\mathcal{A}$ be an antichain that cannot be caught stationarily often. As in the proof of antichain catching for the Woodin towers, we let $D \subseteq \delta$ be a closed unbounded set such that for $\alpha \in D$ there are $B_\alpha \subseteq Z_{\alpha^*}$ and $\mathfrak{A}_\alpha$ showing that $\mathcal{A}$ cannot be caught at α. As before we can assume that for all $\alpha \in D$, $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$ is a maximal antichain in $\mathcal{A} \cap \mathcal{P}_{\mathcal{T}_\alpha}$. We again define a function f by setting $f(\alpha)$ to be the least element of D above α if $\alpha \notin D$; for each $\alpha \in D$, $f(\alpha)$ is some regular β sufficiently large so that:

1. β is above the next inaccessible element of D above α^*, and
2. there is a $b \in \mathcal{A} \cap V_{f(\alpha)}$ such that $b \cap B_\alpha$ is stationary.

Since δ is Woodinized supercompact, we can find an $\alpha \in D$ and a $j : V \to M$ with critical point α such that:

1. α^* is the least element of $j(U)$ above α,
2. if we write $j(\langle Z_\alpha : \alpha \in U\rangle) = \langle Z^j_\alpha : \alpha \in j(U)\rangle$, then $Z^j_{\alpha^*} = Z_{\alpha^*}$, and
3. M is closed under $2^{2^{\alpha^*}}$ sequences.

The proof now follows the proof of Theorem 9.25, with α playing the role of δ. The strength of the embedding is used to find N' closed under $j{\restriction}H((2^{\alpha^*})^+)$. $\dashv$

9.4. Self-Genericity for Towers

The definition of self-genericity for individual ideals (Definition 3.45) generalizes easily in the context of towers.

9.29 Definition. Let $\mathcal{T}$ be a tower of height δ and $\theta > 2^\delta$ a regular cardinal. Let $M' \prec \langle H(\theta), \in, \Delta, \mathcal{T}\rangle$ be good, M be the transitive collapse of M', $\mathcal{T}^M$ the image in M of $\mathcal{T}$ and j the inverse of the transitive collapse map. Then M' is self-generic iff $\bigcup_{\alpha\in\delta\cap\mathrm{ran}(j)} U(j, M' \cap H(\alpha))$ is generic over M for $\mathcal{P}^M_{\mathcal{T}}$.

As in Proposition 8.20, M' is *self-generic* iff M' captures an index for every maximal antichain $\mathcal{A} \in M'$ below some $\alpha < \delta$.

We remind the reader that if $\mathcal{T}$ is a tower and $\lambda < \delta$ is a strong limit cardinal with $U \cap \lambda$ unbounded in λ, then we can consider the tower $\mathcal{T}_\lambda = \mathcal{T}{\restriction}\lambda$, consisting of $\langle I_\alpha : \alpha \in U \cap \lambda\rangle$. The next proposition gives a sufficient condition for $\mathcal{P}_{\mathcal{T}_\lambda}$ to be a regular subalgebra of $\mathcal{P}_{\mathcal{T}}$. It says that if the collection of N that are self-generic for $\mathcal{T}_\lambda$ is a condition in T, then below that condition $\mathcal{P}_{\mathcal{T}_\lambda}$ is a regular subalgebra of $\mathcal{P}_{\mathcal{T}}$.

9.30 Proposition. *Let $\alpha \in U$ be at least $(2^\lambda)^+$. Let S be the collection of N such that:*

1. *$N \prec \langle H(\alpha), \in, \Delta, \mathcal{T}_\lambda, \dots\rangle$, and*
2. *for all maximal antichains $\mathcal{A} \subseteq \mathcal{P}_{\mathcal{T}_\lambda}$, if $\mathcal{A} \in N$ then N captures $\mathcal{A}$ below λ.*

If $S \in I_\alpha^+$, then there is a condition $R \in \mathcal{P}_{\mathcal{T}_\lambda}$ such that $\mathcal{P}_{\mathcal{T}_\lambda}/R$ is a regular subalgebra of $\mathcal{P}_{\mathcal{T}}/(S \cap R)$.

Proof. Let $G \subseteq \mathcal{P}_{\mathcal{T}}/S$ be generic and $j : V \to V^Z/G$ be the canonical embedding. The proof of the easy part of Proposition 2.34 shows that V^Z/G is well-founded at least up to δ, so we replace it by an isomorphic structure M that is transitive to rank δ. Then $j``H(\lambda)$ is self-generic. Hence $\bigcup_{\alpha \in \delta \cap \mathrm{ran}(j)} U(j, j``H(\alpha))$ is generic for $\mathcal{P}_{\mathcal{T}_\lambda}$ over $H(\lambda^*)^V$. Let $R \in \mathcal{P}_{\mathcal{T}_\lambda}$ force that the generic object for $\mathcal{P}_{\mathcal{T}_\lambda}$ can be extended to a generic $G \subseteq \mathcal{P}_{\mathcal{T}}/S$. Then $\mathcal{P}_{\mathcal{T}_\lambda}/R$ is a regular subalgebra of $\mathcal{P}_{\mathcal{T}}/(S \cap R)$. ⊣

This proposition generalizes easily to a criterion for one tower to be embedded in a different tower.

9.31 Proposition. *Let $\mathcal{T}_1, \mathcal{T}_2$ be towers of height $\delta_1 < \delta_2$. Suppose that there is a condition $B \in \mathcal{T}_2$ that is a subset of $\{N \subseteq H(\theta) : N$ is self-generic for $\mathcal{T}_1\}$ for some $\theta \geq (2^{\delta_1})^+$. Then there is a condition $R \in \mathcal{P}_{\mathcal{T}_1}$ such that $\mathcal{P}_{\mathcal{T}_1}/R$ is a regular subalgebra of $\mathcal{P}_{\mathcal{T}_2}/B \cap R$.*

Proof. Let $G \subseteq \mathcal{P}_{\mathcal{T}_2}$ be generic with $B \in G$. Then $j``H(\theta)$ is self-generic for $\mathcal{T}_1$. In particular, $G \cap \mathcal{P}_{\mathcal{T}_1}$ is generic over V for $\mathcal{P}_{\mathcal{T}_1}$. ⊣

In analogy with Lemma 3.46 for individual ideals we see the following proposition.

9.32 Proposition. *Suppose that $\mathcal{T}$ is a tower of inaccessible height δ. Then the following are equivalent:*

1. *$\mathcal{P}_{\mathcal{T}}$ is δ-c.c., and*
2. *there is a closed unbounded set $D \subseteq \delta$ for all $\alpha \in D$*

$$\{N : N \textit{ is self-generic for } \mathcal{P}_{\mathcal{T}_\alpha}\} \in \breve{I}_{\alpha^*}.$$

Woodin showed the following for $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$:

9.33 Proposition. *Suppose that $\delta_1 < \delta_2$ are Woodin cardinals, and either:*

1. *$\mathbb{R}_1 = \mathbb{P}_{<\delta_1}$ and $\mathbb{R}_2 = \mathbb{P}_{<\delta_2}$, or*
2. *$\mathbb{R}_1 = \mathbb{Q}_{<\delta_1}$ and $\mathbb{R}_2$ is either $\mathbb{P}_{<\delta_2}$ or $\mathbb{Q}_{<\delta_2}$.*

Then there is an $b \in \mathbb{R}_2$ such that $\mathbb{R}_1$ is a regular subalgebra of $\mathbb{R}_2/b$.

Proof (Sketch). Let S be the set described in Proposition 9.31, and $b = S$ if $\mathbb{R}_1 = \mathbb{P}_{<\delta_1}$ and $\{N \in S : |N| = \omega\}$ if $\mathbb{R}_1 = \mathbb{Q}_{<\delta_1}$. We need to see that b is stationary. This is shown by fixing an arbitrary structure $\mathfrak{A}$ on $H((2^{\delta_1})^+)$ and building a chain $\langle N_i \rangle$ of elementary substructures of $\mathfrak{A}$ together with an increasing sequence of cardinals $\langle \alpha_i \rangle$. The lengths of the sequences are determined by what $\mathbb{R}_1$ is. The sequences will have the properties that:

1. $i < j$ implies $N_j \cap H(\alpha_i) = N_i \cap H(\alpha_i)$,
2. if $\mathcal{A}$ is a maximal antichain in $\mathbb{R}_1$ that belongs to $\mathrm{Sk}^{\mathfrak{A}}(\bigcup_i (N_i \cap H(\alpha_i)))$ then there is an i such that N_{i+1} captures $\mathcal{A}$ below α_i.

Then $\mathrm{Sk}^{\mathfrak{A}}(\bigcup_i (N_i \cap H(\alpha_i)))$ is an elementary substructure of $\mathfrak{A}$ that belongs to S. $\dashv$

Proposition 9.33 can be reformulated in a general context by the use of Definition 9.17. For example using the same technique as the previous sketch, one can show:

9.34 Proposition. *Let $\rho \leq \delta_1 < \delta_2$ regular cardinals with δ_1 and δ_2 inaccessible. Let $\mathcal{T} = \langle I_\alpha : \alpha \in U \rangle$ be a tower of height δ_2 and $A \in \mathcal{P}_{\mathcal{T}}$ be a condition with support at least δ_1^*. Let $\mathcal{S}$ be the tower of height δ_1 determined by projecting $I_{\mathrm{supp}(A)} \restriction A$ to a sequence of ideals on $H(\alpha)$ for $\alpha < \delta_1$. Suppose that*

1. *$\mathcal{T}/A$ is ρ-complete,*
2. *if J is the projection of $I_{\mathrm{supp}(A)} \restriction A$ to an ideal on $H(\delta_1)$, then J concentrates on $[H(\delta_1)]^{<\rho^+}$, and*
3. *$\mathcal{S}$ captures antichains.*

Then there is a condition $B \in \mathcal{T}$ such that $\mathcal{S}$ is a regular subalgebra of $\mathcal{T}/B$.

9.5. Examples of Stationary Tower Forcing

We now consider some examples of stationary tower forcing. Our goal is to give the reader a glimpse of the possibilities inherent in the forcing without exploring the many applications. This is done in the excellent books by Woodin [126] and Larson [80].

9.35 Example (Woodin). Let δ be Woodin and $\mu < \delta$ be a regular uncountable cardinal. For $\mu \leq \alpha < \delta$, let $Z_\alpha = [H(\alpha)]^{<\mu}$ and $\mathcal{T} = \langle \mathrm{NS} \restriction Z_\alpha : \alpha < \delta \rangle$. Then forcing with $\mathcal{P}_{\mathcal{T}}$ yields a generic embedding $j : V \to M$ with $M^{<\delta} \subseteq M$. If η is an ordinal less than δ and $\{z : z \cap \mu \in \mu\}$ and $\{z : \mathrm{cf}(z \cap \eta) = \rho\}$ are in the generic object G, then the critical point of j is μ and in both $V[G]$ and M, the cofinality of η is $j(\rho)$.

In particular, if there are Woodin cardinals then there is a partial ordering that changes the cofinality of $\aleph_{\omega+1}^V$ to $\aleph_n$, while preserving all cardinals less than $\aleph_\omega$. Closely related to this is:

9.36 Example (Foreman and Magidor [42]). Let δ be supercompact and ρ, μ and κ be regular with $\mu^+ \leq \rho < \kappa < \delta$. Let $\eta < \delta$ and $\langle Z_\alpha : \kappa \leq \alpha < \delta \rangle$

be a sequence of stationary sets determining a stationary tower, with Z_α a subset of

$$\{z \in [H(\alpha)]^{<\kappa} : z \cap \kappa \in \kappa,\ z \cap \alpha \text{ is } {<}\mu^+\text{-closed and } \mathrm{cf}(z \cap \eta) = \rho\}.$$

Then in $V[G]$, the critical point of j is κ, the cofinality of η is ρ and for all ordinals ξ, if $\mathrm{cf}(\xi)^{V[G]} \leq \mu$ then $\mathrm{cf}(\xi)^{V[G]} = \mathrm{cf}(\xi)^V$.

If $Z_\alpha = \{z \in [H(\alpha)]^{<\kappa} : z \cap \kappa \in \kappa,\ z \cap \alpha \text{ is } {<}\mu^+\text{-closed and } \mathrm{cf}(z \cap \eta) = \rho\}$, then a Woodin cardinal suffices to draw the same conclusion.

Specializing this example by taking $\kappa = \eta = \aleph_{\omega+1}$, $\mu = \aleph_{16}$, and $\rho = \aleph_{17}$ and assuming the existence of Woodin cardinals we see that there are partial orderings that force $\aleph_{\omega+1}$ to have cofinality $\aleph_{17}$ but preserve all cardinals below $\aleph_\omega$ and the V-cofinality of any cardinal whose cofinality in $V[G]$ is below $\aleph_{17}$.

If a tower preserves cofinalities then the possible fixed points of its generic embedding are restricted. The next result appears in [42], as Proposition 1.5:

9.37 Proposition. *Suppose that $\mathbb{P}$ is δ-presaturated with δ regular, $j : V \to M$ is definable in $V^{\mathbb{P}}$, M is well-founded and $M^{<\delta} \subseteq M$. Suppose that $\alpha < \delta$ is a regular cardinal bigger than the critical point of j and $j(\alpha) = \alpha$. Then $j``\alpha$ is not ω-closed.*

Proof. If $j``\alpha$ is ω-closed, for ordinals below α forcing with $\mathbb{P}$ preserves the properties of having cofinality ω and of having cofinality bigger than ω.[106] Let $\langle S_\gamma : \gamma < \alpha\rangle \in V$ be a partition of $\alpha \cap \mathrm{Cof}(\omega)$ into stationary sets. Each set on the sequence $j(\langle S_\gamma : \gamma < \alpha\rangle)$ is a stationary subset of α in $V[G]$ since $M^{<\delta} \subseteq M$. Let $f : \alpha \cap \mathrm{Cof}(\omega) \to \alpha$ be given by $f(\eta) = \gamma$ iff $\eta \in S_\gamma$. Consider $j``\alpha$. Then $j``\alpha$ is closed under $j(f)$. Since $j``\alpha$ is $\omega$-closed, $j``\alpha$ intersects each set on the list $j(\langle S_\gamma : \gamma < \alpha\rangle)$. Since $j``\alpha$ is closed under $f$, we must have $j``\alpha = \alpha$. But this contradicts $\alpha > \mathrm{crit}(j)$. ⊣

9.38 Corollary. *Suppose that $\mathcal{T}$ is a presaturated tower of height δ such that if $G \subseteq \mathcal{P}_{\mathcal{T}}$ is generic then $\delta \cap \mathrm{Cof}(\omega)^V = \delta \cap \mathrm{Cof}(\omega)^{V[G]}$. Let $j : V \to M$ be the generic embedding from $\mathcal{T}$. Then j has no fixed points whose V-cofinality is between* $\mathrm{crit}(j)$ *and δ.*

The next couple of examples illustrate that if one is willing to change some cardinals to cofinality ω one can have fixed points within the strength of a generic elementary embedding. Note that this is not possible with conventional large cardinal embeddings.[107]

9.39 Example. Let δ be a Woodin cardinal, $G \subseteq \mathbb{P}_{<\delta}$ be generic and $j : V \to M$ be the generic elementary embedding. Then in $V[G]$:

[106] See Proposition 2.32.

[107] The "Kunen Contradiction" implies that if $j : V \to M$ is a large cardinal embedding and λ is a fixed point of j above the critical point of j, then $P(\lambda) \not\subseteq M$.

1. $j(\delta) = \delta$,

2. δ is a regular cardinal and there are unboundedly many measurable $\mu < \delta$ with $j(\mu) = \mu$, and

3. for unboundedly many measurable $\mu \in \delta$ there is a $\gamma < \mu$ and $x \subseteq \gamma$ such that $V_\mu \subseteq L[x]$.

Proof. We first show that the collection of measurable fixed points of j is unbounded in δ. Let $\beta < \mu < \delta$ where μ is measurable and suppose that $S \subseteq P(H(\beta))$ is a stationary set. Standard indiscernibility arguments show that there is a stationary set $T \subseteq P(H(\mu^+))$ such that $T \leq_{\mathbb{P}_{<\delta}} S$, and for all $N \in T$ there is an expansion $\mathfrak{A}$ of $H(\mu^+)$ in a countable language and an unbounded set of indiscernibles $I \subseteq \mu$ such that $N = \mathrm{Sk}^{\mathfrak{A}}((N \cap H(\beta)) \cup I)\}$.

In analogy to Remark 2.27, the function $f : P(H(\mu^+)) \to \mathrm{On}$ given by $f(z) = \mathrm{ot}(z \cap \mu)$ represents μ in every generic ultrapower produced by the tower $\mathcal{T}$. Since for all $N \in T$, $f(N) = \mu$ we see that $T \Vdash \mu = j(\mu)$. Hence the condition T forces in $\mathbb{P}_{<\delta}$ that μ is a measurable fixed point of the generic embedding j.

We have shown that for every $S \in \mathbb{P}_{<\delta}$ and $\beta < \delta$ there is a $T \leq_{\mathbb{P}_{<\delta}} S$ and a $\mu > \beta$ such that T forces that μ is a fixed point of j. Hence, the set of fixed points of j is cofinal in δ.

To see that δ is a fixed point, suppose that $[f]^M$ is an ordinal less than $j(\delta)$ in M. Then f is a function from some $P(H(\alpha))$ into δ, where $\alpha < \delta$. Since $|P(H(\alpha))| < \delta$ and δ is inaccessible, f is bounded in δ by some fixed point of j, call it μ. Thus $[f]^M < j(\mu) = \mu < \delta$, and we see that δ is a fixed point of j.

Since $j(\delta) = \delta$, δ is regular in M. Since $\mathbb{P}_{<\delta}$ is presaturated, M is closed under $<\delta$-sequences, so δ is regular in $V[G]$.

We sketch the last claim. The condition T forces that there is an expansion $\mathfrak{B}$ of $j(H(\mu^+))$ and an unbounded set $J \subseteq \mu$ such that:

1. $j``H(\mu^+) = \mathrm{Sk}^{\mathfrak{B}}(j``H(\beta) \cup J)$, and

2. J is a set of $\mathfrak{B}$ indiscernibles over $j``H(\beta)$.

Let $\gamma = |H(\beta)|$ and $x \subseteq \gamma$ code the structure $\mathrm{Sk}^{\mathfrak{B}}(j``H(\beta))$ together with the "blueprint" of the indiscernibles J. Then $L[x]$ can reconstruct a structure isomorphic to $H(\mu^+)^V$, and hence V_μ. ⊣

9.40 Remark. The only role of γ is clause 3 of Example 9.39 is to work below the condition S. If we start below the trivial condition then we can take $x \subseteq \omega$. Thus we can force to keep any given $\mu < \delta$ measurable and code V_μ by a real.

Stationary tower forcings typically add sequences $\langle \gamma_i : i \in \omega \rangle$ that are V-generic for Prikry forcing through a measurable cardinal $\mu < \delta$. To see this we use an indiscernible argument similar to the one in the previous example.

Let $S \subseteq P(H(\beta))$ be stationary for some $\beta < \delta$. Let $\mu > \beta$ be measurable. We show that the collection of $M^* \prec H((2^{2^\mu})^+)$ such that $M \cap H(\beta) \in S$ and there is a sequence $\langle \gamma_i : i \in \omega\rangle$ in V that is Prikry generic over M^* is stationary.

Let $\mathfrak{A}$ be any structure expanding $\langle H((2^\delta)^+), \in, \Delta, F, \{\beta\}\rangle$, where F is a normal ultrafilter on some $\mu > \beta$. It suffices to find an $M \prec \mathfrak{A}$ such that $M \cap H(\beta) \in S$ and a Prikry sequence through F over M. Let $N \prec \mathfrak{A}$ with $N \cap H(\beta) \in S$. Choose $\gamma_1 \in \bigcap(F \cap N)$. If $N_1 = \mathrm{Sk}^{\mathfrak{A}}(N \cup \{\gamma_1\})$ then $N_1 \cap H(\mu)$ end extends $N \cap H(\mu)$. This process can be repeated to build sequences $\langle N_i : i \in \omega\rangle$ and $\langle \gamma_i : i \in \omega\rangle$ such that $N_i \prec \mathfrak{A}$ and $N_{i+1} \cap H$ end extends $N_i \cap H(\mu)$. If we let $M = \bigcup N_i$ then $M \prec \mathfrak{A}$ and $\langle \gamma_i : i \in \omega\rangle$ is a Prikry sequence through F over M. Moreover $M \cap H(\beta) = N \cap H(\beta) \in S$.

We have sketched:

9.41 Example. Suppose that $\mathcal{P}_\mathcal{T}$ is a stationary tower given by $\langle Z_\alpha : \alpha < \delta\rangle$ where δ is Woodin and either each $Z_\alpha = P(H(\alpha))$ or there is an $\eta \geq \omega_1$ so that each Z_α is of the form $[H(\alpha)]^{<\eta}$. Then for each $S \in \mathcal{P}_\mathcal{T}$, we can find arbitrarily large measurable $\mu < \delta$ for which there are $T \leq_{\mathcal{P}_\mathcal{T}} S$ such that $T \subseteq \{N \prec H((2^{2^\mu})^+) :$ there is a sequence $\langle \gamma_i : i \in \omega\rangle$ Prikry generic over N for a ultrafilter F on $\mu\}$. This T forces that $\mathcal{P}_\mathcal{T}$ adds a Prikry generic sequence over V through a ultrafilter on μ. Thus $\mathcal{P}_\mathcal{T}$ adds Prikry generic sequences to a cofinal set of cardinals below δ.

We remark that this same technique can add "longer" Prikry sequences to μ.

Woodin proved the following theorem about "Σ_2 resurrection":

9.42 Theorem (Woodin; see [80]). *Suppose that there is a proper class of Woodin cardinals. Then for all Σ_2 formulas $\phi(\vec{x})$, all $\vec{a} \in V$, and all partial orderings $\mathbb{P}$ there is a partial ordering $\mathbb{Q} \in V^\mathbb{P}$ such that:*

$$\phi^V(\vec{a}) \quad \textit{implies} \quad \phi^{\mathbb{P}*\mathbb{Q}}(\vec{a}).$$

We illustrate this theorem by giving a special case that shows that if there is a huge cardinal and a proper class of Woodin cardinals then after any forcing $\mathbb{P}$ one can do a further forcing $\mathbb{Q}$ to restore a huge cardinal to the universe.

9.43 Example. Suppose that $j : V \to M$ is a huge embedding with critical point κ and $j(\kappa) = \lambda$. Suppose that $\mathbb{P}$ is a partial ordering and there is a Woodin cardinal bigger than both λ and $|\mathbb{P}|$. Then in $V^\mathbb{P}$ there is a partial ordering $\mathbb{Q}$ such that

$$V^{\mathbb{P}*\mathbb{Q}} \models \text{ there is a huge cardinal.}$$

Proof. Fix a Woodin cardinal $\delta > \max(|\mathbb{P}|, \lambda)$. We first argue that if $H \subseteq \mathbb{P}$ is generic then we can find a partial ordering $\mathbb{Q} \in V[H]$ such that for all $V[H]$-generic $H' \subseteq \mathbb{Q}$, there is a V-generic $G \subseteq \mathbb{P}_{<\delta}$ such that $V[H * H'] = V[G]$.

Let $\mu = |\mathbb{P}|$. We force with the Woodin tower $\mathbb{P}_{<\delta}$ below the condition $S = \{z : |z \cap H(2^{2^\mu})| = \aleph_0\}$ to get a generic object G. This collapses $|2^{2^\mu}|$ to be countable. Hence for each condition $p \in \mathbb{P}$ there is a V-generic object $H \subseteq \mathbb{P}$ with $p \in H$ that belongs to $V[G]$. Applying standard forcing arguments we see that there is a regular embedding

$$e : \mathbb{P} \to \mathcal{B}(\mathbb{P}_{<\delta}/S).$$

The forcing fact highlighted in the introduction shows that for each generic $H \subseteq \mathbb{P}$ there is a partial ordering $\mathbb{Q} \in V[H]$ such that

$$\mathbb{P}_{<\delta}/S \sim \mathbb{P} * \mathbb{Q}.$$

To see that this $\mathbb{Q}$ works, it suffices to show that for all generic $G \subseteq \mathbb{P}_{<\delta}$, there is a huge cardinal in $V[G]$. We showed in Example 9.39 that in $V[G]$ there is a generic elementary embedding $k : V \to M$ where $M^{<\delta} \subseteq M$ and $k(\delta) = \delta$. In particular, M is a model of "there is a huge embedding j' with critical point κ' and $j'(\kappa') < \delta$". Since this huge embedding comes from a normal, fine, ultrafilter U on $[\lambda']^{\kappa'}$, and $P([\lambda']^{\kappa'})^M = P([\lambda']^{\kappa'})^{V[G]}$, we see that κ' is huge in $V[G]$. ⊣

The next result is a very small improvement of a result of Burke's in his [12]:

9.44 Proposition. *Suppose that I is a countably complete ideal on a set Z and suppose that $\delta > |Z|$ is a Woodin cardinal. Then I is pre-precipitous.*[108]

Proof. We need to see that there is a partial ordering $\mathbb{P}$ and a $\mathbb{P}$-term i and an elementary embedding $j : V \to M$ definable in $V^{\mathbb{P}}$ such that $A \in I$ iff $\|i \in j(A)\| = 0$.

We first show Burke's result that using $\mathbb{P}_{<\delta}$ we can force the existence of an ultrafilter $U \supseteq \breve{I}$ such that V^Z/U is well-founded. For some $\alpha > 2^{2^{|Z|}}$ let $S = \{z \subset H(\alpha) : |z \cap 2^{2^{|Z|}}|$ is countable$\}$. Let $G \subseteq \mathbb{P}_{<\delta}$ be generic with $S \in G$. If $j : V \to M \subseteq V[G]$ is the generic ultrapower, then

$$\textstyle\bigcap_{A \in \breve{I}} j(A) \in j(\breve{I}).$$

In particular, $\bigcap_{A \in \breve{I}} j(A) \neq \emptyset$. If we choose $i \in \bigcap_{A \in \breve{I}} j(A)$ then $U(j, i) \supseteq \breve{I}$ and $V^Z/U(j, i)$ is well-founded.

To see the full result, we apply Burke's technique to the ideals $I \restriction B$ for each $B \in I^+$. Choose a maximal antichain $\mathcal{A} \subset \mathbb{P}_{<\delta}$ of size at least $2^{|Z|}$. Partition $\mathcal{A}$ into non-empty sets $\langle \mathcal{A}_B : B \in I^+ \rangle$. Burke's technique gives us a collection of terms i_T for $T \in \mathcal{A}$ such that if $T \in \mathcal{A}_B$, then

$$T \Vdash i_T \in \textstyle\bigcup_{A \in \breve{I}} j(A) \cap j(B).$$

Let i be a term such that for all $T \in \mathcal{A}$, $T \Vdash i = i_T$. Then $B \in I$ iff $\|i \in j(B)\| = 0$. ⊣

[108] The definition of pre-precipitous is given in Definition 3.39.

9.6. A Tower that is not Precipitous

In this subsection we present an example due to Burke [11], of a tower that has height a supercompact cardinal and is not precipitous. This stands in contrast to Theorems 9.25 and 9.28.

The example in Burke's paper uses a hypothesis that is a consequence of a supercompact cardinal and Schimmerling showed holds in some inner models of the form $L[E]$:

9.45 Definition. Let $*(\kappa, \delta)$ be the statement that κ is a regular cardinal, $\delta > \kappa$ is an inaccessible cardinal and there is an increasing sequence of transitive models of ZFC, $\langle N_\xi : \xi < \lambda\rangle$ that belongs to V_δ such that for all $x \in V_\delta$ there is an $a \prec V_\delta$ with $x \in a$, $|a| < \kappa$, and $a \cap \kappa \in \kappa$ and for some $\xi < \lambda$ the transitive collapse of a is a rank initial segment of N_ξ.

9.46 Theorem (Burke [11]). *Suppose that $*(\kappa, \delta)$. Then there is a tower of height δ with critical point κ that is not precipitous.*

Burke noted that if κ is supercompact and $\delta > \kappa$ is inaccessible then the sequence of length one $\langle V_\kappa\rangle$ is a witness to $*(\kappa, \delta)$. Hence he deduces:

9.47 Corollary. *Suppose that κ is supercompact and $\delta > \kappa$ is inaccessible. Then there is a tower of height δ and critical point κ that is not precipitous.*

We start with two preliminary results.

9.48 Proposition. *Suppose that $\mathcal{T}$ is a precipitous tower of height δ where δ is an inaccessible cardinal, $G \subseteq \mathcal{P}_{\mathcal{T}}$ is generic and $j : V \to M$ is the associated elementary embedding. If $j(\delta) > \delta$, then $\mathcal{P}_{\mathcal{T}}$ is not in M.*

Proof. If the proposition fails then we may assume that for all $G \subseteq \mathcal{P}_{\mathcal{T}}$, $j(\delta) > \delta$ and $\mathcal{P}_{\mathcal{T}} \in M$. Let $N = L(V_\delta, \mathcal{T})$. Then N may not be a model of AC, but $\mathcal{P}_{\mathcal{T}}$ belongs to N as do all functions into V_δ with support below δ.

Fix $G \subseteq \mathcal{P}_{\mathcal{T}}$ generic over V, and let $j : V \to M$ be the elementary embedding coming from the ultrapower map. Let $\delta^* = j(\delta)$. Then G is also generic over N. If j_G is the generic ultrapower constructed over N using G, we have $j_G(\delta) \geq \delta^*$. As this can be expressed in the forcing language, there is a $p \in \mathcal{P}_{\mathcal{T}}$ such that $N \models p \Vdash_{\mathcal{P}_{\mathcal{T}}} j_G(\delta) \geq \check{\delta}^*$.

Let $[f]^M = \delta$ and $[g]^M = \mathcal{T}$. Without loss of generality we can assume that:

1. $\mathrm{supp}(f) = \mathrm{supp}(g) = \alpha$ for some $\alpha < \delta$, and
2. for all $a \in H(\alpha)$, $f(a) < \delta$ and $g(a)$ is a tower of height $f(a)$ that belongs to $L(V_{f(a)}, g(a))$.

Since δ is inaccessible, $|{}^{H(f(\alpha))}f(a)| < \delta$ for all $a \in H(\alpha)$. If $G_a \subseteq \mathcal{P}_{g(a)}$ is generic over $L(V_{f(a)}, g(a))$, then $|j_{G_a}(f(a))| \leq |{}^{H(f(\alpha))}f(a)|$. Hence $L(V_{f(a)}, g(a)) \models$"for all generic $G_a \subseteq \mathcal{P}_{g(a)}$, $j_{G_a}(f(a)) < \check{\delta}$".

Passing to M, and using the fact that $N = L(V_{[f]^M}, [g]^M)$ we see that $N \models$"for all generic $G, j_G(\delta) < \check{\delta^*}$". This contradicts the existence of the condition p. ⊣

9.49 Lemma. *Suppose that δ is an inaccessible cardinal, U is an unbounded subset of δ and $\langle Z_\alpha \subseteq P(H(\alpha)) : \alpha \in U\rangle$ is a sequence of stationary sets such that for $\alpha < \beta \in U$, $\mathrm{NS}\restriction Z_\beta$ projects to a superset of $\mathrm{NS}\restriction Z_\alpha$. Then there is a tower $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ such that for all $\alpha \in U$, $Z_\alpha \in \breve{I}_\alpha$.*

Proof. Let $I_{\alpha,\beta}$ be the projection of $\mathrm{NS}\restriction Z_\beta$ to $P(H(\alpha))$. Then the $I_{\alpha,\beta}$ increase with β and hence stabilize in a proper ideal at some $\beta(\alpha) < \delta$. Then $I_\alpha = I_{\alpha,\beta(\alpha)}$ is the desired sequence. ⊣

We now prove Theorem 9.46.

Proof. Assume $*(\kappa, \delta)$ and let $\langle N_\xi : \xi < \lambda\rangle$ be a witness. To follow Burke's proof closely, it is easier to work with V_α's instead of $H(\alpha)$'s. Since δ is inaccessible, there is a closed unbounded set of $\alpha \in \delta$ where $V_\alpha = H(\alpha)$, so this is primarily a notational distinction.

For α a limit ordinal between κ and δ, let $Z_\alpha = \{a \subseteq V_\alpha : |a| < \kappa,\ a \cap \kappa \in \kappa$ and there is a $\xi < \lambda$ such that the transitive collapse of a is a rank initial segment of $N_\xi\}$.

We claim that:

1. each Z_α is stationary, and

2. for $\alpha < \beta$ the projection of $\mathrm{NS}\restriction Z_\beta$ to α is a superset of $\mathrm{NS}\restriction Z_\alpha$.

To see that Z_α is stationary, fix an algebra $\mathfrak{A}$ with domain V_α. Apply $*(\kappa, \delta)$ with $x = \mathfrak{A}$, to find an $a \prec V_\delta$ with $x \in a$ and a ξ such that the transitive collapse of a is a rank initial segment of N_ξ. Then $a \cap V_\alpha \prec \mathfrak{A}$. Further, the transitive collapse of $a \cap V_\alpha$ is a rank initial segment of the transitive collapse of a and hence of N_ξ.

To see the second claim, it suffices to show that the projection of the closed unbounded filter on V_β restricted to Z_β is a superset of the closed unbounded filter on V_α restricted to Z_α. Fix an algebra $\mathfrak{A}$ on V_α. Let $\mathfrak{B}$ be any fully Skolemized structure expanding $(V_\beta, \in, \delta)$ containing a constant symbol whose interpretation is $\mathfrak{A}$. Then any elementary substructure $b \prec \mathfrak{B}$ has $b \cap V_\alpha \prec \mathfrak{A}$. For $b \prec \mathfrak{B}$ with transitive collapse a rank initial segment of N_ξ, we know that the transitive collapse of $b \cap V_\alpha$ is also a rank initial segment of N_ξ. Thus the projection to V_α of those $b \in Z_\beta$ that are elementary substructures of $\mathfrak{B}$ is contained in the collection of elementary substructures of $\mathfrak{A}$ that belong to Z_α.

By Lemma 9.49 we can find a tower $\mathcal{T} = \langle I_\alpha : \alpha \in U\rangle$ such that for all $\alpha \in U$, $Z_\alpha \in \breve{I}_\alpha$.

Indirectly assume that $\mathcal{P}_\mathcal{T}$ is precipitous. Let $G \subseteq \mathcal{P}_\mathcal{T}$ be generic, and let $j : V \to M \subseteq V[G]$ be the generic ultrapower. Note that $j(\kappa) \geq \delta$, so in particular, $j(\delta) > \delta$. We establish our contradiction by showing that $\mathcal{T} \in M$ and hence $\mathcal{P}_{\mathcal{T}} \in M$. Since $\mathcal{T}$ is definable in V_δ from the sequence $\langle N_\xi : \xi < \lambda \rangle$, this is equivalent to showing that $V_\delta \in M$.

Let $\alpha \in U$. Since $Z_\alpha \in \check{I}_\alpha$, we know that the transitive collapse of $j``V_\alpha$ is a rank initial segment of some element of $j(\langle N_\xi : \xi < \lambda \rangle)$. Denote $j(\langle N_\xi : \xi < \lambda \rangle)$ by $\langle N^j_\xi : \xi < j(\lambda) \rangle$. Let $\xi(\alpha)$ be the least ordinal such that V_α is a rank initial segment of $N^j_{\xi(\alpha)}$ and $\gamma = \sup\{\xi(\alpha) + 1 : \alpha < \delta\}$. Note that the map $\alpha \mapsto \xi(\alpha)$ is monotone. In M, let $R = \{x : \mathrm{rank}(x) < \delta$ and $x \in N^j_\xi$ for some $\xi < \gamma\}$.

We claim that $R = V_\delta$. Clearly $V_\delta \subseteq R$. Suppose that $x \in R$. Let $\beta < \delta$ be a limit ordinal such that x has rank less than β and $\xi < \gamma$ be such that $x \in N^j_\xi$. Since the sequence $\langle N^j_\xi : \xi < j(\lambda) \rangle$ is increasing and $\xi < \gamma$, we can assume that $\xi = \xi(\beta')$ for some $\beta' \geq \beta$. Since $V_{\beta'}$ is a rank initial segment of N^j_ξ, we see that $x \in V_\beta$. $\dashv$

10. Consistency Strength of Ideal Assumptions

In this section we describe some progress towards clarifying the relationships between generic ultrapowers and conventional large cardinals. We begin with a brief survey of the results showing that generic embeddings yield fine structural inner models. Relevant information on these results can be found in the chapters by Mitchell, by Schindler and Zeman, and by Steel in this Handbook and in the papers referenced in the text. We shall confine ourselves to telegraphic remarks here. It is important to note as well that there are direct equiconsistency results between ideal assumptions and determinacy hypotheses. For example, Woodin showed that $2^{\aleph_0} = \omega_2$ + "NS_{ω_1} is $\aleph_1$-dense" is equiconsistent with $\mathrm{AD} + V = L(\mathbb{R})$. This type of result is beyond the scope of this chapter.

Some new results that show the existence of very large cardinals are presented. The underlying point of the new results is that knowing the image of just a few sets by the generic ultrapower embedding is sufficient to show that there is a conventional large cardinal in an inner model whose embedding agrees with the generic ultrapower embedding. The proofs of the new results in this section can be found in the forthcoming paper [32].

Throughout this section we will use the notation π_N to be the unique isomorphism between an extensional set N and its transitive collapse.

10.1. Fine Structural Inner Models

We begin by describing conventional theorems using core model theory to build inner models with large cardinals from generic embedding assumptions. As we will see, the gap between the lower bounds on consistency strength provided by core model theory and the upper bounds given by forcing are, in many cases, quite large.

Early work of Kunen was used by Jech and Prikry (see [65]) to show that if U is the dual of a normal, precipitous ideal on some cardinal κ then $L[U] \models U$ is a normal ultrafilter on κ. In particular, this gave one direction of the equiconsistency between real-valued measurable cardinals and (two-valued) measurable cardinals.[109]

As pointed out earlier, in Theorems 8.6, 8.7 and 8.8, work of Gitik [53, 54] showed that having NS_{ω_2} be precipitous is equiconsistent with the existence of a measurable cardinal of Mitchell order 2. More generally, if $\mu > \omega_1$ then CH + "NS_{μ^+} is precipitous" is equiconsistent with μ^+ being an $(\omega, \mu+1)$ repeat point for the normal ultrafilters on μ^+ in the core model K. The property "κ is an inaccessible cardinal and NS_κ is precipitous" is equiconsistent with κ having an $(\omega, \kappa+1)$-repeat point for the normal ultrafilters on κ in K.

For saturated ideals, the situation is less clear. Steel [114] showed that if there is a presaturated ideal on a cardinal κ and a measurable cardinal $\mu > \kappa$, then there is an inner model with a Woodin cardinal.

Steel also showed that if there is a homogeneous presaturated ideal on ω_1 and CH holds, then Projective Determinacy holds. Hence, for all $n \in \omega$ there is an inner model with n Woodin cardinals. This result was superseded by Woodin who showed that CH was not necessary and was able to get inner models with ω-many Woodin cardinals. This is an exact equiconsistency result, as the Steel-Van Wesep ideal (Theorem 8.1) is homogeneous.

Steel and Zoble showed that the assumption "NS_{ω_1} is $\aleph_2$-saturated and every pair of stationary subsets of $\omega_2 \cap \mathrm{Cof}(\omega)$ simultaneously reflect" implies $\mathrm{AD}^{L(\mathbb{R})}$, with the consequent inner model implications. We note that the Steel-Zoble hypotheses follows from many standard propositions such as Martin's Maximum. Somewhat weaker versions of these results appear in Zoble's thesis [131].

We note that for saturated or cardinal preserving ideals above ω_1, there is an enormous gap between the known lower bounds (some number of Woodin cardinals) and the known upper bounds (almost huge cardinals).

There is a similar situation for Chang's Conjecture type properties. The classical Chang's Conjecture $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ is equiconsistent with the existence of an ω_1-Erdős cardinal, as shown by results of Silver, Baumgartner, Donder and Levinski, among others [70, 24]. For the property $(\aleph_{n+2}, \aleph_{n+1}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ with $n > 0$, the known upper bound on consistency strength is a huge cardinal. A lower bound with $n = 1$ and CH was given by Schindler [99, 100] as a κ with Mitchell order $\kappa^{+\omega}$. For $n \in [2, \omega)$ with $2^{\aleph_{n-1}} = \aleph_n$, Schindler showed that there is an inner model with a strong cardinal. Without the assumptions on cardinal arithmetic, Jensen showed 0-sword exists.

For "gap two" Chang Conjectures the distance between the known lower and upper bounds is even greater, as the best known upper bound is a 2-huge cardinal [33].

[109] The consistency result was proved previously by Solovay [111].

10.2. Getting Very Large Cardinals from Ideal Hypotheses

In this section we discuss various non-standard ways of getting inner models with very large cardinals.

Constructing from Stationary Sets and the Nonstationary Ideal

We begin by illustrating how to construct models with arbitrarily large cardinals by constructing relative to the nonstationary ideal.

Recall the following theorem of Burke (Corollary 4.21):

Theorem. *Suppose that I is a normal, fine, countably complete ideal on a stationary set $Z \subseteq P(X)$. Then for all Y with $|Y| \geq 2^X$, there is a stationary set $A \subseteq P(Y)$ such that I is the projection of the nonstationary ideal on Y restricted to A.*

Suppose that $X = \lambda$ and $Y = \lambda'$ for cardinals λ, λ'. An examination of the argument for Burke's theorem yields the following observation: If $\breve{I}$ is an ultrafilter and U is a normal, fine ultrafilter on $P(\lambda')$ projecting to $\breve{I}$, then $A \in U$. Hence, if U is (say) a supercompact ultrafilter, we can take A to be canonically well-ordered.

The A produced in Burke's theorem can be taken to have many of the properties typical to sets in $\breve{I}$. Arguments from [42] show that if there is a set of measure one C for I such that every element $z \in C$ is ω-closed as a set of ordinals, then we can find an A such that every $N \in A$ is ω-closed as a set of ordinals. Moreover, if I concentrates on sets of size less than κ (resp. equal to κ or greater than κ etc.), then Z can be taken to concentrate on sets of size less than κ (resp. equal to κ or greater than κ).

Many large cardinal properties of a cardinal κ can be defined in terms of the existence of certain kinds of ultrafilters U on a subset of $P(\lambda)$ for $\lambda \geq \kappa$. In attempting to construct a canonical inner model theory for such cardinals, a major obstacle is finding a suitable set of measure one to build into the model. Typically, doing simple *relative constructibility* one finds that $L[U]$ is a very small model and the method fails. Alternatively, one can simply "throw in" a set of measure one $A \in U$, making sure that A is canonically well-ordered. A theorem of Solovay [112] says that if U is a supercompact or strongly compact ultrafilter then A can be taken so that distinct elements of A have different suprema. The cost of the second approach is that the model seems in no way canonical.

Let $A \subseteq [\lambda]^{<\kappa}$ be a set such that the function $\sup : A \to \lambda$ is one-to-one. For β in the range of the sup function, let $a_\beta \in A$ be such that $\sup(a_\beta) = \beta$. Define $A^* \subseteq \lambda \times \lambda$ by setting $(\alpha, \beta) \in A^*$ iff $\alpha \in a_\beta$. Then $A \in L[A^*]$.

From Burke's theorem we get:

10.1 Corollary. *Suppose that κ is $[2^\lambda]^{<\kappa}$-supercompact. Then there is a stationary set $A \subseteq [2^\lambda]^{<\kappa}$ such that $L[\mathrm{NS}{\restriction}[2^\lambda]^{<\kappa}, A^*]$ is a model of* ZFC + *"κ is λ-supercompact".*

We note that there are many stationary sets A that work for Corollary 10.1. One possible definition of such an A is:

$$\{x \in [H((2^\lambda)^+)]^{<\kappa} : x \cap \kappa \in \kappa \text{ and } \mathrm{tc}(x) \subseteq H(\lambda_x)\},$$

where λ_x is the order type of $\lambda \cap x$ and $\mathrm{tc}(x)$ is the transitive closure of x.

Moreover, using arguments similar to Theorem 1.3 of [42], one sees that if U is a supercompact ultrafilter on $[\lambda]^{<\kappa}$ then there is always a stationary set A of good structures in $H((2^\lambda)^+)$ such that the sup function is one-to-one on A. Hence, the assumption of Corollary 10.1 can be reduced to λ-supercompactness. Indeed one sees:

10.2 Corollary. *Let $\kappa < \lambda$ be regular cardinals. Then κ is λ-supercompact iff there is a stationary set $A \subseteq [2^\lambda]^{<\kappa}$ such that the supremum function is one-to-one on A and $L[\mathrm{NS}{\restriction}[2^\lambda]^{<\kappa}, A^*]$ is a model of* ZFC + *"κ is λ-supercompact".*

There is nothing particularly remarkable about supercompact cardinals in the previous arguments. It generalizes easily to essentially any type of very large cardinal such as huge cardinals, n-huge cardinals or towers of n-huge cardinals. In the case of the huge cardinal, to see that the Axiom of Choice holds in the inner model, we note that if U is a huge ultrafilter on $[\lambda]^\kappa$, then there is a set $A \in U$ such that the map $z \mapsto (z \cap \kappa, \sup(z \cap \lambda))$ is a one-to-one map into $\kappa \times \lambda$.[110] Similar tricks for generalizations to larger cardinals such as n-huge cardinals.

Decisive Ideals

Next we give a definition that seems to adequately distinguish between ideals that arise as induced ideals in models built after collapsing large cardinals by forcing and the natural ideals whose generic embeddings are not the traces of large cardinal embeddings in inner models.

10.3 Definition. Let $Z \subseteq P(X)$ and J be an ideal on Z. Let $X' \subseteq X$ and I be the projection of J to an ideal on $P(X')$ via the map $\pi(z) = z \cap X'$. Then J *decides* I iff there is a set $A \in \breve{I}$ and a well-ordering W of A and sets A', W', O' and I' such that for all generic $G \subseteq P(Z)/J$,

1. an initial segment of the ordinals of V^Z/G is well-founded and isomorphic to $(|A'|^+)^V$, and
2. if $j : V \to M$ is the canonical elementary embedding determined by replacing the ultrapower V^Z/G by an isomorphic model M transitive up to $|A'|^+$, then

$$j(A) = A', \qquad j(W) = W', \qquad j``|A| = O', \qquad I' = j(I) \cap P(A')^V.$$

[110] There is even a large set such that the supremum function is one-to-one. This is not necessarily true in the examples we consider later in this section, so we mention the weaker property now.

We will say that J is *decisive* if J decides itself.[111]

Some of the hypotheses in Definition 10.3 are easily satisfied. For example, if J is normal and fine, and $|X| \geq |A'|^+$, then the first clause is automatically satisfied. Moreover, if A has a $\Delta_1^{\mathrm{ZF}^-}$ well-ordering in $H(|A|^+)$ then W and W' automatically exist. Often A has a simple well-ordering given by the properties of the characteristic function of its members.

10.4 Remark. For most induced ideals J produced by collapsing a large cardinal and extending the large cardinal embedding, it is routine to check decisiveness.

The definition can easily be extended to arbitrary generic embeddings, rather than just generic ultrapowers V^Z/G: If $j : V \to M$ is a generic elementary embedding defined in a forcing extension $V^{\mathbb{P}}$ we can ask that M be well-founded up to $|A'|^+$ and that the sets A, A', W', O', I' exist in the inner model M. In this case we say that $\mathbb{P}$ *decides* I.

10.5 Theorem. *Let $\mu \leq \lambda$ be cardinals. Let $\pi : P(\lambda) \to P(\mu)$ be defined by $\pi(z) = z \cap \mu$. Suppose that J is a normal, fine ideal on a set $Z \subseteq P(\lambda)$ that decides a countably complete ideal $I \subseteq P(Z')$ for some $Z' \subseteq P(\mu)$. Suppose that A, W witness the fact that J decides I and W well-orders A as $\langle a_\beta : \beta < \gamma < |A|^+ \rangle$. Let $A^* = \{(\alpha, \beta) : \alpha \in a_\beta\} \subseteq \mu \times \mu$. Then either:*

$$L[A^*, I] \models \check{I} \text{ is an ultrafilter on } A$$

or for some generic $G \subseteq P(Z)/J$

$$L[j(A^*), j(I)] \models j(\check{I}) \text{ is an ultrafilter on } j(A).$$[112]

We can replace clause 1 in the definition of "decisive" by the demand that "if $a = |j(A)|^M$ and $b = (a^+)^M$, then M is well-founded up to b". This change in the hypothesis yields the stronger conclusion that $L[A^*, I] \models \check{I}$ is an ultrafilter.

10.6 Corollary. *Suppose that I is a normal, fine, κ-complete decisive ideal on λ with witnesses A, W. Then:*

1. *If $\alpha < \kappa$ and $A \subseteq [\kappa^{+\alpha}]^{<\kappa}$, then there is an inner model of V with a cardinal μ that is $\mu^{+\alpha}$-supercompact.*

2. *If $A \subseteq [\lambda]^\kappa$, then there is an inner model of V with a huge cardinal.*

3. *If $A \subseteq \{z : \mathrm{ot}(z) = \lambda_1 \text{ and } \mathrm{ot}(z \cap \lambda_1) = \kappa\}$, then there is an inner model of V with a 2-huge cardinal.*

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111 I.e. if we take π to be the identity map and $I = J$.

112 The two alternatives are not equivalent because M is not necessarily well-founded.

Moreover if I is precipitous then:

4. *If $A \subseteq [\lambda]^{<\kappa}$, then κ is λ-supercompact in an inner model of V.*
5. *If $A \subseteq [\lambda]^{\kappa}$, then κ is huge in an inner model.*
6. *If $A \subseteq \{z : \mathrm{ot}(z) = \lambda_1$ and $\mathrm{ot}(z \cap \lambda_1) = \kappa\}$, then κ is 2-huge in an inner model.*

In particular, the following are equiconsistent:

a. *for $n < m \in \omega$ there is normal, fine, decisive ideal on $[\omega_m]^{\omega_n}$,*

b. *there is a huge cardinal,*

as are:

a. *for $n < m \in \omega$ there is normal, fine, decisive ideal on $[\omega_m]^{<\omega_n}$,*

b. *there is a $\kappa^{+(m-n)}$-supercompact cardinal κ.*

Note that there are obvious analogous corollaries for n-huge cardinals.

In unpublished work, starting from a measurable cardinal, Gitik has given an example of an indecisive κ-complete precipitous ideal on a cardinal κ. Another example of a nowhere decisive ideal is the following.

Start in a model with a Woodin cardinal. Collapse the first Ramsey cardinal to be ω_2 so that $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$. Now collapse a Woodin cardinal to be ω_3. Then Theorem 8.37 shows that the Chang ideal on $[\omega_2]^{\omega_1}$ is precipitous. Were it decisive it would give an inner model with a huge cardinal. However, if the model we start in is the minimal inner model with one Woodin cardinal this is impossible.

Chang's Conjecture and Huge Cardinals

In this section we will use Proposition 3.9 and the definitions of the Chang ideals.

Recall that Lemma 4.34 implies that if $\kappa > \mu$ and there are finitely many cardinals between κ and μ then any ideal I concentrating on $\{N \in P(\kappa) : N \cap \mu \in \mu$ and for all cardinals λ with $\mu \le \lambda \le \kappa$ we have $\mathrm{cf}(N \cap \lambda) > \omega\}$, has a canonically well-ordered set $A \in \breve{I}$ of cardinality κ. Thus any Chang ideal involving cardinals at least as big as ω_1 has a canonically well-ordered set of measure one. Hence clause 2 of the hypotheses of the next theorem is satisfied by most Chang ideals. The clause 1 implies that $|N \cap \kappa_1| = \kappa_0$ and hence clause 1 is a version of the Chang's Conjecture $(\kappa_1, \kappa_0) \twoheadrightarrow (\kappa_0, {<}\kappa_0)$. Clauses 3 and 4 are condensation properties that hold in huge type embeddings.

10.7 Theorem. *Suppose that $\kappa_2 > \kappa_1 > \kappa_0$ are cardinals and there is a regular θ and a stationary set $S \subseteq P(H(\theta))$ and sets $A \subseteq [\kappa_1]^{\kappa_0}$, $A' \subseteq [\kappa_2]^{\kappa_1}$, O', I' such that for all $N \in S$:*

1. *$N \cap \kappa_0 \in \kappa_0$ and $N \cap \kappa_1 \in A$,*

2. *$A, A' \in N$ and $\pi_N(A') = A$, and the map $z \mapsto (z \cap \kappa_0, \sup(z))$ is one-to-one on A,*

3. *$O' \in N$ and $N \cap \kappa_1 = \pi_N(O')$, and*

4. *$I' \in N$ and $\pi_N(I') = (\mathrm{CC}((\kappa_1, \kappa_0), (\kappa_0, {<}\kappa_0))\restriction A) \cap N$.*

Then there is an inner model with a huge cardinal.

Theorem 10.7 is proved using a decisive elementary embedding that has critical point κ_0 and $j(\kappa_0) = \kappa_1$ and $\kappa_2 = j(\kappa_1)$. Since the ideal $I = \mathrm{CC}((\kappa_1, \kappa_0), (\kappa_0, {<}\kappa_0))$ is a definable ideal, we have the fact that $j(I) = \mathrm{CC}((\kappa_2, \kappa_1), (\kappa_1, <\kappa_1))^M$. If we assume the principle $(\kappa_2, \kappa_1, \kappa_0) \twoheadrightarrow (\kappa_1, \kappa_0, {<}\kappa_0)$ then the Chang ideals $\mathrm{CC}((\kappa_2, \kappa_1), (\kappa_1, {<}\kappa_1))$ and $\mathrm{CC}((\kappa_1, \kappa_0), (\kappa_0, {<}\kappa_0))$ are both proper.

With this remark in mind we note that we can replace clause 4 of Theorem 10.7 with the property that all $N \in S$ be *correct*, which we define to be the following condensation property:

10.8 Definition. Let

$$N \prec \langle H(\theta), \in, \{\kappa_2, \kappa_1, \kappa_0\}, \Delta, \{A, A'\}\rangle$$

be a structure such that $\mathrm{ot}(N \cap \kappa_2) = \kappa_1$, $\mathrm{ot}(N \cap \kappa_1) = \kappa_0$, and $|N \cap \kappa_0| < \kappa_0$. We will say that N is *correct* for $\mathrm{CC}((\kappa_1, \kappa_0), (\kappa_0, {<}\kappa_0))\restriction A$ iff whenever $\pi_N : N \to \bar{N}$ is the transitive collapse map, we have

$$\pi_N(\mathrm{CC}((\kappa_2, \kappa_1), (\kappa_1, {<}\kappa_1))\restriction A') = (\mathrm{CC}((\kappa_1, \kappa_0), (\kappa_0, {<}\kappa_0))\restriction A) \cap \bar{N}.$$

To make these results concrete we give a corollary. By choosing $n = 0$, the corollary provides a consistent statement about $H(\omega_4)$ that gives an inner model with a huge cardinal:

10.9 Corollary. *Suppose that $n \in \omega$ and there are $A \subseteq [\omega_{n+2}]^{\omega_{n+1}}$, $A' \subseteq [\omega_{n+3}]^{\omega_{n+2}}$, and $O' \in [\omega_{n+3}]^{\omega_{n+2}}$, such that for all structures $\mathfrak{A}$ expanding the structure $\langle H(\omega_{n+4}), \in, \Delta, \{A, A'\}\rangle$ there is an $N \prec \mathfrak{A}$:*

1. *$|N \cap \omega_{i+1}| = \omega_i$ for $i = n, n+1, n+2$, $\omega_n \subseteq N$, and $N \cap \omega_{n+2} \in A$,*

2. *$\pi_N(A') = A$,*

3. *The map $z \mapsto (z \cap \omega_{n+1}, \sup(z))$ is one-to-one on A,*

4. *$O' \in N$ and $N \cap \omega_{n+2} = \pi_N(O')$, and*

5. *N is correct for $\mathrm{CC}((\omega_{n+2}, \omega_{n+1}), (\omega_{n+1}, {<}\omega_{n+1}))\restriction A$.*

Then there is an inner model with a huge cardinal.

Note that if $n \geq 1$ in the previous two corollaries, then Lemma 4.34 gives a canonically well-ordered set A of measure one for the Chang ideal. This set is the collection of $N \cap \omega_{n+4}$ where $N \prec H(\omega_{n+4})$ is a Chang elementary substructure that has uniform cofinality bigger than ω. The issue in satisfying the hypotheses becomes determining where that set goes.

The following theorem gives a partial converse to Corollary 10.9:

10.10 Theorem. *If there is a 2-huge cardinal, then there is a generic extension satisfying* GCH *and the hypotheses of Corollary* 10.9.

A Martin's Maximum Result

It is a standard fact that if κ is supercompact, $\lambda > \kappa$ is regular, and $\langle S_\alpha : \alpha < \lambda\rangle$ is a partition of $\lambda \cap \mathrm{Cof}(\omega)$ then for all supercompact measures U on $[\lambda]^{<\kappa}$, the collection of $z \in [\lambda]^{<\kappa}$ such that

$$z \cap \kappa \in \kappa \text{ and for all } \alpha \in \lambda,\ \alpha \in z \text{ iff } S_\alpha \cap \sup(z) \text{ is stationary in} \sup(z)$$

belongs to U.

Assuming that $|H(\lambda)| = \lambda$ we can index the partition of $\lambda \cap \mathrm{Cof}(\omega)$ by elements of $H(\lambda)$, $\langle S_x : x \in H(\lambda)\rangle$, and hence we can state that for all supercompact measures U on $[H(\lambda)]^{<\kappa}$, the collection of $z \in [H(\lambda)]^{<\kappa}$ such that:

$$z \cap \kappa \in \kappa \text{ and for all } x \in H(\lambda),\ x \in z \text{ iff } S_x \cap \sup(z) \text{ is stationary in } \sup(z)$$

belongs to U.

By our earlier results, if κ is supercompact we can take a stationary subset of $[2^\lambda]^{<\kappa}$ with this property and construct from it with the nonstationary ideal as a predicate to get a model where κ is λ-supercompact. We note that the next theorem is a variation of Theorem 10 of [47].

10.11 Theorem. *Assume Martin's Maximum. Suppose that $\lambda \geq \omega_2$ and $H \subseteq H(\lambda)$ has cardinality λ and $\lambda \subseteq H$. Let $\langle S_x : x \in H\rangle$ be a partition of $\lambda \cap \mathrm{Cof}(\omega)$. Then there is a stationary subset $A \subseteq [H]^{<\omega_2}$ such that for all $N \in A$:*

$$x \in N \textit{ iff } S_x \textit{ is stationary in } \sup(N \cap \lambda).$$

Let κ be a supercompact cardinal and suppose that $\mathbb{P}$ is some standard iteration for creating a model $V[H]$ that satisfies Martin's Maximum. Using the techniques of Corollary 10.1, we can see that there is a set $A \in U$ where U is a supercompact filter on 2^λ, such that in $V[H]$, $L[\mathrm{NS}{\restriction}[2^\lambda]^{<\omega_2}, A] \models \omega_2^{V[H]}$ is λ-supercompact. This might be construed as evidence that a set A of the form produced in Theorem 10.11 could be used to construct an inner model with a supercompact cardinal. Concretely, assuming Martin's Maximum one can ask whether there is a set $A \subseteq H(\omega_4)$ such that the model $L[A, \mathrm{NS}{\restriction}[\omega_4]^{\omega_2}] \models \mathrm{ZFC} +$ "κ is κ^+-supercompact", where $\kappa = \omega_2^V$?

10.3. Consistency Hierarchies Among Ideals

While it is not known how the consistency strength of saturated n-huge ideals compares to the hierarchies of consistency strength of conventional large cardinals, one can prove that they form a hierarchy of consistency strength among themselves. We give a sample result from [35] (Theorem 15).

10.12 Theorem. *Suppose that κ is a successor of a regular cardinal, $\kappa \geq \omega_2$ and there is a κ-complete, κ^+-saturated (resp. κ-centered, or κ-dense) ideal on $[\kappa^{+n+1}]^{\kappa^{+n}}$ for some $n \in \omega$. There is a transitive set model N of* ZFC *such that $\kappa^{+n} \subseteq N$ and $N \models$ "there is a κ-complete, κ^+-saturated (resp. κ-centered, or κ-dense) ideal on $[\kappa^{+n}]^{\kappa^{+n-1}}$."*

Note that this is a consistency strength hierarchy along one of the three axes determining a generic large cardinal, namely the closure of the generic ultrapower M. It is not known how to find hierarchies that involve different axes. For example the following is a typical open question along these lines:

> Suppose that there is an $\aleph_3$-complete, normal, fine, precipitous ideal on $[\omega_5]^{\omega_4}$ concentrating on the collection of z such that the order type of $z \cap \omega_4$ is ω_3. Is it consistent that there is an $\aleph_4$-saturated ideal on ω_3?

11. Ideals as Axioms

In this section we discuss extending ZFC by asserting the existence of generic elementary embeddings. The attempt is to put the discussion in the context of current ideas about methods for evaluating axiom systems. This necessitates terse and incomplete summaries of these ideas. The author wrote a more complete version of this section in [38], but the mathematical situation has changed somewhat since that article was published.

To be explicit from the outset of the discussion, in the collection of axioms that derive from combinatorial assumptions on ideals, there is an anomaly that does not fit the general pattern: the assertion that NS_{ω_1} is $\aleph_2$-saturated. This assumption, combined with the assertion of conventional large cardinals, implies CH fails and that $P(\omega_1)$ is very close to $L(\mathbb{R})$. This situation, and deep elaborations of the situation, are discussed extensively in Woodin's work [127–129].[113]

One might speculate that the assumption NS_{ω_1} is $\aleph_2$-saturated is less analogous to conventional large cardinals than are the "generalized large cardinal" assumptions described in the next section that are derived from the "three parameters".[114] Some slight support for this view comes from the

[113] See also Dehornoy's discussion [19] of Woodin's work.
[114] It is not known to the author how to state the saturation of the nonstationary ideal on ω_1 in a non-circular way using the three parameters.

fact that it is not known if it is possible that NS_{ω_1} be $\aleph_2$-saturated and also be the induced ideal from a conventional large cardinal.[115]

It also seems worthwhile pointing out again explicitly that there are no known generalizations of the assumption that NS_{ω_1} is $\aleph_2$-saturated to other cardinals. By Corollary 6.11, it is inconsistent that NS_{ω_2} be $\aleph_3$-saturated. If one assumes that "NS_{ω_2} restricted to $\mathrm{Cof}(\omega_1)$ is $\aleph_3$-saturated" (an assumption not known to be consistent relative to large cardinals) then Theorem 5.20 implies that $\Theta^{L(\mathbb{R})} < \omega_2$, and hence NS_{ω_1} is not $\aleph_2$-saturated. Thus the natural generalizations of "NS_{ω_1} is $\aleph_2$-saturated" appear to be either outright inconsistent or inconsistent with the assumption itself.

11.1. Generalized Large Cardinals

As has been heralded throughout this chapter, generic elementary embeddings are rather straightforward generalizations of large cardinals. We now focus and elaborate on the connections.

In [113], essentially all conventional large cardinal assumptions[116] are shown to be equivalent to the assertion of the existence of definable elementary embeddings $j : N \to M$, where N and M are transitive classes. The strength of these assertions is determined by two parameters, or axes:

W: Where j sends the ordinals.

Cl: How big N and M are.

Typically, the more one asserts about what j does to the ordinals and the larger N and M are, the stronger the axiom. A remarkable feature of this collection is a form of convexity: asserting the existence of one such j with suitable parameters does not contradict the assertion of other j's that have strengthened the parameters along a fixed given axis.[117] Moreover, by suitable strengthening along the axis, the axioms form a hierarchy both in outright strength and in consistency strength.

Generic large cardinals are straightforward generalizations of large cardinals in that they assert the existence of elementary embeddings of N to M; however, these elementary embeddings are only required to exist in generic extensions of the universe V. As such they introduce a third parameter of potential strength: the nature of the forcing. Thus the three parameters are now:

W: Where j sends the ordinals.

Cl: How big N and M are.

F: The nature of the forcing.

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[115] Explicitly: It is not known how to force over an arbitrary model containing a conventional large cardinal, such as a huge cardinal, and make the nonstationary ideal on ω_1 both be $\aleph_2$-saturated and be the induced ideal from the large cardinal embedding.

[116] In consistency strength there is a cofinal collection of large cardinals of this form.

[117] Subject of course to the Kunen limitations.

As an informal working definition, we take a "generalized large cardinal axiom" to mean an axiom whose statement posits an elementary embedding or system of elementary embeddings by specifying where ordinals go, the closure of N and M, and the isomorphism type of the Boolean algebra used to force the generic embedding.[118]

By adjusting the parameters "W", "Cl" and "F", one gets a cofinal set of generalized large cardinal axioms. These are often studied by fixing one of the parameters and taking the "level set" determined by that parameter. For example, one studies $\aleph_1$-dense ideals on:

$$\omega_1,\ \omega_2,\ [\omega_2]^{\omega_1}, \ldots .$$

In Sect. 6, we showed that one cannot adjust these parameters arbitrarily: there are limitations analogous to the "Kunen contradiction" for large cardinals, as well as limitations on the additional parameter of which forcing can yield elementary embeddings with a given amount of closure.

11.2. Flies in the Ointment

The assertion of a collection Σ of axioms has as a meta-assumption the assertion that Σ is consistent.

Conventional large cardinal axioms have a well-ordered spine that give a cofinal collection of assumptions that are believed to be consistent. Since this collection of axioms is linearly ordered, a finite set of axioms that are individually consistent are mutually consistent. This is not the case with generalized large cardinals, due to the additional axis "F".

Most, although not all, consistency results for generalized large cardinal properties show the consistency of an ideal with a particular property on a particular cardinal. Since the generalized large cardinals are not linearly ordered in strength this leaves open the possibility of mutual inconsistency.

This possibility is unfortunately realized,[119] though examples of this type are remarkably rare at the time of writing.[120] The only example known involves the partition relation

$$\binom{\omega_2}{\omega_1} \to \binom{\omega}{\omega_1}_2 .$$

Namely, as discussed in Remark 5.39, if there is a huge cardinal then there is a forcing extension in which there is an $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal [83] and another one in which there is an ideal $[\lambda]^{\omega_1}$ with quotient isomorphic to

[118] This mechanism appears to define away the anomaly of NS_{ω_1}.

[119] See Corollary 5.38.

[120] As this chapter went to press Woodin discovered more mutually inconsistent generalized large cardinal axioms. At this very moment, it is not known if there is a mutually inconsistent pair of axioms that are the result of varying the parameters along just one of the axes "W", "Cl" or "F".

$\mathrm{Col}(\omega, {<}\lambda)$. The former implies the partition relation and the latter implies its failure.

This same example shows that some particularly attractive sounding axioms are inconsistent. For example, it is not consistent to have the assumption:

> For all $\aleph_2$-c.c. Boolean algebras $\mathbb{B}$ of cardinality less than or equal to 2^{ω_2} that collapse ω_1, there is a normal fine ideal I on $[\omega_2]^{\omega_1}$ such that $P([\omega_2]^{\omega_1})/I \cong \mathbb{B}$.

While the counterexample to mutual consistency is certainly very troubling, it may not be fatal to the program of looking to generalized large cardinals for true extensions of ZFC. The general "picture" given by the axioms is remarkably coherent; conventional large cardinals actually imply the existence of generalized large cardinal embeddings with small critical points and the "mutual inconsistency phenomenon" seems rare. At the moment the question "which generalized large cardinals are true" seems to be narrower, more focussed and less arbitrary than the broader question "which combinatorial and cardinal arithmetic statements are true".

It is also a possibility, albeit somewhat unlikely in the author's opinion, that the axiom "there is a countably complete, normal, fine, $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$" is inconsistent.[121] The axiom collection "for all $n \in \omega \setminus \{0\}$ there is a countably complete, normal, fine $\aleph_1$-dense ideal on $[\omega_{n+1}]^{\omega_n}$" together with the assertions that the ω_n's are generically supercompact answer many classical set theoretic questions. While this or any particular theory may be inconsistent, it does not negate the more general point that generalized large cardinals provide a well-motivated and systematic framework for the development of new axioms that are effective in settling classical independent questions. This framework fits well into traditional Gödelian ideas of axiom development: reflection and generalization. In the author's opinion, the question is not *whether* generalized large cardinals are relevant, but *which* generalized large cardinal axioms are true.

Defining the boundaries of the region of consistency for generalized large cardinals is likely to be a topic of research for some time. The two facets of this research consist in proving relative consistency results for generalized large cardinals from conventional large cardinal axioms and further exploring limitations on what axioms one can consistently assert.

It is certain that there are imaginative versions of these axioms that still have not been considered; moreover these will have interesting and far-reaching consequences. In the course of exploring the possibilities for these axioms it is inconceivable that there will not be more limitations discovered on what generic large cardinals can exist. This should not be viewed as a disaster. Indeed it can be argued that the axiomaticians are being too cautious if they rarely consider axioms that turn out to be inconsistent.

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[121] This property is not known to be consistent relative to conventional large cardinals.

11.3. First-Order Statements in the Language of ZFC

A boundary condition for extending the axioms of ZFC is that the extensions be able to be expressed in a language compatible with the first-order language expressing ZFC. This condition is imposed for the same epistemically convenient reasons that first-order logic is used to state the axioms of ZFC.

A priori, a statement of the form "there is an elementary embedding $j : V \to M, \ldots$" is not a statement in the first-order language of ZFC. It asserts the existence of a proper class, it appears to require truth predicates and so forth. As generalized large cardinals appear to assert as well the existence of virtual sets in a proper extension of the universe, the situation appears even more critical.

In [113] for example, this obstacle was overcome for conventional large cardinals by finding first-order statements in the language of set theory that were provably equivalent by metamathematical methods to the intended second-order axiom. For example, the assertion that κ is supercompact is equivalent to the assertion that for all $\lambda \geq \kappa$ there is a normal, fine, κ-complete ultrafilter on $[\lambda]^{<\kappa}$.

The language of ideals, together with the mechanics of forcing provide the same kind of vehicle for stating generalized large cardinal axioms in the language of set theory. Assuming the existence of a proper class of Woodin cardinals, Burke's Proposition 9.44 shows that every countably complete ideal is pre-precipitous. More directly: the existence of an elementary embedding $j : V \to M \subseteq V[G]$ where $G \subseteq \mathbb{P}$ is generic and $j``\lambda \in M$ is easily seen to be equivalent to the existence of a $\mathbb{P}$-term for an ultrafilter $\dot{U} \subseteq P(P(\lambda))^V$ that is normal for regressive functions in V and fine and is such that there is no descending ω-sequence of U-equivalence classes of functions from V. The idea of an *induced ideal* allows us to restate this combinatorially as a normal, fine, precipitous ideal I on $P(\lambda)$ such that the quotient algebra $P(P(\lambda))/I$ inherits some of the properties of the original partial ordering $\mathbb{P}$. Finally, moving along the "F" axis in the direction of greater strength, the saturation properties of ideals play exactly the same role for generalized large cardinals as ultrafilters do for conventional large cardinals.

11.4. Some Examples of Axioms

In this subsection we name some examples of axioms. Many of the axioms have strong consequences but have not, as yet, been shown to be consistent from conventional large cardinals.

11.1 Definition. A cardinal $\kappa = \mu^+$ is *minimally generically n-huge* iff there is a normal, fine, κ-complete ideal I such that $P([\kappa^{+n}]^{\kappa^{+n-1}})/I$ has a dense set isomorphic to $\mathrm{Col}(\mu, \kappa)$.

The "n-huge" is there because if $j : V \to M \subseteq V[G]$ is the generic embedding, then $M^{j^n(\kappa)} \cap V[G] \subseteq M$. The quotient algebra $\mathrm{Col}(\mu, \kappa)$ is in many ways the simplest quotient, hence "minimal".[122]

11.2 Definition. A successor cardinal $\kappa = \mu^+$, where μ is regular, is *generically supercompact* iff for all $\lambda > \kappa$, and all generic $G \subseteq \mathrm{Col}(\mu, \lambda)$, there is a generic elementary embedding $j : V \to M \subseteq V[G]$ with critical point κ such that $j(\kappa) > \lambda$, $j``\lambda \in M$, and $\lambda > \sup(j``\lambda)$.

Again the name comes from the closure of the embedding and the quotient algebra is asked to be simple.

11.3 Definition. The *Axiom of Resemblance* is the statement that for all regular $\kappa < \lambda$, $n \in \omega$, there is a generic elementary embedding $j : V \to M$ such that $j(\kappa^{+i}) = \lambda^{+i}$ for all $i < n$ and $j``\kappa^{+n} \in M$.

This axiom suggests that the regular cardinals are in some weak sense, indiscernible. It suffices to show many instances of Chang's Conjecture and thus, to transfer instances of GCH from cardinal to cardinal.

The author would be remiss if he did not mention the particularly interesting, if slightly technical axiom "indestructible generic supercompactness". This axiom was discussed by Cummings in [17]:

11.4 Definition. Let μ be a regular cardinal and $\kappa = \mu^+$. Then κ is *indestructibly generically supercompact* iff for all $<\kappa$-directed closed partial orderings $\mathbb{R}$ and all generic $G \subseteq \mathbb{R}$, regular $\lambda > \kappa$, there is a $\tilde{\mathbb{R}} \in V[G]$ that is μ-closed such that if $H \subseteq \tilde{\mathbb{R}}$ is $V[G]$ generic, then there is a

$$j : V[G] \to M \subseteq V[G * H]$$

with:

1. $\mathrm{crit}(j) = \kappa$,
2. $j(\kappa) > \lambda$,
3. $j``\lambda \in M$,
4. $\sup(j``\lambda) < j(\lambda)$, and
5. $M \models \mathrm{cf}(\lambda) = \mu$.

This axiom holds when an indestructible supercompact cardinal[123] is collapsed to be the successor of μ. It plays an important role in the interplay between generalized large cardinal axioms and combinatorial properties of successors of singular cardinals.

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[122] As this article went to press, Woodin showed that it is inconsistent to have ω_1 minimally generically 3-huge, and simultaneously for ω_3 to be minimally generically 1-huge.

[123] That is, κ is supercompact and remains so after any forcing extension via a $<\kappa$-directed closed forcing. Laver [82] showed how to make a supercompact cardinal indestructible, and his construction is given in Cummings's chapter in this Handbook. We applied this concept after Theorem 8.42.

11.5. Coherence of Theories, Hierarchies of Strength and Predictions

The coherence and predictive value of axioms has been considered relevant to the evaluation of axioms by several commentators, notably Gödel [57] and Martin [92].

Predictions

The notion of prediction used here requires some explication.[124] Given a sequence of axiom systems $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \cdots$, the predictive capacity of the sequence can mean at least two things:

A. There are consequences σ of Σ_i are that are proved first (temporally) by some Σ_j with $j > i$, or perhaps have significantly easier proofs in Σ_j than in Σ_i.

B. There is mathematical structure uncovered in the study of Σ_j that has analogues in the theory Σ_i for $i < j$.

While these are necessarily vague and perhaps artificial formulations, some observations can be made. The most trivial remark is that the collections of axioms systems we are studying are often not linearly ordered by inclusion. For coherent collections of axiom systems, the theories do form a directed system. For the purposes of calibration of strength, it seems to this author that it suffices to have a directed collection of axiom systems that has a linearly ordered cofinal subcollection.

A more difficult problem is that both of these criteria are sociological in nature; the order of discovery of consequences structure depends heavily on human events that can proceed in arbitrary and capricious ways. This is particularly evident if, as is usually in axiomatic discussions, the theories Σ_i with small i have a much longer history of historical study than the Σ_j for large j. Concrete examples of this are when Σ_0 may be the theory of Peano Arithmetic, second-order number theory, or ZFC.

Perhaps the most challenging problem is weighing the significance of purported examples of "predictions". It is clearly possible to create artificial "predictions" by concocting consequences of Σ_0 that no previous investigator would have bothered to look at. A type of prediction with perhaps more weight is that many families of axioms, including generalized large cardinals, are determined by varying parameters in a fixed conceptual framework. The axioms with larger/stronger parameters can then be said to "predict" in some sense the axioms with smaller/weaker parameters.

A less artificial kind of prediction is when Σ_{i+1} "predicts" the consistency of Σ_i. Indeed the linearity of the consistency hierarchy for conventional large

[124] The exposition given here is due the author. This terse account does not reflect the views of Martin or Gödel.

cardinals has often been held to be one of its main attributes, particularly in regard to the calibration of strength of properties independent of ZFC.

In the case of generalized large cardinals, issues of consistency strength are largely open, except for the weakest of the cardinals. There are however examples of these hierarchies among the stronger axioms. A notable example of this is Theorem 10.12, where it is shown that the consistency of generically n-huge cardinals follows from the existence of generically $(n+1)$-huge cardinals. The author suspects that many more such results can be proved with sufficient effort and attention.

We now turn to "predictions" in the more commonly used meaning. An oft-cited example of such a "prediction" made by large cardinals involves the structural feature of the subsets of the real numbers known as the Wadge hierarchy. This hierarchy of complexity of sets of real numbers was first shown to have the correct structure using large cardinals [92]. This correctly predicted the analogous structure for Borel sets of reals, which is a theorem of ZFC.

There are such examples in the realm of generalized large cardinals as well. The first proof of Silver's theorem that "if GCH holds below $\aleph_{\omega_1}$ then it holds at $\aleph_{\omega_1}$" was given by Magidor, under the assumption that there was a precipitous ideal on ω_1.[125] Indeed the proof assuming the existence of a precipitous ideal is so short and elegant we give it here:

> Let $\alpha = \omega_1^V$, $\kappa = \aleph_{\omega_1}^V$. Let I be a precipitous ideal on ω_1. Then $|P(\omega_1)| < \aleph_\omega$ so forcing with $P(\omega_1)/I$ preserves all cardinals above $\aleph_\omega$. Let G be generic, and $j : V \to M \subseteq V[G]$ be the generic elementary embedding.
>
> For all $f \in (\kappa^{\omega_1})^V$, $j(f)\restriction\alpha \in M$. Hence $V[G] \models |\kappa^\alpha|^M \geq |\kappa^\alpha|^V$. Since $V[G] \models \alpha < \omega_1$ and $\kappa < \aleph_{\omega_1}$, we know $V[G] \models |\kappa^\alpha|^V \leq \kappa^+$. Since $(\kappa^+)^{V[G]} = (\kappa^+)^V$ we must have $V \models |\kappa^\alpha| \leq \kappa^+$.

This example appears to satisfy the criterion for a Gödel-style prediction: the result was proved first under the assumption of the existence of a precipitous ideal on ω_1, and then in ZFC. The proof, assuming the existence of a precipitous ideal is elegant and much shorter than the ZFC proof.

An example that carries perhaps less weight as it is internal to the theory of generalized large cardinals is the postulate that there is a normal, fine, $\aleph_n$-complete, $\aleph_\omega$-saturated ideal on $[\aleph_\omega]^{\aleph_\omega}$. In [35] there is an argument of Woodin that the existence of such an ideal implies $2^{\aleph_0} \geq \omega_n$. Indeed, if ω_1 is generically n-huge with quotient $\mathrm{Col}(\omega, \omega_1)$, then there can be no such ideal.

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125 The author relates this anecdote from personal interaction with Magidor, who stated that this proof preceded Silver's [109] by several weeks. Other contemporary accounts state that the initial assumption was the existence of a non-regular ultrafilter on ω_1 rather than a precipitous ideal. Indeed the key combinatorial element was a result of Kanamori that there is a "least" function $f : \omega_1 \to \omega_1$ relative to a non-regular ultrafilter; i.e. a non-regular ultrafilter is incompressible. Silver credits Kanamori and Magidor in his paper.

Viewing this as a prediction, it can be argued that Theorem 6.9,[126] which shows in ZFC that there are no such ideals, is confirmation of the prediction.

The author points out that there are numerous cases in this chapter where he was able to show consequences of rather strong ideal axioms[127] that Woodin was later able to show from weaker assumptions. These are also "predictions", though not of ZFC results. Indeed Theorem 5.42, a very early result showing that all sets in $L(\mathbb{R})$ are Lebesgue measurable, preceded the result that "CH together with the existence of an $\aleph_1$-dense ideal on ω_1 implies $\mathrm{AD}^{L(\mathbb{R})}$" by more than 10 years.

Gradations of Consequence

A more subtle type coherence comes in the form of *gradations of consequence.* This idea is that among a coherent family of axioms indexed by a parameter, stronger axioms should prove genuinely stronger natural consequences. This is a strange kind of "prediction", but it has the advantage that it is not related to accidents of mathematical history. Here are some examples from generalized large cardinals.

In Example 5.7, it is shown that if there is an $\aleph_2$-complete ideal I on ω_2 such that $P(\omega_2)/I$ has a dense countably closed subset, then CH holds. In a universe with ambient large cardinals, Theorem 5.20 shows that an $\aleph_2$-complete ideal I on ω_2 that has the weaker property that it concentrates on the approachable ordinals and that $P(\omega_2)/I$ is reasonable, has the weaker consequence that $\Theta^{L(\mathbb{R})} < \omega_2$.

In Theorem 5.9, Woodin showed that if there is a uniform, countably complete $\aleph_1$-dense ideal on ω_2, then CH holds. It follows from Theorem 7.14 that if you add at least ω_2 Cohen reals to a model where there is an $\aleph_1$-dense ideal on ω_2, one gets a model with an $\aleph_2$-saturated ideal on ω_2. Since CH fails in the resulting model, the axiom that there is an $\aleph_1$-dense ideal on ω_2 is strictly stronger than the axiom that there is an $\aleph_2$-saturated ideal on ω_2. Theorem 5.21 shows that the weaker axiom still proves a vestige of CH. It shows that if there is a uniform, countably complete, $\aleph_2$-saturated ideal on ω_2, then $\Theta < \omega_2$. Thus a weaker axiom has a weaker consequence.

Another example of "gradation of consequences" has to do with partition properties. In Sect. 5.6, it is shown that as one passes from normal, κ-complete

1. prime,
2. κ-dense,
3. $(\kappa^+, \kappa^+, \kappa)$-saturated

ideals, one gets the partition properties:

[126] Proved several years later.

[127] Such as CH or the existence of Suslin trees from $\aleph_1$-dense ideals on $[\omega_2]^{\omega_1}$.

1. $\kappa^+ \to (\rho)^2_m$ for all $\rho < \Omega(\kappa)$,
2. $\kappa^+ \to (\kappa^2 + 1)^2_2$,
3. $\kappa^+ \to (\kappa \times 2 + 1)^2_2$,

respectively. While these results have not been shown to be optimal, they nonetheless exhibit the kind of coherence expected from genuine axiom systems.

Methodological Predictions

We mention very briefly yet another possible type of "prediction". The use of "prediction" by Gödel or Martin usually is taken as a prediction of results; e.g. a number-theoretic fact proved using infinitary tools that is verified by a proof in Peano Arithmetic. One can also see predictions in light of methods developed using strong assumptions that can be specialized and applied with weaker assumptions. The fact that the strong assumptions suggested methods that were fruitful in weaker theory could be construed as a kind of inductive evidence for the stronger theory. The Axiom of Determinacy has this type of flavor: it can be specialized to $L(\mathbb{R})$, or (in ZFC) to Borel sets.

This type of "prediction" has occurred in more subtle ways with generic elementary embeddings. In [35] it was first shown that the existence of an $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$ implies that every set in $L(\mathbb{R})$ is Lebesgue measurable, has the property of Baire and the partition property holds in $L(\mathbb{R})$. Later work [47] showed that supercompact large cardinals imply the existence of generic large cardinals sufficiently strong to show the same consequences. Woodin later showed that the existence of sufficiently many Woodin cardinals suffices to show the existence of the generic embeddings needed to prove the same results.

This may be interpreted as follows: the method of generic elementary embeddings can be adapted from their use with the very strong assumption of the existence of an $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$, to more standard contexts of conventional large cardinals and still have some of the same consequences.

Thus: from generic large cardinals one shows certain consequences; these consequences can then be shown to follow from conventional large cardinals using the vehicle of generic large cardinals.

11.6. A Final Relevant Issue

Many authors who propose axioms claim that their axioms are, in some sense, "maximality principles". Unlike other heuristics for the development of axioms (such as reflection, or elementary embedding principles) the notion of a "maximality principle" does not seem sufficiently precise to allow general agreement on what qualifies. For some, CH limits how many reals there are. For others, the failure of CH limits what subsets of ω_1 exist.

The situation is complicated by the existence of models M and N satisfying ZFC such that:

1. M and N have the same cardinals,
2. M and N have the same real numbers,
3. CH holds in M and fails in N.

Of course, M and N differ on $P(\omega_1)$.

A very weak related concept might be the following: ϕ is *not limiting* if any model containing sufficiently large cardinals has a forcing extension in which ϕ is true. Ideas like this have been informally discussed in set theory for some time. Indeed Woodin's work on the Ω-conjecture deals, in part, with exactly these issues. The author conjectures that most generalized large cardinal axioms are not limiting.

11.7. Conclusion

Generalized large cardinal axioms form a natural family of assumptions that are very similar in nature to conventional large cardinals. There are problematic aspects to this generalization due to examples of mutually contradictory ideal assumptions. Nonetheless, even among the contradictory examples, there is a general picture that settles most important examples of independent statements in set theory.

The arguments involving intuitive or aesthetic judgments and prediction or confirmation that have been advanced for large cardinals seem to apply as well to generalized large cardinals; there does not seem to be an abstract conceptual basis to distinguish between the narrower family of axioms and its generalization. Indeed large cardinals[128] *imply* the existence of generic elementary embeddings with small critical points such as ω_1.

The theory is currently immature and it is expected that there will be many surprises before the story is completely told. The author speculates that this avenue is likely to eventually provide reasonable solutions to many of the puzzles of independence in set theory.

12. Open Questions

Some open problems are given in this section. They are organized roughly in the order that material is presented in the paper, rather than in any order of significance. They range from technical questions that the author was curious about to questions that may be fundamental set theoretic problems. None of the problems are guaranteed to be hard or deep. The theory is so underdeveloped that virtually all theorems contain hypotheses whose necessity has not been validated by appropriate examples. The emphasis has been

[128] Via the mechanism of stationary tower forcing.

on enumerating novel problems rather than restating well-known problems such as the question of the consistency of the property: "$\aleph_\omega$ is Jonsson". The reader can thus be trusted to create his or her own problems.

1. What is the consistency strength of "Every function $f : \omega_n \to \omega_n$ is bounded by a canonical function modulo the (strongly) nonstationary ideal on $P(\omega_n)$"? What if we only require that every function is bounded by the nonstationary ideal restricted to ω_n? The nonstationary ideal restricted to $\omega_n \cap \mathrm{Cof}(\omega_{n-1})$?

2. Is there an example of a $Z \subseteq P(X)$ and a normal, fine, countably complete ideal on Z that is $|Z|^+$-saturated, but not $|X|^+$-saturated?

3. What can one say about P-, Q-, and selective ideals on $[\lambda]^{<\kappa}$ or $[\lambda]^\kappa$? (See Proposition 2.7.)

4. In spite of the close ties with the nonstationary ideal, proper forcing and Hungarian combinatorics, almost nothing is known about the various Chang's Conjectures. The most basic consistency results are open, for example, it is not known if it is consistent that $(\aleph_4, \aleph_1) \twoheadrightarrow (\aleph_3, \aleph_0)$. Assuming that $2^{\aleph_0} < \aleph_\omega$, Silver showed that the cardinal $\aleph_\omega$ is Jónsson iff there is an infinite subsequence $\langle \kappa_n : n \in \omega\rangle$ of the $\aleph_n$'s such that the infinitary Chang conjecture of the form $(\dots, \kappa_n, \kappa_{n-1}, \dots, \kappa_1) \rightarrow$ $\rightarrow (\dots, \kappa_{n-1}, \kappa_{n-2}, \dots, \kappa_0)$ holds. It is not known how to get such a sequence of length 4.

5. Essentially equivalent questions are whether there can be countably complete, normal, fine, precipitous ideals on $[\omega_n]^{\omega_m}$. The positive sets for such ideals determine instances of Chang's Conjecture.

6. Is it consistent that there is a normal, fine, countably complete ideal on $[\aleph_{\omega+1}]^{\omega_n}$ for $n \geq 2$? The existence of such an ideal is equivalent to the statement that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_n, \aleph_{n-1})$. Cummings has made some progress on this problem in a negative direction.

7. Are any global Chang's Conjecture type properties consistent? For example is it consistent that for all successor μ and all regular $\kappa < \mu$ one has $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$?

8. Is every κ-complete, cardinal preserving ideal on a regular cardinal κ precipitous? One could also consider a generalization of this to normal, fine, countably complete ideals on κ. (Note that this is known under GCH and various other assumptions.)

9. Does the existence of a precipitous ideal on ω_1 imply the existence of a normal precipitous ideal on ω_1? For arbitrary κ? For ideals on $P(X)$?

10. As noted in the text there are very interesting consequences when a natural ideal has nice generic embeddings. For most natural ideals, the

possible generic ultrapower properties have not been explored. We note here just some of the possible questions.

Can I be precipitous when I is:

(a) The ideal of null sets for Lebesgue measure and $Z = P(\mathbb{R})$?
(b) The ideal of meager sets and $Z = P(\mathbb{R})$?
(c) $I[\lambda]$ for a regular cardinal $\lambda \geq \omega_2$?
(d) $I_d(\lambda, \kappa)$ (see Example 3.17)?
(e) the weak diamond ideal on ω_1?
(f) any of the uniformization ideals?

Some of these ideals are known not to be saturated, but others are not. However, other results are possible. For example, the algebra $P(\mathbb{R})/\{\text{null sets}\}$ is never saturated. It seems possible that it could be densely often c.c.c. Similar possibilities exist for the meager ideal. The saturation possibilities of the weak diamond and uniformization ideals are not known.

11. The author conjectures that for all normal, $\aleph_2$-complete, $\aleph_3$-saturated ideals J on ω_2:

    (a) $I_d(\omega_2, \omega_1) \subseteq J$, and
    (b) $I[\omega_2] \nsubseteq J$ for such a J.

    In particular, there is a set in $I[\omega_2] \cap \breve{J}$. These remarks are not intended to be special to ω_2: suppose that I is a normal, κ-complete, κ^+-saturated ideal on κ. Is there a set $A \in I[\kappa]$ that belongs to $\breve{I}$?

12. Another conjecture in the same spirit is that a normal, fine, ideal I on $P(Z)$ with $Z \subseteq H(\theta)$ that is $|Z|^+$-saturated and $\mathrm{comp}(I) = \omega_2$ must concentrate on $\mathrm{IA}(\mathrm{Cof}(\omega_1))$.[129] This may be the correct generalization of Shelah's theorem that a saturated ideal on ω_2 must concentrate on $\mathrm{Cof}(\omega_1)$. In this problem ω_2 can be replaced by any μ^+ and ω_1 by μ for any regular cardinal μ.

13. Suppose that J is an ideal on $Z \subseteq P(\kappa^{+(n+1)})$, and I is the projected ideal on the projection of Z to $Z' \subseteq P(\kappa^{+n})$. Suppose that the canonical homomorphism from $P(Z')/I$ to $P(Z)/J$ is a regular embedding. Is I $\kappa^{+(n+1)}$-saturated?

14. Is "presaturation" closed under projections?

15. Can there be countably complete, $\aleph_1$-dense, uniform ideals on ω_n for $n > 2$? What are the possible quotient algebras of uniform, countably complete ideals on ω_n?

[129] $\mathrm{IA}(\mathrm{Cof}(\omega_1))$ is the class of internally approachable structures of cofinality ω_1.

16. Is it consistent that there is a uniform ultrafilter on ω_3 such that ω^{ω_3}/U has cardinality ω_3? Is it consistent that there is a uniform ultrafilter U on $\aleph_{\omega+1}$ such that $\omega^{\aleph_{\omega+1}}/U$ has cardinality $\aleph_{\omega+1}$? Give a characterization of the possible cardinalities of ultrapowers.

17. Using the techniques of [36], the previous question could be answered by showing that sufficiently long sequences of consecutive cardinals can carry very strongly layered ideals. A typical question here is whether it is consistent to have an $\aleph_1$-dense ideal on ω_1 and strongly layered ideals on ω_2 and ω_3. A closely related question is whether it is consistent to have three consecutive cardinals carry very strongly layered ideals.

18. Can there be a cardinal $\kappa = \mu^+ \geq \omega_2$ such that the nonstationary ideal on κ restricted to $\mathrm{Cof}(\mu)$ is κ^+-saturated? Is it consistent for the nonstationary ideal on ω_2 restricted to $\mathrm{Cof}(\omega_1)$ to be $\aleph_3$-saturated?

19. Is there a large cardinal axiom such that forcing over an arbitrary model of that axiom makes an induced ideal be the nonstationary ideal on ω_1 and simultaneously $\aleph_2$-saturated in the generic extension?

20. Conjecture: If every ω_n carries a very strongly layered ideal, then $\aleph_\omega$ is Jónsson.

21. Is it consistent for there to be a countably complete, normal, fine, $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$? $[\omega_n]^{\omega_{n-1}}$? These questions are particularly relevant as these ideal properties are a “fly in the ointment” for generalized large cardinal axioms.

22. Find other hierarchies of consistency strength among ideal axioms similar to those described in Theorem 10.12. What are the relations among the various “axes” of the three parameters?

    As a sample of this kind of problem one might ask: from the assumption that there is an $\aleph_3$-complete, normal, fine precipitous ideal on $[\omega_5]^{\omega_4}$ concentrating on the collection of z such that the order type of $z \cap \omega_4$ is ω_3, can one show that it is consistent that there is an $\aleph_4$-saturated ideal on ω_3?

23. Given an example of a model of set theory with an ideal I on $P(Z)$ with $Z \subseteq P(X)$, and a set Y with $|Y| > |Z|$ such that the closed unbounded filter on $PP(Y)$ conditioned on I is not the nonstationary ideal restricted to a single set.

24. Which ideal assumptions imply the existence of Suslin trees on ω_2?

25. We note that if I is a normal, κ-complete, κ^+-saturated ideal on κ, then there is no $\diamondsuit(I)$ sequence (i.e. a $\diamondsuit$-sequence that guesses every subset of κ on an I-positive set). However, the following is open: Does CH together with the existence a normal, countably complete, $\aleph_1$-dense,

ideal on ω_1 imply $\diamondsuit_{\omega_1}$? Is there an ideal assumption on ω_2 that implies $\diamondsuit_{\omega_2}(\mathrm{Cof}(\omega_1))$? Does the existence of a normal, fine, countably complete $\aleph_1$-dense ideal on $[\omega_2]^{\omega_1}$ imply $\diamondsuit_{\omega_1}$?

26. Give a direct proof of $\mathrm{AD}^{L(\mathbb{R})}$ from ideal hypotheses.

27. Does the existence of an $\aleph_2$-complete, $\aleph_2$-dense ideal on ω_3 imply that $2^{\omega_1} = \omega_2$?

28. Does some generic supercompactness property of all of the ω_n's imply that $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$? (or even just full stationary set reflection?)

29. (From [55]) Can there be a non-trivial generic elementary embedding in a forcing extension by a Boolean algebra that is countably generated and proper?

30. (From [55]) Can there be a κ complete ideal on κ, that whose quotient is isomorphic to a forcing of size κ and is α-c.c. for some $\alpha < \kappa$?

31. Can there be a normal $\aleph_2$-complete ideal $I \subseteq P(\omega_2)$ that is $\aleph_3$-saturated in every $\aleph_2$-c.c. forcing extension? Other κ?

32. Can there be a normal, fine, κ-complete, κ-saturated ideal on $[\lambda]^\kappa$ for a non-huge κ? For a non-measurable κ?

33. Can the weakly compact filter on a cardinal κ be κ^+-saturated?

34. Recall the definition of the "forbidden intervals" $\mathcal{C}_\kappa$ (see Definition 6.25). Is there a model of set theory such that for all regular uncountable cardinals κ and all $\lambda \notin \mathcal{C}_\kappa$, there is a κ-complete, uniform, κ^+-saturated ideal on λ? (See Proposition 6.26 and Theorem 7.59.)

35. While the nonstationary ideal on $[\lambda]^{<\kappa}$ cannot be λ^+-saturated, the local saturation properties are relatively unexplored, the results of Gitik and Krueger being the only positive consistency results.

36. There are many open problems about mutually stationary sequences of sets. Probably the easiest to state is: Is there a mutually stationary sequence of sets $\langle S_n \subseteq \omega_n : n \in \omega \setminus k\rangle$ such that whenever $S_n = T_n^0 \cup T_n^1$ with T_n^0 and T_n^1 disjoint, either $\langle T_n^0 : n \in \omega \setminus k\rangle$ or $\langle T_n^1 : n \in \omega \setminus k\rangle$ is not mutually stationary?

37. Explore the properties of natural towers different from the usual stationary towers. Typical problems might be: give an example of a model with an inaccessible cardinal δ such that a stationary tower of height δ concentrating on $\mathrm{IA}(\mathrm{Cof}(\omega_1))$ is presaturated. Give an example of a tower generated by club guessing ideals that is presaturated.

38. Is it consistent to have CH and NS_{ω_1} is $\aleph_2$-saturated?

39. (Woodin) Does the existence of a countably complete $\aleph_1$-dense ideal on ω_1 imply that the continuum is less than ω_3? (If so, combined with Corollary 5.58, it shows that if NS_{ω_1} is $\aleph_1$-dense then $2^{\aleph_0} = \omega_2$.) Does the existence of an $\aleph_1$-dense ideal on ω_1 and the failure of CH imply that $\underset{\sim}{\delta}^1_2 > \omega_2$?

40. Is it consistent with large cardinals that $\Theta^{L(\mathbb{R})} > \omega_3$?

41. Generalize Theorem 7.72. Is it true in ZFC that there is a forcing $\mathbb{Q}$ such that for all generic $G \subseteq \mathbb{Q}$, there is no saturated ideal on ω_3? $\aleph_{\omega+1}$?

42. (Jech) Suppose that there is a supercompact cardinal. Is there a precipitous ideal on ω_1? Any successor cardinal?

43. Is it true in an "$L[E]$ model" model for some large cardinal that there is a precipitous ideal on a successor cardinal?

44. Suppose that J is an ideal on $Z \subseteq P(X)$, that $\mathbb{P}$ is a $|X|^+$-c.c. partial ordering, and that $\bar{J}$ is a $|X|^+$-saturated, normal, fine ideal in $V^{\mathbb{P}}$. Is it true that there are $A \in J^+$ and $L \in \bar{J}^+$ such that

$$\mathrm{id} : \mathbb{P} \times P(Z)/(J{\restriction}A) \to \mathbb{P} * \overline{P(Z)}/(\bar{J}{\restriction}L)$$

is a regular embedding? Note that $\bar{J}$ is precipitous.

45. Assume Martin's Maximum. Is there a set $A \subseteq \omega_4$ such that the model $L[A, \mathrm{NS}{\restriction}[\omega_4]^{\omega_2}] \models \kappa$ is κ^+-supercompact, where $\kappa = \omega_2^V$?

46. Is it consistent to have the stationary tower forcing up to an inaccessible concentrating on $\mathrm{IA}(\mathrm{Cof}({>}\omega_1))$ presaturated? (This is shown consistent from quite exotic large cardinal assumptions in [42]. The example given in Proposition 9.4 of an induced saturated tower concentrates on $\mathrm{IA}(\mathrm{Cof}({>}\omega))$.)

47. Is there a cardinal δ so large that it implies there are two presaturated towers $\mathcal{T}_0$, and $\mathcal{T}_1$ such that:

    (a) If j_i is the elementary embedding induced by $\mathcal{T}_i$, then $\mathrm{crit}(j_0) = \omega_4$ and $\mathrm{crit}(j_1) = \omega_3$.

    (b) The forcing for $\mathcal{T}_0$ can be regularly embedded into the forcing for $\mathcal{T}_1$?

    If there is such a cardinal, then Theorem 3.14 of [42] implies that $\Theta^{L(\mathbb{R})} < \omega_3$. Weaker conditions for $\Theta^{L(\mathbb{R})} < \omega_3$ exist as well [42].

48. Is there a cardinal δ so large that it implies there is an inaccessible κ and two towers $\mathcal{T}_0, \mathcal{T}_1$ yielding generic objects G_0, G_1 and embeddings j_0, j_1 with the property that:

(a) $j_0(\omega_1) = j_1(\omega_1) = \kappa$,

(b) Every real in $V[G_i]$ is generic over V by a forcing of size less than κ, and

(c) $j_0(\omega_3) \neq j_1(\omega_3)$?

If so, again $\Theta^{L(\mathbb{R})} < \omega_3$.

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14. Cardinal Arithmetic

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Contents

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1. Introduction

Cardinal arithmetic is the study of rules and properties of arithmetic operations mainly on infinite cardinal numbers. Since sums and products are trivial in the sense that

$$m + n = m \cdot n = \max\{m, n\}$$

holds for infinite cardinals, cardinal arithmetic refers mainly to exponentiation m^n. If M is a set of cardinality m and $[M]^n$ is the collection of all subsets of M of cardinality n, then m^n is equal to the cardinality of $[M]^n$. Thus exponentiation is intimately connected with the power set operation and hence lies at the heart of set theory. Classical and basic properties of cardinal arithmetic can be found, for example, in the Levy [12] and Jech [8] textbooks (the latter contains more advanced material). The aim of this introduction is to mention some elementary results and to put our chapter in its context—not to give a historical introduction to the subject of cardinal arithmetic, for which the reader is referred to these textbooks, to [7], and to [9] for a more general perspective.

A theorem of Zermelo generalizing a result of J. König says that if $\langle \kappa_i \mid i \in I\rangle$ and $\langle \lambda_i \mid i \in I\rangle$ are sequences of cardinals such that $\kappa_i < \lambda_i$ holds for every $i \in I$, then

$$\textstyle\sum_{i\in I}\kappa_i < \prod_{i\in I}\lambda_i.$$

A theorem of Bukovský and of Hechler says that if μ is a singular cardinal and the values 2^γ for cardinals $\gamma < \mu$ stabilize, then $2^\mu = 2^{\gamma_0}$, where $\gamma_0 < \mu$ is such that $2^{\gamma_0} = 2^\gamma$ for all $\gamma_0 \le \gamma < \mu$.

Building on earlier results (of Hausdorff, Tarski, Bernstein and others) Bukovský (1965) and Jech show how cardinal exponentiation can be computed from the gimel function (which takes κ to $\kappa^{\mathrm{cf}(\kappa)}$). Applications of Solovay and Easton of the forcing method of Cohen (1963) show that for regular cardinals κ there is no restriction on 2^κ except that which follows from the Zermelo-König theorem, namely that $\mathrm{cf}(2^\kappa) > \kappa$ (see [8, Chap. 3] for details). Thus the question about the possible values of $\kappa^{\mathrm{cf}(\kappa)}$ is most interesting from our point of view when κ is a singular cardinal. It was evident that it is much harder to apply the forcing method to singular cardinals. Involving large cardinals, work of Prikry and of Silver showed that it is possible for a strong limit singular cardinal μ to satisfy $2^\mu > \mu^+$ in some generic extension. Using large cardinals Magidor proved the consistency of $\aleph_\omega$ being the first cardinal κ for which $2^\kappa > \kappa^+$ holds. For a long time it was believed that large cardinal and more complex applications of the forcing method should yield greater flexibility for values of the power set of singular cardinals. A first indication that there are possible limitations was the theorem of Silver (1974): If κ is a singular cardinal with uncountable cofinality and if $2^\delta = \delta^+$ for all cardinals $\delta < \kappa$, then $2^\kappa = \kappa^+$. This result paved the way for further investigations by Galvin and Hajnal (1975). A representative result of their work is the following: If $\aleph_\delta$ is a strong limit singular cardinal with uncountable cofinality, then

$$2^{\aleph_\delta} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

For example, if $\aleph_{\omega_1}$ is a strong limit cardinal, then

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}.$$

The method of proof of these results relied in an essential way on the assumption that $\mathrm{cf}(\delta) > \aleph_0$. Shelah (1978) was able to prove similar results for singular cardinals with countable cofinality. For example, if $\aleph_\omega$ is a strong limit cardinal, then

$$2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}. \tag{14.1}$$

In a series of papers culminating in his book [15], Shelah developed a powerful theory with many applications, pcf theory, which changed our view of cardinal arithmetic. A remarkable result of this theory is the following. If $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \aleph_{\omega_4}. \tag{14.2}$$

If $2^{\aleph_0} \leq \aleph_2$, then (14.1) is a better bound than (14.2), but in general, since $(2^{\aleph_0})^+$ can be arbitrarily high, ω_4 seems to be a firmer bound.

The major definition in pcf theory is the set $\mathrm{pcf}(A)$ of possible cofinalities defined for every set A of regular cardinals, as the collection of all cofinalities of ultraproducts $\prod A/D$ with ultrafilters D over A. This basic and rather simple definition appears in many places and is the basis of a very fruitful investigation. It is a basic definition also in the sense that while the power set can be easily changed by forcing, it is very hard to change $\mathrm{pcf}(A)$.

Our aim in this chapter is to give a self-contained development of pcf theory and to present some of its important applications to cardinal arithmetic. Unless stated otherwise, all theorems and results in this chapter are due to Shelah.

The fullest development of pcf theory is in Shelah's book [15], and the interested reader can access newer articles (and the survey paper "Analytical Guide") in the archive maintained at Rutgers University.

In addition to this material, we have profited from expository papers (Burke-Magidor [2], Jech [7], and unpublished notes by Hajnal), and in particular a recently published book [6] which is very detailed, complete and carefully written.

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2. Elementary Definitions

An *ideal* over a set A is a collection $I \subseteq \mathcal{P}(A)$ such that (1) I is closed under subsets, that is, $X \in I$ and $Y \subseteq X$ implies $Y \in I$, and (2) I is closed under finite unions, that is, $X_1, X_2 \in I$ imply $X_1 \cup X_2 \in I$ (and thus the union of any finite sequence of members of I is in I). If $A \notin I$, then I is said to be *proper*. We do not require that ideals be proper (see the definition of $J_{<\lambda}$ in Sect. 3.1).

The dual notion, that of a filter, is also used in this chapter. A collection $F \subseteq \mathcal{P}(A)$ is a *filter* over A if (1) F is closed under supersets, that is, $X \in F$ and $X \subseteq Y \subseteq A$ imply $Y \in F$, and (2) F is closed under finite intersections. However, usually a filter is proper, that is $\emptyset \notin F$.

If I is an ideal over A, then $I^* = \{X \subseteq A \mid A \setminus X \in I\}$ is its *dual* filter. Sets belonging to an ideal are intuitively "small" or "null", whereas those of a filter are "big" or "of measure one". If I is an ideal over A, then subsets of A not in I are called "positive", and the collection of positive sets is denoted I^+.

$$I^+ = \{X \subseteq A \mid X \notin I\}.$$

On denotes the class of ordinals and Reg denotes the class of regular cardinals.

We shall deal in this section with functions from a fixed, infinite set A into the ordinals. The class of these ordinal functions is denoted On^A. If $f, g \in \mathrm{On}^A$, then $f \leq g$ means that $f(a) \leq g(a)$ for all $a \in A$, and similarly $f < g$ means that $f(a) < g(a)$ for all $a \in A$ (this is called the everywhere dominance ordering).

If $F \subseteq \mathrm{On}^A$ is a set, then the supremum function $h = \sup F$ is defined on A by

$$h(a) = \sup\{f(a) \mid f \in F\}.$$

If $f, g \in \mathrm{On}^A$, then we define

$$<(f, g) = \{a \in A \mid f(a) < g(a)\},$$

and similarly

$$\leq(f, g) = \{a \in A \mid f(a) \leq g(a)\}.$$

and

$$=(f, g) = \{a \in A \mid f(a) = g(a)\}.$$

If I is an ideal over A, then we define a relation $\leq_I$ over On^A by

$$f \leq_I g \quad \text{iff} \quad \{a \in A \mid g(a) < f(a)\} \in I.$$

In general, for any relation R on the ordinals, we define R_I over On^A by

$$f \; R_I \; g \quad \text{iff} \quad \{a \in A \mid \neg(f(a) \; R \; g(a))\} \in I.$$

That is, the set of exceptions to the relation is null. In this way $<_I$ and $=_I$ are defined over On^A. We remark that $\leq_I$ is weaker than "$<_I$ or $=_I$". Note that $\leq_I$ is a quasi-ordering, and that $<_I$ is irreflexive (if I is a proper ideal) and transitive.

The notations $X \subseteq_I Y$ and $X =_I Y$ are also used for subsets $X, Y \subseteq A$, with the obvious meaning. For example, $X \subseteq_I Y$ iff $X \setminus Y \in I$.

For a filter F over A, the dual definitions $f <_F g$, $f \leq_F g$ etc. will be used as well. For example, $f <_F g$ means that $\{a \in A \mid f(a) < g(a)\} \in F$. If F is the dual of an ideal I, then $<_F$ and $<_I$ are the same relation of course.

Products of Sets

Suppose that A is an index set and $S = \langle S(a) \mid a \in A\rangle$ is a sequence of non-empty sets of ordinals. Then the product, denoted $\prod S$ or $\prod_{a\in A} S(a)$, is defined as

$$\prod S = \{f \mid f \in \mathrm{On}^A \text{ and } \forall a \in A\ f(a) \in S_a\}.$$

In particular, if $h : A \to \mathrm{On}$ is any ordinal function defined on A, then $\prod h$ (or $\prod_{a\in A} h(a)$) denotes the set of all ordinal functions f defined on A such that $f(a) \in h(a)$ for all $a \in A$.

If A is a set of cardinals, then $\prod A$ (or $\prod_{a\in A} a$) denotes the set of all ordinal functions f defined on A such that $f(a) \in a$ for all $a \in A$. That is, $\prod A$ is $\prod h$ where $h(a) = a$ is the identity function on A.

For an ideal I over A, the relations $<_I$, $\leq_I$, and $=_I$ are defined on $\prod h$, and the reduced product $\prod h/I$ consisting of all $=_I$ equivalence classes is obtained. If $g \in \mathrm{On}^A$, then we may write (somewhat informally) $g \in \prod h/I$ rather than $[g] \in \prod h/I$, that is $\prod h/I$ is considered as a class of functions rather than equivalence classes.

For a filter F over A, the reduced product $\prod h/F$ is defined in a similar way.

A sequence of functions $f = \langle f_\xi \mid \xi < \lambda\rangle$ in $\prod A$ is said to be *$<_I$-increasing* if $\xi_1 < \xi_2$ implies that $f_{\xi_1} <_I f_{\xi_2}$. For typographical reasons we also say that f is I-increasing, or "increasing modulo I" instead of $<_I$-increasing. A sequence is a function, and if f_ξ denotes a value of that function then the sequence itself is denoted f, not $\bar{f}$ or F.

Partial Orderings

We say that $(P, \leq_P)$ is a *quasi-ordering* iff $\leq_P$ is a reflexive and transitive relation on P. A *strict* partial ordering is a transitive and irreflexive relation $<_P$ on P. In this chapter we consider both the quasi-ordering $\leq_I$ and the strict partial ordering $<_I$, defined on ordinal-valued functions. A typical example is $P = \prod h$ with the orderings $<_I$ and $\leq_I$ where $h \in \mathrm{On}^A$ is such that $h(a) > 0$ is a limit ordinal for every $a \in A$. Thus every function in P is $<_I$ bounded by another function there (for every $f \in \prod h$, $f <_I f + 1$, where $f + 1$ is the function taking a to $f(a) + 1$). So our setting is a structure $(P, <_P, \leq_P)$ where $<_P$ is a strict partial ordering, and $\leq_P$ is a quasi-ordering. The following properties of $(P, <_P, \leq_P)$ are obvious for our typical example:

P1. $a <_P b$ or $a = b$ implies $a \leq_P b$, but this implication is not necessarily reversible.

P2. If $a <_P b \leq_P c$ or $a \leq_P b <_P c$, then $a <_P c$.

P3. There is no $<_P$ maximal member: for every $p \in P$ there exists some $p' \in P$ with $p <_P p'$.

The following definitions apply whenever P is a set or a class, $<_P$ is a strict partial ordering, and $\leq_P$ a quasi-ordering on P.

A collection $B \subseteq P$ is said to be *cofinal* in P iff for all $x \in P$ there is some $y \in B$ with $x \leq_P y$. B is $<_P$*-cofinal* if $\forall x \in P \exists y \in B(x <_P y)$. If B is cofinal and $p \in P$, then we can first find $p' \in P$ such that $p <_P p'$ (by property P3 above) and then find $y \in B$ such that $p' \leq_P y$. Then $p <_P y$. Thus we can replace $\leq_P$ with $<_P$ in the definition of "cofinal". The *cofinality*, $\mathrm{cf}(P, \leq_P)$, of the partial ordering is the smallest cardinality of a cofinal subset. (Again $\mathrm{cf}(P, <_P)$ is similarly defined and these two cardinals are equal if properties P1–P3 above hold.) This cardinal need not be regular, if the ordering is not total (linear). We say that $(P, <_P)$ has "true" cofinality if it has a totally ordered subset $B \subseteq P$ that is cofinal. In this case the cofinality of B itself, and hence of P, is a regular cardinal. Observe that if $(P, <_P)$ has a linear cofinal subset whose order-type is a regular cardinal λ, then λ is the cofinality of P (because no cofinal subset of P is of smaller cardinality, even if non-linear subsets are considered). When $(P, <_P)$ has true cofinality, we write

$$\mathrm{tcf}(P, <_P) = \lambda$$

(or just $\mathrm{tcf}(P) = \lambda$ with the $<_P$ understood) to express both the fact that a totally ordered cofinal set exists, and that λ is the minimal cardinality of such a cofinal set.

In cases (when P3 above is not assumed) that $(P, \leq_P)$ has a greatest element, then the cofinality of P is defined to be 1 and its true cofinality is also 1, but since we assume that there are no $<_P$ maximal elements the cofinality and true cofinality (when it exists) are always infinite cardinals.

The following observation was made by Pouzet. For any infinite cardinal λ, $\mathrm{tcf}(P, <_P) = \lambda$ if and only if the following conditions hold:

1. $(P, <_P)$ has a cofinal set of size λ.

2. $(P, <_P)$ is λ-directed: any set $X \subseteq P$ of size $< \lambda$ has an upper bound in $(P, <_P)$.

It follows that if $\mathrm{tcf}(P, <_P) = \lambda$ and $G \subseteq P$ is any cofinal subset, then $\mathrm{tcf}(G) = \lambda$ as well.

A sequence $\langle p_\xi \mid \xi < \lambda \rangle$ of members of P is defined to be *persistently cofinal* iff

$$\forall h \in P \; \exists \xi_0 < \lambda \; \forall \xi \; (\xi_0 \leq \xi < \lambda \implies h <_P p_\xi). \tag{14.3}$$

Clearly every $<_P$-increasing and cofinal sequence is persistently cofinal. If $\langle p_\xi \mid \xi < \lambda\rangle$ is persistently cofinal and $p_\xi \leq_P p'_\xi$ for every $\xi < \lambda$, then $\langle p'_\xi \mid \xi < \lambda\rangle$ is persistently cofinal as well.

If $(P, \leq_P)$ is a quasi-ordering and $X \subseteq P$, then an *upper bound* of X is some $a \in P$ such that $x \leq_P a$ for all $x \in X$. If a is an upper bound of X and $a \leq_P a'$ for every upper bound $a' \in P$ of X, then we say that a is a *least upper bound* of X. We say that an upper bound a of X is a *minimal upper bound* if there is no upper bound a' of X such that $a' \leq_P a \wedge \neg(a \leq_P a')$.

Suppose that $(P, <_P, \leq_P)$ is as above and $X \subseteq P$ is such that for every $x \in X$ there is an $x' \in X$ with $x <_P x'$ (for example X is an increasing sequence in $<_P$). Then $a \in P$ is an *exact upper bound* of X iff

1. a is a least upper bound of X, and
2. X is cofinal in $\{p \in P \mid p <_P a\}$. Namely $p <_P a$ implies $\exists x \in X$ $(p \leq_P x)$.

Exercises are natural places to stop and think, but it is not an absolute requirement to solve them on first encounter. In fact, they often become easy with later material.

2.1 Exercise. Let $\lambda > |A|$ be a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda\rangle$ an increasing sequence of functions in On^A in the $<$ ordering (of everywhere dominance). Then f has an exact upper bound h and $\mathrm{cf}(h(a)) = \lambda$ for every $a \in A$. In fact sup f is the required upper bound.

We repeat the definitions given above, for $(\mathrm{On}^A, <_I, \leq_I)$ where I is a proper ideal over A. So, if $F \subseteq \mathrm{On}^A$ then

$$h \in \mathrm{On}^A \text{ is an upper bound of } F \quad \text{iff} \quad f \leq_I h \text{ for every } f \in F.$$

A function h is a least upper bound of F if it is an upper bound and $h \leq_I h'$ for every upper bound $h' \in \mathrm{On}^A$ of F. Here, the notions of least upper bound and minimal upper bound coincide.

If $h \in \mathrm{On}^A$ and $h(a) = 0$ for some $a \in A$, then $\prod h = \emptyset$. So, to avoid triviality $h(a) > 0$ is assumed for all $a \in A$, whenever the expression $\prod h$ is used. Hence if I is an ideal over A then every $g \in \mathrm{On}^A$ such that $g <_I h$ is $=_I$ equivalent to some function in $\prod h$. In fact, we shall usually consider reduced products $\prod h/I$ for functions h such that $h(a) > 0$ is always a limit ordinal, and hence every function in $\prod h$ is $<$-bounded (everywhere dominated) by some function in $\prod h$.

Suppose that F is a (non-empty) set of functions in On^A such that for every $f \in F$ there exists some $f' \in F$ with $f <_I f'$. Then $h \in \mathrm{On}^A$ is an exact upper bound of F if h is a least upper bound of F and for every $g <_I h$ there is some $f \in F$ with $g <_I f$ (namely F is cofinal in the lower $<_I$ cone determined by h). Actually it is not necessary to require that h is a least upper bound of F since this follows from the assumptions that h is an upper bound of F and F is cofinal in $\prod h/I$. Thus if $F \subseteq \prod h/I$ then h is an exact upper bound of F iff F is cofinal in $\prod h/I$.

If h is an exact upper bound of F and $A_0 \in I^+$ then $h \restriction A_0$ is an exact upper bound of $\langle f \restriction A_0 \mid f \in F \rangle$ with respect to the proper ideal $I \cap \mathcal{P}(A_0)$.

If h is an exact upper bound of F with respect to some ideal I over A and $J \supseteq I$ is a larger ideal over A, then h is an exact upper bound of F modulo J as well.

The definition of "true cofinality" of a reduced product is so important for pcf theory that we restate it for this case.

2.2 Definition. We say that $\mathrm{tcf}(\prod h/I) = \lambda$ iff λ is a regular cardinal and there exists a $<_I$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda \rangle$ in $\prod h$ that is cofinal in $\prod h/I$.

Projections

We shall often encounter the following situation.

1. A is a non-empty set of indices, and $S = \langle S(a) \mid a \in A \rangle$ is a sequence of sets of ordinals. The sup_of_S function is defined on A by taking $a \in A$ to $\sup S(a)$.

2. An ordinal function $f \in \mathrm{On}^A$ is given that is bounded by the sup_of_S, namely $f(a) < \sup S(a)$ for every $a \in A$.

Then we define the *projection* of f onto $\prod S$, denoted $\mathrm{proj}(f, S)$, as the function $f^+ \in \prod S$ defined by

$$f^+(a) = \min(S(a) \setminus f(a)).$$

So $f^+(a) = f(a)$ in case $f(a) \in S(a)$, and otherwise $f^+(a)$ is the least ordinal in $S(a)$ above $f(a)$. (There is such an ordinal by our assumption.) It is clear that f^+ is the least function in $\prod S$ that bounds f, and that $f_1 \leq f_2$ implies $f_1^+ \leq f_2^+$.

We shall apply projections in the presence of an ideal I over A. If $f \in \mathrm{On}^A$ is any function, not necessarily bounded by sup_of_S, we define $f^+ = \mathrm{proj}(f, S)$ as follows. For $a \in A$ such that $f(a) < \sup S(a)$, we define $f^+(a) = \min(S(a) \setminus f(a))$ as before, and for $a \in A$ such that $f(a) \geq \sup S(a)$ we define $f^+(a) = 0$. Clearly, $f_1 =_I f_2$ implies that $f_1^+ =_I f_2^+$. It follows, in case $f <_I \sup S$, that f^+ is the $\leq_I$-least function in $\prod_{a \in A} S(a)$ that $\leq_I$-bounds f, up to $=_I$ equivalence.

Given an ideal I over a set A and an ordinal function $h \in \mathrm{On}^A$, we are interested in the existence and value of the true cofinality of $\prod h/I$. Our first step is to reduce this question to ultraproducts of regular cardinals, and we can proceed as follows. Choose for every $a \in A$ a cofinal set $S(a) \subseteq h(a)$ of order-type $\mathrm{cf}(h(a))$. By our assumption that $h(a) > 0$ is always a limit, non-zero ordinal, the order-type of $S(a)$ is a regular infinite cardinal. Then the collections $\prod h$ and $\prod_{a\in A} S(a)$ are cofinally equivalent. That is for every $f \in \prod h$ there is a $g \in \prod S$ with $f \leq g$ (namely it projection), and vice versa.

Next, $\prod_{a\in A} S(a)$ is isomorphic to $\prod_{a\in A} |S(a)| = \prod_{a\in A} \mathrm{cf}(h(a))$. This is also the case when an ideal I over A is introduced and the relation $\leq_I$ is considered. Then $\prod h/I$ has the same cofinality and true cofinality as $\prod_{a\in A} \mathrm{cf}(h(a))/I$. Hence it suffices to consider reduced products $\prod_{a\in A} k(a)/I$ of functions k such that $k(a)$ are infinite regular cardinals. As the following lemma shows, in some cases we may even take k to be one-to-one. Recall that Reg denotes the class of regular cardinals.

2.3 Lemma. *Suppose that $c : A \to \mathrm{Reg}$ is a function and $B = \{c(a) \mid a \in A\}$ is its range. Suppose I is any ideal over A, and J is its Rudin-Keisler projection on B defined by*

$$X \in J \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}X \in I,$$

where $c^{-1}X = \{a \in A \mid c(a) \in X\}$. Then there is an order-preserving isomorphism $h : \prod B/J \to \prod_{a\in A} c(a)/I$ defined by $h([e]_J) = [e \circ c]_I$ for every $e \in \prod B$. If $|A| < \min B$, then

$$\mathrm{tcf}\big(\textstyle\prod B/J\big) = \mathrm{tcf}\big(\textstyle\prod_{a\in A} c(a)/I\big) \tag{14.4}$$

in the sense that existence of the true cofinality for one of $\prod B/J$ and $\prod c/I$ implies existence for the other poset as well, and these cofinalities are equal.

Proof. For every $e \in \prod B$ define $\bar{e} \in \prod c$ by $\bar{e}(a) = e(c(a))$. That is, $\bar{e} = e \circ c$. Then $e_1 =_J e_2$ iff $\bar{e}_1 =_I \bar{e}_2$, and $e_1 <_J e_2$ iff $\bar{e}_1 <_I \bar{e}_2$. Thus $h([e]_J) = \bar{e}/I$ induces an isomorphism from $\prod B/J$ into $\prod c/I$. Hence $\mathrm{tcf}(\prod B/J)$ is the same as the true cofinality of

$$G = \big\{h([e]_J) \mid e \in \textstyle\prod B\big\}$$

in $<_I$. If $|A| < \min B$, then G will be shown to be cofinal in $\prod c/I$ and this implies (14.4). (In general, if G is any cofinal subset of a partial ordering $(P, <_P)$, then G and P have the same true cofinality.)

Now G is cofinal in $\prod c/I$, because any $g \in \prod c$ is bounded by $\bar{f}$ where $f \in \prod B$ is defined by

$$f(b) = \sup\{g(a) \mid a \in A \text{ and } c(a) = b\}.$$

The fact that $|A| < b$ is used here to deduce that this supremum is below the regular cardinal $b \in B$, and hence that $f \in \prod B$. Since $\bar{f}/I \in G$, G is cofinal in $\prod c$. ⊣

To see how this lemma is applied, suppose that λ is a regular cardinal and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a $<_I$ increasing sequence of functions $f_\xi \in \mathrm{On}^A$. Then (as we have said) $h \in \mathrm{On}^A$ is an exact upper bound of f iff f is cofinal in $\prod h/I$. In this case it follows that $\mathrm{tcf}(\prod h/I) = \lambda$ and hence that the true cofinality of $\prod_{a\in A} \mathrm{cf}(h(a))$ is λ. Let $B = \{\mathrm{cf}(h(a)) \mid a \in A\}$ be the set of cofinalities of the range of h. The preceding lemma shows that λ is the true cofinality of a reduced product of B, if $|A| < \mathrm{cf}(h(a))$ for every $a \in A$.

2.1. Existence of Exact Upper Bounds

An important piece of pcf theory is the determination of conditions that ensure the existence of exact upper bounds. Recall that an exact upper bound of a $<_I$-increasing sequence $\langle f_\xi \mid \xi < \lambda \rangle$ of functions in On^A is a function $g \in \mathrm{On}^A$ that bounds every f_ξ in the $\leq_I$ relation and satisfies the additional requirement that if $d <_I g$ then $d <_I f_\xi$ for some $\xi < \lambda$. The following and Definition 2.8 are central in our presentation of pcf theory.

2.4 Definition (Strongly increasing). Suppose that I is an ideal over A and $f = \langle f_\xi \mid \xi \in L \rangle$ is a $<_I$-increasing sequence of functions $f_\xi \in \mathrm{On}^A$, where L is a set of ordinals. Then f is said to be *strongly increasing* if there are sets $Z_\xi \in I$, for $\xi \in L$, such that whenever $\xi_1 < \xi_2$ are in L

$$a \in A \setminus (Z_{\xi_1} \cup Z_{\xi_2}) \quad \Longrightarrow \quad f_{\xi_1}(a) < f_{\xi_2}(a).$$

2.5 Exercise. An even stronger property would be to require that there are sets $Z_\xi \in I$ for $\xi \in L$ such that whenever $\xi_1 < \xi_2$

$$a \in A \setminus Z_{\xi_2} \to f_{\xi_1}(a) < f_{\xi_2}(a).$$

Prove that a sequence $f = \langle f_\xi \mid \xi \in L \rangle$ satisfies this stronger property iff for every $\xi \in L$

$$\sup\{f_\alpha + 1 \mid \alpha \in L \cap \xi\} \leq_I f_\xi. \tag{14.5}$$

(Recall that $f + 1$ is the function that takes x to $f(x) + 1$.)

2.6 Exercise. Let I be an ideal over A, $\lambda > |A|$ be a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ be a $<_I$ increasing sequence of functions in On^A. Then the following conditions are equivalent:

1. f contains a strongly increasing subsequence of length λ.
2. f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for (I-almost) all $a \in A$.
3. f is cofinally equivalent to some $<$ (i.e. everywhere) increasing sequence of length λ.

Hint. If f (or a subsequence) is strongly increasing, let $Z_\xi \in I$ be the null sets associated with f_ξ and define

$$h(a) = \sup\{f_\xi(a) \mid a \notin Z_\xi\}.$$

Prove that h is an exact upper bound as required to prove that 1 implies 2.

Since $|A| < \lambda$, it is obvious that 2 implies 3. (For every $a \in A$ choose a cofinal subset of $h(a)$ of order-type λ, and let d_ξ be the "flat" function which assigns to $d_\xi(a)$ the ξth point in the $h(a)$ cofinal subset.)

To prove that 3 implies 1, use the following lemma.

2.7 Lemma (The Sandwich Argument). *Suppose that $d = \langle d_\xi \mid \xi \in \lambda\rangle$ is strongly increasing and $f_\xi \in \mathrm{On}^A$ is such that*

$$d_\xi <_I f_\xi \leq_I d_{\xi+1} \quad \textit{for every } \xi \in \lambda.$$

Then $\langle f_\xi \mid \xi \in \lambda\rangle$ is also strongly increasing.

Proof. Let $Z_\xi \in I$ be the sets that affirm that the sequence d is strongly increasing. For every f_ξ, sandwiched between d_ξ and $d_{\xi+1}$, there exists a $W_\xi \in I$ such that

$$d_\xi(a) < f_\xi(a) \leq d_{\xi+1}(a) \quad \text{for all } a \in A \setminus W_\xi.$$

Define $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi+1}$. Then $Z^\xi \in I$, and if $\xi_1 < \xi_2$ then for every $a \in A \setminus (Z^{\xi_1} \cup Z^{\xi_2})$

$$f_{\xi_1}(a) \leq d_{\xi_1+1}(a) \leq d_{\xi_2}(a) < f_{\xi_2}(a).$$

⊣

2.8 Definition. Suppose that I is an ideal over a set A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions $f_\xi \in \mathrm{On}^A$. For any regular cardinal κ such that $\kappa \leq \lambda$ the following crucial property of κ (and f etc.) is denoted $(*)_\kappa$:

$(*)_\kappa$ Whenever $X \subseteq \lambda$ is unbounded, then for some $X_0 \subseteq X$ of order-type κ, $\langle f_\xi \mid \xi \in X_0\rangle$ is strongly increasing.

Thus $(*)_\kappa$ is a kind of partition relation, saying that any unbounded subsequence $\langle f_\xi \mid \xi \in X\rangle$ contains a strongly increasing subsequence of length κ. Clearly $(*)_\kappa$ implies $(*)_{\kappa'}$ for all regular $\kappa' < \kappa$.

2.9 Exercise.

1. Assume $\kappa < \lambda$. Prove that $(*)_\kappa$ holds iff the set of ordinals $\delta \in \lambda$ with $\mathrm{cf}(\delta) = \kappa$ and such that $\langle f_\xi \mid \xi \in X_0\rangle$ is strongly increasing for some unbounded set $X_0 \subseteq \delta$ is stationary in λ.

2. Use the Erdős-Rado Theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ to prove that if $\lambda \geq (2^{|A|})^+$ and f is a $<_I$ increasing sequence of functions as above, of length λ, then $(*)_{|A|^+}$ holds.

Hint for 2. For $i < j$, if there exists some $a \in A$ such that $f_i(a) > f_j(a)$, then define $c(i,j) = a$ for such an a. Otherwise define $c(i,j) = -1$. The homogeneous set must be of color -1, and $(*)_{|A|^+}$ can be derived by taking a subsequence.

We shall give (in Lemma 2.19) conditions that ensure property $(*)_\kappa$ (without any assumptions on 2^κ), but meanwhile the following lemma and theorem explain the main use of $(*)_\kappa$.

2.10 Definition (Bounding projection). Suppose that I is an ideal over A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in On^A. Let $\kappa \le \lambda$ be any regular cardinal. We say that f has the *bounding projection property* for κ if whenever $S = \langle S(a) \mid a \in A \rangle$ with $S(a) \subseteq \mathrm{On}$ and $|S(a)| < \kappa$ is such that the sequence f is $<_I$-bounded by the function sup_of_S, then there exists a $\xi < \lambda$ such that $f^+_\xi = \mathrm{proj}(f_\xi, \langle S(a) \mid a \in A \rangle)$ is an upper bound of f in the $<_I$ relation. (Recall that sup_of_$S(a) = \sup S(a)$ for all $a \in A$.)

2.11 Exercise.

1. If $f = \langle f_\xi \mid \xi < \lambda \rangle$ has the bounding projection property for κ and $f' = \langle f'_\xi \mid \xi < \lambda \rangle$ is such that $f'_\xi =_I f_\xi$ for every ξ, then f' too has the bounding projection property for κ.

2. A seemingly weaker property results if we require that the sup_of_S map $<$-bounds (i.e. everywhere) each f_ξ. Prove that these two definitions are equivalent.

2.12 Lemma (The Bounding Projection Lemma). *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \le \lambda$. Then f satisfies the bounding projection property for κ.*

Later on, we shall see that $(*)_\kappa$ is, in fact, *equivalent* to the bounding projection property for κ (see Theorem 2.15 for an exact formulation).

Proof. Suppose that the lemma is false and S is a counter-example, and we shall obtain a contradiction. By changing each f_ξ on an I set, we do not spoil the $(*)_\kappa$ property, and we may assume that $f_\xi(a) < \sup S(a)$ for all $a \in A$. Then define

$$f^+_\xi = \mathrm{proj}(f_\xi, \langle S(a) \mid a \in A \rangle).$$

Since f^+_ξ is not a $<_I$-upper bound, there exists a $\xi' < \lambda$ such that $\le(f^+_\xi, f_{\xi'}) \in I^+$. That is, $f^+_\xi(a) \le f_{\xi'}(a)$ for an I-positive set of a. Hence $<(f^+_\xi, f_{\xi''}) \in I^+$ for every ξ'' above ξ'. This enables the definition of an unbounded set $X \subseteq \lambda$ such that

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi' \quad \text{then } <(f^+_\xi, f_{\xi'}) \in I^+.$$

Since $(*)_\kappa$ holds, there exists a set $X_0 \subseteq X$ of order-type κ such that $\langle f_\xi \mid \xi \in X_0 \rangle$ is strongly increasing. Let $Z_\xi \in I$ for $\xi \in X_0$ be as in the definition of strong increase (2.4).

For every $\xi \in X_0$ let $\xi' = \min(X_0 \setminus (\xi + 1))$ be the successor of ξ in X_0, and pick

$$a_\xi \in <(f^+_\xi, f_{\xi'}) \setminus (Z_\xi \cup Z_{\xi'}).$$

As $\kappa > |A|$, we may find a single $a \in A$ such that $a = a_\xi$ for a subset X_1 of X_0 of cardinality κ. Now for $\xi_1 < \xi_2$ in X_1

$$f^+_{\xi_1}(a) < f_{\xi'_1}(a) \le f_{\xi_2}(a) \le f^+_{\xi_2}(a).$$

(The first inequality is a consequence of $a_{\xi_1} \in <(f^+_{\xi_1}, f_{\xi'_1})$, the second follows from $\xi'_1 \leq \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A \setminus (Z_{\xi'_1} \cup Z_{\xi_2}),$$

and the third inequality is obvious from the definition of $f^+_{\xi_2}$.)

But now $f^+_\xi(a) \in S(a)$ turns out to be strictly increasing with $\xi \in X_1$, which is absurd since $|S(a)| < \kappa$. ⊣

2.13 Theorem (Exact Upper Bounds). *Suppose that I is an ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in On^A that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the $|A|^+$ bounding projection property for a sequence f that is $<_I$-increasing of length a regular cardinal $\lambda > |A|^+$. We shall prove first that there exists a minimal upper bound to f, and then prove that this bound is necessarily an exact upper bound. Seeking a contradiction, suppose that f has no minimal upper bound. So for every $h \in \mathrm{On}^A$, if h is an upper bound to the sequence f then it is not a minimal upper bound, and there is another upper bound $h' \in \mathrm{On}^A$ to f such that $h' \leq h$ and $<(h', h) \in I^+$.

We shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) \mid a \in A\rangle$ of sets of ordinals satisfying $|S^\alpha(a)| \leq |A|$, and such that:

1. The sequence of functions f is bounded by the map $a \mapsto \sup S^\alpha(a)$. So, the projections can always be defined.

2. The sets $S^\alpha(a)$ are increasing with α: if $\alpha < \beta$ then $S^\alpha(a) \subseteq S^\beta(a)$ for every $a \in A$. For a limit ordinal δ, $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

To define S^0, we pick a function h_0 that bounds f and define $S^0(a) = \{h_0(a)\}$.

Suppose that $S^\alpha = \langle S^\alpha(a) \mid a \in A\rangle$ has been defined. Since the bounding projection property for $|A|^+$ holds and the cardinality of $S^\alpha(a)$ is $\leq |A|$, there exists some $\xi = \xi(\alpha) < \lambda$ such that $h_\alpha = \mathrm{proj}(f_\xi, S^\alpha)$ is an upper bound of f. It follows for every ξ' satisfying $\xi \leq \xi' < \lambda$ that $h_\alpha =_I \mathrm{proj}(f_{\xi'}, S^\alpha)$.

Since h_α is not a minimal upper bound, there exists an upper bound u to the sequence f such that $u \leq h_\alpha$ and

$$<(u, h_\alpha) \in I^+.$$

Define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{u(a)\}$. Then $\mathrm{proj}(f_\xi, S^{\alpha+1}) =_I u$ for all $\xi(\alpha) \leq \xi < \lambda$.

Now let $\xi < \lambda$ be a fixed ordinal greater than every $\xi(\alpha)$ for $\alpha < |A|^+$ (recall that λ is a regular cardinal above $|A|^+$). Consider the functions $H_\alpha = \mathrm{proj}(f_\xi, S^\alpha)$ for $\alpha < |A|^+$. Since f_ξ is above $f_{\xi(\alpha)}$, $H_\alpha =_I h_\alpha$. Thus $<(H_{\alpha+1}, H_\alpha) \in I^+$. Since $\alpha_1 < \alpha_2 < |A|^+$ implies that $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$, the sequence of projections $\langle H_\alpha \mid \alpha < |A|^+\rangle$ thus obtained satisfies the following property:

$$\text{If } \alpha_1 < \alpha_2 < |A|^+, \quad \text{then } H_{\alpha_2} \leq H_{\alpha_1} \text{ and } <(H_{\alpha_2}, H_{\alpha_1}) \in I^+.$$

Yet this is impossible and leads immediately to a contradiction. For every $\alpha < |A|^+$ pick some $a \in A$ such that $H_{\alpha+1}(a) < H_\alpha(a)$. Then the same fixed $a \in A$ is picked for an unbounded set of indices $\alpha \in |A|^+$. Yet as the functions H_α are $\leq$-decreasing, this yields an infinite strictly descending sequence of ordinals! ⊣

Now that the existence of a minimal upper-bound is established, the following lemma concludes the theorem.

2.14 Lemma. *Suppose that I is an ideal over A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in* On^A *that satisfies the bounding projection property for $\kappa = 3$. Let h be a minimal upper bound of f. Then h is an exact upper bound.*

Proof. Assume that f satisfies the bounding projection property for 3, and h is a minimal upper bound of f. Suppose that $g \in \mathrm{On}^A$ is such that $g <_I h$. We must find f_ξ in the sequence f with $g <_I f_\xi$. For simplicity, and without loss of generality, we can assume that $g(a) < h(a)$ for all $a \in A$.

Define $S(a) = \{g(a), h(a)\}$ for every $a \in A$. The bounding projection property implies the existence of $\xi < \lambda$ for which $f_\xi^+ = \mathrm{proj}(f_\xi, \langle S(a) \mid a \in A\rangle)$ is an upper bound of the sequence f. We shall prove that $g <_I f_\xi$ as required. Observe that

$$f_\xi^+ =_I h \tag{14.6}$$

or else $f_\xi^+(a) = g(a) < h(a)$ for an I-positive set of a's in A. But then f_ξ^+ is an upper-bound of f that is smaller than the minimal upper bound h on an I-positive set of indices, and this is impossible. Hence (14.6). Yet, for every a such that $f_\xi^+(a) = h(a)$, $g(a) < f_\xi(a)$ follows from the fact that $g(a) \in S(a)$. Thus $g <_I f_\xi$. This proves the lemma. ⊣

The Bounding Projection Lemma 2.12 and the Exact Upper Bounds Theorem 2.13 show together that a $<_I$-increasing sequence of length a regular cardinal $\lambda > |A|^+$ and which satisfies $(*)_{|A|^+}$ has necessarily an exact upper bound h. As we shall see in the following theorem it can be deduced that

$$\forall a \in A \ \ \mathrm{cf}(h(a)) \geq |A|^+.$$

2.15 Theorem. *Suppose that I is an ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in* On^A*. Then for every regular cardinal κ such that $|A|^+ \leq \kappa \leq \lambda$ the following are equivalent.*

1. *$(*)_\kappa$ holds for f.*
2. *f satisfies the bounding projection property for κ.*

3. *The sequence f has an exact upper bound g for which*

$$\{a \in A \mid \mathrm{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let κ be a regular cardinal such that $|A|^+ \leq \kappa \leq \lambda$. Implication $1 \implies 2$ was proved in Lemma 2.12, and so we next establish $2 \implies 3$.

Since f satisfies the bounding projection property for some cardinal that is $\geq |A|^+$, it satisfies the bounding projection property for $|A|^+$. Theorem 2.13 above implies that f has an exact upper bound g. This exact upper bound is determined up to $=_I$, and we may assume that $g(a)$ is never 0 or a successor ordinal (recall that the sequence f is $<_I$-increasing).

Suppose that $P = \{a \in A \mid \mathrm{cf}(g(a)) < \kappa\} \in I^+$, in contradiction to 3. Choose, for every $a \in P$, $S(a) \subseteq g(a)$ cofinal in $g(a)$ and such that order-type$(S(a)) < \kappa$. For $a \in A \setminus P$ define $S(a) = \{g(a)\}$. Then the bounding projection property for κ gives some $\xi < \lambda$ such that the projection

$$f^+_\xi = \mathrm{proj}(f_\xi, \langle S(a) \mid a \in A\rangle)$$

is an upper bound of f in $\prod_{a\in A} S(a)$. But this is impossible since $f^+_\xi \restriction P < g\restriction P$ (everywhere on P) is in contradiction to our assumption that g is the $\leq_I$-minimal upper bound of f.

We now proceed with $3 \implies 1$. Suppose that g is an exact upper bound for f such that $\mathrm{cf}(g(a)) \geq \kappa$ for all $a \in A$ (change g on a null set if necessary). Choose $S(a) \subseteq g(a)$ cofinal in $g(a)$, closed, and with order-type $\mathrm{cf}(g(a))$. So order-type$(S(a)) \geq \kappa$. We prove that $(*)_\kappa$ holds. Assuming that $X \subseteq \lambda$ is unbounded, we shall find $X_0 \subseteq X$ of order-type κ over which f is strongly increasing. For this we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \prod_{a\in A} S(a) = \prod S$ and an index $\xi(\alpha) \in X$ such that

1. $h_\alpha <_I f_{\xi(\alpha)} <_I h_{\alpha+1}$.
2. The sequence $\langle h_\alpha \mid \alpha < \kappa\rangle$ is $<$ increasing (increasing everywhere). And hence it is certainly strongly increasing.

Then the Sandwich Argument (Lemma 2.7) will show that $\{f_{\xi(\alpha)} \mid \alpha < \kappa\}$ is strongly increasing.

The functions h_α are defined as follows.

1. $h_0 \in \prod_{a\in A} S(a)$ is any function.
2. If $\delta < \kappa$ is a limit ordinal, then define

   $$h_\delta = \sup\{h_\alpha \mid \alpha < \delta\}.$$

   That is

   $$h_\delta(a) = \bigcup\{h_\alpha(a) \mid \alpha < \delta\}$$

   for every $a \in A$. Since each $S(a)$ has regular order-type $\geq \kappa$, and as $\delta < \kappa$, clearly $h_\delta \in \prod_{a\in A} S(a)$.

3. If $h_\alpha \in \prod_{a\in A} S(a)$ is defined then it is bounded by g (since $S(a) \subseteq g(a)$) and hence (as g is an exact upper bound) $h_\alpha <_I f_\xi$ for some $\xi \in X$, which we denote $\xi(\alpha)$. Now let $f^+_{\xi(\alpha)} = \mathrm{proj}(f_{\xi(\alpha)}, S)$ be the projection function, and define $h_{\alpha+1} \in \prod S$ so that $h_{\alpha+1} > \sup\{h_\alpha, f^+_{\xi(\alpha)}\}$.

Thus $h_{\alpha+1} > h_\alpha$ and since $f_{\xi(\alpha)} \leq_I f^+_{\xi(\alpha)}$ we have

$$h_\alpha <_I f_{\xi(\alpha)} <_I h_{\alpha+1}, \quad \text{for every } \alpha. \tag{14.7}$$

Hence

$$X_0 = \{\xi(\alpha) \mid \alpha \in \kappa\} \subseteq X$$

is an increasing enumeration, and it is an evidence for $(*)_\kappa$ (by the Sandwich Argument (Lemma 2.7) and since $\langle h_\alpha \mid \alpha < \kappa\rangle$ is strongly increasing). $\dashv$

We shall give in Lemma 2.19 below a useful condition on f from which $(*)_\kappa$ follows. But first we need a combinatorial theorem.

2.16 Definition. If $S \subseteq \lambda$ is a stationary set, then a *club guessing sequence* is a sequence $\langle C_\delta \mid \delta \in S\rangle$, where each $C_\delta \subseteq \delta$ is closed unbounded in δ, such that for every closed unbounded $D \subseteq \lambda$ there exists some $\delta \in S$ with $C_\delta \subseteq D$.

We shall use the notation $S^\lambda_\kappa = \{\delta \in \lambda \mid \mathrm{cf}(\delta) = \kappa\}$. Clearly for regular infinite cardinals $\kappa < \lambda$, S^λ_κ is stationary in λ.

2.17 Theorem (Club Guessing). *For every regular cardinal κ, if λ is a cardinal such that $\mathrm{cf}(\lambda) \geq \kappa^{++}$, then any stationary set $S \subseteq S^\lambda_\kappa$ has a club-guessing sequence $\langle C_\delta \mid \delta \in S\rangle$ (such that $C_\delta \subseteq \delta$ is closed unbounded of order-type κ).*

Proof. We shall prove this for uncountable κ's, though the theorem holds for $\kappa = \aleph_0$ as well.

Let $S \subseteq S^\lambda_\kappa$ be any stationary set. Fix a sequence $C = \langle C_\delta \mid \delta \in S\rangle$ such that $C_\delta \subseteq \delta$ is closed unbounded of order type κ, for every $\delta \in S$. If $E \subseteq \lambda$ is any closed unbounded set, define

$$C|E = \langle C_\delta \cap E \mid \delta \in S \cap E'\rangle.$$

Here $E' = \{\delta \in E \mid E\cap\delta \text{ is unbounded in } \delta\}$ is the set of accumulation points of E. Clearly $E' \subseteq E$ is closed unbounded. The sequence $C|E$ is defined on $S \cap E'$ in order to ensure that $C_\delta \cap E$ is closed unbounded in δ.

We claim that for some closed unbounded set $E \subseteq \lambda$, $C|E$ is club-guessing. (The theorem demands a sequence defined on *every* $\delta \in S$, but this is trivially obtained once a guessing sequence is defined on a closed unbounded set intersected with S.)

To prove this claim, assume that it is false, and for every closed unbounded set $E \subseteq \lambda$ there is some closed unbounded set $D_E \subseteq \lambda$ not guessed by $C|E$. That is, for every $\delta \in S \cap E'$

$$C_\delta \cap E \not\subseteq D_E.$$

So we can define a decreasing (under inclusion) sequence of closed unbounded sets $E^\alpha \subseteq \lambda$ for $\alpha < \kappa^+$ by induction on α as follows.

1. $E^0 = \lambda$.

2. If $\gamma < \kappa^+$ is a limit ordinal, and E^α for $\alpha < \gamma$ are already defined, let $E^\gamma = \bigcap\{E^\alpha \mid \alpha < \gamma\}$. Clearly $E^\gamma \subseteq \lambda$ is closed unbounded.

3. If E^α is defined, then $E^{\alpha+1} = (E^\alpha \cap D_{E^\alpha})'$. So for every $\delta \in S \cap E^{\alpha+1}$, $C_\delta \cap E^\alpha \nsubseteq E^{\alpha+1}$.

Let $E = \bigcap\{E^\alpha \mid \alpha < \kappa^+\}$. Again $E \subseteq \lambda$ is closed unbounded because $\operatorname{cf}(\lambda) > \kappa^+$.

Now we get the contradiction. Take any $\delta \in S \cap E$. There exists some $\alpha < \kappa^+$ such that $C_\delta \cap E = C_\delta \cap E^\alpha$ (since the sets E^α are decreasing in $\subseteq$ and C_δ has cardinality κ). So $C_\delta \cap E^\alpha = C_\delta \cap E^{\alpha'}$ for every $\alpha' > \alpha$, and in particular for $\alpha' = \alpha + 1$. But as $\delta \in S \cap E^{\alpha+1}$, $C_\delta \cap E^\alpha \nsubseteq E^{\alpha+1}$. ⊣

2.18 Exercise.

1. Club guessing is a relative of the diamond principle which gives much stronger guessing properties. For example, prove that $\diamondsuit^+_{\omega_2}$ implies a sequence $\langle C_\delta \mid \delta \in S^{\omega_2}_{\omega_1}\rangle$ with C_δ closed unbounded in δ such that, for every closed unbounded set $E \subseteq \omega_2$, there exists a closed unbounded set $D \subseteq \omega_2$ such that for every $\delta \in S^{\omega_2}_{\omega_1} \cap D$, C_δ is almost contained in E (i.e. except a bounded set). Prove that it is not possible to have full guessing at a closed unbounded set. That is, it is not possible to require that $C_\delta \subseteq E$ for every $\delta \in S^{\omega_2}_{\omega_1} \cap D$.

2. Prove the club-guessing theorem for $\kappa = \aleph_0$ as well.

Hint. For $S \subseteq S^\lambda_{\aleph_0}$ fix $C = \langle C_\delta \mid \delta \in S\rangle$ where each C_δ is an ω-sequence unbounded in δ. For every closed unbounded set $E \subseteq \lambda$ define the "gluing to E" sequence $C|E = \langle C^*_\delta \mid \delta \in S \cap E^*\rangle$ by

$$C^*_\delta(n) = \max(E \cap (C_\delta(n) + 1)).$$

Try to prove that for some club $E \subseteq \lambda$, $C|E$ is club guessing. Have enough patience for ω_1 trials.

Club guessing is used in the following lemma which produces sequences that satisfy $(*)_\kappa$.

2.19 Lemma. *Suppose that*

1. *I is a proper ideal over A.*

2. *κ and λ are regular cardinals such that $\kappa^{++} < \lambda$.*

3. *$f = \langle f_\xi \mid \xi < \lambda\rangle$ is a sequence of length λ of functions in On^A that is $<_I$-increasing and satisfies the following requirement: For every $\delta < \lambda$ with $\operatorname{cf}(\delta) = \kappa^{++}$, there is a closed unbounded set $E_\delta \subseteq \delta$ such that for some $\delta' \geq \delta$ in λ*

$$\sup\{f_\alpha \mid \alpha \in E_\delta\} <_I f_{\delta'}. \tag{14.8}$$

Then $(*)_\kappa$ holds for f.

Proof. Let $S = S^{\kappa^{++}}_\kappa$ be the stationary subset of κ^{++} consisting of all ordinals with cofinality κ. Fix a club-guessing sequence on S: $\langle C_\alpha \mid \alpha \in S\rangle$. So for every $\alpha \in S$, $C_\alpha \subseteq \alpha$ is closed unbounded, of order-type κ, and for every closed unbounded set $C \subseteq \kappa^{++}$ there is a $\delta \in S$ such that $C_\delta \subseteq C$.

Now let $U \subseteq \lambda$ be an unbounded set, and we shall find an $X_0 \subseteq U$ of order-type κ such that $\langle f_\xi \mid \xi \in X_0\rangle$ is strongly increasing. For this we first define an increasing and continuous sequence $\langle \xi(i) \mid i < \kappa^{++}\rangle \subseteq \lambda$ of order-type κ^{++} by the following recursive procedure.

We start with an arbitrary $\xi(0)$. For i limit, $\xi(i) = \sup\{\xi(k) \mid k < i\}$.

Suppose for some $i < \kappa^{++}$ that $\{\xi(k) \mid k \leq i\}$ has been defined.

For every $\alpha \in S$ define

$$h_\alpha = \sup\{f_{\xi(k)} \mid k \leq i \wedge k \in C_\alpha\}. \tag{14.9}$$

Then ask: is there an ordinal $\sigma > \xi(i)$ below λ such that $h_\alpha <_I f_\sigma$? If the answer is positive, let σ_α be the least such $\sigma < \lambda$, and, if negative, let σ_α be $\xi(i) + 1$.

Since $\lambda > \kappa^{++}$ is regular, we can define

$$\xi(i+1) > \sup\{\sigma_\alpha \mid \alpha \in S\} \quad \text{with } \xi(i+1) \in U.$$

It follows, in case the answer for h_α is positive, that

$$h_\alpha <_I f_{\xi(i+1)}.$$

Finally $D = \{\xi(k) \mid k \in \kappa^{++}\}$ is closed and has order-type κ^{++}. Let $\delta = \sup D$. Then D is closed unbounded in $\delta < \lambda$, and $\mathrm{cf}(\delta) = \kappa^{++}$. By assumption there is a closed unbounded set $E_\delta \subseteq \delta$ such that (14.8) holds. Thus for some $f_{\delta'}$

$$\sup\{f_\xi \mid \xi \in E_\delta\} <_I f_{\delta'}. \tag{14.10}$$

Observe that $D \cap E_\delta$ is closed unbounded in δ, and thus $C = \{i \in \kappa^{++} \mid \xi(i) \in E_\delta\}$ is closed unbounded. Hence for some $\alpha \in S$, $C_\alpha \subseteq C$. So (14.10) implies that

$$\sup\{f_{\xi(i)} \mid i \in C_\alpha\} <_I f_{\delta'}. \tag{14.11}$$

Let $N_\alpha \subseteq C_\alpha$ be the set of non-accumulation points of C_α, that is those $i \in C_\alpha$ for which $C_\alpha \cap i$ is bounded in i. We shall prove that $\{f_{\xi(i)} \mid i \in N_\alpha\}$ is strongly increasing. Since $\xi(i+1) \in U$ for every i, the sandwich argument (2.7) gives a strongly increasing subsequence of $\{f_\alpha \mid \alpha \in U\}$ of order-type κ.

Claim. *For every $i < j$ both in C_α*

$$\sup\{f_{\xi(k)} \mid k \leq i \wedge k \in C_\alpha\} <_I f_{\xi(j)}. \tag{14.12}$$

Proof of Claim. Recall how $f_{\xi(i+1)}$ was defined. We considered (14.9) and asked if h_α is $<_I$ dominated by some f_σ. The answer was positive, since $f_{\delta'}$ is such a bound. Hence the claim and the lemma follow. $\dashv$

2.20 Exercise. Let κ and λ be regular cardinals with $\kappa^{++} < \lambda$, and let F be any function with $\mathrm{dom}(F) \subseteq [\lambda]^{<\kappa}$ and such that $F(X) \in \lambda$ for $X \in \mathrm{dom}(F)$. Suppose that for every $\delta \in S^\lambda_{\kappa^{++}}$ there exists a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dom}(F)$. Then the following set S is stationary: the set of all ordinal $\alpha \in S^\lambda_\kappa$ for which there exists a closed unbounded set $D \subseteq \alpha$ with the property that, for any $a < b$ both in D, $F(\{d \in D \mid d \leq a\}) < b$.

A typical application of Lemma 2.19 is the following.

2.21 Theorem. *Suppose that I is a proper ideal over a set of regular cardinals A, and λ is a regular cardinal such that $\prod A/I$ is λ-directed. If $\langle g_\xi \mid \xi < \lambda\rangle$ is any sequence in $\prod A$, then there exists a $<_I$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ of length λ in $\prod A/I$, such that $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$ and $(*)_\kappa$ holds for f for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Hence if $\kappa = |A|^+$ is such a cardinal, then by Theorem 2.15 and the fact that $(*)_\kappa$ holds, we have an exact upper bound g to the sequence f so that $\{a \in A \mid \mathrm{cf}(g(a)) < \kappa\} \in I$.*

Proof. We shall define a $<_I$-increasing sequence $\langle f_\xi \mid \xi < \lambda\rangle$ in $\prod A/I$ as follows. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in $\prod A$ that $<$-extends f_ξ and g_ξ.

1. At limit stages $\delta < \lambda$ there are two cases. In the first $\mathrm{cf}(\delta) = \kappa^{++}$ where κ is regular and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Then fix some $E_\delta \subseteq \delta$ closed unbounded and of order-type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i \mid i \in E_\delta\}.$$

   Then $f_\delta(a) < a$ when $a > \kappa^{++}$, and thus $f_\delta \in \prod A/I$ since $\{a \in A \mid a \leq \kappa^{++}\} \in I$.

2. If $\delta < \lambda$, but case 1 above does not hold, let $f_\delta \in \prod A$ be any $\leq_I$ upper bound of $\langle f_\xi \mid \xi < \delta\rangle$ guaranteed by the λ-directedness assumption.

Now Lemma 2.19 implies that $(*)_\kappa$ holds for every regular cardinal κ of the required form. $\dashv$

In the following, we shall apply Lemma 2.19 (or rather its consequence Theorem 2.21 above) and obtain an important representation of successors of singular cardinals with uncountable cofinality. But first we introduce a notation.

2.22 Notation. Let X be a set of cardinals, then

$$X^{(+)} = \{\alpha^+ \mid \alpha \in X\}$$

denotes the set of successors of cardinals in X.

2.23 Theorem (Representation of μ^+ as True Cofinality). *Suppose that μ is a singular cardinal with uncountable cofinality. Then there exists a closed unbounded set $C \subseteq \mu$ such that*

$$\mu^+ = \mathrm{tcf}\big(\prod C^{(+)}/J^{bd}\big)$$

where J^{bd} is the ideal of bounded subsets of $C^{(+)}$.

Proof. Let $C_0 \subseteq \mu$ be any closed unbounded set of limit cardinals such that $|C_0| = \mathrm{cf}(\mu)$ and all cardinals in C_0 are above $\mathrm{cf}(\mu)$. Clearly all cardinals in C_0 that are limit points of C_0 are singular cardinals, and hence we can assume that C_0 consists only of singular cardinals.

Observe that $\prod C_0^{(+)}/J^{bd}$ is μ-directed, and in fact is μ^+-directed since μ is a singular cardinal. Indeed, suppose that $F \subseteq \prod C_0^{(+)}$ has cardinality $< \mu$ and define $h(a)$ by $h(a) = \sup\{f(a) \mid f \in F\}$ for every $a \in C_0^{(+)}$ above $|F|$ (so that $h(a) \in a$), and $h(a)$ is arbitrarily defined on smaller a's. This proves that every subset of $\prod C_0^{(+)}$ of cardinality $< \mu$ is bounded in $<_{J^{bd}}$. But then it follows that subsets of $\prod C_0^{(+)}$ of cardinality μ are also bounded: decompose any such subset $F = \bigcup_{\alpha<\mathrm{cf}(\mu)} F_\alpha$ where each F_α has cardinality $< \mu$, then bound each F_α, and finally bound the sequence of bounds.

Thus $\prod C_0^{(+)}/J^{bd}$ is μ^+-directed and we may construct a J^{bd} increasing sequence $f = \langle f_\xi \mid \xi < \mu^+\rangle$ in $\prod C_0^{(+)}$ such that $(*)_\kappa$ holds for every regular cardinal $\kappa < \mu$ (apply Theorem 2.21 in its simpler form in which there is no need to extend a given sequence g).

Theorem 2.15 implies that f has an exact upper bound $h : C_0^{(+)} \to \mathrm{On}$ such that

$$\{a \in C_0^{(+)} \mid \mathrm{cf}(h(a)) < \kappa\} \in J^{bd} \tag{14.13}$$

for every regular $\kappa < \mu$. We may assume that $h(a) \le a$ for every $a \in C_0^{(+)}$, since the identity function is clearly an upper bound to f.

2.24 Claim. *The set $\{\alpha \in C_0 \mid h(\alpha^+) = \alpha^+\}$ contains a closed unbounded set.*

Proof of Claim. Suppose toward a contradiction that for some stationary set $S \subseteq C_0$, $h(\alpha^+) < \alpha^+$ for every $\alpha \in S$. Since all cardinals of C_0 are singular, $\mathrm{cf}(h(\alpha^+)) < \alpha$ for every $\alpha \in S$. Hence (by Fodor's theorem) $\mathrm{cf}(h(\alpha^+))$ is bounded by some $\kappa < \mu$ on a stationary subset of α in S. But this is in contradiction to (14.13) above.

Thus we have proved the existence of a closed unbounded set $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for every $\alpha \in C$. We claim that $\mu^+ = \mathrm{tcf}(\prod C^{(+)}/J^{bd})$. But this is clear since $h\restriction C^{(+)}$, which is the identity function, is an exact upper bound to the sequence $\langle f_\xi\restriction C^{(+)} \mid \xi < \mu^+\rangle$ which is J^{bd} increasing and of length μ^+. This ends the proof of the claim and Theorem 2.23. ⊣

A somewhat stronger form of this theorem is in Exercise 4.17.

2.25 Exercise. Prove the following representation theorem for μ^+ in case $\mathrm{cf}(\mu) = \aleph_0$.

2.26 Theorem. *If μ is a singular cardinal of countable cofinality then for some unbounded set $D \subseteq \mu$ (of order-type ω) of regular cardinals*

$$\mu^+ = \mathrm{tcf}\big(\textstyle\prod D/J^{bd}\big)$$

where J^{bd} is the ideal of bounded subsets of D. For example, there exists a set $B \subseteq \{\aleph_n \mid n < \omega\}$ such that $\mathrm{tcf}(\prod B/J^{bd}) = \aleph_{\omega+1}$.

Hint. Let C_0 be any ω sequence converging to μ, consisting of regular cardinals. Repeat the proof above and define $D = \{\mathrm{cf}(h(a)) \mid a \in C_0\}$. Then use Lemma 2.3.

The theory of exact upper bounds, which is the basis of pcf theory, can be developed in various ways. For example [10] presents Shelah's Trichotomy Theorem, and extends it to further analyze the set of flat points. Suitably interpreted, Theorem 2.15 is equivalent to Theorem 18 of [10]. The following exercise establishes the connection between the trichotomy and the bounding projection property.

2.27 Exercise (The Trichotomy Theorem). Suppose that $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a $<_I$ increasing sequence. Consider the following properties of f and a regular cardinal κ such that $|A| < \kappa \leq \lambda$:

Bad$_\kappa$ There are sets of ordinals $S(a)$ for $a \in A$ such that $|S(a)| < \kappa$ and sup_of_S $<_I$-dominates f, and there is an ultrafilter D over A, extending the dual of I, such that for every $\alpha < \lambda$, $f^+_\alpha <_D f_\beta$ for some $\beta < \lambda$ (where $f^+_\alpha = \mathrm{proj}(f_\alpha, S)$).

Ugly There exists a function $g \in \mathrm{On}^A$ such that, forming $t_\alpha = \{a \in A \mid g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha \mid \alpha < \lambda\rangle$ which we know to be $\subseteq_I$ increasing does not stabilize modulo I. That is, for every α there is some $\beta > \alpha$ in λ such that $t_\beta \setminus t_\alpha \in I^+$.

Good$_\kappa$ There exists an exact upper bound g to the sequence f such that $\mathrm{cf}(g(a)) \geq \kappa$ for every $a \in A$.

Prove that the bounding projection property for κ is equivalent to $\neg$**Bad**$_\kappa$ $\wedge$ $\neg$**Ugly**. Hence the Trichotomy Theorem which says that if neither **Bad**$_\kappa$ nor **Ugly**, then **Good**$_\kappa$.

2.28 Exercise ([15] Lemma 0.D, Chap. V). If λ is a regular cardinal with $\forall\mu < \lambda\ \mu^{|A|} < \lambda$ and $f_\alpha \in \mathrm{On}^A$ for $\alpha < \lambda$, then for some unbounded $E \subseteq \lambda$, for all $\alpha < \beta$ both in E, $f_\alpha \leq f_\beta$ and $\{a \in A \mid f_\alpha(a) = f_\beta(a)\}$ does not depend on α, β in E.

Hence, if I is an ideal over A and $f_\alpha <_I f_\beta$ for all $\alpha < \beta$, then $(*)_\lambda$ holds.

Hint. For $a \in A$ fix some $\gamma_a > \sup\{f_\alpha(a) \mid \alpha \in \lambda\}$. For $\alpha < \lambda$, $a \in A$, define $s^\alpha(a) = \{f_\beta(a) \mid \beta < \alpha\} \cup \{\gamma_a\}$. Define $g_\alpha = \mathrm{proj}(f_\alpha, s^\alpha)$. Let $T = \{\delta < \lambda \mid \mathrm{cf}(\delta) = |A|^+\}$. For $\alpha \in T$ there exists a $\mu_\alpha < \alpha$ such that $g_\alpha = \mathrm{proj}(f_\alpha, s^{\mu_\alpha})$. By Fodor's theorem we may assume $\mu = \mu_\alpha$ is fixed on a stationary set $T' \subseteq T$. Moreover, since s^μ has cardinality $|\mu|$ and $|\mu|^{|A|} < \lambda$, we may assume that $g_\alpha = g$ is fixed for $\alpha \in T'' \subseteq T'$, stationary.

2.2. Application: Silver's Theorem

One form of Silver's Theorem says that if κ is a singular cardinal of uncountable cofinality such that $2^\delta = \delta^+$ for a stationary set of δ's in κ, then $2^\kappa = \kappa^+$. A slightly more general form is the following

2.29 Theorem (Silver [18]). *Let κ be a singular cardinal with uncountable cofinality: $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$. Suppose that there exists a stationary set of cardinals $S \subseteq \kappa$ such that, for every $\delta \in S$, $\delta^{\mathrm{cf}(\kappa)} = \delta^+$. Then*

$$\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

as well.

Proof. Assume that $S \subseteq \kappa$, of order-type $\mathrm{cf}(\kappa)$, is a stationary set of cardinals such that for every $\delta \in S$

$$\delta^{\mathrm{cf}(\kappa)} = \delta^+.$$

We have established the existence of a closed unbounded subset $C \subseteq \kappa$ with $\kappa^+ = \mathrm{tcf}(\prod C^{(+)}/J^{bd})$. So, by taking $S \cap C$ for S, we may conclude that $\prod S^{(+)}/J_{bd}$ has true cofinality κ^+ and let $f = \langle f_\xi \mid \xi \in \kappa^+\rangle$ be J^{bd} increasing and cofinal there.

Since $\lambda^{\mathrm{cf}(\kappa)} = \lambda^+$ for all $\lambda \in S$, there exists an encoding of all pairs $\langle \lambda, X\rangle$ where $X \in [\lambda]^{\mathrm{cf}(\kappa)}$ by ordinals in λ^+. Hence we can encode each $X \in [\kappa]^{\mathrm{cf}(\kappa)}$ by a function $h_X \in \prod S^{(+)}$, where $h_X(\lambda^+)$ gives the code of $X \cap \lambda$. Thus, if $X \neq Y$ then h_X and h_Y are eventually disjoint. Since each h_X is J^{bd} dominated by some f_ξ for $\xi \in \kappa^+$, the following lemma concludes the proof of our theorem.

2.30 Lemma. *For every function $g \in \prod S^{(+)}$, the collection*

$$F = \{X \in [\kappa]^{\mathrm{cf}(\kappa)} \mid h_X <_{J^{bd}} g\}$$

has cardinality $\leq \kappa$.

Proof. Suppose that, on the contrary, $|F| \geq \kappa^+$. For each $\delta \in S$ fix an enumeration of $g(\delta^+) \in \delta^+$ that has order-type $\leq \delta$. Using this enumeration, $h_X(\delta^+)$ is "viewed" as an ordinal in δ, denoted $k_X(\delta)$ whenever $h_X(\delta^+) < g(\delta^+)$. Thus for every $X \in F$, h_X is translated into a pressing down function defined on a final segment of S.

By Fodor's theorem, for some stationary set $S_X \subseteq S$, k_X is bounded on S_X, say by $\delta_X < \kappa$. Now the number of subsets of S is bounded by $2^{\mathrm{cf}(\kappa)} < \kappa$, and hence there exists a subset $F_0 \subseteq F$ of cardinality κ^+, a fixed stationary set S_0, and a fixed cardinal $\delta_0 \in S$ such that $S_X = S_0$ and $\delta_X = \delta_0$ for every X in F_0. Moreover the translation function taking $\delta \in S_0$ to that ordinal in δ_0 that indirectly encodes $X \cap \delta$ can also be assumed to be independent of $X \in F_0$, since there are at most $\delta_0^{\mathrm{cf}(\kappa)} = \delta_0^+$ such functions. Yet this is absurd because the translation function of h_X completely determines $X = \bigcup\{X \cap \delta \mid \delta \in S_0\}$. This is a contradiction which proves the lemma and the theorem. $\dashv$

2.31 Exercise. Show that the following form of Silver's Theorem is equivalent to Theorem 2.29 (cf. [8]). Let κ be a singular cardinal with uncountable cofinality: $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$. Suppose that $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$, and there exists a stationary set of cardinals $S \subseteq \kappa$ such that, for every $\delta \in S$, $\delta^{\mathrm{cf}(\delta)} = \delta^+$. Then

$$\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

as well.

2.32 Exercise. The proof given by Baumgartner and Prikry (in [1]) to Silver's Theorem simplifies the original proof, and is actually simpler than the proof given here which serves to illustrate some of the pcf concepts. In addition the Baumgartner-Prikry proof relies on very elementary notions. The following exercises describes that proof. Assume that κ is a singular cardinal with uncountable cofinality.

1. If $S \subseteq \kappa$ is a stationary set such that $\delta^{\mathrm{cf}(\kappa)} = \delta^+$ for $\delta \in S$, define on $\prod S^{(+)}$ a relation R by $f\ R\ g$ iff $\{\alpha \in S \mid f(\alpha^+) < g(\alpha^+)\}$ is stationary. Prove that for every g the cardinality of $R^{-1}g = \{X \in \kappa^{\mathrm{cf}(\kappa)} \mid h_X\ R\ g\}$ is $\leq \kappa$.

2. Prove that for every $f, g \in \prod S^{(+)}$ that are eventually different either $f\ R\ g$ or $g\ R\ f$. Take any collection $X_i \in [\kappa]^{\mathrm{cf}(\kappa)}$, $i < \kappa^+$, of different subsets and consider $H = \bigcup_{i \in \kappa^+} R^{-1} h_{X_i}$. If $\kappa^{\mathrm{cf}(\kappa)} > \kappa^+$ there must be some $g \notin H$, and hence $h_{X_i}\ R\ g$ for every $i < \kappa^+$. This is a contradiction.

2.3. Application: A Covering Theorem

In this subsection V denotes the universe of all sets, and U a transitive subclass containing all ordinals and satisfying the axioms of ZFC. (See, for example, Levy [12] for the meaning of statements concerning classes.) If X and Y are sets of ordinals in V and U (respectively) and $X \subseteq Y$, then we say that Y *covers* X.

The *countable covering property* of U (or between U and V) is the statement that any countable set of ordinals X is covered by some countable set of ordinals Y in U (that is, Y is in U, and Y is countable in V). Similarly, for any cardinal κ, the $\leq \kappa$ covering property is that any set of cardinals X of cardinality $\leq \kappa$ is covered by some set in U that has cardinality $\leq \kappa$ in V.

If the $\leq \kappa$ covering property holds for every cardinal κ, then we say that the *full covering property* holds for U: every set of ordinals X is covered by some set Y in U such that X and Y are equinumerous in V. The following theorem gives conditions by which the full covering property can be deduced from the countable covering property.

2.33 Theorem (Magidor). *Suppose U is a transitive class containing all ordinals and satisfying all the* ZFC *axioms. Moreover, assume that*

1. GCH *holds in U,*
2. *U and the universe V have the same cardinals, and moreover every regular cardinal in U remains regular in V.*

Then the countable covering property for U implies the full covering property.

Proof. Observe first that if U and V have the same regular cardinals, they have the same cardinals and $\mathrm{cf}^U(\kappa) = \mathrm{cf}^V(\kappa)$ for every ordinal. Also, for every set $X \in U$, $|X|^U = |X|^V$. We prove by induction on $\lambda \in \mathrm{On}$ that every $X \subseteq \lambda$ is covered by some Y in U of the same V cardinality. Of course if X is bounded in λ then the inductive assumption applies, and hence we can consider only sets that are unbounded in λ.

If λ is not a cardinal, let $|\lambda|$ be its cardinality. So $|\lambda| < \lambda < \lambda^+$ in U as well since V and U have the same cardinals. Since λ and $|\lambda|$ are equinumerous in U, the inductive assumption for $|\lambda|$ implies that any subset of λ can be covered by a set in U of the same cardinality.

So we assume that λ is a cardinal. If it is a regular cardinal, then any unbounded $X \subseteq \lambda$ is covered by λ itself. Hence we are left with the case that λ is a singular cardinal, in V and hence in U since both universes have the same regular/singular predicate. Again, if $X \subseteq \lambda$ has cardinality λ then λ itself is a covering as required, and hence we may assume that $|X| < \lambda$.

Assume first that $\mathrm{cf}(\lambda) = \omega$. Then $\mathrm{cf}^U(\lambda) = \omega$ as well. Suppose that an unbounded set $X \subseteq \lambda$ of cardinality $< \lambda$ is given. Take in U an increasing cofinal in λ sequence $\langle \lambda_i \mid i \in \omega \rangle$. Since $2^{<\lambda} = \lambda$ is assumed in U, there exists in U an enumeration of length λ of all *bounded* subsets of λ of cardinality $\leq |X|$. Now consider the sequence $X \cap \lambda_i$, $i \in \omega$ (where X is the set to be covered) and cover first each $X \cap \lambda_i$ by some $Y_i \in U$ with $|Y_i| = |X \cap \lambda_i|$. Then define $\alpha_i \in \lambda$ to be the ordinal that encodes Y_i in U, and form the countable set $A = \{\alpha_i \mid i \in \omega\}$ of ordinals that encode X. This countable set A can be covered by some countable set A' in U, and we can define in U a cover

$$Y = \bigcup \{E \subseteq \lambda_i \mid \ E \text{ is encoded by some ordinal in } A' \text{ and } |E| \leq |X|\}.$$

Clearly $X \subseteq Y$ and $|Y| = |X|$.

Finally suppose that λ is a singular cardinal with uncountable cofinality, and it is here that the theory developed so far is employed. Since V and U have the same regular cardinals, $\mathrm{cf}^U(\lambda) = \mathrm{cf}(\lambda)$.

We work for a while in U and apply the Representation Theorem 2.23 to λ. In fact, we must analyze the proof and use the construction rather than the theorem. Recall that we took an arbitrary closed unbounded set $C_0 \subseteq \lambda$ consisting of singular cardinals such that $|C_0| = \mathrm{cf}(\lambda) < \min C_0$. Then we constructed a J^{bd} increasing sequence $f = \langle f_\xi \mid \xi < \lambda^+ \rangle$ in $\prod C_0^{(+)}$ such that for all limit ordinals $\delta < \lambda^+$ a closed unbounded set $E_\delta \subseteq \delta$ was chosen with $|E_\delta| = \mathrm{cf}(\delta) < \lambda$ and then

$$f_\delta =_{J^{bd}} \sup\{f_i \mid i \in E_\delta\}$$

was defined. All of this is done in U, but now we pass to V and deduce that $(*)_\kappa$ holds for every regular $\kappa < \lambda$ (by Lemma 2.19). Hence f has an exact upper bound h such that $\{a \in C_0^{(+)} \mid \mathrm{cf}(h(a)) < \kappa\} \in J^{bd}$ for every $\kappa < \lambda$. Now the argument of Claim 2.24 applies, and there exists (in V) a closed unbounded set $C \subseteq C_0$ such that $\{f_\xi \restriction C^{(+)} \mid \xi < \lambda^+\}$ is cofinal in $\prod C^{(+)}/J^{bd}$.

We continue now the proof that any set $X \subseteq \lambda$ of cardinality $\lambda_0 < \lambda$ can be covered in U by a set of the same cardinality. Since X is unbounded in λ, $\lambda_0 \geq \mathrm{cf}(\lambda)$. For every $\alpha \in C_0$ cover $X \cap \alpha$ by some $Y_\alpha \in U$ (a subset of α) of cardinality $\leq \lambda_0$. We assume in U an enumeration of length α^+ of all subsets of α of size $\leq \lambda_0$. There is an index $< \alpha^+$ that encodes Y_α in U. The function d taking $\alpha^+ \in C_0^{(+)}$ to that coding ordinal is defined in V and is bounded by some $f_\xi \in U$. Namely $d \restriction C^{(+)} <_{J^{bd}} f_\xi \restriction C^{(+)}$. In U, choose for every $\alpha \in C_0$ a function $g_\alpha : f_\xi(\alpha^+) \to \alpha$ that is one-to-one. Then in V look at the values $g_\alpha(d(\alpha^+)) < \alpha$, and find a stationary set $S \subseteq C$ on which these values are bounded, say by κ. The set $\{g_\alpha(d(\alpha)) \mid \alpha \in S\} \subseteq \kappa$ can be covered by some set Y in U that has the same cardinality (namely $\mathrm{cf}(\lambda)$). Now look in U at the set $\bigcup_{\alpha \in C_0} g_\alpha^{-1} Y$. Every index in $g_\alpha^{-1} Y$ represents a subset of α of cardinality $\leq \lambda_0$, and hence this yields a cover of X of cardinality λ_0. $\dashv$

2.34 Exercise. There is actually no need to start with countable covering in order to deduce covering for all higher cardinals. The following generalization is left as an exercise.

2.35 Theorem. *Assume as in Theorem 2.33 that $U \subseteq V$ have the same regular cardinals, and* GCH *holds in U. Let λ_0 be any cardinal such that every countable set of ordinals is covered by some set in U of cardinality $\leq \lambda_0$. Then any set of ordinals X is covered by some set in U of cardinality $|X| + \lambda_0$.*

3. Basic Properties of the pcf Function

For any set A of regular uncountable cardinals define

$$\mathrm{pcf}(A) = \{\lambda \mid \text{for some ultrafilter } U \text{ over } A,\ \lambda = \mathrm{cf}(\textstyle\prod A/U)\}.$$

Some easily verifiable properties:

1. If $\lambda = \mathrm{tcf}(\prod A/F)$ for some filter F over A, then $\lambda \in \mathrm{pcf}(A)$. (For any ultrafilter U that extends F, $\lambda = \mathrm{tcf}(\prod A/U)$.)

2. $A \subseteq \mathrm{pcf}(A)$. For every $a \in A$ we can take the principal ultrafilter over A concentrating on $\{a\}$.

3. $A \subseteq B$ implies $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(B)$. Because every ultrafilter D over A can be extended to D' over B, and the ultraproducts $\prod A/D$ and $\prod B/D'$ are the same.

4. For any sets A and B, $\mathrm{pcf}(A \cup B) = \mathrm{pcf}(A) \cup \mathrm{pcf}(B)$. Indeed, if $\lambda \in \mathrm{pcf}(A \cup B)$, and D is an ultrafilter over $A \cup B$ with ultraproduct of cofinality λ, then either $A \in D$ or $B \in D$ (or both) and hence $\lambda \in \mathrm{pcf}(A)$ or $\lambda \in \mathrm{pcf}(B)$. For the other direction use the previous item.

We say that A is an *interval of regular cardinals* if for some cardinals $\alpha < \beta$, A is the set of all regular cardinals κ such that $\alpha \leq \kappa < \beta$. This term is slightly misleading because one may misinterpret it as saying that all cardinals between α and β are regular.

3.1 Theorem (The "No Holes" Argument). *Assume that A is an interval of regular cardinals satisfying $|A| < \min A$, and λ is a regular cardinal with $\sup A < \lambda$. Let I be a proper ideal over A such that $\prod A/I$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that every proper initial segment of A is in I (or else substitute for A its first initial segment that is not in I). It now follows that A is infinite and unbounded (without a maximum).

Theorem 2.21 gives an $<_I$-increasing sequence $f = \langle f_\xi \mid \xi \in \lambda\rangle$ in $\prod A/I$ that satisfies $(*)_\kappa$ for every regular cardinal κ in A (and thus for smaller cardinals of course). In particular $(*)_{|A|^+}$ holds, and f has an exact upper bound $h \in \mathrm{On}^A$ such that

$$\{a \in A \mid \mathrm{cf}(h(a)) < \kappa\} \in I \tag{14.14}$$

for every $\kappa \in A$ (this by Theorem 2.15). Since the identity function $\mathrm{id} : A \to A$ taking a to a is clearly an upper bound of f, $h(a) \leq a$ for I-almost all $a \in A$. Yet (14.14) implies that

$$\{a \in A \mid \mathrm{cf}(h(a)) < \min A\} \in I,$$

and hence we have $\min(A) \leq \mathrm{cf}(h(a)) \leq a$ for I-almost all $a \in A$. Changing h on a null set, we may assume for simplicity that this holds for every $a \in A$, namely that

$$\mathrm{cf}(h(a)) \in A \quad \text{for all } a \in A$$

(as A is an interval of regular cardinals). Since the sequence f has length λ, $\prod h/I$ has true cofinality λ. Consequently $\prod_{a \in A} \mathrm{cf}(h(a))/I$ has true cofinality λ as well. Since $|A| < \min A$, Lemma 2.3 gives a proper ideal J on

$B = \{\mathrm{cf}(h(a)) \mid a \in A\} \subseteq A$, such that $\prod B/J$ has true cofinality λ as well. So $\lambda \in \mathrm{pcf}(A)$. We note in addition that J is the Rudin-Keisler projection obtained via $\mathrm{cf} \circ h$, and hence (14.14) implies for every $\kappa < \sup A$ that $B \cap \kappa \in J$. $\dashv$

Upon examination of the proof, the reader will notice that the following slightly stronger formulation of the theorem can be obtained. In this formulation the requirement that A is an interval is relaxed.

3.2 Theorem. *Assume that A is a set of regular cardinals such that $|A| < \min A$, and λ is a regular cardinal such that $\sup A < \lambda$. Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $\prod A/I$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A')$ for some set A' of regular cardinals such that*

1. *$A' \subseteq [\min A, \sup A)$, and A' is cofinal in $\sup A$.*
2. *$|A'| \leq |A|$.*

In fact, λ is the true cofinality of $\prod A'/J$ for an ideal J over A' that contains all bounded subsets of A'.

Proof. Follow the previous proof and let A' be the set $\{\mathrm{cf}(h(a)) \mid a \in A\}$. $\dashv$

3.3 Notation. The property $|A| < \min A$ assumed for the set of regular cardinals appearing in the theorem is so pervasive in the pcf theory that it ought to be given a name. Following [6] we say that a set of regular cardinals A is *progressive* if $|A| < \min A$.

3.1. The Ideal $J_{<\lambda}$

Let A be a set of regular cardinals. For any cardinal λ define

$$J_{<\lambda}[A] = \{X \subseteq A \mid \mathrm{pcf}(X) \subseteq \lambda\}.$$

In plain words, $X \in J_{<\lambda}[A]$ iff for every ultrafilter D over A such that $X \in D$, $\mathrm{cf}(\prod A/D) < \lambda$. That is, X "forces" the cofinalities of its ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal over A, but it is not necessarily a proper ideal since $A \in J_{<\lambda}[A]$ is possible. However, if $\lambda \in \mathrm{pcf}(A)$, then $J_{<\lambda}[A]$ *is* proper ($A \notin J_{<\lambda}[A]$, or else $\mathrm{pcf}(A) \subseteq \lambda$ shows that $\lambda \notin \mathrm{pcf}(A)$). When the identity of A is obvious from the context, we write $J_{<\lambda}$ instead of $J_{<\lambda}[A]$. Note that if $A \subseteq B$ then $J_{<\lambda}[A] = J_{<\lambda}[B] \cap \mathcal{P}(A)$.

Let $J^*_{<\lambda}[A]$ be the dual filter over A. Then

$$J^*_{<\lambda}[A] = \bigcap\bigl\{D \mid D \text{ is an ultrafilter and } \mathrm{cf}(\textstyle\prod A/D) \geq \lambda\bigr\}.$$

3.4 Theorem (λ-directedness). *Suppose that A is a progressive set of regular cardinals. Then for every cardinal λ, $\prod A/J_{<\lambda}[A]$ is λ-directed: any set of fewer than λ functions is bounded in $\prod A/J_{<\lambda}[A]$.*

Proof. The theorem holds trivially if $A \in J_{<\lambda}[A]$, since $|\prod A/J_{<\lambda}| = 1$ in this case. So we assume that $J_{<\lambda}$ is a proper ideal over A. Let $\kappa_0 = \min A$ be the first cardinal of A, and κ_1, κ_2 be the second, third etc. cardinals of A. The case $\lambda \le \kappa_n$ for n finite is quite obvious: if $\lambda = \kappa_n$ then $J_{<\lambda} = \mathcal{P}(\{\kappa_0, \dots, \kappa_{n-1}\})$ and for every family $F \subseteq \prod A$ of cardinality $< \lambda$, $\sup F \in \prod A$, because $(\sup F)(a) = \bigcup\{f(a) \mid f \in F\} < a$, since $|F| < \lambda \le a$ for every $a \notin \{\kappa_0, \dots, \kappa_{n-1}\}$). So we can certainly assume that $\lambda > \kappa_n$ for all $n \in \omega$, and hence that $\{\kappa_n\} \in J_{<\lambda}$.

Since any null subset of A can be removed without changing the structure of $\prod A/J_{<\lambda}$, we may assume that $|A|^+, |A|^{++}, |A|^{+3} \notin A$. That is we can assume that

$$|A|^{+3} < \min A < \lambda.$$

We shall prove by induction on $\lambda_0 < \lambda$ that $\prod A/J_{<\lambda}$ is λ_0^+-directed: for every $F = \{f_i \mid i \in \lambda_0\} \subseteq \prod A$ a family of functions of cardinality λ_0, F has an upper bound in $\prod A/J_{<\lambda}$. The case $\lambda_0 < \min A$ is obvious as we saw.

So let $F = \{f_i \mid i \in \lambda_0\} \subseteq \prod A$ be a subset of $\prod A$ where $\lambda_0 < \lambda$ and assume that $\prod A/J_{<\lambda}$ is λ_0-directed. Our aim is to bound F in $\prod A/J_{<\lambda}$.

In case λ_0 is singular, we take $\langle \alpha_i \mid i < \mathrm{cf}(\lambda_0)\rangle$ increasing and cofinal in λ_0, and obtain $g_i \in \prod A$ for every $i < \mathrm{cf}(\lambda_0)$ that bounds $\{f_\xi \mid \xi < \alpha_i\}$. Then we apply the inductive assumption again to the sequence $\{g_i \mid i < \mathrm{cf}(\lambda_0)\}$, and obtain a bound to F.

Thus λ_0 is assumed to be a regular cardinal above $|A|^{+3}$. We shall replace F by a $<_{J_{<\lambda}}$-increasing sequence that satisfies $(*)_\kappa$ for $\kappa = |A|^+$. That is, using Theorem 2.21 we define a $<_{J_{<\lambda}}$-increasing sequence $\langle f'_\xi \mid \xi < \lambda_0\rangle$ satisfying $(*)_\kappa$ and such that $f_i \le f'_i$.

Hence we can assume that the sequence $f = \langle f_i \mid i < \lambda_0\rangle$ that we want to dominate satisfies $(*)_{|A|^+}$ and thus has an exact upper bound $g \in \mathrm{On}^A$ in $<_{J_{<\lambda}[A]}$ (by Theorem 2.15).

Since the identity function taking $a \in A$ to a is an upper bound of our sequence f, we may assume that $g(a) \le a$ for all $a \in A$ (by possibly changing g on a null set). We intend to prove that $B = \{a \in A \mid g(a) = a\} \in J_{<\lambda}[A]$, and thus that $g =_{J_{<\lambda}} g'$ for some $g' \in \prod A$ which will show that g bounds f in $\prod A/J_{<\lambda}[A]$.

Assume toward a contradiction that $B \notin J_{<\lambda}[A]$. Then (by definition of $J_{<\lambda}$) there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\prod A/D) \ge \lambda$. Clearly $D \cap J_{<\lambda} = \emptyset$, or else $\mathrm{cf}(\prod A/D) < \lambda$. The sequence f of length $\lambda_0 < \lambda$ is necessarily bounded in $\prod A/D$ and we let $h \in \prod A/D$ be such a bound. So $h(a) < g(a)$ for every $a \in B$ (since $g(a) = a$ for $a \in B$). Hence (by definition of an exact upper bound) there is some f_i in f such that $h \restriction B <_{J_{<\lambda}[A]} f_i \restriction B$. But this would imply $h <_D f_i$, which contradicts the definition of h as an upper bound. ⊣

3.5 Corollary. *Suppose that A is a progressive set of regular cardinals. Then for every ultrafilter D over A*

$$\mathrm{cf}\big(\textstyle\prod A/D\big) < \lambda \quad \textit{iff} \quad J_{<\lambda}[A] \cap D \neq \emptyset.$$

Hence $\mathrm{cf}(\prod A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \neq \emptyset$ *and* $J_{<\lambda} \cap D = \emptyset$. *Namely,* $\mathrm{cf}(\prod A/D) = \lambda$ *iff* λ^+ *is the first cardinal* μ *such that* $J_{<\mu} \cap D \neq \emptyset$.

Proof. If $J_{<\lambda}[A] \cap D \neq \emptyset$ and $X \in J_{<\lambda}[A] \cap D$, then by definition of $X \in J_{<\lambda}$

$$\mathrm{cf}\big(\textstyle\prod A/D\big) < \lambda.$$

On the other hand, if $J_{<\lambda} \cap D = \emptyset$, then the above theorem stating that $\prod A/J_{<\lambda}$ is λ-directed gives that $\prod A/D$ is λ-directed as well. Thus $\mathrm{cf}(\prod A/D) < \lambda$ is impossible in this case. The additional conclusion of the corollary is easily derived. ⊣

This corollary allows us to investigate the relationship between $J_{<\lambda}[A]$ and $J_{<\lambda^+}[A]$. By definition $X \in J_{<\lambda^+}[A]$ iff $X \subseteq A$ and for every ultrafilter D over A containing X, $\mathrm{cf}(\prod A/D) \leq \lambda$. For this reason, $J_{<\lambda^+}[A]$ is also denoted $J_{\leq\lambda}[A]$.

If $\lambda \notin \mathrm{pcf}(A)$, for example when λ is singular, then $J_{<\lambda} = J_{\leq\lambda}$. However, if $\lambda \in \mathrm{pcf}(A)$ then $J_{<\lambda} \subset J_{\leq\lambda}$ (where $\subset$ is the strict inclusion relation). Indeed, if D is an ultrafilter over A such that $\mathrm{cf}(\prod A/D) = \lambda$, then by Corollary 3.5 applied to λ^+, $J_{\leq\lambda} \cap D \neq \emptyset$, and certainly $J_{<\lambda} \cap D = \emptyset$. This argument shows that there is a one-to-one mapping from $\mathrm{pcf}(A)$ into $\mathcal{P}(A)$. Namely choosing $X_\lambda \in J_{\leq\lambda} \setminus J_{<\lambda}$ for every $\lambda \in \mathrm{pcf}(A)$. Thus we have the following theorem which is not evident from the definition of pcf.

3.6 Theorem. *If* A *is a progressive set of regular cardinals, then*

$$|\,\mathrm{pcf}(A)| \leq |\mathcal{P}(A)|.$$

Another consequence of Theorem 3.4 is that $\max \mathrm{pcf}(A)$ exists.

3.7 Corollary (max pcf). *If* A *is a progressive set of regular cardinals, then the set* $\mathrm{pcf}(A)$ *contains a maximal cardinal.*

Proof. Observe that if $\lambda_1 < \lambda_2$ are cardinals, then $J_{<\lambda_1}[A] \subseteq J_{<\lambda_2}[A]$. Define

$$I = \bigcup\{J_{<\lambda}[A] \mid \lambda \in \mathrm{pcf}(A)\}.$$

For every $\lambda \in \mathrm{pcf}(A)$ $J_{<\lambda}[A]$ is a proper ideal on A, and hence I (being the union of a chain of proper ideals) is also a proper ideal.

Since I is a proper ideal it can be extended to a maximal proper ideal, and we let E be any ultrafilter over A and disjoint to I. Let $\mu = \mathrm{cf}(\prod A/E)$. Since E is disjoint to I, it is disjoint to every $J_{<\lambda}[A]$ for $\lambda \in \mathrm{pcf}(A)$, and hence $\mathrm{cf}(\prod A/E) \geq \lambda$ by the previous corollary. That is $\mu = \mathrm{cf}(\prod A/E) = \max \mathrm{pcf}(A)$. As an important consequence we note that $\mu = \sup \mathrm{pcf}(A) = \max \mathrm{pcf}(A)$ is a regular cardinal (since it is in $\mathrm{pcf}(A)$). ⊣

3.8 Exercise. If λ is a limit cardinal then

$$J_{<\lambda}[A] = \textstyle\bigcup_{\theta<\lambda} J_{<\theta}[A].$$

Another way of writing this statement is that for every cardinal λ (not necessarily limit)

$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{<\theta^+}[A] = \bigcup_{\theta<\lambda} J_{\leq\theta}[A].$$

The no holes argument has the following consequence.

3.9 Theorem. *Suppose that A is a progressive interval of regular cardinals. Then* $\mathrm{pcf}(A)$ *is again an interval of regular cardinals.*

Proof. We may assume that A is infinite, as the finite case is clear. We may also assume that A has no last cardinal (and deduce the general theorem in a short argument). Let $\lambda_0 = \max \mathrm{pcf}(A)$. We must show that every regular cardinal in the interval $[\min A, \lambda_0]$ is in $\mathrm{pcf}(A)$. Say $\mu = \sup A$. Since $\mu \notin A$ (A has no maximum), μ is a singular cardinal (because A is progressive). Since $A \subseteq \mathrm{pcf}(A)$ and A is an interval of regular cardinals, the substantial part of the proof is in showing that any regular cardinal in $(\mu, \lambda_0]$ is in $\mathrm{pcf}(A)$. But if λ is a regular cardinal and $\mu < \lambda \leq \lambda_0$, then $J_{<\lambda}$ is a proper ideal (since $\lambda \leq \max \mathrm{pcf}(A)$). By Theorem 3.4, $\prod A/J_{<\lambda}$ is λ-directed. Hence Theorem 3.1 applies, and $\lambda \in \mathrm{pcf}(A)$. ⊣

We can get some information even when A is not progressive.

3.10 Definition. Suppose that A is a set of regular cardinals and $\kappa < \min A$ is a cardinal. We define

$$\mathrm{pcf}_\kappa(A) = \bigcup\{\mathrm{pcf}(X) \mid X \subseteq A \text{ and } |X| = \kappa\}.$$

That is, $\mathrm{pcf}_\kappa(A)$ is the collection of all cofinalities of ultraproducts of A over ultrafilters that concentrate on subsets of A of power κ (or less).

Similarly to the previous theorem stating that $\mathrm{pcf}(A)$ of a progressive interval A is again an interval of regular cardinals, we have the following.

3.11 Theorem. *If A is an interval of regular cardinals, and $\kappa < \min A$, then* $\mathrm{pcf}_\kappa(A)$ *is an interval of regular cardinals.*

Proof. Define $\lambda_0 = \sup \mathrm{pcf}_\kappa(A)$, and let λ be a regular cardinal such that $\min A < \lambda < \lambda_0$. Then for some $X \subseteq A$ such that $|X| = \kappa$, $\lambda \leq \max \mathrm{pcf}(X)$. Hence $J_{<\lambda}[X]$ is proper, and we may assume that every initial segment of X is in $J_{<\lambda}$. As X *is* progressive, $\prod X/J_{<\lambda}$ is λ-directed, Theorem 3.2 can be applied, and it yields that $\lambda \in \mathrm{pcf}(X')$ for some $X' \subseteq A$ of cardinality $\leq |X|$. Thus $\lambda \in \mathrm{pcf}_\kappa(A)$. ⊣

Yet another consequence of the λ-directedness of $\prod A/J_{<\lambda}$ is the following

3.12 Theorem. *Suppose that A is a progressive set of regular cardinals and* $B \subseteq \mathrm{pcf}(A)$ *is also progressive. Then*

$$\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A).$$

Hence if $\mathrm{pcf}(A)$ *is progressive, then* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. Suppose that $\mu \in \mathrm{pcf}(B)$, and let E be an ultrafilter over B such that

$$\mu = \mathrm{cf}\big(\textstyle\prod_{b\in B} b/E\big). \tag{14.15}$$

For every $b \in B$ fix an ultrafilter D_b over A such that

$$b = \mathrm{cf}\big(\textstyle\prod A/D_b\big).$$

Define an ultrafilter D over A by

$$X \in D \quad \text{iff} \quad \{b \in B \mid X \in D_b\} \in E. \tag{14.16}$$

We shall prove that $\mu = \mathrm{cf}(\prod A/D)$, and hence that $\mu \in \mathrm{pcf}(A)$.

Consider (14.15). If, for every $b \in B$, $(b', <_{b'})$ is an ordering that has true cofinality b, then $\mu = \mathrm{cf}(\prod_{b\in B} b'/E)$ as well. Hence

$$\mu = \mathrm{cf}\big(\textstyle\prod_{b\in B}\big(\prod A/D_b\big)/E\big). \tag{14.17}$$

It remains to implement this iterated ultraproduct as an ultraproduct of A over D. For this aim consider the Cartesian product $B \times A$ and the ultrafilter P defined on $B \times A$ by

$$H \in P \quad \text{iff} \quad \{b \in B \mid \{a \in A \mid \langle b, a\rangle \in H\} \in D_b\} \in E.$$

For any pair $\langle b, a\rangle$ let $r(\langle b, a\rangle) = a$ be its right projection. The reader should prove the following isomorphism

3.13 Claim. $\prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)/P \cong \prod_{b\in B}(\prod A/D_b)/E$.

Thus μ (an arbitrary cardinal in $\mathrm{pcf}(B)$) is the cofinality of the ultraproduct $\prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)/P$. But the projection map $r : B \times A \to A$, shows that the ultrafilter D defined in (14.16) is the Rudin-Keisler projection of P, and we are almost in the situation of Lemma 2.3, which concludes that $\mu = \mathrm{cf}(\prod A/D)$. However Lemma 2.3 cannot be used verbatim because $|B \times A| < \min A$ is not assumed. All we know is that $|B| < \min B$. Recall (Lemma 2.3) that we had a map from $\prod A$ into $\prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)$ carrying $h \in \prod A$ to $\bar{h} \in \prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)$ defined by

$$\bar{h}(\langle b, a\rangle) = h(a).$$

We have proved that this map induces an isomorphism denoted L of $\prod A/D$ into $\prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)$, but the problem is to prove that the image of L is cofinal there. Let $\lambda = \min B$. We have assumed that $|B| < \lambda$, and we shall use the fact that the reduced product modulo $J_{<\lambda}[A]$ is λ-directed as follows. Given any $g \in \prod_{\langle b,a\rangle\in B\times A} r(\langle b, a\rangle)$ define for every $b \in B$ the map $g_b \in \prod A$ by

$$g_b(a) = g(b, a).$$

Then $\{g_b \mid b \in B\}$ is bounded in $\prod A/J_{<\lambda}[A]$ by some function $h \in \prod A$, and we thus have that $g_b <_{J_{<\lambda}[A]} h$ for every $b \in B$. Hence

$$g_b <_{D_b} h$$

since $J_{<\lambda} \cap D_b = \emptyset$ (because $\mathrm{cf}(\prod A/D_b) = b$ and $\lambda \leq b$). So $g <_P \bar{h}$ is concluded. $\dashv$

4. Generators for $J_{<\lambda}$

A very useful property of the $J_{<\lambda}$ ideals is that for every cardinal $\lambda \in \mathrm{pcf}(A)$ there is a set $B_\lambda \subseteq A$ such that

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda$$

which means that the ideal $J_{<\lambda^+}[A]$ is generated by the sets in $J_{<\lambda}[A] \cup \{B_\lambda\}$. That is, for every $X \subseteq A$, $X \in J_{<\lambda^+}$ iff $X \setminus B_\lambda \in J_{<\lambda}$. So B_λ is a maximal set in $J_{\leq\lambda}[A]$ in the sense that if $B_\lambda \subseteq C \in J_{\leq\lambda}$ then $C \setminus B_\lambda \in J_{<\lambda}$. The property that $J_{\leq\lambda}[A]$ is generated from $J_{<\lambda}[A]$ by the addition of a single set is called *normality*.

Normality of $\lambda \in \mathrm{pcf}(A)$ is obtained by means of a universal sequence for λ, and these sequences are studied first.

4.1 Definition. Suppose that $\lambda \in \mathrm{pcf}(A)$. A sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ of functions in $\prod A$, increasing in $<_{J_{<\lambda}}$, is a *universal* sequence for λ if and only if for every ultrafilter D over A such that $\lambda = \mathrm{cf}(\prod A/D)$, f is cofinal in $\prod A/D$.

4.2 Theorem (Universally Cofinal Sequences). *Suppose that A is a progressive set of regular cardinals. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. The proof is obvious in the case $\lambda = \min A$. (The functions f_ξ defined by $f_\xi(a) = \xi$ will do.) Therefore we shall assume that $|A|^+ < \min A < \lambda$.

Suppose that there is no universal sequence for λ. This means that for every $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ but f is bounded in $\prod A/D$.

The proof is typical in that it makes $|A|^+$ steps and obtains a contradiction from the continuous failure at every step.

So for each $\alpha < |A|^+$ we shall define a $<_{J_{<\lambda}}$-increasing sequence $f^\alpha = \langle f^\alpha_\xi \mid \xi < \lambda\rangle$ in $\prod A$, and assume that no f^α is universal. The definition is by recursion on $\alpha < |A|^+$ and the fact that $\prod A/J_{<\lambda}$ is λ-directed is used in this construction.

If we visualize the functions f^α_ξ as lying on a matrix $\langle \xi, \alpha\rangle \in \lambda \times |A|^+$, then in each column α the functions f^α_ξ are $<_{J_{<\lambda}}$ increasing with ξ, and in each row ξ the functions f^α_ξ are $\leq$ increasing with α.

To begin with $f^0 = \langle f^0_\xi \mid \xi < \lambda\rangle$ is an arbitrary $<_{J_{<\lambda}}$-increasing sequence in $\prod A/J_{<\lambda}$ of length λ.

At limit stages $\delta < |A|^+$ we define $f^\delta = \langle f^\delta_\xi \mid \xi < \lambda \rangle$ by induction on $\xi < \lambda$ so that for every $\xi < \lambda$

1. $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$.

2. $\sup\{f^\alpha_\xi \mid \alpha < \delta\} \leq f^\delta_\xi$.

Suppose now that f^α is defined. Since it is not universal, there exists an ultrafilter D_α over A such that

1. $\mathrm{cf}(\prod A/D_\alpha) = \lambda$, and

2. the sequence f^α is bounded in $<_{D_\alpha}$.

So we can choose an $f^{\alpha+1}_0$ that bounds the sequence f^α in $<_{D_\alpha}$. The sequence $f^{\alpha+1}_i$ for $0 < i < \lambda$ is defined recursively by requiring that

1. $f^{\alpha+1}$ is $<_{J_{<\lambda}}$-increasing and cofinal in $\prod A/D_\alpha$, and

2. $f^{\alpha+1}_i \geq f^\alpha_i$ (everywhere) for every $i < \lambda$.

To sum up, we have constructed $<_{J_{<\lambda}}$-increasing sequences f^α, each of length λ, and ultrafilters D_α over A, for $\alpha < |A|^+$ so that:

1. for every $i < \lambda$, $\langle f^\alpha_i \mid \alpha < |A|^+ \rangle$ is increasing in $\leq$ (i.e. for $\alpha_1 < \alpha_2 < |A|^+$, $f^{\alpha_1}_i(a) \leq f^{\alpha_2}_i(a)$ for every a).

2. $f^\alpha = \langle f^\alpha_\xi \mid \xi < \lambda \rangle$ is bounded in $\prod A/D_\alpha$ by $f^{\alpha+1}_0$.

3. $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

Now let $h = \sup\{f^\alpha_0 \mid \alpha < |A|^+\}$. Then $h \in \prod A$, because $|A|^+ < \min A$. Find for every $\alpha < |A|^+$ an index $i_\alpha < \lambda$ such that $h <_{D_\alpha} f^{\alpha+1}_{i_\alpha}$. This is possible since $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$. Now pick an ordinal $i < \lambda$ such that $i > i_\alpha$ for every $\alpha < |A|^+$. This is possible since $\lambda > |A|^+$ is regular. So $h <_{D_\alpha} f^{\alpha+1}_i$ for every $\alpha < |A|^+$.

Define

$$A^\alpha = \leq(h, f^\alpha_i).$$

The sets $A^\alpha \subseteq A$ are increasing with α, that is $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$ (since $f^\alpha_i \leq f^\beta_i$).

The contradiction is obtained when we show that $A^\alpha \subset A^{\alpha+1}$ (strict inclusion) for every $\alpha < |A|^+$ (and contrast this with $A^\alpha \subseteq A$). For this, observe the following two statements.

1. $A^\alpha \notin D_\alpha$, because $f^\alpha_i <_{D_\alpha} f^{\alpha+1}_0 \leq h$.

2. $A^{\alpha+1} \in D_\alpha$, because $h <_{D_\alpha} f^{\alpha+1}_i$.

$\dashv$

If $\lambda \in \text{pcf}(A)$ and D is an ultrafilter over A such that $\text{cf}(\prod A/D) = \lambda$, then $A \cap (\lambda + 1) \in D$ because otherwise $\{a \in A \mid a > \lambda\} \in D$ and then $\text{cf}(\prod A/D) > \lambda$. Thus, if $\langle f_\xi \mid \xi \in \lambda\rangle$ is a universal sequence for λ, we may assume that $f_\xi(a) = \xi$ for all $a \in A \setminus \lambda$.

4.3 Exercise. If $\lambda = \max \text{pcf}(A)$, then any universal sequence for λ is cofinal in $\prod A/J_{<\lambda}$.

Universal sequences can be used to prove the following

4.4 Theorem. *For every progressive set A of regular cardinals,*

$$\text{cf}\big(\prod A, <\big) = \max \text{pcf}(A).$$

Hence $\text{cf}(\prod A, <)$ *is a regular cardinal.*

Proof. The partial ordering $<$ in this theorem refers to the everywhere dominance relation on $\prod A$. The required equality is obtained by first proving $\geq$ and then $\leq$.

Suppose that $\lambda = \max \text{pcf}(A)$, and D is an ultrafilter over A such that $\lambda = \text{cf}(\prod A/D)$. Then $<_D$ extends $<$ on $\prod A$. That is, for $f, g \in \prod A$, $f < g$ implies $f <_D g$. This shows that any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and hence that $\text{cf}(\prod A, <) \geq \text{cf}(\prod A, <_D) = \lambda$.

Now we must exhibit a cofinal subset of $(\prod A, <)$ of cardinality λ in order to conclude the proof.

Fix for every $\mu \in \text{pcf}(A)$ a universal sequence $f^\mu = \langle f^\mu_i \mid i < \mu\rangle$ for μ. Let F be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of cardinals in $\text{pcf}(A)$ (with possible repetitions) and $i_k < \mu_k$ are arbitrary indices. (Recall the definition of $\sup\{g_1, \ldots, g_n\}$: at every $a \in A$ it returns $\max\{g_1(a), \ldots, g_n(a)\}$). Clearly $|F| = \lambda$.

4.5 Claim. *F is cofinal in* $(\prod A, <)$.

Proof of Claim. Let $g \in \prod A$ be any function there. Consider the following collection of subsets of A:

$$I = \{>(f, g) \mid f \in F\}.$$

(Recall that $>(f, g) = \{a \in A \mid f(a) > g(a)\}$.) This collection is closed under unions, that is

$$>(f_1, g) \cup >(f_2, g) = >(\sup\{f_1, f_2\}, g).$$

If $A \in I$, namely if $>(f, g) = A$ for some $f \in F$, then evidently $g < f$ as required. But otherwise we obtain a contradiction by extending I to a proper maximal ideal J, and considering $\mu = \text{cf}(\prod A/J)$. Then f^μ, the universal sequence for μ, is cofinal in $\prod A/J$, and at the same time it is $\leq_J$ bounded by g since $f \leq_I g$ for all $f \in F$. Yet this is obviously impossible, and thus the theorem is proved. $\dashv$

If $f' = \langle f'_\xi \mid \xi < \lambda \rangle$ is universal sequence for λ, and if $f = \langle f_\xi \mid \xi < \lambda \rangle$ is another sequence in $\prod A$, $<_{J_{<\lambda}}$-increasing and dominating f' (for all $\xi' < \lambda$ there is a $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$) then clearly f is also universal for λ. Hence we can use Theorem 2.21 and deduce the following

4.6 Lemma. *Suppose that A is a progressive set of regular cardinals, and $\lambda \in \mathrm{pcf}(A)$. Let μ be the least ordinal such that $A \cap \mu \notin J_{<\lambda}[A]$. Then there is a universal sequence for λ that satisfies $(*)_\kappa$ with respect to $J_{<\lambda}[A]$ for every regular cardinal κ such that $\kappa < \mu$, and in particular for $\kappa = |A|^+$.*

Proof. Observe first that $\mu \leq \lambda + 1$. (Let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $A \cap (\lambda + 1) \in D$, or else $\{a \in A \mid a > \lambda\} \in D$ and then $\mathrm{cf}(\prod A/D) > \lambda$. Thus $\lambda \in \mathrm{pcf}(A \cap (\lambda + 1))$.) Observe also that $\mu = \lambda$ is impossible, since λ is regular and $A \cap \lambda$ is necessarily bounded in λ as $|A| < \min A \leq \lambda$. The case $\mu = \lambda + 1$ is rather trivial: $\lambda \in A$ and $J_{<\lambda}[A] = \mathcal{P}(A \cap \lambda)$. In this case the functions defined by $f_\xi(a) = \xi$ for all $a \in A \setminus \lambda$ are as required (and $(*)_\lambda$ holds). So we assume that $\mu < \lambda$ and $A \cap \mu$ is unbounded in μ.

Let $\langle f'_\xi \mid \xi < \lambda \rangle$ be any universal sequence for λ. Theorem 2.21 can be applied to this sequence and to $I = J_{<\lambda}$. This gives a sequence $f_\xi \in \prod A$ that dominates f'_ξ and that satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Thus $(*)_\kappa$ holds for every regular $\kappa < \mu$. ⊣

We intend to prove next the existence of a generating set for $J_{<\lambda^+}$. For this we need first the following characterization of generators for $J_{<\lambda^+}$.

4.7 Lemma. *Suppose that A is a progressive set of regular cardinals and $B \subseteq A$. Then*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B \tag{14.18}$$

if and only if

$$B \in J_{<\lambda^+}[A] \quad \textit{and} \tag{14.19}$$

$$\begin{aligned} &\textit{If } D \textit{ is any ultrafilter over } A \textit{ with } \mathrm{cf}(\textstyle\prod A/D) = \lambda \\ &\quad \textit{then } B \in D. \end{aligned} \tag{14.20}$$

Proof. Assume first that (14.18) holds. Then (14.19) is obvious. We prove (14.20). If D is any ultrafilter over A with $\mathrm{cf}(\prod A/D) = \lambda$, then $D \cap J_{<\lambda^+}[A] \neq \emptyset$, and if $X \in D \cap J_{<\lambda^+}[A]$ is any set in the intersection then (14.18) implies that $X \setminus B \in J_{<\lambda}[A]$. Since $D \cap J_{<\lambda} = \emptyset$, $B \in D$ follows.

Now assume that (14.19) and (14.20) hold, and we prove that $J_{<\lambda^+}[A] = J_{<\lambda}[A] + B$.

Since $B \in J_{<\lambda^+}[A]$, $J_{<\lambda^+}[A] \supseteq J_{<\lambda}[A] + B$.

To prove $J_{<\lambda^+}[A] \subseteq J_{<\lambda}[A] + B$ assume $X \in J_{<\lambda^+}[A]$ and prove that $X \setminus B \in J_{<\lambda}$ as follows. Let D be any ultrafilter over A such that $X \setminus B \in D$. Since $X \in J_{<\lambda^+}$, $\mathrm{cf}(\prod A/D) < \lambda^+$. But $\mathrm{cf}(\prod A/D) = \lambda$ is impossible as $B \notin D$ and we assume (14.20). Hence $\mathrm{cf}(\prod A/D) < \lambda$. ⊣

4.8 Theorem (Normality). *If A is a progressive set of regular cardinals, then every cardinal $\lambda \in \mathrm{pcf}(A)$ is normal: there exists a set $B_\lambda \subseteq A$ such that*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda.$$

Proof. By Lemma 4.6, there exists a universal sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ for λ that satisfies $(*)_{|A|^+}$. Hence f has an exact upper bound h in $\mathrm{On}^A / J_{<\lambda}$. Since the identity function is an upper bound of f, we can assume that $h(a) \leq a$ for every $a \in A$. Now define

$$B = \{a \in A \mid h(a) = a\}.$$

We are going to prove that B satisfies the two properties (14.19) and (14.20) which concludes the theorem and shows that B is a generator for $J_{<\lambda^+}$. We first prove that $B \in J_{<\lambda^+}[A]$. If D is any ultrafilter over A containing B then

$$\mathrm{cf}\big(\prod A/D\big) \leq \lambda \tag{14.21}$$

is deduced in two steps. If $D \cap J_{<\lambda} \neq \emptyset$, then the strict inequality of (14.21) holds by definition of $J_{<\lambda}$. But if $D \cap J_{<\lambda} = \emptyset$, then h remains the exact upper bound of the $<_D$ increasing sequence f in $<_D$ (just because D extends the dual filter of $J_{<\lambda}$). So $\mathrm{cf}(\prod h/D) = \lambda$. As h is $=_D$ equivalent to the identity function over A, $\prod A/D$ has cofinality λ.

To prove (14.20), suppose that D is an ultrafilter over A and $\mathrm{cf}(\prod A/D) = \lambda$. If $B \notin D$ then $\{a \in A \mid h(a) < a\} \in D$, and thus $[h]_D$ (the $=_D$-equivalence class of h) is in $\prod A/D$. Yet $D \cap J_{<\lambda}[A] = \emptyset$ (or else $\mathrm{cf}(\prod A/D) < \lambda$), and this implies that $f_\xi <_D h$ for every $\xi < \lambda$ (because $f_\xi <_{J_{<\lambda}} h$). So f is *not* cofinal in $\prod A/D$, in contradiction to f being a universal sequence for λ. ⊣

The generator set B_λ is not uniquely determined, but if B_1 and B_2 are two generators (they both satisfy 14.18), then the symmetric difference $B_1 \triangle B_2$ is in $J_{<\lambda}[A]$. So generators *are* uniquely determined modulo $J_{<\lambda}$, and we can use a "generic" notation.

4.9 Notation. For a progressive set of regular cardinals A and for any cardinal $\lambda \in \mathrm{pcf}(A)$, $B_\lambda[A]$ denotes a subset $B \subseteq A$ such that (14.19) and (14.20) hold, or equivalently

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B. \tag{14.22}$$

We also use the expression "B is a $B_\lambda[A]$ set" if (14.22) holds for B. We often write B_λ instead of $B_\lambda[A]$, when the identity of A is obvious.

The sequence $\langle B_\lambda[A] \mid \lambda \in \mathrm{pcf}(A)\rangle$ is called a "generating sequence" for A, because the ideal $J_{<\lambda}$ is generated by the collection $\{B_{\lambda_0} \mid \lambda_0 < \lambda\}$ (see Corollary 4.12). It is convenient to write $B_\lambda = \emptyset$ when $\lambda \notin \mathrm{pcf}(A)$.

The following conclusion will be needed later on.

4.10 Lemma. *Suppose that A is a progressive set of regular cardinals. If $A_0 \subseteq A$ and $\lambda \in \mathrm{pcf}(A_0)$, then $B_\lambda[A_0] =_{J_{<\lambda}[A_0]} A_0 \cap B_\lambda[A]$. (This justifies our inclination to write B_λ instead of $B_\lambda[A_0]$.)*

Proof. We prove (14.19) and (14.20) for $A_0 \cap B_\lambda[A]$. Clearly $A_0 \cap B_\lambda[A] \in J_{\leq\lambda}[A_0]$. If D_0 is any ultrafilter over A_0 such that $\mathrm{cf}(\prod A_0/D_0) = \lambda$, then $A_0 \cap B_\lambda[A] \in D_0$ can be argued as followed. Assume $A_0 \setminus B_\lambda[A] \in D_0$, and extend D_0 to an ultrafilter over A, still denoted D_0. Then $\mathrm{cf}(\prod A/D_0) = \lambda$ and $B_\lambda[A] \notin D_0$ is in contradiction to (14.20). ⊣

For a progressive set A with $\lambda = \max\mathrm{pcf}(A)$ and B a $B_\lambda[A]$ set, we have by (14.20) that

$$A \setminus B \in J_{<\lambda} \tag{14.23}$$

since $A \in J_{<\lambda^+}[A]$. Hence we can take $B_{\max\mathrm{pcf}(A)} = A$.

We will conclude that the ideal $J_{<\lambda}[A]$ is (finitely) generated by the sets $\{B_\mu[A] \mid \mu < \lambda\}$ using the following "compactness" theorem, which says that any set $X \in J_{<\lambda}$ is covered by a finite collection of B_μ's for $\mu < \lambda$.

4.11 Theorem (Compactness). *Suppose that A is a progressive set of regular cardinals and $\langle B_\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A. Then for any $X \subseteq A$*

$$X \subseteq B_{\lambda_1} \cup B_{\lambda_2} \cup \cdots \cup B_{\lambda_n}$$

for some finite set $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathrm{pcf}(X)$.

Proof. This can be proved by induction on $\lambda = \max\mathrm{pcf}(X)$, since $X \setminus B_\lambda \in J_{<\lambda}$ and so $\max\mathrm{pcf}(X \setminus B_\lambda) < \lambda$. ⊣

4.12 Corollary. *If A is a progressive set of regular cardinals, then for every cardinal λ and every set $X \subseteq A$, $X \in J_{<\lambda}[A]$ iff X is included in a finite union of sets from $\{B_{\lambda'} \mid \lambda' < \lambda\}$.*

Observe that $\lambda \notin \mathrm{pcf}(A \setminus B_\lambda[A])$. For let D_0 be any ultrafilter over $A_0 = A \setminus B_\lambda[A]$. Extend D_0 to an ultrafilter D over A. Since $\prod A_0/D_0$ is isomorphic to $\prod A/D$, it suffices to prove that $\mathrm{cf}(\prod A/D) \neq \lambda$. But since A_0 is disjoint to $B_\lambda[A]$, $B_\lambda[A] \notin D$. So (14.20) implies this, and we have obtained the following result. A set $B \in J_{<\lambda^+}[A]$ is a B_λ set if and only if $\lambda \notin \mathrm{pcf}(A \setminus B)$.

If $\lambda \in \mathrm{pcf}(A)$ and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a universal sequence for λ, then the definition of $B_\lambda[A] = \{a \in A \mid h(a) = a\}$, where h is an exact upper bound of f, shows that $\langle f_\xi \restriction B_\lambda \mid \xi < \lambda\rangle$ is cofinal in $\prod B_\lambda/J_{<\lambda}$. This result is sufficiently interesting to be isolated as a theorem (and we give a somewhat different proof).

4.13 Theorem. *If A is a progressive set of regular cardinals and $\lambda \in \mathrm{pcf}(A)$, then for some set $B \subseteq A$ we have $\mathrm{tcf}(\prod B/J_{<\lambda}[B]) = \lambda$. In fact, any universal sequence for λ is cofinal in $\prod B_\lambda/J_{<\lambda}$ and thus shows that*

$$\mathrm{tcf}\big(\textstyle\prod B_\lambda/J_{<\lambda}\big) = \lambda. \tag{14.24}$$

Proof. We know that there exists a universal sequence for λ and that there exists a generating set B_λ. We will prove that any universal sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ for λ is cofinal in $\prod B_\lambda / J_{<\lambda}$. That is, if $h \in \prod B_\lambda$ is any function then

$$\leq(f_\xi \restriction B_\lambda, h) \in J_{<\lambda} \quad \text{for some } \xi < \lambda.$$

Otherwise the sets $\leq(f_\xi \restriction B_\lambda, h)$ are positive and decreasing with $\xi < \lambda$ (mod $J_{<\lambda}$). Hence there is a filter over B_λ containing them all and extending the dual filter of $J_{<\lambda}[B_\lambda]$. Extending this filter to an ultrafilter D over A, $D \cap J_{<\lambda}[A] = \emptyset$ and the ultraproduct $\prod A/D$ has cofinality λ (as $B_\lambda \in D$ and $D \cap J_{<\lambda} = \emptyset$). In this ultrapower h bounds all functions in f, in contradiction to the assumption that f is universally cofinal. Thus the restriction to $B_\lambda[A]$ of any universal sequence for λ is cofinal in $\prod B_\lambda / J_{<\lambda}$. ⊣

In particular, (14.24) shows (again) that $\lambda = \max \operatorname{pcf}(B_\lambda)$ whenever $\lambda \in \operatorname{pcf}(A)$. We have, more generally, the following characterization.

4.14 Lemma. *The following are equivalent for every filter F over a progressive set of regular cardinals A and for every cardinal λ.*

1. $\operatorname{tcf}(\prod A/F) = \lambda$.
2. *$B_\lambda \in F$, and F contains the dual filter of $J_{<\lambda}[A]$.*
3. *$\operatorname{cf}(\prod A/D) = \lambda$ for every ultrafilter D that extends F.*

In particular we get for every ultrafilter D that

$$\operatorname{cf}\big(\prod A/D\big) = \lambda \quad \textit{iff} \quad B_\lambda \in D \textit{ and } D \cap J_{<\lambda} = \emptyset. \tag{14.25}$$

Equivalently,

$$\operatorname{cf}\big(\prod A/D\big) = \lambda \quad \textit{iff} \quad \lambda \textit{ is the least cardinal such that } B_\lambda \in D. \tag{14.26}$$

Proof. Fix a filter F and a cardinal λ. $1 \implies 3$ is obvious. Assume 3 and we prove 2. Since $\operatorname{cf}(\prod A/D) = \lambda$ for every ultrafilter D that extends F, $B_\lambda \in D$ for every such ultrafilter (by (14.20)). Hence $B_\lambda \in F$. It is clear that F contains the dual filter of $J_{<\lambda}$, or else an extension of F can be found that intersects $J_{<\lambda}$ and thus has an ultraproduct with cofinality below λ.

Assume now 2, and then the fact already proved that $\operatorname{tcf}(\prod B_\lambda / J_{<\lambda}) = \lambda$ shows that $\operatorname{tcf}(\prod A/F) = \lambda$ (as $\prod A/F$ and $\prod B_\lambda/F$ are isomorphic, since $B_\lambda \in F$).

In particular, if D is an ultrafilter over A, then $D \cap J_{<\lambda} = \emptyset$ iff the dual filter of $J_{<\lambda}$ is contained in D. So the equivalence of 1 and 2 of the lemma establishes (14.25). ⊣

4.15 Exercise.

1. If D is an ultrafilter over a progressive set A, and λ is the least cardinal such that $B_\lambda \in D$, then $\lambda = \operatorname{cf}(\prod A/D)$.

2. Suppose that A is a progressive set of regular cardinals and $E = \mathrm{pcf}(A)$ is also progressive. Then

$$\mathrm{pcf}(B_\lambda[A]) =_{J_{<\lambda}[E]} B_\lambda[\mathrm{pcf}(A)].$$

(Use Theorem 3.12.)

4.16 Exercise. If A is a progressive set of regular cardinals, then for every cardinal λ, $\lambda = \max \mathrm{pcf}(A)$ iff $\lambda = \mathrm{tcf}(\prod A/J_{<\lambda})$ iff $\lambda = \mathrm{cf}(\prod A/J_{<\lambda})$.

In Theorem 2.23 we have proved for μ, a singular cardinal with uncountable cofinality, that $\mu^+ = \mathrm{tcf}(\prod C^{(+)}/J^{bd})$ for some closed unbounded set of cardinals $C \subseteq \mu$. Since $J_{<\mu} = J_{<\mu^+} \subseteq J^{bd}$, an apparently stronger claim is obtained by asserting $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$.

4.17 Exercise (The Representation Theorem). If μ is a singular cardinal with uncountable cofinality, then for some closed unbounded set of cardinals $C \subseteq \mu$, $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$. Thus $\mu^+ = \max \mathrm{pcf}(C^{(+)})$.

Hint. Let $C_0 \subseteq \mu$ be a closed unbounded set of limit cardinals such that $\mu^+ = \mathrm{tcf}(\prod C_0^{(+)}/J^{bd})$. Then define $C \subseteq C_0$ so that $C^{(+)} = B_{\mu^+}[C_0^{(+)}]$. Prove that $C_0 \setminus C$ is bounded in μ. Then use Theorem 4.13.

4.18 Exercise. For any filter F over a progressive set A of regular cardinals, define

$$\mathrm{pcf}_F(A) = \{\mathrm{cf}(\textstyle\prod A/D) \mid D \text{ an ultrafilter over } A \text{ that extends } F\}.$$

1. Prove that $\max \mathrm{pcf}_F(A)$ exists.

   Hint. Look at the minimal λ such that $F \cap J_{\leq\lambda} \neq \emptyset$.

2. Deduce that $\mathrm{cf}(\prod A/F) = \max \mathrm{pcf}_F(A)$, so that the cofinality of this partial ordering is a regular cardinal.

3. If $B \subseteq \mathrm{pcf}_F(A)$ is progressive, then $\mathrm{pcf}(B) \subseteq \mathrm{pcf}_F(A)$.

4. Suppose that A is a progressive *interval* of regular cardinals, and let F be the filter of co-bounded subsets of A ($X \in F$ iff $A \setminus X$ is bounded in A). Then $\mathrm{pcf}_F(A)$ is an interval of regular cardinals.

5. The Cofinality of $[\mu]^\kappa$

Some of the most important applications of pcf theory will be described in this section. For example, we will prove that $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$. For this result we investigate obedient universal sequences and their relationship with characteristic functions of elementary substructures. Some of the theorems about obedient sequences proved and used in this section will be applied in the following section to "elevated" sequences. These sequences are not obedient, but they share enough properties with the obedient sequences to enable uniform proofs. This explains our desire to deal here with the shared properties (14.32) and (14.33) rather than with obedient sequences.

As usual, A is a progressive set of regular cardinals. Recall how $B_\lambda[A]$ was obtained. First a universal sequence $\langle f_\xi \mid \xi < \lambda\rangle$ for λ was defined which satisfied $(*)_{|A|^+}$, then an exact upper bound h was constructed, and finally the set $B_\lambda = \{a \in A \mid h(a) = a\}$ was shown to generate $J_{<\lambda^+}[A]$ over $J_{<\lambda}[A]$. Once this is done, we have greater flexibility in tuning-up B_λ by using elementary substructures, and we therefore say first a few words about these structures.

5.1. Elementary Substructures

Elementary substructures are extensively used in pcf theory and its applications, and in this section we study some basic properties of their characteristic functions.

Let Ψ be a "sufficiently large" cardinal, and H_Ψ be the $\in$-structure whose universe is the collection H_Ψ of all sets hereditarily of cardinality less than Ψ (which means having transitive closure of size $< \Psi$). The expression "sufficiently large" depends on the context and means that Ψ is regular and is sufficiently large to include in H_Ψ all sets that were discussed so far. We also add to the structure H_Ψ a well-ordering $<^*$ of its universe. We shall seldom mention $<^*$ explicitly, but it allows us to assume that the objects we talk about are uniquely determined.

For the rest of this section κ denotes a regular cardinal such that $|A| < \kappa < \min A$.

5.1 Definition. An increasing and continuous chain of length λ of elementary substructures of H_Ψ is a sequence $\langle M_i \mid i < \lambda\rangle$ such that

1. Each M_i is an elementary substructure of H_Ψ,
2. $i_1 < i_2 < \lambda$ implies that $M_{i_1} \subseteq M_{i_2}$, and
3. for every limit ordinal $\delta < \lambda$, $M_\delta = \bigcup_{i<\delta} M_i$ (this is continuity).

We say in this chapter that an elementary substructure $M \prec H_\Psi$ is "κ-presentable" if and only if $M = \bigcup_{i<\kappa} M_i$ where $\langle M_i \mid i < \kappa\rangle$ is an increasing and continuous chain of length κ such that

1. M has cardinality κ and $\kappa + 1 \subseteq M$.
2. For every $i < \kappa$, $M_i \in M_{i+1}$. (Thus $M_i \in M_j$ for $i < j$.)

We do not make any assumption on the cardinality of M_i for $i < \kappa$, which may be κ or smaller than κ.

In order to define a κ-presentable elementary substructure define, recursively, the approaching structures M_α, and observe that each M_α (and even the sequence $\langle M_\alpha \mid \alpha \leq \beta \rangle$) is an element of H_Ψ and thus can be incorporated in $M_{\beta+1} \prec H_\Psi$.

We shall use the following observation. Let $\bar{M}_\alpha$ denote the ordinal closure of $M_\alpha \cap \mathrm{On}$. That is $\gamma \in \bar{M}_\alpha$ iff $\gamma \in M_\alpha \cap \mathrm{On}$ or γ is a limit of ordinals in M_α. Since $M_\alpha \in M_{\alpha+1}$ and $M_\alpha \subseteq M_{\alpha+1}$, $\bar{M}_\alpha \in M_{\alpha+1}$, and $\bar{M}_\alpha \subseteq M_{\alpha+1}$.

For any structure N, we let Ch_N be the "characteristic function" of N. That is, the function defined on any regular cardinal μ such that $\|N\| < \mu$ by

$$\mathrm{Ch}_N(\mu) = \sup(N \cap \mu).$$

Then $\mathrm{Ch}_N(\mu) \in \mu$ since μ is regular and N is of smaller cardinality.

A very useful fact that we are going to prove is that if M is κ-presentable, then for cardinals $\kappa \leq \lambda < \mu$, $M \cap \mu$ can be reconstructed from $M \cap \lambda$ and the characteristic function of M restricted to the successor cardinals in the interval $(\lambda, \mu]$. We shall use the following form of this fact.

5.2 Theorem. *Suppose that M and N are elementary substructures of H_Ψ. Let $\kappa < \mu$ be any cardinals (κ is always regular uncountable).*

1. *If $M \cap \kappa \subseteq N \cap \kappa$, and, for every successor cardinal $\alpha^+ \in M \cap \mu + 1$,*

$$\sup(M \cap \alpha^+) = \sup(M \cap N \cap \alpha^+), \tag{14.27}$$

   then $M \cap \mu \subseteq N \cap \mu$.

2. *Therefore, if M and N are both κ-presentable and for every successor cardinal $\alpha^+ \in \mu + 1$*

$$\sup(M \cap \alpha^+) = \sup(N \cap \alpha^+), \tag{14.28}$$

   then $M \cap \mu = N \cap \mu$.

Proof. This is a bootstrapping argument. We prove by induction on δ, a cardinal in the interval $[\kappa, \mu]$, that $M \cap \delta \subseteq N \cap \delta$. For $\delta = \kappa$ this is an assumption. If δ is a limit cardinal, then $M \cap \delta \subseteq N \cap \delta$ is an immediate application of the inductive assumption that $M \cap \delta' \subseteq N \cap \delta'$ for every cardinal δ' in the interval $[\kappa, \delta)$. Assume now that $M \cap \delta \subseteq N \cap \delta$, and we shall argue for $M \cap \delta^+ \subseteq N \cap \delta^+$. If $\delta^+ \notin M$, then M contains no ordinals from the interval $[\delta, \delta^+]$ and the claim is obvious. So assume that $\delta^+ \in M$. (And hence $\delta^+ \in N$ since $[\delta, \delta^+] \cap N \neq \emptyset$.) Let $\gamma = \sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$. Now if $\alpha \in M \cap \gamma$, then there exists some $\beta \in M \cap N \cap \gamma$ such that $\alpha < \beta$. Consider the structure $(H_\Psi, \in, <^*)$ of which M and N are elementary substructures, and pick an injection $f : \beta \to \delta$ that is minimal with respect to the well-ordering $<^*$ of H_Ψ. Then $f \in M \cap N$ because f is definable from the parameter β. Since $\alpha \in M$, $f(\alpha) \in M$, and hence $f(\alpha) \in N$. But then, applying f^{-1} in N, we get $\alpha \in N$. Thus $M \cap \beta \subseteq N \cap \beta$.

For the second part of the theorem, let $M = \bigcup_{\xi<\kappa} M_\xi$ and $N = \bigcup_{\xi<\kappa} N_\xi$ be presentations for M and N. Observe that $M \cap \kappa = N \cap \kappa = \kappa$. Let α^+ be any successor cardinal in the interval $(\kappa, \mu]$. We assume that $\gamma = \mathrm{Ch}_M(\alpha^+) = \mathrm{Ch}_N(\alpha^+)$. We claim that there is a subset of $M \cap N$ that is closed and unbounded in γ. Indeed, the approaching substructures M_ξ provide a closed unbounded sequence $\langle \sup(M_\xi \cap \alpha^+) \mid \xi \in \kappa\rangle$ which is cofinal in γ. Likewise, N contains a closed unbounded sequence of order-type κ cofinal in γ. The intersection of these closed unbounded sets is as required. Hence $\sup(M \cap \alpha^+) = \sup(N \cap \alpha^+) = \sup(M \cap N \cap \alpha^+)$ holds and $M \cap \mu = N \cap \mu$ is obtained by the first part of the theorem. ⊣

Recall that a sequence of functions in $\prod A$ is universal for λ if it is $J_{<\lambda}$ increasing and cofinal in $\prod A/D$ whenever $\mathrm{cf}(\prod A/D) = \lambda$. Recall also (14.3) that a sequence $\langle p_\xi \mid \xi < \lambda\rangle$ of members of a partial ordering $(P, <_P)$ is persistently cofinal iff every member of P is dominated by all members of the sequence with a sufficiently large index.

5.3 Definition. We say that a sequence $\langle f_\xi \mid \xi < \lambda\rangle$ of functions in $\prod A$ is *persistently cofinal for* λ if their restrictions to B_λ form a persistently cofinal sequence in $\prod B_\lambda/J_{<\lambda}$. Namely if for every $h \in \prod A$ there exists a $\xi_0 < \lambda$ such that

$$h \restriction B_\lambda <_{J_{<\lambda}} f_\xi \restriction B_\lambda$$

for all $\xi_0 \le \xi < \lambda$ (where $B_\lambda = B_\lambda[A]$).

For example, if $\langle f_\xi \mid \xi < \lambda\rangle$ is universal for λ then it is persistently cofinal (see Theorem 4.13), and if the functions F_ξ are such that $f_\xi \le_{J_{<\lambda}} F_\xi$ for all $\xi < \lambda$, then $\langle F_\xi \mid \xi < \lambda\rangle$ is also persistently cofinal, although it is not necessarily $J_{<\lambda}$ increasing. Clearly, an arbitrary λ sequence in $\prod A$ is universal for λ iff it is $J_{<\lambda}$ increasing and persistently cofinal.

A basic observation which is used later to define the transitive generators is the following.

5.4 Lemma. *Suppose that the progressive set A and the cardinal $\lambda \in \mathrm{pcf}(A)$ belong to an elementary substructure $N \prec H_\Psi$ so that $N = \bigcup_{\alpha<\kappa} N_\alpha$ where $|A| < \kappa < \min A$ is a regular cardinal, $|N| = \kappa$, $\kappa+1 \subseteq N$, and $\langle N_\alpha \mid \alpha < \kappa\rangle$ is an increasing chain of elementary substructures of H_Ψ. If a sequence of functions $f = \langle f_\xi \mid \xi < \lambda\rangle \in N$, with $f_\xi \in \prod A$, is persistently cofinal for λ, then for every $\xi \ge \sup(N \cap \lambda)$*

$$\le(\mathrm{Ch}_N, f_\xi) = \{a \in A \mid \mathrm{Ch}_N(a) \le f_\xi(a)\} \text{ is a } B_\lambda[A] \text{ set.} \tag{14.29}$$

Proof. We first make some preliminary observations. Since $\kappa < \min A$, we have that $\mathrm{Ch}_N \restriction A \in \prod A$. Since $A, \lambda \in N = \bigcup_{\alpha<\kappa} N_\alpha$, we may as well assume that $A, \lambda, f \in N_0$ (or else re-enumerate the structures). Since $|A| < \kappa$ and $\kappa \subseteq N$, $A \subseteq N$ and we can assume that $A \subseteq N_0$. Since Ψ is sufficiently large, all the pcf theory involved in defining $B_\lambda[A]$ etc. can be done in H_Ψ and hence in N_0. We may assume again that a generating set $B = B_\lambda[A]$ is in N_0. Suppose that $\xi \ge \sup(N \cap \lambda)$. To prove (14.29) we need two inclusions:

1. $\mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}} f_\xi \restriction B$, which shows that $B \subseteq_{J_{<\lambda}} \leq(\mathrm{Ch}_N, f_\xi)$.
2. $\leq(\mathrm{Ch}_N, f_\xi) \cap (A \setminus B) \in J_{<\lambda}$, which shows that $\leq(\mathrm{Ch}_N, f_\xi) \subseteq_{J_{<\lambda}} B$.

We prove 1. For every $a \in A$, if $f_\xi(a) < \mathrm{Ch}_N(a)$ then there exists an index $\alpha = \alpha(a) < \kappa$ such that $f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a)$. Since $|A| < \kappa$ there exists a single index $\alpha < \kappa$ such that, for every $a \in A$, $f_\xi(a) < \mathrm{Ch}_N(a)$ implies that $f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a)$. Hence for every $a \in A$

$$f_\xi(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a). \tag{14.30}$$

But the sequence f is persistently cofinal in $\prod B/J_{<\lambda}$, and hence $h\restriction B <_{J_{<\lambda}} f_\xi \restriction B$ for every $h \in N \cap \prod A$, because $\xi \geq \sup(N \cap \lambda)$. In particular, for $h = \mathrm{Ch}_{N_\alpha} \in N$, we get

$$\mathrm{Ch}_{N_\alpha} \restriction B \leq_{J_{<\lambda}} f_\xi \restriction B \quad (\text{in fact } <_{J_{<\lambda}}).$$

That is, $\{b \in B \mid f_\xi(b) < \mathrm{Ch}_{N_\alpha}(b)\} \in J_{<\lambda}$. Hence, by (14.30), $\{b \in B \mid f_\xi(b) < \mathrm{Ch}_N(b)\} \in J_{<\lambda}$. Thus

$$\mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}} f_\xi \restriction B.$$

This proves 1.

Now we prove 2. That is

$$\{a \in A \setminus B \mid \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}. \tag{14.31}$$

As $\lambda \notin \mathrm{pcf}(A \setminus B)$, $J_{<\lambda}[A \setminus B] = J_{<\lambda^+}[A \setminus B]$ and hence $\prod(A \setminus B)/J_{<\lambda}$ is λ^+-directed, and f (with functions restricted to $A \setminus B$) has an upper bound. Since $f \in N$, we have this upper bound in N. Let $h \in N \cap \prod(A \setminus B)$ be an upper bound in $<_{J_{<\lambda}}$ of the sequence f restricted to $A \setminus B$. Then

$$f_\xi \restriction (A \setminus B) <_{J_{<\lambda}} h < \mathrm{Ch}_N \restriction (A \setminus B).$$

But this is exactly (14.31). $\dashv$

5.2. Minimally Obedient Sequences

Suppose that δ is a limit ordinal and $f = \langle f_\xi \mid \xi < \delta \rangle$ is a sequence of functions $f_\xi \in \prod A$, where A is a set of regular cardinals and $|A| < \mathrm{cf}(\delta) < \min A$ holds. For every closed unbounded set $E \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ let

$$h_E = \sup\{f_\xi \mid \xi \in E\}.$$

That is, $h_E(a) = \sup\{f_\xi(a) \mid \xi \in E\}$. Since $\mathrm{cf}(\delta) < \min A$, $h_E \in \prod A$. We say that h_E is the "supremum along E of the sequence f". Observe that if $E_1 \subseteq E_2$ then $h_{E_1} \leq h_{E_2}$. The following lemma says that among all functions obtained as suprema along closed unbounded subsets of δ there is a minimal one in the $\leq$ ordering.

5.5 Lemma. *Let δ and f be as above (so $|A| < \mathrm{cf}(\delta) < \min A$ and f is a sequence of length δ of functions in $\prod A$). There is a closed unbounded set $C \subseteq \delta$ of order-type* $\mathrm{cf}(\delta)$ *such that*

$$h_C(a) \leq h_E(a)$$

for every $a \in A$ and $E \subseteq \delta$ closed and unbounded of order-type $\mathrm{cf}(\delta)$.

Proof. Assume that there is no such closed unbounded set $C \subseteq \delta$ as required. We construct a decreasing sequence $\langle E_\alpha \mid \alpha < |A|^+\rangle$ of closed unbounded subsets of δ of order-type $\mathrm{cf}(\delta)$ such that for every $\alpha < |A|^+$, $h_{E_\alpha} \not\leq h_{E_{\alpha+1}}$. (Since $|A| < \mathrm{cf}(\delta)$, at limit stages of the construction we may take the intersection of the clubs so far constructed.) Then find a single $a \in A$ such that $h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)$ for an unbounded set of indices α. Yet this is obviously impossible. ⊣

In applications of this lemma, an ideal J over A is assumed and the sequence $\langle f_\xi \mid \xi < \delta\rangle$ is $<_J$-increasing. In that case, the minimal function $f_C = \sup\{f_\xi \mid \xi \in C\}$ $\leq$-bounds each f_ξ, for $\xi \in C$, and hence $\leq_J$-bounds all f_ξ's for $\xi < \delta$. This function f_C is called "minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta\rangle$".

5.6 Definition (Minimally obedient universal sequence). Suppose that λ is in $\mathrm{pcf}(A)$ and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a universal sequence for λ. Let κ be a fixed regular cardinal such that $|A| < \kappa < \min A$. We say that f is *minimally obedient (at cofinality κ)* if for every $\delta < \lambda$ such that $\mathrm{cf}(\delta) = \kappa$, f_δ is the minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta\rangle$.

The universal sequence f is said to be *minimally obedient* if $|A|^+ < \min A$ and it is minimally obedient for every regular κ such that $|A| < \kappa < \min A$.

Suppose that $|A|^+ < \min A$ and $\lambda \in \mathrm{pcf}(A)$. In order to arrange a minimally obedient universal sequence for λ, start with an arbitrary universal sequence $\langle f^0_\xi \mid \xi < \lambda\rangle$ and define the functions f_ξ by induction on $\xi < \lambda$ such that:

1. $f_0 = f^0_0$, and $f_{\xi+1}$ is such that
$$\max\{f_\xi, f^0_\xi\} < f_{\xi+1}.$$
2. At limit stages $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa$ and such that $|A| < \kappa < \min A$ let f_δ be the minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta\rangle$.
3. At limit stages $\delta < \lambda$ with $\mathrm{cf}(\delta)$ not of that form use the fact that $\prod A/J_{<\lambda}$ is λ-directed to define f_δ as a $<_{J_{<\lambda}}$ bound of $\langle f_\xi \mid \xi < \delta\rangle$.

Minimally obedient sequences will be used in conjunction with κ-presentable elementary substructures.

5.7 Lemma. *Let A be a progressive set of regular cardinals, and κ be a regular cardinal such that $|A| < \kappa < \min A$. Suppose that*

1. *$\lambda \in \mathrm{pcf}(A)$,*

2. *$f = \langle f_\xi \mid \xi < \lambda \rangle$ is a minimally obedient at cofinality κ, universal sequence for λ, and*

3. *$N \prec H_\Psi$ (for Ψ sufficiently large) is an elementary, κ-presentable substructure of H_Ψ such that $\lambda, f, A \in N$. (So $A \subseteq N$.)*

Let $\bar{N}$ denote the ordinal closure of $N \cap \mathrm{On}$, that is, the set of ordinals that are in N or that are limit of ordinals in N. Then for every $\gamma \in (\bar{N} \cap \lambda) \setminus N$ there is a closed unbounded set $C \subseteq \gamma \cap N$ (of order-type κ) such that $f_\gamma = \sup\{f_\xi \mid \xi \in C\}$ and thus

$$f_\gamma(a) \in \bar{N} \cap a \quad \text{for every } a \in A.$$

In particular, for $\gamma = \mathrm{Ch}_N(\lambda)$, $\gamma \in \bar{N} \setminus N$, and $f_\gamma = \sup\{f_\xi \mid \xi \in C\}$ for a closed unbounded set $C \subseteq N \cap \gamma$ such that

1. *each f_ξ is in N, and*

2. *for every $h \in N \cap \prod A$*

$$h \restriction B_\lambda[A] <_{J_{<\lambda}} f_\xi \restriction B_\lambda[A]$$

for some $\xi \in C$.

Proof. Since N is κ-presentable, $N = \bigcup_{\alpha<\kappa} N_\alpha$ is the union of an increasing and continuous chain such that $N_\alpha \in N_{\alpha+1}$. It follows for every $\gamma \in \bar{N}$, that either $\gamma \in N$ or $\mathrm{cf}(\gamma) = \kappa$. Indeed, if $\gamma \in \bar{N} \setminus N$, then γ is a limit point of ordinals in N and yet γ is not a limit point of ordinals in any N_α (or else $\gamma \in \bar{N}_\alpha \subseteq N$). Hence $\sup(N_\alpha \cap \gamma) < \gamma$ and

$$E = \{\sup(N_\alpha \cap \gamma) \mid \alpha < \kappa\}$$

is closed unbounded in γ and of order-type κ. Thus $\mathrm{cf}(\gamma) = \kappa$. Observe that $E \subseteq N$, because $N_\alpha \in N$ implies that $\bar{N}_\alpha \subseteq N$ and in particular $\sup(N_\alpha \cap \gamma) \in N$.

Now take $\gamma \in (\bar{N} \cap \lambda) \setminus N$ and consider $f_\gamma(a)$ for $a \in A$. Since $\mathrm{cf}(\gamma) = \kappa$, f_γ is the minimal club-obedient bound of $\langle f_\xi \mid \xi < \gamma \rangle$, and there is thus a closed unbounded set $C \subseteq \gamma$ such that $f_\gamma = f_C$. It follows from the minimality of f_C that $f_C = f_{C \cap E}$ and we may thus assume at the outset that $C \subseteq N$. So $f_\gamma = f_C = \sup\{f_\xi \mid \xi \in C\}$ is the supremum of a set of functions that are all in N. (As $C \subseteq N$ implies that $f_\xi \in N$ for $\xi \in C$.) This shows that $f_\gamma(a) \in \bar{N}$ for every $a \in A$.

In particular, if $\gamma = \mathrm{Ch}_N(\lambda)$, then $\gamma < \lambda$ because $\kappa < \lambda$ and N has cardinality κ. So $\gamma \in \bar{N} \setminus N$. Item 2 is a consequence of the fact that f is a universal sequence (see Theorem 4.13) and that C is unbounded in $N \cap \lambda$. ⊣

The conclusions of Lemmas 5.4 and 5.7 will be given names (14.32) and (14.33) so that we can easily refer to these properties in the future. Let A be a progressive set of regular cardinals and suppose that κ is a regular cardinal such that $|A| < \kappa < \min A$. Suppose that $\lambda \in \mathrm{pcf}(A)$, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a sequence of functions in $\prod A$. We shall refer to the following two properties of a κ-presentable $N \prec H_\Psi$ and a sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ such that $f \in N$.

$$\boxed{\begin{array}{l}\text{If } \gamma = \mathrm{Ch}_N(\lambda), \text{ then} \\ \qquad\qquad \{a \in A \mid \mathrm{Ch}_N(a) \le f_\gamma(a)\} \\ \text{is a } B_\lambda[A] \text{ set.}\end{array}} \tag{14.32}$$

$$\boxed{\begin{array}{l}\text{If } \gamma = \mathrm{Ch}_N(\lambda), \text{ then} \\ \quad 1.\ f_\gamma \le \mathrm{Ch}_N. \\ \quad 2.\ \text{For every } h \in N \cap \prod A \text{ there exists some } d \in \\ \qquad N \cap \prod A \text{ such that} \\ \qquad\qquad h{\restriction}B <_{J_{<\lambda}} d{\restriction}B \text{ and } d \le f_\gamma, \\ \qquad \text{where } B = B_\lambda[A].\end{array}} \tag{14.33}$$

We have seen that any persistently cofinal sequence for λ satisfies (14.32) (this is Lemma 5.4), and that any minimally obedient universal sequence satisfies (14.33) as well (by Lemma 5.7).

Suppose that f is a sequence of length λ and $N \prec H_\Psi$ is κ-presentable and such that $f \in N$ (so $A, \lambda \in N$). Suppose that both (14.32) and (14.33) hold. If $\gamma = \mathrm{Ch}_N(\lambda)$, then $f_\gamma \le \mathrm{Ch}_N$ by (14.33), and hence

$$\{a \in A \mid \mathrm{Ch}_N(a) = f_\gamma(a)\} \tag{14.34}$$

is a $B_\lambda[A]$ set by (14.32). We shall use this observation in the following.

5.8 Lemma. *Suppose that A is a progressive set of regular cardinals and κ is a regular cardinal such that $|A| < \kappa < \min A$. Suppose that $\lambda_0 \in \mathrm{pcf}(A)$ and $f^{\lambda_0} = \langle f^{\lambda_0}_\xi \mid \xi < \lambda_0\rangle$ is a sequence of functions in $\prod A$. Let $N \prec H_\Psi$ be a κ-presentable elementary substructure (Ψ is a sufficiently large cardinal) such that $A, \lambda_0, f^{\lambda_0} \in N$. Suppose that N and $f^{\lambda_0} \in N$ satisfy properties (14.32) and (14.33) for $\lambda = \lambda_0$. Put $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A \mid \mathrm{Ch}_N(a) = f^{\lambda_0}_{\gamma_0}(a)\}.$$

Then the following hold.

1. b_{λ_0} *is a* $B_{\lambda_0}[A]$ *set, namely*

$$J_{\leq\lambda_0}[A] = J_{<\lambda_0}[A] + b_{\lambda_0}.$$

2. There exists a set $b'_{\lambda_0} \subseteq b_{\lambda_0}$ *such that*

(a) $b'_{\lambda_0} \in N$.

(b) $b_{\lambda_0} \setminus b'_{\lambda_0} \in J_{<\lambda_0}[A]$ *(hence* b'_{λ_0} *is also a* B_{λ_0} *set).*

Proof. Note that since $f^{\lambda_0}_{\gamma_0} \leq \mathrm{Ch}_N$, $b_{\lambda_0} = \{a \in A \mid \mathrm{Ch}_N(a) \leq f^{\lambda_0}_{\gamma_0}(a)\}$. We have already observed in the paragraph preceding the lemma that 1 holds.

We prove 2. As the definition of b_{λ_0} involves N and $f^{\lambda_0}_{\gamma_0}$, we do not expect that $b_{\lambda_0} \in N$. However we shall find an inner approximation b'_{λ_0} of b_{λ_0} that lies in N. If $a \in A$ and $f^{\lambda_0}_{\gamma_0}(a) < \mathrm{Ch}_N(a)$, then there exists some $\alpha < \kappa$ such that $f^{\lambda_0}_{\gamma_0}(a) < \mathrm{Ch}_{N_\alpha}(a)$ (because $N = \bigcup_{\alpha<\kappa} N_\alpha$). Since $|A| < \kappa$, there is a sufficiently large $\alpha < \kappa$ such that

$$f^{\lambda_0}_{\gamma_0}(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f^{\lambda_0}_{\gamma_0}(a) < \mathrm{Ch}_{N_\alpha}(a)$$

holds for every $a \in A$. Or equivalently (by negating both sides)

$$a \in b_{\lambda_0} \quad \text{iff} \quad \mathrm{Ch}_{N_\alpha}(a) \leq f^{\lambda_0}_{\gamma_0}(a).$$

That is, we have replaced the parameter N with N_α in the definition of b_{λ_0}, but γ_0 is still a parameter not in N.

Since f^{λ_0} satisfies (14.33), there exists (for $h = \mathrm{Ch}_{N_\alpha}$) some function $d \in N$ such that

1. $\mathrm{Ch}_{N_\alpha} \restriction B_{\lambda_0} <_{J_{<\lambda_0}} d \restriction B_{\lambda_0}$ and
2. $d \leq f^{\lambda_0}_{\gamma_0}$.

Define

$$b'_{\lambda_0} = \{a \in A \mid \mathrm{Ch}_{N_\alpha}(a) \leq d(a)\}.$$

Now all parameters are in N and clearly $b'_{\lambda_0} \in N$. Property 1 above implies that for almost all $a \in B_{\lambda_0}$, $\mathrm{Ch}_{N_\alpha}(a) < d(a)$ (i.e. except on a $J_{<\lambda_0}$ set). Hence $B_{\lambda_0} \subseteq_{J_{<\lambda_0}} b'_{\lambda_0}$. Property 2 implies that $b'_{\lambda_0} \subseteq b_{\lambda_0}$. $\dashv$

Suppose that for every $\lambda \in \mathrm{pcf}(A)$ we attach a certain $B_\lambda[A]$ set b^*_λ. Then the Compactness Theorem 4.11 gives a finite set $\lambda_0, \ldots, \lambda_{n-1}$ of $\mathrm{pcf}(A)$ cardinals such that $A = b^*_{\lambda_0} \cup \cdots \cup b^*_{\lambda_{n-1}}$. Now let $N \prec H_\Psi$ be such that $A \in N$ and assume that the sets b^*_λ are chosen in N for each $\lambda \in \mathrm{pcf}(A) \cap N$. Then the covering cardinals $\lambda_0, \ldots, \lambda_{n-1}$ can be found in N, even when the map $\lambda \mapsto b^*_\lambda$ is not in N. To prove that, we define a descending sequence of cardinals $\lambda_0 > \cdots > \lambda_i$ by induction on i, starting with $\lambda_0 = \max \mathrm{pcf}(A)$, so that the following two conditions hold.

1. $\lambda_i \in N$.
2. If $A_k = A \setminus (b^*_0 \cup \cdots \cup b^*_{k-1}) \neq \emptyset$, then $\lambda_k = \max \mathrm{pcf}(A_k)$.

Since $b^*_0, \ldots, b^*_{k-1}$ are in N, $A_k \in N$ as well, and hence $\lambda_k \in N$ (whenever $A_k \neq \emptyset$ and λ_k is defined). It follows from Lemmas 4.14 and 4.10 that $\lambda_0 > \lambda_1 > \cdots > \lambda_k$. Hence, for some k, $A_k = \emptyset$, and then $A = b^*_0 \cup \cdots \cup b^*_{k-1}$.

Here is a main result saying that the number of characteristic functions $\mathrm{Ch}_N \restriction A$ is bounded by $\max \mathrm{pcf}(A)$.

5.9 Corollary. *Suppose that A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa < \min A$, and N with $A \in N$ is a κ-presentable elementary substructure $N \prec H_\Psi$ and containing, for every $\lambda \in N \cap \mathrm{pcf}(A)$, a sequence $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda\rangle$ that satisfies properties (14.32) and (14.33). Then there are cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(A)$ such that*

$$\mathrm{Ch}_N \restriction A = \sup\{f^{\lambda_0}_{\gamma_0}, \ldots, f^{\lambda_n}_{\gamma_n}\}, \tag{14.35}$$

where $\gamma_i = \mathrm{Ch}_N(\lambda_i)$.

Proof. We employ Lemma 5.8, which assigns $B_\lambda[A]$ sets, $b'_\lambda \in N$, for every $\lambda \in \mathrm{pcf}(A) \cap N$, so that

$$b'_\lambda \subseteq \{a \in A \mid \mathrm{Ch}_N(a) = f^\lambda_{\mathrm{Ch}_N(\lambda)}(a)\}. \tag{14.36}$$

By the inductive covering procedure explained above, for some $\lambda_0, \ldots, \lambda_{n-1}$ in $\mathrm{pcf}(A) \cap N$

$$A = b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_{n-1}}.$$

Since property (14.33) ensures that $f^\lambda_{\mathrm{Ch}_N(\lambda)} \leq \mathrm{Ch}_N$, (14.36) implies that (14.35) holds. ⊣

Application: The Cofinality of $([\mu]^\kappa, \subseteq)$

For cardinals $\kappa \leq \mu$, let $[\mu]^\kappa$ denote the collection of all subsets of μ of size κ. Under the inclusion relation $\subseteq$ this collection is a partial ordering, and we denote its cofinality by $\mathrm{cf}([\mu]^\kappa, \subseteq)$. Likewise, $[\mu]^{<\kappa}$ is the collection of all subsets of μ of cardinality less than κ. For example, if μ is a regular cardinal then the collection of all proper initial segments of μ is cofinal in $[\mu]^{<\mu}$.

One reason for the importance of studying $\mathrm{cf}([\mu]^\kappa, \subseteq)$ is the relationship

$$|[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa \tag{14.37}$$

and its applications to cardinal arithmetic (which we shall see). The proof of (14.37) is quite simple. Suppose that $\mathrm{cf}([\mu]^\kappa, \subseteq) = \lambda$ and let $Y = \{Y_i \in [\mu]^\kappa \mid i < \lambda\}$ be cofinal. A one-to-one map from $[\mu]^\kappa$ to $Y \times 2^\kappa$ can be defined as follows. For every $E \in [\mu]^\kappa$ find some $E \subseteq Y_i$. Since Y_i has cardinality κ, E is isomorphic to some subset S of κ, and then we map E to $\langle Y_i, S\rangle$.

We record some relatively simple facts about cofinalities of $[\mu]^\kappa$.

5.10 Lemma. *For any cardinal μ:*

1. *If $\kappa_1 < \kappa_2$ then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \leq \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

2. *If $\mu_1 < \mu_2$ then* $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \leq \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.

3. *Suppose that $\kappa \leq \mu$ and $E \subseteq [\mu]^\kappa$ is cofinal. Then there exists a cofinal set in $([\mu^+]^\kappa, \subseteq)$ of cardinality $|E| \cdot \mu^+$.*

Proof. We prove the third item. For every $\mu \leq \gamma < \mu^+$ let f_γ be a bijection from γ to μ. Then the collection of all sets of the form $f_\gamma^{-1}X$, where $X \in E$, is cofinal and of cardinality $|E| \cdot \mu^+$. ⊣

A consequence (which can be proved by induction) is that for every $n < \omega$, $\mathrm{cf}([\aleph_n]^{\aleph_0}, \subseteq) = \aleph_n$.

The first application of pcf theory to the subset cofinality question is the following:

5.11 Theorem. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ is an infinite cardinal such that the interval A of regular cardinals in (κ, μ) has size $\leq \kappa$. Then*

$$\mathrm{cf}([\mu]^\kappa, \subseteq) = \max \mathrm{pcf}(A).$$

Proof. Let μ and κ be as in the theorem. Define

$$A = \{\gamma \mid \gamma \text{ is a regular cardinal and } \kappa < \gamma < \mu\}.$$

We assume that $|A| \leq \kappa$, so that A is a progressive interval of regular cardinals. To prove the theorem, we first prove the easier inequality $\geq$. Let $\{X_i \mid i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality $\mathrm{cf}([\mu]^\kappa, \subseteq)$. Define $h_i = \mathrm{Ch}_{X_i} \restriction A$. That is, $h_i(a) = \sup(X_i \cap a)$ for $a \in A$. Then $\{h_i \mid i \in I\}$ is cofinal in $(\prod A, <)$. (Since for every $f \in \prod A$ the range of f is a subset of μ of size $\leq |A| \leq \kappa$, and is hence covered by some X_i. So $f \leq h_i$.) But we know that the cofinality of $(\prod A, <)$ is $\max \mathrm{pcf}(A)$, and hence $|I| \geq \max \mathrm{pcf}(A)$.

Now we prove the $\leq$ inequality. We assume first that $|A| < \kappa$ and prove the $\leq$ inequality for this case. Then we can obtain the $|A| = \kappa$ case by applying the first case to κ^+ (instead of κ) and using

$$\mathrm{cf}([\mu]^\kappa, \subseteq) \leq \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+.$$

So assume that $|A| < \kappa$ (and hence κ is uncountable). We plan to present a cofinal subset of $[\mu]^\kappa$ of cardinality $\max \mathrm{pcf}(A)$. Fix for every $\rho \in \mathrm{pcf}(A)$ a minimally obedient (at cofinality κ) universal sequence for ρ, and let $f = \{f^\rho \mid \rho \in \mathrm{pcf}(A)\}$ be the resulting array of sequences. In fact, we let f be the minimal such array in the well-ordering $<^*$ of H_Ψ, so that $f \in M$ for every $M \prec H_\Psi$ such that $A \in M$. Let $\mathcal{M}$ be the collection of all substructures $M \prec$

H_Ψ that are κ-presentable and such that $A \in M$ (so $A \subseteq M$). We know that (14.32) and (14.33) hold. Consider the collection $F = \{M \cap \mu \mid M \in \mathcal{M}\}$. This collection is clearly cofinal in $[\mu]^\kappa$, since for any set $X \in [\mu]^\kappa$ a structure $M \in \mathcal{M}$ can be defined so that $X \subseteq M$ (or even $X \in M$). We shall prove that $|F| \leq \max \operatorname{pcf}(A)$. We know (by Corollary 5.9) that for every $M \in \mathcal{M}$, $\mathrm{Ch}_M \restriction A$ is the supremum of a finite number of functions taken from the array $\{f^\rho \mid \rho \in \operatorname{pcf}(A)\}$, which contains $\max \operatorname{pcf}(A)$ functions. Hence it suffices to prove that whenever $M, N \in \mathcal{M}$ are such that $\mathrm{Ch}_M \restriction A = \mathrm{Ch}_N \restriction A$, then $M \cap \mu = N \cap \mu$. But this is exactly Theorem 5.2. ⊣

The theorem just proved (Theorem 5.11) has important consequences for cardinal arithmetic which we shall explore now. Look, for example, at $\mu = \aleph_\omega$, $\kappa = \aleph_0$, and $A = \{\aleph_n \mid 1 < n < \omega\}$. Then

$$\operatorname{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max \operatorname{pcf}(A).$$

So $\aleph_\omega^{\aleph_0} = (\max \operatorname{pcf}(A)) + 2^{\aleph_0}$. If $\aleph_\omega$ is a strong limit cardinal then $[\aleph_\omega]^\omega$ has cardinality $2^{\aleph_\omega}$, and this cardinal turns out to be regular since it is $\max \operatorname{pcf}(A)$. Similarly, for every $n < \omega$, $\operatorname{cf}([\aleph_\omega, \subseteq]^{\aleph_n}, \subseteq) = \max \operatorname{pcf}(A)$. Hence

$$\operatorname{cf}([\aleph_\omega]^{\aleph_n}, \subseteq) = \operatorname{cf}([\aleph_\omega]^{\aleph_m}, \subseteq)$$

for every $n, m < \omega$.

Since A is an interval of regular cardinals, $\operatorname{pcf}(A)$ is also an interval of regular cardinals (Theorem 3.9) containing all regular cardinals from $\aleph_2$ to $\max \operatorname{pcf}(A)$. Hence if we write $\max \operatorname{pcf}(A) = \aleph_\alpha$, then $|\alpha| = |\operatorname{pcf}(A)|$ follows. Yet $|\operatorname{pcf}(A)| \leq 2^{\aleph_0}$ (Theorem 3.6). Thus $\operatorname{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \aleph_\alpha$ for an $\alpha < (2^{\aleph_0})^+$. Thus we have proved the following theorem.

5.12 Theorem. $\operatorname{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{(2^{\aleph_0})^+}$.

An immediate conclusion is

5.13 Theorem. $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$.

Proof. If $2^{\aleph_0} > \aleph_\omega$ (equality is impossible by Zermelo-König theorem) then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$, and then $2^{\aleph_0} \leq \aleph_{2^{\aleph_0}}$ implies the theorem as a triviality. So we assume that $2^{\aleph_0} < \aleph_\omega$.

Suppose that $\aleph_\alpha = \operatorname{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$. We have proved in the preceding theorem that $\alpha < (2^{\aleph_0})^+$. Let $\{X_i \mid i < \aleph_\alpha\} \subseteq [\aleph_\omega]^{\aleph_0}$ be cofinal. So $[\aleph_\omega]^{\aleph_0} \subseteq \bigcup\{\mathcal{P}(X_i) \mid i < \aleph_\alpha\}$. Hence $|[\aleph_\omega]^{\aleph_0}| \leq 2^{\aleph_0} \cdot \aleph_\alpha = \aleph_\alpha < \aleph_{(2^{\aleph_0})^+}$. ⊣

We want to generalize this theorem to arbitrary singular cardinals $\aleph_\delta$ such that $\delta < \aleph_\delta$. A straightforward generalization gives the following which we leave as an exercise: If δ is a limit ordinal such that $\delta < \aleph_\delta$, then

$$\operatorname{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(2^{|\delta|})^+}$$

and hence

$$\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

We shall describe now a tighter bound: $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$.

As in the proof for bounding $\aleph_\omega^{\aleph_0}$, which consists in first evaluating the cofinality of $([\aleph_\omega]^{\aleph_0}, \subseteq)$, here too we first investigate cardinalities of covering sets. For cardinals $\mu \geq \tau$ a *cover* for $[\mu]^{<\tau}$ is a collection $\mathcal{C}$ of subsets of μ such that for every $X \in [\mu]^{<\tau}$ there exists a $Y \in \mathcal{C}$ with $X \subseteq Y$. For cardinals $\mu \geq \theta \geq \tau$, $\mathrm{cov}(\mu, \theta, \tau)$ is the least cardinality of a cover for $[\mu]^{<\tau}$ consisting of sets taken from $[\mu]^{<\theta}$. So $\mathrm{cov}(\mu, \theta, \tau)$ measures how many sets, each of cardinality $< \theta$, are needed to cover every subset of μ of cardinality $< \tau$. For example, $\mathrm{cf}([\mu]^\kappa, \subseteq) = \mathrm{cov}(\mu, \kappa^+, \kappa^+)$. We shall prove the following.

5.14 Theorem. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ a regular cardinal. Let A be the set of all regular cardinals in the interval $[\kappa^{++}, \mu)$. If $|A| \leq \kappa$, then*

$$\mathrm{cov}(\mu, \kappa^+, \mathrm{cf}(\mu)^+) = \sup \mathrm{pcf}_{\mathrm{cf}(\mu)}(A).$$

(See Definition 3.10 for $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$*.)*

Before proving this theorem, let's see how it can be employed.

5.15 Corollary. *Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Then*

$$\mathrm{cov}(\aleph_\delta, |\delta|^+, \mathrm{cf}(\delta)^+) < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$$

and hence

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

Proof. Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Let $\mu = \aleph_\delta$, and $\kappa = |\delta|^+$. Define A as the set of all regular cardinals in the interval $(\kappa^{++}, \mu]$. So $|A| \leq |\delta|$. By Theorem 5.14, there exists a collection $\{X_i \mid i \in I\}$, where $X_i \in [\mu]^\kappa$ and $|I| = \sup \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$, such that for every $Z \in [\aleph_\delta]^{\mathrm{cf}(\mu)}$, $Z \subseteq X_i$ for some $i \in I$. Yet, by Theorem 3.11, $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ is also an interval of regular cardinals, containing all regular cardinals in the interval $[\kappa^{++}, \aleph_\alpha)$ where $\aleph_\alpha = \sup \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$. Now $|\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)| \leq |[A]^{\mathrm{cf}(\mu)}| \cdot 2^{\mathrm{cf}(\mu)} \leq |\delta|^{\mathrm{cf}(\mu)}$. It follows (see the proof in the following paragraph) that $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. That is, $|I| < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ (as $\mathrm{cf}(\mu) = \mathrm{cf}(\delta)$). Hence $|[\aleph_\delta]^{\mathrm{cf}(\delta)}| < \kappa^{\mathrm{cf}(\delta)} \cdot \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$. Thus $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ as required.

We prove that $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. Since $\delta < |\delta|^+ \leq (|\delta|^{\mathrm{cf}(\mu)})^+$, it follows that the interval $(\aleph_\delta, \aleph_{(|\delta|^{\mathrm{cf}(\mu)})^+})$ contains $(|\delta|^{\mathrm{cf}(\mu)})^+$ regular cardinals. But the interval of regular cardinals in $(\aleph_\delta, \aleph_\alpha)$ is included in $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ and contains $\leq |\delta|^{\mathrm{cf}(\mu)}$ regular cardinals. Hence $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. $\dashv$

Proof of Theorem 5.14. Let $\kappa < \mu$ and $|A| \leq \kappa$ be as in the theorem. Since A is cofinal in μ and $|A| \leq \kappa$, $\mathrm{cf}(\mu) \leq \kappa$. Let $\rho = \mathrm{cf}(\mu)$ be the cofinality of μ. We shall prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) = \sup \mathrm{pcf}_\rho(A)$.

For the $\geq$ inequality, we must prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) \geq \lambda$ for every $\lambda \in \mathrm{pcf}_\rho(A)$. That is, if $A_0 \subseteq A$ is of cardinality ρ we want to prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) \geq \max \mathrm{pcf}(A_0)$. So let $\{X_i \mid i \in I\}$ be a covering of $[\mu]^{\mathrm{cf}(\mu)}$ with sets X_i of cardinality $\leq \kappa$. For each X_i define $h_i = \mathrm{Ch}_{X_i} \restriction A_0$. Then $\{h_i \mid i \in I\}$ is cofinal in $(\prod A_0, <)$, and hence $|I| \geq \max \mathrm{pcf}(A_0)$.

For the $\leq$ inequality, we must provide a covering set for $\mathrm{cov}(\mu, \kappa^+, \rho^+)$ of cardinality $\sup \mathrm{pcf}_\rho(A)$.

For every $\lambda \in \mathrm{pcf}_\rho(A)$, $\lambda \in \mathrm{pcf}(A)$ as well, and we fix a minimally obedient at cofinality ρ^+ sequence $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda \rangle$ of functions in $\prod A$ that is universal for λ.

For every $\alpha < \mu$ such that $\mathrm{cf}(\alpha) = \rho^+$, let $E_\alpha \subseteq \alpha$ be a closed unbounded subset of α of order-type ρ^+.

Define $\mathcal{F}$ as the collection of all functions of the form $\sup\{f^{\lambda_1}_{\alpha_1}, \ldots, f^{\lambda_n}_{\alpha_n}\}$ where $\lambda_i \in \mathrm{pcf}_\rho(A)$ and $\alpha_i < \lambda_i$. Clearly $\mathcal{F}$ has cardinality $\sup \mathrm{pcf}_\rho(A)$. For $f \in \mathcal{F}$ let

$$E(f) = \bigcup \{E_{f(a)} \mid a \in A \text{ and } \mathrm{cf}(f(a)) = \rho^+\}.$$

Then the cardinality of $E(f)$ is at most κ^+. Let

$$K(f) = \mathrm{Skolem}(E(f) \cup \kappa^+) \prec H_\Psi$$

be the Skolem hull (closure) of $E(f) \cup \kappa^+$. We remind the reader that the structure H_Ψ includes a class well-ordering $<^*$ of all sets, and hence there is a countable set of Skolem functions for H_Ψ so that $X \prec H_\Psi$ iff X is closed under all of these Skolem functions. The cardinality of $K(f)$ is κ^+.

Clearly $\mathcal{K} = \{K(f) \mid f \in \mathcal{F}\}$ has cardinality $\leq \sup \mathrm{pcf}_\rho(A)$. Our aim now is to show that

$$\mathcal{K} \text{ covers } [\mu]^{\mathrm{cf}(\mu)}.$$

Since $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$, this yields that

$$\mathrm{cov}(\mu, \kappa^+, \mathrm{cf}(\mu)^+) = \sup \mathrm{pcf}_{\mathrm{cf}(\mu)}(A).$$

Let $Z \subseteq \mu$ be of size $\rho = \mathrm{cf}(\mu)$. Define $\langle M_i \mid i < \rho^+ \rangle$ an increasing and continuous chain of elementary substructures $M_i \prec H_\Psi$, each of cardinality ρ, such that $A, Z \in M_0$, $M_i \in M_{i+1}$, and $Z \subseteq M_0$. Let $M = \bigcup_{i<\rho^+} M_i$ be the resulting ρ^+-presentable structure.

For every $a \in A \cap M$ (and in fact for every $a \in A$), $\mathrm{Ch}_M(a)$ has cofinality ρ^+. Indeed $\langle \mathrm{Ch}_{M_i}(a) \mid i < \rho^+ \rangle$ is increasing, continuous and with limit $\mathrm{Ch}_M(a)$. There is another closed unbounded sequence in $\mathrm{Ch}_M(a)$ which interests us, namely $E_{\mathrm{Ch}_M(a)}$, and we consider the intersection of these two closed unbounded sets. So there exists a closed unbounded set $D_a \subseteq \rho^+$ such that for every $i \in D_a$

$$\mathrm{Ch}_{M_i}(a) \in E_{\mathrm{Ch}_M(a)}. \tag{14.38}$$

For every $i < \rho^+$, M_i has cardinality ρ and hence $D(i) = \bigcap \{D_a \mid a \in A \cap M_i\}$ is closed unbounded in ρ^+. Form the diagonal intersection $D =$

$\{j \in \rho^+ \mid \forall i < j(j \in D(i))\}$. Fix any $j_0 \in D'$ (a limit point of D). For every $a \in A \cap M_{j_0}$ there exists some $j_1 < j_0$ such that $a \in A \cap M_{j_1}$. If $j_1 < i \le j_0$ and $i \in D$, then $i \in D(j_1)$ and hence $i \in D_a$. So $\mathrm{Ch}_{M_i}(a) \in E_{\mathrm{Ch}_M(a)}$. Thus $\langle \mathrm{Ch}_{M_i}(a) \mid j_1 < i < j_0 \wedge i \in D\rangle$ is a sequence of ordinals in $M_{j_0} \cap E_{\mathrm{Ch}_M(a)}$ that tends to $\mathrm{Ch}_{M_{j_0}}(a)$ (whenever $j_0 \in D'$ and $a \in A \cap M_{j_0}$).

Define $A_0 = A \cap M_{j_0}$. Then $A_0 \in [A]^{\mathrm{cf}(\mu)}$, and $A_0 \in M$. We plan to apply Corollary 5.9 to A_0, ρ^+ and M (substituting A, κ, and N there). For every $\lambda \in \mathrm{pcf}(A_0)$, $\lambda \in \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ and the sequence $\langle f^\lambda_\xi \restriction A_0 \mid \xi < \lambda\rangle$ is, in M, universal for λ and minimally obedient at ρ^+. Hence, by Corollary 5.9,

$$\mathrm{Ch}_M \restriction A_0 = f \restriction A_0 \quad \text{for some } f \in \mathcal{F}. \tag{14.39}$$

Since $Z \subseteq M_{j_0}$, the following proves that $Z \subseteq K(f)$.

Claim. *$M_{j_0} \cap \mu \subseteq K(f)$.*

By Lemma 5.2, this is a consequence of the following

5.16 Lemma. *For every successor cardinal $\sigma^+ \in M_{j_0} \cap \mu$*

$$\sup(M_{j_0} \cap \sigma^+) = \sup(M_{j_0} \cap K(f) \cap \sigma^+).$$

Proof. Assume that $\sigma^+ \in M_{j_0} \cap \mu$. If $\sigma^+ \le \kappa^+$, then $\kappa^+ \subseteq K(f)$ implies the lemma immediately. So assume that $\sigma^+ > \kappa^+$, and hence that $\sigma^+ \in A \cap M_{j_0} = A_0$. Now (14.39) implies that $\mathrm{Ch}_M(\sigma^+) = f(\sigma^+) = \alpha$. Hence $\mathrm{cf}(\alpha) = \rho^+$ and $E_\alpha \subseteq E(f) \subseteq K(f)$. The sequence $\langle \mathrm{Ch}_{M_i}(\sigma^+) \mid j_1 < i < j_0 \wedge i \in D\rangle$ is unbounded in $\mathrm{Ch}_{M_{j_0}}(\sigma^+)$, as we have observed above, and thus shows that the lemma is correct. ⊣

This completes the proof of Theorem 5.14. ⊣

5.17 Exercise.

1. Let μ, κ, and A be as in Theorem 5.14. Suppose that $|A| \le \kappa$. Prove that
$$\mathrm{cov}(\mu, \kappa^+, \aleph_1) = \sup \mathrm{pcf}_{\aleph_0}(A).$$
Conclude that if $\delta < \aleph_\delta$ is a limit ordinal, then
$$\aleph_\delta^{\aleph_0} < \aleph_{(|\delta|^{\aleph_0})^+}.$$
Hint. By induction on μ.

2. Suppose that δ is a limit ordinal such that for every cardinal $\mu < \delta$ $\mu^{\aleph_0} < \delta$. Then $\aleph_\delta$ satisfies the same property, namely for every $\mu < \aleph_\delta$, $\mu^{\aleph_0} < \aleph_\delta$.

   Hint. Without loss of generality, $\delta < \aleph_\delta$. Prove that $\mu^{\aleph_0} < \aleph_\delta$ by induction on $\mu < \aleph_\delta$.

6. Elevations and Transitive Generators

Given a progressive set A of regular cardinals, we have proved the existence of generating sets $B_\lambda = B_\lambda[A]$. Suppose that N is such that $A \subseteq N \subseteq \mathrm{pcf}(A)$ and $B = \langle B_\lambda \mid \lambda \in N \rangle$ is a generating sequence (defined only for λ in N). Then B is said to be *smooth* (or transitive) if for every $\lambda \in N$ and $\theta \in B_\lambda$, $B_\theta \subseteq B_\lambda$.

This definition is trivial when $B_\theta = \{\theta\}$ (that is when $\theta \notin \mathrm{pcf}(A \cap \theta)$). However, we shall be interested in A's for which $\theta \in \mathrm{pcf}(A \cap \theta)$ is possible for $\theta \in A$. The reason for considering subsets N of $\mathrm{pcf}(A)$ in this definition, rather than the whole $\mathrm{pcf}(A)$ (which would be most desirable) is that we only know how to prove the existence of smooth sequences for sets N of cardinality $\min A$.

Our aim is to obtain transitive generators; they will be useful in proving, for example, that for every progressive interval of regular cardinals A, $|\mathrm{pcf}(A)| < |A|^{+4}$. However, there is still some material to cover beforehand.

Fix a progressive set A of regular cardinals and let κ be a regular cardinal such that $|A| < \kappa < \min A$. For every $\lambda \in \mathrm{pcf}(A)$ let $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda \rangle$ be a universal sequence for λ which is minimally obedient (at cofinality κ). It is convenient to assume that for $a \in A \setminus \lambda$, $f^\lambda_\xi(a) = \xi$. The *elevation* of the array $\langle f^\lambda \mid \lambda \in \mathrm{pcf}(A) \rangle$ is another array $\langle F^\lambda \mid \lambda \in \mathrm{pcf}(A) \rangle$ of persistently cofinal sequences defined below, and which will be shown to satisfy properties (14.32) and (14.33).

For every finite, decreasing sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of cardinals such that $\lambda_0 \in \mathrm{pcf}(A)$ and $\lambda_{i+1} \in A \cap \lambda_i$ for $i < n$, and for every ordinal $\gamma_0 \in \lambda_0$, define a sequence $\gamma_1 \in \lambda_1, \ldots, \gamma_n \in \lambda_n$ by

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1}). \tag{14.40}$$

So $\gamma_1 = f^{\lambda_0}_{\gamma_0}(\lambda_1)$, $\gamma_2 = f^{\lambda_1}_{\gamma_1}(\lambda_2)$, etc. until $\gamma_n = f^{\lambda_{n-1}}_{\gamma_{n-1}}(\lambda_n)$. Now define the elevation function $\mathrm{El}_{\lambda_0,\ldots,\lambda_n}$ on λ_0 by

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

We say that the last value obtained, γ_n, is reached from $f^{\lambda_0}_{\gamma_0}$ via the descending sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$.

Fix a cardinal $\lambda_0 \in \mathrm{pcf}(A)$. We want to define the elevated sequence F^{λ_0}, first on $A \cap \lambda_0$. Given any $\lambda \in A \cap \lambda_0$, let $F_{\lambda_0,\lambda}$ be the set of all finite, descending sequences $\langle \lambda_0 > \lambda_1 > \cdots > \lambda_n \rangle$ leading from λ_0 to $\lambda_n = \lambda$, such that λ_i for $i > 0$ are all in A. For every $\gamma_0 \in \lambda_0$ we ask whether there is a maximal value among

$$\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \mid \langle \lambda_0, \ldots, \lambda_n \rangle \in F_{\lambda_0,\lambda}\}.$$

If this set contains a maximum, let $F^{\lambda_0}_{\gamma_0}(\lambda)$ be that maximum, and otherwise put $F^{\lambda_0}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. In case $\lambda \in A$ and $\lambda \geq \lambda_0$, define $F^{\lambda_0}_{\gamma_0}(\lambda) = \gamma_0$. So $F^{\lambda_0} = \langle F^{\lambda_0}_\gamma \mid \gamma < \lambda_0 \rangle$ with $F^{\lambda_0}_\gamma \in \prod A$ is defined.

The elevated array $\langle F^{\lambda_0} \mid \lambda_0 \in \mathrm{pcf}(A)\rangle$ thus defined will give the required transitive generating sequence. Observe first that

$$f^{\lambda_0}_\gamma \leq F^{\lambda_0}_\gamma \quad \text{for every } \gamma < \lambda_0.$$

This is so because $\mathrm{El}_{\lambda_0,\lambda}(\gamma_0) = f^{\lambda_0}_{\gamma_0}(\lambda)$ for every $\lambda \in A \cap \lambda_0$; so that this original value is among the values considered for maximum. Hence

$$F^{\lambda_0} \text{ is persistently cofinal for } \lambda_0.$$

This shows that Lemma 5.4 can be applied and property (14.32) holds whenever $F^\lambda \in N$ (and N is κ-presentable).

Another observation concerns any κ-presentable elementary substructure $N \prec H_\Psi$ such that $A, \langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle \in N$. Being definable, the elevated array is also in N. Even though each f^λ is assumed to be minimally obedient, the elevated sequence F^λ is not anymore club-obedient. We have however the following consequence of Lemma 5.7.

6.1 Lemma. *If $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$, then for every $\lambda \in A \cap \lambda_0$, $F^{\lambda_0}_{\gamma_0}(\lambda) \in \bar{N} \cap \lambda$ (where $\bar{N}$ is the ordinal closure of N). Thus the elevated sequence F^{λ_0} satisfies (14.33). Namely,*

1. *$F^{\lambda_0}_{\gamma_0}(\lambda) \leq \mathrm{Ch}_N(\lambda)$ for every $\lambda \in A$, and*
2. *for every $h \in N \cap \prod A$ there exists some $d \in N \cap \prod A$ such that*

$$h{\restriction}B <_{J_{<\lambda_0}} d{\restriction}B \quad \textit{and} \quad d \leq F^{\lambda_0}_{\gamma_0}$$

where $B = B_{\lambda_0}[A]$.

Proof. Observe first that $A \subseteq N$, $\lambda \in N$, and $F_{\lambda_0,\lambda} \subseteq N$. Consider any $\langle \lambda_0, \ldots, \lambda_n\rangle \in F_{\lambda_0,\lambda}$ and the ordinals γ_i defined by (14.40). It follows from Lemma 5.7 that $\gamma_i \in \bar{N}$. If $\gamma_i \in N$ then obviously $\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1}) \in N$. If, however, $\gamma_i \in \bar{N} \setminus N$, then Lemma 5.7 yields that $f^{\lambda_i}_{\gamma_i}(a) \in \bar{N}$ for every $a \in A$, and in particular $\gamma_{i+1} \in \bar{N}$. Thus $\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \in \bar{N}$ and hence $F^{\lambda_0}_{\gamma_0}(\lambda) \in \bar{N} \cap \lambda$.

Thus (14.33)(1) holds for F^{λ_0}. Since $f^{\lambda_0} \leq F^{\lambda_0}$, where f^{λ_0} is universal and minimally obedient at κ, (14.33)(2) holds as well. ⊣

6.2 Lemma. *Let A, f, and N be as in the previous lemma. Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and $\lambda \in A \cap \lambda_0$.*

1. *If for some descending sequence $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F_{\lambda_0,\lambda}$*

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \mathrm{Ch}_N(\lambda).$$

Then $\mathrm{Ch}_N(\lambda)$ is the maximal value in $\{\mathrm{El}_{\bar{\lambda}}(\gamma_0) \mid \bar{\lambda} \in F_{\lambda_0,\lambda}\}$ and hence

$$\mathrm{Ch}_N(\lambda) = F^{\lambda_0}_{\gamma_0}(\lambda).$$

2. *Suppose that*

$$F^{\lambda_0}_{\gamma_0}(\lambda) = \gamma.$$

For any $a \in A \cap \lambda$, *if*

$$F^{\lambda}_{\gamma}(a) = \mathrm{Ch}_N(a),$$

then

$$F^{\lambda_0}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$$

as well.

Proof. Item 1 says that if some descending sequence leading from λ_0 to λ reaches $\mathrm{Ch}_N(\lambda)$, then no sequence reaches a higher value. But this is clear from Lemma 6.1 since $\mathrm{Ch}_N(\lambda)$ is the maximal possible value.

Item 2 uses item 1. It says that if γ can be reached from $f^{\lambda_0}_{\gamma_0}$ by a finite descending sequence leading to λ, and if there is another sequence leading from λ to a, so that $\mathrm{Ch}_N(a)$ can be reached from f^{λ}_{γ}, then $\mathrm{Ch}_N(a)$ can be reached already from $f^{\lambda_0}_{\gamma_0}$ via the concatenation of these descending sequences (and no higher value can be reached—by 1). $\dashv$

Now we can get our transitive generating sequence.

6.3 Theorem (Transitive Generators). *Suppose that A is a progressive set of regular cardinals, and $|A| < \kappa < \min A$ is a regular cardinal. Let $\langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ be an array of minimally obedient (at cofinality κ) universal sequences. Let $N \prec H_\Psi$ be an elementary substructure that is κ-presentable and such that $A, \langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle \in N$. Let $\langle F^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = Ch_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A \mid \mathrm{Ch}_N(a) = F^{\lambda_0}_{\gamma_0}(a)\}.$$

then the following hold:

1. *Every b_{λ_0} is a $B_{\lambda_0}[A]$ set, namely*

$$J_{\leq\lambda_0}[A] = J_{<\lambda_0}[A] + b_{\lambda_0}.$$

2. *There exists sets $b'_{\lambda_0} \subseteq b_{\lambda_0}$, for $\lambda_0 \in \mathrm{pcf}(A) \cap N$, such that*

   (a) $b_{\lambda_0} \setminus b'_{\lambda_0} \in J_{<\lambda_0}[A]$.

   (b) $b'_{\lambda_0} \in N$ (but the sequence $\langle b'_{\lambda_0} \mid \lambda_0 \in \mathrm{pcf}(A) \cap N\rangle$ is not claimed to be in N).

3. *The collection $\langle b_\lambda \mid \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive; which means that if $\lambda_1 \in b_\lambda$ then $b_{\lambda_1} \subseteq b_\lambda$.*

Proof. The elevated sequence F^{λ_0} satisfies properties (14.32) (because it is persistently cofinal) and (14.33) (as shown in Lemma 6.1). Thus items 1 and 2 of our lemma follow from Lemma 5.8. Observe that $b_{\lambda_0} \subseteq \lambda_0 + 1$, since $B_{\lambda_0} \in J_{<\lambda_0^+}$.

Transitivity (item 3) relies on Lemma 6.2. Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\lambda_1 \in b_{\lambda_0}$. This means

$$\mathrm{Ch}_N(\lambda_1) = F^{\lambda_0}_{\gamma_0}(\lambda_1)$$

where $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$. Say $\mathrm{Ch}_N(\lambda_1) = \gamma_1$. We have to show that $b_{\lambda_1} \subset b_{\lambda_0}$ in this case. So assume that $a \in b_{\lambda_1}$. This means

$$\mathrm{Ch}_N(a) = F^{\lambda_1}_{\gamma_1}(a).$$

Now Lemma 6.2(2) applies and yields

$$F^{\lambda_0}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$$

which gives $a \in b_{\lambda_0}$. ⊣

Localization

Localization is the following property of the pcf function which will be proved in this subsection.

> If A is a progressive set of regular cardinals and $B \subseteq \mathrm{pcf}(A)$ is also progressive, then for every $\lambda \in \mathrm{pcf}(B)$ there exists a $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.

The localization property implies that there exists no $B \subseteq \mathrm{pcf}(A)$ with $|B| = |A|^+$ and such that $b > \max \mathrm{pcf}(B \cap b)$ for every $b \in B$. For indeed if there were such a B it would be progressive, and if we define $\lambda = \max \mathrm{pcf}(B)$, then λ is not in the pcf of any proper initial segment of B. In fact, $\lambda > \max \mathrm{pcf}(B_0)$ for any proper initial segment B_0 of B. It is this conclusion, the simplest case of localization, which is proved first.

6.4 Theorem. *Assume that A is a progressive set of regular cardinals. Then there is no set $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$, and, for every $b \in B$, $b > \max \mathrm{pcf}(B \cap b)$.*

Proof. Assume on the contrary that A is as in the theorem and yet, for some $B \subseteq \mathrm{pcf}(A)$ of cardinality $|A|^+$, $b > \max \mathrm{pcf}(B \cap b)$ for every $b \in B$. Since A is progressive $|A| < \min A$, and in case $|A|^+ \in A$ we may remove the first cardinal of A and assume that $|A|^+ < \min A$. The set $E = A \cup B$ of cardinality $|A|^+$ thus satisfies $|E| < \min E$ and the Transitive Generators Theorem 6.3 can be applied to E.

Find a κ-presentable elementary substructure, $N \prec H_\Psi$, that contains A and B where $\kappa = |E|$. Let $\langle b_\lambda \mid \lambda \in \mathrm{pcf}(E) \cap N \rangle$ be the set of transitive generators (subsets of E) as guaranteed by Theorem 6.3. Let $b'_\lambda \in N$ be such that $b'_\lambda \subseteq b_\lambda$ and $b_\lambda \setminus b'_\lambda \in J_{<\lambda}$.

Since $|A| < |B|$ we can find an initial segment $B_0 \subseteq B$ of cardinality $|A|$ such that if an arbitrary $a \in A$ is in some b_β, $\beta \in B$, then it is already in some b_β with $\beta \in B_0$. Namely

$$\forall a \in A[(\exists \beta \in B) a \in b_\beta \implies (\exists \beta \in B_0) a \in b_\beta]. \tag{14.41}$$

Let $\beta_0 = \min(B \setminus B_0)$. So $B_0 = B \cap \beta_0$ and $B_0 \in N$.

Claim. *There exists a finite descending sequence of cardinals $\lambda_0 > \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(B_0)$ such that*

$$B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}. \tag{14.42}$$

Proof. In fact we shall find a finite sequence $\lambda_0, \ldots, \lambda_n \in N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n}$. The proof is the same as that of Theorem 4.11, but one must be a little bit more careful to ensure that the pcf index-cardinals are in N.

So let $\lambda_0 = \max \mathrm{pcf}(B_0)$. Clearly $\lambda_0 \in N$ and hence $b'_{\lambda_0} \in N$. So $B_1 = B_0 \setminus b'_{\lambda_0} \in N$, and $\lambda_1 = \max \mathrm{pcf}(B_1) \in N \cap \lambda_0$. Next define $B_2 = B_1 \setminus b'_{\lambda_1}$ etc. The point is that we have $B_i \in N$ since $b'_{\lambda_{i-1}} \in N$, and we must stop with $B_{n+1} = \emptyset$ after a finite number of steps since $\lambda_0 > \lambda_1 > \cdots$. Since $b'_{\lambda_i} \subseteq b_{\lambda_i}$, (14.42) holds. $\dashv$

The following claim will bring the desired contradiction and thus prove the theorem. Recall that $\beta_0 = \min(B \setminus B_0)$ and thus $\beta_0 > \max \mathrm{pcf}(B_0) \geq \lambda_0, \ldots, \lambda_n$. Since $\beta_0 \in \mathrm{pcf}(A)$, $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$ (or else $\beta_0 \in \mathrm{pcf}(A \setminus b_{\beta_0})$ which is impossible by Lemma 4.14). Yet the following inclusion shows that this is impossible.

6.5 Claim. $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$.

Proof. Consider any cardinal $a \in b_{\beta_0} \cap A$. Then

$$a \in b_\beta$$

for some $\beta \in B_0$ (by 14.41). As $B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$, $\beta \in b_{\lambda_i}$ for some $0 \leq i \leq n$. But transitivity implies

$$b_\beta \subseteq b_{\lambda_i}$$

and hence

$$a \in b_{\lambda_i}$$

as required. This claim shows that $\max \mathrm{pcf}(b_{\beta_0} \cap A) < \beta_0$, and yet $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$ which is a contradiction! $\dashv$

Thus Theorem 6.4 is proved. $\dashv$

Now we pass to the general case and prove the localization theorem.

6.6 Theorem (Localization). *Suppose that A is a progressive set of regular cardinals. If $B \subseteq \operatorname{pcf}(A)$ is also progressive, then for every $\lambda \in \operatorname{pcf}(B)$ there exists a $B_0 \subseteq B$ with $|B_0| \leq |A|$ and such that $\lambda \in \operatorname{pcf}(B_0)$.*

Proof. We prove by induction on λ that for every A and B as in the theorem the conclusion holds for λ. Replacing B with $B_\lambda[B]$, we may assume that $\lambda = \max \operatorname{pcf}(B)$.

6.7 Claim. *We may assume that the set $\lambda \cap \operatorname{pcf}(B)$ has no maximal cardinal.*

Proof. Suppose on the contrary the existence of some $\lambda_0 = \max(\lambda \cap \operatorname{pcf}(B))$. It is easy to remove λ_0 by defining

$$B_1 = B \setminus B_{\lambda_0}[B].$$

Then $\lambda \in \operatorname{pcf}(B_1)$ still holds since $B_{\lambda_0} \in J_{<\lambda}$. We can now replace B with B_1, and repeat, if necessary, this procedure a finite number of times until the claim holds (for some B_k which is renamed B). ⊣

We shall find now a set $C \subseteq \lambda \cap \operatorname{pcf}(B)$ of cardinality $\leq |A|$ such that $\lambda \in \operatorname{pcf}(C)$. Such C is necessarily progressive. Together with the inductive hypothesis this will conclude the proof; because for every $\gamma \in C$ we can pick $B(\gamma) \subseteq B$ of cardinality $\leq |A|$ and such that $\gamma \in \operatorname{pcf} B(\gamma)$, and then define $B_0 = \bigcup_{\gamma \in C} B(\gamma)$. Since $C \subseteq \operatorname{pcf}(B_0)$, $\lambda \in \operatorname{pcf}(B_0)$ will then follow from $\lambda \in \operatorname{pcf}(C)$ (by Theorem 3.12). So the following is the last piece of the proof.

6.8 Claim. *There exists a set $C \subseteq \lambda \cap \operatorname{pcf}(B)$ of cardinality $\leq |A|$ and such that $\lambda \in \operatorname{pcf}(C)$.*

Proof. Assume no such C exists. We shall construct a sequence $\langle \gamma_i \mid i \in |A|^+ \rangle$ of cardinals in $\operatorname{pcf}(B)$ such that

$$\gamma_i > \max \operatorname{pcf}\{\gamma_j \mid j < i\}.$$

This will contradict Theorem 6.4.

So suppose that $C = \{\gamma_j \mid j < i\}$ have been defined. Then

$$\lambda > \max \operatorname{pcf}(C).$$

Indeed $\lambda = \max \operatorname{pcf}(C)$ is impossible by our assumption that no such C exists, and $\lambda < \max \operatorname{pcf}(C)$ is impossible since $\operatorname{pcf}(C) \subseteq \operatorname{pcf}(B)$ and $\lambda = \max \operatorname{pcf}(B)$. We can find now $\gamma_i \in \operatorname{pcf}(B)$ above $\max \operatorname{pcf}(C)$ (recall that $\operatorname{pcf}(B)$ has no maximum below λ). ⊣

The proof of the theorem is complete. ⊣

7. Size Limitation on pcf of Intervals

This relatively short section is devoted to a theorem which occupies a central place in pcf theory and to a famous application:

$$\aleph_\omega^{\aleph_0} < \max\{(2^{\aleph_0})^+, \aleph_{\omega_4}\}.$$

The reader will notice that many of the ingredients developed so far appear in its proof. We know that for any A progressive set of regular cardinals the cardinality of $\mathrm{pcf}(A)$ does not exceed $2^{|A|}$, and it is an open question whether $|\,\mathrm{pcf}(A)| \le |A|$ or not. At present the following theorem with its enigmatic appearance of the number four is the best result.

7.1 Theorem. *Let A be an interval of regular cardinals such that $|A| < \min A$. Then*

$$|\,\mathrm{pcf}(A)| < |A|^{+4}.$$

Proof. Suppose that A is as in the theorem a progressive interval of regular cardinals, but $|\,\mathrm{pcf}(A)| \ge |A|^{+4}$. Say $|A| = \rho$. The following proof provides a sequence B of length ρ^+ of cardinals in $\mathrm{pcf}(A)$ such that each cardinal $b \in B$ is above $\max\mathrm{pcf}(B \cap b)$. This, of course, will be in contradiction to Theorem 6.4.

Let $S = S^{\rho^{+3}}_{\rho^+}$ be the set of ordinals in ρ^{+3} that have cofinality ρ^+. Choose a club guessing sequence $\langle C_k \mid k \in S\rangle$. So for every closed unbounded set $E \subseteq \rho^{+3}$ there exists some $k \in S$ such that $C_k \subseteq E$.

Consider the cardinal $\sup A$, and let σ be that ordinal such that $\aleph_\sigma = \sup A$. Since $\mathrm{pcf}(A)$ is an interval of regular cardinals (by Theorem 3.9), and since we assume that $\mathrm{pcf}(A)$ has cardinality at least ρ^{+4}, any regular cardinal in $\{\aleph_{\sigma+\alpha} \mid \alpha < \rho^{+4}\}$ is in $\mathrm{pcf}(A)$.

We intend to define a closed set $D \subseteq \rho^{+4}$ of order-type ρ^{+3}, $D = \{\alpha_i \mid i < \rho^{+3}\}$, and the impossible sequence of length ρ^+, B, will be a subset of $\{\aleph^+_{\sigma+\alpha} \mid \alpha \in D\}$. The definition of the ordinal α_i is by induction on $i < \rho^{+3}$.

1. For $i = 0$, $\alpha_0 = 0$.

2. If $i < \rho^{+3}$ is a limit ordinal, then $\alpha_i = \sup\{\alpha_j \mid j < i\}$.

3. Suppose that $\{\alpha_j \mid j \le i\}$ has been defined for some $i < \rho^{+3}$, and we shall define α_{i+1}. Consider $i + 1 \subset \rho^{+3}$ as an isomorphic copy of $\{\alpha_j \mid j \le i\}$. For every $k \in S$ look at the set $C_k \cap (i + 1)$ and define the set of cardinals $e_k = \{\aleph_{\sigma+\alpha_j} \mid j \in C_k \cap (i + 1)\}$. Then the set of successors $e_k^{(+)} = \{\gamma^+ \mid \gamma \in e_k\}$ is a set of regular cardinals, and we ask whether $\max\mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\rho^{+4}}$ or not. There are ρ^{+3} such questions, and therefore we can define $\alpha_{i+1} < \rho^{+4}$ so that $\alpha_i < \alpha_{i+1}$ and the following holds. For every $k \in S$, if $\max\mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\rho^{+4}}$, then $\max\mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\alpha_{i+1}}$.

So $D = \{\alpha_i \mid i < \rho^{+3}\}$ is defined. Let $\delta = \sup D$. Then $\mu = \aleph_{\sigma+\delta}$ is a singular cardinal of uncountable cofinality (that is, of cofinality ρ^{+3}). The Representation Theorem (Exercise 4.17) can be applied now. So there exists a closed unbounded set $C \subseteq D$ such that

$$\mu^+ = \max \mathrm{pcf}(\{\aleph^+_{\sigma+\alpha} \mid \alpha \in C\}). \tag{14.43}$$

The closed unbounded set D is isomorphic to ρ^{+3}, and C is transformed under this isomorphism to a closed unbounded set $E \subseteq \rho^{+3}$. That is

$$E = \{i \in \rho^{+3} \mid \alpha_i \in C\}.$$

By the club-guessing property, there exists a $k \in S$ such that $C_k \subseteq E$. If C'_k denotes the non-accumulation points of C_k, we claim that $B = \{\aleph^+_{\sigma+\alpha_j} \mid j \in C'_k\}$ has the (impossible) property excluded by Theorem 6.4. Since the order-type of C_k is ρ^+, that of C'_k is also ρ^+. It suffices to prove for every $i \in C_k$ that

$$\max \mathrm{pcf}(\{\aleph^+_{\sigma+\alpha_j} \mid j \in C_k \cap (i+1)\}) < \aleph_{\sigma+\alpha_{i+1}}. \tag{14.44}$$

Consider the definition of α_{i+1}. The set $e_k = \{\aleph_{\sigma+\alpha_j} \mid j \in C_k \cap (i+1)\}$ was defined, and since $e_k^{(+)} \subseteq \{\aleph^+_{\sigma+\alpha} \mid \alpha \in C\}$, (14.43) implies that $\max \mathrm{pcf}(e_k^{(+)}) \leq \mu^+$. So the answer to the question for e_k was "yes", and as a result (14.44) holds. ⊣

This theorem leads to surprising applications. Consider for example $A = \{\aleph_n \mid n \in \omega\}$. Then $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max \mathrm{pcf}(A)$ by Theorem 5.11. But $\mathrm{pcf}(A)$ is an interval of regular cardinals of size $< \aleph_4$. Hence if we write $\max \mathrm{pcf}(A) = \aleph_\alpha$, then $\alpha < \omega_4$. Thus

$$\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}.$$

This result holds even if $2^{\aleph_0}$ is larger than $\aleph_{\omega_4}$. It follows now immediately that if $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$. Shelah emphasizes that the former result (concerning the cofinality of $[\aleph_\omega]^{\aleph_0}$) is more basic, and hence one should ask questions concerning cofinalities rather than cardinalities, if one wants to get (absolute) answers.

Generalizing this, we have:

7.2 Theorem. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$ then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(|\delta|^{+4})}.$$

Proof. Write $|\delta| = \kappa$. Then $\kappa < \aleph_\delta$ and if A is the interval of regular cardinals in $(\kappa, \aleph_\delta)$ then $|A| \leq |\delta| = \kappa$ and A is a progressive set. Theorem 5.11 applies with $\mu = \aleph_\delta$ and it yields

$$\mathrm{cf}([\aleph_\delta]^\kappa, \subseteq) = \max \mathrm{pcf}(A).$$

But A is an interval of regular cardinals, and hence $|\mathrm{pcf}(A)| < |A|^{+4}$, by Theorem 7.1. This implies that $\max \mathrm{pcf}(A) < \aleph_{\delta+(|A|^{+4})} \leq \aleph_{|\delta|^{+4}}$. Hence $\mathrm{cf}([\aleph_\delta]^\kappa, \subseteq) < \aleph_{(|\delta|^{+4})}$. ⊣

We are now able to deduce the following application to cardinal arithmetic.

7.3 Theorem. *Suppose that δ is a limit ordinal and $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$. Then*

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{+4})}.$$

Proof. Since $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$, $\delta < \aleph_\delta$. It follows from the cofinality theorem above that

$$\aleph_\delta^{\mathrm{cf}(\delta)} \leq |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_\delta \cdot \aleph_{(|\delta|^{+4})}. \tag{14.45}$$

⊣

8. Revised GCH

The Generalized Continuum Hypothesis (GCH) saying that $2^\kappa = \kappa^+$ for every (infinite) cardinal κ is readily seen to be equivalent to the statement that for every two regular cardinals $\kappa < \lambda$ we have $\lambda^\kappa = \lambda$. In [17] Shelah considers a "revised power set" operation $\lambda^{[\kappa]}$ defined as follows:

$$\lambda^{[\kappa]} = \min\bigl\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subseteq \mathcal{P}\bigl(|\mathcal{P}_0| < \kappa \wedge u = \bigcup \mathcal{P}_0\bigr)\bigr\}.$$

An inductive proof shows that GCH is equivalent to the statement that for all regular cardinals $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$. The "revised" GCH theorem says that for "many" pairs of regular cardinals we have $\lambda^{[\kappa]} = \lambda$.

8.1 Theorem (Shelah's Revised GCH). *If θ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, for some $\kappa_0 < \theta$, for every $\kappa_0 \leq \kappa < \theta$*

$$\lambda^{[\kappa]} = \lambda.$$

The proof that we give here is adopted from a later article [14] of Shelah, and it relies on two notions that we have to investigate first, $\mathrm{pcf}_{\sigma\text{-com}}(A)$ and $T_D(f)$.

8.2 Definition. Let $\lambda > \theta \geq \sigma = \mathrm{cf}(\sigma)$ be cardinals.

1. We say that $\mathcal{P} \subseteq [\lambda]^{\leq\theta}$ is a $(<\sigma)$-*base* for $[\lambda]^{\leq\theta}$ if every $u \in [\lambda]^{\leq\theta}$ is the union of fewer than σ members of $\mathcal{P}$. That is, for some $\mathcal{P}_0 \subseteq \mathcal{P}$, $|\mathcal{P}_0| < \sigma$, and $u = \bigcup \mathcal{P}_0$.

2. We define $\lambda^{[\sigma,\theta]} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\theta}$ is a $(<\sigma)$-base for $[\lambda]^{\leq\theta}\}$. Another notation for $\lambda^{[\sigma,\theta]}$ is $\lambda^{[\sigma,\leq\theta]}$. We have $\lambda^{[\sigma]} = \lambda^{[\sigma,\sigma]}$. In a similar fashion define $\lambda^{[\sigma,<\theta]}$. It is the minimal cardinality of a set $\mathcal{P} \subseteq [\lambda]^{<\theta}$ so that every $u \in [\lambda]^{<\theta}$ is a union of fewer than σ members of $\mathcal{P}$.

3. We say that $\mathcal{P} \subseteq [\lambda]^\theta$ is $(<\sigma)$-*cofinal* in $[\lambda]^\theta$ if every $u \in [\lambda]^\theta$ is included in the union of fewer than σ members of $\mathcal{P}$. That is, for some $\mathcal{P}_0 \subseteq \mathcal{P}$, $|\mathcal{P}_0| < \sigma$, and $u \subseteq \bigcup \mathcal{P}_0$.

4. We define $\lambda^{\langle\sigma,\theta\rangle} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^\theta$ is $({<}\sigma)$-cofinal in $[\lambda]^\theta\}$. Define $\lambda^{\langle\sigma\rangle} = \lambda^{\langle\sigma,\sigma\rangle}$.

For a regular infinite cardinal σ and a set A of regular cardinals define

$$\mathrm{pcf}_{\sigma\text{-com}}(A) = \bigl\{\mathrm{tcf}\bigl(\textstyle\prod A/F\bigr) \mid F \text{ is a } \sigma\text{-complete filter over } A \text{ and } \mathrm{tcf}\bigl(\textstyle\prod A/F\bigr) \text{ exists}\bigr\}. \quad (14.46)$$

(A filter is σ-complete if it is closed under the intersections of less than σ members of the filter.)

Clearly, $A \subseteq \mathrm{pcf}_{\sigma\text{-com}}(A) \subseteq \mathrm{pcf}(A)$.

Define $J^{\sigma\text{-com}}_{<\lambda}[A] \subseteq \mathcal{P}(A)$ by the formula $X \in J^{\sigma\text{-com}}_{<\lambda}[A]$ iff $X \subseteq A$ and whenever F is a σ-complete filter over A with $X \in F$ and such that $\mathrm{tcf}(\prod A/F)$ exists, then $\mathrm{tcf}(\prod A/F) < \lambda$. Equivalently,

$$J^{\sigma\text{-com}}_{<\lambda}[A] = \{X \subseteq A \mid \mathrm{pcf}_{\sigma\text{-com}}(X) \subseteq \lambda\}.$$

Clearly, $J_{<\lambda}[A] \subseteq J^{\sigma\text{-com}}_{<\lambda}[A]$.

8.3 Lemma. *$J^{\sigma\text{-com}}_{<\lambda}[A]$ is a σ-complete ideal.*

Proof. Suppose that $X_i \in J^{\sigma\text{-com}}_{<\lambda}[A]$ for every $i < \sigma^*$ where $\sigma^* < \sigma$. We prove that $X = \bigcup_{i<\sigma^*} X_i \in J^{\sigma\text{-com}}_{<\lambda}[A]$. So let F be a σ-complete filter over A containing X and such that $\mathrm{tcf}(\prod A/F) = \tau$ exists. We must show that $\tau < \lambda$. Assume that F is proper (the cofinality of a reduced product by a improper filter is 1). For every $i < \sigma^*$ consider the filter $F + X_i$ (defined as the collection of all subsets of A that contain a set of the form $A \cap X_i$ for $A \in F$). If for some $i < \sigma^*$, $F_i = F + X_i$ is proper, then it is a σ-complete filter containing X_i and such that $\mathrm{tcf}(\prod A/F_i) = \tau$ (extending the filter F will not change the cofinality of the existing reduced product). But as $X_i \in J^{\sigma\text{-com}}_{<\lambda}[A]$, we get $\tau < \lambda$.

If, for every $i < \sigma^*$, $F + X_i$ is improper, then $X \setminus X_i \in F$. Hence the intersection of these sets which is the empty set is in F, and thus F is improper. ⊣

8.4 Lemma. *Suppose that A is a progressive set of regular cardinals and $\lambda = \max \mathrm{pcf}(A)$. Then $X \in J^{\sigma\text{-com}}_{<\lambda}[A]$ iff X is a union of fewer than σ members of $J_{<\lambda}[A]$. That is, $J^{\sigma\text{-com}}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.*

Proof. Let J be the σ-completion of $J_{<\lambda}[A]$. It is the collection of all sets that are union of fewer than σ members of $J_{<\lambda}[A]$. By the previous lemma, $J^{\sigma\text{-com}}_{<\lambda}[A]$ is σ-complete, and hence it contains J. It remains to prove that $J^{\sigma\text{-com}}_{<\lambda}[A] \subseteq J$. So no assumptions on A were needed in this direction.

Assume for a contradiction that $X \in J^{\sigma\text{-com}}_{<\lambda}[A] \setminus J$. Then $J + (A \setminus X)$, the ideal generated by J and $A \setminus X$, is proper. It is easily seen to be a σ-complete ideal. Let F be the dual filter of that ideal. Then F is σ-complete and $X \in F$. Hence the cofinality of $\prod A/F$ is smaller than λ.

Since $\lambda = \max \mathrm{pcf}(A)$, there are f_ζ for $\zeta < \lambda$ that are increasing and cofinal in $\prod A/J_{<\lambda}[A]$ (Exercise 4.3, or Theorem 4.13). But this sequence is also increasing and cofinal in $\prod A/F$, and this is an obvious contradiction. $\dashv$

We now strengthen the lemma by removing the assumption that $\lambda = \max \mathrm{pcf}(A)$.

8.5 Theorem. *Let A be a progressive set of regular cardinals, and σ a regular cardinal. Then $J^{\sigma\text{-com}}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.*

Proof. We prove by induction on μ that for every progressive set A of regular cardinals with $\mu = \max \mathrm{pcf}(A)$, for all cardinals λ and σ (regular), $J^{\sigma\text{-com}}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.

We know already that $J_{<\lambda}[A] \subseteq J^{\sigma\text{-com}}_{<\lambda}[A]$ and that $J^{\sigma\text{-com}}_{<\lambda}[A]$ is σ-complete. It remains to prove that any $X \in J^{\sigma\text{-com}}_{<\lambda}[A]$ is a union of fewer than σ sets from $J_{<\lambda}[A]$. If $\mu < \lambda$ then $X \in J_{<\lambda}[A]$ and this case is uninteresting. In case $\lambda \leq \mu$, $X \in J^{\sigma\text{-com}}_{<\mu}[A]$. So by the previous lemma, X is a union of less than σ sets from $J_{<\mu}[A]$. But the inductive assumption can be applied to each one of these sets, and the lemma follows since σ is regular. $\dashv$

Another characterization of the ideal $J^{\sigma\text{-com}}_{<\lambda}[A]$ is provided by the following theorem dealing with the cofinality of product of cardinals under the $<$ relation: $f < g$ iff for every $a \in \mathrm{dom}(f)$, $f(a) < g(a)$.

We know (Theorem 4.4) that $X \in J_{<\lambda}[A]$ iff $\mathrm{cf}(\prod X) < \lambda$. For a similar characterization of $J^{\sigma\text{-com}}_{<\lambda}[A]$ we need the following definition. Let σ be a regular cardinal and X a set of regular cardinals. If $\mathcal{F} \subseteq \prod X$, we say that $\mathcal{F}$ is ($<\sigma$)-cofinal iff for every $f \in \prod X$ there is a set $\mathcal{F}_0 \subseteq \mathcal{F}$ with $|\mathcal{F}_0| < \sigma$ and such that $f < \sup \mathcal{F}_0$. In other words, the functions formed by taking the supremum of fewer than σ functions from $\mathcal{F}$ form a cofinal set in $\prod X$. The ($<\sigma$)-cofinality of $\prod X$ is the smallest cardinality of a ($<\sigma$)-cofinal subset. It makes sense to assume that $\sigma \leq \min X$ when inquiring about the ($<\sigma$)-cofinality of X.

8.6 Theorem. *Suppose that A is a progressive set of regular cardinals, $\sigma \leq \min A$ is a regular cardinal, and $\sigma \leq \mathrm{cf}(\lambda)$. Define*

$$J = \big\{B \subseteq A \mid B = \emptyset \text{ or } \textstyle\prod B \text{ has } (<\sigma)\text{-cofinality } < \lambda\big\}.$$

Then $J = J^{\sigma\text{-com}}_{<\lambda}[A]$.

Proof. We first prove that $J \subseteq J^{\sigma\text{-com}}_{<\lambda}[A]$. Suppose $B \in J$ and let D be a σ-complete filter over A containing B and such that $\mathrm{tcf}(\prod A/D)$ exists and is equal to $\lambda' \geq \lambda$. This will lead to a contradiction, thereby proving that $B \in J^{\sigma\text{-com}}_{<\lambda}[A]$. Since $\mathrm{tcf}(\prod A/D) = \lambda'$, λ' is a regular cardinal and there is an increasing sequence S in $\prod A/D$ of length λ' that is cofinal in $\prod A/D$. By definition of $B \in J$, there is a set $\mathcal{F} \subseteq \prod B$ of cardinality $< \lambda$ that is ($<\sigma$)-cofinal. For every $f \in \mathcal{F}$ there is a function $s \in S$ such that $f <_D s$ (f is defined on B and s on A, but as $B \in D$, this makes sense). Since λ'

is regular and bigger than $|\mathcal{F}|$, there is a single $s \in S$ such that $f <_D s$ for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is $({<}\sigma)$-cofinal, $s <_D \sup \mathcal{F}_0$ for some $\mathcal{F}_0 \subseteq \mathcal{F}$ of size $< \sigma$. But as D is σ-complete, and $f <_D s$ for every $f \in \mathcal{F}_0$, $\sup \mathcal{F}_0 \leq_D s$ as well. This is a contradiction, and thus $J \subseteq J^{\sigma\text{-com}}_{<\lambda}[A]$.

Clearly $J_{<\lambda}[A] \subseteq J$ (by Theorem 4.4). If we prove that J is σ-complete then $J^{\sigma\text{-com}}_{<\lambda}[A] \subseteq J$ follows from the previous theorem.

So let $\sigma^* < \sigma$ and $X_i \in J$ for $i < \sigma^*$ be given. We shall prove that $X = \bigcup_{i<\sigma^*} X_i \in J$. For every $i < \sigma^*$ we have a $({<}\sigma)$-cofinal set $P_i \subseteq \prod X_i$ of cardinality $< \lambda$. Then $P = \bigcup_{i<\sigma^*} P_i$ has cardinality $< \lambda$ because $\sigma \leq \mathrm{cf}(\lambda)$. The domain of each function in P_i is X_i, but we can extend it arbitrarily on X and then P can be considered as a subset of $\prod X$. Clearly P is $({<}\sigma)$-cofinal. ⊣

We shall apply this theorem to $J^{\sigma\text{-com}}_{\leq\lambda}[A]$ rather than $J^{\sigma\text{-com}}_{<\lambda}[A]$. That is, replacing λ with λ^+ in the theorem, we get the following corollary in which $\sigma \leq \mathrm{cf}\,\lambda$ is no longer required.

8.7 Corollary. *Suppose that A is a progressive set of regular cardinals, $\sigma \leq \min A$ is a regular cardinal, and $\sigma \leq \lambda$. Define*

$$J = \big\{B \subseteq A \mid B = \emptyset \textit{ or } \textstyle\prod B \textit{ has } ({<}\sigma)\textit{-cofinality } \leq \lambda\big\}.$$

Then $J = J^{\sigma\text{-com}}_{\leq\lambda}[A]$.

8.8 Theorem. *Suppose that:*

1. *$\lambda > \theta > \sigma > \aleph_0$ are given, where θ and σ are regular cardinals, and $2^{<\theta} \leq \lambda$.*
2. *For every $A \subseteq \mathrm{Reg} \cap \lambda \setminus \theta$, if $|A| < \theta$ then $A \in J^{\sigma\text{-com}}_{\leq\lambda}[A]$.*

Then $\lambda = \lambda^{[\sigma,<\theta]}$.

Proof. Fix χ sufficiently large, and let $M \prec H_\chi$ be an elementary substructure of cardinality λ and such that $\lambda + 1 \subseteq M$. We shall prove the following claim which yields the theorem:

$$M \cap [\lambda]^{<\theta} \text{ is a } ({<}\,\sigma)\text{-base for } [\lambda]^{<\theta}.$$

For this, we need the following lemma.

8.9 Lemma. *With the same assumptions of the theorem and on M, let $g : \kappa \to \lambda$ and $f : \kappa \to \lambda + 1$ be given with $\kappa < \theta$, $f \in M$, and such that $\forall a \in \kappa\ g(a) \leq f(a)$. Then there is a collection $\Phi \subseteq M$ of functions from κ to λ such that the following hold:*

1. *$|\Phi| < \sigma$.*
2. *For every $p \in \Phi$, $g \leq p \leq f$ (that is, for all $a \in \kappa$, $g(a) \leq p(a) \leq f(a)$).*

3. For every $a \in \kappa$, if $g(a) < f(a)$, then for some $p \in \Phi$ $g(a) \leq p(a) < f(a)$.

Proof. Think of f as an "approximation from above" in M to the function g (which is not in M, or else the theorem is trivial). The set Φ is not required to be a member of M, and each function of Φ (if different from f) is a better approximation that lies in M. For each $a \in \kappa$, if $f(a)$ is not the best approximation, then Φ contains a function that gets a better value at a.

Fix in M a sequence $\langle C_\delta \mid \delta \leq \lambda, \delta \in \lim \lambda \rangle$ such that $C_\delta \subseteq \delta$ is unbounded in δ and of order-type $\mathrm{cf}(\delta)$.

Define the following subsets of κ:

$$
\begin{aligned}
E_0 &= \{a < \kappa \mid g(a) = f(a)\} \\
E_1 &= \{a < \kappa \mid g(a) < f(a),\ f(a) \text{ is a successor ordinal}\} \\
E_2 &= \{a < \kappa \mid g(a) < f(a), f(a) \text{ is a limit and } \mathrm{cf}(f(a)) < \theta\} \\
E_3 &= \kappa \setminus (E_0 \cup E_1 \cup E_2).
\end{aligned}
$$

Since $2^{<\theta} \leq \lambda$, any bounded subset of θ is in M. So each E_ℓ is in M. We define h on κ as follows. For $a \in E_0$, $h(a) = f(a)$. For $a \in E_1$, $h(a)+1 = f(a)$. For $a \in \kappa \setminus (E_0 \cup E_1)$, $h(a) = \min C_{f(a)} \setminus g(a)$.

Obviously $h{\restriction}E_0 \cup E_1 \in M$. We prove that $h{\restriction}E_2 \in M$ as well. By definition $h{\restriction}E_2$ is a function in $\prod_{\delta \in E_2} C_{f(\delta)}$. But θ is regular, and since $|E_2| < \theta$ and $\mathrm{cf}(f(\delta)) < \theta$, there is a bound below θ on the values of $\{\mathrm{cf}(f(a)) \mid a \in E_2\}$, and hence $|\prod_{\delta \in E_2} C_{f(\delta)}| \leq 2^{<\theta} \leq \lambda$.

So $\prod_{\delta \in E_2} C_{f(\delta)} \subseteq M$, and hence $h{\restriction}E_2 \in M$.

There is no reason to assume that $h{\restriction}E_3$ is in M, but we shall find a set Φ of size $< \sigma$ as required by the lemma. Define $A = \{\mathrm{cf}(f(a)) \mid a \in E_3\}$. Then $A \subseteq \lambda + 1 \setminus \theta$ is a set of regular cardinals of size $\leq \kappa$, and so $A \in J^{\sigma\text{-com}}_{\leq\lambda}[A]$. There is by Corollary 8.7 a family $\mathcal{F}$ of size $\leq \lambda$ that is $({<}\sigma)$-cofinal in $\prod A$. Since $A \in M$ we can have $\mathcal{F} \in M$ and $\mathcal{F} \subseteq M$. Since $\kappa < \min A$ and $A \subseteq \mathrm{Reg}$, $\mathcal{F}$ yields a family of functions, $\mathcal{F}' \subseteq \prod_{\delta \in E_3} C_{f(\delta)} = P$ that is $({<}\sigma)$-cofinal in P. As $h{\restriction}E_3 \in P$, there is a set $\mathcal{F}_0 \subseteq \mathcal{F}'$ of size $< \sigma$ such that $h{\restriction}E_3 < \sup \mathcal{F}_0$. If $e \in \mathcal{F}_0$, then $e(\delta) < f(\delta)$ but $e(\delta) < g(\delta)$ is possible. So we correct each $e \in \mathcal{F}_0$ and define:

$$
e'(\delta) = \begin{cases} e(\delta) & \text{if } g(\delta) \leq e(\delta), \\ f(\delta) & \text{otherwise.} \end{cases}
$$

Then $e' \in M$ because $e, f \in M$ and every subset of κ is in M. The collection $\{h{\restriction}(E_0 \cup E_1 \cup E_2)^\frown e' \mid e \in \mathcal{F}_0\}$ is as required, and the lemma is proved. $\dashv$

We continue now with the proof of the theorem. So let $u \in [\lambda]^{<\theta}$ be given and we shall find a subset of $M \cap [\lambda]^{<\theta}$ of cardinality $< \sigma$ whose union is u. Let $\kappa = |u| < \theta$ be the cardinality of u and take an enumeration $g : \kappa \to u$. We shall define by induction on $n \in \omega$ a set Φ_n of functions from κ to λ such that the following holds.

1. Let $f_0 : \kappa \to \lambda + 1$ be defined by $f_0(a) = \lambda$. Then $\Phi_0 = \{f_0\}$.

2. For every n, $\Phi_n \subseteq M$ and $|\Phi_n| < \sigma$. If $f \in \Phi_n$ then $g \leq f$.

3. For every $f \in \Phi_n$ and $a \in \kappa$ such that $g(a) < f(a)$ there exists a $p \in \Phi_{n+1}$ such that $g(a) \leq p(a) < f(a)$.

This is easily obtained by the lemma.

Let $\Phi = \bigcup_{n<\omega} \Phi_n$. Then $|\Phi| < \sigma$. For any $f \in \Phi$, the set

$$E(f) = \{f(a) \mid a \in \kappa \text{ and } f(a) = g(a)\}$$

is in M (because f is, and any subset of κ). We have $u = \bigcup\{E(f) \mid f \in \Phi\}$ because if $x \in u$ then $x = g(a)$ for some $a \in \kappa$, and $g(a) < f_0(a)$. There exists a sequence $f_n \in \Phi_n$ so that if $g(a) < f_n(a)$ then $f_{n+1}(a) < f_n(a)$. And necessarily for some n $g(a) = f_n(a)$. So $a \in E(f_n)$. This ends the proof of Theorem 8.8. $\dashv$

The following corollary shows that the theorem above can also be applied when $\mathrm{cf}(\theta) < \sigma$.

8.10 Corollary. *Suppose that:*

1. $\lambda > \theta > \sigma = \mathrm{cf}(\sigma) > \aleph_0$ are given, where $\mathrm{cf}(\theta) < \sigma$ and $2^{<\theta} \leq \lambda$.

2. For every $A \subseteq \mathrm{Reg} \cap \lambda + 1 \setminus \theta$, if $|A| < \theta$ then $A \in J^{\sigma\text{-com}}_{\leq\lambda}[A]$.

Then $\lambda = \lambda^{[\sigma, \leq\theta]}$.

Proof. Fix a sequence $\langle \theta_i \mid i < \mathrm{cf}(\theta)\rangle$ of regular cardinals that is cofinal in θ and such that $\sigma < \theta_i$ for all i. We claim for every $i < \mathrm{cf}(\theta)$ that the assumptions of Theorem 8.8 hold for $\lambda > \theta_i > \sigma$, and hence $\lambda = \lambda^{[\sigma,<\theta_i]}$ follows. But this clearly implies that $\lambda = \lambda^{[\sigma,\leq\theta]}$.

For the claim, we must prove that if $\theta' < \theta$ is regular then for every $A \subseteq \mathrm{Reg} \cap \lambda + 1 \setminus \theta'$, if $|A| < \theta'$ then $A \in J^{\sigma\text{-com}}_{\leq\lambda}[A]$. Suppose for a contradiction that this is not the case, and for some σ-complete filter D over $A \subset \mathrm{Reg} \cap \lambda + 1 \setminus \theta'$ we have $\mathrm{tcf}(\prod A/D) = \lambda_0 > \lambda$. We may assume that $A \subseteq \theta$, that is, we may assume that $A \cap \theta \in D$, or else $A \setminus \theta$ is not D-null and then it can be added to D without changing the true cofinality of the reduced product, which contradicts the assumptions of the theorem.

If for every $i < \mathrm{cf}(\theta)$ $A \setminus \theta_i \in D$, then by the σ-completeness of D and the fact that $\mathrm{cf}(\theta) < \sigma$, we get a contradiction. So for some i $A \cap \theta_i$ is not D-null. But then $D' = D + A \cap \theta_i$ is σ-complete and it follows that the true cofinality of $\prod A/D'$ remains λ_0. Yet this is impossible since $(\theta_i)^{|A\cap\theta_i|} \leq 2^{<\theta} \leq \lambda$. $\dashv$

8.1. $T_D(f)$

Let J be an ideal over a cardinal κ. We recall some definitions. The collection of J-positive sets is denoted J^+. The corresponding dual filter is denoted J^*. If R is a relation, if f and g are functions defined on κ, then we define $f\ R_J\ g$ if and only if $\{i \in \kappa \mid f(i)\ R\ g(i)\} \in J^*$. We also write $f\ R_{J^+}\ g$ for $\{i \in \kappa \mid f(i)\ R\ g(i)\} \notin J$. That is, $f(i)\ R\ g(i)$ occurs positively.

Thus $f \neq_J g$ means that $\{i \in \kappa \mid f(i) = g(i)\} \in J$, and $f =_{J^+} g$ means that $\neg f \neq_J g$.

Let κ be a cardinal and D a filter over κ. Consider the $<_D$ ordering on On^κ. For $f \in \mathrm{On}^\kappa$, $\prod_{i<\kappa} f(i)$ is denoted $\prod f$, and $\prod_{i<\kappa} f(i)/D$ is denoted $\prod f/D$. (We consider only functions f such that $f(i) > 0$ for $i \in \kappa$.)

For $\mathcal{F} \subseteq \prod f$, we say that $\mathcal{F}$ is a set of pairwise "D-different" functions, if for every distinct $f_1, f_2 \in \mathcal{F}$ we have $f_1 \neq_D f_2$. For any $f \in \mathrm{On}^\kappa$, define

$$T_D(f) = \sup\bigl\{|\mathcal{F}| \mid \mathcal{F} \subseteq \textstyle\prod f \text{ is a set of pairwise } D\text{-different functions}\bigr\}.$$

(Shelah investigated several different definitions, and this cardinal is denoted T_D^0 in [14].)

8.11 Theorem. *Suppose that D is a filter over κ, $f \in \mathrm{On}^\kappa$ and $T_D(f) = \lambda$. If $2^\kappa < \lambda$ then the supremum in the definition of $T_D(f)$ is attained. In fact, if $2^\kappa < \lambda$ and $\mathcal{F} \subseteq \prod f$ is any maximal family of pairwise D-different functions, then $|\mathcal{F}| = \lambda$.*

Proof. Suppose on the contrary that $\mathcal{F} \subseteq \prod f$ is maximal but $|\mathcal{F}| < \lambda$. Let $\mathcal{G} \subseteq \prod f$ be a collection of pairwise D-different functions such that

$$|\mathcal{G}| > |\mathcal{F}| + 2^\kappa. \tag{14.47}$$

For every $g \in \mathcal{G}$ we can find $f = f(g) \in \mathcal{F}$ such that $X(g) = \{i \in \kappa \mid f(i) = g(i)\}$ is not D-null. As (14.47), there are two distinct functions g_1 and g_2 in $\mathcal{G}$ such that $f(g_1) = f(g_2)$ and $X(g_1) = X(g_2)$. But this implies that g_1 and g_2 agree on a non-null set which is a contradiction to the assumption that the functions in $\mathcal{G}$ are pairwise D-different. ⊣

An obvious observation which turns out to be crucial is the following.

8.12 Lemma. *If $\mathrm{tcf}(\prod f/D)$ exists, then $T_D(f) \geq \mathrm{tcf}(\prod f/D)$.*

Proof. If $\mathrm{tcf}(\prod f/D) = \lambda$, then there exists a $<_D$ increasing sequence of length λ, and hence a set of cardinality λ of pairwise D-different functions. ⊣

Assume now that σ is a regular uncountable cardinal, and D is a σ-complete filter over κ. Then $\prod f/D$ is well-founded. This is used in the following.

8.13 Lemma. *Suppose that σ is a regular uncountable cardinal and D is a σ-complete filter over κ. Suppose $f \in \mathrm{On}^\kappa$ and $T_D(f) \geq \lambda$ where $2^\kappa < \lambda$. Then for some $g \leq_D f$ we have $T_D(g) = \lambda$.*

Proof. Let $g \leq_D f$ be minimal in the $\leq_D$ ordering such that $T_D(g) \geq \lambda$. Suppose for a contradiction that $T_D(g) > \lambda$. There is a set $\{f_\alpha \mid \alpha < \lambda^+\}$ of pairwise D-different functions in $\prod g$. For $\alpha < \lambda^+$ define

$$u_\alpha = \{\beta < \lambda^+ \mid f_\beta <_D f_\alpha\}.$$

If, for some α, $|u_\alpha| \geq \lambda$, then u_α proves that $T_D(f_\alpha) \geq \lambda$ in contradiction to the minimality of g. Hence $|u_\alpha| < \lambda$ for every $\alpha < \lambda^+$.

But now we can apply the Free Mapping theorem of Hajnal and obtain $F \subseteq \lambda^+$, of cardinality λ^+ such that $\alpha \notin u_\beta$ (and $\beta \notin u_\alpha$) for every $\alpha \neq \beta$ in F. (The argument in short is the following. First, we can find $\lambda_0 < \lambda$ such that $|u_\alpha| = \lambda_0$ for unboundedly many $\alpha < \lambda^+$. Re-enumerating, we may assume $|u_\alpha| = \lambda_0$ for every α. On those $\alpha < \lambda^+$ with cofinality λ_0^+ we bound $u_\alpha \cap \alpha$ in α, and use Fodor's lemma.)

Hence there are $f_\alpha \in \mathrm{On}^\kappa$ for $\alpha < (2^\kappa)^+$ such that $f_\alpha \not\leq_D f_\beta$ whenever $\alpha \neq \beta$. But this is impossible in view of the Erdős-Rado partition theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. Indeed, for $\alpha < \beta < \kappa^+$ define $h(\alpha, \beta)$ as some $i < \kappa$ such that $f_\beta(i) < f_\alpha(i)$. Then h has no infinite homogeneous set, which contradicts the theorem. Thus $T_D(g) = \lambda$.

Observe that since $2^\kappa < \lambda$, $L = \{a \in \kappa \mid g(a) \in \lim\}$ is not null in D, and hence we may assume without loss of generality that it is in D. (Or else let $h <_D g$ be such that $g(a) = h(a) + 1$ for every $a \notin L$, and $h(a) = g(a)$ on L. Let f_α, for $\alpha < \lambda$, exemplify $T_D(g) = \lambda$. By minimality of g, there are λ functions f_α that are equal to h on a positive subset of $\kappa \setminus L$. Since $2^\kappa < \lambda$, two such functions are equal on a positive set, which is impossible.) ⊣

The following is one of the two main arguments used in the proof of the revised GCH theorem.

8.14 Theorem. *Assume that $\lambda > \theta \geq \sigma = \mathrm{cf}(\sigma) > \kappa$ are cardinals such that:*

1. *$\theta^\kappa = \theta$.*
2. *If $\tau < \sigma$ then $\tau^\kappa < \sigma$.*
3. *J is an ideal on κ.*
4. *There is a sequence $\bar{\lambda} = \langle \lambda_i \mid i < \kappa \rangle$, $\lambda_i < \lambda$, such that*
   (a) $T_J(\bar{\lambda}) = \lambda$,
   (b) $\lambda_i^{\langle \sigma, \theta \rangle} = \lambda_i$ for every $i < \kappa$.

Then $\lambda^{\langle \sigma, \theta \rangle} = \lambda$. (If we also assume $2^\theta \leq \lambda$, then evidently $\lambda^{[\sigma, \theta]} = \lambda$.)

Proof. In the proof, we actually weaken the requirement $T_J(\bar{\lambda}) = \lambda$ to the following conjunction.

1. There are $f_\alpha \in \prod_{i<\kappa} \lambda_i$, for $\alpha < \lambda$, such that $\alpha \neq \beta \longrightarrow f_\alpha \neq_J f_\beta$.

2. There are $g_\alpha \in \prod_{i<\kappa} \lambda_i$, for $\alpha < \lambda$, such that for every $f \in \prod_{i<\kappa} \lambda_i$ there exists $\alpha < \kappa$ with $f =_{J^+} g_\alpha$.

Fix a sequence of pairwise D-different functions $f_\alpha \in \prod_{i<\kappa} \lambda_i$, for $\alpha < \lambda$, as in 1 above.

For every $i < \kappa$ we assume $\lambda_i^{\langle \sigma,\theta \rangle} = \lambda_i$, so there exists a family $\mathcal{P}_i \subseteq [\lambda_i]^\theta$ of cardinality λ_i that is $({<}\sigma)$-cofinal in $[\lambda_i]^\theta$.

Since $|\mathcal{P}_i| = \lambda_i$, $\prod_{i\in\kappa} \mathcal{P}_i$ is isomorphic to $\prod_{i<\kappa} \lambda_i$. So there is (by 2 above) a family $\{g_\alpha \mid \alpha < \lambda\} \subseteq \prod_{i\in\kappa} \mathcal{P}_i$ such that for every $g \in \prod_{i\in\kappa} \mathcal{P}_i$ there is $\alpha < \lambda$ with $g =_{J^+} g_\alpha$.

For every $g \in \prod_{i\in\kappa} \mathcal{P}_i$ and $A \in J^+$, let $g{\restriction}A$ be the restriction of g to A, and $\prod g{\restriction}A$ is $\prod_{i\in A} g(i)$. We define

$$\mathcal{F}(g{\restriction}A) = \{\zeta \in \lambda \mid \forall i \in A\ f_\zeta(i) \in g(i)\}.$$

In other words, $\mathcal{F}(g{\restriction}A)$ is the set of $\zeta \in \lambda$ such that $f_\zeta{\restriction}A \in \prod g{\restriction}A$. Observe that if $A \subseteq B \subseteq \kappa$, then $\mathcal{F}(g{\restriction}A) \supseteq \mathcal{F}(g{\restriction}B)$.

8.15 Claim. *For every $g \in \prod_{i\in\kappa} \mathcal{P}_i$ and $A \in J^+$, $|\mathcal{F}(g{\restriction}A)| \leq \theta$.*

Since $g(i) \in \mathcal{P}_i \subseteq [\lambda_i]^\theta$, $|\prod_{i\in A} g(i)| \leq \theta^\kappa = \theta$. So, if $|\mathcal{F}(g{\restriction}A)| > \theta$, we would have $\zeta \neq \zeta'$ in λ with $f_\zeta{\restriction}A = f_{\zeta'}{\restriction}A$. But as $A \in J^+$, this contradicts $f_\zeta \neq_J f_{\zeta'}$ and proves the claim.

8.16 Claim. *Every $u \in [\lambda]^\theta$ is included in a union of fewer than σ sets of the form $\mathcal{F}(g_\alpha{\restriction}A)$. That is, the collection $\mathcal{F} = \{\mathcal{F}(g_\alpha{\restriction}A) \mid \alpha < \lambda, A \in J^+\}$ is $({<}\sigma)$-cofinal in $[\lambda]^\theta$.*

Observe first that as $|\mathcal{F}| \leq \lambda \cdot 2^\kappa = \lambda$, this claim proves the theorem.

Given $u \in [\lambda]^\theta$ define for every $i < \kappa$

$$u_i = \{f_\alpha(i) \mid \alpha \in u\}.$$

Then $u_i \in [\lambda_i]^{\leq\theta}$ and hence there is a $\mathcal{P}_i^u \subseteq \mathcal{P}_i$ with $|\mathcal{P}_i^u| < \sigma$ and such that $u_i \subseteq \bigcup \mathcal{P}_i^u$. Since σ is regular, some $\tau < \sigma$ bounds all the cardinals $\sigma_i = |\mathcal{P}_i^u|$, and, as $\tau^\kappa < \sigma$, we have that $|\prod_{i\in\kappa} \sigma_i| < \sigma$. So

$$\mathcal{G} = \textstyle\prod_{i\in\kappa} \mathcal{P}_i^u$$

is a subset of $\prod_{i\in\kappa} \mathcal{P}_i$ of size $< \sigma$. The following two lemmas finish the proof of our claim.

8.17 Lemma. $u \subseteq \bigcup\{\mathcal{F}(g) \mid g \in \mathcal{G}\}$.

Proof. If $\zeta \in u$ then $f_\zeta(i) \in u_i$ for every $i \in \kappa$. Thus $f_\zeta(i) \in \bigcup \mathcal{P}_i^u$ for every $i < \kappa$, and we can find $g \in \mathcal{G}$ such that $f_\zeta(i) \in g(i)$ for all $i < \kappa$. Namely, $\zeta \in \mathcal{F}(g)$ as required. ⊣

8.18 Lemma. *For every $g \in \mathcal{G}$ there is an $\alpha < \lambda$ and $A \in J^+$ such that $\mathcal{F}(g) \subseteq \mathcal{F}(g_\alpha{\restriction}A)$. Thus as $|\mathcal{G}| < \sigma$, u is contained in the union of fewer than σ sets of the form $\mathcal{F}(g_\alpha{\restriction}A)$.*

Proof. For every $g \in \mathcal{G}$ there is some $\alpha < \lambda$ such that $g =_{J^+} g_\alpha$. That is, for some $A \in J^+$, $g \restriction A = g_\alpha \restriction A$. We already observed that $\mathcal{F}(g) \subseteq \mathcal{F}(g \restriction A)$, and hence the lemma follows. So Theorem 8.14 is proved. ⊣

We shall use a variant of Theorem 8.14 in which the assumption $\theta^\kappa = \theta$ is replaced with the assumption that θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma$.

8.19 Corollary. *Assume that $\lambda > \theta > \sigma = \mathrm{cf}(\sigma) > \kappa$ are cardinals such that:*

1. *θ is a strong limit cardinal and $\mathrm{cf}(\theta) < \sigma$.*
2. *If $\tau < \sigma$ then $\tau^\kappa < \sigma$.*
3. *J is an ideal on κ.*
4. *There is a sequence $\bar{\lambda} = \langle \lambda_i \mid i < \kappa \rangle$, $\lambda_i < \lambda$, such that*
   (a) $T_J(\bar{\lambda}) = \lambda$,
   (b) $\lambda_i^{\langle \sigma, \theta \rangle} = \lambda_i$ for every $i < \kappa$.

Then $\lambda^{\langle \sigma, \theta \rangle} = \lambda$.

Proof. Fix a cofinal in θ sequence $\langle \theta_\epsilon \mid \epsilon < \mathrm{cf}(\theta) \rangle$ such that $\theta_\epsilon^\kappa = \theta_\epsilon$ and $\sigma < \theta_\epsilon$ for every ϵ. (Start with any cofinal sequence, and replace θ_ϵ with $(\theta_\epsilon)^\kappa$ if necessary.)

Consider any $\epsilon < \mathrm{cf}(\theta)$. Observe that for every $i < \kappa$ we have $\lambda_i = \lambda_i^{[\sigma, \theta_\epsilon]}$. This follows immediately from the assumptions that θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma$, and such that $\lambda_i = \lambda_i^{\langle \sigma, \theta \rangle}$. Hence Theorem 8.14 is applicable (with θ_ϵ in the role of θ) and $\lambda = \lambda^{\langle \sigma, \theta_\epsilon \rangle}$. Since this holds for every $\epsilon < \mathrm{cf}(\theta)$, we get $\lambda = \lambda^{\langle \sigma, \theta \rangle}$. ⊣

8.2. Proof of the Revised GCH

We prove the following form of the revised GCH.

8.20 Theorem. *If θ is a strong limit singular cardinal, then for every $\lambda \geq \theta$, for some $\sigma < \theta$,*

$$\lambda = \lambda^{[\sigma, \theta]}.$$

Proof. Let $\sigma_0 = (\mathrm{cf}\,\theta)^+$.

The theorem is proved by induction on λ. For $\lambda = \theta$, $\lambda = \lambda^{[\sigma_0, \theta]}$, and the family of all bounded subsets of θ is an evidence for this equality. (Any subset of θ is a union of $\mathrm{cf}(\theta)$ bounded subsets.)

We note for clarification that the induction can easily proceed in case $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\theta)$, and so we may assume that $\mathrm{cf}(\lambda) = \mathrm{cf}(\theta)$. However, we shall not make any use of this in the following proof.

Case 1: For every $A \subseteq \mathrm{Reg} \cap \lambda \setminus \theta$, if $|A| < \theta$ then $A \in J^{\sigma_0\text{-com}}_{\leq\lambda}[A]$.

In this case the inductive assumption is dispensable and Corollary 8.10 yields immediately that $\lambda = \lambda^{[\sigma_0, \leq\theta]}$.

Case 2: Not Case 1.

$$\text{For some } A \subseteq \mathrm{Reg} \cap \lambda \setminus \theta \text{ with } |A| < \theta,\ A \notin J^{\sigma_0\text{-com}}_{\leq\lambda}[A].$$

Hence there is a σ_0-complete filter D over A, where $|A| = \kappa < \theta$, such that $\mathrm{tcf}(\prod A/D) > \lambda$. Say $f : \kappa \to A$ enumerates A. By Lemma 8.12, $T_D(f) \geq \mathrm{tcf}(\prod A/D) > \lambda$. By Lemma 8.13, there exists a $g \leq f$ defined over κ so that $T_D(g) = \lambda$.

We claim that $\{i < \kappa \mid g(i) \geq \theta\} \in D$. If not, then $\{i < \kappa \mid g(i) < \theta\}$ is D-positive. But since $\mathrm{cf}(\theta) < \sigma_0$ and D is σ_0-complete, there is a $\theta' < \theta$ so that $X = \{i < \kappa \mid g(i) < \theta'\}$ is D-positive. Hence $T_D(g \restriction X) = \lambda$. But this is impossible since θ is strong limit and $(\theta')^\kappa < \theta$.

So we can assume now that for every $i < \kappa$, $g(i) \geq \theta$. Hence by the inductive assumption there is a $\sigma(i) < \theta$ such that

$$g(i) = g(i)^{[\sigma(i),\theta]}. \tag{14.48}$$

Since $\mathrm{cf}(\theta) < \sigma_0$ and D is σ_0-complete, there is σ, such that $\kappa, \sigma_0 < \sigma < \theta$ and $\{i < \kappa \mid \sigma(i) < \sigma\}$ is D-positive. For notational simplicity we assume that $\sigma(i) < \sigma$ for all $i < \kappa$. Take $\sigma_1 = (\sigma^\kappa)^+$. Now apply Corollary 8.19 to $\lambda > \theta > \sigma_1 > \kappa$. This yields $\lambda^{\langle\sigma_1,\theta\rangle} = \lambda$, but since θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma_1$ we obtain $\lambda^{[\sigma_1,\theta]} = \lambda$.

$\dashv$

We note that Theorem 8.1 did not make the assumption that θ is a singular cardinal, but Theorem 8.20 did. To see how Theorem 8.1 can be derived from Theorem 8.20, we argue as follows in case θ is a regular uncountable strong limit cardinal. There is a stationary set $S \subseteq \theta$ of strong limit singular cardinals. So if $\lambda \geq \theta$, then Theorem 8.20 applies to each $\theta' \in S$, and $\lambda = \lambda^{[\sigma(\theta'),\theta']}$ follows for some $\sigma(\theta') < \theta'$. By Fodor's theorem, there is a fixed $\sigma < \theta$ such that $\sigma = \sigma(\theta')$ for a stationary set of cardinals $\theta' \in S$. This gives $\lambda = \lambda^{[\sigma,<\theta]}$. So obviously for every $\sigma \leq \kappa < \theta$, we get $\lambda = \lambda^{[\kappa]}$.

8.3. Applications of the Revised GCH

Two applications are given here, the first to the existence of diamond sequences and the second to cellularity of Boolean algebras. Both use the following immediate corollary of the revised GCH theorem.

$$\text{If } \alpha \geq \beth_\omega \text{ then for some regular uncountable } \sigma < \beth_\omega \text{ there is a collection } P_\alpha \subseteq [\alpha]^\sigma \text{ where } |P_\alpha| = |\alpha| \text{ and such that for each } x \in [\alpha]^\sigma, \text{ for some } p \in P_\alpha,\ p \subseteq x. \tag{14.49}$$

To begin this section we recall that for a stationary set $S \subseteq \lambda^+$, $\diamondsuit^-_{\lambda^+}(S)$ is the following diamond statement: there is a sequence $\langle S_\alpha \mid \alpha \in S\rangle$ where $S_\alpha \subseteq \mathcal{P}(\alpha)$, $|S_\alpha| \leq \lambda$, and for every $A \subseteq \lambda^+$, $\{\alpha \in S \mid A \cap \alpha \in S_\alpha\}$ is a stationary set. If $|S_\alpha| = 1$, that is essentially $S_\alpha \subseteq \alpha$, then the sequence is the usual diamond sequence on S, and the resulting statement is the classical diamond $\diamondsuit_{\lambda^+}(S)$. An intriguing theorem of Kunen's (see [11]) states that $\diamondsuit^-_{\lambda^+}(S)$ is equivalent to $\diamondsuit_{\lambda^+}(S)$. (Somewhat more generally, this holds for an arbitrary regular cardinal μ not necessarily a successor cardinal, where $\diamondsuit^-_\mu(S)$ is the diamond statement obtained by restricting S_α to have cardinality not greater than that of α.) When $S = \lambda^+$, we write $\diamondsuit^-_{\lambda^+}$ instead of $\diamondsuit^-_{\lambda^+}(S)$ etc.

A beautiful argument of Gregory [4] proves that if $2^\lambda = \lambda^+$ and $\lambda^{\aleph_0} = \lambda$, then $\diamondsuit^-_{\lambda^+}(S^{\lambda^+}_\omega)$ where $S^{\lambda^+}_\omega$ is the stationary set of ordinals in λ^+ of cofinality ω. (There are stronger formulations, but this suffices to demonstrate the application we have in mind.) To prove this theorem, let $\{X_i \mid i < \lambda^+\}$ be an enumeration of all bounded subsets of λ^+. For every $\alpha < \lambda^+$ define S_α as the collection of all subsets of α that are formed by taking countable unions of sets from $\{X_i \mid i < \alpha\}$. Since $|\alpha|^{\aleph_0} \leq \lambda$, $|S_\alpha| \leq \lambda$. Now, if $A \subseteq \lambda^+$ is given, then the set, C, of $\alpha < \lambda^+$ for which $\forall \zeta < \alpha\ \exists i < \alpha\ (A \cap \zeta = X_i)$ is closed unbounded in λ^+. If $\alpha \in C$ and $\mathrm{cf}(\alpha) = \omega$ then $A \cap \alpha \in S_\alpha$. Applying Kunen's theorem, we can obtain $\diamondsuit_{\lambda^+}(S^{\lambda^+}_\omega)$.

The revised GCH enables in many cases a stronger theorem in which $\lambda^{\aleph_0} = \lambda$ is not required.

8.21 Theorem. *If $\lambda \geq \beth_\omega$ and $2^\lambda = \lambda^+$, then $\diamondsuit^-_{\lambda^+}$ holds. (Hence $\diamondsuit_{\lambda^+}$ is in fact equivalent to $2^\lambda = \lambda^+$ for every $\lambda \geq \beth_\omega$.)*

Proof. As before, let $\{X_i \mid i < \lambda^+\}$ enumerate all bounded subsets of λ^+. $\beth_\omega$ is the first strong limit cardinal, and the revised GCH theorem applies to $\lambda \geq \beth_\omega$. So there is a $\sigma < \beth_\omega$ such that (14.49) holds for some family $P \subseteq [\lambda]^\sigma$.

For every α in the interval $[\lambda, \lambda^+)$, $|\alpha| = \lambda$ and hence P can be transformed into a family $P_\alpha \subseteq [\alpha]^\sigma$ such that (14.49) holds (same σ for all α's). Now we define S_α as the collection of all subsets of α obtained as unions of the form $\bigcup\{X_i \mid i \in B\}$ where $B \in P_\alpha$. So $|S_\alpha| \leq \lambda$.

The argument to prove that $\langle S_i \mid i < \lambda^+\rangle$ is a diamond sequence is now familiar. Let $A \subseteq \lambda^+$ be any set. There is a closed unbounded $C \subseteq \lambda^+$ as before so that for $\alpha \in C$ and $\zeta < \alpha$ there is $i < \alpha$ such that $A \cap \zeta = X_i$. Now pick any $\alpha \in C$ such that $\mathrm{cf}(\alpha) = \sigma$. Pick an increasing sequence $\langle \alpha_\epsilon \mid \epsilon < \sigma\rangle$ cofinal in α, and for each $\epsilon < \kappa$ find $i(\epsilon) < \alpha$ such that $A \cap \alpha_\epsilon = X_{i(\epsilon)}$. Define $u = \{i(\epsilon) \mid \epsilon < \sigma\}$. Observe that if $K \subseteq \sigma$ is any unbounded subset of σ then $\bigcup\{X_{i(\epsilon)} \mid \epsilon \in K\} = A \cap \alpha$. For some $B \in P_\alpha$, $i(\epsilon) \in B$ for unboundedly many $\epsilon < \sigma$. Hence $A \cap \alpha = \bigcup\{X_i \mid i \in B\} \in S_\alpha$. $\dashv$

This kind of result about sufficiently large λ was first seen in this context. Very recently, Shelah [13] has, through a short and ingenious proof having aspects of club guessing, improved the result by weakening the hypothesis

$\lambda \geq \beth_\omega$ to $\lambda \geq \aleph_1$. It had been well known that CH + $\neg\diamondsuit_{\omega_1}$ is consistent by a result of Jensen. Thus, a long story has come to a surprising yet fitting conclusion, that except for the single and focal case $\lambda = \omega$, $\diamondsuit_{\lambda^+}$ is actually equivalent to just the cardinal hypothesis $2^\lambda = \lambda^+$.

We now begin the second application.

8.22 Definition. A subset X of a Boolean algebra is *μ-linked* iff there is a function $h : X \to \mu$ such that $x \wedge y \neq 0_B$ whenever $h(x) = h(y)$.

Our aim is to prove the following theorem from [16]. (For background and motivation and additional results consult [16] and [5].)

8.23 Theorem. *Assume that $\mu = \mu^{<\beth_\omega}$. If B is a c.c.c. Boolean algebra of cardinality $\leq 2^\mu$, then B is μ-linked.*

The proof which follows is an example of an induction that relies on the revised GCH. Since B satisfies the countable chain condition, its completion has cardinality $\leq |B|^{\aleph_0} \leq 2^\mu$, and so we can assume that B is a complete Boolean algebra (and when we prove that it is μ-linked then the original algebra which is embedded in its completion is also μ-linked).

We prove by induction on λ, a cardinal such that $\mu \leq \lambda \leq 2^\mu$, that any subset of B of cardinality λ is μ-linked. This is obvious for $\lambda = \mu$, or when $\mathrm{cf}(\lambda) \leq \mu$ (and the inductive claim holds for smaller cardinals), and so we may assume that $\mathrm{cf}(\lambda) > \mu$. There are several ingredients in the proof of this theorem, and so it is postponed until the required preparations are made.

8.24 Definition. Let C be a Boolean algebra, and $D \subseteq C$ a subalgebra. For any $x \in C$ let $F_x = \{d \in D \mid x \leq d\}$ be the filter generated by x. For a cardinal θ the following property is denoted $(**)_\theta$ (for the pair D and C):

$(**)_\theta$ For every $x \in C$ there is an $F \subseteq F_x$ of cardinality $\leq \theta$ such that for every $b \in F_x$ there is an $a \in F$ such that $a \leq b$.

In other words, F_x is generated by a subset of cardinality $\leq \theta$.

8.25 Lemma. *Let θ, μ, and κ be cardinals such that $\theta, \mu \leq \kappa$. Suppose that C is a Boolean algebra with a decomposition $C = \bigcup_{\alpha<\kappa} C_\alpha$, where the sequence of Boolean subalgebras C_α is increasing and continuous (for limit δ, $C_\delta = \bigcup_{i<\delta} C_i$). Assume the following:*

1. *$C_0 = \emptyset$.*
2. *Each C_α is μ-linked.*
3. *Property $(**)_\theta$ holds for each of the pairs C_α, C.*

Let χ be a sufficiently large cardinal and consider the structure H_χ (with some well ordering of its universe, and with C and its decomposition as constants). Suppose that M_1 and M_2 are two elementary substructures of H_χ that are isomorphic with an isomorphism $g : M_1 \to M_2$ that is the identity *on* $\kappa \cap M_1 \cap M_2$*. Suppose in addition that* $\theta \subseteq M_1 \cap M_2$*, and that* $M_1 \cap \mu = M_2 \cap \mu$*.*

Then for every non-zero $x \in M_1 \cap C$*,*

$$x \wedge g(x) \neq 0_C.$$

Proof. The rank of an element $c \in C$ is the least ordinal τ such that $c \in C_\tau$. Since $C_0 = \emptyset$, the rank of c is a successor ordinal (below κ) such that $c \in C_{\alpha+1} \setminus C_\alpha$. Take $x \in M_1$ of minimal rank $\alpha + 1$ such that $x \wedge g(x) = 0_C$ and we shall obtain a contradiction.

Case 1. $\alpha \in M_1 \cap M_2$. So $g(\alpha) = \alpha$. Let $h : C_{\alpha+1} \to \mu$ be the least function (in the well-ordering of H_χ) given by the assumption that $C_{\alpha+1}$ is μ-linked. So $h \in M_1 \cap M_2$, and since h is definable from α we have $g(h) = h$ (as $g(\alpha) = \alpha$). Say $h(x) = \eta \in \mu$. As $M_1 \cap \mu = M_2 \cap \mu$, we have $g(h(x)) = g(\eta) = \eta$. But $g(h(x)) = g(h)(g(x)) = h(g(x))$. So $h(g(x)) = \eta$, and hence $h(x) = h(g(x))$ which implies that x and $g(x)$ have non-zero meet in C.

Case 2. $\alpha \in M_1 \setminus M_2$, and hence $\alpha \neq g(\alpha)$ and $g(\alpha) \in M_2 \setminus M_1$. Suppose that $g(\alpha) < \alpha$ (case $g(\alpha) > \alpha$ is symmetric). Say $g(x) = y$, and $g(\alpha) = \beta$. Then $\beta + 1$ is the rank of y. Let $\alpha_1 \leq \alpha$ be the least ordinal in M_1 that is strictly above β. Since $\beta + 1 \leq \alpha_1$,

$$y \in C_{\alpha_1}.$$

Let $F_x \subseteq C_{\alpha_1}$ be the filter generated by x. Property $(**)_\theta$ of the pair C_{α_1} and C implies the existence of $F \subseteq F_x$ of cardinality $\leq \theta$ that generates F_x. As x and y are disjoint, the complement, $-y$, of y is in F_x (since it is in C_{α_1}) and hence there is an $a \in F$ that is disjoint from y. Since α_1 and x are in M_1, we have F_x and F in M_1 as well. But as θ is included in M_1, $F \subseteq M_1$ and hence $a \in M_1$ follows. The rank of a is $\alpha_2 + 1 \leq \alpha_1$. The minimality of α_1 implies that $\alpha_2 < \beta$ (equality is impossible because β is not in M_1). But now we can apply a similar argument to F_y (for the pair C_β, C) and discover $b \in C_\beta \cap M_2$ that is disjoint to a. Say $u \in M_1$ is such that $g(u) = b$. Then $u \in C_\alpha$ and hence $x_0 = u \wedge a$ is in C_α. Since $b \in F_y$, $u \in F_x$, and hence x_0 is in F_x too. In particular, $x_0 \neq 0_C$. But $g(x_0) = b \wedge g(a)$ and x_0 is disjoint to $b \wedge g(a)$ because already a is disjoint to b. So x_0 is disjoint to $g(x_0)$, in contradiction to the minimality of the rank of x. ⊣

Here is a lemma which is an immediate consequence of the Engelking-Karlowicz theorem [3]; we state it for reference and will return to its proof later on.

8.26 Lemma. *If* $\mu^\theta = \mu$ *then there is a map* $\tau : [2^\mu]^\theta \to \mu$ *such that if* $\tau(M_1) = \tau(M_2)$ *then* M_1 *and* M_2 *have the same order-type (as subsets of the ordinal* 2^μ*) and the order isomorphism* $g : M_1 \to M_2$ *is the identity on* $M_1 \cap M_2$*.*

8.27 Corollary. *Suppose that $\theta < \mu < \kappa \leq 2^\mu$ are cardinals such that $\mu^\theta = \mu$. Let C be a Boolean algebra of cardinality $\leq 2^\mu$, and suppose that $C = \bigcup_{\alpha<\kappa} C_\alpha$ where the C_α form an increasing and continuous sequence of subalgebras such that: $C_0 = \emptyset$, each C_α is μ-linked, and $(**)_\theta$ holds for each pair C_α, C. Then C is μ-linked.*

Proof. Let χ be sufficiently large and H_χ be the structure of sets of cardinality hereditarily less than χ, with a well-ordering of the universe and C as a constant. For every $a \in C$ find $M(a) \prec H_\chi$ of cardinality θ and such that $\theta \subseteq M(a)$. With each $M = M(a)$ we associate the following three parameters.

1. $M \cap \mu \in [\mu]^\theta$. So there are μ such parameters.

2. $\tau(M \cap 2^\mu)$, where $\tau : [2^\mu]^\theta \to \mu$ is the map from the lemma above.

3. The isomorphism type of $M(a)$ (with a as a parameter). Since $2^\theta \leq \mu$ there are $\leq \mu$ such types.

The map taking $a \in C$ to the three parameters associated with $M(a)$ proves that C is μ-linked. For if $M(a)$ and $M(b)$ have the same parameters then $a \wedge b \neq 0_C$ by the following argument. Let $g : M(a) \to M(b)$ be the isomorphism given by item 3. Then $g(a) = b$, and we plan to apply Lemma 8.25. This is possible because (1) $\tau(M(a) \cap 2^\mu) = \tau(M(b) \cap 2^\mu)$ implies that g is the identity on $2^\mu \cap M(a) \cap M(b)$, (2) $\theta \subseteq M(a) \cap M(b)$ by assumption, and (3) $M(a) \cap \mu = M(b) \cap \mu$ because this is the first of the three parameters. ⊣

We continue the inductive proof of Theorem 8.23. Recall that $\lambda \leq 2^\mu$, B is a complete c.c.c. Boolean algebra of cardinality $\leq 2^\mu$, and every subset of B of cardinality $< \lambda$ is μ-linked. Our aim is to prove that any $X \subseteq B$ of cardinality λ is μ-linked. We intend to use Corollary 8.27, and we must find a $C \subseteq B$ with $X \subseteq C$ and such that the premises of Corollary 8.27 hold.

For every α such that $\beth_\omega \leq \alpha < \lambda$ we have a regular uncountable cardinal $\sigma(\alpha) < \beth_\omega$ and a family $P_\alpha \subseteq [\alpha]^{\sigma(\alpha)}$ such that (14.49) holds. Since $\mathrm{cf}(\lambda) \neq \omega$ (in fact $\mathrm{cf}(\lambda) > \mu$) there is an unbounded set $E \subseteq \lambda$ such that for some fixed σ we have $\sigma = \sigma(\alpha)$ for every $\alpha \in E$. The symbols E and σ retain this meaning throughout the proof. We define $\theta = 2^{<\sigma}$.

8.28 Lemma. *Let $\chi > 2^\mu$ be sufficiently large. Suppose that δ is an ordinal and $\langle M_i \prec H_\chi \mid i < \delta \rangle$ is such that:*

1. $\mathrm{cf}(\delta) > \sigma$.

2. $B, E \in M_0$ *and* $\beth_\omega \subseteq M_0$.

3. $M_i \subseteq M_j$ *for* $i < j$ *and* $M_i \in M_{i+1}$.

4. $|M_i| < \lambda$, *and* $M_i \cap \lambda \in \lambda$.

Then for $M = \bigcup_{i<\delta} M_i$ and $B_0 = B \cap M$, $(**)_{2^{<\sigma}}$ holds for the pair B_0 and B.

Proof. Given $x \in B$ consider $F_x \subseteq B_0$, the filter of members of B_0 that are greater than x. We want to find a $F \subseteq F_x$ of cardinality $\leq \theta = 2^{<\sigma}$ that generates F_x. We choose $a_\zeta \in F_x$ for $\zeta < \sigma$ by the following inductive procedure. Suppose that $A_\zeta = \{a_\epsilon \mid \epsilon < \zeta\}$ is already chosen. Let $G_\zeta = \{\wedge Z \mid Z \subseteq A_\zeta \text{ and } Z \in M\}$. So G_ζ is the collection of all elements of B that can be formed by taking meets of subsets of A_ζ that happen to be in M. Clearly $A_\zeta \subseteq G_\zeta \subseteq F_x$. Since $|A_\zeta| < \sigma$, $|G_\zeta| \leq 2^{<\sigma}$. If there exists $a \in F_x$ not covering any $b \in G_\zeta$, then let a_ζ be such a. If there is no such a, then the procedure stops and $F = G_\zeta$ is as required. We shall prove that the construction cannot proceed for every $\zeta < \sigma$. Suppose it does, and consider $A = \{a_\zeta \mid \zeta < \sigma\}$. Since $\mathrm{cf}(\delta) > \sigma$ there is an $i < \delta$ with $A \subseteq M_i$. As $|M_i| < \lambda$ there is, already in M_{i+1} an ordinal $\alpha \in E$ such that $|M_i| < \alpha$. So $\alpha + 1 \subseteq M_{i+1}$ and hence also $P_\alpha \subseteq M_{i+1}$ (where $P_\alpha \subseteq [\alpha]^\sigma$ satisfies (14.49)). Viewing the universe of M_i as a copy of an ordinal $< \alpha$, the set A is a subset of α of cardinality σ, and we have some $p \in P_\alpha$ such that $p \subseteq A$. Since $\beth_\omega \subset M_{i+1}$, each subset of p is also in M_{i+1}. It follows that for every $a_\zeta \in p$, $A_\zeta \cap p \in M$ and hence $a_\zeta \not\geq \wedge(A_\zeta \cap p)$. Thus $\wedge(A_\zeta \cap p) - a_\zeta \neq 0_B$ is a sequence of σ pairwise disjoint members of B, which contradicts the c.c.c. since σ is uncountable. ⊣

We can complete now the proof of Theorem 8.23. We are assuming that $\lambda \leq 2^\mu$, $\mathrm{cf}(\lambda) > \mu$, $\mu^{<\beth_\omega} = \mu$, and every subset of B of cardinality smaller than λ is μ-linked. A set $X \subseteq B$ of cardinality λ is given, which we want to show is μ-linked. Pick χ sufficiently large and define $M_i \prec H_\chi$, for $i < \mathrm{cf}(\lambda) = \kappa$, such that

1. M_i is increasing and continuous with i. $|M_i| < \lambda$, $\lambda \cap M_i \in \lambda$, and $M_i \in M_{i+1}$.
2. $B, X \in M_0$, $\mu + 1 \subseteq M_0$, $\beth_\omega \subseteq M_0$, and $X \subseteq M = \bigcup_{i<\kappa} M_i$.

We shall prove that $B \cap M$ is μ-linked, and hence that X is μ-linked. For any set R of ordinals, let $\mathrm{nacc}(R)$ denotes those $\alpha \in R$ that are not accumulation points of R (for some $\beta < \alpha$ $R \cap (\beta, \alpha) = \emptyset$).

Let $R \subseteq \kappa$ be a closed unbounded set such that every $\alpha \in \mathrm{nacc}(R)$ is a limit ordinal with $\mathrm{cf}(\alpha) > \sigma$. Then, for $\delta \in \mathrm{nacc}(R)$, Lemma 8.28 applies to the sequence $\langle M_i \mid i < \delta\rangle$ and hence the pair $B \cap M_\delta$, B satisfies $(**)_\theta$ ($\theta = 2^{<\sigma}$). But, then it follows that $(**)_\theta$ holds for *every* $\delta \in R$ for the pair $B \cap M_\delta$, B. Because if $\mathrm{cf}(\delta) > \sigma$ then the lemma applies, and if $\mathrm{cf}(\delta) \leq \sigma$ then δ is a limit of $\leq \sigma$ non-accumulation points of R, and hence $(**)_\theta$ holds for $B \cap M_\delta$ by accumulating $\leq \sigma$ sets, each of cardinality $\leq \theta$.

Now let $\langle \rho_i \mid i < \kappa\rangle$ be an increasing and continuous enumeration of R, and define $C_i = B \cap M_{\rho_i}$, $C = B \cap M$. Then Corollary 8.27 applies with $\theta = 2^{<\sigma}$ and yields that $B \cap M$ is μ-linked. This proves Theorem 8.23.

For completeness we review the theorem of Engelking and Karlowicz that was used in the proof.

8.29 Theorem (Engelking and Karlowicz [3]). *Assume that θ and μ are cardinals such that $\mu^\theta = \mu$. Then there are functions $f_\xi : 2^\mu \to \mu$, for $\xi < \mu$, such that if $A \subseteq 2^\mu$, $|A| \leq \theta$, and $f : A \to \mu$, then there is a $\xi < \mu$ such that $f \subseteq f_\xi$.*

Proof. It is convenient for the proof to see 2^μ as the set of functions from μ to 2. A "template" is a triple (D, S, F) where $D \in [\mu]^\theta$, $S \subseteq 2^D$ and $|S| \leq \theta$ (S is a set of functions from D to 2), and $F : S \to \mu$. The number of possible templates is μ.

For any template $T = (D, S, F)$ we define f_T on 2^μ. If $\alpha \in 2^\mu$ and $\alpha \restriction D \in S$, then we define $f_T(\alpha) = F(\alpha \restriction D)$ (if $\alpha \restriction D \notin S$ then $f_T(\alpha)$ is any value).

Given any $A \subseteq 2^\mu$, $|A| \leq \theta$, and $f : A \to \mu$, find $D \in [\mu]^\theta$ such that $\alpha_1 \restriction D \neq \alpha_2 \restriction D$ whenever $\alpha_1 \neq \alpha_2$ are in A. $S = \{a \restriction D \mid a \in A\}$. For every $s \in S$ there is a unique $a \in A$ such that $s = a \restriction D$ and we define $F(s) = f(a)$. Then $f \subseteq f_T$. ⊣

We can prove now Lemma 8.26. Clearly the map assigning to each $X \in [2^\mu]^\theta$ its order-type (in θ^+) ensures that two sets are isomorphic if they have the same value. The problem is to ensure that two isomorphic sets have an isomorphism that is the identity on their intersection. Given $X \in [2^\mu]^\theta$, let f_X be the collapsing map which assigns to each $x \in X$ the order-type of $x \cap X$. Then there is some $\xi < \mu$ such that $f_X \subseteq f_\xi$ (by the Engelking-Karlowicz theorem). Let's color X with ξ (say the first one). Now if X and Y in $[2^\mu]^\theta$ have the same order-type and the same color ξ, then the isomorphism of X onto Y is the identity on $X \cap Y$ since it is equal to $g_2^{-1} \circ g_1$ where $g_1 = f_\xi \restriction X$ and $g_2 = f_\xi \restriction Y$.

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15. Successors of Singular Cardinals

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Contents

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1. Introduction

1.1. Three Problems

The regular cardinals are divided into three classes—those that are themselves successors of regular cardinals, those that are limit cardinals, and those that are successors of singular cardinals. This chapter is concerned with the peculiar nature of the cardinals in the last of these classes, a nature that combines aspects of the other two classes in surprising ways, and whose investigation involves almost all the tools of modern set theory.

Our chapter begins with a brief discussion of three examples, chosen to illustrate some of the difficulties encountered when working with successors of singular cardinals. In each case, we consider a question concerning regular cardinals whose solution is quite easy to obtain for each $\aleph_n$ $(n < \omega)$ but whose resolution at $\aleph_{\omega+1}$ touches on much deeper issues. This shows us how the strange nature of successors of singular cardinals makes itself felt at the first place possible.

The Cofinality of $[\omega_{\omega+1}]^{\aleph_0}$

Our first example involves the cofinality of $[\omega_{\omega+1}]^{\aleph_0}$—the collection of all countable subsets of $\omega_{\omega+1}$. Recall that the cofinality of $[\kappa]^{\aleph_0}$ is defined to be the minimum cardinality of a family $\mathcal{C} \subseteq [\kappa]^{\aleph_0}$ with the property that every countable subset of κ is covered by a member of $\mathcal{C}$. The cofinality of $[\omega_1]^{\aleph_0}$ is trivially equal to $\aleph_1$ (let $\mathcal{C}$ be the collection of initial segments of ω_1), and a simple induction establishes that the cofinality of $[\omega_n]^{\aleph_0}$ is $\aleph_n$ for $n < \omega$. This leads us to the natural question: What can be said about the cofinality of $[\omega_{\omega+1}]^{\aleph_0}$?

We see immediately that our induction will not work, and the reason why is clear—the answer depends on the cofinality of $[\omega_\omega]^{\aleph_0}$, and this is something that ZFC does not decide. Problems concerning cardinal arithmetic in the neighborhood of a singular cardinal are beyond the scope of this chapter—for more information, we refer to the reader to the chapter by Gitik [38] and that by Abraham and Magidor [1] in this Handbook.

Reflection of Stationary Sets

Our next example is a problem considered in Jech's chapter [49] in this Handbook which will be treated at length in our chapter—the problem of *stationary reflection*. Let us recall the relevant definitions.

1.1 Definition. Suppose that λ is an uncountable regular cardinal, and let S be a stationary subset of λ.

1. Given $\alpha < \lambda$, say that S *reflects at* α if α has uncountable cofinality and $S \cap \alpha$ is a stationary subset of α.
2. S *reflects* if there is some $\alpha < \lambda$ such that S reflects at α.
3. If $S \subseteq \lambda$ is stationary, then $\mathrm{Refl}(S)$ means that every stationary subset of S reflects.

1.2 Definition. If $\kappa < \lambda$ are regular cardinals, then we define

$$S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}.$$

Variations of this notation (along the lines of $S^\lambda_{\leq\kappa}$) should be given the obvious interpretation.

If κ is a regular cardinal, then every ordinal less than κ^+ has a closed unbounded subset consisting of ordinals of cofinality less than κ. This implies immediately that $S^{\kappa^+}_\kappa$ does not reflect, so successors of regular cardinals (in particular, the cardinals $\aleph_n$ for $n < \omega$) always have non-reflecting stationary subsets. However, this simple argument says nothing about $\aleph_{\omega+1}$ (or other successors of singular cardinals) and so we are led at once to the question of whether or not $\aleph_{\omega+1}$ has a non-reflecting stationary subset.

The answer to this question is a typical one. If the universe is sufficiently "L-like", then an argument of Jensen establishes that $\mathrm{Refl}(\aleph_{\omega+1})$ fails. On the other hand, granted the existence of infinitely many supercompact cardinals (which of course, implies that the universe is far from being "L-like"), a construction of Magidor [62] provides a model in which every stationary subset of $\aleph_{\omega+1}$ reflects.

Is $\aleph_{\omega+1}$ a Jónsson cardinal?

The last problem we consider in this introductory section is motivated by a question from the partition calculus, in particular, on whether or not a very weak version of Ramsey's Theorem can hold for colorings of the finite subsets of a given cardinal.

1.3 Definition. A cardinal λ is a *Jónsson cardinal* if for every function $F : [\lambda]^{<\omega} \to \lambda$, there is a set $H \subseteq \lambda$ of size λ such that the range of $F \restriction [H]^{<\omega}$ is a proper subset of λ.

Note that if we consider the function $f : [\omega]^{<\omega} \to \omega$ given by

$$f(s) = |s|,$$

we see that $\aleph_0$ is certainly not a Jónsson cardinal, and next claim establishes that none of the cardinals $\aleph_n$ for $n < \omega$ are either.

1.4 Claim. *If κ is not a Jónsson cardinal, then neither is κ^+.*

Proof. Assume κ is not a Jónsson cardinal, and let $F : [\kappa]^{<\omega} \to \kappa$ attest to this fact. Given an ordinal α with $\kappa \leq \alpha < \kappa^+$, we may combine F together with a bijection between α and κ to build a function

$$F_\alpha : [\alpha]^{<\omega} \to \alpha$$

with the property that for any $A \subseteq \alpha$ of cardinality κ and any $\beta < \alpha$, there is an $s \in [A]^{<\omega}$ such that $F_\alpha(s) = \beta$.

Define a function $G : [\kappa^+]^{<\omega} \to \kappa^+$ by pasting together the functions F_α in a straightforward way—given $\alpha_0 < \cdots < \alpha_n < \kappa^+$, define

$$G(\{\alpha_0, \ldots, \alpha_n\}) := \begin{cases} 0 & \text{if } \alpha_n < \kappa, \text{ and} \\ F_{\alpha_n}(\{\alpha_0, \ldots, \alpha_{n-1}\}) & \text{if } \kappa \leq \alpha_n < \kappa^+. \end{cases}$$

We claim now that the function G shows that κ^+ is not a Jónsson cardinal. To see this, suppose that $A \subseteq \kappa^+$ has cardinality κ^+ and let an ordinal $\beta < \kappa^+$ be given. We produce an $s \in [A]^{<\omega}$ such that $G(s) = \beta$ as follows:

First, locate an $\alpha \in A$ such that $\beta < \alpha$ and $|A \cap \alpha| = \kappa$; it is clear that such an α exists and $\alpha \geq \kappa$. Using the salient property of F_α, we can choose $t \in [A \cap \alpha]^{<\omega}$ such that $F_\alpha(t) = \beta$. If we define $s := t \cup \{\alpha\}$, then $x \in [A]^{<\omega}$ and the definition of G implies $G(s) = \beta$, as required. ⊣

The situation at $\aleph_{\omega+1}$ seems to require a different idea, however, because the question of whether $\aleph_\omega$ can be a Jónsson cardinal is still unresolved. If there is a Jónsson cardinal, then $0^\#$ exists by an argument of Kunen. (See Sect. 18 of [51]. It was known much earlier that Jónsson cardinals imply $V \neq L$ [54].) Thus, if $V = L$ we know that $\aleph_{\omega+1}$ is not a Jónsson cardinal for the simple reason that there are no Jónsson cardinals at all. On the other hand, a theorem of Rowbottom [72] establishes that Ramsey cardinals are Jónsson cardinals. Thus Jónsson cardinals are consistent relative to the existence of moderately-sized large cardinals and so we cannot get a cheap answer regarding $\aleph_{\omega+1}$.

We will discuss this question in more detail in the final section of this chapter, but we point out that in the particular case of $\aleph_{\omega+1}$, Shelah's pcf theory provides us with an answer—$\aleph_{\omega+1}$ is *not* a Jónsson cardinal (see Theorem 5.7), but his methods do not generalize to arbitrary successors of singular cardinals. In particular, the question of whether or not it is possible for the successor of a singular cardinal to be a Jónsson cardinal is still open, and this is still an active area of research in set theory.

One can summarize the situation in all three of the preceding examples quite succinctly—if the universe is sufficiently "L-like", then successors of singular cardinals behave much like successors of regular cardinals, but in the presence of sufficiently large cardinals, one can sometimes find quite different behavior. As we shall see, these opposing forces are typified by the non-reflection implicit in $\square$-sequences and the abundant reflection supplied by large cardinals, and their interplay gives this area of set theory its special character.

1.2. Conventions and Notation

Elementary Submodels

Elementary submodels will be one of our main tools. Our conventions are standard—we assume χ is some ambient regular cardinal much larger than any cardinals under discussion, and by $H(\chi)$ we mean the collection of all sets whose transitive closure has cardinality less than χ. It is well known that $\langle H(\chi), \in\rangle$ is a model of all axioms of ZFC except perhaps for the power set axiom; for all intents and purposes we can pretend that all relevant set theory is done inside of $H(\chi)$. The book [46] contains a gentle development of elementary submodels of $H(\chi)$ in the context of cardinal arithmetic and pcf theory.

Throughout this chapter, $\mathfrak{A}$ will denote some expansion of the structure $\langle H(\chi), \in, <_\chi\rangle$ by at most countably many functions, constants, and relations. Note that we will indulge in a bit of sloppiness by sometimes referring to elementary submodels of $H(\chi)$, and sometimes referring to elementary submodels of $\mathfrak{A}$—both phrases are used to mean exactly the same thing. The symbol $<_\chi$ denotes some fixed well-ordering of $H(\chi)$; we include such a well-ordering in our structure because it gives us canonical Skolem functions. In more detail, for each formula $\varphi(v_0, \ldots, v_n)$ of the language of $\mathfrak{A}$, we can define an n-ary function f_φ with domain $H(\chi)$ by letting $f_\varphi(x_1, \ldots, x_n)$ be the $<_\chi$-least $x \in H(\chi)$ such that $\mathfrak{A} \models \varphi[x, x_1, \ldots, x_n]$ if such an x exists, and setting $f_\varphi(x_1, \ldots, x_n) = 0$ otherwise. We obtain the set of *Skolem terms* for $\mathfrak{A}$ by closing the collection of Skolem functions under composition. The following definition establishes our notational convention.

1.5 Definition. Let $B \subseteq H(\chi)$. Then $\mathrm{Sk}^{\mathfrak{A}}(B)$ denotes the Skolem hull of B in the structure $\mathfrak{A}$. More precisely,

$$\mathrm{Sk}^{\mathfrak{A}}(B) = \{\tau(b_0, \ldots, b_n) : \tau \text{ a Skolem term for } \mathfrak{A}, \text{ and } b_0, \ldots, b_n \in B\}.$$

The Tarski criterion tells us that $\mathrm{Sk}^{\mathfrak{A}}(B)$ is an elementary substructure of $\mathfrak{A}$, and it is the smallest such structure containing every element of B. The following technical lemma (due to Baumgartner [6]) captures a fact about Skolem hulls that is incredibly useful in our context.

1.6 Lemma. *Assume that $M \prec \mathfrak{A}$ and let $\theta \in M$ be a cardinal. If we define $N = \mathrm{Sk}^{\mathfrak{A}}(M \cup \theta)$, then for all regular cardinals $\sigma \in M \setminus \theta^+$,*

$$\sup(M \cap \sigma) = \sup(N \cap \sigma).$$

Proof. Suppose that $\alpha \in N \cap \sigma$; we must produce a $\beta > \alpha$ in $M \cap \sigma$. Since $\alpha \in N$, there is a Skolem term τ and parameters $\alpha_0, \ldots, \alpha_i, \beta_0, \ldots, \beta_j$ such that $\alpha = \tau(\alpha_0, \dots, \alpha_i, \beta_0, \dots, \beta_j)$, where each α_ℓ is less than θ and each β_ℓ is an element of $M \setminus \theta$. We define a function F with domain $[\theta]^{i+1}$ by

$$F(x_0, \dots, x_i) = \begin{cases} \tau(x_0, \dots, x_i, \beta_0, \dots, \beta_k) & \text{if this is an ordinal less than } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

The function F is an element of M as it is definable from parameters in M and so $\beta := \sup(\mathrm{ran}(F))$ is in M as well. Since σ is a regular cardinal, it is clear that $\beta < \sigma$. Now $\alpha \in \mathrm{ran}(F)$ and therefore $\alpha < \beta$ as required. $\dashv$

We will also make great use of sequences of elementary submodels, and the concept captured by the following definition (first explicitly formulated in [36]) will have a prominent role.

1.7 Definition. Let γ be a limit ordinal, and let $\mathfrak{A}$ be as above. An *internally approachable* (IA) *chain of substructures of* $\mathfrak{A}$ *of length* γ is a continuous and increasing sequence of elementary substructures of $\mathfrak{A}$ such that $\langle M_j : j \le i \rangle \in M_{i+1}$ for all $i < \gamma$.

We remark that in circumstances of the preceding definition that for $j < i < \gamma$ we have $i \subseteq M_i$ and $M_j \in M_i$. The following definition isolates one class of IA chains that appears in almost every section of this work. The terminology goes back to Shelah's original investigations in [83].

1.8 Definition. Let λ be a regular cardinal. A λ-*approximating sequence* is an IA chain of substructures $\langle M_i : i < \lambda \rangle$ of length λ such that $\lambda \in M_0$, and for each $\alpha < \lambda$ we have $|M_\alpha| < \lambda$ and $M_\alpha \cap \lambda$ is an initial segment of λ. A λ-approximating sequence is said to be *over* x if $x \in \bigcup_{\alpha<\lambda} M_\alpha$.

The following proposition is simple to prove, but it captures another property of elementary submodels that is used in almost every part of this chapter.

1.9 Proposition. *Let λ be a cardinal, and suppose that M is an elementary submodel of $H(\chi)$ for which $M \cap \lambda$ is an initial segment of λ. If $A \in M$ satisfies $|A| < \lambda$, then $A \subseteq M$.*

Proof. Since $A \in M$, so is the cardinality of A. Since $|A| < \lambda$ and $M \cap \lambda$ is an initial segment of λ, it follows that $|A| \subseteq M$. The model M must contain a bijection b mapping $|A|$ onto A. Given $a \in A$, there is an $\alpha < |A|$ such that $a = b(\alpha)$. Since both b and α are in M, we conclude that a is as well. $\dashv$

For a typical example of how the above proposition is used, consider the case where $\lambda = \mu^+$ for μ a strong limit singular cardinal. If $A \in M$ is of cardinality less than μ, then it follows that every subset of A is also in M. This sort of argument is used without comment throughout the sequel, so it seemed worthwhile to call the reader's attention to it here in the beginning.

λ-Filtration Sequences

Suppose now that $\lambda = \mu^+$, where μ is a singular cardinal. Many combinatorial arguments make use of the fact that every $\alpha < \lambda$ is the union of $\mathrm{cf}(\mu)$ sets of size less than μ. The following definition gives us a systematic way of speaking of this.

1.10 Definition. Suppose that $\lambda = \mu^+$ for μ singular. A matrix

$$\bar{b} = \langle b_{\alpha,i} : \alpha < \lambda, i < \mathrm{cf}(\mu)\rangle$$

of subsets of λ is a λ-*filtration sequence* if for each $\alpha < \lambda$,

1. $b_{\alpha,i} \subseteq \alpha$,
2. the sequence $\langle b_{\alpha,i} : i < \mathrm{cf}(\mu)\rangle$ is continuous and increasing, and
3. $\alpha = \bigcup_{i<\mathrm{cf}(\mu)} b_{\alpha,i}$.

We note that with each λ-filtration sequence $\bar{b}$ we may associate a natural coloring of the pairs from λ, namely

$$d(\beta, \alpha) = \min\{i < \mathrm{cf}(\mu) : \beta \in b_{\alpha,i}\}.$$

On the other hand, if we are given a function $d : [\lambda]^2 \to \mathrm{cf}(\mu)$, then we can construct a λ-filtration sequence by defining

$$b_{\alpha,i} = \{\beta < \alpha : d(\beta, \alpha) < i\}$$

It is straightforward to build λ-filtration sequences that satisfy additional requirements. For example, if we fix an increasing sequence of regular cardinals $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ cofinal in μ, then an easy inductive argument lets us define a λ-filtration sequence with the additional properties that

$$|b_{\alpha,i}| \le \mu_i,$$

and

$$\beta \in b_{\alpha,i} \quad \Longrightarrow \quad b_{\beta,i} \subseteq b_{\alpha,i}.$$

pcf Theory

Our notation concerning pcf theory is fairly consistent with that presented in the chapter by Abraham and Magidor [1] in this Handbook. We take a moment to recall the most important definitions and establish our conventions.

1.11 Definition. Let $(P, \le)$ be a partially ordered set, and let θ be a cardinal. We say that $(P, \le)$ is θ-*directed* if every subset $A \subseteq P$ of cardinality less than θ has an upper bound, that is, there is a $p \in P$ such that $q \le p$ for all $q \in A$. The *cofinality of* $(P, \le)$ (denoted $\mathrm{cf}(P)$ when $\le$ is understood) is the least cardinal θ for which there is a family $A \subseteq P$ of cardinality θ such that for all $p \in P$, there is a $q \in A$ such that $p \le q$.

1.12 Definition. Let I be an ideal on some index set X, and let R be some relation on the class of ordinals On. Given functions f and g mapping X to On, we define

$$fR_Ig \iff \{x \in X : \neg(f(x)Rg(x))\} \in I. \tag{15.1}$$

In particular, we have

$$f <_I g \iff \{x \in X : g(x) \leq f(x)\} \in I, \tag{15.2}$$

and

$$f \leq_I g \iff \{x \in X : g(x) < f(x)\} \in I. \tag{15.3}$$

Note that in general "$f \leq_I g$" is *not* the same as "either $f <_I g$ or $f =_I g$", but the equivalence holds if I is a maximal ideal.

In some situations, we will work with the filter dual to the ideal I, so for example if D is an ultrafilter on X we may say $<_D$ instead of $<_I$ for I the ideal dual to D.

It is quite natural to consider least upper bounds in this context, and the following definition translates this concept into our framework.

1.13 Definition. A function f is a *$\leq_I$-least upper bound for $\langle f_i : i < \alpha\rangle$* if

1. f is a $<_I$-upper bound for $\langle f_i : i < \alpha\rangle$, and
2. if $\{x \in X : g(x) < f(x)\} \notin I$, then there is an $i < \alpha$ such that $\{x \in X : g(x) < f_i(x)\} \notin I$.

There is also a strengthening of "least upper bound" that is a crucial ingredient in many proofs.

1.14 Definition. A $<_I$-increasing sequence of functions $\langle f_i : i < \alpha\rangle$ has a *$<_I$-exact upper bound* (or *$<_I$-eub*) if there is a function $f \in {}^X\mathsf{On}$ such that

1. f is a $<_I$-upper bound for $\langle f_i : i < \alpha\rangle$, and
2. for all $g <_I f$, there is an i for which $g <_I f_i$.

If there is no opportunity for confusion, we may omit explicitly mentioning the order $<_I$.

In general a given sequence need not have an exact upper bound, but if one exists then it is unique up to equivalence modulo the ideal I. We also remark that a $<_I$-exact upper bound is also a $<_I$-least upper bound, but the converse is not true in general.

1.15 Definition. Let I be an ideal on κ, and let $\langle \mu_i : i < \kappa\rangle$ be an increasing sequence of regular cardinals. We say that $(\prod_{i<\kappa} \mu_i, <_I)$ has *true cofinality* θ, or

$$\operatorname{tcf}\left(\prod_{i<\kappa}\mu_i, <_I\right) = \theta$$

if θ is regular and there is a $<_I$-increasing sequence $\langle f_\alpha : \alpha < \theta\rangle$ in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$ such that

$$\big(\forall g \in \textstyle\prod_{i<\kappa}\mu_i\big)(\exists\alpha < \theta)[g <_I f_\alpha].$$

Note that not every structure of the form $(\prod_{i<\kappa} \mu_i, <_I)$ has a true cofinality, but this problem disappears if I is dual to an ultrafilter.

Of course, it is only a cosmetic change to move from products of the form $\prod_{i<\kappa} \mu_i$ to products of the form $\prod A$ for A a set of regular cardinals. This small change in emphasis moves us into pcf theory.

1.16 Definition. A set A of regular cardinals is *progressive* if $|A| < \min(A)$. If A is a progressive set of regular cardinals, then $\mathrm{pcf}(A)$ is the set of all cardinals θ for which there is an ultrafilter D on A such that

$$\mathrm{cf}\big(\textstyle\prod A/D\big) = \theta.$$

If λ is a cardinal and A is a progressive set of regular cardinals, then $J_{<\lambda}[A]$ is the collection of all $B \subseteq A$ such that

$$\mathrm{cf}\big(\textstyle\prod A/D\big) < \lambda \quad \text{for every ultrafilter } D \text{ on } A \text{ containing } B.$$

The name *progressive* comes originally from [46], and seems well on its way to standard usage.

1.17 Fact. Let A be a progressive set of regular cardinals.

1. $J_{<\lambda}[A]$ is an ideal (not necessarily proper) on A.

2. $\mathrm{pcf}(A)$ has a maximal element, $\max\mathrm{pcf}(A)$. It is the least λ such that $A \in J_{<\lambda^+}[A]$.

3. If $\lambda \in \mathrm{pcf}(A)$, then there is a single set $B_\lambda[A] \subseteq A$ (called a *generator for* λ) such that
$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda[A],$$
that is, the ideal $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda}[A]$ together with the set $B_\lambda[A]$. The set $B_\lambda[A]$ is unique modulo the ideal $J_{<\lambda}[A]$.

4. If $\lambda \in \mathrm{pcf}(A)$, then
$$\mathrm{tcf}\big(\textstyle\prod B_\lambda[A], <_{J_{<\lambda}[A]}\big) = \lambda.$$
In particular, $\mathrm{tcf}(\prod B_\lambda[A], <_{J_{<\lambda}[A]})$ is defined. Of course, the same statement holds if we replace $B_\lambda[A]$ by any member of $J_{<\lambda^+}[A] \setminus J_{<\lambda}[A]$.

The preceding gathers together several results of Shelah, and it encapsulates most of what we will be using from pcf theory. The chapter [1] is a good reference for these results, and [46, 55, 10], and [89] provide a selection of external sources for the material.

One topic not covered in [1] is the pseudopower $\mathrm{pp}(\mu)$ of a singular cardinal. A few of the theorems we prove later are most naturally stated in terms of the pseudopower function, so we recall the definition. Readers interested in a more comprehensive study of pseudopowers should consult [46, Chap. 9] for the preliminaries, and [89] for an exhaustive analysis.

1.18 Definition. Suppose that μ is a singular cardinal, and let $\mathrm{PP}(\mu)$ be the set of all cardinals of the form $\mathrm{cf}(\prod A/D)$, where A is a set of regular cardinals cofinal in μ of order-type $\mathrm{cf}(\mu)$ and D is an ultrafilter on A disjoint to the ideal $J^{\mathrm{bd}}[A]$ of bounded subsets of A. We define the *pseudopower of* μ (denoted $\mathrm{pp}(\mu)$) by the formula

$$\mathrm{pp}(\mu) = \sup(\mathrm{PP}(\mu)). \tag{15.4}$$

Our use of $\mathrm{pp}(\mu)$ is limited in that we are almost exclusively concerned with the statement $\mathrm{pp}(\mu) = \mu^+$ and its negation. With this in mind, it seems reasonable to include the following proposition, which is really just a restatement of the definitions involved.

1.19 Proposition. *Let μ be a singular cardinal. Then the following statements are equivalent:*

1. $\mathrm{pp}(\mu) > \mu^+$.
2. *There is an increasing sequence $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ of regular cardinals with limit μ for which*

   $$\left(\prod_{i<\mathrm{cf}(\mu)} \mu_i, <^*\right) \quad \textit{is } \mu^{++}\textit{-directed,}$$

   where $<^*$ abbreviates $<_I$ for I the ideal of bounded subsets of $\mathrm{cf}(\mu)$.

Proof. Assume $\mathrm{pp}(\mu) > \mu^+$. Let θ be the least cardinal greater than μ^+ for which there are A and D as in Definition 1.18 with $\mathrm{cf}(\prod A/D) = \theta$. Since θ is in $\mathrm{pcf}(A)$, we know that there is a generator $B = B_\theta[A]$ corresponding to θ. Since $B \in D$, it must be the case that B is an unbounded subset of A, and we also know

$$\mathrm{tcf}\left(\prod B/J_{<\theta}[A]\right) = \theta.$$

It follows that $(\prod B, <_{J_{<\theta}[A]})$ is θ-directed, hence μ^{++}-directed. If B_0 is a subset of B in $J_{<\theta}[A]$, then B_0 must be bounded in B—otherwise, any ultrafilter on B_0 disjoint to the ideal of bounded sets would contradict the choice of θ. Thus, it follows that

$$\left(\prod B, <_{J^{\mathrm{bd}}[B]}\right) \quad \text{is } \mu^{++}\text{-directed.}$$

If we enumerate B as $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$, then

$$\left(\prod_{i<\mathrm{cf}(\mu)} \mu_i, <^*\right) \quad \text{is } \mu^{++}\text{-directed,}$$

as required.

For the other direction, we note that $A = \{\mu_i : i < \mathrm{cf}(\mu)\}$ is a set of cardinals as in Definition 1.18. If D is any ultrafilter on A disjoint to the ideal of bounded sets, then clearly

$$\mathrm{cf}\big(\prod A/D\big) \geq \mu^+,$$

but since $\prod A/J^{\mathrm{bd}}[A]$ is μ^{++}-directed and $J^{\mathrm{bd}}[A] \cap D = \emptyset$, we know

$$\mathrm{cf}\big(\prod A/D\big) \neq \mu^+.$$

Since $\mathrm{cf}(\prod A/D) \in \mathrm{PP}(\mu)$, we conclude $\mathrm{pp}(\mu) > \mu^+$. $\dashv$

Large Cardinals

Large cardinals make several appearances in this chapter. Our notation is consistent with that of Kanamori's book [53]. We assume that the reader is familiar with the definitions of the most common large cardinals (Mahlo, measurable, supercompact, etc.), as well as with the idea of an elementary embedding. However, there are many equivalent characterizations of some of the cardinals, so we will take a moment to fix the definitions which we shall use.

1.20 Definition.

1. A cardinal κ is *strongly compact* if for any set S, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S.
2. A cardinal κ is *γ-supercompact* (for $\gamma \geq \kappa$) if there is an elementary embedding $j : V \to M$ such that
   (a) $\mathrm{crit}(j) = \kappa$,
   (b) $\gamma < j(\kappa)$, and
   (c) M is closed under sequences of length $\leq \gamma$.

   We will refer to an embedding j with the above properties as a *γ-supercompact embedding.*
3. A cardinal κ is *supercompact* if it is γ-supercompact for all $\gamma \geq \kappa$.

In most cases, we use only a few basic properties of γ-supercompact embeddings. In particular, if we have reason to use a γ-supercompact embedding $j : V \to M$ (with $\kappa = \mathrm{crit}(j)$), then our primary concern tends to be the function $j \restriction \gamma$ and the ordinal

$$\rho := \sup\{j(\alpha) : \alpha < \gamma\}.$$

We note that ρ is an ordinal of cofinality $\mathrm{cf}(\gamma)$, and that $\{j(\alpha) : \alpha < \gamma\}$ is a $<\kappa$-closed unbounded subset of η. Furthermore, since M is closed under sequences of length $\leq \gamma$, the set $\{j(\alpha) : \alpha < \gamma\}$ is an element of M.

It is well-known that strongly compact cardinals can be characterized in many different ways (see [53, Sect. 22] or [51, Chap. 20]). In particular, they can be characterized in terms of elementary embeddings in such a way that it becomes clear that a supercompact cardinal must be strongly compact. The following result taken from [95] gives the characterization.

1.21 Proposition. *A cardinal κ is strongly compact if and only if for every $\gamma \geq \kappa$ there is an elementary embedding $j : V \to M$ such that*

1. $\mathrm{crit}(j) = \kappa$,
2. $\gamma < j(\kappa)$, *and*
3. *for every $X \subseteq M$ with $|X| \leq \gamma$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.*

If we require the above only for a specific $\gamma \geq \kappa$, then κ is said to be γ-strongly compact.

Forcing

In order to keep our chapter to manageable length, we have chosen to keep our use of forcing to a minimum; in the few places we do use it, our notation is standard, that is, as one finds in [51] or [59]. We follow the convention that $p \leq q$ means "p extends q".

By far the most important notion of forcing we use is the Levy collapse, so we take a moment to collect notation.

1.22 Definition. $\mathrm{Col}(\kappa, \lambda)$ is the cardinal collapse to make λ have cardinality κ, i.e. a condition is a partial function from κ into λ with domain of cardinality less than κ, and $p \leq q$ if and only if $q \subseteq p$. $\mathrm{Col}(\kappa, {<}\lambda)$ is the Levy collapse to make all cardinals in $[\kappa, \lambda)$ have cardinality κ. Here a condition p is a partial function from $\lambda \times \kappa$ into λ with $|\operatorname{dom}(p)| < \kappa$, $p(0, i) = 0$, and $p(\alpha, i) < \alpha$ for $\alpha \neq \emptyset$.

Remarks

Finally, I am going to drop the authorial "we" for the remainder of the introduction so that I can comment a bit on the chapter. There were of course two major decisions to make—what should be the scope of the chapter, and for whom should it be written.

The first of these questions was by far the most difficult for me to answer because the assigned topic touches every aspect of set theory. The second question was easier—I decided early on that I would attempt to write the type of survey that I wish had been available when I first started learning about the topic.

The chapter consists of five major divisions, of which the first is taken up with introductory material. The second division is an exposition of Magidor's proof of the consistency of $\mathrm{Refl}(\aleph_{\omega+1})$; I chose this as my starting point because stationary reflection is a major theme in the chapter, and his proof can be used to motivate the idea of approachability and $I[\lambda]$.

The next division deals extensively with $I[\lambda]$ and some of its applications. The section is quite long, and deliberately so, because much of the material covered is scattered through papers of Shelah going back almost three decades. I thought it worthwhile to gather it together in a single place since the lack of such a resource caused considerable aggravation when I was first learning the material. Most of the material in that section is based on notes I took in the Jerusalem Logic Seminar during lectures by Shelah in the summer of 1998 [90].

The fourth division of the paper is primarily based on more recent work of Cummings, Foreman, and Magidor. In contrast to the $I[\lambda]$ material, most of the results in this portion of the paper have appeared in papers where the exposition is masterful—in particular, the papers [36, 16], and [65] are highly recommended. In addition, Cummings has recently written a survey on the same subject [14]—we have benefited immensely from his paper, and in several places we reference his paper for proofs that we have not included in the current chapter.

The final major division of the paper concerns Jónsson cardinals and club guessing, and is for the most part independent of the other parts of the chapter. I chose to include this material for the same reason I devoted so much time to the development of $I[\lambda]$—it is a very interesting topic, but the basic results are scattered through several difficult papers and in Shelah's book [89].

In summary, the chapter could easily have been three times as long as it is, and I reluctantly left out much that is interesting. It is my hope that the level of exposition is sufficient that any set-theorist with a casual interest in the topic can catch a glimpse of the major ideas, and that those people interested in "really learning" the material will receive enough background and guidance for a comfortable transition to current literature.

Finally, a few words of thanks are in order. I would like to thank Matt Foreman and James Cummings for patiently answering my queries as I wrote the chapter. Thanks are due as well to Aki Kanamori and an anonymous referee for many helpful remarks, while Andres Caicedo read through the entire chapter carefully and did yeoman's work rooting out misstatements and typographical errors. The chapter is a much stronger work thanks to their contributions.

2. On Stationary Reflection

We now turn to the question of stationary reflection, mentioned briefly in the first subsection of the introduction. First, we use stationary reflection as a lens to study an important phenomenon in set theory—the tension between

$\Box$-like principles and large cardinals. We then briefly investigate indecomposable ultrafilters and their influence on stationary reflection before plunging into Magidor's proof [62] that $\mathrm{Refl}(\aleph_{\omega+1})$ is consistent relative to the existence of infinitely many supercompact cardinals. This proof occupies the last three subsections of this part of the chapter, and some of the issues that arise therein will lead us to topics considered in later sections.

2.1. Squares and Supercompact Cardinals

In the first part of the introduction, we pointed out that it is quite easy to see that $\mathrm{Refl}(\kappa^+)$ fails whenever κ is a regular cardinal. In particular, if κ is regular then the set $S^{\kappa^+}_\kappa$ is a stationary subset of κ^+ that does not reflect. On the other hand, $\mathrm{Refl}(\kappa)$ holds if κ is weakly compact—this is an easy corollary to the Hanf-Scott characterization [45] of weakly compact cardinals as precisely the Π^1_1-indescribable cardinals. Jech's chapter [49] in this Handbook contains further discussion and references for stationary reflection in the context of successors of regular cardinals and "small" large cardinals. None of this, however, sheds any light on how successors of singular cardinals behave with respect to stationary reflection, and it is to this subject we now turn. Our treatment begins with the $\Box$ principle of Jensen [52].

We affirm some notation for the chapter: For a set X of ordinals, $\mathrm{ot}(X)$ denotes its order-type; $\mathrm{acc}(X)$ denotes the set of accumulation (i.e. limit) points of X other than $\sup(X)$; and $\mathrm{nacc}(X)$ denotes $X \setminus \mathrm{acc}(X)$.

2.1 Definition. Let κ be an uncountable cardinal. A $\Box_\kappa$*-sequence* is a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$ such that for all $\alpha < \kappa^+$,

1. C_α is closed and unbounded in α,
2. $\mathrm{ot}(C_\alpha) \le \kappa$, and
3. if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

$\Box_\kappa$ is the assertion that there is such a sequence.

It is not difficult to prove that $\Box_\kappa$ is equivalent to the *prima facie* stronger principle in which condition (2) is replaced by

$$\mathrm{cf}(\alpha) < \kappa \quad \Longrightarrow \quad \mathrm{ot}(C_\alpha) < \kappa. \tag{15.5}$$

Given a $\Box_\kappa$-sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$, we can construct a $\Box_\kappa$-sequence satisfying (15.5) by fixing a closed unbounded subset C of κ of order-type $\mathrm{cf}(\kappa)$, and replacing each C_α with

$$\{\beta \in C_\alpha : \mathrm{ot}(C_\alpha \cap \beta) \in C\}$$

whenever $\mathrm{ot}(C_\alpha)$ is in $\mathrm{acc}(C) \cup \{\kappa\}$.

Jensen [52] proved that $\Box_\kappa$ holds for all κ in the constructible universe L. Note as well that $\Box_\kappa$ is quite persistent—if $\Box_\kappa$ holds for some cardinal in a model of set theory W, then it will continue to hold in any extension of W in which κ and κ^+ are preserved. These two facts lead us to the following well-known corollary of Jensen's Covering Lemma.

2.2 Theorem. *If $0^\#$ does not exist, then $\square_\mu$ holds for all singular cardinals μ.*

Proof. Let μ be a singular cardinal in V, and assume $0^\#$ does not exist. It suffices to prove that $(\mu^+)^L = \mu^+$, as a $\square_\mu$-sequence from L will then serve to witness that $\square_\mu$ continues to hold in V.

Suppose that $\theta = (\mu^+)^L$, and assume by way of contradiction that $\theta < \mu^+$. It must be the case that $|\theta| = \mu$ and hence $\mathrm{cf}(\theta) < \mu$. Now let X be a cofinal subset of θ of cardinality $\mathrm{cf}(\theta)$. By the Covering Lemma, there is a set $Y \in L$ such that $X \subseteq Y$ and $|Y| \leq |X| + \aleph_1 < \mu = |\theta|$. But since the set Y is cofinal in θ and θ is regular in L, it must be the case that $|Y| = |\theta|$, and we have a contradiction. Thus, $(\mu^+)^L = \mu^+$ and consequently $\square_\mu$ must hold as well. $\dashv$

The failure of $\square_\mu$ for singular μ is not an easy thing to arrange—we have just seen that it necessarily involves large cardinals, and further research has shown that the large cardinals needed are quite large indeed:

2.3 Theorem (Steel [96]). *If there is a singular strong limit cardinal μ for which $\square_\mu$ fails, $L(\mathbb{R})$ determinacy holds and hence there is an inner model with infinitely many Woodin cardinals.*

We cannot hope to treat the relationship between squares and inner models of set theory in this chapter, but we do point out that this is a reason why large cardinals are always in the background when one considers successors of singular cardinals. We send a reader looking for more information on this topic to the chapters by Welch and by Schimmerling in this Handbook. We now leave inner models and turn to combinatorial consequences of $\square_\kappa$.

A $\square_\kappa$-sequence is a prototypical example of a "non-compact" object of size κ^+—if $\langle C_\alpha : \alpha < \kappa^+\rangle$ is such a sequence, then there is no closed unbounded $C \subseteq \kappa^+$ such that $C \cap \alpha = C_\alpha$ for all $\alpha \in \mathrm{acc}(C)$. Much of the strength of $\square_\kappa$ lies in the fact that it is useful for constructing other non-compact objects of size κ^+. In particular, non-reflecting stationary sets are a natural example of such objects, and the following well-known argument (credited to Magidor in [5] and to Jensen in [51]) shows us that they are quite abundant in the presence of squares.

2.4 Theorem. *Suppose that $\square_\kappa$ holds for a given uncountable cardinal κ. Then* $\mathrm{Refl}(S)$ *fails for every stationary $S \subseteq \kappa^+$.*

Proof. Let S be a stationary subset of κ^+, and suppose that $\langle C_\alpha : \alpha < \kappa^+\rangle$ is a $\square_\kappa$-sequence. For each limit ordinal $\alpha < \kappa^+$, let $f(\alpha) = \mathrm{ot}(C_\alpha)$. Clearly this is a regressive function on $S \setminus \kappa + 1$, and so we can find a stationary $T \subseteq S$ on which f is constant.

For $\alpha < \kappa^+$ of uncountable cofinality and $\beta \in \operatorname{acc}(C_\alpha)$, we have

$$f(\beta) = \operatorname{ot}(C_\beta) = \operatorname{ot}(C_\alpha \cap \beta).$$

Thus the function f is one-to-one on $\operatorname{acc}(C_\alpha)$, hence $|T \cap \operatorname{acc}(C_\alpha)| \leq 1$. This immediately implies that $T \cap \alpha$ is not stationary in α and therefore T is a non-reflecting stationary subset of S. ⊣

This is only one example of the ways in which a $\square_\kappa$-sequence can be used to construct non-compact objects of size κ^+. Jensen's original paper [52] uses such sequences together with the Generalized Continuum Hypothesis to construct κ^+-Souslin trees for every uncountable κ. Devlin's book [19] contains a nice exposition of this result, and Todorčević's chapter [98] in this Handbook contains other examples of applications of square in combinatorial set theory.

In light of Theorem 2.4 and Theorem 2.3, it is natural to ask about the effect that large cardinals have on the problem of stationary reflection. We begin with an argument of Solovay that forms part of the background for Magidor's paper [62].

2.5 Theorem. *Suppose that κ is a supercompact cardinal, and $\lambda > \kappa$ is regular. Then* $\operatorname{Refl}(S)$ *holds for every stationary subset of $S^\lambda_{<\kappa}$.*

Proof. Let $S \subseteq S^\lambda_{<\kappa}$, and let j be a λ-supercompact embedding with critical point κ. Let $\rho = \sup\{j(\alpha) : \alpha < \lambda\}$, and note that since

$$M \models \operatorname{cf}(\rho) = \lambda < \rho \quad \text{and} \quad j(\lambda) \text{ is regular},$$

it follows that $\rho < j(\lambda)$.

Thus, if we can show

$$M \models j(S) \cap \rho \text{ is stationary in } \rho \tag{15.6}$$

then the elementarity of j allows us to conclude that $S \cap \alpha$ is stationary in α for some $\alpha < \lambda$. We actually achieve a bit more than (15.6); we show that $j(S) \cap \rho$ is a stationary subset of ρ even in the larger universe V.

Let $C \subseteq \rho$ be closed and unbounded, and consider $D := j^{-1}[C]$. Since $\kappa = \operatorname{crit}(j)$, it follows easily that D is $<\kappa$-club in λ. Since S is a stationary subset of $S^\lambda_{<\kappa}$, we can find $\xi \in D \cap S$. Clearly $j(\xi) \in j(S) \cap C$, so (15.6) follows. ⊣

A few remarks are in order here. First, we note that the above proof only serves to prove that S reflects at an ordinal θ whose cofinality is less than κ. This result is sharp, as we shall see later in Corollary 4.9.

Second, we note that essentially the same proof shows us that given a sequence $\langle S_i : i < \epsilon\rangle$ such that $\epsilon < \kappa$ and each S_i is a stationary subset of $S^\lambda_{<\kappa}$, there is an ordinal $\theta \in S^\lambda_{<\kappa}$ with $S_i \cap \theta$ stationary for all $i < \epsilon$.

Finally, we have the following corollary (also due to Solovay) which gives us our first examples of singular cardinals μ for which $\operatorname{Refl}(\mu^+)$ holds.

2.6 Corollary. *Let μ be a singular limit of supercompact cardinals. Then every stationary subset of μ^+ reflects.*

Proof. Since μ is singular, any stationary subset of μ^+ contains a stationary subset of some fixed cofinality $\theta < \mu$. There is a supercompact cardinal κ with $\theta < \kappa < \mu^+$ and the result follows immediately from Theorem 2.5. $\dashv$

2.2. Reflection and Indecomposable Ultrafilters

The paper [95] credits Gregory with the observation that the conclusion of Theorem 2.5 still holds even if κ is only strongly compact. This is fairly easy to see—essentially the same proof works because of Proposition 1.21. However, the proof given in [95] applies in many more situations, as it establishes a relationship between stationary reflection and indecomposable ultrafilters. In this subsection we examine this line of reasoning.

The following definition is due to Keisler, though Prikry's [71] seems to be the first place where it appears in the literature.

2.7 Definition. Let $\mathcal{U}$ be a uniform ultrafilter on a set A, and let τ be a cardinal. We say that $\mathcal{U}$ is *τ-decomposable* if there is a sequence $\langle A_i : i < \tau \rangle$ of sets such that

$$A = \bigcup_{i<\tau} A_i$$

but

$$\bigcup_{i\in B} A_i \notin \mathcal{U} \quad \text{for each } B \in [\tau]^{<\tau}.$$

We say that $\mathcal{U}$ is *τ-indecomposable* if it is not τ-decomposable. The ultrafilter $\mathcal{U}$ is *indecomposable* if it is τ-indecomposable for all τ with $\aleph_0 < \tau < |A|$.

Indecomposability can be viewed as a weak form of completeness. This is made clear by the following elementary proposition from [71], which connects indecomposability of filters with the closely related notion of descendingly complete filters first studied in [11, 60], and [71].

2.8 Proposition. *If τ is a regular cardinal and $\mathcal{U}$ is a uniform ultrafilter on some cardinal κ, then $\mathcal{U}$ is τ-indecomposable if and only if $\mathcal{U}$ is closed under decreasing intersections of length τ.*

The reason we bring up this topic is that there is a connection between indecomposable ultrafilters and stationary reflection, and the following theorem of Silver and Prikry [71] makes this clear.

2.9 Theorem. *If $\kappa < \lambda$ are regular cardinals and there is a uniform κ-indecomposable ultrafilter on λ, then* $\mathrm{Refl}(S^\lambda_\kappa)$ *holds.*

Before we give the proof, we point out a couple of corollaries illustrating how strongly compact cardinals serve just as well as supercompact cardinals with regard to the reflection results presented in the previous subsection.

2.10 Corollary. *Suppose that κ is strongly compact. Then* $\mathrm{Refl}(S^\lambda_\theta)$ *holds for all regular cardinals θ and λ with $\theta < \kappa < \lambda$.*

Proof. This is easy given our previous work. Since κ is strongly compact, there is a uniform κ-complete ultrafilter on λ. Such an ultrafilter is trivially θ-indecomposable for $\theta < \kappa$, and so we apply Theorem 2.9 to get the conclusion. ⊣

2.11 Corollary. *Let μ be a singular cardinal that is a limit of compact cardinals. Then every stationary subset of μ^+ reflects.*

To prove Theorem 2.9, we rely on an argument attributed by the authors of [95] to an anonymous referee.

2.12 Lemma. *Suppose that S is a non-reflecting (not necessarily stationary) subset of S^λ_κ for some regular $\kappa < \lambda$, and that for each $\delta \in S$, A_δ is a cofinal subset of δ of order-type κ. Then for each $\alpha < \lambda$, there is a regressive function F_α with domain $S \cap \alpha$ such that the family $\{A_\delta \setminus F_\alpha(\delta) : \delta \in S \cap \alpha\}$ is disjoint. In plain terms, for each $\alpha < \lambda$ the family $\{A_\delta : \delta \in S \cap \alpha\}$ can be made pairwise disjoint by removing a bounded set from each A_δ.*

Proof. The proof is by induction on α, with the cases where $\alpha = 0$ or $\alpha = \beta+1$ for $\beta \notin S$ being straightforward.

If α is a limit ordinal, then we choose a closed unbounded $C \subseteq \alpha$ disjoint to S. Thus, every $\delta \in S \cap \alpha$ will fall into an interval of the form (γ, ϵ) where γ and ϵ are consecutive members of C. It is straightforward to check that if we define

$$F_\alpha(\delta) = \max\{F_\epsilon(\delta), \gamma\},$$

then everything works as it should.

The only other troublesome case is when $\alpha = \delta + 1$ for some $\delta \in S$. In this case, we use the fact that A_δ has order-type κ, as this means that for $\gamma \in S \cap \delta$, the set $A_\gamma \cap A_\delta$ is bounded below γ. So we may define

$$F_\alpha(\gamma) = \max\{F_\delta(\gamma), \sup(A_\delta \cap A_\gamma)\}$$

for $\gamma \in S \cap \delta$, and let $F_\alpha(\delta) = 0$. ⊣

The preceding lemma has some importance in general topology, for it is a crucial ingredient in Fleissner's proof [29, 30] of the necessity of large cardinals in Nyikos' celebrated result [70] (assuming the existence of a strongly compact cardinal) that consistently all normal Moore spaces are metrizable.

In the next lemma, we show that uniform κ-indecomposable ultrafilters allow us to patch together the functions F_α from the previous lemma in order to obtain a single function that works simultaneously for all A_δ.

2.13 Lemma. *Let $\kappa < \lambda$ be regular cardinals, and assume $\mathcal{U}$ is a uniform κ-indecomposable ultrafilter on λ. If we are given a family $\{A_\alpha : \alpha < \lambda\}$ of sets, each of order-type κ, with the property that for each $\beta < \lambda$ there is a function F_β with domain β such that*

1. *$F_\beta(\alpha)$ is a proper initial segment of A_α, and*

2. *the family $\{A_\alpha \setminus F_\beta(\alpha) : \alpha < \beta\}$ is disjoint,*

then in fact there is a function F with domain λ such that

1. *$F(\alpha)$ is a proper initial segment of A_α, and*

2. *the family $\{A_\alpha \setminus F(\alpha) : \alpha < \lambda\}$ is disjoint.*

Proof. Fix $\alpha < \lambda$. For each $\epsilon < \kappa$, let B^α_ϵ consist of all those $\beta < \lambda$ for which $F_\beta(\alpha)$ is contained in the first epsilon elements of A_α, that is,

$$B^\alpha_\epsilon := \{\beta < \lambda : \mathrm{ot}(F_\beta(\alpha)) < \epsilon\}.$$

Since the sequence $\langle B^\alpha_\epsilon : \epsilon < \kappa\rangle$ is increasing with union $\lambda \setminus \alpha$, the κ-indecomposability of $\mathcal{U}$ tells us there is an $\epsilon(\alpha) < \kappa$ such that

$$B^\alpha_{\epsilon(\alpha)} \in \mathcal{U}.$$

Now let us define $F(\alpha)$ to be the first $\epsilon(\alpha)$ elements of A_α. We claim that the collection $\{A_\alpha : \backslash F(\alpha) : \alpha < \lambda\}$ is pairwise disjoint.

To see this, suppose that $\alpha < \gamma < \lambda$. Both of the sets $B^\alpha_{\epsilon(\alpha)}$ and $B^\gamma_{\epsilon(\gamma)}$ are in $\mathcal{U}$, so we can find β in their intersection. By the definition of F_β, we know

$$(A_\alpha \setminus F_\beta(\alpha)) \cap (A_\gamma \setminus F_\beta(\gamma)) = \emptyset,$$

and since $F_\beta(\alpha) \subseteq F(\alpha)$ and $F_\beta(\gamma) \subseteq F(\gamma)$, the result follows. ⊣

We are now ready to give the proof of Theorem 2.9.

Proof of Theorem 2.9. Let $\mathcal{U}$ be a uniform κ-indecomposable ultrafilter on λ, and assume by way of contradiction that S is a non-reflecting stationary subset of S^λ_κ. For each $\delta \in S$, we choose a cofinal A_δ of order-type κ, and note that by Lemma 2.12, the family $\{A_\delta : \delta \in S\}$ satisfies the assumptions of Lemma 2.13. Since S is stationary, the function provided by Lemma 2.13 must be constant on some stationary $T \subseteq S$, say with value β. We can assume that $\beta < \min(T)$, and so the sets of the form $A_\delta \setminus \beta$ for $\delta \in T$ are non-empty and pairwise disjoint. However, A_δ is a subset of δ, and so any choice function for this collection contradicts Fodor's Theorem. ⊣

Theorem 2.9 shows us that the existence of indecomposable uniform ultrafilters at the successor of a singular cardinal has high consistency strength, for such ultrafilters are incompatible with $\square$. However, Ben-David and Magidor [7] were able to produce a model in which $\aleph_{\omega+1}$ carries a uniform indecomposable ultrafilter by starting with a supercompact cardinal.

2.14 Theorem (Ben-David and Magidor [7]). *Let V be a model of* ZFC *with a cardinal κ that is κ^+-supercompact. Then there is a forcing extension V' of V such that*

1. $V' \models \kappa = \aleph_\omega$ and $\kappa^+ = \aleph_{\omega+1}$, and

2. in V', $\aleph_{\omega+1}$ carries an indecomposable ultrafilter.

In particular, in their model every stationary subset of $\aleph_{\omega+1}$ made up of ordinals of uncountable cofinality must reflect. This "near miss" brings us to Magidor's construction of a model in which *every* stationary subset of $\aleph_{\omega+1}$ reflects.

2.3. Reflection at $\aleph_{\omega+1}$—Introduction

2.15 Theorem (Magidor [62]). *If the existence of infinitely many supercompact cardinals is consistent, then it is consistent to assume that every stationary subset of $\aleph_{\omega+1}$ reflects.*

Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Define $\mu = \sup_{n<\omega} \kappa_n$ and $\lambda = \mu^+$. Corollary 2.6 tells us that every stationary subset of λ reflects, and our strategy to transfer this down to $\aleph_{\omega+1}$ is a natural one—we define a notion of forcing $\mathbb{P}$ so that in the generic extension $V[G_\omega]$, κ_n becomes $\aleph_{n+1}$, μ becomes $\aleph_\omega$, and λ becomes $\aleph_{\omega+1}$.

This is done in a straightforward fashion, with the iteration $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n < \omega \rangle$ defined by setting

$$\mathbb{P}_0 = \mathrm{Col}(\omega, {<}\kappa_0)$$

and

$$V^{\mathbb{P}_n} \models \dot{\mathbb{Q}}_n = \mathrm{Col}(\kappa_n, {<}\kappa_{n+1}).$$

We let $\mathbb{P}_\omega$ be the inverse limit of this iteration.

Each κ_n is inaccessible so it follows easily that $\mathbb{P}_n$ satisfies the κ_n-chain condition. Note as well that for each n, we can factor $\mathbb{P}_\omega$ as $\mathbb{P}_n * \mathbb{P}^n$ where $\mathbb{P}^n$ is κ_n-closed. Thus, every sequence of ordinals of length less than κ_n added by forcing with $\mathbb{P}_\omega$ must lie in the initial generic extension obtained by forcing with $\mathbb{P}_n$. From this, we conclude that after forcing with $\mathbb{P}_\omega$, each κ_n has been collapsed to $\aleph_{n+1}$, μ has become $\aleph_\omega$, and λ has become $\aleph_{\omega+1}$.

The supercompactness of each κ_n survives in a vestigial way—in the terminology of [31], each κ_n is *generically supercompact*. We give a more precise description of this below.

Let $G \subseteq \mathbb{P}_\omega$ be generic, and let θ be a regular cardinal larger than κ_n. In $V[G]$, there is an $\aleph_n$-closed notion of forcing $\mathbb{R}$ such that if H is a $V[G]$-generic subset of $\mathbb{R}$, then in the extension $V[G * H]$ there is an elementary embedding $j : V[G] \to M \subseteq V[G * H]$ (for some transitive M) satisfying

1. $\mathrm{crit}(j) = \kappa_n$,

2. $j \restriction \theta \in M$, and

3. $j(\kappa_n) > \theta$.

Thus, in a sense the $\aleph_n$-closed forcing $\mathbb{R}$ resurrects the θ-supercompactness of κ_n and the corresponding elementary embedding is called a *generic elementary embedding.* Rather than digressing into the topic of forcing and large cardinals, we will use the preceding fact as a so-called black box. We refer the reader to Magidor's original paper [62] or Cummings' [13] for more details.

We proceed by trying to imitate the proof of Theorem 2.5 using a generic elementary embedding instead of an honest-to-goodness one. In $V[G]$, suppose that S is a stationary subset of $\lambda = \aleph_{\omega+1}$ consisting of ordinals of cofinality $\aleph_n$ for some $n < \omega$. By the generic supercompactness of $\aleph_{n+2} = \kappa_{n+1}$, there is an $\aleph_{n+1}$-closed notion of forcing $\mathbb{R}$ such that, letting $H \subseteq \mathbb{R}$ be $V[G]$-generic, in the model $V[G][H]$ we can find a transitive M and elementary $j : V[G] \to M$ satisfying

- $\mathrm{crit}(j) = \kappa_{n+1} = \aleph_{n+2}$,
- $j \restriction \lambda \in M$, and
- $j(\kappa_{n+1}) > \lambda$.

Just as in the proof of Theorem 2.5, one can argue

$$\rho := \sup\{j(\alpha) : \alpha < \lambda\} < j(\lambda), \tag{15.7}$$

but we run into a problem trying to show

$$M \models j(S) \cap \rho \text{ is stationary in } \rho. \tag{15.8}$$

The difficulty arises because we need to know that S remains stationary in $V[G * H]$. The question of whether this must be so will occupy us in the next subsection.

2.4. κ^+-closed Forcing and Stationary Subsets of S^λ_κ

Our investigations in the preceding subsection have led us to the following question:

> Suppose that $\kappa < \lambda$ are regular cardinals, and let S be a stationary subset of S^λ_κ, and let $\mathbb{P}$ be a κ^+-closed notion of forcing. Is the stationarity of S preserved when forcing with $\mathbb{P}$?

If $\lambda = \kappa^+$, then the answer is an easy "yes"—a simple argument establishes that λ-closed forcing preserves all stationary subsets of λ. Thus, for example, $\aleph_1$-closed forcing preserves the stationary of subsets of $\aleph_1$. However, in this case, a little more work establishes a much stronger result due to Baumgartner [5].

2.16 Proposition. *Let λ be an uncountable regular cardinal, and let $\mathbb{P}$ be an $\aleph_1$-closed notion of forcing. Then every stationary subset of $S^\lambda_{\aleph_0}$ remains stationary in the $\mathbb{P}$-generic extension.*

Proof. Suppose that $\mathbb{P}$ is $\aleph_1$-closed, and suppose that $S \subseteq S^\lambda_{\aleph_0}$ is stationary. Let $\dot{C}$ be a $\mathbb{P}$-name such that

$$V^{\mathbb{P}} \models \dot{C} \text{ is a closed unbounded subset of } \lambda.$$

We show that the set of conditions in $\mathbb{P}$ which force $S \cap \dot{C}$ to be non-empty is dense in $\mathbb{P}$. Toward this end, let $p \in \mathbb{P}$ be arbitrary.

Since S is stationary, we can find a model $M \prec H(\chi)$ containing everything relevant such that $M \cap \lambda$ is some ordinal $\delta \in S$. Let $\langle \alpha_n : n < \omega \rangle$ be increasing and cofinal in δ. By induction on $n < \omega$, we choose a sequence $\langle p_n : n < \omega \rangle$ as follows:

Let $p_0 = p$. Given p_n, let p_{n+1} be the $<_\chi$-least extension of p_n in $\mathbb{P}$ that decides a value for $\min(\dot{C} \setminus \alpha_n)$, say

$$p_{n+1} \Vdash \min(\dot{C} \setminus \alpha_n) = \delta_n \tag{15.9}$$

for some $\delta_n < \lambda$.

Note that p_{n+1} is definable in $H(\chi)$ from p_n and parameters in M. Thus an easy induction shows that each p_n is an element of M, and it follows that each δ_n is in M as well. The sequence $\langle \delta_n : n < \omega \rangle$ is non-decreasing, and the range is cofinal in δ. Since $\mathbb{P}$ is $\aleph_1$-closed, the sequence $\langle p_n : n < \omega \rangle$ has a lower bound q, and q forces δ to be an accumulation point of $\dot{C}$, hence

$$q \Vdash \delta \in S \cap \dot{C}, \tag{15.10}$$

as required. $\dashv$

Now what goes wrong when we replace $S^\lambda_{\aleph_0}$ with S^λ_κ for κ uncountable? Suppose that S is a stationary subset of S^λ_κ (with κ an uncountable regular cardinal) and let $\mathbb{P}$ be a κ^+-closed notion of forcing. Suppose that $\dot{C}$ is a $\mathbb{P}$-name for a closed unbounded subset of λ, and let p be a condition in $\mathbb{P}$. We can certainly find $M \prec H(\chi)$ containing everything relevant such that $M \cap \lambda$ is some ordinal $\delta \in S$. We can also fix an increasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ cofinal in δ. Just as before, we define a decreasing sequence of conditions $\langle p_\alpha : \alpha < \kappa \rangle$ so that $p_0 = p$ and

$$p_{\alpha+1} \Vdash \min(\dot{C} \setminus a_\alpha) = \delta_\alpha \tag{15.11}$$

for some $\delta_\alpha < \lambda$. The problem arises in that we cannot guarantee that $\delta = \sup\{\delta_\alpha : \alpha < \kappa\}$—in the earlier construction we were always assured that $\delta_n < \delta$ because $p_{n+1} \in M$. In our current situation, this need not happen. Consider what happens at stage ω—in order to keep the construction going inside M, we need to know that $\langle p_\alpha : \alpha < \omega \rangle \in M$, and we do not know this unless it happens that $\langle a_\alpha : \alpha < \omega \rangle \in M$. In order for our proof to generalize, we need to be able to choose the sequence $\langle a_\alpha : \alpha < \kappa \rangle$ in such a way that every initial segment lies in M. This will be possible under mild cardinal arithmetic assumptions—for example, in certain circumstances we can arrange for M to be closed under sequences of length less than κ—but it need not happen in general. We will deal with the question of when such a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ exists a bit later on, but for now we will be content with the following *ad hoc* definitions and theorem implicit in [83]:

2.17 Definition. Let $\kappa < \lambda$ be regular cardinals, and let S be a stationary subset of S^λ_κ.

1. We say that S is κ^+*-closed indestructible* if for every κ^+-closed notion of forcing $\mathbb{P}$,
$$V^{\mathbb{P}} \models S \text{ is a stationary subset of } \lambda. \tag{15.12}$$
2. S is said to satisfy the κ^+*-closed indestructibility condition* if for every $x \in H(\chi)$, there are $M \prec H(\chi)$, $\delta \in S$, and cofinal $A \subseteq \delta$ such that
   - $x \in M$,
   - $M \cap \lambda = \delta$,
   - $\mathrm{ot}(A) = \mathrm{cf}(\delta) = \kappa$, and
   - every initial segment of A is in M.

2.18 Theorem. *Let κ and λ be regular cardinals, with $\kappa^+ < \lambda$. A stationary $S \subseteq S^\lambda_\kappa$ satisfies the κ^+-closed indestructibility condition if and only if S is κ^+-closed indestructible.*

Proof. If S satisfies the κ^+-closed indestructibility condition, then the argument that S is actually κ^+-closed indestructible is a simple modification of that given in Proposition 2.16. We leave this to the reader, and concentrate on the other direction. Our first step is an easy lemma.

2.19 Lemma. *Let $\kappa < \lambda$ be regular. A stationary $S \subseteq S^\lambda_\kappa$ satisfies the κ^+-indestructibility condition if and only if it satisfies the weaker version where instead of* $\mathrm{ot}(A) = \kappa$ *we demand only that* $\mathrm{ot}(A) < \delta$*.*

Proof. Note that the only difference between the two conditions is that we have weakened the demands on $A \subseteq \delta$. Suppose that we are given M, δ, and $A = \langle a_\alpha : \alpha < \mathrm{ot}(A)\rangle$ satisfying the weaker requirements.

Since $\mathrm{ot}(A) < \delta$, it follows that $\mathrm{ot}(A) \in M$. Since $\mathrm{cf}(\mathrm{ot}(A)) = \kappa$, we can find an increasing function $f : \kappa \to \mathrm{ot}(A)$ in M with range cofinal in $\mathrm{ot}(A)$. Clearly $\langle a_{f(\alpha)} : \alpha < \kappa\rangle$ is cofinal in δ and of order-type κ. To finish, note that every initial segment of $\langle a_{f(\alpha)} : \alpha < \kappa\rangle$ is in M because each such segment is definable from f and an initial segment of A. $\dashv$

To finish the proof of Theorem 2.18, we assume that $S \subseteq S^\lambda_\kappa$ does not satisfy the κ^+-indestructibility condition and we show that there is a κ^+-closed notion of forcing $\mathbb{P}$ that destroys the stationarity of S.

By assumption, there is a large regular χ and an $x \in H(\chi)$ so that whenever $M \prec H(\chi)$ is chosen with $x \in M$ and $M \cap \lambda = \delta \in S$, there is no cofinal $A \subseteq \delta$ of order-type κ with every initial segment of A in M.

Temporarily assume that $\lambda = \lambda^{<\lambda}$, and let $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$ enumerate $[\lambda]^{<\lambda}$ so that every set in $[\lambda]^{<\lambda}$ appears unboundedly often.

A forcing condition p is an increasing continuous function such that

- $\mathrm{dom}(p)$ is an initial segment of κ^+,
- $\mathrm{ran}(p) \subseteq \lambda \setminus \kappa^+$, and
- for all $\alpha \in \mathrm{nacc}(\mathrm{dom}(p))$, $\{p(\beta) : \beta < \alpha\} = A_{p(\alpha)}$.

A condition q *extends* p if and only if p is an initial segment of q.

It should be clear that $\mathbb{P}$ is κ^+-closed. An easy density argument establishes that forcing with $\mathbb{P}$ adjoins a function $f : \kappa^+ \to \lambda$ with range cofinal in λ. Since $\mathbb{P}$ is κ^+-closed, we can conclude that in the generic extension λ is an ordinal of cofinality κ^+.

Let G be a generic subset of $\mathbb{P}$, and step into the model $V[G]$. As mentioned before, the generic object is essentially an increasing function $f : \kappa^+ \to \lambda$. Let us define $C := \mathrm{acc}(\mathrm{ran}(f))$, and let $\dot{C}$ be a $\mathbb{P}$-name for C. It is clear

$$V^{\mathbb{P}} \models \dot{C} \text{ is closed and unbounded in } \lambda. \tag{15.13}$$

Back in the ground model, let $E \subseteq \lambda$ consist of all $\delta < \lambda$ for which there is an $M \prec H(\chi)$ with $\{x, \lambda, \mathbb{P}, \dot{C}, S, \bar{A}\} \in M$ and $M \cap \lambda = \delta$. Since E contains a closed unbounded subset of λ, it suffices to prove

$$V[G] \models S \cap E \cap \dot{C} = \emptyset. \tag{15.14}$$

If this fails, then we can find a condition $p \in \mathbb{P}$ and ordinal $\delta < \lambda$ such that

$$p \Vdash \delta \in S \cap \dot{C} \cap E. \tag{15.15}$$

Since δ must actually be an element of $S \cap E$, in the ground model we can find a model $M \prec H(\chi)$ confirming δ's membership in E.

Since p forces δ to be in $\dot{C}$, there must be a $\theta \leq \mathrm{dom}(p)$ such that $B := \{p(\alpha) : \alpha < \theta\}$ is cofinal in δ—otherwise, we could extend p to a condition forcing that δ is not in $\dot{C}$. Since $\theta < \kappa^+$, we certainly know $\mathrm{ot}(B) < \delta$. For $\alpha < \theta$, we have $p(\alpha) < \delta$ hence $p(\alpha) \in M$. Since $\bar{A} \in M$, we conclude that $A_{p(\alpha)} \in M$. But $A_{p(\alpha)} = B \cap \alpha$, and thus every initial segment of B lies in the model M. This contradicts our assumption that S does not satisfy the κ^+-closed indestructibility condition.

Our requirement that $\lambda = \lambda^{<\lambda}$ in the above discussion is not an obstacle, as we can first force with the λ-closed notion of forcing $\mathrm{Col}(\lambda, \lambda^{<\lambda})$, and in this extension define $\mathbb{P}$ as above. The composition of these two notions of forcing will destroy the stationarity of S. ⊣

The following lemma gives us a slightly more general condition guaranteeing that a stationary set satisfies the κ^+-indestructibility criterion, at least in the presence of some cardinal arithmetic considerations. We will make use of this in the next subsection.

2.20 Lemma. *Suppose that $\kappa < \lambda$ are regular cardinals with $\lambda = \mu^+$ for a strong limit singular μ. A stationary $S \subseteq S^\lambda_\kappa$ satisfies the κ^+-indestructibility condition if and only if for every $x \in H(\chi)$, there are $M \prec H(\chi)$, $\delta \in S$, and cofinal $A \subseteq \delta$ such that*

- $x \in M$,
- $M \cap \lambda = \delta$,
- $\mathrm{ot}(A) < \delta$, *and*
- *every initial segment of A is covered by a set in $M \cap [\lambda]^{<\mu}$.*

Proof. Let M, δ, and A be as above. Without loss of generality, each of μ, κ, and λ is in M, and so $\mu \subseteq M$. Given $\beta < \delta$, our hypotheses give us a set $B \in M \cap [\lambda]^{<\mu}$ such that $A \cap \beta \subseteq B$. We assume μ is a strong limit, so $2^{|B|} < \mu$. In M, there is a function f mapping $2^{|B|}$ onto the power set of B, and since $2^{|B|} \subseteq M$, it follows that $\mathcal{P}(B) \subseteq M$ as well. In particular, $A \cap \beta \in M$. $\dashv$

As a final remark, we note that the restriction to $\kappa^+ < \lambda$ in the results of this subsection is reasonable, as λ-closed forcing preserves all stationary subsets of λ. This said, we return now to Magidor's proof.

2.5. Reflection at $\aleph_{\omega+1}$—Conclusion

Our goal is this subsection is to finish the proof that in the model from Sect. 2.3, every stationary subset of $\aleph_{\omega+1}$ reflects. The main point is to verify that in $V[G_\omega]$, every stationary subset of $\aleph_{\omega+1}$ satisfies the appropriate version of the indestructibility condition from Definition 2.17.

Recall that in the ground model we are given the following:

- $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of supercompact cardinals
- $\mu = \sup_{n<\omega} \kappa_n$, and
- $\lambda = \mu^+$.

Our notion of forcing $\mathbb{P}_\omega$ collapses each κ_n to $\aleph_{n+1}$, makes $\mu = \aleph_\omega$, and makes $\lambda = \aleph_{\omega+1}$. We had no occasion to mention it before, but it is clear that $\aleph_\omega$ is a strong limit cardinal in the generic extension—this is important for us because we will be using Lemma 2.20.

Goal: In the model $V[G_\omega]$, if S is a stationary subset of $S^{\aleph_{\omega+1}}_{\aleph_n}$ then S satisfies the $\aleph_{n+1}$-indestructibility condition.

We will reach our goal in a roundabout fashion. Back in the ground model, let us fix a λ-filtration system $\langle b_{\alpha,i} : \alpha < \lambda, i < \omega \rangle$ with $|b_{\alpha,i}| \leq \kappa_i$. Let d be the associated coloring of $[\lambda]^2$, that is,

$$d(\beta, \alpha) = \text{least } i \text{ such that } \beta \in b_{\alpha,i}$$

It is clear that $d : [\lambda]^2 \to \omega$, and the following lemma of Shelah establishes a crucial property of d in $V[G_\omega]$.

2.21 Proposition. *In $V[G_\omega]$, if $\delta \in \lambda \setminus \mu$ is a limit ordinal of uncountable cofinality, then there is an unbounded $H_\delta \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ that is homogeneous for the coloring d.*

Proof. Suppose that in $V[G_\omega]$, $\delta < \aleph_{\omega+1}$ has uncountable cofinality and $|\delta| = \aleph_\omega$. Back in the ground model, there is an $n < \omega$ such that

$$V \models \kappa_n \le \mathrm{cf}(\delta) < \kappa_{n+1}. \tag{15.16}$$

Since κ_n is supercompact, there is a κ_n-complete ultrafilter $\mathcal{U}$ on δ containing all co-bounded subsets of δ. In particular, $\mathcal{U}$ is countably complete so that for each $\beta < \delta$, there is a unique $i(\beta) < \omega$ such that

$$B_\beta := \{\alpha \in (\beta, \delta) : d(\beta, \alpha) = i(\beta)\} \in \mathcal{U}. \tag{15.17}$$

2.22 Claim. *In $V[G_\omega]$, if $\langle \beta_\zeta : \zeta < \xi \rangle$ is a sequence of ordinals less than δ of length less than κ_n, then $\bigcap_{\zeta<\xi} B_{\beta_\zeta}$ contains a set from $\mathcal{U}$.*

Proof. Note that this is not automatic—in $V[G_\omega]$ the filter generated by $\mathcal{U}$ need not be κ_n-complete. We note, however, that $\langle \beta_\zeta : \zeta < \xi \rangle$ already lies in the P_n-generic extension $V[G_n]$, and hence the sequence $\langle B_{\beta_\zeta} : \zeta < \xi \rangle$ is in $V[G_n]$ as well. Since P_n satisfies the κ_n-chain condition, it follows that in $V[G_n]$ the filter on δ generated by $\mathcal{U}$ is still κ_n-complete and the conclusion is immediate. ⊣

We are now ready to construct the required H_δ in $V[G_\omega]$. We define a sequence $\langle \beta_\xi : \xi < \kappa_n \rangle$ by induction on ξ. To start, let $\beta_0 < \delta$ be arbitrary. Given $\langle \beta_\zeta : \zeta < \xi \rangle$, define

$$\beta_\xi := \text{least member of } \textstyle\bigcap_{\zeta<\xi} B_{\beta_\zeta} \text{ greater than } \alpha_\xi. \tag{15.18}$$

Note that $\bigcap_{\zeta<\xi} B_{\beta_\zeta}$ contains a set from $\mathcal{U}$ by Claim 2.22, and so a suitable β_ξ can always be defined.

It is clear that $\langle \beta_\xi : \xi < \kappa_n \rangle$ is cofinal in δ, and the sequence is constructed so that for $\zeta < \xi < \kappa_n$ the value of $d(\beta_\zeta, \beta_\xi)$ depends only on β_ζ. Since $\kappa_n = \aleph_{n+1}$ is uncountable, there is an unbounded $H_\delta \subseteq \{\beta_\xi : \xi < \kappa_n\}$ such that $d \restriction [H_\delta]^2$ is constant, and the proof of Proposition 2.21 is complete. ⊣

Our next task is to relate Proposition 2.21 to the problem of preserving stationary sets in forcing conditions. Suppose that $n < \omega$ is given, and let $S \subseteq S^{\aleph_{\omega+1}}_{\aleph_n}$ be stationary. If $n = 0$, then S automatically remains stationary in an $\aleph_1$-closed forcing extension, so assume $n > 0$. In $V[G_\omega]$, let $x \in H(\chi)$ be given and fix a model $M \prec H(\chi)^{V[G_\omega]}$ such that

- $\{x, S, \langle b_{\alpha,i} : i < \omega, \alpha < \lambda\rangle\} \in M$, and
- $M \cap \lambda = \delta \in S$.

By Proposition 2.21, there is an unbounded $H \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ homogeneous for the coloring c, say $c \restriction [H]^2$ is constant with value i. Given $\beta \in H$, it follows that

$$H \cap \beta \subseteq b_{\beta,i}. \tag{15.19}$$

Now $b_{\beta,i} \in M \cap [\aleph_{\omega+1}]^{<\aleph_\omega}$, and since $\aleph_\omega$ is a strong limit cardinal in $V[G_\omega]$, we apply Lemma 2.20 and conclude that S satisfies the $\aleph_{n+1}$-closed indestructibility condition. By our comments at the end of Sect. 2.3, it follows that in $V[G_\omega]$ every stationary subset of $\aleph_{\omega+1}$ reflects.

3. On $I[\lambda]$

The goal of this section is to re-examine issues surrounding the somewhat *ad hoc* notion of the "κ^+-closed indestructibility condition" (Definition 2.17). The investigation will lead us to several important concepts, with $I[\lambda]$ and approachability being the most prominent. We will develop the theory of $I[\lambda]$ in some detail in order to demonstrate its importance for combinatorial set theory. All unattributed results in this section are due to Shelah, although in some cases the theorems as stated may only be implicit in his work.

Let us begin our discussion by recalling the definition of the κ^+-closed indestructibility condition. Let $\kappa < \lambda$ be regular, and let χ be a sufficiently large regular cardinal. A stationary $S \subseteq S^\lambda_\kappa$ satisfies the κ^+-closed indestructibility condition if and only if for any $x \in H(\chi)$, there is a model $M \prec H(\chi)$ such that

- $x \in M$,
- $M \cap \lambda$ is an ordinal $\delta \in S$, and
- there is a cofinal $A \subseteq \delta$ of order-type κ, all of whose initial segments lie in M.

One way to look at this is that the set A is built from initial segments that are simpler (in some vague way) than δ. If it happens that $\lambda = \lambda^{<\lambda}$, then we can give a precise meaning to "simpler", as illustrated by the following proposition.

3.1 Proposition. *Assume $\lambda = \lambda^{<\lambda}$, and fix an enumeration $\bar{a}$ of $[\lambda]^{<\lambda}$ in order-type λ. A set $S \subseteq S^\lambda_\kappa$ satisfies the κ^+-indestructibility condition if and only if there are stationarily many $\delta \in S$ for which there is a cofinal $A_\delta \subseteq \delta$ of order-type κ with every initial segment of A_δ enumerated prior to stage δ.*

Proof. First, assume that S satisfies the indestructibility condition. Let χ be a sufficiently large regular cardinal, and suppose that we are given $x \in H(\chi)$ and a closed unbounded $C \subseteq \lambda$. By Definition 2.17, we know there is a model $M \prec H(\chi)$ such that

- $\{x, \bar{a}, C\} \in M$,
- $M \cap \lambda = \delta \in S$, and
- there is a cofinal $A \subseteq \delta$ of order-type κ with every initial segment lying in M.

Since $C \in M$, we know $\delta \in S \cap C$. Since $\bar{a} \in M$ and $M \cap \lambda = \delta$, we know that $\langle a_\alpha : \alpha < \delta\rangle$ enumerates $M \cap [\lambda]^{<\lambda}$. In particular, since every initial segment of A lies in $M \cap [\lambda]^{<\lambda}$, we have that each such initial segment is enumerated by $\bar{a}$ before stage δ.

For the other direction, assume we are given a sufficiently large regular χ and $x \in H(\chi)$. Let $\langle M_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence over $\{x, \kappa, S, \bar{a}\}$. The set of δ with $M_\delta \cap \lambda = \delta$ is closed and unbounded, so we can find such $\delta \in S$ for which a suitable A_δ exists. Since every initial segment of A_δ is enumerated by $\bar{a}$ before stage δ, it follows immediately that every initial segment of A_δ lies in M_δ, as required. $\dashv$

Note that if $\lambda = \mu^+$ for μ singular, $\kappa < \lambda$ is regular, and $\lambda = \lambda^{<\lambda}$, then there is a set $S \subseteq S^\lambda_\kappa$ such that a set $T \subseteq S^\lambda_\kappa$ satisfies the κ^+-closed indestructibility condition if and only if $T \cap S$ is stationary. Why? Simply fix an enumeration $\langle a_\alpha : \alpha < \lambda\rangle$ of $[\lambda]^{<\lambda}$ and let S be the set of all $\delta \in S^\lambda_\kappa$ for which there is a cofinal $A_\delta \subseteq \delta$ of order-type κ with every initial segment of A_δ enumerated before stage δ. In addition, the set S is unique modulo the nonstationary ideal, because if $\langle b_\alpha : \alpha < \lambda\rangle$ is any other enumeration of $[\lambda]^{<\lambda}$ then the set of δ for which $\{a_\alpha : \alpha < \delta\} = \{b_\alpha : \alpha < \delta\}$ is closed and unbounded in λ.

We will not make use of the above observation, but we shall be looking at similar results throughout this section.

3.1. The Ideal $I[\lambda]$

The ideas introduced in the preceding discussion still make sense even without the assumption that $\lambda = \lambda^{<\lambda}$—all that is required is a suitable enumeration $\bar{a}$.

3.2 Definition. Let $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ be a sequence of bounded subsets of λ. A limit ordinal $\delta < \lambda$ is said to be *approachable with respect to the sequence* $\bar{a}$ if there is an unbounded $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that every initial segment of A is enumerated by $\bar{a}$ before stage δ, that is,

$$\{A \cap \beta : \beta < \delta\} \subseteq \{a_\beta : \beta < \delta\}. \tag{15.20}$$

Now comes the surprising part—the concept isolated in the preceding definition turns out to be incredibly useful in combinatorial set theory, and most of our time in the current section will be spent supporting this thesis. Our first task is to take Definition 3.2 and use it to define an ideal of subsets of λ.

3.3 Definition. A set $S \subseteq \lambda$ *is in* $I[\lambda]$ if and only if there is a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ of bounded subsets of λ and a closed unbounded $C \subseteq \lambda$ such that every $\delta \in S \cap C$ is singular and approachable with respect to the sequence $\bar{a}$.

With the preceding definition, we have isolated the most important concept in this section of the chapter. Our first task is to prove that $I[\lambda]$ is in fact an ideal of subsets of λ.

3.4 Proposition. *For λ a regular cardinal, the collection $I[\lambda]$ is a (possibly improper) normal ideal on λ.*

Proof. We verify that $I[\lambda]$ is closed under diagonal unions; the rest of the proof is quite trivial. Suppose that we are given a sequence $\langle S_i : i < \lambda \rangle$ of elements of $I[\lambda]$. For each $i < \lambda$, there are a corresponding enumeration $\bar{a}_i$ and closed unbounded E_i witnessing $S_i \in I[\lambda]$. Using a pairing function, we can fold all of the enumerations $\bar{a}_i = \langle a^i_\alpha : \alpha < \lambda \rangle$ into a single enumeration $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ for which there is a closed unbounded $E^* \subseteq \lambda$ such that for $\delta \in E^*$ and $i < \delta$, we have

$$\{a^i_\alpha : \alpha < \delta\} \subseteq \{a_\beta : \beta < \delta\}. \tag{15.21}$$

Let $E := E^* \cap \triangle_{i<\lambda} E_i$. We claim that $\bar{a}$ and E serve as witnesses to the membership of $S := \triangledown_{i<\lambda} S_i$ in $I[\lambda]$.

Suppose now that $\delta \in E \cap S$. By the definition of diagonal union, there is an $i < \delta$ such that $\delta \in S_i$. By definition of diagonal intersection, we know that $\delta \in E_i$ as well. Thus, there is an unbounded $A \subseteq \delta$ of order-type $\operatorname{cf}(\delta)$ such that every initial segment of A is enumerated by $\bar{a}_i$ before stage δ. Since $\delta \in E^*$, we know that every initial segment of A is enumerated by $\bar{a}$ before stage δ as well, and the proof is finished. ⊣

It is not true that $I[\lambda]$ must be a *proper* ideal on λ. In fact, we shall see that the assumption that $I[\lambda]$ is not a proper ideal is a combinatorial statement of some power. Before dealing with such matters, we will invest a little time in deriving some equivalent formulations of the ideal $I[\lambda]$. We begin with an observation that will be used throughout this section.

3.5 Definition. Let $\mathfrak{M} = \langle M_i : i < \lambda \rangle$ be a λ-approximating sequence. We define $S[\mathfrak{M}]$ to be the set of $\delta < \lambda$ such that δ is singular, $M_\delta \cap \lambda = \delta$, and there is a cofinal $a \subseteq \delta$ of order-type $\operatorname{cf}(\delta)$ with the property that every (proper) initial segment of a is in M_δ.

3.6 Theorem. *$I[\lambda]$ is the ideal on λ generated by the nonstationary sets together with all sets of the form $S[\mathfrak{M}]$ with $\mathfrak{M}$ a λ-approximating sequence.*

Proof. It is not difficult to prove that $S[\mathfrak{M}]$ is in $I[\lambda]$—if we enumerate $[\lambda]^{<\lambda} \cap \bigcup_{i<\lambda} M_i$ as $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ and let E be the closed unbounded set of $\delta < \lambda$ for which $\delta = M_\delta \cap \lambda$ and $\langle a_\alpha : \alpha < \delta \rangle$ enumerates $M_\delta \cap [\lambda]^{<\lambda}$, then each $\delta \in S[\mathfrak{M}] \cap E$ is approachable with respect to $\bar{a}$.

Conversely, suppose that $S \in I[\lambda]$ and $\mathfrak{M}$ is a λ-approximating sequence over S. We claim $S[\mathfrak{M}]$ contains almost all members of S. To see this, note that M_0 will contain an enumeration $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ and a closed unbounded $C \subseteq \lambda$ such that each element of $S \cap C$ is approachable with respect to $\bar{a}$. We claim now that any $\delta \in S$ satisfying $M_\delta \cap \lambda = \delta$ must be in $S[\mathfrak{M}]$. This is quite easy, as any such δ must be in C as well, and $a_\alpha \in M_\delta$ for each $\alpha < \delta$. $\dashv$

We remark that the above proof establishes something slightly stronger than what is claimed in the theorem—a minor adjustment shows us that a set S is in $I[\lambda]$ if and only if there is a λ-approximating sequence $\mathfrak{M}$ with $S \setminus S[\mathfrak{M}]$ nonstationary.

Theorem 3.6 is the characterization of $I[\lambda]$ that we tend to use in the sequel, but the following characterization due to Shelah [88] is also of interest because of its similarity to square sequences.

3.7 Theorem (Shelah [88]). *Let λ be a regular cardinal. Then the following two conditions are equivalent for a set $S \subseteq \lambda$:*

1. *$S \in I[\lambda]$.*
2. *There is a sequence $\langle C_\alpha : \alpha < \lambda \rangle$ and a closed unbounded $E \subseteq \lambda$ such that*
   - *(a) C_α is a closed (but not necessarily unbounded) subset of α,*
   - *(b) if $\beta \in \mathrm{nacc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$, and*
   - *(c) if $\delta \in E \cap S$ then δ is singular, and C_δ is a closed unbounded subset of δ of order-type $\mathrm{cf}(\delta)$.*

Proof. The proof that (2) implies (1) is by far the easier direction, so we dispose of it first. Assume that S satisfies (2), and let $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda \rangle$ be a λ-approximating sequence over $\{S, \bar{C}, E\}$. We claim that if $\delta \in S \cap E$ and $M_\delta \cap \lambda = \delta$, then C_δ will attest to δ's membership in $S[\mathfrak{M}]$.

The proof is elementary, but it is worthwhile to pursue it in order to further understand the relationship between $I[\lambda]$ and squares. Given such a δ, we need to verify that each (proper) initial segment of C_δ lies in M_δ, so fix $\zeta < \delta$. Let β be the least member of C_δ greater than ζ (so $\beta \in \mathrm{nacc}(C_\delta)$). Then $C_\beta = C_\delta \cap \beta$ hence $C_\delta \cap \beta \in M_\delta$. We are almost done—if it happens that $C_\delta \cap \beta$ is not $C_\delta \cap \zeta$, then it can only be because $\zeta \in C_\delta$. But then

$$C_\delta \cap \zeta = C_\delta \cap \beta \setminus \{\zeta\} \in M_\delta \tag{15.22}$$

as required. Since $S \setminus S[\mathfrak{M}]$ is nonstationary, we conclude $S \in I[\lambda]$.

The journey from (1) to (2) will also make use of λ-approximating sequences. Suppose that $S \in I[\lambda]$, and let $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda \rangle$ be a fixed λ-approximating sequence with $S \in M_0$. It suffices to define a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ with the following properties:

- $A_\alpha \subseteq \alpha$, and if $A_\alpha \neq \emptyset$ then A_α consists entirely of successor ordinals,
- if $\beta \in A_\alpha$, then $A_\beta = A_\alpha \cap \beta$, and
- if $\delta \in S[\mathfrak{M}]$, then A_δ is unbounded in δ of order-type $\mathrm{cf}(\delta)$.

If we can accomplish this, then letting C_α equal the closure of A_α in α gives us a sequence $\langle C_\alpha : \alpha < \lambda\rangle$ with all the requisite properties. The following *ad hoc* definition isolates a key aspect of the construction.

3.8 Definition. A set $x \in [\lambda]^{<\lambda}$ is said to be an $\mathfrak{M}$-*candidate* if for all $\zeta \in x$, there is an $\alpha < \lambda$ such that $x \cap \zeta \in M_\alpha$, and, letting

$$\alpha_\zeta := \text{least } \alpha < \kappa \text{ with } x \cap \zeta \in M_\alpha,$$

the sequence $\langle \alpha_\zeta : \zeta \in x\rangle$ is strictly increasing.

For each $\alpha < \lambda$, let us define

$$P_\alpha := M_\alpha \cap \mathcal{P}(\alpha),$$

and

$$\delta_\alpha := M_\alpha \cap \lambda.$$

Given $\alpha < \lambda$, fix a function F_α mapping $P_\alpha \times \delta_\alpha$ in a one-to-one manner into the set of successor ordinals between δ_α and $\delta_{\alpha+1}$. Note that this is always possible because $M_\alpha \in M_{\alpha+1}$ and hence $\delta_{\alpha+1}$ is much larger than δ_α. We construct the sequence $\langle A_\alpha : \alpha < \lambda\rangle$ using the following recipe:

Case 1. α a successor ordinal.

If $\alpha \in M_0$, then define $A_\alpha = \emptyset$. Otherwise, there is a unique $i < \lambda$ such that $\delta_i < \alpha < \delta_{i+1}$. We now ask the following question:

> Is there a candidate $x \in M_i$ and a regular cardinal $\beta < \min(x)$ such that $\alpha = F_i(x, \beta)$ and $|x| \leq \beta$?

If no such x and β exist, then we set $A_\alpha = \emptyset$. If on the other hand we do find such x and β, then they must be unique and so we can define

$$A_\alpha = \{F_{\alpha_\zeta}(x \cap \zeta, \beta) : \zeta \in x\}, \tag{15.23}$$

where α_ζ is as in Definition 3.8 for our particular x.

Case 2. α is a limit ordinal.

If $M_\alpha \cap \lambda$ is not equal to α, we let $A_\alpha = \emptyset$. Otherwise, we ask the question:

> Is there a candidate $x \in M_{\alpha+1}$ with x cofinal in α such that $\mathrm{ot}(x) = \mathrm{cf}(\alpha) < \min(x)$?

If no such x exists, we define $A_\alpha = \emptyset$. Otherwise, we choose such an x and define

$$A_\alpha = \{F_{\alpha_\zeta}(x \cap \zeta, \mathrm{cf}(\alpha)) : \zeta \in x\}. \tag{15.24}$$

It is worth remarking that the set A_α will be cofinal in α of order-type $\mathrm{cf}(\alpha)$, as the fact that x is a candidate from $M_{\alpha+1}$ implies $\alpha_\zeta < \alpha$ for all $\zeta \in x$. We use this remark in the proof of Claim 3.10.

The rest of the proof consists in verifying that the sequence $\langle A_\alpha : \alpha < \lambda\rangle$ we constructed above has all the required properties.

3.9 Claim. *If $\alpha < \lambda$ and $\gamma \in A_\alpha$ then $A_\gamma = A_\alpha \cap \gamma$.*

Proof. Our assumptions imply $A_\alpha \neq \emptyset$, and so there exist a candidate x and a regular cardinal β such that $\beta < \min(x)$, $|x| \leq \beta$, and

$$A_\alpha = \{F_{\alpha_\zeta}(x \cap \zeta, \beta) : \zeta \in x\}.$$

Now fix $\zeta^* \in x$ such that

$$\gamma = F_{\alpha_{\zeta^*}}(x \cap \zeta^*, \beta).$$

The ordinal γ is a successor ordinal, and thus A_γ is defined by the procedure outlined in the first case in the construction. Clearly γ cannot be in M_0, and when we ask the question associated with Case 1, the answer is "yes" with $x \cap \zeta^*$ and β serving as the unique witnesses. Thus,

$$A_\gamma = \{F_{\alpha_\zeta}(x \cap \zeta) : \zeta \in x \cap \zeta^*\} = A_\alpha \cap \delta_{\alpha_{\zeta^*}} = A_\alpha \cap \gamma,$$

as required. $\dashv$

3.10 Claim. *If $\delta \in S[\mathfrak{M}]$, then A_δ is cofinal in δ with order-type* $\mathrm{cf}(\delta)$.

Proof. We note that this follows easily provided we can establish that the answer to the question asked in Case 2 is "yes", as the set A_δ produced will have the required properties.

From the definition of $S[\mathfrak{M}]$, it follows that δ is singular and there is a cofinal $a \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ with the property that every initial segment of a is in M_δ. The proof of this claim consists of an argument that a can be "thinned out" so that it becomes a candidate of the required sort. The problem is that we cannot just throw away elements of a willy-nilly because we need to preserve that fact that all proper initial segments are in M_δ. The argument given below is not difficult, but we include it for completeness.

We start by noting that we *are* allowed to throw away initial segments of a—closure properties of elementary submodels (and the fact that each $M_i \cap \lambda$ is an initial segment of λ) tells us that the resulting tail of a has the property that all initial segments lie in M_δ. Thus without loss of generality $\mathrm{ot}(a) < \min(a)$.

Let $\kappa = \mathrm{ot}(a) = \mathrm{cf}(\delta)$; we define an increasing sequence of elements of a by the following recursion:

Construction

Given $\langle x_j : j < i\rangle$, we ask if $\langle x_j : j < i\rangle$ is in M_δ. If not, the construction terminates; if so, we define

$$\alpha_i = \text{least } \gamma < \delta \text{ such that } \langle x_j : j < i\rangle \in M_\gamma \tag{15.25}$$

and define

$$x_i = \text{least element of } a \text{ that is not in } M_{\alpha_i}. \tag{15.26}$$

We claim that our construction will not terminate until stage κ. To see this, suppose by way of contradiction that it terminates at stage $i < \kappa$. Since κ is regular, there is a $\zeta \in a$ such that $\{x_j : j < i\} \subseteq a \cap \zeta$. Furthermore, $\sup\{\alpha_j : j < i\} < \delta$. Thus we can find a $\gamma < \delta$ such that M_γ contains ζ, $a \cap \zeta$, and $\sup\{\alpha_j : j < i\}$. Now we can reconstruct $\langle x_j : j < i\rangle$ inside of the model M_γ, so $\langle x_j : j < i\rangle \in M_\delta$ and we have contradicted the fact that our construction was forced to terminate at stage i. Note that since a is cofinal in δ, we will always be able to find an x_i as in (15.26).

Clearly the sequence $\langle x_i : i < \kappa\rangle$ is increasing and cofinal in a. Furthermore, the above argument shows that every initial segment of $\langle x_i : i < \kappa\rangle$ lies in M_δ. Taken with (15.26), this means that

$$x := \{x_i : i < \kappa\} \text{ is a candidate.} \tag{15.27}$$

To finish the proof of Claim 3.10, we note that the construction of x can be carried out inside the model $M_{\delta+1}$, so without loss of generality x itself is in $M_{\delta+1}$. Thus, the answer to the question asked in Case 2 of the construction of $\langle A_\alpha : \alpha < \lambda\rangle$ is "yes", and therefore A_δ is cofinal in δ of order-type $\mathrm{cf}(\delta)$. $\dashv$

This completes the proof of Theorem 3.7. $\dashv$

We close this section with one more characterization of $I[\lambda]$, this time due to Foreman and Magidor. This characterization will be used in Sect. 3.6 when we show that scales exist.

3.11 Theorem. *Let $\mathfrak{M} = \langle M_i : i < \lambda\rangle$ be a λ-approximating sequence over S. Then $\delta \in S[\mathfrak{M}]$ if and only if we can find an IA chain $\langle N_\alpha : \alpha < \mathrm{cf}(\delta)\rangle$ with $|N_\alpha| < \mathrm{cf}(\delta)$ for each $\alpha < \mathrm{cf}(\delta)$, and, letting $N = \bigcup_{\alpha<\mathrm{cf}(\delta)} N_\alpha$, such that*

1. $\sup(N \cap \lambda) = \delta$,
2. *$M_i \in N$ for unboundedly many $i < \delta$, and*
3. *$N \subseteq M_\delta$.*

Proof. Let $\langle \delta_i : i < \mathrm{cf}(\delta)\rangle$ be a sequence affirming that δ is a member $S[\mathfrak{M}]$. By thinning out this sequence along the lines of what we did in the proof of Claim 3.10, we may assume

$$\langle \delta_j : j \le i\rangle \in M_{\delta_{i+1}}.$$

Let $x_i = \langle M_{\delta_j} : j \leq i\rangle$ and note that $x_i \in M_{\delta_{i+1}}$. We define

$$N_i = \mathrm{Sk}^{\mathfrak{A}}(\{x_j : j < i\}).$$

N_i can be computed in $M_{\delta_{i+1}}$ from the parameter M_{δ_i}, and so

$$\langle N_j : j \leq i\rangle \in N_{i+1}.$$

The rest follows easily as well. ⊣

3.2. The Approachability Property

In this section, we turn our attention to applications of the ideal $I[\lambda]$. For simplicity, we restrict ourselves to the situation of $\lambda = \mu^+$ for μ a singular cardinal. Our investigation begins with isolating the following combinatorial principle:

3.12 Definition. Let μ be a singular cardinal. We say that *the Approachability Property holds at* μ (abbreviated by AP_μ) if the ideal $I[\mu^+]$ is improper, that is, if μ^+ itself is a member of $I[\mu^+]$.

The ideal $I[\lambda]$ is normal, so clearly AP_μ holds if and only if $I[\lambda]$ contains a closed unbounded subset of λ.

3.13 Theorem. *For singular μ, AP_μ follows from $\square_\mu$.*

Proof. Let $\lambda = \mu^+$, let $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$ be a $\square_\mu$-sequence, and let $\mathfrak{M} = \langle M_i : i < \lambda\rangle$ be a λ-approximating sequence over $\bar{C}$. We show that $S[\mathfrak{M}]$ contains the closed unbounded set of all $\delta < \lambda$ for which $M_\delta \cap \lambda = \delta$.

Suppose now that $M_\delta \cap \lambda = \delta$. Then C_δ is a closed unbounded subset of δ, $\mathrm{ot}(C_\delta) < \delta$, and

$$\alpha \in \mathrm{acc}(C_\delta) \quad \Longrightarrow \quad C_\alpha = C_\delta \cap \alpha.$$

If $\alpha < \delta$, then C_α is an element of M_δ; from this, it follows easily that every initial segment of C_δ must be in M_δ. This is not quite enough to conclude that $\delta \in S[\mathfrak{M}]$ because it need not be the case that $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$, but if this happens an argument like that of Lemma 2.19 will yield a set A with the properties necessary to witness δ's membership in $S[\mathfrak{M}]$. ⊣

An examination of Theorem 3.7 gives us a good idea of the relationship between $\square_\mu$ and AP_μ. If α is a limit point of C_δ and we are dealing with a $\square_\mu$-sequence, then we know $C_\delta \cap \alpha = C_\alpha$, but in the case of AP_μ, we need to know *the next element of C_δ beyond α* in order to determine $C_\delta \cap \alpha$—if $\beta = \min(C_\delta \setminus \alpha + 1)$, then

$$C_\beta = (C_\delta \cap \alpha) \cup \{\alpha\}.$$

This is reminiscent of the difference between $\diamondsuit$ and CH—both of these axioms involve enumerations, but the stronger axiom has a bit of "promptness" built into it.

3.14 Definition. Let μ be a singular cardinal. An AP_μ*-sequence* is a sequence $\bar{C} = \langle C_\alpha : \alpha < \mu^+\rangle$ such that

1. C_α is a closed (not necessarily unbounded) subset of α,
2. $|C_\alpha| < \mu$, and
3. for a closed unbounded set of $\delta < \mu^+$,
   (a) C_δ is a closed unbounded subset of δ,
   (b) $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta)$, and
   (c) for all $\beta < \delta$, $C_\beta \cap \delta \in \{C_\alpha : \alpha < \delta\}$.

The reader may find the above definition a little surprising because on the face of it we have weakened considerably the conclusion of Theorem 3.7. However, it is not hard to prove that AP_μ holds if and only if there is an AP_μ-sequence, and the above definition has the advantage of being already ensconced in the literature.

We turn now to the task of demonstrating the utility of AP_μ, at least in the case where μ is a strong limit singular cardinal. For the example we have in mind, it is useful to recast AP_μ as a statement about elementary submodels of $H(\chi)$. The following definition from [4] gives us the terminology needed to state the result.

3.15 Definition. Let τ be a regular cardinal, and let M be an elementary submodel of $H(\chi)$ for some sufficiently large regular χ.

1. M is τ*-closed* if $[M]^{<\tau} \subseteq M$.
2. M is *weakly* τ*-closed* if for every $I \in [M]^\tau$, there exists a $J \in [I]^\tau$ such that $[J]^{<\tau} \subseteq M$.

The following theorem is a reformulation of one of the main results of [4], but the argument we use has appeared several times in the literature (see Remark 13.11 of Todorčević's chapter [98] in this Handbook).

3.16 Theorem. *Let $\lambda = \mu^+$ for μ a strong limit singular cardinal. Then AP_μ holds if and only if for every $x \in H(\chi)$, there is a λ-approximating sequence $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda\rangle$ over x such that M_δ is weakly τ-closed for all regular cardinals $\tau < \mu$ and for all $\delta < \lambda$ satisfying $M_\delta \cap \lambda = \delta$.*

Proof. Let $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$ and E be as in the definition of AP_μ. Set $\kappa = \mathrm{cf}(\mu)$, and let $\langle \mu_i : i < \kappa\rangle$ be an increasing sequence of regular cardinals, cofinal in μ, such that $2^{\mu_i} < \mu_{i+1}$.

We build a matrix $\langle M^i_\alpha : \alpha < \lambda, i < \kappa\rangle$ of elementary submodels of $H(\chi)$ such that (letting $M_\alpha := \bigcup_{i<\kappa} M^i_\alpha$)

1. $\langle M_\alpha : \alpha < \lambda\rangle$ is a λ-approximating sequence over $\{x, \bar{C}, E\}$,
2. $|M^i_\alpha| = \mu_i$,

3. $M^i_\alpha \subseteq M^j_\alpha$ for $i < j$,
4. $\mu_{i+1} \cup \mathcal{P}(M^i_{\alpha+1}) \subseteq M^{i+1}_{\alpha+1}$, and
5. for each i, $\mathcal{P}(\bigcup_{\beta\in C_\alpha} M^i_\beta) \subseteq M_{\alpha+1}$.

The construction is straightforward; we note with regard to condition (5) that

$$\left|\mathcal{P}\left(P\bigcup\nolimits_{\beta\in C_\alpha} M^i_\beta\right)\right| \leq 2^{|C_\alpha|+\mu_i} < \mu,$$

and so $\mathcal{P}(\bigcup_{\beta\in C_\alpha} M^i_\beta)$ can be "swallowed" by $M^j_{\alpha+1}$ for some $j < \kappa$.

Now assume $\tau = \mathrm{cf}(\tau) < \mu$. We show first that if $\alpha = \beta + 1$, then M_α is weakly τ-closed. Assume $A \subseteq M_\alpha$ is of cardinality τ. If $\tau = \kappa$, then every member of $[A]^{<\tau}$ is a subset of $M^i_{\alpha+1}$ for some $i < \kappa$, and therefore

$$[A]^{<\tau} \subseteq M_{\alpha+1}$$

by (4). If on the other hand $\tau \neq \kappa$, then there is an $i < \kappa$ such that

$$|A \cap M^i_{\alpha+1}| = \tau,$$

and then

$$[A \cap M^i_{\alpha+1}]^{<\tau} \subseteq M^{i+1}_{\alpha+1} \subseteq M_{\alpha+1},$$

again by (4).

Now suppose that $\delta = M_\delta \cap \lambda$, and let $A \in [M_\delta]^\tau$. If $\mathrm{cf}(\delta) \neq \tau$ then there is an $\alpha < \delta$ such that

$$|A \cap M_{\alpha+1}| = \tau$$

and we can then take advantage of the fact that $M_{\alpha+1}$ is weakly τ-closed. Thus we may assume that $\tau = \mathrm{cf}(\delta)$. If $\tau = \mathrm{cf}(\delta) = \kappa$, then $[A]^{<\tau} \subseteq M_\delta$ as any bounded subset of A is a subset of M^i_δ for some $i < \kappa$. Thus, our one remaining case is when $\tau = \mathrm{cf}(\delta) \neq \kappa$.

Since $M_\delta = \bigcup_{i<\kappa}\bigcup_{\alpha\in C_\delta} M^i_\alpha$, there is an $i < \kappa$ for which

$$B := A \cap \bigcup\nolimits_{\alpha\in C_\delta} M^i_\alpha$$

has cardinality τ. We will finish the proof by showing that $[B]^{<\tau} \subseteq M_\delta$. To see this, suppose that $K \in [B]^{<\tau}$. Since $\mathrm{ot}(C_\delta) = \mathrm{cf}(\delta) = \tau$, there is an $\alpha \in C_\delta$ such that

$$K \subseteq \bigcup\nolimits_{\beta\in C_\delta\cap\alpha} M^i_\beta.$$

Since $\delta \in E$, there is a $\gamma < \delta$ such that $C_\delta \cap \alpha = C_\gamma$. But condition (5) of our construction guarantees that $K \in M_{\gamma+1} \subseteq M_\delta$. This finishes the proof that M_δ is weakly τ-closed whenever τ is a regular cardinal less than μ and $M_\delta \cap \lambda = \delta$.

The other direction of the theorem is easily established—if $\mathfrak{M}$ is any λ-approximating sequence satisfying the assumptions given, then $S[\mathfrak{M}]$ contains the closed unbounded set of δ for which $M_\delta \cap \lambda = \delta$. $\dashv$

The application we give here of AP_μ is reminiscent of Lemma 2.13—we obtain the same conclusion from different hypotheses. Again, the following theorem is taken from [4].

3.17 Theorem. *Let $\lambda = \mu^+$ for μ a singular strong limit cardinal, and assume AP_μ holds. Suppose that for some $\sigma < \mu$ we are given a family*

$$\mathcal{A} = \langle A_\alpha : \alpha < \lambda \rangle$$

of sets from $[\lambda]^{\sigma^+}$ satisfying the following two conditions:

1. *For all $\beta < \lambda$, there is a function $F_\beta : \beta \to [\lambda]^\sigma$ such that*

$$\langle A_\alpha \setminus F_\beta(\alpha) : \alpha < \beta \rangle \text{ is disjoint.} \tag{15.28}$$

2. *For all $Z \in [\lambda]^\sigma$, $|\{\alpha < \lambda : Z \subseteq A_\alpha\}| \leq \mu$.*

Then there exists a function $F : \lambda \to [\lambda]^\sigma$ such that

$$\langle A_\alpha \setminus F(\alpha) : \alpha < \lambda \rangle \text{ is disjoint.} \tag{15.29}$$

Proof. By Theorem 3.16, there is a λ-approximating sequence $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda \rangle$ over $\mathcal{A}$ with the property that M_δ is weakly τ-closed for all regular $\tau < \mu$ whenever $M_\delta \cap \lambda = \delta$.

We claim that if $M_\delta \cap \lambda = \delta$ and $\alpha \geq \delta$, then it must be the case that

$$|A_\alpha \cap \delta| \leq \sigma. \tag{15.30}$$

To see this, suppose that it fails for δ and α. Clearly we may choose $J \subseteq A_\alpha \cap \delta$ of cardinality σ^+. Since M_δ is weakly σ^+-closed, we may assume that $[J]^\sigma \subseteq M_\delta$. Therefore, there is a set $Z \in M_\delta$ of cardinality σ such that $Z \subseteq A_\alpha$.

Since $\alpha \geq \delta$, it follows that for each $\beta \in M_\delta \cap \lambda$, we know

$$H(\chi) \models (\exists \gamma > \beta)[Z \subseteq A_\gamma],$$

hence

$$M_\delta \models (\exists \gamma > \beta)[Z \subseteq A_\gamma].$$

The preceding holds for all $\beta \in M_\delta \cap \lambda$, and so

$$M_\delta \models (\forall \beta < \lambda)(\exists \gamma > \beta)[Z \subseteq A_\gamma].$$

Another application of elementarity tells us

$$H(\chi) \models (\forall \beta < \lambda)(\exists \gamma > \beta)[Z \subseteq A_\gamma],$$

and this contradicts the second assumption we made about $\mathcal{A}$.

Now let $\langle \delta_i : i < \lambda \rangle$ enumerate the closed unbounded set E of δ for which M_δ is weakly σ^+-closed. We define the function $F : \lambda \to [\lambda]^\sigma$ by induction.

We begin by defining $F \restriction \delta_0$ by

$$F(\alpha) = F_{\delta_0}(\alpha),$$

where F_{δ_0} is the function from (15.28) for $\beta = \delta_0$. Next, given $i < \lambda$, we assume that $F \restriction \delta_i$ has already been defined. If $\delta_i \leq \alpha < \delta_{i+1}$, we define

$$F(\alpha) = F_{\delta_{i+1}}(\alpha) \cup (A_\alpha \cap \delta_i).$$

Note that $F(\alpha)$ has cardinality σ, and since E is closed unbounded in λ, our recipe defines $F(\alpha)$ for each $\alpha < \lambda$.

Does F work? All that remains to be shown is that

$$(A_\alpha \setminus F(\alpha)) \cap (A_\beta \setminus F(\beta)) = \emptyset \tag{15.31}$$

for $\alpha < \beta < \lambda$. If there is an i such that $\delta_i \leq \alpha < \beta < \delta_{i+1}$, then (15.31) holds because

$$A_\gamma \setminus F(\gamma) \subseteq A_\gamma \setminus F_{\delta_i}(\gamma)$$

for all $\gamma < \delta_i$. Otherwise, there is an i such that $\alpha < \delta_i \leq \beta$. In this case, note that $A_\alpha \subseteq M_{\delta_i}$ because $M_{\delta_i} \cap \lambda = \delta_i$, $A_\alpha \in M_{\delta_i}$, and $|A_\alpha| = \sigma^+ < \mu$. Thus, (15.31) holds

$$A_\alpha \cap A_\beta \setminus F(\beta) = \emptyset.$$

$\dashv$

The reader may wonder if it is necessary for the family $\mathcal{A}$ to satisfy condition (2) in order for the conclusion to hold. To see that it cannot simply be dropped, let us assume that μ is a singular cardinal for which $\square_\mu$ (and hence AP_μ) holds. Given $\sigma < \mu$, we can apply Theorem 2.4 to find a nonreflecting stationary $S \subseteq \mu^+$ consisting of ordinals of cofinality σ^+. If for each $\delta \in S$ we choose a cofinal A_δ of order-type σ^+, then by Lemma 2.12 the family $\mathcal{A} = \langle A_\delta : \delta \in S \rangle$ satisfies the first assumption of Theorem 3.17, but clearly the conclusion of Theorem 3.17 cannot hold just as in the proof of Theorem 2.9. These issues are studied in much greater detail in [4], as well as in joint work of Hajnal, Juhász, and Shelah [44, 43].

3.3. The Extent of $I[\lambda]$

This section is dedicated to analyzing the extent of the ideal $I[\lambda]$. In particular, we show that the ideal $I[\lambda]$ is quite large—it contains many stationary sets. In the other direction, we show that although $I[\lambda]$ contains stationary sets, if $\mathrm{cf}(\mu) < \kappa < \mu$ for some supercompact cardinal κ then AP_μ fails.

3.18 Theorem. *Suppose that $\kappa^+ < \sigma < \lambda$ for regular cardinals κ, σ, and λ. There is a set $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$ such that $S \cap \theta$ is stationary in θ for stationarily many $\theta \in S^\lambda_\sigma$. In particular, $I[\lambda]$ contains a stationary subset of S^λ_κ.*

Proof. Since $\kappa^+ < \sigma$, one of the basic results of club guessing (see Theorem 2.17 and Exercise 2.18 in the chapter [1]) tells us that there is a sequence $\bar{C} = \langle C_\eta : \eta \in S^\sigma_\kappa \rangle$ such that

- C_η is club in η of order-type κ, and
- for every club $E \subseteq \sigma$, the set of $\eta \in S^\sigma_\kappa$ with $C_\eta \subseteq E$ is stationary.

Let $\mathfrak{M} = \langle M_i : i < \lambda \rangle$ be a λ-approximating sequence over $\{\kappa, \sigma, \bar{C}\}$, and let $S = S[\mathfrak{M}] \cap S^\lambda_\kappa$. We know that S is in $I[\lambda]$, so we need to prove that the set of $\theta \in S^\lambda_\sigma$ for which $S \cap \theta$ is stationary in θ is stationary.

Assume by way of contradiction that this is not true. We can then build an IA chain $\mathfrak{N} = \langle N_i : i < \sigma \rangle$ of elementary submodels of $H(\chi)$ such that

- $|N_i| = \sigma$,
- $\{\kappa, \sigma, S, \bar{C}, \mathfrak{M}\} \in N_0$, and
- $S \cap \theta$ is nonstationary in $\theta := \sup_{i<\sigma}(N_i \cap \lambda)$.

Let $\theta_i = \sup(N_i \cap \lambda)$ for $i < \sigma$. The sequence $\langle \theta_i : i < \sigma \rangle$ is increasing, continuous, and $\langle \theta_j : j \leq i \rangle \in N_{i+1}$ for all $i < \sigma$.

Inside the model $M_{\theta+1}$, we fix a strictly increasing a continuous function g mapping σ onto a closed unbounded subset of θ. Since σ is uncountable, we know the set

$$E := \{i < \sigma : g(i) = \theta_i\}$$

is closed and unbounded in σ. Our goal is to prove that $\theta_\eta \in S[\mathfrak{M}]$ for all $\eta \in S^\sigma_\kappa$ with $C_\eta \subseteq E$. Since the set of such η is a stationary subset of σ, this contradicts the assumption that $S \cap \theta$ is a nonstationary subset of θ.

Thus, let $\eta < \sigma$ be such that $C_\eta \subseteq E$ and for each $\zeta < \theta$ let g_ζ denote the function $g \restriction (C_\eta \cap \zeta)$. It suffices to prove that g_ζ is in M_{θ_η} for all $\zeta < \eta$, as the range of $g \restriction C_\eta$ will then witness that θ_η is a member of $S[\mathfrak{M}]$.

Given $\zeta < \theta_\eta$, we know that g_ζ is in $M_{\theta+1}$ and thus the statement

$$(\exists \alpha < \lambda)[g_\zeta \in M_\alpha] \tag{15.32}$$

holds in $H(\chi)$.

The function g_ζ is definable from ζ, C_η, and a proper initial segment of $\langle N_i : i < \theta_\eta \rangle$, and so there must be an $i_0 < \theta_\eta$ with $g_\zeta \in N_{i_0}$. Now $\mathfrak{M} \in N_{i_0}$, and hence the statement (15.32) must hold in N_{i_0} as well because of elementarity.

In particular, g_ζ must be in M_α for some $\alpha \in N_{i_0} \cap \lambda$. But clearly such an α satisfies

$$\alpha < \sup(N_{i_0} \cap \lambda) < \theta_\eta,$$

and therefore g_ζ is in M_{θ_η} too, as required. $\dashv$

3.19 Corollary. *If $\lambda = \mu^+$ for μ singular and $\kappa < \mu$ is a regular cardinal, then $I[\lambda]$ contains a stationary subset of S^λ_κ.*

Why do we restrict to successors of singular cardinals in the preceding corollary? The answer is that much more is known in the case where $\lambda = \kappa^+$ for κ regular. Later in the chapter (see Corollary 4.6), we show that if κ is regular, then $S^{\kappa^+}_{<\kappa} \in I[\kappa^+]$. Mitchell [69] has announced that it is consistent that no stationary subset of $S^{\aleph_2}_{\aleph_1}$ is in $I[\aleph_2]$. (His argument is outlined in the paper [68].)

We have seen that $I[\lambda]$ contains many large sets, but the next result of Shelah [83] shows that supercompact cardinals impose limits on how big $I[\lambda]$ can be.

3.20 Theorem. *If κ is a supercompact cardinal, the* AP_μ *fails for all singular cardinals μ with* $\mathrm{cf}(\mu) < \kappa < \mu$.

Proof. Assume by way of contradiction that AP_μ holds, and let $\bar{C}$ and E be as in Definition 3.12. By a classical result of Solovay [94], we know that $\theta^{<\kappa} = \theta$ for all regular $\theta > \kappa$.

This means that we can enumerate $[\lambda]^{<\kappa}$ in a one-to-one fashion as $\bar{x} = \langle x_\alpha : \alpha < \lambda\rangle$. Moreover, it follows that $\theta^{<\kappa} \leq \theta + \kappa < \mu$ for all regular $\theta < \mu$ and so there is a function $F : \lambda \to \lambda$ with the property that for all $\alpha < \lambda$,

$$\gamma \leq \alpha \quad \Longrightarrow \quad [C_\gamma]^{<\kappa} \subseteq \{x_\beta : \beta < F(\alpha)\}. \tag{15.33}$$

Let $j : V \to M$ be a λ-supercompact embedding with $\kappa = \mathrm{crit}(j)$. Define

$$\begin{aligned} j(\bar{C}) &= \langle C^j_\alpha : \alpha < j(\lambda)\rangle, \\ j(\bar{x}) &= \langle x^j_\alpha : \alpha < j(\lambda)\rangle, \quad \text{and} \\ \rho &= \sup\{j(\alpha) : \alpha < \lambda\}. \end{aligned}$$

The set $\{j(\alpha) : \alpha < \lambda\}$ is a $<\kappa$-closed subset of ρ, and since C^j_ρ is a club subset of ρ, it follows that

$$D := \{\alpha < \lambda : j(\alpha) \in C^j_\rho\}$$

is a $<\kappa$-closed unbounded subset of λ. Choose $\delta < \lambda$ for which $|D \cap \delta| = \mu$. Since $j(\bar{C})$ is a $\mathrm{AP}_{j(\mu)}$ sequence in M, there is a $\gamma < \rho$ such that

$$C^j_\rho \cap j(\delta) = C^j_\gamma,$$

and we can find $\epsilon < \lambda$ such that $\gamma < j(\epsilon)$. Finally, let us define

$$\mathcal{A} = [j``(D \cap \delta)]^{<\kappa} \subseteq [C^j_\rho \cap j(\delta)]^{<\kappa} = [C^j_\gamma]^{<\kappa}.$$

By the definition of F, we know

$$x \in \mathcal{A} \quad \Longrightarrow \quad x = x^j_\xi \text{ for some } \xi < j(F)(j(\epsilon)) = j(F(\epsilon)). \tag{15.34}$$

However, every element x of $\mathcal{A}$ is also of the form $j(y)$ for some $y \in [D \cap \delta]^{<\kappa}$ and therefore there is a unique $\zeta < \lambda$ such that

$$x = j(x_\zeta) = x^j_{j(\zeta)}. \tag{15.35}$$

From (15.34), an application of elementarity allows us to conclude that the ζ from (15.35) must be less than $F(\epsilon)$. Thus, there is a one-to-one mapping from $\mathcal{A}$ to $F(\epsilon)$ given by

$$x \mapsto \text{ the unique } \zeta < F(\epsilon) \text{ with } x = j(x_\zeta).$$

This is a contradiction, as $|A| = \mu^{<\kappa} = \lambda$, while $F(\epsilon) < \lambda$. $\dashv$

The above argument can easily be modified to prove that if κ is supercompact and $\mathrm{cf}(\mu) < \kappa < \mu$, there is a singular $\theta < \kappa$ with $\mathrm{cf}(\theta) = \mathrm{cf}(\mu)$ such that $S^{\mu^+}_{\theta^+} \notin I[\mu^+]$.

3.4. Weak Approachability and $\mathrm{AP}_{\aleph_\omega}$

What about the situation at $\aleph_{\omega+1}$? Can we (assuming the existence of large cardinals) force the failure of $\mathrm{AP}_{\aleph_\omega}$? The answer is affirmative, and even though the proof is not difficult (we give it at the end of the next section), it will give us an opportunity to re-connect $I[\lambda]$ with some of the ideas used in the proof of Theorem 2.15. The reader may recall that we utilized λ-filtration sequences in order to establish that certain stationary sets were indestructible by nice forcings. Our goal in this section is to deepen our understanding of this connection. The results in this section are all due to Shelah, and most are formulations of ideas and results from [83] and [90].

3.21 Definition. Let $\lambda = \mu^+$ for μ singular, and let $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ be a sequence of elements of $[\lambda]^{<\mu}$. A limit ordinal $\delta < \lambda$ is said to be *weakly approachable with respect to the sequence* $\bar{a}$ if there is an unbounded $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that every initial segment of A is covered by a_β for some $\beta < \delta$, that is, if $\alpha < \delta$ then there is a $\beta < \delta$ such that $A \cap \alpha \subseteq a_\beta$.

The use of the adverb *weakly* should not be surprising, as the above is really a weakening of Definition 3.2. There is a slight discrepancy between the two definitions, as Definition 3.2 uses enumerations of elements of $[\lambda]^{<\lambda}$ instead of $[\lambda]^{<\mu}$, but this difference is irrelevant to the question of deciding whether an ordinal is approachable or not, because the initial segments that must appear in the enumeration in Definition 3.2 are all of size less than μ.

3.22 Definition. Assume $\lambda = \mu^+$ for μ singular. A set $S \subseteq \lambda$ *is in* $I[\lambda;\mu]$ if and only if there is a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ of elements of $[\lambda]^{<\mu}$ and a closed unbounded $C \subseteq \lambda$ such that every $\delta \in S \cap C$ is singular and weakly approachable with respect to the sequence $\bar{a}$.

The proof that $I[\lambda;\mu]$ is a normal ideal is no different from the corresponding proof for $I[\lambda]$. In fact, the ideals are quite closely related—it should be clear that $I[\lambda] \subseteq I[\lambda;\mu]$, while the following proposition tells us that in many situations the ideals actually coincide (compare with Lemma 2.20, for example).

3.23 Proposition. *Suppose that $\lambda = \mu^+$ where μ is a strong limit singular cardinal. Then $I[\lambda] = I[\lambda; \mu]$.*

Proof. It is clear that $I[\lambda] \subseteq I[\lambda; \mu]$, so assume $S \in I[\lambda; \mu]$ with $\bar{a}$ and E the corresponding parameters. Let $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda \rangle$ be a λ-approximating sequence over $\{\bar{a}, E\}$; we show that any $\delta \in S$ satisfying $M_\delta \cap \lambda = \delta$ is in fact in $S[\mathfrak{M}]$. Since $I[\lambda]$ is a normal ideal and $S[\mathfrak{M}] \in I[\lambda]$, this suffices.

Given such a δ, there exists a cofinal $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that every initial segment of A is covered by a_α for some $\alpha < \delta$. So fix $\beta < \delta$, and fix $\alpha < \delta$ such that

$$A \cap \beta \subseteq a_\alpha.$$

Since $\alpha < \delta$ and $\bar{a} \in M_\delta$, it follows that both a_α and $\mathcal{P}(a_\alpha)$ are in M_δ. Since $|a_\alpha| < \mu$ and μ is a strong limit, we know $|\mathcal{P}(a_\alpha)| < \mu$ as well. Since $M_\delta \cap \lambda = \delta$, we conclude

$$\mathcal{P}(a_\alpha) \subseteq M_\delta,$$

and therefore $A \cap \beta \in M_\delta$. Thus, $\delta \in S[\mathfrak{M}]$ as required. $\dashv$

We now bring in yet another notion of approachability—this one is tied to colorings associated with certain λ-filtration sequences, and we have already seen it in disguise in Sect. 2.5. The following notation is due to Shelah [83].

3.24 Definition. Let $\lambda = \mu^+$ for μ singular. A function $d : [\lambda]^2 \to \mathrm{cf}(\mu)$ is said to be *normal* if

$$i < \mathrm{cf}(\mu) \quad \Longrightarrow \quad \sup_{\alpha<\lambda} |\{\beta < \alpha : d(\beta, \alpha) < i\}| < \mu. \tag{15.36}$$

The function d is *transitive* if

$$\alpha < \beta < \gamma < \lambda \quad \Longrightarrow \quad d(\alpha, \gamma) \leq \max\{d(\alpha, \beta), d(\beta, \gamma)\}. \tag{15.37}$$

We note here that a normal transitive coloring $d : [\mu^+]^2 \to \mathrm{cf}(\mu)$ is essentially a λ-filtration sequence $\bar{b} = \langle b_{\alpha,i} : \alpha < \lambda, i < \mathrm{cf}(\mu) \rangle$ with nice properties—if we define

$$b_{\alpha,i} = \{\beta < \alpha : d(\beta, \alpha) \leq i\},$$

then the normality condition on d corresponds to

$$\sup\{|b_{\alpha,i}| : \alpha < \lambda\} < \mu \quad \text{for all } i < \mathrm{cf}(\mu), \tag{15.38}$$

while transitivity translates as

$$\beta \in b_{\alpha,i} \quad \Longrightarrow \quad b_{\beta,i} \subseteq b_{\alpha,i}. \tag{15.39}$$

As noted after Definition 1.10, it is straightforward to build such λ-filtration sequences, and hence there are plenty of normal transitive colorings defined on λ.

3.25 Definition. Suppose that $d : [\mu^+]^2 \to \mathrm{cf}(\mu)$ is a normal transitive function, where μ is a singular cardinal. A limit ordinal $\delta < \mu^+$ is *d-approachable* if there is a cofinal $A \subseteq \delta$ such that for every $\alpha \in A$,

$$\sup\{d(\beta,\alpha) : \beta \in A \cap \alpha\} < \mathrm{cf}(\mu).$$

3.26 Proposition. *Let μ and d be as in the preceding definition, and suppose that $\delta < \mu^+$.*

1. *If $\mathrm{cf}(\delta) \leq \mathrm{cf}(\mu)$, then δ is d-approachable.*
2. *If δ is d-approachable and $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, then there is a cofinal $H \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that*

$$|\operatorname{ran} d \restriction [H]^2| < \mathrm{cf}(\mu).$$

Proof. The first statement is obvious from cardinality considerations. For the second statement, let $A \subseteq \delta$ be as in the definition of d-approachability. Without loss of generality, $\mathrm{ot}(A) = \mathrm{cf}(\delta)$, as shrinking A causes no harm. For $\alpha \in A$, let

$$i_\alpha = \sup\{d(\beta,\alpha) : \beta \in A \cap \alpha\}.$$

Our choice of A implies $i_\alpha < \mathrm{cf}(\mu)$. Since $\mathrm{cf}(\mu) \neq \mathrm{cf}(\delta)$, there is an $i < \mathrm{cf}(\mu)$ for which

$$H := \{\alpha \in A : i_\alpha \leq i\}$$

is unbounded in δ. Clearly H has the required properties. ⊣

3.27 Corollary. *Let d and μ be as in the previous proposition. An ordinal $\delta < \mu^+$ is d-approachable if and only if $\mathrm{cf}(\delta) \leq \mathrm{cf}(\mu)$ or there is a cofinal $H \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ and $i < \mathrm{cf}(\mu)$ such that*

$$\beta < \alpha \text{ in } H \quad \Longrightarrow \quad d(\beta,\alpha) \leq i.$$

The next theorem establishes the connection between the two concepts we have been considering in this section.

3.28 Theorem. *Let $\lambda = \mu^+$ for μ singular. Then the following two conditions are equivalent for a set $S \subseteq \lambda$:*

1. *$S \in I[\lambda;\mu]$.*
2. *There is a normal transitive $d : [\lambda]^2 \to \mathrm{cf}(\mu)$ and a closed unbounded $E^* \subseteq \lambda$ such that all $\delta \in E^* \cap S$ are d-approachable.*

Proof. Assume first that condition (2) holds for S, as witnessed by d and E^*. Any ordinal $\alpha < \lambda$ has a unique representation as $\gamma + i$ where γ is divisible by $\mathrm{cf}(\mu)$ and $i < \mathrm{cf}(\mu)$. With this in mind, we define

$$a_{\gamma+i} = \{\beta < \gamma : d(\beta, \gamma + j) \leq i \text{ for some } j \leq i\}, \tag{15.40}$$

where γ is divisible by $\mathrm{cf}(\mu)$ and $i < \mathrm{cf}(\mu)$. Note that the normality condition on d guarantees that $|a_\alpha| < \mu$ for all α, and so our construction generates an enumeration $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ of sets from $[\lambda]^{<\mu}$. Now let E be the closed and unbounded subset of E^* made up of those elements that are limits of ordinals divisible by $\mathrm{cf}(\mu)$. We claim that any $\delta \in E \cap S$ is weakly approachable with respect to $\bar{a}$.

Given $\delta \in E \cap S$, we know that δ is d-approachable, so there is a cofinal $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that for all $\alpha \in A$,

$$\sup\{d(\beta, \alpha) : \beta \in A \cap \alpha\} < \mathrm{cf}(\mu).$$

Any unbounded subset of A also enjoys this property, so without loss of generality if $\beta < \alpha$ in A, then there is an ordinal γ such that

- $\beta < \gamma \leq \alpha$, and
- γ is divisible by $\mathrm{cf}(\mu)$.

Given $\alpha \in A$, there is a unique ordinal γ such that

- γ is divisible by $\mathrm{cf}(\mu)$, and
- $\alpha = \gamma + j$ for some $j < \mathrm{cf}(\mu)$.

Furthermore, we know

- $A \cap \alpha \subseteq \gamma$, and
- $i_\alpha := \sup\{d(\beta, \alpha) : \beta \in A \cap \alpha\} < \mathrm{cf}(\mu)$.

If we define $i = i_\alpha + j$ (so $\gamma + i < \delta$) then it is straightforward to verify

$$A \cap \alpha \subseteq a_{\gamma+i},$$

and we are done.

Now what about the other direction? Assume that there is a closed unbounded E and an enumeration $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ of elements of $[\lambda]^{<\mu}$ such that each $\delta \in E \cap S$ is weakly approachable with respect to $\bar{a}$.

Let $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ be an increasing sequence of regular cardinals cofinal in μ. A straightforward construction will give us a λ-filtration system $\bar{b}$ such that

- $|b_{\alpha,i}| \leq \mu_i$
- $\beta \in b_{\alpha,i} \implies b_{\beta,i} \subseteq b_{\alpha,i}$, and
- $|a_\alpha| \leq \mu_i \implies a_\alpha \cap \alpha \subseteq b_{\alpha,i}$.

Let d be the coloring

$$d(\beta, \alpha) = \min\{i : \beta \in b_{\alpha,i}\}; \tag{15.41}$$

it is easily check that d is normal and transitive.

Let us assume that δ is weakly approachable with respect to $\bar{a}$, as shown by the set $A \subseteq \delta$. Let $\langle \alpha_\epsilon : \epsilon < \mathrm{cf}(\delta) \rangle$ be the increasing enumeration of A. By our assumption on A, for each $\epsilon < \mathrm{cf}(\delta)$ there is a $\beta_\epsilon < \delta$ such that

$$\{\alpha_\xi : \xi < \epsilon\} \subseteq a_{\beta_\epsilon} \cap \beta_\epsilon.$$

By thinning out A as necessary, we may assume that for all $\epsilon < \mathrm{cf}(\delta)$,

$$\sup\{\alpha_\xi : \xi < \epsilon\} < \beta_\epsilon < \alpha_\epsilon.$$

By Corollary 3.27, we may assume $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$. Thus, for some $i < \mathrm{cf}(\mu)$, the set

$$B := \{\epsilon < \mathrm{cf}(\delta) : d(\beta_\epsilon, \alpha_\epsilon) \leq i \text{ and } |a_{\beta_\epsilon}| < \mu_i\}$$

is unbounded in $\mathrm{cf}(\delta)$.

Next we define $H := \{\beta_\epsilon : \epsilon \in B\}$. Note that the set H is cofinal in δ, and if $\epsilon_1 < \epsilon_2$ are in B, then $\beta_{\epsilon_1} < \alpha_{\epsilon_1} < \beta_{\epsilon_2}$. This implies

$$d(\beta_{\epsilon_1}, \beta_{\epsilon_2}) \leq \max\{d(\beta_{\epsilon_1}, \alpha_{\epsilon_1}), d(\alpha_{\epsilon_1}, \beta_{\epsilon_2})\} \leq i,$$

and so δ is d-approachable. $\dashv$

Using this characterization of $I[\lambda; \mu]$, we can quickly deduce some corollaries concerning $I[\lambda]$.

3.29 Corollary. *If $\lambda = \mu^+$ for μ a strong limit singular cardinal, then $S^\lambda_{\leq \mathrm{cf}(\mu)} \in I[\lambda]$.*

Proof. This is immediate, since it is clear from Theorem 3.28 that $S^\lambda_{\leq \mathrm{cf}(\mu)}$ is in $I[\lambda; \mu]$. $\dashv$

3.30 Corollary. $\mathrm{AP}_{\aleph_\omega}$ *holds in the model from Theorem 2.15. In particular,* $\mathrm{AP}_{\aleph_\omega}$ *does not imply $\square_{\aleph_\omega}$, nor does it imply the existence of a non-reflecting stationary subset of $\aleph_{\omega+1}$.*

Proof. This follows from Proposition 2.21, as without loss of generality, the λ-filtration system used in the proof satisfies the requirements necessary for the coloring d to be normal and transitive. $\dashv$

We shall see later (Corollary 3.41 in the next subsection) that $\mathrm{Refl}(\aleph_{\omega+1})$ implies $\mathrm{AP}_{\aleph_\omega}$, so the above corollary is not simply an artifact of Magidor's construction.

3.5. The Structure of $I[\lambda]$

In the introduction to this part of the chapter, we pointed out that in certain circumstances the question of whether or not a stationary set satisfies the κ^+-indestructibility condition can be reduced to checking whether it has stationary intersection with a certain special subset of S^λ_κ. The result we prove next can be viewed as a version of this phenomenon—the ideal $I[\lambda]$ very often has a simple structure. As in the last subsection, all results presented here are due to Shelah.

3.31 Theorem. *If $\kappa < \lambda$ are uncountable regular cardinals with $2^{<\kappa} < \lambda$, then $I[\lambda] \restriction S^\lambda_\kappa$ is generated over the nonstationary ideal by a single set, that is, there is a set $A^\lambda_\kappa \subseteq S^\lambda_\kappa$ such that $S \subseteq S^\lambda_\kappa$ is in $I[\lambda]$ if and only if $S \setminus A^\lambda_\kappa$ is nonstationary.*

Proof. Let $\mathfrak{M}$ be a λ-approximating sequence with $\kappa \in M_0$. Note that if there is a generator as in the conclusion of the theorem, then there is such a generator in M_0. A simple argument then shows that this set must be equal to $S[\mathfrak{M}] \cap S^\lambda_\kappa$ modulo the nonstationary ideal. Thus, it makes sense to define

$$A^\lambda_\kappa := S[\mathfrak{M}] \cap S^\lambda_\kappa,$$

and work to show that $S \cap A^\lambda_\kappa \neq \emptyset$ for every $S \subseteq S^\lambda_\kappa$ in $I[\lambda]$.

Let $\mathfrak{N} = \langle N_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence over $\{\kappa, S, \mathfrak{M}\}$, and choose $\delta \in S$ such that $N_\delta \cap \lambda = \delta$. Since $S \in N_0$, it follows that $\delta \in S[\mathfrak{N}]$; our goal is to prove that $\delta \in S[\mathfrak{M}]$ as well.

Let c be a closed unbounded subset of δ with $\mathrm{ot}(c) = \kappa$ such that every proper initial segment of c is in N_δ. Let $\langle \alpha_\epsilon : \epsilon < \kappa\rangle$ be the increasing enumeration of c, and in the model $M_{\delta+1}$, let $\langle \beta_\epsilon : \epsilon < \kappa\rangle$ enumerate another closed unbounded subset of δ.

Since κ has uncountable cofinality, the set

$$e := \{\epsilon < \kappa : \alpha_\epsilon = \beta_\epsilon\}$$

is closed unbounded in κ. Furthermore, since $2^{<\kappa} < \lambda$ and $M_0 \cap \lambda$ is an initial segment of λ, we know that every bounded subset of e is in M_0. By the choice of $\{\beta_\epsilon : \epsilon < \kappa\}$, it follows that for any $\zeta < \kappa$ the set $\{\beta_\epsilon : \epsilon \in e \cap \zeta\}$ is in $M_{\delta+1}$; our goal is to show that in fact it is in M_δ—this is enough to prove that $\delta \in A^\lambda_\kappa$.

To do this, note that $\{\alpha_\epsilon : \epsilon \in e \cap \zeta\}$ is in N_δ, as it is definable from an initial segment of c and an initial segment of e. Thus

$$\{\beta_\epsilon : \epsilon \in e \cap \zeta\} = \{\alpha_\epsilon : \epsilon \in e \cap \zeta\} \in N_\delta \cap M_{\delta+1}. \tag{15.42}$$

However, we know that $\mathfrak{M} \in N_\delta$ and $N_\delta \cap \lambda = \delta$. From this it follows

$$N_\delta \cap \bigcup\nolimits_{i<\lambda} M_i = M_\delta. \tag{15.43}$$

Together (15.42) and (15.43) tell us that $\{\beta_\epsilon : \epsilon \in e \cap \zeta\}$ is in M_δ, and so the set $\{\beta_\epsilon : \epsilon \in e\}$ puts δ into A^λ_κ. $\dashv$

3.32 Corollary. *If $\lambda = \mu^+$ for μ strong limit singular, then $I[\lambda]$ contains a maximal set modulo the nonstationary ideal.*

We call this maximal set (should it exist) *the set of approachable points* of λ. In general, the sets A^λ_κ from Theorem 3.31 are unique modulo the nonstationary ideal, so we refer to A^λ_κ as the set of approachable points of cofinality κ. In some of Shelah's work, A^λ_κ is called *the good set of cofinality* κ, but the adjective good will be reserved for a related concept that we shall explore later.

Our goal for the rest of this section is to prove another theorem relating $I[\lambda]$ to colorings of pairs from λ, and then to derive several results illuminating the connection between $I[\lambda]$ and the problem of stationary reflection.

3.33 Theorem. *Suppose that $\kappa < \lambda$ are regular cardinals with $2^{<\kappa} < \lambda$, and let $d : [\lambda]^2 \to \theta$ for some $\theta < \kappa$. If $\mathfrak{M}$ is a λ-approximating sequence over $\{\theta, \kappa, d\}$, then for every δ in $S[\mathfrak{M}] \cap S^\lambda_\kappa$, there exists a cofinal $H \subseteq \delta$ of order-type κ homogeneous for d.*

Proof. If $\delta \in S[\mathfrak{M}] \cap S^\lambda_\kappa$, then we can find a set $a = \{\alpha_i : i < \kappa\}$ cofinal in δ such that $\{a_i : i < \zeta\} \in M_\delta$ for all $\zeta < \kappa$. By induction on $i < \kappa$, we define objects ϵ_i and f_i as follows:

1. $\epsilon_0 = \alpha_0$,
2. $f_i : i \to \theta$ is given by $f_i(j) = d(\epsilon_j, \delta)$,
3. we ask if there is an ordinal α greater than α_i and ϵ_j for all $j < i$ such that $d(\epsilon_j, \alpha) = f_i(j)$ for all $j < i$. If the answer is yes, then ϵ_i is the least such α; otherwise, the construction terminates.

The above construction generates an increasing sequence $\{\epsilon_i : i < i^*\}$ for some $i^* \leq \kappa$. For any $i < i^*$, the sequence $\{\epsilon_j : j \leq i\}$ is definable from the parameters $\{\alpha_j : j < i\}$, d, and f_i. The first of these is in M_δ by our choice of $\{\alpha_i : i < \kappa\}$, while $f_i \in M_\delta$ as well because $2^{<\kappa} < \lambda$. Thus, every proper initial segment of $\{\epsilon_i : i < i^*\}$ lies in M_δ. From this, we conclude that $i^* = \kappa$—this is because at every stage of our construction, the answer to the question posed in (3) above is "yes", as demonstrated by the ordinal δ. The rest of the proof is standard—let $f : \kappa \to \theta$ be defined by

$$f(i) = d(\epsilon_i, \delta).$$

Since $\theta < \kappa$ and κ is regular, there is a single ζ such that $\{i < \kappa : f(i) = \zeta\}$ is unbounded in κ. It is routine to check that

$$H_\delta := \{\epsilon_i : f(i) = \eta\}$$

has all the required properties. ⊣

The next definition is natural in light of the preceding theorem.

3.34 Definition. Suppose that $d : [\lambda]^2 \to \theta$ for some θ and λ. We define $S(d)$ to be the set of those $\delta < \lambda$ that have a cofinal subset H_δ of order-type $\mathrm{cf}(\delta)$ homogeneous for d.

3.35 Corollary. *Suppose that $\lambda = \mu^+$ for μ strong limit singular, and let d be a normal transitive coloring. Then $S(d) \cup S^\lambda_{\le \mathrm{cf}(\mu)}$ generates $I[\lambda]$ over the nonstationary ideal.*

Proof. By Corollary 3.29 and Theorem 3.33. ⊣

3.36 Corollary. *If $\kappa < \lambda$ are regular cardinals with $2^{<\kappa} < \lambda$ and d is a function from $[\lambda]^2$ to θ for some $\theta < \kappa$, then $S(d)$ includes A^λ_κ modulo the nonstationary ideal.*

We now return once more to the topic of stationary reflection, and investigate the extent to which the structure of $I[\lambda]$ limits the patterns of reflection that can arise. Our first step is the following result of Shelah [83, 90].

3.37 Theorem. *Suppose that $\sigma < \tau < \lambda$ are regular cardinals with $2^{<\sigma} < \tau$, and let $d : [\lambda]^2 \to \theta$ for some $\theta < \sigma$. If $\langle \delta_i : i < \tau \rangle$ is a strictly increasing and continuous sequence of ordinals with supremum $\delta < \lambda$, then*

$$\{\delta_i : i \in A^\tau_\sigma\} \setminus S(d) \text{ is nonstationary.} \tag{15.44}$$

Proof. Note that A^τ_σ exists because $2^{<\sigma} < \tau$, so (15.44) makes sense. Define $d^* : [\tau]^2 \to \theta$ by

$$d^*(i, j) = d(\delta_i, \delta_j).$$

We apply Corollary 3.36 to σ, τ, and d^* and conclude that $A^\tau_\sigma \setminus S(d^*)$ is a nonstationary subset of τ. Note as well that if $i \in S(d^*)$ then $\delta_i \in S(d)$ because of the way d^* was defined, so (15.44) follows. ⊣

Our intent is to mine Theorem 3.37 for more information concerning the connection between $I[\lambda]$ and the problem of stationary reflection. For the time being, let us assume the following:

- $\lambda = \mu^+$ for μ a strong limit singular cardinal,
- $d : [\lambda]^2 \to \theta$ for some $\theta < \mu$, and
- $S^*(d) := \lambda \setminus S(d)$ is stationary.

This situation occurs if AP_μ fails, for example. In the remainder of this section, we show that the above assumptions have strong implications for stationary reflection. We begin with a easy result.

3.38 Proposition. *$S^*(d)$ cannot reflect outside of itself.*

Proof. Suppose that $\delta \in S(d)$, and let H be a cofinal subset of δ homogeneous for the coloring d. Then any $\alpha < \delta$ with $\alpha = \sup(H \cap \alpha)$ is also in $S(d)$, and so $S(d)$ contains a closed unbounded subset of δ. The result now follows immediately. ⊣

The next theorem illustrates that there are connections between $I[\lambda]$ and $I[\tau]$ for certain regular $\tau < \mu$. In the statement of the theorem, $S^*_\sigma(d)$ denotes $S^*(d) \cap S^\lambda_\sigma$.

3.39 Theorem. *Let $\sigma < \tau$ be regular cardinals below μ with $2^{<\sigma} < \tau$. If $S^*_\sigma(d) \cap \delta$ is stationary in δ for some $\delta \in S^\lambda_\tau$, then $S^\tau_\sigma \notin I[\tau]$.*

Proof. If $S^\tau_\sigma \in I[\tau]$, then Corollary 3.36 implies that $S(d)$ contains a closed unbounded subset of δ relative to the ordinals of cofinality σ. This is impossible if $S^*_\sigma(d) \cap \delta$ is stationary in δ. ⊣

The following lemma tells us that cardinal arithmetic assumptions have some influence on the structure of $I[\lambda]$. We state the result in terms of τ because we will apply it in conjunction with Theorem 3.39.

3.40 Lemma. *Let $\sigma < \tau$ be regular cardinals such that $\epsilon^{<\sigma} < \tau$ for all $\epsilon < \tau$. Then $S^\tau_\sigma \in I[\tau]$.*

Proof. Let $\mathfrak{M} = \langle M_\alpha : \alpha < \tau \rangle$ be a τ-approximating sequence. We know that $M_\alpha \cap \tau$ is an initial segment of τ for each $\alpha < \tau$. Since $|M_\alpha|^{<\sigma} < \tau$ for each α, it follows (since $M_\alpha \in M_{\alpha+1}$) that

$$[M_\alpha]^{<\sigma} \subseteq M_{\alpha+1}. \tag{15.45}$$

Now suppose that $\delta \in S^\tau_\sigma$ satisfies $\delta = M_\delta \cap \lambda$, and let $A \subseteq \delta$ be any cofinal set of order-type σ. Each initial segment of A is in $[M_\alpha]^{<\sigma}$ for some $\alpha < \sigma$. Since δ is a limit ordinal, it follows from (15.45) that each initial segment of A is in M_δ and therefore $\delta \in S[\mathfrak{M}]$. Since $I[\tau]$ is a normal ideal and $S[\mathfrak{M}] \in I[\tau]$, we conclude that $S^\tau_\sigma \in I[\tau]$ as well. ⊣

We are now in a position to deduce a somewhat unexpected corollary of Theorem 3.39. This is a reformulation of one of the main theorems in [83], and it shows that the fact that $\mathrm{AP}_{\aleph_\omega}$ holds in the model of Theorem 2.15 is no accident.

3.41 Corollary. *If $\aleph_\omega$ is a strong limit and $\mathrm{AP}_{\aleph_\omega}$ fails, then* $\mathrm{Refl}(\aleph_{\omega+1})$ *fails as well.*

Proof. Assume $\aleph_\omega$ is a strong limit and $\mathrm{AP}_{\aleph_\omega}$ fails. Let $d : [\aleph_{\omega+1}]^2 \to \omega$ be a normal transitive function, and let $S^*(d) = \aleph_{\omega+1} \setminus S(d)$. Since $\mathrm{AP}_{\aleph_\omega}$ fails, we know that $S^*(d)$ is stationary and hence there is an $n < \omega$ for which the set S of ordinals in $S^*(d)$ of cofinality $\aleph_{n+1}$ is stationary. Note that

$$2^{\aleph_n} \leq \sigma < \aleph_\omega \quad \Longrightarrow \quad \sigma^{\aleph_n} = \sigma^{<\aleph_{n+1}} = \sigma, \tag{15.46}$$

and so an application of Theorem 3.39 tells us that S cannot reflect in any cofinality greater than $2^{\aleph_n}$.

Now we claim that S has a non-reflecting stationary subset. To see this, fix $k < \omega$ such that $2^{\aleph_n} = \aleph_{n+k}$ and consider the sequence of sets defined by

$$S_0 = S,$$
$$S_{i+1} = \{\delta < \aleph_{\omega+1} : S_i \cap \delta \text{ is stationary in } \delta\}.$$

Standard arguments tell us that

$$\delta \in S_i \quad \Longrightarrow \quad \mathrm{cf}(\delta) \geq \aleph_{n+1+i}, \tag{15.47}$$

and that $S_{i+1} \subseteq S_i$ for $i \geq 1$ (though S_1 need not be a subset of S_0).

We know that S does not reflect in an ordinal of cofinality greater than $\aleph_{n+k}$ and therefore S_k is empty. Let i^* be the first natural number less than k for which S_{i^*+1} is nonstationary. The set S_{i^*} is therefore a stationary set that does not reflect stationarily often. By removing a nonstationary set from S_{i^*}, we can obtain a stationary $T \subseteq S_{i^*}$ that does not reflect at all. Thus $\mathrm{Refl}(\aleph_{\omega+1})$ fails. ⊣

Shelah's paper [87] uses similar arguments to establish the following curious result which shows that supercompact cardinals actually impose a limit on the amount of stationary reflection present.

3.42 Theorem (Shelah [87]). *If* GCH *holds and κ is a $\kappa^{+\omega+1}$-supercompact cardinal, then there are singular cardinals $\zeta < \eta < \kappa$ of countable cofinality for which* $\mathrm{Refl}(S^{\eta^+}_{\zeta^+})$ *fails.*

The bulk of [87] is devoted to showing that the large cardinal assumption in the preceding theorem is sharp.

3.43 Theorem (Shelah [87]). *If the universe contains "sufficiently many" supercompact cardinals, then in some forcing extension there is a κ which is $\kappa^{+\omega}$-supercompact,* GCH *holds, and* $\mathrm{Refl}(S^{\tau}_{\sigma})$ *holds for every regular σ and τ with $\sigma^+ < \tau$.*

We close this section with yet another application of the $S(d)$ characterization of $I[\lambda]$—we sketch a proof that consistently $\mathrm{AP}_{\aleph_\omega}$ fails.

3.44 Theorem. $\neg\,\mathrm{AP}_{\aleph_\omega}$ *is consistent relative to the existence of a supercompact cardinal.*

Proof. Let κ be a supercompact cardinal, and assume that GCH holds. Set $\mu = \kappa^{+\omega}$ and $\lambda = \mu^+$, and let $d : [\lambda]^2 \to \omega$ be a normal transitive function. We know from Theorem 3.20 that there is a cardinal $\theta < \kappa$ for which $S^{\lambda}_{\theta} \notin I[\lambda]$. Since μ is a strong limit cardinal, it follows that

$$A^{\lambda}_{\theta} = S(d) \cap S^{\lambda}_{\theta} \tag{15.48}$$

modulo the nonstationary ideal, and $B := S^\lambda_\theta \setminus S(d)$ is stationary.

Our model of $\neg\,\mathrm{AP}_{\aleph_\omega}$ is obtained as a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P} = \mathrm{Col}(\aleph_0, {<}\theta)$ makes all cardinals less than θ countable using finite conditions, and

$$V^{\mathbb{P}} \models \dot{\mathbb{Q}} = \mathrm{Col}(\aleph_1, \kappa),$$

so forcing with $\dot{\mathbb{Q}}$ collapses κ to $\aleph_1$ using countable conditions.

The iteration $\mathbb{P} * \dot{\mathbb{Q}}$ preserves the stationarity of subsets of λ using a chain condition argument analogous to the standard argument that $\aleph_1$-chain condition forcings preserve stationary subsets of ω_1. Also, after forcing with $\mathbb{P} * \dot{\mathbb{Q}}$ the cardinal θ becomes $\aleph_1$, while μ becomes $\aleph_\omega$ and λ becomes $\aleph_{\omega+1}$. The function d is still normal and transitive after the forcing, so it suffices to prove that our forcing cannot change the truth-value of "$\delta \in S(d)$" for $\delta \in S^\lambda_\theta$.

We show this first for $\mathbb{P}$—the relevant property of $\mathbb{P}$ that we need is that among any θ conditions, there is a pairwise compatible subfamily of size θ. Why does this suffice? It is clear that if $\delta \in S(d)$ in the ground model, then δ remains in $S(d)$ in the extension, so assume $\delta \in S^\lambda_\theta$ and

$$p \Vdash \text{“}\delta \in S(d)\text{”}.$$

What this means is that δ has a cofinal d-homogeneous set H of order-type θ in $V^{\mathbb{P}}$; we claim that such a set must exist already in the ground model. Without loss of generality, there is a specific $n < \omega$ and a $\mathbb{P}$-name $\dot{H}$ such that

$$p \Vdash \text{“}\dot{H} \text{ is a cofinal subset of } \delta \quad \text{and } d \restriction [\dot{H}]^2 \text{ is constant with value } n\text{”}.$$

Let $\langle \delta_\alpha : \alpha < \theta \rangle$ be an increasing sequence with limit δ. For each $\alpha < \theta$, we can find a condition $p_\alpha \leq p$ and an ordinal $\beta_\alpha < \delta$ such that

$$p_\alpha \Vdash \text{“}\beta_\alpha \text{ is the least element of } \dot{H} \text{ above } \delta_\alpha\text{”}.$$

There is a set $I \subseteq \theta$ of size θ with the property that p_α and p_γ are compatible whenever $\alpha < \gamma$ in I. It is clear that $\delta_\alpha < \beta_\alpha$ and so $\{\beta_\alpha : \alpha \in I\}$ is cofinal in δ; by thinning out I we can assume that the sequence $\langle \beta_\alpha : \alpha \in I \rangle$ is increasing as well. Given $\alpha < \gamma$ in I, there is a condition q extending both p_α and p_γ. Since q extends p as well, it must be the case that

$$q \Vdash \text{“}d(\beta_\alpha, \beta_\gamma) = n\text{”},$$

but then $d(\beta_\alpha, \beta_\gamma) = n$ in the ground model as well. Thus, $\{\beta_\alpha : \alpha \in I\}$ puts δ into $S(d)$ in the ground model.

We leave the corresponding argument for $\dot{\mathbb{Q}}$ to the reader—the property that one uses is

$$V^{\mathbb{P}} \models \dot{\mathbb{Q}} \text{ is } \aleph_1\text{-closed}.$$

Thus, if G is a generic subset of $\mathbb{P} * \dot{\mathbb{Q}}$, it follows that

$$V[G] \models S^{\aleph_{\omega+1}}_{\aleph_1} \notin I[\aleph_{\omega+1}],$$

and hence $\mathrm{AP}_{\aleph_\omega}$ fails. $\dashv$

The proof of the preceding theorem furnishes us with a model in which $\aleph_\omega$ is a strong limit and

$$S^{\aleph_{\omega+1}}_{\aleph_1} \notin I[\aleph_{\omega+1}];$$

it does not generalize to get us a model where, for example, $S^{\aleph_{\omega+1}}_{\aleph_2}$ fails to be in $I[\aleph_{\omega+1}]$. The question of whether or not $I[\aleph_{\omega+1}]$ must contain a closed unbounded set relative to the ordinals of cofinality $\aleph_2$ is a major open question in this area. Foreman's survey [32] contains more information on this, as well as many other related open questions.

3.6. An Application—the Existence of Scales

We come now to another extremely important tool for investigating successors of singular cardinals, namely *scales*. The importance of scales in this context was noticed very early by Shelah in his investigations of Jónsson cardinals. For example, the paper [82] dates to 1978.

3.45 Definition. Let μ be a singular cardinal. A *scale of length β for μ* is a triple $(\vec{\mu}, \vec{f}, I)$ where

1. $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals such that $\sup_{i<\mathrm{cf}(\mu)} \mu_i = \mu$.
2. I is an ideal on $\mathrm{cf}(\mu)$.
3. $\vec{f} = \langle f_\alpha : \alpha < \beta\rangle$ is a sequence of functions such that
   (a) $f_\alpha \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$.
   (b) If $\gamma < \delta < \beta$ then $f_\gamma <_I f_\beta$.
   (c) If $f \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$ then there is an $\alpha < \beta$ such that $f <_I f_\alpha$.

The preceding is just a special case of notions studied in pcf theory—the statement "$(\vec{\mu}, \vec{f}, I)$ is a scale of length β for μ" says exactly the same thing as

$$\vec{f} \text{ witnesses } \mathrm{tcf}\big(\textstyle\prod_{i<\mathrm{cf}(\mu)}\mu_i, <_I\big) = \beta.$$

In this chapter, we will concern ourself almost exclusively with a special case of the previous definition—scales of length μ^+ where I is the ideal of bounded sets.

3.46 Definition. Let μ be a singular cardinal. A *scale for* μ is a pair $(\vec{\mu}, \vec{f})$ such that $(\vec{\mu}, \vec{f}, J^{\mathrm{bd}})$ is a scale of length μ^+ for μ, where J^{bd} denotes the ideal of bounded subsets of $\mathrm{cf}(\mu)$. Just as in Proposition 1.19, we let $<^*$ stand for $<_{J^{\mathrm{bd}}}$.

Our goal in this subsection is to prove the fundamental result of Shelah that scales exist for every singular cardinal μ. Before showing this, our plan is to investigate how some of the concepts isolated in the presentation of the theory of exact upper bounds as presented in Sect. 2.1 of Abraham and Magidor's chapter [1] in this Handbook simplify in the context of scales. In particular, we will look closely at their notion of *strongly increasing sequences* and $(*)_\kappa$:

3.47 Definition (Definition 2.4 in [1]). Suppose that I is an ideal over A, L is a set of ordinals, and $\vec{f} = \langle f_\xi : \xi \in L\rangle$ is a sequence of functions with $f_\xi : A \to \mathsf{On}$. We say that $\vec{f}$ is *strongly increasing* if for each $\xi \in L$, there is a set $Z_\xi \in I$ such that for $\xi_1 < \xi_2$ in L,

$$a \in A \setminus (Z_{\xi_1} \cup Z_{\xi_2}) \implies f_{\xi_1}(a) < f_{\xi_2}(a).$$

In the context of interest to us, we shall see that strongly increasing sequences have a much simpler description.

3.48 Definition. Let μ be a singular cardinal, and suppose that we are given a pair $(\vec{\mu}, \vec{f})$ such that $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals with limit μ, and $\vec{f} = \langle f_\alpha : \alpha < \gamma\rangle$ is a $<^*$-increasing sequence of functions in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$. An ordinal $\delta < \gamma$ is said to be *good for* $\vec{f}$ (or simply *good* if $\vec{f}$ is clear from context) if $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, and there is a cofinal $A \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ and an $i^* < \mathrm{cf}(\mu)$ such that

$$f_\beta(i) < f_\alpha(i) \quad \text{for } \beta < \alpha \text{ in } A \text{ and } i > i^*.$$

Note that if $\delta < \gamma$ is of cofinality less than $\mathrm{cf}(\mu)$, then a set A as in the above definition automatically exists. This helps to explain why we restrict ourselves to considering only δ with $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$. Also, one can define good points for $<_I$-increasing sequences where I is not necessarily the ideal of bounded subsets of I—the equivalent formulations of goodness exhibited in Theorem 3.50 below show us how to formulate this.

Our aim is to explore the relationship between good points and strongly increasing sequences. Before we undertake this investigation, we need the following definition.

3.49 Definition. Suppose that $\langle f_i : i < \gamma\rangle$ and $\langle g_j : j < \delta\rangle$ are two $<^*$-increasing sequences of limit length. We say that they are *cofinally interleaved* if every function in one sequence is $<^*$ some function in the other sequence. Equivalently,

$$h <^* f_i \text{ for some } i < \gamma \iff h <^* g_j \text{ for some } j < \delta.$$

3.50 Theorem. *Let μ be a singular cardinal, and let $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ be an increasing sequence of regular cardinals cofinal in μ. Furthermore, assume that $\vec{f} = \langle f_\alpha : \alpha < \gamma\rangle$ is a $<^*$-increasing sequence of functions in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$. Then the following statements are equivalent for an ordinal $\delta < \gamma$ of cofinality greater than $\mathrm{cf}(\mu)$.*

1. *δ is good for $\vec{f}$.*
2. *$\langle f_\alpha : \alpha < \delta\rangle$ has an exact upper bound h such that $\mathrm{cf}(h(i)) = \mathrm{cf}(\delta)$ for all $i < \mathrm{cf}(\mu)$.*
3. *$\langle f_\alpha : \alpha < \delta\rangle$ is cofinally interleaved with an increasing sequence $\langle h_\xi : \xi < \mathrm{cf}(\delta)\rangle$.*
4. *If X is a cofinal subset of δ, then there is a set $X_0 \subseteq X$, cofinal in δ of order-type $\mathrm{cf}(\delta)$, such that $\langle f_\alpha : \alpha \in X_0\rangle$ is strongly increasing.*

Proof. (1) $\to$ (2). Let $A \subseteq \delta$ be cofinal of order-type $\mathrm{cf}(\delta)$ and suppose that $i^* < \mathrm{cf}(\mu)$ satisfies

$$f_\beta(i) < f_\alpha(i) \quad \text{for } \beta < \alpha \text{ in } A \text{ and } i^* < i < \mathrm{cf}(\mu).$$

Define

$$h(i) = \begin{cases} \delta & \text{if } i \leq i^*, \\ \sup\{f_\alpha(i) : \alpha \in A\} & \text{otherwise.} \end{cases}$$

Clearly $\mathrm{cf}(h(i)) = \mathrm{cf}(\delta)$ for all $i < \mathrm{cf}(\mu)$, and h is an upper bound for $\langle f_\alpha : \alpha < \delta\rangle$. Why is it an exact upper bound?

It suffices to prove that if $g < h$, then $g <^* f_\alpha$ for some α. Given such a g, for each $i > i^*$ we define

$$\alpha(i) = \text{least } \alpha \in A \text{ such that } g(i) < f_{\alpha(i)}(i).$$

Since $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, we know

$$\alpha^* := \sup_{i<\mathrm{cf}(\mu)} \alpha(i) < \delta,$$

and clearly $g <^* f_{\alpha^*}$. We remark that (2) explains why good points are sometimes referred to as *flat points* in the literature.

(2) $\to$ (3). Suppose that h is as in (2). For each $i < \mathrm{cf}(\mu)$, let $\langle e_i(\xi) : \xi < \mathrm{cf}(\delta)\rangle$ be the increasing enumeration of a cofinal subset of $h(i)$. For $\xi < \mathrm{cf}(\delta)$, define

$$h_\xi(i) = e_i(\xi).$$

Clearly the sequence $\langle h_\xi : \xi < \mathrm{cf}(\delta)\rangle$ is increasing, and $h_\xi(i) < h(i)$ for all $\xi < \mathrm{cf}(\delta)$ and $i < \mathrm{cf}(\mu)$. We claim that $\langle f_\alpha : \alpha < \delta\rangle$ and $\langle h_\xi : \xi < \mathrm{cf}(\delta)\rangle$ are cofinally interleaved.

Given $\xi < \mathrm{cf}(\delta)$, we know there is an $\alpha < \delta$ with $h_\xi <^* f_\alpha$ because h is an exact upper bound of $\langle f_\alpha : \alpha < \delta \rangle$.

Given $\alpha < \delta$, we know there is an $i_\alpha < \mathrm{cf}(\mu)$ such that $f_\alpha(i) < h(i)$ for $i > i_\alpha$. Thus, for each $i > i_\alpha$ there is a $\xi(\alpha, i) < \mathrm{cf}(\delta)$ such that

$$f_\alpha(i) < e_i(\xi(\alpha, i)).$$

Since $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, it follows that

$$\xi_\alpha := \sup\{\xi(\alpha, i) : i_\alpha < i < \mathrm{cf}(\mu)\} < \mathrm{cf}(\delta).$$

For $i > i_\alpha$, we have

$$f_\alpha(i) < e_i(\xi(\alpha, i)) \leq e_i(\xi_\alpha) = h_{\xi_\alpha}(i)$$

and so $f_\alpha <^* h_{\xi_\alpha}$, and we have finished the proof that the two sequences are cofinally interleaved.

(3) $\to$ (4). Assume $\langle f_\alpha : \alpha < \delta \rangle$ is cofinally interleaved with the increasing sequence $\langle h_\xi : \xi < \mathrm{cf}(\delta) \rangle$. Since X is cofinal in δ, then clearly $\langle f_\alpha : \alpha \in X \rangle$ is cofinally interleaved with $\langle h_\xi : \xi < \mathrm{cf}(\delta) \rangle$ as well. Thus we can choose sequences $\langle \alpha(\zeta) : \zeta < \mathrm{cf}(\delta) \rangle$ and $\langle \xi(\zeta) : \zeta < \mathrm{cf}(\delta) \rangle$ increasing to δ and $\mathrm{cf}(\delta)$ respectively such that $\{\alpha(\zeta) : \zeta < \mathrm{cf}(\delta)\} \subseteq X$, and for each $\xi < \mathrm{cf}(\delta)$,

$$f_{\alpha(\eta)} <^* h_\xi(\zeta) <^* f_\alpha(\zeta) \quad \text{for all } \eta < \zeta.$$

For each $\zeta < \mathrm{cf}(\delta)$, there is an $i(\zeta)$ such that

$$h_{\xi(\zeta)}(i) < f_{\alpha(\zeta)}(i) < h_{\xi(\zeta+1)}(i) \quad \text{for all } i > i(\zeta).$$

Since $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, we can find an unbounded $A \subseteq \delta$ and $i^* < \mathrm{cf}(\mu)$ such that $i(\zeta) = i^*$ for all $\zeta \in A$. In particular, for $\eta < \zeta$ in A and $i > i^*$, we have

$$f_{\alpha(\eta)}(i) < h_{\xi(\eta+1)}(i) \leq h_{\xi(\zeta)}(i) < f_{\alpha(\zeta)}(i).$$

If we let $X_0 = \{\alpha(\zeta) : \zeta \in A\}$ then clearly $\langle f_\alpha : \alpha \in X_0 \rangle$ is strongly increasing.

(4) $\to$ (1). Suppose that X_0 is a cofinal subset of δ of order-type $\mathrm{cf}(\delta)$ such that $\langle f_\alpha : \alpha \in X_0 \rangle$ is strongly increasing. For each $\alpha \in X_0$, there is a corresponding set Z_α bounded below $\mathrm{cf}(\mu)$; we note that without loss of generality $Z_\alpha = [0, i_\alpha]$ for some $i_\alpha < \mathrm{cf}(\mu)$. Again, since $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$ there is an unbounded $A \subseteq X_0$ and $i^* < \mathrm{cf}(\mu)$ such that $i_\alpha = i^*$ for all $\alpha \in A$. From the definition of strongly increasing, it follows that

$$f_\beta(i) < f_\alpha(i) \quad \text{for } \beta < \alpha \text{ in } A \text{ and } i > i^*.$$

Thus δ is good for $\vec{f}$. $\dashv$

The preceding theorem allows us to characterize when $<^*$-increasing sequences have nice exact upper bounds.

3.51 Theorem. *Let μ be a singular cardinal, and suppose that $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals with limit μ. Let $\vec{f} = \langle f_i : i < \mu^+\rangle$ be a $<^*$-increasing sequence of functions in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$, and suppose that $\mathrm{cf}(\mu) < \kappa = \mathrm{cf}(\kappa) < \mu$. Then the following two statements are equivalent:*

1. *$\vec{f}$ has an exact upper bound h such that*

$$|\{i < \mathrm{cf}(\mu) : \mathrm{cf}(h(i)) < \kappa\}| < \mathrm{cf}(\mu).$$

2. *The set $\{\delta \in S^{\mu^+}_{\kappa} : \delta$ is good for $\vec{f}\}$ is stationary.*

Proof. The key here is the concept denoted $(*)_\kappa$ in the chapter of Abraham and Magidor (see [1, Definition 2.8]) in this Handbook. We recall the definition:

3.52 Definition. Suppose that I is an ideal over a set A, λ is a regular cardinal $> |A|$, and $\vec{f} = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions mapping A to On. If $|A| < \kappa \leq \lambda$, then $(*)_\kappa$ denotes the following statement:

$(*)_\kappa$ Whenever $X \subseteq \lambda$ is unbounded, then for some $X_0 \subseteq X$ of order-type κ, $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing.

In Theorem 2.13 of [1], it is shown (among other things) that statement (1) of the current theorem is equivalent to $(*)_\kappa$; we will show that our statement (2) is equivalent to $(*)_\kappa$.

Suppose that $(*)_\kappa$ holds, where κ is a regular cardinal such that $\mathrm{cf}(\mu) < \kappa < \mu$. Let S be the set of $\delta \in S^{\mu^+}_\kappa$ for which there is a cofinal $X_0 \subseteq \delta$ of order-type κ with $\langle f_\xi : \xi \in X_0\rangle$ strongly increasing. Clearly $(*)_\kappa$ implies that S must be stationary, and (2) follows from Theorem 3.50.

Conversely, suppose that (2) holds and X is a cofinal subset of μ^+. Our assumptions imply that there is a good δ of cofinality κ for which $\delta = \sup(X \cap \delta)$. We apply Theorem 3.50 once more to find $X_0 \subseteq X \cap \delta$ with the required properties. ⊣

Now at last we are in a position to prove Shelah's theorem (Theorem 1.5 from Chap. II of [89]) that scales exist for every singular cardinal.

3.53 Theorem. *If μ is singular, then there exists a scale for μ.*

Proof. Let $\lambda = \mu^+$ for μ singular, and let $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ be a strictly increasing sequence of regular cardinals cofinal in μ with $\mathrm{cf}(\mu) < \mu_0$. Let I be the ideal of bounded subsets of $\mathrm{cf}(\mu)$; we shall consider the structure

$$\mathfrak{B} = \prod_{i<\mathrm{cf}(\mu)} \mu_i$$

ordered by $<_I$. It is quite easy to see that $(\mathfrak{B}, <_I)$ is λ-directed, that is, if we are given a family $\mathcal{F} \subseteq \mathfrak{B}$ of size $\leq \mu$, then there is a single function $f \in \mathfrak{B}$ that is above (in the sense of $<_I$) all members of $\mathcal{F}$.

Let $\langle M_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence over $\mathfrak{B}$, and for each $\alpha < \lambda$ let g_α be the $<_\chi$-least function in $\mathfrak{B}$ that dominates $M_\alpha \cap \mathfrak{B}$. (Recall that $<_\chi$ is the well-ordering of $H(\chi)$ built into the ambient structure $\mathfrak{A}$.) Note that $g_\alpha \in M_{\alpha+1}$, and the sequence $\langle g_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing.

3.54 Claim. *The sequence $\langle g_\alpha : \alpha < \lambda\rangle$ has an exact upper bound g with the property that for each regular $\kappa < \mu$,*

$$\{i < \mathrm{cf}(\mu) : \mathrm{cf}(g(i)) < \kappa\} \text{ is bounded below } \mu.$$

Proof of Claim. Given a regular cardinal $\kappa < \mu$, we know that S^λ_κ has a stationary subset in $I[\lambda]$. In fact, we know that there is a stationary $S \subseteq S^\lambda_\kappa$ with the property that for each $\delta \in S$, there is an IA chain of submodels $\langle N_i : i < \kappa\rangle$ such that

- $|N_\xi| < \kappa$,
- $N_\xi \subseteq M_\delta$ for all $\xi < \kappa$, and
- $M_\alpha \in N$ for unboundedly many $\alpha < \delta$, where $N := \bigcup_{\xi<\kappa} N_\xi$.

(This follows from Theorem 3.11.) Assume now that $\mathrm{cf}(\mu) < \kappa$ and fix such a δ and sequence $\langle N_i : i < \kappa\rangle$.

For $\xi < \kappa$ and $i < \mathrm{cf}(\mu)$, we define

$$h_\xi(i) := \sup(N_\xi \cap \mu_i),$$

and

$$h(i) := \sup(N \cap \mu_i).$$

For each $i < \mathrm{cf}(\mu)$, the sequence $\langle h_\xi(i) : \xi < \kappa\rangle$ is increasing with limit $h(i)$, and it follows easily that h is an exact upper bound for $\langle h_\xi : \xi < \kappa\rangle$.

We claim now that h is an exact upper bound for $\langle g_\alpha : \alpha < \delta\rangle$. Note that h is an upper bound for $\langle g_\alpha : \alpha < \delta\rangle$ because N contains M_α for cofinally many $\alpha < \delta$. If $f <_I h$ then $f <_I h_\xi$ for some ξ. But $h_\xi \in N \subseteq M_\delta$ hence $h_\xi \in M_\alpha$ for some $\alpha < \delta$. It follows that

$$f <_I h_\xi <_I g_\alpha,$$

and hence h is an exact upper bound for the sequence $\langle g_\alpha : \alpha < \delta\rangle$.

We conclude that the sequence $\langle g_\alpha : \alpha < \lambda\rangle$ has stationarily many good points of cofinality κ for each regular $\kappa < \mu$ greater than $\mathrm{cf}(\mu)$. The conclusion of the claim follows from Theorem 3.51. $\dashv$

To recap, we have constructed a $<^*$-increasing sequence $\langle g_\alpha : \alpha < \mu^+\rangle$ in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$ with an exact upper bound g such that for all $\kappa < \mu$,

$$|\{i < \mathrm{cf}(\mu) : \mathrm{cf}(g(i)) < \kappa\}| < \mathrm{cf}(\mu). \tag{15.49}$$

By making inessential modifications to g and each g_α, we may assume that

- $\mathrm{cf}(\mu) < \mathrm{cf}(g(i))$ for all $i < \mathrm{cf}(\mu)$, and
- $g_\alpha(i) < g(i)$ for all $\alpha < \mu^+$ and $i < \mathrm{cf}(\mu)$.

Let B be the set of cardinals of the form $\mathrm{cf}(g(i))$ for some $i < \mathrm{cf}(\mu)$. The set B is cofinal in μ of order-type $\mathrm{cf}(\mu)$ by (15.49). Let $\vec{\theta} = \langle \theta_\xi : \xi < \mathrm{cf}(\mu) \rangle$ enumerate B in increasing order, and for $\xi < \mathrm{cf}(\mu)$ let

$$B_\xi := \{i < \mathrm{cf}(\mu) : \mathrm{cf}(g(i)) = \theta_\xi\}.$$

Our goal is to construct a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$ in $\prod_{\xi < \mathrm{cf}(\mu)} \theta_\xi$ so that $(\vec{\theta}, \vec{f})$ is a scale for μ^+. To do this, we use the fact that there is a natural mapping

$$\Phi : \textstyle\prod_{i<\mathrm{cf}(\mu)} g(i) \to \prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$$

defined as follows:

For each $i < \mathrm{cf}(\mu)$, let $\langle e_i(\epsilon) : \epsilon < \mathrm{cf}(g(i)) \rangle$ be the increasing enumeration of a cofinal subset of $g(i)$. For $h \in \prod_{i<\mathrm{cf}(\mu)} g(i)$, we define

$$\Phi[h](\xi) = \text{the least } \epsilon < \theta_\xi \text{ such that } h(i) \leq e_i(\epsilon) \text{ for all } i \in B_\xi.$$

Note that $\Phi[h] \in \prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$, and clearly

$$h \leq^* h' \quad \Longrightarrow \quad \Phi[h] \leq^* \Phi[h'].$$

Thus, $\langle \Phi[g_\alpha] : \alpha < \mu^+ \rangle$ is a $\leq^*$-increasing sequence in $\prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$.

We claim now that if h is in $\prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$, then there is an α such that $h \leq^* \Phi[g_\alpha]$. Given such an h, define $h' \in \prod_{i<\mathrm{cf}(\mu)} g(i)$ by

$$h'(i) = e_i(h(\xi)) \quad \text{for the unique } \xi < \mathrm{cf}(\mu) \text{ with } i \in B_\xi.$$

Clearly $h' \in \prod_{i<\mathrm{cf}(\mu)} g(i)$ and furthermore $\Phi[h'] = h$. Since g is an exact upper bound for the g_α's, there is an $\alpha < \mu^+$ with $h' \leq^* g_\alpha$ and so

$$h = \Phi[h'] \leq^* \Phi(g_\alpha).$$

As an immediate corollary, it follows that for all $h \in \prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$ there is a $\gamma < \mu^+$ such that $h <^* \Phi[g_\gamma]$ (and not just $h \leq^* \Phi[g_\gamma]$). Now we define $\vec{f}$ in the obvious manner:

Given $\langle f_\beta : \beta < \alpha \rangle$, we let $f_\alpha = \Phi[g_\gamma]$ where $\gamma < \mu^+$ is the least ordinal greater than or equal to α such that $f_\beta <^* \Phi[g_\gamma]$ for all $\beta < \alpha$. The verification that $(\vec{\theta}, \vec{f})$ is a scale for μ^+ is now routine. $\dashv$

We remark that in the above context, if $S \in I[\lambda]$ is a stationary set of ordinals of cofinality larger than $\mathrm{cf}(\mu)$, then the preceding construction gives us a scale that is good at almost every point of S. In fact, much more is true—given such an $S \in I[\lambda]$ and *any* scale $(\vec{\mu}, \vec{f})$ for μ, a similar argument establishes that almost every point of S is good for $\vec{f}$.

3.55 Theorem (Cummings, Foreman, and Magidor [17]). *Let $\lambda = \mu^+$ where μ is singular, and suppose that $(\vec{\mu}, \vec{f})$ is a scale for μ. If $S \in I[\lambda]$, then almost all points of S with cofinality greater than* $\mathrm{cf}(\mu)$ *are good for $\vec{f}$.*

The above theorem can be summarized by the statement "approachable points are good". With regard to the converse, we mention that Gitik and Sharon [39] have constructed a model in which $\aleph_{\omega^2+1}$ carries a scale for which all points are good, but $\mathrm{AP}_{\aleph_{\omega^2}}$ fails. We discuss this model more at the end of Sect. 4.7. Whether this can happen at $\aleph_\omega$ is still open, but the following result of Cummings, Foreman, and Magidor [17] gives some information.

3.56 Theorem. *If $\square_{\aleph_n}$ holds for all finite n, then in $\aleph_{\omega+1}$ all good points of cofinality greater than $\aleph_1$ are approachable.*

4. Applications of Scales and Weak Squares

4.1. Weakenings of $\square$—Part I

If one views a $\square_\mu$-sequence as a sequence of sets endowed with strong global coherence properties, then two natural means of weakening this combinatorial principle become apparent—one might require the coherence properties to hold only some of the time, or one might weaken the amount of coherence required. Both of these ideas are important and we look at them in this subsection and the next.

4.1 Definition. Suppose that λ is a regular cardinal and η is an ordinal. We say that a set $S \subseteq \lambda$ *carries a partial square as a subset of* λ if there is a sequence $\langle C_\delta : \delta \in S\rangle$ and an ordinal $\eta < \lambda$ such that

1. C_δ is a closed and unbounded subset of C_δ for all limit $\delta \in S$,
2. $\mathrm{ot}(C_\delta) < \eta$, and
3. if $\gamma \in \mathrm{acc}(C_\alpha) \cap \mathrm{acc}(C_\beta)$, then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

We suppress reference to λ if λ is clear from context. We call the sequence $\langle C_\alpha : \alpha < \lambda\rangle$ a partial square for S, and if we wish to emphasize the value of η then we say S *carries a partial square type-bounded by* η. If S is a subset of S^λ_κ for some $\kappa < \lambda$, then it is standard practice to require $\eta = \kappa + 1$.

Note that if S carries a partial square, then it is straightforward to extend the partial square sequence to the set

$$S^+ := S \cup \bigcup_{\delta \in S} \mathrm{acc}(C_\delta). \tag{15.50}$$

We note as well that if a set $S \subseteq S^\lambda_\kappa$ carries a partial square, then S can be written as a union of fewer than λ sets, each of which carries a partial square type-bounded by $\kappa+1$. This means that in practice, the demand of the last sentence from Definition 4.1 is not a serious one.

For $\lambda = \mu^+$, it is clear that $\square_\mu$ holds if and only if λ carries a partial square sequence type-bounded by $\mu+1$. On the other hand, if $S \subseteq \lambda$ carries a partial square, then an argument similar to that of Theorem 3.13 shows that S is in $I[\lambda]$. The following theorem shows us that a result analogous to Theorem 2.4 holds in the presence of partial squares.

4.2 Theorem. *Suppose that $S \subseteq \lambda$ carries a partial square. Then every stationary $T \subseteq \lambda$ has a stationary subset T^* which does not reflect at any ordinal in S.*

Proof. Let $\langle C_\delta : \delta \in S\rangle$ be a partial square on S, type-bounded by some $\eta < \lambda$. By the comments after Definition 4.1, we may assume

$$\delta \in S \implies \operatorname{acc}(C_\delta) \subseteq S.$$

This implies that $S \cap \delta$ contains a closed unbounded subset of δ for all $\delta \in S$ of uncountable cofinality.

Let T be a stationary subset of λ. If $T \setminus S$ is stationary then the remarks in the preceding paragraph imply that $T \setminus S$ cannot reflect at an ordinal in S and we are done.

If on the other hand $T \setminus S$ is nonstationary, then without loss of generality $T \subseteq S \setminus \eta$. Just as in the proof of Theorem 2.4, there is a stationary $T^* \subseteq T$ and an ordinal $\xi < \eta$ such that $\operatorname{ot}(C_\alpha) = \xi$ for all $\alpha \in T^*$. If $\delta \in S$ has uncountable cofinality, then $T^* \cap \operatorname{acc}(C_\delta)$ has at most one element and so $T^* \cap \delta$ is not stationary in δ. ⊣

The next theorem is due to Shelah [87], and it gives us information about $I[\lambda]$ in the case where λ is the successor of a regular cardinal.

4.3 Theorem. *If $\tau < \sigma$ are regular cardinals, then $S^{\sigma^+}_\tau$ can be written as the union of σ sets, each of which carries a partial square.*

Proof. As usual, let χ be a sufficiently large regular cardinal, and let $\mathfrak{A}$ be the structure $\langle H(\chi), \in, <_\chi\rangle$. Given $\alpha \in S^{\sigma^+}_\tau$ and $\zeta < \sigma$, we define

$$M(\alpha,\zeta) = \operatorname{Sk}^{\mathfrak{A}}(\{\alpha\} \cup \zeta).$$

For each $\alpha \in S^{\sigma^+}_\tau$ there is a least $\zeta = \zeta(\alpha) \geq \tau$ such that

- $M(\alpha,\zeta) \cap \sigma$ is an initial segment of σ, and
- $\operatorname{cf}(M(\alpha,\zeta) \cap \sigma) > \aleph_0$,

and we define

$$N_\alpha := M(\alpha, \zeta(\alpha)).$$

4.4 Claim. *$N_\alpha \cap \alpha$ is an ω-closed unbounded subset of α.*

Proof. We know $N_\alpha \cap \alpha$ is unbounded in α since $\mathrm{cf}(\alpha) = \tau \leq \zeta(\alpha) \subseteq N_\alpha$ and $\alpha \in N_\alpha$. Thus, we need only prove that $N_\alpha \cap \alpha$ is ω-closed.

Let x be a countable subset of $N_\alpha \cap \alpha$, and assume by way of contradiction that $\beta := \sup x \notin N_\alpha$. Note that this assumption implies that β is an ordinal of countable cofinality strictly less than α, so we can define

$$\gamma := \min(N_\alpha \cap \alpha \setminus \beta).$$

Since $\gamma \in N_\alpha$, it follows that $\beta < \gamma$, and $N_\alpha \cap \gamma$ is bounded below γ. Note as well that $\mathrm{cf}(\gamma) = \sigma$, for if it were less than σ, then N_α would contain every member of a cofinal subset of γ because $\mathrm{cf}(\gamma)$ would be in an element of N_α and $N_\alpha \cap \sigma$ is an initial segment of σ.

Since both σ and γ are in N_α, we can find an increasing continuous function $f : \sigma \to \gamma$ in N_α whose range is unbounded in γ. If we step outside of N_α, we see that $f \restriction N_\alpha \cap \sigma$ is an increasing function from $N_\alpha \cap \sigma$ to a cofinal subset of β. This gives a contradiction, as $N_\alpha \cap \sigma$ has uncountable cofinality, while the cofinality of β is countable. ⊣

For each $\alpha \in S^{\sigma^+}_\tau$, let D_α be the closure of $N_\alpha \cap \alpha$ in α. Given ρ and ϵ less than σ, we define

$$S(\rho, \epsilon) := \{\alpha \in S^{\sigma^+}_\tau : N_\alpha \cap \sigma = \rho \text{ and } \mathrm{ot}(D_\alpha) = \epsilon\}.$$

We claim that $\langle D_\alpha : \alpha \in S(\rho, \epsilon)\rangle$ is a partial square sequence on $S(\rho, \epsilon)$, and the following claim suffices.

4.5 Claim. *If α and β are in $S(\rho, \epsilon)$, then*

$$\gamma \in \mathrm{acc}(D_\alpha) \cap \mathrm{acc}(D_\beta) \quad \Longrightarrow \quad D_\alpha \cap \gamma = D_\beta \cap \gamma. \tag{15.51}$$

Proof. Assume first that γ is of countable cofinality. Then by Claim 4.4, γ is an element of both N_α and N_β. We know $|\gamma| \leq \sigma$, and since $N_\alpha \cap \sigma = N_\beta \cap \sigma = \rho$, it follows that $N_\alpha \cap \gamma = N_\beta \cap \gamma$.

If $\mathrm{cf}(\gamma) > \aleph_0$, then γ is a limit of ordinals from $\mathrm{acc}(D_\alpha) \cap \mathrm{acc}(D_\beta)$ of countable cofinality, and so the result follows by the previous paragraph. ⊣

Since $S^{\sigma^+}_\tau = \bigcup_{\rho, \epsilon < \sigma} S(\rho, \epsilon)$, the proof of the theorem is complete. ⊣

The preceding theorem has the following corollary, which was promised back in Sect. 3.3.

4.6 Corollary. *If σ is a regular cardinal, then $S^{\sigma^+}_{<\sigma} \in I[\sigma^+]$.*

Proof. We know that $S^{\sigma^+}_{<\sigma}$ is the union of σ sets each carrying a partial square. By the remarks preceding Theorem 4.2, each of these sets is in $I[\sigma^+]$, and now the result follows immediately because $I[\sigma^+]$ is a normal ideal. ⊣

The following theorem due to Džamonja and Shelah [21] serves as the counterpart to Theorem 4.3 at the successor of a singular cardinal.

4.7 Theorem. *Suppose that μ is singular, $\lambda = \mu^+$, and $\kappa < \mu$ is an uncountable cardinal such that* $\mathrm{cf}([\mu]^\kappa, \subseteq) = \mu$. *Then the set $S^\lambda_{\leq\kappa}$ is the union of μ sets, each of which carries a partial square type-bounded by κ^+.*

There are many open questions concerning the notion of partial square. In particular, the question of whether S^λ_κ has a stationary subset which carries a partial square is still open in many cases—for example, if λ is inaccessible, or $\lambda = \mu^+$ for μ singular with $\mathrm{cf}(\mu) \leq \kappa$ (see [91, Question 5.9]). We shall see in Sect. 4.3 that these matters are related to an open problem involving $\diamondsuit$, while Foreman and Todorčević [35] have investigated some consequences of the conjecture that for all successor λ, the set $S^\lambda_{\aleph_1}$ contains a stationary subset on which there is a partial square.

We close this subsection with a discussion of partial squares above a supercompact cardinal κ. Foreman and Magidor [34] used ideas of Baumgartner to establish the following theorem:

4.8 Theorem. *If κ is supercompact, then there is a forcing extension which preserves the supercompactness of κ and in which, letting λ denote $\kappa^{+\omega+1}$, we have*

$$S^\lambda_{\geq\kappa} \textit{ carries a partial square type-bounded by } \kappa^{+\omega}. \tag{15.52}$$

The preceding theorem tells us that the reflection result from Theorem 2.5 is sharp, for we have the following corollary.

4.9 Corollary. *Let κ is a supercompact cardinal for which (15.52) holds, and let S be a stationary subset of $\lambda = \kappa^{+\omega+1}$. Then* $\mathrm{Refl}(S)$ *holds if and only if $S \cap S^\lambda_{\geq\kappa}$ is nonstationary.*

Proof. If $S \cap S^\lambda_{\geq\kappa}$ is nonstationary, then $\mathrm{Refl}(S)$ holds by way of Theorem 2.5. If $S \cap S^\lambda_{\geq\kappa}$ is stationary, then we can find a regular σ such that $\kappa < \sigma < \kappa^{+\omega}$ and $T := S \cap S^\lambda_\sigma$ is stationary. By Theorem 4.2, T has a stationary subset T^* that does not reflect in any ordinal of cofinality κ or greater. But S^λ_σ only reflects in ordinals of cofinality greater than σ, and therefore T^* cannot reflect at all. ⊣

4.2. Weakenings of □—Part II

In this subsection, we examine yet another way to weaken $\square_\kappa$—we reduce the amount of coherence required by the sequence. We begin with the classic definition (due to Jensen [52]) of *weak square*.

4.10 Definition. Let κ be a cardinal. A $\square^*_\kappa$*-sequence* or *weak square sequence for* κ is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ such that

1. $\mathcal{C}_\alpha$ is a family of closed unbounded subsets of α,

2. $|\mathcal{C}_\alpha| \leq \kappa$,

3. $C \in \mathcal{C}_\alpha \Longrightarrow \mathrm{ot}(C_\alpha) \le \kappa$, and
4. $C \cap \beta \in \mathcal{C}_\beta$ for $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{acc}(C)$.

$\square^*_\kappa$ is the assertion that there is such a sequence.

An elementary argument establishes that $\square^*_\kappa$ follows if $\kappa^{<\kappa} = \kappa$, so weak square is of greatest interest in the case of singular cardinals. Before we get to some applications, we make a few remarks on the structure of weak square sequences.

First, note that for $\alpha < \kappa^+$ we have

$$\mathrm{cf}(\alpha) < \kappa \quad \Longrightarrow \quad \mathrm{ot}(C) < \kappa \text{ for all } C \in \mathcal{C}_\alpha. \tag{15.53}$$

This follows from the simple fact that such an α has no closed unbounded subset of order-type κ.

Next, we may assume the $\square^*_\kappa$ sequence has the property that each $\mathcal{C}_\alpha$ contains a set of order-type $\mathrm{cf}(\alpha)$. To arrange this, we choose for each limit $\epsilon < \kappa$ a closed unbounded $D_\epsilon \subseteq \epsilon$ of order-type $\mathrm{cf}(\epsilon)$. For each C in $\mathcal{C}_\alpha$, check if the order-type of C is in $\mathrm{acc}(D_\epsilon) \cup \{\epsilon\}$. If so, then add the set $\{\beta \in C : \mathrm{ot}(C_\beta) \in D_\epsilon\}$ to $\mathcal{C}_\alpha$. It is straightforward to verify that this modification does the job.

The preceding observation helps us situate $\square^*_\kappa$ in the hierarchy of combinatorial principles we have been studying. It is clear that $\square_\kappa$ implies $\square^*_\kappa$, and the following easy proposition shows us that $\square^*_\kappa$ implies AP_κ.

4.11 Proposition. *If $\square^*_\kappa$ holds, then so does AP_κ.*

Proof. Let $\bar{\mathcal{C}}$ be a $\square^*_\kappa$ sequence such that $\mathcal{C}_\alpha$ contains a set of order-type $\mathrm{cf}(\alpha)$ for each limit $\alpha < \kappa^+$. Let $\mathfrak{M} = \langle M_i : i < \kappa^+ \rangle$ be a κ^+-approximating sequence over $\bar{\mathcal{C}}$. If $\delta < \kappa^+$ satisfies $\delta = M_\delta \cap \kappa^+$, then any set in $\mathcal{C}_\delta$ of order-type $\mathrm{cf}(\delta)$ establishes that δ is in $S[\mathfrak{M}]$. Thus, $S[\mathfrak{M}]$ contains a closed unbounded set and therefore AP_κ holds. ⊣

None of the implications in the chain $\square_\kappa \Longrightarrow \square^*_\kappa \Longrightarrow \mathrm{AP}_\kappa$ can be reversed. The model of Ben-David and Magidor [7] mentioned in Theorem 2.14 provides a model in which $\square^*_{\aleph_\omega}$ holds and $\square_{\aleph_\omega}$ fails. (Apter and Henle [3] have obtained similar results starting from a κ^+-strongly compact cardinal instead of a κ^+-supercompact cardinal.) On the other hand, an argument outlined in [34, Sect. 5] yields a model in which $\mathrm{AP}_{\aleph_\omega}$ holds and $\square^*_{\aleph_\omega}$ fails.

It turns out that weak square is compatible with strong forms of stationary reflection. For example, we have the following theorem of Cummings, Foreman, and Magidor [16]:

4.12 Theorem. *If the existence of infinitely many supercompact cardinals is consistent, then there is a model of* ZFC *in which $\square^*_{\aleph_\omega}$ holds, and*

> *For $1 \le m \le n < \omega$, if $\langle S_i : i < \aleph_m \rangle$ is a sequence of stationary subsets of $\{\alpha < \aleph_{\omega+1} : \mathrm{cf}(\alpha) < \aleph_m\}$, then there is an ordinal $\delta < \aleph_{\omega+1}$ of cofinality $\aleph_n$ such that the S_i all reflect at δ.*

In the above model, it is clear that $\square_{\aleph_\omega}$ must fail. Ben-David and Shelah [8] claimed to construct a model of $\square^*_{\aleph_\omega}$ in which $\mathrm{Refl}(\aleph_{\omega+1})$ holds, but more recent work of Cummings, Foreman, and Magidor has demonstrated that the purported proof cannot be completed along the lines indicated, as weak square is incompatible with certain forms of generic supercompactness. We refer the reader to [18, Sect. 10] for the actual statement of this result.

Finally, we point out that even though weak square is compatible with stationary reflection, it is still a powerful principle for constructing non-compact objects. This can be seen in the following result of Jensen, which we state without proof. Section 5 of Cummings' paper [14] contains a very nice proof taken from unpublished notes of Magidor [63] concerning Todorčević's paper [97].

4.13 Theorem. *If κ is a cardinal, then $\square^*_\kappa$ is equivalent to the existence of a special κ^+-Aronszajn tree.*

Proof. See [14, Sect. 5]. $\dashv$

We turn our attention now to the gap between $\square_\kappa$ and $\square^*_\kappa$. In the course of investigating the core model for one Woodin cardinal, Schimmerling [74] isolated a spectrum of combinatorial principles that form a natural hierarchy between $\square_\kappa$ and $\square^*_\kappa$.

4.14 Definition. Let $\sigma \leq \kappa$ be cardinals. (We assume κ is infinite, but σ is allowed to be finite.) A $\square^\sigma_\kappa$*-sequence* is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+\rangle$ such that

1. $\mathcal{C}_\alpha$ is a family of closed unbounded subsets of α,
2. $1 \leq |\mathcal{C}_\alpha| \leq \sigma$,
3. $C \in \mathcal{C}_\alpha \implies \mathrm{ot}(C) \leq \kappa$, and
4. for all $C \in \mathcal{C}_\alpha$,
$$\beta \in \mathrm{acc}(C) \quad\implies\quad C \cap \beta \in \mathcal{C}_\beta.$$

$\square^\sigma_\kappa$ is the assertion that there is such a sequence. One should give $\square^{<\sigma}_\kappa$ the obvious meaning along the lines above.

It follows immediately that $\square^1_\kappa = \square_\kappa$, $\square^\kappa_\kappa = \square^*_\kappa$, and

$$\square^\sigma_\kappa \;\Rightarrow\; \square^\tau_\kappa \quad \text{if } 1 \leq \sigma \leq \tau \leq \kappa.$$

Note that the requirement that $\sigma \leq \kappa$ is natural, as the principle $\square^{\kappa^+}_\kappa$ is true in ZFC—for each $\alpha < \kappa^+$ choose a closed unbounded $C_\alpha \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha)$, and then define $\mathcal{C}_\alpha = \{C_\beta \cap \alpha : \alpha \leq \beta < \kappa^+\}$. This combinatorial principle is more commonly known as *silly square*.

Before continuing our investigation into these principles, it is natural to ask if there are non-trivial implications between them. The following theorem from [16] gives a satisfactory answer.

4.15 Theorem. *Let κ be supercompact, and suppose that $2^{(\kappa^{+\omega})} = \kappa^{+\omega+1}$. Let μ and ν be two cardinals (one or both can be finite) such that*

$$1 \leq \mu < \nu < \aleph_\omega.$$

Then there is a generic extension in which

1. *all cardinals $\leq \nu$ are preserved,*
2. *$\aleph_\omega = \kappa_V^{+\omega}$,*
3. *$\square^{\nu}_{\aleph_\omega}$ holds, and*
4. *$\square^{\mu}_{\aleph_\omega}$ fails.*

With regard to applications of these principles, we restrict attention to their impact on stationary reflection, and a reader seeking more information concerning these principles and their relation to core models is referred to [73, 75, 77, 78], and [15].

We begin our discussion of stationary reflection with two results of Schimmerling.

4.16 Proposition. *If $\kappa^{<\sigma} = \kappa$ and $\square^{<\sigma}_{\kappa}$ holds, then every stationary subset S of κ^+ has a stationary subset that does not reflect at ordinals of cofinality σ or greater.*

Proof. Suppose that $\bar{\mathcal{C}}$ is a $\square^{<\sigma}_{\kappa}$-sequence, and define

$$F(\alpha) = \{\mathrm{ot}(C) : C \in \mathcal{C}_\alpha\}$$

for limit ordinals $\alpha < \kappa^+$. Since $\kappa^{<\sigma} = \kappa$, we note that F has only κ possible values. Thus, given a stationary set $S \subseteq \kappa^+$, there must be a stationary $T \subseteq S$ on which F is constant, say with value X.

If $\mathrm{cf}(\alpha) \geq \sigma$ and $C \in \mathcal{C}_\alpha$, then for each $\beta \in T \cap \mathrm{acc}(C)$ we know the order-type of $C \cap \beta$ is in X. This means that $\mathrm{acc}(C) \cap T$ is bounded below α and therefore $T \cap \alpha$ is not stationary in α. ⊣

As a quick corollary, we see that $\square^{<\omega}_{\kappa}$ implies that $\mathrm{Refl}(S)$ fails for all stationary subsets S of κ^+. Another modification of the argument yields the following proposition at successors of singular cardinals.

4.17 Proposition. *Suppose that μ is singular and $\square^{<\mathrm{cf}(\mu)}_{\mu}$ holds. Then for every stationary subset S of μ^+, there is a stationary $T \subseteq S$ and a $\beta < \mu$ such that T does not reflect at ordinals of cofinality $> \beta$.*

Proof. Let $\bar{\mathcal{C}}$ be a $\square^{<\mathrm{cf}(\mu)}_{\mu}$-sequence, and define $F : \mu^+ \to \mu$ by

$$F(\alpha) = \sup\left(\{\mathrm{ot}(C) : C \in \mathcal{C}_\alpha\}\right).$$

If $S \subseteq \mu^+$ is stationary, then there is a stationary $T \subseteq S$ for which $F \restriction T$ is constant, say with value $\beta < \mu$. If $\mathrm{cf}(\gamma) > \beta$ and $C \in \mathcal{C}_\gamma$, then

$$\beta < \mathrm{cf}(\gamma) \leq \mathrm{ot}(C)$$

so for all sufficiently large $\delta \in \mathrm{acc}(C)$ we have $F(\delta) > \beta$ and hence $\delta \notin T$. $\dashv$

The preceding result of Schimmerling is sharp, as Cummings, Foreman, and Magidor have shown [16, Sect. 10] that it is consistent for a strengthening of $\square^\omega_{\aleph_\omega}$ to hold simultaneously with the statement "every stationary subset of $\aleph_{\omega+1}$ reflects in all sufficiently large cofinalities".

With regard to simultaneous reflection, we have the following result of Cummings, Foreman, and Magidor [16].

4.18 Theorem. *Let μ be a singular cardinal, and assume $\square^\sigma_\mu$ holds for some $\sigma < \mu$. Given a stationary set $T \subseteq \mu^+$, we can find a sequence $\langle S_i : i < \mathrm{cf}(\mu)\rangle$ of stationary subsets of T that do not reflect simultaneously in any ordinal of cofinality greater than $\mathrm{cf}(\mu)$.*

This theorem follows from results presented later in the chapter—see Theorem 4.66. Note that by Theorem 4.12, the conclusion of Theorem 4.18 does not follow from $\square^*_\mu$. The following theorem, again taken from [16], shows that the conclusion of the preceding theorem cannot be strengthened to rule out simultaneous reflection of fewer than $\mathrm{cf}(\mu)$ sets.

4.19 Theorem. *If the existence of infinitely many supercompact cardinals is consistent, then there is a model of* ZFC *in which $\square^\omega_{\aleph_\omega}$ holds, and*

> *For every finite sequence $\langle S_i : i < n\rangle$ of stationary subsets of $\aleph_{\omega+1}$, there is an $M < \omega$ such that for each $m \geq M$ there is an ordinal δ of cofinality $\aleph_m$ such that all S_i reflect at δ.*

The next theorem of Cummings and Schimmerling [15] shows us that $\square^*_{\aleph_\omega}$ cannot be strengthened to $\square^{<\aleph_\omega}_{\aleph_\omega}$ in Theorem 4.12.

4.20 Theorem. *If μ is a singular strong limit cardinal for which $\square^{<\mu}_\mu$ holds, and T is a stationary subset of μ^+, then there is a sequence of stationary sets $\langle S_i : i < \mathrm{cf}(\mu)\rangle$ and a cardinal $\theta < \mu$ such that*

1. *$S_i \subseteq T \cap S^{\mu^+}_{<\theta}$, and*
2. *the sets S_i do not simultaneously reflect at any ordinal of cofinality θ or greater.*

We will give a proof of this theorem because it fits in nicely with other proofs in this subsection, but we need the following easy observation.

4.21 Lemma. *Let μ be a singular cardinal and $\sigma \leq \mu^+$. If $\square^{<\sigma}_\mu$ holds, then there is a $\square^{<\sigma}_\mu$-sequence $\langle \mathcal{D}_\alpha : \alpha < \mu^+\rangle$ with the property that all elements of each $\mathcal{D}_\alpha$ are of order-type less than μ.*

Proof. Let D be a closed unbounded subset of μ of order-type $\mathrm{cf}(\mu)$, and let $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$ be a $\square_\mu^{<\sigma}$-sequence. If $C \in \mathcal{C}_\delta$, then $\mathrm{ot}(C) \leq \mu$ and we define a set $C^* \subseteq C$ according to the following cases:

- If $\mathrm{ot}(C) \in \mathrm{acc}(D) \cup \{\mu\}$, then $C^* = \{\delta \in C : \mathrm{ot}(C \cap \delta) \in D\}$.
- Otherwise, set $C^* = \{\delta \in C : \max(\mathrm{ot}(C) \cap \mathrm{acc}(D))\}$.

If we define $\mathcal{D}_\alpha$ to be the collection $\{C^* : C \in \mathcal{C}_\alpha\}$, then it is routine to verify that this sequence has the required properties. ⊣

In contrast to the situation with weak square, we cannot guarantee that each $\mathcal{D}_\alpha$ will contain a set of order-type $\mathrm{cf}(\alpha)$. It is shown in [16, Sect. 5] that these "improved" sequences exert slightly more influence on stationary reflection than their unimproved versions.

Proof of Theorem 4.20. Let $\mathcal{C}$ be a $\square_\mu^{<\mu}$-sequence satisfying the conclusion of the preceding lemma, and let $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ be an increasing sequence of regular cardinals cofinal in μ. Given a stationary $T \subseteq \mu^+$, we can pass to a stationary $T^* \subseteq T$ such that for some σ and τ,

$$\delta \in T^* \quad \Longrightarrow \quad |\mathcal{C}_\delta| = \sigma \text{ and } \mathrm{cf}(\delta) = \tau. \tag{15.54}$$

Next, let us suppose that $\alpha < \mu^+$ and $j < \mathrm{cf}(\mu)$, and define

$$A(\alpha, j) = \{\mathrm{ot}(C) : C \in \mathcal{C}_\alpha\} \cap \mu_j. \tag{15.55}$$

For each j, there are at most $2^{\mu_j} < \mu$ possible values for $A(\alpha, j)$, and thus we can find a stationary $S_j \subseteq T$ and a set A_j such that

$$\alpha \in S_j \quad \Longrightarrow \quad A(\alpha, j) = A_j. \tag{15.56}$$

Assume now that the sets $\langle S_j : j < \mathrm{cf}(\mu) \rangle$ reflect simultaneously at an ordinal δ, and choose $C \in \mathcal{C}_\delta$.

Since $\mathrm{ot}(C) < \mu$, there is a $j < \mathrm{cf}(\mu)$ with $\mathrm{ot}(C) < \mu_j$. If $\beta < \gamma$ are members of $\mathrm{acc}(C) \cap S_j$, then $\mathrm{ot}(C \cap \beta)$ and $\mathrm{ot}(C \cap \gamma)$ are distinct elements of A_j. Since $|A_j| < \mu_i$, it follows that $|\,\mathrm{acc}(C) \cap S_j| < \mu_i$ and therefore $\mathrm{cf}(\delta) < \mu_i$. ⊣

We close this subsection with commentary on the compatibility of these combinatorial principles with large cardinals. Burke and Kanamori have observed (see [73]) that if κ is μ^+-strongly compact then $\square_\mu^{<\,\mathrm{cf}(\mu)}$ must fail (even if $\mathrm{cf}(\mu) \geq \kappa$). On the other hand, Cummings, Foreman, and Magidor [16, Sect. 9] demonstrate that $\square_\mu^{\mathrm{cf}(\mu)}$ can consistently hold if κ is supercompact and $\kappa \leq \mathrm{cf}(\mu) < \mu$. This line of research is also continued in the paper [2] of Apter and Cummings.

4.3. On $\diamondsuit$

Our goal in this section is to combine ideas from the previous two sections in order to investigate Jensen's axiom $\diamondsuit$ at successors of singular cardinals. We begin by recalling the following classical definitions:

4.22 Definition. Let κ be an infinite cardinal, and let S be a stationary subset of κ^+. We say that $\diamondsuit(S)$ *holds* if there is a sequence $\langle S_\alpha : \alpha \in S\rangle$ such that for all $X \subseteq \kappa^+$, there are stationarily many $\alpha \in S$ for which

$$X \cap \alpha = S_\alpha.$$

We say that $\diamondsuit^*(S)$ *holds* if there is a sequence $\langle \mathcal{S}_\alpha : \alpha \in S\rangle$ such that $\mathcal{S}_\alpha$ is a family of at most κ subsets of α, and for all $X \subseteq \kappa^+$, the set of α for which $X \cap \alpha \in \mathcal{S}_\alpha$ contains the restriction to S of a closed unbounded subset of κ^+.

It is true that $\diamondsuit^*(S)$ implies $\diamondsuit(T)$ for all stationary $T \subseteq S$, but this is not immediately obvious. The way to see this is through the following classical result, which we state without proof:

4.23 Proposition. *Let κ be an infinite cardinal, and let $S \subseteq \kappa^+$ be stationary. Then $\diamondsuit(S)$ holds if and only if there is a sequence $\langle \mathcal{S}_\alpha : \alpha \in S\rangle$ such that such that $\mathcal{S}_\alpha$ is a family of at most κ subsets of α, and whenever $X \subseteq \kappa^+$, the set $\{\alpha \in S : X \cap \alpha \in \mathcal{S}_\alpha\}$ is stationary in κ^+.*

Proof. See Lemma III.3.4 and Lemma IV.2.6 of [19]. $\dashv$

It is a well-known result of Jensen that $\diamondsuit(\omega_1)$ is independent of ZFC + CH (see [20]). For cardinals above $\aleph_1$, the situation is different. Gregory [41] was the first to note this—he showed that $\diamondsuit^*(S^{\kappa^+}_\sigma)$ holds if $2^\kappa = \kappa^+$ and $\kappa^\sigma = \kappa$. Thus, we have instances of $\diamondsuit$ following from simple cardinal arithmetic assumptions. The question of particular interest to us in this subsection arises from the following related result of Shelah [83].

4.24 Theorem. *Assume μ is a strong limit singular cardinal with $2^\mu = \mu^+$. Then $\diamondsuit^*(\{\delta < \mu^+ : \mathrm{cf}(\delta) \neq \mathrm{cf}(\mu)\})$ holds.*

Proof. To simplify our notation a bit, let λ denote μ^+, and let S denote $\{\delta < \lambda : \mathrm{cf}(\delta) \neq \mathrm{cf}(\mu)\}$. Fix a λ-filtration sequence $\langle b_{\alpha,i} : i < \mathrm{cf}(\mu), \alpha < \lambda\rangle$, and let $\langle A_\alpha : \alpha < \lambda\rangle$ enumerate $[\lambda]^{<\lambda}$ in such a way that each set in $[\lambda]^{<\lambda}$ appears unboundedly often.

Given $\delta \in S$ and $i < \mathrm{cf}(\mu)$, we define

$$\mathcal{S}_{\delta,i} = \left\{\bigcup\nolimits_{\alpha \in I} A_\alpha : I \subseteq b_{\delta,i}\right\}$$

and

$$\mathcal{S}_\delta := \bigcup\nolimits_{i<\mathrm{cf}(\mu)} \mathcal{S}_{\delta,i}.$$

Since μ is a strong limit cardinal, we know that $|\mathcal{S}_\delta| \leq \mu$ for each $\delta \in S$. Now suppose that $A \subseteq \lambda$; we produce a closed unbounded $C \subseteq \lambda$ such that

$$\delta \in C \cap S \quad \Longrightarrow \quad A \cap \delta \in \mathcal{S}_\delta. \tag{15.57}$$

For $\alpha < \lambda$, let $f(\alpha)$ be the least $\beta > \alpha$ such that $A \cap \alpha = A_\beta$. Let $C \subseteq \lambda$ be the closed unbounded set of ordinals that are closed under f; we will show that (15.57) holds for this choice of C.

Give $\delta \in C \cap S$, let $\langle \delta_j : j < \mathrm{cf}(\delta) \rangle$ be an increasing sequence cofinal in δ such that $f(\delta_j) < \delta_{j+1}$. Since $\mathrm{cf}(\mu) \neq \mathrm{cf}(\delta)$, there is some $i^* < \mathrm{cf}(\mu)$ such that

$$|b_{\delta,i^*} \cap \{f(\delta_j) : j < \mathrm{cf}(\delta)\}| = \mathrm{cf}(\delta).$$

Let us define

$$J := \{j < \mathrm{cf}(\delta) : f(\delta_j) \in b_{\delta,i^*}\}$$

and

$$I := \{f(\delta_j) : j \in J\}.$$

Since J is unbounded in $\mathrm{cf}(\delta)$ and $A \cap \delta_j = A_{f(\delta_j)}$, it follows that

$$A \cap \delta = \textstyle\bigcup_{i \in I} A_\alpha.$$

Since $I \subseteq b_{\delta,i^*}$, it follows that $A \cap \delta \in \mathcal{S}_\delta$ as required. $\dashv$

Given Theorem 4.24, it is natural to ask if $\diamondsuit(S^{\mu^+}_{\mathrm{cf}(\mu)})$ follows if μ is a strong limit singular cardinal for which $2^\mu = \mu^+$. This question is still very much open, but we can use ideas of the sort we have been considering in this chapter to give sufficient conditions for this to hold.

4.25 Definition. Suppose that $\lambda = \mu^+$ for μ singular. A set $S \subseteq \lambda$ is said to be *diamond-friendly* if there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \lambda \rangle$ such that

1. $\mathcal{C}_\alpha$ is a family of $\leq\mu$ closed unbounded subsets of α, and

2. there is a closed unbounded $E \subseteq \lambda$ such that for $\delta \in E \cap S$, there is a closed unbounded $C_\delta \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that

$$\alpha \in \mathrm{acc}(C_\delta) \quad \Longrightarrow \quad C_\delta \cap \alpha \in \mathcal{C}_\alpha.$$

If one were to write down a definition analogous to Definition 4.1 for what it means for a set to carry a weak square, one might very well end up with something like the above. We have stayed away from this notation, though, as Džamonja and Shelah have already utilized it in [21] for a related concept. We leave the proof of the following proposition as an easy exercise for the reader—the proposition establishes a connection between diamond-friendliness and the various weak forms of square we have been considering.

4.26 Proposition. *Let $\lambda = \mu^+$ for some singular cardinal μ.*

1. *If $S \subseteq \lambda$ is diamond-friendly, then $S \in I[\lambda]$.*
2. *If $S \subseteq \lambda$ carries a partial square, then S is diamond-friendly.*
3. *If $\square^*_\mu$ holds, then every subset of λ is diamond-friendly.*

We come now to the main theorem of this subsection, the proof of which is a substantial reworking of Shelah's original proof from [85]. Before we state the theorem, we introduce some notation.

4.27 Definition. Let κ be an infinite cardinal.

1. A *potential $\diamondsuit$-sequence for κ^+* is a sequence $\bar{\mathcal{S}} = \langle \mathcal{S}_\alpha : \alpha < \kappa^+\rangle$ such that each $\mathcal{S}_\alpha$ is a collection of at most κ subsets of α.
2. If $\bar{\mathcal{S}}$ is a potential $\diamondsuit$-sequence for κ^+ and $X \subseteq \kappa^+$, then we define

$$\mathrm{Trap}_{\bar{\mathcal{S}}}(X) = \{\alpha < \kappa^+ : X \cap \alpha \in \mathcal{S}_\alpha\}. \tag{15.58}$$

If κ^+ and $\bar{\mathcal{S}}$ are clear from context, then we may suppress explicit reference to them in the notation.

4.28 Theorem. *Let μ be a strong limit singular cardinal with $2^\mu = \mu^+$. If $\{\delta < \mu^+ : \mathrm{cf}(\delta) > \mathrm{cf}(\mu)\}$ contains a diamond-friendly stationary subset S, then there is a potential $\diamondsuit$-sequence $\bar{\mathcal{S}}$ such that for any $X \subseteq \mu^+$, there is a closed unbounded $E \subseteq \mu^+$ with the property that*

$$\delta \in E \cap S \quad \Longrightarrow \quad \mathrm{Trap}_{\bar{\mathcal{S}}}(X) \text{ contains a closed unbounded subset of } \delta. \tag{15.59}$$

Proof. Let λ denote μ^+. We are going to need several auxiliary objects, so we present them as a list:

- $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals cofinal in μ.
- $\bar{A} = \langle A_i : i < \lambda\rangle$ enumerates $[\lambda]^{<\lambda}$.
- $\bar{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha < \lambda\rangle$ attests to the diamond-friendliness of S.
- $\bar{e} = \langle e_\alpha : \alpha < \lambda\rangle$ is a sequence of one-to-one functions such that e_α maps α into μ.
- $\mathfrak{M}$ is a λ-approximating sequence over $\{\bar{A}, \bar{\mathcal{C}}, \bar{e}\}$.
- $\bar{\mathcal{S}} = \langle \mathcal{S}_\alpha : \alpha < \lambda\rangle$ is defined by $\mathcal{S}_\alpha := M_{\alpha+1} \cap \mathcal{P}(\alpha)$.

Given $X \subseteq \lambda$, we choose $\mathfrak{N}$ to be a λ-approximating sequence over $\{X, \mathfrak{M}\}$ and let E^* denote the closed unbounded subset of λ consisting of those δ for which $N_\delta \cap \lambda = \delta$. If $\delta \in E^*$, then $M_\delta \cap \lambda = \delta$ and therefore M_δ and N_δ contain the same bounded subsets of λ.

We now work to prove that (15.59) holds for this choice of E. To do this, we fix $\delta \in S \cap E$. The proof consists in defining a certain closed and unbounded $C \subseteq \delta$, and then verifying that this C is contained in $\mathrm{Trap}_{\bar{\mathcal{S}}}(X)$.

Note that since $\bar{\mathcal{C}}$ is in M_0, we can apply the diamond-friendliness of S to find a closed unbounded $C_\delta \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that

$$\alpha \in \mathrm{acc}(C_\delta) \quad \Longrightarrow \quad C_\delta \cap \alpha \in \mathcal{C}_\alpha.$$

Let $\langle \delta_\zeta : \zeta < \mathrm{cf}\,\delta \rangle$ be the increasing enumeration of $C_\delta \cap E^*$.

For $\zeta < \mathrm{cf}\,\delta$, we know $X \cap \delta_\zeta \in N_{\delta_\zeta + 1}$. Since $\delta_\zeta + 1 \leq \delta_{\zeta+1}$ and $\delta_{\zeta+1} \in E^*$, it follows that

$$X \cap \delta_\zeta \in M_{\delta_{\zeta+1}}.$$

This means

$$X \cap \delta_\zeta = A_{\beta_\zeta}$$

for some $\beta_\zeta < \delta_{\zeta+1}$.

In the model $M_{\delta_{\zeta+1}+1}$, we have available the function $e_{\delta_{\zeta+1}} : \delta_{\zeta+1} \to \mu$, and so we can define

$$i(\zeta) := \text{least } i < \mathrm{cf}(\mu) \quad \text{such that} \quad e_{\delta_{\zeta+1}}(\beta_\zeta) \leq \mu_i.$$

Since $\mathrm{cf}(\mu) < \mathrm{cf}(\delta)$, it follows that there is a single $i^* < \mathrm{cf}(\mu)$ such that

$$D := \{\zeta < \mathrm{cf}(\delta) : i(\zeta) \leq i^*\}$$

is unbounded in $\mathrm{cf}(\delta)$. Finally, we define

$$C := \{\delta_\zeta : \zeta = \sup(D \cap \zeta)\}.$$

The set C just constructed is closed and unbounded in δ, so the following claim will finish the proof.

4.29 Claim. $C \subseteq \mathrm{Trap}_{\bar{\mathcal{S}}}(X)$.

Proof. Given $\alpha \in C$, we must prove that $X \cap \alpha$ is in $\mathcal{S}_\alpha$. In light of the definition of $\mathcal{S}_\alpha$, this reduces to showing that $X \cap \alpha$ is in the model $M_{\alpha+1}$. We do this by demonstrating that $X \cap \alpha$ is definable from parameters available in $M_{\alpha+1}$.

By definition, our α is of the form δ_{ζ^*} for some $\zeta^* \in \mathrm{acc}(D)$. In light of this, it suffices to prove that the sequence $\langle X \cap \delta_\zeta : \zeta \in D \cap \zeta^* \rangle$ is an element of $M_{\alpha+1}$.

By the definition of "diamond-friendly", we know that $C_\delta \cap \alpha$ is in $M_{\alpha+1}$, and since $|C_\delta \cap \alpha| < \mathrm{cf}(\delta) < \mu$, it follows that $M_{\alpha+1}$ contains every subset of $C_\delta \cap \alpha$ as well as every increasing enumeration of such a set. Thus, we conclude that $M_{\alpha+1}$ contains the sequence $\langle \delta_\zeta : \zeta < \zeta^* \rangle$.

Similarly, $M_{\alpha+1}$ contains $D \cap \zeta^*$, as well as every function from $D \cap \zeta^*$ to μ_{i^*}. This tells us that the sequence

$$s := \langle e_{\delta_{\zeta+1}}(\beta_\zeta) : \zeta \in D \cap \zeta^* \rangle$$

must be in $M_{\alpha+1}$.

It follows that the sequence $\langle \beta_\zeta : \zeta \in D \cap \zeta^* \rangle$ is in $M_{\alpha+1}$ as it is definable from s and the sequence $\bar{e}$. Since $\bar{A} \in M_0$, we conclude

$$\langle A_{\beta_\zeta} : \zeta \in D \cap \zeta^* \rangle \in M_{\alpha+1},$$

which finishes the proof as $A_{\beta_\zeta} = X \cap \delta_\zeta$. $\dashv$

This claim establishes that $\mathrm{Trap}_{\bar{S}}(X)$ contains a closed unbounded subset of δ for all $\delta \in S \cap E$, and the theorem is proved. $\dashv$

We get the following as an immediate corollary.

4.30 Corollary. *Let μ be a strong limit singular cardinal with $2^\mu = \mu^+$. If $\square^*_\mu$ holds, then $\diamondsuit(S)$ holds for every stationary $S \subseteq \mu^+$ that reflects stationarily often in ordinals of cofinality greater than* $\mathrm{cf}(\mu)$. *In particular, the hypotheses imply $\diamondsuit(S^{\mu^+}_{\mathrm{cf}(\mu)})$.*

The following theorem of Shelah [85] gives a consistency result about the failure of $\diamondsuit$ in the situation we have been considering.

4.31 Theorem (see Conclusion 10 of [85]). *Suppose that μ is a singular strong limit cardinal with $2^\mu = \mu^+$, and S is a non-reflecting stationary subset of $\{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}$. Then there is a notion of forcing $\mathbb{P}$ such that*

1. *$\mathbb{P}$ adds no new sequences of length less than μ,*
2. *$\mathbb{P}$ satisfies the μ^+-chain condition,*
3. *$\mathbb{P}$ preserves the stationarity of S, and*
4. *$V^{\mathbb{P}} \models \neg\diamondsuit(S)$.*

Note that the above consistency result does not require large cardinals; in light of Corollary 4.30, it is not surprising that the techniques used to obtain the model of Theorem 4.31 cannot make $\diamondsuit$ fail on a stationary set that reflects stationarily often. It is also clear that ZFC still has things to say about this problem, for the following theorem of Shelah [93] shows us that non-trivial consequences of $\diamondsuit(S^{\mu^+}_{\mathrm{cf}(\mu)})$ follow from instances of the Generalized Continuum Hypothesis.

4.32 Theorem. *Assume $\lambda = \mu^+$, where μ is a strong limit singular cardinal. Further assume that $2^\mu = \lambda$, and $S \subseteq \lambda$ is a stationary subset of $S^\lambda_{\mathrm{cf}(\mu)}$. Then we can find a family $\{A_\delta : \delta \in S\}$ such that each A_δ is cofinal in δ of order-type* $\mathrm{cf}(\mu)$, *and, letting $\langle \alpha_{\delta,i} : i < \mathrm{cf}(\mu) \rangle$ enumerate A_δ in increasing order, we have that for every function f with domain λ and range a bounded subset of μ,*

$$\{\delta \in S : f(\alpha_{\delta,2i}) = f(\alpha_{\delta,2i+1}) \text{ for all } i < \mathrm{cf}(\mu)\} \text{ is stationary.}$$

Finally, we remark the Shelah has very recently established an important result on the existence of diamonds. Using pcf theory, he had actually shown the following equivalence:

4.33 Theorem. *If $\beth_\omega \leq \kappa$, then $\diamondsuit(\kappa^+)$ holds if and only if $2^\kappa = \kappa^+$.*

For a proof see Theorem 8.21 of the chapter of Abraham and Magidor in this Handbook. In his recent [80], Shelah used club guessing ideas in an ingenious, short proof to show that the $\beth_\omega$ can be replaced by ω_1, specifically:

4.34 Theorem (Shelah [80]). *If $\omega_1 \leq \kappa$, then $2^\kappa = \kappa^+$ implies that $\diamondsuit(S)$ holds for any stationary $S \subseteq \{\delta < \kappa^+ \mid \mathrm{cf}(\delta) \neq \mathrm{cf}(\kappa)\}$.*

Hence, $\diamondsuit_\kappa$ is actually equivalent to $2^\kappa = \kappa^+$ for all $\kappa \geq \omega_1$. This is optimal in light of the well-known independence result for $\diamondsuit(\omega_1)$ of Jensen, and moreover, the cofinality restriction of the theorem is optimal in light of Theorem 4.31. However, Theorem 4.24 is still germane because of the stronger conclusion of $\diamondsuit^*$; $\diamondsuit^*(\kappa^+)$ fails if a Cohen subset of κ^+ is added, and so one each easily get the consistency of $\diamondsuit(\kappa^+) + \neg\diamondsuit^*(\kappa^+)$.

4.4. Very Weak Square

The purpose of this brief subsection is to establish a connection between the ideas we have been considering and some combinatorial principles discussed in Todorčević's chapter [98] in this Handbook. If μ is singular, then we have seen how the landscape between $\square_\mu$ and AP_μ is filled by a natural hierarchy of square-like principles. In this section, we take a look at combinatorial principles even weaker than AP_μ (in fact, so weak that some of them are consistent with supercompact cardinals) that are still strong enough to have important consequences.

To set the stage, let $\lambda = \mu^+$ for a singular cardinal μ. In our investigation of $I[\lambda]$, we saw that this ideal is the normal ideal generated by sets of the form $S[\mathfrak{M}]$ for $\mathfrak{M}$ a λ-approximating sequence. Recall that $S[\mathfrak{M}]$ is defined to be the set of $\delta < \lambda$ such that

- $M_\delta \cap \lambda = \delta$, and
- there is a cofinal $a \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ with the property that every initial segment of a is M_δ.

In non-specific terms, the ordinal δ is "singularized" by a set a that is in some sense "captured by M_δ". If we vary the meaning of "captured by M_δ", we end up with a family of combinatorial principles related to $I[\lambda]$ and AP_μ.

For example, let us consider the following definition.

4.35 Definition. Suppose that $\lambda = \mu^+$ for some singular cardinal μ, and let $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda\rangle$ be a λ-approximating sequence. Let $S^{\mathrm{VWS}}[\mathfrak{M}]$ be the set of all $\delta < \lambda$ such that

1. $M_\delta \cap \lambda = \delta$, and
2. there is a cofinal $a \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$ such that all bounded countable subsets of a are elements of M_δ.

Some remarks are in order here. First, the designation "VWS" stands for *very weak square*, for reasons which will be made clear below. Second, if $\mathrm{cf}(\delta) > \aleph_0$ and $\delta \in S^{\mathrm{VWS}}[\mathfrak{M}]$, then it must be the case that $(\mathrm{cf}\,\delta)^{\aleph_0} \leq \mu$ because $|M_\delta| = \mu$. Thus, the preceding definition will be of most interest when μ is *countably closed*, that is, when $\tau^{\aleph_0} < \mu$ for all $\tau < \mu$.

Assume now that μ is a countably closed singular cardinal. The sets of the form $S^{\mathrm{VWS}}[\mathfrak{M}]$ generate a normal ideal on λ, which we denote by $I^{\mathrm{VWS}}[\lambda]$. The following proposition captures some of its basic properties.

4.36 Proposition. *Let $\lambda = \mu^+$ for μ a countably closed singular cardinal.*

1. $S^\lambda_{\aleph_0} \in I^{\mathrm{VWS}}[\lambda]$.
2. *If $S \subseteq S^\lambda_{\aleph_1}$, then $S \in I[\lambda] \iff S \in I^{\mathrm{VWS}}[\lambda]$.*
3. $I[\lambda] \subseteq I^{\mathrm{VWS}}[\lambda]$.
4. *If μ has uncountable cofinality, then $\lambda \in I^{\mathrm{VWS}}[\lambda]$.*

Proof. All are easy. For (4), notice that since $\mu^{<\mu} = \mu$, if $\langle M_\alpha : \alpha < \lambda\rangle$ is a λ-approximation sequence and δ has uncountable cofinality, then any countable subset of $M_\delta \cap \lambda$ is in M_δ. ⊣

The next proposition gives us a characterization of $I^{\mathrm{VWS}}[\lambda]$ that does not require the language of elementary submodels. The result is easy, and taken from [23].

4.37 Proposition. *Let $\lambda = \mu^+$ for μ a countably closed singular cardinal. A set $S \subseteq \lambda$ is in $I^{\mathrm{VWS}}[\lambda]$ if and only if there is a sequence $\langle A_\alpha : \alpha < \lambda\rangle$ such that $A_\alpha \subseteq \alpha$ and for some closed unbounded $E \subseteq \lambda$, if $\delta \in E \cap S$ then*

1. *A_δ is cofinal in δ of order-type $\mathrm{cf}(\delta)$, and*
2. *if $\mathrm{cf}(\delta) > \aleph_0$ then $[A_\delta]^{\aleph_0} \subseteq \{A_\alpha : \alpha < \delta\}$.*

Proof. See [23, Claim 2.8]. ⊣

Now we have seen that for singular cardinals μ, the combinatorial principle AP_μ has strength. Since AP_μ is just the statement that $I[\mu^+]$ is an improper ideal, it is natural to use $I^{\mathrm{VWS}}[\mu^+]$ in such a fashion as well. This is a fruitful idea, and it leads us to the *very weak square* principle studied previously by Foreman and Magidor [34].

4.38 Definition. Let $\lambda = \mu^+$ for a countably closed singular cardinal μ. A sequence $\langle A_\alpha : \alpha < \mu^+\rangle$ is a *very weak square sequence for* μ if there is a closed unbounded $E \subseteq \mu^+$ such that for all $\alpha \in E$ of uncountable cofinality,

1. A_α is unbounded in α, and
2. $[A_\alpha]^{\aleph_0} \subseteq \{A_\beta : \beta < \alpha\}$.

We say that *very weak square holds at* μ (denoted VWS_μ) if there is a very weak square sequence for μ.

The reader may note that the sets A_α are not required to be closed. This is quite important, and the change gives rise to a combinatorial principle known as *not-so-very-weak square* studied briefly by Foreman and Magidor in [34]. The paper [34] contains several applications of the very weak square principle. We limit ourselves to the following characterization of VWS_μ, which recalls the characterization of AP_μ given in Theorem 3.16. The proof is similar, and we remark that such arguments have been used many times in the literature, going back to work of Jensen and Silver in [52].

4.39 Theorem. *Let* μ *be a countably closed of countable cofinality, and let* $\lambda = \mu^+$. *Then* VWS_μ *holds if and only if for every* $x \in H(\chi)$, *there is a* λ*-approximating sequence* $\mathfrak{M} = \langle M_\alpha : \alpha < \lambda\rangle$ *over* x *with the property that if we are given* δ *satisfying* $M_\delta \cap \lambda = \delta$ *and a set* $A \subseteq M_\delta$ *satisfying* $\aleph_0 < |A| = \mathrm{cf}\,|A| < \mu$, *then there is* $B \subseteq A$ *such that*

1. $|B| = |A|$, *and*
2. $[B]^{\aleph_0} \subseteq M_\delta$.

Proof. We mirror the proof of Theorem 3.16. Let $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$ be a very weak square sequence for λ, with E the associated closed unbounded set. Let $\langle \mu_i : i < \omega\rangle$ be an increasing sequence of regular cardinals with limit μ such that $\mu_i^{\aleph_0} < \mu_{i+1}$.

Now build a matrix $\langle M^i_\alpha : \alpha < \lambda, i < \omega\rangle$ of elementary submodels of $H(\chi)$ such that (letting $M_\alpha := \bigcup_{i<\omega} M^i_\alpha$)

1. $\langle M_\alpha : \alpha < \lambda\rangle$ is a λ-approximating sequence over $\{x, \bar{C}, E\}$,
2. $|M^i_\alpha| = \mu_i$,
3. $M^i_\alpha \subseteq M^j_\alpha$ for $i < j$,
4. $\mu_{i+1} \cup [M^i_{\alpha+1}]^{\aleph_0} \subseteq M^{i+1}_{\alpha+1}$, and
5. if C_α is countable, then for each $i < \omega$,

$$\left[\bigcup_{\beta \in C_\alpha} M^i_\beta\right]^{\aleph_0} \subseteq M_{\alpha+1}. \tag{15.60}$$

Now let τ be an uncountable regular cardinal below μ and suppose that A is in $[M_\alpha]^\tau$ for some $\alpha < \lambda$. There are three cases that arise.

Case 1. α a successor ordinal.

In this case, we fix an $i < \omega$ for which $B := A \cap M^i_\alpha$ has cardinality τ. Our construction then guarantees that every countable subset of B is in M^{i+1}_α, which is a subset of M_α.

Case 2. α a limit ordinal with $\mathrm{cf}(\alpha) \neq \tau$.

For such an α, there must be a $\beta < \alpha$ for which $A \cap M_{\beta+1}$ has cardinality τ. The argument for Case 1 then gives us $B \in [A]^\tau$ with $[B]^{\aleph_0} \subseteq M_\alpha$.

Case 3. α a limit ordinal of cofinality τ.

In this situation, we know $M_\alpha = \bigcup_{i<\omega} \bigcup_{\gamma \in C_\alpha} M^i_\gamma$, and so there is an $i < \omega$ for which the set

$$B := A \cap \bigcup\nolimits_{\gamma \in C_\alpha} M^i_\gamma$$

has cardinality τ.

If K is a countable subset of B, then there is a countable $J \subseteq C_\alpha$ with

$$K \subseteq \bigcup\nolimits_{\gamma \in J} M^i_\gamma.$$

Since $\alpha \in E$, we know that J appears as C_β for some $\beta < \alpha$. But then

$$K \subseteq \bigcup\nolimits_{\gamma \in C_\beta} M^i_\gamma \subseteq M_{\beta+1} \subseteq M_\alpha.$$

Thus, B is a subset of A of cardinality τ with $[B]^{\aleph_0} \subseteq M_\alpha$, as required.

For the converse, let $\mathfrak{M}$ be a λ-approximating sequence with the properties claimed. We show that $S^{\mathrm{VWS}}[\mathfrak{M}]$ contains the closed unbounded set of ordinals δ with $\delta = M_\delta \cap \lambda$. Clearly we may assume $\mathrm{cf}(\delta)$ is uncountable, so let $A \subseteq \delta$ be cofinal of order-type $\mathrm{cf}(\delta)$. Then our choice of $\mathfrak{M}$ gives us a cofinal $B \subseteq A$, all of whose countable subsets are in M_δ. Thus, $\delta \in S^{\mathrm{VWS}}[\mathfrak{M}]$ and VWS_μ holds. $\dashv$

We will not pursue combinatorial applications of VWS_μ here. We remark that VWS_μ is equivalent to the existence of a Jensen matrix (see [98, Definition 11.14]) in many situations—one can find a proof of this in [34]; it is similar to the proof of the preceding theorem. Todorčević demonstrates the relationship between Jensen matrices and cofinal Kurepa families; the reader is referred to his chapter [98] (in particular, Sect. 11) in this Handbook for more information.

With regard to the relationship between VWS_μ and AP_μ, we mention the following theorem of Foreman and Magidor [34]:

4.40 Theorem. *Suppose that* GCH *holds and κ is a supercompact cardinal. Then there is a class forcing extension of V that preserves cardinals and cofinalities in which* VWS_μ *holds for all singular μ and where κ remains supercompact.*

Thus, through successive weakenings, we have finally arrived at versions of square that are compatible with supercompact cardinals. Oddly enough, even though modifications of Foreman and Magidor's argument proves that very weak square is consistent even with huge cardinals, the principle not-so-very-weak square mentioned earlier necessarily fails at the successor of a limit of infinitely many supercompact cardinals. Thus, the two principles are not equivalent. We will not pursue this, but the proof appears in [34]. Returning to the topic at hand, we note the following corollary of the preceding theorem.

4.41 Corollary. *If κ is supercompact, then it is consistent that* $\mathrm{VWS}_{\kappa^{+\omega}}$ *holds even though* $\mathrm{AP}_{\kappa^{+\omega}}$ *necessarily fails.*

The situation at $\aleph_\omega$ is a bit strange, though, for Shelah has shown that under GCH, the two principles are equivalent at $\aleph_\omega$. The following theorem from [23] gives a tighter formulation.

4.42 Theorem. *Let $\lambda = \mu^+$ for μ a singular strong limit cardinal of countable cofinality. If $\kappa < \mu$ is countably closed, then $I[\lambda] \restriction S^\lambda_\kappa = I^{\mathrm{VWS}} \restriction S^\lambda_\kappa$.*

Proof. By the third statement in Proposition 4.36, we need only take an $S \subseteq S^\lambda_\kappa$ in $I^{\mathrm{VWS}}[\lambda]$ and prove it is in $I[\lambda]$. By way of Proposition 4.37, we fix a sequence $\bar{A} = \langle A_\alpha : \alpha < \lambda\rangle$ and closed unbounded $E \subseteq \lambda$ such that for $\delta \in E \cap S$, we have

1. A_δ is cofinal in δ with $\mathrm{ot}(A_\delta) = \kappa$, and
2. $[A_\delta]^{\aleph_0} \subseteq \{A_\alpha : \alpha < \delta\}$.

By thinning out A_δ if necessary, we may assume as well that

3. for $\delta \in E \cap S$, if $\alpha \in A_\delta$ then $[A_\delta \cap \alpha]^{\aleph_0} \subseteq \{A_\gamma : \gamma < \alpha\}$.

Notice that the above makes use of the facts that κ is countably closed and A_δ need not be a closed subset of δ.

Let $\langle \mu_n : n < \omega\rangle$ be an increasing sequence of regular cardinals cofinal in μ. An easy induction lets us define a λ-filtration sequence $\bar{b} = \langle b_{\alpha,n} : n < \omega\rangle$ with $|b_{\alpha,n}| \le \mu_n$, and such that for all $\alpha < \lambda$

$$\beta \in b_{\alpha,n} \quad \Longrightarrow \quad b_{\beta,n} \subseteq b_{\alpha,n} \tag{15.61}$$

and

$$|A_\alpha| = \aleph_0 \quad \Longrightarrow \quad A_\alpha \subseteq b_{\alpha,0}. \tag{15.62}$$

Let $\mathfrak{M}$ be a λ-approximating sequence over $\{\bar{A}, S, E, \bar{B}\}$. We will show that $S \subseteq S[\mathfrak{M}]$.

With this in mind, suppose that $\delta \in S$ satisfies $\delta = M_\delta \cap \lambda$. Since E is in M_0, it is clear that δ must be in E as well. Our goal is to establish that every initial segment of A_δ is in the model M_δ. As a first step toward this, we claim the following:

$$\alpha \in A_\delta \quad \Longrightarrow \quad A_\delta \cap \alpha \subseteq b_{\alpha,n} \text{ for some } n < \omega. \tag{15.63}$$

Assume by way of contradiction that (15.63) fails. Then for each $i < \omega$ we can choose an ordinal β_i such that

$$\beta_i \in (A_\delta \cap \alpha) \setminus b_{\alpha,i}. \tag{15.64}$$

Since $\delta \in E \cap S$, by condition (3) above, there is an ordinal $\gamma < \alpha$ with $\{\beta_i : i < \omega\} = A_\gamma$.

Choose n such that $\gamma \in b_{\alpha,n}$. By (15.61), we know $b_{\gamma,n} \subseteq b_{\alpha,n}$. But

$$\{\beta_i : i < \omega\} = A_\gamma \subseteq b_{\gamma,0} \subseteq b_{\gamma,n} \subseteq b_{\alpha,n},$$

and this contradicts (15.64) for the choice $n = i$. Therefore, (15.63) must hold.

Now suppose that $\alpha \in A_\delta$. Choose an $n < \omega$ such that $A_\delta \cap \alpha \in b_{\alpha,n}$. The set $b_{\alpha,n}$ is in M_δ and of cardinality at most μ_n. Since μ is a strong limit cardinal, it follows that every subset of $b_{\alpha,n}$ is in M_δ. In particular, $A_\delta \cap \alpha \in M_\delta$ and the proof is finished. ⊣

4.43 Corollary. *If* GCH *holds, then* $I[\aleph_{\omega+1}] = I^{\mathrm{VWS}}[\aleph_{\omega+1}]$.

Proof. By assumption, $S^{\aleph_{\omega+1}}_{\aleph_0}$ lies in both ideals. By the second claim in Proposition 4.36, the ideals are the same when restricted to ordinals of cofinality $\aleph_1$. Under GCH, the cardinals $\aleph_n$ for $1 < n < \omega$ are all countably closed, and so the preceding theorem applies. It follows easily now that the ideals coincide. ⊣

4.5. NPT and Good Scales

This section deals with yet another failure of compactness of interest in combinatorial set theory.

4.44 Definition.

1. If A is a family of non-empty sets, then a *transversal* for A is a one-to-one choice function on A.

2. A family A of non-empty sets is said to be κ*-free* if every subfamily of cardinality less than κ has a transversal. We say that A is *free* if the entire family has a transversal.

3. $\mathrm{PT}(\kappa, \theta)$ means that every κ-free family $A = \langle A_\alpha : \alpha < \kappa \rangle$ with each A_α of size less than θ is free.

4. $\mathrm{NPT}(\kappa, \theta)$ denotes the negation of $\mathrm{PT}(\kappa, \theta)$.

We begin with an easy application of the compactness theorem for propositional logic.

4.45 Proposition. *An* ω*-free family of finite sets is free. Thus,* $\mathrm{PT}(\kappa, \aleph_0)$ *holds for any infinite* κ.

Proof. Suppose that $\{A_\alpha : \alpha < \kappa\}$ is an ω-free family of finite sets, and fix an enumeration of each A_α. For each $\alpha < \kappa$ and $i < |A_\alpha|$, we let $p_{\alpha,i}$ be a propositional variable and consider the theory consisting of the following propositional sentences:

1. $\bigvee_{i<|A_\alpha|} p_{\alpha,i}$ for each $\alpha < \kappa$,
2. $p_{\alpha,i} \to \neg p_{\alpha,j}$ for each $\alpha < \kappa$ and $i \neq j$, and
3. $p_{\alpha,i} \to \neg p_{\beta,j}$ whenever $\alpha \neq \beta$ and the ith element of A_α is the jth element of A_β.

Any valuation v satisfying this theory gives rise to a transversal F for the family $\{A_\alpha : \alpha < \kappa\}$—for each $\alpha < \kappa$ it follows from (1) and (2) that there is a unique $i < |A_\alpha|$ with $v(p_{\alpha,i}) = 1$, and so we can define a function F by the rule

$$F(\alpha) = \text{ the } i\text{th element of } A_\alpha \text{ for the unique } i \text{ with } v(p_{\alpha,i}) = 1.$$

Since our valuation satisfies the sentences from (3), it follows that F is one-to-one.

With the preceding interpretation in mind, it is clear that our collection of formulas is finitely satisfiable because the family $\{A_\alpha : \alpha < \kappa\}$ is ω-free. An application of the compactness theorem for propositional logic gives us a valuation satisfying the entire theory, and therefore the family has a transversal. ⊣

In the other direction, we show that instances of NPT arise from non-reflecting stationary sets.

4.46 Proposition. *Let $\kappa < \lambda$ be regular cardinals. If* $\mathrm{Refl}(S^\lambda_\kappa)$ *fails, then* $\mathrm{NPT}(\lambda, \kappa^+)$ *holds.*

Proof. Let $\kappa < \lambda$ be regular cardinals, and assume that there is a non-reflecting stationary set $S \subseteq S^\lambda_\kappa$. For each $\delta \in S$, let A_δ be a cofinal subset of δ of order-type κ. The family $\{A_\delta : \delta \in S\}$ is λ-free by Lemma 2.12, but the existence of a transversal would contradict Fodor's Theorem. ⊣

As a corollary, we see that $\mathrm{NPT}(\kappa^+, \kappa^+)$ holds for all regular κ. The following theorem of Milner and Shelah [67] shows that non-reflecting stationary sets can be used to lift examples of NPT to larger cardinals.

4.47 Theorem. *Let $\theta < \kappa < \lambda$ be regular cardinals. If* $\mathrm{NPT}(\kappa, \theta)$ *holds and there is a non-reflecting stationary $S \subseteq S^\lambda_\kappa$, then* $\mathrm{NPT}(\lambda, \theta)$ *holds as well.*

Proof. Let $\mathcal{A} = \{A_\gamma : \gamma < \kappa\}$ be an instance of $\mathrm{NPT}(\kappa, \theta)$. For each $\alpha \in S$ we choose an increasing function $e_\alpha : \kappa \to \alpha$ with range cofinal in α. Given $\alpha \in S$ and $\gamma < \kappa$, we define

$$B_{\alpha,\gamma} = (\{\alpha\} \times A_\gamma) \cup \{e_\alpha(\gamma)\}.$$

We claim that the family

$$\mathcal{B} = \{B_{\alpha,\gamma} : \alpha \in S, \gamma < \kappa\}$$

will witness $\mathrm{NPT}(\lambda, \theta)$.

First, we show that $\mathcal{B}$ does not have a transversal. Suppose by way of contradiction that the family $\mathcal{B}$ has a transversal F. Since $\mathcal{A}$ does not have a transversal, it follows that for each $\alpha \in S$ there must be a $\gamma_\alpha < \kappa$ with

$$F(\alpha, \gamma_\alpha) = e_\alpha(\gamma_\alpha).$$

An application of Fodor's Theorem gives us a stationary $T \subseteq S$ and $\gamma^* < \kappa$ such that $e_\alpha(\gamma_\alpha) = \gamma^*$ for all $\alpha \in T$, and this means that our F cannot be one-to-one.

To finish the proof, we must prove that the family $\mathcal{B}$ is λ-free. It suffices to prove that for each $\alpha < \lambda$, the family $\{B_{\beta,\gamma} : \beta \in S \cap \alpha, \gamma < \kappa\}$ has a transversal.

By Lemma 2.12, we can find for each $\beta \in S \cap \alpha$ an ordinal $\eta_\beta < \kappa$ so that all sets of the form $\{e_\beta(\gamma) : \eta_\beta < \gamma < \kappa\}$ for $\beta \in S \cap \alpha$ are disjoint. We then choose $F(\beta, \gamma)$ so that

$$F(\beta, \gamma) = e_\beta(\gamma) \quad \text{for } \eta_\beta < \gamma < \kappa$$

and such that

$$\langle F(\beta, \gamma) : \gamma \leq \eta_\beta \rangle \text{ is a transversal for } \{\beta\} \times \{A_\gamma : \gamma \leq \eta_\beta\}.$$

There are no problems in doing this, and evidently this gives the desired transversal. $\dashv$

4.48 Corollary. $\mathrm{NPT}(\aleph_n, \aleph_1)$ *holds for* $1 \leq n < \omega$.

Proof. We see that $\mathrm{NPT}(\aleph_1, \aleph_1)$ holds by way of Proposition 4.46, and a trivial induction argument using Theorem 4.47 lifts this to $\mathrm{NPT}(\aleph_n, \aleph_1)$ for $1 \leq n < \omega$. $\dashv$

The following corollary will be superseded by results later in the subsection, but we record it for later use.

4.49 Corollary. *If* $\mathrm{Refl}(\aleph_{\omega+1})$ *fails, then* $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ *holds.*

Proof. If $\mathrm{Refl}(\aleph_{\omega+1})$ fails, then $\mathrm{Refl}(S^{\aleph_{\omega+1}}_{\aleph_n})$ must fail for some $n < \omega$. The conclusion follows from the preceding corollary and Theorem 4.47. $\dashv$

For the rest of this subsection we focus our attention on $\mathrm{NPT}(\kappa, \theta)$ for the special case $\theta = \aleph_1$. Such questions are of particular interest, because Shelah [81] has shown that $\mathrm{NPT}(\kappa, \aleph_1)$ is equivalent to the existence of a κ-free Abelian group (that is, all subgroups of cardinality less than κ are free) of size κ that is not itself free. It is still an open problem whether the "Abelian" can be dropped in the preceding result—it is known that $\mathrm{NPT}(\kappa, \aleph_1)$ implies the existence of a κ-free non-free group of cardinality κ, but whether the reverse implication holds is still unsolved. We refer the reader to the introduction of [65] for a much more extensive discussion of such matters and how they fit into a much more general setting.

The following result of Shelah settles our question for the case of singular cardinals. The result is a special case of his theorem on singular compactness [81].

4.50 Theorem. $\mathrm{PT}(\kappa, \aleph_1)$ *holds for every singular* κ.

Proof. We refer the reader to [14, Sect. 12] for a nice proof of this result. $\dashv$

Thus, the behavior of $\aleph_\omega$ stands in contrast to that of $\aleph_n$ for $1 \leq n < \omega$. At successors of singular cardinals the situation is much more delicate. Magidor and Shelah [65] have shown that $\mathrm{NPT}(\aleph_{\omega \cdot n+1}, \aleph_1)$ holds for each $n < \omega$, while $\mathrm{PT}(\aleph_{\omega^2+1}, \aleph_1)$ is consistent relative to large cardinals. Their arguments make use of the combinatorics of scales, and though a presentation of the consistency of $\mathrm{PT}(\aleph_{\omega^2+1}, \aleph_1)$ is beyond the scope of the paper, we will develop the techniques needed to see the relationship between scales and transversals. The following definition is the key to unlocking this connection.

4.51 Definition. Let μ be a singular cardinal. A scale $(\vec{\mu}, \vec{f})$ is *good* if every ordinal α of cofinality greater than $\mathrm{cf}(\mu)$ is a good point for $\vec{f}$, that is, if $\alpha < \mu^+$ satisfies $\mathrm{cf}(\alpha) > \mathrm{cf}(\mu)$, there is an unbounded $A \subseteq \alpha$ and $i < \mathrm{cf}(\mu)$ such that the sequence $\langle f_\beta(j) : \beta \in A \rangle$ is strictly increasing for all $j > i$.

The following easy proposition shows that good scales follow from the Approachability Property.

4.52 Proposition. *Let* μ *be a singular cardinal. If* AP_μ *holds, then all scales* $(\vec{\mu}, \vec{f})$ *for* μ *are good.*

Proof. Suppose that $(\vec{\mu}, \vec{f})$ is a scale for μ, and let $\mathfrak{M}$ be a λ-approximating sequence over $(\vec{\mu}, \vec{f})$. Given $\delta \in S[\mathfrak{M}]$ with cofinality greater than $\mathrm{cf}(\mu)$, the proof of Claim 3.54 shows that δ is a good point for $\vec{f}$. We have assumed AP_μ, and therefore $S[\mathfrak{M}]$ contains a closed unbounded $E \subseteq \mu^+$. If we enumerate E as $\langle \alpha_i : i < \mu^+ \rangle$ and define $g_i = f_{\alpha_i}$, then it is not hard to show that $\vec{g}$ is a good scale. $\dashv$

We now move on to the application of goodness to questions about NPT. The following theorem is due to Magidor and Shelah [65], though our presentation is based on that in Cummings' survey [14].

4.53 Theorem. *If* μ *is a singular cardinal of countable cofinality and there is a good scale* $(\vec{\mu}, \vec{f})$ *for* μ*, then* $\mathrm{NPT}(\mu^+, \aleph_1)$.

Proof. Let S denote the set $S^{\mu^+}_{\aleph_1}$, and for each $\alpha \in S$, define

$$A_\alpha = \{(m, f_\alpha(m)) : m < \omega\}.$$

The following lemma captures a key property of the family $\{A_\alpha : \alpha \in S\}$.

4.54 Lemma. *For every $\alpha < \mu^+$, we can choose objects B_β and D_β for $\beta \in S \cap \alpha$ such that*

1. *B_β is a cofinite subset of A_β,*
2. *D_β is a closed unbounded subset of β, and*
3. *$\{B_\beta \times D_\beta : \beta \in S \cap \alpha\}$ is a disjoint family of sets.*

Before given the proof, we show that the lemma implies $\mathrm{NPT}(\mu^+, \aleph_1)$. We start by choosing, for each $\alpha \in S$, a closed unbounded $E_\alpha \subseteq \alpha$ of order-type ω_1. By Lemma 4.54, the collection $\{A_\alpha \times E_\alpha : \alpha \in S\}$ is an instance of $\mathrm{NPT}(\mu^+, \aleph_2)$, but we need to work harder to obtain a witness for $\mathrm{NPT}(\mu^+, \aleph_1)$.

To do this, choose for each limit $\delta < \omega_1$ a cofinal set C_δ of order-type ω. Let $\langle e_\alpha(\gamma) : \gamma < \omega_1 \rangle$ be the increasing enumeration of E_α. Finally, for $\alpha \in S$ and limit $\delta < \omega_1$, we define

$$B_{\alpha,\delta} = (C_\delta \times \{\alpha\}) \cup (A_\alpha \times \{e_\alpha(\delta)\})$$

and set

$$\mathcal{B} = \{B_{\alpha,\delta} : \alpha \in S,\ \delta \text{ countable limit}\}.$$

4.55 Claim. *The collection $\mathcal{B}$ is an instance of* $\mathrm{NPT}(\mu^+, \aleph_1)$.

Proof. It is clear that each $B_{\alpha,\delta}$ is countable. To see that there is no transversal for $\mathcal{B}$, assume by way of contradiction that F is a one-to-one choice function for the collection.

For a fixed $\alpha \in S$, Fodor's Theorem implies that $F(\alpha, \delta)$ is of the form $(x, e_\alpha(\delta))$ for almost all δ, and therefore we can find a countable limit ordinal δ_α such that

$$F(\alpha, \delta_\alpha) = (x_\alpha, e_\alpha(\delta_\alpha))$$

for some $x_\alpha \in A_\alpha$. There are only μ possibilities for x_α, and therefore we can find a stationary $S^* \subseteq S$ and a single $x^* \in \omega \times \mu$ such that

$$\alpha \in S^* \quad \Longrightarrow \quad x_\alpha = x^*.$$

Since $e_\alpha(\delta_\alpha) < \alpha$, another application of Fodor's Theorem contradicts our assumption that F is one-to-one. Thus, there can be no transversal for the family $\mathcal{B}$.

To see that $\mathcal{B}$ is μ^+-free, it suffices to show for each $\alpha < \mu^+$ that a transversal exists for the collection

$$\{B_{\beta,\delta} : \beta \in S \cap \alpha \text{ and } \delta < \omega_1 \text{ is a limit}\}. \tag{15.65}$$

With this end in mind, let us fix an $\alpha < \lambda$ and work toward defining a transversal F for the collection (15.65). Given $\beta \in S \cap \alpha$, the set of $\delta < \omega_1$ with $e_\beta(\delta) \notin D_\beta$ is nonstationary. An appeal to Lemma 2.12 gives us a transversal G_β for the collection $\{C_\delta : e_\beta(\delta) \notin D_\beta\}$, and we define $F(\beta, \delta)$ for such δ by

$$F(\beta, \delta) = (G_\beta(\delta), \beta).$$

It remains to define $F(\beta, \delta)$ for those δ with $e_\beta(\delta) \in D_\beta$. This is done by choosing $x_\beta \in B_\beta$ and defining

$$F(\beta, \delta) = (x_\beta, e_\beta(\delta)).$$

In either case, $F(\beta, \delta)$ is an element of $B_{\beta,\delta}$, and the proof that F is one-to-one is straightforward. $\dashv$

We now turn our attention to the heart of the matter—the proof of Lemma 4.54.

Proof of Lemma 4.54. We prove by induction on $\alpha < \mu^+$ that for any $\delta < \alpha$ we can find B_β and D_β for $\beta \in S \cap (\delta, \alpha]$ such that

1. B_β is a cofinite subset of A_β,
2. D_β is closed and unbounded in β,
3. $\{B_\beta \times D_\beta : \beta \in S \cap (\delta, \alpha]\}$ is a disjoint family, and
4. $D_\beta \cap \delta = \emptyset$.

If we do this, then Lemma 4.54 follows immediately. It is worth pointing out that in general,

$$(B \times D) \cap (B^* \times D^*) = \emptyset \iff \text{either } B \cap B^* = \emptyset \text{ or } D \cap D^* = \emptyset. \quad (15.66)$$

We will use the above observation implicitly in the proof.

Case 1. α is a successor.

Since $\alpha \notin S$, the result follows from the induction hypothesis applied to the predecessor of α.

Case 2. $\mathrm{cf}(\alpha) = \aleph_0$.

Given $\delta < \alpha$, let $\langle \epsilon_n : n < \omega\rangle$ enumerate a cofinal ω-sequence in α with $\epsilon_0 = \delta$. For each n, we apply our induction hypothesis to the ordinals $\epsilon_n < \epsilon_{n+1}$. This gives us B_β and D_β for each $\beta \in S \cap (\epsilon_n, \epsilon_{n+1}]$ with the additional property that $D_\beta \cap \epsilon_n = \emptyset$, and from this it follows that the entire family $\{B_\beta \times D_\beta : \beta \in S \cap (\delta, \alpha]\}$ is disjoint.

Case 3. $\mathrm{cf}(\alpha) = \aleph_1$.

Given $\delta < \alpha$, we proceed very much as in the previous case with the added complication that B_α and D_α must also be defined. Notice as well that it suffices to prove the result for the case where δ is a successor ordinal, so we may assume that $\delta \notin S$.

Let D_α be a closed unbounded subset of α of order-type ω_1 chosen so that $\min(D_\alpha) = \delta$ and $D_\alpha \cap S = \emptyset$. Let $\langle \epsilon_i : i < \omega_1\rangle$ be the increasing enumeration of D_α.

Given $i < \omega_1$, we apply our induction hypothesis to the ordinals $\epsilon_i < \epsilon_{i+1}$ to obtain B_β and D_β for all $\beta \in S \cap (\epsilon_i, \epsilon_{i+1}]$. Since removing an initial segment of D_β causes no harm, we may assume that D_β is completely contained in the interval $(\epsilon_i, \epsilon_{i+1})$. In particular, $D_\beta \cap D_\alpha = \emptyset$.

Since $D_\alpha \cap S = \emptyset$, the preceding paragraph defines B_β and D_β for all $\beta \in S \cap (\delta, \alpha)$, and the family $\{B_\beta \times D_\beta : \beta \in S \cap (\delta, \alpha)\}$ is disjoint. We have arranged things so that $D_\alpha \cap D_\beta = \emptyset$ for all $\beta \in S \cap (\delta, \alpha)$, and so if we define B_α to be A_α, we have what we need.

Case 4. $\mathrm{cf}(\alpha) > \aleph_1$.

This is the interesting case, for it is here that the goodness of our scale becomes important. In order to remove a bit of clutter, we omit reference to the ordinal δ (that is, we give the proof for $\delta = 0$) as the obvious modifications give the result in full generality.

We begin by recalling that α is a good point for our scale, and therefore we can find a cofinal $A \subseteq \alpha$ and an $i^* < \omega$ with the property that the sequence $\langle f_\beta(i) : \beta \in A\rangle$ is strictly increasing for all $i > i^*$.

Define C to be the set of $\beta < \alpha$ with $\sup(A \cap \beta) = \beta$. We say that $\beta \in S \cap \alpha$ is of *Type I* if β is not in $\mathrm{acc}(S)$, while members of $S \cap \mathrm{acc}(C)$ are said to be of *Type II*.

The choice of B_β and C_β for β of Type I is along the lines of what was done in previous cases—if $\epsilon < \gamma$ are consecutive elements of C, then our induction hypothesis gives us B_β and D_β for all $\beta \in S \cap (\epsilon, \gamma]$. Since we are free to discard initial segments of D_β, we can also assume that D_β is entirely contained in the interval (ϵ, γ).

The preceding paragraph defines B_β and D_β for all β of Type I, and it does so in such a fashion that $D_\beta \cap C = \emptyset$ for all β of Type I. We now turn our attention to the β of Type II.

For each $\beta \in S \cap \mathrm{acc}(C)$, let $D_\beta := C \cap \beta$. This guarantees that D_β and D_{β^*} are disjoint whenever β is of Type II and β^* is of Type I. We now show that it is possible to choose B_β for each β of Type II in such a way that each B_β is a cofinite subset of A_β, and the family $\{B_\beta : \beta \in S \cap \mathrm{acc}(C)\}$ is disjoint.

Thus, let β be a fixed ordinal of Type II. Recall that $i^* < \omega$ has the property that for each $i > i^*$, the sequence $\langle f_\eta(i) : \eta \in A\rangle$ is strictly increasing. Since $\beta \in S$, we know $\mathrm{cf}(\beta) = \omega_1$ and thus there exists an $m_\beta < \omega$ and a cofinal $E \subseteq A \cap \beta$ such that

$$f_\eta(i) < f_\beta(i) \quad \text{for } i > m_\beta \text{ and } \eta \in E.$$

By the choice of A, this means that

$$f_\eta(i) < f_\beta(i) \quad \text{for all } \eta \in A \cap \beta \text{ and } i > \max\{i^*, m_\beta\}.$$

Let $\epsilon(\beta)$ be the least member of A above β, and choose

$$n(\beta) > \max\{i^*, m_\beta\}$$

so large that

$$f_\beta(i) < f_{\epsilon(\beta)}(i) \quad \text{for all } i > n(\beta).$$

Finally, we define

$$B_\beta := \{(m, f_\beta(m)) : m > n(\beta)\}.$$

Note that B_β is a cofinite subset of A_β, and the following claim shows us that this choice of B_β has the desired property.

4.56 Claim. *The family* $\{B_\beta : \beta \in S \cap \operatorname{acc}(C)\}$ *is disjoint.*

Proof. By way of contradiction, assume that $\beta < \beta^*$ are two elements of $S \cap \operatorname{acc}(C)$ for which $B_\beta \cap B_{\beta^*} \neq \emptyset$. A point in this intersection must have the form (m, η) for some $m > \max\{n(\beta), n(\beta^*)\}$ with

$$\eta = f_\beta(m) = f_{\beta^*}(m). \tag{15.67}$$

Because $\epsilon(\beta)$ is in $A \cap \beta^*$, it follows that

$$f_\beta(m) < f_{\epsilon(\beta)}(m) < f_{\beta^*}(m). \tag{15.68}$$

The statements (15.67) and (15.68) are contradictory, therefore B_β and B_{β^*} are disjoint. ⊣

Thus, we have defined B_β and D_β for all $\beta \in S \cap \alpha$ in the case where $\operatorname{cf}(\alpha) > \aleph_1$. Using our observation (15.66), it is straightforward to verify that $\{B_\beta \times D_\beta : \beta \in S \cap \alpha\}$ is a disjoint collection. ⊣

This completes the proof of Lemma 4.54, and Theorem 4.53. ⊣

4.57 Corollary. AP_μ *implies* $\mathrm{NPT}(\mu^+, \aleph_1)$ *for singular* μ *of countable cofinality.*

We are in a position to deduce the following corollary (essentially a restatement of one of the main results of [83]), but we shall soon see that the strong limit assumption is unnecessary.

4.58 Corollary. *If* $\aleph_\omega$ *is a strong limit, then* $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ *holds.*

Proof. Since $\aleph_\omega$ is assume to be a strong limit, Corollary 3.41 implies that either $\mathrm{AP}_{\aleph_\omega}$ holds or $\mathrm{Refl}(\aleph_{\omega+1})$ fails. In the former case, the result follows from Corollary 4.57, while in the latter case, the result follows from Corollary 4.49. ⊣

Our attention now focuses on a result of Magidor and Shelah [65] stating that $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ always holds, whether or not $\aleph_\omega$ is a strong limit. This will give us the opportunity to delve a bit more deeply into the structure of scales, and the key result we use is *Shelah's Trichotomy Theorem.*

4.59 Theorem (Shelah's Trichotomy Theorem). *Let I be an ideal on the set A, and let $\langle f_\alpha : \alpha < \delta\rangle$ be a $<_I$-increasing sequence with* $\operatorname{cf}(\delta) > |A|^+$. *Then exactly one of the following possibilities holds:*

1. *There is an exact upper bound h such that $\mathrm{cf}(h(x)) > |A|$ for all $x \in A$.*

2. *There is an ultrafilter U on A disjoint to I, and for each $x \in A$ a set S_x of cardinality at most $|A|$ such that some subfamily of $\prod_{x \in A} S_x$ is cofinally interleaved with $\langle f_\alpha : \alpha < \delta \rangle$ modulo U.*

3. *There is a function h such that the sequence*

$$\langle \{x \in A : f_\alpha(x) < h(x)\} : \alpha < \delta \rangle$$

is not eventually constant modulo I.

The preceding theorem is central in some presentations of the theory of exact upper bounds, for example, that found in Kojman [55, 56]. In Shelah [89], it appears as Claim 1.2 on page 41, while in Abraham and Magidor [1], one finds it as Exercise 2.27. One sometimes sees the three alternatives presented under the names good, bad, and ugly; we have stayed away from this because "good" conflicts with the notion of "good point" (they are related, but not the same thing), and even in [65] the adjective "good" is used for yet another related concept.

We will not prove the Trichotomy Theorem, but we do make a couple of observations before moving on to the proof of $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$. First, we remark that in the situation of the Trichotomy Theorem, if $\mathrm{cf}(\delta) > 2^{|A|}$ then the first alternative necessarily holds by cardinality considerations. Second, if the second (respectively, third) alternative holds for the sequence $\langle f_\alpha : \alpha < \delta \rangle$ then there is a closed unbounded $C \subseteq \delta$ such that the second (respectively, third) alternative holds for all initial segments $\langle f_\alpha : \alpha < \epsilon \rangle$ of our sequence where ϵ is in C and $\mathrm{cf}(\epsilon) > |A|$.

We now turn our attention to the special case of scales for $\aleph_\omega$. To simplify our notation a bit, we note that for such scales it makes sense to speak of a pair $(A, \vec{f})$ where $A \subseteq \omega$ instead of $(\vec{\mu}, \vec{f})$, that is, we view $\vec{f}$ as a $<^*$-increasing sequence of functions in $\prod_{n \in A} \aleph_n$ instead of re-indexing and dealing with a sequence $\langle \mu_i : i < \omega \rangle$.

4.60 Lemma. *Let $(A, \vec{f})$ be a scale for $\aleph_\omega$, and let $\delta < \aleph_{\omega+1}$ be an ordinal of uncountable cofinality for which the sequence $\langle f_\alpha : \alpha < \delta \rangle$ has an exact upper bound g with $\mathrm{cf}(g(n)) > \aleph_0$ for all $n \in A$ (that is, the first alternative of the Trichotomy Theorem holds at δ). Then δ is a good point for $\vec{f}$.*

Proof. By Theorem 3.50, it suffices to prove that $\mathrm{cf}(g(n)) = \mathrm{cf}(\delta)$ for all sufficiently large $n \in A$. Assume by way of contradiction that this fails, so $\mathrm{cf}(g(n)) \neq \mathrm{cf}(\delta)$ for unboundedly many $n \in A$. We split into two cases:

Case 1. $\mathrm{cf}(\delta) < \mathrm{cf}(g(n))$ for unboundedly many $n \in A$.

Let $B \subseteq A$ be the set of n for which $\mathrm{cf}(\delta) < \mathrm{cf}(g(n))$. Let $\langle e(\epsilon) : \epsilon < \mathrm{cf}(\delta) \rangle$ enumerate an unbounded subset of δ. Since $\mathrm{cf}(\delta) > \aleph_0$, we may assume that there is a fixed n^* such that

$$f_{e(\epsilon)}(n) < g(n) \quad \text{for all } \epsilon < \mathrm{cf}(\delta) \text{ and } n \in A \setminus n^*.$$

Now define $h \in \prod_{n \in A} \aleph_n$ by

$$h(n) = \begin{cases} \sup\{f_{e(\epsilon)}(n) : \epsilon < \mathrm{cf}(\delta)\} & \text{if } n \in B \setminus n^*, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $h < g$, so there must exist an $\epsilon < \mathrm{cf}(\delta)$ with $h <^* f_{e(\epsilon)}$. Clearly this is absurd.

Case 2. $\aleph_0 < \mathrm{cf}(g(n)) < \mathrm{cf}(\delta)$ for unboundedly many $n \in A$.

Let B be the set of $n \in A$ for which $\aleph_0 < \mathrm{cf}(g(n)) < \mathrm{cf}(\delta)$. There are only finitely many infinite cardinals below $\mathrm{cf}(\delta)$, and so by thinning out B, we may assume that there is a regular cardinal $\kappa < \mathrm{cf}(\delta)$ such that

$$\aleph_0 < \mathrm{cf}(g(n)) = \kappa < \mathrm{cf}(\delta)$$

for all $n \in B$.

For each $n \in B$, we let $\langle e_n(\epsilon) : \epsilon < \kappa \rangle$ be the increasing enumeration of a cofinal subset of $g(n)$ and define $g_\epsilon \in \prod_{n \in B} \aleph_n$ by

$$g_\epsilon(n) = e_n(\epsilon).$$

The sequence $\langle g_\epsilon : \epsilon < \kappa \rangle$ is increasing, and cofinally interleaved with $\langle f_\alpha \restriction B : \alpha < \delta \rangle$. This is impossible, because $\mathrm{cf}(\delta) \neq \kappa$. ⊣

We remark that it is possible for an exact upper bound to exist at a non-good point in a scale (we will see this phenomenon in the proof of Theorem 4.63 below), so the first alternative in the Trichotomy Theorem is not the same as "goodness" in general.

The following corollary translates some of our prior observations into the situation at hand.

4.61 Corollary. *Let $(A, \vec{f})$ be a scale for $\aleph_\omega$.*

1. *If $2^{\aleph_0} < \aleph_\omega$, then all ordinals $\delta < \aleph_{\omega+1}$ with $\mathrm{cf}(\delta) > 2^{\aleph_0}$ are good.*

2. *Let S be the set of good points for $\vec{f}$ of cofinality $\aleph_1$. If δ is not good and $\mathrm{cf}(\delta) > \aleph_1$, then $S \cap \delta$ is a nonstationary subset of δ.*

Proof. The first part follows because as we noted above, the first alternative of the Trichotomy must hold for the sequence $\langle f_\alpha : \alpha < \delta \rangle$, and by Lemma 4.60 this means that δ is a good point for the scale. The second statement of the corollary is just a reformulation of one of the observations made after the statement of Theorem 4.59 in light of the information given by Lemma 4.60. ⊣

We now come to the promised result of Magidor and Shelah [65] that lets us drop the strong limit condition from the statement of Corollary 4.58.

4.62 Theorem. $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ *holds.*

Proof. Let $(A, \vec{f})$ be a scale for $\aleph_\omega$. The construction is essentially the same as that given in the proof of Theorem 4.53, so we will only outline the required changes. (A full proof of a much more general theorem can be found in the first section of [65].)

The first change is to replace $S^{\aleph_{\omega+1}}_{\aleph_1}$ by the set S of good points for $\vec{f}$ of cofinality $\aleph_1$. The proof of Lemma 4.54 goes through for this choice of S, but now the fourth case (where $\mathrm{cf}(\alpha) > \aleph_1$) splits into two subcases, depending on whether or not α is a good point for the scale.

If α is good, then the same proof works. If α is not good, then by an appeal to Corollary 4.61 we find a closed unbounded $C \subseteq \alpha$ such that $C \cap S = \emptyset$, and an argument as in the second and third cases of Theorem 4.53 completes the proof. ⊣

We finish this section by investigating situations where no good scales exist. Since AP_μ implies that all scales for μ are good, it is clear that large cardinals must be involved. It should be no surprise by now that supercompact cardinals are the place to look, and we have the following result of Shelah [89].

4.63 Theorem. *Suppose that κ is supercompact and μ is a singular cardinal such that $\mathrm{cf}(\mu) < \kappa < \mu$. Then there is no good scale for μ.*

Proof. Let $(\vec{\mu}, \vec{f})$ be a scale for μ, and let $j : V \to M$ be a μ^+-supercompact embedding with $\mathrm{crit}(j) = \kappa$. Let

$$j(\vec{\mu}) = \langle \mu^j_i : i < \mathrm{cf}(\mu)\rangle,$$

and

$$j(\vec{f}) = \langle f^j_\alpha : \alpha < j(\mu^+)\rangle.$$

We know that $(j(\vec{\mu}), j(\vec{f}))$ is a scale for $j(\mu)$.

As usual, let $\rho = \sup\{j(\alpha) : \alpha < \mu^+\}$ (which is less than $j(\mu^+)$). We claim that ρ is not a good point for $j(\vec{f})$; we do this by proving that the sequence $\langle f^j_\alpha : \alpha < \rho\rangle$ has an exact upper bound g with the property that $\mathrm{cf}(g(i)) = \mu_i$ for all $i < \mathrm{cf}(\mu)$. Since exact upper bounds are (modulo the ideal) unique, it follows from Theorem 3.50 that ρ is not a good point for $j(\vec{f})$.

It is clear that if we define

$$g(i) = \sup\{j(\epsilon) : \epsilon < \mu_i\},$$

then g is an upper bound for $\langle f^j_\alpha : \alpha < \rho\rangle$; we claim that g is indeed an exact upper bound for $\langle f^j_\alpha : \alpha < \rho\rangle$.

Given $h <^* g$ in $\prod_{i<\mathrm{cf}(\mu)} j(\mu_i)$, we can find $h' \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$ such that $h <^* j(h')$. Since $(\vec{\mu}, \vec{f})$ is a scale, there is an $\alpha < \mu^+$ such that $h' <^* f_\alpha$, hence

$$h <^* j(f_\alpha) = f^j_{j(\alpha)}.$$

Since $j(\alpha) < \rho$, we are done.

To finish the proof, we apply the elementarity of j to conclude that $(\vec{\mu}, \vec{f})$ is not a good scale. $\dashv$

We note that essentially the same proof we gave for the consistency of $\neg\, \mathrm{AP}_{\aleph_\omega}$ yields the consistency of there being no good scales for $\aleph_\omega$—a reader seeking more details can find them in Sect. 18 of Cummings' survey [14]. Magidor [64] has shown that Martin's Maximum implies that no scale for $\aleph_\omega$ is good (again, [14] contains a proof), while Foreman and Magidor [34] give a proof that there is no good scale for $\aleph_\omega$ if the model-theoretic transfer principle

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$$

holds. This version of Chang's Conjecture was proved consistent by Levinski, Magidor, and Shelah—see [61].

4.6. Varieties of Nice Scales

The previous section illustrated that for a singular cardinal μ, the existence of a "well-behaved" scale can function as a construction principle along the lines of $\square_\mu$ or AP_μ. We continue with this theme in the current section by considering scales with even stronger properties and analyzing the consequences of their existence. We begin with a strengthening of goodness studied by Cummings, Foreman, and Magidor in [16].

4.64 Definition. Let $(\vec{\mu}, \vec{f})$ be a scale for the singular cardinal μ. We say $(\vec{\mu}, \vec{f})$ is a *very good scale for* μ if for each $\alpha < \mu^+$ with $\mathrm{cf}(\alpha) > \mathrm{cf}(\mu)$, there are a closed unbounded $C \subseteq \alpha$ and $i < \mathrm{cf}(\mu)$ such that

$$f_\beta(j) < f_\gamma(j) \quad \text{for all } \beta < \gamma \text{ in } C \text{ and all } j > i. \tag{15.69}$$

We let VGS_μ abbreviate the statement that a very good scale for μ exists.

It is clear that a very good scale is also good, and the next few results (taken from [16]) show us that very good scales fit into the hierarchy of combinatorial principles we have been considering in this chapter.

4.65 Theorem. *Suppose that μ is singular and $\square^\sigma_\mu$ holds for some $\sigma < \mu$. Then VGS_μ holds.*

Proof. Let $(\vec{\mu}, \vec{f})$ be any scale for μ, and let $\langle \mathcal{C}_\alpha : \alpha < \mu^+\rangle$ be a $\square^\sigma_\mu$ sequence with the property that all members of each $\mathcal{C}_\alpha$ are of order-type less than μ (see Lemma 4.21). We also assume without loss of generality that σ is less than μ_0, the first cardinal in the sequence $\vec{\mu}$.

Build a sequence $\vec{g} = \langle g_\alpha : \alpha < \mu^+\rangle$ of functions in $\prod_{i<\mathrm{cf}(\mu)} \mu_i$ according to the following recipe:

$$g_0 = f_0,$$
$$g_{\alpha+1}(i) = \max\{f_{\alpha+1}(i), g_\alpha(i)\}.$$

For α limit, we choose $g_\alpha \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$ such that

1. $f_\alpha < g_\alpha$,
2. $g_\gamma <^* g_\alpha$ for all $\gamma < \alpha$, and
3. $g_\alpha(i) > \sup\big(\{\sup_{\beta \in C} g_\beta(i) : C \in \mathcal{C}_\alpha, |C| < \mu_i\}\big)$.

There are no obstacles in the above construction, and clearly $(\vec{\mu}, \vec{g})$ is a scale for μ. Given a limit ordinal $\alpha < \mu^+$ such that $\mathrm{cf}(\alpha) > \mathrm{cf}(\mu)$, we can choose $C \in \mathcal{C}_\alpha$ and $i < \mathrm{cf}(\mu)$ such that $|C| < \mu_i$. If $j \geq i$ and $\beta < \gamma$ in C, then $C \cap \beta \in \mathcal{C}_\beta$, $\gamma \in C \cap \beta$, and $|C \cap \beta| < \mu_j$. This means $g_\gamma(j) < g_\beta(j)$, as required. $\dashv$

4.66 Theorem. *Suppose that μ is singular, and* VGS_μ *holds. Then for every stationary $S \subseteq \mu^+$, there is a sequence $\langle S_i : i < \mathrm{cf}(\mu)\rangle$ of stationary subsets of S with the property that for all $\delta < \mu^+$ of cofinality greater than* $\mathrm{cf}(\mu)$*,*

$$|\{i < \mathrm{cf}(\mu) : S_i \cap \delta \textit{ is stationary in } \delta\}| < \mathrm{cf}(\mu).$$

Proof. Let $(\vec{\mu}, \vec{f})$ be a very good scale for μ, and let $S \subseteq \mu^+$ be stationary. For each $i < \mathrm{cf}(\mu)$, we can find $\gamma_i < \mu_i$ and a stationary $S_i \subseteq S$ such that

$$\alpha \in S_i \quad \Longrightarrow \quad f_\alpha(i) = \gamma_i. \tag{15.70}$$

By way of contradiction, suppose that we can find an ordinal $\delta < \mu^+$ such that $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$ and $S_i \cap \delta$ is stationary in δ for unboundedly many $i < \mathrm{cf}(\mu)$. Since $\vec{f}$ is a very good scale, there is a closed unbounded $C \subseteq \delta$ and an $i < \mathrm{cf}(\mu)$ such that $f_\alpha(j) < f_\beta(j)$ for all $\alpha < \beta$ in C and all $j > i$.

If we choose $j > i$ with $S_j \cap \delta$ stationary in δ, then we can find $\alpha < \beta$ in $C \cap S_j$. By choice of C and i, it must be the case that $f_\alpha(j) < f_\beta(j)$. On the other hand, the choice of S_j implies $f_\alpha(j) = f_\beta(j)$, and so we have a contradiction. $\dashv$

The preceding two theorems now yield the proof of Theorem 4.18, which states that the conclusion of Theorem 4.66 follows if $\square^\sigma_\mu$ holds for some $\sigma < \mu$. Before leaving the topic of very good scales, we mention two corollaries of the preceding theorem.

4.67 Corollary. *If* $\mathrm{VGS}_{\aleph_\omega}$ *holds, then there is a family $\{S_n : n < \omega\}$ of stationary subsets of $\aleph_{\omega+1}$ with the property that no infinite subfamily can reflect simultaneously.*

4.68 Corollary. $\mathrm{VGS}_{\aleph_\omega}$ *does not follow from* $\square^*_{\aleph_\omega}$.

Proof. This follows from Corollary 4.67 and Theorem 4.12. $\dashv$

We mention that question of whether VGS_μ follows from $\square_\mu^{<\mu}$ is still open. The result of Cummings in Theorem 4.20 shows us that $\square_\mu^{<\mu}$ has similar consequences with regard to stationary reflection.

We have seen that for singular μ, AP_μ implies the existence of good scales, while $\square_\mu^\sigma$ for some $\sigma < \mu$ implies the existence of very good scales. Taken together with Corollary 4.68, these facts suggest the question of whether one can formulate a weaker version of VGS_μ that will be a consequence of $\square_\mu^*$, yet strong enough to have interesting consequences. Again the paper [16] provides us with a satisfying answer.

4.69 Definition. A scale $(\vec{\mu}, \vec{f})$ for a singular cardinal μ is *better* if for every $\alpha < \mu^+$ with $\mathrm{cf}(\alpha) > \mathrm{cf}(\mu)$, there is a closed unbounded $C \subseteq \alpha$ such that

1. $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$, and
2. for all $\beta \in C$, there is an $i < \mathrm{cf}(\mu)$ such that $f_\gamma(j) < f_\beta(j)$ for all $j > i$ and $\gamma \in C \cap \beta$.

We note that ordinals $\alpha < \mu^+$ with $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\mu)$ automatically enjoy the "betterness" property of the preceding definition—we will need this fact in the proof of Theorem 4.72 given below.

As noted by the authors of [16], better scales can be constructed from weak square.

4.70 Theorem. *Let μ be a singular cardinal. If $\square_\mu^*$ holds, then there is a better scale for μ.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$ be a $\square_\mu^*$ sequence with the property that each $\mathcal{C}_\alpha$ contains a set of order-type $\mathrm{cf}(\alpha)$. Given a scale $(\vec{\mu}, \vec{g})$ for μ, we construct a new scale $(\vec{\mu}, \vec{f})$ by induction on $\alpha < \mu^+$:

Case 1. $\alpha = 0$.
In this case, we simply define f_0 to be g_0.

Case 2. α is a successor.
In this case, we define $f_{\beta+1} = g_\gamma$ where γ is chosen so large that $f_\beta <^* g_\gamma$.

Case 3. α is a limit.
If α is a limit ordinal, then for each $C \in \mathcal{C}_\alpha$ we define a function f_C by

$$f_C(i) = \begin{cases} \sup_{\gamma \in C} f_\gamma(i) & \text{if } \mathrm{ot}(C) < \mu_i, \\ 0 & \text{otherwise.} \end{cases} \tag{15.71}$$

We let $f_\alpha = g_\gamma$ for $\gamma < \mu^+$ so large that

- $f_C <^* g_\gamma$ for all $C \in \mathcal{C}_\alpha$, and
- $f_\beta <^* g_\gamma$ for all $\beta < \alpha$.

This is possible because $\prod_{i < \mathrm{cf}(\mu)} \mu_i$ is μ^+-directed under $<^*$ and $|\mathcal{C}_\alpha| \leq \mu$.

The above construction produces a $<^*$-increasing sequence $\vec{f}$, and clearly

$(\vec{\mu}, \vec{f})$ is a scale for μ. It remains to verify that $(\vec{\mu}, \vec{f})$ is a better scale.

Given $\delta < \mu$ with $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$, we fix a $C \in \mathcal{C}_\delta$ with $\mathrm{ot}(C) < \mu$. If $\alpha \in \mathrm{acc}(C)$, then $C \cap \alpha \in \mathcal{C}_\alpha$ and by our construction, we have

$$f_{C\cap\alpha} <^* f_\alpha.$$

Since $\mu_i > \mathrm{ot}(C \cap \alpha)$ for all sufficiently large $i < \mathrm{cf}(\mu)$, there is an $i < \mathrm{cf}(\mu)$ such that

$$f_{C\cap\alpha}(j) = \sup_{\gamma \in C\cap\alpha} f_\gamma(j) \quad \text{for all } j > i.$$

Now let D be any closed unbounded subset of $\mathrm{acc}(C)$ with $\mathrm{ot}(D) = \mathrm{cf}(\delta)$; it follows easily that for all $\alpha \in D$ there is an $i < \mathrm{cf}(\mu)$ such that $f_\beta(j) < f_\alpha(j)$ whenever $\beta \in C \cap \alpha$ and $j > i$. ⊣

We now turn to the topic of the utility of better scales as a construction principle. The following concept (studied first by Shelah in [84]) will help us give show that better scales are indeed quite useful objects.

4.71 Definition. Let κ be a cardinal. A sequence $\langle A_\alpha : \alpha < \kappa^+\rangle$ is an ADS-*sequence for* κ if each A_α is a unbounded subset of κ, and for every $\beta < \kappa^+$ there is a function $F_\beta : \beta \to \kappa$ such that the sets $\langle A_\alpha \setminus F_\beta(\alpha) : \alpha < \beta\rangle$ are pairwise disjoint. We say that ADS_κ *holds* if there is an ADS-sequence for κ.

The designation ADS stands for "almost disjoint sets". Note as well that such sequences always exist if κ is regular, as any collection of κ^+ almost disjoint subsets of κ gives rise to an ADS-sequence for κ. On the other hand, if μ is singular and $\langle A_\alpha : \alpha < \mu^+\rangle$ is an ADS_μ-sequence, then we get an example of $\mathrm{NPT}(\mu^+, (\mathrm{cf}\,\mu)^+)$ by choosing for each $\alpha < \mu^+$ an unbounded subset of A_α of order-type $\mathrm{cf}(\mu)$. It follows that ADS_μ does not automatically hold in the case of singular μ, but the following result from [16] shows that the existence of a better scale for μ is strong enough to imply that it does.

4.72 Theorem. *If there is a better scale for* μ*, then* ADS_μ *holds.*

Proof. Let $(\vec{\mu}, \vec{f})$ be a better scale. Without loss of generality, we assume $\mathrm{cf}(\mu) < \mu_0$ and

$$\sup_{j<i} \mu_j < f_\alpha(i) \quad \text{for all } \alpha < \mu^+ \text{ and } i < \mathrm{cf}(\mu). \tag{15.72}$$

We claim that $\langle \mathrm{ran}(f_\alpha) : \alpha < \mu^+\rangle$ is an ADS-sequence for μ. To see this, we first prove the following statement by induction on $\alpha < \mu^+$:

$\oplus_\alpha$ There is function $F_\alpha : \alpha \to \mathrm{cf}(\mu)$ such that if $\gamma < \beta < \alpha$ and $i > \max\{F_\alpha(\gamma), F_\alpha(\beta)\}$, then $f_\gamma(i) < f_\beta(i)$.

Once we have this, it is not difficult to finish the proof. To see this, fix $\alpha < \mu^+$ and let F_α be a function as in $\oplus_\alpha$. For $\beta < \alpha$, define

$$B_\beta := \{f_\beta(i) : i > F_\alpha(\beta)\}.$$

If $\gamma < \beta < \alpha$ and $x \in B_\gamma \cap B_\beta$, then

$$x = f_\gamma(i) = f_\beta(j)$$

for some $i > F_\alpha(\gamma)$ and $j > F_\alpha(\beta)$. Because of condition (15.72), it must be the case that $i = j$ and therefore

$$i \geq \max\{F_\alpha(\gamma), F_\alpha(\beta)\}.$$

This gives us a contradiction, as $\oplus_\alpha$ implies

$$f_\gamma(i) < f_\beta(i),$$

so we conclude that $\langle B_\beta : \beta < \alpha\rangle$ is a disjoint family.

The proof of $\oplus_\alpha$ is by induction on α, with the case $\alpha = 0$ being trivial.

Case 1. $\alpha = \beta + 1$.

Let $F_\beta : \beta \to \mathrm{cf}(\mu)$ witness $\oplus_\beta$. We define $F_\alpha(\beta)$ to be 0, and for $\gamma < \beta$ we choose $F_\alpha(\gamma) < \mathrm{cf}(\mu)$ so large that

- $F_\alpha(\gamma) \geq F_\beta(\gamma)$, and
- $f_\gamma(i) < f_\beta(i)$ for all $i > F_\alpha(\gamma)$

It is straightforward to verify that F_α satisfies the demands of $\oplus_\alpha$.

Case 2. α is a limit.

Since $(\vec{\mu}, \vec{f})$ is a better scale, there is a closed unbounded $C \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha)$ such that for $\beta \in C$, there is an $i < \mathrm{cf}(\mu)$ with $f_\gamma(j) < f_\beta(j)$ for all $\gamma \in C \cap \beta$ and $j > i$.

For each $\delta \in \mathrm{nacc}(C)$, let F_δ be a function with the properties required by $\oplus_\delta$. If $\gamma < \alpha$, then γ lies in an interval of the form $[\sup(C \cap \delta), \delta)$ for some unique $\delta \in \mathrm{nacc}(C)$. Let $\epsilon = \sup(C \cap \delta)$, and note that $\epsilon \in C$ except for the case where $\delta = \min(C)$.

We now define $F_\alpha(\gamma) < \mathrm{cf}(\mu)$ to be so large that

1. $F_\alpha(\gamma) \geq F_\delta(\gamma)$,
2. $f_\epsilon(i) \leq f_\gamma(i) < f_\delta(i)$ for all $i > F_\alpha(\gamma)$, and
3. $f_{\epsilon^*}(i) < f_\epsilon(i)$ for all $\epsilon^* \in C \cap \epsilon$ and $i > F_\alpha(\gamma)$.

Note that condition (3) is where our choice of C is important, and the conjunction of (2) and (3) guarantees

$$f_\zeta(i) \leq f_\gamma(i) \quad \text{for all } \zeta \in C \cap \gamma \text{ and } i > F_\alpha(\gamma). \tag{15.73}$$

All that remains is to prove that the function F_α defined above satisfies all the requirements of $\oplus_\alpha$. Given $\gamma < \beta < \alpha$, we need to prove

$$i > \max\{F_\alpha(\gamma), F_\alpha(\beta)\} \quad \Longrightarrow \quad f_\gamma(i) < f_\beta(i). \tag{15.74}$$

We break the verification of (15.74) into two subcases.

Subcase 1. There is a $\delta \in \mathrm{nacc}(C)$ such that $\sup(C \cap \delta) \leq \gamma < \beta < \delta$.

Our construction guarantees that $F_\alpha(\gamma) \geq F_\delta(\gamma)$ and $F_\alpha(\beta) \geq F_\delta(\beta)$. Since F_δ satisfies the requirements of $\oplus_\delta$, (15.74) follows immediately.

Subcase 2. Subcase 1 fails.

In this situation, there must be ordinals $\delta^* < \delta$ in $\mathrm{nacc}(C)$ such that

$$\gamma < \delta^* \leq \sup(C \cap \delta) \leq \beta < \delta.$$

If $i > F_\alpha(\gamma)$, then $f_\gamma(i) < f_{\delta^*}(i)$ by requirement (2) in our definition of $F_\alpha(\gamma)$. If $i > F_\alpha(\beta)$, then an appeal to (15.73) tells us that $f_{\delta^*}(i) \leq f_\beta(i)$. Thus, (15.74) holds for this subcase as well.

Now that we have established $\oplus_\alpha$ for all $\alpha < \mu^+$, the proof of Theorem 4.72 is complete. ⊣

In the remainder of this subsection, we discuss some applications of the principle ADS_μ for μ a singular cardinal. Our first stop is the following theorem of Shelah (originally appearing in [84], but see also Lemma 4.9 in Chap. VII of [89]).

4.73 Theorem. *If $W \subseteq V$ are models of* ZFC *such that*

$$W \models \theta \textit{ is a cardinal and } \mathrm{ADS}_\theta \textit{ holds,}$$

and $(\theta^+)^W$ remains a cardinal in V, then

$$V \models \mathrm{cf}(\theta) = \mathrm{cf}(|\theta|). \tag{15.75}$$

Proof. In W, let $\langle A_\alpha : \alpha < (\theta^+)^W \rangle$ be an ADS_θ-sequence and suppose by way of contradiction that

$$V \models \mathrm{cf}(\theta) \neq \mathrm{cf}(|\theta|). \tag{15.76}$$

We begin by stepping into the model V and assessing the situation. First, it is clear from (15.76) that θ is not a cardinal, and so there is a cardinal κ such that $\kappa < \theta < \kappa^+ = (\theta^+)^W$. Thus, the ADS_θ-sequence from W looks like a sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ of subsets of the ordinal θ from the point of view of V.

Still inside of V, let σ be the cofinality of the ordinal θ and let τ be the cofinality of κ. There is a increasing sequence of sets $\langle B_i : i < \tau \rangle$ with union θ such that $|B_i| < \kappa$ for each $i < \tau$. Given $\alpha < \kappa^+$, we know that A_α has cofinality σ (as a set of ordinals). Since $\tau \neq \sigma$ and the sequence $\langle B_i : i < \tau \rangle$ is increasing, it follows that there is an ordinal $i(\alpha) < \tau$ with $A_\alpha \cap B_{i(\alpha)}$ unbounded in θ.

Fix $i^* < \tau$ so that $Z = \{\alpha < \kappa^+ : i(\alpha) = i^*\}$ is of size κ^+, and let α^* be the κth member of Z. By choice of $\langle A_\alpha : \alpha < \kappa^+ \rangle$, there is a function

$F : \alpha^* \to \theta$ such that the family $\{A_\alpha \setminus F(\alpha) : \alpha < \alpha^*\}$ is disjoint. Since $A_\alpha \cap B_{i^*}$ is unbounded in A_α for all $\alpha \in Z$, it follows that

$$\{A_\alpha \cap B_{i^*} \setminus F(\alpha) : \alpha \in Z \cap \alpha^*\} \tag{15.77}$$

is disjoint family of κ non-empty subsets of B_{i^*}. However, this is absurd as $|B_{i^*}| < \kappa$. ⊣

The preceding theorem also sheds light on a question of Bukovský and Copláková-Hartová [9], who ask if there can exist two models $W \subseteq V$ of ZFC with $\aleph^W_{\omega+1} = \aleph^V_2$. This problem and its natural generalizations have been studied by Cummings [12]. He has shown that the assumption of ADS_θ in Theorem 4.73 can be replaced, in the interesting case where θ is singular, by the existence of a good scale for θ. He also discusses how these results connect to other problems concerning singular cardinal combinatorics.

We end this subsection with an application of ADS to reflection of generalized stationary sets in the sense of Jech [50]. We use the following definition, which Kueker [57] has shown to be equivalent to Jech's original definition.

4.74 Definition. Let X be an uncountable set. A set $S \subseteq [X]^{\aleph_0}$ is *stationary* if and only if for all $F : {}^{<\omega}X \to X$, there is an $A \in S$ closed under F.

Jech's chapter [49] in this Handbook contains a much more comprehensive treatment of generalized stationary sets. We, however, will rest content with just the following definitions.

4.75 Definition. Let X be an uncountable transitive set.

1. A stationary set $S \subseteq [X]^{\aleph_0}$ *reflects to* $Y \subseteq X$ if $|Y| \subseteq Y$ and $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.
2. If S is stationary in $[X]^{\aleph_0}$, then we say $\mathrm{Refl}^*(S)$ *holds* if every stationary $T \subseteq S$ reflects to some $Y \in [X]^{\aleph_1}$ with $\mathrm{cf}(\mathrm{ot}(Y \cap \mathsf{On})) = \aleph_1$.

The vocabulary of the preceding definition now allows us to state the following theorem from [16].

4.76 Theorem. *Suppose that μ is singular of countable cofinality. If* ADS_μ *holds then* $\mathrm{Refl}^*([\mu^+]^{\aleph_0})$ *fails.*

Proof. Let $\langle A_\alpha : \alpha < \mu^+\rangle$ be an ADS_μ-sequence. Clearly we may assume that each A_α is of order-type ω, and we define

$$S = \{X \in [\mu^+]^{\aleph_0} : A_{\sup(X)} \subseteq X\}.$$

4.77 Claim. *The set S is stationary in $[\mu^+]^{\aleph_0}$.*

Proof. Let $F : {}^{<\omega}\mu^+ \to \mu^+$ be a finitary function. There is a closed unbounded set of $\alpha < \mu^+$ that are closed under F. Suppose now that $\mu < \alpha < \mu^+$, $\mathrm{cf}(\alpha) = \aleph_0$, and α is closed under F. Let C be an unbounded ω-sequence in α, and define A to be the closure of $C \cup A_\alpha$ under the function F. Since $\sup(A) = \alpha$, it follows that $A \in S$ and therefore S is stationary in $[\mu^+]^{\aleph_0}$. ⊣

Let us now assume by way of contradiction that $\mathrm{Refl}^*([\mu^+]^{\aleph_0})$ holds. This gives us a $Y \in [\mu^+]^{\aleph_1}$ such that

- $\omega_1 \subseteq Y$,
- $\mathrm{ot}(Y)$ has cofinality $\aleph_1$, and
- $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.

Note that if $X \in [Y]^{\aleph_0}$ then $\sup(X) < \sup(Y)$.

By our assumptions, there is a function $F : \sup(Y) \to \mu$ such that the sequence $\langle A_\alpha \setminus F(\alpha) : \alpha < \sup(Y)\rangle$ is disjoint. For $X \in S \cap [Y]^{\aleph_0}$, let

$$h(X) = \min(A_{\sup(X)} \setminus F(\sup(X))).$$

Note that $h(X) \in X$ because $A_{\sup(X)} \subseteq X$, and

$$\sup(X_0) \neq \sup(X_1) \implies h(X_0) \neq h(X_1) \tag{15.78}$$

because of the properties of F.

By Jech's generalization of Fodor's Theorem (see [49, Sect. 4]), there is a stationary $T \subseteq S \cap [Y]^{\aleph_0}$ on which h is constant. It follows immediately from (15.78) that there is a β such that $\sup(X) = \beta$ for all $X \in T$.

Since $\mathrm{cf}(\beta) = \aleph_0 < \aleph_1 = \mathrm{cf}(\sup(Y))$, it follows that $\beta < \sup(Y)$. But then it is easy to see that $\{X \in [Y]^{\aleph_0} : \sup(X) = \beta\}$ is nonstationary—a contradiction. $\dashv$

By work of Foreman, Magidor, and Shelah [36], Martin's Maximum implies $\mathrm{Refl}^*([\lambda]^{\aleph_0})$ for all regular $\lambda \geq \aleph_2$. As a consequence, Martin's Maximum implies that $\mathrm{ADS}_{\aleph_\omega}$ fails. Since $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ always holds, we see that $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ does not imply $\mathrm{ADS}_{\aleph_\omega}$.

4.7. Some Consequences of $\mathrm{pp}(\mu) > \mu^+$

In the Analytical Guide appendix to [89], Shelah writes

> **14.6** Up to now we have many consequences of GCH (or instances of it) and few of the negations of such statements. We now begin to have consequences of the negation ... so we can hope to have proofs by division to cases. For example, let λ be strong limit singular; if $\mathrm{pp}\,\lambda > \lambda^+$ then $\mathrm{NPT}(\lambda^+, \mathrm{cf}\,\lambda)$ [$\mathrm{NPT}(\lambda^*, (\mathrm{cf}\,\lambda)^+)$ in the conventions adopted in this chapter] and if $\mathrm{pp}(\lambda) \leq \lambda^+$ then $2^\lambda = \lambda^+$ (and $\diamondsuit^*_{\{\delta<\lambda^+:\mathrm{cf}(\delta)\neq\mathrm{cf}(\lambda)\}}$) and so various constructions are possible.

In this section, we take advantage of better scales to illustrate his point. The following theorem is a restatement of part of Claim 1.3 from Chap. II of [89].

4.78 Theorem. *Let μ be a singular cardinal. If $\mathrm{pp}(\mu) > \mu^+$, then there is a better scale for μ.*

Proof. By Proposition 1.19, there is an increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ of regular cardinals with limit μ such that

$$\left(\prod_{i<\mathrm{cf}(\mu)}\mu_i, <^*\right) \text{ is } \mu^{++}\text{-directed.} \tag{15.79}$$

For each limit $\delta < \mu^+$, choose a closed unbounded $C_\delta \subseteq \delta$ of order-type $\mathrm{cf}(\delta)$, and define

$$\mathcal{C}_\delta = \{C_\alpha \cap \delta : \delta \leq \alpha < \mu^+,\ \alpha \text{ limit}\}.$$

Note that $|\mathcal{C}_\delta| \leq \mu^+$, so our sequence is essentially a $\square_\mu^{\mu^+}$-sequence. As mentioned before, such a sequence is referred to as a *silly square sequence* in the literature.

If $\prod_{i<\mathrm{cf}(\mu)} \mu_i$ we construct a $<^*$-increasing sequence $\langle g_\alpha : \alpha < \mu^+\rangle$ such that for every limit $\delta < \mu^+$ and $C \in \mathcal{C}_\delta$,

$$f_C <^* g_\delta,$$

where f_C is defined as in (15.71). Note that this construction is the place where (15.79) is used in a crucial manner.

Note as well that our construction guarantees that every limit $\delta < \mu^+$ satisfying $\mathrm{cf}(\delta) > \mathrm{cf}(\mu)$ is a good point for $\vec{g}$. From our earlier work, it follows that $\vec{g}$ has an exact upper bound g with the property that

$$|\{i < \mathrm{cf}(\mu) : \mathrm{cf}(g(i)) < \kappa\}| < \mathrm{cf}(\mu)$$

for all $\kappa < \mu$.

The rest of the proof parallels that of Theorem 3.53—one uses g to find an increasing sequence $\vec{\theta} = \langle \theta_\xi : \xi < \mathrm{cf}(\mu)\rangle$ of regular cardinals with limit μ with the property that there is a natural operation Φ that transforms the sequence $\vec{g}$ into a sequence $\vec{f}$ of functions in $\prod_{\xi<\mathrm{cf}(\mu)} \theta_\xi$ in such a way that $(\vec{\theta}, \vec{f})$ is a scale for μ. The only detail that must be checked is that this transformation preserves the "betterness" that is built into $\vec{g}$, and this is straightforward. $\dashv$

4.79 Corollary. *If μ is a strong limit singular cardinal and $2^\mu > \mu^+$, then there is a better scale for μ. Hence* $\mathrm{ADS}(\mu)$ *and* $\mathrm{NPT}(\mu^+, (\mathrm{cf}\,\mu)^+)$ *hold as well.*

Proof. The proof of the corollary follows immediately once we know that the assumptions imply $\mathrm{pp}(\mu) > \mu^+$. If μ has uncountable cofinality, then this follows from Conclusion 5.5 on page 93 of Shelah's [89] (or see Theorem 9.1.3 on page 271 of [46] for a more explicit statement). If $\mathrm{cf}(\mu) = \aleph_0$, then we get what we need (that is, $\mathrm{pp}(\mu) > \mu^+$) from Conclusion 5.10 on page 410 of [89]. $\dashv$

We conclude this section with a result designed to complement Proposition 1.19 from the introduction to this chapter.

4.80 Theorem. *The following statements are equivalent for a singular cardinal μ.*

1. $\mathrm{pp}(\mu) = \mu^+$.
2. *If $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals with limit μ, then there is an unbounded $I \subseteq \mathrm{cf}(\mu)$ such that $\vec{\mu} \restriction I$ admits a scale for μ, that is, we can find $\vec{f}$ such that $(\vec{\mu} \restriction I, \vec{f})$ is a scale for μ.*

If in addition we know $\mathrm{cf}(\mu) = \aleph_0$, then we can add

3. *Every increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ of regular cardinals with limit μ admits a scale for μ.*

Proof. Note that the implication from (2) to (1) follows immediately from Proposition 1.19 by taking the contrapositive, so we concentrate on the implication from (1) to (2). Assume now that $\mathrm{pp}(\mu) = \mu^+$, and let $\vec{\mu}$ be given.

We may assume that $\mathrm{cf}(\mu) < \mu_0$, so that the set $A = \{\mu_i : i < \mathrm{cf}(\mu)\}$ is a progressive set of regular cardinals cofinal in μ. It is clear from the definitions involved that $\mu^+ \in \mathrm{pcf}(A)$, so let $B = B_{\mu^+}[A]$ be the corresponding generator and recall that

$$\mathrm{tcf}\big(\prod B, <_{J_{<\mu^+}[A]}\big) = \mu^+. \tag{15.80}$$

The set B is unbounded in A (this follows easily from the assumption that $\mathrm{pp}(\mu) = \mu^+$) and so there is an unbounded $I \subseteq \mathrm{cf}(\mu)$ for which $B = \{\mu_i : i \in I\}$. Observe as well that any subset of B in $J_{<\mu^+}[A]$ must be bounded in B—if D is any ultrafilter A disjoint to $J^{\mathrm{bd}}[A]$, then $\mathrm{cf}(\prod A/D) \geq \mu^+$. When we combine this observation with (15.80), it follows that

$$\mathrm{tcf}\big(\prod B, <_{J^{\mathrm{bd}}[B]}\big) = \mu^+,$$

and from this we see that $\vec{\mu} \restriction I$ admits a scale.

It is clear that (1) follows from (3) by the same reason that it follows from (2). For the other direction, we actually prove something a little stronger than required—we show that for any singular μ with $\mathrm{pp}(\mu) = \mu^+$ (regardless of cofinality), if we are given an increasing sequence of regular cardinals $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ with limit μ that in addition satisfies

$$\max \mathrm{pcf}\{\mu_j : j < i\} < \mu_i \quad \text{for all } i < \mathrm{cf}(\mu), \tag{15.81}$$

then $\vec{\mu}$ admits a scale. If $\mathrm{cf}(\mu) = \aleph_0$, then any relevant $\vec{\mu}$ will satisfy this property, and so this will be sufficient to prove that (1) implies (3) under those circumstances.

Suppose now that $\vec{\mu}$ satisfies (15.81), and let $A = \{\mu_i : i < \mathrm{cf}(\mu)\}$. If D is any ultrafilter on A, then either D contains a bounded subset of A (and hence $\mathrm{cf}(\prod A/D) < \mu$ by (15.81), or D is disjoint to $J^{\mathrm{bd}}[A]$ (in which case

$\mathrm{cf}(\prod A/D) = \mu^+$ because $\mathrm{pp}(\mu) = \mu^+$). We conclude that $\max \mathrm{pcf}(A) = \mu^+$ and hence A is the pcf generator $B_{\mu^+}[A]$ corresponding to μ^+. Thus,

$$\mathrm{tcf}\bigl(\prod A, <_{J_{<\mu^+}[A]}\bigr) = \mu^+.$$

It follows from work earlier in the current proof that

$$J_{<\mu^+}[A] \subseteq J^{\mathrm{bd}}[A],$$

and clearly

$$J^{\mathrm{bd}}[A] \subseteq J_{<\mu^+}[A]$$

because of our assumption (15.81). Putting these three statements together, we see that

$$\mathrm{tcf}\bigl(\prod A, <_{J^{\mathrm{bd}}[A]}\bigr) = \mu^+,$$

and therefore $\vec{\mu}$ admits a scale. ⊣

In closing the section, we mention that the relationship between failures of the Singular Cardinals Hypothesis and reflection phenomena is a very active area of current research, as evidenced by the following recent results:

- (Sharon [79]) It is consistent that there is a strong limit singular cardinal μ such that $2^\mu > \mu^+$ and $\mathrm{Refl}(\mu^+)$ holds.
- (Gitik and Sharon [39]) It is consistent that $\aleph_{\omega^2}$ is a strong limit cardinal, $2^{\aleph_{\omega^2}} > \aleph_{\omega^2+1}$, $\mathrm{VGS}_{\aleph_{\omega^2}}$ holds, but $\mathrm{AP}(\aleph_{\omega^2})$ fails.

The latter result gives us an example where goodness is not the same as approachability. Their work leaves open the question of whether it is consistent that $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega} > \aleph_{\omega+1}$, and $\neg\Box^*_{\aleph_\omega}$, as well as the question of whether it is consistent to have a strong limit singular μ with $2^\mu > \mu^+$ such that $\Box^*_\mu$ fails *and* such that GCH holds beneath μ.

4.8. Trees at Successors of Singular Cardinals

Early on in the chapter we lamented that some material that should be covered in such a survey had to be sacrificed in the interest of keeping the chapter to a manageable length. Thus, this subsection will be quite short, and little more than a list of open problems, but the topic we cover is too important to be left out altogether. We speak of trees at successors of singular cardinals.

Throughout the earlier parts of the chapter, we have used reflection properties of stationary sets as a lens to study the various combinatorial principles uncovered in the course of the narrative. We could just as easily have used trees to study the fine gradations in strength in the hierarchy of concepts we have examined.

Let us start with Jensen's result [52] that $\Box_\kappa$ together with the GCH implies the existence of a κ^+-Souslin tree. Schimmerling [76] notes that in fact, a simple modification of Jensen's argument shows that it suffices to assume only the principle $\square_\kappa^{<\omega}$ in addition to GCH. He goes on to suggest the following family of questions:

> Assume μ is a singular cardinal and GCH holds. Find the least θ such that $\square_\mu^{<\theta}$ does not imply the existence of a μ^+-Souslin tree. Does the existence of an $\aleph_{\omega+1}$-Souslin tree follow from GCH and $\square_{\aleph_\omega}^*$? What about GCH and $\square_{\aleph_\omega}^\omega$?

With regard to Aronszajn trees, recall that $\square_\kappa^*$ is equivalent to the existence of a special κ^+-Aronszajn tree (see Theorem 4.13). Shelah proved that if μ is a singular limit of strongly compact cardinals, then there are no μ^+-Aronszajn trees at all. This result appears in [66], along with his joint work with Magidor wherein starting with a 2-huge cardinal, they prove the consistency of there being no $\aleph_{\omega+1}$-Aronszajn trees at all.

Foreman's collection of open questions [32] lists several problems concerning Aronszajn trees at successors of singular cardinals. At the forefront is the general task of discerning the relationship between the failure of the singular cardinals hypothesis, weak versions of square, and the existence of Aronszajn trees. We offer the following list of questions taken from [32] as an example of some specific manifestations of the general problem:

- If $\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$, does there exist an $\aleph_{\omega+1}$-Aronszajn tree?
- If $\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$, does $\square_{\aleph_\omega}^*$ holds?
- Does the existence of a μ^+-Aronszajn tree follow from the existence of a very good scale for μ?
- If $\mathrm{cf}(\mu) = \aleph_0$ and AP_μ holds, does there exist a μ^+-Aronszajn tree?

With these questions, we leave the topic of trees and push on into the last section of the chapter.

5. Square-Bracket Partition Relations

In the opening of this chapter, we dealt briefly with the following well-known open question:

$$\text{If } \mu \text{ is singular, is there a Jónsson algebra on } \mu^+? \tag{15.82}$$

This question has been responsible for a great deal of the development of pcf theory and club guessing technology by Shelah. As we shall see, there are many partial results, but a final resolution remains elusive. In this final section of the chapter, we consider this open question and some of its special cases.

5.1. Colorings of Finite Subsets

We begin with our "official" definition of Jónsson cardinals—we gave a different definition in Definition 1.3, but we shall see that the two are equivalent.

5.1 Definition.

1. An *algebra* is a structure $\mathcal{A} = \langle A, f_n \rangle_{n<\omega}$, where each f_n is a finitary function mapping A to A.

2. A *Jónsson algebra* is an algebra without a proper subalgebra of the same cardinality.

3. A cardinal λ is a *Jónsson cardinal* if there is **no** Jónsson algebra of cardinality λ, that is, every algebra of cardinality λ has a proper subalgebra of cardinality λ.

We refer the reader to [53, Chap. 12] for the basic theory of Jónsson cardinals, for we do not have room to explore in detail the ways in which such cardinals fit into the general scheme of large cardinals. We also cite the follow result from [53], which appears as Exercise 8.12.

5.2 Theorem. *The following are equivalent:*

1. *λ is a Jónsson cardinal.*

2. *$\lambda \to [\lambda]^{<\omega}_{\lambda}$, that is, whenever $F : [\lambda]^{<\omega} \to \lambda$, we can find an $H \subseteq \lambda$ of size λ such that the range of $F \restriction [H]^{<\omega}$ is a proper subset of λ.*

3. *Any structure for a countable first-order language with domain of cardinality λ has a proper elementary substructure with domain of the same cardinality.*

The equivalence of (1) and (2) is due to Erdős and Hajnal [26], while the equivalence of (1) and (3) was first shown by Keisler and Rowbottom [54]. We will need one more standard fact about Jónsson cardinals; we give a proof to illustrate the technique of utilizing Skolem functions.

5.3 Theorem. *The following statements are equivalent for a cardinal λ:*

1. *λ carries a Jónsson algebra.*

2. *For every sufficiently large regular $\chi > \lambda$, if M is a elementary submodel of $\langle H(\chi), \in, <_\chi \rangle$ such that*

   (a) $\lambda \in M$, and

   (b) $|M \cap \lambda| = \lambda$,

   then $\lambda \subseteq M$.

Proof. Assume that λ carries a Jónsson algebra. By condition (2) of Theorem 5.2, there is a function $F : [\lambda]^{<\omega} \to \lambda$ with the property that

$$A \in [\lambda]^\lambda \quad \Longrightarrow \quad \operatorname{ran}(F \restriction [A]^{<\omega}) = \lambda.$$

Given a model M as above, by elementarity there is such a function F inside of M. Given $\alpha < \lambda$, there must be a set $a \in [M \cap \lambda]^{<\omega}$ such that $F(a) = \alpha$. Since a is also an element of M (any finite sequence of elements from M is in M as well!), we see that α is definable from parameters in M and therefore $\alpha \in M$. Thus $\lambda \subseteq M$ as required.

For the other direction, let $\mathfrak{A}$ be the structure $\langle H(\chi), \in, \lambda, <_\chi \rangle$ and let $\langle f_n : n < \omega \rangle$ be a complete set of Skolem functions for $\mathfrak{A}$. We assume without loss of generality that f_n is k_n-ary for some $k_n \leq n$. We define a function F with domain $[\lambda]^{<\omega}$ as follows:

$$F(\alpha_1, \ldots, \alpha_n) = \begin{cases} f_n(\alpha_1, \ldots, \alpha_{k_n}) & \text{if this is an ordinal below } \lambda \text{, and} \\ 0 & \text{otherwise.} \end{cases} \tag{15.83}$$

The definition of F guarantees that for every subset B of λ, the range of F restricted to $[B]^{<\omega}$ coincides with $\mathrm{Sk}^{\mathfrak{A}}(B) \cap \lambda$. Thus, statement (2) implies that this particular F witnesses that $\lambda \nrightarrow [\lambda]^{<\omega}_\lambda$. By Fact 5.2, λ is not a Jónsson cardinal. ⊣

Note that the proof of the preceding theorem actually establishes more than is claimed—in addition, we may require that the models M in (2) contain some fixed $x \in H(\chi)$. Theorem 5.3 is quite useful, and as an illustration we prove a theorem (due independently to Tryba [99] and Woodin) that shows that (15.82) is connected to stationary reflection.

5.4 Theorem. *If λ is a regular Jónsson cardinal, then every stationary subset of λ reflects.*

Proof. Let λ be a regular Jónsson cardinal, and suppose that $M \prec H(\chi)$ (for a sufficiently large regular χ) is such that $\lambda \in M$, $|M \cap \lambda| = \lambda$, but $\lambda \nsubseteq M$. It suffices to prove that every stationary $S \in M$ reflects—if λ had a non-reflecting stationary subset then there would be one in M.

Given such an S, we claim that $S \setminus M$ must be stationary. To see why, assume by way of contradiction that there is a closed unbounded $E \subseteq \lambda$ with $E \cap S \subseteq M$. In the model M, we can find a function $f : S \to \lambda$ such that $S_\alpha := f^{-1}(\{\alpha\})$ is stationary for each $\alpha < \lambda$. Fix $\alpha < \lambda$ such that $\alpha \notin M$. Since $S_\alpha \subseteq S$, we know $E \cap S_\alpha \subseteq M$. Given $\beta \in E \cap S_\alpha$, since $f \in M$ and $\beta \in M$ we conclude $\alpha = f(\beta) \in M$, a contradiction.

Thus, we can find $\delta \in S \setminus M$ such that $\delta = \sup(M \cap \delta)$. Let us define

$$\beta_\delta := \min(M \cap \lambda \setminus \delta). \tag{15.84}$$

Since δ is not in M, we clearly have $\delta < \beta_\delta$. Furthermore, an easy argument establishes that β_δ is a limit ordinal of uncountable cofinality. To finish the proof, we establish that $S \cap \beta_\delta$ is a stationary subset of β_δ.

Assume by way of contradiction that this fails. Since S and β_δ are both members of M, the model M contains a closed unbounded subset C of β_δ disjoint to S. Now $M \cap \delta$ is cofinal in δ, and so for any $\alpha < \delta$ there is a $\beta \in M \cap \lambda$ with $\alpha < \beta < \delta$. Furthermore, since

$$M \models C \text{ is unbounded in } \beta_\delta,$$

there is a $\gamma \in M \cap C$ with $\beta < \gamma$. By our choice of β_δ, it must be the case that $\gamma < \delta$. Thus, δ is a limit point of C. But C is closed, and so δ must be in C as well. This contradicts our assumption that S and C are disjoint. $\dashv$

As an immediate corollary, we see that successors of regular cardinals carry Jónsson algebras.

5.5 Corollary. *If κ is regular, then κ^+ carries a Jónsson algebra.*

The fact that successors of regular cardinals cannot be Jónsson cardinals is put to good use in the following lemma which provides a strengthening of Theorem 5.3.

5.6 Lemma. *The following two statements are equivalent:*

1. *λ is a Jónsson cardinal.*
2. *For every sufficiently large regular $\chi > \lambda$ whenever we are given a cardinal κ satisfying $\kappa^+ < \lambda$, there is an $M \prec \langle H(\chi), \in, <_\chi\rangle$ such that*
   (a) *$\{\lambda, \kappa\} \in M$,*
   (b) *$|M \cap \lambda| = \lambda$,*
   (c) *$\lambda \nsubseteq M$, and*
   (d) *$\kappa + 1 \subseteq M$.*

Proof. The implication (2) $\to$ (1) is immediate by the comments following Theorem 5.3. For the other direction, assume that λ is a Jónsson cardinal, but χ and κ provide a counterexample to (2). Let $\mathfrak{A}$ be the structure $\langle H(\chi), \in, \lambda, \kappa, <_\chi\rangle$. Since λ is a Jónsson cardinal, we can find and elementary submodel M of $\mathfrak{A}$ such that $|M \cap \lambda| = \lambda$ but $\lambda \nsubseteq M$. Let $B = M \cap \lambda$; without loss of generality $M = \mathrm{Sk}^{\mathfrak{A}}(B)$.

Since κ is in M, our choice of κ implies $\kappa \nsubseteq M$. The cardinal κ^{++} is in M as well, and since κ^{++} carries a Jónsson algebra by Corollary 5.5, it follows that

$$\sup(M \cap \kappa^{++}) < \kappa^{++}.$$

Now let $N = \mathrm{Sk}^{\mathfrak{A}}(B \cup \kappa)$. Since $\kappa + 1 \subseteq N$, our choice of κ implies $\lambda \subseteq N$. In particular, $\sup(N \cap \kappa^{++}) = \kappa^{++}$. This is a contradiction, as Lemma 1.6 tells us that $\sup(M \cap \kappa^{++}) = \sup(N \cap \kappa^{++}) < \kappa^{++}$. $\dashv$

We now fulfill a promise made in the introductory section of the chapter—we prove that $\aleph_{\omega+1}$ cannot be a Jónsson cardinal. The proof is due to Shelah [82] and furnishes yet another instance of the utility of scales.

5.7 Theorem. *Suppose that $\lambda = \mu^+$ where μ is a singular cardinal, and assume $(\vec{\mu}, \vec{f})$ is a scale for μ with the property that μ_i carries a Jónsson algebra for each i. Then there is a Jónsson algebra on λ.*

Proof. Let M be an elementary submodel of $H(\chi)$ such that

- both λ and $(\vec{\mu}, \vec{f})$ are in M,
- $\mathrm{cf}(\mu) \subseteq M$, and
- $|M \cap \lambda| = \lambda$.

The conclusion follows from Lemma 5.6 provided that we establish $\lambda \subseteq M$.

This will follow provided we can show $|M \cap \mu_i| = \mu_i$ for arbitrarily large $i < \mathrm{cf}(\mu)$—since each μ_i is in M and no μ_i is a Jónsson cardinal, we could conclude $\mu \subseteq M$ and then an easy argument yields $\lambda = \mu^+ \subseteq M$ as well.

Thus, suppose by way of contradiction that $|M \cap \mu_i| < \mu_i$ for all sufficiently large $i < \mathrm{cf}(\mu)$, and let $\mathrm{Ch}_M^{\vec{\mu}}$ be the *characteristic function of M on $\vec{\mu}$*, defined by

$$\mathrm{Ch}_M^{\vec{\mu}}(i) := \begin{cases} \sup(M \cap \mu_i) & \text{if } \sup(M \cap \mu_i) < \mu_i, \\ 0 & \text{otherwise.} \end{cases} \tag{15.85}$$

Clearly $\mathrm{Ch}_M^{\vec{\mu}} \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$, and $\mathrm{Ch}_M^{\vec{\mu}}(i) = \sup(M \cap \mu_i)$ for all sufficiently large $i < \mathrm{cf}(\mu)$ by our assumptions.

Since $M \cap \lambda$ is unbounded in λ and $(\vec{\mu}, \vec{f})$ is a scale, it follows that there is an $\alpha \in M \cap \lambda$ such that

$$\mathrm{Ch}_M^{\vec{\mu}}(i) < f_\alpha(i) \quad \text{for all sufficiently large } i < \mathrm{cf}(\mu). \tag{15.86}$$

Since $f_\alpha \in M$ and $\mathrm{dom}(f_\alpha) \subseteq M$, it follows that $f_\alpha(i) \in M$ for all $i < \mathrm{cf}(\mu)$, and this contradicts (15.86). $\dashv$

5.8 Corollary. *$\aleph_{\omega+1}$ is not a Jónsson cardinal.*

Theorem 5.7 is quite powerful, especially when taken in conjunction with results in pcf theory. The following corollary collects a few of the more important consequences of Theorem 5.7.

5.9 Corollary. *Let μ be a singular cardinal, and assume μ^+ is a Jónsson cardinal.*

1. μ is a limit of regular Jónsson cardinals.

2. $\mathrm{pp}(\mu) > \mu^+$.

3. If $\mathrm{cf}(\mu)$ *is uncountable, then* $\{\theta < \mu : \theta^+$ *is a Jónsson cardinal*$\}$ *is closed and unbounded in* μ*.*

Proof. If μ^+ is a Jónsson cardinal, then we conclude from Theorem 5.7 that whenever $(\vec{\mu}, \vec{f})$ is a scale for μ, it must be the case that μ_i is a Jónsson cardinal for all sufficiently large $i < \mathrm{cf}(\mu)$. From this, (1) follows easily.

We regard to (2), note that Theorem 5.7 implies that if $\vec{\mu}$ is an increasing sequence of successors of regular cardinals with limit μ, then $\vec{\mu}$ does not admit a scale for μ. By Theorem 4.80, it follows that $\mathrm{pp}(\mu) > \mu^+$.

Statement (3) follows for essentially the same reason as (1); the missing ingredient here is Theorem 2.23 in the chapter of Abraham and Magidor in this Handbook on cardinal arithmetic [1]. ⊣

Concerning statement (2) in Corollary 5.9, we mention that here is a relatively easy "pcf free" proof due to Erdős, Hajnal, and Rado that 2^μ must be greater than μ^+ if μ^+ is a Jónsson cardinal. Since their argument makes use of colorings of pairs, we examine it in the next subsection (see Proposition 5.11). Also, Shelah [92] has proved a generalization of Theorem 5.7 that implies, among other things, that the cardinal $\beth_\omega^+$ carries a Jónsson algebra.

5.2. Colorings of Pairs

The second part of Theorem 5.2 showed us that Jónsson cardinals can be characterized by a certain *square-bracket partition relation.* The notation below is due originally to Erdős, Hajnal, and Rado [27], and we refer the reader to Chap. XI of [28] for a survey of the elementary theory of square-bracket partition relations.

5.10 Definition. Given cardinals κ, λ, and μ, and an ordinal γ, the notation

$$\kappa \to [\mu]^\lambda_\gamma$$

means that given a partition $f : [\kappa]^\lambda \to \gamma$, there is a set $H \subseteq \kappa$ of cardinality μ such that

$$\mathrm{ran}(f \restriction [H]^\lambda) \text{ is a proper subset of } \gamma.$$

The negation of this assertion is denoted

$$\kappa \nrightarrow [\mu]^\lambda_\gamma.$$

Expressions such as $\kappa \to [\mu]^{<\lambda}_\gamma$ should be given the obvious meaning.

It is a theorem of Erdős and Hajnal [26] that $\kappa \nrightarrow [\kappa]^\omega_\kappa$ for every infinite cardinal κ (see [53, Sect. 23] for a nice proof due to Galvin and Prikry [37]). This result is a key ingredient in Kunen's proof [58] that there is no elementary embedding $j : V \to V$.

In this subsection, we will concentrate on colorings of pairs, that is, the case where $\lambda = 2$ in Definition 5.10. Our first result is an easy proposition due to Erdős, Hajnal, and Rado [27] which we mentioned in the discussion following Corollary 5.9.

5.11 Proposition. *If $2^\kappa = \kappa^+$, then $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$.*

Proof. Assume $2^\kappa = \kappa^+$, and let $\langle X_\alpha : \alpha < \kappa^+\rangle$ enumerate the bounded subsets of κ^+ in such a way that $X_\alpha \subseteq \alpha$. An easy construction yields a function $f : [\kappa^+]^2 \to \kappa^+$ with the property that whenever $\kappa \leq \alpha < \beta$ and $\gamma < \beta$, there is a $\xi \in X_\alpha$ such that $f(\xi, \beta) = \gamma$. If A is any unbounded subset of κ^+ and $\gamma < \kappa^+$, we can choose α with $X_\alpha \subseteq A$. If β is the least element of A greater than γ and α, then our construction guarantees that there is a $\xi \in A$ with $f(\xi, \beta) = \gamma$. ⊣

Todorčević [97] obtained the following significant strengthening of Theorem 5.4 using his technique of minimal walks. Shelah's paper [86] also contains a proof of this theorem, and Sect. 20 of Hajnal and Hamburger's book [42] gives a nice exposition.

5.12 Theorem. *If λ has a non-reflecting stationary subset, then $\lambda \nrightarrow [\lambda]^2_\lambda$.*

Proof. See [42, Sect. !20]. ⊣

5.13 Corollary. *If κ is a regular cardinal, then $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$.*

The following counterpart of Theorem 5.7 is also due to Todorčević, and Burke and Magidor [10] contains a nice treatment. The proof we give is based on their paper.

5.14 Theorem. *Let $(\vec{\mu}, \vec{f})$ be a scale for the singular cardinal μ. If*

$$\mu_i \nrightarrow [\mu_i]^2_{\mu_i} \tag{15.87}$$

for all $i < \mathrm{cf}(\mu)$, then

$$\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}. \tag{15.88}$$

Proof. Assume $(\vec{\mu}, \vec{f})$ is a scale satisfying (15.87), with c_i the corresponding coloring on μ_i for each $i < \mathrm{cf}(\mu)$. Define $d : [\mu^+]^2 \to \mathrm{cf}(\mu)$ by setting $d(\alpha, \beta)$ equal to the maximal $i^* < \mathrm{cf}(\mu)$ for which $f_\beta(i^*) < f_\alpha(i^*)$ if such an i^* exists, and setting $d(\alpha, \beta)$ equal to 0 otherwise.

Now define $c : [\mu^+]^2 \to \mu^+$ by

$$c(\alpha, \beta) = c_{i^*}(f_\beta(i^*), f_\alpha(i^*))$$

where $\alpha < \beta$ and $i^* = d(\alpha, \beta)$. We now work to prove that c has the properties needed for us to conclude that $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$.

Suppose that we are given a set $X \subseteq \mu^+$ of cardinality μ^+, and a color $\gamma < \mu^+$; we must produce $\alpha < \beta$ in X with $c(\alpha, \beta) = \gamma$. Let M be an elementary submodel of $H(\chi)$ such that

- μ, $(\vec{\mu}, \vec{f})$, and X are all in M,
- $\mathrm{cf}(\mu) \subseteq M$, and
- $|M| < \mu$

and let $\mathrm{Ch}^{\vec{\mu}}_M$ be defined as in (15.85).

There is an α_0 such that $\mathrm{Ch}^{\vec{\mu}}_M <^* f_\alpha$ for all $\alpha \geq \alpha_0$. Since μ^+ is regular and $|X| = \mu^+$, it follows that there is an $i_0 < \mathrm{cf}(\mu)$ and a set $Y \subseteq X$ of cardinality μ^+ such that

$$\sup(M \cap \mu_i) = \mathrm{Ch}^{\vec{\mu}}_M(i) < f_\alpha(i) \quad \text{for all } i \geq i_0 \text{ and } \alpha \in Y.$$

For each $\alpha < \mu^+$ and $i \geq i_0$, define

$$A(\alpha, i) = \{f_\xi(i) : \xi \in Y \setminus \alpha\}.$$

Note that $A(\alpha, i) \subseteq \mu_i$, and for each i the sequence $\langle A(\alpha, i) : \alpha < \mu^+\rangle$ is non-increasing, hence eventually constant. Thus, for $i_0 \leq i < \mathrm{cf}(\mu)$ there is a set $A_i \subseteq \mu_i$ and $\alpha_i < \mu^+$ such that

$$\alpha \geq \alpha_i \quad \Longrightarrow \quad A(\alpha, i) = A_i.$$

Since μ^+ is regular, it follows that there is a single $\alpha^* < \mu^+$ satisfying

$$\alpha \geq \alpha^* \quad \Longrightarrow \quad A(\alpha, i) = A_i \text{ for } i \geq i_0. \tag{15.89}$$

We can assume that $\alpha^* \geq \alpha_0$, as increasing α^* preserves (15.89).

Next, note that A_i is unbounded in μ_i for all sufficiently large $i < \mathrm{cf}(\mu)$. This has a straightforward proof by contradiction—if A_i is bounded below μ_i for arbitrarily large $i < \mathrm{cf}(\mu)$, then one easily contradicts the fact that $\vec{f}$ is a scale.

Choose $i^* \geq i_0$ large enough so that $\gamma < \mu_{i^*}$ and A_{i^*} is unbounded in μ_{i^*}. Our choice of c_{i^*} guarantees the existence of $\xi < \zeta$ in A_{i^*} for which

$$c_{i^*}(\xi, \zeta) = \gamma.$$

The definition of A_{i^*} implies that there is a $\beta \in Y$ such that

$$f_\beta(i^*) = \xi.$$

Now let $N = \mathrm{Sk}^{\mathfrak{A}}(M \cup \mu_{i^*})$. An application of Lemma 1.6 from the introduction tells us that

$$\mathrm{Ch}^{\vec{\mu}}_N(i) = \sup(N \cap \mu_i) = \mathrm{Ch}^{\vec{\mu}}_M(i) \quad \text{for } i > i^*,$$

and since $\beta \in Y$, it follows that

$$\mathrm{Ch}^{\vec{\mu}}_N(i) < f_\beta(i) \quad \text{for } i > i^*. \tag{15.90}$$

The ordinal ζ is in N, and by elementarity

$$N \models (\exists \alpha \in X)[f_\alpha(i^*) = \zeta],$$

so we can find a $\alpha \in N \cap X$ such that $f_\alpha(i^*) = \zeta$.

Since $f_\alpha(i) \in N \cap \mu_i$ for all i, it follows from (15.90) that

$$i > i^* \implies f_\alpha(i) < f_\beta(i).$$

Since $f_\beta(i^*) = \xi < \zeta = f_\alpha(i^*)$, we conclude that $d(\alpha, \beta) = i^*$ and therefore

$$c(\alpha, \beta) := c_{i^*}(f_\beta(i^*), f_\alpha(i^*)) = c_{i^*}(\xi, \zeta) = \gamma,$$

as desired. ⊣

The proof of the theorem just completed exploits a crucial property of the function d defined in the course of the demonstration. As we shall see, that particular function implies that certain square-bracket partition relations fail dramatically at successors of singular cardinals. To make this precise we need the following definition of Shelah, which is only one of an imposing family of related concepts. We refer to the reader to the first appendix of [89] for more information.

5.15 Definition. Let μ be a singular cardinal. $\mathrm{Pr}_1(\mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ says that there is a function $c : [\mu^+]^2 \to \mathrm{cf}(\mu)$ such that whenever we are given a sequence $\langle t_i : i < \mu^+\rangle$ of disjoint sets from $[\mu^+]^{<\mathrm{cf}(\mu)}$ and an ordinal $\gamma < \mathrm{cf}(\mu)$, we can find $\epsilon_1 < \epsilon_2$ such that $c(\alpha, \beta) = \gamma$ whenever $\alpha \in t_{\epsilon_1}$ and $\beta \in t_{\epsilon_2}$.

We point out that in the above definition, the coloring takes on only $\mathrm{cf}(\mu)$ possible values, and not μ^+. Also, note that

$$\mathrm{Pr}_1(\mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu)) \implies \mu^+ \nrightarrow [\mathrm{cf}(\mu)]^2_{\mathrm{cf}(\mu)}$$

because each t_i can be taken to be a singleton. The following theorem is due to Shelah (see Conclusion 4.1 in Chap. II of [89]).

5.16 Theorem. $\mathrm{Pr}_1(\mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ *holds for every singular cardinal* μ.

Proof. Let $\lambda = \mu^+$ and $\kappa = \mathrm{cf}(\mu)$. By Theorem 3.53, we can find a scale $(\vec{\mu}, \vec{f})$ for μ with $\kappa < \mu_0$. Let us define $d : [\lambda]^2 \to \kappa$ just as in the proof of Theorem 5.14, that is $d(\alpha, \beta)$ is set (for $\alpha < \beta < \lambda$) equal to the maximal $i^* < \mathrm{cf}(\mu)$ for which $f_\beta(i^*) < f_\alpha(i^*)$ if such an i^* exists, and $d(\alpha, \beta) = 0$ otherwise.

Next, let $h : \kappa \to \kappa$ be a partition of κ into sets of size κ. We define (again for $\alpha < \beta < \lambda$)

$$c(\alpha, \beta) = h(d(\alpha, \beta)).$$

We claim that c has the property required by $\mathrm{Pr}_1(\lambda, \kappa, \kappa)$.

To see this, suppose that we are given a sequence $\langle t_i : i < \lambda\rangle$ of subsets of λ, each of size less than κ. Without loss of generality, we assume that all t_i have the same order-type $\xi < \kappa$, $i \leq \min(t_i)$, and that $\sup(t_i) < \min(t_j)$ for

$i < j < \kappa$. Given $\gamma < \kappa$, our task is to find $\epsilon_1 < \epsilon_2 < \lambda$ such that $c \restriction t_{\epsilon_1} \times t_{\epsilon_2}$ is constant with value γ.

For each $\beta < \lambda$, we define functions $f_\beta^{\inf}$ and $f_\beta^{\sup}$ mapping κ to μ by

$$f_\beta^{\inf}(i) = \inf\{f_\alpha(i) : \alpha \in t_\beta\}$$

and

$$f_\beta^{\sup}(i) = \sup\{f_\alpha(i) : \alpha \in t_\beta\}.$$

Note that $f_\beta^{\sup} \in \prod_{i<\kappa} \mu_i$ as $\xi < \kappa < \mu_0$ and each μ_i is regular, while

$$f_\beta^{\inf}(i) = f_{\min(t_\beta)}(i) \quad \text{for all sufficiently large } i < \kappa \tag{15.91}$$

as $\xi < \kappa$ and κ is regular.

Let M be an elementary submodel of $H(\chi)$ of size κ with $\kappa \subseteq M$ and containing λ, $(\vec{\mu}, \vec{f})$, and $\langle t_i : i < \lambda \rangle$. Finally, let $\mathrm{Ch}_M^{\vec{\mu}}$ be the characteristic function of M on $\vec{\mu}$, defined as in (15.85).

Since $\vec{f}$ forms a scale, by (15.91) there is an ordinal $\epsilon_1 < \lambda$ such that $\mathrm{Ch}_M^{\vec{\mu}} <^* f_{\epsilon_1}^{\inf}$. Also, we note that for all sufficiently large $i < \kappa$ we have

$$\mu_i = \sup\{f_\beta^{\inf}(i) : \beta < \lambda\}.$$

This follows easily from the fact that $(\vec{\mu}, \vec{f})$ is a scale.

After putting these two observations together, it follows that we can choose an $i^* < \kappa$ such that the following conditions hold:

1. $h(i^*) = \gamma$,
2. $\mu_{i^*} = \sup\{f_\beta^{\inf}(i^*) : \beta < \lambda\}$, and
3. $\mathrm{Ch}_M^{\vec{\mu}}(i) < f_{\epsilon_1}^{\inf}(i)$ for all $i \geq i^*$.

By condition (2), we can find an $\epsilon^* < \lambda$ such that

$$f_{\epsilon_1}^{\sup}(i^*) < \delta := f_{\epsilon^*}^{\inf}(i^*).$$

We now define $N = \mathrm{Sk}^{\mathfrak{A}}(M \cup \mu_{i^*})$. By Lemma 1.6, we know that

$$\sup(N \cap \sigma) = \sup(M \cap \sigma)$$

for every regular cardinal in M greater than μ_{i^*}. In particular,

$$\mathrm{Ch}_M^{\vec{\mu}} \restriction [i^* + 1, \kappa) = \mathrm{Ch}_N^{\vec{\mu}} \restriction [i^* + 1, \kappa).$$

By elementarity,

$$N \models (\exists \beta < \lambda)[f_\beta^{\inf}(i^*) = \delta],$$

and so there is an $\epsilon_2 \in N$ with $f_{\epsilon_2}^{\inf}(i^*) = \delta$. Thus

$$f_{\epsilon_1}^{\sup}(i^*) < f_{\epsilon_2}^{\inf}(i^*). \tag{15.92}$$

On the other hand, for $i > i^*$ we have

$$f^{\sup}_{\epsilon_2}(i) < \mathrm{Ch}^{\vec{\mu}}_N(i) = \mathrm{Ch}^{\vec{\mu}}_M(i) < f^{\inf}_{\epsilon_1}(i). \tag{15.93}$$

Thus for $\alpha \in t_{\epsilon_1}$ and $\beta \in t_{\epsilon_2}$, (15.92) and (15.93) imply

$$d(\alpha, \beta) = i^*,$$

and so

$$c(\alpha, \beta) = h(e(\alpha, \beta)) = h(i^*) = \gamma,$$

as required. ⊣

As a special case of the preceding theorem, we obtain the following.

5.17 Corollary. *If μ is singular, then $\mu^+ \nrightarrow [\mu^+]^2_{\mathrm{cf}(\mu)}$.*

Much more is known about colorings of pairs at successors of singular cardinals, but the techniques used involve club guessing in an essential way—see [25] and [24], for example. We discuss the interplay between club guessing and the existence of complicated colorings in the next subsection.

5.3. Colorings and Club Guessing

In the previous subsection, we have exploited the existence of scales to shed light on square-bracket partition relations at successors of singular cardinals. In this subsection, we get more results using a different technique—club guessing. We remark that all the results presented in this section are due to Shelah, and they can be found in his book [89]. Club guessing has already made a few brief appearances in earlier chapters, e.g., [49, Theorem 1.16] and [1, Theorem 2.17]. In the following subsections we develop the theory just enough to shed light on the question (15.82). We begin with some terminology.

5.18 Definition. Let κ be a regular cardinal and let $S \subseteq \kappa$ be a stationary set. A family $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is an S-*club system* if the set of $\delta \in S$ for which C_δ is not closed and unbounded in δ is nonstationary.

Given an S-club system $\bar{C}$ for a cardinal κ, there are several senses in which $\bar{C}$ can be said to "guess clubs", but they all have the same general form—they all require that for every closed unbounded $E \subseteq \kappa$, the set $E \cap C_\delta$ is "large" for a "large" set of $\delta \in S$. For example, one might require for every closed unbounded $E \subseteq \kappa$ that $C_\delta \subseteq E$ for stationarily many $\delta \in S$—we used this sort of club guessing in our proof of Theorem 3.18. The hypotheses of the following theorem include another example of club guessing, and we shall use the theorem and its proof to motivate further investigations.

5.19 Theorem. *Suppose that $\lambda = \mu^+$ where μ is singular, and let $S \subseteq \lambda$ be stationary. Further suppose that there is an S-club system $\bar{C} = \langle C_\delta : \delta \in S\rangle$ with the property that for every closed unbounded $E \subseteq \lambda$, there is a stationary set of $\delta \in S$ such that for all $\beta < \delta$ and $\gamma < \mu$, there is an $\alpha \in E \cap \mathrm{nacc}(C_\delta)$ such that*

1. $\beta < \alpha$,

2. $\gamma < \mathrm{cf}(\alpha)$, *and*

3. $\mathrm{cf}(\alpha)$ *carries a Jónsson algebra.*

Then λ *carries a Jónsson algebra.*

Proof. Let M by an elementary submodel of $H(\chi)$ for some sufficiently large regular χ containing S and $\bar{C}$. Further suppose that $\lambda \in M$ and $|M \cap \lambda| = \lambda$. We must establish that $\lambda \subseteq M$.

Given $\delta \in S$, let us define β_δ to be the least member of $M \cap \lambda$ greater than or equal to δ. Clearly, $\delta < \beta_\delta$ if and only if $\delta \notin M$. Let $E \subseteq \lambda$ be the closed unbounded set of all $\alpha \in \lambda \setminus \mu$ satisfying $\alpha = \sup(M \cap \alpha)$, and fix $\delta \in S$ such that $E \cap \mathrm{nacc}(C_\delta)$ satisfies the assumptions of the theorem. The following crucial lemma tells us that M must contain many points of C_δ, even though δ itself need not be in M.

5.20 Lemma. *M contains all members of $E \cap \mathrm{nacc}(C_\delta)$ of cofinality greater than $\mathrm{cf}(\beta_\delta)$.*

Proof. We dispose of the easy case first—if $\delta \in M$ then $E \cap \mathrm{nacc}(C_\delta) \subseteq M$ as well. To see why, suppose that $\beta \in E \cap \mathrm{nacc}(C_\delta)$. Since $\beta = \sup(\beta \cap M)$, there is an ordinal $\beta^* \in M$ with $\sup(C_\delta \cap \beta) < \beta^* < \beta$. Thus, β is the least member of C_δ above β^*. Since C_δ and β^* are both in M (recall that we assumed $\delta \in M$), it follows that β must be in M as well.

The case where δ is not in M is more difficult as we do not have C_δ available in the model. In this case, recall that $\delta < \beta_\delta$.

Inside M, let d be a closed unbounded subset of β_δ of order-type $\mathrm{cf}(\beta_\delta)$. The same argument we used in the proof of Theorem 5.4 shows us that δ is an element of d.

Now let $\beta \in E \cap \mathrm{nacc}(C_\delta)$ satisfy $\mathrm{cf}(\beta) > |d| = \mathrm{cf}(\beta_\delta)$. It follows that β cannot be a limit point of d, and so (using the definition of E) there exists an ordinal β_0 with

- $\beta_0 \in M$,
- $\sup(C_\delta \cap \beta) < \beta_0 < \beta$, and
- $\sup(d \cap \beta) < \beta_0$.

Given β_0, we define A to be the set $\{\min(C_\epsilon \setminus \beta_0) : \epsilon \in d \cap S\}$. Notice that A is in M because all parameters used in its definition are available there. Also, $|A| \leq |d| < \mathrm{cf}(\beta)$ and so $A \cap \beta$ is bounded below β. Thus, there is an ordinal β_1 such that

- $\beta_1 \in M$,
- $\beta_0 < \beta_1 < \beta$, and
- $A \cap [\beta_1, \beta) = \emptyset$.

Let us now define

$$d^* = \{\epsilon \in d \cap S \setminus \beta_1 : \min(C_\epsilon \setminus \beta_0) = \min(C_\epsilon \setminus \beta_1)\}.$$

The set d^* is an element of M as it is definable from parameters in M. Clearly δ is in the set d^* as

$$\min(C_\delta \setminus \beta_0) = \min(C_\delta \setminus \beta_1) = \beta. \tag{15.94}$$

If $\epsilon \in d \cap S$ is such that $\min(C_\epsilon \setminus \beta_0) < \beta$, then our choice of β_1 implies $\min(C_\epsilon \setminus \beta_0) < \beta_1$ and hence $\epsilon \notin d^*$. Thus

$$\epsilon \in d^* \implies \beta \leq \min(C_\epsilon \setminus \beta_0). \tag{15.95}$$

Putting (15.94) and (15.95) together, we conclude

$$\beta = \min\left(\{\min(C_\epsilon \setminus \beta_0) : \epsilon \in d^*\}\right).$$

Thus β is definable from parameters in M, and so $\beta \in M$ as required. $\dashv$

To finish the proof that λ carries a Jónsson algebra, we note that it suffices to prove that $\sigma \subseteq M$ for arbitrarily large $\sigma < \mu$, as this implies $\mu \subseteq M$ which implies $\lambda = \mu^+ \subseteq M$.

Choose an ordinal $\delta \in S$ as guaranteed by the assumptions of the theorem. Given $\sigma < \mu$, our choice of δ together with the preceding lemma lets us find an $\alpha \in E \cap \mathrm{nacc}(C_\delta)$ such that

- $\sigma < \mathrm{cf}(\alpha)$,
- $\mathrm{cf}(\alpha)$ carries a Jónsson algebra, and, most importantly,
- $\alpha \in M$.

Since $M \cap \alpha$ is unbounded in α, it follows immediately that $M \cap \mathrm{cf}(\alpha)$ is unbounded in $\mathrm{cf}(\alpha)$. However, $\mathrm{cf}(\alpha)$ is in M and it also carries a Jónsson algebra. By Theorem 5.3, we must conclude $\mathrm{cf}(\alpha) \subseteq M$, and the proof is complete. $\dashv$

We mentioned prior to the statement of Theorem 5.19 that its hypotheses require the S-club sequence $\bar{C}$ to guess clubs in a certain sense. We will invest a little time in unpacking some notation of Shelah in order to make this more precise.

5.21 Definition. Let κ be a regular cardinal, and let $\bar{C}$ be an S-club system for some stationary $S \subseteq \kappa$. A sequence $\bar{I} = \langle I_\delta : \delta \in S\rangle$ is said to be an *ideal sequence associated with* $\bar{C}$ if each I_δ is a (not necessarily proper) ideal on C_δ extending $J^{\mathrm{bd}}[C_\delta]$, the ideal of bounded subsets of C_δ. When we write "let $(\bar{C}, \bar{I})$ be an S-club system" it is to be understood that $\bar{I}$ is an ideal sequence associated with $\bar{C}$.

Now associated to every pair $(\bar{C}, \bar{I})$ as in the preceding definition, we have two natural club guessing ideals.

5.22 Definition. Let κ be a regular cardinal, let $S \subseteq \kappa$ be stationary, and let $(\bar{C}, \bar{I})$ be an S-club system.

1. $\mathrm{id}^a(\bar{C}, \bar{I})$ is defined to be the set of all $A \subseteq \kappa$ for which there is a club $E \subseteq \lambda$ such that $\{\delta \in S \cap A : C_\delta \setminus E \in I_\delta\}$ is nonstationary.
2. $\mathrm{id}^p(\bar{C}, \bar{I})$ is defined to be the set of all $A \subseteq \kappa$ for which there is a club $E \subseteq \lambda$ such that $\{\delta \in S \cap A : E \cap C_\delta \notin I_\delta\}$ is nonstationary.

We pause here for a moment to consider the preceding definitions. Note that an S-club system $(\bar{C}, \bar{I})$ can be said to "guess clubs" if one or both of these ideals are proper—for example, if $I_\delta = J^{\mathrm{bd}}[C_\delta]$ for all $\delta \in S$ and $\mathrm{id}^a(\bar{C}, \bar{I})$ is a proper ideal, then for every closed unbounded $E \subseteq \kappa$ the set of δ such that E contains a tail of C_δ is stationary. With the same choice of $\bar{I}$, we see that $\mathrm{id}^p(\bar{C}, \bar{I})$ is a proper ideal if and only if for every closed unbounded $E \subseteq \kappa$ there is a stationary set of $\delta \in S$ with $E \cap C_\delta$ unbounded in δ. (This last statement is true for any $\delta \in S \cap \mathrm{acc}(E)$ of uncountable cofinality, so in practice we usually require the ideal I_δ to contain $\mathrm{acc}(C_\delta)$.) Note that in the case of $\mathrm{id}^a(\bar{C}, \bar{I})$, we require that $E \cap C_\delta$ contains almost all (in the sense of I_δ) elements of C_δ, while in the case of $\mathrm{id}^p(\bar{C}, \bar{I})$ we only require that $E \cap C_\delta$ is I_δ-positive.

We do not have room here for a full exposition of club guessing results, and accordingly we focus on the special case that is extremely relevant to square-bracket partition relations at successors of singular cardinals. In particular, we will not have an opportunity to discuss recent work of Foreman and Komjath [33] and Ishiu [47, 48] in this area; the reader is encouraged to look in these papers for more information.

5.23 Definition. Suppose that $\lambda = \mu^+$ for some singular cardinal μ, and let $\delta < \lambda$ be a limit ordinal. For C_δ closed and unbounded in δ, we define $J^{b[\mu]}[C_\delta]$ to be the ideal of subsets of C_δ generated sets of the following forms:

- $\mathrm{acc}(C_\delta)$
- $C_\delta \cap \beta$ for all $\beta < \delta$
- $\{\alpha \in C_\delta : \mathrm{cf}(\alpha) < \gamma\}$ for all $\gamma < \mu$

There are three comments that should be made here. First, the notation is due to Shelah, who first realized the relevance of these ideals to the problems we are considering. Second, note that a set A is in $J^{b[\mu]}[C_\delta]$ if and only if there are $\beta < \delta$ and $\gamma < \mu$ such that

$$A \subseteq \mathrm{acc}(C_\delta) \cup \beta \cup \{\epsilon < \delta : \mathrm{cf}(\epsilon) < \gamma\}. \tag{15.96}$$

This makes it clear that $J^{b[\mu]}[C_\delta]$ is τ-complete where $\tau = \min\{\mathrm{cf}(\delta), \mathrm{cf}(\mu)\}$. Finally, we point out that the ideal $J^{b[\mu]}[C_\delta]$ is perhaps easier to visualize when one considers what it means for a set to be $J^{b[\mu]}[C_\delta]$-positive—a set $A \subseteq C_\delta$ is *not in* $J^{b[\mu]}[C_\delta]$ if for any $\beta < \delta$ and $\gamma < \mu$, there is an ordinal $\alpha \in \mathrm{nacc}(C_\delta)$ such that $\beta < \alpha$ and $\mathrm{cf}(\alpha) > \gamma$.

In this terminology, we see that the hypothesis concerning $\bar{C}$ in Theorem 5.19 becomes "for every closed unbounded $E \subseteq \lambda$, there are stationarily many $\delta \in S$ such that the set $\{\alpha \in E \cap C_\delta : \mathrm{cf}(\alpha)$ carries a Jónsson algebra$\}$ is not in $J^{b[\mu]}[C_\delta]$". We now state without proof a club guessing result of Shelah taken from [89, Chap. III].

5.24 Theorem. *Let $\lambda = \mu^+$ for μ a singular cardinal, and let $S \subseteq S^{\lambda}_{\mathrm{cf}(\mu)}$ be stationary. There is an S-club system $\bar{C}$ such that for every closed unbounded $E \subseteq \lambda$, the set of $\delta \in S$ with $E \cap C_\delta \notin J^{b[\mu]}[C_\delta]$ is stationary.*

Said another way, the previous theorem states that $\mathrm{id}^p(\bar{C}, \bar{I})$ is a proper ideal, where I_δ is defined to be $J^{b[\mu]}[C_\delta]$. We remark that the theorem is never explicitly stated in [89], but it follows from piecing together various club guessing results appearing in Chap. III of that book, and the paper [40] contains an exposition of related results. From this theorem, we can see that the hypotheses of Theorem 5.19 are satisfied for any singular cardinal μ that is not a limit of regular Jónsson cardinals.

5.25 Theorem. *If μ is a singular cardinal that is not a limit of regular Jónsson cardinals, then μ^+ carries a Jónsson algebra.*

Proof. Let $\langle C_\delta : \delta \in S^{\mu^+}_{\mathrm{cf}(\mu)}\rangle$ be a club system as in Theorem 5.24. The hypotheses of Theorem 5.19 are then satisfied because of the definition of $J^{b[\mu]}[C_\delta]$. $\dashv$

Plowing through the notation associated with club guessing may seem a steep price to pay for the preceding theorem, especially as we obtained a similar result in a much neater way using scales. However, the club guessing proof seems to give more information, and we will spend the rest of the subsection illustrating this.

The ideals defined in Definition 5.22 are certainly natural ones to consider in association with club guessing, but for our purposes we need to also consider the following ideals:

5.26 Definition. Let κ be a regular cardinal, let $S \subseteq \kappa$ be stationary, and let $(\bar{C}, \bar{I})$ be an S-club system.

1. $\mathrm{id}_a(\bar{C}, \bar{I})$ is defined to be the set of all $A \subseteq \kappa$ for which there is a club $E \subseteq \lambda$ such that $\{\delta \in S : A \cap C_\delta \setminus E \in I_\delta\}$ is nonstationary.

2. $\mathrm{id}_p(\bar{C}, \bar{I})$ is defined to be the set of all $A \subseteq \kappa$ for which there is a club $E \subseteq \lambda$ such that $\{\delta \in S : E \cap A \cap C_\delta \notin I_\delta\}$ is nonstationary.

The notation above is due to Shelah and even though it is not terribly descriptive, we note that the switch from superscripts in $\mathrm{id}^p(\bar{C}, \bar{I})$ to subscripts in $\mathrm{id}_p(\bar{C}, \bar{I})$ is suggestive—the "superscript ideals" concentrate on the set S, while the "subscript ideals" concentrate instead on the set $\bigcup_{\delta\in S} C_\delta$.

To see the difference between the two, let us consider the case where $(\bar{C}, \bar{I})$ is an S-club system for some stationary $S \subseteq \kappa$. The ideal $\mathrm{id}^p(\bar{C}, \bar{I})$ is a proper ideal if and only if $\mathrm{id}_p(\bar{C}, \bar{I})$ is, and both these conditions happen if and only if the pair $(\bar{C}, \bar{I})$ guesses clubs in the "p-sense".

Now, given such $(\bar{C}, \bar{I})$, a set T is in $\mathrm{id}^p(\bar{C}, \bar{I})$ if and only if the sequence $\langle (C_\delta, I_\delta) : \delta \in S\cap T\rangle$ no longer guesses clubs, while T is in $\mathrm{id}_p(\bar{C}, \bar{I})$ precisely when the sequence $\langle (C_\delta \cap T, I_\delta \restriction (C_\delta \cap T)) : \delta \in S\rangle$ no longer guesses clubs. In both cases, T is used to "shrink" the sequence $(\bar{C}, \bar{I})$, but the shrinking is done in different senses.

The importance of this ideal is shown by the following theorem. Although this theorem was first stated explicitly in [22], its proof is the same as that of Theorem 5.19 and it is implicit in Claim 3.7 in Chap. III of [89] (which appears as Corollary 5.28 below).

5.27 Theorem. *Let $\lambda = \mu^+$ for μ a singular cardinal, and let $S \subseteq \lambda$ be stationary. Given an S-club sequence $\bar{C}$ as in the conclusion of Theorem 5.24, there is a function $F : [\lambda]^{<\omega} \to \lambda$ such that for any unbounded $A \subseteq \lambda$, the range of $F\restriction[A]^{<\omega}$ is in the filter dual to $\mathrm{id}_p(\bar{C}, \bar{I})$, that is*

$$A \in [\lambda]^\lambda \quad\Longrightarrow\quad [\lambda \setminus \mathrm{ran}(F\restriction[A]^{<\omega})] \in \mathrm{id}_p(\bar{C}, \bar{I}). \tag{15.97}$$

Proof. Let $x = \{S, \lambda, (\bar{C}, \bar{I})\}$, and let χ be a sufficiently large regular cardinal. The function $F : [\lambda]^{<\omega} \to \lambda$ is to code the Skolem functions of the structure $\mathfrak{A} = \langle H(\chi), \in, x, <_\chi\rangle$ just as in the proof of Theorem 5.3. This means, in particular, that for any $B \subseteq \lambda$ we have

$$\mathrm{ran}(F\restriction[B]^{<\omega}) = \mathrm{Sk}^{\mathfrak{A}}(B) \cap \lambda. \tag{15.98}$$

Now let A be an unbounded subset of λ, and let $B = \mathrm{ran}(F\restriction[A]^{<\omega})$. If $B = \lambda$, there is nothing to prove. If not, then $M = \mathrm{Sk}^{\mathfrak{A}}(A)$ is an elementary submodel of $H(\chi)$ satisfying

- $M \cap \lambda = B = \mathrm{ran}(F\restriction[A]^{<\omega})$,
- λ, S, and $(\bar{C}, \bar{I})$ are all in M,
- $|M \cap \lambda| = \lambda$, and
- $\lambda \not\subseteq M$.

Let E be the closed unbounded set of $\alpha < \lambda$ for which $\alpha = \sup(M \cap \alpha)$. By Lemma 5.20, if $\delta \in S$ is such that C_δ guesses E then M contains all members of $E \cap \mathrm{nacc}(C_\delta)$ of cardinality greater than β_δ. In other words,

$$\delta \in S \quad\Longrightarrow\quad E \cap (\lambda \setminus B) \cap C_\delta \in I_\delta.$$

Therefore $\lambda \setminus B$ is in $\mathrm{id}_p(\bar{C}, \bar{I})$, as was to be shown. $\dashv$

Thus, even though we as yet do not know whether the successor of a singular cardinal must carry a Jónsson algebra, the above theorem gives an approximation to a positive answer—for any unbounded $A \subseteq \lambda$, the function F takes on almost all values (in the sense of $\mathrm{id}_p(\bar{C}, \bar{I})$) when restricted to $[A]^{<\omega}$. As a corollary, we deduce the following result of Shelah.

5.28 Corollary. *Let λ, S, and $(\bar{C}, \bar{I})$ be as in the statement of Theorem 5.27. If there exists a partition of λ into θ sets, each of which is $\mathrm{id}_p(\bar{C}, \bar{I})$-positive, then $\lambda \nrightarrow [\theta]^{<\omega}_{\lambda}$.*

Proof. Let $\langle A_i : i < \theta \rangle$ be a partition of λ into $\mathrm{id}_p(\bar{C}, \bar{I})$ positive sets, and for $s \in [\lambda]^{<\omega}$ we define

$$c(s) = i \iff F(s) \in A_i.$$

Given an unbounded $A \subseteq \lambda$ an $i < \theta$, we know A_i meets the range of $F \restriction [A]^{<\omega}$. In particular, there must be an $s \in [A]^{<\omega}$ with $F(s) \in A_i$, and for this choice of s we have $c(s) = i$, as required. ⊣

If each C_δ is of cardinality less than μ (this can be guaranteed if the cofinality of μ is uncountable, but the question for countable cofinality is still open), then the function F from Theorem 5.27 can be improved to a function defined on pairs. This is shown in [24], where minimal walks are combined with combinatorics of scales to construct a complicated coloring of pairs. It then follows that the conclusion of the corollary can then be improved to $\lambda \nrightarrow [\lambda]^2_{\lambda}$ (and even to $\mathrm{Pr}_1(\lambda, \lambda, \mathrm{cf}(\mu))$ using a stronger argument). The fourth section of Chap. III from [89] also achieves this improvement of the corollary, but Shelah's colorings are constructed directly from a partition of λ into disjoint $\mathrm{id}_p(\bar{C}, \bar{I})$-positive sets.

We now close the section by remarking that the problem of partitioning λ into $\mathrm{id}_p(\bar{C}, \bar{I})$-positive sets is a very concrete one, and Shelah exploits this in [89] as a way of obtaining Jónsson algebras and colorings of pairs.

6. Concluding Remarks

In this final section, we make a few remarks concerning the overarching themes present in this chapter. As mentioned in the introduction, one can view successors of singular cardinals as a battleground between "compactness" and "non-compactness". On the one side, we have all the reflection phenomena associated with supercompact cardinals and their ilk, while squares act as a paradigmatic representative for the other side. In between, we find a gradation of combinatorial principles that can be used to measure the non-reflection inherent in a combinatorial statement.

There is more here to hold our interest than the simple weighing of combinatorial statements on scales of reflection, however, for the theory contains many surprises. We hope that this introduction to the topic adequately communicates the richness of this area of set theory, and that it will serve as an invitation for other researchers to join in the exploration.

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16. Prikry-Type Forcings

Moti Gitik

Contents

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One of the central topics of set theory since Cantor has been the study of the power function $\kappa \to 2^\kappa$. The basic problem is to determine all the possible values of 2^κ for a cardinal κ. Cohen [7] proved the independence of CH and invented the method of forcing. Easton [11] building on Cohen's results showed that the function $\kappa \to 2^\kappa$ for regular κ can behave in any prescribed way consistent with the Zermelo-König inequality, which entails $\mathrm{cf}(2^\kappa) > \kappa$. This reduces the study to singular cardinals.

It turned out that the situation with powers of singular cardinals is much more involved. Thus, for example, a remarkable theorem of Silver states that a singular cardinal of uncountable cofinality cannot be the first to violate GCH. The Singular Cardinals Problem is the problem of finding a complete set of rules describing the behavior of the function $\kappa \to 2^\kappa$ for singular κ's.

There are three main tools for dealing with the problem: pcf theory, inner model theory and forcing involving large cardinals. The purpose of this chapter is to present the main forcing tools for dealing with powers of singular cardinals. We refer to [19] or to [24] for detailed discussion of the Singular Cardinals Problem.

The chapter should be accessible to a reader with knowledge of forcing (say, Chaps. VII, VIII of Kunen's book [30]) and familiarity with ultrapowers and elementary embeddings. Thus §§5, 26 of Kanamori's book [26] will be more than enough. Only Sect. 6 requires in addition a familiarity with iterated forcing (for example Baumgartner's paper [5], §§0–2 of Sect. II of Shelah's book [54], or Cummings' chapter [8] in this Handbook). The following sections can be read independently: Sects. 1 and 2; Sects. 1.1, 3 and 4; Sects. 1.1 and 5.1, 5.2; Sect. 6.

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1. Prikry Forcings

We describe here the classical Prikry forcing and some variations of it. They were all introduced implicitly or explicitly by Karel Prikry in [47].

1.1. Basic Prikry Forcing

Let κ be a measurable cardinal and U a normal ultrafilter over κ.

1.1 Definition. Let $\mathcal{P}$ be the set of all pairs $\langle p, A\rangle$ such that

(1) p is a finite subset of κ,

(2) $A \in U$, and

(3) $\min(A) > \max(p)$.

It is convenient sometimes to view p as an increasing finite sequence of ordinals.

We define two partial orderings on $\mathcal{P}$, the first one conspicuously lacking any useful closure property and the second closed enough to compensate for the lack of closure of the first.

1.2 Definition. Let $\langle p, A\rangle, \langle q, B\rangle \in \mathcal{P}$. We say that $\langle p, A\rangle$ *is stronger than* $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \geq \langle q, B\rangle$ iff

(1) p is an end extension of q, i.e. $p \cap (\max(q)+1) = q$,

(2) $A \subseteq B$, and

(3) $p \setminus q \subseteq B$.

We shall use $\leq$ with the corresponding meaning, and proceed analogously in similar definitions without further comment.

1.3 Definition. Let $\langle p, A\rangle, \langle q, B\rangle \in \mathcal{P}$. We say that $\langle p, A\rangle$ is *a direct (or Prikry) extension* of $\langle q, B\rangle$ and denote this by $\langle p, A\rangle \geq^* \langle q, B\rangle$ iff

(1) $p = q$, and

(2) $A \subseteq B$.

We will force with $\langle \mathcal{P}, \leq\rangle$, and $\langle \mathcal{P}, \leq^*\rangle$ will be used to show that no new bounded subsets are added to κ after the forcing with $\langle \mathcal{P}, \leq\rangle$.

Let us prove a few basic lemmas.

1.4 Lemma. *Let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \leq\rangle$. Then $\bigcup\{p \mid \exists A(\langle p, A\rangle \in G)\}$ is an ω-sequence cofinal in κ.*

Proof. Just note that for every $\alpha < \kappa$ and $\langle q, B\rangle \in \mathcal{P}$ the set

$$D_\alpha = \{\langle p, A\rangle \in \mathcal{P} \mid \langle p, A\rangle \geq \langle q, B\rangle \text{ and } \max(p) > \alpha\}$$

is dense in $\langle \mathcal{P}, \leq\rangle$ above $\langle q, B\rangle$. ⊣

1.5 Lemma. *$\langle \mathcal{P}, \leq\rangle$ satisfies the κ^+-c.c.*

Proof. Note that any two conditions having the same first coordinate are compatible: If $\langle p, A\rangle, \langle p, B\rangle \in \mathcal{P}$, then $\langle p, A \cap B\rangle$ is stronger than both of them. ⊣

Let us now state three lemmas about $\leq^*$ and its relation to $\leq$. The third one contains the crucial idea of Prikry that makes everything work.

1.6 Lemma. $\leq^* \subseteq \leq$.

This is obvious from Definitions 1.2 and 1.3.

1.7 Lemma. *$\langle \mathcal{P}, \leq^* \rangle$ is κ-closed.*

Proof. Let $\langle \langle p_\alpha, A_\alpha \rangle \mid \alpha < \lambda \rangle$ be a $\leq^*$-increasing sequence of length λ for some $\lambda < \kappa$. Then all the p_α's are the same. Set $p = p_0$ and $A = \bigcap_{\alpha<\lambda} A_\alpha$. Then $A \in U$ by κ-completeness of U. So $\langle p, A \rangle \in \mathcal{P}$, and it is stronger than each $\langle p_\alpha, A_\alpha \rangle$ according to $\leq^*$. $\dashv$

1.8 Lemma (The Prikry condition). *Let $\langle q, B \rangle \in \mathcal{P}$ and σ be a statement of the forcing language of $\langle \mathcal{P}, \leq \rangle$. Then there is a $\langle p, A \rangle \geq^* \langle q, B \rangle$ such that $\langle p, A \rangle \parallel \sigma$ (i.e. $\langle p, A \rangle \Vdash \sigma$ or $\langle p, A \rangle \Vdash \neg\sigma$), where, again, we force with $\langle \mathcal{P}, \leq \rangle$ and not with $\langle \mathcal{P}, \leq^* \rangle$.*

Proof. We identify finite subsets of κ and finite increasing sequences of ordinals below κ, i.e. $[\kappa]^{<\omega}$. Define a partition $h : [B]^{<\omega} \to 2$ as follows:

$$h(s) = \begin{cases} 1, & \text{if there is a } C \text{ such that } \langle q \cup s, C \rangle \Vdash \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

U is a normal ultrafilter, so by the Rowbottom theorem (see [26, 7.17] or [25, 70]) there is an $A \in U$, $A \subseteq B$ homogeneous for h, i.e. for every $n < \omega$ and every $s_1, s_2 \in [A]^n$, $h(s_1) = h(s_2)$. Now $\langle q, A \rangle$ will decide σ. Otherwise, there would be

$$\langle q \cup s_1, B_1 \rangle, \langle q \cup s_2, B_2 \rangle \geq \langle q, A \rangle$$

such that $\langle q \cup s_1, B_1 \rangle \Vdash \sigma$ and $\langle q \cup s_2, B_2 \rangle \Vdash \neg\sigma$. By extending one of these conditions if necessary, we can assume that $|s_1| = |s_2|$. But then $s_1, s_2 \in [A]^{|s_1|}$ and $h(s_1) \neq h(s_2)$, which contradicts the homogeneity of A. $\dashv$

The above lemma allows us to implement the κ-closure of $\langle \mathcal{P}, \leq^* \rangle$ in the actual forcing $\langle \mathcal{P}, \leq \rangle$. Thus we can conclude the following:

1.9 Lemma. *$\langle \mathcal{P}, \leq \rangle$ does not add new bounded subsets of κ.*

Proof. Let $t \in \mathcal{P}$, $\underset{\sim}{a}$ is a name, $\lambda < \kappa$ and

$$t \Vdash \underset{\sim}{a} \subseteq \check{\lambda}.$$

For every $\alpha < \lambda$ denote by σ_α the statement "$\check{\alpha} \in \underset{\sim}{a}$". We define by recursion a $\leq^*$-increasing sequence of conditions $\langle t_\alpha \mid \alpha < \lambda \rangle$ such that $t_\alpha \parallel \sigma_\alpha$ for each $\alpha < \lambda$. Let t_0 be a direct extension of t deciding σ_0; one exists by Lemma 1.8. Suppose that $\langle t_\beta \mid \beta < \alpha \rangle$ is defined. Define t_α. First, using Lemma 1.7 we find a direct extension t'_α of $\langle t_\beta \mid \beta < \alpha \rangle$. Then by Lemma 1.8 choose a direct extension t_α of t'_α deciding σ_α. This completes the definition of $\langle t_\alpha \mid \alpha < \lambda \rangle$. Now let t^* be a direct extension of $\langle t_\alpha \mid \alpha < \lambda \rangle$ (again Lemma 1.7 is used). Then $t^* \geq t$ (in fact $t^* \geq^* t$) and $t^* \Vdash \underset{\sim}{a} = \check{b}$ where $b = \{\alpha < \lambda \mid t^* \Vdash \alpha \in \underset{\sim}{a}\}$. $\dashv$

Let us summarize the situation.

1.10 Theorem. *The following holds in $V[G]$:*

(a) κ has cofinality $\aleph_0$.

(b) All the cardinals are preserved.

(c) No new bounded subsets are added to κ.

Proof. (a) Is established by Lemma 1.4, (c) by Lemma 1.9. Finally, (b) follows from (c) and Lemma 1.5. $\dashv$

If $2^\kappa > \kappa^+$ in V, then in $V[G]$ the Singular Cardinal Hypothesis will fail at κ.

Let $C = \bigcup\{p \mid \exists A(\langle p, A\rangle \in G)\}$. By Lemma 1.4, C is an ω-sequence cofinal in κ. It is called a *Prikry sequence for U*. The generic set G can be easily reconstructed from C:

$$G = \{\langle p, A\rangle \in \mathcal{P} \mid p \text{ is an initial segment of } C \text{ and } C \setminus (\max(p)+1) \subseteq A\}.$$

So, $V[G] = V[C]$.

1.11 Lemma. *C is almost contained in every set in U, i.e.*

$(*)$ *for every $A \in U$ the set $C \setminus A$ is finite.*

Proof. Let $A \in U$. Then the set

$$D = \{\langle p, B\rangle \in \mathcal{P} \mid B \subseteq A\}$$

is dense in $\mathcal{P}$. So, there is a $\langle q, S\rangle \in G \cap D$. But then, for every $\langle q', S'\rangle \geq \langle q, S\rangle$, $q' \setminus q \subseteq S \subseteq A$. Hence, also, $C \setminus q \subseteq A$. $\dashv$

The above implies that C generates U, i.e. $X \in U$ iff $X \in V$ and $C \setminus X$ is finite.

Mathias [38] pointed out that $(*)$ of Lemma 1.11 actually characterizes Prikry sequences:

1.12 Theorem. *Suppose that M is an inner model of* ZFC*, U a normal ultrafilter over κ in M. Assume that C is an ω-sequence satisfying* $(*)$*. Then C is a Prikry sequence for U over M.*

Proof. We need to show that the set

$$\begin{aligned} G(C) = \{\langle p, A\rangle \in \mathcal{P} \mid\ & p \text{ is an initial segment of } C \\ & \text{and } C \setminus (\max(p)+1) \subseteq A\} \end{aligned}$$

is a generic subset of $\mathcal{P}$ over M. The only non-trivial property to check is that $G(C) \cap D \neq \emptyset$ for every dense open subset $D \in M$ of $\mathcal{P}$. Let us first point out that the following holds in M:

1.13 Lemma. *Let $\langle q, B\rangle \in \mathcal{P}$ and $D \subseteq \mathcal{P}$ be dense open. Then there are $\langle q, A\rangle \geq^* \langle q, B\rangle$ and $m < \omega$ such that for every n with $m \leq n < \omega$ and every $s \in [A]^n$, we have $\langle q \cup s, A \setminus (\max(s)+1)\rangle \in D$.*

Proof. We define a partition $h : [B]^{<\omega} \to 2$ as in Lemma 1.8 only replacing "$\Vdash \sigma$" by "$\in D$". Let $A' \in U$, $A' \subseteq B$ be homogeneous for h. Then, starting with some m, for every $n \geq m$ and $s \in [A']^n$ we have $h(s) = 1$. Hence there will be a set $A_s \in U$ such that $\langle q \cup s, A_s\rangle \in D$. Set $A = A' \cap \Delta\{A_s \mid s \in [A']^n, m \leq n < \omega\}$, where

$$\begin{aligned}&\Delta\{A_s \mid s \in [A']^n, m \leq n < \omega\}\\ &\quad = \{\alpha < \kappa \mid \forall n \geq m \forall s \in [A']^n(\max(s) < \alpha \to \alpha \in A_s)\}.\end{aligned}$$

Then clearly $A \in U$. The condition $\langle q, A\rangle$ is as desired, since for each $n \geq m$ and $s \in [A]^n$ we have $A \setminus (\max(s)+1) \subseteq A_s$ and, so $\langle q \cup s,\ A \setminus (\max(s)+1)\rangle \in D$. ⊣

Now, let $D \in M$ be a dense open subset of $\mathcal{P}$. For every finite $q \subseteq \kappa$, using Lemma 1.13, we pick $m(q) < \omega$ and $A(q) \in U$ such that $\langle q, A(q)\rangle \geq^* \langle q, \kappa \setminus (\max(q)+1)\rangle$ and for every $n \geq m(q)$ and $s \in [A(q)]^n$, $\langle q \cup s, A(q) \setminus (\max(s)+1)\rangle \in D$. Set

$$A = \Delta\{A(q) \mid q \in [\kappa]^{<\omega}\} = \{\alpha < \kappa \mid \forall q \in [\kappa]^{<\omega}(\max q < \alpha \to \alpha \in A(q))\}.$$

There is a $\tau < \kappa$ such that $C \setminus \tau \subseteq A$. Consider $\langle C \cap \tau, A \setminus \tau\rangle$. Since $C \cap \tau$ is finite, $\langle C \cap \tau, A \setminus \tau\rangle \in \mathcal{P}$. Then, for every $n \geq \max(C \cap \tau)$ and $s \in [C \setminus \tau]^n$ we have

$$\langle (C \cap \tau) \cup s, A \setminus (\max(s)+1)\rangle \in D,$$

since $A \setminus \tau \subseteq A \setminus (C \cap \tau)$. But $C \setminus \tau \subseteq A$, so we can pick $s \in [C \setminus \tau]^n$ for some $n \geq \max(C \cap \tau)$. Then $(C \cap \tau) \cup s \subseteq C$ and $C \setminus (\max(s)+1) \subseteq A \setminus (\max(s)+1)$. Hence, $\langle (C \cap \tau) \cup s, A \setminus (\max(s)+1)\rangle \in G(C) \cap D$. ⊣

1.2. Tree Prikry Forcing

We would now like to eliminate the use of the normality of the ultrafilter U in the previous construction. Note that it was used only once in the proof of the Prikry condition 1.8.

Let us now assume only that U is a κ-complete ultrafilter over κ.

1.14 Definition. A set T is called a *U-tree with a trunk t* iff

(1) T consists of finite increasing sequences of ordinals below κ.

(2) $\langle T, \trianglelefteq\rangle$ is a tree, where $\trianglelefteq$ is the order of end extension of finite sequences, i.e. $\eta \trianglelefteq \nu$ iff $\nu \restriction \mathrm{dom}(\eta) = \eta$.

(3) t is a trunk of T, i.e. $t \in T$ and for every $\eta \in T$, $\eta \trianglerighteq t$ or $t \trianglerighteq \eta$.

(4) For every $\eta \trianglerighteq t$ the set $\mathrm{Suc}_T(\eta) = \{\alpha < \kappa \mid \eta^\frown\langle\alpha\rangle \in T\}$ is in U.

Define $\mathrm{Lev}_n(T) = \{\eta \in T \mid \mathrm{length}(\eta) = n\}$ for every $n < \omega$.

We now define the *tree Prikry forcing.*

1.15 Definition. The set $\mathcal{P}$ consists of all pairs $\langle t, T\rangle$ such that T is a U-tree with trunk t.

1.16 Definition. Let $\langle t, T\rangle, \langle s, S\rangle \in \mathcal{P}$. We say that $\langle t, T\rangle$ *is stronger than* $\langle s, S\rangle$ and denote this by $\langle t, T\rangle \geq \langle s, S\rangle$ iff $S \supseteq T$.

Note that $S \supseteq T$ implies that $t \trianglerighteq s$ and $t \in S$.

1.17 Definition. Let $\langle t, T\rangle, \langle s, S\rangle \in \mathcal{P}$. We say that $\langle t, T\rangle$ is *a direct (or Prikry) extension of* $\langle s, S\rangle$ and denote this by $\langle t, T\rangle \geq^* \langle s, S\rangle$ iff

(1) $S \supseteq T$, and

(2) $s = t$.

As in the previous section we will force with $\langle \mathcal{P}, \leq\rangle$ and the role of $\leq^*$ will be to provide closure.

1.18 Lemma. *Let $\langle T_\alpha \mid \alpha < \lambda\rangle$ be a sequence of U-trees with the same trunk and $\lambda < \kappa$. Then $T = \bigcap_{\alpha<\lambda} T_\alpha$ is a U-tree having that same trunk.*

Proof. Let t be the trunk of T_0 (and so of every T_α). Suppose that $\eta \in T$ and $\eta \trianglerighteq t$. Then

$$\mathrm{Suc}_T(\eta) = \bigcap\nolimits_{\alpha<\lambda} \mathrm{Suc}_{T_\alpha}(\eta).$$

By κ-completeness of U, $\mathrm{Suc}_T(\eta) \in U$. Hence T is a U-tree with trunk t. ⊣

Using Lemma 1.18 it is easy to prove lemmas analogous to Lemmas 1.4–1.7.

1.19 Lemma. *Let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \leq\rangle$. Then*

$$\bigcup\{t \mid \exists T(\langle t, T\rangle \in G)\}$$

is an ω-sequence cofinal in κ.

1.20 Lemma. *$\langle \mathcal{P}, \leq\rangle$ satisfies the κ^+-c.c.*

1.21 Lemma. *$\leq^* \subseteq \leq$.*

1.22 Lemma. *$\langle \mathcal{P}, \leq^*\rangle$ is κ-closed.*

Let us show that $\langle \mathcal{P}, \leq, \leq^*\rangle$ satisfies the Prikry condition. The proof is based on the following Ramsey property:

If T is a U-tree and $f : T \to \lambda < \kappa$, then there is an U-tree $S \subseteq T$ such that $f\restriction\mathrm{Lev}_n(S)$ is constant for each $n < \omega$.

We prefer here and later to give a direct proof instead of deducing first a relevant Ramsey property and then proving it.

1.23 Lemma (The Prikry condition). *Let $\langle t, T\rangle \in \mathcal{P}$ and σ be a statement of the forcing language. Then there is a $\langle s, S\rangle \geq^* \langle t, T\rangle$ such that $\langle s, S\rangle \parallel \sigma$.*

Proof. Suppose otherwise. Consider the set $\mathrm{Suc}_T(t)$. We split it into three sets as follows:

$$\begin{aligned} X_0 &= \{\alpha \in \mathrm{Suc}_T(t) \mid \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown\langle\alpha\rangle \\ &\qquad\qquad \text{such that } \langle t^\frown\langle\alpha\rangle, S_\alpha\rangle \Vdash \sigma\} \\ X_1 &= \{\alpha \in \mathrm{Suc}_T(t) \mid \exists S_\alpha \subseteq T \text{ a } U\text{-tree with trunk } t^\frown\langle\alpha\rangle \\ &\qquad\qquad \text{such that } \langle t^\frown\langle\alpha\rangle, S_\alpha\rangle \Vdash \neg\sigma\} \\ X_2 &= \mathrm{Suc}_T(t) \setminus (X_0 \cup X_1). \end{aligned}$$

Clearly, $X_0 \cap X_1 = \emptyset$, since by Lemma 1.18 any two conditions with the same trunk are compatible. Now U is an ultrafilter and $\mathrm{Suc}_T(t) \in U$, so for some $i < 3$, $X_i \in U$. We shrink T to a tree T_1 with the same trunk t, having $\mathrm{Suc}_{T_1}(t) = X_i$ and: If $i < 2$, then let T_1 be S_α above $t^\frown\langle\alpha\rangle$ for every $\alpha \in X_i$; if $i = 2$, then let T_1 be the same as T above $t^\frown\langle\alpha\rangle$ for every $\alpha \in X_2$. We continue by recursion to shrink the initial tree T level by level. Thus define a decreasing sequence $\langle T_n \mid n < \omega\rangle$ of U-trees with trunk t so that

(1) $T_0 = T$.

(2) For every $n > 0$ and $m > n$, $T_m \restriction (n + |t|) = T_n \restriction (n + |t|)$, i.e. after stage n the n-th level above the trunk remains unchanged in all T_m's for $m \geq n$.

(3) For every $n > 0$, if $i < 2$, $\eta \in \mathrm{Lev}_{n+|t|}(T_n)$ and for some U-tree S with trunk η we have $\langle \eta, S\rangle \Vdash {}^i\sigma$, then

  (3a) $\langle \eta, (T_n)_\eta\rangle \Vdash {}^i\sigma$, and

  (3b) For every $\nu \in \mathrm{Lev}_{n+|t|}(T_n)$ having the same immediate predecessor as η,

$$\langle \nu, (T_n)_\nu\rangle \Vdash {}^i\sigma.$$

Here, ${}^0\sigma$ denotes σ, ${}^1\sigma$ denotes $\neg\sigma$ and for a tree R with $r \in R$,

$$(R)_r = \{r' \in R \mid r' \trianglerighteq r\}.$$

Now we set $T^* = \bigcap_{n<\omega} T_n$. Clearly, T^* is a U-tree with a trunk t by (2) or by Lemma 1.18. Consider $\langle t, T^*\rangle \in \mathcal{P}$. By the assumption, $\langle t, T^*\rangle \nVdash \sigma$. Pick a condition $\langle s, S\rangle \geq \langle t, T^*\rangle$ forcing σ with $n = |s - t|$ as small as possible. Then $s \in \mathrm{Lev}_{n+|t|}(T^*) = \mathrm{Lev}_{n+|t|}(T_n)$. By (3) of the recursive construction,

$$\langle s, (T_n)_s\rangle \Vdash \sigma$$

and for every $s' \in \mathrm{Lev}_{n+|t|}(T_n)$ with the same predecessor as s, $\langle s', (T_n)_{s'}\rangle \Vdash \sigma$. But $T^* \subseteq T_n$, so

$$\langle s, (T^*)_s\rangle \Vdash \sigma \quad \text{and} \quad \langle s', (T^*)_{s'}\rangle \Vdash \sigma$$

for every s' as above.

Let s^* denote the immediate predecessor of s, i.e. s without its last element. Then $\langle s^*, (T^*)_{s^*}\rangle \Vdash \sigma$ since for every $\langle r, R\rangle \geq \langle s^*, (T^*)_{s^*}\rangle$, $r = s'{}^\frown r'$ for some $s' \in \mathrm{Lev}_{n+|t|}(T^*)$ and $s' \rhd s^*$. Hence, $\langle r, R\rangle \geq \langle s', (T^*)_{s'}\rangle \Vdash \sigma$.

But we chose s to be of minimal length such that for some S, $\langle s, S\rangle \Vdash \sigma$, yet $|s^*| = |s| - 1$. Contradiction. ⊣

Now, as in Lemma 1.9 the κ-closure of $\langle \mathcal{P}, \leq^*\rangle$ can be used to derive the following:

1.24 Lemma. *$\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets of κ.*

The conclusions are the same as those of the previous section.

1.25 Theorem. *The following holds in $V[G]$:*

(a) κ has cofinality $\aleph_0$.

(b) All the cardinals are preserved.

(c) No new bounded subsets are added to κ.

1.3. Adding a Prikry Sequence to a Singular Cardinal

Suppose that κ is a limit of an increasing sequence $\langle \kappa_n \mid n < \omega\rangle$ of measurable cardinals. We want to add an ω-sequence dominating every sequence in $\prod_{n<\omega} \kappa_n$, i.e. a sequence $\langle \tau_m \mid m < \omega\rangle \in \prod_{n<\omega} \kappa_n$ such that for every $\langle \rho_m \mid m < \omega\rangle \in (\prod_{n<\omega} \kappa_n) \cap V$ and for all but finitely many m's, $\tau_m > \rho_m$.

Fix a κ_n-complete ultrafilter U_n over κ_n for every $n < \omega$. One can assume normality but it is not necessary.

Let $n < \omega$. We describe first a very simple forcing for adding a one-element Prikry sequence.

1.26 Definition. Let $Q_n = U_n \cup \kappa_n$. If $p, q \in Q_n$ we define $p \geq_n q$ iff either

(1) $p, q \in U_n$ and $p \subseteq q$,

(2) $q \in U_n$ and $p \in q$, or

(3) $p = q \in \kappa_n$.

Thus we can pick a set in U_n, and then shrink it still in U_n or pick an element of this set. In particular, above every condition there is an atomic one. So, the forcing $\langle Q_n, \leq_n\rangle$ is trivial.

Nevertheless we also define a direct extension ordering:

1.27 Definition. Let $p, q \in Q_n$. Set $p \geq_n^* q$ iff $p = q$, or $p, q \in U_n$ and $p \subseteq q$.

The forcing $\langle Q_n, \leq_n, \leq_n^*\rangle$ is called the *one-element Prikry forcing*. The following lemma follows from the κ_n-completeness of U_n.

1.28 Lemma. *$\langle Q_n, \leq_n^*\rangle$ is κ_n-closed.*

1.29 Lemma. *$\langle Q_n, \leq_n, \leq_n^* \rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is a $q \geq_n^* p$ such that $q \parallel \sigma$.*

The proof repeats the first stage of the proof of Lemma 1.23.
We now combine Q_n's together.

1.30 Definition. Let $\mathcal{P}$ be the set of all sequences $p = \langle p_n \mid n < \omega \rangle$ so that

(1) For every $n < \omega$, $p_n \in Q_n$.

(2) There is an $\ell(p) < \omega$ so that for every $n < \ell(p)$, p_n is an ordinal below κ_n and for every $n \geq \ell(p)$, $p_n \in U_n$.

The orderings $\leq$ and $\leq^*$ are defined on $\mathcal{P}$ in obvious fashion:

1.31 Definition. Let $p = \langle p_n \mid n < \omega \rangle$, $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$. We say that $p \geq q$ (resp. $p \geq^* q$) iff for every $n < \omega$, $p_n \geq_n q_n$ (resp. $p_n \geq_n^* q_n$).

For $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$ we denote $\langle p_m \mid m < n \rangle$ by $p{\restriction}n$ and $\langle p_m \mid m \geq n \rangle$ by $p \setminus n$. Let $\mathcal{P}{\restriction}n = \{p{\restriction}n \mid p \in \mathcal{P}\}$ and $\mathcal{P} \setminus n = \{p \setminus n \mid p \in \mathcal{P}\}$.

The following splitting lemma is obvious:

1.32 Lemma. *$\mathcal{P} \simeq \mathcal{P}{\restriction}n \times \mathcal{P} \setminus n$ for every $n < \omega$.*

1.33 Lemma. *For every $n < \omega$, $\langle \mathcal{P} \setminus n, \leq^* \rangle$ is κ_n-closed.*

The above follows from the fact that each U_m with $m \geq n$ is κ_n-complete.

1.34 Lemma. *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p = \langle p_n \mid n < \omega \rangle$ be an element of $\mathcal{P}$ and σ be a statement of the forcing language. Suppose for simplicity that $\ell(p) = 0$. Then let $p_n = A_n \in U_n$ for every $n < \omega$. We want to find a direct extension of p deciding σ. Assume that there is no such extension. Define by recursion on $n < \omega$ a $\leq^*$-increasing sequence $\langle q(n) \mid n < \omega \rangle$ of $\leq^*$-extensions of p such that for every $n < \omega$ the following holds:

(1) If $m \geq n$, then $q(m){\restriction}n = q(n){\restriction}n$.

(2) If $q = \langle q_n \mid n < \omega \rangle \geq q(n)$ decides σ and $\ell(q) = n+1$ then already $\langle q_m \mid m \leq n \rangle^\frown \langle q(n)_m \mid m > n \rangle$ decides σ and in the same way as q; moreover, for every $\tau_n \in q(n)_n$ also $\langle q_m \mid m < n \rangle^\frown \langle \tau_n \rangle^\frown \langle q(n)_m \mid m > n \rangle$ makes the same decision.

The recursive construction is straightforward. At the nth stage, the κ_n-completeness of the U_m's for $m \geq n$ is used in order to take care of the possibilities for initial sequences of length $n-1$ below κ_n. The number of such possibilities is $|\prod_{i \leq n-1} \kappa_i| = \kappa_{n-1} < \kappa_n$. Now define $s = \langle s_n \mid n < \omega \rangle$ to be $\langle q(n)_n \mid n < \omega \rangle$. Clearly, $s \in \mathcal{P}$ and $s \geq^* p$. The conclusion is now as in Lemma 1.23. Thus let $q = \langle q_n \mid n < \omega \rangle$ be an extension of s forcing σ and with $\ell(q)$ as small as possible. By the assumption, $\ell(q) > 0$. Let $n = \ell(q) - 1$. Now, using (2) of the construction, we conclude that

$$\langle q_m \mid m < n\rangle^\frown\langle \tau_n\rangle^\frown\langle s_m \mid m > n\rangle \Vdash \sigma$$

for every $\tau_n \in q(n)_n = s_n$. But then also $\langle q_m \mid m < n\rangle^\frown\langle s_m \mid m \geq n\rangle \Vdash \sigma$, contradicting the minimality of $\ell(q)$. $\dashv$

Combining Lemmas 1.32, 1.33 and 1.34 we obtain the following:

1.35 Lemma. *$\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets to κ.*

Note that for each $n < \omega$, $\mathcal{P}\restriction n$ is just a trivial forcing "adding" a sequence of length n of ordinals in $\prod_{m \leq n-1} \kappa_m$.

1.36 Lemma. *$\langle \mathcal{P}, \leq\rangle$ satisfies the κ^+-c.c.*

Proof. Note that any two conditions $p = \langle p_n \mid n < \omega\rangle$ and $q = \langle q_n \mid n < \omega\rangle$ are compatible provided $\ell(p) = \ell(q)$ and $\langle p_n \mid n < \ell(p)\rangle = \langle q_n \mid n < \ell(q)\rangle$. $\dashv$

Now let $G \subseteq \mathcal{P}$ be generic for $\langle \mathcal{P}, \leq\rangle$. Define an ω-sequence $\langle t_n \mid n < \omega\rangle \in \prod_{n<\omega} \kappa_n$ as follows: $t_n = \tau$ if for some $p = \langle p_m \mid m < \omega\rangle \in G$ with $\ell(p) > n$ $p_n = \tau$.

Using density arguments it is easy to show the following:

1.37 Lemma. *For every $\langle s_n \mid n < \omega\rangle \in (\prod_{n<\omega} \kappa_n) \cap V$ there is an $n_0 < \omega$ such that for every $n \geq n_0$, $t_n > s_n$.*

Combining lemmas together we now obtain the following:

1.38 Theorem. *The following holds in $V[G]$:*

(a) All cardinals and cofinalities are preserved.

(b) No new bounded subsets are added to κ.

(c) There is a sequence in $\prod_{n<\omega} \kappa_n$ dominating every sequence in $(\prod_{n<\omega} \kappa_n) \cap V$.

1.4. Supercompact and Strongly Compact Prikry Forcings

In this section, we present Prikry forcings for supercompact and strongly compact cardinals. The main feature of these forcings is that not only κ changes its cofinality to ω, but also every regular cardinal in the interval $[\kappa, \lambda]$ does so, if we use a λ-supercompact (or strongly compact) cardinal κ. The presentation will follow that of Menachem Magidor who was the first to use these forcings in his celebrated papers [35, 36].

Fix cardinals $\kappa \leq \lambda$. Let $\mathcal{P}_\kappa(\lambda) = \{P \subseteq \lambda \mid |P| < \kappa\}$. Let us recall few basic definitions.

1.39 Definition. An ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ is called *normal* iff

(1) U is κ-complete.

(2) U is fine, i.e. for every $\alpha < \lambda$, $\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\} \in U$.

(3) For every $A \in U$ and every $f : A \to \lambda$ satisfying $f(P) \in P$ for $P \in A$ there are $A' \in U$ and $\alpha' < \lambda$ such that for every $P \in A'$ we have $f(P) = \alpha'$.

1.40 Definition.

(1) κ is called λ-*strongly compact* iff there exists a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

(2) κ is called λ-*supercompact* iff there exists a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

(3) If $P, Q \in \mathcal{P}_\kappa(\lambda)$, then P is *strongly included* in Q iff $P \subseteq Q$ and $\mathrm{otp}(P) < \mathrm{otp}(Q \cap \kappa)$. We denote this by $P \subsetneqq Q$.

Suppose now that κ is λ-supercompact cardinal and U is a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$. The normality of U easily implies the following:

(a) If F is function from a set in U into $\mathcal{P}_\kappa(\lambda)$ such that for all $P \neq \emptyset$ $F(P) \subsetneqq P$, then F is constant on a set in U.

(b) If for every $Q \in \mathcal{P}_\kappa(\lambda)$, $A_Q \in U$, then $\{P \mid \forall Q \subsetneqq P\ (P \in A_Q)\} \in U$. (This last set is called the *diagonal intersection* of the system $\{A_Q \mid Q \in \mathcal{P}_\kappa(\lambda)\}$.)

For $B \subseteq \mathcal{P}_\kappa(\lambda)$, denote by $[B]^{[n]}$ the set of all n element subsets of B totally ordered by $\subsetneqq$; denote $\bigcup_{n<\omega}[B]^{[n]}$ by $[B]^{[<\omega]}$. The following is a straightforward analog of the Rowbottom theorem:

If $F : [\mathcal{P}_\kappa(\lambda)]^{[<\omega]} \to 2$, then there is an $A \in U$ such that for every $n < \omega$, F is constant on $[A]^{[n]}$.

We are now ready to define the *supercompact Prikry forcing* with a normal ultrafilter U over $\mathcal{P}_\kappa(\lambda)$.

The definitions will be the same as in Definition 1.1 with only κ replaced by $\mathcal{P}_\kappa(\lambda)$ and the order on ordinals replaced by $\subsetneqq$.

1.41 Definition. Let $\mathcal{P}$ be the set of all pairs $\langle\langle P_1, \dots, P_n\rangle, A\rangle$ such that

(1) $\langle P_1, \dots, P_n\rangle$ is a finite $\subsetneqq$-increasing sequence of elements of $\mathcal{P}_\kappa(\lambda)$,

(2) $A \in U$, and

(3) for every $Q \in A$, $P_n \subsetneqq Q$.

1.42 Definition. Let $\langle\langle P_1, \dots, P_n\rangle, A\rangle$, $\langle\langle Q_1, \dots, Q_m\rangle, B\rangle \in \mathcal{P}$. Then define $\langle\langle P_1, \dots, P_n\rangle, A\rangle \geq \langle\langle Q_1, \dots, Q_m\rangle, B\rangle$ iff

(1) $n \geq m$,

(2) for every $k \leq m \quad P_k = Q_k$,

(3) $A \subseteq B$, and

(4) $\{P_{m+1}, \dots, P_n\} \subseteq B$.

1.43 Definition. Let $\langle\langle P_1, \dots, P_n\rangle, A\rangle, \langle\langle Q_1, \dots, Q_m\rangle, B\rangle \in \mathcal{P}$. Then $\langle\langle P_1, \dots, P_n\rangle, A\rangle \geq^* \langle\langle Q, \dots, Q_m\rangle, B\rangle$ iff

(1) $\langle P_1, \dots, P_n\rangle = \langle Q_1, \dots, Q_m\rangle$, and

(2) $A \subseteq B$.

The next lemmas are proved as in Definition 1.1 with obvious changes from κ to $\mathcal{P}_\kappa(\lambda)$.

1.44 Lemma. $\leq^* \subseteq \leq$.

1.45 Lemma. *$\langle \mathcal{P}, \leq^*\rangle$ is κ-closed.*

1.46 Lemma (The Prikry condition). *Let $\langle q, B\rangle \in \mathcal{P}$ and σ be a statement of the forcing language (i.e. of $\langle \mathcal{P}, \leq\rangle$). Then there is a $\langle p, A\rangle \geq^* \langle q, B\rangle$ such that $\langle p, A\rangle \parallel \sigma$.*

1.47 Lemma. *$\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets to κ.*

1.48 Lemma. *$\langle \mathcal{P}, \leq\rangle$ satisfies the $(\lambda^{<\kappa})^+$-c.c.*

Proof. As in Lemma 1.5, any two conditions with the same finite sequence, i.e. of the form $\langle p, A\rangle$ and $\langle p, B\rangle$ are compatible. The number of possibilities for p's now is $\lambda^{<\kappa}$. So we are done. $\dashv$

By the theorem of Solovay (see [55] or [27]), $\lambda^{<\kappa} = \lambda$ if λ is regular or of cofinality $\geq \kappa$, and $\lambda^{<\kappa} = \lambda^+$ if $\mathrm{cf}(\lambda) < \kappa$. Note that λ-supercompactness of κ actually implies its $\lambda^{<\kappa}$-supercompactness. We can restate Lemma 1.48 using Solovay's theorem as follows:

1.49 Lemma. *$\langle \mathcal{P}, \leq\rangle$ satisfies the μ^+-c.c., where*

$$\mu = \begin{cases} \lambda, & \text{if } \mathrm{cf}(\lambda) \geq \kappa \\ \lambda^+, & \text{if } \mathrm{cf}(\lambda) < \kappa. \end{cases}$$

Our next lemma presents the main property of the supercompact Prikry forcing. Also, it shows that Lemma 1.49 is sharp.

Let G be a generic subset of $\langle \mathcal{P}, \leq\rangle$ and let $\langle P_n \mid 1 \leq n < \omega\rangle$ be the Prikry sequence produced by G, i.e. the sequence such that for every $n < \omega$, there is an $A \in U$ with $\langle\langle P_1, \dots, P_n\rangle, A\rangle \in G$.

1.50 Lemma. *Every $\delta \in [\kappa, \mu]$ of cofinality $\geq \kappa$ (in V) changes its cofinality to ω in $V[G]$, where*

$$\mu = \begin{cases} \lambda, & \text{if } \operatorname{cf}(\lambda) \geq \kappa \\ \lambda^+, & \text{if } \operatorname{cf}(\lambda) < \kappa. \end{cases}$$

Moreover, for each $\delta \leq \lambda$, $\delta = \bigcup_{n<\omega}(P_n \cap \delta)$, i.e. it is a countable union of old sets each of cardinality less than κ.

Proof. Let $\alpha < \lambda$. The fineness of U implies that $\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\} \in U$. Then, by a density argument, $\alpha \in P_n$ for all but finitely many n's. Hence, for each $\delta \leq \lambda$

$$\delta = \bigcup_{n<\omega}(P_n \cap \delta).$$

This implies that each $\delta \leq \lambda$ of cofinality $\geq \kappa$ in V changes cofinality to ω in $V[G]$, as witnessed by $\langle \sup(P_n \cap \delta) \mid n < \omega \rangle$. In order to finish the proof, we need to deal with λ of cofinality below κ and to show that in this case λ^+ also changes its cofinality to ω. Fix in V a sequence cofinal in λ of regular cardinals $\langle \lambda_i \mid i < \operatorname{cf}(\lambda) \rangle$, a sequence of functions $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ in $\prod_{i<\operatorname{cf}(\lambda)} \lambda_i$ and an ultrafilter D over $\operatorname{cf}(\lambda)$ including all cobounded subsets of $\operatorname{cf}(\lambda)$, so that

(a) $\alpha < \beta < \lambda^+ \implies f_\alpha < f_\beta \pmod D$, and

(b) for every $g \in \prod_{i<\operatorname{cf}(\lambda)} \lambda_i$ there is an $i < \lambda^+$ such that $f_i > g \pmod D$.

Using $\lambda^{<\kappa} = \lambda^+$, it is not hard directly by induction to construct such sequence of f_i's. One can also appeal to general pcf considerations; see [1]. Now, by fineness and density again, for every $\alpha < \lambda^+$ and for all but finitely many $n < \omega$ we will have $P_n \supseteq \operatorname{ran}(f_\alpha)$. Hence, for such n's, $\langle \bigcup(P_n \cap \lambda_i) \mid i < \operatorname{cf}(\lambda) \rangle > f_\alpha$. So, $\{\langle \bigcup(P_n \cap \lambda_i) \mid i < \operatorname{cf}(\lambda) \rangle \mid n < \omega\}$ will be an ω-sequence of functions from $(\prod_{i<\operatorname{cf}(\lambda)} \lambda_i) \cap V$ unbounded in $(\prod_{i<\operatorname{cf}(\lambda)} \lambda_i) \cap V$. This implies that λ^+ should have cofinality ω in $V[G]$. ⊣

Let us now turn to *strongly compact Prikry forcing.* So, we give up normality and assume only that U is a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$. The construction here is completely parallel to the construction of the tree Prikry forcing in Definition 1.2.

1.51 Definition. A set T is called a *U-tree with trunk t* iff

(1) T consists of finite sequences $\langle P_1, \ldots, P_n \rangle$ of elements of $\mathcal{P}_\kappa(\lambda)$ so that $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$.

(2) $\langle T, \trianglelefteq \rangle$ is a tree, where $\trianglelefteq$ is the order of the end extension of finite sequences.

(3) t is a trunk of T, i.e. $t \in T$ and for every $\eta \in T$, $\eta \trianglerighteq t$ or $t \trianglerighteq \eta$.

(4) For every $\eta \trianglerighteq t$,

$$\mathrm{Suc}_T(\eta) = \{Q \in \mathcal{P}_\kappa(\lambda) \mid \eta^\frown\langle Q\rangle \in T\} \in U.$$

The definitions of the forcing notion $\mathcal{P}$ and the orders $\leq$ and $\leq^*$ are now exactly the same as those in Definitions 1.15, 1.16 and 1.17. $\langle \mathcal{P}, \leq, \leq^*\rangle$ here shares all the properties of the tree Prikry forcing of Sect. 1.2 except the κ^+-c.c. Thus the Lemmas 1.18, 1.21–1.24 are valid in the present context with basically the same proofs. Instead of the κ^+-c.c. we will have here the $(\lambda^{<\kappa})^+$-c.c. Also Lemmas 1.48–1.50 hold with the same proofs.

Let us summarize the properties of both supercompact and strongly compact Prikry forcings.

1.52 Theorem. *Let G be a generic set for $\langle \mathcal{P}, \leq, \leq^*\rangle$, where $\langle \mathcal{P}, \leq, \leq^*\rangle$ is either supercompact or strongly compact Prikry forcing over $\mathcal{P}_\kappa(\lambda)$. The following holds in $V[G]$:*

(a) No new bounded subsets are added to κ.

(b) Every cardinal in the interval $[\kappa, \mu]$ of cofinality $\geq \kappa$ (as computed in V) changes its cofinality to ω.

(c) All the cardinals above μ are preserved, where

$$\mu = \begin{cases} \lambda, & \text{if } \mathrm{cf}(\lambda) \geq \kappa \\ \lambda^+, & \text{if } \mathrm{cf}(\lambda) < \kappa. \end{cases}$$

2. Adding Many Prikry Sequences to a Singular Cardinal

In this section we present the extender-based Prikry forcing over a singular cardinal. It is probably the simplest direct way for violating the Singular Cardinal Hypothesis using minimal large cardinal hypotheses. This type of forcing first appeared in [20] in a more complicated form. The presentation here follows [17, Sect. 3].

Let, as in Definition 1.3, $\kappa = \bigcup_{n<\omega} \kappa_n$ with $\langle \kappa_n \mid n < \omega\rangle$ increasing and each κ_n measurable. The Prikry forcing described in Definition 1.3 produces basically one Prikry sequence. More precisely, if GCH holds in the ground model, then κ^+-many new ω-sequences are introduced but all of them are coded by the generic Prikry sequence. Here we present a way for adding any number of Prikry sequences into $\prod_{n<\omega} \kappa_n$. In particular, this will increase the power of κ as large as one likes without adding new bounded subsets and preserving all the cofinalities.

The basic idea is to use many ultrafilters over each of the κ_n's instead of a single one as in Definition 1.3. This leads naturally to extenders over the κ_n's. For the basics about extenders and corresponding large cardinal hypotheses, which are significantly weaker than λ-supercompactness of Lemma 1.4, see the fine structure and inner model chapters of this Handbook.

Assume GCH and let $\lambda \geq \kappa^+$ be a regular cardinal. Suppose that we want to add to κ or into $\prod_{n<\omega} \kappa_n$ at least λ many Prikry sequences. Our basic assumption will now be that each κ_n is a $(\lambda+1)$-strong cardinal. This means that for every $n < \omega$ there is a $(\kappa_n, \lambda+1)$-extender E_n over κ_n whose ultrapower contains $V_{\lambda+1}$ and which moves κ_n above λ. We fix such E_n and let $j_n : V \to M_n \simeq \mathrm{Ult}(V, E_n)$. For every $\alpha < \lambda$ we define a κ_n-complete ultrafilter $U_{n\alpha}$ over κ_n by setting $X \in U_{n\alpha}$ iff $\alpha \in j_n(X)$. Actually only $U_{n\alpha}$'s with $\alpha \geq \kappa_n$ will be important. Note that a lot of $U_{n\alpha}$'s are comparable in the Rudin-Keisler order $\leq_{\mathrm{RK}}$, recalling that

$$\begin{aligned} U \leq_{\mathrm{RK}} W \quad \text{iff} \quad & \exists f : \bigcup W \to \bigcup U \ \forall X \subseteq \bigcup U \\ & (X \in U \leftrightarrow f^{-1}(X) \in W). \end{aligned}$$

Thus for example, if α is a cardinal and $\beta \leq \alpha$, then $U_{n(\alpha+\beta)} \geq_{\mathrm{RK}} U_{n,\alpha}$ and $U_{n(\alpha+\beta)} \geq_{\mathrm{RK}} \ U_{n,\beta}$.

We will need a strengthening of the Rudin-Keisler order. For $\alpha, \beta < \lambda$ define

$$\begin{aligned} \alpha \leq_{E_n} \beta \quad \text{iff} \quad & \alpha \leq \beta \text{ and} \\ & \text{for some } f \in {}^{\kappa_n}\kappa_n,\ j_n(f)(\beta) = \alpha. \end{aligned}$$

Clearly, then $\alpha \leq_{E_n} \beta$ implies $U_{n\alpha} \leq_{\mathrm{RK}} U_{n\beta}$, as witnessed by any $f \in {}^{\kappa_n}\kappa_n$ with $j_n(f)(\beta) = \alpha$: If $A \in U_{n\beta}$, then $\beta \in j_n(A)$. So $\alpha = j_n(f)(\beta) \in j_n(f)``j_n(A) = j_n(f``A)$. Hence $f``A \in U_{n,\alpha}$. Note that, in general, $\alpha < \beta < \lambda$ and $U_{n\alpha} <_{\mathrm{RK}} U_{n\beta}$ does not imply $\alpha <_{E_n} \beta$.

The partial order $\langle \lambda, \leq_{E_n} \rangle$ is κ_n-directed, as we see in Lemma 2.1 below. Actually, it is κ_n^{++}-directed, but for our purposes κ_n-directness will suffice. Thus, using GCH, find some enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ so that for every regular cardinal $\delta < \kappa_n$, $\langle a_\alpha \mid \alpha < \delta \rangle$ enumerates $[\delta]^{<\delta}$ and every element of $[\delta]^{<\delta}$ appears δ many times in the enumeration. Let $j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle) = \langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle$. Then, $\langle a_\alpha \mid \alpha < \lambda \rangle$ will enumerate $[\lambda]^{<\lambda} \supseteq [\lambda]^{<\kappa_n}$ in both M_n and V; this coding will be applied below.

The next lemma is a basic application of commutativity of diagrams corresponding to extenders and their ultrafilters.

2.1 Lemma. *Let $n < \omega$ and $\tau < \kappa_n$. Suppose that $\langle \alpha_\nu \mid \nu < \tau \rangle$ is a sequence of ordinals below λ and $\alpha \in \lambda \setminus (\bigcup_{\nu<\tau} \alpha_\nu + 1)$ codes this sequence, i.e. $a_\alpha = \{\alpha_\nu \mid \nu < \tau\}$. Then $\alpha >_{E_n} \alpha_\nu$ for every $\nu < \tau$.*

Proof. Fix $\nu < \tau$. Consider the following diagram

$$\begin{tikzcd}
& M_n \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] \\
& N_{\alpha_\nu} \arrow[uu, bend right=60, "k_{\alpha_\nu}"']
\end{tikzcd}$$

where $N_\alpha \simeq \mathrm{Ult}(V, U_{n,\alpha})$, $k_\alpha([f]_{U_{n,\alpha}}) = j_n(f)(\alpha)$ and the same with α_ν replacing α. Then $j_n(\langle a_\beta \mid \beta < \kappa_n\rangle) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n\rangle))$ and $k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa_n\rangle)([\mathrm{id}])_{U_{n,\alpha}})) = j_n(\langle a_\beta \mid \beta < \kappa_n\rangle)(\alpha) = a_\alpha = \{\alpha_\mu \mid \mu < \tau\}$. But $\tau < \kappa_n$, so it is fixed by k_α, since $\mathrm{crit}(k_\alpha) \geq \kappa_n$. Hence $i_\alpha(\langle a_\beta \mid \beta < \kappa_n\rangle)([\mathrm{id}]_{U_{n,\alpha}})$ is a sequence of ordinals of length τ. Let α^*_ν denote its ν-th element. Then, by elementarity, $k_\alpha(\alpha^*_\nu) = \alpha_\nu$. We can hence define $k_{\alpha_\nu \alpha} : N_{\alpha_\nu} \longrightarrow N_\alpha$ by setting $k_{\alpha_\nu\alpha}([f]_{U_{\alpha_\nu}}) = i_\alpha(f)(\alpha^*_\nu)$. It is easy to see that $k_{\alpha_\nu\alpha}$ is elementary embedding and the following diagram is commutative.

$$\begin{tikzcd}
& M_n \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] \\
& N_{\alpha_\nu} \arrow[u, "k_{\alpha_\nu\alpha}"'] \arrow[uu, bend right=60, "k_{\alpha_\nu}"']
\end{tikzcd}$$

Finally, we can define the desired projection $\pi_{\alpha\alpha_\nu}$ of $U_{n,\alpha}$ onto U_{n,α_ν}. Thus let $\pi_{\alpha\alpha_\nu} : \kappa_n \to \kappa_n$ be a function such that $[\pi_{\alpha\alpha_\nu}]_{U_{n,\alpha}} = \alpha^*_\nu$. Then, $j_n(\pi_{\alpha\alpha_\nu})(\alpha) = k_\alpha\ ([\pi_{\alpha,\alpha_\nu}]_{U_{n,\alpha}}) = k_\alpha(\alpha^*_\nu) = \alpha_\nu$. So, $\alpha >_{E_n} \alpha_\nu$. ⊣

Hence we obtain the following:

2.2 Lemma. *For every set $a \subseteq \lambda$ of cardinality less than κ_n, there are λ many α's below λ so that $\alpha >_{E_n} \beta$ for every $\beta \in a$.*

For every $\alpha, \beta < \lambda$ such that $\alpha >_{E_n} \beta$ we fix the projection $\pi_{\alpha\beta} : \kappa_n \to \kappa_n$ defined as in Lemma 2.1 witnessing this. Let $\pi_{\alpha\alpha} = \mathrm{id}$, the identity map: $\kappa_n \to \kappa_n$.

The following two lemmas are standard.

2.3 Lemma. *Let $\gamma < \beta \leq \alpha < \lambda$. If $\alpha \geq_{E_n} \beta$ and $\alpha \geq_{E_n} \gamma$, then $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_{n\alpha}$.*

Proof. We consider the following commutative diagram

$$
\begin{tikzcd}
 & M_n \\
V \arrow[ur, "j_n"] \arrow[r, "i_\alpha"] \arrow[dr, "i_\beta"] \arrow[ddr, "i_\gamma"] & N_\alpha \arrow[u, "k_\alpha"'] \\
 & N_\beta \arrow[u, "k_{\beta\alpha}"'] \arrow[uu, bend right, "k_\beta"'] \\
 & N_\gamma \arrow[uu, bend right, "k_{\gamma\alpha}"'] \arrow[uuu, bend right=60, "k_\gamma"']
\end{tikzcd}
$$

where for $\delta', \delta \in \{\alpha, \beta, \gamma\}$

$$i_\delta : V \longrightarrow N_\delta \simeq \mathrm{Ult}(V, U_{n\delta})$$

$$k_\delta([f]_{U_{n\delta}}) = j_n(f)(\delta)$$

and

$$k_{\delta'\delta}([f]_{U_{n\delta'}}) = i_\delta(f)([\pi_{\delta\delta'}]_{U_{n\delta}}).$$

Then $k_\alpha([\pi_{\alpha\beta}]_{U_{n\alpha}}) = k_\alpha(k_{\beta\alpha}([\mathrm{id}]_{U_{n\beta}})) = k_\beta([\mathrm{id}]_{U_{n\beta}}) = j_n(\mathrm{id})(\beta) = \beta$. The same is true for γ, i.e.

$$k_\alpha([\pi_{\alpha\gamma}]_{U_{n\alpha}}) = \gamma.$$

But $M_n \vDash \gamma < \beta$ and k_α is elementary, so $N_\alpha \vDash [\pi_{\alpha\gamma}]_{U_{n\alpha}} < [\pi_{\alpha\beta}]_{U_{n\alpha}}$. Hence

$$\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_{n\alpha}.$$

$\dashv$

2.4 Lemma. *Let $\{\alpha_i \mid i < \tau\} \subseteq \alpha < \lambda$ for some $\tau < \kappa_n$. Assume that $\alpha \geq_{E_n} \alpha_i$ for every $i < \tau$. Then there is a set $A \in U_{n\alpha}$ so that for every $i, j < \tau$: $\alpha_i \geq_{E_n} \alpha_j$ implies $\pi_{\alpha\alpha j}(\nu) = \pi_{\alpha_i\alpha_j}(\pi_{\alpha\alpha_i}(\nu))$ for every $\nu \in A$.*

Proof. It is enough to prove the lemma for $\tau = 2$ and then to use the κ_n-completeness of $U_{n\alpha}$. So, let $\beta, \gamma < \alpha$ and assume that $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$.

Consider the following commutative diagram:

$$\begin{tikzcd}
 & M_n \\
 & N_\alpha \arrow[u, "k_\alpha"'] \\
V \arrow[uur, "j_n"] \arrow[ur, "i_\alpha"] \arrow[dr, "i_\beta"'] \arrow[ddr, "i_\gamma"'] & \\
 & N_\beta \arrow[uu, "k_{\beta\alpha}"'] \arrow[uuu, bend right, "k_\beta"'] \\
 & N_\gamma \arrow[u, "k_{\gamma\beta}"'] \arrow[uuu, bend right, "k_{\gamma\alpha}"'] \arrow[uuuu, bend right=60, "k_\gamma"']
\end{tikzcd}$$

where k's and i's are defined as in Lemma 2.3.

We need to show that

$$[\pi_{\alpha\gamma}]_{U_{n\alpha}} = [\pi_{\beta\gamma} \circ \pi_{\alpha\beta}]_{U_{n\alpha}}.$$

As in Lemma 2.3, $k_\alpha([\pi_{\alpha\gamma}]_{U_{n\alpha}}) = \gamma$. On the other hand, again as Lemma 2.3,

$$\begin{aligned} k_\alpha([\pi_{\beta\gamma} \circ \pi_{\alpha\beta}]_{U_{n\alpha}}) &= j_n(\pi_{\beta\gamma} \circ \pi_{\alpha\beta})(\alpha) = j_n(\pi_{\beta\gamma})(j_n(\pi_{\alpha\beta})(\alpha)) \\ &= j_n(\pi_{\beta\gamma})(\beta) = \gamma. \end{aligned}$$

Since k_α is elementary, we have in N_α the desired equality. ⊣

We are now ready to define our first forcing notion. It will resemble the one-element Prikry forcing considered in Definition 1.3 and will be built from two pieces. Fix $n < \omega$.

2.5 Definition. Let $Q_{n1} = \{f \mid f$ is a partial function from λ to κ_n of cardinality at most $\kappa\}$. We order Q_{n1} by inclusion, which here is denoted by $\leq_1$.

Thus Q_{n1} is basically the usual Cohen forcing for blowing up the power of κ^+ to λ. The only, and minor, change is that the functions take values inside κ_n rather than 2 or κ^+.

2.6 Definition. Let Q_{n0} be the set of triples $\langle a, A, f\rangle$ so that

(1) $f \in Q_{n1}$.

(2) $a \subseteq \lambda$ with

(2a) $|a| < \kappa_n$,

(2b) $a \cap \mathrm{dom}(f) = \emptyset$, and

(2c) a has a $\leq_{E_n}$-maximal element, i.e. an element $\alpha \in a$ such that $\alpha \geq_{E_n} \beta$ for every $\beta \in a$.

(3) $A \in U_{n\,\max(a)}$.

(4) For every $\alpha, \beta, \gamma \in a$, if $\alpha \geq_{E_n} \beta \geq_{E_n} \gamma$, then $\pi_{\alpha\gamma}(\rho) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\rho))$ for every $\rho \in \pi_{\max(a),\alpha}$"$A$.

(5) For every $\alpha > \beta$ in a and every $\nu \in A$,

$$\pi_{\max(a),\alpha}(\nu) > \pi_{\max(a),\beta}(\nu).$$

The last two conditions can be met by Lemmas 2.3, 2.4.

2.7 Definition. Let $\langle a, A, f\rangle$, $\langle b, B, g\rangle \in Q_{n0}$. We say that $\langle a, A, f\rangle$ *is stronger than* $\langle b, B, g\rangle$ and denote this by $\langle a, A, f\rangle \geq_0 \langle b, B, g\rangle$ iff

(1) $f \supseteq g$,

(2) $a \supseteq b$, and

(3) $\pi_{\max(a),\max(b)}$"$A \subseteq B$.

We now define a forcing notion Q_n which is an extender analog of the one-element Prikry forcing of Definition 1.3.

2.8 Definition. $Q_n = Q_{n0} \cup Q_{n1}$.

2.9 Definition. The direct extension ordering $\leq^*$ on Q_n is defined to be $\leq_0 \cup \leq_1$.

2.10 Definition. Let $p, q \in Q_n$. Then $p \leq q$ iff either

(1) $p \leq^* q$, or

(2) $p = \langle a, A, f\rangle \in Q_{n0}$, $q \in Q_{n1}$ and the following holds:

(2a) $q \supseteq f$,

(2b) $\mathrm{dom}(q) \supseteq a$,

(2c) $q(\max(a)) \in A$, and

(2d) for every $\beta \in a$, $q(\beta) = \pi_{\max(a),\beta}(q(\max(a)))$.

Clearly, the forcing $\langle Q_n, \leq\rangle$ is equivalent to $\langle Q_{n1}, \leq_1\rangle$, i.e. Cohen forcing. However, the following basic facts relate it to the Prikry-type forcing notion.

2.11 Lemma. *$\langle Q_n, \leq^*\rangle$ is κ_n-closed.*

2.12 Lemma. *$\langle Q_n, \leq, \leq^*\rangle$ satisfies the Prikry condition, i.e. for every $p \in Q_n$ and every statement σ of the forcing language there is a $q \geq^* p$ deciding σ.*

Proof. Let $p = \langle a, A, f\rangle$. Suppose otherwise. By recursion on $\nu \in A$ define an increasing sequence $\langle p_\nu \mid \nu \in A\rangle$ of elements of Q_{n1} with $\mathrm{dom}(p_\nu) \cap a = \emptyset$ as follows. Suppose that $\langle p_\rho \mid \rho \in A \cap \nu\rangle$ is defined and $\nu \in A$. Define p_ν as follows: Let $u = \bigcup_{\rho<\nu} p_\rho$. Then $u \in Q_{n1}$. Consider $q = \langle a, A, u\rangle$. Let $q^\frown\langle\nu\rangle = u \cup \{\langle \beta, \pi_{\max(a),\beta}(\nu)\rangle \mid \beta \in a\}$. If there is a $p \geq_1 q^\frown\langle\nu\rangle$ deciding σ, then let p_ν be some such p restricted to $\lambda \setminus a$. Otherwise, set $p_\nu = u$. Note that there will always be a condition deciding σ.

Finally, let $g = \bigcup_{\nu\in A} p_\nu$. Shrink A to a set $B \in U_{n\max(a)}$ so that $p_\nu{}^\frown\langle\nu\rangle = p_\nu \cup \{\langle\beta, \pi_{\max(a),\beta}(\nu)\rangle \mid \beta \in a\}$ decides σ the same way or does not decide σ at all, for every $\nu \in B$. By our assumption $\langle a, B, g\rangle \not\parallel \sigma$. However, pick some $h \geq \langle a, B, g\rangle$, $h \in Q_{n1}$ deciding on σ. Let $h(\max(a)) = \nu$. Then, $p_\nu{}^\frown\langle\nu\rangle$ decides σ. But this holds then for every $\nu \in B$. Hence, already $\langle a, B, g\rangle$ decides σ. Contradiction. $\dashv$

Let us now define the main forcing of this section by putting the blocks of Q_n's together. This forcing is called the *extender-based Prikry forcing* over a singular cardinal.

2.13 Definition. The set $\mathcal{P}$ consists of sequences $p = \langle p_n \mid n < \omega\rangle$ so that

(1) For every $n < \omega$, $p_n \in Q_n$.

(2) There is an $\ell(p) < \omega$ so that for every $n < \ell(p)$, $p_n \in Q_{n1}$, and for every $n \geq \ell(p)$, $p_n = \langle a_n, A_n, f_n\rangle \in Q_{n0}$ and $a_n \subseteq a_{n+1}$.

2.14 Definition. Let $p = \langle p_n \mid n < \omega\rangle$ and $q = \langle q_n \mid n < \omega\rangle \in \mathcal{P}$. We set $p \geq q$ (resp. $p \geq^* q$) iff for every $n < \omega$, $p_n \geq_{Q_n} q_n$ (resp. $p_n \geq^*_{Q_n} q_n$).

The forcing $\langle\mathcal{P}, \leq\rangle$ does not satisfy the κ^+-c.c. However:

2.15 Lemma. *$\langle\mathcal{P}, \leq\rangle$ satisfies the κ^{++}-c.c.*

Proof. Let $\{p(\alpha) \mid \alpha < \kappa^{++}\}$ be a set of elements of $\mathcal{P}$, with $p(\alpha) = \langle p(\alpha)_n \mid n < \omega\rangle$ and $p(\alpha)_n = \langle a(\alpha)_n, A(\alpha)_n, f(\alpha)_n\rangle$ for $n \geq \ell(p(\alpha))$. There is an $S \subseteq \kappa^{++}$ stationary such that for every $\alpha, \beta \in S$ the following holds:

(a) $\ell(p(\alpha)) = \ell(p(\beta)) = \ell$.

(b) For every $n < \ell$, $\{\mathrm{dom}(p(\alpha)_n) \mid \alpha \in S\}$ forms a Δ-system with $p(\alpha)_n$ and $p(\beta)_n$ having the same values on its kernel.

(c) For every $n \geq \ell$, $\{(a(\alpha)_n \cup \mathrm{dom}(f(\alpha)_n) \mid \alpha \in S\}$ forms a Δ-system with $f(\alpha)_n$, $f(\beta)_n$ having the same values on the kernel. Also, if $\alpha, \beta \in S$ then $a(\alpha)_n \cap \mathrm{dom}(f(\beta)_n) = \emptyset$.

Now let $\alpha < \beta$ be in S. We construct a condition $q = \langle q_n \mid n < \omega\rangle$ stronger than both $p(\alpha)$ and $p(\beta)$.

For every $n < \ell$ let $q_n = p(\alpha)_n \cup p(\beta)_n$. Now suppose that $n \geq \ell$. q_n will be of the form $\langle b_n, B_n, g_n\rangle$. Set $g_n = f(\alpha)_n \cup f(\beta)_n$. We would like to define

b_n as the union of $a(\alpha)_n$ and $a(\beta)_n$. But Definition 2.6(2(iii)) requires the existence of a maximal element in the $\leq_{E_n}$ order which need not be the case in the simple union of $a(\alpha)_n$ and $a(\beta)_n$. It is easy to fix this. Just pick some $\rho < \lambda$ above $a(\alpha)_n \cup a(\beta)_n$ in the $\leq_{E_n}$ order. Also let $\rho > \sup(\mathrm{dom}(f(\alpha)_n)) + \sup(\mathrm{dom}\ (f(\beta))_n)$. Lemma 2.2 insures that there are such ρ's. Now we set $b_n = a(\alpha)_n \cup a(\beta)_n \cup \{\rho\}$. Let $B'_n = \pi^{-1}_{\rho\alpha^*}(A(\alpha)_n) \cap \pi^{-1}_{\rho\beta^*}(A(\beta)_n)$, where $\alpha^* = \max(a(\alpha)_n)$ and $\beta^* = \max(a(\beta)_n)$. Finally we shrink B'_n to a set $B_n \in U_{n\rho}$ satisfying Definition 2.6((4), (5)). This is possible by Lemmas 2.3, 2.4. ⊣

For $p = \langle p_n \mid n < \omega\rangle \in \mathcal{P}$ set $p{\restriction}n = \langle p_m \mid m < n\rangle$ and $p \backslash n = \langle p_m \mid m \geq n\rangle$. Let $\mathcal{P}{\restriction}n = \{p{\restriction}n \mid p \in \mathcal{P}\}$ and $\mathcal{P} \setminus n = \{p \setminus n \mid p \in \mathcal{P}\}$. Then the following lemmas are obvious:

2.16 Lemma. *$\mathcal{P} \simeq \mathcal{P}{\restriction}n \times \mathcal{P} \setminus n$ for every $n < \omega$.*

2.17 Lemma. *$\langle \mathcal{P} \setminus n, \leq^*\rangle$ is κ_n-closed. Moreover, if $\langle p^\alpha \mid \alpha < \delta < \kappa\rangle$ is a $\leq^*$ increasing sequence with $\kappa_{\ell(p_0)} > \delta$, then there is a $p \geq^* p^\alpha$ for every $\alpha < \delta$.*

We will now turn to the Prikry condition and establish a more general statement which will allow us to deduce in addition that κ^+ is preserved after forcing with $\langle \mathcal{P}, \leq\rangle$.

Let us introduce first some notation. For $p = \langle p_n \mid n < \omega\rangle \in \mathcal{P}$ and m with $\ell(p) \leq m < \omega$, let $p_m = \langle a_m, A_m, f_m\rangle$. Denote a_m by $a_m(p)$, A_m by $A_m(p)$ and f_m by $f_m(p)$. Let $\langle \nu_{\ell(p)}, \dots, \nu_m\rangle \in \prod_{k=\ell(p)}^m A_k(p)$. We denote by

$$p^\frown\langle \nu_{\ell(p)}, \dots, \nu_m\rangle$$

the condition obtained from p by adding the sequence $\langle \nu_{\ell(p)}, \dots, \nu_m\rangle$, i.e. a condition $q = \langle q_n \mid n < \omega\rangle$ such that $q_n = p_n$ for every n, $n < \ell(p)$ or $n > m$, and if $\ell(p) \leq n \leq m$ then $q_n = f_n(p) \cup \{\langle \beta, \pi_{\max(a_n(p)),\beta}(\nu_n)\rangle \mid \beta \in a_n(p)\}$.

We prove the following analog of Lemma 1.13:

2.18 Lemma. *Let $p \in \mathcal{P}$ and D be a dense open subset of $\langle \mathcal{P}, \leq\rangle$ above p. Then there are $p^* \geq^* p$ and $n^* < \omega$ such that for every $\langle \nu_0, \dots, \nu_{n^*-1}\rangle \in \prod_{m=\ell(p)}^{\ell(p)+n^*-1} A_m(p^*)$, $p^{*\frown}\langle \nu_0, \dots, \nu_{n^*-1}\rangle \in D$.*

Let us first deduce the Prikry condition from this lemma.

2.19 Lemma. *Let $p \in \mathcal{P}$ and σ be a statement of the forcing language. Then there is a $p^* \geq^* p$ deciding σ.*

Proof of Lemma 2.19 from Lemma 2.18. Consider $D = \{q \in \mathcal{P} \mid q \geq p$ and $q \parallel \sigma\}$. Clearly, D is dense open above p. Apply Lemma 2.18 to this D and choose n^* as small as possible and $p^* \geq^* p$ such that for every $q \geq p^*$ with $\ell(q) \geq n^*$, $q \in D$. If $n^* = \ell(p)$, then we are done. Suppose otherwise.

Assume for simplicity that $\ell(p) = 0$ and $n^* = 2$. Then let $p^* = \langle p_n^* \mid n < \omega\rangle$ and for every $n < \omega$ let $p_n^* = \langle a_n^*, A_n^*, f_n^*\rangle$. Let $\alpha_0 = \max(a_0^*)$ and $\alpha_1 = \max(a_1^*)$. Then $A_0^* \in U_{0\alpha_0}$ and $A_1^* \in U_{1\alpha_1}$. Let $\nu_0 \in A_0^*$ and $\nu_1 \in A_1^*$. Consider $p^{*\frown}\langle \nu_0, \nu_1\rangle$ the condition obtained from p^* by adding ν_0 and ν_1. Clearly, $\ell(p^{*\frown}\langle \nu_0, \nu_1\rangle) = 2$. Hence it decides σ. Now we shrink A_1^* to $A_{1\nu_0}^*$ so that for every $\nu_1', \nu_1'' \in A_{1\nu_0}^*$ $p^{*\frown}\langle \nu_0, \nu_1'\rangle$ and $p^{*\frown}\langle \nu_0, \nu_1''\rangle$ decide σ the same way. Let $A_1^{**} = \bigcap\{A_{1\nu_0}^* \mid \nu_0 \in A_0^*\}$. We shrink now A_0^* to A_0^{**} so that for every $\nu_0', \nu_0'' \in A_0^{**}$ and for every $\nu_1 \in A_1^{**}$, $p^{*\frown}\langle \nu_0', \nu_1\rangle$ and $p^{*\frown}\langle \nu_0'', \nu_1\rangle$ decide σ in the same way. Let p^{**} be a condition obtained from p^* by replacing in it A_0^* by A_0^{**} and A_1^* by A_1^{**}. Then $p^{**} \geq^* p^*$ and $p^{**} \parallel \sigma$. Contradiction.

Proof of Lemma 2.18. The main objective is to reduce the problem to the point where we can use the argument of the corresponding fact in Sect. 1.3, as if we were forcing using $\langle U_{n\max(a_n)} \mid n < \omega\rangle$.

We first prove the following crucial claim:

Claim. *There is a $p' \geq^* p$, $p' = \langle p_n' \mid n < \omega\rangle$, such that for every $q \geq p'$, $q = \langle q_n \mid n < \omega\rangle$, if $q \in D$, then also*

$$\langle p_n' \mid n < \ell(p)\rangle^\frown\langle q_n\restriction a_n(p') \cup f_n(p') \mid \ell(p') \leq n < \ell(q)\rangle^\frown\langle p_n' \mid n \geq \ell(q)\rangle \in D,$$

where $p_n' = \langle a_n(p'), A_n(p'), f_n(p')\rangle$ for $n \geq \ell(p)$.

Proof of Claim. Choose a function $h : \kappa \leftrightarrow [\kappa]^{<\omega}$, such that for every $n < \omega$, $h\restriction\kappa_n : \kappa_n \leftrightarrow [\kappa_n]^{<\omega}$. Now define by recursion a $\leq^*$-increasing sequence $\langle p^\alpha \mid \alpha < \kappa\rangle$ of direct extensions of p, where $p^\alpha = \langle p_n^\alpha \mid n < \omega\rangle$ and, for $n \geq \ell(p)$, $p_n^\alpha = \langle a_n^\alpha, A_n^\alpha, f_n^\alpha\rangle$. Set $p^0 = p$. Suppose that $\alpha < \kappa$ and $\langle p^\beta \mid \beta < \alpha\rangle$ has been defined. As a recursive assumption we assume the following:

$$(*) \qquad \begin{array}{l}\text{For every } n < \omega \text{ and for } \beta, \gamma, \kappa_n \leq \beta, \gamma < \kappa,\\ \text{if } \ell(p) \leq m \leq n+1, \text{then } a_m^\beta = a_m^\gamma \text{ and } A_m^\beta = A_m^\gamma.\end{array}$$

Let $\tilde{p}^\alpha$ be $p^{\alpha-1}$ if α is successor ordinal, and a direct extension of $\langle p^\beta \mid \beta < \alpha\rangle$ satisfying $(*)$ if α is a limit ordinal. Note that if $n < \omega$ is the maximal such that $\alpha \geq \kappa_n$ then Lemma 2.17 applies, since the parts of p^β's below κ_{n+1} satisfy $(*)$. Now we consider $h(\alpha)$. Let $h(\alpha) = \langle \nu_1, \dots, \nu_k\rangle$. If $\langle \nu_0, \dots, \nu_{k-1}\rangle \notin \prod_{m=\ell(p)}^{\ell(p)+k-1} A_m(\tilde{p}^\alpha)$, then we set $p^\alpha = \tilde{p}^\alpha$, where for $m \geq \ell(p)$, $\tilde{p}_m^\alpha = \langle a_m(\tilde{p}^\alpha), A_m(\tilde{p}^\alpha), f_m(\tilde{p}^\alpha)\rangle$. If not, we consider $q = \tilde{p}^{\alpha\frown}\langle \nu_0, \dots, \nu_{k-1}\rangle$. If there is no direct extension of q inside D, then let $p^\alpha = \tilde{p}^\alpha$. Otherwise, let $s = \langle s_n \mid n < \omega\rangle \geq^* q$ be in D. Define $p^\alpha = \langle p_n^\alpha \mid n < \omega\rangle$ then as follows:

(a) For each n with $n \geq \ell(p) + k$ or $n < \ell(p)$, let $p_n^\alpha = s_n$, and

(b) For each n with $\ell(p) \leq n \leq \ell(p) + k - 1$, $a_n(p^\alpha) = a_n(\tilde{p}^\alpha)$, $A_n(p^\alpha) = A_n(\tilde{p}^\alpha)$, and $f_n(p^\alpha) = f_n(s)\restriction((\mathrm{dom}(f_n(s)) \setminus a_n(\tilde{p}^\alpha))$.

The meaning of this last part of the definition is that we extend for n with $\ell(p) \le n \le \ell(p)+k-1$ only $f_n(\tilde{p}^\alpha)$ and only outside of $a_n(\tilde{p}^\alpha)$. Clearly such defined p^α satisfies $(*)$.

Finally, $(*)$ allows us to put all the $\langle p^\alpha \mid \alpha < \kappa\rangle$ together. Thus we define $p' = \langle p'_n \mid n < \omega\rangle$ as follows:

(i) For $n < \ell(p)$, let $p'_n = \bigcup_{\alpha<\kappa} p^\alpha_n$.

(ii) For $n \ge \ell(p)$, let $f_n(p') = \bigcup_{\alpha<\kappa} f_n(p^\alpha)$, $a_n(p') = a_n(p^{\kappa_n})$, and $A_n(p') = A_n(p^{\kappa_n})$.

Obviously $p' \in \mathcal{P}$ and $p' \ge^* p$. This p' is as desired. Thus, if $q \ge p'$ is in D, then we consider $\alpha = h^{-1}(\langle q_n(\max(a_n(p'))) \mid \ell(p) \le n < \ell(q)\rangle)$. By the construction of $p^\alpha \le^* p'$, $p^{\alpha\frown}\langle q_n(\max(a_n(p'))) \mid \ell(p) \le n < \ell(q)\rangle$ will be in D. Then also $p'^\frown\langle q_n(\max(a_n(p')) \mid \ell(p) \le n < \ell(q)\rangle \in D$, since D is open.

This concludes the proof of the claim.

Now let $p' \ge^* p$ be given by the claim. Assume for simplicity that $\ell(p) = 0$. We would like to shrink the sets $A_n(p')$ in a certain way. Thus define $p(1) \ge^* p'$ such that:

$(*)_1$ For every $m < \omega$ and $\langle \nu_0, \ldots, \nu_{m-1}\rangle \in \prod_{n=0}^{m-1} A_n(p(1))$, if for some $\nu \in A_m(p(1))$, $p(1)^\frown\langle \nu_0, \ldots, \nu_{m-1}, \nu\rangle \in D$, then for every $\nu' \in A_m(p(1))$ $p(1)^\frown\langle \nu_0, \ldots, \nu_{m-1}, \nu'\rangle \in D$.

Let $m < \omega$ and $\vec{\nu} = \langle \nu_0, \ldots, \nu_{m-1}\rangle \in \prod_{n=0}^{m-1} A_n(p')$, where in case of $m = 0$, $\vec{\nu}$ is the empty sequence. Consider the set

$$X_{m,\vec{\nu}} = \{\nu \in A_m(p') \mid p'^\frown\langle \nu_0, \ldots, \nu_{m-1}, \nu\rangle \in D\}.$$

Define $A_{m\vec{\nu}}$ to be $X_{m,\vec{\nu}}$, if $X_{m\vec{\nu}} \in U_{m,\max(a_m(p'))}$ and $A_m(p') \setminus X_{m,\vec{\nu}}$, otherwise. Let $A_m = \bigcap\{A_{m,\vec{\nu}} \mid \vec{\nu} \in \prod_{n=0}^{m-1} A_n(p')\}$. Define now $p(1) = \langle p(1)_n \mid n < \omega\rangle$ as follows: for each $n < \omega$ let $p(1)_n = \langle a_n(p'), A_n, f(p')\rangle$. Clearly, such defined $p(1)$ satisfies $(*)_1$.

Then, in a similar fashion we chose $p(2) \ge^* p(1)$ satisfying:

$(*)_2$ For every $m < \omega$ and $\langle \nu_0, \ldots, \nu_{m-1}\rangle \in \prod_{n=0}^{m-1} A_n(p(2))$, if for some $\langle \nu_m, \nu_{m+1}\rangle \in A_m(p(2)) \times A_{m+1}(p(2))$,

$$p(2)^\frown\langle \nu_0, \ldots, \nu_{m-1}\rangle^\frown\langle \nu_m, \nu_{m+1}\rangle \in D,$$

then for every $\langle \nu'_m, \nu'_{m+1}\rangle \in A_m(p(2)) \times A_{m+1}(p(2))$,

$$p(2)^\frown\langle \nu_0, \ldots, \nu_{m-1}\rangle^\frown\langle \nu'_m, \nu'_{m+1}\rangle \in D.$$

Continue and define for every k with $2 \le k < \omega$ a $p(k) \ge^* p(k-1)$ satisfying $(*)_k$, where $(*)_k$ is defined analogously for k-sequences. Finally, let p^* be a direct extension of $\langle p(k) \mid 1 \le k < \omega\rangle$. Let $s \ge p^*$ be in D. Set

$n^* = \ell(s)$. Consider $\langle s_0(\max(a_0(p^*))), \ldots, s_{n^*-1}(\max(a_{n^*-1}(p^*)))\rangle$. Then the choice of p', $p' \leq^* p^*$ and openness of D imply that

$$p^{*\frown}\langle s_0(\max(a_0(p^*)), \ldots, s_{n^*-1}(\max(a_{n^*-1}(p^*)))\rangle \in D.$$

But $p^* \geq^* p(n^*)$. So, p^* satisfies $(*)_{n^*}$. It follows that for $\langle \nu_0, \ldots, \nu_{n^*-1}\rangle \in \prod_{m=0}^{n^*-1} A_m(p^*)$, $p^{*\frown}\langle \nu_0, \ldots, \nu_{n^*-1}\rangle \in D$. ⊣

Combining these lemmas we obtain the following:

2.20 Proposition. *The forcing $\langle \mathcal{P}, \leq\rangle$ does not add new bounded subsets to κ and preserves all the cardinals above κ^+.*

Actually, it is not hard now to show that κ^+ is preserved as well.

2.21 Lemma. *Forcing with $\langle \mathcal{P}, \leq\rangle$ preserves κ^+.*

Proof. Suppose that $(\kappa^+)^V$ is not a cardinal in a generic extension $V[G]$. Recall that $\mathrm{cf}(\kappa) = \aleph_0$ and by Proposition 2.20 it is preserved. So, $\mathrm{cf}((\kappa^+)^V) < \kappa$ in $V[G]$. Pick $p \in G$, $\delta < \kappa$ and a name $\underset{\sim}{g}$ so that $\kappa_{\ell(p)} > \delta$ and

$$p \Vdash (\underset{\sim}{g} : \check{\delta} \to (\kappa^+)^V \text{ and } \mathrm{ran}(\underset{\sim}{g}) \text{ is unbounded in } (\kappa^+)^V).$$

For every $\tau < \delta$ let

$$D_\tau = \{q \in \mathcal{P} \mid q \geq p \text{ and for some } \alpha < \kappa^+,\ q \Vdash \underset{\sim}{g}(\check{\tau}) = \check{\alpha}\}.$$

Define by recursion, using Lemma 2.17, a $\leq^*$-increasing sequence $\langle p^\tau \mid \tau < \delta\rangle$ of $\leq^*$-extensions of p so that p^τ satisfies the conclusion 2.18 with $D = D_\tau$. By Lemma 2.17, there is a $p^\delta \geq^* p^\tau$ for each $\tau < \delta$.

Now let $\tau < \delta$. By the choice of p^τ there is an $n(\tau) < \omega$ such that for every $\langle \nu_0, \ldots, \nu_{n(\tau)-1}\rangle \in \prod_{m=\ell(p)}^{\ell(p)+n(\tau)-1} A_m(p^\delta)$, $p^{\delta\frown}\langle \nu_0, \ldots, \nu_{n(\tau)-1}\rangle \in D_\tau$. This means that for some $\alpha(\nu_0, \ldots, \nu_{n(\tau)-1}) < \kappa^+$

$$p^{\delta\frown}\langle \nu_0, \ldots, \nu_{n(\tau)-1}\rangle \Vdash \underset{\sim}{g}(\check{\tau}) = \check{\alpha}(\nu_0, \ldots, \nu_{n(\tau)-1}).$$

Set

$$\alpha(\tau) = \sup\Big\{\alpha(\nu_0, \ldots, \nu_{n(\tau)-1}) \mid \langle \nu_0, \ldots, \nu_{n(\tau)-1}\rangle \in \prod_{m=\ell(p)}^{\ell(p)+n(\tau)-1} A_m(p^\delta)\Big\}.$$

Then clearly $\alpha(\tau) < \kappa^+$ and

$$p^\delta \Vdash \underset{\sim}{g}(\check{\tau}) < \check{\alpha}(\tau).$$

Now let $\alpha^* = \bigcup_{\tau<\delta} \alpha(\tau)$. Then again $\alpha^* < \kappa^+$ and

$$p^\delta \Vdash \forall \tau < \check{\delta}(\underset{\sim}{g}(\tau) < \check{\alpha}^*).$$

But this is impossible since $p \leq^* p^\delta$ forced that the range of g was unbounded in κ^+. Contradiction. ⊣

Finally, let us show that this forcing adds λ ω-sequences to κ. Thus, let $G \subseteq \mathcal{P}$ be generic. For every $n < \omega$ define a function $F_n : \lambda \to \kappa_n$ as follows:

$$F_n(\alpha) = \nu \quad \text{if for some } p = \langle p_m \mid m < \omega \rangle \in G \text{ with } \ell(p) > n,\ p_n(\alpha) = \nu.$$

Now for every $\alpha < \lambda$ set $t_\alpha = \langle F_n(\alpha) \mid n < \omega \rangle$. Let us show that the set $\{t_\alpha \mid \alpha < \lambda\}$ has cardinality λ. Notice that we cannot claim that all such sequences are new or even distinct due to the Cohen parts of conditions, i.e. the f_n's.

2.22 Lemma. *For every $\beta < \lambda$ there is an α with $\beta < \alpha < \lambda$ such that t_α dominates every t_γ with $\gamma \leq \beta$.*

Proof. Suppose otherwise. Then there is a $p = \langle p_n \mid n < \omega \rangle \in G$ and $\beta < \lambda$ such that

$$p \Vdash \forall \alpha(\beta < \alpha < \lambda \to \exists \gamma \leq \beta\ (\underset{\sim}{t_\alpha} \text{ does not dominate } \underset{\sim}{t_\gamma})).$$

For every $n \geq \ell(p)$ let $p_n = \langle a_n, A_n, f_n \rangle$. Pick some

$$\alpha \in \lambda \setminus \big(\bigcup_{n<\omega} a_n \cup \bigcup \mathrm{dom}(f_n) \cup (\beta + 1)\big).$$

We extend p to a condition q so that $q \geq^* p$ and for every $n \geq \ell(q) = \ell(p)$, $\alpha \in b_n$, where $q_n = \langle b_n, B_n, g_n \rangle$. Then q will force that t_α dominates every t_γ with $\gamma < \alpha$. This leads to the contradiction. Thus, let $\gamma < \alpha$ and assume that q belongs to the generic subset of $\mathcal{P}$. Then either $t_\gamma \in V$ or it is a new ω-sequence. If $t_\gamma \in V$ then it is dominated by t_α by the usual density arguments. If t_γ is new, then for some $r \geq q$ in the generic set $\gamma \in c_n$ for every $n \geq \ell(r)$, where $r_n = \langle c_n, C_n, h_n \rangle$. But also $\alpha \in c_n$ since $c_n \supseteq b_n$. This implies $F_n(\alpha) > F_n(\gamma)$ (by Definition 2.6(5)) and we are done. ⊣

We now have the following conclusion.

2.23 Theorem. *The following holds in $V[G]$:*

(a) All cardinals and cofinalities are preserved.

(b) No new bounded subsets are added to κ; in particular, GCH *holds below κ.*

(c) There are λ new ω-sequences in $\prod_{n<\omega} \kappa_n$. In particular, $2^\kappa \geq \lambda$.

2.24 Remark. The initial large cardinal assumptions used here are not optimal. We refer to Mitchell's chapter [41] on the Covering Lemma for matters of the consistency strength. In the next section another extender-based Prikry forcing requiring much weaker extenders will be introduced.

It is tempting to extend Lemma 2.22 and claim that $\langle t_\alpha \mid \alpha < \lambda$ and $t_\alpha \notin V \rangle$ is a scale in $\prod_{n<\omega} \kappa_n$, i.e. for every $t \in \prod_{n<\omega} \kappa_n$ there is an $\alpha < \lambda$ such $t_\alpha \notin V$ and t_α dominates t. Unfortunately this is not true in general.

We need to replace $\prod_{n<\omega}\kappa_n$ by the product of a sequence $\langle\lambda_n \mid n<\omega\rangle$ related to λ (basically the Prikry sequence for $U_{n\lambda}$ whenever it is defined). Assaf Sharon [50] made a full analysis of possible cofinalities structure for a similar forcing (the one that will be discussed in the next section). Let us now deal with a special case that cannot be covered by such forcing. Let us assume that for every $n<\omega$, $j_n(\kappa_n)=\lambda$, where $j_n: V\to M_n \simeq \mathrm{Ult}(V,E_n)$ is the canonical embedding. In particular, each κ_n is a superstrong cardinal. Then the following holds.

2.25 Lemma. *Let $t\in\prod_{n<\omega}\kappa_n$ in $V[G]$. Then there is an $\alpha<\lambda$ such that $t_\alpha\notin V$ and for all but finitely many $n<\omega$, $t_\alpha(n)>t(n)$.*

Proof. Let $\underset{\sim}{t}$ be a name of t. Pick $p\in G$ forcing "$\underset{\sim}{t}\in\prod_{n<\omega}\check{\kappa}_n$". Define for every $n<\omega$ a set dense open above p:

$$D_n=\{q\in\mathcal{P}\mid q\geq p \text{ and there is a } \nu_n<\kappa_n \text{ such that } q\Vdash \underset{\sim}{t}(n)=\check{\nu}_n\}.$$

Apply Lemma 2.18 to each of D_n's and construct a $\leq^*$-sequence $\langle p(k)\mid k<\omega\rangle$ of direct extensions of p such that $p(k)$ and D_k satisfy the conclusion of Lemma 2.18. Let p^* be a common direct extension of $p(k)$'s. Then for every k, $1\leq k<\omega$, there is an $n(k)<\omega$ such that for every $\langle\nu_0,\dots,\nu_{n(k)-1}\rangle\in\prod_{m=\ell(p)}^{\ell(p)+n(k)-1}A_m(p^*)$,

$$p^{*\frown}\langle\nu_0,\dots,\nu_{n(k)-1}\rangle\Vdash\underset{\sim}{t}(k-1)=\check{\xi}(\nu_0,\dots,\nu_{n(k)-1})$$

for some $\xi(\nu_0,\dots,\nu_{n(k)-1})<\kappa_{k-1}$. Assume for simplicity of notation that $\ell(p)=0$. Let $1\leq k<\omega$. We can assume that $\xi(\nu_0,\dots,\nu_{n(k)-1})$, defined above, depends really only on $\nu_0,\dots,\nu_{k-1}$, since its values are below κ_{k-1} and ultrafilters over κ_m's are κ_k-complete for $m\geq k$. Also assume that for every $m>0$ $A_m(p^*)\cap\kappa_{m-1}=\emptyset$. Now, we replace ξ by a bigger function η depending only on ν_{k-1}. Thus set

$$\eta(\nu_{k-1})=\bigcup\bigl\{\xi(\nu_0,\dots,\nu_{k-2},\nu_{k-1})\mid\langle\nu_0,\dots,\nu_{k-2}\rangle\in\textstyle\prod_{m=0}^{k-2}A_m(p^*)\bigr\}+\nu_{k-1}.$$

Clearly, $\eta(\nu_{k-1})<\kappa_{k-1}$. So,

$$p^{*\frown}\langle\nu_0,\dots,\nu_{k-1}\rangle\Vdash\underset{\sim}{t}(k-1)<\check{\eta}(\nu_{k-1})$$

for every k, $1\leq k<\omega$ and every $\langle\nu_0,\dots,\nu_{k-1}\rangle\in\prod_{m=0}^{k-1}A_m(p^*)$. For every $n<\omega$ let $\eta_n: A_n(p^*)\to\kappa_n$ be the restriction of η to κ_n. Let $\alpha_n=\max(a_n(p^*))$. Consider $j_n(\eta_n)(\alpha_n)$ where $j_n: V\to M_n$ is the embedding of the extender E_n. Then $j_n(\eta_n)(\alpha_n)<j_n(\kappa_n)=\lambda$. Choose some α below λ and above $\bigcup_{n<\omega}j_n(\eta_n)(\alpha_n)\cup(\mathrm{dom}(f_n(p^*)))$. Now extend p^* to a condition p^{**} such that $p^{**}\geq^* p^*$ and for every $n<\omega$ $\alpha\in a_n(p^{**})$. Then,

$$p^{**}\Vdash\forall n(\underset{\sim}{t_\alpha}(n)>\eta_n((\underset{\sim}{t_{\alpha_n}}(n))>\underset{\sim}{t}(n)).$$

So we are done. ⊣

The extender-based forcing described in this section can also be used with much stronger extenders than those used here. Thus with minor changes we can deal with E_n's such that $j_n(\kappa_n) < \lambda$ but requiring $j_n(\kappa_{n+1}) > \lambda$. Once $j_n(\kappa_{n+1}) \leq \lambda$ for infinitely many n's, the arguments like the one in the proof of the Prikry condition seem to break down completely.

Another probably more exciting direction is to use shorter extenders instead of long ones. Thus it turned out that for λ's below $\kappa^{+\omega_1}$ an extender of length κ_n^{+n} over κ_n for $n < \omega$ suffices. The basic idea is to replace in $p \in \mathcal{P}$ the subset $a_n(p)$ of λ by an order preserving function from λ to κ_n^{+n}. Such defined forcing fails to satisfy κ^{++}-c.c. and actually will collapse λ to κ^+. But using increasing with n similarity of ultrafilters involved in the extenders, it turns out that there is a subforcing satisfying the κ^{++}-c.c. and still producing λ new sequences in $\prod_{n<\omega} \kappa_n$. This approach was implemented in [17] for calculating the consistency strength of various instances of the failure of the Singular Cardinal Hypothesis and, as well, for constructing more complicated cardinal arithmetic configurations.

3. Extender-Based Prikry Forcing with a Single Extender

In this section we present a simplified version of the original extender-based Prikry forcing of [19, Sect. 1]. Our aim is simultaneously to change the cofinality of a regular cardinal to $\aleph_0$ and blow up its power. Recall that the Prikry forcing of Definitions 1.1 and 1.2 does the first part, i.e. change cofinality. As in the previous section, we would like to use an extender instead of a single ultrafilter in order to blow up the power.

Let κ, λ be regular cardinals with $\lambda \geq \kappa^{++}$. Assume that κ is $V_{\kappa+\delta}$-strong for a δ so that $\kappa^{+\delta} = \lambda$. Let E be an extender over κ witnessing this and $j : V \longrightarrow M \simeq \mathrm{Ult}(V, E)$ with $M \supseteq V_{\kappa+\delta}$ be the corresponding elementary embedding. Suppose also that there is a function $f_\lambda : \kappa \to \kappa$ such that $j(f_\lambda)(\kappa) = \lambda$. Notice that such a function always exists for small λ's like $\lambda = \kappa^{++}$, $\lambda = \kappa^{+116}$, $\lambda = \kappa^{+\kappa+1}$ etc., just take mappings $\alpha \to \alpha^{++}$, $\alpha \to \alpha^{+116}$, $\alpha \to \alpha^{+\alpha+1}$. In general, assuming $j(\kappa) > \lambda$, it is not hard to force such f_λ. The idea is to force for every inaccessible $\alpha \leq \kappa$ a generic function from α to α and then to extend the embedding specifying to κ the value λ under the generic function from $j(\kappa)$ to $j(\kappa)$ in M.

If κ is a strong cardinal then for every $\lambda > \kappa$ there is a (κ, λ)-extender E and a function $f : \kappa \to \kappa$ so that $j_E(f)(\kappa) = \lambda$, where $j_E : V \to M \simeq \mathrm{Ult}(V, E)$. The Solovay argument [56], originally used for a supercompact κ, works without change for a strong cardinal κ: Let κ be a strong cardinal and suppose that for some $\lambda > \kappa$ for every (κ, λ)-extender E and every function $f : \kappa \to \kappa$ we have $j_E(f)(\kappa) \neq \lambda$. Let λ be the least such ordinal. Pick a $(\kappa, 2^{2^\lambda})$-extender E^*. Let $j : V \to M \simeq \mathrm{Ult}(V, E^*)$. Then, in M, λ will be the least such that for every (κ, λ)-extender E and every function $f : \kappa \to \kappa$,

$j_E(f)(\kappa) \neq \lambda$, since $M \supseteq V_{2^{2^\lambda}}$. Now define a function $g : \kappa \to \kappa$ as follows: $g(\alpha) =$ the least $\beta > \alpha$ such that for every (α, β)-extender E and every function $f : \alpha \to \alpha$, $j_E(f)(\alpha) \neq \beta$, if there is such a β and let $g(\alpha) = 0$ otherwise. Then, clearly, $j(g)(\kappa) = \lambda$. But then $E^* \restriction \lambda$ and g provide the contradiction.

Suppose for simplicity that V satisfies GCH. Then we will have ${}^{\kappa^+}V_{\kappa+\delta} \subseteq M$. For every $\alpha < \lambda$ define a κ-complete ultrafilter U_α over κ by setting $X \in U_\alpha$ iff $\alpha \in j(X)$. Notice that U_κ will be normal and each U_α with $\alpha < \kappa$ will be trivial; we shall ignore such U_α and refer to U_κ as the least one. As in Sect. 2, we define a partial ordering $\leq_E$ on λ:

$$\alpha \leq_E \beta \quad \text{iff} \quad \alpha \leq \beta \text{ and for some } f \in^\kappa \kappa,\ j(f)(\beta) = \alpha.$$

Again, clearly, $\alpha \leq_E \beta$ implies that $U_\alpha \leq_{\mathrm{RK}} U_\beta$ as witnessed by any $f \in {}^\kappa\kappa$ with $j(f)(\beta) = \alpha$. In the previous section only the κ directedness (more precisely, κ_n directedness for every $n < \omega$) of the ordering was used. Here we will need more—κ^{++}-directedness. Thus, as in Sect. 2, fix an enumeration $\langle a_\alpha \mid \alpha < \kappa \rangle$ of $[\kappa]^{<\kappa}$ so that for every regular cardinal $\mu < \kappa$, $\langle a_\alpha \mid \alpha < \mu \rangle$ enumerates $[\mu]^{<\mu}$ and every element of $[\mu]^{<\mu}$ appears μ many times in the enumeration. Let $j(\langle a_\alpha \mid \alpha < \kappa \rangle) = \langle a_\alpha \mid \alpha < j(\kappa) \rangle$. Then, $\langle a_\alpha \mid \alpha < \lambda \rangle$ will enumerate $[\lambda]^{<\lambda} \supseteq [\lambda]^{<\kappa^{++}}$. For each $\alpha < \lambda$ we consider the following basic commutative diagram:

$$\begin{tikzcd} & M \\ V \arrow[ur, "j"] \arrow[dr, "i_\alpha"'] & \\ & N_\alpha \arrow[uu, "k_\alpha"'] \end{tikzcd}$$

where $i_\alpha : V \longrightarrow N_\alpha \simeq \mathrm{Ult}(V, U_\alpha)$ and $k_\alpha([f]_{U_\alpha}) = j(f)(\alpha)$.

3.1 Lemma. $\mathrm{crit}(k_\alpha) = (\kappa^{++})^{N_\alpha}$.

Proof. It is enough to show that $k_\alpha(\kappa) = \kappa$, since $k_\alpha((\kappa^+)^{N_\alpha}) = \kappa^+$ and $k_\alpha((\kappa^{++})^{N_\alpha}) = \kappa^{++}$ by elementarity. But ${}^\kappa N_\alpha \subseteq N_\alpha$. Hence $(\kappa^+)^{N_\alpha} = \kappa^+$. By $2^\kappa = \kappa^+$, $(\kappa^{++})^{N_\alpha} < \kappa^{++}$. So $(\kappa^{++})^{N_\alpha}$ is the first ordinal moved by k_α.

In order to show that κ is fixed let us use the function $f_\lambda : \kappa \to \kappa$ representing λ in M. Thus by commutativity, $k_\alpha(i_\alpha(f_\lambda)) = j(f_\lambda)$. Clearly, $i_\alpha(f_\lambda) : i_\alpha(\kappa) \to i_\alpha(\kappa)$ and $i_\alpha(f_\lambda) \restriction \kappa = f_\lambda$. Hence

$$N_\alpha \vDash \forall \tau < \kappa\ (i_\alpha(f_\lambda)(\tau) < \kappa).$$

Using k_α we obtain that

$$M \vDash \forall \tau < k_\alpha(\kappa)\ (j(f_\lambda)(\tau) < k_\alpha(\kappa)).$$

But $k_\alpha(\kappa) \le k_\alpha([\mathrm{id}]_{U_\alpha}) = \alpha < \lambda$. Hence,

$$M \vDash \forall \tau < k_\alpha(\kappa)\ (j(f_\lambda)(\tau) < \lambda).$$

But $k_\alpha(\kappa) \ge \kappa$ and $j(f_\lambda)(\kappa) = \lambda$. So, $k_\alpha(\kappa)$ must be equal to κ and we are done. $\dashv$

The following is a consequence of the previous lemma.

3.2 Lemma. *For every α with $\kappa \le \alpha < \lambda$, $\alpha \ge_E \kappa$.*

Proof. By Lemma 3.1, $k_\alpha(\kappa) = \kappa$. So, $k_\alpha([g]_{U_\alpha}) = \kappa$ for $g : \kappa \to \kappa$ representing κ in N_α. Then g projects U_α on U_κ and $j(g)(\alpha) = k_\alpha([g]_{U_\alpha}) = \kappa$. $\dashv$

We can now improve Lemma 2.1 to κ^{++}-directedness.

3.3 Lemma. *Let $\langle \alpha_\nu \mid \nu < \kappa^+ \rangle$ be a sequence of ordinals below λ. Suppose that $\alpha \in \lambda \setminus (\bigcup_{\nu < \kappa^+} \alpha_\nu + 1)$ codes $\{\alpha_\nu \mid \nu < \kappa^+\}$, i.e. $a_\alpha = \{\alpha_\nu \mid \nu < \kappa^+\}$. Then $\alpha >_E \alpha_\nu$ for every $\nu < \kappa^+$.*

Proof. Let $\nu < \kappa^+$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & M \\
V \arrow[ur, "j"] \arrow[r, "i_\alpha"] \arrow[dr, "i_{\alpha_\nu}"'] & N_\alpha \arrow[u, "k_\alpha"'] \\
 & N_{\alpha_\nu} \arrow[uu, bend right, "k_{\alpha_\nu}"']
\end{tikzcd}$$

Then $j(\langle a_\beta \mid \beta < \kappa \rangle) = k_\alpha(i_\alpha(\langle a_\beta \mid \beta < \kappa \rangle))$. Let $\alpha^* = [\mathrm{id}]_{U_\alpha}$. Then $k_\alpha(\alpha^*) = \alpha$. So, $a_\alpha = k_\alpha(a^*_{\alpha^*})$, where $a^*_{\alpha^*} = i_\alpha(\langle a_\beta \mid \beta < \kappa \rangle)(\alpha^*)$. But $a_\alpha = \{\alpha_{\nu'} \mid \nu' < \kappa^+\}$ and, by Lemma 3.1, $k_\alpha(\kappa^+) = \kappa^+$. So, $a^*_{\alpha^*} = \{\alpha^*_{\nu'} \mid \nu' < \kappa^+\}$, where $k_\alpha(\alpha^*_{\nu'}) = \alpha_{\nu'}$. Now we can define an elementary embedding $k_{\alpha_\nu \alpha} : N_{\alpha_\nu} \longrightarrow N_\alpha$. Set

$$k_{\alpha_\nu \alpha}([f]_{U_{\alpha_\nu}}) = i_\alpha(f)(\alpha^*_\nu).$$

Finally, every function representing α^*_ν in N_α will be a projection of U_α onto U_{α_ν} and witness $\alpha_\nu <_E \alpha$. $\dashv$

For $\beta \le_E \alpha < \lambda$ we fix a projection $\pi_{\alpha\beta} : \kappa \to \kappa$ defined as in Lemma 3.3. Let $\pi_{\alpha\alpha} = \mathrm{id}$. The following two lemmas were actually proved in the previous section (Lemmas 2.3 and 2.4).

3.4 Lemma. *Let $\gamma < \beta \leq \alpha < \lambda$. If $\alpha \geq_E \beta$ and $\alpha \geq_E \gamma$, then $\{\nu < \kappa \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in U_\alpha$.*

3.5 Lemma. *Let $\alpha, \beta, \gamma < \lambda$ be so that $\alpha \geq_E \beta \geq_E \gamma$. Then there is an $A \in U_\alpha$ so that for every $\nu \in A$*

$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)).$$

Consider the following set:

$$\begin{aligned}\overline{X} = \{\nu < \kappa \mid \exists \nu^* \leq \nu(&\nu^* \text{ is inaccessible}, \\ &f_\lambda \restriction \nu^* : \nu^* \longrightarrow \nu^*, \text{ and } f_\lambda(\nu^*) > \nu)\}.\end{aligned}$$

Clearly $\overline{X} \in U_\alpha$ for every $\alpha < \lambda$ (ignoring α's below κ). Also the function $g : \overline{X} \to \kappa$ defined by $g(\nu) =$ the maximal inaccessible $\nu^* \leq \nu$ closed under f_λ and with $f_\lambda(\nu^*) > \nu$, projects each U_α onto U_κ. Let us change each $\pi_{\alpha\kappa}$ to g on $\overline{X}$ and for $\nu \in \kappa \setminus \overline{X}$ let $\pi_{\alpha\kappa}(\nu) = 0$. Also change $\pi_{\alpha\beta}$'s a little for $\alpha, \beta > \kappa$. Thus for $\nu \in \kappa \setminus \overline{X}$ let $\pi_{\alpha\beta}(\nu) = 0$. If $\nu \in \overline{X}$ and $\pi_{\alpha\beta}(\nu)$ is below $\pi_{\alpha\kappa}(\nu)$ then change $\pi_{\alpha\beta}(\nu)$ to ν or any ordinal between $\pi_{\alpha\kappa}(\nu)$ and ν. Note that these changes are on a small set since $\kappa \setminus \overline{X} \notin U_\alpha$ for any $\alpha < \lambda$. Hence the changed $\pi_{\alpha\beta}$'s are still projections. The following summarizes the main properties of the U_α's and $\pi_{\alpha\beta}$'s:

(1) $\langle \lambda, \leq_E \rangle$ is a κ^{++}-directed partial ordering.

(2) $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a Rudin-Keisler commutative sequence of κ-complete ultrafilters over κ with projections $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \lambda, \alpha \geq_E \beta \rangle$.

(3) For every $\alpha < \lambda$, $\pi_{\alpha\alpha}$ is the identity on a fixed set $\overline{X}$ which belongs to every U_β for $\beta < \lambda$.

(4) (Commutativity) For every $\alpha, \beta, \gamma < \lambda$ such that $\alpha \geq_E \beta \geq_E \gamma$, there is a $Y \in U_\alpha$ so that for every $\nu \in Y$

$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)).$$

(5) For every $\alpha < \beta$, $\gamma < \lambda$, if $\gamma \geq_E \alpha, \beta$ then

$$\{\nu < \kappa \mid \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in U_\gamma.$$

(6) U_κ is a normal ultrafilter.

(7) $\kappa \leq_E \alpha$ when $\kappa \leq \alpha < \lambda$.

(8) (Full commutativity at κ) For every $\alpha, \beta < \lambda$ and $\nu < \kappa$, if $\alpha \geq_E \beta$ then $\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\nu))$.

(9) (Independence of the choice of projection to κ) For every $\alpha, \beta, \kappa \leq \alpha$, $\beta < \lambda$, and $\nu < \kappa$

$$\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\nu).$$

(10) Each U_α is a P-point ultrafilter, i.e. for every $f \in {}^\kappa\kappa$, if f is not constant mod U_α, then there is a $Y \in U_\alpha$ such that for every $\nu < \kappa$ $|Y \cap f^{-1}\{\nu\}| < \kappa$.

The last property just follows using the set $\overline{X}$ defined above and the normality of U_κ.

A system of ultrafilters and projections satisfying (1)–(10) was called in [19] a *nice system*. Its existence is a bit weaker than the strongness assumption used here. In what follows we will use only such a system in order to define extender-based Prikry forcing over κ.

Let us denote $\pi_{\alpha\kappa}(\nu)$ by ν^0, where $\kappa \le \alpha < \lambda$ and $\nu < \kappa$. By a $^\circ$-increasing sequence of ordinals we mean a sequence $\langle \nu_1, \dots, \nu_n\rangle$ of ordinals below κ so that

$$\nu_1^0 < \nu_2^0 < \cdots < \nu_n^0.$$

For every $\alpha < \lambda$ by $X \in U_\alpha$ we shall always mean that $X \subseteq \overline{X}$, in particular, it will imply that for $\nu_1, \nu_2 \in X$ if $\nu_1^0 < \nu_2^0$ then $|\{\alpha \in X \mid \alpha^0 = \nu_1^0\}| < \nu_2^0$. The following weak version of normality holds, since U_α is a P-point: if $X_i \in U_\alpha$ for $i < \kappa$ then also $X = \Delta^*_{i<\kappa} X_i = \{\nu \mid \forall i < \nu^0\ (\nu \in X_i)\} \in U_\alpha$.

Let $\nu < \kappa$ and $\langle \nu_1, \dots, \nu_n\rangle$ be a finite sequence of ordinals below κ. Then ν is called *permitted* for $\langle \nu_1, \dots, \nu_n\rangle$ if $\nu^0 > \max\{\nu_i^0 \mid 1 \le i \le n\}$.

We shall ignore U_α's with $\alpha < \kappa$ and denote U_κ by U_0.

Let us now define a forcing notion for adding λ ω-sequences to κ.

3.6 Definition. The set of forcing conditions $\mathcal{P}$ consists of all the elements p of the form $\{\langle \gamma, p^\gamma\rangle \mid \gamma \in g \setminus \{\max(g)\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\}$, where

(1) $g \subseteq \lambda$ of cardinality $\le \kappa$ which has a maximal element in $\le_E$-ordering and $0 \in g$. Further let us denote g by $\mathrm{supp}(p)$, $\max(g)$ by $mc(p)$, T by T^p, and $p^{\max(g)}$ by p^{mc} (mc for the maximal coordinate).

(2) For $\gamma \in g$, p^γ is a finite $^\circ$-increasing sequence of ordinals $< \kappa$.

(3) T is a tree with a trunk p^{mc} consisting of $^\circ$-increasing sequences. All the splittings in T are required to be on sets in $U_{mc(p)}$, i.e. for every $\eta \in T$, if $\eta \ge_T p^{mc}$ then the set

$$\mathrm{Suc}_T(\eta) = \{\nu < \kappa \mid \eta^\frown\langle \nu\rangle \in T\} \in U_{mc(p)}.$$

Also require that for $\eta_1 \ge_T \eta_2 \ge_T p^{mc}$

$$\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2).$$

(4) For every $\gamma \in g$, $\pi_{mc(p),\gamma}(\max(p^{mc}))$ is not permitted for p^γ.

(5) For every $\nu \in \mathrm{Suc}_T(p^{mc})$

$$|\{\gamma \in g \mid \nu \text{ is permitted for } p^\gamma\}| \le \nu^0.$$

(6) $\pi_{mc(p),0}$ projects p^{mc} onto p^0 (so p^{mc} and p^0 are of the same length).

Let us give some intuitive motivation for the definition of forcing conditions. We want to add a Prikry sequence for every $U_\alpha(\alpha < \lambda)$. The finite sequences p^γ ($\gamma \in \text{supp}(p)$) are initial segments of such sequences. The support of p has two distinguished coordinates. The first is the 0-coordinate of p and the second is its maximal coordinate. The 0-coordinate or more precisely the Prikry sequence for the normal ultrafilter will be used further in order to push the present construction to $\aleph_\omega$. Also condition (6) will be used for this purpose. The maximal coordinate of p is responsible for extending the Prikry sequences for γ's in the support of p. The tree T^p is a set of possible candidates for extending p^{mc} and by using the projections map $\pi_{mc(p),\gamma}$ for $\gamma \in \text{supp}(p)$, T^p becomes also the set of candidates for extending the p^γ's. Instead of working with a tree, it is possible to replace it by a single set. The proof of the Prikry condition will then be a bit more complicated. Condition (4) means that the information carried by $\max(p^{mc})$ is impossible to project down. The reasons for such a condition are technical. Condition (5) is desired to allow the use of the diagonal intersections.

In contrast to the main forcing of the previous section, we deal with κ coordinates simultaneously (i.e. the support of the condition may have cardinality κ). This causes difficulties since we cannot hope to have full commutativity between κ many ultrafilters.

3.7 Definition. Let $p, q \in \mathcal{P}$. We say that p *extends* q and denote this by $p \geq q$ iff

(1) $\text{supp}(p) \supseteq \text{supp}(q)$.

(2) For every $\gamma \in \text{supp}(q)$, p^γ is an end-extension of q^γ.

(3) $p^{mc(q)} \in T^q$.

(4) For every $\gamma \in \text{supp}(q)$,

$$p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}``((p^{mc(q)} \setminus q^{mc(q)}) \restriction (\text{length}(p^{mc}) \setminus (i+1)),$$

where $i \in \text{dom}(p^{mc(q)})$ is the largest such that $p^{mc(q)}(i)$ is not permitted for q^γ.

(5) $\pi_{mc(p),mc(q)}$ projects $T^p_{p^{mc}}$ into $T^q_{q^{mc}}$.

(6) For every $\gamma \in \text{supp}(q)$ and $\nu \in \text{Suc}_{T^p}(p^{mc})$, if ν is permitted for p^γ, then

$$\pi_{mc(p),\gamma}(\nu) = \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu)).$$

In clause (5) above the following notation is used: let T be a tree and $\eta \in T$, then T_η consists of all finite sequences μ such that $\eta^\frown\mu$ is in T.

Intuitively, we are allowing almost everything to be added on the new coordinates and restrict ourselves to choosing extensions from the sets of measure one on the old coordinates. Actually here we are really extending only the maximal old coordinate and then we are using the projection map. This idea goes back to [13] and further to Mitchell [44].

3.8 Definition. Let $p, q \in \mathcal{P}$. We say that p is a *direct (or Prikry) extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) for every $\gamma \in \operatorname{supp}(q)$, $p^\gamma = q^\gamma$.

Our strategy is to show that $\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition, that $\langle \mathcal{P}, \leq^* \rangle$ is κ-closed, and that $\langle \mathcal{P}, \leq \rangle$ satisfies the κ^{++}-c.c.

The Prikry condition together with κ closedness of $\langle \mathcal{P}, \leq^* \rangle$ insure that no new bounded subsets of κ are added. The κ^{++}-c.c. takes care of cardinals $\geq \kappa^{++}$. Since κ will change its cofinality to $\aleph_0$, an argument similar to Lemma 2.21 will be used to show that κ^+ is preserved. Clause 3.7(4) of the system of ultrafilters and projections insures that at least λ-many ω-sequences will be added to κ.

3.9 Lemma. *The relation $\leq$ is a partial order.*

Proof. Let us check the transitivity of $\leq$. Suppose that $r \leq q$ and $q \leq p$. Let us show that $r \leq p$. Conditions (1) and (2) of Definition 3.7 are obviously satisfied. Let us check (3), i.e. let us show that $p^{mc(r)} \in T^r$. Since $p \geq q \geq r$, $mc(r) \in \operatorname{supp}(q)$, $q^{mc(r)} \in T^r$ and

$$p^{mc(r)} \setminus q^{mc(r)} = \pi_{mc(q),mc(r)}``(p^{mc(q)} \setminus q^{mc(q)}).$$

Also $p^{mc(q)} \in T^q$. By (5) of Definition 3.7 (for q and r) $\pi_{mc(q),mc(r)}$ projects $T^q_{q^{mc}}$ into a subtree of $T^r_{q^{mc(r)}}$. Hence $p^{mc(r)} \in T^r$ and, so condition (3) is satisfied.

Let us check condition (4). Suppose that $\gamma \in \operatorname{supp}(r)$. We need to show that $p^\gamma \setminus r^\gamma = \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus r^{mc(r)})$. In order to simplify the notation, we are assuming here that every element of $p^{mc(r)} \setminus r^{mc(r)}$ is permitted for $r^\gamma$. Since $q \geq r$, $q^\gamma \setminus r^\gamma = \pi_{mc(r),\gamma}``(q^{mc(r)} \setminus r^{mc(r)})$. So, we need to show only that $p^\gamma \setminus q^\gamma = \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus q^{mc(r)})$. Since $p \geq q$, $p^{mc(q)} \in T^q$ and $p^\gamma \setminus q^\gamma = \pi_{mc(q),\gamma}``(p^{mc(q)} \setminus q^{mc(q)})$. Using condition (6) of Definition 3.7 for $q \geq r$ and the elements of $p^{mc(q)} \setminus q^{mc(q)}$, we obtain the following

$$\begin{aligned} p^\gamma \setminus q^\gamma &= \pi_{mc(q),\gamma}``(p^{mc(q)} \setminus q^{mc(q)}) \\ &= \pi_{mc(r),\gamma}``(\pi_{mc(q),mc(r)}``(p^{mc(q)} \setminus q^{mc(q)})) \\ &= \pi_{mc(r),\gamma}``(p^{mc(r)} \setminus q^{mc(r)}). \end{aligned}$$

The last equality holds by condition (4) of Definition 3.7 used for p and q.

Let us check condition (5), i.e. $\pi_{mc(p),mc(r)}$ projects $T^p_{p^{mc}}$ into $T^r_{p^{mc(r)}}$. Since $p \geq q$, $T^p_{p^{mc}}$ is projected by $\pi_{mc(p),mc(q)}$ into $T^q_{q^{mc}}$. Since $q \geq r$, $\pi_{mc(q),mc(r)}$ projects $T^q_{q^{mc}}$ into $T^r_{q^{mc(r)}}$. Now, using condition (6) for p and q with $\gamma = mc(r)$, we obtain condition (5) for p and r.

Finally, let us check condition (6). Let $\gamma \in \mathrm{supp}(r)$, $\nu \in \mathrm{Suc}_{T^p}(p^{mc})$ and suppose that ν is permitted for p^γ. Using condition (5) for p and q, we obtain that $\pi_{mc(p),mc(q)}(\nu) \in \mathrm{Suc}_{T^q}(q^{mc})$. Recall that it was required in Clause 3.6(3) that each splitting has a splitting below it in the tree. Denote $\pi_{mc(p),mc(q)}(\nu)$ by δ. By condition (6) for q and r, $\pi_{mc(q),\gamma}(\delta) = \pi_{mc(r),\gamma}(\pi_{mc(q),mc(r)}(\delta))$. Using (6) for p and q, we obtain

$$\begin{aligned}\pi_{mc(p),\gamma}(\nu) &= \pi_{mc(q),\gamma}(\pi_{mc(p),mc(q)}(\nu))\\ &= \pi_{mc(q),\gamma}(\delta)\\ &= \pi_{mc(r),\gamma}(\pi_{mc(q),mc(r)}(\delta)).\end{aligned}$$

Once more using (6) for p and q,

$$\pi_{mc(q),mc(r)}(\pi_{mc(p),mc(q)}(\nu)) = \pi_{mc(p),mc(r)}(\nu).$$

This completes the checking of (6) and also the proof of the lemma. $\dashv$

The main point of the proof appears in the next lemma.

3.10 Lemma. *Let $q \in \mathcal{P}$ and $\alpha < \lambda$. Then there is a $p \geq^* q$ so that $\alpha \in \mathrm{supp}(p)$.*

Proof. If $\alpha \leq_E mc(q)$, then it is obvious. Thus, if $\alpha \in \mathrm{supp}(q)$, then we can take $p = q$. Otherwise add to q a pair $\langle \alpha, t\rangle$ where t is any $^\circ$-increasing sequence so that $\max(q^{mc})$ is not permitted for t.

Now suppose that $\alpha \not\leq_E mc(q)$. Pick some $\beta < \lambda$ so that $\beta \geq_E \alpha$ and $\beta \geq_E mc(q)$. Without loss of generality we can assume that $\beta = \alpha$. We shall define p to be of the form

$$q' \cup \{\langle \alpha, t, T\rangle\}$$

where q' is derived from q by deleting T^q from the triple $\langle mc(q), q^{mc}, T^q\rangle$, t is an $^\circ$-increasing sequence which projects onto q^0 by $\pi_{\alpha 0}$ and the tree T will be defined below.

First consider the tree T_0 which is the inverse image of $T^q_{q^{mc}}$ by $\pi_{\alpha,mc(q)}$, with t added as the trunk. Then $p_0 = q' \cup \{\langle \alpha, t, T_0\rangle\}$ is a condition in $\mathcal{P}$ which is "almost" an extension and even a direct extension of q. The only concern is that condition (6) of Definition 3.7 may not be satisfied by p_0 and q. In order to repair this, let us shrink the tree T_0 a little.

Denote $\mathrm{Suc}_{T_0}(t)$ by A. For $\nu \in A$ set

$$B_\nu = \{\gamma \in \mathrm{supp}(q) \mid \gamma \neq mc(q) \text{ and } \nu \text{ is permitted for } q^\gamma\}.$$

Then we have $|B_\nu| \leq \nu^0$, since $\pi_{\alpha,mc(q)}(\nu) \in \mathrm{Suc}_{T^q}(q^{mc})$, $\nu^0 = \pi_{\alpha 0}(\nu) = \pi_{mc(q),0}(\pi_{\alpha mc(q)}(\nu))$, and q being in $\mathcal{P}$ satisfies condition (5) of Definition 3.6.

Clearly, for $\nu, \delta \in A$, if $\nu^0 = \delta^0$ then $B_\nu = B_\delta$, and if $\nu^0 > \delta^0$ then $B_\nu \supseteq B_\delta$. Also, if $\nu \in A$ and ν^0 is a limit point of $\{\delta^0 \mid \delta \in A\}$, then $B_\nu = \bigcup\{B_\delta \mid \delta \in A$ and $\delta^0 < \nu^0\}$. So the sequence $\langle B_\nu \mid \nu \in A\rangle$ is increasing and continuous (according to the ν^0's). Obviously, $\bigcup\{B_\nu \mid \nu \in A\} = \mathrm{supp}(q) \setminus \{mc(q)\}$. Let $\langle \xi_i \mid i < \kappa\rangle$ be an enumeration of $\mathrm{supp}(q) \setminus \{mc(q)\}$ such that for every $\nu \in A$

$$B_\nu \subseteq \{\xi_i \mid i < \nu^0\}.$$

Now pick for every $i \in A$ a set $C_i \subseteq A$, with $C_i \in U_\alpha$ so that for every $\nu \in C_i$ $\pi_{\alpha\xi_i}(\nu) = \pi_{mc(q),\xi_i}(\pi_{\alpha,mc(q)}(\nu))$. Let $C = A {}^\frown \Delta^*_{i<\kappa} C_i = \{\nu \in A \mid \forall i < \nu^0 (\nu \in C_i)\}$. Then $C \in U_\alpha$.

Now define T to be the tree obtained from T_0 by intersecting every level of T_0 with C. Let us show that condition (6) of Definition 3.7 is now satisfied. Suppose that $\gamma \in \mathrm{supp}(q)$. If $\gamma = mc(q)$, then everything is trivial. Assume that $\gamma \in \mathrm{supp}(q) \setminus \{mc(q)\}$. Then for some $i_0 < \kappa$, $\gamma = \xi_{i_0}$. Suppose that some $\nu \in C$ is permitted for q^γ. Then $\xi_{i_0} = \gamma \in B_\nu$. Since $B_\nu \subseteq \{\xi_i \mid i < \nu^0\}$, $i_0 < \nu^0$. Then $\nu \in C_{i_0}$. Hence

$$\pi_{\alpha\xi_{i_0}}(\nu) = \pi_{mc(q),\xi_{i_0}}(\pi_{\alpha,mc(q)}(\nu)).$$

So condition (6) is satisfied by p. Hence, $p \geq^* q$. ⊣

3.11 Lemma.

(a) $\langle \mathcal{P}, \geq\rangle$ *satisfies the* κ^{++}*-c.c.*

(b) $\langle \mathcal{P}, \geq^*\rangle$ *is* κ*-closed.*

Proof of (a). Let $\{p_\alpha \mid \alpha < \kappa^{++}\} \subseteq \mathcal{P}$. Without loss of generality, we can assume their supports form a Δ-system and are contained in κ^{++}. Also, we can assume that there are s and $\langle t, T\rangle$ so that for every $\alpha < \kappa^{++}$, $p_\alpha \restriction \alpha = s$ and $\langle p_\alpha^{mc}, T^{p_\alpha}\rangle = \langle t, T\rangle$. Let us then show that p_α and p_β are compatible for every $\alpha, \beta < \kappa^{++}$.

Let $\alpha, \beta < \kappa^{++}$ be fixed. We would like simply to take the union $p_\alpha \cup p_\beta$ and to show that this is a condition stronger than both p_α and p_β. The first problem is that $p_\alpha \cup p_\beta$ may not be in $\mathcal{P}$, since $\mathrm{supp}(p_\alpha \cup p_\beta) = \mathrm{supp}(p_\alpha) \cup \mathrm{supp}(p_\beta)$ may not have a maximal element. In order to fix this, let us add say to p_α some new coordinate δ so that $\delta \geq_E mc(p_\alpha), mc(p_\beta)$. Let p^*_α be the extension of p_α defined in the previous lemma by adding δ as a new coordinate to p_α. Then $p^*_\alpha \cup p_\beta \in \mathcal{P}$. But we do need a condition stronger than both p_α and p_β. The condition $p^*_\alpha \cup p_\beta$ is a good candidate for it. The only concerns here are (5) and (6) of Definition 3.6. Actually, (5) can be easily satisfied by intersecting $T^{p^*_\alpha}_{(p^*_\alpha)^{mc}}$ with $\pi^{-1}_{\delta,mc(p_\beta)}$ “$(T^{p_\beta}_{p^{mc}_\beta})$. In order to satisfy (6), we need to shrink $T^{p^*_\alpha}$ more. The argument of the previous lemma can be used for this.

Proof of (b). Let $\delta < \kappa$ and let $\langle p_i \mid i < \delta\rangle$ be an $\leq^*$-increasing sequence of elements of $\mathcal{P}$. Pick $\alpha < \lambda$ above $\{mc(p_i) \mid i < \delta\}$. Let p be the union of p_i's with T^{p_i} removed. Set $T = \bigcap_{i<\delta} \pi^{-1}_{\alpha,mc(p_i)}$"$T^{p_i}$. Also remove all τ's with $\tau^0 \leq \delta$ from this tree. Let t be a $^\circ$-increasing sequence so that $\pi_{\alpha 0}$"$t = p^0_0$. Consider $p \cup \{\langle \alpha, t, T\rangle\}$. Clearly, it belongs to $\mathcal{P}$. Now, as in Lemma 3.9, shrink T to a tree T^i so that $p \cup \{\langle \alpha, t, T^i\rangle\}^* \geq p_i$, where $i < \delta$. Let $T^* = \bigcap_{i<\delta} T^i$ and consider $r = p \cup \{\langle \alpha, t, T^*\rangle\}$. Then $r \geq^* p_i$ for every $i < \delta$. ⊣

3.12 Lemma. *$\langle \mathcal{P}, \leq, \leq^*\rangle$ satisfies the Prikry condition, i.e. for every statement σ of the forcing language, for every $q \in \mathcal{P}$ there exists a $p \geq^* q$ deciding σ.*

Proof. Let σ be a statement and $q \in \mathcal{P}$. In order to simplify the notation we assume that $q = \emptyset$. The object in this proof is to reduce the problem to that of Prikry forcing on some U_α, so the arguments of Sect. 1.1 can be applied. In order to find a suitable ordinal α, which will be p^{mc}, pick an elementary submodel N of V_μ for μ sufficiently large to contain all the relevant information of cardinality κ^+ and closed under κ-sequences of its elements. Pick $\alpha < \lambda$ above (in the $\leq_E$-ordering) all the elements of $N \cap \lambda$. Let T be a tree so that $\{\langle \alpha, \emptyset, T\rangle\} \in \mathcal{P}$. More precisely, we should write $\{\langle 0, \emptyset\rangle\} \cup \{\langle \alpha, \emptyset, T\rangle\}$. But let us omit the least coordinate when the meaning is clear. If there is a $p \in N$ so that $p \cup \{\langle \alpha, \emptyset, T'\rangle\} \in \mathcal{P}$ and decides σ, for some $T' \subseteq T$, then we are done. Suppose otherwise. Denote $\mathrm{Suc}_T(\langle\rangle)$ by A. We shall define by recursion sequences $\langle p_\nu \mid \nu \in A\rangle$ and $\langle T^\nu \mid \nu \in A\rangle$.

Let $\nu = \min(A)$. Consider $\{\langle \alpha, \langle \nu\rangle, T_{\langle \nu\rangle}\rangle\}$. If there is no $p \in N$ and $T' \subseteq T_{\langle \nu\rangle}$ such that $p \cup \{\langle \alpha, \langle \nu\rangle, T'\rangle\}$ is in $\mathcal{P}$ and decides σ, then set $p_\nu = \emptyset$ and $T^\nu = T_{\langle \nu\rangle}$. Otherwise, pick some p and $T' \subseteq T_{\langle \nu\rangle}$ so that $p \cup \{\langle \alpha, \langle \nu\rangle, T'\rangle\}$ is in $\mathcal{P}$ and decides σ. Set $p_\nu = p$ and $T^\nu = T'$.

Suppose now that p_ξ and T^ξ are defined for every $\xi < \nu$ in A. We shall define p_ν and T^ν. But let us first define p'_ν and p''_ν. Define p''_ν to be the union of all p_ξ's with $\xi \in A \cap \nu$. Let $p'_\nu = \{\langle \gamma, p'^{\gamma}_\nu\rangle \mid \gamma \in \mathrm{supp}(p''_\nu)\}$, where for $\gamma \in \mathrm{supp}(p''_\nu)$, $p'^{\gamma}_\nu = p''^{\gamma}_\nu$ unless ν is permitted for p''^{γ}_ν and then $p'^{\gamma}_\nu = p''^{\gamma}_\nu {}^\frown \langle \pi_{\alpha\gamma}(\nu)\rangle$. If there is no $p \in N$ and T' so that $q = p \cup \{\langle \alpha, \langle \nu\rangle, T'\rangle\} \in \mathcal{P}$, $q \geq^* p'_\nu \cup \{\langle \alpha, \langle \nu\rangle, T_{\langle \nu\rangle}\rangle\}$ and $q \parallel \sigma$, then set $p_\nu = p''_\nu$ and $T^\nu = T_{\langle \nu\rangle}$. Suppose otherwise. Let p, T' be witnessing this. Then set $T^\nu = T'$ and $p_\nu = p''_\nu \cup (p \setminus p'_\nu)$.

This completes the recursive definition. Set $p = \bigcup_{\nu \in A} p_\nu$. For $i < \kappa$ let

$$C_i = \begin{cases} A, & \text{if there is no } \delta \in A \text{ such that } \delta^0 = i, \\ \bigcap\{\mathrm{Suc}_{T^\delta}(\langle \delta\rangle) \mid \delta \in A \text{ and } \delta^0 = i\}, & \text{otherwise.} \end{cases}$$

Note that $C_i \in U_\alpha$ since $A \in U_\alpha$ and this means by our agreement that for $\nu_1, \nu_2 \in A$, if $\nu^0_1 < \nu^0_2$ then $|\{\gamma \in A \mid \gamma^0 = \nu^0_1\}| < \nu^0_2$. Set $A^* = A \cap \Delta^*_{i<\kappa} C_i$. Then for every $\nu \in A^*$ for every $\delta \in A$ if $\delta^0 < \nu^0$ then $\nu \in \mathrm{Suc}_{T^\delta}(\langle \delta\rangle)$. Let S be the tree obtained from T by first replacing $T_{\langle \nu\rangle}$ by T^ν for every $\nu \in A^*$ and then intersecting all levels of it with A^*. ⊣

3.12.1 Claim. *$p \cup \{\langle \alpha, \emptyset, S\rangle\}$ belongs to $\mathcal{P}$.*

Proof. The only non-trivial point here is to show that $p \cup \{\langle \alpha, \emptyset, S\rangle\}$ satisfies condition (5) of the definition of $\mathcal{P}$. So let $\nu \in \mathrm{Suc}_S(\langle\rangle)$. By the definition of $S, \mathrm{Suc}_S(\langle\rangle) = A^*$. Consider the set

$$B_\nu = \{\gamma \in \mathrm{supp}(p) \mid \nu \text{ is permitted for } \underset{\sim}{p}^\gamma\}.$$

For every $\delta \in A$ let $B_{\nu,\delta} = \{\gamma \in \mathrm{supp}(p_\delta) \mid \nu \text{ is permitted for } p^\gamma\}$. Then $B_\nu = \bigcup_{\delta \in A} B_{\nu,\delta}$. But, actually the definition of the sequence $\langle p_\delta \mid \delta \in A\rangle$ implies that $B_\nu = \bigcup \{B_{\nu,\delta} \mid \delta \in A \text{ and } \delta^0 < \nu^0\}$. The number of δ's in A with $\delta^0 < \nu^0$ is $\leq \nu^0$, since for $\delta, \nu \in A$, $\delta^0 < \nu^0$ implies $\delta < \nu^0$, and it means in particular that for every $\xi < \nu^0$, $|\{\delta \in A \mid \delta^0 = \xi\}| < \nu^0$. So it is enough to show that for every $\delta \in A$, $\delta^0 < \nu^0$ implies $|B_{\nu,\delta} | \leq \nu^0$. Fix some $\delta \in A$ such that $\delta^0 < \nu^0$. Since $\nu \in A^*$ and $\delta^0 < \nu^0$, $\nu \in \mathrm{Suc}_{T^\delta}(\langle\delta\rangle)$. But $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\} \in \mathcal{P}$. So, by the definition of $\mathcal{P}, |B_{\nu,\delta}| \leq \nu^0$. $\dashv$

Then, clearly $p \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* \langle \alpha, \emptyset, T\rangle\}$.

For $\delta \in \mathrm{Suc}_S(\langle\rangle) = A^*$ let us denote by $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ the sequence $\{\langle \gamma, (p^\gamma)_{\pi_{\alpha\gamma}(\delta)}\rangle \mid \gamma \in \mathrm{supp}(p)\} \cup \{\langle \alpha, \langle\delta\rangle, S_{\langle\delta\rangle}\rangle\}$, where

$$(p^\gamma)_{\pi_{\alpha\gamma}(\delta)} = \begin{cases} p^{\gamma\frown}\pi_{\alpha\gamma}(\delta), & \text{if } \delta \text{ is permitted for } p^\gamma, \\ p^\gamma, & \text{otherwise.} \end{cases}$$

Note that $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ is a condition and $\pi_{\alpha\gamma}(\delta)$ is added only for γ's which appear in the support of some p_ξ with $\xi^0 < \delta^0$ and hence, with $\xi < \delta$. Also $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta{}^* \geq p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$.

3.12.2 Claim. *For every $\delta \in \mathrm{Suc}_S(\langle\rangle)$ if for some $q, R \in N$,*

$$(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta \leq^* q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\} \quad \textit{and} \quad q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\} \Vdash \sigma (\textit{resp. } \neg\sigma),$$

then $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta \Vdash \sigma$ (resp. $\neg\sigma$).

Proof. Such a $q \cup \{\langle \alpha, \langle\delta\rangle, R\rangle\}$ is a direct extension of $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$. By the choice of p_δ and T^δ, $p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$ decides σ. But $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta{}^* \geq p_\delta \cup \{\langle \alpha, \langle\delta\rangle, T^\delta\rangle\}$. $\dashv$

Let us now shrink the first level of S in order to insure that for every δ_1 and δ_2 in the new first level

$$(p \cup \{\langle \alpha, \emptyset, S\rangle\})_{\delta_1} \Vdash \sigma \text{ (resp. } \neg\sigma) \quad \text{iff} \quad (p \cup \{\langle \alpha, \emptyset, S\rangle\})_{\delta_2} \Vdash \sigma \text{ (resp. } \neg\sigma).$$

Let us denote the shrunken tree again by S.

3.12.3 Claim. *For every $\delta \in \mathrm{Suc}_S(\langle\rangle)$, $(p \cup \{\langle \alpha, \emptyset\rangle\})_\delta \not\parallel \sigma$.*

Proof. Suppose otherwise. Then every δ in $\mathrm{Suc}_S(\langle\rangle)$ will force the same truth value of σ. Suppose, for example, that σ is forced. Then $p \cup \{\langle \alpha, \emptyset, S\rangle\}$ will force σ. Since every $q \geq p \cup \{\langle \alpha, \emptyset, S\rangle\}$ is compatible with one of $(p \cup \{\langle \alpha, \emptyset, S\rangle\})_\delta$ for $\delta \in \mathrm{Suc}_S(\langle\rangle)$. This contradicts the initial assumption. $\dashv$

Now, climbing up level by level in the fashion described above for the first level, construct a direct extension $p^* \cup \{\langle \alpha, \emptyset, S^* \rangle\}$ of $p \cup \{\langle \alpha, \emptyset, S \rangle\}$ so that:

(a) For every $\eta \in S^*$, if for some $q, R \in N$, $(p^* \cup \{\alpha, \emptyset, S^*\rangle\})_\eta \leq^* q \cup \{\langle \alpha, \langle \eta \rangle, R \rangle\}$ and $q \cup \{\langle \alpha, \langle \eta \rangle, R\rangle\} \Vdash \sigma$ (resp. $\neg\sigma$), then $(p^* \cup \{\langle \alpha, \emptyset, S^* \rangle\})_\eta \Vdash \sigma$ (resp. $\neg\sigma$).

(b) If $\eta_1, \eta_2 \in S^*$ are of the same length, then

$$(p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\})_{\eta_1} \Vdash \sigma \text{ (resp. } \neg\sigma) \quad \text{iff} \quad (p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\})_{\eta_2} \Vdash \sigma \text{ (resp. } \neg\sigma).$$

As in Claim 3.12.3, it is impossible to have any $\eta \in S^*$ so that $(p^* \cup \{\langle \alpha, \emptyset, S^* \rangle\})_\eta$ decides σ. Combining this with (a) we obtain the following.

3.12.4 Claim. *For every $q, R, t \in N$, if $q \cup \{\langle \alpha, t, R\rangle\} \geq p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\}$ then $q \cup \{\langle \alpha, t, R \rangle\}$ does not decide σ.*

Proof. Just note that $q \cup \{\langle \alpha, t, R\rangle\} \geq^* (p^* \cup \{\langle \alpha, \emptyset, S\rangle\})_t$ and use (a). ⊣

Pick some $\beta \in N \cap \lambda$ which is $\leq_E$ above every element of $\operatorname{supp}(p^*)$. This is possible since $\operatorname{supp}(p^*) \in N$. Shrink S^* to a tree S^{**}, as in Lemma 3.10 in order to insure the following:

$$\text{For every } \nu \in \operatorname{Suc}_{S^{**}}(\langle\rangle) \text{ and } \gamma \in \operatorname{supp}(p^*), \quad \text{if } \nu \text{ is permitted for } (p^*)^\gamma, \text{ then } \pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)).$$

Let S^{***} be the projection of S^{**} to β via $\pi_{\alpha,\beta}$. Denote $p^* \cup \{\langle \beta, \emptyset, S^{***}\rangle\}$ by p^{**}. Then $p^{**} \in N$ and $p^{**} \cup \{\langle \alpha, \emptyset, S^{**}\rangle\} \geq^* p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\}$. Since N is an elementary submodel there is some $q \in N$ with $q \geq p^{**}$ deciding σ. Let, for example, $q \Vdash \sigma$. Pick some $t \in S^{**}$ so that $\pi_{\alpha\beta}``t = q^\beta$. Such t exists, since by Definition 3.7 q^β belongs to S^{***} which is the image of S^{**} under $\pi_{\alpha\beta}$. Note also that $mc(q) <_E \alpha$ by the choice of N. Let R be a from S^{**}_t by intersecting S^{**}_t with $\pi^{-1}_{\alpha, mc(q)}(T^q)$ and shrinking, if necessary, as in Lemma 3.10 in order to insure the equality of projections $\pi_{\alpha\gamma}$ and $\pi_{mc(q),\gamma} \circ \pi_{\alpha, mc(q)}$ for permitted γ's in $\operatorname{supp}(q)$. Then $q \cup \{\langle \alpha, t, R\rangle\}$ will be a condition stronger than q. Hence, it forces σ. But this contradicts Claim 3.12.4, since $q \cup \{\langle \alpha, t, R\rangle\} \geq p^* \cup \{\langle \alpha, \emptyset, S^*\rangle\}$. This contradiction finishes the proof of Lemma 3.12. ⊣

Let G be a generic subset of $\mathcal{P}$. By Lemma 3.10, for every $\alpha < \lambda$ there is a $p \in G$ with $\alpha \in \operatorname{supp}(p)$. Let us denote $\bigcup\{p^\alpha \mid p \in G\}$ by G^α.

3.13 Lemma.

(a) For every $\alpha < \lambda$, G^α is a Prikry sequence for U_α, i.e. an ω-sequence almost contained in every set in U_α.

(b) G^0 is an ω-sequence unbounded in κ.

(c) If $\alpha \neq \beta$ then $G^\alpha \neq G^\beta$, moreover $\alpha < \beta$ implies that G^β dominates G^α.

Proof. (a) Follows from the definition of $\mathcal{P}$. (b) Is a trivial consequence of (a). For (c) note that there is a $\gamma < \lambda$ such that $\gamma \geq_E \alpha, \beta$. By Lemma 3.4 then $\{\nu < \kappa \mid \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in U_\gamma$. This together with the definition of $\mathcal{P}$ implies that G^α is dominated by G^β. ⊣

3.14 Lemma. *κ^+ remains a cardinal in $V[G]$.*

Proof. Suppose otherwise. Then it changes its cofinality to some $\mu < \kappa$. Let $g : \mu \to (\kappa^+)^V$ be unbounded in $(\kappa^+)^V$. Pick $p \in G$ forcing this. Suppose for simplicity that $\emptyset \Vdash \underset{\sim}{g} : \check{\mu} \to \check{\kappa}^+$ unbounded. Pick an elementary submodel N as in Lemma 3.12. Let $\alpha < \lambda$ be above every element of $N \setminus \lambda$. Pick a tree T so that $\{\langle \alpha, \emptyset, T\rangle\} \in \mathcal{P}$. As in Lemma 3.12, define by induction an $\leq^*$ increasing sequence of direct extensions of $\{\langle \alpha, \emptyset, T\rangle\}$ $\langle q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\} \mid i < \mu\rangle$ so that

(a) $q_i \in N$.

(b) If for some $q, R, t \in N$ and $j < \kappa^+$, $q \cup \{\langle \alpha, t, R\rangle\} \geq q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\}$ and $q \cup \{\langle \alpha, t, R\rangle\} \Vdash \underset{\sim}{g}(\check{i}) = \check{j}$, then
$$(q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\})_t \Vdash \underset{\sim}{g}(\check{i}) = \check{j}.$$

Using Lemma 3.11, find S so that $\bigcup_{i<\mu} q_i \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* q_i \cup \{\langle \alpha, \emptyset, S^i\rangle\}$ for every $i < \mu$. Denote $\bigcup_{i<\mu} q_i$ by p. As in Lemma 3.12, choose $\beta \in N \setminus \lambda$ above $\mathrm{supp}(p)$ and project S into β using $\pi_{\alpha\beta}$. Denote the projection by S^*. Let $p^* = p \cup \{\langle \beta, \emptyset, S^*\rangle\}$. Then $p^* \in N$ and $p^* \cup \{\langle \alpha, \emptyset, S\rangle\} \geq^* p \cup \{\langle \alpha, \emptyset, S\rangle\}$. Since N is an elementary submodel, for every $i < \mu$ there will be a $q \in N$, $q \geq p^*$ forcing a value for $\underset{\sim}{g}(i)$. Then, using (b), as in Lemma 3.12 for some $t \in S$ $(p \cup \{\langle \beta, \emptyset, S\rangle\})_t$ will force the same value for $\underset{\sim}{g}(i)$. But $|S| = \kappa$. So, all such values are bounded in κ^+ by some ordinal δ which is impossible, since $N \supseteq \kappa^+$ and $N \vDash (\phi \Vdash (\underset{\sim}{g} : \check{\mu} \to \check{\kappa}^+ \text{ unbounded}))$. Contradiction. ⊣

Now combining the lemmas together, we obtain the following.

3.15 Theorem. *The following holds in $V[G]$:*

(a) κ has cofinality $\aleph_0$ and $\kappa^{\aleph_0} \geq \lambda$.

(b) All the cardinals are preserved.

(c) No new bounded subsets are added to κ.

If κ is a strong cardinal and $\lambda > \kappa$, then by the Solovay argument, described in the beginning of the section, there is a function $f : \kappa \to \kappa$ and a λ-strong embedding $j : V \to M$ so that $j(f)(\kappa) = \lambda$. Now, having f and j we can use the extender-based Prikry forcing over κ, as it was defined above. So, the following holds.

3.16 Theorem. *Let V be a model of* GCH *and let κ be a strong cardinal. Then for every λ there exists a cardinal preserving set generic extension $V[G]$ of V so that*

(a) No new bounded subsets are added to κ.

(b) κ changes its cofinality to $\aleph_0$.

(c) $2^\kappa \geq \lambda$.

4. Down to $\aleph_\omega$

The forcings of Sects. 2 and 3 produce models with κ of cofinality $\aleph_0$, GCH below κ, and 2^κ arbitrarily large. But such κ are quite large. Thus, in Sect. 2, it is a limit of measurables. In Sect. 3, it is a former measurable and no cardinals below it were collapsed. We should now like to collapse cardinals below κ and to move it to $\aleph_\omega$. Note that it is impossible to keep 2^κ arbitrarily large once κ is $\aleph_\omega$, since by the celebrated results of Shelah [53] $2^{\aleph_\omega} < \min(\aleph_{(2^{\aleph_0})^+}, \aleph_{\omega_4})$ provided that $\aleph_\omega$ is a strong limit. Our goal will be only to produce a finite gap between $\kappa = \aleph_\omega$ and 2^κ. It is possible to generalize this to countable gaps and for this we refer to [19, Sect. 3]. The possibility of getting uncountable gaps between $\aleph_\omega$ and $2^{\aleph_\omega}$ is a major open problem of cardinal arithmetic.

Let $2 \leq m < \omega$. We construct a model satisfying "$2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$" based on the forcing of the previous section.

The basic ideas for moving down to a small cardinal like $\aleph_\omega$ are due to Magidor [35, 36]. Hugh Woodin, see [9] or [15] was able to replace the use of supercompacts and huge cardinals by Magidor in [15] by strong cardinals. We present here a simplified version of [20, Sect. 2]. The main simplification is an elimination of M-generic sets used there. Another simplification, suggested by Assaf Sharon, allows the removal of bounds $b(p, \gamma)$ of [20, Sect. 2].

Fix a $(\kappa, \kappa+m)$-extender E over κ. Let $j : V \to M \simeq \mathrm{Ult}(V, E)$, $\mathrm{crit}(j) = \kappa$, $M \supseteq V_{\kappa+m}$, be the canonical embedding. Assume GCH. Let $\langle U_\alpha \mid \alpha < \lambda\rangle$, $\langle \pi_{\alpha\beta} \mid \alpha, \beta < \lambda,\ \beta \leq_E \alpha\rangle$ be as in the previous section with $\lambda = \kappa^{+m}$ and $f_\lambda : \kappa \to \kappa$ defined by $f_\lambda(\nu) = \nu^{+m}$.

We now define the set of forcing conditions.

4.1 Definition. The set of forcing conditions $\mathcal{P}$ consists of all elements p of the form

$$\begin{aligned}&\{\langle 0, \langle \tau_1, \ldots, \tau_n\rangle, \langle f_0, \ldots, f_n\rangle, F\rangle\}\\ &\quad \cup \{\langle \gamma, p^\gamma\rangle \mid \gamma \in g \setminus \{\max(g), 0\}\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\},\end{aligned}$$

where

(1) $\{\langle 0, \langle \tau_1, \ldots, \tau_n\rangle\rangle\} \cup \{\langle \gamma, p^\gamma\rangle \mid \gamma \in g \setminus \{\max(g), 0\}\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\}$ is as in Definition 3.6. Let us use the notations introduced there.

So, we denote g by $\mathrm{supp}(p), \max(g)$ by $mc(p)$, T by T^p and $p^{\max(g)}$ by p^{mc}. Also, let us denote further $\langle \tau_1, \ldots, \tau_n \rangle$ by p^0, $\langle f_0, \ldots, f_n \rangle$ by f^p, for $i < n$ f_i by f_i^p, n by n^p and F by F^p.

(2) $f_0 \in \mathrm{Col}(\omega, \tau_1)$, $f_i \in \mathrm{Col}(\tau_i^{+m+1}, \tau_{i+1})$ for $0 < i < n$, and $f_n \in \mathrm{Col}(\tau_n^{+m+1}, \kappa)$.

(3) F is a function on the projection of $T_{p^{mc}}$ by $\pi_{mc(p),0}$ so that

$$F(\langle \nu_0, \ldots, \nu_{i-1} \rangle) \in \mathrm{Col}(\nu_{i-1}^{+m+1}, \kappa).$$

Let us denote by $T^{p,0}$ the projection of T by π_{mc0}. For every $\eta \in T^{p,0}_{p^0}$ let F_η be defined by $F_\eta(\nu) = F(\eta^\frown \langle \nu \rangle)$.

Intuitively, the forcing $\mathcal{P}$ is intended to turn κ to $\aleph_\omega$ and simultaneously blowing up its power to κ^{+m+1}. The part of $\mathcal{P}$, which is responsible for blowing up the power of κ is the forcing used in Sect. 3. The function $f_0, \ldots, f_{n-1}$ provides partial information about collapsing already known elements of the Prikry sequence for U_0. F is a set of possible candidates for collapsing between further, still unknown elements of this sequence. Note, that for $i < n$ we are starting the collapse with τ_i^{+m+1}, i.e. we intend to preserve all $\tau_i, \tau_i^+, \ldots, \tau_i^{+m+1}$. The reason for this appears in the proof of the κ^{++}-c.c. and of the Prikry condition. It looks technical but what is hidden behind is that collapsing indiscernibles (i.e. members of Prikry's sequences for U_α's ($\alpha < \lambda$)) causes collapsing generators, i.e. cardinals between κ and λ. Shelah's bounds on the power of $\aleph_\omega$, [53] suggest that there is no freedom in using collapses below κ without effecting the structure of cardinals above κ as well.

4.2 Definition. Let $p, q \in \mathcal{P}$. We say that p *extends* q and denote this by $p \geq q$ iff

(1) $\{\langle 0, p^0 \rangle\} \cup \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \mathrm{supp}(p) \setminus \{mc(p), 0\}\} \cup \{\langle mc(p), p^{mc}, T^p \rangle\}$
extends
$\{\langle 0, q^0 \rangle\} \cup \{\langle \gamma, q^\gamma \rangle \mid \gamma \in \mathrm{supp}(q) \setminus \{mc(q), 0\}\} \cup \{\langle mc(q), q^{mc}, T^q \rangle\}$
in the sense of Definition 3.7.

(2) For every $i < \mathrm{length}(q^0) = n^q$, $f_i^p \geq f_i^q$.

(3) For every $\eta \in T^{p,0}_{p^0}$, $F^p(\eta) \supseteq F^q(\eta)$.

(4) For every i with $n^q \leq i < n^p$,

$$f_i^p \supseteq F^q((p^0 \setminus q^0) \restriction i + 1).$$

(5) $\min(p^0 \setminus q^0) > \sup(\mathrm{ran}(f_{n^q}))$.

4.3 Definition. Let $p, q \in \mathcal{P}$. We say that p is a *direct extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$.

The following lemmas are analogous to the corresponding lemmas of the previous section and they have analogous proofs.

4.4 Lemma. *The relation $\leq$ is a partial order.*

4.5 Lemma. *Let $q \in \mathcal{P}$ and $\alpha < \kappa^{+m}$. Then there is a $p \geq^* q$ so that $\alpha \in \mathrm{supp}(p)$.*

4.6 Lemma. *$\langle \mathcal{P}, \leq \rangle$ satisfies the κ^{++}-c.c.*

For the proof of the last lemma, note only that the number of possibilities for the collapsing part $\langle f_0, , \dots, f_n \rangle, F$ of a condition (in the form of Definition 4.1) is κ^+. It is important that F depends only on the normal ultrafilter of the extender. This way F can be viewed as an element of $\mathrm{Col}(\kappa, i_\kappa(\kappa))$ of $N_\kappa \simeq \mathrm{Ult}(V, U_\kappa)$, which (in V) has cardinality κ^+. Once allowing F to depend on the extender itself, say on the maximal coordinate of a condition, we will have the correspondence to $\mathrm{Col}(\kappa, j(\kappa))$ of $M \simeq \mathrm{Ult}(V, E)$. This set is of cardinality $> \kappa^+$ (in V) and using it, it is easy to produce κ^{++} incompatible conditions.

If $p \in \mathcal{P}$ and $\tau \in p^0$, then the set $\mathcal{P}/p$ of all extensions of p in $\mathcal{P}$ can be split in the obvious fashion into two parts: one everything above τ and the second everything below τ. Denote them by $(\mathcal{P}/p)^{\geq \tau}$ and $(\mathcal{P}/p)^{<\tau}$. Then $\mathcal{P}/p$ can be viewed as $(\mathcal{P}/p)^{\geq \tau} \times (\mathcal{P}/p)^{<\tau}$. The part $(\mathcal{P}/p)^{<\tau}$ consists of finitely many Levy collapses and the part $(\mathcal{P}/p)^{\geq \tau}$ is similar to $\mathcal{P}$ but has a slight advantage, namely the Levy collapses used in it are τ^{+m+1}-closed. Using this observation, one can show the following analog of Lemma 3.11(b):

4.7 Lemma. *If $p \in \mathcal{P}$ and $\tau \in p^0$, then $\langle (\mathcal{P}/p)^{\geq \tau}, \leq^* \rangle$ is τ^{+m+1}-closed.*

Let us now turn to the Prikry condition.

4.8 Lemma. *$\langle \mathcal{P}, \leq, \leq^* \rangle$ satisfies the Prikry condition.*

Proof. Let σ be a statement of the forcing language and $q \in \mathcal{P}$. We shall find $p \geq^* q$ deciding σ. In order to simplify notation, assume that $q = \emptyset$.

Pick an elementary submodel N, $\alpha < \kappa^{+m}$ and T as in Lemma 3.12. Consider condition $\{\langle \alpha, \emptyset, T \rangle\}$. More precisely, we should write $\{\langle 0, \emptyset, \emptyset, \emptyset \rangle \cup \{\langle \alpha, \emptyset, T \rangle\}$. But when the meaning is clear we shall omit $\{\langle 0, \emptyset, \emptyset, \emptyset \rangle\}$. If for some $p \in N$ $\{\langle 0, \emptyset, f, F \rangle\} \cup p \cup \{\langle \alpha, \emptyset, T' \rangle\} \in \mathcal{P}$ and decides σ, for some $T' \subseteq T, f$ and F, then we are done. Suppose otherwise.

As in the proof of Lemma 3.12 we first show that it is possible to deal with conditions having fixed support. Once the support is fixed the proof will be more or less like that of Lemma 1.20, with small complications due to the involvement of collapses.

4.8.1 Claim. *There are p, F and S in N so that*

(a) $\{\langle 0,\emptyset,\emptyset,F\rangle\}\cup p\cup\{\langle\alpha,\emptyset,S\rangle\}\geq^*\{\langle\alpha,\emptyset,T\rangle\}$.

(b) If for some $q\in N$, q^0, q^α, F', T' and $\vec{f}$,

$$\{\langle 0,q^0,\vec{f},F'\rangle\}\cup q\cup\{\langle\alpha,q^\alpha,T'\rangle\}$$

is a an extension of $\{\langle 0,\emptyset,\emptyset,F\rangle\}\cup p\cup\{\langle\alpha,\emptyset,T^*\rangle\}$ and forces σ (or $\neg\sigma$) then also

$$\{\langle 0,q^0,(\vec{f}\restriction\mathrm{length}(q^0))^\frown F(q^0),F\rangle\}\cup(p)_{q^\alpha}\cup\{\langle\alpha,q^\alpha,S_{q^\alpha}\rangle\}$$

forces the same, where $(p)_{q^\alpha}$ is the set $\{\langle\gamma,p^{\gamma\frown}t^\gamma\rangle\mid\gamma\in\mathrm{supp}(p)\}$ with t^γ the maximal final segment of $\pi_{\alpha\gamma}$"q^α permitted for p^γ.

Proof. Let A denote $\mathrm{Suc}_T(\langle\rangle)$. Assume that $A\subseteq\kappa$ and for $\nu_1,\nu_2\in A$, $\nu_1<\nu_2$ implies $\nu_1^0<\nu_2^0$. Also assume that only the elements of A appear in T, i.e. $T\subseteq[A]^{<\omega}$. Let $\{\langle q_i^0,\vec{f}_i,q_i^\alpha\rangle\mid i<\kappa\}$ be an enumeration of all triples $\langle q^0,\vec{f},q^\alpha\rangle$ such that

(i) $q^\alpha\in T$.

(ii) $q^0=\pi_{\alpha 0}$"q^α.

(iii) If $q^0=\langle\tau_0,\dots,\tau_{n-1}\rangle$ then $\mathrm{dom}(\vec{f})=n$ and $\vec{f}(0)\in\mathrm{Col}(\omega,\tau_0)$, $\vec{f}(1)\in\mathrm{Col}(\tau_0^{+m+1},\tau_1),\dots,\vec{f}(n-1)\in\mathrm{Col}(\tau_{n-2}^{+m+1},\tau_{n-1})$. (Note that we do not enumerate the "last" function from $\mathrm{Col}(\tau_{n-1}^{+m+1},\kappa)$.)

For every $\nu\in A$, $|\{\rho\in A\mid\rho^0=\nu^0\}|\leq(\nu^0)^{+m}$. So, the number of such triples satisfying $q^0(i)\leq\nu^0$ for every $i\leq$ length(q^0) is at most $(\nu^0)^{+m}$. We can assume that $\{\langle q_i^0,\vec{f}_i,q_i^\alpha\rangle\mid i<(\nu^0)^{+m}\}=\{\langle q^0,\vec{f},q^\alpha\rangle\mid\langle q^0,\vec{f},q^\alpha\rangle$ satisfy the conditions (i), (ii), (iii) above and $q^0(i)\leq\nu^0$ for every $i\leq\mathrm{length}(q^0)\}$.

Define by recursion sequences $\langle p_i\mid i<\kappa\rangle$, $\langle T^i\mid i<\kappa\rangle,\langle f^i\mid i<\kappa\rangle$ and $\langle F^i\mid i<\kappa\rangle$. Set $p_0=\emptyset$, $T^0=T$, $f^0=\emptyset$ and $F^0=\emptyset$.

Suppose that p_j, T^j and F^j are defined for every $j<i$. Define p_i, T^i, f^i and F^i.

Set first $p_i''=\bigcup_{j<i}p_j$. Let $p_i'=\{\langle\gamma,p'^\gamma\rangle\mid\gamma\in\mathrm{supp}(p_i'')\}$, where for $\gamma\in\mathrm{supp}(p_i'')$, $p_i'^\gamma=p_i''^\gamma$ unless there is a $\nu\in q_i^\alpha$ permitted for $p_i''^\gamma$ and then $p_i'^\gamma=p_i''^{\gamma\frown}$ the maximal final segment of $\pi_{\alpha\gamma}$"q_i^α permitted for $p_i''^\gamma$.

We now wish to define a function F' on the set $q_i^{0\frown}(T_{q_i^\alpha})^0=_{df}\{q_i^{0\frown}\langle\eta\rangle\mid\langle\eta\rangle\in(T_{q_i^\alpha})^0\}$. Let $\langle\eta\rangle\in(T_{q_i^\alpha a})^0$ (it may be just the empty sequence). Consider the set

$$C=\{j<i\mid q_i^{0\frown}\langle\eta\rangle\text{ extends }q_j^0\text{ and }q_i^{0\frown}\langle\eta\rangle\in q_j^{0\frown}(T^j)^0\}.$$

For every $j\in C$ we have

$$q_j^0(\mathrm{length}(q_j^0)-1)\leq q_i^{0\frown}\langle\eta\rangle(\mathrm{length}(q_i^{0\frown}\langle\eta\rangle)-1).$$

Then, by the properties of the enumeration $\{\langle q^0_\nu, \vec{f}_\nu, q^\alpha_\nu\rangle \mid \nu < \kappa\}$ we have $j < (q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1))^{+m}$. So

$$C \subseteq (q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1))^{+m}.$$

Now define

$$F'(q^0_i{}^\frown\langle\eta\rangle) = \bigcup_{j\in C} F^j(q^0_i{}^\frown\langle\eta\rangle).$$

Then

$$F'(q^0_i{}^\frown\langle\eta\rangle) \in \mathrm{Col}(q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1)^{+m+1}, \kappa),$$

since $|C| \le q^0_i{}^\frown\langle\eta\rangle(\mathrm{length}(q^0_i{}^\frown\langle\eta\rangle) - 1)^{+m}$. Define

$$r^i = \{\langle q^0_i, \vec{f}_i{}^\frown F'(q^0_i), F'\rangle\} \cup p'_i \cup \{\langle \alpha, q^\alpha_i, T_{q^\alpha_i}\rangle\}.$$

If $r^i \notin \mathcal{P}$ or it belongs to $\mathcal{P}$ and there is no $p \in N, T', g$ and F so that $\{\langle 0, q^0_i, \vec{f}_i{}^\frown g, F\rangle\} \cup p \cup \{\langle\alpha, q^\alpha_i, T'\rangle\} \in \mathcal{P}$ extends r^i and decides σ, then set $p_i = p''_i$, $T^i = T_{q^\alpha_i}$, $f^i = F'(q^0_i)$ and $F^i = F'$. Otherwise, pick some p, T', g and F witnessing this. Then define $T^i = T'$, $F^i = F$, $f^i = g$, $F^i(q^0_i) = f^i$. Set $p_i = p''_i \cup p^*$, where $p^* = p \setminus p'_i$.

This completes the recursive definition. Set $p = \bigcup_{i<\kappa} p_i$. Now define a subtree S of T by putting together all the T^i's for $i < \kappa$. The definition is level by level. Thus, if S is defined up to level n and t sits in S on this level, then set

$$\begin{aligned}\mathrm{Suc}_S(t) = \{\nu \in A \mid \nu^0 > \max(t),\ &\text{and for every } i < \nu^0,\\ &\nu \in \mathrm{Suc}_{T^i}(\langle\rangle) \text{ and } \nu \in \mathrm{Suc}_{T^i}(t) \text{ when } t \in T^i\}.\end{aligned}$$

So $\mathrm{Suc}_S(t) \in U_\alpha$.

Let us now put together all the F^i's. Define a function F on a tree $(S)^0$.

Thus let $\eta \in S^0$. Consider the set $C = \{j < \kappa \mid q^0_j \subseteq \eta \in q^0_j{}^\frown(T^j)^0\}$. Let $\ell = \mathrm{length}(\eta) - 1$. Then for each $j \in C$ $q^0_j(\mathrm{length}(q^0_j) - 1) \le \eta(\ell)$. So, by the choice of the enumeration $\{\langle q^0_\nu, \vec{f}_\nu, q^\alpha_\nu\rangle \mid \nu < \kappa\}$ we have $j < \eta(\ell)^{+m}$. Hence $C \subseteq \eta(\ell)^{+m}$. Define $F(\eta) = \bigcup_{j\in C} F^j(\eta)$. Then $F(\eta) \in \mathrm{Col}(\eta(\ell)^{+m+1}, \kappa)$.

4.8.2 Subclaim. $\{\langle 0, \emptyset, \emptyset, F\rangle\} \cup p \cup \{\langle\alpha, \emptyset, S\rangle\} \in \mathcal{P}$.

Proof. The only problem is to show that for every $\nu \in \mathrm{Suc}_S(\langle\rangle)$,

$$|\{\gamma \in \mathrm{supp}(p) \mid \nu \text{ is permitted for } p^\gamma\}| \le \nu^0.$$

Thus let $\nu \in \mathrm{Suc}_S(\langle\rangle)$ and $i < \kappa$. If $\langle q^0_i, \vec{f}_i, q^\alpha_i\rangle$ satisfies $\max(q^0_i) < \nu^0$, then $i < \max(q^0_i)^{+m} < \nu^0$. Hence for every $i \ge \nu^0$, ν is not permitted for q^0_i. So after the stage ν^0 we did not add any new coordinate γ with ν permitted for $(p_i)^\gamma$. This means that $\{\gamma \in \mathrm{supp}(p) \mid \nu$ is permitted for $p^\gamma\} = \bigcup_{i<\nu^0}\{\gamma \in \mathrm{supp}(p_i) \mid \nu$ is permitted for $p^\gamma\}$ and we are done. ⊣

Denote $\{\langle 0, \emptyset, \emptyset, F\rangle\} \cup p \cup \{\langle \alpha, \emptyset, S\rangle\}$ by p^*. We now show that p^* is as desired. Clearly, $p^* \geq^* \{\langle \alpha, \emptyset, T\rangle\}$. Suppose that for some $q \in N$, q^0, q^α, G R and $\vec{f}$

$$\{\langle 0, q^0, \vec{f}, G\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, R\rangle\} \geq p^*$$

and

$$\{\langle 0, q^0, \vec{f}, G\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, R\rangle\} \Vdash \sigma \quad (\text{or } \neg\sigma)$$

Let $q^0 = \langle \tau_1, \ldots, \tau_n\rangle$ and $\vec{f} = \langle f_0, \ldots, f_n\rangle$. Obviously, $n > 0$ since otherwise we will have a direct extension of p^* (and hence of $\{\langle \alpha, \emptyset, T\rangle\}$) deciding σ contrary to the initial assumption. Find $i < \tau_n^{+m}$ such that $\langle q^0, \langle f_0, \ldots, f_{n-1}\rangle, q^\alpha\rangle = \langle q_i^0, \vec{f_i}, q_i^\alpha\rangle$. Consider the condition

$$r^i = \{\langle q_i^0, \vec{f_i}{}^\frown F'(q_i^0), F'\rangle\} \cup p_i' \cup \{\langle \alpha, q_i^\alpha, T_{q_i^\alpha}\rangle\},$$

defined at stage i of the construction. We have

$$\{\langle 0, q^0, \vec{f}, G\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, R\rangle\} \geq^* r^i,$$

since $R \subseteq S_{q^\alpha} \subseteq T_{q^\alpha}$, $F'(\eta) \subseteq F(\eta)$ for η's from the common domain, so that in particular, $F'(q_i^0) \subseteq F(q_i^0) \subseteq f_n$. But then

$$\{\langle 0, q_i^0, \vec{f_i}{}^\frown f^i, F^i\rangle\} \cup (p_i)_{q_i^\alpha} \cup \{\langle \alpha, q_i^\alpha, T^i_{q_i^\alpha}\rangle\} \Vdash \sigma \quad (\text{or } \neg\sigma)$$

by the choice of f^i, F^i, T^i and p_i at the stage i of the construction. Hence also

$$\{\langle 0, q^0, \langle f_0, \ldots, f_{n-1}, F(q^0)\rangle, F\rangle\} \cup (p)_{q^\alpha} \cup \{\langle \alpha, q^\alpha, S_{q^\alpha}\rangle\}$$

forces the same. This completes the proof Claim 4.8.1. ⊣

Fix $p^* = \{\langle 0, \emptyset, \emptyset, F\rangle\} \cup p \cup \{\langle \alpha, \emptyset, S\rangle\}$ satisfying the conclusion of Claim 4.8.1.

As in Lemma 3.12, it is possible to show that the assumption "$q \in N$" is not really restrictive. Briefly, if there is some q outside of N which is used to decide σ, then there exists one also inside N. So the following claim will provide the desired contradiction.

4.8.3 Claim. *There is a*

$$p^{**} = \{\langle 0, \emptyset, \emptyset, F{\restriction}T^*\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T^*\rangle\} \geq^* p^*$$

such that the following holds:

$(*)$ *There are no $q \in N, q^0, q^\alpha, \vec{f}, F'$ and T' such that*

$$p^{**} \leq \{\langle 0, q^0, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, T'\rangle\} \parallel \sigma.$$

Proof. We shall construct by recursion a $\le^*$-increasing sequence $\langle p(\ell) \mid \ell \le \omega\rangle$ of direct extensions of p^* satisfying for every $\ell \le \omega$ the following condition:

$(*)_\ell$ There are no $q \in N$, q^0, q^α, $\vec{f}$, F' and T' such that length$(q^0) \le \ell$ and

$$p(\ell) \le \{\langle 0, q^0, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, q^\alpha, T'\rangle\} \parallel \sigma.$$

Clearly, then $p(\omega)$ will be as desired.

Set $p(0) = p^*$. Define $p(1)$ to be a condition of the form $\{\langle 0, \emptyset, \emptyset, F\restriction T_1\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1\rangle\}$ with T_1 defined below. Consider the three sets

$$\begin{aligned} X_i = \big\{&\nu \in \mathrm{Suc}_S(\langle\rangle) \mid \exists f_0^\nu \in \mathrm{Col}(\omega, \nu^0) \\ &\big(\{\langle 0, \langle \nu^0\rangle, f_0^\nu{}^\frown F(\langle \nu^0\rangle), F\rangle\} \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, S_{\langle\nu\rangle}\rangle\} \ \Vdash {}^i\sigma\big)\big\}, \end{aligned}$$

where $i < 2$, ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$, and

$$X_2 = \mathrm{Suc}_S(\langle\rangle) \setminus (X_0 \cup X_1).$$

There is an $i < 3$ such that $X_i \in U_\alpha$. Let T_1' be the tree obtained from S by intersecting all its levels with X_i. Let $r = \{\langle 0, \emptyset, \emptyset, F\restriction T_1'\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1'\rangle\}$. If there is no $q \in N, \vec{f}, \nu, F'$ and T' such that

$$r \le \{\langle 0, \langle \nu^0\rangle, \vec{f}, F'\rangle\} \cup q \cup \{\langle \alpha, \langle\nu\rangle, T'\rangle\} \parallel \sigma,$$

then set $T_1 = T_1'$ and $p(1) = r$. We claim that this is the only possible case. Otherwise, pick $q, \vec{f} = \langle f_0, f_1\rangle, \nu, F'$ and T' witnessing this and, say, forcing σ. By the previous claim, then

$$\{\langle 0, \langle \nu^0\rangle, f_0{}^\frown F(\nu^0), F\rangle\} \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, (T_1')_{\langle\nu\rangle}\rangle\} \Vdash \sigma.$$

By the choice of T_1', then $X_0 \in U_\alpha$. Hence, for every $\nu \in \mathrm{Suc}_{T_1}(\langle\rangle)$ there is an $f_0^\nu \in \mathrm{Col}(\omega, \nu^0)$ such that

$$\{\langle 0, \langle \nu^0\rangle, f_0^\nu{}^\frown F(\langle \nu^0\rangle), F\rangle \cup p_{\langle\nu\rangle} \cup \{\langle \alpha, \langle\nu\rangle, (T_1')_{\langle\nu\rangle}\rangle\} \Vdash \sigma.$$

Note that the function taking ν^0 to f_0^ν is actually a regressive function on $(X_0)^0$. Find $Y \in U_\alpha$ and $f^* \in \mathrm{Col}(\omega, \kappa)$ such that for every $\nu \in Y$, $f_0^\nu = f^*$. Let T_1 be a tree obtained from T_1' by shrinking all its levels to Y. Set $F_1 = F\restriction T_1$. Finally, let

$$p(1) = \{\langle 0, \emptyset, f^*, F_1\rangle\} \cup p \cup \{\langle \alpha, \emptyset, T_1\rangle\}.$$

By the construction, $p^* \le^* p(1) \Vdash \sigma$, which contradicts the assumption that it is impossible to decide σ by direct extensions of p^*.

Let us define $p(2) = \langle 0, \emptyset, \emptyset, F\restriction T_2\rangle \cup p \cup \{\langle \alpha, \emptyset, T_2\rangle\}$ now. Fix a $\nu \in \mathrm{Suc}_{T^1}(\langle\rangle)$. Let $\{\langle f_i, \nu_i\rangle \mid 1 \le i < (\nu^0)^{+m}\}$ be the enumeration of all pairs $\langle f, \rho\rangle$ such that $\rho \in \mathrm{Suc}_{T_1}(\langle\rangle)$, $\rho^0 = \nu^0$ and $f \in \mathrm{Col}(\omega, \nu^0)$. We would first like to define $T_{2\langle\rho\rangle}$ for every $\rho \in \mathrm{Suc}_{T_1}(\langle\rangle)$ with $\rho^0 = \nu^0$. In order to do this define by recursion on $i < (\nu^0)^{+m}$ sets S_i as follows: for $i = 0$ let $S_0 = (T_1)_{\langle\nu\rangle}$. Suppose that S_j is defined for every $j < i$. Set $S = (\bigcap_{j<i} S_j) \cap (T_1)_{\langle\nu_i\rangle}$. Consider a condition

$$r = \{\langle 0, \langle\nu^0\rangle, f_i, F(\langle\nu^0\rangle), F{\restriction}S\rangle\} \cup (p)_{\langle\nu_i\rangle} \cup \{\langle\alpha, \langle\nu_i\rangle, S\rangle\}.$$

Clearly, $r \geq p(1)$. By the choice of $p(1)$, neither r or its direct extensions decide σ. Then, the construction of $p(1)$ from $p(0)$ applied to r (instead of $p(0)$) will produce

$$r^i = \{\langle 0, \langle\nu^0\rangle, f_i, F(\langle\nu^0\rangle), F{\restriction}S_i\rangle\} \cup (p)_{\nu_i} \cup \{\langle\alpha, \langle\nu_i\rangle, S_i\rangle\} \geq^* r$$

satisfying the following: There are no $q \in N, \rho, g_1, g_2, F', S'$ such that

$$r^i \leq \{\langle 0, \langle\nu^0, \rho^0\rangle, \langle f_i, g_1, g_2\rangle, F'\rangle\} \cup (p)_{\langle\nu_i,\rho\rangle} \cup \{\langle\alpha, \langle\nu_i, \rho\rangle, S'\rangle\} \parallel \sigma.$$

Now let $(T_2)_{\langle\nu^0\rangle} = \bigcap_{j<(\nu_0)^{+m}} S_j$. Define T_2 to be the tree obtained from T_1 by replacing $(T_1)_{\langle\nu\rangle}$ by $(T_2)_{\langle\nu^0\rangle}$ for each $\nu \in \mathrm{Suc}_{T_1}(\langle\rangle)$. Set $p(2) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_2\rangle\} \cup p \cup \{\langle\alpha, \emptyset, T_2\rangle\}$. It is easy to see that $p(2)$ satisfies $(*)_2$.

We continue in the same fashion and define $p(n) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_n\rangle\} \cup p \cup \{\langle\alpha, \emptyset, T_n\rangle\}$ satisfying $(*)_n$ for every n, $2 \leq n < \omega$. Finally let $T_\omega = \bigcap_{n<\omega} T_n$. Set $p(\omega) = \{\langle 0, \emptyset, \emptyset, F{\restriction}T_\omega\rangle \cup p \cup \{\langle\alpha, \emptyset, T_\omega\}$. Then $p(\omega)$ will satisfy $(*)_n$ for every $n < \omega$ and hence $(*)$. ⊣

This completes the proof of Lemma 4.8. ⊣

Using Lemma 4.8 as a replacement for Lemma 3.12, the arguments of Lemma 3.12 show the following:

4.9 Lemma. *κ^+ remains a cardinal in $V^{\mathcal{P}}$.*

Lemma 3.13 transfers directly to the present forcing notion. Thus for G a generic subset of $\mathcal{P}$ and $\alpha < \kappa^{+m}$ define and G^α, as in Sect. 3, as to be $\bigcup\{p^\alpha \mid p \in G\}$. Let $G^0 = \langle\kappa_0, \kappa_1, \ldots, \kappa_n, \ldots\rangle$.

4.10 Lemma.

(a) For every $\alpha < \kappa^{+m}$, G^α is a Prikry sequence for U_α.

(b) G^0 is an ω-sequence unbounded in κ.

(c) If $\alpha \neq \beta$ are then $G^\alpha \neq G^\beta$.

Let $\alpha < \kappa^{+m}$ and $G^\alpha = \langle\nu_0, \nu_1, \ldots, \nu_n, \ldots\rangle$. An easy density argument provides an $n(\alpha) < \omega$ such that either

(i) for all but finitely many n's, $\nu^0_{n+n(\alpha)} = \kappa_n$, or

(ii) for all but finitely many n's, $\nu^0_n = \kappa_{n+n(\alpha)}$.

Transform G^α into a sequence $G'^\alpha = \langle \nu'_0, \nu'_1, \ldots, \nu'_n, \ldots\rangle$ defined as follows:

$$\nu'_n = \begin{cases} \nu_{n+n(\alpha)}, & \text{if (a) holds,} \\ \nu_{n-n(\alpha)}, & \text{if (b) holds and } n \geq n(\alpha)\text{, and} \\ \kappa_n, & \text{if (b) holds and } n < n(\alpha). \end{cases}$$

Then, for every $n < \omega$, $(\nu'_n)^0 = \kappa_n$.

Assaf Sharon [50] showed that $\langle G'^\alpha \mid \alpha < \kappa^{+m}\rangle$ is a scale in $\prod_{n<\omega} \kappa_n^{+m}$, i.e. every member of $\prod_{n<\omega} \kappa_n^{+m}$ is dominated by one of the G'^α's and $\alpha < \beta$ implies that G'^β dominates G'^α.

The next lemma is obvious.

4.11 Lemma. *If $\aleph_0 < \tau < \kappa$ and τ remains a cardinal in $V[G]$, then for some n and for some $m' \leq m$, $\tau = \kappa_n^{+m'+1}$.*

Implementing $\mathrm{Col}(\nu, \nu^+)$'s also, Sharon [50] was able to collapse each κ_n^+ as well. Thus in his model $\kappa_n^{+m'+1}$ for $1 \leq m' \leq m$ are the only uncountable cardinals below κ. Notice that $\langle \kappa_n^+ \mid n < \omega\rangle$ and $\langle \kappa_n^{+m+1} \mid n < \omega\rangle$ are Prikry sequences for U_{κ^+} and $U_{\kappa^{+m+1}}$ and so correspond to κ^+ and κ^{+m+1} of the ultrapower M by $(\kappa, \kappa+m)$-extender E. So, in V, $\mathrm{cf}((\kappa^{+m+1})^M) = \kappa^+$. Also $\langle \kappa_{n+1} \mid n < \omega\rangle$ may be viewed as a sequence corresponding to $j(\kappa)$ which again has cofinality κ^+. Hence, the collapses involved collapse between members of the same cofinality.

Now combining all the lemmas, we obtain the following.

4.12 Theorem. *In a generic extension $V[G]$, $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$.*

5. Forcing Uncountable Cofinalities

In the previous sections we dealt with a singular κ of cofinality $\aleph_0$ or changed the cofinality of a regular κ to $\aleph_0$. Here we would like to deal with forcings changing cofinality to an uncountable value. The first such forcing was introduced by Magidor [37]. It changed the cofinality of a regular κ to any prescribed regular value δ below κ. The Magidor forcing adds a closed unbounded in κ sequence of order type δ instead of an ω-sequence added by the Prikry forcing in Definition 1.1. The initial assumption used for this was stronger than just measurability. A measurable cardinal κ of the Mitchell order δ, i.e. $o(\kappa) = \delta$, was used. Later Mitchell [45] showed that this assumption is optimal. Lon Radin [48] defined a forcing of the same flavor which not only could change the cofinality of κ to $\delta < \kappa$ by shooting a closed unbounded δ-sequence, but also adding a closed unbounded κ-sequence preserving regularity and even measurability of κ. It is not a big deal to add a closed unbounded subset to a regular κ preserving its regularity and also measurability. But what is special about the Radin club is that it consists of cardinals which were regular in the ground model and this way combines together a variety of ways of changing cofinalities. This feature allows results of global character in the cardinal arithmetic. Thus, shortly after the discovery of the Radin forcing, Foreman and Woodin [12] constructed a model satisfying $2^\tau > \tau^+$ for every τ and Woodin produced a model with $2^\tau = \tau^{++}$ for every τ. Later James Cummings [9] constructed a model with $2^\tau = \tau^+$ for every regular τ and $2^\tau = \tau^{++}$ for every singular cardinal τ. Recently, Carmi Merimovich [39, 40] obtained additional results of this type introducing extender-based Radin forcing.

5.1. Radin Forcing

Here we will give the basics of Radin forcing. A comprehensive account on the matter containing various beautiful results of Woodin using Radin forcing should appear in the book by Cummings and Woodin [10]. Originally Radin [48] and then Mitchell [42] defined this forcing axiomatically. We will follow a more concrete approach due to Woodin.

Let $j : V \to M$ be an elementary embedding of V into transitive inner model M, with critical point κ. Define a normal ultrafilter $U(0)$ over κ:

$$X \in U(0) \quad \text{iff} \quad \kappa \in j(X).$$

If $U(0) \in M$, then we define a κ-complete ultrafilter $U(1)$, only not over κ but over V_κ:

$$X \in U(1) \quad \text{iff} \quad \langle \kappa,\ U(0)\rangle \in j(X).$$

Such defined $U(1)$ concentrate on pairs $\langle \nu, F\rangle$ so that ν is a measurable cardinal below κ and F is a normal ultrafilter over ν.

If $U(1) \in M$, then we can continue and define a κ-complete ultrafilter $U(2)$ over V_κ:

$$X \in U(2) \quad \text{iff} \quad \langle \kappa, U(0), U(1)\rangle \in j(X).$$

Continue by recursion and define a sequence

$$\vec{U} = \langle \kappa, U(0), U(1), \ldots, U(\alpha), \ldots \mid \alpha < \operatorname{length}(\vec{U})\rangle,$$

where each $U(\alpha)$ will be a κ-complete ultrafilter over V_κ:

$$X \in U(\alpha) \quad \text{iff} \quad \vec{U}\restriction\alpha = \langle \kappa, U(0), U(1), \ldots, U(\beta), \ldots \mid \beta < \alpha\rangle \in j(X),$$

and $\operatorname{length}(\vec{U})$ will be the least α with $\vec{U}\restriction\alpha \notin M$. For example, if $M \supseteq \mathcal{P}(\mathcal{P}(\kappa))$, then $\operatorname{length}(\vec{U})$ will be at least $(2^\kappa)^+$, as we will see below. Let us call $\vec{U}$ and $\vec{U}\restriction\alpha$ $(0 < \alpha < \operatorname{length}(\vec{U}))$ *j-sequences of ultrafilters*.

Fix some α^* with $0 < \alpha^* \le \operatorname{length}(\vec{U})$. Let $\vec{V} = \vec{U}\restriction\alpha^*$. We want to define Radin forcing with the ultrafilter sequence $\vec{V}$. Denote it by $R_{\vec{V}}$. As usual, it will have two orders $\le$ and $\le^*$.

Let us deal first with $\alpha^* = 1$ and $\alpha^* = 2$. Thus, for $\alpha^* = 1$, $\vec{V} = \langle \kappa, U(0)\rangle$. Let $\langle R_{\vec{V}}, \le, \le^*\rangle$ be the usual Prikry forcing with $U(0)$ of Definition 1.1, only instead of writing $\langle t, A\rangle$ (where t is an increasing finite sequence and $A \in U(0)$) we shall write $\langle t, \langle \kappa, U(0)\rangle, A\rangle$.

Now let $\alpha^* = 2$. Then $\vec{V} = \langle \kappa, U(0), U(1)\rangle$. We would like to incorporate both $U(0)$ and $U(1)$ in the process generating the generic cofinal sequence. Thus instead of $A \in U(0)$ in the previous case we allow two sets $A_0 \in U(0)$ and $A_1 \in U(1)$, or equivalently, a set in $U(0) \cap U(1)$. Notice, that we can separate $U(0)$ and $U(1)$ since $U(0)$ concentrates on ordinals and $U(1)$ on pairs $\langle \nu, F\rangle$ with F a normal ultrafilter over ν. An initial condition in $R_{\vec{V}}$ will have a form

$$p = \langle\langle \kappa, U(0), U(1)\rangle, A\rangle$$

with $A \in U(0) \cap U(1)$ and require also that each $a \in A$ is either an ordinal or a pair consisting of a measurable cardinal and a normal ultrafilter over it. In order to extend p pick $a \in A$ and $B \subseteq A$, with $B \in U(0) \cap U(1)$ such that the rank of each member of B is above $\text{rank}(a) + 1$. If a is an ordinal then just add it. We will obtain a one-step extension of p

$$\langle a, \langle\langle \kappa, U(0), U(1)\rangle, B\rangle\rangle.$$

If $a = \langle \nu, F\rangle$, then consider $A \cap \nu$. a can be added to p only if this set is in F. Notice that the set $X_A = \{\langle \nu', F'\rangle \mid A \cap \nu' \in F'\} \in U(1)$ since $A \cap \kappa \in U_0$ and so in M, $\langle \kappa, U_0\rangle \in j(X_A)$. If $A \cap \nu \in F$, then let $B_\nu \in F$ be a subset of $A \cap \nu$. The following will be one-step extension of p:

$$\langle\langle\langle \nu, F\rangle, B_\nu\rangle, \langle\langle \kappa, U(0), U(1)\rangle, B\rangle\rangle.$$

Consider a one-step extension $\langle d, \langle\langle \kappa, U(0), U(1)\rangle, B\rangle\rangle$. If d is an ordinal then repeat the recipe of one-step extension described above. Suppose that $d = \langle\langle \nu, F\rangle, B_\nu\rangle$. We now have two alternatives. The first, just as at step one, is to add an ordinal or a pair but between ν and κ. The second is to add an element of B_ν. Thus $\langle \nu, F\rangle$ will be responsible for producing a Prikry sequence for F. This way, generically a sequence of the type ω^2 will be produced.

We now turn to the general case and give a formal definition of $R_{\vec{V}}$ the Radin forcing with the sequence of ultrafilters $\vec{V}$. First let us introduce some notation. Thus, for a sequence $\vec{F} = \langle F(0), \ldots, F(\tau), \ldots \mid \tau < \text{length}(\vec{F})\rangle$ let $\bigcap \vec{F} = \bigcap\{F(\tau) \mid \tau < \text{length}(\vec{F})\}$. For an ordinal $d = \nu$ or pair $d = \langle \nu, \vec{F}\rangle$ or a triple $d = \langle \nu, \vec{F}, B\rangle$ let us denote ν by $\kappa(d)$. For a triple $d = \langle \nu, \vec{F}, B\rangle$ by $d \in A$ we shall mean that the two first coordinates of d, i.e. $\langle \nu, \vec{F}\rangle$ belong to A.

The main idea behind this forcing is to use members of finite sequences (that it produces) to give rise to separate blocks that are themselves Radin forcings. In order to realize this idea let us first shrink a bit the possibilities of choosing these finite sequences. Let $\vec{F}$ be a sequence of ultrafilters over ν. We would like to use only $\vec{F}$'s which are j-sequences of ultrafilters for some $j : V \to M$. Also, we like to have a set $B \in \bigcap \vec{F}$ such that each member d of it is a j-sequence for some j with critical point $\kappa(d)$.

To achieve this let us define by recursion classes of sequences:

$$A^{(0)} = \{\vec{F} \mid \vec{F} \text{ is a } j\text{-sequence of ultrafilters for some } j : V \to M\}$$

$$A^{(n+1)} = \{\vec{F} \in A^{(n)} \mid \forall\alpha\ 0 < \alpha < \text{length}(\vec{F})\ (A^{(n)} \cap V_{\kappa(\vec{F})} \in F(\alpha))\}$$

$$\overline{A} = \bigcap\nolimits_{n<\omega} A^{(n)}.$$

The main feature of $\overline{A}$ is that if $\vec{F} \in \overline{A}$ then, for $0 < \alpha < \text{length}(\vec{F})$, $F(\alpha)$ concentrates on $\overline{A} \cap V_{\kappa(\vec{F})}$, since then $A^{(n)} \cap V_{\kappa(\vec{F})} \in F(\alpha)$ for every n and hence by countable completeness of $F(\alpha)$, also $\overline{A} \cap V_{\kappa(\vec{F})} \in F(\alpha)$.

Note that each measurable cardinal is in $\overline{A}$. But in the presence of stronger large cardinals, $\overline{A}$ turns out to be much wider. We will need the following statement proved by Cummings and Woodin [10]:

5.1 Lemma. *Let E be a (κ, λ)-extender and $j : V \to M \simeq \text{Ult}(V, E)$ the corresponding elementary embedding, so that $M \supseteq V_{\kappa+2}$ and ${}^{\kappa}M \subseteq M$. Let $\vec{U}$ be the j-sequence of ultrafilters of the maximal length. Then*

(a) $\text{length}(\vec{U}) \geq (2^\kappa)^+$.

(b) For every $\alpha < (2^\kappa)^+$, $\vec{U}{\restriction}\alpha \in \overline{A}$.

Proof. Note that ${}^{\alpha}V_{\kappa+2} \subseteq M$ for every $\alpha < (2^\kappa)^+$. Hence $\vec{U}{\restriction}\alpha \in M$ for every such α.

Let us first show that for every $\alpha < (2^\kappa)^+$, $\vec{U}{\restriction}\alpha \in A^{(1)}$. Equivalently, for every β, $0 < \beta < (2^\kappa)^+$, we need to show that $A^{(0)} \cap V_\kappa \in U(\beta)$. By the definition $\vec{U}$, this means that in M, $\vec{U}{\restriction}\beta \in j(A^0)$. So we need to find in M an embedding constructing $\vec{U}{\restriction}\beta$. Let E' be the extender $E{\restriction}[\beta]^{<\omega}$. Then $E' \in M$, since ${}^{\beta}V_{\kappa+2} \subseteq M$. Consider the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j'"'] & M \arrow[r, "i"] & N \simeq \text{Ult}(M, E') \\ & M' \simeq \text{Ult}(V, E') \arrow[u, "k"'] & \end{tikzcd}$$

Now, it is not hard to see that $i = j'{\restriction}M$, since ${}^{\kappa}M \subseteq M$ and E' is an extender over κ. In particular, $i(\kappa) = j'(\kappa)$. Since ${}^{\kappa}V_{\kappa+2} \cap M = {}^{\kappa}V_{\kappa+2} \cap V$, we have $V_{j'(\kappa)+2} \cap N = V_{j'(\kappa)+2} \cap M'$. In addition, $\beta \subseteq \text{ran}(k)$, so $\text{crit}(k) \geq \max(\beta, \kappa^+)$. Let $\vec{U^*}$ be the i-sequence of ultrafilters constructed in M. We show by induction that $U^*(\gamma) = U(\gamma)$ for every $\gamma < \beta$. First note that $k(U) = U$ for every ultrafilter U over κ. Thus $\text{crit}(k) > \kappa$ implies that $U = k``U$. Also, clearly, $k``U \subseteq k(U)$. Finally, using $V_{\kappa+1} \cap M' = V_{\kappa+1} \cap M = V_{\kappa+1}$ and maximality of U as a filter we have $U = k(U)$.

Suppose now that $\gamma < \beta$ and we have already shown $\vec{U^*}{\restriction}\gamma = \vec{U}{\restriction}\gamma$. Let $X \subseteq V_\kappa$. Then $X \in U^*(\gamma)$ iff $\vec{U^*}{\restriction}\gamma \in i(X)$ iff $\vec{U}{\restriction}\gamma \in i(X)$ iff $\vec{U}{\restriction}\gamma \in j'(X)$ iff

$k(\vec{U}\restriction\gamma) \in j(X)$ (by elementarity of k and since $\vec{U}\restriction\gamma \in M'$) iff $\vec{U}\restriction\gamma \in j(X)$ iff $X \in U(\gamma)$ (since $k(\gamma) = \gamma$ and $k(U(\delta)) = U(\delta)$ for every $\delta < \gamma$).

This concludes the proof of $\vec{U}\restriction\alpha \in A^{(1)}$, for $\alpha < (2^\kappa)^+$. Let us show that $\vec{U}\restriction\alpha \in A^{(n)}$ for every n, $2 \leq n < \omega$ and $\alpha < (2^\kappa)^+$. First, for $n = 2$ we have

$$\begin{aligned} \vec{U}\restriction\alpha \in A^{(2)} \quad & \text{iff} \quad \forall\beta < \alpha\ A^{(1)} \cap V_\kappa \in \vec{U}\restriction\beta \\ & \text{iff} \quad \forall\beta < \alpha\ \vec{U}\restriction\beta \in j(A^{(1)}) \\ & \text{iff} \quad \forall\beta < \alpha \forall\gamma < \beta\ j(A^{(0)}) \cap V_\kappa \in U(\gamma). \end{aligned}$$

It is enough to show that $j(A^{(0)}) \cap V_\kappa = A^{(0)} \cap V_\kappa$, since we already proved that $A^{(0)} \cap V_\kappa \in U(\gamma)$ for every $\gamma < (2^\kappa)^+$. Let $\vec{F} \in V_\kappa$ be an i-sequence of ultrafilters for an embedding of either V or M with critical point $\nu = \kappa(\vec{F}) < \kappa$. The length of $\vec{F}$ is below κ, and κ is an inaccessible, so it is easy to find an extender inside V_κ such that the elementary embedding i' of it agrees with i long enough and constructs $\vec{F}$. Hence i' will witness both $\vec{F} \in j(A^{(0)})$ and $\vec{F} \in A^{(0)}$. The same argument works for any $n \geq 2$. Thus we will have

$$\vec{U}\restriction\alpha \in A^{(n)} \quad \text{iff} \quad \forall\gamma(\gamma + n \leq \alpha \to j^{n-1}(A^{(0)}) \cap V_\kappa \in U(\gamma)),$$

where j^{n-1} is an application of j $n-1$ many times, or equivalently the embedding $j_{0n-1} : V \to M_{n-1}$ of V into the $n-1$ times iterated ultrapower M_{n-1} of V by E. Again, as above $j^{n-1}(A^{(0)}) \cap V_\kappa = A^{(0)} \cap V_\kappa$. ⊣

Note that using stronger j's it is possible to show that longer ultrafilter sequences are in $\overline{A}$.

We are now ready to define Radin forcing. Let $\vec{V} = \langle U(\alpha) \mid \alpha < \text{length}(\vec{V})\rangle$ be a j-sequence of ultrafilters in $\overline{A}$ for some $j : V \to M$ with $\text{crit}(j) = \kappa$.

5.2 Definition. Let $R_{\vec{V}}$ be the set of finite sequences $\langle d_1, \ldots, d_n, \langle\kappa, \vec{V}\rangle, A\rangle$ such that

(1) $A \in \bigcap \vec{V}$ and $A \subseteq \overline{A}$.

(2) $A \cap V_{\kappa(d_n)+1} = \emptyset$.

(3) For every m with $1 \leq m \leq n$, either

(3a) d_m is an ordinal, or

(3b) $d_m = \langle\nu, \vec{F}_\nu, A_\nu\rangle$ for some $\vec{F}_\nu \in \overline{A}$, $A_\nu \subseteq \overline{A}$ and $A_\nu \in \bigcap \vec{F}_\nu$.

(4) For every $1 \leq i < j \leq n$,

(4a) $\kappa(d_i) < \kappa(d_j)$, and

(4b) if d_j is of the form $\langle\nu, F_\nu, A_\nu\rangle$ then $A_\nu \cap V_{\kappa(d_i)+1} = \emptyset$.

Each d_m of the form $\langle \nu, \vec{F}_\nu, A_\nu \rangle$ will give rise to Radin forcing $R_{\vec{F}_\nu}$ with $\vec{F}_\nu$ playing the same role as $\vec{V}$ in $R_{\vec{V}}$.

We define two orders $\leq$ and $\leq^*$ on $R_{\vec{V}}$, where, as usual, $\leq$ will be used to force and $\leq^*$ will provide the closure.

5.3 Definition. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle$, $q = \langle e_1, \ldots, e_m, \langle \kappa, \vec{V} \rangle, B \rangle \in R_{\vec{V}}$. We say that p *is stronger than* q and denote this by $p \geq q$ iff

(1) $A \subseteq B$.

(2) $n \geq m$.

(3) There are $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for $1 \leq k \leq m$, either

  (3a) $e_k = d_{i_k}$, or

  (3b) $e_k = \langle \nu, \vec{F}_\nu, B_\nu \rangle$ and then $d_{i_k} = \langle \nu, \vec{F}_\nu, C_\nu \rangle$ with $C_\nu \subseteq B_\nu$.

(4) Let $i_1, \ldots, i_m$ be as in (3). Then the following holds for every j, $1 \leq j \leq n$:

  (4a) if $j > i_m$, then $d_j \in B$ or d_j is of the form $\langle \nu, \vec{F}_\nu, C_\nu \rangle$ with $\langle \nu, \vec{F}_\nu \rangle \in B$ and $C_\nu \subseteq B \cap \nu$;

  (4b) if $j < i_m$, then for the least k with $j < i_k$, e_k is of the form $\langle \nu, \vec{F}_\nu, B_\nu \rangle$ so that

    (i) if d_j is an ordinal then $d_j \in B_\nu$, and

    (ii) if $d_j = \langle \rho, \vec{T}, S \rangle$ then $\langle \rho, \vec{T} \rangle \in B_\nu$ and $S \subseteq B_\nu$.

5.4 Definition. Let $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle$, $q = \langle e_1, \ldots, e_m, \langle \kappa, \vec{V} \rangle, B \rangle \in R_{\vec{V}}$. We say that p *is a direct extension of* q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) $n = m$.

Intuitively, $\langle R_{\vec{V}}, \leq, \leq^* \rangle$ is like Prikry forcing once some point of the form $\langle \nu, \vec{F}_\nu \rangle$ is produced, when it starts to act completely autonomously and eventually adds its own sequence.

As in the case of the Prikry forcing, any two conditions in $R_{\vec{V}}$ having the same finite sequences are compatible. So we obtain the following analogue of Lemma 1.5:

5.5 Lemma. $\langle R_{\vec{V}}, \leq \rangle$ *satisfies the* κ^+*-c.c.*

Suppose that $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$. Let, for some m with $1 \le m \le n$, $d_m = \langle \nu_m, \vec{V}_m, A_m \rangle$. Set $p^{\le m} = \langle d_1, \dots, d_m \rangle$ and

$$p^{>m} = \langle d_{m+1}, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle.$$

Then $p^{\le m} \in R_{\vec{V}_m}$ and $p^{>m} \in R_{\vec{V}}$. Let for $\vec{W} \in \overline{A}$ and $q \in R_{\vec{W}}$

$$R_{\vec{W}}/q = \{ r \in R_{\vec{W}} \mid r \ge q \}.$$

5.6 Lemma. *$R_{\vec{V}}/p \simeq R_{\vec{V}_m}/p^{\le m} \times R_{\vec{V}}/p^{>m}$.*

5.7 Lemma. *$\langle R_{\vec{V}}/p^{>m}, \le^* \rangle$ is ν_m-closed.*

This together with the Prikry condition (the next lemma) will suffice to prove the preservation of cardinals. Thus let $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$ and ξ be a cardinal. If $\xi > \kappa$, then we use Lemma 5.5. Let $\xi \le \kappa$. Then we pick the last m, $1 \le m \le n$ with d_m of the form $\langle \nu_m, \vec{V}_m, A_m \rangle$ such that $\nu_m < \xi$, if it exists. Work with $R_{\vec{V}}/p^{>m}$ in this case. Otherwise we continue to deal with $R_{\vec{V}}$. Suppose for simplicity that such an m does not exist, i.e. $\xi \le \nu_m$ for every m, $1 \le m \le n$ with $d_m = \langle \nu_m, \vec{V}_m, A_m \rangle$.

Let

$$\rho = \min(\{ \kappa, \kappa(d_m) \mid 1 \le m \le n \text{ and } d_m \text{ is of form } \langle \nu_m, \vec{V}_m, A_m \rangle \} \setminus \xi).$$

Assume for simplicity that $\rho = \kappa$. If $\operatorname{length}(\vec{V}) = 1$, then $R_{\vec{V}}$ is just the Prikry forcing and it preserves cardinals. Suppose that $\operatorname{length}(\vec{V}) > 1$. Let $\delta < \kappa$. Extend p to p^* by shrinking A to $A \setminus V_{\delta+1}$. Then $\langle R_{\vec{V}}/p^*, \le^* \rangle$ will be δ-closed. Using the Prikry condition, one can see that $\langle R_{\vec{V}}/p^*, \le \rangle$ does not add new subsets to δ. But δ was any cardinal below κ. So ξ is not collapsed even if $\xi = \kappa$ and we are done.

Let us now turn to the Prikry condition. The main new point here is that we are allowed to extend a given condition by picking elements from different ultrafilters of the sequence $\vec{V}$. So maybe different choices will decide some statement σ differently. The heart of the matter will be to show that this really does not happen. Actually, we can pass from one choice of an ultrafilter to another, remaining with compatible conditions.

5.8 Lemma. *$\langle R_{\vec{V}}, \le, \le^* \rangle$ satisfies the Prikry condition.*

Proof. Let $p \in R_{\vec{V}}$ and σ be a statement of the forcing language. We need to find $p^* \ge^* p$ that decides σ. Suppose that there is no such p^*. Assume for simplicity that $p = \langle \langle \kappa, \vec{V} \rangle, A \rangle$.

For every $\vec{d} = \langle d_1, \dots, d_n \rangle \in [V_\kappa]^n$ consider

$$\langle d_1, \dots, d_n \rangle^\frown p =_{df} \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \setminus V_{\kappa(d_n)+1} \rangle.$$

Suppose that it is a condition in $R_{\vec{V}}$. Let

$$\widetilde{A}(\vec{d}) = \{d \in A \mid \text{ either } d \text{ is an ordinal and then } \vec{d}^\frown d^\frown p \in R_{\vec{V}} \\ \text{or } d \text{ is of the form } \langle \nu, \vec{F}_\nu \rangle \text{ and then } \vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu \rangle^\frown p \in R_{\vec{V}}\}.$$

Clearly, $\widetilde{A}(\vec{d}) \in \bigcap \vec{V}$. We split $\widetilde{A}$ into three sets: First, set

$$A_0(\vec{d}) = \{d \in \widetilde{A}(\vec{d}) \mid \text{either (i) or (ii)}\}$$

where

(i) d is an ordinal and there is a B_d such that

$$\vec{d}^\frown d^\frown p \leq^* \langle \vec{d}^\frown d, \langle \kappa, \vec{V} \rangle, B_d \rangle \Vdash \sigma, \text{ or}$$

(ii) d is of the form $\langle \nu, \vec{F}_\nu \rangle$ and there are B_d and b_d such that

$$\vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu \rangle^\frown p \leq^* \langle \vec{d}^\frown \langle \nu, \vec{F}_\nu, b_d \rangle, \langle \kappa, \vec{V} \rangle, B_d \rangle \Vdash \sigma.$$

Then, let $A_1(\vec{d})$ be the same as $A_0(\vec{d})$ but with σ replaced by $\neg\sigma$. Finally, set

$$A_2(\vec{d}) = \widetilde{A}(\vec{d}) \setminus (A_0(\vec{d}) \cup A_1(\vec{d})).$$

For every $\alpha < \text{length}(\vec{V})$ choose an $i_\alpha \leq 2$ such that $A_{i_\alpha}(\vec{d}) \in U(\alpha)$. Set $A(\alpha, \vec{d}) = A_{i_\alpha}(\vec{d})$. If $\vec{d}^\frown p \notin R_{\vec{V}}$ then let $A(\alpha, \vec{d}) = A$. Set

$$A(\alpha) = \big\{d \in A \mid \forall \vec{d} = \langle d_1, \ldots, d_n \rangle \in [V_\kappa]^n \\ \big(\text{if } \max\{\kappa(d_k) \mid 1 \leq k \leq n\} < \kappa(d), \text{ then } d \in A(\alpha, \vec{d})\big)\big\}.$$

This is the kind of diagonal intersection which is appropriate for our setting. We claim that $A(\alpha) \in U(\alpha)$. Thus, for every $\vec{d} \in [V_\kappa]^n$ we have $A(\alpha, \vec{d}) \in U(\alpha)$. So, in M, $\langle \kappa, V{\restriction}\alpha \rangle \in j(A(\alpha, \vec{d}))$ for every $\vec{d} \in [V_\kappa]^n$. Clearly $\kappa(\langle \kappa, U{\restriction}\alpha \rangle) = \kappa$. Hence, by the definition of $A(\alpha)$, $\langle \kappa, V{\restriction}\alpha \rangle \in j(A(\alpha))$.

Define now $A^* = \bigcup_{\alpha < \text{length}(\vec{V})} A(\alpha)$. Obviously $A^* \in \bigcap_{\alpha < \text{length}(\vec{V})} U(\alpha)$. Consider $p^* = \langle \langle \kappa, \vec{V} \rangle, A^* \rangle$. By our initial assumption there is no direct extension of p^* deciding σ. Pick $\langle \langle d_1, \ldots, d_{n+1} \rangle, \langle \kappa, \vec{V} \rangle, B \rangle$ to be an extension of p^* deciding σ with n as small as possible. Suppose, for example, that it forces σ. Pick $\alpha < \text{length}(\vec{V})$ such that $d_{n+1} \in A(\alpha)$. Let $\vec{d} = \langle d_1, \ldots, d_n \rangle$. Then $\vec{d}^\frown p \in R_{\vec{V}}$. By the definition of $A(\alpha)$, $d_{n+1} \in A(\alpha, \vec{d})$. By the choice of $A(\alpha, \vec{d})$, then $A(\alpha, \vec{d}) = A_0(\vec{d})$. This means that for every $d \in A(\alpha) \setminus V_{\kappa(d_n)+1}$ there are $\widetilde{d}$ and B such that

$$\vec{d}^\frown d^\frown p \leq^* \langle \vec{d}^\frown \widetilde{d}, \langle \kappa, \vec{V} \rangle, B \rangle \Vdash \sigma.$$

Obviously we can replace p by p^* here. In what follows we show that for some C

$$p^* \leq \langle\langle d_1, \ldots, d_n\rangle, \langle\kappa, \vec{V}\rangle, C\rangle \Vdash \sigma.$$

This will contradict the minimality of n and, in turn, our initial assumption.

We shrink first the sets in $U(\beta)$ for every $\beta < \alpha$ (if there are any). Suppose that $\alpha > 0$. The case $\alpha = 0$ is similar and slightly easier. For every $d \in A(\alpha) \setminus V_{\kappa(d_n)+1}$ of the form $\langle\nu, \vec{F}_\nu\rangle$ pick some b_d and B_d so that $\vec{d}^\frown d^\frown p^* \leq^* \langle\vec{d}, \langle\langle\nu, \vec{F}_\nu\rangle, b_d\rangle, \langle\kappa, \vec{V}\rangle, B_d\rangle \Vdash \sigma$. We take a diagonal intersection of the B_d's. Thus, let

$$B^* = \{e \in A^* \mid \forall d \in V_{\kappa(e)}\,(\text{if } B_d \text{ is defined then } e \in B_d)\}.$$

For every $\beta < \mathrm{length}(\vec{V})$, $B^* \in U(\beta)$, since clearly for every $d \in V_\kappa$ with B_d defined $\langle\kappa, \vec{V}\restriction\beta\rangle \in j(B_d)$ due to $B_d \in \bigcap\vec{V}$, so $\langle\kappa, \vec{V}\restriction\beta\rangle \in j(B^*)$.

Note that by the choice of $\overline{A}$ and Lemma 5.1(3(b)), $b_d \in \bigcap_{W\in\vec{F}_\nu} W$, where each $W \in \vec{F}_\nu$ is a ν-complete ultrafilter over V_ν. Consider $A^{<\alpha} = j(\langle b_d \mid d \in A(\alpha)\rangle)(\vec{V}\restriction\alpha)$ (recall that $A(\alpha) \in U(\alpha)$ implies that $\vec{V}\restriction\alpha \in j(A(\alpha))$). Then, by elementarity, $A^{<\alpha} \in U(\beta)$ for every $\beta < \alpha$. Also, note that the set $A'(\alpha) = \{d \in A(\alpha) \mid A^{<\alpha} \cap V_{\kappa(d)} = b_d\} \in U(\alpha)$, since $j(A^{<\alpha}) \cap V_{\kappa(\vec{V}\restriction\alpha)} = j(A^{<\alpha}) \cap V_\kappa = A^{<\alpha} = j(\langle b_d \mid d \in A(\alpha)\rangle)(\vec{V}\restriction\alpha)$ and hence $\vec{V}\restriction\alpha \in j(A'(\alpha))$.

Set $A^{\leq\alpha} = (A^{<\alpha} \cup A'(\alpha)) \cap A^*$. Then $A^{\leq\alpha} \in U(\beta)$ for every $\beta \leq \alpha$.

Now let us shrink the sets in $U(\beta)$ for all $\beta > \alpha$ (if there are any). Actually, we need to care for only β's with $A^{\leq\alpha} \notin U(\beta)$. Consider the set

$$A^{>\alpha} = \{\langle\nu, \vec{F}\rangle \in A^* \mid \exists\xi < \mathrm{length}(\vec{F})(A'(\alpha) \cap V_\nu \in F(\xi))\}.$$

Then $A^{>\alpha} \in U(\beta)$ for every β with $\alpha < \beta < \mathrm{length}(\vec{V})$. Set

$$A^{**} = (A^{\leq\alpha} \cup A^{>\alpha}) \cap B^*.$$

Clearly, $A^{**} \in \bigcap\vec{V}$. Consider a condition $p^{**} = \langle\langle\kappa, \vec{V}\rangle, A^{**}\rangle$ and $q = \langle d_1, \ldots, d_n\rangle^\frown p^{**}$. By the choice of n, neither q nor its direct extensions can decide σ. Pick some $r \geq q$ forcing $\neg\sigma$. Let $r = \langle e_1, \ldots, e_m, \langle\kappa, \vec{V}\rangle, C\rangle$. There is a $k \leq m$ such that $\kappa(d_n) = \kappa(e_k)$, by Definition 5.2(3). Consider three cases.

Case 1. $k = m$.

Then choose some $d \in A(\alpha) \cap C$ such that $C \cap \nu \in \bigcap\vec{F}_\nu$ where $d = \langle\nu, \vec{F}_\nu\rangle$. By the choice of $A(\alpha)$ and B^* there is a b_d such that

$$\langle d_1, \ldots, d_n, \langle\langle\nu, \vec{F}_\nu\rangle, b_d\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \setminus V_{\kappa(d)+1}\rangle \Vdash \sigma.$$

Clearly we can shrink $A^{**} \setminus V_{\kappa(d)+1}$ to C. Then,

$$\langle e_1, \ldots, e_m, \langle\langle\nu, \vec{F}_\nu\rangle, b_d \cap C\rangle, \langle\kappa, \vec{V}\rangle, A^{**} \cap C \setminus V_{\kappa(d)}\rangle$$

will be a common extension of r and $\langle d_1,\dots,d_n, \langle\langle\nu,\vec{F}_\nu\rangle, b_d\rangle\langle\kappa,\vec{V}\rangle, A^{**}\cap C\setminus V_{\kappa(d)}\rangle$, which is clearly impossible since they disagree about σ.

Case 2. $k<m$ and for $k<j\le m$, $e_j\in A^{<\alpha}$.

Pick $d\in A'(\alpha)\cap C$, $d=\langle\nu,\vec{F}_\nu\rangle$ such that $C\cap A^{<\alpha}\cap V_\nu\in\cap\vec{F}_\nu$. Then, by the choice of $A'(\alpha)$, $b_d=A^{<\alpha}\cap V_\nu$. So,

$$\langle d_1,\dots,d_n,\langle\langle\nu,\vec{F}_\nu\rangle,A^{<\alpha}\cap V_\nu\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\setminus V_{\kappa(d)+1}\rangle\rangle\Vdash\sigma.$$

But

$$\begin{aligned}&\langle\langle e_1,\dots,e_m\rangle,\langle\langle\nu,\vec{F}_\nu\rangle,C\cap A^{<\alpha}\cap V_\nu\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\cap C\setminus V_{\kappa(d)+1}\rangle\\&\quad\ge\langle d_1,\dots,d_n,\langle\langle\nu,\vec{F}_\nu\rangle,A^{<\alpha}\cap\nu\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\cap C\setminus V_{\kappa(d)+1}\rangle,\end{aligned}$$

since $e_j\in A^{<\alpha}$ for every $k<j\le m$, $\kappa(d_n)=\kappa(e_k)$ and $r\ge q$. Also, clearly,

$$\langle\langle e_1,\dots,e_m\rangle,\langle\langle\nu,\vec{F}_\nu\rangle,C\cap A^{<\alpha}\cap\nu\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\cap C\setminus V_{\kappa(d)+1}\rangle\rangle\ge r.$$

But this is impossible, since $r\Vdash\neg\sigma$.

Case 3. $k<m$ and there is a j with $k<j\le m$ such that $e_j\notin A^{<\alpha}$.

Let j^* be the minimal j with $k<j\le m$ and $e_j\notin A^{<\alpha}$. Then $e_{j^*}\in A'(\alpha)\cup A^{>\alpha}$. If $e_{j^*}\in A'(\alpha)$, then

$$\begin{aligned}&\langle e_1,\dots,e_{j^*-1},\langle\langle\nu,\vec{F}_\nu\rangle,E\cap A^{<\alpha}\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\setminus V_{\kappa(e_{j^*})+1}\rangle\\&\quad\ge\langle d_1,\dots,d_n,\langle\langle\nu,\vec{F}_\nu\rangle,A^{<\alpha}\cap V_\nu\rangle\ \langle\kappa,\vec{V}\rangle,A^{**}\setminus V_{\nu+1}\rangle\Vdash\sigma,\end{aligned}$$

by minimality of j^*, where $e_{j^*}=\langle\langle\nu,\vec{F}_\nu\rangle,E\rangle$. But, $\langle e_1,\dots,e_{j^*-1},\langle\langle\nu,\vec{F}_\nu\rangle, E\cap A^{<\alpha}\rangle,\langle\kappa,\vec{V}\rangle,A^{**}\setminus V_{\kappa(e_{j^*})+1}\rangle$ and r are compatible, which is impossible since $r\Vdash\neg\sigma$.

So, assume that $\langle\nu,\vec{F}_\nu\rangle\in A^{>\alpha}\setminus A^{\le(\alpha)}$, where $e_{j^*}=\langle\langle\nu,\vec{F}_\nu\rangle,E\rangle$. We have $E\in\bigcap\vec{F}_\nu$. By the choice of $A^{>\alpha}$, for some $\xi<\text{length}(\vec{F}_\nu)$ $A'(\alpha)\cap V_\nu\in F_\nu(\xi)$. Hence $A'(\alpha)\cap E\in F_\nu(\xi)$. Pick some $\langle\tau,\vec{G}_\tau\rangle\in(A'(\alpha)\cap E)\setminus V_{\kappa(e_{j^*-1})+1}$ such that $E\cap\tau\in\cap\vec{G}_\tau$. This can be done since $\vec{F}_\nu$ is a j'-sequence for some j' and $E\in\bigcap\vec{F}_\nu$. Now we can extend r by adding to it $\langle\tau,\vec{G}_\tau\rangle$. This will reduce the situation to the one considered above, i.e. $e_{j^*}\in A'(\alpha)$. This completes the proof of the lemma. ⊣

Now let G be a generic subset of $R_{\vec{V}}$. Combining the previous lemmas together, we obtain the following:

5.9 Theorem. *$V[G]$ is a cardinal preserving extension of V.*

Consider the following crucial set:

$$C_G=\big\{\kappa(d)<\kappa\mid\exists p\in G\ (d\text{ is one of the elements of the finite sequence of }p)\big\}.$$

5.10 Lemma. *C_G is a closed unbounded subset of κ.*

Proof. C_G is unbounded since for every condition $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle$ and every ordinal $\tau < \kappa$ we can find some $\nu \in A \cap (\kappa \setminus \tau)$ and extend p by adding ν to its finite sequence $\langle d_1, \dots, d_n \rangle$.

Let us show that C_G is closed. Thus, let for some $\tau < \kappa$ some

$$p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \Vdash \check{\tau} \notin \underset{\sim}{C}_G.$$

Clearly, $\tau \neq \kappa(d_i)$ for any i, $1 \leq i \leq n$. If $\tau > \kappa(d_n)$, then we shrink A to $A \setminus (\tau + 1)$. By the definition of the forcing ordering $\leq$,

$$\langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \setminus (\tau + 1) \rangle \Vdash \sup(\underset{\sim}{C}_G \cap \check{\tau}) = \check{\kappa}(d_n).$$

Suppose now that $\tau < \kappa(d_n)$. Let $i^* < n$ be the least such that $\tau < \kappa(d_{i^*+1})$. If d_{i^*+1} is an ordinal, then again by the definition of the forcing ordering $\leq$, p forces that $\underset{\sim}{C}_G$ will not have elements in the open interval $(\kappa(d_{i^*}), d_{i^*+1})$, where $d_0 = 0$. So, let $d_{i^*+1} = \langle \nu, \vec{F}_\nu, B_\nu \rangle$. Then $B_\nu \setminus (\tau+1) \in \bigcap \vec{F}_\nu$ and the extension of p

$$\begin{aligned} &\bigl\langle d_1, \dots, d_{i^*}, \langle \nu, \vec{F}_\nu, B_\nu \setminus (\tau + 1) \rangle, d_{i^*+2}, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \bigr\rangle \\ &\quad \Vdash \sup(\underset{\sim}{C}_G \cap \check{\tau}) = \check{\kappa}(d_{i^*}). \end{aligned}$$

Combining all the cases together we conclude that there is always an extension of p forcing that τ is not a limit of elements of C_G. ⊣

The next question will be crucial for the issue of changing cofinalities:

What is the order type of C_G?

For every τ with $0 < \tau < \kappa$, $U(\tau)$ concentrates on the set $X_\tau = \{\langle \nu, \vec{F}_\nu \rangle \mid \vec{F}_\nu$ is a sequence of ν-complete ultrafilters over V_ν of length $\tau < \nu\}$. Clearly, $\{X_\tau \mid 0 < \tau < \kappa\}$ are disjoint. We can add to them also $X_0 = \kappa$ and $X_\kappa = \{\langle \nu, \vec{F}_\nu \rangle \mid \vec{F}_\nu$ is a sequence of ν-complete ultrafilters over ν of length $\nu\}$. Using this partition and an easy induction it is not hard to see the following.

5.11 Lemma. *Let δ, $0 < \delta < \kappa$, $\mathrm{length}(\vec{V}) = \delta$, and $G \subseteq R_{\vec{V}}$ be generic. Then, in $V[G]$, a final segment of C_G has order type ω^δ, where ω^δ is the ordinal power. Moreover, $\langle \kappa, \vec{V}, \bigcup \{X_\tau \mid 0 < \tau < \delta\} \rangle$ forces the order type of C_G to be ω^δ. In particular, $\mathrm{otp}(C_G) = \delta$ if δ is an uncountable cardinal.*

Combining this with Theorem 5.9 we obtain the following:

5.12 Theorem. *Let $\mathrm{length}(\vec{V}) = \delta < \kappa$ be a cardinal, and let $G \subseteq R_{\vec{V}}$ be generic. Then $V[G]$ is a cardinal preserving extension of V in which κ changes its cofinality to $\mathrm{cf}(\delta)^V$.*

Notice that if $\delta > 0$ then $R_{\vec{V}}$ changes cofinalities also below κ. Hence new bounded subsets are added to κ. Mitchell [45] showed that once one changes the cofinality of κ to some uncountable $\delta < \kappa$ preserving cardinals, then new bounded subsets of κ must appear, provided the ground model was the core model. On the other hand, it is possible to prepare a ground model and then force in order to change cofinality of κ to an uncountable δ without adding new bounded subsets. This was first done by Mitchell [44], combining iterated ultrapowers and forcing. A pure forcing construction was given in [13].

If we force with $R_{\vec{V}}$ having $\mathrm{length}(\vec{V}) = \kappa$, then κ changes its cofinality to ω again.

5.13 Lemma. *Suppose that* $\mathrm{length}(\vec{V}) = \kappa$ *and* $G \subseteq R_{\vec{V}}$ *generic. Then, in* $V[G]$, $\mathrm{cf}(\kappa) = \aleph_0$.

Proof. Let $\langle X_\tau \mid \tau < \kappa\rangle$ be the partition defined before Lemma 5.11. Then, $\bigcup_{\tau<\kappa} X_\tau \in \bigcap \vec{V}$, since for every $\tau < \kappa$ $X_\tau \in U(\tau)$ and $\vec{V} = \langle U(0), \ldots, U(\tau), \ldots \mid \tau < \kappa\rangle$. Let $X = \bigcup_{\tau<\kappa} X_\tau$. Consider

$$Y = \big\{\langle \nu, \vec{F}_\nu\rangle \in X \mid \bigcup\{X_\tau \mid \tau < \mathrm{length}(\vec{F}_\nu)\} \cap V_\nu \in \cap \vec{F}_\nu\big\} \cup \kappa.$$

Clearly, $Y \in \bigcap \vec{V}$. Now pick some $p = \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \in G$ with $A \subseteq Y$. Let

$$C = \big\{\langle \nu, \vec{F}_\nu\rangle \in V_\kappa \mid \exists E \in \bigcap \vec{F}_\nu(\langle \nu, \vec{F}_\nu, E\rangle \text{ appears in a condition in } G)\big\}.$$

Then, $C \setminus (\kappa(d_n) + 1) \subseteq A$. A simple density argument shows that for every $\tau < \kappa$, C will contain unboundedly many members of X_τ. Let

$$C' = \{\nu < \kappa \mid \exists \vec{F}(\langle \nu, \vec{F}\rangle \in C)\}.$$

Clearly, C' is just the set of all limit points of C_G. Also, for every $\nu \in C'$ there is a unique $\vec{F}_\nu$ with $\langle \nu, \vec{F}_\nu\rangle \in C$. We define an increasing sequence $\langle \nu_n \mid n < \omega\rangle$ of elements of C' as follows: $\nu_0 = \min(C')$, $\nu_{n+1} = \min\{\nu \in C' \mid \exists \vec{F}_\nu \langle \nu, \vec{F}_\nu\rangle \in X_{\nu_n}\} \setminus (\nu_n + 1))$.

Set $\nu_\omega = \bigcup_{n<\omega} \nu_n$. We claim that $\nu_\omega = \kappa$. Otherwise there is a $\tau < \kappa$ such that $\langle \nu_\omega, \vec{F}\rangle \in C \cap X_\tau$ for some (unique) $\vec{F}$, since C' is closed and $C \subseteq A \subseteq Y \subseteq X = \bigcup_{\tau<\kappa} X_\tau$. Then there is a $q \geq p$ in G with $\langle\langle \nu_\omega, \vec{F}\rangle, B\rangle$ appearing in q for some $B \in \cap \vec{F}$. We require also $B \subseteq \bigcup\{X_{\tau'} \mid \tau' < \tau\} \cap V_{\nu_\omega}$. This is possible since $q \geq p$, $A \subseteq Y$, $\nu_\omega > \kappa(d_n)$, and hence $\bigcup\{X_{\tau'} \mid \tau' < \tau\} \cap V_{\nu_\omega} \in \bigcap \vec{F}$. Now, by the definition of X_τ, we have $\tau < \nu_\omega$. So, there is an $n < \omega$ with $\nu_n > \max(\tau, \min(B))$. But $\nu_n \in C'$, hence $\langle \nu_n, \vec{F}_{\nu_n}\rangle$ should be in B, for some (unique) $\vec{F}_{\nu_n}$. The same holds for each ν_m with $n \leq m < \omega$. In particular, $\langle \nu_{n+1}, \vec{F}_{\nu_{n+1}}\rangle \in \bigcup_{\tau'<\tau} X_{\tau'}$. But it was picked to be in X_{ν_n} which is disjoint to each $X_{\tau'}$ for $\tau' < \nu_n$. Contradiction. ⊣

Similar arguments show that for every $\delta < \kappa^+$, if $\mathrm{length}(\vec{V}) = \delta$ then the forcing $R_{\vec{V}}$ changes the cofinality of κ. If δ is a successor ordinal, then to $\aleph_0$; if δ is limit and $\mathrm{cf}(\delta) \neq \kappa$ then to $\mathrm{cf}(\delta)$ and, finally, if $\mathrm{cf}(\delta) = \kappa$ then to $\aleph_0$.

Let us now show that if $\vec{V}$ is long enough then $R_{\vec{V}}$ can preserve measurability of κ. Later it will be shown that length$(\vec{V}) = \kappa^+$ suffices to keep κ regular and so inaccessible. The ability of keeping κ regular turned out to be very important in applications to the cardinal arithmetic. Thus a basic common theme used there is to arrange some particular pattern of the power function over C_G, sometimes adding Cohen subsets or collapsing cardinals in between and then to cut the universe at κ. This type of constructions were used by Foreman-Woodin [12], Cummings [9] and recently by Merimovich [40].

5.14 Definition. An ordinal $\gamma <$ length$(\vec{V})$ is called a *repeat point* for $\vec{V}$ if for every δ with $\gamma \leq \delta <$ length$(\vec{V})$ and for every $A \in U(\delta)$, there is a $\delta' < \gamma$ such that $A \in U(\delta')$. Equivalently, $\bigcup \vec{V} = \bigcup \vec{V} \restriction \gamma$.

Note that if $2^\kappa = \kappa^+$ and our sequence has length κ^{++}, then there will be κ^{++} repeat points between κ^+ and κ^{++}. This implies that also $\vec{V} = \vec{U} \restriction \alpha$ will have a repeat point for unboundedly many α's below κ^{++}.

5.15 Theorem. *If γ is a repeat point for $\vec{V}$ and $G \subseteq R_{\vec{V}}$ is generic, then κ remains measurable in $V[G]$.*

Proof. Recall that $\vec{V} = \langle U(\alpha) \mid \alpha < \text{length}(\vec{V})\rangle$ is a j-sequence for some elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$. By the definition of a repeat point, the forcing $R_{\vec{V}}$ and $R_{\vec{V}\restriction\gamma}$ are basically the same (we need only to replace $\langle \kappa, \vec{V}\rangle$ in each condition of $R_{\vec{V}}$ by $\langle \kappa, \vec{V}\restriction\gamma\rangle$ in order to pass to $R_{\vec{V}\restriction\gamma}$). So we can view G as a generic subset of $R_{\vec{V}\restriction\gamma}$. Define now an ultrafilter F over κ in $V[G]$. Let $\underset{\sim}{X}$ be a name of a subset of κ. Set $\underset{\sim}{X}[G] \in F$ iff for some $\langle d_1, \dots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \in G$ the following holds in M: For some $B \in \bigcap j(\vec{V})$,

$$\langle d_1, \dots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}).$$

First note that F is well defined. Thus, let some $\langle d_1, \dots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle \in G$ forces "$\underset{\sim}{X} = \underset{\sim}{Y}$". Then, in M

$$\langle d_1, \dots, d_n, \langle j(\kappa), j(\vec{V}), j(A)\rangle\rangle \Vdash j(\underset{\sim}{X}) = j(\underset{\sim}{Y}).$$

But $A \in \bigcap \vec{V}$. In particular, $A \in U(\gamma)$. Hence, $\langle \kappa, \vec{V}\restriction\gamma\rangle \in j(A)$. Also, $j(A) \cap V_\kappa = A$. So, $\langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle$ is addible to $\langle d_1, \dots, d_n, \langle j(\kappa), j(\vec{V}), j(A)\rangle$. But if for some $B \in \bigcap j(\vec{V})$,

$$\langle d_1, \dots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}),$$

then

$$\begin{aligned}&\langle d_1, \dots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B \cap j(A)\rangle \\ &\quad \Vdash_{R_{j(\vec{V})}} \check{\kappa} \in j(\underset{\sim}{X}) \wedge j(\underset{\sim}{X}) = j(\underset{\sim}{Y}).\end{aligned}$$

Let us establish normality for F. Suppose that $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle\rangle \in G$ and $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle\rangle \Vdash (\{\nu < \kappa \mid \underset{\sim}{f}(\nu) < \nu\} \in \underset{\sim}{F})$. Then, in M, for some $B \in \bigcap j(\vec{V})$

$$\langle d_1, \ldots, d_n, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A\rangle, \langle j(\kappa), j(\vec{V})\rangle, B\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) < \check{\kappa}.$$

Working in M, construct $B' \in \bigcap j(\vec{V})$ such that: If for some $\nu < \kappa$, we have a condition $\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, E\rangle$ forcing "$j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}$", then $\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle$ forces the same.

Back in V, the set

$$\begin{aligned} D = \big\{&\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle\rangle \mid \text{for some } \nu < \kappa, \\ &\langle x_1, \ldots, x_\ell, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, C\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}\big\} \end{aligned}$$

will be dense in $R_{\vec{V}\restriction\gamma}$ above $\langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$. Thus, if some $p \in R_{\vec{V}\restriction\gamma}$ with $p \geq \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$ has no extension in D, then we consider the statement

$$\begin{aligned} \varphi \equiv \text{“There is a } q \in R_{\vec{V}} \text{ stronger than } p, \text{ a } \nu < \kappa, \text{ and} \\ \text{an } r \text{ in } \underset{\sim}{G}(R_{j(\vec{V})\setminus\kappa+1}) \text{ such that } \langle q, r\rangle \Vdash_{R_{j(\vec{V})}} j(\underset{\sim}{f})(\check{\kappa}) = \check{\nu}\text{”}, \end{aligned}$$

where $\underset{\sim}{G}(R_{j(\vec{V})\setminus\kappa+1})$ is the canonical name of a generic subset of $R_{j(\vec{V})\setminus\kappa+1}$. Let, in M, $s \geq^* \langle\langle j(\kappa), j(\vec{V})\rangle, B'\rangle$ deciding φ. Then s must force φ. Find some $s_1 \geq^* s$ deciding the values of ν and q in φ. This leads to the contradiction.

So, pick some $\langle e_1, \ldots, e_m, \langle \kappa, \vec{V}\rangle, A'\rangle \geq \langle d_1, \ldots, d_n, \langle \kappa, \vec{V}\rangle, A\rangle$ in $G \cap D$. There is a $\delta < \kappa$ such that

$$\langle\langle e_1, \ldots, e_m\rangle, \langle\langle \kappa, \vec{V}\restriction\gamma\rangle, A'\rangle, \langle j(\kappa), j(\vec{V})\rangle, B'\rangle \Vdash j(\underset{\sim}{f})(\check{\kappa}) = \check{\delta}.$$

Then $\{\nu < \kappa \mid f(\nu) = \delta\} \in F$, by the definition of F. ⊣

Similar arguments show that it is possible to preserve the degree of strongness and even of supercompactness of j. Notice also that F defined above extends $U(0)$, but the elementary embedding of F does not extend that of $U(0)$. Instead, it extends a certain iterated ultrapower embedding using ultrafilters of $\mathrm{Ult}(V, U(0))$ between κ and $i_{U(0)}(\kappa)$.

We now want to show that κ remains regular in $V^{R_{\vec{V}}}$ when we have $\mathrm{cf}(\mathrm{length}(\vec{V})) \geq \kappa^+$. But first we need to extend a bit the Prikry condition Lemma 5.8 in the spirit of Lemma 2.18. This will allow us to deal with dense sets. The situation here is more involved due to the possibility of extending a given condition by adding to it elements from different ultrafilters $U(\alpha)$'s. We start with the following definition.

5.16 Definition. Let $\vec{F}$ be a sequence of ultrafilters over some $\nu \leq \kappa$. A tree $T \subseteq [V_\nu]^{\leq n}$ with $n < \omega$ levels is called $\vec{F}$-*fat* iff

(1) For every $\langle \nu_1, \dots, \nu_k \rangle \in T$, $\kappa(\nu_1) < \kappa(\nu_2) < \dots < \kappa(\nu_k)$.

(2) For every $\langle \nu_1, \dots, \nu_k \rangle \in T$ with $k < n$, there is an $\alpha < \mathrm{length}(\vec{F})$ so that $\mathrm{Suc}_T(\langle \nu_1, \dots, \nu_k \rangle) \in F(\alpha)$.

Let T be as in Definition 5.16 and η a maximal branch in T. A sequence $\vec{A} = \langle \vec{A}(1), \dots, \vec{A}(n) \rangle \in [V_\nu]^n$ will be called a *sequence of η-measure one* if, for every i, $1 \le i \le n$ with $\eta(i)$ of form $\langle \tau_i, \vec{G}_{\tau_i} \rangle$ we have $\vec{A}(i) \in \bigcap \vec{G}_{\tau_i}$. Let $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$ and $d_i = \langle \langle \nu_i, \vec{F}_i \rangle, A_i \rangle$ or $d_i = \nu_i < \kappa$ for each i, $1 \le i \le n$. Here also denote $\langle \langle \kappa, \vec{V} \rangle, A \rangle$ by $\langle \langle \nu_{n+1}, \vec{F}_{n+1} \rangle, A_{n+1} \rangle$. Let $1 \le i_1 < \dots < i_m \le n+1$ be some elements of the set $\{ i \mid 1 \le i \le n+1, d_i = \langle \langle \nu_i, \vec{F}_i \rangle, A_i \rangle \}$.

Let for each k with $1 \le k \le m$ and some $n_k < \omega$, $T_k \subseteq [V_{\nu_{i_k}}]^{n_k}$ be a $\vec{F}_{\nu_{i_k}}$-fat tree, η_k a maximal branch in T_k, and $\vec{A}_k \in [V_{\nu_{i_k}}]^{\le n_k}$ a sequence of η_k-measure one. Let $q = \langle t_1, \dots, t_\ell, t_{\ell+1} \rangle$ be obtained from p by adding to it between $d_{i_k - 1}$ and d_{i_k}, for each k, $1 \le k \le m$, the following n_k-sequence $\langle s_j \mid 1 \le j \le n_k \rangle$, where $s_j = \eta_k(j)$, if $\eta_k(j)$ is an ordinal, or $s_j = \langle \tau_j, \vec{G}_{\tau_j}, \vec{A}_k(j) \rangle$, if $\eta_k(j) = \langle \tau_i, G_{\tau_i} \rangle$. Denote by $p^\frown \langle \eta_1, \vec{A}_1 \rangle^\frown \cdots ^\frown \langle \eta_m, \vec{A}_m \rangle$ the condition in $R_{\vec{V}}$ obtained from q by the obvious shrinking of sets of measure one needed in order to satisfy Definition 5.2, i.e. for every i with $1 < i \le \ell + 1$, if $t_i = \langle \delta_i, \vec{H}_i, B_i \rangle$, then we replace B_i by $B_i \setminus V_{\kappa(t_{i-1})+1}$.

5.17 Lemma. *Let D be a dense open subset of $R_{\vec{V}}$ and $p = \langle d_1, \dots, d_n, \langle \kappa, \vec{V} \rangle, A \rangle \in R_{\vec{V}}$. Then there are $p^* = \langle d_1^*, \dots, d_n^*, \langle \kappa, \vec{V} \rangle, A^* \rangle \ge^* p$; $1 \le i_1 < \dots < i_m \le n + 1$; and for $1 \le k \le m$, $T_k \subseteq [V_{\nu_{i_k}}]^{n_k}$ $\vec{F}_{\nu_{i_k}}$-fat trees so that the following holds:*

For every sequence $\langle \eta_k \mid 1 \le k \le m \rangle$ such that η_k is a maximal branch in T_k, there exists a sequence $\langle \vec{A}_k \mid 1 \le k \le m \rangle$ such that

(1) $\vec{A}_k \in [V_{i_k}]^{n_k}$ is a sequence of η_k-measure one, and

(2) $p^{*\frown} \langle \eta_1, \vec{A}_1 \rangle^\frown \cdots ^\frown \langle \eta_m, \vec{A}_m \rangle \in D$.

5.18 Remark. Roughly, the meaning of this is that in order to get into D we need to specify certain $U(\alpha)$'s (or $F(\alpha)$'s, if below κ) and sets A_α's in these ultrafilters. Then any choice of elements in A_α's will put us into D.

Proof. The proof is very similar to that of Lemma 5.8. Suppose for simplicity that $p = \langle \langle \kappa, \vec{V} \rangle, A \rangle$. We need to find a direct extension $p^* = \langle \langle \kappa, \vec{V} \rangle, A^* \rangle$ of p and a $\vec{V}$-fat tree T of some finite height m such that the following holds: for every maximal branch $\eta = \langle f_1, \dots, f_m \rangle$ through T there are sets $\vec{A} = \langle a_1, \dots, a_m \rangle$ of η-measure one (i.e. for every i with $1 \le i \le m$, if $f_i = \langle \tau_i, \vec{G}_{\tau_i} \rangle$ then $a_i \in \bigcap \vec{G}_{\tau_i}$) such that $p^{*\frown} \langle \eta, \vec{A} \rangle \in D$, where

$$p^{*\frown} \langle \eta, \vec{A} \rangle = \langle f_1', \dots, f_m', \langle \kappa, \vec{V} \rangle, A^* \setminus V_{\kappa(f_m)} \rangle$$

and for every i with $1 \le i \le m$, either

(α) f_i is an ordinal and then $f_i' = f_i$, or

(β) $f_i = \langle \tau_i, \vec{G}_{\tau_i}\rangle$ and then $f_i' = \langle \tau_i, \vec{G}_{\tau_i}, a_i\rangle$.

If p already has a direct extension in D, then we take such an extension and set $T = \{\langle\rangle\}$. Suppose that this is not the case. Define $\widetilde{A}(\vec{d})$ as in Lemma 5.8. Here we split it only into two sets $A_0(\vec{d}) = \{d \in \widetilde{A}(\vec{d}) |$ either (i) or (ii)$\}$ and $A_1(d) = \widetilde{A}(\vec{d}) \setminus A_0(\vec{d})$, where:

(i) d is an ordinal and then there is a $B_{\vec{d}}$ such that

$$\vec{d}^\frown d^\frown p \leq^* \langle \vec{d}^\frown d, \langle \kappa, \vec{V}\rangle, B_{\vec{d}}\rangle \in D.$$

(ii) d is of form $\langle \nu, \vec{F}_\nu\rangle$ and then there are $B_{\vec{d}}$ and $b_{\vec{d}}$ such that

$$\vec{d}^\frown \langle \nu, \vec{F}_\nu, A \cap V_\nu\rangle^\frown p \leq^* \langle \vec{d}^\frown \langle \nu, \vec{F}_\nu, b_{\vec{d}}\rangle, \langle \kappa, \vec{V}\rangle, B_{\vec{d}}\rangle \in D.$$

As in Lemma 5.8, define $A(\alpha, \vec{d})$'s and $A(\alpha) \in U(\alpha)$ for $\alpha < \text{length}(\vec{V})$. Set $A^1 = \bigcup\{A(\alpha) \mid \alpha < \text{length}(\vec{V})\}$ and $p^1 = \langle\langle \kappa, \vec{V}\rangle, A^1\rangle$. Then p^1 satisfies the following:

$(*)_1$ If $p^1 \leq q = \langle e_0, \ldots, e_m, \langle \kappa, \vec{V}\rangle, B\rangle \in D$, then there is an $\alpha < \text{length}(\vec{V})$ such that for every $e_m' \in A(\alpha) \setminus V_{\kappa(e_{m-1})+1}$, $\langle e_0, \ldots, e_{m-1}, e_m', \langle \kappa, \vec{V}\rangle, A^1\rangle$ has a direct extension of form $\langle e_0, \ldots, e_{m-1}, e_m'', \langle \kappa, \vec{V}\rangle, A''\rangle$ in D.

Just pick α with $e_m \in A(\alpha)$ (more precisely, only $\langle \nu, \vec{F}_\nu\rangle$ if $e_m = \langle \nu, \vec{F}_\nu, B_\nu\rangle$). Then $e_m \in A(\alpha, \langle e_0, \ldots, e_{m-1}\rangle)$ and so by choice of $A(\alpha, \langle e_0, \ldots, e_{m-1}\rangle)$, for every $e_m' \in A(\alpha, \langle e_0, \ldots, e_{m-1}\rangle)$ a direct extension of $\langle e_0, \ldots, e_{m-1}, e_m'$, $\langle \kappa, \vec{V}\rangle, A^1\rangle$ will be in D. But if we were to take $e \in A(\alpha) \setminus V_{\kappa(e_{m-1})+1}$, then $e \in A(\alpha, \langle e_0, \ldots, e_{m-1}\rangle)$, by the definition of the diagonal intersection.

If for some $d \in A^1$, $d^\frown p^1$ has a direct extension in D, then we are done. Thus choose $\alpha < \text{length}(\vec{V})$ with $d \in A(\alpha)$. By the choice of $A(\alpha)$, then for every $d' \in A(\alpha)$ some direct extension of $d'^\frown p^1$ will be in D. Let us fix for every $d \in A(\alpha)$ a direct extension $\langle \widetilde{d}, \langle \kappa, \vec{V}\rangle, B_d\rangle$ of $d^\frown p^1$ in D, where $\widetilde{d}$ is either d, if d is an ordinal or $\langle \nu, \vec{F}_\nu, b_d\rangle$ if $d = \langle \nu, \vec{F}_\nu\rangle$. Set $A^* = \{e \in A^1 \mid \forall e' \in V_e(e \in B_{e'})\}$. Clearly, $A^* \in \bigcap \vec{V}$ and for every $d \in A^*$, $A^* \setminus V_{\kappa(d)+1} \subseteq B_d$. So, for every $d \in A(\alpha) \cap A^*$, $\langle \widetilde{d}, \langle \kappa, \vec{V}\rangle, B_d\rangle \leq^* \langle \widetilde{d}, \langle \kappa, \vec{V}\rangle, A^* \setminus V_{\kappa(d)+1}\rangle$. Hence, also $\langle \widetilde{d}, \langle \kappa, \vec{V}\rangle, A^* \setminus V_{\kappa(d)+1}\rangle$ is in D. Then we can take $p^* = \langle\langle \kappa, \vec{V}\rangle, A^*\rangle$ and T to be a one level tree which level consists of $A(\alpha) \cap A^*$.

Suppose now that there is no $d \in A^1$ with $d^\frown p^1$ having a direct extension in D. We continue to two steps extensions. Replacing A by A^1 we define $\widetilde{A}(\vec{d})$ as above. Let $A_0(\vec{d}) = \{d \in \widetilde{A}(\vec{d}) \mid$ there are $\alpha(\vec{d}) < \text{length}(\vec{V})$ and $C(\vec{d}) \subseteq \widetilde{A}(d) \setminus \kappa(d)$, $C(\vec{d}) \in U(\alpha(\vec{d}))$ such that for every $c \in C(\vec{d})$ there is in D a direct extension of the condition $\vec{d}^\frown d^\frown c^\frown p^1$ (i.e. the one obtained by adding $\vec{d}$, d and c to $p^1)\}$ and $A_1(\vec{d}) = A^1 \setminus A_0(\vec{d})$. Define $A(\alpha, \vec{d})$'s, $A(\alpha)$'s, A^2 and p^2 as was done above. Now, if for some $d_1, d_2 \in A^2$ some direct extension of $d_1{}^\frown d_2{}^\frown p^2$ is in D, then by $(*)_1$ for some $\beta < \text{length}(\vec{V})$, for every $d'_2 \in A^1(\beta) \setminus V_{\kappa(d_1)+1}$, $d_1{}^\frown d'_2{}^\frown p^2$ will have a direct extension in D. But then for $\alpha < \text{length}(\vec{V})$ with $d_1 \in A(\alpha)$ we will have that $d_1 \in A_0(\langle\,\rangle)$, i.e. for every $d'_1 \in A(\alpha)$ for some $\beta' < \text{length}(\vec{V})$ for every $d'_2 \in A^1(\beta') \setminus V_{\kappa(d'_1)+1}$, $d'_1{}^\frown d'_2{}^\frown p'$ will have a direct extension in D. In this case we can define p^* and two levels tree T. The definition is similar to those given above. Otherwise we consider $(*)_2$ the two-step analogue of $(*)_1$. Continue in a similar fashion. Thus at stage n we will have sets $A^n(\alpha) \in U(\alpha)$, $A^n = \bigcup\{A^n(\alpha) \mid \alpha < \text{length}(\vec{V})\}$ and $p^n = \langle\langle\kappa, \vec{V}\rangle, A^n\rangle$. Also the following n-dimension version of $(*)_1$ will hold:

$(*)_n$ If $p^n \leq q = \langle e_0, \ldots, e_{m-1}, d_1, \ldots, d_n, \langle\kappa, \vec{V}\rangle, B\rangle \in D$, then there is an n-levels $\vec{V}$-fat tree T_q such that for every maximal branch $\eta = \langle f_1, \ldots, f_n\rangle$ of T_q there are sets $\vec{A} = \langle a_1, \ldots, a_n\rangle$ of η-measure one and $B_\eta \in \bigcap \vec{V}$ such that

$$\langle e_0, \ldots, e_{m-1}\rangle^\frown\langle\eta, \vec{A}\rangle^\frown\langle\langle\kappa, \vec{V}\rangle, B_\eta\rangle \in D.$$

Again, if for some $d_1, \ldots, d_n \in A^n$, a direct extension q of $\langle d_1, \ldots, d_n\rangle^\frown p^n$ is in D, then we can easily finish. Just use T_q given by $(*)_n$ as T and let $A^* = \{e \in A^n \mid \forall\eta \in V_{\kappa(e)}(e \in B_\eta)\}$.

Suppose that the process does not stop at any $n < \omega$. Set

$$p^* = \big\langle\langle\kappa, \vec{V}\rangle, \textstyle\bigcap_{n<\omega} A_n\big\rangle.$$

Then $p^* \geq^* p$. By our assumption, no direct extension of p (and so of p^*) is in D. Pick some q, $q = \langle d_1, \ldots, d_n, \langle\kappa, \vec{V}\rangle, B\rangle \geq p^*$ and $q \in D$. Then $q \geq^* \langle d_1, \ldots, d_n\rangle^\frown p^n$. So, by the choice of p^n, we were supposed to stop at stage n. Contradiction. ⊣

We are now ready to show the following:

5.19 Theorem. *If* $\text{cf}(\text{length}(\vec{V})) \geq \kappa^+$ *then* κ *remains regular* (*and hence inaccessible*) *in* $V^{R_{\vec{V}}}$.

5.20 Remark. In view of Theorem 5.15 the converse of Theorem 5.19 is false.

Proof. Suppose that $\delta < \kappa$ and $\underset{\sim}{f}$ is an $R_{\vec{V}}$-name so that the weakest condition forces

$$\underset{\sim}{f} : \check{\delta} \longrightarrow \check{\kappa}.$$

Let $t = \langle\mu_1, \ldots, \mu_s, \langle\kappa, \vec{V}\rangle, E\rangle \in R_{\vec{V}}$. We find a $p \geq t$ forcing "$\text{ran}\underset{\sim}{f}$ is bounded in κ". Let $\xi < \delta$. Consider the set

$$D_\xi = \big\{p \in R_{\vec{V}} \mid \text{for some } d \in V_\kappa \setminus V_{\mu_3+1} \text{ appearing in } p,\ (p \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\kappa}(d))\big\}.$$

Clearly, D_ξ is a dense subset of $R_{\vec{V}}$. For every $\vec{d} = \langle d_1, \dots, d_n\rangle \in V_\kappa$ with $\vec{d}^\frown\langle\langle\kappa,\vec{V}\rangle, V_\kappa\backslash V_{\kappa(d_n)+1}\rangle \in R_{\vec{V}}$ apply Lemma 5.17 to $\vec{d}^\frown\langle\langle\kappa,\vec{V}\rangle, V_\kappa\backslash V_{\kappa(d_n)+1}\rangle$ and to D_ξ. We are interested only in the last T_m and only if $i_m = n+1$ there. Such a T_m is a $\vec{V}$-fat tree of the height $n_m < \omega$. Denote T_m further as $T(\xi,\vec{d})$. By Definition 5.16, for every $\eta \in T_m \setminus \mathrm{Lev}_{n_m}(T_m)$ there is an $\alpha(\eta) < \mathrm{length}(\vec{V})$ such that $\mathrm{Suc}_{T_m}(\eta) \in U(\alpha(\eta))$. Define $\alpha(\vec{d}) = \bigcup\{\alpha(\eta) \mid \eta \in T_m \setminus \mathrm{Lev}_{n_m}(T_m)\}$. Then $\alpha(\vec{d}) < \mathrm{length}(\vec{V})$, since $\mathrm{cf}(\mathrm{length}(\vec{V})) = \kappa^+$. Pick $\alpha(\xi) < \mathrm{length}(\vec{V})$ to be larger than each $\alpha(\vec{d})$ with $\vec{d}$ as above. Finally let $\alpha < \mathrm{length}(\vec{V})$ be above each $\alpha(\xi)$. Consider the following set:

$$B = \{\langle\nu,\vec{F}_\nu\rangle \in V_\kappa \mid \forall\xi<\delta\forall\vec{d}\in V_\nu(T(\xi,\vec{d})\cap V_\nu \text{ is } \vec{F}_\nu\text{-fat})\}.$$

By the choice of $\alpha, B \in U(\alpha)$. For every $\xi < \delta$, let $A^*_\xi \in \bigcap\vec{V}$ be the set given by Lemma 5.17 applied to D_ξ and t. Let $A^* = \bigcap_{\xi<\delta} A^*_\xi$. Every condition of $R_{\vec{V}}$ can be extended to one containing elements of $B \setminus V_{\mu_s+1}$. Hence the following will conclude the proof:

Claim. *Let $p \geq \langle\mu_1,\dots,\mu_s,\langle\kappa,\vec{V}\rangle, A^*\setminus V_{\mu_s+1}\rangle$ and some $\langle\nu,\vec{F}_\nu\rangle \in B\setminus V_{\mu_s+1}$ appears in p. Then*

$$p \Vdash \forall\xi < \check{\delta}\ (\underset{\sim}{f}(\xi) < \check{\nu}).$$

Proof. Suppose otherwise. Let

$$p \geq \langle\mu_1,\dots,\mu_s,\langle\kappa,\vec{V}\rangle, A^*\setminus V_{\mu_s+1}\rangle,$$

some $\langle\nu,\vec{F}_\nu\rangle \in B\setminus V_{\mu_s+1}$ appears in p and for some $\xi<\delta$ $p \Vdash \underset{\sim}{f}(\check{\xi}) \geq \check{\nu}$. Let $p = \langle d_1,\dots,d_\ell,\langle\langle\nu,\vec{F}_\nu\rangle,a_\nu\rangle, d_{\ell+2},\dots,d_n,\langle\kappa,\vec{V}\rangle, A\rangle$. Consider

$$p' = \langle d_1,\dots,d_\ell,\langle\kappa,\vec{V}\rangle, A^*\setminus V_{\kappa(d_\ell)+1}\rangle.$$

We would like to apply Remark 5.18. By the definition of B, $T(\xi,\langle d_1,\dots, d_\ell\rangle)\cap V_\nu$ is $\vec{F}_\nu$-fat. Since $a_\nu \in \bigcap\vec{F}_\nu$, we can find a maximal branch $\langle f_1,\dots,f_m\rangle$ through $T(\xi,\langle d_1,\dots,d_\ell\rangle)$ inside a_ν. By Lemma 5.17, there is a $q \geq p'$ with $q\in D_\xi$ of form

$$\langle e_1,\dots,e_i,\widetilde{f}_1,\dots,\widetilde{f}_m, A^*\setminus V_{\kappa(f_m)+1}\rangle$$

where $\kappa(e_i) = \kappa(d_\ell)$ and for every j, $1\leq j\leq m$, $\widetilde{f}_j$ is f_j, if f_j is an ordinal, or $\widetilde{f}_j = \langle f_j, b_j\rangle$ for some b_j, otherwise. $q \in D_\xi$ implies that $q \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\kappa}(f_m)$. Obviously, $\kappa(f_m) < \nu$, since $f_m \in a_\nu \subseteq V_\nu$. On the other hand, q and p are compatible, since $p \geq \langle\mu_1,\dots,\mu_s,\langle\kappa,\vec{V}\rangle, A^*\setminus V_{\mu_s+1}\rangle$,

$$p = \langle d_1,\dots,d_\ell,\langle\langle\nu,\vec{F}_\nu\rangle,a_\nu\rangle, d_{\ell+2},\dots,d_n,\langle\kappa,\vec{V}\rangle, A\rangle$$

and, hence $\langle\nu,\vec{F}_\nu\rangle, d_{\ell+2},\dots,d_n$ come from A^*. So they are addible to q. Hence

$$\langle e_1,\dots,e_i,\widetilde{f}_1,\dots,\widetilde{f}_m,\langle\langle\nu,\vec{F}_\nu\rangle,a_\nu\setminus V_{\kappa(f_m)+1}\rangle, d_{\ell+2},\dots,d_n,\langle\kappa,\vec{V}\rangle, A\rangle$$

is a common extension of q and p. But this is impossible since $p \Vdash \underset{\sim}{f}(\check{\xi}) \geq \check{\nu}$ and $q \Vdash \underset{\sim}{f}(\check{\xi}) < \check{\nu}$. Contradiction. ⊣

5.2. Magidor Forcing and Coherent Sequences of Measures

Magidor [37] invented a forcing for changing the cofinality of a cardinal κ to an uncountable value $\delta < \kappa$. As an initial assumption, his forcing uses a coherent sequence of measures of length δ. Coherent sequences of measures were introduced by Mitchell [43]. In [42] Mitchell showed that it is possible to do Radin forcing with coherent sequences of measures replacing an elementary embedding $j : V \to M$. The main advantage of this approach is reducing initial assumptions to weaker ones that in turn also provide equiconsistency results. This allows the simultaneous treatment of both the Magidor and the Radin forcings.

5.21 Definition. A *coherent sequence of measures* (ultrafilters) $\vec{U}$ is a function with domain of form

$$\{(\alpha, \beta) \mid \alpha < \ell^{\vec{U}} \text{ and } \beta < o^{\vec{U}}(\alpha)\}$$

for an ordinal $\ell^{\vec{U}}$, the length of $\vec{U}$, and a function $o^{\vec{U}}(\alpha)$, called the *order* of $\vec{U}$ at α. For each pair $(\alpha, \beta) \in \mathrm{dom}(\vec{U})$,

(1) $U(\alpha, \beta)$ is a normal ultrafilter over α, and

(2) if $j^\alpha_\beta : V \longrightarrow N^\alpha_\beta \simeq \mathrm{Ult}(V, (\alpha, \beta))$ is the canonical embedding, then

$$j^\alpha_\beta(\vec{U}) \restriction \alpha + 1 = \vec{U} \restriction (\alpha, \beta),$$

where

$$\vec{U} \restriction \alpha = \vec{U} \restriction \{(\alpha', \beta') \mid \alpha' < \alpha \text{ and } \beta' < o^{\vec{U}}(\alpha')\}$$

and

$$\vec{U} \restriction (\alpha, \beta) = \vec{U} \restriction \big\{(\alpha', \beta') \mid (\alpha' < \alpha \text{ and } \beta' < o^{\vec{U}}(\alpha')) \\ \text{or } (\alpha' = \alpha \text{ and } \beta' < \beta)\big\}.$$

Suppose that $\vec{U}$ is a coherent sequence of measures with $\ell^{\vec{U}} = \kappa + 1$ and $o^{\vec{U}}(\kappa) = \delta > 0$. We will now use $\vec{U}$ as a replacement for $\vec{V}$ of the previous section. Thus, over κ, $\vec{U}(\kappa) = \langle \vec{U}(\kappa, \alpha) \mid \alpha < \delta \rangle$ is used. Let $A \in \bigcap \vec{U}(\kappa) = \bigcap_{\alpha < \delta} U(\kappa, \alpha)$. Elements of A are ordinals only, no more pairs of form $\langle \nu, \vec{F}_\nu \rangle$ with ν an ordinal and $\vec{F}_\nu$ a sequence of ultrafilters over V_ν. But actually, if $\nu \in A$ and $o^{\vec{U}}(\nu) > 0$, then we have a sequence of measures $\vec{U}(\nu) = \langle \vec{U}(\nu, \alpha) \mid \alpha < o^{\vec{U}}(\nu) \rangle$ over ν. And it can be used exactly as $\vec{F}_\nu$ of the previous section. Note that here $\vec{U}(\nu)$ is determined uniquely from ν and $\vec{U}$. Also, because of coherence, namely Definition 5.21(2), there is no need to define the set $\overline{A}$ as it was done in the previous section before the definition of $R_{\vec{V}}$ (Lemma 5.1).

Let us denote for an ordinal $d = \nu$ or pair $d = \langle \nu, B \rangle$, ν by $\kappa(d)$. Using the above observations we define $\mathcal{P}_{\vec{U}}$ a coherent sequences analogue of $R_{\vec{V}}$.

5.22 Definition. Let $\mathcal{P}_{\vec{U}}$ be the set of finite sequences $\langle d_1, \dots, d_n, \langle \kappa, A \rangle \rangle$ such that:

(1) $A \in \bigcap \vec{U}(\kappa)$.

(2) $\min(A) > \kappa(d_n)$.

(3) For every m with $1 \leq m \leq n$, either

(3a) d_m is an ordinal and then $o^{\vec{U}}(d_m) = 0$, or

(3b) $d_m = \langle \nu, A_\nu \rangle$ for some ν with $o^{\vec{U}}(\nu) > 0$ and $A_\nu \in \bigcap_{\alpha < 0^{\vec{U}}(\nu)} U(\nu, \alpha)$.

(4) For every $1 \leq i \leq j \leq m$,

(4a) $\kappa(d_i) < \kappa(d_j)$, and

(4b) if d_j is of form $\langle \nu, A_\nu \rangle$ then $\min(A_\nu) > \kappa(d_i)$.

The definition of orders $\leq, \leq^*$ on $\mathcal{P}_{\vec{U}}$ repeats those of $R_{\vec{V}}$ (5.2), only ultrafilter sequences $\vec{F}_\nu$'s and $\vec{V}$ are removed from the conditions there.

5.23 Definition. Let $p = \langle d_1, \dots, d_n, \langle \kappa, A \rangle \rangle, q = \langle e_1, \dots, e_m, \langle \kappa, B \rangle \rangle \in \mathcal{P}_{\vec{U}}$. We say that p *is stronger than* q and denote this by $p \geq q$ iff

(1) $A \subseteq B$.

(2) $n \geq m$.

(3) There are $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that for every k with $1 \leq k \leq m$, either

(3a) $e_k = d_{i_k}$, or

(3b) $e_k = \langle \nu, B_\nu \rangle$ and then $d_{i_k} = \langle \nu, C_\nu \rangle$ with $C_\nu \subseteq B_\nu$.

(4) Let $i_1, \dots, i_m$ be as in (3). Then the following holds for every j with $1 \leq j \leq n$ and $j \notin \{i_1, \dots, i_k\}$:

(4a) If $j > i_m$, then $d_j \in B$ or d_j is of form $\langle \nu, C_\nu \rangle$ with $\nu \in B$ and $C_\nu \subseteq B \cap \nu$.

(4b) If $j < i_m$, then for the least k with $j < i_k$, e_k is of form $\langle \nu, B_\nu \rangle$ so that

(i) if d_j is an ordinal then $d_j \in B_\nu$, and

(ii) if $d_j = \langle \rho, S \rangle$ then $\rho \in B_\nu$ and $S \subseteq B_\nu$.

5.24 Definition. Let $p = \langle d_1, \dots, d_n, \langle \kappa, A \rangle \rangle, q = \langle e_1, \dots, e_m, \langle \kappa, B \rangle \rangle \in \mathcal{P}_{\vec{U}}$. We say that p is a *direct extension* of q and denote this by $p \geq^* q$ iff

(1) $p \geq q$, and

(2) $n = m$.

Now all the results of the previous section are valid in the present context with $\mathcal{P}_{\vec{U}}$ replacing $R_{\vec{V}}$. Also their proofs require only trivial changes.

If $\delta < \kappa$, then $\langle U(\kappa,\alpha) \mid \alpha < \delta\rangle$ can be split. Thus for every $\alpha < \delta$, $U(\kappa,\alpha)$ concentrates on the set $Y_\alpha = \{\nu < \kappa \mid o^{\vec{U}}(\nu) = \alpha\}$. $\mathcal{P}_{\vec{U}}$, above the condition $\langle\langle\kappa, \bigcup_{\alpha<\delta} Y_\alpha\rangle\rangle$ is then the Magidor forcing for changing cofinality of κ to $\mathrm{cf}(\delta)$.

5.3. Extender-Based Radin Forcing

In this section we give a brief description of the extender-based Radin forcing developed by Merimovich [39]. Previously, extender-based Magidor forcing was introduced by Miri Segal [49]. The basic idea is to combine the forcing of Sect. 3 with those of Sect. 5.1.

Assume GCH and let $j : V \longrightarrow M \supseteq V_{\kappa+4}$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. First, as in Sect. 3, but with $\lambda = \kappa^{++}$, for every $\alpha < \kappa^{++}$, we consider U_α an ultrafilter over κ defined by:

$$X \in U_\alpha \quad \text{iff} \quad \alpha \in j(X).$$

Define a partial order $\leq_j$ on λ:

$$\alpha \leq_j \beta \quad \text{iff} \quad \alpha \leq \beta \text{ and for some } f \in {}^\kappa\kappa,\ j(f)(\beta) = \alpha.$$

Let $\langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \kappa^{++},\ \alpha \geq_j \beta\rangle$ be the sequence of projections defined in Sect. 3. The whole system (i.e. the extender)

$$\langle\langle U_\alpha \mid \alpha < \kappa^{++}\rangle, \langle \pi_{\alpha\beta} \mid \beta \leq \alpha < \kappa^{++}, \alpha \geq_j \beta\rangle\rangle$$

is in M, as ${}^{\kappa^{++}}V_{\kappa+3} \subseteq V_{\kappa+3} \subseteq M$. Denote this system by $E(0)$ and U_α by $E_\alpha(0)$ for every $\alpha < \kappa^{++}$. Now, as in Lemma 5.1, we use the fact that $E(0) \in M$ in order to define $E(1)$. Thus for every $\alpha < \kappa^{++}$, we define over V_κ the following ultrafilter:

$$A \in E_{\langle\alpha,E(0)\rangle}(1) \quad \text{iff} \quad \langle\alpha, E(0)\rangle \in j(A).$$

It is possible to use only α as an index instead of $\langle\alpha, E(0)\rangle$, but it turns out that the latter notation is more convenient. Note that $E_{\langle\alpha,E(0)\rangle}(1)$ concentrates on elements of form $\langle\xi, e(0)\rangle$, where $e(0)$ is an extender over ξ^0 (recall, that in the notation of Sect. 3, ξ^0 denotes the projection of ξ to the normal ultrafilter by $\pi_{\alpha\kappa}$) of length $(\xi^0)^{++}$ including projections between its measures. Also note that σ_α defined by $\sigma_\alpha(\xi, e(0)) = \xi$ projects $E_{\langle\alpha,E(0)\rangle}(1)$ onto $E_\alpha(0) = U_\alpha$.

We define projections $\pi_{\langle\alpha,E(0)\rangle,\langle\beta,E(0)\rangle}$ for $\kappa^{++} > \alpha \geq \beta$ with $\alpha \geq_j \beta$ as follows:

$$\pi_{\langle\alpha,E(0)\rangle,\langle\beta,E(0)\rangle}(\langle\xi, e(0)\rangle) = \langle\pi_{\alpha\beta}(\xi), e(0)\rangle.$$

Then, in M,

$$j(\pi_{\langle\alpha,E(0)\rangle,\langle\beta,E(0)\rangle})(\langle\alpha, E(0)\rangle) = \langle\beta, E(0)\rangle.$$

This defines an extender

$$E(1) = \big\langle \langle E_{\langle \alpha, E(0)\rangle}(1) \mid \alpha < \kappa^{++}\rangle, \\ \langle \pi_{\langle \alpha, E(0)\rangle, \langle \beta, E(0)\rangle} \mid \kappa^{++} > \alpha \geq \beta,\ \alpha \geq_j \beta \rangle \big\rangle.$$

Continue by recursion. Suppose that $\tau < \kappa^{+4}$ and a sequence of extenders $\langle E(\tau') \mid \tau' < \tau\rangle$ is already defined. Again, as ${}^{\kappa^{++}}V_{\kappa+4} \subseteq V_{\kappa+4} \subseteq M$, $\langle E(\tau') \mid \tau' < \tau\rangle \in M$. So, for every $\alpha < \kappa^{++}$ we can define an ultrafilter over V_κ as follows

$$A \in E_{\langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau\rangle}(\tau) \\ \text{iff} \quad \langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau\rangle \in j(A).$$

Define projections:

$$\pi_{\langle \alpha, E(0), \ldots, E(\tau), \ldots \mid \tau' < \tau\rangle, \langle \beta, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau\rangle}(\langle \xi, d\rangle) = \langle \pi_{\alpha\beta}(\xi), d\rangle,$$

for every α, β, with $\kappa^+ > \alpha \geq \beta$ and $\alpha \geq_j \beta$. Further, let us suppress these long indexes and use only α and β, i.e. the above projection will be denote by $\pi_{\alpha\beta}$ and $E_{\langle \alpha, E(0), \ldots, E(\tau'), \ldots \mid \tau' < \tau\rangle}(\tau)$ by $E_\alpha(\tau)$. Define

$$E(\tau) = \big\langle \langle E_\alpha(\tau) \mid \alpha < \kappa^{++}\rangle, \langle \pi_{\alpha\beta} \mid \kappa^{++} > \alpha \geq \beta, \alpha \geq_j \beta\rangle \big\rangle.$$

Fix some $\tau^* \leq \kappa^{+4}$. Let $\vec{E} = \langle E(\tau) \mid \tau < \tau^*\rangle$.

In [24] and [25] Merimovich used such $\vec{E}$ to define the extender-based Radin forcing. The general definition is quite complicated and we will not reproduce it here. Instead let us concentrate on the case length$(\vec{E}) = 2$. This exhibits the idea of the Merimovich construction. So let $\vec{E} = \langle E(0), E(1)\rangle$. For each $\alpha < \kappa^{++}$ let $\overline{\alpha} = \langle \alpha, E(0), E(1)\rangle$. Set $\overline{E} = \{\langle \alpha, E(0), E(1)\rangle \mid \alpha < \kappa^{++}\}$.

5.25 Definition. A *basic condition* in $\mathcal{P}_{\vec{E}}$ over κ is one of form

$$p = \{\langle \overline{\gamma}, p^{\overline{\gamma}}\rangle \mid \overline{\gamma} \in s\} \cup \{\langle \overline{\alpha}, p^{\overline{\alpha}}\rangle, T\}$$

so that

(1) $s \subseteq \overline{E}$, $|s| \leq \kappa$ and $\overline{\kappa} \in s$.

This s is the support of the condition and here, instead of just ordinals used as supports in the extender-based Prikry forcing of Sect. 3, its elements are of form $\overline{\gamma} = \langle \gamma, E(0), E(1)\rangle$.

(2) $p^{\overline{\gamma}} \in V_\kappa$ is a finite sequence of elements of form an ordinal ν or a pair $\langle \nu, e_\nu(0)\rangle$ with $e_\nu(0)$ and extender of length $(\nu^0)^{++}$ over ν^0 (recall that, as in Sect. 3, ν^0 denotes the projection of ν by $\pi_{\gamma,\kappa}$, i.e. to the normal measure). We require that the ν^0's of elements of $p^{\overline{\gamma}}$ are increasing. Denote the ν of the last element of $p^{\overline{\gamma}}$ by $\kappa(p^{\overline{\gamma}})$, if $p^{\overline{\gamma}}$ is nonempty and let $\kappa(p^{\overline{\gamma}}) = 0$ otherwise.

(3) $\overline{\alpha}$ is above every $\overline{\gamma} \in s$ in the $\leq_j$ order (i.e. $\gamma \leq_j \alpha$).

(4) $\kappa(p^{\overline{\alpha}}) \leq \kappa(p^{\overline{\gamma}})$.

(5) $T \in E_\alpha(0) \cap E_\alpha(1) \setminus V_{\kappa(p^{\overline{\kappa}})+1}$.

(6) For every $\overline{\nu} \in T$,

$$|\{\overline{\gamma} \in s \mid (\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0\}| \leq (\kappa(\overline{\nu}))^0.$$

(7) For every $\overline{\nu} \in T$, $\overline{\beta}, \overline{\gamma} \in \overline{s}$, if $(\kappa(p^{\overline{\beta}}))^0, (\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0$ and $\overline{\beta} \neq \overline{\gamma}$ then

$$\pi_{\overline{\alpha},\overline{\beta}}(\overline{\nu}) \neq \pi_{\overline{\alpha},\overline{\gamma}}(\overline{\nu}).$$

As in Sect. 3, we write $T^p, mc(p), \mathrm{supp}(p)$ for $T, \overline{\alpha}$ and $s \cup \{\overline{\alpha}\}$ respectively.

5.26 Definition. For basic conditions p, q of $\mathcal{P}_{\vec{E}}$ over κ, define $p \geq^* q$ iff

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.

(2) For every $\overline{\gamma} \in \mathrm{supp}(q)$, $p^{\overline{\gamma}} = q^{\overline{\gamma}}$.

(3) $T^p \subseteq \pi^{-1}_{mc(p),mc(q)}$ "T^q.

(4) For every $\overline{\gamma} \in \mathrm{supp}(q)$ and $\overline{\nu} \in T^p$, if $(\kappa(p^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0$ then

$$\pi_{mc(p),\overline{\gamma}}(\overline{\nu}) = \pi_{mc(q),\overline{\gamma}}(\pi_{mc(p),mc(q)}(\overline{\nu})).$$

Now let p_0 be a basic condition over κ and $\overline{\nu} \in T^{p_0}$. We define $p_0{}^\frown\overline{\nu}$, a one-element extension of p_0 by $\overline{\nu}$.

5.27 Definition. $p_0{}^\frown\langle\overline{\nu}\rangle$ will be of form $p'_1{}^\frown p'_0$ where

(1) $\mathrm{supp}(p'_0) = \mathrm{supp}(p_0)$.

(2) For every $\overline{\gamma} \in \mathrm{supp}(p'_0)$

$$p'^{\overline{\gamma}}_0 = \begin{cases} \pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}), & \\ \qquad \text{if } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0 \text{ and } \overline{\nu} \text{ is of form } \langle \nu, e_\nu(0)\rangle, & \\ p_0^{\overline{\gamma}}{}^\frown \pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}), & \\ \qquad \text{if } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0 \text{ and } \overline{\nu} \text{ is an ordinal}, & \\ p_0^{\overline{\gamma}}, \quad \text{otherwise}. & \end{cases}$$

(3) $T^{p'_0} = T^{p_0} \setminus V_{(\kappa(\overline{\nu}))^0+1}$.

If $\overline{\nu}$ is an ordinal then p'_1 is empty, otherwise the following holds:

(4) $mc(p'_1) = \overline{\nu}$.

(5) $\mathrm{supp}(p_1') =$
$\{\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu}) \mid \overline{\gamma} \in \mathrm{supp}(p_0) \text{ and } (\kappa(p_0^{\overline{\gamma}}))^0 < (\kappa(\overline{\nu}))^0\} \cup \{\overline{\nu}\}$.

(6) $p_1'^{\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})} = p_0^{\overline{\gamma}}$.

(7) $T^{p_1'} = T^{p_0} \cap V_{(\kappa(\overline{\nu}))^0}$.

Definition 5.27 is the crucial step of the definition of $\mathcal{P}_{\vec{E}}$. If $\overline{\nu}$ was an ordinal then $p^\frown\langle\overline{\nu}\rangle = p_0'$ is generated as in Sect. 3. But if $\overline{\nu}$ is of form $\langle \nu, e_\nu(0)\rangle$ then after adding $\overline{\nu}$, p_0 splits into two blocks p_0' and p_1'. p_0' is still a basic condition over κ generated in the fashion of Sect. 3. But p_1' is a new block. We just separate and move to the new block every $p_0^{\overline{\gamma}}$ to which $\overline{\nu}$ can be added. The actual addition, $\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})$, is kept both in the support of p_1' and on the new $p_0'^{\overline{\gamma}}$. T^{p_0} is moved down to ν and p_1' is a basic condition over ν^0. We can extend it further using measures of the extender $e_\nu(0)$. It acts from now autonomously and as a condition in the extender-based Prikry forcing of Sect. 3. Note that we still keep some connection with the upper block p_0'. Thus $\pi_{mc(p_0),\overline{\gamma}}(\overline{\nu})$'s appear in both $\mathrm{supp}(p_1')$ and p_0', as $p_0'^{\overline{\gamma}}$. See the figure below which gives an example of such p_0, p_0', p_1'.

$$\begin{array}{lllll}
 & & \overline{\mu}_{32} & & \\
 & \overline{\mu}_{11} & \overline{\mu}_{31} & & \\
\overline{\mu}_0 & \overline{\mu}_{10} & \overline{\mu}_{30} & \overline{\mu}_4 & T \cap V_\kappa(\overline{\nu}) \\
\hline
\overline{\nu}^0 & \pi_{\alpha_4\alpha_1}(\overline{\nu}) & \pi_{\alpha_4\alpha_3}(\overline{\nu}) & \overline{\nu} &
\end{array} \quad p_1'$$

$$\begin{array}{llllll}
 & & \overline{\mu}_{21} & & & \\
\overline{\nu}^0 & \pi_{\alpha_4\alpha_1}(\overline{\nu}) & \overline{\mu}_{20} & \pi_{\alpha_4\alpha_3}(\overline{\nu}) & \overline{\nu} & T \setminus V_{\kappa(\overline{\nu})+1} \\
\hline
\overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc &
\end{array} \quad p_0'$$

$$\begin{array}{llllll}
 & & & \overline{\mu}_{32} & & \\
 & \overline{\mu}_{11} & \overline{\mu}_{21} & \overline{\mu}_{31} & & \\
\overline{\mu}_0 & \overline{\mu}_{10} & \overline{\mu}_{20} & \overline{\mu}_{30} & \overline{\mu}_4 & T \\
\hline
\overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc &
\end{array} \quad p_0$$

Once we have a two-block condition $p_1{}^\frown p_0$ we can extend it further in the same way by adding either $\overline{\nu} \in T_{p_0}$ or $\overline{\nu} \in T_{p_1}$. In the first case this will generate a new block between p_1 and p_0 and the second below p_1. We are allowed to repeat this any finite number of times. Thus a general condition in $P_{\vec{E}}$ will be of form $p = p_n{}^\frown p_{n-1}{}^\frown \cdots {}^\frown p_0$ where p_0 is a basic condition over κ, p_1 over some $\nu_0 < \kappa, \ldots$ and, p_n over some $\nu_{n-1} < \nu_{n-2}$.

An example of a condition in $P_{\vec{E}}$:

$$\begin{array}{lllllll}
 & \overline{\tau}_{12} & & & & & \\
 & \overline{\tau}_{11} & & & & & \\
\overline{\tau}_0 & \overline{\tau}_{10} & \overline{\tau}_2 & \overline{\tau}_3 & \overline{\tau}_4 & \overline{\tau}_5 & R \\
\hline
\overline{\mu}_0 & \overline{\mu}_1 & \overline{\mu}_2 & \overline{\mu}_3 & \overline{\mu}_4 & \overline{\mu}_5 = mc &
\end{array} \quad p_2$$

$$\begin{array}{ccccccc} & & \overline{\mu}_{71} & & & & \\ \overline{\mu}_0 & \overline{\mu}_1 & \overline{\mu}_{70} & \overline{\mu}_8 & \overline{\mu}_6 & \overline{\mu}_5 & S \\ \hline \overline{\nu}_0 & \overline{\nu}_1 & \overline{\nu}_2 & \overline{\nu}_3 & \overline{\nu}_4 & \overline{\nu}_5 = mc & \end{array} \quad p_1$$

$$\begin{array}{cccccc} & & & \overline{\nu}_{71} & & \\ \overline{\nu}_0 & \overline{\nu}_1 & \overline{\nu}_6 & \overline{\nu}_{70} & \overline{\nu}_4 & T \\ \hline \overline{\kappa} & \overline{\alpha}_1 & \overline{\alpha}_2 & \overline{\alpha}_3 & \overline{\alpha}_4 = mc & \end{array} \quad p_0$$

Each block may grow separately. Thus in the example the maximal coordinate of p_1 changed from $\overline{\nu}_4$, corresponding to $\overline{\alpha}_4$, to a new value $\overline{\nu}_5$. New coordinates $\overline{\nu}_2, \overline{\nu}_3$ were added in p_1 and $\overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4$ in p_2.

The following is a straightforward generalization of Definition 5.27.

5.28 Definition. Let $p, q \in P_{\overline{E}}$. We say that p is a *one-point extension* of q and denote this by $p \geq_1 q$ iff p and q are of form

$$\begin{aligned} p &= p_{n+1}{}^\frown p_n{}^\frown \cdots {}^\frown p_0 \\ q &= q_n{}^\frown \cdots {}^\frown q_0 \end{aligned}$$

and there is a k with $0 \leq k \leq n$ such that

(1) p_i and q_i are basic conditions over some ν_i with $p_i \geq^* q_i$ for $i < k$.

(2) p_{i+1} and q_i are basic conditions over some ν_i with $p_{i+1} \geq^* q_i$ for each $k < i \leq n$.

(3) There is a $\overline{\nu} \in T^{q_k}$ such that $p_{k+1}{}^\frown p_k \geq^* q_k{}^\frown \langle \overline{\nu} \rangle$.

We now define n-point extension for every $n < \omega$.

5.29 Definition. Let $p, q \in P_{\vec{E}}$. We say that p is an *n-point extension* of q and denote this by $p \geq_n q$ iff either $n = 0$ and $p \geq^* q$, or else $n > 0$ and there are $p^n, \ldots, p^0$ such that

$$p = p^n \geq_1 \cdots \geq_1 p^0 = q.$$

Finally, we can define the order $\leq$ on $P_{\vec{E}}$.

5.30 Definition. Let $p, q \in P_{\vec{E}}$. Define $p \geq q$ iff there is $n < \omega$ such that $p \geq_n q$.

Let G be a generic subset of $\langle \mathcal{P}_{\vec{E}}, \leq \rangle$. For every α with $\kappa \leq \alpha < \kappa^{++}$ we want to collect together all the ordinals corresponding to α into a set which we call G^α. Define

$$\begin{aligned} G^\alpha = \bigl\{ \kappa(p^{\vec{E}_\alpha}) \mid \exists p \in G\, (&p \text{ is a basic condition} \\ &\text{over } \kappa \text{ with } \vec{E}_\alpha \in \operatorname{supp}(p) \text{ and } p^{\vec{E}_\alpha} \neq \emptyset) \bigr\}. \end{aligned}$$

It is not hard to see using the definition of the order on $P_{\vec{E}}$ that G^α will be unbounded in κ sequence of order type ω^2. Also $\alpha \neq \beta$ will imply $G^\alpha \neq G^\beta$.

In addition, the sequence G^κ (the one corresponding to the normal ultrafilter) will be closed.

Now let length($\vec{E}$) be any ordinal $\leq \kappa^{+4}$. Merimovich [39] showed that his forcing $P_{\vec{E}}$ shares all the properties of the Radin forcing of Lemma 5.1, only κ^+-c.c. should be replaced by κ^{++}-c.c. This causes a new problem to show that κ^+ is preserved in cases of regular κ. In order to preserve measurability of κ the following variation of repeat point is used:

> $\tau <$ length($\vec{E}$) is called a *repeat point* of $\vec{E}$ if for every
>
> $\xi <$ length($\vec{E}$) and $\alpha < \kappa^{++}$, $A \in E_\alpha(\xi)$ implies that for some
>
> $\xi' < \tau$ $A \in E_\alpha(\xi')$.

That is, τ acts simultaneously as a repeat point of the sequence of ultrafilters $\langle E_\alpha(\xi') \mid \xi < \text{length}(\vec{E})\rangle$ for each $\alpha < \kappa^{++}$. Clearly, there will be lots of repeat points below κ^{+4}. The κ^{++} sets G^α defined above for a generic $G \subseteq P_{\vec{E}}$ will witness $2^\kappa = \kappa^{++}$; G^κ will be a club in κ.

In further work [40], Merimovich added collapses to the extender-based Radin forcing. This allowed him to reprove results of Foreman-Woodin [12], and Woodin and obtain new interesting patterns of global behavior of the power function.

6. Iterations of Prikry-Type Forcing Notions

In this section we present two basic techniques for iterating Prikry-type forcing notions. The first one is called the Magidor or full support iteration and the second, Easton support iteration.

A set with two partial orders $\langle \mathcal{P}, \leq, \leq^* \rangle$ is called a *Prikry-type forcing notion* iff

(a) $\leq \supseteq \leq^*$.

(b) (The Prikry condition) For every $p \in \mathcal{P}$ and statement σ of the forcing language of $\langle \mathcal{P}, \leq \rangle$ there is a $p^* \geq^* p$ deciding σ.

Notice that any forcing $\langle \mathcal{P}, \leq \rangle$ can be turned into a Prikry-type by defining $\leq^* = \leq$. In this case the iterations below coincide with the usual iterations with full or Easton support.

6.1. Magidor Iteration

The presentation below follows [16] and is a bit different from Magidor's original version [34].

Let ρ be an ordinal. We define an iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\alpha \mid \alpha < \rho\rangle$. For every $\alpha < \rho$ define by recursion $\mathcal{P}_\alpha$ to be the set of all p of form $\langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha\rangle$ so that for every $\gamma < \alpha$,

(a) $p\restriction\gamma = \langle \underset{\sim}{p}_\beta \mid \beta < \gamma\rangle \in \mathcal{P}_\gamma$, and

(b) $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma$ is a condition in the forcing $\langle \underset{\sim}{Q}_\gamma, \underset{\sim}{\leq}_\gamma, \underset{\sim}{\leq}^*_\gamma\rangle$ of Prikry-type".

We next define two orderings $\leq_{\mathcal{P}_\alpha}$ and $\leq^*_{\mathcal{P}_\alpha}$ on $\mathcal{P}_\alpha$.

6.1 Definition. Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha\rangle$, $q = \langle \underset{\sim}{q}_\gamma \mid \gamma < \alpha\rangle \in \mathcal{P}_\alpha$. Then $p \geq_{\mathcal{P}_\alpha} q$ iff

(1) For every $\gamma < \alpha$, $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

(2) There exists a finite $b \subseteq \alpha$ such that for every $\gamma \in \alpha \setminus b$, $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \geq^*_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

If the set b in (2) is empty, then we call p a *direct extension* of q and denote this by $p \geq^*_{\mathcal{P}_\alpha} q$.

Thus we use full support iteration here, but in order to pass from a condition $q \in \mathcal{P}_\alpha$ to a stronger one, we are allowed to take non-direct extensions only at finitely many places. A typical example and the one originally used by Magidor in [34], is an iteration of Prikry forcings at each measurable below α. Here, in order to extend a condition we may shrink sets of measure one at each measurable $\beta < \alpha$ but only for finitely many β's is it allowed to add new elements of the Prikry sequence. We further discuss this important example in detail. Let us now show that $\langle \mathcal{P}_\alpha, \leq, \leq^*\rangle$ is itself of Prikry-type.

6.2 Lemma. *Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha\rangle \in \mathcal{P}_\alpha$ and σ be a statement of the forcing language of $\langle \mathcal{P}_\alpha, \leq\rangle$. Then there is a direct extension of p deciding σ.*

Proof. We deal first with the successor case. Let $\alpha = \alpha' + 1$. Assume that $\mathcal{P}_{\alpha'}$ has the Prikry property, $\mathcal{P}_\alpha = \mathcal{P}_{\alpha'} * \underset{\sim}{Q}_{\alpha'}$, and $\Vdash_{\mathcal{P}_{\alpha'}} (\langle \underset{\sim}{Q}_{\alpha'}, \underset{\sim}{\leq}_{\alpha'}, \underset{\sim}{\leq}^*_{\alpha'}\rangle$ has the Prikry property). Let $G_{\alpha'} \subseteq \mathcal{P}_{\alpha'}$ be generic for $\langle \mathcal{P}_{\alpha'}, \leq\rangle$ with $p\restriction\alpha' = \langle \underset{\sim}{p}_\gamma \mid \gamma < \alpha'\rangle \in G_{\alpha'}$. Find $p^*_{\alpha'} \geq^*_{\alpha'} p_{\alpha'}$ in $Q_{\alpha'}$ which decides $\sigma[G_{\alpha'}]$. Back in V, let $\underset{\sim}{p}^*_{\alpha'}$ be a name of such $p^*_{\alpha'}$ so that

$$p\restriction\alpha' \Vdash_{\mathcal{P}_{\alpha'}} \underset{\sim}{p}^*_{\alpha'} \text{ decides } \sigma.$$

Use the Prikry property of $\langle \mathcal{P}_{\alpha'}, \leq, \leq^*\rangle$ to find a $q \geq^* p\restriction\alpha'$ such that $q \Vdash_{\mathcal{P}_{\alpha'}} (\underset{\sim}{p}^*_{\alpha'} \Vdash_{\underset{\sim}{Q}^*_{\alpha'}} {}^i\sigma)$, for some $i < 2$, where ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Then, with $r = q^\frown \underset{\sim}{p}^*_{\alpha'}$, we have $r \Vdash_{\mathcal{P}_\alpha} {}^i\sigma$.

Suppose now that α is a limit ordinal. Assume that there is no direct extension of p deciding σ. We define by recursion on $\beta < \alpha$

$$p(\beta) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta\rangle^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha\rangle \geq^* p$$

so that $p(\beta)\restriction\beta = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta\rangle \Vdash_{\mathcal{P}_\beta} \neg\sigma_\beta$ where $\sigma_\beta \equiv (\exists \underset{\sim}{q} \in \mathcal{P}_\alpha \setminus \beta(\underset{\sim}{q} \geq^* p \setminus \beta$ and $\underset{\sim}{q} \Vert \sigma))$.

Suppose that $\langle p(\gamma) \mid \gamma < \beta\rangle$ are defined and $\leq^*$-increasing. Define $p(\beta)$:

Case 1. $\beta = \beta' + 1$.

Force with $\mathcal{P}_{\beta'} = \mathcal{P}_\alpha \restriction \beta'$, i.e. with $\langle \mathcal{P}_{\beta'}, \leq \rangle$. Let $G_{\beta'} \subseteq \mathcal{P}_{\beta'}$ be generic with $p(\beta') \restriction \beta' \in G_{\beta'}$. At stage β' we use $\langle Q_{\beta'}, \leq_{\beta'}, \leq^*_{\beta'} \rangle$. It satisfies the Prikry condition. So there is a $p^*_{\beta'} \geq^*_{\beta'} p_{\beta'}$ deciding σ_β.

6.2.1 Claim. $p^*_{\beta'} \Vdash_{Q_{\beta'}} \neg \sigma_\beta$.

Proof. Suppose otherwise. Then there is a $p^{**}_{\beta'} \geq^* p^*_{\beta'}$ with

$$p^{**}_{\beta'} \Vdash_{Q_{\beta'}} (\exists \underset{\sim}{q} \in \mathcal{P}_\alpha \setminus \beta (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta} {}^i\sigma))$$

for some $i < 2$, where ${}^o\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Without loss of generality assume $i = 0$. Then there are $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \beta' \rangle \in G_{\beta'}$ and $\underset{\sim}{q}$ such that $p(\beta') \restriction \beta' \leq r$ and

$$r \Vdash_{\mathcal{P}_{\beta'}} (\underset{\sim}{p}_{\beta'} \leq^*_{\beta'} \underset{\sim}{p}^{**}_{\beta'} \Vdash_{\underset{\sim}{Q}_{\beta'}} (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta} \sigma)).$$

Hence, $r \Vdash_{\mathcal{P}_{\beta'}} (\underset{\sim}{p}^{**}_{\beta'} {}^\frown \underset{\sim}{q} \geq^* p \setminus \beta'$ and $\underset{\sim}{p}^{**}_{\beta'} {}^\frown \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta'} \sigma)$. In particular, $p(\beta') \restriction \beta' \leq r \Vdash_{\mathcal{P}_{\beta'}} \sigma_{\beta'}$ which contradicts the choice of $p(\beta')$. ⊣

Now, since $G_{\beta'}$ was arbitrary, we can take a name $\underset{\sim}{p}^*_{\beta'}$ of $p^*_{\beta'}$ such that $p(\beta') \restriction \beta' \Vdash (\underset{\sim}{p}^*_{\beta'} \Vdash_{\underset{\sim}{Q}_{\beta'}} \neg \sigma_\beta)$. Set $p(\beta) = p(\beta') \restriction \beta' {}^\frown \underset{\sim}{p}^*_{\beta'} {}^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha \rangle$.

Case 2. β is a limit ordinal.

Then we need to show that

$$p(\beta) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \beta \rangle {}^\frown \langle \underset{\sim}{p}_\gamma \mid \beta \leq \gamma < \alpha \rangle$$

is as desired, i.e. $p(\beta) \restriction \beta \Vdash \neg \sigma_\beta$. Suppose otherwise; then there is an $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \beta \rangle \in \mathcal{P}_\beta$ such that $r \geq p(\beta) \restriction \beta$ and $r \Vdash \sigma_\beta$. Extend it, if necessary, so that for some $\underset{\sim}{q}$ and $i < 2$

$$r \Vdash (\underset{\sim}{q} \geq^* p \setminus \beta \text{ and } \underset{\sim}{q} \Vdash_{\mathcal{P}_\alpha \setminus \beta} {}^i\sigma)$$

where ${}^0\sigma = \sigma$ and ${}^1\sigma = \neg\sigma$. Let us assume that $i = 0$. By the definition of order on $\mathcal{P}_\beta$ (Definition 6.1(2)), there is a $\beta^* < \beta$ such that for every γ with $\beta^* \leq \gamma < \beta$, $r \restriction \gamma \Vdash \underset{\sim}{r}_\gamma \geq^*_\gamma \underset{\sim}{p}^*_\gamma$. Consider a $\mathcal{P}_{\beta^*}$-name $\underset{\sim}{q}' = \langle \underset{\sim}{r}_\gamma \mid \beta^* \leq \gamma < \beta \rangle {}^\frown \underset{\sim}{q}$. Then, $r \restriction \beta^* \Vdash (\underset{\sim}{q}' \geq^* p \setminus \beta^*$ and $\underset{\sim}{q}' \Vdash_{\mathcal{P}_\alpha \setminus \beta^*} \sigma)$. But $r \restriction \beta^* \geq p(\beta^*) \restriction \beta^* \Vdash \neg \sigma_{\beta^*}$. Contradiction.

This completes the construction. Consider $p(\alpha) = \langle \underset{\sim}{p}^*_\gamma \mid \gamma < \alpha \rangle$. Pick some $r \geq p(\alpha)$ deciding σ. Now we obtain a contradiction as in Case 2. This completes the proof of the lemma. ⊣

Let us now use this type of iteration to prove the following result of Magidor [34]:

6.3 Theorem. *Let κ be a strongly compact cardinal. Then there is a cardinal preserving extension in which κ is the least strongly compact and also the least measurable.*

Proof. We use the Magidor iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ defined by recursion on α as follows:

(a) If $\Vdash_{\mathcal{P}_\alpha}$ (α is not measurable), then take $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ to be the trivial forcing

(b) If $\Vdash_{\mathcal{P}_\alpha}$ (α is a measurable cardinal), then let $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ be the Prikry forcing over α with some normal ultrafilter.

(c) If $\neg(a)$ and $\neg(b)$, then we pick a maximal antichain $\langle p^i \mid i < \tau\rangle$ of elements of $\mathcal{P}_\alpha$ so that each p^i decides measurability of α. Above each p^i forcing (α is not measurable) we take $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ to be the trivial forcing. Above every p^i forcing measurability of α let $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ be the Prikry forcing over α with some normal ultrafilter.

This means that $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is a $\mathcal{P}_\alpha$-name such that $\mathcal{P}_\alpha$ forces: "if α is a measurable then $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is Prikry forcing, and otherwise $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is trivial.

Let us now force with $\langle \mathcal{P}_\kappa, \leq\rangle$. Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. Then, in $V[G_\kappa]$, all measurable cardinals below κ are destroyed. Note that for $\alpha < \kappa$ the iteration past stage $\alpha + 1$ does not add measurables below α, since it is itself a Prikry-type iteration with $\leq^*$-order more than 2^α-closed. So, no new subsets are added to α. We need only show that κ remains strongly compact. This will follow from the next more general statement. ⊣

Note that the above proof is a simplification of Magidor's proof, which showed that the measures to be killed are exactly the unique normal extensions of measures of order 0 in V.

6.4 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is the Magidor iteration of Prikry-type forcing notions such that $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many α's. Then κ is strongly compact in $V^{\mathcal{P}_\kappa}$ provided it was such in V and for every $\alpha < \kappa$, $\Vdash_{\mathcal{P}_\alpha}$ ((a) $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leqslant}_\alpha{}^*\rangle$ is $|\alpha|$-closed, and (b) for all $p, q, r \in \underset{\sim}{Q}_\alpha$, if $p, q \geq^* r$ there is a $t \in \underset{\sim}{Q}_\alpha$ such that $t \geq^* p, q$).*

6.5 Remark. The requirement (a) holds for most of the Prikry-type forcing notions. But we refer the reader to [16] and [46] for doing without closure but still preserving measurability. The requirement (b) is much more restrictive. For example extender-based Prikry forcings of Sects. 2 and 3 do not satisfy it. Also the Easton support iteration that will be defined later fails to satisfy (b). It will be shown in Lemma 6.8 in non trivial cases (a) + (b) imply existence of a measurable cardinal $\geq |\alpha|$.

Proof. Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic, i.e. generic for $\langle \mathcal{P}_\kappa, \leq\rangle$. Let $\lambda \geq \kappa$. We want to establish the λ-strong compactness of κ. In V pick a κ-complete fine ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ (recall that U is fine if for every $\alpha < \lambda$ the set

$\{P \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in P\}$ is in U). Let $j : V \longrightarrow M \simeq \mathrm{Ult}(V, U)$. Back in $V[G_\kappa]$, let us define $U^* \supseteq U$ over $\mathcal{P}_\kappa(\lambda)$ as follows:

$$X \in U^* \quad \text{iff} \quad \begin{array}{l} \text{for some } p \in G_\kappa, \text{ in } M \text{ there is a } q \in \mathcal{P}_{j(\kappa)} \setminus \kappa \text{ with} \\ q \geq^* j(p) \setminus \kappa \text{ so that } p^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\mathrm{id}}]_U \in j(\underset{\sim}{X}) \\ \text{for some name } \underset{\sim}{X} \text{ of } X. \end{array}$$

Note that $\mathcal{P}_\kappa = j(\mathcal{P}_\kappa){\restriction}\kappa$, since, in M, $j(\mathcal{P}_\kappa){\restriction}\kappa \subseteq {}^\kappa V_\kappa$ and $j``V_\kappa = V_\kappa$. So G_κ is an M-generic subset of $\mathcal{P}_\kappa$. Also $j(p){\restriction}\kappa = p$ for every $p \in \mathcal{P}_\kappa$. We need to first check that U^* is well defined. By (b) any two q's as above are compatible. Note also that if p, $p' \in \mathcal{P}_\kappa$ are $\leq$-compatible, then, in M, $j(p) \setminus \kappa$ and $j(p') \setminus \kappa$ are $\leq^*$-compatible. To see this, let $r \in \mathcal{P}_\kappa$, $r \geq p, p'$. Then there is a $\beta < \kappa$ such that for $\beta \leq \alpha < \kappa$, $r{\restriction}\beta \Vdash_{\mathcal{P}_\beta} \underset{\sim}{r}_\beta \geq^*_\beta \underset{\sim}{p}_\beta, \underset{\sim}{p}'_\beta$, where $r = \langle \underset{\sim}{r}_\gamma \mid \gamma < \kappa\rangle$, $p = \langle \underset{\sim}{p}_\gamma \mid \gamma < \kappa\rangle$, and $p' = \langle \underset{\sim}{p}'_\gamma \mid \gamma < \kappa\rangle$. So, in M, the same is true for $j(r)$, $j(p)$ and $j(p')$. Hence, r forces $\leq^*$-compatibility of $j(p) \setminus \kappa$ and $j(p') \setminus \kappa$ witnessed by $j(r) \setminus \kappa$. In particular, this shows using (b) that $q \geq^* j(p) \setminus \kappa$ is $\leq^*$-compatible with every $j(p') \setminus \kappa$ with p, $p' \in G$.

Now applying above, if $p \in G$ forces "$\underset{\sim}{X} \in \underset{\sim}{U}^*$ and $\underset{\sim}{X} = \underset{\sim}{Y}$", then for some $q \in \mathcal{P}_{j(\kappa)} \setminus \kappa$, $q \geq^* j(p) \setminus \kappa$ we have

$$j(p){\restriction}\kappa^\frown q \ \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\mathrm{id}}]_U \in j(\underset{\sim}{X}).$$

But by elementarity, $j(p) \Vdash j(\underset{\sim}{X}) = j(\underset{\sim}{Y})$. Also, $j(p){\restriction}\kappa^\frown q \geq^* j(p)$. Hence

$$j(p){\restriction}\kappa^\frown q \ \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\mathrm{id}}]_U \in j(\underset{\sim}{Y}).$$

Clearly, $U^* \supseteq U$, and so it is fine. Let $\langle X_\nu \mid \nu < \delta < \kappa\rangle$ be a partition of $\mathcal{P}_\kappa(\lambda)$. We need to show that then for some $\nu < \delta$, $X_\nu \in U^*$. Pick some $p \in G_\kappa$ and names $\langle \underset{\sim}{X}_\nu \mid \nu < \delta\rangle$ such that $p \Vdash \langle \underset{\sim}{X}_\nu \mid \nu < \delta\rangle$ is a partition of $\mathcal{P}_\kappa(\lambda)$.

Then in M, $j(p) \Vdash (\langle j(\underset{\sim}{X}_\nu) \mid \nu < \delta\rangle$ is a partition of $\mathcal{P}_{j(\kappa)}(j(\lambda))$. Now we use κ-completeness of $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^*\rangle$ in order to find $\nu^* < \delta$ and $q \in \mathcal{P}_{j(\kappa)} \setminus \kappa$ with $q \geq^* j(p) \setminus \kappa$ such that for some $r \in G_\kappa$,

$$r^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\mathrm{id}}] \in j(\underset{\sim}{X}_{\nu^*}).$$

Hence $X_{\nu^*} \in U^*$ and we are done. $\dashv$

Note that once the ultrafilter U (in the proof above) is normal and the forcing $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^*\rangle$ is λ^+-closed, then the ultrafilter U^* extending U will be normal as well. Just use a regressive function instead of a partition in the proof of Lemma 6.4.

In particular, if we change the cofinality of each measurable cardinal below a measurable cardinal κ using the Magidor iteration of Prikry forcings, then the normal measure U over κ in V extends to a normal measure in the extension, provided $\langle \mathcal{P}_{j(\kappa)} \setminus \kappa, \leq^* \rangle$ is κ^+-closed. In order to insure this degree of closure, we may take U which concentrates on non-measurables, i.e.

$$\{\alpha < \kappa \mid \alpha \text{ is not a measurable }\} \in U.$$

It is still necessary to check that the iteration $\mathcal{P}_\kappa$ does not turn κ into a measurable in M (the ultrapower by U). This will follow from the following general statement. The proof of it is based on [34].

6.6 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is the Magidor iteration of Prikry-type forcing notions such that*

(a) $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many α's.

(b) For every $\alpha < \kappa$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leqslant}_\alpha{}^* \rangle$ is $|\alpha|$-closed, and: for all $p, q, r \in \underset{\sim}{Q}_\alpha$, if $p, q \geq^* r$ there is a $t \in \underset{\sim}{Q}_\alpha$ such that $t \geq^* p, q)$.

(c) The forcing in the interval $[\alpha, (2^\alpha)^+]$ is trivial for stationary many α's.

If κ is measurable in $V^{\mathcal{P}_\kappa}$ then it was measurable in V.

6.7 Remark. We do not know if there is a non-trivial Prikry-type forcing $\langle Q, \leq, \leq^* \rangle$ satisfying the clause 2 for a non-measurable cardinal α, assuming that $\langle Q, \leq^* \rangle$ is not α^+-closed. So, the clause 3 may hold automatically.

Proof. Let G be a generic subset of $\mathcal{P}_\kappa$ and W a κ-complete ultrafilter over κ in $V[G]$. Then, clearly, κ is at least a Mahlo cardinal in V. So, the following set is stationary in V:

$$S = \big\{\alpha < \kappa \mid \mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha, |V_\alpha| = \alpha, \\ \text{the forcing is trivial in the interval } [\alpha, (2^\alpha)^+]\big\}.$$

Suppose for simplicity that $0_{\mathcal{P}_\kappa} = \langle 0_{\underset{\sim}{Q}_\gamma} \mid \gamma < \kappa \rangle \in G$ and it forces that W is a κ-complete ultrafilter over κ in $V[G]$; otherwise, just work above a condition forcing this. Note that in our setting, $0_{\mathcal{P}_\kappa}$ need not be weaker than every other condition in $\mathcal{P}_\kappa$: We may have a $t = \langle \underset{\sim}{t}_\gamma \mid \gamma < \kappa \rangle \in \mathcal{P}_\kappa$ such that for infinitely many γ's $\underset{\sim}{t}_\gamma$ is a non-direct extension of $\underset{\sim}{0}_\gamma$ in $\underset{\sim}{Q}_\gamma$; such a t would be incompatible with $0_{\mathcal{P}_\kappa}$.

Let $\alpha \in S$. Define an ultrafilter U_α over κ in $V[G{\restriction}\alpha]$ as follows:

$$X \in U_\alpha \quad \text{iff} \quad \text{for some } p \in G{\restriction}\alpha \text{ there is a } q \in \mathcal{P}_\kappa \setminus \alpha \text{ with } q \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha \\ \text{so that } p^\frown q \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{X} \in \underset{\sim}{W} \text{ for some name } \underset{\sim}{X} \text{ of } X.$$

Trivially, U_α is well-defined. $\alpha \in S$ implies that U_α is at least a $(2^\alpha)^+$-complete ultrafilter over κ in $V[G{\restriction}\alpha]$ (just use the $\leq^*$-completeness of the forcing $\mathcal{P}_\kappa \setminus \alpha$ to deal with partitions of κ into $\leq (2^\alpha)^+$ many pieces).

We use now the argument of Levy-Solovay [33] to find a condition $t(\alpha) \in \mathcal{P}_\alpha$ with $t(\alpha) \geq 0_{\mathcal{P}_\alpha}$ so that for every set $X \in V$ with $X \subseteq \kappa$, either

$$t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha \quad \text{or} \quad t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \notin \underset{\sim}{U}_\alpha.$$

Thus, suppose that there is no such $t(\alpha)$. Work in V. For each $q \in \mathcal{P}_\alpha$ with $q \geq 0_{\mathcal{P}_\alpha}$, we pick a set $A_q \subseteq \kappa$ such that q does not decide whether $A_q \in \underset{\sim}{U}_\alpha$. Define a function from κ into a set of cardinality at most 2^α as follows:

$$F(\nu) = \langle \langle q, i \rangle \mid q \in \mathcal{P}_\alpha,\ i < 2, \text{ and: } i = 0 \text{ if } \nu \in A_q,\ i = 1 \text{ otherwise} \rangle.$$

Now, in $V[G{\restriction}\alpha]$, U_α is $(2^\alpha)^+$-complete ultrafilter, hence there is an $X \in V \cap U_\alpha$ such that $F(\nu) = F(\mu)$, for any $\nu, \mu \in X$. Pick some $q \in G{\restriction}\alpha$ forcing this. Finally, back in V, there is an $i < 2$ such that for each $\nu \in X$ the pair $\langle q, i \rangle$ appears in $F(\nu)$. Then, $i = 0$ implies $X \subseteq A_q$ and $i = 1$ implies $X \subseteq \kappa \setminus A_q$. But $q \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha$. Hence, either $q \Vdash_{\mathcal{P}_\alpha} A_q \in \underset{\sim}{U}_\alpha$ or $q \Vdash_{\mathcal{P}_\alpha} \kappa \setminus A_q \in \underset{\sim}{U}_\alpha$, which contradicts the choice of A_q.

Set now (in V)

$$U(\alpha) = \{X \subseteq \kappa \mid t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha\}.$$

Clearly, $U(\alpha)$ is a $(2^\alpha)^+$-complete ultrafilter over κ.

We shall find a stationary subset S' of S such that for every $\alpha < \beta \in S'$, $U(\alpha) = U(\beta)$. Then, $\alpha \in S'$ will imply that $U(\alpha)$ is a κ-complete ultrafilter over κ.

Thus, consider the sequence of conditions $\langle t(\alpha) \mid \alpha \in S \rangle$. For each $\alpha \in S$ we have $t(\alpha) \geq 0_{\mathcal{P}_\alpha}$. Hence, by the definition of the order $\leq$, there is a finite set $b(\alpha) \subseteq \alpha$ such that for each $\gamma \in \alpha \setminus b(\alpha)$,

$$t(\alpha){\restriction}\gamma \Vdash_{\mathcal{P}_\gamma} t(\underset{\sim}{\alpha})_\gamma \geq^* \underset{\sim}{0}_\gamma \text{ in the forcing } \underset{\sim}{Q}_\gamma.$$

Now, we shrink S to a stationary set S_1 such that for each $\alpha, \beta \in S_1$, $b(\alpha) = b(\beta)$. Denote $b(\alpha)$ for $\alpha \in S_1$ by b. Let $\delta = \max(b) + 1$. The cardinality of the forcing $\mathcal{P}_\delta$ is less than α, for each $\alpha \in S_1$, since $\alpha = |V_\alpha|$ and $\mathcal{P}_\delta \in V_\alpha$. Hence, there are a stationary $S' \subseteq S_1$ and $t \in \mathcal{P}_\delta$ such that for each $\alpha \in S'$ we have $t(\alpha){\restriction}\delta = t$. It follows that $t(\alpha)$ and $t(\beta)$ are compatible in the order $\leq^*$, for any $\alpha, \beta \in S'$. We claim that $U(\alpha) = U(\beta)$, for each $\alpha, \beta \in S'$.

Recall the definition of $U(\alpha)$. Thus,

$$\begin{aligned} X \in U(\alpha) \quad &\text{iff} \quad t(\alpha) \Vdash_{\mathcal{P}_\alpha} X \in \underset{\sim}{U}_\alpha \\ &\text{iff} \quad \exists q \in \mathcal{P}_\kappa \setminus \alpha \text{ with } q \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha \\ &\qquad\ \text{such that } t(\alpha)^\frown q \Vdash_{\mathcal{P}_\kappa} X \in \underset{\sim}{W}. \end{aligned}$$

Suppose for a moment that there is an $X \in U(\alpha) \setminus U(\beta)$. Find $q_\alpha \in \mathcal{P}_\kappa \setminus \alpha$ with $q_\alpha \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha$ such that $t(\alpha)^\frown q_\alpha \Vdash_{\mathcal{P}_\kappa} X \in \underset{\sim}{W}$ and $q_\beta \in \mathcal{P}_\kappa \setminus \alpha, q_\beta \geq^* 0_{\mathcal{P}_\kappa} \setminus \alpha$ such that $t(\beta)^\frown q_\beta \Vdash_{\mathcal{P}_\kappa} \kappa \setminus X \in \underset{\sim}{W}$. But $t(\alpha)^\frown q_\alpha$ and $t(\beta)^\frown q_\beta$ are $\leq^*$-compatible, which is impossible since they force contradictory assertions. $\dashv$

The next simple observation shows that the conditions (a) and (b) of Lemma 6.4 already imply some strength.

6.8 Lemma. *Let $\langle Q, \leq, \leq^* \rangle$ be a non-trivial Prikry-type forcing notion and κ be an uncountable cardinal such that*

(1) $\langle Q, \leq^* \rangle$ is κ-closed.

(2) For all $p, q, r \in Q$, if $p, q \geq^* r$ there is a $t \in Q$ such that $t \geq^* p, q$.

Then there is a measurable cardinal $\geq \kappa$.

Proof. Let λ be a cardinal which contains a new subset. Fix a name $\underset{\sim}{a}$ of such a subset of λ. We assume that 0_Q already forces this.

Set

$$A = \{\rho < \lambda \mid \exists t \geq^* 0_Q \ t \Vdash_Q \check{\rho} \in \underset{\sim}{a}\}.$$

Then

$$0_Q \Vdash_Q \underset{\sim}{a} \neq \check{A}$$

just since A is old but a is new. Now define U to be the set of all $X \subseteq \lambda$ such that

$$\exists t \geq^* 0_Q \ t \Vdash_Q (\rho \in \check{X} \text{ for the least } \rho \text{ such that } \rho \in \underset{\sim}{a} \Delta \check{A}).$$

Then, clearly, U is a κ-complete ultrafilter over λ. Let us show that it is a non-principal one. Suppose otherwise. Then, for some $\rho < \lambda$ we will have $\{\rho\} \in U$. Hence there is a $t \geq^* 0_Q$ such that $t \Vdash \check{\rho} \in \underset{\sim}{a} \Delta \check{A}$. Extend t to some $s \geq^* t$ such that

$$s \Vdash_Q \check{\rho} \in \underset{\sim}{a} \quad \text{or} \quad s \Vdash_Q \check{\rho} \in \check{A}.$$

The former possibility implies that $\rho \in A$, by the definition of A, which is impossible. If the later possibility occurs, then, again by the definition of A, we will have an $r \geq^* 0_Q$ such that $r \Vdash_Q \check{\rho} \in \underset{\sim}{a}$. But r is compatible with s, so we arrive to a contradiction. Hence, U is non-principal and we are done. $\dashv$

6.9 Example. Let us show how the Magidor iteration may destroy stationarity. Fix a regular cardinal κ, and set $Z = \{\alpha < \kappa \mid \alpha \text{ is a measurable}\}$. Assume that Z is stationary. Change the cofinality of each measurable cardinal below κ to ω using the Magidor iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ of Prikry forcings. By Lemma 6.6, only the members of Z change their cofinality. Let G be a generic subset of $\mathcal{P}_\kappa$ with $0_{\mathcal{P}_\kappa} \in G$. Let C_α denote the Prikry sequence for α deduced from G, where $\alpha \in Z$. Define a function $f : Z \to \kappa$ by setting $f(\alpha) = \min(C_\alpha)$.

6.10 Claim. *There is a finite $b \subseteq \kappa$ such that the elements of the sequence $\langle C_\alpha \mid \alpha \in Z \setminus b \rangle$ are pairwise disjoint. In particular, f is one-to-one on $Z \setminus b$ and, so Z is not stationary in $V[G]$.*

Proof. Work in V. Let $t \in \mathcal{P}_\kappa$ with $t \geq 0_{\mathcal{P}_\kappa}$. Suppose for simplicity that $t \geq^* 0_{\mathcal{P}_\kappa}$; otherwise, we work only with the coordinates where the extension is direct. Let $t = \langle \underset{\sim}{t}_\gamma \mid \gamma < \kappa \rangle$ and for each $\gamma \in Z$ we have $\underset{\sim}{t}_\gamma = \langle \langle \rangle, \underset{\sim}{A}_\gamma \rangle$, where $\underset{\sim}{A}_\gamma$ is a $\mathcal{P}_\gamma$-name of a set in the normal ultrafilter $\underset{\sim}{U}^*_\gamma$ over γ which extends a normal ultrafilter U_γ, as in Lemma 6.4. Note that by Lemma 6.6, the forcing at each $\gamma \in \kappa \setminus Z$ is trivial.

Fix $\gamma \in Z$. Let G_γ be a generic subset of $\mathcal{P}_\gamma$ with $r = t \restriction \gamma \in G_\gamma$. Turn to $V[G_\gamma]$. Let us show that the set

$$B_\gamma = \{\nu \in A_\gamma \mid \forall \delta \in Z \cap \gamma\, (\nu \notin C_\delta)\}$$

must be in U^*_γ. Consider $j(r) \setminus \kappa$ in M, where $j : V \to M$ is the canonical embedding into the ultrapower of V by U_γ. Let $q \geq^* j(r) \setminus \kappa$ be obtained from $j(r) \setminus \kappa$ by replacing each set of measure one $\underset{\sim}{A}_\delta$ (for $\delta \in j(Z) \setminus (\kappa+1)$) of $j(r) \setminus \kappa$ by $\underset{\sim}{A}_\delta \setminus (\kappa+1)$. Then

$$r^\frown q \Vdash_{\mathcal{P}_{j(\kappa)}} [\check{\gamma}] \in j(\underset{\sim}{B}_\gamma).$$

Hence, $B_\gamma \in U^*_\gamma$.

Finally, back in V, we define $t^* \geq^* t$ by replacing each $\underset{\sim}{A}_\gamma$ by $\underset{\sim}{B}_\gamma$. Then t^* will force that C_γ's are pairwise disjoint. ⊣

Suppose now that κ above was measurable and there was a measure U on κ concentrating on measurables. Then $Z \in U$. But in $V[G]$, Z is not stationary any more. Hence U does not extend to a normal ultrafilter.

6.2. Leaning's Forcing

Jeffrey Leaning [32] suggested a new and interesting way to put together Prikry forcings over different cardinals avoiding iteration. Below, we will briefly describe his forcing.

Fix a set Z of measurable cardinals, and set $\kappa = \sup(Z)$. For each $\delta \in Z$ pick a normal ultrafilter U_δ over δ. Set $\vec{U} = \langle U_\delta \mid \delta \in Z\rangle$.

6.11 Definition. Let the *filter of long measure one sets* be

$$\mathfrak{L}(\vec{U}) = \{X \subseteq \kappa \mid X \cap \delta \in U_\delta \text{ for all } \delta \in Z\}.$$

6.12 Definition. Let $\mathbf{D}(\vec{U})$ be the set of all the pairs $\langle s, X\rangle$ such that

(1) $s \in [\kappa]^{<\omega}$.

(2) $X \in \mathfrak{L}(\vec{U})$.

6.13 Definition. Let $\langle s, X\rangle, \langle t, Y\rangle \in \mathbf{D}(\vec{U})$. Then $\langle s, X\rangle \geq \langle t, Y\rangle$ iff

(1) $s \subseteq t$.

(2) $X \subseteq Y$.

(3) $s \setminus t \subseteq Y$.

If $s = t$ then $\langle s, X\rangle \geq^* \langle t, Y\rangle$.

In [32] Leaning showed that $\langle \mathbf{D}(\vec{U}), \leq, \leq^* \rangle$ satisfies the Prikry condition, and so it is a Prikry-type forcing notion. He found a very interesting application of this forcing. Thus, starting from an assumption weaker than $o(\kappa) = 2$, Leaning constructed a forcing extension in which the first measurable cardinal κ may have any number $\lambda \leq \kappa$ normal measures.

Note that if Z does not include its limit points (for example, if there is no measurable which is a limit of measurables), then this forcing is equivalent to the Magidor iteration of Prikry forcings for elements of Z. Crucially for each $\delta \in Z$, the forcing $\mathcal{P}_\delta$ below δ has cardinality less than δ. Hence, it is not hard to replace a name of a set of measure one by an actual set in U_δ; see [33] or just apply the corresponding argument from Lemma 6.6. Also, for each $\delta \in Z$ the set $A_\delta = \delta \setminus \sup(Z \cap \delta)$ is in U_δ and these sets are disjoint. Hence, we can link between finite sequences s and measurable cardinals in Z.

Leaning's forcing is equivalent for a while to a kind of the Magidor "iteration" of Prikry forcings, where instead of names of sets of measure one actual sets of measure one (i.e. those from U_δ's) are used. But once the set Z includes δ such that

$$\text{for all } X \in U_\delta, \text{ there is a } \mu < \kappa \text{ such that } X \cap \mu \in U_\mu,$$

the forcing $\langle \mathbf{D}(\vec{U}), \leq, \leq^* \rangle$ is different. Namely, at this stage the Magidor "iteration" of Prikry forcings without names fails to satisfy the Prikry condition. Thus, for example, there is no direct extension of the condition $\langle \langle \langle \rangle, \gamma \rangle \mid \gamma \in Z \rangle$ which can decide the following statement: "The first element of the Prikry sequence for δ belongs to the Prikry sequence of some $\mu < \delta$".

6.3. Easton Support Iteration

In many applications of iterated forcings it is important to have the κ-c.c. at stage κ of an iteration. The Magidor iteration or full support iteration, as well as the usual full support iterations in different contexts, fail to have this property. The common approach is to replace a full support by an Easton one. In the present section we show how to realize this dealing with iterations of Prikry-type forcing notions. The method was introduced in [13] and simplified in [16]. Shelah [51] found generalizations and applied them to small cardinals.

Let us give one example that illuminates the difference between full and Easton support iteration.

6.14 Example. Suppose that κ is inaccessible and the limit of a set A of measurable cardinals. Assume for simplicity that A does not contain any of its limit points. Either iteration can be used to add a Prikry sequence C_γ for each $\gamma \in A$. In case of the full support iteration this sequence is uniform (below a certain condition) in the sense that if $\langle X_\gamma \mid \gamma \in A \rangle$ is any sequence in V such that X_γ is in a normal ultrafilter U_γ over γ, then $\bigcup_{\gamma \in A} (C_\gamma \setminus X_\gamma)$

is finite. Just the definition of the Magidor iteration and an easy density argument imply this. Thus let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in A\rangle$ be a condition in this iteration. A does not contain its limit points, so we can assume that each $\underset{\sim}{p}_\gamma$ is in V. Then p_γ is just a condition in the Prikry forcing with U_γ. Hence $p_\gamma = \langle t_\gamma, A_\gamma\rangle$, where $t_\gamma \in [\gamma]^{<\omega}$ and $A_\gamma \in U_\gamma$. Suppose now that we force only with extensions of the condition $\{\langle \emptyset, \gamma\rangle \mid \gamma \in A\}$. Then all but finitely many t_γ's are empty. Let $t_{\gamma_1}, \dots, t_{\gamma_n}$ be the only nonempty t_γ's. Extend p to a condition $q = \{\langle t_\gamma, X_\gamma \cap A_\gamma\rangle \mid \gamma \in A\}$. Then $q \Vdash (\bigcup_{\gamma\in A} \underset{\sim}{C}_\gamma \setminus \check{X}_\gamma) \subseteq \bigcup_{i=1}^n t_{\gamma_i}$.

In the case of Easton support iteration this will not be true: for example the set $\{\min(C_\gamma) \mid \gamma \in A\}$ will be essentially an Easton support Cohen subset of κ, and in fact $V[\langle C_\gamma \mid \gamma \in A\rangle]$ will not have uniform sequence of Prikry sequences as in the full support iteration.

Let us now turn to the definition of the Easton iteration of Prikry-type forcing notions.

Let ρ be an ordinal. We define an iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\alpha \mid \alpha < \rho\rangle$ with Easton support. For every $\alpha < \rho$ define by recursion $\mathcal{P}_\alpha$ to be the set of all elements p of form $\langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle$, where

(1) $g \subseteq \alpha$.

(2) g has an Easton support, i.e. for every inaccessible $\beta \le \alpha$, $\beta > |g \cap \beta|$, provided that for every $\gamma < \beta$, $|\mathcal{P}_\gamma| < \beta$.

(3) For every $\gamma \in \mathrm{dom}(g)$,

$$p \restriction \gamma = \langle \underset{\sim}{p}_\beta \mid \beta \in g \cap \gamma\rangle \in \mathcal{P}_\gamma$$

and $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma$ is a condition in the forcing $\langle \underset{\sim}{Q}_\gamma, \underset{\sim}{\le}_\gamma, \underset{\sim}{\le}^*_\gamma\rangle$ of Prikry-type".

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle$ and $q = \langle \underset{\sim}{q}_\gamma \mid \gamma \in f\rangle$ be elements of $\mathcal{P}_\alpha$. Then $p \ge q$ iff

(1) $g \supseteq f$.

(2) For every $\gamma \in f$, $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \ge_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

(3) There exists a finite subset b of f so that for every $\gamma \in f \setminus b$, $p\restriction\gamma \Vdash_{\mathcal{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \ge^*_\gamma \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$".

If the set b in (3) is empty, then we call p a *direct extension* of q, and denote this by $p \ge^* q$.

Notice that in contrast to Definition 6.1, we are allowed to take non-direct extensions in both $\le$ and $\le^*$ orderings for infinitely many coordinates $\gamma < \alpha$ provided that they are outside of the support (i.e. outside of f for extensions of $q = \langle \underset{\sim}{q}_\gamma \mid \gamma \in f\rangle$). Inside the support, as in Definition 6.1, only for finitely many γ's can a non-direct extension be taken.

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g\rangle \in \mathcal{P}_\alpha$ and $\beta < \alpha$. Consider $p\restriction\beta = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g \cap \beta\rangle$. Let $G_\beta \subseteq \mathcal{P}_\beta$ be generic with $p\restriction\beta \in G_\beta$. Then $p \setminus \beta = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g \setminus \beta\rangle \in \mathcal{P} \setminus \beta =$

$\mathcal{P}_\alpha/G_\beta$. Let $t = \langle t_\gamma \mid \gamma \in f\rangle \in \mathcal{P}_\alpha/G_\beta$ be an extension of $p \setminus \beta$. The support f of t need not be in V. But we can always find an $f^* \in V$, $f \subseteq f^* \subseteq \alpha \setminus \beta$ satisfying (2) of the definition of the conditions. Thus let $\underset{\sim}{t}, \underset{\sim}{f}$ be a $\mathcal{P}_\beta$-names of t, f so that

$$p{\restriction}\beta \Vdash \underset{\sim}{t} = \langle \underset{\sim}{t}_\gamma \mid \gamma \in \underset{\sim}{f}\rangle \geq p \setminus \beta.$$

Work in V. Define $f^* \subseteq \alpha$ covering $\underset{\sim}{f}$ and satisfying (2) of the definition of the conditions. The construction of f^* is recursive. Let $f^* \cap \beta = \emptyset$. Suppose that $\beta < \gamma < \alpha$ and $f^* \cap \delta$ is already defined for each $\delta < \gamma$. If γ is a limit ordinal then let $f^* \cap \gamma = \bigcup_{\delta<\gamma} f^* \cap \delta$. If $\gamma = \gamma' + 1$, then we include γ' in f^* only in the case if some extension of $p{\restriction}\beta$ forces (in $\mathcal{P}_\beta$) "$\check{\gamma}' \in \underset{\sim}{f}$". This completes the definition of f^*. It is easy to check that for every $\gamma \lneqq \alpha$,

$$p{\restriction}\beta \Vdash_{\mathcal{P}_\beta} (\check{f}^* \supseteq \underset{\sim}{f} \quad \text{and} \quad |\check{f}^* \cap \check{\gamma}| \leq |\underset{\sim}{f} \cap \check{\gamma}| + |\mathcal{P}_\beta|).$$

Now, if γ with $\beta < \gamma \leq \alpha$ is inaccessible and for every $\delta < \gamma$, $|\mathcal{P}_\delta| < \gamma$, then $|f^* \cap \gamma| < \gamma$, since, back in $V[G_\beta]$ we have $|f \cap \gamma| < \gamma$ and $|\mathcal{P}_\beta| < \gamma$. So γ remains inaccessible and $|f^* \cap \gamma| \leq |f \cap \gamma| + |\mathcal{P}_\beta| < \gamma$. Clearly, in V, $|f^* \cap \gamma| < \gamma$ holds then as well.

Using the observation above, we can establish the Prikry condition for $\langle \mathcal{P}_\alpha, \leq, \leq^*\rangle$ repeating the argument of Lemma 6.2.

6.15 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Then κ is measurable in $V^{\mathcal{P}_\kappa}$, provided:*

(a) κ is measurable in V.

(b) $V \vDash 2^\kappa = \kappa^+$.

(c) For every cardinal $\alpha < \kappa$ we have

(i) $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is $|\alpha|$-closed).

(ii) for every β with $\alpha < \beta < \alpha^+$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is α^+-closed).

(d) For a closed unbounded set of α's below κ,
$\Vdash_{\mathcal{P}_\alpha}$ either

(i) $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha\rangle$ is $|\alpha|^+$-closed, or

(ii) for all $p, q \in \underset{\sim}{Q}^*_\alpha$, if $p, q \underset{\sim}{\geq}^*_\alpha 0_{\underset{\sim}{Q}_\alpha}$ there is a $t \in \underset{\sim}{Q}_\alpha$ such that $t \underset{\sim}{\geq}^*_\alpha p, q$

where $0_{\underset{\sim}{Q}_\alpha}$ is the weakest element of $\underset{\sim}{Q}_\alpha$.

6.16 Remark.

(1) The requirement $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$ for unboundedly many $\alpha < \kappa$ easily implies here that $\mathcal{P}_\alpha \subseteq V_\alpha$, for every inaccessible α in a closed unbounded subset of κ, due to the Easton support of conditions.

(2) If at each $\alpha < \kappa$, $\leq_\alpha = \leq^*_\alpha$, then also $\leq = \leq^*$ for $\mathcal{P}_\kappa$ and the lemma is actually the Kunen-Paris [31] result on preservation of measurability. Also our argument is very close to the Kunen-Paris one.

(3) If κ was a supercompact then as in [4] it is possible to show that κ remains strongly compact. Clearly, the supercompactness may be lost by iterating the Prikry forcing at each measurable below κ.

(4) Even if the alternative (ii) of the conclusion holds for each $\alpha < \kappa$, $\langle \mathcal{P}_\kappa, \leq^* \rangle$ fails to satisfy it, i.e. in $\mathcal{P}_\kappa$ there are lots of incompatible direct extensions of a fixed condition.

Proof. Let U be a κ-complete ultrafilter over κ. Consider its elementary embedding

$$j : V \to M \simeq \mathrm{Ult}(V, U).$$

Then ${}^\kappa M \subseteq M$.

Let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. The set of $\alpha < \kappa$ such that $\mathcal{P}_\alpha \subseteq V_\alpha$ is a member of U. Hence $\mathcal{P}_\kappa \subseteq V_\kappa$, $\mathcal{P}_\kappa = \mathcal{P}_{j(\kappa)} \restriction \kappa$ and for every $p \in \mathcal{P}_\kappa$ we have $j(p) = p$. Using $2^\kappa = \kappa^+$, we chose an enumeration $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^+ \rangle$ of all canonical names of subsets of κ. In M, at κ either $\langle Q_\kappa, \leq^*_\kappa \rangle$ is κ^+-closed, or for every $p, q \in Q_\kappa$, if $p, q \geq^*_\kappa 0_{Q_\kappa}$ then there is a $t \in Q_\kappa$ with $t \geq^*_\kappa p, q$. Suppose first that $\langle Q_\kappa, \leq^*_\kappa \rangle$ is κ^+-closed. Define by recursion a $\leq^*$-increasing sequence $\langle r_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions in $\mathcal{P}_{j(\kappa)} \setminus \kappa$ such that for every $\alpha < \kappa^+$ there is a $p \in G_\kappa$ satisfying

$$p^\frown r_\alpha \parallel \check{\kappa} \in j(\underset{\sim}{A}_\alpha).$$

Let $U^* = \{A_\alpha \mid \alpha < \kappa^+$, for some $p \in G_\kappa$ $p^\frown r_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{A}_\alpha)\}$. It is routine to check that U^* is well-defined and is a normal ultrafilter over κ extending U.

We now turn to the second possibility, i.e. any two $\leq^*_\kappa$-extensions of $0_{\underset{\sim}{Q}_\kappa}$ in Q_κ are $\leq^*_\kappa$-compatible. Define by recursion an $\leq^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+ \rangle$ of conditions in $\mathcal{P}_{j(\kappa)} \setminus (\kappa + 1)$ such that for every $\alpha < \kappa^+$ there are $p \in G_\kappa$ and $\underset{\sim}{t}$ such that $p \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{t} \underset{\sim}{\geq}^*_\kappa 0_{\underset{\sim}{Q}_\kappa}$ and

$$p^\frown \underset{\sim}{t}^\frown \underset{\sim}{r}_\alpha \parallel \kappa \in j(\underset{\sim}{A}_\alpha).$$

Let

$$U^* = \big\{A_\alpha \mid \alpha < \kappa^+, \text{ and for some } p \in G_\kappa \text{ and } \underset{\sim}{t},\; p \Vdash_{\mathcal{P}_\kappa} \underset{\sim}{t} \underset{\sim}{\geq}^*_\kappa 0_{\underset{\sim}{Q}_\kappa} \text{ and } p^\frown \underset{\sim}{t}^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{A})\big\}.$$

Using the compatibility in $\langle Q_\kappa, \leq^* \rangle$ of any two extensions of 0_{Q_κ}, it is routine to check that U^* is well defined and is a κ-complete ultrafilter extending U. Note that U^* need not be normal anymore. $\dashv$

Using a similar idea a bit more general result can be shown.

6.17 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \le \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Let U_1 be a κ-complete ultrafilter over κ and U_0 a normal ultrafilter over κ such that $U_0 \trianglelefteq U_1$ in the Mitchell order (i.e. $U_0 = U_1$ or $U_0 \in \mathrm{Ult}(V, U_1)$). Then U_1 extends to a κ-complete ultrafilter in $V^{\mathcal{P}_\kappa}$ provided:*

(a) $V \vDash 2^\kappa = \kappa^+$.

(b) For every cardinal $\alpha < \kappa$ we have

(i) $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\le}^*_\alpha\rangle$ is $|\alpha|$-closed).

(ii) For every β with $\alpha < \beta < \alpha^+$, $\Vdash_{\mathcal{P}_\alpha} (\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\le}^*_\alpha\rangle$ is α^+-closed).

(c) The set of $\alpha < \kappa$ satisfying the condition below is in U_0: $\Vdash_{\mathcal{P}_\alpha}$ either

(i) $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\le}^*_\alpha\rangle$ is $|\alpha|^+$-closed, or

(ii) for all $p, q \in \underset{\sim}{Q}^*_\alpha$, if $p, q \underset{\sim}{\ge}^*_\alpha 0_{\underset{\sim}{Q}_\alpha}$ there is a $t \in \underset{\sim}{Q}_\alpha$ with $t \underset{\sim}{\ge}^*_\alpha p, q$.

Proof. If $U_1 = U_0$, then this was proved in Lemma 6.15. Suppose then that $U_0 \in \mathrm{Ult}(V, U_1)$. Let $M_1 = \mathrm{Ult}(V, U_1)$ and $j_1 : V \to M_1$ be the corresponding elementary embedding. Consider $M = \mathrm{Ult}(M_1, U_0)$ and $j_{10} : M_1 \to M$ the corresponding elementary embedding. Set $j = j_{10} \circ j_1$. Clearly, $j : V \to M$ is an elementary embedding, $U_0 = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ and $U_1 = \{X \subseteq \kappa \mid j_{10}([\mathrm{id}]_{U_1}) \in j(X)\}$. We use j, M as in the proof of Lemma 6.15 to define a $\le^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$, but now deciding statements "$j_{10}([\mathrm{id}]_{U_1}) \in j(\underset{\sim}{A}_\alpha)$" and not "$\check\kappa \in j(\underset{\sim}{A}_\alpha)$" as in that lemma. The κ-complete ultrafilter defined using this sequence will then be as desired. ⊣

The above lemma turned out to be useful for iterations of extender-based Prikry and Radin forcings for which the $\le^*_\alpha$-compatibility condition (i.e. the alternative (ii) of the conclusion of the lemma) fails.

The next lemma is a basic tool our Easton support iteration and has the same proof as that for the usual Easton support iteration. See Baumgartner [5], Jech [25] or Shelah [54] for the proof.

6.18 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \le \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. If κ is a Mahlo cardinal, then $\mathcal{P}_\kappa$ satisfies the κ-c.c.*

Let us show now an analog of Lemma 6.6 that Easton iterations of Prikry-type forcing notions do not create new measurable cardinals. The proof is based on an argument of Kimchi and Magidor [28]; see also Apter [3].

6.19 Lemma. *Suppose that $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \le \kappa, \beta < \kappa\rangle$ is an Easton iteration of Prikry-type forcing notions such that for unboundedly many α's $\mathcal{P}_\alpha \subseteq {}^\alpha V_\alpha$. Let G be a generic subset of $\mathcal{P}_\kappa$. If κ is a measurable cardinal in $V[G]$, then it was measurable already in V.*

Proof. Let W be a normal ultrafilter over κ in $V[G]$. Fix a $p \in G$ such that $p \Vdash_{\mathcal{P}_\kappa} (\underset{\sim}{W}$ is a normal ultrafilter over $\kappa)$. Work in V. Clearly, κ is a Mahlo cardinal. It is enough to find a $q \geq p$ such that for every $X \subseteq \kappa$, q decides the statement "$\check{X} \in \underset{\sim}{W}$".

Suppose that there is no such q. We build a binary κ-tree T of height κ. Star with $\langle p, \kappa \rangle$.

Successor Levels

Let the pair $\langle r, A \rangle$ be on level α of T. We assume that $r \geq p$, $A \subseteq \kappa$ and $r \Vdash_{\mathcal{P}_\kappa} (\check{A} \in \underset{\sim}{W})$. Pick some partition A_0, A_1 of A and incompatible extensions r_0, r_1 of r such that $r_0 \Vdash_{\mathcal{P}_\kappa} (\check{A}_0 \in \underset{\sim}{W})$ and $r_1 \Vdash_{\mathcal{P}_\kappa} (\check{A}_1 \in \underset{\sim}{W})$. Place both $\langle r_0, A_0 \rangle$ and $\langle r_1, A_1 \rangle$ in T at the level $\alpha + 1$ to be the successors of $\langle r, A \rangle$.

Limit Levels

Let $\alpha < \kappa$ be a limit ordinal. For each branch in T of height α, we take the intersection of all second coordinates of elements along the branch. We thus obtain a partition of κ into at most 2^α many sets. But κ is Mahlo, hence $2^\alpha < \kappa$. Also,

$$p \Vdash_{\mathcal{P}_\kappa} (\underset{\sim}{W} \text{ is a normal ultrafilter over } \kappa).$$

Hence, there are an element A of this partition and $r \geq p$ such that

$$r \Vdash_{\mathcal{P}_\kappa} \check{A} \in \underset{\sim}{W}.$$

For all such A, we place a pair of form $\langle r, A \rangle$ into T at level α as the successor of each element of the branch generating A.

This completes the construction of T.

Turn now to $V[G]$. κ is measurable and so weakly compact. Hence T must have a κ-branch. Let $\langle \langle r_\alpha, A_\alpha \rangle \mid \alpha < \kappa \rangle$ be such a branch. For each $\alpha < \kappa$ set $B_\alpha = A_\alpha \setminus A_{\alpha+1}$. By the construction of T, then there is an s_α such that $\langle s_\alpha, B_\alpha \rangle$ is an immediate successor of $\langle r_\alpha, A_\alpha \rangle$. In addition, $s_\alpha \geq r_\alpha$ and the conditions $r_{\alpha+1}$, s_α are incompatible. Also, for each $\beta > \alpha$, we have $A_\beta \subseteq A_{\alpha+1}$. So, $A_\beta \cap B_\alpha = \emptyset$. But

$$r_\beta \Vdash_{\mathcal{P}_\kappa} \check{A}_\beta \in \underset{\sim}{W} \quad \text{and} \quad s_\alpha \Vdash_{\mathcal{P}_\kappa} \check{B}_\alpha \in \underset{\sim}{W},$$

hence r_β and s_α are incompatible. This implies that s_β and s_α are incompatible as well, since $s_\beta \geq r_\beta$.

Hence, $\langle s_\alpha \mid \alpha < \kappa \rangle$ forms an antichain of size κ in $V[G]$. But this is impossible, since we can run the usual Δ-system argument for the Easton support iteration $(\mathcal{P}_\kappa)^V$ inside $V[G]$ and this will give the κ-c.c. Contradiction. $\dashv$

Let us conclude with two applications. The first one will be a construction of a κ^+-saturated ideal over an inaccessible κ concentrating on cardinals of cofinality $\aleph_0$. Such an ideal was first constructed by Woodin starting from a supercompact and using a beautiful construction involving passing to a model without AC and then restoring the choice by forcing. Mitchell in [44] gave another construction from the optimal assumptions. The construction below follows the lines of [13]. Let $U_0 \triangleleft U_1$ be normal ultrafilters over κ (i.e. $U_0 \in \mathrm{Ult}(V, U_1)$). Suppose GCH for simplicity. Fix a sequence of normal ultrafilters $\langle U(\beta) \mid \beta < \kappa\rangle$ representing U_0 in the ultrapower by U_1. Pick some $A \subseteq \kappa$, $A \in U_1 \setminus U_0$ such that for every $\beta \in A$, $A \cap \beta \notin U(\beta)$. We define $\langle \mathcal{P}_\kappa, \leq, \leq^*\rangle$ by taking the Easton iteration of Prikry forcings with $U(\beta)$ (or more precisely with the extension of $U(\beta)$ defined in Remark 6.7) for every $\beta \in A$. Let $j : V \longrightarrow M_1 \simeq \mathrm{Ult}(V, U_1)$ and let $G_\kappa \subseteq \mathcal{P}_\kappa$ be generic. Fix an enumeration $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^+\rangle$ of all canonical names of subsets of κ. As in Lemma 6.15, we define a $\leq^*$-increasing sequence $\langle \underset{\sim}{r}_\alpha \mid \alpha < \kappa^+\rangle$ of elements of $\mathcal{P}_{j(\kappa)} \setminus \kappa+1$ such that for every $\alpha < \kappa^+$ there are $p \in G_\kappa$ and $t \in Q_\kappa$ with

$$p^\frown t^\frown \underset{\sim}{r}_\alpha \parallel \check{\kappa} \in j(\underset{\sim}{A}_\alpha).$$

Define

$$F_1 = \big\{B \subseteq \kappa \mid \text{there are } p \in G_\kappa \text{ and } \alpha < \kappa^+ \\ \text{such that } p^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j(\underset{\sim}{B})\big\}.$$

It is not hard to see that F_1 is a well-defined normal filter over κ extending U_1.

Let us establish the normality. Suppose that $\langle B_\beta \mid \beta < \kappa\rangle$ is a sequence of elements of F_1. We need to show that $B = \Delta\{B_\beta \mid \beta < \kappa\} \in F_1$. By the definition of F_1, for each $i < \kappa$ there are $p_\beta \in G_\kappa$ and $\alpha_\beta < \kappa^+$ such that

$$p_\beta{}^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown \underset{\sim}{r}_{\alpha_\beta} \Vdash \check{\kappa} \in j(\underset{\sim}{B}_\beta).$$

Let $\alpha \geq \bigcup_{\beta<\kappa} \alpha_\beta$. We would like to show that for some $p \in G_\kappa$,

$$p^\frown 0_{\underset{\sim}{Q}_\kappa}{}^\frown \underset{\sim}{r}_\alpha \ \Vdash \ \check{\kappa} \in j(\underset{\sim}{B}).$$

Suppose otherwise. Then for some $p \in G_\kappa$, $t \in Q_\kappa$ and $\underset{\sim}{r} \geq \underset{\sim}{r}_\alpha$,

$$p^\frown t^\frown \underset{\sim}{r} \ \Vdash \ \check{\kappa} \notin j(\underset{\sim}{B}).$$

Then, by the definition of the diagonal intersection, there would be $\beta < \kappa$, $p' \in G$, $t' \in Q_\kappa$, and $\underset{\sim}{r}'$ such that

$$p^\frown t^\frown \underset{\sim}{r} \leqslant p'^\frown t'^\frown \underset{\sim}{r}' \ \Vdash \ \check{\kappa} \notin j(\underset{\sim}{B}_\beta).$$

But this is impossible, since $p_\beta{}^\frown 0_{Q_\kappa}{}^\frown \underset{\sim}{r}_{\alpha_\beta} \Vdash \check{\kappa} \in j(\underset{\sim}{B}_\beta)$, there is a $q \in G_\kappa$ which is stronger than both p and p_β, and so $q^\frown t^\frown \underset{\sim}{r} \geq p_\beta{}^\frown 0_{Q_\kappa}{}^\frown 0_{Q_\kappa}{}^\frown \underset{\sim}{r}_{\alpha_\beta}$. Contradiction.

Forcing with F_1-positive sets is equivalent to forcing with $\langle Q_\kappa, \leq_\kappa\rangle$. The last forcing is just Prikry forcing with an extension of U_0. Hence it satisfies the κ^+-c.c. Clearly, F_1 concentrates on cardinals of cofinality $\aleph_0$, since each member of A is such a cardinal in $V[G_\kappa]$. Reference [13] contains generalizations of the above construction for cofinalities different from $\aleph_0$ and to the nonstationary ideal. Thus it was shown there that $NS_\kappa \restriction S$ can be κ^+-saturated for a stationary set $S \subseteq \kappa$ so that for every regular cardinal $\delta < \kappa$ $S \cap \{\beta < \kappa \mid \mathrm{cf}(\beta) = \delta\}$ is stationary.

If we define a function $f : A \to \kappa$ by $f(\alpha) = \min(C_\alpha)$, where C_α is the Prikry sequence for α, then for every $\gamma < \kappa$ the set $\{\alpha < \kappa \mid f(\alpha) = \gamma\}$ will be F_1-positive. This is in contrast to a similar construction in Sect. 6.2. There, f is one-to-one. Below, we will see that this f may be a projection function from a non-normal extension of U_1 to a normal extension of U_0.

Let us now turn to the second application. Consider $U_0 \triangleleft U_1$ as above. Perform the same iteration. Let $j_1 : V \longrightarrow M_1 \simeq \mathrm{Ult}(V, U_1)$. In $M_1[G_\kappa]$, at stage κ we are supposed to use the Prikry forcing with a normal ultrafilter U_0^* extending U_0. Clearly, U_0^* is such also in $V[G_\kappa]$. Obviously, any two direct extensions of the weakest condition in Prikry forcing are compatible. Hence, by Lemma 6.6 or Lemma 6.8, there is a κ-complete ultrafilter U_1^* extending U_1. We pick U_1^* as it was defined in Lemma 6.17 using the embedding j_1.

6.20 Lemma. $U_1^* >_{\mathrm{RK}} U_0^*$.

Proof. Define the projection map $f : A \longrightarrow \kappa$ as follows: $f(\alpha) =$ the first element of the Prikry sequence of α, where $A \in U_1 \setminus U_0$ is as in the first application. In order to show that this f projects U_1^* onto U_0^*, it is enough to prove that for every $B \in U_0^*$ and $C \in U_1^*$

$$f^{-1}(B) \cap C \neq \emptyset.$$

So let $C \in U_1^*$ and $B \in U_0^*$. By the definition of U_1^* there are $p \in G_\kappa$, $t = \langle \emptyset, D \rangle \in Q_\kappa$ and $\alpha < \kappa^+$ so that $p^\frown t^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j_1(\underset{\sim}{C})$. Then also, $p^\frown \langle \emptyset, D \cap B \rangle^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j_1(\underset{\sim}{C})$ and in addition $\langle \emptyset, D \cap B \rangle \Vdash_{Q_\kappa}$ (the first element of the Prikry sequence of κ is in B). Hence,

$$p^\frown \langle \emptyset, D \cap B \rangle^\frown \underset{\sim}{r}_\alpha \Vdash \check{\kappa} \in j_1(\underset{\sim}{C}) \cap j_1(\underset{\sim}{f}^{-1})(j_1(\underset{\sim}{B})) = j_1(\underset{\sim}{C} \cap \underset{\sim}{f}^{-1}(\underset{\sim}{B})).$$

So, $C \cap f^{-1}(B) \in U_1^*$. In particular, $C \cap f^{-1}(B) \neq \emptyset$.

Notice now that U_1^* cannot be isomorphic to U_0^* or in other words, f cannot be one-to-one on a set in U_1^*. Thus, by the κ-c.c. every closed unbounded subset of κ in $V[G_\kappa]$ contains a closed unbounded subset of κ which is in V. U_1 was normal in V, hence U_1^* containing U_1 contains as well all closed unbounded subsets of κ. Clearly, f is regressive. So, if it is one-to-one on a set $E \in U_1^*$ then E is nonstationary which is impossible. Hence $U_1^* >_{\mathrm{RK}} U_0^*$ and we are done. $\dashv$

The construction above turns the Mitchell order into the Rudin Keisler order for two ultrafilters. Longer sequences were dealt in [13], and the consistency correlation between these orderings was studied in [14]. In [15], the construction above was extended further in order to turn a Mitchell increasing sequence of length κ^{++} into a Rudin-Keisler increasing sequence of the same length. Such a sequence (with minor changes) can be used in the extender-based Prikry forcing of Sect. 3 for changing the cofinality of κ to $\aleph_0$ blowing simultaneously its power to κ^{++}. This way, the consistency strength of the negation of the Singular Cardinal Hypothesis is reduced to the optimal value $o(\kappa) = \kappa^{++}$, i.e. a measurable cardinal of Mitchell order κ^{++}.

6.4. An Application to Distributive Forcing Notions

We would like to apply the iteration techniques of Sects. 6.1 and 6.2 to distributive forcing notions.

Let $\langle Q, \leq\rangle$ be (κ, ∞)-distributive, i.e. it does not add new sequences of ordinals of length less than κ or, equivalently, the intersection of any less than κ dense open subsets of Q is dense open. If κ is $2^{|Q|}$-supercompact (or $2^{|Q|}$-strongly compact) then it is possible to turn Q into a Prikry-type forcing $\langle Q, \leq, \leq^*\rangle$ with $\langle Q, \leq^*\rangle$ κ-closed.

Recall that a map $\pi : \mathcal{P}_1 \to \mathcal{P}_2$ between forcing notions is called a *projection* if

(a) $q \leq r$ implies $\pi(q) \leq \pi(r)$.

(b) $\pi(0_{\mathcal{P}_1}) = 0_{\mathcal{P}_2}$.

(c) If $p \geq \pi(q)$, then there is an $r \geq q$ with $\pi(r) \geq p$.

If $G_1 \subseteq \mathcal{P}_1$ is generic then $\pi``G_1$ generates a generic subset of $\mathcal{P}_2$. We say that in this case $\mathcal{P}_2$ is a subforcing of $\mathcal{P}_1$.

6.21 Lemma. *Assume that $\langle Q, \leq\rangle$ is a (κ, ∞)-distributive forcing notion where κ is $2^{|Q|}$-supercompact. Let $\langle \mathcal{P}, \leq, \leq^*\rangle$ be the supercompact Prikry forcing with a normal ultrafilter over $\mathcal{P}_\kappa(2^{|Q|})$. Then $\langle Q, \leq\rangle$ is a subforcing of $\langle \mathcal{P}, \leq\rangle$.*

Proof. Let $\lambda = 2^{|Q|}$. Fix $\langle D_\alpha \mid \alpha < \lambda\rangle$ a list of all dense open subsets of Q. Let G be a generic subset of $\mathcal{P}$ and $\langle P_n \mid n < \omega\rangle$ its Prikry sequence. Then, by Lemma 1.50, $\lambda = \bigcup_{n<\omega} P_n$. Each $P_n \in V$ and has cardinality less than κ. Hence, by distributivity, $D(n) = \bigcap\{D_\alpha | \alpha \in P_n\} \in V$ is dense open subset of Q. Also, $D(n+1) \subseteq D(n)$, since $P_{n+1} \supseteq P_n$. Now, we pick an increasing sequence $\langle q_n \mid n < \omega\rangle$ with $q_n \in D(n)$. It will generate a generic subset of Q. ⊣

Let $\pi : \mathcal{P} \to Q$ be a projection map, which exists by the previous lemma. Define now a forcing ordering (quasiorder) $\leq_Q$ over $\mathcal{P}$:

$$p \leq_Q r \quad \text{iff} \quad \pi(p) \leq \pi(r).$$

Then $\langle \mathcal{P}, \leq_Q\rangle$ is a forcing equivalent to $\langle Q, \leq\rangle$.

6.22 Lemma. *$\langle \mathcal{P}, \leq_Q, \leq^* \rangle$ is a Prikry-type forcing notion.*

Proof. Clearly, $\leq_Q \supseteq \leq \supseteq \leq^*$. So we need to check that for every $p \in \mathcal{P}$ and a statement σ of the forcing $\langle \mathcal{P}, \leq_Q \rangle$ there is a $p^* \geq^* p$ deciding σ in $\langle \mathcal{P}, \leq_Q \rangle$. Set

$$A_0 = \{q \in \mathcal{P} \mid q \geq_Q p \text{ and } q \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \sigma\}, \quad \text{and}$$
$$A_1 = \{q \in \mathcal{P} \mid q \geq_Q p \text{ and } q \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \neg\sigma\}.$$

Note that any $q_0 \in A_0$ and $q_1 \in A_1$ are incompatible in $\langle \mathcal{P}, \leq \rangle$, since $\leq \subseteq \leq_Q$. Also, each $r \in \mathcal{P}$ has a $\leq_Q$-extension in A_0 or in A_1. Thus, it must have a $\leq$-extension in one of these sets. Let, for example, $r \leq_Q s \in A_0$. So, $\pi(r) \leq \pi(s)$ and by (3) of the definition of projection there is an $r' \geq r$ such that $\pi(r') \geq \pi(s)$. Hence, $r' \geq_Q s \in A_0$ and so $r' \in A_0$. The above means that $A_0 \cup A_1$ is dense $\langle \mathcal{P}, \leq \rangle$. The Prikry condition for $\langle \mathcal{P}, \leq, \leq^* \rangle$ implies then that there is a $p^* \geq^* p$ forcing in $\langle \mathcal{P}, \leq \rangle$ "$\underset{\sim}{G} \cap A_i \neq \emptyset$" for some $i \in 2$, where $\underset{\sim}{G}$ is the canonical name for a $\langle \mathcal{P}, \leq \rangle$-generic set. Without loss of generality suppose that $i = 0$. Then, $p^* \Vdash_{\langle \mathcal{P}, \leq_Q \rangle} \sigma$. Otherwise, there will be a $q \in A_1$ such that $q \geq_Q p^*$. But, then, using the property (3) of the projection, there will be a $q' \geq p^*$ such that $q' \geq_Q q$. Hence $q' \in A_1$ which means $q' \Vdash_{\langle \mathcal{P}, \leq \rangle} \underset{\sim}{G} \cap A_1 \neq \emptyset$. This contradicts the choice of p^*. $\dashv$

Let us conclude with an example of iterating distributive forcing notions. We refer to [16, 13, 46] and [29] for more sophisticated applications.

A subset E of a regular $\kappa > \aleph_0$ is called *fat* if for every $\delta < \kappa$ and every closed unbounded subset C of κ there is a closed subset $s \subseteq E \cap C$ of order type δ. It is not hard to obtain a fat subset with fat complement. For example, just force a Cohen subset to κ. It will be as desired. Suppose now that $E \subseteq \kappa$ is fat. Consider the usual forcing for adding a club to E: $P[E] = \{d \mid d \text{ is a closed bounded subset of } E\}$ ordered by the end extension, i.e. $d_1 \geq d_2$ iff $d_1 \cap \max(d_2) + 1 = d_2$. By Abraham and Shelah [2], or just directly, the forcing $\langle P[E], \leq \rangle$ is (κ, ∞)-distributive.

Suppose now that for every $n < \omega$, κ_n is a κ_n^+-supercompact cardinal, $2^{\kappa_n} = \kappa_n^+$ and E_n is a fat subset of κ_n. We would like to produce a cardinal preserving extension in which every E_n will contain a club.

By Lemma 6.22, for every $n < \omega$ there is a Prikry-type forcing $\langle Q_n, \leq_n, \leq_n^* \rangle$ such that $\langle Q_n, \leq_n \rangle$ is equivalent to $\langle P[E_n], \leq \rangle$ and $\langle Q_n, \leq_n^* \rangle$ is κ_n-closed. Let $\langle \mathcal{P}_n, \underset{\sim}{Q}_n \mid n < \omega \rangle$, $\langle \mathcal{P}_\omega, \leq, \leq^* \rangle$ be the Magidor iteration (the Easton iteration is just the same in case of ω stages) of $\langle Q_n, \leq_n, \leq_n^* \rangle$'s. It certainly will add clubs to each E_n. We need to show only that cardinals are preserved. Let $m < \omega$. We use an obvious splitting $\mathcal{P}_\omega = \mathcal{P}_{\leq m} * \mathcal{P}_{>m}$ of $\mathcal{P}_\omega$ into the part of the iteration up to m and those above m. Then, $\langle \mathcal{P}_{>m}, \leq^* \rangle$ will be κ_{m+1}-closed. So the Prikry condition will imply that it does not add new bounded subsets to κ_{m+1}. $\mathcal{P}_{\leq m}$ is a finite iteration $P[E_0] * P[E_1] * \cdots * P[E_m]$. For every $k \leq m$, $|\mathcal{P}_{\leq k}| = \kappa_k$. So each E_{k+1} remains fat in $V^{\mathcal{P}_{\leq k}}$. Hence, $\mathcal{P}_{\leq m}$ preserves all the cardinals.

7. Some Open Problems

We conclude this chapter with several open problems on cardinal arithmetic. Some of them are well known; others are less so, but seem to us important for the further understanding of the power function.

The first and probably the most well known:

1 Problem. Suppose that $\aleph_\omega$ is strong limit or even $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$. Is it possible to have $2^{\aleph_\omega} > \aleph_{\omega_1}$?

By Shelah [53], an upper bound is $\min(\aleph_{\omega_4}, \aleph_{(2^{\aleph_0})^+})$. It is shown in [21] that "$2^{\aleph_\omega} > \aleph_{\omega_1}$" implies an inner model with overlapping extenders. Recently this was improved in [23] to Projective Determinacy.

The next problem is probably a bit less well known, but according to Shelah it is the crucial for cardinal arithmetic.

2 Problem. Let a be a set of regular cardinals with $|a| < \min(a)$. Can $|\mathrm{pcf}(a)| > |a|$?

Recall that $\mathrm{pcf}(a) = \{\mathrm{cf}(\prod a/D) \mid D \text{ an ultrafilter over } a\}$. By the basics of pcf theory, $|\mathrm{pcf}(a)\ | \leq 2^{|a|}$ (see [53], [6] or [1]). It is unknown even for countable a's whether "$|\mathrm{pcf}(a)| > |a|$" implies an inner model with a strong cardinal. But in [18], it was shown that if for a set a of regular cardinals above $2^{|a|^+ + \aleph_2}$ we have $|\mathrm{pcf}(a)| > |a| + \aleph_1$, then there is an inner model with a strong cardinal.

Recall that $\mathrm{pp}(\kappa) = \sup\{\mathrm{cf}(\prod a/D) \mid a \subseteq \kappa$ is a set of at most $\mathrm{cf}(\kappa)$ of regular cardinals, unbounded in κ and D an ultrafilter over a including all cobounded subsets of $a\}$. The next problem was proposed by Shelah in [52] and deals with the following strengthening of "$|\mathrm{pcf}(a)| = |a|$" called the Shelah Weak Hypothesis:

For every cardinal λ the number of singular cardinals $\kappa < \lambda$ with $\mathrm{pp}(\kappa) \geq \lambda$ is at most countable.

Also, for uncountable cofinality an even stronger statement is claimed:

For every cardinal λ the number of singular cardinals $\kappa < \lambda$ of uncountable cofinality with $\mathrm{pp}(\kappa) \geq \lambda$ is finite.

3 Problem. Is the negation of the Shelah Weak Hypothesis consistent?

In [22] was shown that much more complicated forcing notions than those of Sects. 2 and 3 seem to be needed in order to deal with the negation of the Weak Hypothesis.

The general formulation of the Singular Cardinals Problem (SCP) is as follows: Find a complete set of rules describing the behavior of the power (or more generally, the pseudo-power (pp)) function on singular cardinals. In terms of core models (see the inner model chapters of this Handbook) we can reformulate SCP in a more concrete form: Given a core model K with certain large cardinals, which functions in K can be realized as the power set function in a set generic extension of K, i.e. if $F : \lambda \to \lambda \in K$ for some ordinal λ, is there a generic extension of K satisfying $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ for all $\alpha < \lambda$?

If we restrict ourselves to finite gaps between singular cardinals and its power then, at present, the most general results on possible behavior of the power function are due Merimovich [40]. They extend previous results by Foreman-Woodin [12], Woodin, Cummings [9] and Segal [49]. However lots of possibilities are still open. Let us state a few of the simplest:

4 Problem. Is it possible to have $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$ and two stationary sets $S_1, S_2 \subseteq \omega_1$ with $S_1 \cup S_2 = \omega_1$ such that

$$\begin{aligned} \alpha \in S_1 \quad &\text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+2} \quad \text{and} \\ \alpha \in S_2 \quad &\text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+3}? \end{aligned}$$

5 Problem. Is it possible to have two stationary sets $S_1, S_2 \subseteq \omega_2$ with $S_1 \cup S_2 = \omega_2$ and $S_2 \cap \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\}$ stationary such that

$$\begin{aligned} \alpha \in S_1 \quad &\text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{and} \\ \alpha \in S_2 \quad &\text{implies} \quad 2^{\aleph_\alpha} = \aleph_{\alpha+2}? \end{aligned}$$

The usual approach via Magidor or Radin forcing produces a club set of αs with the same cardinal behavior, and here we would like to have different behaviors on relatively big sets. The first of these two problems may be the easier one, since we need only GCH on S_1 and, so starting with the GCH in the ground model nothing special should be done on S_1. Note also that in view of Silver's Theorem (see [25, Sect. 1.8]) we must have $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$ in models of Problem 5 and $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1+2}$ in those of Problem 4. Methods of [21] can be used to show that at least a strong cardinal is needed for constructing a model of Problem 4. By [23], the strength of at least Projective Determinacy is needed for constructing a model of Problem 5.

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17. Beginning Inner Model Theory

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Contents

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This chapter provides an introduction to the basic theory of inner models without fine structure. It assumes that the reader is familiar with Gödel's class L of constructible sets; however Sect. 1 begins by recalling the definition and basic theory of L with an emphasis on the condensation property. This discussion leads to a consideration of relative constructibility—that is, models of the form $L[A]$—and then to $L[U]$, the minimal model with a measurable cardinal. A discussion of $0^{\#}$, and of sharps in general, leads up to a brief description of the Dodd-Jensen core model K^{DJ}, which links the models L and $L[U]$.

Sections 2 and 3 discuss generalizations of the ideas of Sect. 1 to larger cardinals. Section 2 looks at sequences of measurable cardinals and the models $L[\mathcal{U}]$ constructed from such sequences. The use of iterated ultrapowers to compare pairs of models, introduced in Sect. 1 for the model $L[U]$, is extended to these models $L[\mathcal{U}]$. Section 3 introduces the notion of an *extender*, a generalized form of ultrafilter used to express cardinal properties stronger than measurability. Extenders are combined with the ideas of Sect. 2 to obtain models $L[\mathcal{E}]$, constructed from a sequence $\mathcal{E}$ of extenders, which can contain cardinals up to a strong cardinal.

The definition of models for still stronger cardinals requires an understanding of iteration trees and fine structure, which are not covered in this chapter. Section 4 gives a brief survey of such larger cardinals, and the current status of their inner model theory.

The principal goal of research in inner models is to define a *core model* K which can coexist with larger cardinals in the universe V. The construction of the core model is not described in this chapter except for a brief description of K^{DJ} (which is the core model if there is no model with a measurable cardinal) in Sect. 1.2. Because of its centrality, however, the core model itself is mentioned frequently. Briefly, the *core model* K should have two properties: (1) it is like L, and (2) it is close to V. The first property is satisfied by defining it as one of the models $L[\mathcal{U}]$ or $L[\mathcal{E}]$ described in this chapter. For the second property we can ask for some form of a *covering lemma*. In the case when L is the core model—that is, when the only large cardinal properties which hold anywhere are those which hold in L—the second criteria is satisfied by Jensen's covering lemma, which states that every uncountable set x of ordinals in V is contained in a set $y \in L$ of the same cardinality. This also holds of K^{DJ} when it is the core model—that is, when there is no model with a measurable cardinal—but for larger cardinals the core model K can only be expected to satisfy some form of the *weak covering lemma*: that $(\lambda^{+})^{K} = \lambda^{+}$ for every singular cardinal λ.

In the final Sect. 5 there is a further discussion of the core model, but from a somewhat different perspective. This is not an attempt to describe the construction of an existing model, but instead is an attempt to answer the questions "how do we decide that a particular model is 'the core model' " and "how will we recognize a model, newly discovered in the future, as the core model". This attempt is, of course, highly speculative: new discoveries may show that models with the properties we are expecting are impossible or even uninteresting, or a newly discovered model with properties substantially different from what we expect may play a critical role with respect to larger cardinals, which demands that it be recognized as the core model.

Most of the topics related to inner model theory which are not covered in this chapter can be found elsewhere in this Handbook. The core model and covering lemma are introduced in the chapters [24, 33, 38]. An excellent source for further information on large cardinals is Kanamori's book [15, 16]. For more information on L, the standard reference is [3]. The more recent book [42] is an excellent introduction to inner models and core model techniques. In this Handbook, fine structure is covered in the chapters [36, 40].

One other approach to inner models which is not covered in this chapter is the class HOD of hereditarily definable sets and its variants. The model HOD has the serious disadvantage that it is not canonical—for example, it can easily be changed by forcing. However it is frequently used for models in which the axiom of choice fails, where it usually gives more readable proofs than do symmetric models, and has been used in studies of determinacy and of cardinals large enough that the inner models described in this chapter are unknown or poorly understood.

The major goal of this introduction is to establish notation and a certain amount of background for other topics in this Handbook. Where sketches of proofs are given, the intention is not so much to present the proof itself as to introduce techniques which are important to the further development of the theory of inner models and core models.

1. The Constructible Sets

1.1 Definition. Gödel's class L of constructible sets is defined to be $L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha$, where the sets L_α are defined by recursion on α as follows:

1. $L_0 = \varnothing$,
2. $L_{\alpha+1} = \operatorname{def}(L_\alpha, \in)$,
3. $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ if λ is a limit ordinal.

Here $\operatorname{def}(L_\alpha, \in)$ is the set of subsets of L_α which are first-order definable in the structure $(L_\alpha, \in)$, using parameters from L_α.

The most basic property of L is the following:

1.2 Lemma. *There is a Π_2 sentence of set theory, which we denote by "$V = L$", such that the transitive models of the sentence "$V = L$" are exactly the sets L_α and the class L itself. Furthermore, if α is any ordinal then $\langle L_\nu : \nu < \alpha\rangle$ is definable in L_α by a Σ_1 formula.*

The main content of the sentence "$V = L$" is the statement $\forall x \exists \alpha \exists y$ $(y = L_\alpha \ \& \ x \in L_\alpha)$. See Jech [14, Lemma 13.17] for a proof of Lemma 1.2 in the case that α is a limit ordinal, which is sufficient for most uses which do not involve fine structure. The use of fine structure goes beyond Lemma 1.2 by splitting the successor interval between L_α and $L_{\alpha+1}$ into infinitely many levels of definability.

The most important property of the constructible hierarchy follows from Lemma 1.2:

1.3 Lemma (Condensation Lemma).

1. *If $X \prec_1 L_\alpha$ for some ordinal α, then there is an ordinal $\alpha' \leq \alpha$ such that $X \cong L_{\alpha'}$.*

2. *If X is a proper class such that $X \prec_1 L$, then $X \cong L$.*

That is, if X is a Σ_1 elementary substructure of L or of any L_α, then X is isomorphic, via its transitive collapse, to L or some $L_{\alpha'}$.

The simplest application of Lemma 1.3 is Gödel's proof that GCH holds in L.

1.4 Definition. If $\mathcal{M}$ is any structure and X is a subset of the universe of $\mathcal{M}$ then the *Skolem hull* of X in $\mathcal{M}$ is the smallest elementary submodel of $\mathcal{M}$ containing X. We write $\mathcal{H}^{\mathcal{M}}(X)$ for the Skolem hull of X in $\mathcal{M}$.

This definition assumes the existence of such a unique minimal submodel of $\mathcal{M}$ containing X. In all of our applications the model $\mathcal{M}$ will have a definable well-ordering which provides Skolem functions that ensure this. Definition 1.4 can also be used in cases when $\mathcal{M}$ is a well-founded class model of ZF. In this case, provided X contains a proper class of ordinals, the Skolem hull $\mathcal{H}^{\mathcal{M}}(X)$ is equal to $\bigcup_{\alpha \in X} \mathcal{H}^{V_\alpha \cap \mathcal{M}}(X \cap V_\alpha)$, and hence is definable in $\mathcal{M}$.

1.5 Theorem. $L \models \mathrm{GCH}$.

Proof. We work inside L. An easy induction on α shows that $|L_\alpha| = |\alpha|$ for all infinite ordinals α. Hence, to establish $2^\kappa = \kappa^+$ it is enough to show that any set $x \subseteq \kappa$ is a member of L_{κ^+}. To this end, pick τ large enough that $x \in L_\tau$ and set $X = \mathcal{H}^{L_\tau}(\kappa \cup \{x\})$. By Lemma 1.3 there is an ordinal α such that $\pi : (X, \in) \cong L_\alpha$ where π is the transitive collapse map. Then $x = \pi(x) \in L_\alpha$, and $|L_\alpha| = |X| = \kappa$ so $\alpha < \kappa^+$. $\dashv$

The aim of the inner model theory which we will outline in this chapter is to extend this result to a more general class of models. We will describe (informally, and without attempting a precise definition) a hierarchy satisfying the analog of Lemma 1.3 as a *hierarchy with condensation*.

1.1. Relative Constructibility

Each of the inner models which we will consider is defined as the class of all sets which are *constructible from* some specified set or class A. Two notions of relative constructibility are in general use:

1.6 Definition.

1. If A is a transitive set then $L(A) = \bigcup_{\alpha\in\mathrm{On}} L_\alpha(A)$, where the sets $L_\alpha(A)$ are defined exactly like the hierarchy L_α except that rule 1.1(1) is replaced by $L_0(A) = A$.

2. If A is any set or class, then $L[A] = \bigcup_{\alpha\in\mathrm{On}} L_\alpha[A]$, where the sets $L_\alpha[A]$ are defined exactly like the hierarchy L_α, except that rule 1.1(2) is replaced by
$$L_{\alpha+1}[A] = \mathrm{def}(L_\alpha[A], \in, A),$$
where $\mathrm{def}(L_\alpha[A], \in, A)$ is the set of subsets of $L_\alpha[A]$ first-order definable with parameters from $L_\alpha[A]$, using $A \cap L_\alpha[A]$ as a predicate.

The class $L(A)$ satisfies ZF and contains the set A, and it can be characterized as the smallest such class which contains the ordinals. It need not satisfy the axiom of choice, and indeed it is usually used in cases where the axiom of choice is intended to fail. The most important example is $L(\mathbb{R})$, the smallest model of ZF containing all the reals.[1] This model is heavily used in studies of the axiom of determinacy (AD), where it reconciles that axiom with the axiom of choice in the sense that the axiom "$L(\mathbb{R}) \models \mathrm{AD}$" implies many of the same consequences as the full axiom of determinacy, but is consistent with the axiom of choice in V. We will not consider models of the form $L(A)$ further in this chapter.

If A is a set then the model $L[A]$ always satisfies ZFC. It need not have A itself as a member, but the restriction $A \cap L[A]$ of A to the model $L[A]$ is in $L[A]$. The model $L[A]$ can be characterized as the smallest model M of ZF which contains all the ordinals and has $A \cap M$ as a member. The case when A is a class is similar, provided that replacement holds for formulas with a predicate for A.

In one sense the models $L[A]$ can be fully as complex as any other model of set theory. This is clear in the case that A is a class, since (assuming the axiom of global choice) the universe V can be coded by a class A of ordinals, so that $L[A] = V$. However, a surprising result of Jensen ([2], see [10, Theorem 5.1]) shows that A need not be a proper class: he defines a class generic extension $V[G]$ of the universe V such that $V[G] = L[a]$ for some $a \subseteq \omega$. Thus any class can be contained in a model of the form $L[a]$, with $a \subseteq \omega$.

[1] Strictly speaking Definition 1.6 does not apply to $L(\mathbb{R})$, since $\mathbb{R}$ is not transitive. Taking $L(\mathbb{R})$ to be $L(V_{\omega+1})$ repairs this defect and also gives a more convenient form to the low levels of the $L_\alpha(\mathbb{R})$ hierarchy.

In another sense the models $L[A]$ are quite simple when A is a set—nearly as simple as L itself. This simplicity appears when working above the set A, for submodels M of $L[A]$ such that $A \cap L[A] \in M$. For example, the sentence "$V = L$" can be generalized straightforwardly to a sentence "$V = L[A]$" which satisfies the following generalization of the Condensation Lemma 1.3:

1.7 Lemma. *Suppose that A is a set, and that $X \prec_1 L_\alpha[A]$, where $\alpha \in \mathrm{On} \cup \{\mathrm{On}\}$, and the transitive closure of $A \cap L_\alpha[A]$ is contained in X. Then there is an $\alpha' \leq \alpha$ such that $X \cong L_{\alpha'}[A]$.*

Hence the sets $L_\alpha[A]$, with $\alpha \geq \mathrm{rank}(A)$, also form a hierarchy with condensation, and it follows that all of the basic properties of L, such as GCH, $\Diamond_\kappa$ and $\Box_\kappa$, hold in $L[A]$—at least above $\mathrm{rank}(A)$—for the same reason that they hold in L

Lemma 1.7 does not give any information about the set A itself, and it says nothing about how models $L[A]$ and $L[A']$ might be related when $A \neq A'$. If we are to use the techniques of inner model theory to study the set A, then we need a version of Lemma 1.7 which does not assume $A \cap L[A] \in X$. Any such lemma will require some restriction on the class of sets A for which it is valid.

Elementary embeddings (or, rather, sets A encoding elementary embeddings) have proved to be especially fruitful for this purpose. One reason for this fruitfulness is that when A and A' encode different elementary embeddings of $L[A]$ and $L[A']$, respectively, then it is possible, under suitable conditions, to use the embeddings themselves to modify the models so that they can be compared. This gives at least a start on the goal of understanding the relationships between distinct models $L[A]$ and $L[A']$. This idea may be seen in the proof of Theorem 1.9 and in the comparison Lemma 2.8 for sequences of measures.

A second reason for this fruitfulness arises in the consideration of the embeddings $\pi : L_{\bar\alpha}[\bar A] \cong X \prec_1 L_\alpha[A]$ arising from a transitive collapse. If embeddings coded by A are suitably chosen, then the embedding π will be closely related to the embeddings encoded into A. In this case an analog of the Condensation Lemma 1.3 may hold without the restriction $A \cap L[A] \subseteq X$ needed for Lemma 1.7. This phenomenon often occurs, and is heavily used, in the analysis of inner models for large cardinals.

1.2. Measurable Cardinals

The simplest, and oldest, example of a model $L[A]$ in which A encodes an embedding of $L[A]$ is $L[U]$, the minimal inner model for a measurable cardinal.

1.8 Definition. Recall that a cardinal κ is *measurable* if there is an elementary embedding $i : V \to M$, where M is a well-founded class and $\kappa = \mathrm{crit}(i)$. Here $\mathrm{crit}(i)$ is the *critical point* of i, that is, the least ordinal α such that $i(\alpha) > \alpha$.

The *ultrafilter* associated with such an embedding i is the set

$$U = \{x \subseteq \kappa : \kappa \in i(x)\}.$$

This set U is a *κ-complete ultrafilter* on κ, where κ-completeness means that $\bigcap X \in U$ whenever $X \subseteq U$ and $|X| < \kappa$. Indeed, U is *normal*, which is a stronger property: for any function $f : \kappa \to \kappa$, if $\{\nu < \kappa : f(\nu) < \nu\} \in U$ then there is an ordinal $\gamma < \kappa$ such that $\{\nu : f(\nu) = \gamma\} \in U$.

A normal ultrafilter is frequently called a *measure*. The analogy with Lebesgue measure on the real line, from which this terminology is derived, is slightly strained since neither normality nor the property of being two-valued has an analog in the real line; however this usage has a strong historical basis (evidenced by the term "measurable cardinal") and it is useful in a context such as the present chapter, in which non-normal ultrafilters never appear.

In the other direction, from an ultrafilter to an embedding, the *ultrapower* construction gives, for any normal ultrafilter U on κ, an embedding $i^U : V \to M = \mathrm{Ult}(V, U)$ with critical point κ such that U is the ultrafilter associated with i^U. The ultrapower has the property that $M = \{i^U(f)(\kappa) : f \in V \cap {}^\kappa V\}$, and as a consequence i^U is minimal among all embeddings related to U in the following sense: Any other embedding $i : V \to N$ with the same associated ultrafilter U can be factored as

$$i : V \xrightarrow{i^U} \mathrm{Ult}(V, U) \xrightarrow{k} N,$$

where the embedding k is defined by $k([f]_U) = k(i^U(f)(\kappa)) = i(f)(\kappa)$.

It is easy to see that if U is a normal ultrafilter on κ, then $U \cap L[U]$ is a normal ultrafilter in $L[U]$. On its face, the model $L[U]$ appears to depend on the choice of the ultrafilter U; however Kunen [18] showed that it depends only on the critical point of U.

The proof, which we outline below, uses iterated ultrapowers. We write $i^U_\alpha : V \to \mathrm{Ult}_\alpha(V, U)$ for the α-fold iterated ultrapower by U, which is defined by setting $\mathrm{Ult}_0(V, U) = V$, $\mathrm{Ult}_{\alpha+1}(V, U) = \mathrm{Ult}(\mathrm{Ult}_\alpha(V, U), i^U_\alpha(U))$, and $\mathrm{Ult}_\alpha(V, U) = \operatorname{dir\,lim}_{\alpha' < \alpha} \mathrm{Ult}_{\alpha'}(V, U)$ if α is a limit ordinal.

We will need the fact that every iterated ultrapower $\mathrm{Ult}_\alpha(L[U], U)$ is well-founded. This is easily proved by induction on α: more generally, let M be any well-founded model containing the ordinals, and suppose that M satisfies that U is a countably complete ultrafilter. A useful observation is that all iterated ultrapowers of M are definable subsets of M, and hence we can work inside M. It is easy to see that $\mathrm{Ult}(M, U)$ is well-founded. For any ordinal α such that $\mathrm{Ult}_\alpha(M, U)$ is well-founded, it then follows, by working inside $\mathrm{Ult}_\alpha(M, U)$, that $\mathrm{Ult}_{\alpha+1}(M, U)$ is also well-founded. Hence the least ordinal α such that $\mathrm{Ult}_\alpha(M, U)$ is ill-founded would be a limit ordinal. Now call an ordinal γ *U-soft in M* if there is an iterated ultrapower i^U_α by U such that the set of ordinals in $\mathrm{Ult}_\alpha(M, U)$ below $i^U_\alpha(\gamma)$ is ill-founded. Let γ be the least U-soft ordinal in M, and let α be least such that i^U_α witnesses that

γ is soft. Now if α' is any ordinal in the interval $0 \leq \alpha' < \alpha$, then $i^U_{\alpha'}(\gamma)$ is, by elementarity, the least $i^U_{\alpha'}(U)$-soft ordinal in $\mathrm{Ult}_{\alpha'}(M, U)$. But this is impossible, since for sufficiently large $\alpha' < \alpha$ there is an ordinal $\xi < i^U_{\alpha'}(\gamma)$ in $\mathrm{Ult}_{\alpha'}(M, U)$ such that $i_{\alpha',\alpha}(\xi)$ is a member of an infinite descending sequence below $i_\alpha(\gamma)$. Then $i^U_{\alpha',\alpha}$ is an iterated ultrapower of $\mathrm{Ult}_{\alpha'}(M, U)$ by $i^U_{\alpha'}(U)$ which witnesses that ξ is $i^U_{\alpha'}(U)$-soft in $\mathrm{Ult}_{\alpha'}(M, U)$.

The proof above can be generalized to any well-founded model M with $\omega_1 \subseteq M$, and to any iterated ultrapower of M by arbitrary measures in M rather than by the single ultrafilter U. We will later see that the situation is much more difficult for iterations involving cardinals beyond a strong cardinal.

1.9 Theorem. *Suppose that U and U' are normal ultrafilters in $L[U]$ and $L[U']$, respectively.*

1. *If* $\mathrm{crit}(U) = \mathrm{crit}(U')$ *then $U = U'$, and hence $L[U] = L[U']$.*
2. *If* $\mathrm{crit}(U) < \mathrm{crit}(U')$ *then $L[U'] = \mathrm{Ult}_\alpha(L[U], U)$ for some ordinal α.*

1.10 Corollary. *The model $L[U]$ has only the one normal ultrafilter U.*

Sketch of Proof of Theorem 1.9(1). The proof of Theorem 1.9 uses the following two observations about the iterated ultrapower $\mathrm{Ult}_\lambda(L[U], U)$, where $\lambda > \kappa^+$ is a cardinal of uncountable cofinality.

(1) The set $C = \{i^U_\alpha(\kappa) : \alpha < \lambda\}$ is a closed, unbounded set of indiscernibles for $\mathrm{Ult}_\lambda(L[U], U)$ which generates its measure $i^U_\lambda(U)$ in the sense that

$$i^U_\lambda(U) = \{x \subseteq \lambda : \sup(C - x) < i^U_\lambda(\kappa)\}, \tag{17.1}$$

and therefore $\mathrm{Ult}_\lambda(L[U], U) = L[i^U_\lambda(U)] = L[\mathcal{C}_\lambda]$ where $\mathcal{C}_\lambda$ is the filter of closed unbounded subsets of λ. To see that (17.1) holds, let x be any subset of $i_\lambda(\kappa)$ in $\mathrm{Ult}_\lambda(L[U], U)$. Then there is some $\alpha_0 < \lambda$ and $x_{\alpha_0} \in \mathrm{Ult}_{\alpha_0}(L[U], U)$ such that $x = i_{\alpha_0,\lambda}(x_{\alpha_0})$. For ordinals α in the interval $\alpha_0 < \alpha < \lambda$ set $x_\alpha = i_{\alpha_0,\alpha}(x_{\alpha_0})$, so that $x = i_{\alpha,\lambda}(x_\alpha)$. Then $i^U_\alpha(\kappa) \in x \iff i^U_\alpha(\kappa) \in x_{\alpha+1} = i_{\alpha,\alpha+1}(x_\alpha) \iff x_\alpha \in i^U_\alpha(U) \iff x \in i^U_\lambda(U)$.

(2) Let Γ be the class of ordinals $\xi > \lambda$ such that $i^U_\lambda(\xi) = \xi$. Then simple cardinal arithmetic shows that Γ is a proper class, and contains all of its limit points of cofinality greater than λ.

Now suppose that the models $L[U]$ and $L[U']$ are as in the hypothesis of Theorem 1.9(1), with $\kappa = \mathrm{crit}(U) = \mathrm{crit}(U')$. Let $\lambda = (2^\kappa)^+$. By the first observation, $\mathrm{Ult}_\lambda(L[U], U) = \mathrm{Ult}_\lambda(L[U'], U') = L[\mathcal{C}_\lambda]$, with $i^U_\lambda(U) = i^{U'}_\lambda(U') = \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]$. By the second observation $\Gamma = \{\xi > \lambda : i^U_\lambda(\xi) = i^{U'}_\lambda(\xi) = \xi\}$ is a proper class.

Let $X = \mathcal{H}^{L[\mathcal{C}_\lambda]}(\kappa \cup \Gamma \cup \{\mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]\}) \prec L[\mathcal{C}_\lambda]$ be the Skolem hull, and let $\pi : M \cong X$ be its transitive collapse. Then $X \subseteq \mathrm{ran}(i^U_\lambda)$, so $M \prec L[U]$. However $U = \pi^{-1}(\mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]) \in M$, and the proper class Γ is a subset of M. It follows by Lemma 1.7 that $M = L[U]$. By the same argument $M = L[U']$ and $U' = \pi^{-1}(\mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda])$, so $L[U] = L[U']$ and $U = U'$. $\dashv$

1.11 Theorem (Silver). $L[U] \models \mathrm{GCH}$.

Sketch of Proof. First we recall Gödel's proof of GCH for L. Assume $V = L$, and fix any set $x \in \mathcal{P}^L(\lambda)$. Now pick some $\tau > \lambda$ such that $x \in L_\tau$, and let $\pi : M_x \cong X \prec_1 L_\tau$ where $|X| = \lambda$, $\lambda \cup \{x\} \subseteq X$, and M_x is transitive. Then the Condensation Lemma 1.3 implies that $M_x = L_\alpha$ for some $\alpha < \lambda^+$, so that $x = \pi^{-1}(x) \in L_\alpha$. Thus $\{z \subseteq \lambda : z \leq^L x\} \subseteq L_\alpha$, and since $|L_\alpha| = \lambda$ it follows that no set in $\mathcal{P}(\lambda)$ has more than λ many $<^L$-predecessors. Hence $\operatorname{otp}(\mathcal{P}(\lambda), \leq^L) = \lambda^+$, so $L \models 2^\lambda = \lambda^+$.

Now assume $V = L[U]$, where U is a normal ultrafilter on κ, and fix a cardinal λ. If $\lambda \geq \kappa$ then Gödel's proof for L can be easily adapted to $L[U]$ by substituting Lemma 1.7 for Lemma 1.3. Thus we only need to consider the case $\lambda < \kappa$.

Fix a set $x \subseteq \lambda$, and pick τ such that $x \in L_\tau[U]$. For convenience, also let $L_\tau[U]$ satisfy ZF^-, the axioms of ZF without the Power Set Axiom; this will be true if τ is any successor cardinal. Now let $X \prec L_\tau[U]$ where $x \in X$, $\lambda \subseteq X$, and $|X| = \lambda$. If $M_x \cong X$ is the transitive collapse of X, then $M_x = L_{\alpha_x}[U_x]$ for some $\alpha_x < \lambda^+$ and some filter U_x which is a normal ultrafilter in M_x.

In order to conclude, as in the proof for L, that $\operatorname{otp}\bigl(\mathcal{P}(\lambda), <^{L[U]}\bigr) = \lambda^+$, we need to show that $\{z \subseteq \lambda : z \leq^{L[U]} x\} \subseteq M_x$. The fact that $U_x \neq U$ is a complication which is not present in L, and we will use the techniques from the proof of Theorem 1.9 to deal with it. The assumption that $L_\tau[U]$, and hence M_x, satisfies ZF^- makes it is easy to verify that the iterated ultrapower $i^{U_x}_\kappa : L_{\alpha_x}[U_x] \to \operatorname{Ult}_\kappa(L_{\alpha_x}[U_x], U_x)$ can be defined and has all of the required properties: In particular, $\operatorname{Ult}_\kappa(L_{\alpha_x}[U_x], U_x) = L_{\alpha'_x}[i^{U_x}_\kappa(U_x)]$ for some $\alpha'_x < \kappa^+$, and $i^{U_x}_\kappa(U_x) \subseteq U$ since $i^{U_x}_\kappa(U_x)$ is generated by the set $C_x = \{i^{U_x}_\nu(\lambda) : \nu < \kappa\}$, which is in U since it is closed and unbounded. Since $i^{U_x}_\lambda {\restriction} \mathcal{P}^{M_x}(\lambda)$ is the identity, it follows that $\{z \subseteq \lambda : z <^{L[U]} x\} \subseteq \mathcal{P}^{L_{\alpha'_x}[U_x]}(\lambda) \subseteq M_x$, as desired. ⊣

This proof can be interpreted as showing that $L[U]$ contains a hierarchy with condensation; however this hierarchy has two flaws: (i) the very existence of the model $L[U]$, and hence of this hierarchy, is conditional on the existence of the normal ultrafilter U, and (ii) unlike the structures L_α, the structures M_x do not actually satisfy condensation. That is, the model M_x is not actually an initial segment of $L[U]$, but only a structure which can be compared to an initial segment of $L[U]$ by means of an iterated ultrapower. The first, and more important, of these two flaws was fixed by the Dodd and Jensen [4, 6, 5] with their introduction of the original core model K^{DJ}. They defined a *mouse* to be a structure $M = L_{\alpha^M}[U_M]$ such that (i) M satisfies the sentence "U_M is a normal ultrafilter", (ii) all of the iterated ultrapowers of M are well-founded, and (iii) M satisfies a fine structure condition which implies that there is a $\rho \leq \operatorname{crit}(U_M)$ such that $L_{\alpha^M+1}[U_M] \models \bigl|\alpha^M\bigr| = \rho$. The Dodd-Jensen core model K^{DJ} is defined to be $L[\mathcal{M}]$, where $\mathcal{M}$ is the class of all mice. With the emergence of a general concept of "the core model" (see Sect. 5), K^{DJ} came to be seen as the core model below $L[U]$, that is, it is the core model provided that there is no model with a measurable cardinal.

The weakest mouse is equi-constructible with $0^{\#}$, which is described in the next subsection. The model M_x in the proof of Theorem 1.11 is an example of a mouse; however its construction required starting with the model $L[U]$ and it is difficult to prove that such a model exists using any assumption weaker than a measurable cardinal. Dodd and Jensen threw out the assumption $M_x \models \mathrm{ZF}^-$ of Theorem 1.11, and replaced it with clauses (i) and (ii); they then used fine structure to show that iterated ultraproducts of mice can still be defined and have the required properties.

The second flaw, the lack of condensation, is only a minor technical problem at the level of one measurable cardinal but leads to serious difficulties at higher levels. This problem is corrected by the modern presentation of the core model. As adapted to the special case of the Dodd-Jensen core model K^{DJ}, this presentation works as follows: First note that K^{DJ} does satisfy a form of condensation, for if $\pi : L_{\alpha'}[\mathcal{M}'] \cong X \prec_1 L_\alpha[\mathcal{M}]$, then π preserves the property of being a mouse. It follows that $\mathcal{M}'$ is contained in $\mathcal{M}$, and since the Dodd-Jensen mice are well-ordered by relative constructibility it follows that $\mathcal{M}'$ is an initial sequence of the class $\mathcal{M}$. We can extend this to $L[U]$ as follows: each mouse is a model $L_{\alpha_M}[U_M]$. Since U_M and $L_{\alpha_M}[U_M]$ are equi-constructible, K^{DJ} can be equivalently written as $L[\langle U_M : M \in \mathcal{M}\rangle]$ instead of as $L[\mathcal{M}]$, and then $L[U]$ is equal to $L[\mathcal{M}, U] = L[\langle U_M : M \in \mathcal{M}\rangle, U]$. If we let $\mathcal{U}$ be the sequence $\langle U_M : M \in \mathcal{M}\rangle^\frown\langle U\rangle$, then $L[U] = L[\mathcal{U}]$, and the transitive collapse of a substructure $X \prec_1 L_\alpha[\mathcal{U}]$ has the form $L_{\alpha'}[\mathcal{U}']$ where, as in the case of K^{DJ}, the sequence $\mathcal{U}'$ is an initial segment of $\mathcal{U}$. Thus $X \cong L_{\alpha'}[\mathcal{U}\restriction\alpha']$, which is an initial segment of $L[\mathcal{U}] = L[U]$.

Notice that this construction has the further advantage of smoothly joining the construction of K^{DJ} with $L[U]$ at one extreme and (taking $\mathcal{U}$ to be empty) L at the other.

1.3. $0^{\#}$, and Sharps in General

This subsection covers the first steps of the core model hierarchy suggested by the proof of Theorem 1.11. They are the first steps historically, since the model $L[0^{\#}]$ was the first canonical inner model to be extensively studied other than L and $L[U]$. They are also the first steps in the sense that they lie at the bottom of the core model hierarchy: $0^{\#}$ is, as we will see later, essentially the same as the first Dodd-Jensen mouse.

Lemma 1.2 implies that if $i : L \to M$, where M is a well-founded class, then $M = L$. As Scott [37] observed, it follows that there are no measurable cardinals in L: otherwise let $U \in L$ be a normal ultrafilter on the least measurable cardinal κ of L. Then $i^U(\kappa) > \kappa$; but this is impossible since $i^U(\kappa)$ is, by the elementarity of the embedding i^U, the least measurable cardinal in $\mathrm{Ult}(L, U) = L$, and that is κ.

Nontrivial embeddings from L into L can, however, exist in V: for example, if U is a normal ultrafilter and $i^U : V \to \mathrm{Ult}(V,U)$ then $i^U \restriction L : L \to L$. Silver's $0^\#$ gives a complete analysis of such embeddings. We say that a class I of ordinals is a class of indiscernibles for a model M if for any formula $\varphi(v_0, \dots, v_{n-1})$ of the language of set theory and any increasing sequences $(c_0, \dots, c_{n-1})$ and $(c'_0, \dots, c'_{n-1})$ of members of I we have $M \models \varphi(c_0, \dots, c_{n-1}) \iff \varphi(c'_0, \dots, c'_{n-1})$.

1.12 Definition. We say that $0^\#$ *exists* if there is closed proper class I of indiscernibles for L. In this case we define $0^\# \subseteq \omega$ to be the set of Gödel numbers of formulas $\varphi(v_0, \dots, v_{n-1})$ such that $L \models \varphi(c_0, \dots, c_{n-1})$ for any increasing sequence $\langle c_0, \dots, c_{n-1} \rangle \in [I]^n$.

Since I is a class of indiscernibles for L, this characterization of the set $0^\#$ does not depend on the choice of the sequence $\vec{c} \in [I]^n$. The fact that I is required to be closed implies that the definition of $0^\#$ does not depend on the choice of the class I. It also implies that the members of I possess the following normality property, which Silver called *remarkability*: if η is any ordinal and $f : \mathrm{On} \to \mathrm{On}$ is any map definable in L from parameters in L_η such that $f(c_0, \dots, c_{n-1}) = \xi < c_0$ for some sequence $\vec{c} = (c_0, \dots, c_{n-1}) \in [I - \eta]^n$, then $f(\vec{d}) = \xi$ for every sequence $\vec{d} \in [I - \eta]^n$.

Silver showed that if $0^\#$ exists then there is a unique maximal class I, the *Silver indiscernibles* such that $L = \mathcal{H}^L(I)$, that is, every set in L is definable in L from parameters in I. This class can be obtained by starting with any remarkable class I' of indiscernibles. Then $\mathcal{H}^L(I') \prec L$ is a proper class and hence is isomorphic to L. If $\pi : \mathcal{H}^L(I') \cong L$ is the transitive collapse map, then $I = \pi``I'$ is a closed class of indiscernibles and $\mathcal{H}^L(I) = L$.

Our Definition 1.12 requires that I be a proper class, but Silver showed that this is not necessary:

1.13 Theorem. *If there is an uncountable set of indiscernibles for L then $0^\#$ exists. Furthermore, there is a Π^1_2 formula ψ such that if a is any subset of ω, then $\psi(a)$ holds if and only if $a = 0^\#$.*

Thus for example, the existence of $0^\#$ is an immediate consequence of the existence of a Ramsey cardinal. The following result shows how $0^\#$ can be used to characterize the elementary embeddings from L into L:

1.14 Theorem (Silver). *Assume that $0^\#$ exists. Then (i) for any strictly increasing map $\pi : I \to I$ there is a unique elementary embedding $i : L \to L$ such that $\pi = i \restriction I$, and (ii) if $i : L \to L$ then $i``I \subseteq I$, and i is determined by $i \restriction I : I \to I$.*

The proof follows easily from the indiscernibility of the members of I and the fact that every constructible set is definable from members of I: if x is the unique set satisfying a formula $\varphi(x, \alpha_0, \dots, \alpha_{n-1})$ for some sequence $(\alpha_0, \dots, \alpha_{n-1}) \in [I]^{<\omega}$, then $i(x)$ must be the unique set x' satisfying $\varphi(x', \pi(\alpha_0), \dots, \pi(\alpha_{n-1}))$.

This leaves open one gap in the use of $0^\#$ to characterize embeddings from L into L: the question of whether the existence of such an embedding implies the existence of $0^\#$. This question was settled by Kunen; the version of the proof which we sketch below is largely due to Silver and is included because it involves ideas which are basic to the proof of the covering lemma:

1.15 Theorem. *If $i : L \to L$ is a nontrivial elementary embedding then $0^\#$ exists.*

Sketch of Proof. Let $\kappa = \mathrm{crit}(i)$. We can assume without loss of generality that i is continuous at every ordinal of cofinality greater than κ: if it is not, then factor the embedding i as $i : L \to X := \{i(f)(\kappa) : f \in L\} \prec L$ and replace i with $i' : L \xrightarrow{i} X \xrightarrow{\pi} L$, where $\pi : X \cong L$ is the transitive collapse.

We will define, for each $\nu \in \mathrm{On}$, a class Γ_ν of ordinals which is unbounded and contains all of its limit points of cofinality greater than κ. If we set $\kappa_\nu = \inf(\Gamma_\nu - \kappa)$ then the class $J = \{\kappa_\nu : \nu \in \mathrm{On}\}$ will be a class of indiscernibles for L, and by Silver's results this implies that $0^\#$ exists.

Set $\Gamma_0 = \mathrm{On} \cap \mathrm{ran}(i)$, and if λ is a limit ordinal then set $\Gamma_\lambda = \bigcap_{\nu<\lambda} \Gamma_\nu$. Now suppose that Γ_ν has been defined, and write $\mathcal{H}^L(X) \prec L$ for the class of sets definable in L from parameters in X. Then $\mathcal{H}^L(\Gamma_\nu) \cong L$ since Γ_ν is a proper class, so consider the map

$$i_\nu : L \cong \mathcal{H}^L(\Gamma_\nu) \prec L.$$

Then $\Gamma_{\nu+1}$ is defined to be the set of ordinals ξ such that $i_\nu(\xi) = \xi$.

Notice that $\Gamma_\nu = \mathrm{On} \cap \mathcal{H}^L(\Gamma_\nu)$, that $\kappa_\nu = \inf(\Gamma_\nu - \kappa) = i_\nu(\kappa)$, and that if $\nu > \nu'$ then $i_{\nu'}(\kappa_\nu) = \kappa_\nu$. Now define, for each pair $\nu' < \nu$ of ordinals, the embedding $i_{\nu',\nu} : L \cong \mathcal{H}^L(\kappa_{\nu'} \cup \Gamma_\nu) \prec L$ to be the inverse of the transitive collapse of $\mathcal{H}^L(\kappa_{\nu'} \cup \Gamma_\nu)$. Thus $i_{\nu',\nu}$ is the identity on $\kappa_{\nu'} \cup \Gamma_{\nu+1}$.

We claim that $i_{\nu',\nu}(\kappa_{\nu'}) = \kappa_\nu$. This claim is equivalent to the statement that $\kappa_\nu \cap \mathcal{H}^L(\kappa_{\nu'} \cup \Gamma_\nu) = \kappa_{\nu'}$, and if it were false then there would be $\vec{\alpha} \in [\Gamma_\nu]^{<\omega}$ and a formula φ such that

$$L \models \exists \eta \in \kappa_\nu - \kappa_{\nu'} \exists \vec{\gamma} \in [\kappa_{\nu'}]^{<\omega} \big(\varphi(\vec{\gamma}, \eta, \vec{\alpha}) \ \& \ \forall \eta' < \eta \, \neg\varphi(\vec{\gamma}, \eta', \vec{\alpha})\big). \quad (17.2)$$

Now the embedding $i_{\nu'} : L \cong \mathcal{H}^L(\Gamma_{\nu'}) \prec L$ is elementary, and $i_{\nu'}(\kappa) = \kappa_{\nu'}$, but $i_{\nu'}(\kappa_\nu) = \kappa_\nu$ and $i_\nu(\vec{\alpha}) = \vec{\alpha}$ since $i_{\nu'} {\restriction} \Gamma_\nu$ is the identity. Thus formula (17.2) implies that

$$L \models \exists \eta \in \kappa_\nu - \kappa \exists \vec{\gamma} \in [\kappa]^{<\omega} \big(\varphi(\vec{\gamma}, \eta, \vec{\alpha}) \ \& \ \forall \eta' < \eta \, \neg\varphi(\vec{\gamma}, \eta', \vec{\alpha})\big).$$

But this is impossible, since any such ordinal η would be in Γ_ν and $\kappa_\nu = \min(\Gamma_\nu - \kappa)$.

This completes the proof of the claim. Now suppose that $\vec{c}$ and $\vec{c}\,'$ are two increasing sequences in $[J]^{<\omega}$ which differ only in the ith place; say that, $c'_i = \kappa_{\nu'} < c_i = \kappa_\nu$ while $c_j = c'_j$ for $j \neq i$. Then $i_{\nu',\nu}(\vec{c}\,') = \vec{c}$, and since $i_{\nu',\nu} : L \to L$ is elementary it follows that $\vec{c}\,'$ and $\vec{c}\,'$ satisfy the same formulas over L. But if $\vec{c}\,'$ and $\vec{c}$ are any two increasing sequences of the same length from $[J]^{<\omega}$, then one can be obtained from the other in a finite sequence of steps in such a way that each step changes only one element of the sequence. Hence J is a class of indiscernibles for L. ⊣

The following result of Silver states that if $0^{\#}$ exists then the class L of constructible sets is much smaller than V:

1.16 Theorem (Silver). *Assume that $0^{\#}$ exists, and let I be the class of Silver indiscernibles. Then (i) every uncountable cardinal κ of V is a member of I, and indeed $|I \cap \kappa| = \kappa$, (ii) every Silver indiscernible is weakly compact in L, (iii) $\forall \eta \ \left|\mathcal{P}^L(\eta)\right| = |\eta|$, and (iv) $\forall \eta \ \mathrm{cf}(\eta^{+L}) = \omega$.*

Clause (ii) can be strengthened by replacing "weakly compact" with any large cardinal property which can consistently hold in L. This fact suggests that the existence of $0^{\#}$ can be viewed as the weakest large cardinal property which cannot consistently be true in L, and further experience has supported this view. Such a statement cannot be proved, or even stated precisely, without a precise definition of "large cardinal property"; however it is true for large cardinals inside the core model, and the covering lemma provides other senses in which $L[0^{\#}]$ is a minimal extension of L. For example, if M is any class model such that $M \models \lambda^+ \neq (\lambda^+)^L$ for some singular cardinal λ of M, then $L[0^{\#}]$ is contained in M.

Solovay once suggested that $L[0^{\#}]$ might be minimal in another sense: that every real $a \in L[0^{\#}]$ such that $0^{\#} \notin L[a]$ would be set generic over L. This suggestion was refuted by Jensen [2], who used class forcing to construct a counterexample. A weaker conjecture might be that $0^{\#}$ is the minimal real which is easily definable; however Friedman [11] has shown that if $0^{\#}$ exists then there is a set a such that $0 <_L a <_L 0^{\#}$ and a is a Π^1_2 singleton; furthermore, the set defined by this Π^1_2 formula remains a singleton in any extension with the same ordinals. See [10, Theorem 6.5] for more on this subject.

1.4. Other Sharps

The process used to define $0^{\#}$ can also be applied to models larger than L. This process is commonly used in two slightly different contexts: in order to define the sharp of a large cardinal property, and in order to define the sharp of a set.

In order to construct the sharp of a large cardinal property, we need to start with a minimal inner model M for the property such that M has a suitable inner model theory. For a measurable cardinal, for example, we could take any model of the form $M = L[U]$ such that U is a normal ultrafilter in M. If J is a closed proper class of indiscernibles for M, then we can define a new real, just as with $0^{\#}$, to be the set of Gödel numbers of formulas $\varphi(x_0, \ldots, x_{n-1})$ such that $M \models \varphi(c_0, \ldots, c_{n-1})$ for any $(c_0, \ldots, c_{n-1}) \in [J]^n$.

By using the inner model theory for the model M in question, together with Silver's techniques from the theory of $0^\#$, it can be shown that this construction yields a unique real even though (as in the case of $L[U]$) the model M may not itself be unique.

This procedure is not limited to properties involving a single cardinal. As we will see shortly, the ideas of $L[U]$ can be extended to a model $M = L[\mathcal{U}]$, having a proper class of measurable cardinals, so that M has an inner model theory similar to that of $L[U]$. The procedure described above, applied to the model M, will then yield a real which is the sharp for a proper class of measurable cardinals.

The sharp construction was first applied to $L[U]$ by Solovay, who gave the name $0^\dagger$ to the resulting sharp for a measurable cardinal. This precedent has had the effect of leading to a proliferation of typographical symbols for sharps of various large cardinal properties, the most common of which is $0^\P$, used for the sharp of a strong cardinal. The use of these symbols, apparently chosen on a whim and with no relation to the cardinals they are supposed to represent, places an unfortunate and unnecessary burden on the reader's memory. Fortunately the most important example, the sharp for a Woodin cardinal, has escaped the use of such symbols. This sharp is important because a number of applications of Woodin cardinals, particularly to inner model theory, appear to require the sharp for a Woodin cardinal, rather than simply a Woodin cardinal itself. It is commonly denoted by $M_1^\#$, where M_1 is the standard symbol for the minimal model with one Woodin cardinal.

It is straightforward to generalize the construction of $0^\#$ to obtain the sharp $A^\#$ for an arbitrary set A of ordinals: If $A \subseteq \gamma$, and J is a closed, proper class of indiscernibles for $L[A]$, then $A^\#$ is the set of pairs $(n, \vec{a})$ where $n = \ulcorner\varphi\urcorner$ is the Gödel number of a formula $\varphi(\vec{v}, z, \vec{u})$, $\vec{a} \in [\gamma]^{<\omega}$, and $L[A] \models \varphi(\vec{c}, A, \vec{a})$ for any $\vec{c} \in [J]^n$. By use of an appropriate coding, we can regard $A^\#$ as a subset of γ.

In particular, this construction can be used to iterate the sharp operation: Starting with $0^{\#^1} = 0^\#$, we define $0^{\#^{\alpha+1}} = (0^{\#^\alpha})^\#$. If α is a limit ordinal then $0^{\#^\alpha}$ is defined to be a set encoding $\langle 0^{\#^\gamma} : \gamma < \alpha\rangle$.

Assuming the existence of a large cardinal (a Ramsey cardinal is much more than enough) it can be shown to be consistent that $0^{\#^\alpha}$ exists for all ordinals α. The model $L[\langle 0^{\#^\alpha} : \alpha \in \mathrm{On}\rangle]$ forms a hierarchy with condensation, and this hierarchy is an initial segment of the core model hierarchy toward which we are working. This process can easily be continued: the model $M = L[\langle 0^{\#^\alpha} : \alpha \in \mathrm{On}\rangle]$ is a minimal model for the large cardinal property "$A^\#$ exists for all sets A", and (given a class of indiscernibles for this model) we can define the sharp for this property. This sharp will be a subset of ω and it is the next step $0^{\#^{\mathrm{On}}}$ in the desired hierarchy.

On the one hand there seems to be no obvious bound determining how far the hierarchy obtained through this process can be extended, but on the other hand it is not clear how to generalize the process to give a uniform definition, using indiscernibles, of such a hierarchy. The core model provides such a definition by replacing the indiscernibles by ultrafilters, as suggested by the proof of Theorem 1.11. We will conclude this section by discussing the relationship between ultrafilters and sets of indiscernibles.

1.5. From Sharps to the Core Model

The Dodd-Jensen core model K^{DJ} was briefly described at the end of Sect. 1.3. This subsection will explain how the concept of a mouse, which they invented for this model, generalizes and extends the concept of sharps.

Recall that they defined a *mouse* to be a model $L_{\alpha^M}[U_M]$, similar to the models M_x used in the proof of Theorem 1.11, but with the strong theory ZF^- replaced with a fine structural condition. One of the uses of this fine structural condition was to allow them to define iterated ultrapowers of mice and to show that they have the required properties. Silver, Magidor and others later gave a construction of K^{DJ} without the need for fine structure, but fine structure is still needed to define core models for larger cardinals.

As in the proof of Theorem 1.11, iterated ultrapowers can be used to compare two mice. This comparison process prewellorders the class of mice, and shows that they form, in an appropriate sense, a hierarchy with condensation. The Dodd-Jensen core model is defined to be the model $K = L[\mathcal{M}]$, where $\mathcal{M}$ is the class of all mice; the well-ordering of mice and their condensation properties imply that $L[\mathcal{M}]$ is a model of ZF + GCH.

This model cannot contain a measurable cardinal, but Dodd and Jensen proved a covering lemma for K which asserts that a model $L[U]$ with a measurable cardinal has approximately the same relation to K that $0^{\#}$ has to L.

The Dodd-Jensen core model can be better understood by considering a translation from mice to sharps and vice versa. Suppose first that $M = L_\alpha[U]$ is a mouse. Then the result of the iterated ultrapower $i^U_{\mathrm{On}} : M \to M^* = \mathrm{Ult}_{\mathrm{On}}(M, U)$ is a well-founded model by clause (ii) of the definition of a mouse. The ordinals On^{M^*} of M^* have length greater than On, and On is the measurable cardinal in M^*. If we write i^U_λ for the embedding from $L_\alpha[U]$ to $\mathrm{Ult}_\lambda(L_\alpha[U], U)$, then the class $I = \{i^U_\lambda(\kappa) : \lambda \in \mathrm{On}\}$ is a class of indiscernibles for M^*.

This class I can be used, as described in the last subsection, to define an initial sequence of the class of sharps. This sequence of sharps can be defined by recursion over On^{M^*}, so that the length of On^{M^*} provides an indication of how long a hierarchy of sharps will be generated before the process generates a model for which I is not a class of indiscernibles. Clause (iii) of the definition of a mouse can be used to show that this final model is M^*.

In discussing the other direction, from sharps to mice, we will use Kunen's notion of a M-ultrafilter:

1.17 Definition. A *normal M-ultrafilter* in Kunen's sense is an ultrafilter on $\mathcal{P}(\kappa) \cap M$, for some κ in M, such that

1. If $f : \kappa \to \kappa$ is in M and $\{\nu < \lambda : f(\nu) < \nu\} \in U$, then there is γ such that $\{\nu : f(\nu) = \gamma\} \in U$, and
2. If $x \subseteq \mathcal{P}(\kappa)$ is a member of M, and $|x|^M = \kappa$, then $U \cap x \in M$.

The second condition enables the ultrapower by U to be iterated, even though $U \notin M$. If $i^U : M \to \mathrm{Ult}(M, U)$ then $U_1 = \{[f]_U : f \in ({}^{\kappa}U) \cap M\}$ is an $\mathrm{Ult}(M, U)$-ultrafilter, which can be used as $i^U(U)$.

A member of the sharp hierarchy is a set which encodes the theory of a model M^*, with parameters taken from a class I of indiscernibles for M^*. A mouse will be a model $L_\alpha[U]$, where α is the least ordinal such that U is not a normal ultrafilter in $L_{\alpha+1}[U]$. We could easily get an M^*-ultrafilter U on any limit point of I by setting $U = \{x \subseteq \lambda : \sup((I \cap \lambda) - x) < \lambda\}$, the filter on λ generated by $I \cap \lambda$. A better construction, however, uses the analog of Theorem 1.14 for M^* to get an M^*-ultrafilter on $\lambda = \min(I)$: let $i : I \to I$ be an increasing map such that $i(\lambda) > \lambda$. By Theorem 1.14 the embedding i extends to a map $i^* : M^* \to M^*$ such that $i = i^* \restriction I$, and $U = \{x \subseteq \lambda : \lambda \in i^*(x)\}$ is a normal M^*-ultrafilter.

2. Beyond One Measurable Cardinal

The next step beyond $L[U]$ is to develop an inner model theory for models with many measurable cardinals. This is straightforward so long as all of the measures have different critical points: If $\mathcal{U} = \langle U_\nu : \nu < \lambda \rangle$ is a sequence of measures, with increasing critical points κ_ν, then the model $L[\mathcal{U}]$ has measures $U_\nu \cap L[\mathcal{U}]$, and (as with the model $L[U]$) no other measures. If it is desired to have several measures on the same cardinal then the answer is less obvious: if U_0 and U_1 are two measures on a cardinal κ then $U_0 \cap L[U_0, U_1] = U_1 \cap L[U_0, U_1]$ by Kunen's Theorem 1.10, so the model $L[U_0, U_1]$ has only one normal ultrafilter.

A way to proceed is suggested by the following observation of Kunen:

2.1 Proposition. *Every measurable cardinal κ has a normal ultrafilter U_κ which concentrates on nonmeasurable cardinals.*

Proof. Suppose as an induction hypothesis that for each measurable cardinal $\lambda < \kappa$ there is a normal ultrafilter U_λ concentrating on nonmeasurable cardinals, and let U be a normal ultrafilter on κ. If U concentrates on nonmeasurable cardinals then set $U_\kappa = U$; otherwise

$$U' = [\langle U_\lambda : \lambda < \kappa \rangle]_U = \{x \subseteq \kappa : \{\lambda < \kappa : x \cap \lambda \in U_\lambda\} \in U\}$$

is a second normal ultrafilter on κ which concentrates on nonmeasurable cardinals. In this case set $U_\kappa = U'$. ⊣

Note that in the second case of the proof, the model $L[\langle U_\lambda : \lambda < \kappa \rangle, U', U]$ is a model with at least two normal ultrafilters U' and U on κ. The following partial order captures the relation between U' and U:

2.2 Definition. If U and U' are normal ultrafilters on a cardinal κ then we write $U' \lhd U$ if $U' \in \mathrm{Ult}(V, U)$.

Thus $U' \lhd U$ if and only if there is a function f such that

$$\{\alpha < \kappa : f(\alpha) \text{ is a normal ultrafilter on } \alpha\} \in U$$

and

$$U' = \{x \subseteq \kappa : \{\lambda < \kappa : x \cap \lambda \in f(\lambda)\} \in U\}.$$

The argument which Scott used to prove that there are no measurable cardinals in L proves the follow proposition:

2.3 Proposition. *The ordering $\lhd$ is well-founded.*

Proof. Assuming the contrary, let κ be the least cardinal such that the normal ultrafilters on κ are not well-founded by $\lhd$. Then there is a normal ultrafilter U on κ so that $\{U' : U' \lhd U\}$ is not well-founded by $\lhd$. The normal ultrafilters on κ in $\mathrm{Ult}(V, U)$ are exactly the normal ultrafilters U' on κ in V such that $U' \lhd U$, and the $\lhd$-ordering on these normal ultrafilters in $\mathrm{Ult}(V, U)$ is the same as in V since V and $\mathrm{Ult}(V, U)$ have the same functions from κ to V_κ. Thus the measures on κ in $\mathrm{Ult}(V, U)$ are not well-founded under $\lhd$. Since $\mathrm{Ult}(V, U)$ is well-founded it follows that $\mathrm{Ult}(V, U)$ satisfies that the measures on κ are not well-founded by $\lhd$, but this is impossible since, by the elementarity of the embedding i, there is no cardinal $\lambda < i(\kappa)$ in $\mathrm{Ult}(V, U)$ such that the measures on λ are not well-founded by $\lhd$. $\dashv$

2.4 Definition. The order $o(U)$ of a normal ultrafilter U is its rank in the ordering $\lhd$, that is, $o(U) = \sup\{o(U') + 1 : U' \lhd U\}$. The order of a cardinal κ is $o(\kappa) = \sup\{o(U) + 1 : U \text{ is a normal ultrafilter on } \kappa\}$.

Thus a measure U has order 0 if and only if the set of smaller measurable cardinals is not a member of U. A cardinal κ has order 0 if it is not measurable, and order 1 if it is measurable, but has no measures concentrating on smaller measurable cardinals. Since each measure $U' \lhd U$ is equal to $[f]_U$ for some $f : \kappa \to V_\kappa$, we have the following upper bound:

2.5 Proposition (Solovay). *If U is a normal ultrafilter on a measurable cardinal κ then $o(U) < (2^\kappa)^+$, and hence $o(\kappa) \leq (2^\kappa)^+$.*

Under the GCH, it follows that $o(\kappa) \leq \kappa^{++}$.

The inner models $L[\mathcal{U}]$ for sequences of measures utilize this ordering $\lhd$. We give here the original presentation of these models as in [25]. This presentation is the simplest way to approach these models, and we will show that it can be generalized with the use of extenders to define inner models with a strong cardinal. However it is not adequate for dealing with cardinals very much larger than this; and in Sect. 3 we will follow up by giving a brief description of the modified presentation now used for the core model and for inner models with larger cardinals.

2.6 Definition. A *coherent sequence of measures* is a function $\mathcal{U}$ such that

1. $\mathrm{dom}(\mathcal{U}) = \{(\kappa,\beta) : \kappa < \mathrm{len}(\mathcal{U}) \text{ and } \beta < o^{\mathcal{U}}(\kappa)\}$, where $\mathrm{len}(\mathcal{U})$ is a cardinal and $o^{\mathcal{U}}$ is a function mapping cardinals $\kappa < \mathrm{len}(\mathcal{U})$ to ordinals.

2. If $(\kappa,\beta) \in \mathrm{dom}(\mathcal{U})$ then $U = \mathcal{U}(\kappa,\beta)$ is a normal ultrafilter on κ.

3. If $U = \mathcal{U}(\kappa,\beta)$ then $o^{i^U(\mathcal{U})}(\kappa) = \beta$ and $i^U(\mathcal{U})(\kappa,\beta') = \mathcal{U}(\kappa,\beta')$ for all $\beta' < \beta$.

The final clause of this definition is the *coherence condition*, which can also be expressed by saying that $(i^{\mathcal{U}(\kappa,\beta)}(\mathcal{U}))\restriction\kappa+1 = \mathcal{U}\restriction(\kappa,\beta)$. Here we write $\mathcal{U}\restriction(\kappa,\beta)$ for the restriction of $\mathcal{U}$ to

$$\{(\kappa',\beta') \in \mathrm{dom}(\mathcal{U}) : \kappa' < \kappa \vee (\kappa' = \kappa \wedge \beta' < \beta)\}$$

and $\mathcal{U}\restriction\lambda$ for $\mathcal{U}\restriction(\lambda,0)$. Notice that the coherence condition implies that $\mathcal{U}(\kappa,\beta') \lhd \mathcal{U}(\kappa,\beta)$ for all $\beta' < \beta < o^{\mathcal{U}}(\kappa)$, so that $o(\mathcal{U}(\kappa,\beta)) \geq \beta$.

The following theorem is the main result of [25]; it is a generalization of the Corollary 1.10 to Theorem 1.9:

2.7 Theorem. *If $\mathcal{U}$ is a coherent sequence of measures in $L[\mathcal{U}]$, then the only normal ultrafilters in $L[\mathcal{U}]$ are the members of the sequence $\mathcal{U}$.*

It follows from Theorem 2.7 that $o(\mathcal{U}(\kappa,\beta))$ is exactly equal to β in $L[\mathcal{U}]$, and that each cardinal κ has exactly $|o^{\mathcal{U}}(\kappa)|$ many normal ultrafilters in $L[\mathcal{U}]$. Theorem 1.9 itself does not generalize to $L[\mathcal{U}]$: starting in a model with κ^+ measurable cardinals, where κ is measurable, it is possible to construct sequences $\mathcal{U}$ and $\mathcal{U}'$ with the same domain such that $L[\mathcal{U}] \neq L[\mathcal{U}']$.

2.1. The Comparison Process

The main difficulty in generalizing the proofs of Theorem 1.9 and Theorem 1.11 to the models $L[\mathcal{U}]$ is in adapting the iterated ultrapowers used in those proofs. Recall that they used the iterated ultrapower $\mathrm{Ult}_\lambda(L_\alpha[U], U)$, where λ is some larger cardinal, so that the image i^U_λ of U is contained in the closed unbounded filter $\mathcal{C}_\lambda$ on λ. Thus two normal ultrafilters U and U' were compared indirectly, via the filter $\mathcal{C}_\lambda$. In adapting this construction to the models $L[\mathcal{U}]$ we use iterated ultrapowers to compare sequences $\mathcal{U}$ and $\mathcal{U}'$ directly, through a process known as *iterating the least difference*.

2.8 Lemma (Comparison). *Suppose that $L_\alpha[\mathcal{U}]$ and $L_{\alpha'}[\mathcal{U}']$ satisfy* ZF$^-$, *and that $\mathcal{U}$ and $\mathcal{U}'$ are coherent in $L_\alpha[\mathcal{U}]$ and $L_{\alpha'}[\mathcal{U}']$ respectively. Then for some sequence $\mathcal{W}$ and some ordinals $\bar\alpha$ and $\bar\alpha'$, there are iterated ultrapowers $i : L_\alpha[\mathcal{U}] \to L_{\bar\alpha}[\mathcal{W}\restriction\bar\alpha]$ and $i' : L_{\alpha'}[\mathcal{U}'] \to L_{\bar\alpha'}[\mathcal{W}\restriction\bar\alpha']$.*

Notice that the sequence $\mathcal{W}$ plays the role of the closed unbounded filter in the proof of Theorem 1.9

For simplicity, this statement of Lemma 2.8 assumes that the models being compared are sets; however the process can also be used if one or both of the models is a proper class, that is, if one or both of $L_\alpha[\mathcal{U}]$ or $L_{\alpha'}[\mathcal{U}]$ is replaced by $L[\mathcal{U}]$ or $L[\mathcal{U}']$. In this case one or both of the iterated ultrapowers may have length On, and one (but never both) of the models $L_{\bar\alpha}[\mathcal{W}\restriction\bar\alpha]$ or $L_{\bar\alpha'}[\mathcal{W}\restriction\bar\alpha']$ may have length larger than On. The simplest example of this is when $L[\mathcal{U}] = L[U]$, with a single normal ultrafilter, and $L[\mathcal{U}'] = L$. Then the comparison consists of iterating U, the only ultrapower available, and U must be iterated On many times to move it past L. Thus $L_{\bar\alpha}[\mathcal{W}\restriction\bar\alpha] = \mathrm{Ult}_{\mathrm{On}}(L[U], U)$, which has On as its measurable cardinal and has length greater than On.

Proof of Lemma 2.8. An iterated ultrapower of length θ of a model M is a family of maps $i_{\nu,\nu'} : M = M_\nu \to M_{\nu'}$, commuting in the sense that $i_{\nu,\nu''} = i_{\nu',\nu''} \circ i_{\nu,\nu'}$ whenever $\nu < \nu' < \nu'' < \theta$, which is defined by recursion on ordinals $\nu < \nu' < \theta$ by setting $M_0 = M$ and $M_\nu = \operatorname{dir\,lim}_{\nu'<\nu} M_{\nu'}$ for each limit $\nu \le \theta$, and for successor ordinals $\nu+1 < \theta$ letting either $M_{\nu+1} = M_\nu$ or else $M_{\nu+1} = \mathrm{Ult}(M_\nu, U_\nu)$ for some M_ν-measure U_ν. The iterated ultrapowers used in this proof are *internal*, which means that $U_\nu \in M_\nu$ for all $\nu < \theta$. We write i_ν for $i_{0,\nu}$.

The proof given before Lemma 1.9 that the model $L[U]$ is iterable relied on the fact that every iterated ultrapower of $L[U]$ is internal to $L[U]$ in the stronger sense that every iterated ultrapower of $L[U]$ is definable in $L[U]$. That is not true for the models $L[\mathcal{U}]$ described here, since the choice of which ultrafilters to use in the iteration may be made externally to $L[\mathcal{U}]$; however, every iterated ultrapower of $L[\mathcal{U}]$ can be embedded into an iterated ultrapower which is definable in $L[\mathcal{U}]$, and thus the argument before Lemma 1.9 shows that every iterated ultrapower of $L[\mathcal{U}]$ is well-founded.

We define two iterated ultrapowers: $i_{\nu',\nu} : M_{\nu'} \to M_\nu$ on $L[\mathcal{U}]$ and $i'_{\nu',\nu} : M'_{\nu'} \to M'_\nu$ on $L[\mathcal{U}']$, as follows: Suppose that $i_\nu : M_0 = L_\alpha[\mathcal{U}] \to M_\nu = L_{\alpha_\nu}[\mathcal{U}_\nu]$ and $i'_\nu : M'_0 = L_{\alpha'}[\mathcal{U}'] \to M'_\nu = L_{\alpha'_\nu}[\mathcal{U}'_\nu]$ have already been defined, where $\mathcal{U}_\nu = i_\nu(\mathcal{U})$ and $\mathcal{U}'_\nu = i'_\nu(\mathcal{U}')$. Let $\gamma = \min\{\alpha_\nu, \alpha'_\nu\}$. If $\mathcal{U}'_\nu\restriction\gamma = \mathcal{U}_\nu\restriction\gamma$ then we are finished, since we can take $\bar\alpha = \alpha_\nu$ and $\bar\alpha' = \alpha'_\nu$ and let $\mathcal{W}$ be the longer of the sequences $\mathcal{U}_\nu$ and $\mathcal{U}'_\nu$. In this case we say that the comparison *terminates at stage* ν.

Otherwise we define $M_{\nu+1}$ and $M'_{\nu+1}$ by the process of *iterating the least difference*: Let (κ_ν, β_ν) be the lexicographically least pair of ordinals such that $\kappa_\nu < \gamma$, $\beta_\nu \le \min\{o^{\mathcal{U}_\nu}(\kappa_\nu), o^{\mathcal{U}'_\nu}(\kappa_\nu)\}$, and

$$\mathcal{U}_\nu(\kappa_\nu, \beta_\nu) \ne \mathcal{U}'_\nu(\kappa_\nu, \beta_\nu), \tag{17.3}$$

where the inequality (17.3) may hold either because $o^{\mathcal{U}_\nu}(\kappa_\nu) \ne o^{\mathcal{U}'_\nu}(\kappa_\nu)$ and $\beta_\nu = \min\{o^{\mathcal{U}_\nu}(\kappa_\nu), o^{\mathcal{U}'_\nu}(\kappa_\nu)\}$ (so that only one side of (17.3) is defined) or because there is a set $x_\nu \in M_\nu \cap M'_\nu$ such that $x_\nu \in \mathcal{U}_\nu(\kappa_\nu, \beta_\nu) \not\Longleftrightarrow x_\nu \in \mathcal{U}'_\nu(\kappa_\nu, \beta_\nu)$. If $\beta_\nu = o^{\mathcal{U}_\nu}(\kappa_\nu)$ then set $M_{\nu+1} = M_\nu$; otherwise set $M_{\nu+1} = \mathrm{Ult}(M_\nu, \mathcal{U}(\kappa_\nu, \beta_\nu))$. Similarly, $M'_{\nu+1} = \mathrm{Ult}(M'_\nu, \mathcal{U}'_\nu(\kappa_\nu, \beta_\nu))$ if $\beta_\nu < o^{\mathcal{U}'_\nu}(\kappa_\nu)$, and $M'_{\nu+1} = M'_\nu$ otherwise.

Note that the ordinals κ_ν are strictly increasing: we have $\mathcal{U}_{\nu+1}{\restriction}\kappa_\nu + 1 = \mathcal{U}_\nu{\restriction}(\kappa_\nu, \beta_\nu) = \mathcal{U}'_\nu{\restriction}(\kappa_\nu, \beta_\nu) = \mathcal{U}'_{\nu+1}{\restriction}\kappa_\nu + 1$, where the outer equalities follow from the coherence condition and the inner equality follows from the minimality of the pair (κ_ν, β_ν). It follows that $i_{\nu+1,\nu'}(\kappa_\nu) = i'_{\nu+1,\nu'}(\kappa_\nu) = \kappa_\nu$ for all $\nu' > \nu$.

In order to complete the proof of the lemma, we need to show that this comparison eventually terminates. The proof relies on the following observation:

2.9 Claim. *Suppose that τ is an infinite regular cardinal and $\langle N_\nu : \nu < \tau\rangle$ is an iterated ultrapower with embeddings $j_{\nu,\nu'} : N_\nu \to N_{\nu'}$. Further suppose that $|N_0| < \tau$, that $S \subseteq \tau$ is stationary, and that $y_\nu \in N_\nu$ for each $\nu \in S$. Then there is a stationary set $S' \subseteq S$ such that $j_{\nu,\nu'}(y_\nu) = y_{\nu'}$ for all $\nu < \nu'$ in S'.*

Proof. For each limit $\nu \in S$ there is an ordinal $\gamma < \nu$ such that $y_\nu \in \operatorname{ran}(j_{\gamma,\nu})$, so by Fodor's Lemma there is a fixed $\gamma_0 < \tau$ such that the set of $\nu \in S$ such that $y_\nu \in \operatorname{ran}(j_{\gamma_0,\nu})$ is stationary. Since $|N_{\gamma_0}| \le \max\{|N_0|, |\gamma_0|\} < \tau$, there is a fixed $\bar{y} \in N_{\gamma_0}$ such that $S' = \{\nu \in S : y_\nu = j_{\gamma_0,\nu}(\bar{y})\}$ is stationary. Now if $\nu < \nu'$ are in S' then $y_{\nu'} = j_{\gamma_0,\nu'}(\bar{y}) = j_{\nu,\nu'} j_{\gamma_0,\nu}(\bar{y}) = j_{\nu,\nu'}(y_\nu)$. ⊣

Set $\tau = (\max\{\alpha, \alpha'\})^+$, and suppose for the sake of contradiction that the comparison does not terminate in fewer then τ steps. By applying the claim successively to the two iterations of the comparison we get a stationary subset S_0 of τ such that for any two ordinals $\nu < \nu'$ in S_0 we have $i_{\nu,\nu'}(\kappa_\nu) = i'_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'}$. It follows that $\beta_\nu < \min\{o^{\mathcal{U}_\nu}(\kappa_\nu), o^{\mathcal{U}'_\nu}(\kappa_\nu)\}$ for all $\nu \in S_0$, for otherwise if we take any $\nu' > \nu$ in S_0, then either $i_{\nu,\nu'}(\kappa_\nu) = \kappa_\nu < \kappa_{\nu'}$ or $i'_{\nu,\nu'}(\kappa_\nu) = \kappa_\nu < \kappa_{\nu'}$.

Thus x_ν is defined for each $\nu \in S_0$, and by applying the claim twice again we get a stationary set $S_1 \subseteq S_0$ such that $i_{\nu,\nu'}(x_\nu) = i'_{\nu,\nu'}(x_\nu) = x_{\nu'}$ for each $\nu < \nu'$ in S_1. But this is impossible, for since $i_{\nu+1,\nu'}(\kappa_\nu) = i'_{\nu+1,\nu'}(\kappa_\nu) = \kappa_\nu$ it follows that

$$\begin{aligned}
x_\nu \in \mathcal{U}_\nu(\kappa_\nu, \beta_\nu) \quad &\iff \quad \kappa_\nu \in i_{\nu,\nu+1}(x_\nu) \\
&\iff \quad \kappa_\nu \in i_{\nu,\nu'}(x_\nu) = x_{\nu'} = i'_{\nu,\nu'}(x_\nu) \\
&\iff \quad \kappa_\nu \in i'_{\nu,\nu+1}(x_\nu) \\
&\iff \quad x_\nu \in \mathcal{U}'_\nu(\kappa_\nu, \beta_\nu),
\end{aligned}$$

contradicting the choice of x_ν and thus completing the proof of Lemma 2.8. ⊣

As an example of the use of Lemma 2.8, we sketch the proof of Theorem 2.7:

Sketch of Proof. Suppose that Theorem 2.7 is false, so that there is a sequence $\mathcal{U}$ such that $\mathcal{U}$ is coherent in $L[\mathcal{U}]$ and $L[\mathcal{U}]$ contains a normal ultrafilter U which is not a member of the sequence $\mathcal{U}$. We can assume that

Lemma 2.7 does hold for every proper initial segment of the sequence $\mathcal{U}$, that κ and $\beta = o(U)$ are the smallest ordinals such that there is a normal ultrafilter U on κ in $L[\mathcal{U}]$ with $o(U) = \beta$ which is not in the sequence $\mathcal{U}$, and that U is the first such ultrafilter in the order of construction of $L[\mathcal{U}]$. Note that all of these statements can be expressed by sentences in $L[\mathcal{U}]$.

Now apply Lemma 2.8 to the models $L[\mathcal{U}]$ and $\mathrm{Ult}(L[\mathcal{U}], U)$ (with $\alpha = \alpha' = \mathrm{On}$). We must also have $\bar{\alpha} = \bar{\alpha}' = \mathrm{On}$; otherwise if, for example, $\mathrm{On} = \bar{\alpha} < \bar{\alpha}'$, then the lemma would fail in $L_{\bar{\alpha}}[\mathcal{W}{\restriction}\bar{\alpha}] = L[\mathcal{W}{\restriction}\mathrm{On}]$, which contradicts the fact that $L_{\bar{\alpha}'}[\mathcal{W}]$ satisfies the sentence stating that Lemma 2.7 does hold for every proper initial segment of $\mathcal{W}$. Thus we have the following diagram:

$$\begin{tikzcd} & \mathrm{Ult}(L[\mathcal{U}], U) \arrow[dr, "i"] & \\ L[\mathcal{U}] \arrow[ur, "i^U"] \arrow[rr, "i'"] & & L[\mathcal{W}] \end{tikzcd} \tag{17.4}$$

This diagram obviously commutes on definable members of $L[\mathcal{U}]$, but since the diagram itself is definable in $L[\mathcal{U}]$, the least element of $L[\mathcal{U}]$ for which it failed to commute would be definable. Hence diagram (17.4) commutes.

In particular $i'(\kappa) = i\, i^U(\kappa)$, so $i'(\kappa) > \kappa$. Since $i{\restriction}\kappa$ and $i'{\restriction}\kappa$ are the identity it follows that i' begins with an ultrapower by a normal ultrafilter on κ; that is, $\beta_0 = \beta = o(U) < o^{\mathcal{U}}(\kappa)$ and $i' = i'_\theta = i'_{1,\theta}\, i^{\mathcal{U}(\kappa,\beta)}$. But now $U = \mathcal{U}(\kappa, \beta)$, for if x is any subset of κ in $L[\mathcal{U}]$ then $i\, i^U(x) = i'(x) = i'_{1,\theta} i^{\mathcal{U}(\kappa,\beta)}(x)$, so

$$\begin{aligned} x \in U \quad &\Longleftrightarrow \quad \kappa \in i i^U(x) \quad \Longleftrightarrow \quad \kappa \in i'_{1,\theta}\, i^{\mathcal{U}(\kappa,\beta)}(x) \\ &\Longleftrightarrow \quad x \in \mathcal{U}(\kappa, \beta). \end{aligned}$$

$\dashv$

Models $L[\mathcal{U}]$ with higher order measures are more difficult to obtain than the model $L[U]$ with one measure. One might try to proceed by analogy with the model $L[U]$, choosing a coherent sequence $\mathcal{U}$ in V and using the model $L[\mathcal{U}]$, but this fails on two counts. In the first place it is not clear that there is a coherent sequence $\mathcal{U}$ in V, for example it is not known whether $o(\kappa) = \omega$ implies that there is a coherent sequence $\mathcal{U}$ of measures in V with $o^{\mathcal{U}}(\kappa) = \omega$. In the second place, if $o(\kappa) > \kappa^+$ then it is not clear that a sequence which is coherent in V need be coherent in $L[\mathcal{U}]$. The first construction of an inner model of $o(\kappa) = \omega$ from the assumption $o^V(\kappa) = \omega$ used the covering lemma; however we outline a proof which avoids this. Call a sequence $\mathcal{U}$ *weakly coherent* if it satisfies conditions 1 and 2 of Definition 2.6, together with the following weakened coherence condition: if $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$ and $U = \mathcal{U}(\kappa, \beta)$ then $o^V(U) = \beta$.

We first show that the comparison process can be modified to use sequences which are only weakly coherent. Notice that this proof requires that $\mathcal{U}$ and $\mathcal{W}$ be sequences of measures in V, not just in $L[\mathcal{U}]$ and $L[\mathcal{W}]$. The example described following Theorem 2.7 shows that this hypothesis cannot be eliminated.

2.10 Lemma. *Suppose that $\mathcal{U}$ and $\mathcal{W}$ are weakly coherent sequences of measures in V with the same domain. Then $L[\mathcal{U}] = L[\mathcal{W}]$, and $\mathcal{U}(\kappa,\beta) \cap L[\mathcal{U}] = \mathcal{W}(\kappa,\beta) \cap L[\mathcal{W}]$ for every (κ,β) in their common domain.*

Proof. We compare the model V with itself, using iterated ultrapowers $i_\nu : V = M_0 \to M_\nu$ and $j_\nu : V = N_0 \to N_\nu$. The comparison process is similar to that in Lemma 2.8 except that we simultaneously compare each of the sequences $i_\nu(\mathcal{U})$ and $i_\nu(\mathcal{W})$ in M_ν with each of the sequences $j_\nu(\mathcal{U})$ and $j_\nu(\mathcal{W})$ in N_ν. Thus, condition (17.3) of the proof of Lemma 2.8 is modified as follows: Suppose that M_ν and N_ν have been defined. Define o^{M_ν} and o^{N_ν} by setting $o^{M_\nu}(\kappa) = o^{i_\nu(\mathcal{U})}(\kappa) = o^{i_\nu(\mathcal{W})}(\kappa)$ and $o^{N_\nu}(\kappa) = o^{j_\nu(\mathcal{U})}(\kappa) = o^{j_\nu(\mathcal{W})}(\kappa)$. Now let (κ_ν, β_ν) be the least pair (κ,β) such that one of the following hold:

1. $\beta < \min\{o^{M_\nu}(\kappa), o^{N_\nu}(\kappa)\}$ and there is a set in $M_\nu \cap N_\nu$ on which the four filters $i_\nu(\mathcal{U})(\kappa,\beta)$, $i_\nu(\mathcal{W})(\kappa,\beta)$, $j_\nu(\mathcal{U})(\kappa,\beta)$ and $j_\nu(\mathcal{W})(\kappa,\beta)$ do not all agree.

2. $\beta = \min\{o^{M_\nu}(\kappa), o^{N_\nu}(\kappa)\}$ and $o^{M_\nu}(\kappa) \neq o^{N_\nu}(\kappa)$.

Now proceed with a slightly modified version of the proof of Lemma 2.8. In case 1 pick U_ν to be one of $\{i_\nu(\mathcal{U})(\kappa,\beta), i_\nu(\mathcal{W})(\kappa,\beta)\}$ and U'_ν to be one of $\{j_\nu(\mathcal{U})(\kappa,\beta), j_\nu(\mathcal{W})(\kappa,\beta)\}$ so that $U_\nu \cap M_\nu \cap N_\nu \neq U'_\nu \cap M_\nu \cap N_\nu$, and set $M_{\nu+1} = \mathrm{Ult}(M_\nu, U_\nu)$ and $N_{\nu+1} = \mathrm{Ult}(N_\nu, U'_\nu)$. In case 2 let $M_{\nu+1} = M_\nu$ if $o^{M_\nu}(\kappa) = \beta$ and $M_{\nu+1} = \mathrm{Ult}(M_\nu, i_\nu(\mathcal{U})(\kappa,\beta))$ if $\beta < o^{M_\nu}(\kappa)$, and define $N_{\nu+1}$ similarly.

Unlike the proof of Lemma 2.8, the sequence of ordinals κ_ν need not be strictly increasing; however the sequence is nondecreasing and the fact that $\beta_{\nu+1} < \beta_\nu$ whenever $\kappa_{\nu+1} = \kappa_\nu$ implies that for each ν there is an $n < \omega$ such that $\kappa_{\nu+n} > \kappa_\nu$. This, together with the weak coherence of $\mathcal{U}$ and $\mathcal{W}$, is enough to show that the comparison terminates at some stage θ.

There is a λ such that either $o^{i_\theta(\mathcal{U})} = o^{j_\theta(\mathcal{U})} \restriction \lambda$ or $o^{j_\theta(\mathcal{U})} = o^{i_\theta(\mathcal{U})} \restriction \lambda$; we may assume the former. We will show that $L[i_\theta(\mathcal{U})] = L[i_\theta(\mathcal{W})]$, and since i_θ is an elementary embedding it follows that $L[\mathcal{U}] = L[\mathcal{W}]$, as was to be shown.

The four sequences $i_\theta(\mathcal{U}) \restriction \lambda$, $i_\theta(\mathcal{W}) \restriction \lambda$, $j_\theta(\mathcal{U})$ and $j_\theta(\mathcal{W})$ agree on sets in $M_\theta \cap N_\theta$, and thus $L[i_\theta(\mathcal{U})] = L[i_\theta(\mathcal{W})]$ will follow if we can show that $L[i_\theta(\mathcal{U})] \subseteq M_\theta \cap N_\theta$. Suppose the contrary, and let α be least such that there is a set in $L_{\alpha+1}[i_\theta(\mathcal{U})]$ which is not in $M_\theta \cap N_\theta$. Now $i_\theta(\mathcal{U})$ and $j_\theta(\mathcal{U})$ agree on all sets in $L_\alpha[i_\theta(\mathcal{U})]$. Thus $L_\alpha[i_\theta(\mathcal{U})] = L_\alpha[j_\theta(\mathcal{U})]$, and the restrictions of $i_\theta(\mathcal{U})$ and $j_\theta(\mathcal{U})$ to this set are equal. However $L_{\alpha+1}[i_\theta(\mathcal{U})]$ is equal to the set of subsets of $L_\alpha[i_\theta(\mathcal{U})]$ definable over $L_\alpha[i_\theta(\mathcal{U})]$ using as a predicate the restriction of $i_\theta(\mathcal{U})$ to $L_\alpha[i_\theta(\mathcal{U})]$, and similarly for $L_{\alpha+1}[j_\theta(\mathcal{U})]$. Hence $L_{\alpha+1}[i_\theta(\mathcal{U})] = L_{\alpha+1}[j_\theta(\mathcal{U})]$, and it follows that $L_{\alpha+1}[i_\theta(\mathcal{U})] \subseteq M_\theta \cap N_\theta$, contradicting the choice of α. This contradiction completes the proof of Lemma 2.10. $\dashv$

2.11 Corollary. *If $\mathcal{U}$ is weakly coherent in V, then either $\mathcal{U}$ is coherent in $L[\mathcal{U}]$ or there is an inner model of $\exists\kappa(o(\kappa) = \kappa^{++})$.*

Sketch of Proof. Let $\mathcal{U}$ be any weakly coherent sequence which is not coherent in $L[\mathcal{U}]$. Since initial segments of $\mathcal{U}$ are also weakly coherent, we may assume that $\mathcal{U}$ has minimal length, so that $\mathcal{U}\restriction(\kappa,\beta)$ is coherent in $L[\mathcal{U}\restriction(\kappa,\beta)]$ for all (κ,β) in the domain of $\mathcal{U}$. In particular, if (κ,β) is the least place at which $\mathcal{U}$ is not coherent in $L[\mathcal{U}]$ then $\mathcal{U}\restriction(\kappa,\beta)$ is coherent in $L[\mathcal{U}\restriction(\kappa,\beta)]$, and it will be sufficient to show that $L[\mathcal{U}\restriction(\kappa,\beta)] \models o(\kappa) = \beta = \kappa^{++}$.

To this end, set $U = \mathcal{U}(\kappa,\beta)$ and consider the following triangle:

$$\begin{tikzcd}
L[\mathcal{U}] \arrow[r, "j"] \arrow[dr, "i^U\restriction L[\mathcal{U}]"'] & \mathrm{Ult}(L[\mathcal{U}], U) \arrow[d, "k"] \\
& L[i^U(\mathcal{U})]
\end{tikzcd}$$

where $i^U : V \to \mathrm{Ult}(V, U)$, j is the ultrapower of $L[\mathcal{U}]$ using functions in $L[\mathcal{U}]$, and k is defined by $k\big(j(f)(\kappa)\big) = i^U(f)(\kappa)$.

We claim that $i^U(\mathcal{U})\restriction\kappa+1$ agrees with $\mathcal{U}\restriction(\kappa,\beta)$ on all sets in $L[\mathcal{U}]$. To see this, let $\mathcal{U}'$ be the sequence obtained from $\mathcal{U}$ by replacing $\mathcal{U}\restriction(\kappa,\beta)$ with $i^U(\mathcal{U})\restriction\kappa+1$. Then $\mathcal{U}'$ is weakly coherent, and has the same domain as $\mathcal{U}$, so by Lemma 2.10 it is equal to $\mathcal{U}$ on sets in $L[\mathcal{U}]$.

This implies that k is not the identity on $o^{j(\mathcal{U})}(\kappa)+1$, since otherwise we would have $k(j(\mathcal{U})\restriction\kappa+1) = i^U(\mathcal{U})\restriction\kappa+1$. Since $L[\mathcal{U}]$ and $L[j(\mathcal{U})]$ have the same subsets of κ, and $i^U(\mathcal{U})\restriction\kappa+1$ agrees with $\mathcal{U}$ on these subsets, this would contradict the assumption that $\mathcal{U}$ is not coherent in $L[\mathcal{U}]$ at (κ,β).

Now let $\eta = \mathrm{crit}(k)$. Then $\eta \le o^{L[j(\mathcal{U})]}(\kappa)$, and since $\beta = k(o^{L[j(\mathcal{U})]}(\kappa))$ it follows that $k(\eta) \le \beta$. Also $\eta > \kappa$, and η is a cardinal in $L[j(\mathcal{U})]$ and hence in $L[\mathcal{U}]$. But $k(\eta)$ is a cardinal in $L[i^U(\mathcal{U})]$, and hence also in $L[i^U(\mathcal{U})\restriction\kappa+1] = L[\mathcal{U}\restriction(\kappa,\beta)]$. Thus $\beta \ge k(\eta) \ge \kappa^{++}$ in $L[\mathcal{U}\restriction(\kappa,\beta)]$. ⊣

2.2. Indiscernibles from Iterated Ultrapowers

We now look at the use of iterated ultrapowers to generate systems of indiscernibles, and at the relation between these indiscernibles and those added generically by Prikry forcing and its variants. Such forcing is covered extensively in chapter [12].

The simplest case is Prikry forcing [31], which involves only one normal ultrafilter. Let U be a normal ultrafilter on a cardinal κ, and let $i^U_\omega : V \to M_\omega = \mathrm{Ult}_\omega(V, U)$ be the iterated ultrapower of length ω. Then the set $C = \{i^U_n(\kappa) : n < \omega\}$ is a set of indiscernibles over M_ω in the following sense: if x is any subset of $i_\omega(\kappa)$ in M_ω, then there are $n < \omega$ and $x' \subseteq i_n(\kappa)$ in M_n such that $x = i_{n,\omega}(x')$. Then for all $m \ge n$ we have $i_m(\kappa) \in x$ if and only if $x' \in i_n(U)$, which is to say if and only if $x \in i_\omega(U)$. Hence C is almost contained in any set $x \in \mathcal{P}^{M_\omega}(i_\omega(\kappa))$ such that $x \in i_\omega(U)$. By

Mathias's genericity criterion [23], this implies that the sequence C is generic for Prikry forcing over M_ω.

In order to extend this construction to the variants of Prikry forcing discovered by Magidor [21] and Radin [32], let $\mathcal{U} = \langle U_\beta : \beta < \eta \rangle$ be a $\triangleleft$-increasing sequence of measures on κ, with $o(U_\beta) = \beta$ for $\beta < \eta$ and define an iterated ultrapower $i_\nu : V \to M_\nu$, of length θ, as follows:

As usual, set $M_0 = V$ and set $M_\nu = \operatorname{dir}\lim_{\nu' < \nu} M_{\nu'}$ whenever ν is a limit ordinal. Now suppose that M_ν has been defined. Set $\kappa_\nu = i_\nu(\kappa)$, and let $M_{\nu+1} = \mathrm{Ult}(M_\nu, i_\nu(\mathcal{U})_{\beta_\nu})$ where $\beta_\nu < i_\nu(\eta)$ is the least ordinal β such that $\{\nu' < \nu : i_{\nu',\nu}(\beta_{\nu'}) = \beta\}$ is bounded in ν. If there is no such ordinal β then set $\theta = \nu$ and stop the process.

Assuming $\eta < \kappa^{++}$ and $2^\kappa = \kappa^+$, Fodor's Lemma implies that $\theta < \kappa^{++}$. If $\eta < \kappa$ then a straightforward induction shows that $\theta = \omega^\eta$, and that β_ν is always the least ordinal β such that $\nu = \nu' + \omega^\beta$ for some ordinal $\nu' < \nu$. In particular $\beta_\nu = 0$ if ν is a successor ordinal.

The set $C = \{i_\nu(\kappa) : \nu < \theta\}$ is a closed unbounded subset of $i_\theta(\kappa)$, since the sequence $\langle i_\nu(\kappa) : \nu < \theta \rangle$ is continuous. If $x \in M_\nu$ and $x \subseteq \kappa_\nu$ then $\kappa_\nu \in i_{\nu,\theta}(x) \iff x \in i_\nu(\mathcal{U})_{\beta_\nu} \iff i_{\nu,\theta}(x) \in i_\theta(\mathcal{U})_\beta$, where $\beta = i_{\nu,\theta}(\beta_\nu)$. Thus the sets $C_\beta = \{\kappa_\nu : \nu < \theta \text{ and } i_{\nu,\theta}(\beta_\nu) = \beta\}$ are sets of indiscernibles for the normal ultrafilters $i_\theta(\mathcal{U})_\beta$ on $i_\theta(\kappa)$.

We have already considered the case $n = 1$, when $C = C_0$ is a Prikry sequence for the normal ultrafilter $i(\mathcal{U}_0)$ on $i_\omega(\kappa)$. If $\eta < \kappa$ is an uncountable regular cardinal then $M_\theta[C]$ is a generic extension of the model M_θ by Magidor's generalization [21] of Prikry forcing: the cardinals of $M_\theta[C]$ are the same as those of M_θ, while κ_θ is regular in M_θ and has cofinality η in $M_\theta[C]$. Notice that $\vec{C} = \langle C_\beta : \beta < \eta \rangle \in M_\theta[C]$, since $C_\beta = \{\lambda \in C : o(\lambda) = \beta\}$.

The covering lemma, which is discussed in a separate chapter [24], implies that these results are the best possible in the sense that if there is a cardinal κ which is regular in the core model but is singular of cofinality $\eta > \omega$ in V, then $o(\kappa) \geq \eta$ in the core model. Furthermore, the singularity of κ is witnessed by a set which is similar to the Prikry-Magidor generic set C described above, but which may be more irregular: it satisfies $o(\nu) \geq \limsup\{o(\nu') + 1 : \nu' \in C \cap \nu\}$, while the Prikry-Magidor generic set satisfies the stronger condition $\forall \nu \in C\ o(\nu) = \limsup\{o(\nu') + 1 : \nu' \in C \cap \nu\}$. The case of $\mathrm{cf}(\kappa) = \omega$ can be somewhat more complicated.

If $\kappa < \eta \leq \kappa^{++}$ then the set C obtained from the iterated ultrapower described above is generic for Radin forcing [32], or rather for the variant of Radin forcing described in [26]. It is a closed unbounded subset of $i_\theta(\kappa)$ and it is eventually contained in every member x of the filter $\bigcap i_\theta(\mathcal{U})$ on $\mathcal{P}^{M_\theta}(i_\theta(\kappa))$. If $\eta \geq \kappa^+$ then the sequence $\langle C_\beta : \beta < i_\theta(\eta) \rangle \notin M_\theta[C]$, and the cardinals of $M_\theta[C]$ are the same as those of M_θ. If $\mathrm{cf}(\eta) = \kappa^+$ then κ_θ remains inaccessible in $M_\theta[C]$, and κ can have stronger larger cardinal properties in $M_\theta[C]$ as the ordinal η becomes larger. For the most important example, define β to be a weak repeat point in the sequence $\mathcal{U}$ if for each set $A \in U_\beta$ there is $\beta' < \beta$ such that $A \in U_{\beta'}$. If $\eta = \beta + 1$, where β is a weak repeat point in $\mathcal{U}$, then $i_\theta(\kappa)$ is measurable in $M_\theta[C]$, with a measure on $i_\theta(\kappa)$ in $M_\theta[C]$ which extends the measure $i(U_\beta)$ in M_θ.

If the set C is obtained by Radin forcing or, equivalently, by an iterated ultrapower as described above, then C is eventually contained in any closed unbounded subset of κ which is a member of the ground model M. It can be shown [28] that if this additional condition is imposed, then neither the hypothesis $o(\kappa) \geq \kappa^+$ for preserving the inaccessibility of κ nor the hypothesis of a weak repeat point for preserving measurability can be weakened. If this condition is removed, however, then work of Gitik [13], improved by Mitchell [28], has shown that if $M \models o(\kappa) = \kappa$ then there is a forcing to add a closed, unbounded set $C \subseteq \kappa$ such that every member of C is inaccessible in M, while κ is still measurable in $M[C]$. Gitik also shows that if $\{\nu < \kappa : o(\kappa) > \beta\}$ is stationary in κ for all $\beta < \kappa$ then κ remains inaccessible in κ. Such sets cannot be obtained by iterated ultrapowers alone, without forcing. Both of these results are the best possible.

3. Extender Models

The next step above the hierarchy of measurable cardinals is the hierarchy leading to a strong cardinal:

3.1 Definition. A cardinal κ is λ-*strong* if there is an elementary embedding $j : V \to M$ such that $\kappa = \operatorname{crit}(j)$, $\lambda < j(\kappa)$, and $\mathcal{P}^\lambda(\kappa) \subseteq M$. A cardinal κ is *strong* if it is λ-strong for every ordinal λ.

A cardinal is 1-strong if and only if it is measurable; however an embedding of the form i^U, where U is an ultrafilter on κ, will never witness that a cardinal κ is 2-strong since $U \in \mathcal{P}^2(\kappa) - \operatorname{Ult}(V, U)$. An *extender* is a generalized ultrafilter designed to represent the stronger embeddings needed for strong cardinals. Extenders can be equivalently defined in either of two different ways, as elementary embeddings or as sequences of ultrafilters. We will begin with the simpler of the two:

3.2 Definition. A (κ, λ)-*extender* is an elementary embedding $\pi : M \to N$ where M and N are transitive models of ZF^-, $\kappa = \operatorname{crit}(\pi)$, and $\lambda \leq \pi(\kappa)$.

The model M need not be a model of ZF; indeed we can typically assume that κ is the largest cardinal in M since $\mathcal{P}^M(\kappa)$ is the only part of M which will be used for the ultrapower construction. Extenders are so called because the embedding π can be extended to an embedding on a full model M' of set theory, provided that the subsets of κ in M' are contained in those of M:

3.3 Definition. Suppose that $\pi : M \to N$ is an extender and M' is a model of set theory such that $\mathcal{P}^{M'}(\kappa) \subseteq \mathcal{P}^M(\kappa)$.

If $a, a' \in [\lambda]^{<\omega}$, and f and f' are functions in M' with domains $[\kappa]^{|a|}$ and $[\kappa]^{|a'|}$ respectively, then we say that $(f, a) \sim_\pi (f', a')$ if and only if

$(a, a') \in \pi(\{(v, v') \in [\kappa]^{|a|} \times [\kappa]^{|a'|} : f(v) = f'(v')\})$. We write $[f, a]_\pi$ for the equivalence class $\{(f', a') : (f, a) \sim_\pi (f', a')\}$.

Finally we write $\mathrm{Ult}(M', \pi)$ for the model with universe

$$\{[f, a]_\pi : f \in {}^\kappa M' \cap M' \;\;\&\;\; a \in {}^{<\omega}\lambda\},$$

and with the membership relation $\in_\pi$ defined by $[f, a]_\pi \in_\pi [f', a']_\pi$ if $(a, a') \in \pi(\{(v, v') : f(v) \in f'(v')\}$.

The ultrapower embedding $i^\pi : M' \to \mathrm{Ult}(M', \pi)$ is defined by $i^\pi(x) = [x, \varnothing]_\pi$. Here x is regarded as a constant, that is, a 0-ary function.

We will only be interested in extenders such that $\mathrm{Ult}(M', \pi)$ is well-founded and hence isomorphic to a transitive model, and we will identify $\mathrm{Ult}(M', \pi)$ with the transitive model to which it is isomorphic.

The ordinal λ is called the *length* of the (κ, λ)-extender π, and is written $\mathrm{len}(\pi)$. The embedding π does not actually itself determine the value of λ, since the same embedding π could be used as to represent a (κ, λ') extender for any $\lambda' < \pi(\kappa)$. When necessary, the ordinal λ may be explicitly specified, for example by writing $\mathrm{Ult}(M', \pi, \lambda)$ instead of $\mathrm{Ult}(M', \pi)$ or $[f, a]_{\pi,\lambda}$ instead of $[f, a]_\pi$.

If $\lambda < \lambda'$ then a natural elementary embedding

$$k : \mathrm{Ult}(M', \pi, \lambda) \to \mathrm{Ult}(M', \pi, \lambda')$$

can be defined by setting $k([f, a]_{\pi,\lambda}) = [f, a]_{\pi,\lambda'}$. It can be that $\mathrm{Ult}(M', \pi, \lambda) = \mathrm{Ult}(M', \pi, \lambda')$ and k is the identity, in which case we will say that the (κ, λ)- and (κ, λ')-extenders defined by π are equivalent. This will happen whenever there is, for each $\nu \in \lambda'$, some $a \in [\lambda]^{<\omega}$ and $f \in M$ such that $[f, a]_\pi = [\mathrm{id}, \nu]_\pi$. For example, the $(\kappa, \lambda + 1)$- and $(\kappa, \lambda + 2)$-extenders determined by π will always be equivalent, since if s is the successor function, $s(\nu) = \nu + 1$, then $[s, \{\lambda\}]_\pi = [\mathrm{id}, \{\lambda + 1\}]_\pi$.

Łoś's Theorem for extender ultrapowers is proved in the same way as the Łoś's Theorem for ultrafilters:

3.4 Proposition (Łoś's Theorem). *Suppose that $\varphi(v_0, \dots, v_{n-1})$ is any formula of set theory, and that $a_i \in [\lambda]^{<\omega}$ for $i < n$ and $f_i : [\kappa]^{|a_i|} \to \lambda$. Then*

$$\mathrm{Ult}(M', \pi) \models \varphi([f_0, a_0]_\pi, \dots, [f_{n-1}, a_{n-1}]_\pi)$$

if and only if

$$(a_0, \dots, a_{n-1}) \in \pi\big(\{(v_0, \dots, v_{n-1}) : M' \models \varphi(f_0(v_0), \dots, f_{n-1}(v_{n-1}))\}\big).$$

This statement suggests the alternate definition of an extender as a sequence E of ultrafilters:

3.5 Definition. The ultrafilter sequence representing a (κ, λ)-extender π is the sequence $E^\pi = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ of ultrafilters defined by

$$E_a = \{x \subseteq {}^a\kappa : a \in \pi(\{\mathrm{ran}(v) : v \in x\})\}. \tag{17.5}$$

Here we write ran(v) for the sequence $\langle v(a_i) : i < |a|\rangle \in {}^{|a|}\kappa$, where $a = \langle a_i : i < |a|\rangle$. The use of ran$(v)$ instead of v in the right side of (17.5) is necessary because a need not be a member of M. This complication could have been avoided by equivalently defining E_a to be an ultrafilter on subsets of $[\kappa]^{|a|}$ or ${}^{|a|}\kappa$ instead of on ${}^a\kappa$; however the use of ${}^a\kappa$ simplifies some later notation.

3.6 Definition. The ultrapower $\mathrm{Ult}(M', E)$ is defined to be the direct limit of the commuting system of maps

$$\big(\langle \mathrm{Ult}(M', E_a) : a \in {}^{<\omega}\lambda\rangle, \langle \pi_{a,a'} : \mathrm{ran}(a) \subseteq \mathrm{ran}(a')\rangle\big),$$

where $\pi_{a,a'} : \mathrm{Ult}(M', E_a) \to \mathrm{Ult}(M, E_{a'})$ is defined by setting $\pi_{a,a'}([f]_{E_a}) = [v \mapsto f(v{\restriction}a)]_{E_{a'}}$.

It can easily be shown that if π is a (κ, λ)-extender then $\mathrm{Ult}(M, E^\pi) = \mathrm{Ult}(M, \pi, \lambda)$.

In the future we will follow the usual practice of using the ultrafilter representation for extenders. This generally makes for clearer notation, which among other things does not tie down the variables M and N. It also has the advantage of explicitly incorporating the length λ of the extender, but requires additional notation for the shortened extender: if $E = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ is a (κ, λ)-extender and $\lambda' < \lambda$, then we write $E|\lambda'$ for the subsequence $\langle E_a : a \in [\lambda']^{<\omega}\rangle$ of E. Thus $E|\lambda'$ is the (κ, λ')-extender represented by the embedding π^E.

It may happen that $\mathrm{Ult}(V, E|\lambda') = \mathrm{Ult}(V, E)$, in which case we say that the two extenders are equivalent. This will be true whenever there is, for each $\alpha \in \lambda - \lambda'$, a function f and finite set $a \in [\lambda']^{<\omega}$ such that $[f, a]_E = [\mathrm{id}, \{a\}]_E$ or, equivalently, such that $\{v \in {}^{a\cup\{\alpha\}}\kappa : f(v{\restriction}a) = v(\alpha)\} \in E_{a\cup\{\alpha\}}$. As a simple example, by taking f to be the successor function we can see that $E|(\lambda + 1)$ is always equivalent to $E|(\lambda + 2)$.

The notion of countable completeness is somewhat more complicated for extenders than for ultrafilters:

3.7 Definition. An (κ, λ)-extender E is *countably complete* if for each sequence $(a_i : i \in \omega)$ of sets $a_i \in [\lambda]^{<\omega}$ and each sequence $(X_i : i < \omega)$ of sets $X_i \in E_{a_i}$ there is a function $v : \bigcup_i a_i \to \kappa$ such that $v{\restriction}a_i \in X_i$ for each $i < \omega$.

As in the case of ultrafilters, countably complete extenders are important because they ensure well-foundedness of iterated ultrapowers.

3.8 Definition. Suppose that M is a model of set theory and $\mathcal{E}$ is a collection of extenders in M. An iterated ultrapower of M by extenders in $\mathcal{E}$ is a pair of sequences $\langle M_\nu : \nu \leq \theta\rangle$ and $\langle E_\nu : \nu < \theta\rangle$, together with a commuting system of elementary embeddings $i_{\nu,\nu'} : M_\nu \to M_{\nu'}$, such that $M_0 = M$, if ν is a limit ordinal then M_ν is the direct limit of the models $\langle M_{\nu'} : \nu' < \nu\rangle$ under the embeddings $i_{\nu',\nu''}$, and if $\nu < \theta$ then $E_\nu \in i_{0,\nu}(\mathcal{E})$ and $i_{\nu,\nu+1} : M_\nu \to \mathrm{Ult}(M_\nu, E_\nu) = M_{\nu+1}$.

3.9 Lemma. *If $\mathcal{E}$ is a collection of countably complete extenders then any iterated ultrapower using extenders in $\mathcal{E}$ is well-founded.*

Proof. Suppose to the contrary that we have an iterated ultrapower as in Definition 3.8 with M_θ ill-founded. The initial model M_0 could be a proper class, but in that case M_0 can be replaced by an initial segment of M_0 satisfying ZF^- which exhibits the ill-foundedness; thus we can assume that M_0 is a set.

Fix a regular cardinal τ such that the ill-founded iterated ultrapower is a member of $H(\tau)$, the set of sets which are hereditarily of size less than τ, let $X \prec H(\tau)$ be a countable elementary substructure containing the iterated ultrapower, and let $\sigma : P \cong X$ be the inverse of the transitive collapse map. Set $\bar\theta = \sigma^{-1}(\theta)$, and set $\bar E_\nu = \sigma^{-1}(E_{\sigma(\nu)})$ and $\bar M_\nu = \sigma^{-1}(M_{\sigma(\nu)})$ for each $\nu < \bar\theta$.

Set $\bar{\mathcal{E}} = \sigma^{-1}(\mathcal{E})$. Then $(\langle \bar M_\nu : \nu \leq \bar\theta\rangle, \langle \bar E_\nu : \nu < \bar\theta\rangle)$ is an ill-founded iterated ultrapower of $\bar M_0$ of countable length $\bar\theta$, using only extenders from $\bar{\mathcal{E}}$.

We will define a commuting sequence of elementary embeddings

$$\begin{tikzcd}
V & & & & \\
\bar M_0 \arrow[u, "\sigma_0"] \arrow[r, "i_{0,1}"] & \bar M_1 \arrow[ul, "\sigma_1"'] \arrow[r, "i_{1,2}"] & \bar M_2 \arrow[ull, "\sigma_2"'] \arrow[r, "i_{2,3}"] & \cdots \arrow[r] & \bar M_{\bar\theta} \arrow[ullll, "\sigma_{\bar\theta}"']
\end{tikzcd} \tag{17.6}$$

with $\sigma_0 = \sigma \restriction \bar M_0$. Thus $\sigma_{\bar\theta}$ embeds $\bar M_{\bar\theta}$ into V, contradicting the assumption that $\bar M_{\bar\theta}$ is ill-founded and thus completing the proof of the lemma.

The embedding σ_0 has already been defined, and the requirement that the diagram (17.6) commutes determines the choice of σ_α for limit ordinals $\alpha \leq \bar\theta$: if $x \in \bar M_\alpha$ then $\sigma_\alpha(x) = \sigma_{\alpha'}(i^{-1}_{\alpha,\alpha'}(x))$ where α' is any ordinal less than α such that $x \in i_{\alpha',\alpha}``\bar M_{\alpha'}$.

To define $\sigma_{\alpha+1}$, supposing that $\sigma_\alpha : M_\alpha \to H(\tau)$ has been defined, set $\bar\lambda = \operatorname{len}(\bar E_\alpha)$, and let $\langle (\bar X_i, \bar a_i) : i < \omega\rangle$ be an enumeration of the set of pairs (X, a) in M_α such that $a \in [\bar\lambda]^{<\omega}$ and $X \in (\bar E_\alpha)_a$. Then $\sigma_\alpha(\bar X_i) \in (\sigma_\alpha(\bar E_\alpha))_{\sigma_\alpha(\bar a_i)}$, and since $\sigma_\alpha(\bar E_\alpha)$ is a member of the collection $\mathcal{E}$ of countably complete extenders there is a function $v : \bigcup_i \sigma_\alpha(\bar a_i) \to \sigma_\alpha(\bar\kappa)$ such that $v \restriction \sigma_\alpha(\bar a_i) \in \sigma_\alpha(\bar X_i)$ for each $i \in \omega$. Then a straightforward induction shows that the map $\sigma_{\alpha+1} : \bar M_{\alpha+1} \to H(\tau)$ defined by setting $\sigma_{\alpha+1}(x) = \sigma_\alpha(f)(v \restriction \sigma_\alpha(a))$ for each $x = [f, a]_{\bar E_\alpha} \in \bar M_{\alpha+1}$ is an elementary embedding such that $\sigma_\alpha = \sigma_{\alpha+1} \circ \bar i_{\alpha,\alpha+1}$. $\dashv$

This completes the preliminary exposition of extenders, and we now discuss sequences of extenders. The following definition is almost the same as that of a coherent sequence of ultrafilters:

3.10 Definition. A *coherent sequence of nonoverlapping extenders* is a function $\mathcal{E}$ with domain of the form $\{(\kappa, \beta) : \beta < o^{\mathcal{E}}(\kappa)\}$ such that

1. if $o^{\mathcal{E}}(\kappa) > 0$ then $o^{\mathcal{E}}(\lambda) < \kappa$ for every $\lambda < \kappa$,

and if $\beta < o^{\mathcal{E}}(\kappa)$ then

2. $\mathcal{E}(\kappa, \beta)$ is a $(\kappa, \kappa+1+\beta)$-extender E, and
3. $i^{\mathcal{E}(\kappa,\beta)}(\mathcal{E})\restriction(\kappa+1) = \mathcal{E}\restriction(\kappa, \beta)$.

Here $\mathcal{E}\restriction(\kappa, \beta)$ is the restriction of $\mathcal{E}$ to those pairs (κ', β') in its domain which are lexicographically less than (κ, β).

The term *nonoverlapping* refers to clause 1. We will show that nonoverlapping sequences are adequate to construct models with a strong cardinal. It is possible to obtain models with somewhat larger cardinals by weakening clause 1 and modifying the comparison iteration; Baldwin [1] describes a general method of constructing such models. Cardinals very much larger than a strong cardinal, however, require extender sequences $\mathcal{E}$ with overlapping extenders, which greatly complicates the theory of iterated ultrapowers on $L[\mathcal{E}]$, and usually requires the use of iteration trees rather than the linear iterations described in Definition 3.8.

Note that the indexing of the sequences described in Definition 3.10 is the same as that used for sequences of ultrafilters: $\mathcal{E}(\kappa, \beta)$ is the βth extender with critical point κ. This indexing works well for nonoverlapping extenders but fails to be meaningful for sequences with overlapping extenders, where there may be a proper class of extenders with the same critical point κ, and there may be extenders which have critical point κ, but which are stronger than all of the extenders with critical point κ.

All sequences of extenders referred to in this section will be nonoverlapping.

One useful difference between sequences of ultrafilters and sequences of extenders is the fact that the coherence functions for extenders are trivial. The coherence property for a sequence $\mathcal{U}$ of measures depends on the presence, for each $\beta' < \beta < o(\kappa)$, of a function f such that $\beta' = [f]_{\mathcal{U}(\kappa,\beta)}$, or equivalently, such that $\beta' = i^{\mathcal{U}(\kappa,\beta)}(f)(\kappa)$; thus the sequence $\mathcal{U}$ may, for example, be coherent in V but not in $L[\mathcal{U}]$. In the case of a sequence $\mathcal{E}$ of extenders, however, the only coherence function needed is the identity function: if $\beta' < \beta < o^{\mathcal{E}}(\kappa)$ then $\beta' = [\mathrm{id}, \{\beta'\}]_{\mathcal{E}(\kappa,\beta)}$, that is, $\beta' = i^{\mathcal{E}(\kappa,\beta)}(\mathrm{id})(\beta')$. The following proposition, which is not true for sequences of measures, follows immediately:

3.11 Proposition. *If $\mathcal{E}$ is a coherent nonoverlapping sequence of extenders in V and M is an inner model such that the restriction of $\mathcal{E}$ to M is a member of M, then $\mathcal{E}$ is coherent in M.*

In order to define the class $L[\mathcal{E}]$ of sets constructible from $\mathcal{E}$, we can code $\mathcal{E}$ as $\{(\kappa, \beta, a, x) : x \in (\mathcal{E}_{\kappa.\beta})_a\}$. Using this coding, if M is an inner model then $\mathcal{E} \cap M$ is the code for the sequence of restrictions $\langle E_a \cap M : a \in \operatorname{dom} E\rangle$ to M of the extenders E in $\mathcal{E}$.

As we did with sequences of ultrafilters, we need to start with a weaker version of coherence in order to obtain long extender sequences which are coherent in $L[\mathcal{E}]$:

3.12 Definition. A sequence $\mathcal{E}$ of extenders is *weakly coherent* if each extender $E = \mathcal{E}(\kappa,\beta)$ is a $(\kappa,\kappa+1+\beta)$-extender such that $o^{i^E(\mathcal{E})}(\kappa) = \beta$.

3.13 Definition. Suppose that N_0 and M_0 are models with countably complete weakly coherent extender sequences $\mathcal{E}_0$ and $\mathcal{F}_0$, respectively. The *comparison iterations* of N_0 and M_0 are defined as follows: Assume $i_\alpha : M_0 \to M_\alpha$ and $j_\alpha : N_0 \to N_\alpha$ have been defined, and let (κ,β) be the least pair such that one of the following holds:

1. $\beta = o^{i_\alpha(\mathcal{E})}(\kappa) < o^{j_\alpha(\mathcal{F})}(\kappa)$.
2. $\beta = o^{j_\alpha(\mathcal{F})}(\kappa) < o^{i_\alpha(\mathcal{E})}(\kappa)$.
3. $\beta < \min\{o^{j_\alpha(\mathcal{F})}(\kappa), o^{i_\alpha(\mathcal{E})}(\kappa)\}$ and there is an $a \in [\kappa+1+\beta]^{<\omega}$ and $x \in \mathcal{P}({}^a\kappa) \cap M_\alpha \cap N_\alpha$ such that $x \in (i_\alpha(\mathcal{E})(\kappa,\beta))_a - (j_\alpha(\mathcal{F})(\kappa,\beta))_a$.

If there is no such pair (κ,β) then the sequences $i_\alpha(\mathcal{E})$ and $j_\alpha(\mathcal{F})$ have the same domain and are equal, at least with respect to sets which are in both models. If κ is greater than the length of one of the sequences $i_\alpha(\mathcal{E})$ or $j_\alpha(\mathcal{E})$, that is, if $o^{i_\alpha(\mathcal{E})}(\mu) = 0$ for all $\mu \geq \kappa$ or $o^{j_\alpha(\mathcal{F})}(\mu) = 0$ for all $\mu \geq \kappa$, then one of the sequences is an initial segment of the other (again, at least with respect to sets which are in both models). In either case the process is terminated at this stage.

Otherwise define $i_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ to be the ultrapower embedding $i^{i_\alpha(\mathcal{E})(\kappa,\beta)} : M_\alpha \to \mathrm{Ult}(M_\alpha, i_\alpha(\mathcal{E})(\kappa,\beta))$ in cases 2 and 3, and in case 1 define $M_{\alpha+1} = M_\alpha$ and let $i_{\alpha,\alpha+1}$ be the identity. Similarly define $N_{\alpha+1}$ by using the extender $j_\alpha(\mathcal{F})(\kappa,\beta)$ in cases 1 and 3, and set $N_{\alpha+1} = N_\alpha$ in case 2.

The proof that this comparison iteration terminates will use the following proposition, which is proved just like Claim 2.9.

3.14 Proposition. *Suppose that θ is an uncountable regular cardinal, and that we have an iterated extender ultrapower $\langle M_\alpha : \alpha < \theta\rangle$ with iteration embeddings $i_{\alpha',\alpha} : M_{\alpha'} \to M_\alpha$. If X is a set in M_0 such that $|i_\lambda(X)| < \theta$ for each $\lambda < \theta$, and $y_\alpha \in i_{0,\alpha}(X)$ for all $\alpha < \theta$, then for every stationary set $S \subseteq \theta$ there is a stationary set $S' \subseteq S$ such that if $\alpha' < \alpha$ are in S' then $y_\alpha = i_{\alpha',\alpha}(y_{\alpha'})$.*

3.15 Lemma. *If M_0, N_0, $\mathcal{E}$ and $\mathcal{F}$ are as in Definition 3.13, and θ is a regular cardinal such that $\theta \geq \sup\{\, 2^\kappa : o^{\mathcal{E}}(\kappa) > 0$ or $o^{\mathcal{F}}(\kappa) > 0\}$, then the comparison process terminates in fewer than θ steps.*

Proof. Assume the contrary, and at each $\alpha < \theta$ let κ_α and β_α be as in the definition of $M_{\alpha+1}$ and $N_{\alpha+1}$. By applying Proposition 3.14 twice, once to the iterated ultrapower of M_0 and then to that of N_0, we can find a stationary set $S \subseteq \theta$ such that if $\alpha' < \alpha$ are in S then $\kappa_\alpha = i_{\alpha',\alpha}(\kappa_{\alpha'}) = j_{\alpha',\alpha}(\kappa_{\alpha'})$.

Now the sequence $\langle \kappa_\alpha : \alpha < \theta\rangle$ is nondecreasing. Furthermore, whenever $\kappa_{\alpha+1} = \kappa_\alpha$ we have $\beta_{\alpha+1} < \beta_\alpha$, and it follows that for each α there is $k < \omega$

such that $\kappa_\alpha < \kappa_{\alpha+k}$. It follows that $\kappa_{\alpha'} < \kappa_\alpha$ whenever $\alpha' < \alpha$ are limit ordinals.

Now $o^{i_{\alpha+1}(\mathcal{E})}(\kappa_\alpha) = o^{j_{\alpha+1}(\mathcal{F})}(\kappa_\alpha) = \beta_\alpha$ for each $\alpha < \theta$, so case 1 or 2 can only occur at stages α such that $\kappa_{\alpha'} < \kappa_\alpha$ for all $\alpha' < \alpha$. In particular, it never happens that cases 1 and 2 both occur at stages with the same critical point κ_α. For ordinals $\alpha < \alpha'$ in S we have $i_{\alpha,\alpha'}(\kappa_\alpha) = j_{\alpha,\alpha'}(\kappa_\alpha) = \kappa_{\alpha'} > \kappa_\alpha$, so if $\alpha \in S$ and $\alpha^* \geq \alpha$ is the last stage for which $\kappa_{\alpha^*} = \kappa_\alpha$ then case 3 must occur at stage α^*. Finally, let $a_{\alpha^*} \in [\beta_{\alpha^*}]^{<\omega}$ and $x_{\alpha^*} \subseteq [\kappa_\alpha]^{|a_{\alpha^*}|}$ be as in the definition of the comparison at stage α^*. Two more applications of Proposition 3.14 give a stationary $S' \subseteq S$ such that if $\alpha < \gamma$ are in S' then $x_{\gamma^*} = i_{\alpha,\gamma}(x_{\alpha^*}) = j_{\alpha,\gamma}(x_{\alpha^*})$. Set $E_\alpha = i_{0,\alpha}(\mathcal{E})(\kappa_\alpha, \beta_\alpha)$ and $F_\alpha = j_{0,\alpha}(\mathcal{F})(\kappa_\alpha, \beta_\alpha)$. Then we have

$$\begin{aligned} x_\alpha \in (E_{\alpha^*})_{a_{\alpha^*}} &\iff a_{\alpha^*} \in i_{\alpha^*,\alpha^*+1}(x_\alpha) \\ &\iff a_{\alpha^*} \in i_{\alpha^*+1,\gamma} \circ i_{\alpha^*,\alpha^*+1} \circ i_{\alpha,\alpha^*}(x_\alpha) \\ &\qquad\quad = i_{\alpha,\gamma}(x_\alpha) = x_\gamma, \end{aligned}$$

since $i_{\alpha,\alpha^*}(x_\alpha) \cap [\kappa_\alpha]^{|a_{\alpha^*}|} = x_\alpha$ and $i_{\alpha^*+1,\gamma}(a_{\alpha^*}) = a_\alpha$. Similarly, $x_\alpha \in (F_{\alpha^*})_{a_{\alpha^*}}$ if and only if $a_{\alpha^*} \in j_{\alpha,\gamma}(x_\alpha) = x_\gamma$, and hence $x_{\alpha^*} \in (E_{\alpha^*})_{a_{\alpha^*}}$ if and only if $x_{\alpha^*} \in (F_{\alpha^*})_{a_{\alpha^*}}$. This contradicts the choice of x_{α^*} and hence completes the proof of the lemma. ⊣

The proof of Lemma 3.15 relied crucially on the fact that $i_{\alpha^*+1,\gamma}(a_{\alpha^*}) = j_{\alpha^*+1,\gamma}(a_{\alpha^*}) = a_{\alpha^*}$ for all $\alpha < \gamma$ in S; that is, none of the generators arising from a use of an extender in the iteration is moved by the remainder of the iteration. This problem of *moving generators* is the reason that linear iterations like those used in the proof of Lemma 3.15 are not adequate for comparisons of sequences having overlapping extenders. Thus iteration trees are needed for the analysis of inner models with larger cardinals.

When the comparison process terminates, it is only guaranteed that the sequences match with respect to sets which are in both models, so it is important to observe that this is true of all relevant sets:

3.16 Proposition. *Suppose that the comparison maps $i_\theta : L[\mathcal{E}] \to L[i_\theta(\mathcal{E})]$ and $j_\theta : L[\mathcal{F}] \to L[i_\theta(\mathcal{F})]$ terminate with $i_\theta(\mathcal{E})$ equal to $j_\theta(\mathcal{F})\restriction\eta$ in the sense that $o^{i_\theta(\mathcal{E})} = o^{j_\theta(\mathcal{F})}\restriction\eta$ and*

$$i_\theta(\mathcal{E}) \cap L[i_\theta(\mathcal{E})] \cap L[j_\theta(\mathcal{F})] = (j_\theta(\mathcal{F}))\restriction\eta \cap L[i_\theta(\mathcal{E})] \cap L[j_\theta(\mathcal{F})].$$

Then $L[i_\theta(\mathcal{E})] \subseteq L[j_\theta(\mathcal{F})]$, so that $i_\theta(\mathcal{E}) = (j_\theta(\mathcal{F})\restriction\eta) \cap L[i_\theta(\mathcal{E})]$.

Proof. We prove by induction on α that $L_\alpha[i_\theta(\mathcal{E})] \subseteq L_\alpha[j_\theta(\mathcal{F})]$ for all ordinals α. It is only the successor case that could be problematic: assume as an induction hypothesis that $\alpha < \eta$ and $L_\alpha[i_\theta(\mathcal{E})] \subseteq L_\alpha[j_\theta(\mathcal{F})]$. Notice that it follows that $i_\theta(\mathcal{E}) \cap L_\alpha[i_\theta(\mathcal{E})] = j_\theta(\mathcal{F}) \cap L_\alpha[j_\theta(\mathcal{F})]$ if $\alpha \leq \eta$, and $i_\theta(\mathcal{E}) \cap L_\alpha[i_\theta(\mathcal{E})] = (j_\theta(\mathcal{F})\restriction\eta) \cap L_\alpha[j_\theta(\mathcal{E})]$ if $\alpha > \eta$. In either case both $L_\alpha[i_\theta(\mathcal{E})]$ and $i_\theta(\mathcal{E}) \cap L[i_\theta(\mathcal{E})]$ are definable in $L_\alpha[j_\theta(\mathcal{F})]$, and it follows that $L_{\alpha+1}[i_\theta(\mathcal{E})] \subseteq L_{\alpha+1}[j_\theta(\mathcal{F})]$. ⊣

3.17 Definition. Suppose that $\mathcal{E}$ is a weakly coherent sequence and $\varphi(\mathcal{E})$ is a sentence in the language of set theory. Then $L[\mathcal{E}]$ is said to be *φ-minimal* if $L[\mathcal{E}] \models \varphi(\mathcal{E})$ but there is no proper initial segment $\mathcal{E}' = \mathcal{E}\restriction(\kappa, \beta)$ of $\mathcal{E}$ such that $L[\mathcal{E}'] \models \varphi(\mathcal{E}')$.

3.18 Proposition. *Suppose that $\mathcal{E}$ is weakly coherent and $L[\mathcal{E}]$ is φ-minimal for some formula φ, and suppose that $\pi : L[\mathcal{E}] \to L[\mathcal{E}']$ is an elementary embedding. Then the comparison of $L[\mathcal{E}]$ and $L[\mathcal{E}']$ gives the following diagram:*

$$\begin{tikzcd} L[\mathcal{E}] \arrow[rr, "j_\theta"] \arrow[dr, "\pi"'] & & L[j_\theta(\mathcal{E})] \\ & L[\mathcal{E}'] \arrow[ur, "k_\theta"'] & \end{tikzcd} \tag{17.7}$$

Furthermore, if π is definable in $L[\mathcal{E}]$ then this diagram commutes.

Proof. If $j_\theta : L[\mathcal{E}] \to L[j_\theta(\mathcal{E})]$ and $k_\theta : L[\mathcal{E}'] \to L[k_\theta(\mathcal{E}')]$ are the two embeddings generated by the comparison process, then Proposition 3.16 implies that one of the two sequences $j_\theta(\mathcal{E})$ and $k_\theta(\mathcal{E}')$ is an initial segment of the other. Since φ-minimality is a first order property, both of the models $L[j_\theta(\mathcal{E})]$ and $L[k_\theta(\mathcal{E}')]$ are φ-minimal; and it follows that neither can be a proper initial segment of the other. Thus $j_\theta(\mathcal{E}) = k_\theta(\mathcal{E}')$.

It follows that the comparison yields the diagram (17.7). To see that the diagram commutes whenever π is definable, suppose the contrary and let x be the least set in the order of construction of $L[\mathcal{E}]$ such that $j_\theta(x) \neq k_\theta \circ \pi(x)$. Since π is definable in $L[\mathcal{E}]$, the set x is also definable, but this is impossible since then $j_\theta(x)$ and $k_\theta \circ \pi(x)$ are both defined in $L[j_\theta(\mathcal{E})]$ by the same formula and hence must be equal. $\dashv$

3.19 Lemma. *Suppose that $\mathcal{E}$ is a weakly coherent extender sequence and that E is a countably complete $(\kappa, \kappa + 1 + \beta)$-extender in $L[\mathcal{E}]$ such that $o^{i^E(\mathcal{E})}(\kappa) = \beta$. Then $E = \mathcal{E}(\kappa, \beta)$.*

Proof. If this fails then we may assume that $\mathcal{E}$ is φ-minimal for the formula φ asserting that it fails. Pick a counterexample $E \in L[\mathcal{E}]$ with (κ, β) as small as possible and let j_θ and k_θ be the maps arising from the comparison of $L[\mathcal{E}]$ with the model $\mathrm{Ult}(L[\mathcal{E}], E)$. By Proposition 3.18 this gives rise to the following commutative diagram:

$$\begin{tikzcd} L[\mathcal{E}] \arrow[rr, "j_\theta"] \arrow[dr, "i^E"'] & & L[j_\theta(\mathcal{E})] \\ & L[i^E(\mathcal{E})] \arrow[ur, "k_\theta"'] & \end{tikzcd} \tag{17.8}$$

Now all of the extenders $i^E(\mathcal{E})(\kappa, \beta')$ for $\beta' < o^{i^E(\mathcal{E})}(\kappa) = \beta$ are members of $L[\mathcal{E}]$, and by the minimality of (κ, β) it follows that $i^E(\mathcal{E})\restriction(\kappa+1) = \mathcal{E}\restriction(\kappa, \beta)$.

If $o^{\mathcal{E}}(\kappa) = \beta$ then this would imply $j_\theta(\kappa) = \kappa < i^E(\kappa)$, contradicting the commutativity of diagram (17.8). Hence the comparison starts with case 1, so that $j_{0,1} = i^{E'}$, where $E' = \mathcal{E}(\kappa, \beta)$, and $k_{0,1}$ is the identity. Furthermore, $i^{E'}(\mathcal{E}){\upharpoonright}\kappa + 1 = \mathcal{E}{\upharpoonright}(\kappa, \beta) = i^E(\mathcal{E}){\upharpoonright}\kappa + 1$, so $\kappa_1 > \kappa$. Now suppose that $a \in [\kappa+1+\beta]^{<\omega}$ and $x \subseteq [\kappa]^{|a|}$. Then $x \in E_a \iff a \in i^E(x) \iff a \in k_\theta \circ i^E(x)$ and $x \in E'_a \iff a \in i^{E'}(x) = j_{0,1}(x) \iff a \in j_{1,\theta} \circ j_{0,1}(x) = j_\theta(x)$. Since $j_\theta(x) = k_\theta \circ i^E(x)$ it follows that $E = E'$, contrary to the choice of E. ⊣

3.20 Corollary. *If $\mathcal{E}$ is a weakly coherent extender sequence of countably complete extenders, then $\mathcal{E}$ is coherent in $L[\mathcal{E}]$.*

Proof. Suppose to the contrary that $\gamma < o^{\mathcal{E}}(\alpha)$ and $i^{\mathcal{E}(\alpha,\gamma)}(\mathcal{E}){\upharpoonright}\gamma \neq \mathcal{E}{\upharpoonright}(\alpha, \gamma)$. Let $\beta < \gamma$ be such that $\mathcal{E}(\alpha, \beta) \neq i^{\mathcal{E}(\alpha,\gamma)}(\mathcal{E})(\alpha, \beta)$, and apply Lemma 3.19 with $E = i^{\mathcal{E}(\alpha,\gamma)}(\mathcal{E})(\alpha, \beta)$. ⊣

It should be noted that the assumption that the extenders in $\mathcal{E}$ are countably complete is used only to assure that any iterated ultrapower using extenders in $\mathcal{E}$ is well-founded.

3.21 Theorem. *If κ is a strong cardinal, then there is a weakly coherent sequence $\mathcal{E}$ of countably complete extenders such that there is a strong cardinal $\kappa' \leq \kappa$ in $L[\mathcal{E}]$.*

Proof. We define the domain $o^{\mathcal{E}}$ of $\mathcal{E}$ and the extenders $\mathcal{E}(\lambda, \beta)$ using recursion on λ with an inner recursion on β. Suppose that $o^{\mathcal{E}}{\upharpoonright}\lambda$ and $\mathcal{E}{\upharpoonright}\lambda$ have been defined. If λ is not measurable, or if there is some $\lambda' < \lambda$ such that $o^{\mathcal{E}}(\lambda') \geq \lambda$, then set $o^{\mathcal{E}}(\lambda) = 0$. Otherwise define extenders $\mathcal{E}(\lambda, \beta)$ by recursion on β. Suppose that $\mathcal{E}{\upharpoonright}(\lambda, \beta)$ has been defined. If there is a countably complete $(\lambda, \lambda + 1 + \beta)$-extender E such that $o^{i^E(\mathcal{E}{\upharpoonright}\lambda)}(\lambda) = \beta$, then let $\mathcal{E}(\lambda, \beta)$ be any such extender. If there is no such extender E then the inner recursion terminates and $o^{\mathcal{E}}(\lambda)$ is defined to be β.

The sequence $\mathcal{E}$ is coherent in $L[\mathcal{E}]$ by Corollary 3.20. Now a cardinal κ' is strong in $L[\mathcal{E}]$ if and only if $o^{\mathcal{E}}(\kappa') = \mathrm{On}$. The necessity follows from the fact that if $o^{\mathcal{E}}(\kappa') \in \mathrm{On}$ then $\mathcal{E}{\upharpoonright}\kappa' + 1$ is a set, but there is no extender E on κ' in $L[\mathcal{E}]$ such that $\mathcal{E}{\upharpoonright}\kappa' + 1 \in \mathrm{Ult}(L[\mathcal{E}], E)$. To see that the condition $o^{\mathcal{E}}(\kappa') = \mathrm{On}$ is sufficient, let X be any set in $L[\mathcal{E}]$ and fix τ so that $X \in L_\tau[\mathcal{E}]$. Now set $E = \mathcal{E}_{\kappa,\tau}$. Then by coherence $\mathcal{E}{\upharpoonright}(\kappa, \tau) = i^E(\mathcal{E}){\upharpoonright}\kappa + 1$, so $X \in L_\tau[\mathcal{E}] = L_\tau[\mathcal{E}{\upharpoonright}(\kappa, \tau)] = L_\tau[i^E(\mathcal{E})] \in \mathrm{Ult}(L[\mathcal{E}], E)$.

To finish the proof we need to show that there is some $\kappa' \leq \kappa$ such that $o^{\mathcal{E}}(\kappa') = \mathrm{On}$. We may suppose that $o^{\mathcal{E}}(\kappa') < \mathrm{On}$ for all $\kappa' < \kappa$. This implies that $o^{\mathcal{E}}(\kappa') < \kappa$ for all $\kappa' < \kappa$: otherwise there is, for all ordinals β, an extender F on κ so that $i(\kappa) > \beta$ and $V_\beta \subseteq \mathrm{Ult}(V, F)$. Then $o^{i^F(\mathcal{E})}(\kappa') = i^F(o^{\mathcal{E}}(\kappa')) > \beta$, but $i^F(\mathcal{E}){\upharpoonright}\beta = \mathcal{E}{\upharpoonright}\beta$. Since β was arbitrary, this implies that $o^{\mathcal{E}}(\kappa') = \mathrm{On}$, contrary to assumption.

Now suppose that $\mathcal{E}{\upharpoonright}(\kappa, \beta)$ has been defined. We must show that there is a countably complete $(\kappa, \kappa + 1 + \beta)$-extender E such that $i^E(o^{\mathcal{E}})(\kappa) = \beta$.

Since κ is strong in V, there is a countably complete extender F on κ such that $\mathcal{E}\upharpoonright(\kappa,\beta) \in \mathrm{Ult}(V,F)$. Now $i^F(\mathcal{E})$ is defined in $\mathrm{Ult}(V,F)$ in the same way as $\mathcal{E}$ is defined in V. Since $\mathcal{E}(\kappa,\gamma) \in \mathrm{Ult}(V,F)$ for each $\gamma < \beta$, and $\mathcal{E}(\kappa,\gamma)$ is a possible choice for $i^F(\mathcal{E})(\kappa,\gamma)$, we must have $o^{i^F(\mathcal{E})}(\kappa) \geq \beta$.

If $o^{i^F(\mathcal{E})} > \beta$ then set $E = i^F(\mathcal{E})(\kappa,\beta)$. Since V and $\mathrm{Ult}(V,F)$ have the same subsets of κ and $E\upharpoonright\kappa = i^F(\mathcal{E})\upharpoonright\kappa$, E is also a countably complete extender on V and satisfies $o^{i^E(\mathcal{E})} = \beta$. Otherwise, if $o^{i^F(\mathcal{E})}(\kappa) = \beta$, let $E = F|(\kappa+1+\beta)$, the $(\kappa,\kappa+1+\beta)$-extender given by the embedding $i^F : V \to \mathrm{Ult}(V,F)$. Since the identity functions serve as coherence functions for extenders, $i^E(o^{\mathcal{E}})(\kappa) = i^F(o^{\mathcal{E}})(\kappa) = \beta$, and hence E is a suitable choice for $\mathcal{E}(\kappa,\beta)$. ⊣

Theorem 3.21 can be generalized to smaller cardinals: If κ is λ-strong in V then there is a sequence $\mathcal{E}$ such that $o^{\mathcal{E}}(\kappa) > (\kappa^{+\lambda})^{L[\mathcal{E}]}$, and this holds if and only if κ is λ-strong in $L[\mathcal{E}]$.

The next result shows that, as was the case for sequences of measures, the sequence $\mathcal{E} \cap L[\mathcal{E}]$ is uniquely determined by its domain, provided that the extenders $\mathcal{E}(\kappa,\beta)$ are countably complete extenders in V, not merely in $L[\mathcal{E}]$.

3.22 Theorem. *Suppose that $\mathcal{E}$ is a weakly coherent sequence of extenders in V, $\beta < o^{\mathcal{E}}(\kappa)$, and F is a countably complete extender of length $\kappa+1+\beta$ such that $o^{i^F(\mathcal{E})}(\kappa) = \beta$. Then $F \cap L[\mathcal{E}] = \mathcal{E}(\kappa,\beta)$.*

Proof. Let $i_\theta : M_0 := L[\mathcal{E},F] \to M_\theta$ and $j_\theta : N_0 := L[\mathcal{E},F] \to N_\theta$ be iterated ultrapowers comparing the model $L[\mathcal{E},F]$ with itself, with the comparison process modified to include F as an alternative to $\mathcal{E}(\kappa,\beta)$. This means that case 3 of Definition 3.13 is modified to allow $M_{\nu+1}$ to be either of $\mathrm{Ult}(M_\nu, i_\nu(\mathcal{E}(\kappa,\beta)))$ or $\mathrm{Ult}(M_\nu, i_\nu(F))$ if the ultrafilter in question differs on a set in $M_\nu \cap N_\nu$ either from $j_\nu(\mathcal{E})(i_\nu(\kappa), i_\nu(\beta))$ or (in the case $i_\nu(\kappa) = j_\nu(\kappa)$ and $i_\nu(\beta) = j_\nu(\beta)$) from $j_\nu(F)$. Similarly, $j_\nu(F)$ is a candidate for use in defining $N_{\nu+1}$.

Lemma 3.15, asserting that the comparison terminates, is still valid for this comparison. Consider the final models $M_\theta = L[i_\theta(\mathcal{E}), i_\theta(F)]$ and $N_\theta = L[j_\theta(\mathcal{E}), j_\theta(F)]$ of this comparison. One of the sequences $i_\theta(\mathcal{E})$ and $j_\theta(\mathcal{E})$ will be an initial segment (possibly proper) of the other; suppose that $i_\theta(\mathcal{E})$ is an initial segment of $j_\theta(\mathcal{E})$. Then we have that $i_\theta(\mathcal{E}(\kappa,\beta))$ and $i_\theta(F)$ agree with $j_\theta(\mathcal{F})(i_\theta(\kappa), i_\theta(\beta))$, and hence with each other, on all sets in $M_\theta \cap N_\theta$. By the elementarity of i_θ there is a set $x \in M_\theta$ on which $i_\theta(F)$ and $i_\theta(\mathcal{E}(\kappa,\beta))$ differ. Let x be the first such set in the order of construction of M_θ, and suppose that $x \in L_{\tau+1}[i_\theta(\mathcal{E}), i_\theta(F)] - L_\tau[i_\theta(\mathcal{E}), i_\theta(F)]$. Then $L_\tau[i_\theta(\mathcal{E}), i_\theta(F)] = L_\tau[i_\theta(\mathcal{E})] = L_\theta[j_\theta(\mathcal{E})]$ so, as in the proof of Lemma 3.19, $x \in L_{\tau+1}[i_\theta(\mathcal{E})] \subseteq N_\theta$. Thus $x \in M_\theta \cap N_\theta$, contradicting the assumption that $i_\theta(F)$ and $i_\theta(\mathcal{E}(\kappa,\beta))$ differ about x. ⊣

3.23 Corollary. *If $\mathcal{E}$ and $\mathcal{E}'$ are weakly coherent sequences of extenders in V with the same domain then $L[\mathcal{E}] = L[\mathcal{E}']$ and $\mathcal{E} \cap L[\mathcal{E}] = \mathcal{E}' \cap L[\mathcal{E}']$.*

It was previously observed that this statement is false, even for ultrapowers of order 0, if the requirement that $\mathcal{E}$ be a sequence of extenders in V is weakened to require only that they be extenders in $L[\mathcal{E}]$.

We conclude this section by showing that the Generalized Continuum Hypothesis holds in $L[\mathcal{E}]$. The same argument shows that other consequences of condensation such as $\Diamond_\kappa$ and $\Box_\kappa$ also hold in $L[\mathcal{E}]$.

3.24 Theorem. *If $\mathcal{E}$ is a coherent sequence of countably complete extenders in $L[\mathcal{E}]$ then $L[\mathcal{E}] \models \mathrm{GCH}$.*

Proof. The proof of Theorem 3.24 will require the proof of a condensation lemma for $L[\mathcal{E}]$. Let us say that a model M is a *coarse mouse* in $L[\mathcal{E}]$ with projectum ρ if $\pi : M \cong X \prec L_\tau[\mathcal{E}]$ where $L_\tau[\mathcal{E}] \models \mathrm{ZF}^-$ and $X = \mathcal{H}^{L_\tau[\mathcal{E}]}(\{\mathcal{E}\} \cup \rho \cup p)$ for some finite set $p \in L_\tau[\mathcal{E}]$ of parameters. As in the proof of GCH for L, every subset of ρ in $L[\mathcal{E}]$ is in some coarse mouse with projectum ρ, and each such mouse has cardinality $|\rho|$. Hence it will be enough to show that if M and N are coarse mice in $L[\mathcal{E}]$ with the same projectum ρ, then either $\mathcal{P}(\rho) \cap M \subseteq N$ or $\mathcal{P}(\rho) \cap N \subseteq M$.

First, suppose that $o^{\mathcal{E}}(\kappa) \leq \rho$ for all $\kappa < \rho$ and let $i_\theta : M \to P$ and $j_\theta : N \to Q$ be the maps arising from the comparison of $M = L_{\tau_0}[\mathcal{F}_0]$ and $N = L_{\tau_1}[\mathcal{F}_1]$. Then $\mathcal{F}_0 \restriction \rho = \mathcal{F}_1 \restriction \rho = \mathcal{E} \restriction \rho$ and hence both $i_\theta \restriction \rho$ and $j_\theta \restriction \rho$ are the identity. Therefore $\mathcal{P}^M(\rho) = \mathcal{P}^P(\rho)$ and $\mathcal{P}^N(\rho) = \mathcal{P}^Q(\rho)$; and since one of P and Q is contained in the other it follows that one of $\mathcal{P}^M(\rho)$ and $\mathcal{P}^N(\rho)$ is contained in the other, as was to be proved.

In particular, the assumption that there is no overlapping in the sequence $\mathcal{E}$ implies that $2^\kappa = \kappa^+$ in $L[\mathcal{E}]$ for any κ such that $o^{\mathcal{E}}(\kappa) > 0$.

Now suppose that there is $\kappa < \rho$ with $o^{\mathcal{E}}(\kappa) > \rho$, and let $M = L_\alpha[\mathcal{F}]$ be any coarse ρ-mouse in $L[\mathcal{E}]$. If we set $\beta = o^{\mathcal{F}}(\kappa)$, then because $L[\mathcal{E}]$ satisfies GCH at κ we have $\mathcal{P}^{L[\mathcal{E}]}(\kappa) \subseteq M$ and hence the extenders $\mathcal{F}(\kappa, \gamma)$ for $\gamma < \beta$ are all extenders in $L[\mathcal{E}]$. It follows by Lemma 3.19 that $\mathcal{F}(\kappa, \gamma) = \mathcal{E}(\kappa, \gamma)$ for all $\gamma < \beta$, and hence $\mathcal{F} \restriction \kappa + 1 = \mathcal{E} \restriction (\kappa, \beta)$.

Now if M and N are two coarse ρ-mice in $L[\mathcal{E}]$ with $\beta^M = \beta^N$, then the same argument as that used for the case when $o^{\mathcal{E}}(\kappa) \leq \rho$ for all $\kappa < \rho$ implies that one of $\mathcal{P}^M(\rho)$ and $\mathcal{P}^N(\rho)$ is a subset of the other. Thus, if we hold β fixed then there are at most ρ^+ many subsets of ρ which are in some coarse mouse M for $L[\mathcal{E}]$ with projectum ρ and which have $\beta^M = \beta$. Now $\beta^M < \rho^+$ in $L[\mathcal{E}]$ for any such coarse mouse with projectum ρ, so there can be at most ρ^+-many subsets of ρ in $L[\mathcal{E}]$. ⊣

The natural well-ordering of $\mathcal{P}^{L[\mathcal{E}]}(\rho)$ suggested by this proof is given by setting $x \prec y$ if there is $\beta < o^{\mathcal{E}}(\kappa)$ such that x, but not y, is a member of $\mathrm{Ult}(L[\mathcal{E}], \mathcal{E}(\kappa, \beta))$; and otherwise setting $x \prec y$ if x is less than y in the order of construction either of $\mathrm{Ult}(L([\mathcal{E}], \mathcal{E}(\kappa, \beta))$ where β is least such that $x, y \in \mathrm{Ult}(L([\mathcal{E}], \mathcal{E}(\kappa, \beta))$, or of $L[\mathcal{E}]$ if there is no such β. Note that $i^{\mathcal{E}(\kappa,\beta)}(\rho) < \rho^+$ for any $\beta < \rho^+$, and hence this well-ordering has ordertype ρ^+.

3.1. The Modern Presentation of $L[\mathcal{E}]$

Almost all of the description of $L[\mathcal{U}]$ and $L[\mathcal{E}]$ given so far has followed the original style of [25]; the only exception being the brief description at the end of Sect. 1.2 of the application of the modern presentation to $L[U]$ and K^{DJ}. This presentation was invented in order to accommodate larger cardinals than those considered here, but it has several advantages even for models with smaller cardinals, especially when core model and fine structural techniques are being used.

We will now outline some aspects of this new presentation. There are three major changes.

(1) As was pointed out previously, the method of indexing used in the models of this chapter breaks down beyond a strong cardinal. Instead we index extenders in the sequence with a single ordinal. In the original indexing of these models, the index γ for a extender $E = \mathcal{E}_\gamma$ on the sequence is given by $\gamma = (\nu^+)^{L[\mathcal{E}\restriction\gamma]}$ where ν is the larger of κ^+ and the length of the extender E. This choice of ν ensures that E can easily be coded as a subset of ν.

As part of this indexing, the class coding the sequence $\vec{\mathcal{E}}$ is chosen so that $L_\gamma[\mathcal{E}] = L_\gamma[\mathcal{E}\restriction\gamma]$, while $L_{\gamma+1}[\mathcal{E}]$ is the collection of subsets of $L_\gamma[\mathcal{E}]$ which are definable in the structure $(L_\gamma[\mathcal{E}], \mathcal{E}\restriction\gamma, \mathcal{E}_\gamma)$.

This indexing is still commonly used, but Jensen and others have also worked with indexing schemes using indices as large as $i^E(\kappa^+)$.

(2) More importantly, an extender $E = \mathcal{E}_\gamma$ of the sequence $\mathcal{E}$ does not measure all of the sets in $L[\mathcal{E}]$, but instead only measures the sets in $L_\gamma[\mathcal{E}\restriction\gamma]$, that is, the sets already constructed at the time E appears. This is in contrast to the models of this chapter, in which an extender E is expected to measure sets in $L[\mathcal{E}]$ which require E, and even larger extenders, for their construction. Note that if $\kappa = \mathrm{crit}(\mathcal{E}_\gamma)$ then the choice of $\gamma = (\nu^+)^{L_\gamma[\mathcal{E}\restriction\gamma]}$ implies that $\mathcal{P}(\nu) \cap L[\mathcal{E}\restriction\gamma] \subseteq L_\gamma[\mathcal{E}\restriction\gamma] = L_\gamma[\mathcal{E}]$. Thus $\mathcal{E}_\gamma \subseteq L_\gamma[\mathcal{E}]$, and hence $\mathcal{E}_\gamma$ is a member of $L_{\gamma+1}[\mathcal{E}]$.

An extender $\mathcal{E}_\gamma$ with critical point κ_γ will be a full extender in the final model $L[\mathcal{E}]$ if and only if no new subsets of κ_γ are constructed in $L[\mathcal{E}] - L_\gamma[\mathcal{E}]$. The other extenders, those extenders $\mathcal{E}_\gamma$ for which $\mathcal{P}(\kappa_\gamma) \cap L[\mathcal{E}] \not\subseteq L_\gamma[\mathcal{E}]$, are only partial extenders in $L[\mathcal{E}]$; however (as in the discussion of K^{DJ} at the end of Sect. 1.2) they serve as full extenders inside the mice by which these new subsets of κ_γ are constructed. In fact these mice turn out to be exactly the initial segments $L_\alpha[\mathcal{E}] = L_\alpha[\mathcal{E}\restriction\alpha]$ of the model $L[\mathcal{E}]$.

(3) This use of the partial extenders in mice requires the definition and use of a fine structure which is essentially identical to Jensen's fine structure for L. Fine structure is beyond the purview of this chapter, but one important consequence has already been mentioned in connection with K^{DJ}: whenever $\rho < \alpha$ and there is a set $x \in \mathcal{P}(\rho) \cap L_{\alpha+1}[\mathcal{E}] - L_\alpha[\mathcal{E}]$, then $L_{\alpha+1}[\mathcal{E}] \models |\alpha| \leq \rho$.

This discussion ignores one further difference: the model $L[\mathcal{E}]$ (like all recent fine structural arguments) is defined using Jensen's rudimentary hierarchy $J_\alpha[\mathcal{E}]$ instead of the hierarchy $L_\alpha[\mathcal{E}]$ used in this chapter. This change yields a substantial technical simplification, but makes no conceptual difference.

The main disadvantage of the newer approach is evident. The use of fine structure makes the newer models $L[\mathcal{E}]$ more complex than the models $L[\mathcal{U}]$, and furthermore, the extra complexity cannot be delayed, since the model $L[\mathcal{E}]$ cannot even be defined without it.[2] Thus one would want to have a good understanding of the simpler models described here, as well as of fine structure in the simpler setting of L, before studying the newer extender models.

We list below some of the advantages which justify the extra complexity. It should be noted that for larger cardinals there is no choice: the inner models require the newer style—which was in fact invented in order to make inner models for these cardinals possible. However it turns out that arguments using the newer $L[\mathcal{E}]$ style models are simpler, even though the older style $L[\mathcal{U}]$ could have been used instead. The discussion below indicates some of the reasons for this.

(1) A much stronger condensation property holds for the new fine structural models than for those discussed in this chapter. This point was briefly touched on during the discussion of the model $L[U]$ in Sect. 1.2.

(2) The coherence property is simpler and more robust in the fine structural models. We have already seen this as an advantage of using extenders instead of ultrafilters, and this sometimes gives reason to use extender models even when all extenders used turn out to be equivalent to ultrafilters. This advantage is strengthened in the fine structural models, in which all relevant functions have already been constructed before the extender is added.

(3) The use of partial extenders helps to simplify and strengthen the comparison process. Suppose that the two sequences $\mathcal{E}$ and $\mathcal{E}'$ being compared differ first at an ordinal γ, so that $\mathcal{E}\restriction\gamma = \mathcal{E}'\restriction\gamma$ but $\mathcal{E}_\gamma \neq \mathcal{E}'_\gamma$. Then $\mathcal{E}_\gamma$ measures only the sets in $L_\gamma[\mathcal{E}] = L_\gamma[\mathcal{E}\restriction\gamma] = L_\gamma[\mathcal{E}'\restriction\gamma] = L_\gamma[\mathcal{E}']$, which contains the sets measured by $\mathcal{E}'_\gamma$. Hence there is no need for the maneuver used in Definition 3.13, in which two extenders are deemed to differ for the purposes of defining the iterated ultrapower only if they differ on a set in the intersection $L[\mathcal{E}] \cap L[\mathcal{E}']$ of the two models: If $\mathcal{E}_\gamma$ and $\mathcal{E}_{\gamma'}$ differ at all, then they disagree on a member of their common domain $L_\gamma[\mathcal{E}] = L_\gamma[\mathcal{E}']$.

(4) The development of the core model is greatly simplified in fine structural models, because there is no need to treat mice and ultrafilters separately. Under the old approach, the core model was a structure of the form $K = L[\mathcal{U}, \mathcal{M}]$ where $\mathcal{U}$ is a coherent sequence of measures and $\mathcal{M}$ is the

[2] There do exist inner models for larger cardinals which do not use fine structure. These include the original Martin-Steel model [22] for a Woodin cardinal, the HOD models having Woodin cardinals which Woodin obtained from determinacy hypotheses, and Woodin's recent models for cardinals beyond a supercompact. However all of these models fall badly short of being the L-like models we are looking for: for example, it is still not known whether the Martin-Steel models satisfy GCH.

class of mice over $\mathcal{U}$. In fine structural models the core model K has the form $L[\mathcal{E}]$, and the mice used to construct the model are simply the initial segments $L_\gamma[\mathcal{E}]$ of $L[\mathcal{E}]$, with some of the partial measures of $\mathcal{E}$ being used as full measures in the mouse $L_\gamma[\mathcal{E}]$.

This point becomes more important for core models for larger cardinals. In order for the construction to work properly, the mice must reflect the properties of the full core model, and in particular they must be allowed to recursively contain smaller mice. This seems almost prohibitively complicated when working with a core model in the form $K = L[\mathcal{U}, \mathcal{M}]$, with the measures and the mice treated separately, but it falls out naturally in the fine structural model $K = L[\mathcal{E}]$ where a mouse $M = L_\gamma[\mathcal{E}]$ will contain as smaller mice all its initial segments $L_{\gamma'}[\mathcal{E}]$ for $\gamma' < \gamma$.

(5) The fine structural models come much closer to satisfying the analog of Theorem 1.9 than do the models described in this chapter. To see why this is so, consider an argument like that given for Lemma 3.22, where $\mathcal{E}\restriction\gamma$ has been defined and $E = \mathcal{E}_\gamma$ and F is a second extender which could have been chosen as $\mathcal{E}_\gamma$. In the fine structural model both of these extenders measure the same collection of sets, namely the members of the structure $L_\gamma[\mathcal{E}\restriction\gamma]$. Thus instead of using an iterated ultrapower of the structure $L[\mathcal{E}, F]$, in which both extenders are used in the construction, one can use the *bicephelus* $(L_\gamma, \mathcal{E}\restriction\gamma, E, F)$, in which both extenders are available as predicates but neither is used in the construction. The only extra hypothesis on E and F which is needed, beyond the requirement that each extender individually satisfies the conditions to be $\mathcal{E}_\gamma$, is that they are jointly iterable in the sense that all ultrapowers of this structure are well-founded.

One further point should be noted: the principal disadvantage of the fine structural approach, the need to introduce the extra complexity of fine structure at the very beginning, is not an issue in the development of the core model because the fine structure will be required in any case. Indeed incorporating fine structure into the initial definition of $L[\mathcal{E}]$ allows for a much more natural presentation and development of the core model and its fine structure.

4. Remarks on Larger Cardinals

In this section we briefly list some of the most important large cardinals above measurability, in increasing order of size. The primary focus is on the inner model theory available for these large cardinal properties; more information on some of these inner models can be found in later chapters in this Handbook.

All of the large cardinal properties described here are defined by elementary embeddings. Throughout this section, i is always an elementary embedding and M is a well-founded class.

Strong cardinals

Strong cardinals, together with their inner models, have already been introduced and an inner model has been described. It was also pointed out that such simple models, with comparison defined by linear iterations, are inadequate to handle very much larger cardinals. The line beyond which iteration trees are needed is not sharp. Baldwin [1] uses modified linear iterations to handle cardinals substantially larger than strong cardinals, and Schindler [35] has used nearly linear iterations to define a fine structural core model up to the sharp of a proper class of strong cardinals. In the other direction, a careful analysis shows that fine structural models actually use a simple form of iteration tree even down at the level of a 2-strong cardinal, that is, one with an extender E on κ such that $\mathcal{P}^2(\kappa) \subseteq \mathrm{Ult}(V, E)$.

Because of the need for iteration trees rather than linear iterations, it is much more difficult to obtain iterable models for larger cardinals in this range. Indeed, it is not known[3] whether a core model larger than those constructed by Schindler in [35] can be constructed without an added assumption of some large cardinal strength in the universe. Chapter [33] covers the core model and the covering lemma up to a Woodin cardinal.

Woodin cardinals

A cardinal δ is said to be *Woodin* if for all functions $f : \delta \to \delta$ there is an embedding $i : V \to M$ with critical point $\kappa < \delta$ such that $f``\kappa \subseteq \kappa$ and $V_{i(f)(\kappa)} \subseteq M$.

Woodin cardinals were defined by Woodin in 1984, following work of Foreman, Magidor and Shelah [8], and are the most important large cardinal property for current research in set theory. The most notable result concerning Woodin cardinals is probably the equiconsistency of the axiom of determinacy with the existence of infinitely many Woodin cardinals, due to Woodin, Martin and Steel, which is discussed in chapters [29] and [17].

This, and other consequences of Woodin cardinals, depend largely on two forcing notions which can be used to prove that inner models for Woodin and stronger cardinals must differ in important respects from those for smaller cardinals. The first of these forcing notions is *stationary tower forcing*, which was defined by Woodin using ideas from Foreman, Magidor and Shelah [9, 8]. In one form, this forcing will preserve a Woodin cardinal δ, while making massive changes to the cardinal structure below δ: for example, there is a stationary subset of singular cardinals below δ whose successors are collapsed by the forcing. Hence there cannot be a core model satisfying the weak covering property for (exactly) a Woodin cardinal, although there is one for the sharp of a Woodin cardinal. In addition, stationary tower forcing

[3] It is now known, by a recent unpublished result of Jensen and Steel, that if there is no model with a Woodin cardinal then the core model K can be constructed with no extra large cardinal hypothesis.

can collapse ω_1, and in the process will add new countable mice. This forcing is discussed in the book [20].

The second forcing notion, invented by Woodin, is the remarkable "all sets are generic" forcing: If M is a model with a Woodin cardinal δ, and M is iterable in V, then there is a forcing notion $P \in M$ of size δ such that for any set x in V there is a tree iteration of M, with final model N and embedding $i: M \to N$, such that $N[x]$ is a generic extension of N by the forcing $i(P)$. This forcing can be used to show that the minimal model M for a Woodin cardinal cannot satisfy the sentence asserting that M is iterable, even when M is iterable in the universe V. The implications of this for the core model are discussed further at the end of this chapter.

At present it is not known how to construct core models for cardinals in this range without some large cardinal properties holding in the universe. Jensen has shown that a subtle cardinal, a property weak enough to hold in L, is enough to show prove that the core model exists and satisfies the covering lemma; however it is an open question whether this assumption is needed. Other than this gap, the core model theory through ω many Woodin cardinals is well understood [39]. The strongest current result on existence of iterable inner models is due to Neeman, who has constructed [30] iterable extender models with a Woodin limit of Woodin cardinals. These models, however, are not fine structural, and no core model results are known in this region.

Superstrong Cardinals

A cardinal κ is *superstrong* if there is an embedding $i: V \to M$ with critical point κ such that $V_{i(\kappa)} \subseteq M$.

As was pointed out previously, a superstrong cardinal is at the outer limits of our understanding of inner models. Much of the basic inner model theory is understood up to a superstrong cardinal: for example it is known [34] that $\square_\kappa$ holds in any extender model up through a superstrong cardinal. Indeed they show that $\square_\kappa$ holds in an extender model $L[\mathcal{E}]$ for any cardinal κ short of what Jensen has labeled a *subcompact cardinal.* Jensen has shown that $\square_\kappa$ cannot hold if κ is subcompact. However it is not known, under any large cardinal assumption, that there are any iterable extender models with anything near a superstrong cardinal.

Supercompact Cardinals

A cardinal κ is λ-*supercompact* if there is an embedding $i: V \to M$ with critical point κ such that ${}^{\lambda}M \subseteq M$, and κ is *supercompact* if κ is λ-supercompact for all cardinals λ.

None of the models described in this chapter give any promise of yielding models with a supercompact cardinal. However Woodin has recently proposed a form of model, using what he calls *suitable extender sequences* which can include supercompact cardinals and even the larger cardinals discussed in the next paragraph, and which he hopes to show have many of the properties enjoyed by the extender models $L[\mathcal{E}]$ which have been discussed in this chapter.

Like these models, Woodin's models have the form $L[\mathcal{E}]$, the class of sets constructible from a sequence of extenders. An important difference is that not all of the extenders witnessing large cardinal properties are members of the sequence $\mathcal{E}$; in fact all of the critical points of extenders on the sequence are below the first supercompact cardinal. It is still not known whether these models have an analog of the comparison process of Lemma 2.8, and no proofs are known for their iterability.

Larger Cardinals

A number of cardinals larger than supercompact have been defined. Some of these have important consequences, notably huge cardinals and variants of these. A cardinal κ is *huge* if there is an elementary embedding $i : V \to M$ with critical point κ such that ${}^{i(\kappa)}M \subseteq M$.

Catalogs of large cardinal properties, such as this one, traditionally end with a nontrivial elementary embedding from V into V, which Kunen proved in [19] to be inconsistent. It is still open whether such an embedding is consistent with ZF without the axiom of choice.

5. What is the Core Model?

This section is not intended to be a description of existing core models, but rather an examination of the term "core model" itself. We will try to determine the meaning of the phrase "the core model", and in particular explain the difference between it and the term "extender model". The structure, construction and properties of known core models is described elsewhere in this chapter and in chapters [24, 33, 38] and [36]. In addition the reader may want to look at [27], which discusses from a relatively non-technical point of view the use of iteration trees and the construction of the Steel core model up to a Woodin cardinal.

Our first approach will be to look at the history of the term "core model", which was introduced by Dodd and Jensen [5, 6] for the model which is variously referred to as the Dodd-Jensen core model, K^{DJ}, or the core model below a measurable cardinal. The history, however, begins earlier—at least as far back as Jensen's discovery of the covering lemma for L, since the Dodd-Jensen core model generalizes this result. The model $L[U]$ also predated K^{DJ}, and although $L[U]$ is not contained in the structure K^{DJ} which Dodd and Jensen referred to as the core model, they proved [7] the covering lemma for $L[U]$ and hence brought this model into the modern pantheon of core models. Their work was extended by Mitchell to include sequences of measures. The core model to this point is described in chapter [24]. The use of extenders as a generalization of normal ultrafilters, and of iteration trees as a generalization of iterated ultrapowers, led to the Steel core model, which is described in chapters [33, 38]. This model is currently at the frontier of the subject.

Of the two terms under consideration, only "extender model" has a precise meaning: an extender model is a model of the form $L[\mathcal{E}]$ where $\mathcal{E}$ is a good sequence of extenders as defined in chapter [38]. Every known core model is an extender model, but this should not be assumed to be true for larger cardinals; indeed it seems unwise to be dogmatic about the properties of as yet unknown core models until we have a better idea of what is possible.

Even keeping this caveat in mind, "the core model" is always singular: there is at most one core model in any given model of set theory, and in particular there is at most one true core model in the true universe of sets.

Some authors have used the term "core model" to mean the same as "extender model". While it is true that every known core model is an extender model, and that generally, or arguably always, an extender model is its own core model, the distinction between the terms is important and should be preserved. The term *extender model* describes the interior structure of the model, while the term *core model* refers to the relation between the model and the class of all sets.

Some illumination on this point can be gained by looking at cases in which we find it useful, in apparent contradiction to the dictum in the last paragraph, to speak of "*a* core model". It is often useful to refer to a model as "a core model" if it is the core model as defined inside some model which is of particular interest, but is not necessarily the universe of all sets. In a related usage, the term "core model" is often used for a model obtained by a particular construction which is known to yield the core model under additional assumptions such as the nonexistence of some large cardinal property. The Dodd-Jensen core model K^{DJ} is an example of both usages: It is characterized by its mode of construction, which is an initial segment of the core model construction in every model for which such a construction is known. It is also characterized by the fact that it (or at least $K^{\mathrm{DJ}} \cap M$) is *the* core model inside *any* model M, so long as M does not have an inner model with a measurable cardinal. Core models for larger cardinals are less clear cut, since the core model for a model M varies with the particular extenders which are members of M, even though the large cardinal strength of the model is held fixed. There is a unique core model, however, for the sharps of such large cardinal properties.

The second approach to understanding the term "core model" is through consideration of the properties of the known core models. These properties fall into two classes. The properties in the first class are those which hold in any extender model: These models are built up from below, in a manner analogous to the construction of L, and as a consequence they satisfy some sort of condensation. They satisfy combinatorial principles such as $\Diamond_\kappa$ and (for cardinals small enough that we currently have a core model) $\Box_\kappa$. They satisfy the generalized continuum hypothesis, they satisfy the global axiom of choice and their well-ordering, both of their reals, and of their full universe, has a logical form which is as simple as possible in any model with the same large cardinal properties.

The other class of properties of the core models are those which might be seen as asserting that the model is close to V. The most important of these is the covering lemma, or at the least some form of the weak covering lemma. A second is rigidity: there is no nontrivial elementary embedding $i : K \to K$. A third is absoluteness: the core model is absolute for a class of sentences which falls just short of including the sentence asserting that there is a set not in that core model.

It is unclear to what extent we should assume that these properties will necessarily hold for larger core models. Even down at the level of a Woodin cardinal, without the sharp of a Woodin cardinal, there is no inner model which satisfies both weak covering and invariance under forcing; and properties which seem close to rigidity fail well below a Woodin cardinal.

A final property bridges these two classes: The core models are uniquely defined by a formula which is absolute under set generic extensions. This formula says on the one hand that the model is built up from below as an extender model $L[\mathcal{E}]$, and on the other that the construction is greedy, including everything appropriate into the sequence $\mathcal{E}$. If we take the first class of properties as evidence of minimality then we could take something like the following as the definition of the core model:

5.1 Definition. The *core model* is the minimal class inner model of ZF which contains all of the large cardinal structure which exists in the universe.

We could modify the statement by requiring ZFC rather than ZF, but it seems better to regard the axiom of choice as a consequence (so far, at least) of minimality.

Although it is labeled a “definition”, Definition 5.1 is not intended to be a precise mathematical definition. Neither “minimal” nor “large cardinal structure” have a precise meaning. The phrase “minimal class inner model of ZF” is, perhaps, reasonably clear. We can take “minimal” to mean $\subseteq$-minimal, which works for all known core models—provided a suitable meaning for the term “large cardinal structure” is understood.

The meaning of this term is somewhat more problematic. One important point is that “large cardinal structure” is not the same as “large cardinal properties”. The model $L[U]$ is not $\subseteq$-minimal among all models having a measurable cardinal; for example $\mathrm{Ult}(L[U], U)$ is a proper subclass of $L[U]$. However $L[U]$ is the minimal model containing the filter $U \cap L[U]$, and it seems quite clear that the ultrafilter U should be included as part of the large cardinal structure. There are more doubtful cases in which Definition 5.1 may be at least potentially circular: once a particular model K has been anointed as “the core model” there will be a tendency to take the “large cardinal structure” of the universe to be just that structure which is contained in K.

As a case study to illustrate how the line might be drawn, we consider the situation when there is a Woodin cardinal δ, but no sharp for a model with a Woodin cardinal. There is an obvious candidate for the core model in this case, namely the extender model $L[\mathcal{E}]$ given by Steel's core model construction described in chapter [33]. It might be objected that this model is not really obtained by Steel's construction of the core model, but rather as a limiting case of that construction: Steel's construction gives a sequence of models $K_\theta = L_\theta[\mathcal{E}_\theta]$ for measurable cardinals $\theta < \delta$. Each of the models K_θ is unequivocally the core model in V_θ, and the extender sequences $\mathcal{E}_\theta$ of the models K_θ agree so as to yield a combined sequence $\mathcal{E} = \bigcup_\theta \mathcal{E}_\theta$ such that δ is Woodin in $L[\mathcal{E}]$. This objection is a reason for caution, but is irrelevant to the application of Definition 5.1, which deliberately avoids specifying a particular means of construction. A second objection to the model $L[\mathcal{E}]$ is that it is not iterable: Woodin's "all sets are generic" forcing demonstrates that there is an iteration tree of height δ which can be defined in $L[\mathcal{E}]$, but which has no well-founded branch in $L[\mathcal{E}]$. Again this is a reason for caution but is not necessarily fatal: the iterability of the model might well be considered as large cardinal structure, but it is large cardinal structure which does not exist in the universe and thus cannot be expected to exist in the core model. In fact, for example, the existence of a model $L[\mathcal{E}]$ with a Woodin cardinal such that every iteration tree in $L[\mathcal{E}]$ has a well-founded branch in V implies the existence of a class of indiscernibles for $L[E]$.

A more significant question is raised by Woodin's stationary tower forcing, which massively violates the weak covering lemma. The cardinal δ is still Woodin in the generic extension, but it is possible to arrange (and is possibly impossible to avoid) that every sufficiently large successor cardinal below δ is collapsed. This probably should not bother us: we can consider this to be analogous to Prikry forcing at a measurable cardinal, which shows that if there is a measurable cardinal then no core model will satisfy the covering lemma in all generic extensions. It is true that the situation at a measurable cardinal is well understood while that at a Woodin cardinal is quite hazy, but the analogy seems reasonable.

It has been argued that it is not really necessary to give up the weak covering lemma because there is a second candidate for the core model. If we assume that the ground model is $L[\mathcal{E}]$, then an $L[\mathcal{E}]$-generic set G for the stationary tower forcing is essentially an extender which gives an elementary embedding $i^G : L[\mathcal{E}] \to L[i^G(\mathcal{E})]$ with the property that $V_\delta \cap L[\mathcal{E}][G] \subseteq L[i^G(\mathcal{E})]$. In particular, $L[i^G(\mathcal{E})]$ does satisfy the covering lemma in the generic extension $L[\mathcal{E}][G]$, and furthermore, $L[i^G(\mathcal{E})]$ is the model obtained as described above using Steel's construction inside $L[\mathcal{E}][G]$. We could take $L[i^G(\mathcal{E})]$ as the core model, provided that we are willing to give up invariance under forcing. In favor of $L[\mathcal{E}]$, we could assert that $i : L[\mathcal{E}] \to L[i^G(\mathcal{E})]$ should be regarded as analogous to $\mathrm{Ult}(L[U], U) = L[i^U(U)]$, and note that $L[i^U(U)]$ is certainly not the core model. This view is supported by Woodin's [41] extensive and fruitful theory of iterated ultrapowers using generic embeddings such as i^G, but it is weakened by the fact that it throws no light on the failure of the weak covering lemma.

The model $L[\mathcal{E}]$ is certainly the core model according to Definition 5.1, at least inside the ground model $L[\mathcal{E}]$ itself. The question is whether the mice in $L[i^G(\mathcal{E})] - L[\mathcal{E}]$ should be included as part of the large cardinal structure of $L[\mathcal{E}][G]$. For an answer to this question we consider another analogy with $L[U]$: Jensen has proved (see Theorem 3.43 in chapter [24]) that if H is a $L[U]$ generic Levy collapse, then in $L[U][H]$ there is an embedding $i : K^{\mathrm{DJ}} \to K^{\mathrm{DJ}}$ such that $\mathrm{crit}(i)$ is smaller than $\mathrm{crit}(U)$. The embedding is constructed from a model $N = L_\alpha[U^N]$ in $L[U][H] - L[U]$ which is iterable and satisfies ZF^-. In fact N is a mouse; however it is not a mouse in the sense of K^{DJ} because there is no subset of $\mathrm{crit}(U^N)$ in $L_{\alpha+1}[U^N] - L_\alpha[U^N]$. Now U^N is not an ultrafilter in $L[U^N]$, so let $\alpha' > \alpha$ be the least ordinal such that there is a subset of $\mathrm{crit}(U^N)$ in $L_{\alpha'+1}[U^N] - L_{\alpha'}[U^N]$. If $N' = L_{\alpha'}[U^N]$ were iterable then it would be a member of K^{DJ}, and that is not true because U^N measures all sets in K^{DJ} while there is a set in $L_{\alpha'+1}[U^N]$ which U^N does not measure. Thus N' is not iterable; in fact the set in $L_{\alpha'+1}[U^N]$ which is not measured by U^N can be constructed from a sequence of functions in N' which witnesses that $\mathrm{Ult}(N', U^N)$ is not well-founded.

The extra information given by the ordinal $\alpha' > \alpha$ shows that $L_\alpha[U^N]$ is, in an extended sense, not really iterable. Similarly, the information given by the extender sequence $\mathcal{E}$ shows that the supposed mice M which are in $L[i^G(\mathcal{E})]$ but not in $L[\mathcal{E}]$ are not really iterable: if we attempt to compare M with $L[\mathcal{E}]$ then the tree on M has height δ and has no well-founded cofinal branch, as any such branch could be used to construct the sharp for a Woodin cardinal. Thus it seems appropriate to conclude that M is not part of the large cardinal structure of $L[\mathcal{E}][G]$, and hence that $L[\mathcal{E}]$ is the core model in $L[\mathcal{E}][G]$.

Why then does Steel's construction seem to go wrong here? As was suggested earlier, it is not the construction which is in error: If θ is a measurable cardinal below δ then every mouse in the model $K_\theta = L_\theta[i^G(\mathcal{E})\restriction\theta]$ is iterable in $V_\theta^{L[\mathcal{E}][G]}$, and hence K_θ really is the core model in the universe $V_\theta^{L[\mathcal{E}][G]}$. The only error is in assuming that the limit of these local core models will be a core model in $V^{L[\mathcal{E}][G]}$: it is not, because its "mice" are not iterable there.

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18. The Covering Lemma

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Contents

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1. The Statement

Ronald Jensen's discovery of the covering lemma arose out of work on the singular cardinals problem. Paul Cohen published his proof of the independence of the continuum hypothesis [4, 5] in 1963, and one year later William Easton's thesis [13, 14] completely settled the question of the size of the continuum for regular cardinals. The continuum problem for singular cardinals remained open, and the *Singular Cardinal Hypothesis* (SCH), stating (in its simplest form) that $2^\lambda = \lambda^+$ for every singular strong limit cardinal, became one of the most important problems in set theory. It was ten years before Jack Silver made the first significant advance on the problem: In a sharp contrast to Easton's result, which stated that the only constraints on the size of the continuum for regular cardinals are the obvious ones, Silver [56] proved that SCH cannot fail at a singular cardinal of uncountable cofinality unless it already fails at all but a nonstationary set of smaller cardinals. Silver's proof, which depends heavily on the use of the filter of closed unbounded subsets of λ, fails badly at cardinals of cofinality ω and attention turned immediately to understanding this case. A year later, in 1974, Jensen distributed a series of handwritten notes titled *Marginalia to a Theorem of Silver*.[1] These notes, later revised by Keith Devlin and Jensen and published [7] under the same title, stated and proved the basic covering lemma for L:

1.1 Theorem (Covering Lemma for L). *If $0^\#$ does not exist then for any set x of ordinals there is a set $y \in L$ such that $y \supseteq x$ and $|y| = |x| + \aleph_1$.*

It is an immediate corollary that $\neg 0^\#$ implies SCH: Theorem 1.1 implies that any function $f : \mathrm{cf}(\lambda) \to \lambda$ is determined by a covering set $y \supseteq \mathrm{ran}(f)$ in L of size at most $\max\{\aleph_1, \mathrm{cf}(\lambda)\}$, together with a function from $\mathrm{cf}(\lambda)$ into y. Thus $\lambda^{\mathrm{cf}(\lambda)} \leq (\lambda^{\mathrm{cf}(\lambda)})^L \, \tau^{\mathrm{cf}(\lambda)} = \lambda^+ 2^{\mathrm{cf}(\lambda)} = \max\{\lambda^+, 2^{\mathrm{cf}(\lambda)}\}$, where $\tau = \max\{\aleph_1, \mathrm{cf}(\lambda)\}$. This implies the more general form of SCH, $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+ \, 2^{\mathrm{cf}(\lambda)}$ for every singular cardinal λ, and this in turn implies $2^\lambda = \lambda^+$ if λ is a singular strong limit cardinal.

The most obvious direction in which to extend the covering lemma is by weakening the assumption $\neg 0^\#$ to allow larger cardinals in the universe. The first step in this direction was due to Anthony Dodd and Jensen, who constructed a core model K^{DJ} under the assumption that there is no inner model with a measurable cardinal [9, 10, 8]. The Dodd-Jensen core model is, in many ways, similar to L: it satisfies GCH along with most of the combinatorial properties of L, and it satisfies an analogous covering lemma:

1.2 Theorem (Covering Lemma for K^{DJ}). *Assume that there is no inner model with a measurable cardinal, and let K^{DJ} be the Dodd-Jensen core model. Then for any set x of ordinals there is a set $y \in K^{\mathrm{DJ}}$ such that $y \supseteq x$ and $|y| = |x| + \aleph_1$.*

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[1] It is worth pointing out that about 20 years later, during the 1990's, this same problem led to another of the major advances in set theory, Saharon Shelah's pcf theory ([53], see chapter [1]).

The statement cannot be extended directly to larger cardinals, as Prikry forcing [46] gives a counterexample. However, Dodd and Jensen generalize Theorem 1.2 to show that Prikry forcing is the only possible counterexample [11]:

1.3 Theorem (Covering Lemma for $L[U]$). *Assume that $0^\dagger$ does not exist but that there is an inner model with a measurable cardinal, and that the model $L[U]$ is chosen so that $\kappa = \mathrm{crit}(U)$ is as small as possible. Then one of the following two statements holds:*

1. *For every set x of ordinals there is a set $y \in L[U]$ with $y \supseteq x$ and $|y| = |x| + \aleph_1$.*
2. *There is a sequence $C \subseteq \kappa$, which is Prikry generic over $L[U]$, such that for all sets x of ordinals there is a set $y \in L[U, C]$ such that $y \supseteq x$ and $|y| = |x| + \aleph_1$.*

Furthermore, the sequence C of clause 2 is unique up to finite initial segments.

Theorem 1.3 can easily be generalized to models with no inaccessible limit of measurable cardinals, but two problems have to be overcome to extend it to larger cardinals: (i) it is necessary to construct a core model which can consistently contain larger cardinals and for which the basic argument of the proof of the covering lemma can be made to work, and (ii) it is necessary to find a useful statement of the covering lemma for this core model which can be proved from the basic argument. The construction and basic properties of the core model are given in chapters [47] and [57], or in [29]. In addition the current chapter includes, in Sect. 4, an outline of the theory of the core model for sequences of measures and for non-overlapping extenders.

A statement of the full covering lemma for these models will be deferred to Sect. 4 of this chapter, but Theorem 1.8 below states a simplified version which generalizes the result of Dodd and Jensen by showing that a singular cardinal which is regular in K is made singular by a set which approximates a Prikry-Magidor generic set (see [32, Sect. 2.2]). This statement requires some preliminary definitions.

Say that a cardinal κ is *μ-measurable* if there is an embedding $i : V \to M$ such that the measure $\{x \subseteq \kappa : \kappa \in i(x)\}$ associated with i is a member of M. This is the weakest large cardinal property which requires the existence of something more than normal ultrafilters.

For the rest of this subsection we assume that there is no inner model with a μ-measurable cardinal, and we assume that K is the core model. If κ is a cardinal of K and $\beta < o(\kappa)$ then use $\mathcal{U}(\kappa, \beta)$ to denote the measure of order β on κ in K. First we define what appears to be a rather weak notion of indiscernibility:

1.4 Definition. Assume that κ is a singular cardinal which is regular in K. A closed unbounded subset C of κ is a *weak Prikry-Magidor set* for K if (i) $|C| < \kappa$, and (ii) if x is any closed unbounded subset of κ with $x \in K$ then $C - x$ is bounded in κ.

Any Prikry-Magidor generic subset of κ is a weak Prikry-Magidor set.

1.5 Theorem. *If there is no model with a μ-measurable cardinal then any weak Prikry-Magidor set $C \subseteq \kappa$ for K has the following two properties:*

1. *C is eventually contained in any set $a \in K$ such that $a \in \mathcal{U}(\kappa, \beta)$ for all $\beta < o(\kappa)$.*
2. *$C \cap \lambda$ is a weak Prikry-Magidor set over K for every sufficiently large limit point λ of C.*

1.6 Definition. A function σ is an *assignment function in K* for C if

1. There is an $h \in K$ such that $\sigma(\nu) = h(\nu)$ for all sufficiently large $\nu \in C$.
2. C is a set of indiscernibles for $\mathcal{U}(\kappa, \sigma(\nu))$ in the sense that for any sequence $\langle a_\xi : \xi < \kappa \rangle \in K \cap {}^\kappa\mathcal{P}(\kappa)$ of subsets of κ, and for all sufficiently large $\nu \in C$, we have $\forall \xi < \nu \, (\nu \in a_\xi \iff a_\xi \in \mathcal{U}(\kappa, \sigma(\nu)))$.

If $o(\kappa) < \kappa$, as in Prikry-Magidor forcing, then we can always take σ to be the function $\sigma(\nu) = o(\nu)$. If $o(\kappa) \geq \kappa^+$ then Radin forcing (cf. chapter [15]) can be used to add a set C which satisfies the definition of a weak Prikry-Magidor set except that κ remains regular, and hence $|C| = \kappa$. Clearly such a set does not have an assignment function, since any assignment function would be bounded in κ^+. Thus the following theorem would be false if the requirement that $|C| < \kappa$ were dropped from Definition 1.4.

1.7 Theorem. *Any weak Prikry-Magidor set $C \subseteq \kappa$ for K has an assignment function in K. Furthermore*

1. *The assignment function σ is unique except for initial segments.*
2. *The assignment function is weakly increasing in the sense that $\sigma(\nu) \geq \limsup\{\sigma(\xi) + 1 : \xi \in C \cap \nu\}$ for every sufficiently large limit point ν of C.*

Any weak Prikry-Magidor set C which satisfies the stronger version of clause 2 obtained by changing the inequality to an equality, and in particular has $\sigma(\xi) = 0$ for each successor member ξ of C, is a Prikry-Magidor generic set.

We cannot hope to actually cover subsets of κ using indiscernibles for only a single cardinal κ, but the following theorem, which is our promised version of the covering lemma, generalizes Dodd and Jensen's Theorem 1.3 to say that any small subset of κ can be approximated by a weak Prikry-Magidor set:

1.8 Theorem. *If a singular cardinal κ is regular in K then for any set $x \subseteq \kappa$ with $|x| < \kappa$ there is a weak Prikry-Magidor set $C \subseteq \kappa$ for K and a function $g : \kappa \to \kappa$ in K such that $x - \bigcup_{\nu \in C} (g(\nu) - \nu)$ is bounded in κ.*

The Weak Covering Lemma

No satisfactory statement of the full covering lemma is known for cardinals much larger than a single strong cardinal: the indiscernibles are too complicated to use to approximate arbitrary sets in the manner of Theorem 1.3 or Theorem 1.8. What remains is known as the weak covering lemma, which is proved by using the same basic proof as that used below a strong cardinal, but applying it only to subsets of the interval $(\lambda, (\lambda^+)^K)$, in which there cannot be any indiscernibles.

1.9 Definition. A class model M of set theory satisfies the *weak covering property* if $(\lambda^+)^M = \lambda^+$ for every singular cardinal λ of V.

The weak covering lemma, stating that K has the weak covering property, is among the most important consequences of the covering lemma. If K contains more than a few measurable cardinals then the weak covering property is needed to prove the basic properties of the core model, including the full covering lemma; indeed the weak covering property may be taken as part of the definition of what it means to be a "core model". The best results known to date are as follows:[2]

1.10 Theorem.

1. *If the sharp for a model with a class of strong cardinals does not exist, then there is a core model K of the form $L[\mathcal{E}]$ which satisfies the weak covering property (see [50]).*

2. *If there are no inner models with a Woodin cardinal and there is a subtle cardinal θ, then the Steel core model $K_\theta = L_\theta[\mathcal{E}]$ below θ exists, and satisfies the weak covering property for $\lambda < \theta$ (see chapter [47]).*

The proof of clause 1 will be sketched in Sect. 4. The proof of clause 2 is given in [45].

It is not clear what, if anything, can be done in the actual vicinity of a Woodin cardinal. Mitchell [42] reports some unsatisfactory results from

[2] It is now known, by recent unpublished work of Jensen and Steel, that if there is no model with a Woodin cardinal then there is a core model K as in clause 1. This new result supersedes clauses 1 and 2.

The proof uses a new technique which was discovered by Jensen and is described in the recent paper [27], which deals with a model K^c of the form $L[\mathcal{E}]$ under the assumption that there is no model with a cardinal κ which is simultaneously a limit of Woodin cardinals and of cardinals strong to κ. Although this paper does not actually show that the weak covering lemma holds in K^c, it does show that a weak covering property holds in the structures which Jensen calls "stacks". This is sufficient to show that many core model arguments can be extended to such large cardinals. Indeed the technique works up to a superstrong cardinal; however, it is not known that the model K^c exists for such cardinals.

applying the standard proof at a Woodin cardinal, but the following result of Woodin may suggest a more useful direction. The theorem only applies below $2^{\aleph}$ (assuming AC in V) but that is the region where the large cardinals implied by AD exist. This result also goes beyond the large cardinal limit of $\neg$AD imposed by Theorem 1.10.

1.11 Theorem (Woodin [64]). *Suppose that the nonstationary ideal on ω_1 is $\aleph_2$-saturated, and suppose that M is a transitive inner model of* ZF + DC + AD *containing all reals and ordinals such that every set of reals in M is, in V, weakly homogeneously Souslin. Let X be a bounded subset of Θ^M such that $|X| = \aleph_1$. Then there exists a $Y \supseteq X$ in M such that $|Y|^M = \aleph_1$.*

Here Θ^M is the supremum of the ordinals δ in M such that there is a map in M from the reals onto δ.

The Strong Covering Lemma

This concludes, until Sect. 4, the discussion of cardinals larger than a measurable cardinal. We now return to the models L and $L[U]$ in order to look at another direction in which the original covering lemma has been extended. The *strong covering lemmas* use Jensen's proof but show that more can be extracted from it. Theorem 1.12, which is our version of the strong covering lemma for the Dodd-Jensen core model, is essentially taken from unpublished notes of Timothy Carlson, who proved it for L by using a variant, influenced by ideas of Silver, of Jensen's proof. The idea, as well as the name, comes from work of Shelah (see [53, Theorem VII.0.1] and [54]) who obtains the strong covering property in a more general setting by assuming the ordinary covering property together with some extra combinatorial structure. We will describe his main application in the next section.

1.12 Theorem (Strong Covering Lemma). *Assume that there is no inner model with a measurable cardinal. Then there is a class $\mathbf{C} \subseteq K^{\mathrm{DJ}}$, definable in K^{DJ}, such that the following statements hold:*

1. *If x is any uncountable set of ordinals then there is a set $X \in \mathbf{C}$ such that $x \subseteq X$ and $|x| = |X|$.*

2. *The class $\mathbf{C}$ is closed in V under increasing unions of uncountable cofinality; that is, if $\langle X_\nu : \nu < \eta \rangle$ is an increasing sequence of members of $\mathbf{C}$ and $\mathrm{cf}(\eta) > \omega$ then $\bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$.*

Notice that clause 2 holds for all sequences $\langle X_\nu : \nu < \eta \rangle$, not only for those which are members of K.

The statement of Theorem 1.12 remains valid if $L[U]$ exists but $0^\dagger$ does not, provided that K is replaced by the appropriate model $L[U]$ or $L[U, C]$ from Theorem 1.3. The following statement, however, is stronger and is easier to generalize to larger core models (see Sect. 4).

1.13 Theorem. *Assume that $0^\dagger$ does not exist, and that the measure U and Prikry sequence C are as in Theorem 1.3(2). Then there is a class $\mathbf{C} \subseteq L[U, C]$ which satisfies clauses 1 and 2 of Theorem 1.12, and in addition*

3. *For each set $X \in \mathbf{C}$ there is an ordinal $\rho < \max\{\aleph_2, |X|^+\}$ and a function $h \in L[U]$ such that $X = \mathcal{H}^h(\rho \cup C)$, the smallest set containing $\rho \cup C$ and closed under h.*

4. *The class $\mathbf{C}$ is definable in $L[U]$ in the sense that there is a formula φ such that a set X is in $\mathbf{C}$ if and only if there is a set $A \in U$, a function $h \in L[U]$ and an ordinal ρ such that $L[U] \models \varphi(A, h, \rho)$ and $X = \mathcal{H}^h(\rho \cup (C \cap A))$.*

Clause 4 follows from the definability of forcing: the formula $\varphi(A, h, \rho)$ asserts that $(\varnothing, A) \Vdash \mathcal{H}^h(\check{\rho} \cup \dot{C}) \in \dot{\mathbf{C}}$, where the forcing is Prikry forcing for the measure U, $\dot{C}$ is a name for the resulting Prikry sequence, and $\dot{\mathbf{C}}$ is a name, derived from the proof of the covering lemma, for the class $\mathbf{C}$.

The following proposition gives a very useful property of the function h. It is also true for L, for the Dodd-Jensen core model, and for the core model for sequences of measures.

1.14 Proposition. *Let h be as in Theorem 1.13 for $X \in \mathbf{C}$. Then h can be written as $h = \bigcup_{\nu<\alpha} h_\alpha$ for some functions $h_\nu \in X$ such that $h_\nu \subseteq h_{\nu'}$ whenever $\nu < \nu' < \alpha$.*

The Covering Lemma without Second-Order Closure

The strong covering lemma can be viewed as asserting that if $0^\#$ does not exist then every sufficiently closed set is a member of L. The precise statement of the requirement that X be sufficiently closed has both first-order and second-order components. Magidor's covering lemma [31] for L weakens the conclusion of the covering lemma in order to eliminate the second-order components:

1.15 Theorem (Magidor [31]). *If $0^\#$ does not exist and x is a set of ordinals which is closed under the primitive recursive set functions, then there are sets $y_n \in L$ for $n < \omega$ such that $x = \bigcup_{n<\omega} y_n$.*

Magidor also extends Theorem 1.15 to the Dodd-Jensen core model by requiring closure under a larger set of functions in K^{DJ} and assuming that there is no inner model with an ω_1-Erdős cardinal. He points out that this assumption is necessary, since if there is an ω_1-Erdős cardinal in K then there is a generic extension M of K such that for any countable set $\mathcal{F}$ of functions in K there is a set $X \in M$ which is closed under the functions in $\mathcal{F}$, but is not a countable union of sets in K.

The following theorem was proved independently of Theorem 1.15, but the same idea lies behind both theorems.

1.16 Theorem (Mitchell [33, 41], Jensen [12]). *If there is no model with a Woodin cardinal then every regular Jónsson cardinal is Ramsey in the core model K. Furthermore, if κ is δ-Jónsson for some uncountable ordinal $\delta < \kappa$ then κ is δ-Erdős in K.*

A cardinal κ is said to be *δ-Jónsson* if any structure with universe κ and countably many predicates has an elementary substructure with order type δ; and κ is said to be *δ-Erdős* [3] if for any such structure and any closed unbounded subset C of κ there is a normal set of indiscernibles of order type δ contained in C.

A similar proof shows that Chang's conjecture implies that ω_2 is ω_1-Erdős in K, and together with a result of Silver (1967, unpublished) proves the equiconsistency of the two notions.

This concludes our discussion of the various statements of the covering lemma. In Sect. 2 we will briefly describe some of the basic applications of the core model, and in Sect. 3 we will outline the basic proof of the covering lemma and its variants under the hypothesis that $0^\dagger$ does not exist. The final section looks at larger cardinals, giving the statement and an outline of the proof of the covering lemma for sequences of ultrafilters or extenders. The basic proof is taken almost unchanged from Sect. 3, but the analysis of the resulting system of indiscernibles is much more difficult.

2. Basic Applications

We pointed out earlier that the source of the covering lemma, as well as its first application, is the Singular Cardinal Hypothesis:

2.1 Theorem (Jensen [7]). *If $0^\#$ does not exist, $\lambda^{\mathrm{cf}(\lambda)} = \max\{\lambda^+, 2^{\mathrm{cf}(\lambda)}\}$ for every singular cardinal λ and hence $2^\lambda = \lambda^+$ for every singular strong limit cardinal λ.*

Jensen's proof can be generalized to larger cardinals, but the full strength of the failure of SCH was not discovered until Moti Gitik combined the covering lemma with Shelah's pcf theory:

2.2 Theorem (Gitik [17, 20]). *The failure of the Singular Cardinal Hypothesis is equiconsistent with $\exists\kappa(o(\kappa) = \kappa^{++})$.*

In Sect. 4.3 we present Gitik's proof that the failure of the singular cardinal hypothesis implies that there is an inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$. Gitik's proof that this is sufficient is given in [16]; in this Handbook [15] he describes a later method of forcing which is simpler and more general, but which gives slightly weaker results in this case.

The following theorem is Shelah's main application of the Strong Covering Lemma 1.12. The sufficiency of slightly stronger large cardinal assumptions is proved in [55].

2.3 Theorem. *If M is a model containing K such that $M \models \mathrm{GCH}$, and r is a real such that $M[r] \models \neg\mathrm{CH}$, then there is an inner model with an inaccessible cardinal. If, in addition, the cardinals of $M[r]$ are the same as those of M then there is an inner model with a measurable cardinal.*

The Weak Covering Lemma

By far the most important consequence of the covering lemma is the weak covering property, Definition 1.9. Indeed it is arguably more accurate to turn the statement around: the covering lemma is an application, and not necessarily the most important application, of the weak covering lemma. Below a strong cardinal the proof of the weak covering lemma is a special case of the proof of the full covering lemma, so that the importance of the weak covering lemma is not immediately apparent. Beyond a strong cardinal, in Steel's core model, we do not know how to even begin the proof of the covering lemma without first proving, by an entirely different method using a weak large cardinal hypothesis, a slightly weaker version of the weak covering lemma.

Among the most important properties of the core model K (stated under the assumption that $0^\P$ does not exist) which follow from the weak covering lemma are the following:

- The construction of K from the countably complete core model K^{c}.
- If $i : K \to M$ is an elementary embedding, where M is well-founded, then i is an iterated ultrapower of K.
- If U is a normal K-ultrafilter on κ and $\mathrm{Ult}(K,U)$ is well-founded then $U \in K$. If $\mathrm{crit}(U) > \omega_2$ then the hypothesis that $\mathrm{Ult}(K,U)$ is well-founded can be omitted.

Many results which are usually regarded as consequences of the covering lemma in fact use only these basic properties of the core model. Among such results are the lower bounds in the following theorem:

2.4 Theorem.

1. *The failure of* GCH *at a measurable cardinal κ is equiconsistent with $\exists\kappa(o(\kappa) = \kappa^{++})$.*
2. *If κ is weakly compact and $o^K(\kappa) < \kappa^{++}$ then $(\kappa^+)^K = \kappa^+$.*
3. *If κ is Jónsson, there is no model with a Woodin cardinal, and the Steel core model exists (in particular, if there is no model with a strong cardinal), then $(\kappa^+)^K = \kappa^+$; furthermore $(\lambda^+)^K = \lambda^+$ for stationarily many $\lambda < \kappa$ [62].*
4. *The consistency of a Woodin cardinal implies that of the existence of a saturated ideal on ω_1. If the Steel core model exists then the existence*

of such an ideal implies in turn that there is a Woodin cardinal in an inner model [58].

Sketch of Proof. We prove, as an example, the lower bound for clause 1. The upper bound is proved in Gitik [16]. Suppose that U is a measure on κ and $2^\kappa \geq \kappa^{++}$, but that $o(\kappa) < \kappa^{++}$ in K. Let $i^U : V \to M = \mathrm{Ult}(V, U)$, and consider $i = i^U \restriction K : K \to K^M$. Then i is an iterated ultrapower of K, so let $i = i_{0,\theta}$ where $i_{\nu,\nu'} : N_\nu \to N_{\nu'}$. If $\nu < \theta$ is a limit ordinal then there are $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{Ult}(N_\nu, i_{\xi_\nu,\nu}(U_\nu))$. Since $o(\kappa) < \kappa^{++} \leq \theta$, there is a stationary class $S \subseteq \kappa^{++}$ of ordinals of cofinality ω such that $\xi_\nu = \bar{\xi}$ and $U_\nu = \bar{U}$ are constant for $\nu \in S$. Now fix $\nu \in S \cap \lim(S)$. If $\vec{\kappa} = \langle \nu_n : n < \omega \rangle$ is a cofinal sequence in $S \cap \nu$ and $\kappa_n = \mathrm{crit}(i_{\nu_n,\nu})$, then the sequence $\vec{\kappa}$ generates the measure $i_{\bar{\xi},\nu}(\bar{U})$. Since ${}^\omega M \subseteq M$, the sequence $\vec{\kappa}$ and hence the measure $i_{\bar{\xi},\nu}(\bar{U})$ is a member of M. It follows that $i_{\bar{\xi},\nu}(\bar{U}) \in K^M$; but this is impossible since $i_{\bar{\xi},\nu}(\bar{U})$ is not in $N_{\nu+1}$ and hence not in $N_\theta = K^M$. This contradiction completes the proof that $o(\kappa) \not< \kappa^{++}$ in K. ⊣

The naïve proof of clause 2 uses the fact that κ is inaccessible; however, Ralf Schindler [49] has adapted it to successor cardinals, showing that if κ has the tree property and $2^\kappa > \kappa^+$ then κ is strong in K.

The main reason for the importance of the weak covering property is that it can be used to adapt to K techniques which Kunen (see [28, §21]) originally used in proving that $0^\#$ follows from the existence of a nontrivial elementary embedding from L into L. As applied to L these techniques make use of the fact that any proper class $\Gamma \prec L$ is isomorphic to L. The corresponding fact for K is that any class $\Gamma \prec K$ is isomorphic to K, provided that the class

$$\{\lambda : \mathrm{ot}(\Gamma \cap (\lambda^+)^K) = (\lambda^+)^K\} \tag{18.1}$$

is stationary. Cardinal calculations show that the classes Γ used in Kunen's arguments satisfy that

$$\{\lambda : 2^{<\lambda} = \lambda \wedge \mathrm{cf}(\lambda) < \lambda \wedge |\Gamma \cap \lambda^+| = \lambda^+\} \tag{18.2}$$

is stationary. The weak covering lemma implies that the class (18.2) is contained in the class (18.1) and hence implies that $\Gamma \cong K$.

The Full Covering Lemma

The Singular Cardinal Hypothesis has already been mentioned as a result which requires the full covering lemma. We now look at other such results.

2.5 Theorem (Dodd-Jensen [11], Mitchell [37]). *Let κ be a singular cardinal of cofinality λ which is regular in K. Then κ is measurable in K, and if $\lambda > \omega$ then $o(\kappa) \geq \lambda$ in K.*

The proof, using Theorems 1.7 and 1.8, is easy, and a more careful analysis yields a classification of singular cardinals [38]:

2.6 Theorem (Mitchell [38]). *Assume that* $\neg\exists\kappa\,(o(\kappa) = \kappa^{++})$. *Let* κ *be a singular cardinal which is regular in* K. *Then there is a cofinal set* $C \subseteq \kappa$ *of ordertype* $\mathrm{cf}(\kappa)$ *such that*

1. *If* $\mathrm{cf}(\kappa) > \omega$ *then* C *is a weak Prikry-Magidor set (Definition 1.4).*
2. *If* $\mathrm{cf}(\kappa) = \omega$ *then let* $\beta \leq o(\kappa)$ *be the least ordinal such that* $o(\nu) < \beta$ *for all but boundedly many* $\nu \in C$. *Then*
   - *(a) If* β *is a successor ordinal then* C *is Prikry generic over* K.
   - *(b) If* $\mathrm{cf}^K(\beta) < \kappa$ *then* $\mathrm{cf}(\beta) = \omega$, *and* C *is a weak Prikry-Magidor sequence.*
   - *(c) If* $\mathrm{cf}^K(\beta) = \kappa$, *witnessed by* $\tau : \kappa \to \beta$, *then there is a weak Prikry-Magidor sequence* D *with assignment function* σ *such that the increasing enumeration* $\langle c_n : n < \omega\rangle$ *of* C *is definable recursively from* D *by letting* c_{n+1} *be the least member* c *of* D *such that* $\sigma(c) \geq \tau(c_n)$.
   - *(d) If* $\mathrm{cf}^K(\beta) = \kappa^+$ *then* C *is a sequence of accumulation points for* κ *(the definition of an accumulation point is given in Definition 4.18).*

Further, the set C *can be chosen to be maximal in a sense which makes it definable up to initial segment, except in case (2d) where any two such sequences eventually alternate.*

A measure U on κ is a *weak repeat point* if for every set $A \in U$ there is a $U' \lhd U$ with $A \in U'$. Results similar to the following theorem have been proved by Gitik [19, 22] for the nonstationary ideal.

2.7 Theorem (Mitchell [34]). *If the closed, unbounded filter on* ω_1 *is an ultrafilter, then there is a weak repeat point in* K.

2.8 Theorem (Sureson [60], Mitchell [36]). *The following four statements are equiconsistent, where* $\delta < \kappa$ *is a regular cardinal.*

1. *There is a* κ*-complete ultrafilter* U *on* κ *extending the closed, unbounded filter such that* $\{\alpha : \mathrm{cf}(\alpha) = \delta\} \in U$.
2. *There is a* κ*-complete ultrafilter* U *on* κ *with* δ *skies; that is, there is an increasing sequence* $\langle\alpha_\nu : \nu < \delta\rangle$ *of ordinals between* κ *and* $i^U(\kappa)$ *with the property that* $i^U(f)(\alpha_\nu) < \alpha_{\nu'}$ *for all* $\nu < \nu' < \delta$ *and all* $f : \kappa \to \kappa$.
3. *There is a* κ^+*-saturated normal filter* F *with* $\{\alpha : \mathrm{cf}(\alpha) = \delta\} \in F$.
4. $o(\kappa) = \delta + 1$ *if* $\delta > \omega$, *and* $o(\kappa) = 2$ *if* $\delta = \omega$.

The covering lemma is used to prove that each of clauses 1–3 imply that clause 4 holds in K. The forcing used in [36] to prove the other direction has been simplified and extensively generalized by Gitik; in particular it is used to give the upper bounds for the consistency strength of the failure of SCH.

The $\mathbf{\Sigma}^1_3$-absoluteness theorem, Theorem 2.9 below, was originally proved by Jensen assuming $\neg 0^\dagger$; Magidor (unpublished, see [59, §4]) has given a simpler proof but one which gives slightly less information. Clause 1 was proved under the assumption that $\neg\exists\kappa\,(o(\kappa)=\kappa^{++})$ by Mitchell [39] using Jensen's method. Steel and Welch [59] later proved clause 1 using Magidor's method, and Steel, using results of Hjorth, extended it [58, Theorem 7.9] to prove clause 2.

We say that a model M is $\mathbf{\Sigma}^1_3$-*correct* if for any Σ^1_3 formula φ and any real $r\in M$ we have $M\models\varphi(r)$ if and only if $V\models\varphi(r)$.

2.9 Theorem ($\mathbf{\Sigma}^1_3$-absoluteness).

1. *Suppose that there is no inner model of $\exists\kappa\,(o(\kappa)=\kappa^{++})$ and that $r^\#$ exists for every real r. Then any model M of* ZFC *such that $M\supseteq K$ is $\mathbf{\Sigma}^1_3$-correct.*

2. *Assume that there are two measurable cardinals and no inner model with a Woodin cardinal. Then any model M of* ZFC *such that $M\supseteq K$ is $\mathbf{\Sigma}^1_3$-correct.*

The conclusion can be equivalently stated as "$\mathbf{\Sigma}^1_3$ formulas are absolute for models containing K".

3. The Proof

This section outlines the proof of the Jensen and Dodd-Jensen covering lemmas up through a single measurable cardinal. Section 4 will continue, using the same basic ideas, to describe the covering lemmas for larger cardinals.

Section 3.1 briefly describes the basic tools, including fine structure, needed for the proof. Section 3.2 gives the proof of Jensen's covering lemma for L, Theorem 1.1 (including the proof of the strong variant, Theorem 1.12). Section 3.3 extends this proof to the Dodd-Jensen covering lemma, Theorems 1.2 and 1.3. Finally Sect. 3.4 looks at the two major variants on the covering lemma: Magidor's Theorem 1.15 and Theorem 1.16, stating that Jónsson cardinals are Ramsey in K.

The proofs given in this section are not complete, but enough details are given so that a reader with some understanding of fine structure should be able to fill in the rest. Complete proofs may be found in the original sources, [7–11, 26, 31], or in later references such as Devlin [6].

3.1. Fine Structure and Other Tools

This section has two incompatible aims: the first is to be accessible to a reader without a sophisticated knowledge of fine structure, and the second is to present a proof which is sufficiently complete that a reader with a understanding of fine structure can fill in the details.

One very interesting approach to this dilemma was invented by Silver (see [28, 31]), who gave a proof of the Jensen covering lemma which essentially eliminates any need for fine structure. He has extended this method to yield the Dodd-Jensen covering lemma, and it has been further extended and publicized by Magidor. In unpublished work, Magidor and Silver have used this approach at least up a model with a cardinal α such that $o(\alpha)$ is measurable. It is not known whether this approach works up to $o(\alpha) = \alpha^{++}$, and it seems unlikely that it will work for the newer models containing cardinals up to a Woodin cardinal. This rules out its use here, since this section is intended to serve as an introduction to covering lemmas for larger models.

The approach we have used is very close than that presented by Schindler and Zeman earlier in this Handbook [52]. We have attempted to make this section accessible without such an introduction: The hope is that this presentation will be sufficiently generic that a knowledgeable reader will be readily able to translate it to his preferred version, while at the same time it is sufficiently specific (without being too detailed) that it is understandable to a naïve reader. However, any reader wanting a full understanding of the subject is encouraged to read [52] before or after this section.

Our presentation of fine structure, like Jensen's original papers, is based directly on master code structures. We follow current practice in using Jensen's J_α hierarchy, rather than the L_α hierarchy. This newer hierarchy yields substantial advantages, some of which will be pointed out in the text, for a complete exposition of the fine structure; however, the differences are not apparent at this level of detail and the naïve reader will lose little, if anything, by simply reading J_α as L_α.

One unfortunate exception to this equivalence comes from the fact that members $M = J_\alpha$ of the J_α hierarchy are conventionally indexed by $\alpha = \mathrm{On}(M)$, which is always a limit ordinal. Thus the γth member of this hierarchy is $J_{\omega\cdot\gamma}$, which is nearly the same as L_γ. In particular $J_{\omega\cdot\gamma+n}$ does not exist for $0 < n < \omega$: the successor of a member $J_{\omega\cdot\gamma}$ of the hierarchy is $J_{\omega\cdot\gamma+\omega}$.

At some points in the arguments, primarily those involving the Downward Extension Lemma, it did not seem possible to give the full proof without being more specific about the fine structure; in these cases we restrict ourselves to giving the proof in the simplest case, which is Σ_1 definability over J_α for a limit ordinal α. This case may seem very special, but in fact it essentially contains the general case. See [52] for a more complete discussion of fine structure.

As stated earlier, we take the basic models of our fine structure to be the sets J_α. We will call these models *mice* in anticipation of larger core models.

Two concepts are basic to the fine structure of a mouse $M = J_\alpha$: the Σ_n projectum ρ^M_n and the Σ_n-Skolem function h^M_n. A third concept which is central to the proof of the covering lemma is the Σ_n-ultrafilter $\mathrm{Ult}_n(M, \pi, \kappa)$ of M, obtained by using the embedding π as an extender. A fourth concept, the use of substructures of mice, is used in the definition of fine structure and is central to the proof of $\square_\kappa$ and other combinatorial applications of fine structure; however, it is more peripheral to the proof of the covering lemma, where it is only needed for the non-countably closed case.

We discuss these four concepts further before beginning the actual proof. We begin with the definition of the fine structure of J_α in the special case when $n = 1$ and α is a limit ordinal.

3.1 Definition. Assume that α is a limit ordinal, and that $M = (J_\alpha, A)$ is *amenable*, that is, $A \cap x \in J_\alpha$ for all $x \in J_\alpha$.

1. The Σ_1 *projectum* ρ^M_1 of an amenable structure $M = (J_\alpha, A)$ is the least ordinal ρ such that there is a Σ_1 subset x of ρ which is not a member of J_α, but is Σ_1-definable in M using a finite set $p \subseteq \alpha$ as a parameter.

2. The Σ_1 *standard parameter* p^M_1 of M is the least finite sequence $p \in [\alpha]^{<\omega}$ of ordinals such that there is some set $x \subseteq \rho^M_1$ so that $x \notin J_\alpha$, but x is Σ_1-definable in M from parameters in $\rho^M_1 \cup p$.

   The ordering of the parameters is lexicographical on descending sequences of ordinals; that is, $p < p'$ if $\max(p \triangle p') \in p'$.

3. The Σ_1 *standard master code* is the set A^M_1 of pairs $(\ulcorner\varphi\urcorner, \xi)$ such that $\xi < \rho^M_1$ and $\ulcorner\varphi\urcorner$ is the Gödel number of a Σ_1 formula φ over M, with parameter p^M_1, such that $M \models \varphi(\xi)$.

4. The Σ_1-*Skolem function* h^M_1 of M is defined as follows: fix an enumeration $\langle \exists z \varphi_n : n < \omega \rangle$ of the Σ_1 formulas of set theory. Then $h^M_1(\langle n, x \rangle)$ is defined if and only if there are z and y such that $M \models \varphi_n(x, y, z, p^M_1)$. In this case $h^M_1(\langle n, x \rangle) = y$ where (α', z, y) is the lexicographically least triple such that $(J_{\alpha'}, A \cap \alpha') \models \varphi_n(x, y, z, p^M_1)$.

5. The Σ_1-*code* $\mathfrak{C}_1(M)$ of M is the structure $(J_{\rho^M_1}, A^M_1)$.

It should be noticed that the Σ_1-Skolem function is itself Σ_1-definable over M. The Σ_1-Skolem function is a function of one variable; however, we will frequently abuse the notation by writing it as a function with a variable number of arguments. Thus $h^M_1(x_1, x_2, x_3)$ should be understood to mean $h^M_1(\ulcorner x_1, x_2, x_3 \urcorner)$ where $\ulcorner \ldots \urcorner$ is an appropriate coding of finite sequences. In addition, we will abuse notation by writing h_1"x to mean h_1"$^{<\omega}x$, the closure of x under the Skolem function h_1.

We will rarely be using the function h_1^M as a Skolem function for any particular formula φ_n, and so will not normally mention the parameter n explicitly, regarding it instead as being coded into the stated parameters.

If α is a successor ordinal, $\alpha = \gamma + 1$, then the definitions are the same, except that the hierarchy $\langle J_{\alpha'} : \alpha' < \alpha \rangle$ used for the definition of h_1^M is replaced by a hierarchy, with length ω, of the sets in $J_{\omega\cdot\gamma+\omega} - J_{\omega\cdot\gamma}$. The hierarchy depends on the specific fine structure being used. Jensen originally used the Levy hierarchy on L_γ, the kth level of which contains the subsets of L_γ which are Σ_k-definable in (L_γ, A). Later he invented the rudimentary functions and the hierarchy of sets $J_{\omega\cdot\alpha}$ in order to avoid technical complications caused by the use of the Levy hierarchy. See chapter [52] for a detailed presentation of the rudimentary functions and their use in setting up the fine structure.

One major advantage of the J_α hierarchy over the L_α hierarchy is that $[J_\alpha]^{<\omega} \subseteq J_\alpha$ even for successor ordinals α. Thus a finite set of ordinals $\alpha_0, \ldots, \alpha_{k-1}$ can be freely treated as a single parameter, the finite sequence $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$. In the case of the L_α hierarchy some awkward and painful coding is necessary to achieve the same result.

We will not say more about the successor case, except to mention that arguments involving fine structure generally treat the case of successor α as simpler special cases of the arguments for limit ordinals α. This characterization of the case of successor α as "simpler" assumes, of course, an understanding of the detailed definition of the fine structure.

We now turn to consider the fine structure for $n > 1$. The central theme of fine structure is that it is never necessary to deal directly with Σ_{n+1} definability for any n greater than zero; instead a Σ_{n+1} formula is reduced to an equivalent Σ_1 formula over the Σ_n-code of J_α. The definition of the Σ_n-code $\mathfrak{C}_n(J_\alpha)$ is itself a good example of this theme.

3.2 Definition. We define the Σ_n-codes of J_α by recursion on $n < \omega$. We set $\mathfrak{C}_0(J_\alpha) = (J_\alpha, \varnothing)$, and for $n \geq 0$

$$\rho_{n+1}^{J_\alpha} = \rho_1^{\mathfrak{C}_n(J_\alpha)} \qquad p_{n+1}^{J_\alpha} = p_1^{\mathfrak{C}_n(J_\alpha)} \qquad h_{n+1}^{J_\alpha} = h_1^{\mathfrak{C}_n(J_\alpha)}$$
$$A_{n+1}^{J_\alpha} = A_1^{\mathfrak{C}_n(J_\alpha)} \qquad \mathfrak{C}_{n+1}(J_\alpha) = \mathfrak{C}_1(\mathfrak{C}_n(J_\alpha))$$

Finally, the projectum of J_α is defined to be $\mathrm{proj}(J_\alpha) = \rho^{J_\alpha} = \inf_n \rho_n^{J_\alpha}$. Since the sequence of projecti $\langle \rho_n^{J_\alpha} : n < \omega \rangle$ is nonincreasing, $\rho_n^{J_\alpha} = \mathrm{proj}(J_\alpha)$ for all sufficiently large $n < \omega$.

Note that if $n > 1$ then the Σ_n-Skolem function h_n^M need not be Σ_n-definable over M. Jensen's Σ_n-uniformization theorem states that this construction can be used to define a Skolem function for Σ_n formulas over J_α which is Σ_n-definable in J_α (though not uniformly so). This Σ_n-uniformization theorem was an important part of the motivation for Jensen's invention of fine structure, but it has turned out to have little direct importance because it is simpler and more useful to work directly with the fine structure.

We now consider the problem of recovering the original structure $M = (J_\alpha, A)$ from its code $M_1 = \mathfrak{C}_1(M) = (J_{\rho_1^M}, A_1^M)$. In order to do so, recall that the Skolem function h_1^M is Σ_1-definable in M from the parameter p_1^M, and that $A_1^M = \{\ulcorner \varphi_n(a, p_1^M) \urcorner : n < \omega \;\; \& \;\; a \in (\rho^{M_1})^{<\omega}\}$, the set of Gödel numbers of the Σ_1 theory of M with parameters from $\rho_1^M \cup p_1^M$. Now define X to be the set of equivalence classes $[\xi]_\sim$, where $\xi \in \rho_1^M \cap \mathrm{dom}(h_1^M)$ and $\xi \sim \xi'$ if and only if $\ulcorner h_1^M(\xi) = h_1^M(\xi') \urcorner \in A_1^M$. The membership relation E on X is defined by $[\xi] \mathrel{\mathrm{E}} [\xi']$ if and only if $\ulcorner h_1^M(\xi) \in h_\xi^M(\xi') \urcorner \in A_1^M$; and the subset $\bar{A} \subseteq X$ is defined by setting $[\xi] \in \bar{A}$ if and only if $\ulcorner h_1^M(\xi) \in A \urcorner \in A_1^M$.

It is straightforward to verify that we can define a Σ_1-elementary embedding $i : (X, \mathrm{E}, \bar{A}) \to (J_\alpha, \in, A)$ by setting $i([\xi]) = h_1^M(\xi)$. This embedding i is an isomorphism if and only if $M = h_1^M$“ρ_1^M, in which case we can say that this construction *recovers M from its Σ_1-code* $\mathfrak{C}_1(M)$.

3.3 Definition. The structure $M = (J_\alpha, A)$ is said to be *1-sound* if $J_\alpha = h_1^M$“ρ_1^M. Further, M is said to be *n-sound* if it is $(n-1)$-sound and $\mathfrak{C}_{n-1}(M)$ is 1-sound; and M is said to be *sound* if M is n-sound for all n.

We will say that the model J_α is *n-sound* or *sound*, respectively, if the structure $(J_\alpha, \varnothing)$ is n-sound or sound.

Notice that if M is sound then one can repeat the process described above n times in order to recover M from any of its codes $\mathfrak{C}_n(M)$. Thus the following lemma is the basic fact of fine structure:

3.4 Lemma. *If α is any ordinal then the structure J_α is sound.*

We will only consider the case when α is a limit ordinal, and begin with the proof that J_α is 1-sound.

Sketch of Proof. Let $Z = h_1^M$“$\rho_1^M \prec_1 J_\alpha$, and let i be the collapse map $i : \bar{M} \cong Z \prec_1 J_\alpha$. Since $\bar{M} \models$ “$V = L$”, we must have $\bar{M} = J_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$. Since i is Σ_1-elementary and $\rho_1^M \cup p_1^M \subseteq Z$, the set $A_1^{J_\alpha}$ is Σ_1-definable in $J_{\bar{\alpha}}$. Since $A_1^{J_\alpha} \notin J_\alpha$ it follows that $\bar{\alpha} = \alpha$.

Similarly, p_1^M is the least parameter which can be used to define A_1^M in J_α, and $i^{-1}(p_1^M) \leq p_1^M$, so $p_1^M = i(p_1^M)$. But every member of $\mathrm{dom}(i)$ is Σ_1-definable in $J_{\bar{\alpha}}$ from parameters in $\rho_1^M \cup p_1^M$, and it follows that i is the identity. Thus h_1^M“$\rho_1^M = Z = \mathrm{ran}(i) = J_\alpha$. This completes the proof that J_α is 1-sound. $\dashv$

It should be noted that this proof is closely related to the proof that GCH holds in L. Both rely on the following condensation lemma:

3.5 Lemma. *If $Z \prec_0 J_\alpha$ then there is an $\bar{\alpha} \leq \alpha$ such that $Z \cong J_{\bar{\alpha}}$.*

The proof here is somewhat more delicate than that of GCH, as the sentence “$V = L$” needs to be carefully formulated so that it is satisfied by J_α

even for successor α. The hypothesis of Lemma 3.5, stating that Z is Σ_0-elementary, is meant be interpreted in the terms of the J_α hierarchy as meaning that Z is closed under rudimentary functions. This is slightly stronger than the assertion that it is Σ_0-elementary in the sense of the Levy hierarchy. This observation is another example of the superiority of the J_α hierarchy: when using the L_α hierarchy, the hypothesis must be strengthened to require that Z be Σ_1-elementary.

In order to prove Lemma 3.4 for arbitrary n we need an extension of Lemma 3.5 in which the model J_α is replaced by the Σ_{n-1}-code $\mathfrak{C}_{n-1}(J_\alpha)$. This generalization is given by the *downward extension* or *condensation* property, stated in the following lemma, and is central to many applications of fine structure.

3.6 Lemma (Downward Extension Lemma). *Suppose that* $i : (J_{\rho'}, A') \prec_0 \mathfrak{C}_n(J_\alpha)$. *Then there is an* $\alpha' \leq \alpha$ *such that* $(J_{\rho'}, A') = \mathfrak{C}_n(J_{\alpha'})$, *and* i *extends to a* Σ_n*-embedding* $\tilde{\imath} : J_{\alpha'} \to J_\alpha$. *Furthermore* $\tilde{\imath}$ *preserves the first* n *stages of the fine structure, so that* $\tilde{\imath}\, h_k^{J_{\alpha'}} = h_k^{J_\alpha}\, \tilde{\imath}$ *for all* $k \leq n$.

Sketch of Proof. Lemmas 3.4 and 3.6 are proved by a joint induction on n. First we assume that Lemma 3.6 is true for $\mathfrak{C}_n(J_\alpha)$, and use this to prove that J_α is $(n+1)$-sound. The proof is essentially identical to the proof given above that J_α is 1-sound. The collapse map $i : J_{\bar{\alpha}} \cong Z \prec_0 J_\alpha$ becomes $i : (J_{\bar{\rho}}, \bar{A}) \cong Z \prec_0 (J_{\rho_n^M}, A_n^M) = \mathfrak{C}_n(J_\alpha)$. Since Lemma 3.6 holds for $\mathfrak{C}_n(J_\alpha)$ this can be written as $i : \mathfrak{C}_n(J_{\alpha'}) \to \mathfrak{C}_n(J_\alpha)$ for some $\alpha' \leq \alpha$. Since $A_{n+1}^{J_\alpha}$ is Σ_1-definable in $\mathfrak{C}_n(J_{\alpha'})$ we must have $\alpha' = \alpha$, and since $i^{-1}(p_{n+1}^{J_\alpha}) \leq p_{n+1}^{J_\alpha}$, which is the least parameter which can be used to define $A_{n+1}^{J_\alpha}$, we must have $i(p_{n+1}) = p_{n+1}$. Hence i is the identity on $\mathfrak{C}_n(J_\alpha)$, so J_α is $(n+1)$-sound.

To complete the proof, we show that if J_α is $(n+1)$-sound, and Lemma 3.6 holds for $\mathfrak{C}_n(J_\alpha)$, then Lemma 3.6 also holds for $\mathfrak{C}_{n+1}(J_\alpha)$. Suppose that $i : (J_{\rho'}, A') \prec_0 \mathfrak{C}_{n+1}(J_\alpha)$.

Apply to the structure $(J_{\rho'}, A')$ the construction described before Definition 3.3 to recover a structure (J_α, A) from its Σ_1-code $\mathfrak{C}_1(J_\alpha, A)$. The assumption that i is Σ_0-elementary implies that the construction succeeds to the extent of defining a model $(X, \mathrm{E}, \bar{A})$ and an embedding $i' : (X, \mathrm{E}, \bar{A}) \to \mathfrak{C}_n(J_\alpha)$. The existence of the embedding i' ensures that (X, E) is well-founded, and therefore $X \cong J_{\rho''}$ for some ordinal ρ''. If A'' is the image of $\bar{A}$ under this isomorphism, then i' induces an embedding $\tilde{\imath}_n : (J_{\rho''}, A'') \to \mathfrak{C}_n(J_\alpha)$.

By the construction, the set A' encodes the Σ_1 theory of $(J_{\rho''}, A'')$, and since i is Σ_0-elementary and A encodes the Σ_1 theory of $\mathfrak{C}_n(J_\alpha)$ it follows that $\tilde{\imath}_n$ is a Σ_1-elementary embedding. By the induction hypothesis it follows that there is an ordinal α' such that $(J_{\rho''}, A'') = \mathfrak{C}_n(J_{\alpha'})$, and an embedding $\tilde{\imath}_0 : J_{\alpha'} \to J_\alpha$ which extends $\tilde{\imath}_n$ and which preserves the first n stages of the fine structure, as far as $\mathfrak{C}_n(J_{\alpha'})$.

Now it only remains to verify that $(J_{\rho'}, A') = \mathfrak{C}_1(J_{\rho''}, A'')$, which entails verifying that $\rho_1^{(J_{\rho''},A'')} = \rho'$ and $p_1^{(J_{\rho''},A'')} = \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$. The inequality $\rho_1^{(J_{\alpha''},A'')} \leq \rho'$ follows from the fact that $A' \notin J_{\rho''}$, which is proved by the argument of the Russell paradox: If $A' \in J_{\rho''}$ then so is $y = \{\nu : \ulcorner \nu \notin h(\nu)\urcorner \in A'\}$, where $h : \rho' \to J_{\rho''}$ is the Skolem function coded by A'. But this is impossible, as then $y = h(\nu)$ for some ν and then $\nu \in y \iff \nu \notin y$.

The inequality $\rho' \leq \rho_1^{(J_{\alpha''},A'')}$ follows from the fact that $A' \cap \xi \in J_{\rho'}$ for $\xi < \rho'$, which follows from the assumption that $(J_{\rho'}, A') \prec_0 \mathfrak{C}_{n+1}(J_\alpha)$. Thus $\rho' = \rho_1^{(j_{\alpha''},A'')}$, and this implies the inequality $p_1^{(J_{\rho''},A'')} \leq \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$ since $p_1^{(J''_\rho,A'')}$ is, by definition, the least parameter which can be used to define A''.

The final inequality $p_1^{(J_{\rho''},A'')} \geq \tilde{\imath}_n^{-1}(p_1^{\mathfrak{C}_n(J_\alpha)})$ is the point in the proof of Lemma 3.6 which requires the joint induction with Lemma 3.4: assume for the sake of contradiction that $p_{n+1}^{J_{\bar\alpha}} < \tilde{\imath}_n^{-1}(p_{n+1})^{J_\alpha}$ and apply Lemma 3.4 to $J_{\bar\alpha}$. This implies that $\tilde{\imath}_n^{-1}(p_{n+1}^{J_\alpha})$ is Σ_1-definable in $(J_{\alpha'}, A')$ from $p_{n+1}^{J_{\bar\alpha}}$. It follows that $p_{n+1}^{J_\alpha}$ is Σ_1-definable in $\mathfrak{C}_n(J_\alpha)$ from $\tilde{\imath}_{n+1}(p_{n+1}^{J_{\bar\alpha}}) < p_{n+1}^{J_\alpha}$, but this contradicts the definition of $p_{n+1}^{J_\alpha}$.

This completes the proof of Lemma 3.6, except for the claim that $\tilde{\imath}$ is Σ_{n+1}-elementary. To see this, notice that the embedding i'' constructed in the induction step is one quantifier stronger than i'. The map $\tilde{\imath}$ is obtained by repeating this process $n+1$ times, and hence the original Σ_0 embedding is strengthened to a Σ_{n+1}-elementary embedding $\tilde{\imath} : J_{\alpha'} \to J_\alpha$. $\dashv$

It should be noted that the statement that the embedding $\tilde{\imath}$ preserves the fine structure is stronger—and usually more useful—than the statement that $\tilde{\imath}$ is Σ_n-elementary.

If we define $\bar{h}_n^{J_\alpha} = h_1^{J_\alpha} \ldots h_n^{J_\alpha}$, then $\bar{h}^{J_\alpha} : \rho_n^{J_\alpha} \to J_\alpha$ and an induction using Lemma 3.4 shows that $J_\alpha = \bar{h}_n^{J_\alpha} ``\rho_n^{J_\alpha}$. In order to avoid considering detailed fine structure as much as possible, we make the following convention:

Notation. Unless stated otherwise, we abuse notation by using $h_n^{J_\alpha}$ to denote the function $\bar{h}_n^{J_\alpha}$ described above, and we call it the Σ_n-Skolem function of J_α.

We end the discussion of Lemma 3.6 with Lemma 3.7, which is frequently useful in applications of the covering lemma and in particular proves, used along with the proof of the covering lemma itself, Proposition 1.14 from the introduction.

3.7 Lemma. *The Σ_n-Skolem function $h_n^{J_\alpha}$ of J_α can be written as an increasing union $h_n^{J_\alpha} = \bigcup_{\nu<\eta} g_\nu$ of functions $g_\nu \in J_\alpha$, with $\eta \leq \rho_n^{J_\alpha}$.*

Sketch of Proof. First consider the case when $n = 1$ and α is a limit ordinal. Pick a sequence of ordinals α_ν cofinal in α, and define g_ν to be the function defined in J_{α_ν} by the same Σ_1 formula (with the same parameter $p_1^{J_\alpha}$) as was used to define $h_1^{J_\alpha}$ in J_α. Thus $g_\nu(x) = y$ if and only if $h_1^{J_\alpha}(x) = y$, both x and y are in J_{α_ν}, and in addition the witness to the Σ_1 fact "$h_1^{J_\alpha}(x) = y$" is a member of J_{α_ν}.

For $n > 1$, apply the construction above to $\mathfrak{C}_{n-1}(J_\alpha)$, noting that $\rho_n^{J_\alpha}$ is always a limit ordinal for $n > 0$. ⊣

The importance of Lemma 3.6 to fine structure theory extends far beyond the arguments above; however, its importance in the proof of the covering lemma is secondary to the that of the upward extension property, Lemma 3.10, described below.

Embeddings of Mice

In this subsection we define a generalized ultrapower which is central to the proof of the covering lemma. This ultrapower, which is used to extend a given embedding $\pi : J_{\bar\kappa} \to J_\kappa$ to an embedding $\tilde\pi : J_{\bar\alpha} \to J_{\alpha'}$ with a larger domain, can be described in modern terms as the ultrapower by the extender $E_{\pi,\beta}$ of length β which is associated with the embedding π. It should be noted, however, that this construction of Jensen is older than, and in fact is ancestral to, the modern notion of an extender. Extenders are more completely described in chapter [32].

We first explain the extender construction by defining the Σ_0-ultrapower $\mathrm{Ult}(M, \pi, \beta)$ of a model M.

3.8 Definition. Assume that M and N are transitive models of a fragment of set theory, and that $\pi : N \to N'$ is a Σ_0-elementary embedding such that $\mathcal{P}(\nu) \cap M \subseteq N$ for all $\nu < \mathrm{On}(N)$ such that $\sup(\pi``\nu) < \beta$. Then

$$\begin{aligned}\mathrm{Ult}(M, \pi, \beta) = \{[a, f]_\pi : f \in M \text{ and } \mathrm{dom}(f) \in \mathrm{dom}(\pi) \\ \text{and } a \in [\beta]^{<\omega} \cap \pi(\mathrm{dom}(f))\}\end{aligned} \tag{18.3}$$

where $[a, f]_\pi$ is the equivalence class of the pair (a, f) under the relation

$$(a, f) \sim_\pi (a', f') \iff (a, a') \in \pi(\{(\vec\nu, \vec\nu') : f(\vec\nu) = f'(\vec\nu')\}). \tag{18.4}$$

The membership relation E_π and any other predicates of $\mathrm{Ult}(M, \pi, \beta)$ are defined similarly, and the embedding $i : M \to \mathrm{Ult}(M, \pi, \beta)$ is defined as usual by $i(x) = [a, C_x]_\pi$ where a is arbitrary and C_x is the constant function, $\forall z\, C_x(z) = x$.

The embedding $i : M \to \mathrm{Ult}(M, \pi, \beta)$ satisfies Łos's theorem for Σ_0 formulas:

3.9 Proposition. *If φ is a Σ_0 formula, then for any $f_0, \dots, f_n$ in M and $a_0, \dots, a_n$ in β we have $\mathrm{Ult}(M, \pi, \beta) \models \varphi([f_0, a_0], \dots, [f_n, a_n])$ if and only if $\langle a_0, \dots, a_n\rangle \in \pi(\{\langle u_0, \dots, u_n\rangle : M \models \varphi(f_0(u_0), \dots, f_n(u_n))\})$.*

If $M = J_\alpha$ for some ordinal α then $\mathrm{ran}(i)$ is cofinal in $\mathrm{Ult}(M, \pi, \beta)$, and it follows that i is Σ_1-elementary. In particular, i preserves the Σ_1-Skolem function of J_α.

We will need to define Σ_n ultrapowers, for arbitrary $n \in \omega$, so that they preserve Σ_{n+1}-Skolem functions. The obvious way to define such an ultrapower is to modify Definition 3.8 by replacing the condition "$f \in M$" of (18.3) with "f is Σ_n-definable in J_α"; however, doing so would require first proving Jensen's uniformization theorem, which states that there is a Σ_n-definable Skolem function for Σ_n formulas on J_α. A second possible approach is that of Silver, who showed that it is possible to define $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ by using compositions of the naïve Σ_n-Skolem function, and that the naïve Skolem function is preserved by the resulting embedding even though it is not defined by a Σ_n formula. This is the simplest approach, as it avoids the use of fine structure, but it appears to have difficulties with models for larger cardinals.

Our approach will be closer to the first one, but will use the fine structure directly. The notion of Σ_n ultrapower which we use can be defined in two different, but equivalent, ways. One way is to define $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ directly, using Definition 3.8, but allowing any function f of the form $f(x) = h_n(x, q)$ where h_n is the Σ_n-Skolem function mapping a subset of $J_{\rho_n^{J_\alpha}}$ onto J_α, and $q \in J_{\rho_n^{J_\alpha}}$ is an arbitrary parameter. The other way is indirect, by taking the ordinary Σ_0 ultrapower $i : \mathfrak{C}_n(J_\alpha) \to \mathrm{Ult}(\mathfrak{C}_n(J_\alpha), \pi, \beta)$ of the Σ_n-code of J_α, and then extending this to a map $\tilde{i} : J_\alpha \to J_{\tilde{\alpha}}$. This approach has the advantage that most arguments can be carried out at the level of the Σ_n-code of J_α, which involves the easily understandable Σ_0 ultrapower and Σ_1-Skolem function.

The extension of π to an embedding $\tilde{\pi}$ with the larger domain J_α depends on Lemma 3.10 below, which is the counterpart of the Downward Extension Lemma 3.6 given earlier. One major difference between the Upward and Downward Extension Lemmas concerns the well-foundedness of the new structure. In the Downward Extension Lemma, this structure is a substructure of a given well-founded structure and hence is automatically well-founded. In the Upward Extension Lemma the well-foundedness of $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ must be explicitly assumed.

3.10 Lemma (Upward Extension Lemma). *Suppose that $\pi : J_{\bar{\kappa}} \to J_\kappa$, that $\beta \leq \kappa$, and either $\rho_n^{J_\alpha} > \min\{\nu : \pi(\nu) \geq \beta\}$ or $\mathrm{ran}(\pi)$ is cofinal in β and $\pi(\rho_n^{J_\alpha}) \geq \beta$. Set $M_n = \mathfrak{C}_n(J_\alpha)$ and $\widetilde{M}_n = \mathrm{Ult}(M_n, \pi, \beta)$.*

1. *There is a structure $\widetilde{M}_0$ such that $\widetilde{M}_n$ is, formally, equal to $\mathfrak{C}_n(\widetilde{M}_0)$. If this structure $\widetilde{M}_0$ is well-founded then there is an ordinal $\tilde{\alpha}$ such that $\widetilde{M}_0 = J_{\tilde{\alpha}}$ and $\widetilde{M}_n = \mathfrak{C}_n(J_{\tilde{\alpha}})$.*

2. *There is an embedding $\tilde{\pi} : J_\alpha \to \widetilde{M}_0$ such that $\pi{\restriction}J_{\bar{\beta}} = \tilde{\pi}{\restriction}J_{\bar{\beta}}$, where $\bar{\beta}$ is the least ordinal such that $\pi(\bar{\beta}) \geq \beta$ if $\beta < \kappa$, or $\bar{\beta} = \bar{\kappa}$ if $\beta = \kappa$.*

3. *The embedding $\tilde{\pi}$ preserves the Σ_k codes for $k \leq n$: in particular, $\tilde{\pi} \circ h_k^{J_\alpha}(x) = h_k^{\widetilde{M}_0} \circ \tilde{\pi}(x)$ for all x for which either side is defined.*

4. *The embedding $\tilde{\pi}$ preserves the Σ_1-Skolem function of M_n in the sense*

that there is a function $\tilde{h}$, which is Σ_1-definable over $\widetilde{M}_n$, such that $\tilde{\pi}\, h^M_{n+1}(x) = \tilde{h}\,\tilde{\pi}(x)$ for all $x \in M$ such that either side is defined.

The proof of the Upward Extension Lemma is nearly the same as that of the Downward Extension Lemma 3.6. The models $\widetilde{M}_{n-k}$, and embeddings $\tilde{\pi}_{n-k} : \mathfrak{C}_{n-k}(J_\alpha) \to \widetilde{M}_{n-k}$ are defined by recursion on $k \leq n$: The embedding $\tilde{\pi}_0 : \mathfrak{C}_n(J_\alpha) \to \widetilde{M}_n$ is the Σ_0 ultrapower, and $\widetilde{M}_{n-(k+1)}$ is constructed from $\widetilde{M}_{n-k}$ by the same recovery process as was used for the Downward Extension Lemma. This process uses the fact that $\widetilde{M}_{n-k}$ has the form $(\widetilde{M}_{n-k}, \mathrm{E}, \tilde{A}_{n-k})$ and satisfies the first-order sentences asserting that $(\widetilde{M}_{n-k}, \mathrm{E})$ is a model of $V = L$, and that $\tilde{A}_{n-k}$ is the Σ_n theory of a larger model $\widetilde{M}_{n-(k+1)}$ of which $\widetilde{M}_{n-k}$ is the Σ_1-code.

The embedding $\tilde{\pi}$ does not, in general, preserve fine structure below ρ^M_n; for example if β is a cardinal in L then $\rho^{\widetilde{M}} \geq \beta$, since every bounded subset of β in L is a member of J_β, but it may happen that $\rho^M_{n+1} < \bar{\beta}$. In this case $\tilde{\pi}(\rho^M_{n+1}) = \pi(\rho^M_{n+1}) < \beta$. It follows that the function $\tilde{h}$ will not, in general, be the Σ_1-Skolem function $h_1^{\widetilde{M}_n}$. It is defined by the same formula as $h_1^{\widetilde{M}_n}$, but using the image $\tilde{\pi}_0(p_1^{J_{M_n}})$ of the standard parameter of M_n instead of the standard parameter $p_1^{\widetilde{M}_n}$ of $\widetilde{M}_n$.

In order for the ultrapower $\mathrm{Ult}_n(M, \pi, \beta)$ described above to be defined, the set $\{(\vec{\nu}, \vec{\nu}') : f(\vec{\nu}) = f'(\vec{\nu}')\}$ must be in the domain of π for each pair (f, f') of functions in $\Sigma_n(M)$. This yields the following condition for the existence of $\mathrm{Ult}_n(M, \pi, \beta)$:

3.11 Proposition. *Let M, π and β be as above, and let $\bar{\beta}$ be the least ordinal such that $\pi(\bar{\beta}) \geq \beta$ (or $\bar{\beta} = \mathrm{On}(N)$ if $\beta = \sup(\mathrm{ran}(\pi))$). Then the ultrapower $\mathrm{Ult}_n(M, \pi, \beta)$ is defined if and only if either $\rho^M_n > \bar{\beta}$, or else $\rho^M_n \geq \bar{\beta}$ and $\mathrm{ran}(\pi)$ is cofinal in β.*

Equivalently, the ultrapower is defined if and only if either

1. *Every subset of $\bar{\kappa}$ which is Σ_n-definable in M is a member of $J_{\bar{\kappa}}$, or*
2. *$\mathrm{ran}(\pi)$ is cofinal in β and every bounded subset of $\bar{\kappa}$ which is Σ_n-definable in M is a member of $J_{\bar{\kappa}}$.*

3.2. Proof of the Covering Lemma for L

The main part of the proof of the covering lemma for L is a construction which shows that any set $X \prec_1 J_\kappa$, which is suitable in a sense to be made precise in Definition 3.14, is a member of L. The concluding part of the proof is an analysis of the notion of suitability showing that every uncountable set of ordinals is contained in a suitable set of the same cardinality.

The construction, together with the Definition 3.14 of suitability, will prove the following lemma:

3.12 Lemma.

1. *If* $X \prec_1 J_\kappa$ *is suitable then there is a cardinal* $\rho < \kappa$ *of* L *and a function* $h \in L$ *such that* $X = h``(\rho \cap X)$.

2. *If* $X \prec_1 J_\kappa$ *is suitable and* $\rho < \kappa$ *is a cardinal of* L *then* $X \cap J_\rho$ *is also suitable.*

3.13 Corollary. *Any suitable set* $X \prec_1 J_\kappa$ *is a member of* L.

Proof. The proof is by induction on κ. Let $X \subseteq \kappa$ be suitable, and let h and ρ be as in clause 1. Then $X \cap J_\rho$ is suitable by clause 2 and hence is in L by the induction hypothesis, but then $X = h``(X \cap \rho) \in L$. ⊣

In order to describe the basic construction, we fix a cardinal κ of L and a set $X \prec_1 J_\kappa$ with $\sup(X) = \kappa \not\subseteq X$. Let $\pi : N \to X$ be the collapse map, so that $N = J_{\bar\kappa}$ for some ordinal $\bar\kappa$, and let (α, n) be the lexicographically largest pair such that $\mathrm{Ult}_n(J_\alpha, \pi, \kappa)$ is defined. There are two cases:

1. If $\mathcal{P}(\delta) \cap L \subseteq N$ for all $\delta < \bar\kappa$ then $J_\alpha = L$ and $n = 0$.

2. Otherwise α is the least ordinal such that there is a bounded subset of $\bar\kappa$ in $J_{\alpha+\omega} - J_{\bar\kappa}$, and n is the least integer such there is such a subset which is Σ_{n+1}-definable in J_α. That is, $\rho_{n+1}^{J_\alpha} < \bar\kappa \leq \rho_n^{J_\alpha}$, and $\rho_m^{J_{\alpha'}} \geq \bar\kappa$ whenever $\bar\kappa \leq \alpha' < \alpha$ and $m < \omega$.

The basic construction will succeed whenever $\widetilde{M} = \mathrm{Ult}_n(J_\alpha, \pi, \kappa)$ is well-founded; the Definition 3.14 of suitability, given in the next subsection, is a generalization of this requirement. If case (1) occurs for some suitable set X then $\tilde\pi : L \to \mathrm{Ult}(L, \pi, \kappa) = L$ is a nontrivial embedding from L into L, which implies by Kunen's theorem (see chapter [32, Theorem 1.13]) that $0^\#$ exists. This contradicts our current assumption that the core model is equal to L, so we can assume that case (2) occurs for all suitable sets X. Then by Lemma 3.10, $\mathrm{Ult}_n(J_\alpha, \pi, \kappa) = J_{\tilde\alpha}$ for some ordinal $\tilde\alpha$, and the following diagram commutes:

$$\begin{tikzcd} J_\alpha \arrow[r, "\tilde\pi"] & \widetilde{\mathcal{M}} = \mathrm{Ult}_n(J_\alpha, \pi, \kappa) = J_{\tilde\alpha} \\ J_{\bar\kappa} \arrow[u, hook] \arrow[r, "\pi"] & X \prec_1 J_\kappa \arrow[u, hook] \end{tikzcd} \tag{18.5}$$

Now let $\bar\rho = \rho_{n+1}^{J_\alpha}$. Then $\bar\rho < \bar\kappa$, and $J_\alpha = \bar h``\bar\rho$ where $\bar h = h_{n+1}^{J_\alpha}$, so

$$X = \pi``J_{\bar\kappa} = \pi``(\bar\kappa \cap \bar h``(\bar\rho)), \tag{18.6}$$

and furthermore

$$\tilde h \circ \tilde\pi = \tilde\pi \circ \bar h \tag{18.7}$$

where $\tilde{h}$ is the function given by Lemma 3.10(4). Putting equations (18.6) and (18.7) together, we get

$$X = J_\kappa \cap (\widetilde{\pi} \circ \bar{h}``\bar{\rho}) = J_\kappa \cap (\tilde{h} \circ \widetilde{\pi}``\bar{\rho}) = J_\kappa \cap \tilde{h}``(X \cap \rho) \tag{18.8}$$

where $\rho = \sup(\pi``\bar{\rho}) < \kappa$. Since $\tilde{h} \in L$, this completes the basic construction.

Suitable Sets

Here is the formal definition of suitability:

3.14 Definition. Suppose $X \subseteq L$ and let $\pi : N \cong X$ be the inverse of the transitive collapse. Then X is *suitable* if $X \prec_1 J_\kappa$ for some ordinal κ and $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is well-founded for all triples (α, n, β) such that the ultrapower is defined.

We write $\mathbf{C}$ for the class of suitable sets.

Proof of Lemma 3.12. If $X \prec_1 J_\kappa$ is any set in $\mathbf{C}$ then the basic construction succeeds for X, and hence clause 1 of Lemma 3.12 holds for X. Clause 2 of that lemma is clear. ⊣

It follows by Corollary 3.13 that every suitable set is in L, so Jensen's covering Lemma 1.1 for L will follow if we can show that every uncountable set is contained in a suitable set of the same cardinality. For the Strong Covering Lemma 1.12 we additionally need to show that the class $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Notice that Definition 3.14 is absolute, so that the class $\mathbf{C}$ is definable in L.

The countably closed sets give a easy, but useful, special case:

3.15 Definition. We will call a set $X \prec_1 J_\kappa$ *countably closed* if there is a set $Y \prec H(\lambda)$, for some $\lambda \geq \kappa$, such that ${}^\omega Y \subseteq Y$ and $X = Y \cap J_\kappa$.

If $|x|^\omega < \kappa$ then it is always possible to find a countably closed $X \supseteq x$ with $|X| = |x|^\omega$, so the following easily proved observation is often all that is needed.

3.16 Proposition. *Every countably closed set $X \prec_1 J_\kappa$ is suitable, and hence is a member of L.*

It follows that if $0^\#$ does not exist then every set x is contained in a set $y \in L$ such that $|y| \leq |x|^\omega$. This result gives much of the strength of the covering lemma, and moreover its proof highlights the most important ideas of the proof of the full covering lemma while omitting the most delicate part of the argument. This by itself would be sufficient reason to consider the countably closed case, but for core models involving measurable cardinals or non-overlapping extenders the countably closed case of the covering lemma is a necessary step in the proof of the full lemma: it is used to prove that the weak covering lemma, Definition 1.9, holds in a variant K^{c} of the core model. The weak covering lemma for K^{c} is then used to prove the existence and essential properties of the true core model K, and only after this can the full covering lemma be proved for K.

The following lemma will conclude the proof of Theorem 1.1 and of Theorem 1.12 in the case $0^{\#}$ does not exist: the covering lemma and the strong covering lemma for L:

3.17 Lemma. *The class* $\mathbf{C}$ *is unbounded in* $[J_\kappa]^\delta$ *for every uncountable cardinal* δ*, and* $\bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$ *whenever* $\langle X_\nu : \nu < \eta\rangle$ *is an increasing sequence of sets in* $\mathbf{C}$ *with* $\mathrm{cf}(\eta) > \omega$*.*

The proof of this lemma will take up the remainder of Sect. 3.2.

Fix, for the moment, a set X which is not suitable, and let α, n and β be such that $\widetilde{M} = \mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is defined but not well-founded. This ill-foundedness is witnessed by a descending E-chain, $\dots \mathrel{\mathrm{E}} z_2 \mathrel{\mathrm{E}} z_1 \mathrel{\mathrm{E}} z_0$, of members of $\widetilde{M}$, where E is the membership relation of $\widetilde{M}$. In order to prove Lemma 3.17 we need to incorporate additional structure into such a witness:

3.18 Definition. A *witness* w *to the unsuitability* of $X \prec_1 J_\kappa$ is a ω-chain of Σ_0-elementary embeddings $i_k : m_k \to m_{k+1}$ such that

1. $i_k \in X$ and $m_k \in X$ for each $k < \omega$.
2. $\operatorname{dir\,lim}(\pi^{-1}[w]) = \mathfrak{C}_n(J_\alpha)$ for some ordinal α and some $n \in \omega$.
3. $\operatorname{dir\,lim}(w)$ is not the Σ_n-code of any well-founded model $J_{\tilde{\alpha}}$.
4. Write β_k for the critical point of i_k. Then the sequence $\langle \beta_k : k < \omega\rangle$ is nondecreasing.
5. For each k we have $m_k \in m_{k+1}$, and there is a function $f \in m_{k+1}$ such that $f``\beta_k = i_k``m_k$.

We will call $\beta = \sup_k(\beta_k)$ the *support* of the witness w, and we will call the pair (α, n) the *height* of w in X. We will say that a witness w is *minimal* in X if it has minimal height in X among all witnesses with the same support β.

There may be more than one minimal witness for X with the same support β. It is possible, with some care, to modify the definition so that this minimal witness is unique; however, we do not need to do so.

3.19 Lemma. *A set* $X \prec_1 J_\kappa$ *is unsuitable if and only if it has a witness to its unsuitability. Furthermore, if* w *is a witness to the unsuitability of* X *then*

1. *If* $w \subseteq X' \prec_1 X$ *then* w *is also a witness to the unsuitability of* X'*.*
2. *If, in clause 1,* w *is a minimal witness for* X *then it is also a minimal witness for* X'*, and furthermore any other minimal witness for* X' *with the same support is also a minimal witness for* X*.*
3. *If* $X = Y \cap J_\kappa$*, where* $Y \prec_1 H(\tau)$ *for some cardinal* $\tau > \kappa$*, then* $w \notin Y$*.*

We will give some immediate consequences of Lemma 3.19 and then use it to finish the proof of the covering lemma. We will then give the proof of Lemma 3.19.

3.20 Corollary. *If $\langle X_\nu : \nu < \eta \rangle$ is an increasing sequence of sets in $\mathbf{C}$, with $\mathrm{cf}(\eta) > \omega$, then $X = \bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$.*

Proof. Otherwise there would be a witness w to the unsuitability of X; but since $\mathrm{cf}(\eta) > \omega$ this would imply that $w \subseteq X_\nu$ for some $\nu < \eta$ and hence w is a witness to the unsuitably of X_ν by clause 1. $\dashv$

We can use Lemma 3.19 to give a proof of Proposition 3.16, although a direct proof is somewhat simpler.

3.21 Corollary. *Every countably closed set $X \prec_1 J_\kappa$ is suitable, and hence is a member of L.*

Proof. By definition, X is countably closed if and only if $X = Y \cap J_\kappa$ for some $Y \prec_1 H(\tau)$ where ${}^\omega Y \subseteq Y$. Then any witness to the unsuitability of X would have to be a member of Y, contrary to clause 3 of Lemma 3.19. $\dashv$

The following lemma will complete the proof of the covering lemma except for the proof of Lemma 3.19.

3.22 Lemma. *The class $\mathbf{C}$ is unbounded in $[J_\kappa]^\delta$ for any cardinal δ with $\omega < \delta < \kappa$.*

Proof. Jensen's proof of this result begins by generically collapsing the cardinal κ onto δ^+. The proof given here is essentially the same, but the presentation is slightly different: instead of carrying out the generic collapse we work with the set $\mathrm{Col}(\delta^+, J_\kappa)$ of forcing conditions for the collapse. The members of $\mathrm{Col}(\delta^+, J_\kappa)$ are functions $\sigma : \xi \to J_\kappa$ with $\xi < \delta^+$. With the obvious notions of "closed" and "unbounded" this space satisfies Fodor's Lemma: if $S \subseteq \mathrm{Col}(\kappa^+, J_\kappa)$ is a stationary set and F is a function with domain S such that $F(\sigma) \in \mathrm{ran}(\sigma)$ for all $\sigma \in S$, then F is constant on a stationary subset of S. The reason for using the space $\mathrm{Col}(\delta^+, J_\kappa)$ instead of $[J_\kappa]^\kappa$ is that $\mathrm{Col}(\delta^+, J_\kappa)$ also satisfies the following variant of Fodor's Lemma:

3.23 Proposition. *Suppose that $S \subseteq \mathrm{Col}(\delta^+, J_\kappa)$ is a stationary set such that $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ for all $\sigma \in S$, and that F is a function defined on S such that $F(\sigma)$ is a countable subset of $\mathrm{ran}(\sigma)$ for all $\sigma \in S$. Then there is a stationary subset S' of S and a function $\sigma_0 \in S'$ such that*

$$\forall \sigma \in S' \, (\sigma_0 \subseteq \sigma \text{ and } F(\sigma) \subseteq \mathrm{ran}(\sigma_0)).$$

Proof. Let $f(\sigma) < \mathrm{dom}(\sigma)$ be the least ordinal η such that $F(\sigma) \subseteq \sigma``\eta$, and let $S_0 \subseteq S$ be a stationary set on which $f(\sigma)$ is constant. Pick any $\sigma_0 \in S_0$ and let $S' = \{\sigma \in S_0 : \sigma_0 \subseteq \sigma\}$. Then S' and σ_0 are as required. $\dashv$

Let S_0 be the set of functions $\sigma \in \mathrm{Col}(\delta^+, J_\kappa)$ such that $\mathrm{ran}(\sigma) \notin \mathbf{C}$, $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma) \prec_1 J_\kappa$. We will prove that S_0 is nonstationary, which implies that $\mathbf{C}$ is unbounded in $[J_\kappa]^\delta$.

Suppose to the contrary that S_0 is stationary. It follows by Lemma 3.19 that there is, for each $\sigma \in S_0$, a minimal witness w^σ to the unsuitability of $\mathrm{ran}(\sigma)$. Let β^s be the support of w^σ. By the ordinary Fodor's Lemma there is a stationary set $S_1 \subseteq S_0$ such that $\beta = \beta^\sigma$ is constant for $\sigma \in S_1$, and by Proposition 3.23 there is a stationary set $S_2 \subseteq S_1$ and $\sigma_0 \in S_2$ such that $\sigma_0 \subseteq \sigma$ and $w^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_2$. It follows that w^{σ_0} is a minimal witness to the unsuitability of $\mathrm{ran}(\sigma)$ for each $\sigma \in S_2$. Now consider the class $\mathcal{Y}$ of sets $Y \prec_{\Sigma_1} H(\kappa^+)$ such that $w^{\sigma_0} \in Y$. Then

$$\mathcal{X} = \{\sigma \in \mathrm{Col}(\delta^+, J_\kappa) : \exists Y \in \mathcal{Y}\ \mathrm{ran}(\sigma) = Y \cap J_\kappa\}$$

contains a closed unbounded subset of $\mathrm{Col}(\delta^+, J_\kappa)$, and hence $S_2 \cap \mathcal{X} \neq \varnothing$. However, this contradicts Lemma 3.19(3), and this contradiction completes the proof of Lemma 3.22. $\dashv$

This completes the proof of the covering lemma, except for the proof of Lemma 3.19:

Proof of Lemma 3.19. First, notice that if w is a witness with support β to the unsuitability of X, then clauses 4 and 5 imply that $\mathrm{dir\,lim}(w) = \mathrm{Ult}(\mathrm{dir\,lim}(\pi^{-1}[w]), \pi, \beta)$, and hence clause 3.18(3) implies that X is in fact unsuitable.

Now suppose that X is unsuitable, so that there are α, n and β such that $\mathrm{Ult}_n(J_\alpha, \pi, \beta)$ is defined, but not well-founded. If we write $M_n = \mathfrak{C}_n(J_\alpha)$ then this means that $\mathrm{Ult}(M_n, \pi, \beta)$ is defined, but is not the Σ_n-code of any well-founded structure $J_{\tilde{\alpha}}$. We will find a witness w to the unsuitability of X, such that w has height and support less than or equal to (α, n) and β, respectively.

If $\mathrm{Ult}(M_n, \pi, \beta)$ is not well-founded then there are $f_k \in M_n$ and $a_k \in \beta$ so that $z_{k+1}\ \mathrm{E}\ z_k$, where $z_k = [a_k, f_k]_\pi = \tilde{\pi}(f_k)(a_k)$, and E is the membership relation of $\mathrm{Ult}(M_n, \pi, \beta)$. If on the other hand $\mathrm{Ult}(M_n, \pi, \beta)$ is well-founded, then, since $\tilde{\pi} : M_n \to \widetilde{M}_n = \mathrm{Ult}(M_n, \pi, \beta)$ is Σ_1-elementary, there is a (ill-founded) structure $\widetilde{M}$ such that $\widetilde{M}_n = \mathfrak{C}_n(\widetilde{M})$, along with a map $\tilde{h}_n$, the Σ_n-Skolem function of $\widetilde{M}$, mapping $\widetilde{M}_n$ onto $\widetilde{M}$. Then we can find $z_k = [a_k, f_k]_\pi$ so that $\tilde{h}_n(z_{k+1})\ \mathrm{E}\ \tilde{h}_n(z_k)$ for each $k < \omega$.

Write $M_n = (J_{\rho_n}, A_n)$ (if $n = 0$ then $\rho_n = \alpha$ and $A_n = \varnothing$, in which case we assume as usual that α is a limit ordinal). Let $\alpha_k < \rho_n$ be the least ordinal $\xi > \alpha_{k-1}$ such that $\{f_1, \dots, f_k\} \subseteq J_\xi$, and let β_k be the least member of X such that $\{a_0, \dots, a_k\} \subseteq \beta_k$. Finally let $\bar{\beta}_k = \pi^{-1}(\beta_k)$ and let $\bar{j}_k : \bar{m}_k \cong \mathcal{H}_{\Sigma_1}^{(J_{\alpha_k}, A_n \cap J_{\alpha_k})}(\bar{\beta}_k \cup \{f_1, \dots, f_k\})$ be the transitive collapse of the Σ_1 hull of $\bar{\beta}_k \cap \{f_1, \dots, f_k\}$ in $(J_{\alpha_k}, A_n \cap J_{\alpha_k})$, with $\bar{i}_k = \bar{j}_{k+1}^{-1} \bar{j}_k : \bar{m}_k \to \bar{m}_{k+1}$.

Then $\bar{m}_k, \bar{i}_k \in J_{\bar{\kappa}}$ for each $k < \omega$. Set $w = \langle \pi(\bar{m}_k), \pi(\bar{i}_k) : k < \omega \rangle$, with $m_k = \pi(\bar{m}_k)$ and $i_k = \pi(\bar{i}_k)$, and set $\beta' = \sup_k(\beta_k) \leq \beta$.

If $\bar{w} = \langle \bar{m}_k, \bar{i}_k : k < \omega \rangle = \pi^{-1}[w]$ then $\operatorname{dir}\lim(\bar{w}) \prec_0 M_n = \mathfrak{C}_n(J_\alpha)$ and hence, by Lemma 3.6, $\operatorname{dir}\lim(\bar{w})$ is the Σ_n-code of $J_{\alpha'}$ for some $\alpha' \leq \alpha$, but w was constructed so that $\operatorname{dir}\lim(w)$ is not the Σ_n-code of any well-founded model. Finally, since $\alpha_{k+1} > \alpha_k$, the Skolem function mapping β_k onto j_k"$m_k \prec_{\Sigma_1} J_{\alpha_k}$, with parameters $\{f_1, \ldots, f_k\}$, is a member of $J_{\alpha_{\kappa+1}}$. This gives the functions f required by clause 3.18(5).

Thus w is the desired witness to the unsuitability of X.

To prove clause 3.19(3), note that by the absoluteness of well-foundedness we can find, working in Y, a sequence $a'_k < \beta'$ of ordinals and a sequence $f'_k \in m_{k+1}$ of functions such that if f''_k is the image $j_k(f'_k)$ of f'_k in $\operatorname{dir}\lim(w)$ then the sets $z'_k = f''_k(a'_k)$ demonstrate, in the same way that $\langle z_k : k < \omega \rangle$ above did for $\widetilde{M}_n$, that $\operatorname{dir}\lim(w)$ is not the Σ_n-code of a well-founded structure. Then the sets a'_k and f'_k are members of $Y \cap J_\kappa = X$, so the sets $\bar{z}'_k = \bar{i}_k\, \pi^{-1}(f'_k)(\pi^{-1}(\alpha'_k))$ demonstrate that $\operatorname{dir}\lim(\pi^{-1}[w])$ is not the Σ_n-code of a well-founded structure, contradicting clause 3.18(2).

Clause 3.19(1), stating that any witness $w \subseteq X' \prec_1 X$ to the unsuitability of X is also a witness that X' is not suitable, is straightforward. Finally, to prove clause 3.19(2), suppose that w is minimal, and that w' is a minimal witness for X' having the same support β.

Let (α', n') and (α'', n) be the heights of w' and w, respectively, in X', and let $\bar{\pi} = (\pi^X)^{-1}\pi^{X'}$. Then $(\alpha', n') \leq (\alpha'', n)$ since w' is minimal, so

$$\begin{aligned}
\operatorname{dir}\lim((\pi^X)^{-1}\text{``}w') &= \operatorname{Ult}_{n'}(\operatorname{dir}\lim((\pi^{X'})^{-1}\text{``}w'), \bar{\pi}, \bar{\beta}) \\
&= \operatorname{Ult}_{n'}(J_{\alpha'}, \bar{\pi}, \bar{\beta}) \\
&\subseteq \operatorname{Ult}_n(J_{\alpha''}, \bar{\pi}, \bar{\beta}) = \operatorname{dir}\lim((\pi^X)^{-1}\text{``}w).
\end{aligned}$$

Hence $\operatorname{dir}\lim((\pi^X)^{-1}\text{``}w')$ is well-founded, and it follows that w' is a witness to the unsuitability of X with support β, and by the minimality of w we must have $\operatorname{Ult}_{n'}(J_{\alpha'}, \bar{\pi}, \bar{\beta}) = J_\alpha$ and $n' = n$. Hence the height of w' in X is (α, n), so that w' is also a minimal witness for X. ⊣

This completes the proof of the covering lemma for L. In the rest of this section we consider two variations on this proof. The first, and most important, extends the argument to models with a measurable cardinal in order to obtain the Dodd-Jensen covering lemma; the second variation applies the argument to unsuitable sets X, obtaining Magidor's Covering Lemma 1.15 and the absoluteness theorem for Jónsson cardinals, Theorem 1.16.

3.3. Measurable Cardinals

The primary aim of this subsection is to prove the Dodd-Jensen covering lemma, and an important secondary aim is to prepare the way for Sect. 4 which describes the covering lemma for larger core models. In accordance with this secondary aim we do not assume $\neg 0^\dagger$ except when it is explicitly specified. For simplicity we do assume that there is no model of $\exists\kappa\, o(\kappa) =$

κ^{++}, but much of our discussion is true in general (though not always in detail) for the larger core models described in chapter [57].

In Dodd and Jensen's original papers [9–11], the minimal model $L[U]$ for a measurable cardinal is treated separately from the Dodd-Jensen core model K^{DJ}. The model $L[U]$ is the simplest natural analogue of L and was already well understood long before the core model was invented. While $L[U]$ has many of the properties of L, there is one vital difference: The existence of the model L is implied by the axioms of set theory, but the construction of the model $L[U]$ depends on being first given the filter U which will be the measure in the model $L[U]$.

If $L[U]$ does exist then K^{DJ} can easily be obtained by "iterating the measure U out of the universe":

$$K^{\mathrm{DJ}} = \bigcap_{\nu\in\mathrm{On}} \mathrm{Ult}^{\nu}(L[U],U) = \bigcup_{\nu\in\mathrm{On}}(\mathrm{Ult}^{\nu}(L[U],U) \cap V_{i_\nu(\kappa)})$$

where $i_\nu : L[U] \to \mathrm{Ult}^{\nu}(L[U],U)$ is the ν-fold iteration of the ultrapower by U. In order to define an inner model which would exist even in the absence of a model $L[U]$, Dodd and Jensen defined the core model K^{DJ} to be $L[\mathcal{M}]$, where $\mathcal{M}$ is a class of approximations, called *mice*, to models of the form $L[U]$. The mice are structures $M = J_\alpha[W]$ with the properties (i) $M \models$ "W is a measure", (ii) M is iterable (in the sense that every iterated ultrapower of M is well-founded) and (iii) M is sound and has projectum smaller than $\mathrm{crit}(W)$. Note that condition (iii) implies that $J_{\alpha+\omega}[W] \models |\alpha| < \mathrm{crit}(W)$, so that W is not a measure in any model larger than M.

In [35], the Dodd-Jensen core model was extended to obtain a core model for sequences of measures. This extended core model had the form $K[\mathcal{U}] = L[\mathcal{U},\mathcal{M}]$, where $\mathcal{U}$ was the sequence of measures in $K[\mathcal{U}]$ and $\mathcal{M}$ was a class of mice. The mice $M \in \mathcal{M}$ were models of the form $M = J_\alpha[\mathcal{U}']$ where the sequence $\mathcal{U}'$ was a concatenation $\mathcal{U}' = \mathcal{U}^\frown\mathcal{W}$ of the sequence of measures of $K[\mathcal{U}]$ with a sequence $\mathcal{W}$ of filters which are measures in M but not in $J_{\alpha+\omega}[\mathcal{U}']$. The sequence $\mathcal{W}$ corresponded to the measure W in a Dodd-Jensen mouse $J_\alpha[W]$.

The modern approach to mice, which we follow here, originated in attempts to extend the core model to cardinals approaching a supercompact cardinal. This program has many difficulties, some of which are still not solved, but a key to making a beginning was the observation that the original notion of a mouse was too simple. A key fact in the theory of the constructible sets is that all of the models J_α, as well as L itself, have the same structure, so that they differ only in length. It became clear in course of the investigation that the mice for an extended core model should similarly have the same structure as the full core model. That is, the mice are themselves constructed from smaller mice, like a well-founded version of Swift's well known flea, which

> hath smaller fleas that on him prey
> And these have smaller still to bit 'em;
> And so proceed *ad infinitum.*

This seemed prohibitively complicated, but a suggestion of S. Baldwin made it possible to realize the desired situation while simplifying, instead of complicating, the construction: The mice and the core model would be structures of the form $M = J_\alpha[\mathcal{E}]$ or $M = L[\mathcal{E}]$, respectively. The members of the sequence $\mathcal{E}$ would be extenders, but some would be only partial extenders, not measuring all of the sets in M. These partial extenders would be the full extenders of those mice which are members of M; thus the sequence $\mathcal{E}$ codes both the mice and the extenders in the structure M.

For the rest of this section we limit ourselves to sequences of measures, with no extenders, and we mark this restriction by using the letter $\mathcal{U}$ to denote the sequence instead of $\mathcal{E}$.

As hoped, this approach leads to a feasible fine structure for the extended core models, but surprisingly it also simplifies the fine structure for the previously existing core models. This is particularly surprising for the Dodd-Jensen core model, the mice of which have at most one measurable cardinal. It would seem at first glance that nothing could be simpler than a mouse of the form $J_\alpha[U]$, but the apparent simplicity of this model hides a complicated fine structure. For example, consider the key fact of the fine structure of L—and even of Gödel's proof of the continuum hypothesis—that every constructible subset of ω is a member of J_{ω_1}. This fact fails badly in the model $L[U]$: if $\kappa = \mathrm{crit}(U)$ then $J_\alpha[U] = J_\alpha$ for all $\alpha \leq \kappa + 1$. The first nonconstructible set to be constructed is $0^\#$, which is a subset of ω and is Δ_1-definable over $J_{\kappa+\omega}[U]$, so that $0^\# \in J_{\kappa+\omega\cdot 2}[U] - J_{\kappa+\omega}[U]$.

The newer fine structure avoids this problem because the subsets of ω in $L[\mathcal{U}]$ are all in $J_{\omega_1}[\mathcal{U}]$ and hence are constructed from the restriction $\mathcal{U}\restriction\omega_1$ of $\mathcal{U}$ to (partial) measures on countable ordinals. In fact, the first nontrivial member of $\mathcal{U}$ is the L-ultrafilter on the first Silver indiscernible c_0 which is induced by, and which constructs, the real $0^\#$. In the case $L[\mathcal{U}] = L[U]$, the sequence $\mathcal{U}$ has as its last nontrivial member the measure U, which is the only member of the sequence $\mathcal{U}$ which is a full measure on $L[\mathcal{U}]$.

The benefit of the new approach to the core model is suggested by the fact that the following two modifications are all that is necessary to adapt Definition 3.1 of fine structure for L to the core model.

1. An added predicate is needed to represent the sequence of measures, so that the Σ_1-code is a structure of the form $(J_\alpha[\mathcal{U}], \mathcal{U}\restriction\alpha, A)$ instead of (J_α, A).

2. For ordinals α such that $\mathcal{U}_\alpha \neq \varnothing$ it is necessary to begin the construction with a special *amenable code*, defined to be $\mathfrak{C}_0(J_\alpha[\mathcal{U}]) = (J_{\rho_0}, \mathcal{U}\restriction\rho_0, \mathcal{U}_\alpha)$ where, if $\kappa = \mathrm{crit}(\mathcal{U}_\alpha)$, then ρ_0 is defined to be κ^+ of $L[\mathcal{U}\restriction\alpha]$.

The "Skolem function" $h_0^{J_\alpha[\mathcal{U}]}$ mapping $\rho_0 = \rho_0^{J_\alpha[\mathcal{U}]} = \kappa^{+J_\alpha[\mathcal{U}]}$ onto $J_\alpha[\mathcal{U}]$ is derived from the function mapping functions $f : \kappa \to J_\kappa[\mathcal{U}]$ in $J_{\rho_0}[\mathcal{U}]$ to their equivalence classes $[f]_{\mathcal{U}_\alpha} \in J_\alpha[\mathcal{U}] = J_\alpha[\mathcal{U}\restriction\alpha] \subseteq \mathrm{Ult}(J_\kappa[\mathcal{U}], \mathcal{U}_\alpha)$.

The analogous structure in the fine structure of L was simply $\mathfrak{C}_0(J_\alpha) = (J_\alpha, \varnothing)$. The amenable code is needed here because the obvious structure $(J_\alpha[\mathcal{U}], \in, \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is not amenable: $\mathcal{P}(\alpha)^{J_\alpha[\mathcal{U}]} \in J_\alpha[U]$, but $\mathcal{U}_\alpha \cap \mathcal{P}(\alpha)^{J_\alpha[\mathcal{U}]} = \mathcal{U}_\alpha \notin J_\alpha[\mathcal{U}]$. If $\mathcal{U}_\alpha = \varnothing$ then the amenable code is simply $(J_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \varnothing)$, as in L.

Some additional change is necessary for cardinals larger than measurable cardinals:

3. In models where iteration trees are needed instead of linear iterated ultrapowers, the standard parameter is augmented to included a *witness* to its minimality. This witness, which is discussed later, is used in the models of this section, but does not need to be explicitly included in the structure.

4. The amenable code is somewhat more complicated in the case of sequences $L[\mathcal{E}]$ involving extenders instead of only measures. See chapter [47] for details.

The proof that the fine structure given by this definition satisfies the necessary properties is, of course, more complicated than the proof in L. We begin with the definition of a mouse. Recall that U is a M*-ultrafilter* on κ if it is a normal M-ultrafilter in the sense of Kunen, that is, U is a normal ultrafilter on $\mathcal{P}^M(\kappa)$ and $U \cap X \in M$ whenever $X \in M$ and $M \models |X| = \kappa$. If M is a structure $J_\alpha[\mathcal{U}]$ and $\gamma < \alpha$ then we write $M|\gamma$ for the initial segment $J_\gamma[\mathcal{U}]$ of M.

3.24 Definition. A *mouse* is a premouse which is iterable and sound.

We define the terms *premouse* and *sound* by a simultaneous recursion on α:

1. A *premouse* is a model $J_\alpha[\mathcal{U}]$ (or $L[\mathcal{U}]$, allowing $\alpha = \mathrm{On}$) which satisfies the following three conditions:

   (a) For each γ such that $\mathcal{U}_\gamma \neq \varnothing$, there is a cardinal κ of $J_\alpha[\mathcal{U}{\restriction}\gamma]$ such that $(J_\gamma[\mathcal{U}{\restriction}\gamma], \mathcal{U}_\gamma) \models (\gamma = \kappa^{++}$ and $\mathcal{U}_\gamma$ is a normal $J_\gamma[\mathcal{U}{\restriction}\gamma]$-measure on $\kappa)$.

   (b) (Coherence) If $\mathcal{U}_\gamma \neq \varnothing$ then $(i^{\mathcal{U}_\gamma}(\mathcal{U}{\restriction}\gamma)){\restriction}\gamma + 1 = \mathcal{U}{\restriction}\gamma$.

   (c) (Soundness) The structure $(J_{\alpha'}[\mathcal{U}], \mathcal{U}{\restriction}\alpha', \mathcal{U}_{\alpha'})$ is sound for every ordinal $\alpha' < \alpha$.

   We say that a sequence $\mathcal{U}$ is *good* if $L[\mathcal{U}]$ is a premouse.

2. A premouse $\mathcal{M} = (J_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is said to be n*-sound* if $h^{\mathcal{M}}_m$“$\rho_m = J_\alpha[\mathcal{U}]$ for each $m \leq n$, where $h^{\mathcal{M}}_m$ and $\rho^{\mathcal{M}}_m$ are the Σ_m-Skolem functions and Σ_m projectum of $\mathcal{M}$, respectively. The model $\mathcal{M}$ is *sound* if it is n-sound for all $n \in \omega$.

   We will say that $\mathcal{M}$ is *sound above* η if either $\mathcal{M}$ is sound or there is n such that $\rho^{\mathcal{M}}_{n+1} \leq \eta$, $\mathcal{M}$ is n-sound and $h^{\mathcal{M}}_{n+1}$“$\eta = J_\alpha[\mathcal{U}]$.

3. (Iterability) A premouse $J_\alpha[\mathcal{U}]$ is *iterable* if every iterated ultrapower of $J_\alpha[\mathcal{U}]$ is well-founded.

   Note that this definition of the term *iterable* needs to be supplemented by Definition 3.30, given later, of an iterated ultrapower of a premouse.

Again, see chapter [47] for the somewhat more complicated conditions on the sequence $\mathcal{U}$ when it is allowed to contain extenders.

3.25 Remark. Notice that any premouse satisfies GCH, since the soundness condition implies that whenever $x \subseteq \eta < \alpha$ and $x \in J_{\alpha+\omega}[\mathcal{U}] - J_\alpha[\mathcal{U}]$, then $J_{\alpha+\omega}[\mathcal{U}] \models |\alpha| \leq \eta$. This property is often called *acceptability* in the literature, where it is used as a placeholder for soundness in the definition of a premouse in order to avoid the use of simultaneous recursion as in Definition 3.24.

If $J_\alpha[\mathcal{U}]$ is a n-sound premouse then the ultrapower $\mathrm{Ult}_n(J_\alpha[\mathcal{U}], \pi, \beta)$ of $M = J_\alpha[\mathcal{U}]$ by the extender derived from an embedding π is defined just like that for L, by taking the ultrapower $\mathrm{Ult}_0(\mathfrak{C}_n(M), \pi, \beta)$ of the Σ_n-code using functions in $\mathfrak{C}_n(M)$ and then using the upward extension property to extend the embedding to all of M. In particular the upward extension property, Lemma 3.10 (which we will not restate here) is still valid for these models. Since not every premouse $J_\alpha[\mathcal{U}]$ is sound, the Proposition 3.11 giving the conditions for the existence of $\mathrm{Ult}_n(J_\alpha[\mathcal{U}], \pi, \beta)$ needs to be supplemented with the requirement that $J_\alpha[\mathcal{U}]$ be n-sound. In addition, we now have the possibility of taking an ultrapower by one of the ultrafilters $U = \mathcal{U}_\gamma$ in $M = J_\alpha[\mathcal{U}]$. The ultrapower $\mathrm{Ult}_n(M, U)$, like $\mathrm{Ult}_n(M, \pi, \beta)$, is obtained by taking the ordinary ultrapower $\mathrm{Ult}(\mathfrak{C}_n(M), U)$ of the nth code of M and using Lemma 3.10.

3.26 Lemma. *Suppose that $M = J_\alpha[\mathcal{U}]$ is an n-sound iterable premouse, and U is a M-ultrafilter with* $\mathrm{crit}(U) \geq \rho^M_{n+1}$*. Then the embedding $i^U : M \to M' = \mathrm{Ult}_n(M, U)$ satisfies the following two properties:*

1. $A^M_{n+1} \notin M'$, and hence $\rho^{M'}_{n+1} = \rho^M_{n+1}$.

2. $i^U(p^M_{n+1}) = p^{M'}_{n+1}$.

Furthermore, any embedding $i : M \to M'$ which is given by the Upward Extension Lemma from a Σ_0-elementary embedding from $\mathfrak{C}_n(M)$ and which satisfies clause 1 also satisfies clause 2.

The thing which makes the conclusion stronger than that of Lemma 3.10 is the assertion that $\rho^{M'}_{n+1} = \rho^M_{n+1}$. This may be contrasted with diagram (18.5) in the proof of the covering lemma, in which $\tilde{\pi} : J_\alpha \to J_{\tilde{\alpha}} = \mathrm{Ult}(J_\alpha, \pi, \kappa)$ and $\rho^{J_\alpha} < \bar{\kappa}$, while $\rho^{J_{\tilde{\alpha}}}_{n+1} = \kappa = \tilde{\pi}(\bar{\kappa})$.

A part of the proof of clause 2 will be deferred until after the discussion of iterated ultrapowers.

Sketch of Proof. To see that $\rho^{M'}_{n+1} = \rho^{M}_{n+1}$ we need to verify that the master code $A = A^M_{n+1} \subseteq \rho^M_{n+1}$ is not a member of M'. Suppose to the contrary that $A = [f]_U \in M' = \mathrm{Ult}_n(M, U)$. Then A can be written as $\{\beta < \rho^M_{n+1} : \{\xi : \beta \in f(\xi)\} \in U\}$, which is a member of M since the assumptions that U is a M-ultrafilter and $\rho^M_{n+1} \leq \kappa$ imply that

$$U \cap \{\{\xi < \kappa : \beta \in f(\xi)\} : \beta < \rho^M_{n+1}\} \in M.$$

This contradiction concludes the proof that $A \notin M'$.

One direction of clause 2 is straightforward: clearly $i^U(p^M_{n+1}) \geq p^{M'}_{n+1}$, since A^M_{n+1} can be defined using the parameter $i^U(p^M_{n+1})$. The hard part is to see that $i^U(p^M_{n+1}) \leq p^{M'}_{n+1}$. The proof proceeds by induction on n, and we will only present the case $n = 0$. If to the contrary $p^{M'}_1 < i^U(p^M_1)$, then there is an ordinal $i^U(\nu) \in i^U(p^M_1) - p^{M'}_1$ such that $p^{M'}_1 - i^U(\nu) = i^U(p^M_1) - (i^U(\nu)+1)$. Set $p = p^M - (\nu+1)$ and $p' = i^U(p) = p^{M'}_1 - i^U(\nu)$, and set $A' = \{\xi < i^U(\nu) : M' \models \Phi(\xi, p')\}$ where Φ is the universal Σ_1 formula. Any subset of $i(\nu)$ which is Σ_1-definable in M' from p' is rudimentary in A', so if we can show that $A' \in M'$, then it will follow that any set Σ_1-definable from parameters in $p' \cup \nu$ is also a member of M'. This will contradict the assumption that $p' = p^{M'}_1 - \nu$.

Set $A = \{\xi < \nu : \Phi(\xi, p)\}$. Then $A \in M$, so $i^U(A) \in M'$. Unfortunately it may not be the case that $A' = i^U(A)$, so we need to analyze this set further. Let $M = J^{\mathcal{U}}_\alpha$, and define a prewellordering R on A by $\xi' \, R \, \xi$ if $\xi \in A$ and $\exists\gamma \, (J^{\mathcal{U}}_\gamma \models \Phi(\xi', p) \,\&\, \forall\gamma' < \gamma \; J^{\mathcal{U}}_\gamma \not\models \Phi(\xi, p))$. Then R is also Σ_1-definable from p, and hence is a member of M. Define R' in M' similarly. Then the prewellordering R' on A' is an initial segment of the preordering $i(R)$ on $i(A)$. Now $i(R) \in M'$, and if $i(R)$ is a prewellordering then all initial segments of $i(R)$ are also in M'. The following definition will be used to show that $i(R)$ is a prewellordering:

3.27 Definition. A *solidity witness that* $\nu \in p^M_1$ is a function $\tau \in M$ which maps A into the ordinals of M so that $\forall \xi', \xi \in A \; (\xi' \, R \, \xi \iff \tau(\xi') \leq \tau(\xi))$.

If τ is a solidity witness that ν is in p^M_{n+1} then $i(\tau)$ is an order preserving embedding from $i(R)$ into the ordinals of M'. Since M' is well-founded it follows that $i(R)$ is a prewellordering.

The general proof of the existence of a solidity witness will be deferred until after the introduction of iterated ultrapowers; however, we note here that the construction of a solidity witness τ can be carried out in any admissible set containing R; in fact this is the central element of the standard proof that well-foundedness is absolute. This leads to two easy cases, in which the solidity witness for a mouse $M = J_\alpha(\mathcal{U}^M)$ can be found in an admissible initial segment $M|\gamma = J_\gamma(\mathcal{U}^M \restriction \gamma)$ of M. If M has a measurable cardinal $\mu \geq \nu$ then $(\nu^+)^M$ exists, so there is a solidity witness in the admissible set $M|(\nu^+)^M$; and if M has a full measure $\mathcal{U}_\gamma$ with $\mathrm{crit}(\mathcal{U}_\gamma) < \nu < \gamma$ and $R \in \mathrm{Ult}(M, \mathcal{U}_\gamma)$ then M has a solidity witness in the admissible set $M|\gamma$. $\dashv$

Note that the hypothesis that U is a normal M-ultrafilter holds whenever $U = \mathcal{U}^M_\gamma$ for some $\gamma \leq \alpha$. If $\gamma < \alpha$ then a slightly stronger result holds, since the hypothesis that $\kappa \geq \rho^M_{n+1}$ can be eliminated (with some adjustment to the conclusion). Even then, however, not all of the fine structure of M is preserved by the ultrapower i^U. First, and most important, the ultrapower $M' = \mathrm{Ult}_n(M, U)$ is never sound above $\kappa = \mathrm{crit}(U)$, even if M is, since $\kappa \notin (i^U h^M_{n+1})``\mathrm{crit}(U) = h^{M'}_{n+1}``\mathrm{crit}(U) = h^{M'}_{n+1}``\rho_{n+1}$. The model $M'$ *is* sound above $\kappa + 1$. Second, the two projecti $\rho^M_n$ and $\rho^M_{n+1}$ need not be preserved by the embedding $i$: if $\mathrm{crit}(U) = \rho^M_{n+1}$ then $\rho^{M'}_{n+1} = \rho^M_{n+1} < i^U(\rho^M_{n+1})$; and in any case $\rho^{M'}_n = \sup(i^U\, ``\rho^M_n)$, which may be smaller then $i^U(\rho^M_n)$.

The existence of unsound premice is an important difference between the fine structure of L and that of larger core models. The counterpart of Lemma 3.4, which states that J_α is sound, is given by the following lemma:

3.28 Lemma. *Any iterable premouse $M = J_\alpha[\mathcal{U}]$ is an iterated ultrapower of a mouse.*

The mouse is given by the following definition:

3.29 Definition. The nth *core* of a premouse M, written $\mathrm{core}_n(M)$, is the model obtained by decoding the nth code $M_n = \mathfrak{C}_n(M)$ of M. The *core* of M, written $\mathrm{core}(M)$, is $\mathrm{core}_n(M)$ where n is least such that $\rho^M_n = \rho^M$.

Note that the definition of $\mathfrak{C}_n(M)$ is not hindered by the possibility that M is not sound. The structure $\mathrm{core}(M)$ will be equal to the transitive collapse of the substructure of M containing those elements which are, in an appropriate sense, definable in M. In particular, if we define $\mathrm{core}_1(M)$ to be the model obtained by decoding the Σ_1-code $M_1 = \mathfrak{C}_1(M)$, and then decoding M_1, then $\mathrm{core}_1(M)$ is the transitive collapse of $h^M_1``\rho^M_1$, the set of $x \in M$ which are Σ_1-definable using parameters from $\rho^M_1 \cup p^M_1$.

Any further sketch of the proof will clearly depend on the definition and properties of iterated ultrapowers. These were described in chapter [32], but are complicated here by the fact that they may involve ultrapowers of differing degrees and since they may involve filters $\mathcal{U}^{M_\nu}_\gamma$ which are not full ultrafilters on M_ν. Both of these situations result in the *drops* mentioned in the following definition.

The situation is slightly simpler in the case when $0^\dagger$ does not exist, so that the premice $J_\alpha[\mathcal{U}]$ have at most one full ultrafilter, than it is in the more general case needed in Sect. 4. At stage ν of the iterated ultrapowers being considered here there are only two possible choices. One is to use the single full ultrafilter in the model M_ν, which will be the last member of the sequence $\mathcal{U}^{\mathcal{M}_\nu}$; this is case 3a of the definition. The other is to use one of the earlier filters in the sequence $\mathcal{U}^M$. This earlier filter is not a full ultrafilter in M_ν and hence must be applied to a smaller mouse in M_ν on which it is an ultrafilter; this is case 3c. Case 3b does not arise in the absence of $0^\dagger$.

3.30 Definition. An *iterated ultrapower of a premouse* $M = J_\alpha[\mathcal{U}]$ is a sequence of models M_ν for $\nu \leq \theta$, together with a finite set $D \subseteq \theta+1$, called the set of *drops*, and embeddings $i_{\nu,\nu'} : M_\nu \to M_{\nu'}$ defined for all pairs $\nu < \nu' \leq \theta$ such that $D \cap (\nu, \nu'] = \varnothing$.

All of these are determined by a sequence of filters $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} \in M_\nu$, with a strictly increasing sequence of critical points $\operatorname{crit}(U_\nu)$, as follows:

1. $M_0 = J_\alpha[\mathcal{U}]$.
2. If ν is a limit ordinal then $M_\nu = \operatorname{dir\,lim}_{\nu_0 \leq \nu' < \nu} M_{\nu'}$ where $\nu_0 = \sup(D \cap \nu)$. Note that this direct limit exists since the finiteness of D implies that $D \cap \nu$ is bounded in ν, so that the embeddings $i_{\nu'',\nu'} : M_{\nu''} \to M_{\nu'}$ exist for all sufficiently large $\nu'' < \nu' < \nu$.
3. If $\nu + 1 \leq \theta$ then $M_{\nu+1}$ is determined by the choice of the ultrafilter $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu}$, where $\gamma_\nu > \gamma_{\nu'}$ for all $\nu' < \nu$. There are three cases:
   (a) If U_ν is a full ultrafilter on M_ν then set $n_\nu = n$ where n is the largest number such that $\operatorname{Ult}_n(M_\nu, U_\nu)$ is defined. If $n_\nu = n_{\nu'}$ for all sufficiently large $\nu' < \nu$, then $M_{\nu+1} = \operatorname{Ult}_{n_\nu}(M_\nu, U_\nu)$. In this case $i_{\nu,\nu+1}$ is the canonical embedding.

       Note that since the critical points of the ultrafilters U_ν are increasing, M_ν is sound above $\operatorname{crit}(U_\nu)$ and hence the soundness hypothesis of Lemma 3.26 is satisfied.

   (b) If U_ν is a full ultrafilter on M_ν, but $n_\nu < n_{\nu'}$ for all $\nu' \in \nu - \max(D \cap \nu)$, then $M_{\nu+1} = \operatorname{Ult}_{n_\nu}(M_\nu, U_\nu)$. In this case we add $\nu + 1$ to D, so that $i_{\nu,\nu+1}$ is not defined.

       Note that this happens when $\operatorname{crit}(U_\nu) \geq \rho^{M_\nu}_{n_{\nu'}}$, but $\operatorname{crit}(U_{\nu'}) < \rho^{M_{\nu'}}_{n_{\nu'}}$, for $\sup(D \cap \nu) < \nu' < \nu$. This case is known as a *drop in degree*.

   (c) If U_ν is not a full ultrafilter on M_ν, then let $M^*_{\nu+1}$ be the largest initial segment of M_ν on which U_ν is an ultrafilter. Thus $M^*_{\nu+1} = J_{\alpha^*_\nu}[\mathcal{U}_\nu]$ where α^*_ν is the least ordinal $\beta < \alpha$ such that there is a subset x of $\operatorname{crit}(U_\nu)$ in $J_{\beta+\omega}[\mathcal{U}_\nu] - J_\beta[\mathcal{U}_\nu]$ which is not measured by U_ν.

       In this case, which is known as a *normal drop*, we set $M_{\nu+1} = \operatorname{Ult}(M^*_{\nu+1}, U_\nu)$, and we add $\nu+1$ to D so that $i_{\nu,\nu+1}$ is not defined.

We say that M is *iterable* if every model in any iterated ultrapower of M is well-founded and no attempt to create an iterated ultrapower leads to infinitely many drops.

Here again the situation becomes more complicated in the case of extenders, where *iteration trees* are needed instead of the linear iterated ultrapowers described above. See chapter [47] or [57].

3.31 Remark. In practice we will frequently make the trivial modification that $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} = \varnothing$ is also allowed in an iterated ultrapower, and set $M_{\nu+1} = M_\nu$ in this case. This gives what is known as *padded* iterated ultrapowers.

3.32 Lemma. *The formula asserting that a set M is a mouse is absolute for models N containing ω_1.*

Proof. The statement that M is a premouse is first-order over M, as is the assertion that M is sound, so we only need verify that the iterability of M is absolute. If M is countable in N then this can be proved using the Shoenfield absoluteness theorem, as the statement that there is an ill-founded iterated ultrapower of M, with the iterated ultrapower indexed by a countable well-order, is a Σ^1_2 statement. The proof for general M is similar to the proof of Shoenfield's theorem: for each countable ordinal α one builds a "tree of attempts to find a ill-founded iterated ultrapower of length at most α", that is to say, a tree T_α such that the infinite branches of T_α correspond exactly to the ill-founded iterated ultrapowers of M of length at most α. If there is an ill-founded iteration of length $\alpha < \omega_1$ in V, then T_α has an infinite branch and hence is ill-founded. Assuming that the tree T_α can be constructed in N just as it was constructed in V (this relies on the fact that $\omega_1 \subseteq N$, so that α is countable in N) the tree T_α is in N. By the absoluteness of well-foundedness it is ill-founded there, so there is, in N, an infinite branch of T_α which specifies an ill-founded iterated ultrapower of M.

Since this is a very important technique, we suggest here one method of constructing such a tree. In order to simplify the construction we first ignore the possibility of drops. A node at the nth level of the tree T_α will be a 4-tuple $p = \langle x, \vec{U}, \vec{M}, \vec{\xi}\rangle$ such that

1. $\{0, \alpha\} \subseteq x \in [\alpha+1]^{<\omega}$,
2. $\vec{M}$ is a finite iterated ultrapower of $M_0 = M$ indexed by the ordinals in x and using the ultrafilters $\vec{U}$. That is, if $\nu \in x$ and $\nu' = \min(x-(\nu+1))$ then $U_\nu \in M_\nu$ and $M_{\nu'} = \mathrm{Ult}(M_\nu, U_\nu)$.
3. $\vec{\xi}$ is a descending sequence of ordinals, $\vec{\xi}$ has length n, and $\vec{\xi} \in M_\alpha$.

We will say that a node $p' = \langle x', \vec{U}', \vec{M}', \vec{\xi}\rangle$ at the $n+1$st level of T_α is *below p in T_α* if $x' \supseteq x$ and for each $\nu \in x$ there is $\sigma_\nu : M_\nu \to M'_\nu$ such that

1. σ_0 is the identity.
2. $\sigma_\nu(U_\nu) = U'_\nu$.
3. If $\nu \in x$, $\nu' = \min(x - (\nu+1))$ and $\nu'' = \min(x' - (\nu+1))$ then $\sigma_{\nu'}([f]_{U_\nu}) = i'_{\nu',\nu''}([\sigma_\nu(f)]_{U'_\nu})$, where $i'_{\nu',\nu''} : M_{\nu'} \to M_{\nu''}$ is the embedding associated with the iteration $\vec{M}'$.
4. $\xi'_k = i'_{\nu,\nu'}(\sigma_\nu(\xi_k))$ for each $k < n$, where $\nu = \max(x)$ and $\nu' = \max(x')$.

In order to see how an ill-founded iteration $\langle M_\nu : \nu < \alpha\rangle$ yields an infinite branch in T_α we need the concept of a *support*:

3.33 Definition. If $\langle M_\nu : \nu \le \eta\rangle$ is an iterated ultrapower, then the notion of a *support* is defined by recursion on η: a finite set $y \subseteq \eta + 1$ is a support for $z \in M_\eta$ if

1. $\{0, \eta\} \subseteq y$.
2. If η is a limit ordinal then there are $\nu < \eta$ in y and $z' \in M_\nu$ such that $z = i_{\nu,\eta}(z')$ and $y \cap (\nu + 1)$ is a support for z' in $\vec{M}\restriction(\nu + 1)$.
3. If $\eta = \nu + 1$ then $\nu \in y$ and $y \cap \eta$ is a support for $\{U_\nu, f\}$, where $M_\eta = \mathrm{Ult}(M_\nu, U_\nu)$ and $z = [f]_{U_\nu}$.

Suppose that $y \subseteq \eta + 1$ is a support in $\langle M_\nu : \nu \le \eta\rangle$. We will construct a finite iterated ultrapower $\langle M'_\nu : \nu \in x\rangle$, together with embeddings $\sigma_\nu : M'_\nu \to M_\nu$ for $\nu \in x$, with the key property that the range of each embedding σ_ν is exactly the set of $z \in M_\nu$ such that $y \cap (\nu + 1)$ is a support for z.

The index set x for the iteration is the set of $\nu \in y$ such that $\nu = 0$, $\nu = \eta$, or $\nu + 1 \in y$. The models M'_ν and embeddings $\sigma_\nu : M'_\nu \to M_\nu$ for $\nu \in x$, and the ultrafilters $U'_\nu \in M'_\nu$ for $\nu \in x \cap \eta$, are defined by recursion on ν. To start out, $M'_0 = M_0$ and σ_0 is the identity. If $\nu \in x \cap \eta$ then $U'_\nu = \sigma_\nu^{-1}(U_\nu)$, which exists because $y \cap (\nu + 1)$ is a support for U_ν in M_ν, and $M'_{\nu'} = \mathrm{Ult}(M_\nu, U_\nu)$ where $\nu' = \min(x - (\nu + 1))$. If $\nu' = \nu + 1 \in x$ then $\sigma_{\nu+1}$ is defined by setting $\sigma_{\nu+1}([f]_{U'_\nu}) = [\sigma_\nu(f)]_{U_\nu}$. Otherwise ν' is a limit ordinal and $\sigma_{\nu'}([f]_{U'_\nu}) = i_{\nu+1,\nu'}([\sigma_\nu(f)]_{U_\nu})$.

Now if $\langle M_\nu : \nu \le \alpha\rangle$ is an ill-founded iterated ultrapower, then let $\vec{\xi} = \langle \xi_n : n < \omega\rangle$ be an infinite descending sequence of ordinals in the final model M_α, and pick an increasing sequence $\{y_n : n < \omega\}$ such that y_n is a support for $\vec{\xi}\restriction n$. Then the construction in the last paragraph gives a sequence $\langle (x_n, \vec{U}_n, \vec{M}_n, \vec{\xi}_n) : n \in \omega\rangle$ which is an infinite branch in the tree T_α.

To go the other direction, if $\langle\langle x_n, \vec{U}_n, \vec{M}_n, \vec{\xi}_n\rangle : n \in \omega\rangle$ is an infinite branch of T_α then we can obtain, by a direct limit construction, an iterated ultrapower which is indexed by $\bigcup_n x_n$ and hence has length at most α. Clause 4 of the definition implies that the direct limit maps $\sigma_{n,\alpha} : M_{n,\alpha} \to M_\alpha$ satisfy $\sigma_{n,\alpha}(\vec{\xi}_n) = \sigma_{n',\alpha}(\vec{\xi}_{n'})\restriction n$ for all $n < n' < \omega$. Thus $\bigcup_n \sigma_{n,\alpha}(\vec{\xi}_n)$ is an infinite descending sequence of ordinals which witnesses that the final model M_α is ill-founded.

In order to allow for iterated ultrapowers including drops, the definition of T_α must be modified: first, the definition of a node allows iterations with drops, and in the definition of the tree ordering the maps σ_ν is required to preserve the drops; furthermore any $\nu \in x' - x$ at which $\vec{M}'_\nu$ drops are required to be larger than $\max(x \cap \alpha)$. Finally, clause 4 of that definition is modified to state that either clause 4 holds as stated previously, or else the iteration $\vec{M}'$ has a drop in $x' - x$.

An infinite branch of T_α describes a presumptive iteration which is indexed by a subset of α, and hence has length at most α. If a new drop is added at infinitely many levels in the branch then the presumptive iteration has infinitely many drops; otherwise the presumptive iteration is a real iteration and hence has a last model, but the levels of the branch beyond the last drop provide a witness $\vec{\xi}$ that the last model of the iteration is ill-founded. In either case this presumptive iteration demonstrates that the model M_0 is not iterable. ⊣

Comparisons of Mice

In the case of L, the only mice are the structures J_α, and hence it is trivial that, given two mice M and N, one is an initial segment of the other. Under appropriate conditions the same crucial fact is true of the mice for higher core models, but the proof requires the use of iterated ultrapowers to compare the two mice. We describe this process below, using the notation of Definition 3.30 for the iterated ultrapowers. Superscripts M and N are used to distinguish the iterated ultrapower on M from that on N.

3.34 Definition (Comparison for Premice). We will say that two premice M and N *strongly agree up to* τ if $\operatorname{crit}(\mathcal{U}^M_{\gamma'}) \geq \tau$ and $\operatorname{crit}(\mathcal{U}^N_{\gamma'}) \geq \tau$ for all $\gamma' \geq \gamma$, where γ is the least ordinal such that $\mathcal{U}^M_\gamma \neq \mathcal{U}^N_\gamma$. Assume strongly agree up to τ. Then the comparison of M and N is defined by the use of iterated ultrapowers on M and N, which we distinguish by means of superscripts M and N.

Start the comparison by setting $M_0 = M$ and $N_0 = N$. Now suppose that M_ν and N_ν have been defined. If either of the models N_ν or M_ν is an initial segment of the other then the comparison is complete, and the iterated ultrapower is terminated with $\theta = \nu$. Otherwise let γ_ν be the least ordinal γ such that $\mathcal{U}^{N_\nu}_\gamma \neq \mathcal{U}^{M_\nu}_\gamma$, and set $U^M_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu}$ and $U^N_\nu = \mathcal{U}^{N_\nu}_{\gamma_\nu}$. Now use these ultrafilters to define $M_{\nu+1}$ and $N_{\nu+1}$ as in Definition 3.30 of an iterated ultrapower.

Note that Definition 3.34 uses padded iterated ultrapowers, since it may be that $\mathcal{U}^{N_\nu}_\gamma \neq \mathcal{U}^{M_\nu}_\gamma$ because one of the two is equal to $\varnothing$.

The coherence property of premice ensures that both the indices γ_ν and the critical points of the ultrafilters U^M_ν and U^N_ν are strictly increasing.

3.35 Lemma. *This comparison process always stops after fewer than τ^+ steps, with one of M_θ and N_θ an initial segment of the other.*

Proof. Suppose to the contrary that the comparison continues for τ^+ steps, and let $i_{\nu,\nu'} : N_\nu \to N_{\nu'}$ and $j_{\nu,\nu'} : M_\nu \to M_{\nu'}$ be the iteration embeddings. Set $\nu_0 = \max(D^N \cap D^M)$, the last place at which either iteration drops. Then for each $\nu < \tau^+ - \nu_0$ at least one of $i_{\nu,\nu+1}$ and $j_{\nu,\nu+1}$ is nontrivial; let κ_ν be the critical point of this embedding. If one of these embeddings is trivial then set $x_\nu = \varnothing$; otherwise pick $x_\nu \subseteq \kappa_\nu$ so that $x_\nu \in M_\nu \cap N_\nu$ and $x_\nu \in U^M_\nu \not\Longleftrightarrow x_\nu \in U^N_\nu$.

Now for each limit ordinal $\nu \in \tau^+ - \nu_0$ there is some $\eta_\nu < \nu$ such that κ_ν and x_ν are in the range of $i_{\eta_\nu,\nu}$, say $\kappa_\nu = i_{\eta_\nu,\nu}(\kappa'_\nu)$ and $x_\nu = i_{\eta_\nu,\nu}(x'_\nu)$. By Fodor's Lemma there is a stationary set $S_0 \subseteq \tau^+$ on which η_ν is constant, say $\eta_\nu = \eta$, and since $|N_\eta| < \tau^+$ there is a stationary $S_1 \subseteq S_0$ on which κ'_ν and x'_ν are also constant, say $\kappa'_\nu = \kappa'$ and $x'_\nu = x'$. Then for any $\nu' < \nu$ in S_1 we have $i_{\nu,\nu'}(\kappa_\nu) = i_{\nu,\nu'} i_{\eta,\nu}(\kappa') = i_{\eta,\nu'}(\kappa') = \kappa_{\nu'}$, and similarly $i_{\nu,\nu'}(x_\nu) = x_{\nu'}$. In particular $i_{\nu,\nu+1}$ is not the identity for $\nu \in S_1$, since otherwise we would have $\kappa_{\nu'} = i_{\nu,\nu'}(\kappa_\nu) = i_{\nu+1,\nu'} i_{\nu,\nu+1}(\kappa_\nu) = i_{\nu+1,\nu'}(\kappa_\nu) = \kappa_\nu$. Similarly there is a stationary set $S_2 \subseteq S_1$ such that if $\nu < \nu'$ are in S_2 then $j_{\nu,\nu'}(\kappa_\nu) = \kappa_{\nu'}$ and $j_{\nu,\nu'}(x_\nu) = x_{\nu'}$. But this is impossible, for if $\nu < \nu'$ are in S_2 then $x_\nu \in U^N_\nu \Longleftrightarrow \nu \in i_{\nu,\nu'}(x_\nu) = x_{\nu'} = j_{\nu,\nu'}(x_\nu) \Longleftrightarrow \kappa_\nu \in U^M_\nu$, contrary to the choice of x_ν. ⊣

The next few results analyze some of the possible outcomes of this comparison. All results assume that M and N satisfy the requirements for Definition 3.34: that is, they strongly agree up to τ and are sound above τ.

3.36 Lemma. *Suppose that M_θ is a proper initial segment of N_θ. Then M is sound, and the only ultrafilters U^M_ν used in the iteration of M are full ultrafilters with* $\operatorname{crit}(U^M_\nu) < \rho^{M_\nu}$. *Thus $D^M = \varnothing$.*

Sketch of Proof. Since M_θ is an initial segment of the premouse N_θ, the definition of a premouse implies that it is sound. Any model $\operatorname{Ult}(M_\nu, U)$ obtained by taking the ultrapower by an ultrafilter U with $\operatorname{crit}(U) \geq \rho^{M_\nu}$ is unsound, and this unsoundness is preserved by any further iterated ultrapowers.

If $D^M \neq \varnothing$ then let $\nu + 1 = \max(D^M)$. Then $M_{\nu+1} = \operatorname{Ult}(M^*_{\nu+1}, U^M_\nu)$ where $\operatorname{crit}(U^M_\nu) \geq \rho^{M^*_{\nu+1}}$, and hence $M_{\nu'}$ is unsound for all $\nu' \geq \nu + 1$. Similarly, if any of the models $M_{\nu+1}$ arise as ultrapowers by an ultrafilter $U^M_\nu \in M_\nu$ with $\operatorname{crit}(U^M_\nu) \geq \rho^{M_\nu}$, or if $M = M_0$ is unsound, then all succeeding models $M_{\nu'}$ are unsound. In either case this contradicts the fact that M_θ is sound. ⊣

3.37 Lemma. *Suppose that $M_\theta = N_\theta$, that $D^M = D^N = \varnothing$, and that $\tau \geq \max\{\rho^M, \rho^N\}$. Then $M = N$.*

Sketch of Proof. Since $D^M = D^N = \varnothing$, both $i^M_{0,\theta} : M \to M_\theta$ and $i^N_{0,\theta} : N \to N_\theta$ are defined. Since $\tau \geq \rho^M$ we have $M = h^M “\tau$. By Lemma 3.26, we have $i^M_{0,\theta} h^M = h^{M_\theta}$ so it follows that $M \cong h^{M_\theta} “\tau$. Similarly $N \cong h^{N_\theta} “\tau$, and since $N_\theta = M_\theta$ it follows that $M = N$. ⊣

3.38 Corollary. *At least one of D^M and D^N are empty.*

Sketch of Proof. Assume to the contrary that both iterated ultrapowers drop. Then Lemma 3.36 implies that neither of M_θ and N_θ is a proper initial segment of the other, so $M_\theta = N_\theta$. Now let $\nu + 1$ be the largest member of

$D^M \cup D^N$, and suppose for example that $\nu + 1 \in D^M$. Then the remainder of the iterated ultrapower can be regarded as a comparison of $M^*_{\nu+1}$ with either N_ν or $N^*_{\nu+1}$, depending on whether $\nu + 1 \in D^N$. Furthermore, since D^N contains some ordinal $\nu' \leq \nu+1$, this comparison satisfies the hypothesis of Lemma 3.37, with $\tau = \operatorname{crit}(U^M_\nu)$, so $M^*_{\nu+1} = N_\nu$ or $M^*_{\nu+1} = N^*_{\nu+1}$. This is impossible since $U^M_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} \neq \mathcal{U}^{N_\nu}_{\gamma_\nu}$. ⊣

3.39 Lemma. *Suppose that M and N are mice with* $\max\{\rho^N, \rho^M\} \leq \tau$*, and that M and N strongly agree up to τ. Then one of M and N is an initial segment of the other.*

Sketch of Proof. We prove the lemma by induction on the lengths of the mice M and N. First suppose that $M_\theta = N_\theta$. If $D^M = D^N = \varnothing$ then $M = N$ by Lemma 3.37. Otherwise, suppose that $D^N \neq \varnothing$ and let $\nu + 1$ be the largest member of D^N. If this is a drop in degree, then the remainders $\langle M_\xi : \nu \leq \xi < \theta\rangle$ and $\langle N_\xi : \nu \leq \xi < \theta\rangle$ of the two iterated ultrapowers form the comparison of M_ν with N_ν. This comparison has no drops, so we can apply Lemma 3.37 to conclude that $M_\nu = N_\nu$, contradicting the fact that $\nu < \theta$. Similarly, if this is a normal drop, then the remainders of the two iterated ultrapowers form the comparison of M_ν with the mouse $N^*_{\nu+1}$ to which the iteration on N drops at that point. Again, Lemma 3.37 shows that $M_\nu = N^*_{\nu+1}$, which is an initial segment of N_ν, so that the comparison would have terminated at $\nu < \theta$.

Thus we can assume without loss of generality that M_θ is a proper initial segment of N_θ. It follows by Lemma 3.36 that the iteration of M uses only ultrafilters with critical point smaller than the projectum; however, since the M and N strongly agree to τ the comparison uses only ultrafilters with critical point larger than τ, which in turn is larger then the projectum ρ^M. Thus M is never moved in the comparison, that is, $M_\theta = M$. ⊣

We are now ready to sketch a proof of Lemma 3.28 together with the existence of solidity witnesses:

3.40 Lemma. *If M is an iterable premouse then M is an iterated ultrapower of the mouse* $\operatorname{core}(M)$*, and M has a solidity witness for each $\nu \in p^M_k$ and $k < \omega$.*

Sketch of Proof of Lemmas 3.28 and 3.40. The proof is an induction over n, showing for each n that $\operatorname{core}_n(M)$ is an iterated ultrapower of $\operatorname{core}_{n+1}(M)$ and that p^M_{n+1} has solidity witnesses. We will give the proof for the case $n = 0$, beginning by showing that $M = \operatorname{core}_0(M)$ is an iterated ultrapower of $N = \operatorname{core}_1(M)$ (with critical point at least ρ^M_1) and will assume for the moment that N has solidity witnesses for all $\nu \in p^N_1$. Let $\pi : N \to M$ be the collapse map, and let $i : N \to N_\theta$ and $j : M \to M_\theta$ be the iterated ultrapowers comparing N and M. We begin by assuming that both maps have critical point at least ρ^M_1. Then neither side of the comparison can drop: if, say, the iterated ultrapower on M dropped then M_θ would be a proper initial segment of N_θ since A_1^M is definable in N_θ, but not in M_θ; however, this contradicts Lemma 3.36. Furthermore $N_\theta = M_\theta$ since A_1^M is definable in, but not a member of, each. Now the existence of solidity witnesses for p_1^N implies that $i(p_1^N) = p_1^{N_\theta} = j\pi(p_1^N)$, but this implies that $ih_1^N(\xi) = j\pi h_1^N(\xi)$ for all $\xi < \rho_1^M$. Since N is 1-sound it follows that $i = j\pi$.

If M is not an iterated ultrapower of N, then $M \neq M_\theta$ and j is not the identity. Let ν be the least stage in the ultrapower that $j_{\nu,\nu+1}$ is nontrivial, let U'_ν be the ultrafilter used at this point, and let $\eta = \mathrm{crit}(j)$ be its critical point. Then $\eta \notin j\pi h_1^N$"$\eta = ih_1^N$"η, and it follows that η is also the critical point of an ultrapower in i: that is, $i_{\nu,\nu+1} = i^{U_\nu}$ for an ultrafilter U_ν in N_ν, as in diagram (18.9):

$$\begin{tikzcd} N \arrow[r, "i_{0,\nu}"] \arrow[dr, "\pi"] & N_\nu \arrow[r, "i^{U_\nu}"] & N_{\nu+1} \arrow[r, "i_{\nu+1,\theta}"] & N_\theta \arrow[d, equal] \\ & M \arrow[r, "j_{0,\nu}"] & M_\nu \arrow[r, "j^{U'_\nu}"] & M_{\nu+1} \arrow[r, "j_{\nu+1,\theta}"] & M_\theta \end{tikzcd} \tag{18.9}$$

Let $x \subseteq \eta$ be a set in $N_\nu \cap M_\nu$ such that $x \in U_\nu \not\Longleftrightarrow x \in U'_\nu$. Then $x = h_1^{N_\nu}(\xi)$ for some $\xi < \eta$. We claim that $x = h_1^{M_\nu}(\xi)$ as well: to see this, note that $j_{\nu,\theta}(h_1^{M_\nu}(\xi)) = h_1^{M_\theta}(\xi) = h_1^{N_\theta}(\xi) = j_{\nu,\theta}(x)$, and each of $i_{\nu,\theta}$ and $j_{\nu,\theta}$ are the identity on η. Thus $i_{\nu,\theta}(x) = j_{\nu,\theta}(x)$, but this is impossible since then $x \in U_\nu \iff \nu \in i_{\nu,\theta}(x) \iff \nu \in j_{\nu,\theta}(x) \iff x \in U'_\nu$, contradicting the choice of x.

This completes the proof that M is an iterated ultrapower of $\mathfrak{C}_1(M)$, except for verifying that neither i nor j have critical point smaller than ρ_1^M. If this is false, then it must be that $\rho_1^M = \mu^{+M}$, where $\mu = \mathrm{crit}(\mathcal{U}_\gamma^M)$ for some $\gamma > \rho_1^M$, and $(\mu^{++})^N = (\mu^{++})^M \cap h_1^M$"$\rho_1^M < (\mu^{++})^M$. If $\gamma \geq (\mu^{++})^N$ is least such that $\mathcal{U}_\gamma^M$ is a measure on μ, then the measures in N with critical point μ are exactly the measures in $\mathcal{U}^M \restriction \gamma$. It follows that none of these measures will be applied in the comparison, and hence $\mathrm{crit}(i) \geq \rho_1^M$.

To see that $\mathrm{crit}(j) \geq \rho_1^M$ we need to use another basic result, the proof of which will be delayed until after the current proof is completed.

3.41 Lemma (Dodd-Jensen Lemma). *Suppose that N is an iterable premouse, $i : N \to P$ is an iterated ultrapower, and $k : N \to P$ is any Σ_0-elementary embedding. Then the range of k is cofinal in P, i does not drop, and $i(\alpha) \geq k(\alpha)$ for all $\alpha \in N$.*

Now if $\mathrm{crit}(j) = \mu < \mathrm{crit}(i)$, then $i(\mu) = \mu < j(\mu) = j\pi(\mu)$. This contradicts Lemma 3.41 with $P = N_\theta$, $k = i\pi$, and $\alpha = \mu$, and this contradiction completes the proof that M is an iterated ultrapower of $\mathrm{core}_1(M)$.

The proof that M has solidity witnesses is similar. Fix $\nu \in p_1^M$, set $p = p_1^M - (\nu + 1)$, and let $\pi : N \to M$ be the transitive collapse of the Σ_1

hull h_1^M"ν of $\nu \cup p$ in M. Recall that $A = \{\xi < \nu : M \models \Phi(\xi, p)\}$, the Σ_1 theory of $\nu \cup p$ in M, and R is the prewellordering of A defined by letting $\xi \, R \, \xi'$ if there is γ such that $J_\gamma^{\mathcal{U}} \models \Phi(\xi, p)$ and $\forall \gamma' < \gamma \; J_{\gamma'}^{\mathcal{U}} \models \neg\Phi(\xi', p)$ where Φ is the universal Σ_1 formula. Clearly A and R are Σ_1-definable in N. Let $i : N \to N_\theta$ and $j : M \to M_\theta$ be the iterated ultrapowers comparing N and M. By the same argument as before, $\mathrm{crit}(i) \geq \nu$. We claim that N_θ is a proper initial segment of M_θ, for if not, then Lemma 3.41 implies $\mathrm{crit}(j) \geq \nu$ by the same argument as before, and hence A is a member of M_θ; however, $A \notin N_\theta$ and hence $M_\theta \not\subseteq N_\theta$.

Now there is a $\tau \in M_\theta$ mapping A into the ordinals of N_θ such that $\forall \xi, \xi' \in A \; (\tau(\xi) \leq \tau(\xi') \iff \xi \, R \, \xi')$, namely the map defined by letting $\tau(\xi)$ be the least ordinal γ such that $J_\gamma^{\mathcal{U}^{M_\theta}} \models \Phi(\xi, i\pi^{-1}(p))$. This map is Σ_1-definable in N_θ and hence is a member of M_θ.

If j is the identity, then $M_\theta = M$ and hence $\tau \in M$ is the desired solidity witness. If j is not the identity, then it does not drop by Lemma 3.36 so either M has a full measure U with $\mathrm{crit}(U) \geq \nu$ or else there is a full measure $U \in M$ with $\mathrm{crit}(U) < \nu$ but $R \in \mathrm{Ult}(M, U)$. We observed following Definition 3.27 that either of these implies that M has a solidity witness that ν is in p_1^M, and this completes the proof of Lemma 3.40. $\dashv$

We outline the proof of Lemma 3.41. A full proof is given in Sect. 4 of chapter [57].

Sketch of the Proof of Lemma 3.41. Suppose that i and k are as in the hypothesis. We will define iterations $i_n : N_n \to N_{n+1}$ so that the direct limit $i^* = \dots i_3 i_2 i_1 i_0 : N_0 \to N_\omega$ is an iterated ultrapower on N. We will then see that if the conclusion fails then the iteration i^* contradicts the assumption that N is iterable.

The maps i_n and k_n are defined by recursion on n, starting by setting $N_1 = P$, $i_0 = i$ and $k_0 = k$. The recursion step uses a general copy process: using the notation of diagram (18.10), we set $i_{n+1} = k_n(i_n)$ and $k_{n+1} = i_n * k_n$.

To describe the copy process leading to diagram (18.10), suppose that $e : N \to P$ is an iterated ultrapower and $j : N \to P$ is a Σ_0-elementary embedding. We will define the copy map $j(e)$ and a map $e * j$ so that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "j(e)"] & N' \\
N \arrow[u, "j"] \arrow[r, "e"] & P \arrow[u, "e*j"']
\end{tikzcd} \tag{18.10}$$

The definition is by recursion on length of the iterated ultrapower e, with the induction step using the following diagram:

$$\begin{tikzcd} P_\nu \arrow[r, "e^{j_\nu(U_\nu)}"] & P_{\nu+1} \\ N_\nu \arrow[u, "j_\nu"] \arrow[r, "e^{U_\nu}"] & N_{\nu+1} \arrow[u, "j_{\nu+1}"'] \end{tikzcd} \tag{18.11}$$

Here $e^{U_\nu} : N_\nu \to N_{\nu+1}$ is the νth stage of the iteration e on N, $j_0 = j$, and $j_{\nu+1}$ is defined by $j_{\nu+1}([f]_{U_\nu}) = [j_\nu(f)]_{j_\nu(U_\nu)}$. The maps of diagram (18.10) are defined by letting $j(e)$ be the iteration using the maps $e^{j_\nu(U_\nu)}$ on the top row of diagram (18.11), and setting $e * j = j_\theta$ where θ is the length of the iteration.

Now suppose that the map i does contain a drop. Then each of the maps i_n contains a drop, and hence $i^* = \dots i_3 i_2 i_1 i_0 : N_0 \to N_\omega$ is a presumptive iteration on N containing infinitely many drops, contrary to the assumption that N is iterable. Hence i does not drop.

Now we will show that if either of the other clauses of the conclusion is false, then there are ordinals $\alpha_n \in N_n$ such that $i_n(\alpha_n) > \alpha_{n+1}$, so that the ordinals $\alpha'_n = \dots i_{n+1} i_n(\alpha_n)$ form an infinitely descending sequence of ordinals in N_ω, again contradicting the assumption that N is iterable. In the case that the range of k is bounded in $P = N_1$ a simple induction shows that the ordinals $\alpha_{n+1} = \sup(\mathrm{ran}(k_n)) \in N_{n_1}$ are as required. If $i(\alpha) > k(\alpha)$ for some $\alpha \in N$, then the ordinals $\alpha_n \in N_n$ are defined by recursion on n, setting $\alpha_0 = \alpha$ and $\alpha_{n+1} = k_n(\alpha_n)$. Then by induction on n we have $\alpha_{n+1} = k_n(\alpha_n) < i_n(\alpha_n)$ and $k_{n+1}(\alpha_{n+1}) = k_{n+1}k_n(\alpha_n) < k_{n+1}i_n(\alpha_n) = i_{n+1}k_n(\alpha_n) = i_{n+1}(\alpha_{n+1})$. ⊣

The Dodd-Jensen Core Model

The discussion above is valid for the core model up through $\exists\kappa\,(o(\kappa) = \kappa^{++})$. We now turn our attention to the Dodd-Jensen core model, and for this purpose we assume that $0^\dagger$ does not exist, and hence no mouse has any full measures except possibly for the final nontrivial measure on its sequence.

3.42 Definition. We will define the Dodd-Jensen core $K^{\mathrm{DJ}} = L[\mathcal{U}]$ by recursion as follows: Suppose $K^{\mathrm{DJ}}_\kappa = J_\kappa[\mathcal{U}\restriction\kappa]$ has been defined. Let $\mathcal{M}$ be the set of mice $M = J_{\alpha^M}[\mathcal{U}^M]$ such that M has no full measures, $\rho^M = \kappa$ and $\mathcal{U}^M\restriction\kappa = \mathcal{U}\restriction\kappa$. Then $K^{\mathrm{DJ}}_{\bar\kappa} = \bigcup\mathcal{M}$, where $\bar\kappa = \sup\{\alpha^M : M \in \mathcal{M}\}$.

To justify this definition of K^{DJ}, notice that Lemma 3.39 implies that of any two members M_0 and M_1 of $\mathcal{M}$, one is an initial segment of the other. Thus $K^{\mathrm{DJ}}_{\bar\kappa} = J_{\bar\kappa}[\mathcal{U}\restriction\bar\kappa]$ where $\mathcal{U}\restriction\bar\kappa = \bigcup\{\mathcal{U}^M : M \in \mathcal{M}\}$. The ordinal $\bar\kappa$ will be equal to κ^+ in K^{DJ}.

If there is no inner model with a measurable cardinal, then the core model K is equal to K^{DJ}. If there is an inner model with a measurable cardinal, but $0^\dagger$ does not exist, then $K = L[U]$, where the model $L[U]$ is chosen so that the critical point of the measure U of $L[U]$ is as small as possible.

Note that $K^{\mathrm{DJ}} = L$ if $0^{\#}$ does not exist. If $K = L[U]$, with U a measure on κ, then we can write $L[U] = L[\mathcal{U}]$ where, setting $\gamma = (\kappa^{++})^{K^{\mathrm{DJ}}}$, the sequence $\mathcal{U}$ has a largest member $\mathcal{U}_\gamma = U$ and $\mathcal{U}\restriction\gamma = \mathcal{U}^{\mathrm{DJ}}\restriction\gamma$.

The fact that a mouse is required to be iterable, while a level J_α of the L hierarchy is only required to be well-founded, makes the definition of a mouse logically more complicated than that of the sets J_α. For example, the statement $\exists\alpha\ (\omega, \mathrm{E}) \cong (J_\alpha, \in)$ is a Π^1_1 statement about the set $\mathrm{E} \subseteq \omega^2$, while the statement that (ω, E) is isomorphic to a mouse $J_\alpha[\mathcal{U}]$ is Π^1_2.

However, $(K^{\mathrm{DJ}})^M = K^{\mathrm{DJ}} \cap M$ whenever M is a transitive model of ZF containing ω_1, since the definition of a mouse is absolute for models containing ω_1 by Lemma 3.32.

It is not the case, as it is for L, that there is a sentence $V = K$ such that every class model of $V = K$ is equal to K. This can be seen by considering the class L, which may or may not be equal to K^{DJ}, depending on whether $0^{\#}$ exists. A more general example can be obtained by taking any mouse M in K^{DJ} with a measure U on κ, and let $\widetilde{M} = \mathrm{Ult}_{\mathrm{On}}(M, U)$. Then $i^U_{\mathrm{On}}(\kappa) = \mathrm{On}$, and the initial segment $\widetilde{M}|\,\mathrm{On} = V^{\widetilde{M}}_{\mathrm{On}}$ of $\widetilde{M}$ is a class model of ZFC $+\ V = K$ which is not equal to K^{DJ}. Notice, for example, that the critical points of the iteration form a closed and unbounded class of indiscernibles for $\widetilde{M}|\,\mathrm{On}$ which is definable in $L[M]$, and hence in K^{DJ}.

Vickers and Welch have observed [61] that the existence of a Ramsey cardinal implies that it is consistent that there is a proper class $X \prec K^{\mathrm{DJ}}$ such that $X \not\cong K^{\mathrm{DJ}}$.

The proof of Jensen's covering lemma for L relied on Kunen's result that the existence of a nontrivial embedding $i : L \to L$ implies that $0^{\#}$ exists. The proof of the covering lemma for K^{DJ} will rely on an analogous result, Lemma 3.47, stating that if $0^{\dagger}$ does not exist and there is a nontrivial elementary embedding $i : K^{\mathrm{DJ}} \to M$, then $K = L[U]$ for some measure U in K. In particular, if there is a nontrivial embedding $i : K^{\mathrm{DJ}} \to K^{\mathrm{DJ}}$, then $K \neq K^{\mathrm{DJ}}$, so that (assuming $0^{\dagger}$ does not exist) $K = L[U]$ for some measure U in K. However, Jensen has shown that it is not necessarily true, as one might expect, that U is the ultrafilter associated with i, or even that $\mathrm{crit}(U) = \mathrm{crit}(i)$. Notice that the model N of this proof gives an example of a mouse which is not a member of K, and of a mouse which is added by a set forcing.

3.43 Theorem. *Suppose that $L[U]$ satisfies that U is a measure on κ, and let G be $L[U]$-generic for the collapse of $\lambda = (\kappa^+)^{L[U]}$ onto ω. Then in $L[U, G]$ there is a fully iterable premouse N with measurable cardinal less than κ such that $J_\lambda(U)$ is an ultrapower of N. Hence there is an elementary embedding $i : K^{\mathrm{DJ}} \to K^{\mathrm{DJ}}$ with* $\mathrm{crit}(i) < \kappa$.

Proof. We will prove that the theorem is true in $\mathrm{Ult}(L[U], U)$, so that by elementarity it is true in $L[U]$ as well. Let $j : L[U] \to M = \mathrm{Ult}(L[U], U)$, let

$$\kappa' = j(\kappa), \qquad \lambda' = j(\lambda), \quad \text{and} \quad U' = j(U),$$

and let G be M-generic for the collapse of λ' onto ω. Let $\mathcal{L}$ be the infinitary language with a constant $\mathbf{x}$ for each member x of $J_{\lambda'}[U', G]$ and one additional constant $\mathbf{W}$ which will denote a measure on κ. Let T be the theory with sentences

$$\forall z\big(z \in \mathbf{x} \iff \textstyle\bigvee_{y \in x}(z = \mathbf{y})\big)$$

for each $x \in J_{\lambda'}[U', G]$, together with a sentence asserting that $\mathbf{W}$ is an amenable measure in $J_\lambda[\mathbf{W}]$ such that $\mathrm{Ult}(J_\lambda[\mathbf{W}], \mathbf{W}) \cong J_{\lambda'}[U']$. This theory is consistent because it is true in $L[U]$, so by the Barwise compactness theorem it has a model $\mathcal{A}$. Let $W = \mathbf{W}^{\mathcal{A}}$ and let N be the premouse $J_\lambda[W]$. Then $\mathrm{Ult}(N, W) = J_{\lambda'}[U']$ by the construction of W, and it follows that N is fully iterable since $J_{\lambda'}[U']$ is fully iterable.

Now K^{DJ} is equal to the initial segment of $\mathrm{Ult}_{\mathrm{On}}(N, W)$ below On, its measurable cardinal, and hence it follows easily that there is an embedding from K^{DJ} into K^{DJ} with critical point κ. $\dashv$

A major difference between the proof of the covering lemma for L and the proof of the covering lemma for $L[U]$ is that the proof of the analog Lemma 3.47 of Kunen's result itself uses the covering lemma. In order to avoid circularity we prove the covering lemma in two steps: the first step proves enough of the strength of the covering lemma to prove Lemma 3.47, using the following observation:

3.44 Lemma. *If U is a countably complete normal K^{DJ}-ultrafilter then U is a measure in $L[U]$, and $K^{\mathrm{DJ}} \subseteq L[U]$.*

Proof. Let $\kappa = \mathrm{crit}(U)$ and set $\gamma = (\kappa^{++})^{K^{\mathrm{DJ}}}$. Take $\mathcal{U}$ to be the sequence such that $K^{\mathrm{DJ}} = L[\mathcal{U}]$ and let $\mathcal{U}'$ be the good sequence defined by $\mathcal{U}' \restriction \gamma = \mathcal{U} \restriction \gamma$ and $\mathcal{U}'_\gamma = U$. We will show that $\mathcal{P}(\kappa)^{L[\mathcal{U}']} = \mathcal{P}(\kappa)^{K^{\mathrm{DJ}}}$, which implies that U is a measure in $L[\mathcal{U}']$, and hence in $L[U]$.

Suppose the contrary, and let α be the least ordinal such that there is a set $x \in \mathcal{P}(\kappa) \cap J_{\alpha+\omega}[\mathcal{U}'] - K^{\mathrm{DJ}}$. Now the model $J_\alpha[\mathcal{U}']$ need not be sound, but the iterability of $L[\mathcal{U}]$ and the countable completeness of U ensure that $J_{\alpha'}[\mathcal{U}]$ is iterable. Hence Lemma 3.28 implies that $J_\alpha[\mathcal{U}']$ is an iterated ultrapower of a mouse N, but this is impossible since any such mouse would be a member of K^{DJ}.

To see that $K^{\mathrm{DJ}} \subseteq L[U]$, suppose the contrary and use iterated ultrapowers to compare K^{DJ} with $K^{L[U]}$. Note that the latter is equal to $L[U]$, and can be written $L[\mathcal{U}'']$, where $\mathcal{U}''$ is constructed like the sequence $\mathcal{U}'$ above, but using $(K^{\mathrm{DJ}})^{L[U]}$ instead of the true K^{DJ}. If the iterated ultrapower beginning with K^{DJ} does not drop, then it must be trivial since there are no full measures in K^{DJ}, and this would imply $K^{\mathrm{DJ}} \subseteq L[U]$. If it does drop, on the other hand, then the iterated ultrapower of $L[U]$ does not drop, and the final model $L[U_\theta]$ of that iteration is an initial segment of the final model of the iteration on K^{DJ}.

Then that iteration would construct a class of indiscernibles for $L[U_\theta]$, which implies that $0^\dagger$ exists and in fact is a member of K^{DJ}, which is absurd. $\dashv$

The second part of the proof will use the following analog of the Condensation Lemma for L:

3.45 Lemma. *Assume that there is no inner model with a measurable cardinal, and that $X \prec K^{\mathrm{DJ}}$ is a class such that* $\mathrm{ot}(X \cap (\lambda^+)^{K^{\mathrm{DJ}}}) = (\lambda^+)^{K^{\mathrm{DJ}}} = \lambda^+$ *for a proper class of cardinals λ of K^{DJ}. Then $X \cong K^{\mathrm{DJ}}$.*

Sketch of Proof. Let N be the transitive collapse of X, and suppose to the contrary that $N \neq K^{\mathrm{DJ}}$. Then there is a mouse $M \in K^{\mathrm{DJ}} - N$. Now use iterated ultrapowers to compare the models M and N, and let M_θ and N_θ be the final models of the iterated ultrapower of M and N, respectively. Then N_θ must be a proper initial segment of M_θ, since there is a set $x \subseteq \rho^M$ which is definable in M but not in N. Then x is definable in M_θ, but not in N_θ so N_θ is a proper initial segment of M_θ and it follows by Corollary 3.38 that the iterated ultrapower on N does not drop. Since N does not contain any full ultrafilters it follows that the iterated ultrapower on N must be trivial, that is, $N = N_\theta$, and this implies that $\mathrm{ot}(X \cap (\lambda^+)^{K^{\mathrm{DJ}}}) = (\lambda^+)^N = (\lambda^+)^{M_\theta} < \lambda^+$ for every cardinal $\lambda > |M_\theta|$, contradicting the hypothesis. This contradiction completes the proof of Proposition 3.45. $\dashv$

The assumption that $(\lambda^+)^{K^{\mathrm{DJ}}} = \lambda^+$ is actually unnecessary here: any iterated ultrapower of K^{DJ} is a member of K^{DJ}, since it is a finite sequence of drops, separated by an iterated ultrapower by a single ultrafilter. In particular $M_\theta \in K^{\mathrm{DJ}}$.

The set Γ from Kunen's proof satisfies $|\Gamma \cap \lambda^+| = \lambda^+$ on a stationary set. Hence the importance of the weak covering property:

3.46 Definition. A cardinal λ is *countably closed* if $\eta^\omega < \lambda$ for all $\eta < \lambda$, and a model M has the *countably closed weak covering property* if for all sufficiently large countably closed singular cardinals λ we have $(\lambda^+)^M = \lambda^+$.

3.47 Lemma. *Suppose that $0^\dagger$ does not exist, and that K satisfies the countably closed weak covering property. If there is a nontrivial elementary embedding $i : K \to M$, then $K = L[U]$ where U is a measure in $L[U]$ with* $\mathrm{crit}(U) = \mathrm{crit}(i)$.

Proof. Set $\kappa = \mathrm{crit}(i)$. We can assume that $M = \{i(f)(\kappa) : f \in K\}$, for otherwise we could factor i as

$$i : K \xrightarrow{i'} M' \cong \{i(f)(\kappa) : f \in K\} \prec M,$$

and work with $i' : K \to M'$ instead of $i : K \to M$.

First we show that $K \neq K^{\mathrm{DJ}}$, so that $K = L[U]$ for some measure U. Suppose to the contrary that $K = K^{\mathrm{DJ}}$. We claim that in this case M also equals K^{DJ}: to see this, assume $M \neq K^{\mathrm{DJ}}$ and use iterated ultrapowers to compare M with K^{DJ}. The iteration on M is trivial, and the iteration on K^{DJ} drops and generates a closed and unbounded class I of indiscernibles for M. Then our assumption on i implies that $i^{-1}[I]$ is a class of indiscernibles for K^{DJ}, which would generate a countably complete K^{DJ}-measure U_{ω_1} on the limit of the first ω_1 members of I. By Lemma 3.47, it follows that U_{ω_1} is a measure in $L[U_{\omega_1}]$, contrary to our assumption that $K = K^{\mathrm{DJ}}$.

Thus $i : K^{\mathrm{DJ}} \to K^{\mathrm{DJ}}$, and we can apply the proof, sketched in chapter [32, Theorem 1.13], of Kunen's corresponding result for L. This involves defining a continuously descending sequence of classes Γ_α, beginning with $\Gamma_0 = \mathrm{ran}(i)$ and setting $\Gamma_{\alpha+1} = \{x \in \Gamma_\alpha : i_\alpha(x) = x\}$ where i_α is the transitive collapse of $\Gamma_\alpha \prec K$. The classes Γ_α contain all of their limit points of cofinality greater than κ, and if $\kappa < \nu \in \Gamma_\alpha$ then $|\Gamma_\alpha \cap \nu^+| = \nu^+$. Since $\nu^+ = \nu^{+K}$ by the weak covering property, it follows by Lemma 3.45 that $\Gamma_\alpha \cong K^{\mathrm{DJ}}$ for each ordinal α. Now the same argument as for L shows that if we set $\kappa = \mathrm{crit}(i)$ and $\kappa_\alpha = \min(\Gamma_\alpha - \kappa)$ then the class of ordinals $(\kappa_\alpha : \alpha \in \mathrm{On})$ is a closed and unbounded class of indiscernibles for K^{DJ}. It follows that $\{\kappa_\alpha : \alpha < \omega_1\}$ generates a normal K^{DJ}-measure U_{ω_1} on κ_{ω_1}, and since U_{ω_1} is countably complete it follows by Lemma 3.44 that U_{ω_1} is a measure in $L[U_{\omega_1}]$.

This completes the proof that $K = L[U]$ for some measure U. We must have $\mathrm{crit}(U) \geq \mathrm{crit}(i)$, for otherwise Kunen's argument for L implies directly that $0^\dagger$ exists. To see that $\mathrm{crit}(U) \leq \mathrm{crit}(i)$, assume to the contrary that $\lambda = \mathrm{crit}(U) > \kappa = \mathrm{crit}(i)$ and observe that it is true for $L[U]$, as it is for L, that $\mathcal{H}^{L[U]}(\Gamma) \cong L[U]$ for any proper class Γ of ordinals. Now $M = L[U']$, with $\lambda' = \mathrm{crit}(U') \geq \lambda = \mathrm{crit}(U)$, so there is an iterated ultrapower $j : K \to M$. Let $\Gamma = \{\nu : i(\nu) = j(\nu)\}$. Then Γ is a proper class, it contains its limit points of cofinality greater than λ, and $\kappa \notin \mathcal{H}^K(\Gamma) = \mathrm{On} \cap \Gamma$.

We will complete the proof by showing that this is impossible. First, note that the family of proper classes $\Gamma \prec L[U]$ which contain all of their limit points of cofinality greater than λ is closed under intersections of size at most λ. Hence there is such a class Γ' such that $\Gamma' \cap \lambda$ is as small as possible. Now if $k : L[U''] \cong \mathcal{H}^K(\Gamma')$ is the transitive collapse then, since $\mathrm{crit}(U)$ is as small as possible, $\mathrm{crit}(U'') = \mathrm{crit}(U) = \lambda$. However, there is some $\eta < \lambda$ so that $k(\eta) > \eta$. Then $k(\eta) \in \kappa \cap \mathcal{H}^K(\Gamma') - \mathcal{H}^K(\Gamma'')$, where $\Gamma'' = \{\nu : k(\nu) = \nu\}$, contrary to the choice of Γ'. $\dashv$

Part 1 of the Proof

The proof of the Dodd-Jensen covering lemma can be divided into two parts. The first part is a direct generalization of the proof of the covering lemma for L: it involves defining the basic construction for suitable sets $X \prec K_\kappa$, and showing that the class of suitable sets is unbounded. One of the two major novelties in this stage of the proof is the possible use of an iterated ultrapower in the construction. The second part of the proof, which has no analog for L, is used to analyze the indiscernibles generated by this iterated ultrapower.

In the case when this iterated ultrapower is infinite, these indiscernibles will yield a sequence C which is Prikry generic over $K = L[U]$.

The other major novelty arises from the fact that Lemma 3.47, which is needed in the proof, has the hypothesis that K satisfies the countably closed weak covering property. Thus we will, during part one of the proof, simultaneously prove two results, the first of which is the hypothesis to the second.

3.48 Lemma. *Assume that* $0^\dagger$ *does not exist.*

1. *The core model* K *satisfies the countably closed weak covering property.*
2. *If* K *satisfies the countably closed weak covering property, then it also satisfies the full Dodd-Jensen covering lemma, Theorems 1.12 and 1.13.*

Most of this proof will be reused in proving the covering lemma for sequences of measures; however, certain segments of the proof are substantially simplified by our assumption that $0^\dagger$ does not exist. This extra assumption is equivalent to the assumption that no premouse has more than one full ultrafilter.

The following definition will be valid up to a strong cardinal. We write K_κ for $J_\kappa[\mathcal{U}] = K \cap V_\kappa$.

3.49 Definition. Let $X \prec_1 K_\kappa$, with transitive collapse $\pi : \overline{K} \to X$, where $\overline{K} = L_{\bar\kappa}[\overline{\mathcal{U}}]$. We say that X is *suitable* if $\mathrm{Ult}_n(M, \pi, \beta)$ is iterable whenever $n \in \omega$, $\beta \leq \kappa$, and $M = J_\alpha[\mathcal{U}']$ is an iterable premouse (possibly with $\alpha = \mathrm{On}$) such that $\mathrm{Ult}_n(M, \pi, \beta)$ is defined and $\mathcal{U}'{\upharpoonright}\bar\beta = \overline{\mathcal{U}}{\upharpoonright}\bar\beta$ where $\bar\beta$ is the least ordinal such that $\pi(\bar\beta) \geq \beta$.

As in the proof of the covering lemma for L, we say that $X \prec_1 K_\kappa$ is *countably closed* if $X = Y \cap K_\kappa$, where ${}^\omega Y \subseteq Y$ and $Y \prec H(\tau)$ for some $\tau > \kappa$.

3.50 Lemma.

(i) Every countably closed set $X \prec_1 K_\kappa$ *is suitable, and*

(ii) the class of suitable sets X *is closed under increasing unions of uncountable cofinality, and is unbounded in* $H(\delta)^{(K_\kappa)}$ *for any uncountable cardinal* δ.

Sketch of Proof. The only difference between the proof of this lemma and the corresponding lemma for L is that we need to check that the model $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \beta)$ is iterable rather than merely well-founded. This is straightforward for clause (i). For clause (ii) this involves changing Definition 3.18 of a witness to the unsuitability of X: Clause 3.18(2), stating that $\operatorname{dir\,lim}(\pi^{-1}(w)) = \mathfrak{C}_n(J_\alpha)$ for some ordinal α, is modified to require that $\operatorname{dir\,lim}(\pi^{-1}(w))$ be the Σ_n-code for some mouse. Clause 3.18(3), is modified to state that the witness w either the structure $\widetilde{M}_0$ of which $\operatorname{dir\,lim}(w)$ is the Σ_n-code is ill-founded, or else there is an ill-founded iteration of this structure.

The proof for both clauses relies on the fact that if there is an ill-founded iteration then there is one of countable length. $\dashv$

We are now ready to describe the basic construction. As in the proof of the covering lemma for L, we are given a suitable set $X \prec_1 K_\kappa$, and we let $\pi : \overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}] \cong X \prec_1 K_\kappa$ be the transitive collapse. We assume that X is not transitive, so that π is not the identity, and furthermore we assume that either X is countably closed or else K satisfies the countably closed weak covering property.

In order to postpone some complications which arise in the proof of the covering lemma for sequences of measures, we make the following additional assumption:

$$\begin{gathered}\text{If } K = L[U]\text{, where } U \text{ is a measure on a cardinal } \mu \text{ of } K\text{,}\\ \text{then either } (\mu^+)^K \subseteq X \text{ or else } \kappa \le (\mu^+)^K.\end{gathered} \tag{18.12}$$

This assumption does not involve any loss of generality: the case $(\mu^+)^K \subseteq X$ shows that any set x of size at most $(\mu^+)^K$ is contained in a set $y' \in K$ of size $(\mu^+)^K$, and then the case $\kappa \le (\mu^+)^K$ shows that x can be covered by a set $y \subseteq y'$ which satisfies Theorem 1.3. The case $(\mu^+)^K \subseteq X$ is a relativization of the proof for L and requires no new ideas.

In the proof for L, the next step was to set $M = J_\alpha$, where $\alpha \ge \bar\kappa$ was the least ordinal such that there is a bounded subset of $\bar\kappa$ in $J_\alpha - J_{\bar\kappa}$. If it happens that $\overline{\mathcal{U}} = \mathcal{U}{\restriction}\bar\kappa$ then we can similarly take $J_\alpha[\mathcal{U}]$, but in general we need to modify the construction by using iterated ultrapowers to compare the models $\overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}]$ and $K = L[\mathcal{U}]$. A key step of the proof is showing (see Lemma 3.51(2)) that $\overline{K}$ is never moved in this comparison, so that the final model of the iterated ultrapower on K is a model $M_\theta = J_{\alpha_\theta}[\mathcal{U}_\theta]$ such that $\overline{\mathcal{U}} = \mathcal{U}_\theta{\restriction}\bar\kappa$.

Thus we obtain the following variant of diagram (18.5), where each of the subset symbols indicate containment as an initial segment:

$$\begin{tikzcd} K \arrow[r, squiggly, "i"] & M_\theta \supseteq M \arrow[r, "\tilde\pi"] & \mathrm{Ult}_n(M,\pi,\kappa) = \widetilde{M} \\ & \overline{K} = J_{\bar\kappa}[\overline{\mathcal{U}}] \arrow[u, hook] \arrow[r, "\pi"] & X \prec K_\kappa \arrow[u, hook] \end{tikzcd} \tag{18.13}$$

We will write $M = J_\alpha[\mathcal{U}^M]$ and $\widetilde{M} = J_{\tilde\alpha}[\widetilde{\mathcal{U}}]$. As in diagram (18.5), $\overline{K}$ is the transitive collapse of the set X and π is the inverse of the collapse map. The iterated ultrapower is indicated by the wavy line from K to M_θ. Since this iteration drops whenever it is nontrivial (see Lemma 3.51(3)), the wavy line does not represent an embedding.

Once the model $M_\theta = J_{\alpha_\theta}[\mathcal{U}^{M_\theta}]$ has been constructed, diagram (18.13) is completed like diagram (18.5): Let $M = J_\alpha[\mathcal{U}^{M_\theta}]$, where (α, n) is the largest pair $(\alpha, n) \leq (\alpha_\theta, n_\theta)$ such that $\mathrm{Ult}_n(J_\alpha[\mathcal{U}^{M_\theta}], \pi, \kappa)$ is defined. Thus α is the least ordinal such that there is a bounded subset x of κ which is definable in $J_\alpha[\mathcal{U}_\theta]$ but is not a member of $\overline{K}$. Finally, set $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa)$.

3.51 Lemma. *Assume that $0^\dagger$ does not exist. Let $X \prec_1 K_\kappa$ be a suitable set which is not transitive, so that the collapse $\pi : \overline{K} \cong X \prec_1 K_\kappa$ is not the identity. Finally, assume that either X is countably complete or else K has the countably closed weak covering property.*

1. *$\mathcal{P}^K(\eta) \not\subseteq \overline{K}$, where $\eta = \mathrm{crit}(\pi)$.*
2. *In the comparison of K and $\overline{K}$, the iterated ultrapower on the model $\overline{K}$ is trivial.*
3. *Either $\overline{K}$ is an initial segment of K, or else $1 \in D$, so that the iterated ultrapower on K drops immediately.*
4. *$\overline{K}$ is an initial segment of the final model M_θ of the iteration of K.*
5. *$\widetilde{M} \in K$.*

Sketch of Proof. If clause 1 fails then $U' = \{x \in \mathcal{P}^{\overline{K}}(\eta) : \eta \in \pi(x)\}$ is a K-ultrafilter. If X is countably closed then U' is countably complete, so Lemma 3.44 implies that U' is a measure in $L[U']$. If K has the countably closed weak covering property then $\mathrm{Ult}(K, U')$ is well-founded since it can be embedded into $\mathrm{Ult}(K, \pi, \kappa)$, which is well-founded by the definition of suitability, so Lemma 3.47 implies that $K = L[U]$ where $\mathrm{crit}(U) \leq \mathrm{crit}(U')$. Thus, under the hypothesis of either clause of Lemma 3.48, $K = L[U]$ for some measure U with $\mathrm{crit}(U) \leq \mathrm{crit}(i)$; but this is impossible: if $\mathrm{crit}(U) < \mathrm{crit}(i)$ then it would follow that $0^\dagger$ exists, contrary to the hypothesis, while if $\mathrm{crit}(U) = \mathrm{crit}(i)$ then $\mathrm{crit}(U)$ would be definable in K_κ as the only measurable cardinal, and hence would be in X.

To see that the ultrapower on $\overline{K}$ is trivial, first note that the extra assumption (18.12) on X implies that any full measure in $\overline{K}$ is contained in K_η. Thus the iterated ultrapower on $\overline{K}$ must be trivial unless it drops. By Lemma 3.38 this would imply that the iterated ultrapower on K does not drop, and its final model M_θ is an initial segment of the final model above $\overline{K}$; but this is absurd, since $\overline{K}$ is a set and M_θ is a proper class.

To verify clause 3, note that if $\overline{K}$ is not an initial segment of K then the iterated ultrapower on K is nontrivial; however, again using (18.12), any full ultrafilter in K with critical point smaller than η would also be in $\overline{K}$, and hence would not be used in the iteration. Thus the iterated ultrapower on K must drop immediately.

Clause 4 follows from clause 2, so it only remains to check clause 5, stating that $\widetilde{M} \in K$. If $\widetilde{M}$ has no full measure $U = \mathcal{U}^{\widetilde{M}}_\gamma$ with $\mathrm{crit}(U) < \kappa$, then $\widetilde{M}$

is iterable because of the suitability of X. Now $\widetilde{M}$ is sound above κ, because of its construction as an ultrapower $\mathrm{Ult}(M, \pi, \kappa)$. On the other hand, the projectum ρ of $\widetilde{M}$ cannot be smaller that κ, as otherwise $\widetilde{M}$ would be an iterated ultrapower of a mouse M' of size at most ρ, but this is impossible since then $M' \in K_\kappa \subseteq M$. It follows that $\widetilde{M}$ is sound, and hence is a member of K.

Now it will be sufficient to show that there is no full measure U in $\widetilde{M}$ with $\mu = \mathrm{crit}(U) < \kappa$. First, we observe that any such measure U would have to satisfy $\kappa = {\mu^+}^K$: otherwise $\kappa > (\mu^+)^K$, so $U = \mathcal{U}^{\widetilde{M}}_\gamma$ for some $\gamma < \kappa$. Then $\tilde{\pi}^{-1}(U) = \mathcal{U}^M_{\pi^{-1}(\gamma)}$, with $\pi^{-1}(\gamma) < \bar{\kappa}$. By the construction it follows that $\tilde{\pi}^{-1}(U) \in \overline{K}$, so $U \in K$, contradicting the special assumption (18.12).

Hence the following lemma will prove that there is no full measure U in $\widetilde{M}$ with $\mu = \mathrm{crit}(U) < \kappa$, and hence will complete the proof of Lemma 3.51:

3.52 Lemma. *κ is not a successor cardinal in K.*

Proof. We will first assume that there is no measure in $\widetilde{M}$ with critical point below κ. We will show that if κ is a successor in K then there is an $\eta < \kappa$ such that $X = \tilde{h}``(X \cap \eta)$, which implies that κ is singular in K and hence is not a successor. To do so we will need to consider the indiscernibles generated by the iteration i.

If $M \neq M_\theta$ then M is a proper initial segment of the potential premouse M_θ. It follows that M is sound and is hence a mouse. In this case the proof proceeds exactly as in that of the covering lemma for L, and leads to the conclusion that $X = \tilde{h}``(X \cap \rho)$, where $\tilde{h}$ comes from Lemma 3.10 and $\rho = \pi(\rho^M_{m+1})$ where m is least such that $\rho^M_{m+1} < \bar{\kappa}$.

Thus we can assume that $M = M_\theta$. Then Lemma 3.51(3) states that $1 \in D$, so $D \neq \varnothing$. Let $\nu_0 + 1 < \theta$ be the largest member of D. Then M_θ is an iterated ultrapower (without drops) of the potential premouse $M^*_{\nu_0+1}$, which is an initial segment of M_{ν_0}. All of the remaining ultrapowers have the same degree n, and $M^*_{\nu_0+1}$ is n-sound. Let $\overline{C} = \{i_{\nu_0,\nu}(\kappa_{\nu_0}) : \nu_0 < \nu < \theta\}$, where $i_{\nu_0,\nu} : M^*_{\nu_0+1} \to M_\nu$. Then $\overline{C}$ is a sequence of indiscernibles for M_θ.

Let $\bar{\rho}$ be the Σ_n projectum of M, which is equal to the Σ_n projectum of $M^*_{\nu_0+1}$, and let $\bar{h}$ be the Σ_n-Skolem function of M_θ. Then

$$M_\theta = \bar{h}``(\bar{\rho} \cup \overline{C}) \tag{18.14}$$

by the soundness of $M^*_{\nu_0+1}$ and Lemma 3.26. Now let $\rho = \sup(\pi``\bar{\rho})$ and $C = \pi``\overline{C}$. If $\tilde{h} = \tilde{h}^X$ is the function given by Lemma 3.10, then it follows that $X = K_\kappa \cap \tilde{\pi}``M_\theta = K_\kappa \cap \tilde{\pi}h``(\bar{\rho} \cup \overline{C}) = \tilde{h}\pi``(\bar{\rho} \cup \overline{C}) \subseteq \tilde{h}``(\rho \cup C)$.

Now $\overline{C}$ cannot be unbounded in $\bar{\kappa}$: $\bar{\kappa}$ is not a limit cardinal in M since κ is not a limit cardinal in K, but each member of $\overline{C}$ is a cardinal in M. Thus $X \subseteq \tilde{h}``\eta$ where $\eta = \sup(\rho \cup C) < \kappa$, as claimed.

This completes the proof in the case that there is no measure $U \in \widetilde{M}$ with $\mathrm{crit}(U) < \kappa$. If there is such a measure then, as was pointed out at the end of the proof of Lemma 3.51, $\kappa = (\mu^+)^K$ where $\mu = \mathrm{crit}(U)$. In this case set $M' = \mathrm{Ult}_n(M, \overline{U})$ where $\overline{U} = \tilde{\pi}^{-1}(U)$. The same argument as above shows that $\widetilde{M'} = \mathrm{Ult}(M', \pi, \kappa) \in K$. In this case M' and $\widetilde{M'}$ should be used in place of M and $\widetilde{M}$ in proof above. Note that M' is the result of carrying out one more step in the iteration i of which M_θ is the last model. ⊣

This completes the proof of Lemma 3.52, and hence of Lemma 3.51, and we can now finish the proof of clause 1 of Lemma 3.48:

3.53 Corollary. *If* $0^\dagger$ *does not exist then* K *satisfies the countably closed weak covering property.*

Sketch of Proof. Suppose to the contrary that λ is a countably closed singular cardinal, and that $\kappa = \lambda^{+K} < \lambda^+$. Then $\mathrm{cf}(\kappa) \leq \lambda$, and since λ is singular it follows that $\mathrm{cf}(\kappa) < \lambda$. Since λ is countably closed it follows that $\mathrm{cf}(\kappa)^\omega < \lambda$, so there is a set $Y \prec H(\kappa^+)$ with $Y^\omega \subseteq Y$ and $|Y| = \mathrm{cf}(\lambda)^\omega < \kappa$ such that $Y \cap \kappa$ is cofinal in κ. Thus $X = Y \cap K_\kappa$ is countably closed, and hence is suitable, contradicting Lemma 3.52. ⊣

Part 2 of the Proof: Analyzing the Indiscernibles

We have now constructed all of the elements of diagram (18.13) and we have proved the countably closed weak covering lemma. In order to complete the proof of Lemma 3.48(2), and hence of Theorems 1.12 and 1.13, the strong covering lemma below $0^\dagger$, we need to study in more detail the indiscernibles C introduced in the proof of Lemma 3.52. The use of indiscernibles from an iterated ultrapower as a Prikry sequence is discussed in Sect. 2.2 of chapter [32].

Fix, for the moment, an arbitrary suitable set X. We need to find $f \in K$ and $\eta < \kappa$ such that either $X = f``(\eta \cap X)$ or else $C$ is a Prikry sequence and $X = f``(C \cup (\eta \cap X))$. Furthermore, we want to show that the Prikry sequence C, if it is exists, is unique modulo finite differences.

Equation (18.14) states that $M = h``(\rho \cup \overline{C})$. This statement can be strengthened:

$$\forall \xi \in \bar{\kappa} - \overline{C} \;\; \xi \in \bar{h}``(\bar{\rho} \cup (\overline{C} \cap \xi)). \tag{18.15}$$

Now let $\rho = \sup(\pi``\bar{\rho})$ and $C = \pi``\overline{C}$. If $\tilde{h} = \tilde{h}^X$ is the function given by Lemma 3.10, then it follows that $X = \tilde{h}``((X \cap \rho) \cup C)$, and if $\xi \in X \cap \kappa$ then $\xi \in \tilde{h}``((X \cap \rho) \cup (C \cap \xi + 1))$.

If $\overline{C}$ is finite then we can define $f(x) = \tilde{h}^X(x, C)$, so that $f \in K$ and $X = f``(X \cap \rho^X)$. Thus the first of the desired alternatives hold.

For the remainder of the proof, we will assume that C is infinite. We use a superscript X to designate the results of applying this construction to the arbitrary suitable set X.

3.54 Definition. Let $\mathbf{C}$ be the class of suitable sets X such that either C^X is finite or else $K = L[U]$, the set C^X is a Prikry sequence for U, and C^X is maximal in the sense that $C - C^X$ is finite whenever C is any other Prikry sequence for $L[U]$.

Notice that C^X and $C^{X'}$ differ only finitely for any two sets $X, X' \in \mathbf{C}$ such that C^X and $C^{X'}$ are both infinite.

The following lemma will complete the proof of the Dodd-Jensen covering lemma:

3.55 Lemma. *If $0^\dagger$ does not exist then the class $\mathbf{C}$ is closed under increasing unions of uncountable cofinality, and is unbounded in $[K_\kappa]^\delta$ whenever κ is a cardinal of K and δ is an uncountable regular cardinal.*

The proof of this lemma will take up the rest of Sect. 3.3. We already know that the class of unsuitable sets is nonstationary, and by the comments above, we can assume that C^X is infinite for all but a nonstationary set of sets X.

First we show, assuming Lemma 3.55 is true for all cardinals $\mu < \kappa$, that $\mathrm{ot}(C^X) = \omega$ on all but a nonstationary set. Assume the contrary; then there is a $\mu < \kappa$ such that the ω^{th} member C^X of C^X is equal to μ for stationarily many sets X. Since the induction hypothesis states that the covering lemma holds for $C \cap K_\mu$, there is a X with $\mu^X = \mu$ such that $X \cap K_\mu \in \mathbf{C}$, but this implies that $K = L[U]$ where U is a measure on μ. Now this measure U must be in X, and is generated by $C^X \cap \mu$. Thus the iterated ultrapower from diagram (18.13) which was used to generate $\overline{C}$ would not continue past $(\pi^X)^{-1}(\mu)$, and hence $\overline{C}^X \subseteq (\pi^X)^{-1}(\mu)$. This contradicts the assumption that $C^X \not\subseteq \mu$, and completes the proof that C^X has order type ω except on a nonstationary set.

The proof of Lemma 3.55 is based on the following observation:

3.56 Proposition. *Suppose that X is suitable and $X = Y \cap K_\kappa$ where $Y \prec H(\kappa^+)$ and $C^X \in Y$. Then $X \in \mathbf{C}$.*

Proof. First we show that U is a measure in $L[U]$. Since the members of $\overline{C} = \pi^{-1}(C^X)$ come from the iteration of the unique full measure of M_θ, they generate the final measure $\overline{U}$ of the measure sequence of that model. Thus the filter U is the final measure in the measure sequence $\widetilde{\mathcal{U}}$ of $\widetilde{M} = J_{\tilde{\alpha}}[\widetilde{\mathcal{U}}]$. It follows that $\widetilde{\mathcal{U}} \in Y$ since $U \in Y$. If γ is the least ordinal such that U is not a measure in $J_\gamma[\widetilde{\mathcal{U}}]$, then $\gamma \in Y$ and hence $(\pi^N)^{-1}(J_\gamma[\widetilde{\mathcal{U}}]) = J_{\bar{\gamma}}[\mathcal{U}^M]$ is in the transitive collapse N of Y. Evidently $\bar{\gamma} \geq \mathrm{On}(M)$ since $\overline{U}$ is a measure in M, but this is impossible since there is a bounded subset of $\bar{\kappa}$ which is definable in M but is not in $\overline{K} = H(\bar{\kappa})^N$.

Thus C^X is a Prikry sequence for the measure U, and if C^X fails to satisfy Definition 3.54 then it is because there is another Prikry sequence C' such that $C' - C$ is infinite. Then by elementarity there is such a sequence C' which is a member of Y. Then $C' \subseteq X$, so any member α of $C' - C^X$ is in $\tilde{h}^X``(\tilde{\rho}^X \cup (C \cap \alpha)) \subseteq h^X``\alpha$, and since $\tilde{h}^X \in K = L[U]$ it follows that $C' - C^X$ is finite since C' is a Prikry sequence. ⊣

3.57 Corollary. *If X is countably closed then $X \in \mathbf{C}$.*

Now we will deal with the proof of Lemma 3.55 for non-countably closed sets X. This proof uses the ideas of the proof of Lemma 3.50, stating that the class of suitable sets is unbounded, but is significantly more difficult. We use the notation $a \subseteq^* b$ to mean that $a - b$ is finite, and $a =^* b$ to mean that $a \subseteq^* b$ and $b \subseteq^* a$.

3.58 Lemma. *If X_0, X_1 are suitable and $X_0 \subseteq X_1$, then $C^{X_1} \cap X_0 \subseteq^* C^{X_0}$.*

Proof. We will use subscripts 0 and 1 to distinguish objects defined from X_0 or X_1; for example we write $\pi_0 = \pi^{X_0}$ and $\pi_1 = \pi^{X_1}$. We will find a function h^*, definable in $M_1 = M^{X_1}$, such that $\xi \in h^*``\xi$ for all but boundedly many $\xi \in \pi_1^{-1}((C_1 \cap X_0) - C_0)$. Since this can only hold for finitely many $\xi \in \pi_1^{-1}(C_1)$, this will imply that $C_1 \cap X_0 \subseteq^* C_0$.

To this end, let ν be any member of $X_0 \cap (C_1 - C_0)$ and set $\nu_0 = \pi_0^{-1}(\nu)$. Then $\nu_0 \notin \overline{C}_0$, so $\nu_0 \in h_0``\nu_0$ where h_0 is the Skolem function of $M_0 = M^{X_0}$. Now let $\tau = \pi_1^{-1} \circ \pi_0 : \overline{K}_0 \to \overline{K}_1$, and let

$$\tilde{\tau} : M_0 \to M^* = \mathrm{Ult}(M_0, \tau, \bar{\kappa}_1).$$

Then $\nu_1 = \tau(\nu_0) \in h^*``\nu_1$ where h^* is given by Lemma 3.10. But M^* is sound above $\bar{\kappa}_1$, and agrees with $\bar{K}_1$ up to $\bar{\kappa}_1$, so by Lemma 3.39 one of M^* and M_1 is an initial segment of the other. Since every bounded subset of $\bar{\kappa}$ in M^* is a member of $\bar{K}_1$, it must be that M^* is an initial segment of M_1 and it follows that h^* is definable in M_1 from some parameter q. Since $\nu \in X_0 \cap (C_1 - C_0)$ was arbitrary, it follows that $X_0 \cap (C_1 - C_0)$ is finite, that is, $C_1 \cap X_0 \subseteq^* C_0$. ⊣

3.59 Corollary. *The class $\mathbf{C}$ is uncountably upward closed.*

Proof. Suppose that $X = \bigcup_{\xi<\eta} X_\xi$ is an increasing union of sets $X_\xi \in \mathbf{C}$ such that $\mathrm{cf}(\eta) > \omega$. Then X is suitable since the class of suitable sets is closed under uncountable increasing unions, and $C^X \subseteq X_\xi$ for some $\xi < \eta$ so $C^X \subseteq^* C^{X_\xi}$ by Lemma 3.58. In particular, the fact that C^{X_ξ} is a Prikry sequence for the measure U implies that C^X is a Prikry sequence for the same measure.

To complete the proof that $X \in \mathbf{C}$ we need to show that C^X is maximal, and since C^{X_ξ} is maximal it is sufficient to show that $C^{X_\xi} \subseteq^* C^X$. Now if ν is any member of $C^{X_\xi} - C^X$, then $\nu \in \tilde{h}``\nu$ where $\tilde{h}$ is the function given by Lemma 3.10. But since $C^{X_\xi}$ is a Prikry sequence over $K$ and $\tilde{h} \in K$, it follows that $C^{X_\xi} - C^X \subseteq \{\nu \in C^{X_\xi} : \nu \in \tilde{h}``\nu\}$ is finite, so $C^{X_\xi} \subseteq^* C^X$. ⊣

The following lemma completes the proof of the Dodd-Jensen covering lemma. We give a proof which is somewhat different from that given by Dodd and Jensen in [11], as that proof does not easily adapt to larger core models.

3.60 Lemma. *If δ is a regular cardinal and κ is a cardinal of K then $\mathbf{C}$ is unbounded in $[K_\kappa]^\delta$.*

Proof. As in the proof of the covering lemma for L, we work in the space $\mathrm{Col}(\delta, K_\kappa)$. If $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ then we will sometimes identify σ with $\mathrm{ran}(\sigma)$, especially when it appears as a superscript. Let S be the set of functions $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ such that $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma)$ is suitable, but $\mathrm{ran}(\sigma) \notin \mathbf{C}$, and suppose for the sake of contradiction that S is stationary in $\mathrm{Col}(\delta, K_\kappa)$. By Lemma 3.23 there is a $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ such that $\sigma \supseteq \sigma_0$ and $C^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_0$. Thus $C^\sigma \subseteq^* C^{\sigma_0}$ for all $\sigma \in S_0$ by Lemma 3.58. Set $C_0 = C^{\sigma_0}$.

As in the proof of the covering lemma for L, we define, for each member of S_0, a witness $w(\sigma)$ to the fact that $\mathrm{ran}(\sigma) \notin \mathbf{C}$:

3.61 Claim. *There is a function w mapping each member σ of S_0 to a countable subset of $\mathrm{ran}(\sigma)$ such that for any $\sigma_1, \sigma_2 \in S_0$ such that $\sigma_1 \subseteq \sigma_2$ and $w(\sigma_2) \subseteq \mathrm{ran}(\sigma_1)$ we have $C^{\sigma_1} \subseteq^* C^{\sigma_2}$.*

First we show that the lemma follows from this claim. By applying Lemma 3.23 a second time, we can find an $\sigma_1 \in S_0$ and a stationary set $S_1 \subseteq S_0$ so that $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \mathrm{ran}(\sigma_1)$ for all $\sigma \in S_1$. If σ is any member of S_1 then $C^{\sigma_1} \subseteq^* C^\sigma$ by Claim 3.61 and $C^\sigma \subseteq^* C^{\sigma_1}$ by Lemma 3.58. Thus $C^\sigma =^* C^{\sigma_1}$ for all $\sigma \in S_1$.

Since S_1 is stationary, there is a $\sigma \in S_1$ such that $\mathrm{ran}(\sigma) = Y \cap K_\kappa$ for some $Y \prec H(\kappa^+)$ with $C^{\sigma_1} \in Y$. Since $C^\sigma =^* C^{\sigma_1}$ it follows that $C^\sigma \in Y$. This implies $\mathrm{ran}(\sigma) \in \mathbf{C}$ by Lemma 3.56, contradicting the fact that $\sigma \in S_1 \subseteq S_0$. This contradiction shows that S_0 is not stationary, and hence $\mathbf{C}$ is unbounded.

Proof of Claim 3.61. We will fix $\sigma \in S_1$ for the moment in order to define $w(\sigma)$. The critical point is that $C^\sigma \subseteq^* C^{\sigma_0}$, so that C^σ is determined, up to a finite set, by $D = C^{\sigma_0} - C^\sigma$. If D is finite then we can set $w(\sigma) = \varnothing$, so we will assume that D is infinite. Let $\langle d_k : k < \omega \rangle$ enumerate D in increasing order, and set $\bar{d}_k = \pi^{-1}(d_k)$. Then $\bar{d}_k \in h^M$ "d_k, where h^M is the Skolem function for the premouse M of diagram (18.13).

To define the function w we modify Definition 3.18 of a witness to the unsuitability of X by replacing clause 3 with the statement that there is a function h which is Σ_n-definable over $\operatorname{dir\,lim}(w)$ such that $d \in h$"d for all $d \in D$.

To see that this witness function $w(\sigma)$ satisfies Claim 3.61, let $\sigma_1 \subseteq \sigma$ be a member of S_0 with $w(\sigma) \subseteq \mathrm{ran}(\sigma_1)$. Write π_1 for π^{σ_1}, and set $\tau = \pi_1^{-1}\pi : \overline{K}^{\sigma_1} \to \overline{K}^{\sigma}$. If $\bar{\mathfrak{m}} = \operatorname{dir\,lim}(\pi_1^{-1}(w(\sigma)))$ then the map τ extends to an elementary embedding $\tilde{\tau} : \bar{\mathfrak{m}} \to \mathfrak{m}$, so $\bar{\mathfrak{m}}$ is also a premouse. The measure on $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is generated by the indiscernibles $\tau^{-1}(\overline{C}^{\sigma}) = \pi_1^{-1}(C^\sigma)$, and since $C^\sigma \subseteq^* C^{\sigma_1}$ it follows that this measure is equal to the measure in M^{σ_1}. Thus $\bar{\mathfrak{m}}$ strongly agrees with M^{σ_1} up to $\bar{\kappa}_1$. Since both premice are sound above $\bar{\kappa}_1$ it follows that one is an initial segment of the other. Now if M^{σ_1} were a proper initial segment of $\bar{\mathfrak{m}}$, then there would be a bounded subset x of $\bar{\kappa}_1$

in $\bar{\mathfrak{m}} - \overline{K}^{\sigma_1}$. This is not the case, since every bounded subset of $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is in some $\bar{\mathfrak{m}}_k \in \overline{K}^{\sigma_1}$, so $\bar{\mathfrak{m}}$ must be an initial segment of M^{σ_1}. Hence, the Skolem function of $\bar{\mathfrak{m}}$ is definable in M^{σ_1}, and thus every sufficiently large member d of $\pi_1^{-1}(D)$ is in $h^{M^{\sigma_1}}$ "d. It follows that $D \cap C^{\sigma_1}$ is finite, which is to say that $C^{\sigma_1} \subseteq^* C^{\sigma}$, as was to be proved. ⊣

This completes the proof of Lemma 3.60, and hence of the Dodd-Jensen covering lemma, Theorems 1.2 and 1.3. ⊣

3.4. Unsuitable Covering Sets

As we have seen, the proof of the covering lemma for L shows, assuming $\neg 0^{\#}$, that every suitable set is in L. This striking fact suggests that the proof may also have something to say about sets X which are not suitable. Some restrictions on X are certainly needed: for example, if X is a Cohen generic subset of some uncountable regular cardinal τ then any unbounded $y \subseteq \tau$ in L intersects both X and its complement $\tau - X$. Thus we will retain the first order part of the definition of suitability: we assume that $X \prec_1 J_\kappa$ (or $X \prec_1 K_\kappa$ in the case of larger core models) but omit the second-order condition that $\widetilde{M}^X$ is well-founded and (in the case of K) that C^X is a maximal Prikry sequence. This one idea leads to two separate results: For L, or more generally if there is no ω_1-Erdős cardinal in K, it gives Magidor's Covering Lemma, 1.15, while in the presence of larger cardinals it gives Theorem 1.16 stating that Jónsson and Ramsey cardinals relativize to K.

We recall the statement of Magidor's Covering Lemma 1.15 for L. This statement follows [31] in using the hypothesis that X is primitive recursively closed instead of $X \prec_1 J_\kappa$, but we do not verify that this weaker condition is sufficient.

3.62 Theorem. *Suppose that $0^{\#}$ does not exist and that X is a primitive recursively closed subset of J_κ, and let $\delta = \inf(\kappa - X)$. Then there are functions $h_i \in L$ for $i < \omega$ such that $X \cap \kappa = \bigcup_{i<\omega} h_i$ "δ.*

Sketch of Proof. Like the covering lemma, this theorem is proved by induction on κ. Suppose that $X \prec_1 J_\kappa$. If $\operatorname{cf}(\kappa) = \omega$, say $\kappa = \bigcup_{n<\omega} \kappa_n$, then $X = \bigcup_{n<\omega}(J_{\kappa_n} \cap X)$, so the truth of the theorem for X follows from its truth for each of the sets $X \cap J_{\kappa_n}$ for $n < \omega$. Thus we can assume that $\operatorname{cf}(\kappa) > \omega$. In addition we can assume that κ is a cardinal in L, that X is cofinal in κ, and that $\kappa \not\subseteq X$. Note that we do not assume that $|X| < \kappa$.

The proof begins exactly like that of the covering lemma, with the transitive collapse $\pi : N = J_{\bar{\kappa}} \cong X \subseteq J_\kappa$. Thus $\delta = \operatorname{crit}(\pi) < \kappa$.

If X is suitable then $X \in L$ by the proof of the covering lemma, so we can assume that X is not suitable. We recall the construction, given in Lemma 3.19, of a witness to the unsuitability of X. There is a triple (α, n, β), with $\beta < \kappa$, such that $\operatorname{Ult}_n(J_\alpha, \pi, \beta)$ is defined but not well-founded. Let (α, n, β) be the least such triple, in the lexicographic ordering, and pick

$f_i \in J_\alpha$ and $a_i \in [\beta]^{<\omega}$ for $i < \omega$ so that $[a_{i+1}, f_{i+1}]_\pi \mathrel{\mathrm{E}} [a_i, f_i]_\pi$, where E is the membership relation in the ultrapower. Then $\beta = \sup(\bigcup_i a_i)$, and since $\mathrm{cf}(\kappa) > \omega$ it follows that $\beta < \kappa$. We will show that there are functions $h_i \in L$ such that $X = \bigcup_{i<\omega} h_i``(X \cap \beta)$. The truth of the theorem for X then follows by applying the induction hypothesis to the set $X \cap \beta$.

In order to simplify notation we will assume that $n = 0$ and that α is a limit ordinal. We make two observations:

1. We can choose $\langle f_i : i < \omega\rangle$ so that $J_\alpha = \mathcal{H}^{J_\alpha}_{\Sigma_1}(\beta \cup \{f_i : i < \omega\})$. If this is not true for the original choice of functions f_i, then let $\mathcal{M}' = J_{\alpha'}$ be the transitive collapse of $\mathcal{H}^{J_{\alpha_i}}_{\Sigma_1}(\beta \cup \{f_i : i < \omega\})$. Then $\alpha' \le \alpha$ and $\mathrm{Ult}(J_{\alpha'}, \pi, \beta)$ is ill-founded, so $\alpha' = \alpha$ by the minimality of α. The original functions f_i may be moved in the collapse, but we can replace them by their images under the collapse.

2. If $\alpha_i < \alpha$ is the least ordinal such that $f_i \in J_{\alpha_i}$ and $\mathcal{M}_i$ is the transitive collapse of $\mathcal{H}^{J_{\alpha_i}}_{\Sigma_1}(\bar\kappa \cup \{f_0, \ldots, f_i\})$, then $\widetilde{\mathcal{M}}_i = \mathrm{Ult}(\mathcal{M}_i, \pi, \kappa)$ is well-founded. To see this, note that $\mathrm{Ult}(J_{\alpha_i}, \pi, \kappa)$ is well-founded by the minimality of the triple (α, n, β), and $\widetilde{\mathcal{M}}_i$ is a substructure of $\mathrm{Ult}(J_{\alpha_i}, \pi, \kappa)$.

It follows that $\widetilde{\mathcal{M}}_i = \mathrm{Ult}(\mathcal{M}_i, \pi, \kappa) \in L$ for each $i < \omega$, and $X = \pi``\bar\kappa = J_\kappa \cap \bigcup_{i<\omega} h_i``(X \cap \beta)$ where $h_i \in L$ is the function given by Lemma 3.10. This completes the proof of Theorem 3.62. $\dashv$

This argument, applied to K^{DJ} in the absence of a model with a ω_1-Erdős cardinal, yields Magidor's generalization of Theorem 1.15 to K^{DJ}, while applied to larger core models K it yields the absoluteness to K of Jónsson and Ramsey cardinals, Theorem 1.16. This extension of the argument to K requires that the iterated ultrapower constructed in Definition 3.30 be modified by adding a second type of drop: Suppose that $M_\nu = J_{\alpha_\nu}[\mathcal{U}_\nu]$ has been defined, and let $\bar\beta_\nu$ be the largest ordinal such that $\mathcal{U}_\nu \restriction \bar\beta_\nu = \overline{\mathcal{U}} \restriction \bar\beta_\nu$. The next model, $M_{\nu+1}$, is defined normally, following Definition 3.30, except in the special case when $D \cap \nu = \varnothing$ and there is a triple (α, n, β) with $\beta \le \pi(\beta_\nu)$ such that $\mathrm{Ult}_n(J_\alpha[\mathcal{U}_\nu], \pi, \beta)$ is defined but not iterable. In this case put ν into D and set $M_{\nu+1} = J_\alpha[\mathcal{U}_\nu]$, where (α, n, β) is the least such triple. It is still true that if $X = Y \cap K_\kappa$ and N is the transitive collapse of Y then K^N is an initial segment of the final model M_θ of this iteration: either as in the original proof because M_θ defines a bounded subset of $\bar\kappa$ which is not in $H(\kappa)^{(N)} = \overline{K}$, or else because $\mathrm{Ult}_n(M_\theta, \pi, \kappa)$ is not iterable, while $\mathrm{Ult}(K^N, \pi, \kappa)$ can be embedded into $K_{\sup(Y)}$ which is iterable.

Let C^X be the set of putative indiscernibles generated by this proof, that is, the image under π of the critical points (after the last drop) of the iterated ultrapower. Then we get, as in the proof of Theorem 3.62, a set of functions

$h_k \in K$ for $k < \kappa$ so that

$$\begin{aligned}\widetilde{M} &= \bigcup\{\mathcal{H}^{h_k}(\rho \cup C^X) : k \in \omega\}\\ &= \bigcup\{h_k``(\rho \cup \vec{c}) : k < \omega \wedge \vec{c} \in [C^X]^{<\omega}\}\end{aligned}$$

where $\rho = \inf(\kappa - X)$.

If C^X is finite or countable then this gives X as a countable union of sets in K, so we can assume that C^X is uncountable. There is no reason to expect C^X to be a set of indiscernibles for K, but it *is* a set of indiscernibles for any structure in the range of $\tilde{\pi}$. This observation explains the importance of the following proposition:

3.63 Proposition. *Suppose that $X = Y \cap K_\kappa$ where $\kappa \in Y$, Y is cofinal in κ, and $Y \cap K_\lambda \prec_1 K_\lambda$ for some cardinal $\lambda > \kappa$. Then $\mathcal{P}(\kappa) \cap Y \subseteq \operatorname{ran}(\tilde{\pi}^X)$.*

Proof. Let $\pi^Y : N^Y \cong Y$ be the transitive collapse of Y, so that $N^Y \cap K_{\bar{\kappa}} = \overline{K}$ and $\pi^Y \restriction \overline{K} = \pi^X$. Fix any member z of $\mathcal{P}^K(\kappa) \cap Y$, let $\mathfrak{m} \in Y$ be the least mouse such that $z \in \mathfrak{m}$, and set $\bar{\mathfrak{m}} = (\pi^X)^{-1}(\mathfrak{m}) \in N^Y$. By Lemma 3.39 one of $\bar{\mathfrak{m}}$ and M^X is an initial segment of the other. Every bounded subset of $\bar{\kappa}$ in $\bar{\mathfrak{m}}$ is a member of N^X since $X = Y \cap K_\kappa$, but there is a bounded subset of $\bar{\kappa}$ which is definable in M^X and not a member of N^X. It follows that M^X is not a proper initial segment of $\bar{\mathfrak{m}}$, so $\bar{\mathfrak{m}}$ must be an initial segment of M^X. It follows that $\bar{z} = (\pi^Y)^{-1}(z) \in M^X$, and hence $z = \tilde{\pi}^Y(\bar{z}) = \tilde{\pi}^X(\bar{z}) \in \operatorname{ran}\tilde{\pi}^X$. ⊣

3.64 Corollary. *Suppose that $X = Y \cap K_\kappa$ where $Y \prec H(\lambda)$ for some $\lambda > \kappa$, and that $\mathcal{A} \in Y \cap K$ is a structure with universe κ. Then C^X is a set of indiscernibles for $\mathcal{A}$.*

Furthermore, if $D \in K \cap Y$ is a closed and unbounded subset of κ then $C^X - D$ is bounded in $\sup(C^X)$.

We will use this proposition to show that any Jónsson cardinal is Ramsey in K. The argument that every δ-Jónsson cardinal is δ-Erdős in K is similar, as is Magidor's argument that Theorem 1.15 holds for K^{DJ} unless there is an ω_1-Erdős cardinal in K^{DJ}.

Let $\mathcal{A}$ be any structure in K with universe κ. Since X is Jónsson there are sets Y and X as in the hypothesis of Proposition 3.64 such that $|X| = \kappa$ but $\kappa \not\subseteq X$.

It follows from the construction of the set C^X that $|C^X| = \kappa$. To show that κ is Ramsey in K we will show that there is a $\rho < \kappa$ and a set $C \in K$ of indiscernibles for $\mathcal{A}$ such that $(C^X - \rho) \subseteq C$. To this end let U be the filter on κ generated by C^X, that is, $z \in U$ if and only if $C^X - z$ is bounded in κ. Let $\mathfrak{m}$ be the least mouse with projectum κ such that $\mathcal{A} \in \mathfrak{m}$. Then $\mathfrak{m} \in Y$, so U is a normal ultrafilter on $\mathfrak{m}$. Furthermore $\operatorname{Ult}(\mathfrak{m}, U)$ is iterable since U is countably complete, so $\operatorname{Ult}(\mathfrak{m}, U) \in K$ and hence $U \cap \mathfrak{m} \in K$. Let h be the Skolem function of $\mathfrak{m}$ and define C to be the set of $\nu < \kappa$ such that, for each

$k < \omega$ and each set $z \in \kappa^{1+k} \cap h``\nu$,

$$\begin{array}{rcll} z \in U & \iff & \nu \in z & \text{if } k = 0 \\ z \in U^{1+k} & \iff & \{\vec{\gamma} \in (\kappa - \nu)^k : \nu^\frown\vec{\gamma} \in z\} \in U^k & \text{if } k > 0. \end{array}$$

Then C is a set of indiscernibles for $\widetilde{M}^X$, and hence for $\mathcal{A}$. Furthermore $C \in K$, and $C^X - C$ is bounded in κ since $h \in Y$. Then C is the required set of indiscernibles in K for $\mathcal{A}$, and since $\mathcal{A}$ was arbitrary this completes the proof that every Jónsson cardinal is Ramsey in K.

4. Sequences of Measures

This section concerns the covering lemma in the presence of models containing large cardinals. Most of the section will concentrate on the core model for sequences of measures; the remainder will describe, with less detail, what is known about the covering lemma up to a strong cardinal and then for overlapping extenders in the Steel core model up to and beyond a Woodin cardinal. We begin with a general survey, which is followed by a precise statement of the covering lemma for sequences of measures and some indications as to its proof.

The two large cardinal properties which critically affect the statement of the covering lemma are measurable cardinals and Woodin cardinals. Measurable cardinals are critical because they provide, via Prikry forcing, the first counterexample to the full covering property. Woodin cardinals are critical because they provide, via stationary tower forcing, a counterexample to the weak covering property as described in Sect. 4 of chapter [32].

The Covering Lemma and Sequences of Measures

The Dodd-Jensen covering lemma neatly accommodates the covering lemma to models $L[U]$ with a single measure; indeed the hypotheses $\neg\exists 0^\dagger$ and $K = L[U]$ are as well understood as is the hypothesis $\neg\exists 0^\#$ of the Jensen covering lemma. The situation for larger core models is both more complicated and less elegant. We begin this section by describing some of these complications, in rough order of the size of the core model at which they first appear.

The first three observations are relevant even in models in which $o(\kappa) \leq 1$ for all κ, that is, when no cardinal has more than one measure. To simplify the notation for this case we use an increasing enumeration $\vec{\kappa} = \langle \kappa_\nu : \nu < \theta \rangle$ of the measurable cardinals in $K = L[\mathcal{U}]$, and write U_ν for the full measure on κ_ν. A system of indiscernibles for this model K is a sequence $\mathcal{C} = \langle C_\nu : \nu < \theta \rangle$, with $C_\nu \subseteq \kappa_\nu$. Each set C_ν is either finite or a Prikry sequence, but in addition the sequence $\mathcal{C}$ as a whole is uniformly a system of indiscernibles:

$$\forall \vec{x} \in K \ \big((\forall \nu < \theta \, x_\nu \in U_\nu) \implies \big|\bigcup\{C_\nu - x_\nu : \nu < \theta\}\big| < \omega \big). \tag{18.16}$$

This leads to our first observation:

1. The sets C_ν need not be infinite: formula (18.16) is meaningful even if some or all of the sets C_ν are finite.

The only constraint on the function $f(\nu) = |C_\nu|$ when $o(\kappa_\nu) = 1$ for all ν is that ot $C_\nu \leq \omega$. For any predetermined function f, there is a straightforward modification of Prikry forcing which can be used to obtain a sequence such that $|C_\nu| = f(\nu)$ for all $\nu \in \mathrm{dom}(\mathcal{U})$: the conditions are pairs $(a, \vec{A})$ such that $A_\nu \in U_\nu$, $a_\nu \subseteq \kappa_\nu$ and $|a_\nu| \leq f(\nu)$ for each $\nu < \theta$, and $\bigcup_\nu a_\nu$ is finite. The order is defined by $(a', \vec{A}') \leq (a, \vec{A})$ if $a'_\nu \supseteq a_\nu$, $A'_\nu \subseteq A_\nu$, and $a'_\nu - a_\nu \subseteq A_\nu$ for each $\nu < \theta$.

As a consequence the relation between $L[\mathcal{U}]$ and $L[\mathcal{U}, \mathcal{C}]$ is more complicated than that between $L[U]$ and $L[U, C]$:

2. The function $f(\nu) = |C_\nu|$ need not be a member of K.

As an example, suppose that $\theta \geq \omega$ and let $a \subseteq \omega$ be a real which is Cohen generic over K. Then each of the measures U_n can be extended to a measure in $K[a]$, so we can modify Prikry forcing as described above to obtain a system $\mathcal{C}$ of indiscernibles for $K[a]$ such that $|C_n| = 1$ (or, alternatively, $|C_n| = \omega$) for each $n \in a$ and $C_n = \varnothing$ for each $n \in \omega - a$. Thus $a \in K[\mathcal{C}]$. If $|C_n| = \omega$ for each $n \in \omega$ then a is definable in $K[\mathcal{C}]$ as $\{n \in \omega : \mathrm{cf}(\kappa_n) = \omega\}$. If $|C_n| = 1$ for $n \in a$ then the covering lemma can be used to show that the system $\mathcal{C}$, and hence the set a, is definable in $K[\mathcal{C}]$ up to a finite set.

Note that a can be any set so long as the measures U_n can be extended to measures in $K[a]$.

In the Dodd-Jensen covering lemma for $L[U]$, the Prikry sequence C, if it exists, does not depend on the set x to be covered. This is not true for longer sequences:

3. If there is an inaccessible limit of measurable cardinals in K, then there is a cardinal preserving generic extension $K[G]$ of K in which each measure in K has a Prikry sequence, but there is no sequence $\mathcal{C} = \langle C_\nu : \nu < \kappa \rangle$ of Prikry sequences which satisfies (18.16) [36, Theorem 1.3].

An inaccessible limit of measurable cardinals is needed to obtain such a sequence: it is shown in [40, Theorem 4.1] that if there is no model with an inaccessible limit of measurable cardinals then, as in the Dodd-Jensen covering lemma, there is a single sequence $\mathcal{C}$ which can be used to cover any set x.

Since the remaining observations only apply in the presence of cardinals κ with $o(\kappa) > 1$, we now revert to the notation for sequences of measures described in chapter [32] and in the last section: the core model K is a structure of the form $L[\mathcal{U}]$, where $\mathcal{U}$ is a sequence of filters such that each member $\mathcal{U}_\gamma$ of the sequence is a normal measure on $L[\mathcal{U}{\restriction}\gamma] \cap \mathcal{P}(\kappa)$, where $\gamma = \kappa^{++}$ in $L[\mathcal{U}{\restriction}\gamma]$. Not all of the filters $\mathcal{U}_\gamma$ are full measures in K, but we only need to consider those measures which are full.

We frequently write $\mathcal{U}(\alpha,\beta)$ for the βth full measure on α in $L[\mathcal{U}]$; that is, $\mathcal{U}(\alpha,\beta)=\mathcal{U}_{\gamma_\beta}$ where $\langle\gamma_\nu : \nu < o(\alpha)\rangle$ is the increasing enumeration of the ordinals γ such that $\mathcal{U}_\gamma$ is a full measure on α in $L[\mathcal{U}]$. We write $o(\alpha)$, as above, for the least ordinal β such that $\mathcal{U}(\alpha,\beta)$ is undefined. The sequence $\mathcal{U}$ has the following *coherence* property: if $i : K \to \mathrm{Ult}(K,\mathcal{U}(\alpha,\beta))$ then $o^{i(\mathcal{U})}(\alpha)=\beta$ and $i(\mathcal{U})(\alpha,\beta')=\mathcal{U}(\alpha,\beta')$ for all $\beta'<\beta$.

For the next three observations we assume that the core model does not contain any extenders, so that K always satisfies $o(\alpha)\le\alpha^{++}$.

Corresponding to a sequence $\mathcal{U}$ of measures we will use $\mathcal{C}$ to denote a system of indiscernibles: if $\gamma\in\mathrm{dom}(\mathcal{C})$ then $\mathcal{C}_\gamma\subseteq\mathrm{crit}(\mathcal{U}_\gamma)$ is a set of indiscernibles for the measure $\mathcal{U}_\gamma$ (or, in the other notation, $\mathcal{C}(\alpha,\beta)\subseteq\alpha$ is a set of indiscernibles for $\mathcal{U}(\alpha,\beta)$). The precise definition of a system of indiscernibles will be given later, in Definitions 4.15 through 4.18 and in the covering lemma, Theorem 4.19.

4. The sets $\mathcal{C}(\kappa,\beta)$ may have order type greater than ω. In general, the set $\bigcup_{\beta<o(\kappa)}\mathcal{C}(\kappa,\beta)$ of indiscernibles for measures on a cardinal κ is a closed subset of κ which may have any order type up to $\min\{\kappa,\omega^{o(\kappa)}\}$.

In [30] Magidor generalizes Prikry forcing in order to add such a sequence of indiscernibles and hence change the cofinality of a cardinal κ to any smaller regular cardinal λ, provided that $o(\kappa)\ge\lambda$ in the ground model. This forcing is discussed briefly in chapter [32] and extensively in chapter [15].

For longer sequences of measures, and in particular when $o(\kappa)>\kappa$, it is important in applications that the domain of the sequence $\mathcal{C}^X$ is contained in the covering set X. For this reason, we assume a slightly different context for the covering lemma for sequences than was used for the Dodd-Jensen covering lemma. Let $\kappa=\sup(x)$ be a cardinal of K, where x is the set which we are trying to cover. We will look for a covering set $X\supseteq x$ such that $X\prec_1 K_{\breve\kappa}$, where $\breve\kappa$ is the least cardinal of K such that $\breve\kappa\ge\max\{\kappa,o(\kappa)\}$.

This requirement that $\mathrm{dom}(\mathcal{C}^X)\subseteq X$ leads to two somewhat technical problems in the study of longer sequences of measures:

5. It need not be that every suitable set X can be written as $X=h``(\rho;\mathcal{C}^X)$ where $\rho=\min(\kappa-X)$ and $h\in K$.

The notation $h``(\rho;\mathcal{C}^X)$ (which is defined in Definition 4.17) corresponds to the notation $h``(\rho\cup C)$ used when $K=L[U]$, but takes account of the fact that $\mathcal{C}^X$ is a function rather than a set. Recall that the strong version of the Jensen covering lemma states that if $0^\#$ does not exist then every suitable set X is in L, and the Dodd-Jensen covering lemma for $L[U]$ states (assuming there is a Prikry sequence C) that any such set can be written as $h``(\rho\cup C)$ where $\rho=\min(\kappa-X)$.

The covering lemma for longer sequences states that $X=h``(\rho\cap X;\mathcal{C}^X)$ for some $\rho<\kappa$, however, the induction used to show that ρ can be taken to be $\inf(\kappa-X)$ breaks down for sequences of measures: it depends on the fact that $X \cap K_\rho$ is suitable as a subset of K_ρ, but in the case of sequences of measures it may be that ρ is measurable, but is not a member of X. In this case $X' = X \cap K_{\breve{\rho}}$ is not suitable since $\operatorname{dom}(\mathcal{C}^{X'})$ is not contained in X'.

So far this limitation has not caused problems in applications, nor has the next difficulty:

6. If $o(\alpha) \geq \alpha^+$ then it is not known whether countable completeness of a set X is enough to ensure that X is a suitable covering set. What is known is stated in Theorem 4.19, which requires that the set X be $\operatorname{cf}(\kappa)$-closed. In particular it is not known whether there always exist suitable covering sets of size less than $\operatorname{cf}(\kappa)^+$.

The problem, again, comes from the requirement that $\operatorname{dom}(\mathcal{C}) \subseteq X$, but in this case it is the measures on κ which are in question. These measures are generated by cofinal subsets of $\bigcup_\beta \mathcal{C}(\kappa, \beta)$, so the assumption that X contains its subsets of size at most $\operatorname{cf}(\kappa)$ implies that these subsets, and hence the corresponding measures, are in X.

Extenders

If $o(\kappa) > \kappa^{++}$ then the core model is built using extenders, and we will write $K = L[\mathcal{E}]$ to denote the core model. Below $0^\P$, the sharp for a strong cardinal, the extenders do not overlap and the covering lemma as stated for sequences of measures remains true with two modifications. One of these is primarily notational, but the following situation is unexpected:

7. If $\operatorname{cf}(\kappa) = \omega$ and $\{\alpha < \kappa : o(\alpha) \geq \alpha^{+n}\}$ is unbounded in κ for all $n < \omega$, then the fact that a set X is countably closed does not ensure that X contains all of the extenders on κ which are generated by the system $\mathcal{C}^X$ of indiscernibles for X.

   If $\operatorname{cf}(\kappa) \geq \omega_1$ and ${}^{\operatorname{cf}(\kappa)}X \subseteq X$, however, in this case X does contain all such extenders. As a result the covering lemma up to $0^\P$ is similar to the result of substituting "ω_1" and "countable" for "ω" and "finite" in the covering lemma for sequences of measures.

Both parts of this observation are due to Gitik. In [18] he defines a game which can be used to reconstruct a extender E on a cardinal of uncountable cofinality from the sequences of ordinals which generate the constituent ultrafilters, and in [21] he constructs a model in which this is not possible for extenders on a cardinal of cofinality ω.

The set $0^\P$ marks the introduction of overlapping extenders, and thus of a dramatic shift in our understanding of the covering lemma:

8. If $0^\P$ exists then we cannot prove much more than the weak covering lemma and the absoluteness theorem for Jónsson and Ramsey cardinals (Theorem 1.16).

The basic construction of the proof of the covering lemma does still go through for overlapping extenders, with considerably increased technical difficulties [45, 43], but it uses iteration trees rather than the linear iteration of Definition 3.30. The indiscernibles generated by such iterations are very poorly understood, and the proofs for the known results above $0^\P$ rely on avoiding indiscernibles rather than on analyzing them.

9. No core model with cardinals very much larger than $0^\P$ is known to exist and satisfy the weak covering lemma without an additional assumption that there is a subtle cardinal in the universe. It is known that there is no model for a Woodin cardinal which satisfies the weak covering lemma in set generic extensions.

Even the weak covering lemma is false for any model with a Woodin cardinal δ, since stationary tower forcing can be used to collapse successors of many singular cardinals below δ. The situation between $0^\P$ and a Woodin cardinal is still under investigation.

Of course any statement such as these must rely on implicit assumptions about what it means to be a "core model". Section 4 of chapter [32] explores the assumptions lying behind the statement here.

The use of $0^\P$ as the dividing line is an oversimplification: it is possible to use tricks to push some of the results somewhat further. In fact Schindler [50] has constructed a core model under the assumption that there is no sharp for a model with a class of strong cardinals. More importantly, there are some suggestions that it is the presence of actual overlapping extenders in K which cause the difficulty, not partial extenders such as those which appear in the countable mouse $0^\P$. Schimmerling and Woodin have shown that in certain special cases the core model can be proved to have the full covering property, even though it contains inner models with several Woodin cardinals. See [48], where Schimmerling and Woodin show that this result is not limited to the Steel core models, but has consequences for the existence of core-like models.

This concludes our summary. The next subsection contains a more detailed discussion of the covering lemma for sequences of measures.

4.1. The Core Model for Sequences of Measures

See chapter [32] for a discussion of the inner models for sequences of measures, and Sect. 3.3 of this chapter on the Dodd-Jensen core model for its discussion of the core model K in particular. Recall that $K = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequence with members $\mathcal{U}_\gamma$ which are $J_\gamma[\mathcal{U}{\restriction}\gamma]$-measures. We will define the sequence $\mathcal{U}$, and hence the core model, by recursion on γ. The main problem in designing this recursion is to ensure that the final model $L[\mathcal{U}]$ is iterable: when $\mathcal{U}{\restriction}\gamma$ has been defined, then the decision whether to set $\mathcal{U}_\gamma = U$ for some measure U must take into account the requirement that any iterated ultrapower of the as yet undefined model $L[\mathcal{U}]$ must be well-founded. This is accomplished by defining two core models: the first, the *countably complete* core model K^c, has the weak covering property and is iterable because its full measures are countably complete; the second, the *true* core model K, has the full covering property and is iterable because it is an elementary substructure of K^c.

4.1 Definition. Either the core model K, or the countably complete core model K^c, are defined as $L[\mathcal{U}]$ where the sequence $\mathcal{U}$ is defined by recursion on γ as follows. Assume that $\mathcal{U}\restriction\gamma$ has already been defined:

1. If there is a mouse $M = J_{\gamma'}[\mathcal{U}']$ such that $\mathcal{U}'\restriction\gamma = \mathcal{U}\restriction\gamma$, the projectum of M is smaller than γ, and no measure in $\mathcal{U}' - \mathcal{U}$ is full in M, then set $\mathcal{U}\restriction\gamma' = \mathcal{U}'$.

2. If there is no mouse as in clause 1, and if $J_\gamma[\mathcal{U}] \models \gamma = \kappa^{++}$ for some $\kappa < \gamma$ such that there is a $J_\gamma[\mathcal{U}]$-ultrafilter U on κ with $i^U(\mathcal{U})\restriction\gamma+1 = \mathcal{U}\restriction\gamma$, then set $\mathcal{U}_\gamma = U$, provided it satisfies an iterability condition depending on which model is being constructed:

   (a) For the model K^c, the ultrafilter U is added to the sequence only if U is countably complete and $\operatorname{cf}(\operatorname{crit}(U)) = \omega_1$.

   (b) For the true core model K, the ultrafilter U is added to the sequence only if $\operatorname{Ult}(L[\mathcal{W}], U)$ is well-founded for every iterable inner model $L[\mathcal{W}]$ such that $\mathcal{W}\restriction\gamma = \mathcal{U}$.

3. If neither of the previous clauses apply then $\mathcal{U}_\gamma = \varnothing$.

The construction in clause 1 apparently depends on the choice of the mouse M to be added; however, if two mice $J_{\alpha_0}[\mathcal{W}_0]$ and $J_{\alpha_1}[\mathcal{W}_1]$ satisfy clause 1, then one of them is an initial segment of the other. Thus clause 1 could be equivalently restated by specifying that $\mathcal{U}\restriction\gamma$ is to be extended to the longest good sequence $\mathcal{U}' \supseteq \mathcal{U}\restriction\gamma$ such that $L_{\gamma'}[\mathcal{U}']$ is iterable, the largest cardinal in $L[\mathcal{U}']$ below $\sup(\operatorname{dom}(\mathcal{U}'))$ is smaller than γ, and no measures in $\mathcal{U}' - \mathcal{U}\restriction\gamma$ are full in $L[\mathcal{U}']$.

It can be shown that there is never more than one choice of the measure $\mathcal{U}_\gamma$ satisfying clause 2. One way of doing so is to pick a mouse M with projectum κ containing a set x on which two candidate measures U and U' differ, and compare $\operatorname{Ult}(M, U)$ and $\operatorname{Ult}(M, U')$. Another is by using a *bicephelus*, which is a structure $\mathcal{B} = (J_\gamma[\mathcal{U}], \mathcal{U}, U, U')$ which is like a mouse except that both of U and U' are used as the top measure $\mathcal{U}_\gamma$ of $\mathcal{U}$. As in the proof of [32, Theorem 3.22], an iterated ultrapower is used to compare $\mathcal{B}$ with itself and conclude that in fact U is equal to U'. The construction is simpler than that of [32, Theorem 3.22] since $\mathcal{B}$ is a perfectly normal mouse except for the doubled top measures U and U', which are used only as predicates, not in the construction of $J_\gamma[\mathcal{U}]$.

4.2 Lemma. *The model K^c is iterable.*

Proof. We present a proof which seems slightly oblique compared to the original proof, but which extends naturally to models with sequences of extenders. First we show that if $\sigma : M \to K^{\mathrm{c}}_\theta$ is an elementary embedding, where M is a countable transitive set and θ is a sufficiently large cardinal, and if U is a full measure in M, then σ can be extended to obtain an elementary embedding $\tilde\sigma : \mathrm{Ult}(M,U) \to K^{\mathrm{c}}_\theta$ such that $\sigma = \tilde\sigma i^U$. To define $\tilde\sigma$ let $A = \bigcap \sigma``U$. Then $A \neq \varnothing$ since $\sigma``U$ is a countable subset of the countably complete ultrafilter $\sigma(U)$, so choose $\lambda \in A$ and define $\tilde\sigma([f]_U) = \sigma(f)(\lambda)$. Then $\tilde\sigma$ is elementary because $\mathrm{Ult}(M,U) \models \varphi([f]_U)$ if and only if $M \models \{\nu : \varphi(f(\nu))\} \in U$, and by the choice of λ this holds if and only if $K^{\mathrm{c}}_\theta \models \varphi(f(\lambda))$.

Now suppose that K^{c} is not iterable. Then for sufficiently large θ there is a countable elementary substructure $X \prec V_\theta$ containing an iteration witnessing this failure. If $\sigma : M \cong X \cap K^{\mathrm{c}}_\theta$ is the transitive collapse then M is not iterable, and there is a countable iteration of M witnessing this failure. If this iteration does not contain any drops then the construction of the last paragraph can be repeated countably many times to obtain embeddings $\tilde\sigma_\nu : M_\nu \to K^{\mathrm{c}}_\theta$ of the models M_ν of this iteration into K^{c}_θ, but this is impossible because the final ill-founded model M_δ of the iteration is embedded by $\tilde\sigma_\delta$ into the well-founded set K^{c}_θ. If the iteration does contain a drop, with the first drop occurring at ν_0, then $\tilde\sigma_{\nu_0} \restriction M^*_{\nu_0+1}$ embeds $M^*_{\nu_0+1}$ into an iterable mouse $\widetilde{M} = \tilde\sigma_{\nu_0}(M^*_{\nu_0+1})$ of K^{c}_θ. The remainder of the iteration on $M = M_0$ can then be copied to obtain an ill-behaved iteration on $\widetilde{M}$, which contradicts the fact that $\widetilde{M}$ is iterable. ⊣

We will show that K is iterable by giving, in Lemma 4.11, a characterization of K as the transitive collapse of an elementary substructure of K^{c}. This characterization, which depends on the weak covering lemma for K^{c}, begins with the following preliminary definitions generalizing the fact that the weak covering lemma for K^{DJ} implies that any elementary substructure $X \prec K^{\mathrm{DJ}}$ with $|X \cap \lambda^+| = \lambda^+$ is isomorphic to K^{DJ}.

4.3 Definition. An iterable premouse $M = L[\mathcal{U}]$ is said to be *universal* if, whenever M is compared with any other iterable premouse M', the iterated ultrapower on M' does not drop and the final model in that iteration is a (possibly proper) initial segment of the final model of the iteration on M.

Note that a universal premouse M must be a proper class, since if $M = J_\alpha[\mathcal{U}]$ is an iterable premouse which is a set then Lemma 3.28 implies that M is the iterated ultrapower of a mouse $M' = J_{\alpha'}[\mathcal{U}']$. Thus $L[\mathcal{U}']$ is an iterable premouse, and M comes out shorter than $L[\mathcal{U}']$ when they are compared because the iteration on $L[\mathcal{U}']$ consists of an initial drop to the mouse M', followed by the iterated ultrapower of M' to M.

4.4 Proposition. *If M is a iterable class premouse and $\lambda^{+M} = \lambda^+$ for a stationary class of cardinals, then M is universal.*

Proof. If M is not universal then there is an iterable premouse M' and iterated ultrapowers $i : M \to P$ and $i' : M' \to P'$ such that P is a proper initial segment of P'. Thus the class of ordinals of P is the class Ω of actual ordinals, and since P' is longer, we have $\Omega \in P'$. The iteration i does not drop by Lemma 3.36. We can assume that i' also does not drop, since if it did drop we could consider only the tail of the iteration starting after the last drop. If ν_0 and κ are chosen so that $\Omega = i'_{\nu_0,\Omega}(\kappa)$ then the class Γ of ordinals λ such that $i_{0,\lambda}``\lambda \subseteq \lambda$ and $\lambda = i'_{\nu_0,\lambda}(\kappa)$ is closed and unbounded. Fix $\lambda \in \Gamma$ such that $(\lambda^+)^M = \lambda^+$. Then $\lambda^{+^{P'}} = \lambda^{+^{M'_\lambda}} = i'_{\nu_0,\lambda}(\kappa^{+^{M'_{\nu_0}}}) < \lambda^+$. This implies that $\lambda^{+^P} = \lambda^{+^{M_\lambda}} < \lambda^+$. We will obtain a contradiction by showing that $\kappa^{+^{M_\lambda}} = \lambda^+$. If $i_{0,\lambda}(\lambda) = \lambda$ then this follows immediately since $\lambda^{+^{M_\lambda}} = i_{0,\lambda}(\lambda^+) \geq \lambda^+$. Otherwise there is a unique $\nu < \lambda$ such that $i_{0,\nu}(\lambda) = \lambda$ and $i_{\nu+1,\lambda}(\lambda) = \lambda$, but $\mathrm{crit}(i_{\nu,\nu+1}) = \mathrm{cf}^{M_\nu}(\lambda)$. Then $\lambda^{+^{M_\lambda}} = \lambda^{+^{M_{\nu+1}}} = \lambda^{+^{M_\nu}} = \lambda^{+^M}$, with the first and last equalities being proved as in the case when λ is never moved. ⊣

We will prove the following lemma later, simultaneously with the full covering lemma:

4.5 Lemma (Weak Covering Lemma for K^{c}). *Suppose that there is no inner model of $\exists\kappa\,(o(\kappa) = \kappa^{++})$ and that λ is a singular strong limit cardinal. Then $(\lambda^+)^{K^c} = \lambda^+$.*

4.6 Definition. A class Γ is *thick* if Γ is a proper class, there is some τ such that Γ contains its limit points of cofinality greater than τ, and $|\Gamma \cap \lambda^+| = \lambda^+$ for all sufficiently large singular strong limit cardinals $\lambda \in \Gamma$.

4.7 Proposition. *Any set sized intersection of thick classes is thick.*

The following observation explains the definition of a thick class, and also the decision to require that every measurable cardinal in K^{c} have cofinality ω_1.

4.8 Proposition. *Suppose that W is an iterable, class length premouse, τ is an ordinal, and Γ is a thick class such that for any singular cardinal $\lambda \in \Gamma$ with $\mathrm{cf}(\lambda) > \tau$ we have that $\mathrm{cf}^W(\lambda)$ is not measurable, and $\lambda^{+^W} = \lambda^+$.*

Then any (nondropping) iterated ultrapower $i : W \to N$ is continuous at points $\xi \in \Gamma$ of cofinality greater than τ. Hence, the class Γ' of ordinals $\xi \in \Gamma$ such that $i(\xi) = \xi$ is thick.

Proof. If $i(\xi) > \sup(i``\xi)$, then it must be that at some stage ν in the iteration, $W_{\nu+1} = \mathrm{Ult}(W_\nu, U_\nu)$ where U_ν is a measure in W_ν on a cardinal κ_ν with $\mathrm{cf}^{W_\nu}(i_\nu(\xi)) = \kappa_\nu$. Since $\kappa_\nu = i_\nu(\mathrm{cf}^W(\xi))$ it follows that $\mathrm{cf}^W(\xi)$ is measurable in W. By the assumption on W, it follows that ξ is not a member of Γ of cofinality greater than τ.

For the final sentence, consider the class X of cardinals ξ such that ξ is a limit point of Γ of cofinality at least τ, and $i``\xi \subseteq \xi$. Then X is a proper class, and Γ' contains all of the limit points of X of cofinality at least τ.

Finally, if $\lambda \in \Gamma'$ and $\lambda^{+W} = \lambda^+$ then $i(\lambda^+) = \lambda^+$, and an argument like that in the last paragraph shows that $|\{\xi \in \lambda^+ : i(\xi) = \xi\}| = \lambda^+$. Thus $|\Gamma' \cap \lambda^+| = \lambda^+$, so Γ' is thick. $\dashv$

4.9 Lemma. *Assume that K^c satisfies the weak covering lemma, that λ is a limit cardinal and Γ is thick. Then for each $x \in \mathcal{P}^{K^c}(\lambda)$ there is a $y \in \mathcal{H}^{K^c}(\Gamma \cup \lambda)$, the Skolem hull of $\Gamma \cup \lambda$, such that $x = y \cap \lambda$.*

Proof. Let $\pi : N \cong \mathcal{H}^{K^c}(\Gamma \cup \lambda)$ be the transitive collapse. Then $\xi^{+N} \geq |\Gamma \cap \xi^+| = \xi^+$ for all singular limit cardinals ξ of sufficiently large cofinality, so N is universal by Lemma 4.4. It follows that the comparison of N with K^c will result in iterated ultrapowers with no drops and a common final model P. Furthermore, the critical point of each of the associated embeddings $i : N \to P$ and $j : K^c \to P$ is at least λ, since N and K^c agree on all measures with critical point below λ. Thus $x \in \mathcal{P}^P(\lambda) = \mathcal{P}^N(\lambda) \subseteq \mathrm{dom}(\pi)$, and the set $y = \pi(x)$ satisfies the requirements. $\dashv$

4.10 Lemma. *If K^c satisfies Lemma 4.5, Γ is thick and $\lambda > \omega_1$ is a strong limit cardinal of cofinality ω_1, then $\lambda \in \mathcal{H}^{K^c}(\lambda \cup \Gamma)$.*

Proof. Let $X = \mathcal{H}^{K^c}(\lambda \cup \Gamma)$, and suppose to the contrary that $\lambda \notin X$. Let $\pi : N \cong X$ be the transitive collapse, and let $U = \{x \subseteq \lambda : \lambda \in i(x)\}$. Then U is a K^c-ultrafilter, since $\mathcal{P}^{K^c}(\lambda) \subseteq N$. Furthermore U is countably complete. To see this, let A be any countable subset of U. Since $\mathrm{cf}(\lambda^{+K^c}) = \mathrm{cf}(\lambda^+) = \lambda^+ > \omega$ there is $B \in K^c$ of size κ such that $A \subseteq B$, and since $\mathrm{cf}(\lambda) > \omega$ it follows that there is $B' \in K^c$ of size less than κ such that $A \subseteq B' \subseteq B$. Then $\bigcap A \supseteq \bigcap(B' \cap U)$, but the latter is nonempty since $B' \cap U \in K^c$ and U is a K^c-ultrafilter.

Thus $U \in K^c$, which is impossible since $o(U) = o^{K^c}(\lambda)$. The contradiction shows that $\lambda \in X$. $\dashv$

4.11 Lemma. *If K^c satisfies Lemma 4.5 then K is isomorphic to the class $X = \bigcap\{\mathcal{H}^{K^c}(\Gamma) : \Gamma$ is thick$\}$.*

Sketch of Proof. Let $\pi : \widetilde{K} \cong X$ be the transitive collapse of X. First we show that X is a proper class. Suppose to the contrary that X is a set and let $\kappa = \sup(X \cap \mathrm{On})$. Now use Proposition 4.7 to define a descending sequence $\langle \Gamma_\nu : \nu < \omega_1 \rangle$ of thick classes such that $X = \mathcal{H}^{K^c}(\Gamma_0) \cap V_{\kappa+1}$, the sequence of ordinals $\kappa_\nu = \inf(\mathcal{H}^{K^c}(\Gamma_\nu) - \kappa)$ is strictly increasing, and $\lambda = \sup\{\kappa_\nu : \nu < \omega_1\}$ is a strong limit cardinal. We will show that $\lambda \in X$, contradicting the choice of κ.

Suppose the contrary, and pick $\Gamma_{\omega_1} \subseteq \bigcap_{\nu<\omega_1} \Gamma_\nu$ so that $\lambda \notin \mathcal{H}^{K^c}(\Gamma_{\omega_1})$. By Lemma 4.10 there is a parameter $a \in \Gamma_{\omega_1}$, an ordinal $\xi < \lambda$, and a formula φ such that λ is the unique η such that $K^c \models \varphi(a, \xi, \eta)$. Let $\tau \in K$ be the

Skolem function for φ (with parameter a) so that $K^{\mathrm{c}} \models \forall\iota\,(\exists\eta\,\varphi(a,\iota,\eta) \implies \varphi(a,\iota,\tau(\iota)))$.

Now, notice that if $\nu < \omega_1$ then $\tau``\kappa_\nu \cap \lambda \subseteq \kappa_\nu$: otherwise there is some $\nu' > \nu$ and $\xi' < \kappa_\nu$ so that $\kappa_\nu \leq \tau(\xi') < \kappa_{\nu'}$. Then the least such ξ' is definable from κ_ν, $\kappa_{\nu'}$ and a, so $\xi' \in \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_\nu) \cap \kappa_\nu = X \cap \kappa$, but this is impossible since in that case $\tau(\xi') \in \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_{\nu'})$.

Now let ξ_0 be the least ordinal ξ such that $\lambda = \tau(\xi)$, and fix $\nu_0 < \omega_1$ so $\xi_0 < \kappa_{\nu_0}$. Then $\lambda = \min(\tau``\kappa_{\nu_0} - \kappa_{\nu_0}) \in \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_{\nu_0})$. However, this implies that $\xi_0 \in \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_{\nu_0}) \cap \kappa_{\nu_0} = X \cap \kappa \subseteq \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_{\omega_1})$. Hence $\lambda \in \mathcal{H}^{K^{\mathrm{c}}}(\Gamma_{\omega_1})$, contrary to the choice of Γ_{ω_1}.

Now we show that $\widetilde{K} = K$. Else, fix λ such that $\widetilde{K}_\lambda \neq K_\lambda$, and fix a thick class Γ small enough that if $\pi : W \cong \mathcal{H}^{K^{\mathrm{c}}}(\Gamma)$ is the collapse map then $\pi``\widetilde{K}_\lambda = X \cap K^{\mathrm{c}}_{\pi(\lambda)}$. Note that W, with the class $\pi^{-1}[\Gamma]$, satisfies the conditions of Proposition 4.8. If we consider the least place at which K differs from $\widetilde{K}$, and hence from W, then there are two possibilities: $\widetilde{K}$ is missing a mouse which is in K, or $\widetilde{K}$ is missing a full measure which is in K. The first is impossible, since it would contradict the universality of W. Thus there must be a measure $U = \mathcal{U}_\gamma$ on some cardinal $\kappa < \lambda$ in K such that $U \notin W$, but K and W agree up to $\gamma = \kappa^{++W}$. Now consider the following diagram:

$$\begin{tikzcd} W \arrow[r, "i^U"] \arrow[dr, "j"'] & \mathrm{Ult}(W,U) \arrow[d, "k"] \\ & \overline{N} \end{tikzcd}$$

where $\overline{N}$ is the common final model of the iterated ultrapowers coming from the comparison of the universal models W and $\mathrm{Ult}(W,U)$, and j and k are the embeddings from these iterated ultrapowers. Let Γ' be the set of $\xi \in \pi^{-1}[\Gamma]$ such that $ki^U\pi^{-1}(x) = j\pi^{-1}(x)$. Then Γ' is thick, but $\pi(\kappa) \notin \mathcal{H}^W(\Gamma')$, contradicting the assumption that $\pi(\kappa) \in X$. ⊣

The trick used at the end of the last proof, using an approximation W which agrees with the relevant initial segment of K but which satisfies the hypothesis of Proposition 4.8, is often necessary. The following theorem gives another example:

4.12 Theorem. *If K^{c} satisfies the weak covering lemma then any universal iterable premouse M is an iterated ultrapower of K.*

Proof. Suppose that M is a counterexample, and let ν be the first stage in the comparison with K at which the iterated ultrapower on M becomes nontrivial. Thus $M_\nu = M$ and $M_{\nu+1} = \mathrm{Ult}(M,U)$, and the ultrafilter $U = \mathcal{U}^{M_\nu}_\gamma$ is not in the νth model N_ν in the iterated ultrapower on K.

Fix η large enough that $i_\nu(\eta) > \gamma$, where $i_\nu : K = N_0 \to N_\nu$ is the embedding coming from the iterated ultrapower on K, and as in the last proof choose W satisfying the hypothesis of Proposition 4.8 which agrees with K

up to η, so that $W_\eta = K_\eta$. Thus the description in the last paragraph of the comparison between K and M applies equally to the comparison between W and M, and for the rest of the proof we use the latter.

Since both W and M are universal, the comparison of these two models gives iterated ultrapowers of length $\theta \leq \mathrm{On}$ with no drops and with a common final model P, as in the left half of the following diagram:

$$\begin{tikzcd} W = N_0 \arrow[rr, "i_\nu"] & & N_\nu \arrow[r, "i^U"] \arrow[d, "i_{\nu,\theta}"'] \arrow[dr, "s"] & \mathrm{Ult}(N_\nu, U) \arrow[d, "t"] \\ M \arrow[r, equal, "j_\nu"] & M_\nu \arrow[r, "j_{\nu,\theta}"'] & P & Q \end{tikzcd} \tag{18.17}$$

Since there are no drops, the models N_ν, P and M_ν all have the same subsets of κ, so U is an N_ν-ultrafilter. Furthermore, $\mathrm{Ult}(N_\nu, U)$ is well-founded: otherwise $\mathrm{Ult}(P, U)$ would be ill-founded, but the iterated ultrapower $j_{\nu,\theta}$ can be copied to an iterated ultrapower on $\mathrm{Ult}(M_\nu, U)$ with last model $\mathrm{Ult}(P, U)$, so the iterability of M_ν implies that $\mathrm{Ult}(P, U)$ is well-founded. Now compare N_ν and $\mathrm{Ult}(N_\nu, U)$. Again, both models are universal so this comparison gives embeddings s and t as in diagram (18.17) with the same final model Q.

Let $\Gamma = \{\xi > \eta : ti^U i_\nu(\xi) = si_\nu(\xi)\}$. Then Γ is thick, so Theorem 4.11 implies that $K_\eta = W_\eta \subseteq \mathcal{H}^W(\Gamma)$. It follows that $i_\nu(K_\eta) \subseteq \mathcal{H}^{N_\nu}(\Gamma \cup \kappa)$, so $ti^U \restriction W_{i_\nu(\eta)} = s \restriction W_{i_\nu(\eta)}$. In particular, $s(\kappa) > \kappa$ since $ti^U(\kappa) > \kappa$, so the iteration s begins with an ultrapower by some measure $U' = \mathcal{U}_\gamma^{N_\nu} \in N_\nu$ with critical point κ. But then $U = U'$, since if x is any subset of κ in N_ν then $x \in U \iff \kappa \in ti^U(x) = s(x) \iff x \in U'$. This contradicts the assumption that $U \notin N_\nu$. ⊣

4.13 Corollary. *If K^c satisfies the weak covering property and U is a normal ultrafilter on K such that $\mathrm{Ult}(K, U)$ is well-founded, then there is some γ such that $U = \mathcal{U}_\gamma$, where $K = L[\mathcal{U}]$.*

Proof. Apply Theorem 4.12 to $\mathrm{Ult}(K, U)$. ⊣

Note that the hypothesis that M is universal cannot be eliminated from Theorem 4.12: The model N constructed in the proof of Jensen's Theorem 3.43 provides a counterexample in which N is a set, and a similar argument, starting with an assumption somewhat weaker than two measurable cardinals, gives a counterexample in which N is a proper class. However, such situations can only occur below ω_2: If $\lambda \geq \omega_2$ then the covering lemma implies that $\mathrm{cf}((\lambda^+)^K) > \omega$, and it follows that any K-ultrafilter on λ is countably complete.

4.2. The Covering Lemma up to $o(\kappa) = \kappa^{++}$

We use the following setting for the covering lemma for sequences of measures: We take κ to be a cardinal of K which is singular in V, and we consider covering sets $X \prec_1 K_{\breve{\kappa}}$ where $\breve{\kappa}$ is a cardinal of K with $\breve{\kappa} \geq \max\{\kappa, o(\kappa)\}$. The covering lemma asserts that for suitable sets X there is a system $\mathcal{C}^X$ of indiscernibles, a function h in K, and a $\rho < \kappa$ such that $X = h``(X \cap \rho; \mathcal{C})$, the smallest set containing $X \cap \rho$ and closed under h and $\mathcal{C}$. We will call such a set X a *covering set.*

Some definitions are required before we can give a precise statement of the covering lemma. Here is a general definition for a system of indiscernibles:

4.14 Definition.

1. If U is a measure, then $\operatorname{crit}(U)$ is the cardinal κ such that U is a measure on κ.

2. If $\gamma < \gamma'$, with $\gamma' \in \operatorname{dom}(\mathcal{U})$, then $\operatorname{Coh}_{\gamma,\gamma'}$ is the least function f in the ordering of $L[\mathcal{U}]$ such that $\gamma = [f]_{\mathcal{U}_{\gamma'}}$ in $\operatorname{Ult}(L[\mathcal{U}], \mathcal{U}_{\gamma'})$.

4.15 Definition. If $\mathcal{U}$ is a sequence of measures then a *system of indiscernibles* for $M = L[\mathcal{U}]$ is a function $\mathcal{C}$ such that

1. $\operatorname{dom}(\mathcal{C}) \subseteq \operatorname{dom}(\mathcal{U})$, and $\mathcal{C}_\gamma \subseteq \operatorname{crit}(\mathcal{U}_\gamma)$ for all $\gamma \in \operatorname{dom}(\mathcal{C})$.

2. For any function $f \in M$ there is a finite set a of ordinals such that if $\gamma \in \operatorname{dom}(\mathcal{U})$ and $\lambda = \operatorname{crit}(\mathcal{U}_\gamma)$ then

$$\forall \nu \in (\mathcal{C}_\gamma - \sup(a \cap \lambda)) \forall x \in f``(\nu \times \{\lambda\}) \, (\nu \in x \iff x \cap \lambda \in \mathcal{U}_\gamma).$$

The indiscernible sequences rising from the covering lemma have some additional structure:

4.16 Definition. If $\mathcal{C}$ is a system of indiscernibles for M, then $\mathcal{C}$ is said to be *h-coherent* if $h \in M$ is a function and the following conditions hold:

1. For all $\nu \in \bigcup_\gamma \mathcal{C}_\gamma$ there is a unique $\gamma \in h``\nu$ such that $\nu \in \mathcal{C}_\gamma$.

2. Suppose that $\nu \in \mathcal{C}_\gamma \cap \mathcal{C}_{\gamma'}$ and $\gamma \in h``\nu$. If $\gamma' \neq \gamma$ then $\operatorname{crit}(\mathcal{U}_{\gamma'}) < \operatorname{crit}(\mathcal{U}_\gamma)$, and $\operatorname{crit}(\mathcal{U}_{\gamma'}) \in \mathcal{C}_{\gamma''}$ for some $\gamma'' < \gamma$ with $\operatorname{crit}(\mathcal{U}_{\gamma''}) = \operatorname{crit}(\mathcal{U}_\gamma)$.

3. Suppose $\gamma_\nu = \operatorname{Coh}_{\gamma',\gamma}(\nu)$, where $\gamma' < \gamma$ with $\operatorname{crit}(\mathcal{U}_{\gamma'}) = \operatorname{crit}(\mathcal{U}_\gamma)$; and suppose that $\nu \in \mathcal{C}_\gamma$ with $\gamma' \in h``\nu$. Then $\mathcal{C}_{\gamma_\nu} = \mathcal{C}_{\gamma'} \cap (\nu - \nu')$, where $\nu'$ is least such that $\gamma \in h``\nu'$.

For a simple example, consider a set $C \subseteq \kappa$ which is Magidor generic over M, making $\operatorname{cf}(\kappa) = o^M(\kappa) = \lambda$ for some cardinal $\lambda < \kappa$. In this case we can take h to be the function such that $h(\beta)$ is the index of the βth full measure on κ, that is, such that $\mathcal{U}(\kappa, \beta) = \mathcal{U}_{h(\beta)}$ for all $\beta < o(\kappa)$. Then $\mathcal{C}_{h(\beta)} = \{\nu \in C : o(\nu) = \beta\}$. If we take C to be Radin generic, with $o(\kappa) < \kappa^+$, then we could define h so that $\mathcal{U}(\kappa, \sigma(\xi)) = \mathcal{U}_{h(\xi)}$, where σ is the canonical function taking κ onto $o(\kappa) < \kappa^+$. If C is Radin generic with $o(\kappa) = \kappa^+$, on the other hand, then there is no $h \in M$ and h-coherent system

$\mathcal{C}$ of indiscernibles such that $C = \bigcup_\gamma \mathcal{C}_\gamma$, for having such a system $\mathcal{C}$ would require that h maps κ onto κ^+.

The function $h``(x;\mathcal{C})$ provides a weak sense in which a covering set X is generated by a function $h \in K$ and a sequence $\mathcal{C}$ of indiscernibles:

4.17 Definition. Suppose that $\mathcal{C}$ is a system of indiscernibles and x is a set. Then $h``(x;\mathcal{C})$ is the smallest set $X$ such that $x \subseteq X$ and $X = h``(X \cup \bigcup_{\gamma \in X} \mathcal{C}_\gamma)$.

Definition 4.17 is too weak, since it does not provide any bounds on the size of the sets $\mathcal{C}_\gamma$. The functions defined below are used in clause 4 of Theorem 4.19 to describe a stronger sense in which X is generated by $\mathcal{C}$:

4.18 Definition. If $\mathcal{C}$ is a g-coherent system of indiscernibles and X is a set then we define

1. $s^{\mathcal{C}}(\gamma, \xi)$ is the least member of $\mathcal{C}_\gamma - (\xi + 1)$.
2. $s_*^{\mathcal{C}}(\gamma, \xi)$ is the least member of $\bigcup_{\gamma' \geq \gamma} \mathcal{C}_{\gamma'} - (\xi + 1)$.
3. If λ is measurable in K and $\gamma \leq \lambda^{++K}$ then an ordinal ξ is an *accumulation point* of $\mathcal{C}$ in X for γ if the ordinals γ, ξ are in X, and the set $\bigcup\{\mathcal{C}_{\gamma''} : \mathrm{crit}(\mathcal{U}_{\gamma''}) = \lambda \text{ and } \gamma'' \geq \gamma'\}$ is unbounded in $X \cap \xi$ for all $\gamma' < \gamma$ in $X \cap g``\nu$.
4. $a^{\mathcal{C},X}(\gamma, \xi)$ is the least accumulation point of $\mathcal{C}$ in X for γ above ξ.

This definition of an accumulation point does not seem to be entirely satisfactory, since it depends on the set X and the function g as well as on the system $\mathcal{C}$; however, clause 5 of Theorem 4.19 gives a sense in which the functions $s^{\mathcal{C}}$ and $a^{\mathcal{C},X}$ are, up to finite differences, independent of g, X and $\mathcal{C}$.

4.19 Theorem (Covering Lemma for Sequences of Measures). *Assume there is no inner model of $\exists\kappa\,(o(\kappa) = \kappa^{++})$. Let κ be a cardinal of the core model K, and let $\breve{\kappa}$ be a cardinal of K such that $\breve{\kappa} \geq \max\{\kappa, o(\kappa)\}$. Finally, let X be a set such that $\kappa \not\subseteq X = Y \cap K_{\breve{\kappa}}$ for some set Y such that $Y \prec_1 H(\breve{\kappa}^+)$ and ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$.*

Then there is an ordinal $\rho < \kappa$, a function $h \in K$, and a function $\mathcal{C}$ such that

1. *$\mathcal{C}$ is an h-coherent system of indiscernibles for K.*
2. *$\mathrm{dom}(\mathcal{C}) \subseteq X$ and $\bigcup_\gamma \mathcal{C}_\gamma \subseteq X$.*
3. *$X = h``(X \cap \rho; \mathcal{C})$, and hence $X \subseteq h``(\rho; \mathcal{C})$.*
4. *For all $\nu \in X \cap \kappa$, either $\nu \in h``(X \cap \nu)$ or else there is a γ such that $\nu \in \mathcal{C}_\gamma$. In the latter case there is $\xi \in X \cap \nu$ such that either*

    (a) $\nu = s^{\mathcal{C}}(\gamma, \xi) = s_*^{\mathcal{C}}(\gamma, \xi)$, or else

(b) $\nu = a^{\mathcal{C},X}(\gamma', \xi)$ *for some* $\gamma' > \gamma$ *in* $h``(X \cap \nu)$*.*

Furthermore, clause (a) holds if ν *is a limit point of* X*.*

5. *If* X' *is another set satisfying the hypothesis of the theorem, then there is a finite set* a *of ordinals such that for any* $\xi, \gamma \in X \cap X'$ *with* $a \cap \mathrm{crit}(\mathcal{U}_\gamma) \subseteq \xi$ *and* $\xi > \max\{\rho^X, \rho^{X'}\}$ *we have*

$$s^{\mathcal{C}}(\gamma, \xi) = s^{\mathcal{C}'}(\gamma, \xi)$$
$$s_*^{\mathcal{C}}(\gamma, \xi) = s_*^{\mathcal{C}'}(\gamma, \xi)$$
$$a^{\mathcal{C},X}(\gamma, \xi) = a^{\mathcal{C}',X'}(\gamma, \xi)$$

whenever either is defined.

To see that Theorem 4.19 implies the Dodd-Jensen covering lemma as a special case, notice that if $K = L[U]$ then $\mathcal{C}$ contains only a single set C of indiscernibles for the unique measure U. Then clause 4 asserts that $\mathrm{ot}(C) \leq \omega$, and clause 5 asserts that C is maximal.

4.20 Remark. As with the Dodd-Jensen covering lemma, the hypothesis ${}^{\mathrm{cf}(\kappa)}X \subseteq X$ can be weakened: If $\mathrm{cf}(\kappa) < \delta < \kappa$ and δ is the successor of a regular cardinal, then there is an unbounded class $\mathbf{C} \subseteq \mathcal{P}_\delta(K_{\breve{\kappa}})$ of sets X satisfying the conclusion of Theorem 4.19 such that if $\vec{X}$ is an increasing chain of members of $\mathbf{C}$ such that $\mathrm{cf}(\kappa) < \mathrm{cf}(\mathrm{len}(\vec{X})) < \kappa$ then $\bigcup \vec{X} \in \mathbf{C}$.

4.21 Remark. The assumption that ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ is used to ensure that the measures on κ generated by $\mathcal{C}$ are members of X. As was pointed out in observation 6 at the beginning of this section, this assumption can be weakened to ${}^{\omega}Y \subseteq Y$ if $o(\kappa) < \kappa^+$.

Similarly, if $o(\kappa) < \kappa^+$ then Remark 4.20 can be improved to state that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality.

4.22 Remark. If every measurable limit point of X is a member of X then the condition $\rho < \kappa$ can be strengthened to $\rho = \inf(\kappa - X)$, so that $X = h``(\rho; \mathcal{C})$. In particular, $\rho = \inf(\kappa - X)$ whenever $o(\alpha) < \inf(\kappa - X)$ for all $\alpha < \kappa$.

Introduction to the Proof

Before beginning to sketch the proof of the covering lemma we pause to look at three complications and digressions:

1. It was pointed out earlier that in order to ensure that $\mathrm{dom}(\mathcal{C}) \subseteq X$ we are assuming that $X \prec_1 K_{\breve{\kappa}}$, rather than $X \prec_1 K_\kappa$ as in the last section. This change, however, appears only in the very last step of the proof: until then we work only with $X \cap K_\kappa$ and use arguments which closely parallel those of the Dodd-Jensen covering lemma.

Similarly, this final step is the only place where the closure condition ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ is used: up until then countable closure, ${}^{\omega}Y \subseteq Y$, is all that is needed.

2. The proof we give is for the $\mathrm{cf}(\kappa)$-closed case, with ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$, as in the statement of Lemma 4.19. With one exception, the extension of the proof to the stronger result of Remark 4.20 is relatively straightforward, using the ideas outlined in the proof of the Dodd-Jensen covering lemma. The exception is Lemma 4.28, and we will digress from the main line of the proof to state Lemma 4.29, the analogue of Lemma 4.28 for the unclosed case, and to sketch its proof. The reader may, if desired, skip this digression.

3. As was explained in Sect. 4.1 an essential complication arises from the special role which the weak covering lemma plays in the definition of the core model. Beyond $0^\dagger$, the core model K is constructed in two stages: The first stage constructs the *countably complete* core model K^{c}, for which iterability is guaranteed (below the sharp for a class of strong cardinals) by the fact that every full measure of K^{c} is countably complete in V. After Lemma 4.5 is proved for K^{c}, the true model K is shown to be an elementary substructure of K^{c}, so that the iterability of K^{c} implies the iterability of K.

Part 1 of the Proof

Here we give the first part of the proof of the covering lemma for the true core model K. At the end of this subsection we will show how to adapt this proof to prove the Weak Covering Lemma 4.5 for K^{c}. As in the proof of the Dodd-Jensen covering lemma we begin with the following diagram:

$$\begin{tikzcd} M_0 \arrow[r, rightsquigarrow, "i"] & M_\theta \supseteq M \arrow[r, "\tilde{\pi}"] & \widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa) \\ & \overline{K} \arrow[u, hook] \arrow[r, "\pi"] & X \cap K_\kappa \prec_1 K_\kappa \arrow[u, hook] \end{tikzcd} \tag{18.18}$$

The construction of this diagram is identical to the construction for the Dodd-Jensen covering lemma: M_θ is obtained as the last model of the iterated ultrapower of $M_0 = K$ arising from the comparison of K with the transitive collapse $\overline{K}$ of $X \cap K_\kappa$; and $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa)$ where M is the largest initial segment of M_θ, and n is the largest integer, such that the ultrapower is defined.

The proof of the analogue of Lemma 3.51, which states that the construction of diagram (18.18) succeeds, is the same as for the Dodd-Jensen covering lemma except for two items. The first is clause 3.51(3):

4.23 Claim. *Either $\theta = 0$ and M is a proper initial segment of $M_0 = K$, or else 1 is in the set D of drops in the iteration on M_0. That is, either the iteration is trivial or it drops immediately.*

Proof. Set $\eta = \mathrm{crit}(\pi)$ and let ρ be least such that $\mathcal{P}^K(\rho) \not\subseteq \overline{K}$. As in the proof of Lemma 3.17 it will be sufficient to show that $\rho \leq \eta$ and that any ultrafilter U in $K - \overline{K}$ has critical point $\mathrm{crit}(U) \geq \rho$.

We will show the second half first: suppose to the contrary that $U \in K - \overline{K}$ and $\tau = \mathrm{crit}(U) < \min\{\eta, \rho\}$. Evidently $\eta > \tau^{+\overline{K}}$, since otherwise $\tau^{+K} > \tau^{+\overline{K}}$, which contradicts the assumption that $\mathcal{P}^K(\tau) \subseteq \overline{K}$. Then $\eta \geq \tau^{++\overline{K}}$, and since $K \models o(\tau) < \tau^{++}$ it follows that $o^{\overline{K}}(\tau) < \eta$. Thus $o(\tau) = \pi(o^{\overline{K}}(\tau)) = o^{\overline{K}}(\tau)$ and $o(\tau) \subseteq \mathrm{ran}(\pi)$, which implies that every measure on τ in K is in $\overline{K}$, contradicting the choice of U.

Now suppose that $\rho > \eta$, that is, that $\mathcal{P}^K(\eta) \subseteq \overline{K}$. Then the filter $U = \{x \subseteq \tau : \eta \in \pi(x)\}$ is a normal ultrafilter on $\mathcal{P}^K(\tau)$, and hence is a member of K. Now factor π into $\pi : \overline{K} \xrightarrow{i^U} \mathrm{Ult}(\overline{K}, U) \xrightarrow{k} K$ and apply the argument from the last paragraph to the map k to conclude that every ultrafilter on η in K is in $\mathrm{Ult}(\overline{K}, U)$. In particular $U \lhd U$, which is impossible since $\lhd$ is well-founded. $\dashv$

The second item to consider is clause 3.51(2):

4.24 Claim. *The model $\overline{K}$ is not moved in the comparison of $\overline{K}$ with K.*

Proof. Since the iterated ultrapower on K drops, that on $\overline{K}$ does not. Suppose for the sake of contradiction that the claim is false, and let ν be the least stage at which the iterated ultrapower on $\overline{K}$ is nontrivial. Thus $N_\nu = N_0 = \overline{K}$, and $N_{\nu+1} = \mathrm{Ult}(\overline{K}, \overline{U})$ for some measure $\overline{U} = \mathcal{U}^{\overline{K}}_\gamma$ which is not in the νth model M_ν of the iteration on K. If $\overline{U}$ is an ultrafilter on M_ν then set $\overline{M}_\nu = M_\nu$; otherwise let $\overline{M}_\nu$ be the largest initial segment of M_ν such that every set in $\overline{M}_\nu$ is measured by $\overline{U}$. In either case $\overline{U}$ is an $\overline{M}_\nu$-ultrafilter, and $\overline{M}_\nu$ is a mouse with projectum at most $\mathrm{crit}(\overline{U})$.

First we show that $\mathrm{Ult}_n(\overline{M}_\nu, \overline{U})$ is iterable, where n is largest for which the ultrapower is defined. To see this, let $\mu = \mathrm{crit}(\overline{U})$ and note that $\pi(\overline{U}) = \mathcal{U}_{\pi(\gamma)}$ is a measure on $\pi(\mu)$ in K, while $\widetilde{M}_\nu = \mathrm{Ult}_n(\overline{M}_\nu, \pi, \pi(\mu)+1)$ is an initial segment of K by the same argument as for $\widetilde{M} = \widetilde{M}_\theta$. Since $\mathcal{U}_{\pi(\gamma)}$ is a full measure in K it follows that $\mathrm{Ult}_n(\widetilde{M}_n, \mathcal{U}_{\pi(\gamma)})$ is iterable. Then $\mathrm{Ult}_n(M_\nu, \overline{U})$ must also be iterable, since it can be embedded into $\mathrm{Ult}_n(\widetilde{M}_n, \mathcal{U}_{\pi(\gamma)})$ and hence any witness to the contrary could be copied to a witness that $\mathrm{Ult}_n(\widetilde{M}_\nu, \mathcal{U}_{\pi(\gamma)})$ is not iterable.

Thus we can use iterated ultrapowers to compare the models M_ν and $\mathrm{Ult}_n(M_\nu, \overline{U})$. An argument like that for Lemma 3.39 shows that neither of the two iterated ultrapowers drops and that they have the same last model N, giving rise to the following diagram, where s and t are the embeddings of the two iterated ultrapowers.

$$\begin{tikzcd} M_\nu \arrow[rr, "s"] \arrow[dr, "i^U"'] & & N \\ & \mathrm{Ult}_n(M_\nu, \overline{U}) \arrow[ur, "t"'] & \end{tikzcd} \tag{18.19}$$

Furthermore diagram (18.19) commutes, since every member of M_ν can be written as $h^{M_\nu}_{n+1}(\xi)$ for some $\xi < \mu$, and both of the embeddings s and $ti^{\overline{U}}$ are the identity on μ and both embeddings map $h^{M_\nu}_{n+1}$ to h^{N}_{n+1} by Lemma 3.26. It follows that $\mathrm{crit}(s) = \mathrm{crit}(ti^{\overline{U}}) = \mu$, so that the ultrapower s on M_ν starts with an ultrapower using a measure $\mathcal{U}^{M_\nu}_\gamma$. Furthermore, for every set $x \subseteq \kappa$ in M_ν we have $x \in \overline{U} \iff \mu \in ti^{\overline{U}}(x) \iff \mu \in i(x) \iff x \in \mathcal{U}^{M_\nu}_\gamma$. Thus $U = \mathcal{U}^{M_\nu}_\gamma \in M_\nu$. ⊣

This completes part one of the proof of the covering lemma for K, and we are now ready to prove the Weak Covering Lemma 4.5 for K^{c}:

Proof of Lemma 4.5. The proof is similar to the proof of the weak covering lemma for K^{DJ}. Suppose to the contrary that λ is a singular cardinal with $\mu^\omega < \lambda$ for all $\mu < \lambda$, and that $\kappa = \lambda^{+K^{\mathrm{c}}} < \lambda^+$. Then $\mathrm{cf}(\kappa) < \lambda$, and hence there is a set $X \prec_1 K_\kappa$, cofinal in κ, such that $\lambda \not\subseteq X$, ${}^\omega X \subseteq X$ and if $\eta = \min(\lambda - X)$ then $\mathrm{cf}(\eta) = \omega_1$. The final condition is obtained by constructing X as the union of an increasing chain of sets of length ω_1.

Now apply the construction above of part one of the proof to the set X, using K^{c} for K. The constraint $\mathrm{cf}(\eta) = \omega_1$ is needed to ensure that the measure U of the first paragraph of the proof of Claim 4.23 would be in K^{c} if it existed.

Now, as in the proof of the weak covering lemma for K^{DJ}, the fact that $\kappa = (\lambda^+)^{K^{\mathrm{c}}}$ implies that the set of indiscernibles generated by the construction is bounded by $\lambda + 1$. It follows that $X = h^X``(X \cap \lambda)$, which is impossible since it would imply that $\mathrm{cf}^{K^{\mathrm{c}}}(\lambda^{+K^{\mathrm{c}}}) \leq \lambda$. This contradiction completes the proof of Lemma 4.5. ⊣

We now turn to the main subject of this section, the analysis of indiscernibles which will complete the proof of the full covering lemma.

Part 2 of the Proof: Analyzing the Indiscernibles

As in the proof of the Dodd-Jensen covering lemma, the $\widetilde{M} = \mathrm{Ult}_n(M, \pi, \kappa)$ of diagram (18.18) is a mouse in K. It follows that $\widetilde{M}$ is an initial segment of K; that is, $\widetilde{M} = J_{\tilde{\alpha}}[\mathcal{U}{\restriction}\tilde{\alpha}]$ for some ordinal $\tilde{\alpha} < \kappa^+$.

Still following the proof of the Dodd-Jensen covering lemma, let ν_0 be the largest member of the set D of drops, and let $\bar{\rho} < \bar{\kappa}$ be the Σ_n projectum of $M^*_{\nu_0+1}$ and hence of M_ν for every ordinal ν in the interval $\nu_0 < \nu \leq \theta$. The ordinal ρ required by Lemma 4.19 must satisfy

$$\rho \geq \sup(\pi``\bar{\rho}). \tag{A}$$

In the proof of the Dodd-Jensen covering lemma we could set $\rho = \sup(\pi``\bar{\rho})$, but in the present proof there are several other things which can go wrong, and each of these will determine a separate lower bound for ρ. Rather than specifying ρ at this point we will, at various points during the course of the proof, specify a series (A)–(E) of lower bounds on ρ. At any point in the proof we will assume that ρ is an ordinal less than κ which satisfies all the lower bounds specified up to that point.

Let $\overline{\mathcal{C}}$ be the system of indiscernibles on $\overline{K}$ given by the iteration of K, and define $\widetilde{\mathcal{C}}$ with $\mathrm{dom}(\widetilde{\mathcal{C}}) = \tilde{\pi}``\mathrm{dom}(\overline{\mathcal{C}})$ by setting $\widetilde{\mathcal{C}}_{\tilde{\pi}(\gamma)} = \pi``\overline{\mathcal{C}}_\gamma - \rho$ for each $\gamma \in \mathrm{dom}(\overline{\mathcal{C}})$. This is nearly the desired set of indiscernibles: it is an $\tilde{h}$-coherent system of indiscernibles at least for $\mathrm{ran}(\tilde{\pi})$, and $X \cap K_\kappa = K_\kappa \cap \pi``h(\bar{\rho} \cap X; \overline{\mathcal{C}}) = K_\kappa \cap \tilde{h}(X \cap \rho; \widetilde{\mathcal{C}})$ where $\tilde{h}$ is the Skolem function of $\widetilde{M}$.

In order to convert $\widetilde{\mathcal{C}}$ into a system of indiscernibles for K we will show that $\widetilde{\mathcal{C}}$ generates a sequence of normal ultrafilters $\mathcal{U}^*_\gamma$ on K such that $\mathrm{Ult}(K, U)$ is well-founded. It will follow that $\mathcal{U}^*_\gamma$ is equal to some full measure $\mathcal{U}_{\tau(\gamma)}$ in K, and will define a sequence $\mathcal{C}$ of indiscernibles for K by setting $\mathcal{C}_{\tau(\gamma)} = \mathcal{C}^*_\gamma$. Finally, in order to show that $\mathcal{C}$ is a sequence of indiscernibles, we will use the assumption ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ to show that the range of τ is contained in Y, and obtain the required function h by combining $\tilde{h}$ with a function g obtained by applying the covering lemma to $X \cap (\breve{\kappa} - \kappa)$.

The coherence function $\mathrm{Coh}^{\gamma'}_{\gamma,}$ was defined in Definition 4.14. Note that this definition does make sense even though the measures $\mathcal{U}_{\gamma'}$ and $\mathcal{U}_\gamma$ are partial in K, and are full measures only in $\widetilde{M}$.

4.25 Definition. Define the relation $\nu \in_\gamma x$, for $x \in K$ and γ an ordinal, as follows:

$$\nu \in_\gamma x \quad \Longleftrightarrow \quad \begin{cases} \nu \in x & \text{if } \nu \in \widetilde{\mathcal{C}}_\gamma \\ x \cap \nu \in \mathcal{U}_{\gamma''} & \text{if } \nu \in \mathcal{C}_{\gamma'} \text{ where } \gamma < \gamma' \text{ and } \gamma'' = \mathrm{Coh}_{\gamma,\gamma'}(\nu) \\ \text{undefined} & \text{otherwise} \end{cases}$$

4.26 Definition. If $\gamma \in \mathrm{dom}(\widetilde{\mathcal{C}})$ then define

$$\mathcal{C}^+_\gamma = \bigcup\{\widetilde{\mathcal{C}}_{\gamma'} : \gamma' \geq \gamma \ \ \& \ \ \mathrm{crit}(\widetilde{\mathcal{U}}_{\gamma'}) = \mathrm{crit}(\mathcal{U}_\gamma)\}.$$

If $\mathcal{C}^+_\gamma$ is cofinal in $\mathrm{crit}(\mathcal{U}_\gamma)$, then we write $\mathcal{U}^*_\gamma$ for the set of $x \in \mathcal{P}^K(\mathrm{crit}(\mathcal{U}_\gamma))$ such that $\nu \in_\gamma x$ for all sufficiently large $\nu \in \mathcal{C}^+_\gamma$.

In order to show that the filters $\mathcal{U}^*_\gamma$ are K-ultrafilters we use the idea of an *indiscernible sequence*:

4.27 Definition. A sequence $\vec{\alpha} = \langle \alpha_n : n < \omega \rangle$ is a *$\widetilde{\mathcal{C}}$-indiscernible sequence* for $\vec{\gamma} = \langle \gamma_n : n < \omega \rangle$ if $\vec{\alpha}$ is strictly increasing, $\alpha_n \in \widetilde{\mathcal{C}}_{\gamma_n}$ for all but finitely many $n < \omega$, and either (i) $\sup_n(\gamma_n) = \sup_n(\alpha_n)$, or (ii) $\mathrm{crit}(\mathcal{U}_{\gamma_n}) = \sup_{n\in\omega}(\alpha_n)$ for all $n < \omega$.

The following lemma corresponds to the argument that C^X is a Prikry sequence in the proof of the Dodd-Jensen covering lemma.

4.28 Lemma. *If $\vec{\alpha}$ is a $\widetilde{\mathcal{C}}$-indiscernible sequence for $\vec{\gamma}$ in $\widetilde{\mathcal{C}}$ then for any function $f \in K$ there is an $n_0 < \omega$ such that:*

1. *If* $n_0 \leq n < n' < \omega$ *and* $\operatorname{crit}(\mathcal{U}_{\gamma_{n'}}) < \gamma_m \leq \min\{\gamma_n, \gamma_{n'}\}$, *then for all* $\xi < \alpha_n$ *we have* $\alpha_n \in_{\gamma_m} f(\xi) \iff \alpha_{n'} \in_{\gamma_m} f(\xi)$.

2. *If* $n_0 < n$ *and* $\gamma_n < \kappa$ *then for all* $\xi < \alpha_n$ *we have* $\alpha_n \in f(\xi)$ *if and only if* $f(\xi) \cap \operatorname{crit}(\mathcal{U}_{\gamma_n}) \in \mathcal{U}_{\gamma_n}$.

Proof. Suppose that the lemma fails for some $\widetilde{\mathcal{C}}$-indiscernible sequence $\vec{\alpha}$ for $\vec{\gamma}$. The assertion that clause 2 fails uses parameters $\vec{\alpha}$ and $\{\gamma_n : \gamma_n < \kappa\}$, both of which are contained in X, and since ${}^{\omega}Y \subseteq Y$ it follows that both parameters are members of Y. By elementarity it follows that there is such a function f which is a member of Y. Then $f \in \operatorname{ran}(\tilde{\pi})$ by Proposition 3.63, so $\tilde{\pi}^{-1}(f)$ is in M and contradicts the fact that $\bar{\mathcal{C}}$ is a sequence of indiscernibles for M.

For clause 1, define $\gamma_{n',n} = \operatorname{Coh}_{\gamma_{n'},\gamma_n}(\alpha_n)$ whenever this is defined. Then the statement "$\alpha_n \in_{\gamma_m} f(\xi)$" is equivalent to the statement "Either $\gamma_n = \gamma_m$ and $\alpha_n \in f(\xi)$ or else $\gamma_{m,n}$ is defined and $f(\xi) \cap \alpha_n \in \mathcal{U}_{\gamma_{m,n}}$", so the statement that the lemma does not hold for $\vec{\alpha}$, $\vec{\gamma}$ and f can be stated using as parameters $\vec{\alpha}$, the ordinals $\gamma_{n',n}$, and $\{(n,n') \in \omega^2 : \gamma_n = \gamma_{n'}\}$. All of these are contained in X, so the same argument as in the last paragraph yields a contradiction. $\dashv$

Before using Lemma 4.28 to show that the sets $\mathcal{U}^*$ are K-ultrafilters, we digress to look at the analog of Lemma 4.28 for the case when X is not countably closed.

Digression for Non-Countably Closed Sets X

It was pointed out in the introduction to the proof of Theorem 4.19 that Lemma 4.28 is the one point in the proof where a new idea, beyond those presented in the proof of the Dodd-Jensen covering lemma, is needed in order to strengthen Theorem 4.19 as in Remark 4.20 by removing the assumption that $X = Y \cap K_{\breve{\kappa}}$ for some countably closed set Y. Lemma 4.29 below substitutes for Lemma 4.28 in this case. Lemma 4.29 and its proof may be skipped without affecting the proof of the covering lemma as stated in Theorem 4.19.

4.29 Lemma. *Suppose that* $\delta = \tau^+$ *where* τ *is a uncountable regular cardinal, and let* $\mathbf{C}$ *be the set of* $X \in \mathcal{P}_\delta(K_{\breve{\kappa}})$ *such that* $\mathcal{C}^X$ *satisfies Lemma 4.28. Then* $\mathbf{C}$ *is unbounded in* $X \in \mathcal{P}_\delta(K_{\breve{\kappa}})$ *and is closed under unions of increasing sequences of uncountable cofinality.*

Note that the requirement on δ is stronger than is needed for the corresponding results in the proof of the Dodd-Jensen covering lemma, for which δ could be any uncountable cardinal.

Sketch of Proof. The proof of the following analogue of Lemma 3.58 is direct:

4.30 Lemma. *If $X_0 \subseteq X_1$, and $\vec{\alpha}$ and $\vec{\gamma}$ are sequences with range contained in X_0 such that $\vec{\alpha}$ is a $\mathcal{C}^{X_1}$-indiscernible sequence for $\vec{\gamma}$, then $\vec{\alpha}$ is also a $\mathcal{C}^{X_0}$-indiscernible sequence for $\vec{\gamma}$.*

As in the Dodd-Jensen covering lemma, it easily follows that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Thus we only need to prove that $\mathbf{C}$ is unbounded.

Let S be the set of $\sigma \in \operatorname{Col}(\delta, K_\lambda)$ such that $\operatorname{cf}(\operatorname{dom}(\sigma)) = \tau$ and $\operatorname{ran}(\sigma)$ fails to satisfy Lemma 4.28. As in the Dodd-Jensen covering lemma, we will finish the proof of Lemma 4.29 by showing that S is nonstationary. Suppose toward a contradiction that S is stationary, and for each function $\sigma \in S$ let $\vec{\alpha}^\sigma$ and $\vec{\gamma}^\sigma$ be sequences which witness that Lemma 4.29 fails for $X^\sigma = \operatorname{ran}(\sigma)$; that is, $\vec{\alpha}^\sigma$ is a $\mathcal{C}^\sigma$-indiscernible sequence for $\vec{\gamma}^\sigma$, but $\vec{\alpha}$ and $\vec{\gamma}$ fail to satisfy one of clauses 1 or 2 of Lemma 4.28. Now continue following the proof of Lemma 3.60, which was the analog in the proof of the Dodd-Jensen covering lemma of Lemma 4.29: Let A^σ be the set

$$\bigl\{\vec{\alpha}^\sigma, \{\gamma^\sigma_n : \gamma^\sigma_n < \kappa\}, \{(n', n) : \gamma^\sigma_{n',n} \text{ is defined}\}, \{(n', n) : \gamma^\sigma_{n'} = \gamma^\sigma_n\}\bigr\}$$

of parameters used in the proof of Lemma 4.28, and find $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ so that $\sigma \supseteq \sigma_0$ and $A^\sigma \subseteq \operatorname{ran}(\sigma_0)$ for all $\sigma \in S_0$.

Recall that the key point in the proof of Lemma 3.60 was that, because $C^\sigma \subseteq^* C^{\sigma_0}$ for every $\sigma \in S_0$, each of the sets C^σ were determined (up to a finite set) by the subset $D^\sigma = C^{\sigma_0} - C^\sigma$ of C^{σ_0}. The key step in the current proof is to use Fodor's Lemma and the hypothesis that $\delta = \tau^+$ to find a set Z which fills the role of C^{σ_0}. Toward this end, choose an function $k : \tau \cong \operatorname{dom}(\sigma_0) < \tau^+$. Since $\operatorname{cf}(\tau) > \omega$ there is, for each $\sigma \in S$, an ordinal $\xi^\sigma < \tau$ such that $\bigcup A^\sigma \subseteq \sigma_0 k``\xi^\sigma$. By Fodor's Lemma there is a stationary subset $S_0' \subseteq S_0$ such that $\xi^\sigma$ is constant, say $\xi^\sigma = \xi$ for each $\sigma \in S_0'$. Set $Z = (\sigma_0 \circ k)``\xi$, so that $|Z| < \tau$ and $\bigcup A^\sigma \subseteq Z$ for every $\sigma \in S_0'$.

The rest of the proof parallels the proof of Lemma 3.60 for the Dodd-Jensen covering lemma. First define, for each $\sigma \in S_0'$, a set $w(\sigma)$ which witnesses that the restriction of $\widetilde{\mathcal{C}}$ to Z is as large as possible. This set is obtained by modifying Definition 3.18 as follows: Set the support β^σ of $w(\sigma)$ to be $\beta^\sigma = \max\{\sup(Z), \rho^\sigma + 1\} < \kappa$, and replace the requirement that $w(\sigma)$ be countable with the condition $|w(\sigma)| = |Z|$. Finally, modify clause 3 of Definition 3.18 to state that $\mathfrak{m}^\sigma = \operatorname{dir\,lim}(w(\sigma))$ is the Σ_n-code of a mouse of K, and there is a function $f = f^\sigma$ which is Σ_1-definable in $\mathfrak{m}^\sigma$ such that (i) for any $\alpha, \gamma \in Z$ such that $\gamma < \kappa$ and $\alpha \notin \widetilde{\mathcal{C}}^\sigma(\gamma)$, there is a set $x \in f``\alpha$ such that $x \in \mathcal{U}(\gamma)$ but $\alpha \notin x$, and (ii) for any $\alpha < \alpha'$ in $Z$ which are not members of the same set $\mathcal{C}^\sigma(\gamma)$, there is $x \in f``(\alpha \cap w(\sigma))$ such that $\alpha \in x$ and $\alpha' \notin x$.

Thus $w(\sigma)$ gives a complete description of the restriction of $\mathcal{C}^\sigma$ to Z.

Now, since $|w(\sigma)| = |Z| < \tau$ and $\operatorname{cf}(\operatorname{dom}(\sigma)) = \tau$ for every member σ of S_0', Lemma 3.23 implies that there is $\sigma_1 \in S_0'$ and a stationary set $S_1 \subseteq S_0'$ such that if $\sigma \in S_1$ then $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$. By shrinking S_1 further, if necessary, we can ensure that $\beta^\sigma = \beta$ is constant for $\sigma \in S_1$.

Now since S_1 is unbounded and the sequences $\vec{\alpha}^{\sigma_1}$ and $\vec{\gamma}^{\sigma_1}$ do not satisfy the conclusion of Lemma 4.28, there is some $\sigma \in S_1$ with a function $f \in \operatorname{ran}(\sigma)$ which witnesses this failure. It follows that $\vec{\alpha}^{\sigma_1}$ is not a $\widetilde{\mathcal{C}}^{\sigma}$-indiscernible sequence, and by the definition of $w(\sigma)$ it follows that there is a function f' which is Σ_1-definable in $\operatorname{dir lim}(w(\sigma))$ which witnesses this failure. Now $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$, and it follows that $\operatorname{dir lim}(w(\sigma)) \subseteq \widetilde{M}^{\sigma_1}$. To see this, notice that $\bar{\mathfrak{m}} = \operatorname{dir lim}((\pi^{\sigma_1})^{-1} w(\sigma)) \subseteq M$, since every subset of $\bar{\rho} = (\pi^{\sigma_1})^{-1}(\rho^{\sigma_1})$ in $\mathfrak{m}$ is a member of M and there is a subset of $\bar{\rho}$ definable in M which is not a member of M.

Thus $\vec{\alpha}^{\sigma_1}$ is not a $\widetilde{\mathcal{C}}^{\sigma_1}$-indiscernible sequence for $\vec{\gamma}^{\sigma_1}$. This contradicts the choice of $\vec{\alpha}^{\sigma_1}$ and $\vec{\gamma}^{\sigma_1}$, and hence completes the proof of Lemma 4.29. ⊣

Continuation of the Main Proof

This completes the digression for non-countably closed covering sets X, and we now return to the basic proof of the covering lemma.

4.31 Lemma. *Suppose that $\mathcal{C}^+(\gamma)$ is cofinal in $\alpha = \operatorname{crit}(\mathcal{U}_\gamma)$. Then $\mathcal{U}^*_\gamma$ is a normal ultrafilter on K, and $\operatorname{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. Hence $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$ in K for some ordinal $\tau(\gamma)$.*

Furthermore, for any function $f \in K$ there is an $\eta < \alpha$ such that

$$\begin{aligned} &\forall \nu, \gamma \big(\eta < \nu < \alpha < \gamma \ \ \& \ \ \nu \in \mathcal{C}^+_\gamma \\ &\qquad \implies \ \forall \xi < \nu \,(\nu \in_\gamma f(\xi) \iff f(\xi) \in \mathcal{U}^*_\gamma)\big). \end{aligned} \tag{18.20}$$

We break up the proof of Lemma 4.31 into two parts, depending on the cofinality of α.

4.32 Lemma. *The conclusion of Lemma 4.31 holds whenever $\operatorname{cf}(\alpha) = \omega$. Furthermore if $\alpha < \kappa$ then $\tau(\gamma) = \gamma$, and if $\alpha = \kappa$ then $\tau(\gamma) \in Y$.*

Proof. In this case everything except the existence of $\tau(\gamma)$ follows immediately from Lemma 4.28, as any counterexample could be witnessed by a $\widetilde{\mathcal{C}}$-indiscernible sequence. The assertion that $\tau(\gamma) = \gamma$ if $\alpha < \kappa$ follows from clause 2 of that lemma, and the assertion that $\tau(\gamma) \in Y$ if $\gamma > \kappa$ follows from its proof.

The existence of $\tau(\gamma)$ follows from Corollary 4.13, which states that $\mathcal{U}^*_\gamma$ is equal to some full measure $\mathcal{U}_{\gamma'}$ on the K-sequence provided that $\operatorname{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. If it is not well-founded then there are functions $f_n \in K \cap H(\alpha^+)$ such that $[f_{n+1}]_{\mathcal{U}^*_\gamma} < [f_n]_{\mathcal{U}^*_\gamma}$ for each $n < \omega$. As in the proof of Lemma 4.28 we can assert this condition on the functions f_n by a statement in Y, and by elementarity there must be such a sequence in Y. This is impossible, as it would imply that $\operatorname{Ult}(M, \tilde{\pi}^{-1}(\mathcal{U}_\gamma))$ is ill-founded, but $\tilde{\pi}^{-1}(\mathcal{U}_\gamma) = \mathcal{U}^M_{\pi^{-1}(\gamma)} \in M$, and hence $\operatorname{Ult}(M, \tilde{\pi}^{-1}(\mathcal{U}_\gamma))$ must be well-founded since M is an iterable model obtained by an iteration on K. ⊣

Before proving Lemma 4.31 when $\operatorname{cf}(\alpha) > \omega$ we need to make the following important observation:

4.33 Lemma. *Suppose $\vec{\alpha}$ is an increasing sequence with $\alpha_n \in \widetilde{\mathcal{C}}_{\gamma_n}$ for each $n < \omega$, and that* $\operatorname{crit}(\mathcal{U}_{\gamma_n}) = \alpha$ *for all $n < \omega$. If $\alpha' = \lim_n \alpha_n < \alpha$ then $\alpha' \in \widetilde{\mathcal{C}}_{\gamma'}$ for some $\gamma' \geq \limsup\{\gamma_n + 1 : n < \omega\}$.*

Proof. We can assume that $\gamma_n < \limsup_{m<\omega}(\gamma_m + 1)$ for all $n < \omega$. We want to use Lemma 4.32, using α' for α. This can be done by using $X \cap K_{\breve{\alpha}'}$, which is a suitable set for the covering lemma at α'. The iteration used in the construction of diagram (18.18) for $X \cap K_{\breve{\alpha}'}$ is an initial segment of that for X: let θ' be the least ordinal such that $\operatorname{crit}(i_{\theta',\theta}) > \pi^{-1}(\alpha')$ where the embeddings $i_{\xi',\xi}$ come from the iteration of K with last model M_θ. The first θ' stages of this iteration are exactly those which are used in the proof of the covering lemma for α', using the suitable set $X \cap K_{\breve{\alpha}'}$. By Lemma 4.32 it follows that this sequence generates measures $\mathcal{U}_{\gamma'}$, with critical point α', on the K-sequence; and furthermore $\mathcal{U}_{\gamma'} \in X$ since it is generated by countable sequences contained in X. Now the embedding $i_{\theta',\theta'+1}$ comes from an ultrapower of $M_{\theta'}$ using a measure $\mathcal{U}_{\bar{\gamma}'}^{M'_\theta}$ larger than all of those in N. Thus $\bar{\gamma}' > \pi^{-1}(\gamma'_n)$ for each $n < \omega$. But $\gamma_n = \tilde{\pi} i_{\theta',\theta}(\pi^{-1}(\gamma'_n))$ and $\alpha' \in \widetilde{\mathcal{C}}_{\gamma'}$ where $\gamma' = \tilde{\pi} i_{\theta',\theta}(\bar{\gamma}')$. Thus $\gamma' > \gamma_n$ for each $n < \omega$. ⊣

Proof of 4.31 for $\operatorname{cf}(\alpha) > \omega$. Suppose that $\operatorname{cf}(\alpha) > \omega$. We will first prove that for any function $f \in K$ there is an η satisfying (18.20).

Suppose to the contrary that f is a function for which no η exists as required. Define sequences ξ_n, ν_n and γ_n so that $\vec{\gamma}$ is nondecreasing, $\xi_n < \nu_n \in \mathcal{C}^+_{\gamma_n}$, and $\nu_n \in_{\gamma_n} f(\xi_n) \not\Longleftrightarrow f(\xi_n) \in \mathcal{U}^*_{\gamma_n}$ but for all $\nu \in \mathcal{C}^+_{\gamma_n} - \nu_{n+1}$ we have $\nu \in_{\gamma_n} f(\xi_n) \Longleftrightarrow f(\xi_n) \in \mathcal{U}^*_{\gamma_n}$.

Now set $\alpha' = \sup_n(\nu_n)$. By Lemma 4.33 $\alpha' \in \mathcal{C}_{\gamma'}$ for some $\gamma' \geq \sup_n(\gamma_n + 1)$, and if we set $\gamma'_n = \operatorname{Coh}_{\gamma_n,\gamma'}(\nu_n)$ then $\vec{\nu}$ is a $\widetilde{\mathcal{C}}$-indiscernible sequence for $\vec{\gamma}'$. Hence the lemma fails at α', contradicting Lemma 4.32. A similar argument shows that $\mathcal{U}^*_\gamma$ is normal.

Finally $\mathcal{U}^*_\gamma$ is countably complete when $\operatorname{cf}(\alpha) > \omega$, so $\operatorname{Ult}(K, \mathcal{U}^*_\gamma)$ is well-founded. Hence Corollary 4.13 implies that $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$ for some ordinal $\tau(\gamma)$. ⊣

We are now ready to specify the second and third of the lower bounds on ρ:

$$\rho > \sup\{\gamma \in \operatorname{dom}(\mathcal{U}^*) \cap \kappa : \mathcal{U}_\gamma \neq \mathcal{U}^*_\gamma\}. \tag{B}$$

Condition (B) holds for all sufficiently large $\rho < \kappa$ since Lemma 4.28 implies that $\{\operatorname{crit}(\mathcal{U}^*_\gamma) : \mathcal{U}_\gamma \neq \mathcal{U}^*_\gamma\}$ is finite.

$$\rho > \sup \bigcup\{\widetilde{\mathcal{C}}_\gamma : \kappa < \gamma \ \& \ \mathcal{U}^*_\gamma \text{ is not defined}\}. \tag{C}$$

To see that the right-hand side of condition (C) is smaller than κ, suppose to the contrary that there is a cofinal set $C \subseteq \kappa$ such that for each $\nu \in C$ there is $\gamma_\nu > \kappa$ such that $\nu \in \widetilde{\mathcal{C}}_{\gamma_\nu}$ and $\mathcal{U}^*_{\gamma_\nu}$ is not defined. By taking a subsequence if necessary, we can assume that $\langle \gamma_\nu : \nu \in C \rangle$ is nondecreasing, but this implies that $\mathcal{U}^*_{\gamma_\nu}$ is defined for each $\nu \in C$.

Conditions (B) and (C) enable us to complete the definition of $\mathcal{C}$:

4.34 Definition. If $\gamma \in \operatorname{dom}\widetilde{\mathcal{C}}$ then let $\tau(\gamma)$ be the ordinal such that $\mathcal{U}^*_\gamma = \mathcal{U}_{\tau(\gamma)}$.

Define $\mathcal{C}$ by setting $\mathcal{C}_{\tau(\gamma)} = \widetilde{\mathcal{C}}_\gamma$ for all γ such that $\tau(\sigma)$ is defined.

Condition (A) ensures that $\tau(\gamma) = \gamma$ for all $\gamma < \kappa$. We now make our single use of the assumption that X is $\mathrm{cf}(\kappa)$-closed, that is, that $X = Y \cap K_{\breve\kappa}$ for some $Y \prec H(\lambda)$ with ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$

4.35 Claim. *If $\mathcal{U}^*_\gamma$ is defined then $\mathcal{U}^*_\gamma \in X$, and hence $\tau(\beta) \in X$.*

Proof. As in the proof of Lemma 4.28, the filter $\mathcal{U}^*_\gamma$ is generated in X by any cofinal subsequence of $\mathcal{C}^+_\gamma$. Since ${}^{\mathrm{cf}(\kappa)}\kappa \subseteq Y$, there is such a subsequence in Y. ⊣

4.36 Remark. If $o(\kappa) < \kappa^{+K}$ then the assumption that X is $\mathrm{cf}(\kappa)$-closed is unnecessary, for in that case there is a partition of κ into disjoint sets $\langle A_\beta : \beta < o(\kappa)\rangle$ such that $A_\beta \in \mathcal{U}(\kappa,\beta)$ for each $\beta < o(\kappa)$. If $\vec{A} \in Y$ then there is an $\eta < \kappa$ so that $\mathcal{C}(\kappa,\beta) - A_\beta \subseteq \eta$ for all $\beta < o(\kappa)$. If $\nu \in \mathcal{C}(\kappa,\beta) - \eta$ then β is definable from $\vec{A}$ as the unique ordinal β such that $\nu \in A_\beta$, so Claim 4.35 holds for all $X = Y \cap K_{\breve\kappa}$ with $\vec{A} \in Y$.

It is easy to see from the construction that $\mathcal{C}$ is a sequence of indiscernibles for K. Thus $\mathcal{C}$ satisfies clauses 1 and 2 of Theorem 4.19.

4.37 Claim. *There is a function $g \in K$ such that $X \subseteq g``(X \cap \kappa)$.*

Proof. Apply the proof of the covering lemma to the full set $X \prec_1 K_{\breve\kappa}$ (rather than to $X \cap K_\kappa$). Notice that, as in the proof of Lemma 4.5, there are no measurable cardinals in the interval $(\kappa, \breve\kappa]$ and hence any indiscernibles which come up in the construction must be smaller than κ. It follows, just as in the proof of the covering lemma for L, that there is a function $g \in K$ such that $X = g``(X \cap \kappa)$. ⊣

We now put the fourth lower bound on ρ:

$$\rho > \sup\bigl\{\nu : \exists\beta\,(\nu \in \mathcal{C}(\kappa,\beta) \wedge \beta \notin g``(X \cap \nu))\bigr\}. \tag{D}$$

The following claim justifies this bound:

4.38 Claim. *There is an ordinal $\eta < \kappa$ such that $\gamma \in g``(X \cap \nu)$ whenever $\gamma > \kappa$ and $\eta < \nu \in \mathcal{C}_\gamma$*

Proof. Define, in K, a disjoint sequence of sets $\langle A_\gamma : \gamma \in \operatorname{ran}(g)\rangle$ such that $A_\gamma \in \mathcal{U}_\gamma$ whenever $\gamma \in \operatorname{ran}(g)$ and $\mathcal{U}_\gamma$ is a full ultrafilter on κ in K. By Lemma 4.31 there is an $\eta < \kappa$ so that for all $\gamma \in \operatorname{dom}(\mathcal{C}) - \kappa$ and all $\nu \in \mathcal{C}_\gamma - \eta$ and $\xi < \nu$ we have $\nu \in A_{g(\xi)} \iff A_{g(\xi)} \in \mathcal{U}_\gamma$. Since the diagonal union $B = \{\nu < \kappa : \exists\xi < \nu \;\; \nu \in A_{g(\xi)}\}$ is a member of each measure $\mathcal{U}_{g(\xi)}$, we can also assume $\bigcup\{\mathcal{C}_\gamma - \eta : \gamma \in \operatorname{ran}(g)\} \subseteq B$. It follows that this choice of η will satisfy the statement of the lemma. ⊣

This completes the proof of the first three clauses of Theorem 4.19. For the rest of the proof of the theorem it will be convenient to use the notation $\mathcal{U}(\alpha, \beta)$, which explicitly names the critical point of the measure, rather than the notation $\mathcal{U}_\gamma$. In doing so we will consistently adjust the notation described earlier by replacing γ with the pair (α, β): for example, we will write $s^{\mathcal{C}}(\alpha, \beta, \xi)$ instead of $s^{\mathcal{C}}(\gamma, \xi)$, we will say that $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ instead of $\vec{\gamma}$, and we will write $\mathrm{Coh}_{\alpha,\beta,\beta'}$ for the coherence function relating $\mathcal{U}(\alpha, \beta)$ and $\mathcal{U}(\alpha, \beta')$.

It will also be useful to have a notion of an indiscernible sequence which, like that of a Prikry sequence, depends directly on the sequence $\mathcal{U}$ of measures rather than on a system of indiscernibles.

4.39 Definition. We say that $\vec{\nu}$ is an *indiscernible sequence* for $(\vec{\alpha}, \vec{\beta})$ if (i) $\vec{\nu}$ is a strictly increasing sequence of ordinals of length ω, (ii) either $\sup_n(\nu_n) = \sup_n(\alpha_n)$ or else $\alpha_n = \sup_n(\nu_n)$ for all n, and (iii) for any function $f \in K$ there is $n_0 < \omega$ such that $\forall n > n_0 \forall \xi < \alpha_n\, (\nu_n \in f(\xi) \iff f(\xi) \cap \alpha_n \in \mathcal{U}_{\alpha_n, \beta_n})$.

Notice that Lemma 4.28 implies that any $\widetilde{\mathcal{C}}$-indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta}')$ where $\beta'_n = \tau(\beta_n)$.

In the rest of this proof we will say $\vec{\nu} <^* \vec{\nu}'$ to mean that $\nu_n < \nu'_n$ for all but finitely many $n < \omega$; and we will use $>^*$, $\leq^*$ and $\geq^*$ similarly.

We will first prove clause 4 of Theorem 4.19 in the case when $\mathrm{cf}(\nu) = \omega$. Suppose that $\nu \in \mathcal{C}_{\alpha,\beta}$; we want to show that there is an $\xi < \nu$ so that $\nu = s(\alpha, \beta, \xi) = s_*(\alpha, \beta, \xi)$. If this is not so then there is a cofinal sequence of ordinals $\nu_n \in \mathcal{C}_{\alpha,\beta_n}$ with $\beta_n \geq \beta$, and this contradicts Lemma 4.33 which implies that $\beta \geq \limsup_n(\beta_n + 1)$.

Now let $\nu \in \mathcal{C}_{\alpha,\beta_0}$ be arbitrary and let β_1 be the largest ordinal such that ν is an accumulation point in X for (α, β_1). Then $\bigcup_{\beta_1 \leq \beta < o(\alpha)} \mathcal{C}(\alpha, \beta)$ is bounded in $\nu \cap X$, say by $\xi \in X \cap \nu$. If $\beta_0 \geq \beta_1$ then $\nu = s(\alpha, \beta_0, \xi) = s_*(\alpha, \beta_0, \xi)$, so we can suppose that $\beta_0 < \beta_1$. We claim that there are only finitely many accumulation points in X for (α, β_1) in the interval (ξ, ν), so that $\nu = a(\alpha, \beta_1, \xi')$ for some ξ' in $[\xi, \nu) \cap X$. If, to the contrary, there are infinitely many such accumulation points, then let ν' be the least member of the interval $(\xi, \nu]$ which is a limit of accumulation points for (α, β_1). Then $\mathrm{cf}(\nu') = \omega$ and it follows from the last paragraph that $\nu' \in \mathcal{C}(\alpha, \beta')$ for some $\beta' \geq \beta_1$, contradicting the choice of ξ. This contradiction completes the proof of clause 4, except for the last sentence which states that $\nu = s(\alpha, \beta_0, \xi_0) = s_*(\alpha, \beta_0, \xi_0)$ whenever ν is a limit point of X. We will defer the proof of this for the case $\mathrm{cf}(\nu) > \omega$ until after the proof of clause 5, on which its proof depends.

Notice that any increasing ω-sequence $\vec{\nu}$ of indiscernibles from $\mathcal{C}$ is an indiscernible sequence for some $\vec{\alpha}, \vec{\beta}$. To see this, suppose that $\nu_n \in \mathcal{C}(\alpha_n, \beta_n)$, with $\alpha_n, \beta_n \in g``\nu_n$. If $\alpha_n \leq \sup_n(\nu_n)$ for each n, then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$. Otherwise $\alpha_n = \alpha$ is constant for sufficiently large n with $\alpha_n > \alpha' = \sup_n(\nu_n)$, and Lemma 4.33 implies that $\alpha' \in \mathcal{C}(\alpha, \beta)$ for some $\beta < o(\alpha)$ such that $\beta_n > \beta$ for all sufficiently large $n < \omega$. Then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}', \vec{\beta}')$ where $\alpha'_n = \alpha'$ and $\beta'_n = \mathrm{Coh}_{\alpha,\beta_n,\beta}(ga')$ for all n such that $\alpha_n = \alpha$.

The proof of clause 5 relies on Lemma 4.28(2), which implies for any sequences $\vec{\nu}, (\vec{\alpha}, \vec{\beta}) \in X$ that $\vec{\nu}$ is a $\mathcal{C}^X$-indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ if and only if it is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in the sense of Definition 4.39; and similarly for X'. Suppose that clause 5 is false for the function $s^{\mathcal{C}}$. Then we can assume, without loss of generality, that there are infinite sequences $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$ in $X \cap X'$ such that for each $n < \omega$ we have (i) $\nu'_n = s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$ exists, (ii) $\xi_{n+1} \geq \nu'_n$, and (iii) $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ either does not exist or is strictly larger than $s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$. Then $\vec{\nu}'$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$, and since $Y \prec H(\lambda)$ it follows that Y satisfies that there is an indiscernible sequence $\vec{\nu}$ for $(\vec{\alpha}, \vec{\beta})$ such that $\nu_n > \xi_n$ for all n. Thus, by Lemma 4.28, $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ exists for all but finitely many $n < \omega$, so we can set $\nu_n = s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$. By the choice of $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$, we must have $\vec{\nu} >^* \vec{\nu}'$, but then again Y satisfies that there is an indiscernible sequence ν'' for $(\vec{\alpha}, \vec{\gamma})$ such that $\vec{\xi} <^* \vec{\nu}'' <^* \vec{\nu}$, so that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in $\mathcal{C}$, which contradicts the choice of $\vec{\nu}$.

The proof of clause 5 for the function $s^{\mathcal{C}}_*$ is similar, except that $\vec{\nu}'$ is an indiscernible sequence for some $(\vec{\alpha}, \vec{\beta}')$ with $\vec{\beta}' \geq^* \vec{\beta}$, instead of for $(\vec{\alpha}, \vec{\beta})$ itself.

The proof that clause 5 holds for the function $a^{\mathcal{C}}$ is similar but slightly more complicated. We say that $\vec{\nu}$ is an *accumulation point sequence* for $(\vec{\alpha}, \vec{\gamma})$ if for all sequences $\vec{\gamma}' <^* \vec{\gamma}$ and $\vec{\nu}' <^* \vec{\nu}$ there are sequences $\vec{\nu}''$ and $\vec{\beta}''$ with $\vec{\nu}' <^* \vec{\nu}'' <^* \vec{\nu}$ and $\vec{\beta}' \leq \vec{\beta}''$ such that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta}'')$. By using the elementarity of Y and the fact that being an indiscernible sequence is absolute between Y and V, it follows that being an accumulation point sequence is also absolute between Y and V. The rest of clause 5 follows as for the functions $s^{\mathcal{C}}$ and $s^{\mathcal{C}}_*$.

This completes the proof of clause 5, and we now return to the proof of the final sentence of clause 4, which states that if ν is any limit point of X such that $\nu \in \mathcal{C}(\alpha, \beta)$ for some $\beta < o(\alpha)$ then $\nu = s(\alpha, \beta, \xi)$ for some $\xi \in X \cap \nu$. Let Z be the set of ordinals $\nu \in X \cap \lim(X)$ such that $\nu \notin h``\nu$ and there is no α, β and ξ in X such that $\nu = s(\alpha, \beta, \xi) = s_*(\alpha, \beta, \xi)$.

We specify the last lower bound on ρ:

$$\rho \geq \sup(Z). \tag{E}$$

This is justified by the following claim:

4.40 Claim. *The set Z is finite.*

Sketch of Proof. Suppose to the contrary that $\vec{\nu}$ is an increasing ω-sequence of members of Z. Then $\alpha = \sup_n(\nu_n) \notin Z$ since clause 4 holds for ordinals of cofinality ω, so $\vec{\nu}$ is an indiscernible sequence in $\mathcal{C}$ for some pair $(\vec{\alpha}, \vec{\beta})$ with

$\alpha_n \leq \alpha$. Since $\nu_n \in Z$, $\nu_n = a^{\mathcal{C}}(\alpha_n, \beta'_n, \xi_n)$ for some β'_n with $\beta_n < \beta'_n \leq o(\alpha_n)$ and $\xi_n \in X \cap \nu_n$.

Now proceed as in the proof of Lemma 4.33 for each of the ordinals ν_n. Let $C_n = \bigcup\{\mathcal{C}(\alpha_n, \beta) : \beta_n \leq \beta < \beta'_n\}$ and for each $\nu \in C_n$ let $\beta_{n,\nu}$ be the ordinal β such that $\nu \in \mathcal{C}(\alpha_n, \beta)$. Set $\vec{\beta}_n = \langle \beta_{n,\nu} : \nu \in C_n \rangle$. If the sequences $\vec{C}_n$ and $\vec{\beta}_n$ are in X then we can use the argument of Lemma 4.33 to conclude that $\beta_n \geq \sup_{\nu \in C_n} \beta_{n,\nu}$, contrary to assumption.

To deal with the general case, pick a set $X' = Y' \cap K_{\breve{\kappa}}$ as in the hypothesis of the Covering Lemma 4.19 so that $C_n \in Y'$ and $\vec{\beta}_n \in Y'$ for each $n < \omega$. Then the argument in the last paragraph shows that it cannot be true that $\nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and at the same time $\nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for unboundedly many $\nu \in C_n$. But by clause 5, for sufficiently large $n < \omega$ we have $\nu_n \in \mathcal{C}^{X}(\alpha_n, \beta_n) \implies \nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and $\nu \in \mathcal{C}^{X}(\alpha_n, \beta_{n,\nu}) \implies \nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for all $\nu \in C_n$. This contradiction completes the proof of Claim 4.40. $\dashv$

This completes the proof of the last sentence of clause 4, which is the end of the proof of Theorem 4.19, the covering lemma for sequences of measures.

4.3. The Singular Cardinal Hypothesis

We will now use Theorem 4.19 to establish the lower bound for the strength of a failure of the Singular Cardinal Hypothesis:

4.41 Theorem (Gitik [17]). *If there is a singular cardinal κ with $2^\kappa > \max\{\kappa^+, 2^{\mathrm{cf}(\kappa)}\}$ then there is a cardinal κ with $o(\kappa) \geq \kappa^{++}$ in K.*

The proof combines the use of the covering lemma with two theorems from Shelah's pcf theory. The first can be found as Conclusion 5.10(2) on page 410 of [53].

4.42 Theorem. *If κ is the least cardinal satisfying $\kappa^{\mathrm{cf}(\kappa)} > \kappa^+ + 2^{\mathrm{cf}(\kappa)}$ then $\mathrm{pp}(\kappa) \geq \kappa^{++}$, $\mathrm{cf}(\kappa) = \omega$, and $\forall \mu < \kappa\, \mu^\omega \leq \max\{\mu^+, 2^\omega\}$.*

We will assume that $o^K(\kappa) < \kappa^{++}$, where κ is given by the conclusion of Theorem 4.42, and derive a contradiction. Note that the conclusion implies that $\kappa > 2^\omega$ and $\mu^\omega = \mu$ for each cardinal μ of uncountable cofinality in the interval $2^\omega \leq \mu < \kappa$.

The statement that $\mathrm{pp}(\kappa) \geq \kappa^{++}$ implies that there is a sequence $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$ of regular cardinals smaller than κ, together with a sequence $\vec{f} = \langle f_\alpha : \alpha < \kappa^{++} \rangle$ of functions in $\prod \vec{\kappa}$ which is $<^*$-increasing and $<^*$-cofinal in $\prod \vec{\kappa}$. We will call such a sequence a *scale* and will use it to derive the contradiction. The first part of the proof will use the covering lemma to obtain from the given scale a scale in which each of the functions f_α is what Gitik calls a *diagonal sequence*. The exact meaning of this term will be given in Lemma 4.44 after some notation has been established, but a typical example, requiring $o(\kappa_{n+1}) \geq \kappa_n$ for each n, would be a sequence $f \in \prod \vec{\kappa}$

such that $f(n+1) = s_*(\kappa_{n+1}, f(n), \kappa_n)$ for each $n < \omega$. This construction requires separate covering sets for each sequence f_α, and relies heavily on the fact that any two such covering sets agree (on their common domain) for all but finitely many κ_n. The final contradiction, however, requires finding an appropriate collection of covering sets which agree for some particular fixed κ_n, and for this a second result of Shelah will be needed. A proof is in Jech [25, Lemma 24.10].

4.43 Lemma. *If $\langle f_\alpha : \alpha < \kappa^{++} \rangle$ is a scale, then for each $\alpha < \kappa^{++}$ with* $\mathrm{cf}(\alpha) = \kappa^+$ *there is an* exact upper bound *(eub) of $\langle f_{\alpha'} : \alpha' < \alpha \rangle$; that is, a function $g \in \prod \vec{\kappa}$ such that $f_{\alpha'} <^* g$ for all $\alpha' < \alpha$, and for any function $g' <^* g$ there is an $\alpha' < \alpha$ such that $g' <^* f_{\alpha'}$.*

In what follows we say that a set X is a *covering set* if it satisfies the hypothesis of the covering lemma, Theorem 4.19. All covering sets have cardinality 2^ω unless stated otherwise. We will be using a number of different covering sets, and will heavily use the next indiscernible function $s^X_*(\kappa, \beta, \xi)$ and next accumulation point function $a^X(\gamma, \beta, \xi)$ from that lemma. These functions depend on the choice of covering set X, but by clause 5 of Theorem 4.19 there is, for any two covering sets X and X', an $n_0 < \omega$ such that the functions defined using the two sets agree (whenever the arguments are in both sets) above κ_{n_0}. Keeping this in mind, we will normally simplify the notation by omitting the superscripts X. In addition we will use a standard, fixed covering set X for many of our calculations, but we will want this set to include a number of objects which are not defined until later in the course of the proof. To see that we can do so without loss of generality, note that if some desired object is not a member of X then we can choose a new, larger covering set X' which does include it. If we were to redo the proof up to this point using X' instead of X then there is some $n < \omega$ such that the X agrees with X' about indiscernibles above κ_n, and hence about everything defined in the proof so far which lies above κ_n. In this case we can throw out a finite initial segment $\vec{\kappa} \restriction n+1$ of the sequence $\vec{\kappa}$. By restricting the functions f_α in the original scale to this reduced sequence we obtain a scale for which X and X' agree. This will cause no problems so long as it occurs only finitely often.

We begin by assuming that $\{\vec{\kappa}, \vec{f}\} \subseteq X$. Set $\kappa'_n = \min(h^X``\kappa_n)$, so $\kappa_n \leq \kappa'_n \leq \kappa$. If $\kappa'_n > \kappa_n$ then κ_n will be an indiscernible for κ'_n. Let $\beta_n \leq o(\kappa_n)$ be the largest ordinal such that κ_n is an accumulation point for (κ'_n, β_n) in $\mathcal{C}^X$, noting that Definition 4.18 of an accumulation point makes perfectly good sense even if $\kappa'_n = \kappa_n$. Pick $g^*(n) < \kappa_n$ in X large enough that $\kappa'_n \in h^X``g^*(n)$, $\bigcup_{\beta \geq \beta_n} \mathcal{C}_{\kappa'_n, \beta} \subset g^*(n)$, and $\mathcal{C}^X$ has no accumulation points for (κ'_n, β_n) in $\kappa_n - g^*(n)$. The latter is possible because it follows from $\mathrm{cf}(\kappa_n) > \omega$ that there are only boundedly many accumulation points for (κ'_n, β_n) below κ_n

Now choose, for each $\alpha < \kappa^{++}$, a covering set X_α with $f_\alpha \in X_\alpha$. Since $o(\kappa) < \kappa^{++}$ implies that there are only κ^+ many possible Skolem functions h^{X_α}, there is a function h such that $\{\alpha < \kappa^+ : h^{X_\alpha} = h\}$ is cofinal in κ^{++}.

By throwing away the rest of the sequence $\vec{f}$ we can assume without loss of generality that $h^{X_\alpha} = h$ for all $\alpha < \kappa^{++}$. Similarly we can assume that there is an $n_0 < \omega$ such that $\rho^{X_\alpha} = \rho$ is constant and that the ordinals κ'_n, β_n and $g^*(n)$ computed using any X_α are the same as those computed using X for all $n > n_0$. By cutting off the start of the sequence $\vec{\kappa}$ we can assume that $n_0 = 0$. We will also assume that $h \in X$.

Define, for each $\alpha < \kappa^{++}$, a function f'_α by taking $f'_\alpha(n)$ to be the least ordinal $\xi \leq f_\alpha(n)$ such that $\kappa'_n \cap h^X``(\{\kappa'_n\} \cup (\xi+1)) \not\subseteq f_\alpha(n)$. The functions $f'_\alpha$ are unbounded in $\prod_n \kappa_n$: to see this, let $g$ be any member of $\prod \vec{\kappa}$ and pick $\alpha$ so that $f_\alpha(n) > \sup(\kappa_n \cap h``g(n))$ for almost all n. Then $f'_\alpha >^* g$.

Thus we can assume that $f'_\alpha = f_\alpha$ for all α, which implies that $f_\alpha(n)$ is an indiscernible in $\mathcal{C}_{\kappa'_n,\beta}$ for some $\beta < o(\kappa'_n)$. We now show that we can assume that the functions f_α are what Gitik calls *diagonal sequences*:

4.44 Lemma. *Under the assumptions of Theorem 4.42 there is a sequence $\vec{\kappa}$ of regular cardinals and a scale $\langle f_\alpha : \alpha < \kappa^{++}\rangle$ in $\prod \vec{\kappa}$ such that, using the notation introduced above, $\operatorname{cf}(\beta_n) = \kappa_{n-1}$. Furthermore, if we fix continuous, cofinal functions $t_n : \kappa_{n-1} \to \beta_n$, then each of the functions f_α satisfies $f_\alpha(n) = s_*(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n))$ for almost all n.*

Note that the cofinalities are computed in V, and the maps t_n need not be in K.

Proof. Each of the ordinals β_n is a limit ordinal, for if $\beta_n = \beta + 1$ then $\mathcal{C}_{\kappa'_n,\beta}$ is cofinal in κ_n and $\mathcal{C}_{\kappa'_n,\beta+1} \cap \kappa_n \subseteq g^*(n)$, but this implies that $\operatorname{cf}(\kappa_n) = \omega$, contrary to assumption.

For any $\alpha < \kappa^{++}$ we know that each of the ordinals $f_\alpha(n)$ is equal to either $a(\kappa'_n, \beta, \gamma)$ or $s(\kappa'_n, \beta, \gamma)$ for some $\gamma < f(n)$ and $\beta \in h``f_\alpha(n)$. Since there is always some $\beta' \in \beta_n \cap h``f_\alpha(n)$ such that $s_*(\kappa'_n, \beta, g^*(n))$ is larger than either of these, we can assume that $f_\alpha(n) = s_*(\kappa'_n, \beta_{\alpha,n}, g^*(n))$ for some $\beta_{\alpha,n} \in \beta_n \cap h``f_\alpha(n)$.

4.45 Claim. *For any $\delta < \kappa$ there are at most finitely many $n < \omega$ such that $\operatorname{cf}(\beta_n) < \delta$; and there are at most finitely many n such that $\operatorname{cf}(\beta_n) = \kappa_n$.*

Proof. First suppose that $\operatorname{cf}(\beta_n) < \delta < \kappa$ for all n in an infinite set A, and let $\sigma_n : \operatorname{cf}(\beta_n) \to \beta_n$ be cofinal maps. For each $s \in \prod_{n\in A} \delta_n$ define $g_s \in \prod_{n \in A} \kappa_n$ (up to a finite set) by $g_s(n) = s_*(\kappa'_n, \sigma_n(s(n)), g^*(n))$. Then the maps g_s are cofinal in $\prod_{n\in A} \kappa_n$, but this is impossible since $\prod_{n\in A} \kappa_n$ has cofinality κ^{++} and there are at most $\delta^\omega < \kappa$ many functions g_s.

Now suppose that $\operatorname{cf}(\beta_n) = \kappa_n$ for all n in an infinite set A. We will use the assumption that $h \in X$ to show that for all $\alpha < \kappa^{++}$ we have $f_\alpha(n) < \sup(X \cap \kappa_n)$ for all but finitely many $n \in A$. This is impossible since $|X| = 2^\omega < \kappa_n = \operatorname{cf}(\kappa_n)$, and hence $X \cap \kappa_n$ is bounded in κ_n.

Recall that each f_α is covered by the covering set X_α in the sense that $f_\alpha(n) = \gamma_k$ for some sequence $\langle \gamma_0, \ldots, \gamma_k\rangle$ of indiscernibles in $\mathcal{C}^{X_\alpha}$ such that for each $i \leq k$ either $\gamma_i = s(\alpha_i, \eta_i, \xi_i)$ or $\gamma_i = a(\alpha_i, \eta_i, \xi_i)$ for some α_i, η_i and ξ_i

in $h``(\rho \cup \vec{\gamma}\restriction i)$. Let $i \leq \kappa$ be least such that $\gamma_i \geq \sup(X \cap \kappa_n)$. If $\alpha_i < \kappa_n$ then $\gamma_i < \alpha_i < \sup(X \cap \kappa_n)$, contrary to the choice of i, so it must be that $\alpha_i \geq \kappa_n$ which implies $\alpha_i = \kappa'_n$. In that case we have $\eta_i < \sup(\beta_n \cap h``(X \cap \kappa_n))$ and $\xi_i < \sup(X \cap \kappa_n)$. Thus we can find $\beta > \eta_i$ in $X \cap \beta_n$ and $\xi > \xi_i$ in $X \cap \kappa_n$, and then $\gamma_i \leq s_*(\kappa'_n, \beta, \xi) < \sup(X \cap \kappa_n)$ for all n sufficiently large that X agrees with X_α at κ_n. This again contradicts the choice of i. ⊣

Now define D to be the smallest set such that each of the ordinals κ_n is in D and D is closed under the function σ defined as follows: Suppose that $\gamma \in D$, let γ' be largest such that either $\gamma' = \gamma$ or $\gamma \in \mathcal{C}_{\gamma',\beta}$ for some β, and let $\beta = \beta(\gamma) > 0$ be the largest ordinal such that γ is an accumulation point for (γ', β). If $\mathrm{cf}(\beta) > \omega$ then define $\sigma(\gamma) = \mathrm{cf}(\beta)$; otherwise leave $\sigma(\gamma)$ undefined.

4.46 Claim. $\mathrm{ot}(D) = \omega$.

Proof. Otherwise let δ be the least limit point of D. Then the set $D' = \{\gamma \in D - \delta : \sigma(\gamma) < \delta\}$ is infinite. Now consider a covering set $X' \supseteq X$ of size δ^ω with $\delta \subseteq X$. Then $\sigma^{X'}(\gamma) = \sigma^X(\gamma)$ for infinitely many of the $\gamma \in D'$, and X' is cofinal in any γ with $\sigma^{X'}(\gamma) < \delta$. This contradicts the fact that every $\gamma \in D$ is regular. ⊣

If $\gamma \in D$ and $\sigma(\gamma)$ is defined then let $t_\gamma : \sigma(\gamma) \to \beta(\gamma)$ be continuous, increasing and cofinal. Note that we do not assume that $t_\gamma \in K$. Also let $g^{**}(\gamma) < \gamma$ be large enough that $\mathcal{C}_{\gamma',\beta} \cap \gamma \subseteq g^{**}(\gamma)$ for all $\beta \geq \beta(\gamma)$. Define $\mathcal{F}$ to be the set of functions $f \in \prod D$ such that for all but finitely many $\gamma \in D$ such that $\sigma(\gamma)$ is defined we have $f(\gamma) = s(\gamma', t_\gamma(f(\sigma(\gamma)), g^{**}(\gamma)))$.

4.47 Claim. *For each $\alpha < \kappa^{++}$ there is an $f \in \mathcal{F}$ such that $f_\alpha <^* f\restriction\vec{\kappa}$.*

Proof. Fix $\alpha < \kappa^{++}$, and work in the covering set X_α for f_α. Define a sequence of functions $g_k \in \prod D$ by recursion on $k \in \omega$ as follows: set $g_0(\kappa_n) = f_\alpha(\kappa_n)$ for $n < \omega$, and $g_0(\gamma) = 0$ for $\gamma \in D - \vec{\kappa}$. Now define g_{k+1} so that the inequalities

$$\begin{aligned} g_{k+1}(\gamma) &\geq s_*(\gamma', t_\gamma(g_k(\sigma(\gamma))), g^{**}(\gamma)) \\ s_*(\gamma', t_\gamma(g_{k+1}(\sigma(\gamma))), g^{**}(\gamma)) &\geq g_k(\gamma) \end{aligned}$$

hold for all $\gamma \in D$ such that $\sigma(\gamma)$ is defined. Then the function $f \in \prod D$ defined by $f(\gamma) = \sup_k(g_k(\gamma))$ is as required. ⊣

Define a tree order on D by letting the immediate successors of a ordinal $\gamma' \in D$ be the ordinals $\gamma \in D$ such that $\gamma' = \sigma(\gamma)$. This tree is infinite and finitely branching, so it has an infinite branch. This branch is an infinite subset $D' \subseteq D$ such that $\sigma(\gamma) = \max(D' \cap \gamma)$ for each $\gamma \in D' - \{\min(D')\}$. This set D' satisfies all of the original assumptions on $\vec{\kappa}$, and $\prod D'$ has a scale $\vec{f}'$ of functions satisfying the equation

$$f'_\alpha(n) = s_*(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n)) \tag{18.21}$$

for all $\alpha < \kappa^{++}$ and all sufficiently large $n < \omega$, so the scale $\vec{f}'$ consists of diagonal sequences. This completes the proof of Lemma 4.44. $\dashv$

Now modify the sequence $\vec{f}$ by replacing the functions f_α such that $\mathrm{cf}(\alpha) = \kappa^+$ with an exact upper bound (given by Lemma 4.43) of the sequence $\langle f_{\alpha'} : \alpha' < \alpha\rangle$. These functions need not satisfy (18.21), but they do satisfy the following:

4.48 Claim. *If* $\mathrm{cf}(\alpha) = \kappa^+$ *then* $f_\alpha(n) = a(\kappa'_n, t_n(f_\alpha(n-1)), g^*(n))$ *for all but finitely many* $n < \omega$. *Furthermore* $\mathrm{cf}(f_\alpha(n)) > \mathrm{cf}(f_\alpha(n-1))$ *for infinitely many* $n < \omega$.

Note that we do not exclude the possibility that $f_\alpha(n)$ is also equal to $s_*(\kappa'_n, \beta_{\alpha,n}, g^*(n))$.

Proof. Let $\vec{\xi}$ be any sequence such that $\xi_{n-1} < f_\alpha(n-1)$ for almost all $n < \omega$. Since f_α is an exact upper bound of $\vec{f}\upharpoonright\alpha$, there is an $\alpha' < \alpha$ so that $f_\alpha(n-1) > f_{\alpha'}(n-1) > \xi_n$ for almost all n. Then $f_\alpha(n) > f_{\alpha'}(n) \geq s_*(\kappa'_n, t_n(\xi_n), g^*(n))$ for almost all n.

This shows that there is an accumulation point sequence $\vec{\eta}$ for the sequence $\langle(\kappa'_n, t_n(f_\alpha(n-1))) : n < \omega\rangle$ such that $\vec{\eta} \leq^* f_\alpha$. We now show that no accumulation point sequence $\vec{\eta}$ can satisfy $g^*(n) <^* \eta(n) <^* f_\alpha(n)$ for infinitely many n. To this end let $\vec{\eta}$ be any sequence such that $f_\alpha(n) > \eta_n$ for almost all n, and pick $\alpha' < \alpha$ so large that $f_{\alpha'}(n) > s_*(\kappa_n, t_n(\eta_{n-1}), g^*(n))$ for almost all n. Then $s_*(\kappa'_n, t_n(f_{\alpha'}(n-1)), g^*(n)) = f_{\alpha'}(n) > \eta_n$ for almost all n.

Finally, if $\mathrm{cf}(f_\alpha(n)) \leq \mathrm{cf}(f_\alpha(n-1))$ for all but finitely many $n < \omega$ then the set $\{\mathrm{cf}(f_\alpha(n)) : n < \omega\}$ is bounded by some $\delta < \kappa$, but in this case $\prod_n f_\alpha(n)$ would have cofinality at most $\delta^\omega < \kappa$, when in fact it has true cofinality $\mathrm{cf}(\alpha) = \kappa^+$. $\dashv$

There is a certain tension implicit in the statement of Claim 4.48: the first sentence appears to say that $\{s_*(\kappa'_n, t_n(\xi), g^*(n)) : \xi < f_\alpha(n-1)\}$ is cofinal in $f_\alpha(n)$, but that would contradict the second sentence. This is not yet an actual contradiction because the covering set in which $s_*(\kappa'_n, t_n(\xi), g^*(n))$ is evaluated varies with ξ. In the remainder of the proof we will realize this contradiction. Towards this end, pick a set $A \subseteq \omega$ such that

$$\big\{\alpha < \kappa^{++} : \mathrm{cf}(\alpha) = \kappa^+ \ \ \& \ \ A = \{n : \mathrm{cf}(f_\alpha(n)) > \mathrm{cf}(f_\alpha(n-1))\}\big\} \quad (18.22)$$

has cardinality κ^{++}, and then use the case $(2^\omega)^+ \to (\omega_1)^2_\omega$ of the Erdős-Rado theorem to find an uncountable subset S of the set (18.22) and an $n_0 \in \omega$ such that $f_{\alpha'}(n) < f_\alpha(n)$ for all $\alpha' < \alpha$ in S and all $n > n_0$. Let $\langle \delta_\iota : \iota < \omega_1\rangle$ enumerate S, set $g_\iota = f_{\delta_\iota}$ and write τ_n for $\sup_{\iota<\omega_1}(g_\iota(n))$.

Enlarge the covering set X, if necessary, so that $S \cup \{S\} \subseteq X$. Note that $\{g_\iota(n) : \iota < \omega_1\} \subseteq X$ for each n, so X is cofinal in τ_n. It follows by the final sentence of Theorem 4.19.4 that $\tau_n = s(\kappa'_n, \beta'_n, g^*(n))$ for some β'_n, for all but finitely many $n < \omega$, and that $\beta'_n \geq \sup_{\iota<\omega_1}(t_n(g_\iota(n-1)))$. As a consequence we can work with indiscernibles for τ_n rather than for κ'_n, as follows: Let $t^*_n : \tau_n \to \beta'_n$ be defined by $t^*_n(\xi) = \mathrm{Coh}_{\kappa'_n, t_n(\xi), \beta'_n}(\tau_n)$. Then $s_*(\kappa'_n, t_n(\gamma), g^*_n) = s_*(\tau_n, t^*_n(\gamma), g^*_n)$.

For each $n < \omega$ let Y_n be a covering set containing all of the data so far which has $\tau_{n-1} \subseteq Y_n$ and $|Y_n| = |\tau_{n-1}|^\omega \leq \tau^+_{n-1} \leq \kappa_{n-1} < \tau_n$. Define maps $d_{\iota,n}(\gamma)$ by setting $d_{\iota,n}(\gamma) = s^{Y_n}_*(\tau_n, t^*_n(\gamma), g^*(n))$ for each ordinal $\gamma < g_\iota(n-1)$, and set $d^*_{\iota,n} = \sup\{d_{\iota,n}(\gamma) : \gamma < g_\iota(n-1)\}$. Notice that $d^*_{\iota,n} < g_\iota(n)$ for all $n \in A$. This is clear if the nondecreasing sequence $\langle d_{\iota,n}(\gamma) : \gamma < g_\iota(n)\rangle$ is eventually constant; and if it is not constant then $\mathrm{cf}(d^*_{\iota,n}) = \mathrm{cf}(g_\iota(n-1)) < \mathrm{cf}(g_\iota(n))$, and since $d^*_{\iota,n} \leq g_\iota(n)$ it follows that $d^*_{\iota,n} < g_\iota(n)$.

Fix, for each $\iota < \omega_1$, some $\alpha_\iota < \delta_\iota$ and $n_\iota < \omega$ so that $d^*_{\iota,n} < f_{\alpha_\iota}(n) < g_\iota(n)$ for all $n \geq n_\iota$ in A. Then for each $\iota < \omega_1$ we have, for sufficiently large $n < \omega$,

$$\begin{aligned} &s^{Y_n}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) \\ &\quad = d_{\iota,n}(f_{\alpha_\iota}(n-1) \leq d^*_{n,\gamma} < f_{\alpha_\iota} = s^{X_{\alpha_\iota}}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)). \end{aligned} \tag{18.23}$$

By enlarging X if necessary, we can assume that all ordinals mentioned in the inequality (18.23) are in X. Then for all $n < \omega$ there is ι_n such that for all $\iota > \iota_n$

$$s^{Y_n}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) = s^X_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) \tag{18.24}$$

and for every $\iota < \omega_1$ there is $n_\iota < \omega$ such that for all $n > n_\iota$

$$s^{X_{\alpha_\iota}}_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)) = s^X_*(\tau_n, t^*_n(f_{\alpha_\iota}(n-1)), g^*(n)). \tag{18.25}$$

Now fix $\iota > \sup_{n<\omega}(\iota_n)$, and then pick $n > n_\iota$ large enough that inequality (18.23) holds for this n and ι. Then all three of (18.23), (18.24) and (18.25) hold, and this contradiction completes the proof of Theorem 4.41.

4.4. The Covering Lemma for Extenders

This subsection is unevenly divided into three parts. The largest part concerns the covering lemma up to $0^\P$, which is understood nearly as well as that for sequences of measures. A smaller part covers the covering lemma for the Steel core model, for which little is known beyond the weak covering lemma, and the final part describes what is known beyond this. No proofs are given. See [28] or chapter [57] for definitions and basic properties of extenders.

Up to a Strong Cardinal

This subsection covers the covering lemma when $o(\kappa) > \kappa^{++}$ but $0^\P$ does not exist; that is, when the core model contains extenders, but not overlapping extenders. More information may be found in [23].

It was remarked in the introduction to Sect. 4 that the extension of the covering lemma to this region involves two significant changes: one which is easy and largely notational, and another which is rather surprising. We will begin with the notational considerations, which come into play whenever extenders are present. These considerations are all that is needed for Theorem 4.50, which deals with extenders of length less than $\kappa^{+\omega}$ where κ is the critical point of the extender. We will consider the more surprising change following this theorem.

The first observation is that since there are no overlapping extenders, the notations $\mathcal{E}(\alpha,\beta)$ and $o(\alpha)$ are still meaningful: $\mathcal{E}(\alpha,\beta)$ is the βth full extender on α, and $o(\alpha)$ is the order type of the set of full extenders on α. Some care is required in the use of this notation: it is not true, as it is for sequences of measures, that $E = \mathcal{E}(\alpha,\beta)$ implies $o^{i^E(\mathcal{E})}(\alpha) = \beta$. For an example of this, let $E = \mathcal{E}(\alpha,\beta)$ where $\beta \geq \alpha^{++}$, and let U be the associated ultrafilter, that is, $x \in U$ if and only if $\alpha \in i^E(x)$. Then $U = \mathcal{E}(\alpha,\beta')$ for some $\beta' < \beta$. In fact $\mathcal{E}(\alpha,\beta') = \mathcal{E}_{\gamma'}$ where (since $\mathcal{E}(\alpha,\beta')$ is a measure) $\gamma' = (\alpha^{++})^{L[i^U(\mathcal{E})]}$. There are β' many ordinals $\gamma'' < \gamma'$ such that $\mathcal{E}_{\gamma''}$ is a full extender, so we must have $\beta' \leq (\alpha^{++})^{L[i^U(\mathcal{E})]}$. Now $\{\nu : o(\nu) > \nu^{++}\} \in U$, so $o^{i^U(\mathcal{E})}(\alpha) > (\alpha^{++})^{L[i^U(\mathcal{E})]}$. Thus $o^{i^U(\mathcal{E})}(\alpha) > (\alpha^{++})^{L[i^U(\mathcal{E})]} \geq \beta'$.

Because of this, it is not strictly true that comparisons of these models use only linear iterations; however, the tree iterations which they do use have a particularly simple form: there is a single trunk with no side branches of length more than one, and furthermore each extender used in the iteration tree is a member of (though not necessarily on the extender sequence of) the model to which it is applied. These simple trees can be modified to obtain linear iterations (cf. [2, 23]) or they can be handled directly without using the stronger techniques required for larger extenders (cf. [59]). Schindler [50] has extended such linearization techniques to work for cardinals below the sharp for a class of strong cardinals, and it is not known how much further they can be stretched.

Before we can state a covering lemma for models with extenders, we need to develop some notation for dealing with indiscernibles for extenders. Consider for contrast the more familiar case of indiscernibles for measures. If U is an ultrafilter on κ in some model M, and $i = i^U : M \to \mathrm{Ult}(M,U)$ is the canonical embedding, then κ is an indiscernible for $i(U)$ in the sense that

$$\forall x \in i``\mathcal{P}(\kappa)\bigl(\kappa \in x \iff i^{-1}(x) \in U \iff x \in i(U)\bigr),$$

and κ generates $\mathrm{Ult}(M,U)$ in the sense that

$$\mathrm{Ult}(M,U) = \{i^U(f)(\kappa) : f \in M\}.$$

Now let E be a (κ,λ)-extender, and let $i = i^E : M \to \mathrm{Ult}(M,E)$ be the canonical embedding. In this case the role previously played by the ordinal κ is played by the interval $[\kappa,\lambda)$: if we write E_a for the ultrafilter corresponding to $a \in [\lambda]^{<\omega}$ then

$$\forall x \in i``\mathcal{P}(\kappa^{|a|})\bigl(a \in x \iff i^{-1}(x) \in E_a \iff x \in i(E_a)\bigr),$$

and

$$\mathrm{Ult}(M,E) = \{i(f)(a) : f \in M \wedge a \in [\kappa,\lambda)^{<\omega}\}.$$

Thus a plays the role of an indiscernible for $i(E_a) = i(E)_{i(a)}$. Since $i(a) = i``a$ it will be sufficient to consider individual ordinals in the interval $[\kappa,\lambda)$.

In the example above we regard the critical point κ of i as a *principal indiscernible* for $i(E)$, and we will call an ordinal $\alpha \in [\kappa,\lambda)$ the *indiscernible for* $i(E)_{i(\alpha)}$ *belonging to* κ. In order to extend these concepts to an iterated ultrapower, we will write a system of indiscernibles for a model $M = L[\mathcal{E}]$ as a pair $(\mathcal{C}, b)$ of functions, where $\mathcal{C}_\gamma$ is the set of principal indiscernibles for the extender $\mathcal{E}_\gamma$ and $b(\gamma,\alpha,\xi)$ is the indiscernible (if there is one) for $(\mathcal{E}(\gamma))_\xi$ which belongs to α. Here is the precise definition:

4.49 Definition. If $i_{0,\theta} : M_0 \to M_\theta = M$ is an iterated ultrapower then the system $(\vec{\mathcal{C}}, b)$ of indiscernibles for M_θ generated by $i_{0,\theta}$ is defined as follows:

1. $\alpha \in \mathcal{C}_\gamma$ if and only if there are $\nu < \nu' \le \theta$ such that $\alpha = \mathrm{crit}(i_{\nu,\nu'})$ and $\mathcal{E}_\gamma = \mathcal{E}_\gamma^{M_{\nu'}} = i_{\nu,\nu'}(E_\nu)$ where E_ν is the extender such that $M_{\nu+1} = \mathrm{Ult}(M_\nu, E_\nu)$.

2. If $\alpha \in \mathcal{C}(\gamma)$, with ν and ν' as in clause 1, then $b(\gamma,\alpha,\eta)$ is defined if and only if $\eta \in i``[\alpha,\lambda)$ where E_ν is a (α,λ)-extender. In this case $b(\gamma,\alpha,\eta) = i^{-1}_{\nu,\nu'}(\eta)$.

In order to obtain an abstract definition of a system of indiscernibles for a model $M = L[\mathcal{E}]$, without any assumption that the system came from an iterated ultrapower, we replace clause 4.15(2) of Definition 4.15 of a system of indiscernibles for sequences of measures with clause $2'$ below.

$2'$ For any function $f \in M$ there is a finite sequence $\vec{a}$ of ordinals such that if $\alpha \in \mathcal{C}(\gamma)$, with $\vec{a} \cap [\alpha,\gamma) = \varnothing$, and $\bar{b} = b(\gamma,\alpha,b)$, then $\bar{b} \in x \iff x \cap V_\gamma \in (\mathcal{E}(\gamma))_b$.

Some obvious changes need to be made to the definition of a h-coherent system of indiscernibles, and the definition of $X = h``(\rho;\mathcal{C})$ needs to be modified to $h``(\rho;\mathcal{C},b)$.

4.50 Theorem (Covering Lemma for Short Extenders). *Assume that $n < \omega$ and that there is no inner model M such that $\{\alpha < \kappa : o^M(\alpha) = \alpha^{+n}\}$ is unbounded in κ for any cardinal κ. Let κ be a cardinal of K, set $\lambda = \kappa^{+n}$, and suppose $X = Y \cap K_\lambda$ where $Y \prec H(\lambda^+)$ and ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$. Then there is a pair $(\mathcal{C}, b)$, a function $h \in K$, and an ordinal $\rho < \kappa$ such that*

1. *The pair $(\mathcal{C}, b)$ is a h-coherent system of indiscernibles for K.*

2. $\mathrm{dom}(\mathcal{C}) \cup \mathrm{dom}(b) \subseteq X$, *and* $\mathrm{ran}(b) \cup \bigcup \mathrm{ran}(\mathcal{C}) \subseteq X$.

3. *For all $\nu \in X - \rho$, one of the following four conditions hold:*

   (a) $\nu \in h``(X \cap \nu)$.

(b) $\nu = s^{\mathcal{C}}(\gamma, \xi)$ *for some* $\xi \in X \cap \nu$ *and* $\gamma \in h``(X \cap \nu)$.

(c) $\nu = a^{\mathcal{C},X}(\gamma, \xi)$ *for some* $\xi \in X \cap \nu$ *and* $\gamma \in h``(X \cap \nu)$. *Furthermore, this clause never holds if* ν *is a limit point of* X.

(d) $\nu = b(\gamma, \alpha, a)$ *for some* $\alpha \in X \cap \nu$ *and* $\gamma, a \in h``\alpha$.

4. *If* X' *is another set satisfying the hypothesis of the theorem, and* $\mathcal{C}'$, b', X' *satisfy clauses 1–4, then there is a finite set* $\vec{d}$ *of ordinals such that if* $\xi, \gamma \in X \cap X'$ *with* $[\xi, \gamma) \cap \vec{d} = \varnothing$, *then*

$$s^{\mathcal{C}}(\gamma, \xi) \le s^{\mathcal{C}'}(\gamma, \xi)$$
$$a^{\mathcal{C},X}(\gamma, \xi) \le a^{\mathcal{C}',X'}(\gamma, \xi)$$
$$b(\gamma, \alpha, \nu) = b'(\gamma, \alpha, \nu).$$

In particular, the left side of the above relations is defined whenever the right side is defined.

Longer extenders require the second, and more interesting, modification to the covering lemma which was alluded to in observation 7 at the beginning of this section: an extender can not necessarily be reconstructed from its countable sequences of indiscernibles. Again we contrast indiscernibles for extenders with those for measures. Suppose that M is a model of set theory and $C \subseteq \kappa$ is an ω-sequence of indiscernibles for M, in the sense that $U = \{x \subseteq \kappa : C - x \text{ is finite}\}$ is a normal M-ultrafilter on κ. Then the added hypothesis ${}^{\omega}M \subseteq M$ implies that $C \in M$, so that $U \in M$ and hence $U \in K^M$.

Now if (C, b) is similarly a system of indiscernibles for a M-extender E, then the situation is more complicated. Again, C is an ω-sequence of indiscernibles which generates the normal measure $U = E_\kappa$ associated with E. In fact all of the ultrafilters E_a are members of M, since E_a is generated by the ω-sequence $\langle b(\nu, \alpha, a) : \nu \in C\rangle$). It is not clear, however, that these ultrafilters E_a can be reassembled in M to obtain the extender E, and in fact Gitik showed in [21] (see chapter [15]) that this reassembly is not always possible. He also showed that it is possible under the stronger hypothesis of Theorem 4.50, namely that $\{\alpha < \kappa : o(\alpha) > \alpha^{+n}\}$ is bounded in κ for some $n < \omega$. In addition he discovered a game which does provide the desired reassembly, provided that it is applied to a sequence (C, b) of indiscernibles such that $\mathrm{ot}(C)$ has uncountable cofinality. Hence a version of the covering lemma can be obtained for these longer extenders by systematically replacing ω with ω_1 [23]:

4.51 Theorem (Covering Lemma up to $0^{\P}$). *Assume that* $0^{\P}$ *does not exist. Let* κ *be a cardinal of* K *with* $\mathrm{cf}(\kappa) > \omega$, *and set* $\lambda = o(\kappa)^{+K}$. *Then if* $\kappa \not\subseteq X = Y \cap K_\lambda$, *where* $Y \prec H(\lambda)$ *and* ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$, *then there is a pair* $(\mathcal{C}, b)$ *such that the conclusion of Theorem 4.50 holds, except that clause 3c is modified as follows, where we write* $a^{\mathcal{C},X}_\iota(\gamma, \xi)$ *for the* ι*th accumulation point for* E_γ *above* ξ*:*

(3c′) $\nu = a^{\mathcal{C},X}_{\iota}(\gamma,\xi)$ *for some* $\xi \in X \cap \nu$*, some* $\gamma \in h``(X \cap \nu)$*, and some* $\iota < \omega_1$*.*

Furthermore clause (3c′) *does not hold for any limit point* ν *of* X *with* $\mathrm{cf}(\nu) > \omega$*.*

For details, see [23], which shows that these results are strong enough to give the correct lower bound for the consistency strength of a failure of the Singular Cardinal Hypothesis at a cardinal of cofinality greater than ω.

Up to a Woodin Cardinal

The following form of the weak covering lemma is proved for countably closed cardinals in chapter [47]. This proof originally appeared in [45], and the general case is proved in [43].

4.52 Theorem (Weak Covering Lemma up to a Woodin). *Suppose that there is no inner model with a Woodin cardinal, and that the Steel core model* K *exists.*[3] *Then* $(\lambda^+)^K = \lambda^+$ *for every singular cardinal* λ*.*

By "the Steel core model K exists" we mean that Steel's construction of the core model up to a Woodin cardinal, described in chapter [47], succeeds in constructing a class model K satisfying the weak covering lemma. It is known that this follows from the assumption that there is a class of subtle cardinals.

The proof involves several technical difficulties which either do not occur or are easily dealt with below $0^\P$, but it closely parallels the earlier proofs. Like the first part of the proof of Theorem 4.19 it gives, for any suitable covering set X, a mouse $\widetilde{M}$, a system $\widetilde{\mathcal{C}}$ of indiscernibles for $\widetilde{M}$, and an ordinal $\rho < \kappa$ such that $X = h^{\widetilde{M}}``(X \cap \rho; \widetilde{\mathcal{C}})$. However, the system $\widetilde{\mathcal{C}}$ of indiscernibles comes from an iteration tree, not a linear iteration, and no known analysis of such indiscernibles yields any useful information. The proof of Lemma 4.52 sidesteps this problem: like the proof of the Covering Lemma 4.5 for sequences of measures, it relies on the observation that there are no measures, and hence no indiscernibles, in the interval $(\lambda, (\lambda^+)^K]$.

Theorem 4.52 is actually weaker than it appears at first: its hypothesis that the Steel core model exists has no parallel in the covering lemmas for smaller cardinals. Recall that the proof of the full covering lemma for sequences of measures involved first proving the weak covering lemma for the model K^{c} constructed using countably complete measures, and then defining the true core model K as a elementary submodel of K^{c}. There are at least two problems in extending this procedure past $0^\P$. The most important of

[3] Jensen and Steel have recently shown that this extra assumption that K exists in unnecessary, as it follows from the assumption that there is no inner model with a Woodin cardinal. This work is as of yet unpublished.

these is the fact that countable completeness is not, so far as is known, sufficient to ensure iterability of extender sequences significantly beyond a strong cardinal. Steel, in his construction of K^{c}, replaces countable completeness with a stronger notions, which he calls *countable certification*. However, the proof of the weak covering lemma, as given for sequences of measures, does not work for K^{c} as defined from countably certified measures. Instead Steel defines K^{c} by using a measurable cardinal in V, which provides the certification needed to prove that K^{c} satisfies a form of the weak covering lemma which is slightly weaker than the countably closed weak covering property, Definition 3.46, but is sufficiently strong to support the definition of K and the proof of the full covering lemma. Further work by Steel, Jensen and others has weakened the strength required to a subtle cardinal; however, there is no clear strategy for obtaining the weak covering property with any weaker assumptions. In contrast, Mitchell and Schindler [44] have obtained a model which is iterable and (in what appear to be the appropriate senses) universal with no large cardinal assumptions.[4]

The second problem involves the proof that the iteration from the basic construction in the proof of the core model drops immediately, that is, that $1 \in D$. In the case of the covering lemma for extenders below $0^{\P}$ this argument splits. The argument used to show that the weak covering lemma holds for countably complete cardinals λ is similar to that for sequences of measures but requires an extra assumption that $o(\alpha) < \lambda$ for all $\alpha < \lambda$. The proof of the full covering lemma, on the other hand, uses a different proof relying on the weak covering lemma; it does not show that the iteration drops, but instead shows that even when the iteration does not drop there is still a Skolem function $g^X \in K$ (derived, for a suitable set X, from an extender $\mathcal{E}_\gamma$ of length κ and critical point less than $\inf(\kappa - X)$) such that $X = g^X``(X \cap \rho^X; \mathcal{C}^X)$ for some $\rho^X < \lambda$. Beyond $0^{\P}$ the notion of $o(\alpha)$ is not meaningful, so only the second argument, which requires the weak covering lemma, is usable.

A few other results are known which use the ideas of the covering lemma. One of these is Theorem 1.16, asserting that any Jónsson cardinal κ is Ramsey in K. This proof avoids a measurable cardinal at κ, since if κ were measurable then it would be Ramsey, and it avoids smaller measurable cardinals by selecting a set of indiscernibles witnessing that κ is Ramsey which contains only nonmeasurable cardinals. Others such results demonstrate that certain properties of the smaller core models extend to larger cardinals: Schindler proves in [51] that if M is a model which contains all of its countable subsets then the core model K^M defined inside M is an iterated ultrapower of K, and Gitik, Schindler and Shelah prove in [24] that if $\kappa > \omega_2$ is a cardinal in K then any sound mouse M extending $K\|\kappa$ and projecting to κ is an initial segment of K.

[4] In recent work to appear in [27], Jensen has shown that such a model also has a form of the weak covering property.

Beyond a Woodin Cardinal

As was pointed out earlier, not even the weak covering lemma is valid for a model containing a Woodin cardinal δ: Woodin has defined a notion of forcing, the stationary tower forcing [63], such that the cardinal δ is still Woodin in the generic extension $L[\mathcal{E}][G]$ but there are cofinally many singular cardinals $\lambda < \delta$ such that $(\lambda^+)^{L[\mathcal{E}]} < (\lambda^+)^{L[\mathcal{E}][G]}$. Indeed it seems likely that every sufficiently large successor cardinal less than δ is collapsed by this forcing.

It is possible that this situation is analogous to that of Prikry forcing at a measurable cardinal, in that one could hope for an analogue of the Dodd-Jensen lemma stating that any failure of the weak covering lemma is achieved by some variant of stationary tower forcing. Some very weak results in this direction are proved in [42], but there are many more questions than theorems. One difficulty is that the stationary tower forcing, unlike Prikry forcing, has a number of variants; furthermore there are other forcings, notably Woodin's "all sets generic" forcing, which require a Woodin cardinal in the universe and which may be relevant to this question.

More promising developments deal with core models which do not contain Woodin cardinals, but which are large in the sense that they have inner models with Woodin cardinals. The best result so far is due to Schimmerling and Woodin, in [48]:

4.53 Theorem. *Suppose that $\mathcal{E}$ is a good extender sequence and the model $W = L[\mathcal{E}, x]$ is sufficiently iterable. Then either there is an amenable ultrafilter U on W with* $\mathrm{crit}(W) > \mathrm{rank}(x)$ *such that* $\mathrm{Ult}(W, U)$ *is well-founded, or else W has the weak covering property above* $\mathrm{rank}(x)$.

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19. An Outline of Inner Model Theory

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Contents

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1. Introduction

This chapter is an exposition of the theory of canonical inner models for large cardinal hypotheses, or *extender models*. We hope to convey the most important ideas and methods of this theory without sinking into the morass of fine-structural detail surrounding them. The resulting outline should be accessible to anyone familiar with the theory of iterated ultrapowers and $L[U]$ contained in Kenneth Kunen's paper [14], and with the fine structure theory for L contained in Ronald Jensen's paper [10].

We shall present basic inner model theory in what is roughly the greatest generality in which it is currently known. This means that the theory we shall outline applies to extender models which may satisfy large cardinal hypotheses as strong as "There is a Woodin cardinal which is a limit of Woodin cardinals". Indeed, granted the iterability Conjecture 6.5, the theory applies to extender models satisfying "There is a superstrong cardinal". Measuring the scope of the theory descriptive-set-theoretically, we can say that it applies to any extender model containing only reals which are ordinal definable over $L(\mathbb{R})$, and in fact to extender models containing somewhat more complicated reals. One can obtain a deeper analysis of a smaller class of inner models by restricting to models satisfying at most "There is a strong cardinal" (and therefore having only Δ^1_3 reals). The basic theory of this smaller class of models is significantly simpler, especially with regard to the structure of the iterated ultrapowers it uses. One can find expositions of this special case in the papers [19] and [20], and in the book [50].

Our outline of basic inner model theory occupies Sects. 2 through 6 of this chapter. In Sects. 7 and 8 we present an application of this theory in descriptive set theory: we show that the model $\mathrm{HOD}^{L(\mathbb{R})}$ of all sets hereditarily ordinal definable in $L(\mathbb{R})$ is (essentially) an extender model.

The reader can find in [15] an exposition of basic inner model theory which is similar to this one, but somewhat less detailed. That paper then turns toward applications of inner model theory in the realm of consistency-strength lower bounds, an important area driving much of the evolution of the subject which we shall, nevertheless, avoid here. There is a more thorough and modern exposition of this area in [32]. We shall also abstain here from any extended discussion of the history of inner model theory. The reader can find philosophical/historical essays on the subject in the introductory sections of [18] and [15], and in [11, 24, 44], and in the chapter notes of [50].

2. Premice

The models we consider will be of the form $L[\vec{E}]$, where $\vec{E}$ is a coherent sequence of extenders. This framework seems quite general; indeed, it is plausible that there are models of the $L[\vec{E}]$ form for all the known large cardinal hypotheses. The framework is due, for the most part, to William Mitchell [21, 22].

2.1. Extenders

An *extender* is a system of ultrafilters which fit together in such a way that they generate a single elementary embedding. The concept was originally introduced by Mitchell [22], and then simplified to its present form by Jensen.

2.1 Definition. Let $\kappa < \lambda$ and suppose that M is transitive and rudimentarily closed. We call E a (κ, λ)-*extender over* M iff there is a nontrivial Σ_0-elementary embedding $j : M \to N$, with N transitive and rudimentarily closed, such that $\kappa = \text{crit}(j)$, $\lambda < j(\kappa)$, and

$$E = \{(a, x) \mid a \in [\lambda]^{<\omega} \wedge x \subseteq [\kappa]^{|a|} \wedge x \in M \wedge a \in j(x)\}.$$

We say in this case that E is *derived from* j, and write $\kappa = \text{crit}(E)$, $\lambda = \text{lh}(E)$.

If the requirement that N be transitive is weakened to $\lambda \subseteq \text{wfp}(N)$, where $\text{wfp}(N)$ is the wellfounded part of N, then we call E a (κ, λ)-*pre-extender over* M. For the most part, this weakening is important only in the sort of details we intend to suppress.

If E is a (κ, λ)-pre-extender over M and $a \in [\lambda]^{<\omega}$, then setting $E_a = \{x \mid (a, x) \in E\}$, we have that E_a is an M, κ-complete nonprincipal ultrafilter on the field of sets $P([\kappa]^{|a|}) \cap M$. Thus we can form the ultrapower $\text{Ult}(M, E_a)$. The fact that all the E_a's come from the same embedding implies that there is a natural direct limit of the $\text{Ult}(M, E_a)$'s, and we call this direct limit $\text{Ult}(M, E)$. We can present $\text{Ult}(M, E)$ more concretely as follows.

Let E be a (κ, λ)-pre-extender over M. Let us identify finite sets of ordinals with their increasing enumerations. Let $a, c \in [\lambda]^{<\omega}$ with $a \subseteq c$, and let s be the increasing enumeration of $\{i \mid c(i) \in a\}$. For $x \subseteq [\kappa]^{|a|}$, we set

$$x_{ac} = \{u \in [\kappa]^{|c|} \mid u \circ s \in x\}.$$

If we think of x as a $|a|$-ary predicate on κ, then x_{ac} is just the result of blowing it up to a $|c|$-ary predicate by adding dummy variables at spots corresponding to ordinals in $c \setminus a$. It is easy to see that

$$x \in E_a \iff x_{ac} \in E_c.$$

That this is true of all x, a, c is a property of E known as *compatibility*. Notice that it really is a property of E alone; M only enters in through $P(\kappa) \cap M$, and E determines $P(\kappa) \cap M$. Similarly, if f is a function with domain $[\kappa]^{|a|}$, then f_{ac} is the function with domain $[\kappa]^{|c|}$ given by $f_{ac}(u) = f(u \circ s)$, which comes from f by adding the appropriate dummy variables. It is easy to see that E has the following property, known as *normality*: if $a \in [\lambda]^{<\omega}$, $i < |a|$, $f \in M$ is a function with $\text{dom}(f) = [\kappa]^{|a|}$, and[1]

$$\text{for } E_a \text{ a.e. } u, \quad f(u) \in u(i),$$

[1] Here and in the future we use the "almost every" quantifier: given a filter F, we say $\phi(u)$ holds for F a.e. u iff $\{u \mid \phi(u)\} \in F$.

then

$$\exists\xi < a(i)(f_{a,a\cup\{\xi\}}(v) = v(j) \text{ for } E_{a\cup\{\xi\}} \text{ a.e. } v),$$

where j is such that ξ is the jth element of $a \cup \{\xi\}$. (Just take $\xi = j(f)(a)$, where E is derived from j.) Again, normality is a property of E alone.

Suppose that M is transitive and rudimentarily closed, and that $E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$ is a family of M-κ-complete ultrafilters E_a on $[\kappa]^{|a|}$, having the compatibility and normality properties. We construct $\mathrm{Ult}(M, E)$ as follows. Suppose that $a, b \in [\kappa]^{<\omega}$ and f, g are functions in M with domains $[\kappa]^{|a|}$ and $[\kappa]^{|b|}$; then we put

$$\langle a, f\rangle \sim \langle b, g\rangle \quad \text{iff} \quad \text{for } E_{a\cup b} \text{ a.e. } u\ (f_{a,a\cup b}(u) = g_{b,a\cup b}(u)).$$

It is easy to check that $\sim$ is an equivalence relation; we use $[a, f]^M_E$ to denote the equivalence class of $\langle a, f\rangle$, and omit the subscript and superscript when context permits. Let

$$[a, f] \mathrel{\tilde{\in}} [b, g] \quad \text{iff} \quad \text{for } E_{a\cup b} \text{ a.e. } u\ (f_{a,a\cup b}(u) \in g_{b,a\cup b}(u)).$$

Then $\mathrm{Ult}(M, E)$ is the structure consisting of the set of all $[a, f]$ together with $\tilde{\in}$. We shall identify the wellfounded part of $\mathrm{Ult}(M, E)$ with its transitive isomorph, so that $\tilde{\in} = \in$ on the wellfounded part.

Suppose also that M satisfies the Axiom of Choice, as will indeed be the case in our applications. We then have Łoś's theorem for Σ_0 formulae, in that if φ is Σ_0 and $c = \bigcup_{i=1}^n a_i$, then

$$\mathrm{Ult}(M, E) \models \varphi[[a_1, f_1], \ldots, [a_n, f_n]]$$

if and only if

$$\text{for } E_c \text{ a.e. } u\ (M \models \varphi[(f_1)_{a_1 c}(u), \ldots, (f_n)_{a_n c}(u)]).$$

(The full Łoś's theorem may fail, as M may not satisfy enough ZFC.) It follows that the canonical embedding

$$i^M_E : M \to \mathrm{Ult}(M, E)$$

is Σ_1-elementary, where i^M_E is given by $i^M_E(x) = [\{0\}, c_x]$, with $c_x(\alpha) = x$ for all α.

We have $[a, \mathrm{id}] = a$ for all $a \in [\lambda]^{<\omega}$ by an easy induction using the normality of E. From this and Łoś's theorem we get

$$x \in E_a \iff a \in i^M_E(x),$$

for all a, x, and

$$[a, f] = i^M_E(f)(a),$$

for all a, f. The first of these facts implies that E is the (κ, λ)-pre-extender over M derived from i^M_E. Thus compatibility and normality are equivalent to pre-extenderhood; moreover, if E is a (κ, λ)-pre-extender over Q, then E is also a (κ, λ)-pre-extender over any transitive, rudimentarily closed M such that $P(\kappa) \cap M = P(\kappa) \cap Q$. It is definitely not the case, however, that the wellfoundedness of $\mathrm{Ult}(Q, E)$ implies the wellfoundedness of $\mathrm{Ult}(M, E)$.

If E is derived from $j : M \to N$, then there is a natural embedding $k : \mathrm{Ult}(M, E) \to N$ given by $k([a, f]) = j(f)(a)$, and the diagram

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "i_E"'] & & N \\ & \mathrm{Ult}(M, E) \arrow[ur, "k"'] & \end{tikzcd}$$

commutes. It is easy to see that $k{\restriction}\lambda = \mathrm{id}$.

If E is a (κ, λ)-pre-extender over M and $\xi \leq \lambda$, then we set $E{\restriction}\xi = \{(a, x) \in E \mid a \subseteq \xi\}$. There is a natural embedding σ from $\mathrm{Ult}(M, E{\restriction}\xi)$ into $\mathrm{Ult}(M, E)$ given by: $\sigma([a, f]^M_{E{\restriction}\xi}) = [a, f]^M_E$. We call ξ a *generator* of E just in case $\xi = \mathrm{crit}(\sigma)$; that is, $\xi \neq [a, f]^M_E$ for all $f \in M$ and $a \subseteq \xi$. The idea is that in this case $E{\restriction}(\xi + 1)$ has more information than $E{\restriction}\xi$, in that it determines a "bigger" ultrapower. The smallest generator of E is κ. All other generators are $> \kappa^{+M}$.

2.2 Definition. If E is a (κ, λ)-pre-extender over M, then

$$\nu(E) = \sup(\kappa^{+M} \cup \{\xi + 1 \mid \xi \text{ is a generator of } E\}).$$

We call $\nu(E)$ the *support* of E.

The (κ, λ)-extender derived from j can capture significantly more of the strength of j than the normal measure (that is, $(\kappa, \kappa + 1)$-extender) derived from j. For example, if $|V^N_\alpha|^N \leq \lambda$, then the existence of the factor map k implies that $V^N_\alpha = V^{\mathrm{Ult}(M,E)}_\alpha$. So if there is an embedding $j : V \to N$ such that $V_{\mathrm{crit}(j)+2} \subseteq N$, then there is an extender whose ultrapower gives rise to such an embedding. Indeed, if we remove the requirement that $\lambda < j(\kappa)$ from the definition of "extender", the results just discussed still go through, and we see that any embedding can be fully captured by such a generalized extender. We have included the restriction $\lambda < j(\kappa)$ in Definition 3.1 only because nothing we shall prove here requires these "long" extenders, and it simplifies the exposition.

2.2. Fine Extender Sequences

Our models are to be constructed from *coherent sequences* of extenders. Roughly speaking, this means that each E_α is either trivial (i.e. $E_\alpha = \emptyset$), or is an extender over $L[\vec{E}{\restriction}\alpha]$ satisfying certain conditions. The extenders in a coherent sequence must appear in order of increasing strength, in that $\beta < \alpha$ implies $i_{E_\alpha}(\vec{E})_\beta = \vec{E}_\beta$. There can be no gaps, in that $i_{E_\alpha}(\vec{E})_\alpha = \emptyset$.

These two conditions constitute *coherence*, a key idea which goes back to [21]. There are further conditions on the extender sequences we consider which insure that if $E_\alpha \neq \emptyset$, then α is completely determined by the embedding coded in E_α; this prevents us from coding random information into our model via the indexing of its extenders. There are different ways of handling the details here, all of which lead to the same class of models in the end. We shall adopt the indexing scheme of [25].

We shall use the Jensen J hierarchy to stratify our models. If A is any set or class,

$$L[A] = \bigcup_{\alpha \in \mathrm{On}} J^A_\alpha,$$

where $J^A_0 = \emptyset$, $J^A_\lambda = \bigcup_{\alpha<\lambda} J^A_\alpha$ for λ limit, and

$$J^A_{\alpha+1} = \ \mathrm{rud}^A(J^A_\alpha),$$

the closure of $J^A_\alpha \cup \{J^A_\alpha\}$ under rudimentary functions and the function $x \mapsto A \cap x$. If $\vec{E}$ is a sequence, then we shall abuse notation slightly by writing $J^{\vec{E}}_\alpha$ for J^A_α, where $A = \{(\beta, z) \mid z \in E_\beta\}$. In the case of interest to us, each E_α is either $\emptyset$ or a pre-extender over $J^{\vec{E}}_\alpha$ of length α, and $E_\alpha = \emptyset$ if α is a successor ordinal. It follows then that $J^A_\alpha = J^{\vec{E}\restriction\alpha}_\alpha$ and $E_\alpha \subseteq J^{\vec{E}\restriction\alpha}_\alpha$; from this we get that for all $X \subseteq J^{\vec{E}}_\alpha$,

$$X \in J^{\vec{E}}_{\alpha+1} \quad \text{iff} \quad X \text{ is definable over } (J^{\vec{E}}_\alpha, \in, \vec{E}\restriction\alpha, E_\alpha),$$

where the definition of X may use parameters from $J^{\vec{E}}_\alpha$. (See [38, 1.4].) Although we are officially using the J hierarchy, we might have used Gödel's L hierarchy instead, and the reader who prefers can change the J's to L's in what follows. (The advantages of using the J hierarchy show up in details we shall suppress.)

There is one important point here: in our setup, if $E_\alpha \neq \emptyset$, then E_α is an extender *over* $J^{\vec{E}}_\alpha$; it only measures the subsets of its critical point constructed before stage α. There may or may not be subsets of $\mathrm{crit}(E_\alpha)$ constructed in $L[\vec{E}]$ after stage α; if there are, then E_α does not measure them, and so fails to be an extender over all of $L[\vec{E}]$. The idea of adding such "partial" extenders to our sequences $\vec{E}$ is due to Stewart Baldwin and Mitchell. It leads to a stratification of core models much simpler than the sort studied previously. In particular, the hierarchies we shall study are (strongly) acceptable in the sense of [6].

2.3 Definition. A set A is *acceptable at* α iff

$$\forall\beta < \alpha\forall\kappa((P(\kappa) \cap (J^A_{\beta+1} \setminus J^A_\beta) \neq \emptyset) \longrightarrow J^A_{\beta+1} \models |J^A_\beta| \leq \kappa).$$

Notice that if A is acceptable at α and $J^A_\alpha \models$ "κ^+ exists", then $J^A_\alpha \models$ "$P(\kappa)$ exists and $P(\kappa) \subseteq J^A_{\kappa^+}$". It follows that GCH is true in J^A_α.

It is a basic fact in the fine structure of L that $\emptyset$ is acceptable at all α. On the other hand, if μ is a normal measure on κ, then μ is not acceptable at $\kappa + 2$, since there are subsets of ω in $J^\mu_{\kappa+2} \setminus J^\mu_{\kappa+1}$ (such as $0^\#$), while κ is not countable in $J^\mu_{\kappa+2}$ (or anywhere else).

Suppose that E is a pre-extender over M, and that $M \models \kappa^+$ exists, where $\kappa = \mathrm{crit}(E)$. Let $\nu = \nu(E)$ and $\eta = (\nu^+)^{\mathrm{Ult}(M,E)}$ be in the wellfounded part of $\mathrm{Ult}(M, E)$. We shall use the ordinal η to index E in extender sequences. Let E^* be the (κ, η)-pre-extender derived from E. It is easy to check that $\nu = \nu(E^*)$ and $E{\restriction}\nu = E^*{\restriction}\nu$, so that E and E^* are equivalent. For a minor technical reason, it is E^* which we shall index at η. We call E^* the *trivial completion* of E.

We shall need the following very technical concept. Let E be an extender over M. We say that E is of *type Z* iff $\nu(E) = \lambda + 1$ for some limit ordinal λ such that (a) $\lambda = \nu(E{\restriction}\lambda)$, and (b) $(\lambda^+)^{\mathrm{Ult}(M,E)} = (\lambda^+)^{\mathrm{Ult}(M,E{\restriction}\lambda)}$. Notice that our indexing convention would require that the trivial completions E^* and $(E{\restriction}\lambda)^*$ be indexed at the same place, if E is type Z. We resolve this conflict by giving $(E{\restriction}\lambda)^*$ preference, and therefore putting no type Z extenders on our sequences.

We are ready for one of the most important definitions in this chapter.

2.4 Definition. A *fine extender sequence* is a sequence $\vec{E}$ such that for each $\alpha \in \mathrm{dom}(\vec{E})$, $\vec{E}$ is acceptable at α, and either $\vec{E}_\alpha = \emptyset$, or E_α is a (κ, α)-pre-extender over $J^{\vec{E}}_\alpha$ for some κ such that $J^{\vec{E}}_\alpha \models \kappa^+$ exists, and:

1. E_α is the trivial completion of $E_\alpha{\restriction}\nu(E_\alpha)$, and hence $\alpha = (\nu(E_\alpha)^+)^{\mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)}$, and E_α is not of type Z,
2. (Coherence) $i(\vec{E}{\restriction}\kappa){\restriction}\alpha = \vec{E}{\restriction}\alpha$ and $i(\vec{E}{\restriction}\kappa)_\alpha = \emptyset$, where $i : J^{\vec{E}}_\alpha \to \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ is the canonical embedding, and
3. (Closure under initial segment) for any η such that $(\kappa^+)^{J^{\vec{E}}_\alpha} \leq \eta < \nu(E_\alpha)$, $\eta = \nu(E_\alpha{\restriction}\eta)$, and $E_\alpha{\restriction}\eta$ is not of type Z, one of the following holds:
   (a) there is a $\gamma < \alpha$ such that E_γ is the trivial completion of $E_\alpha{\restriction}\eta$, or
   (b) $E_\eta \neq \emptyset$, and letting $j : J^{\vec{E}}_\eta \to \mathrm{Ult}(J^{\vec{E}}_\eta, E_\eta)$ be the canonical embedding and $\mu = \mathrm{crit}(j)$, there is a $\gamma < \alpha$ such that $j(\vec{E}{\restriction}\mu)_\gamma$ is the trivial completion of $E_\alpha{\restriction}\eta$.

2.5 Remarks. Let $\vec{E}$ be a fine extender sequence, $E_\alpha \neq \emptyset$, and let $i : J^{\vec{E}}_\alpha \to \mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ be the canonical embedding.

1. Although $\mathrm{Ult}(J^{\vec{E}}_\alpha, E_\alpha)$ may be illfounded, it must be that $\alpha + 1$ is contained in the wellfounded part of the ultrapower, and this is enough to make sense of the conditions in Definition 2.4. Also, $\vec{E}{\restriction}\beta \in J^{\vec{E}}_\alpha$ for all $\beta < \alpha$, and it is natural then to set $i(\vec{E}{\restriction}\alpha) = \bigcup_{\beta<\alpha} i(\vec{E}{\restriction}\beta)$.

2. Let $\nu = \nu(E_\alpha)$. By coherence, $J_\alpha^{i(\vec{E}\restriction\alpha)} = J_\alpha^{\vec{E}}$. Since $\alpha = \nu^+$ in $\mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$, and since $i(\vec{E}\restriction\alpha)$ is acceptable at all $\beta < \sup_{\gamma<\alpha} i(\gamma)$ by Łoś's theorem (acceptability being a Π_1 property of $\vec{E}\restriction\alpha$ whenever α is a limit), there are no cardinals $> \nu$ in $J_\alpha^{\vec{E}}$. The ordinal ν itself may be a successor ordinal. It is not hard to show that if ν is a limit ordinal, then ν is a cardinal in both $J_\alpha^{\vec{E}}$ and $\mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$.

3. Let $\kappa = \mathrm{crit}(E_\alpha)$. By clause 1 of Definition 2.4, there is a map of $(P(\kappa) \cap J_\nu^{\vec{E}}) \times [\nu]^{<\omega}$ onto α, the map being in $J_{\alpha+1}^{\vec{E}}$. Thus α is not a cardinal in $J_{\alpha+1}^{\vec{E}}$.

4. For the fine sequences $\vec{E}$ we construct, E_α is an extender over $L[\vec{E}\restriction\alpha]$, and $\alpha = \nu(E_\alpha)^+$ in both $L[\vec{E}\restriction\alpha]$ and $\mathrm{Ult}(L[\vec{E}\restriction\alpha], E_\alpha)$. This in fact follows from the clauses of Definition 2.4 if we can iterate from $J_\alpha^{\vec{E}}$ via E_α and its images On times.

Definition 2.4 diverges slightly from the definition of "good extender sequence" in [25, Sect. 1]. The latter definition is wrong, in that the extender sequences constructed in Sect. 11 of [25] and Sect. 6 of the present chapter do not satisfy it. This was shown by Martin Zeman. The problem lies in the initial segment condition of [25], which does not contain the proviso in clause 3 of Definition 2.4 that $E_\alpha\restriction\eta$ is not of type Z. Zeman showed that on any reasonably rich sequence of the sort constructed in [25] or Sect. 6 of this chapter, there must be extenders E such that for some $\eta < \nu(E)$, $\eta = \nu(E\restriction\eta)$ and $E\restriction\eta$ is of type Z.[2] Our indexing scheme implies that the conclusion of clause 3 of Definition 2.4 must then fail for one of $E\restriction\eta$ and $E\restriction(\eta-1)$. Ralf Schindler and Hugh Woodin independently found the correct axiomatization of the properties of the extender sequences constructed in [25] and here: one simply adds that type Z extenders do not occur on the sequence, and weakens the initial segment condition to take this into account.[3]

It might be hoped that alternative 3(b) of Definition 2.4 could be dropped, but Farmer Schlutzenberg has recently proved that if $\vec{E}$ is a fine extender sequence such that $L[\vec{E}]$ has two strong cardinals, then case 3(b) does occur somewhere in $\vec{E}$. The initial segment condition in Definition 2.4 is crucial in the proof that the comparison process terminates. We need some form of it as an axiom on our extender sequences in order to get a decent theory going. Other forms of this axiom are discussed in [39].

Following a suggestion of Sy Friedman, Jensen has investigated an indexing of extenders different from the sort described in Definition 2.4 (cf. [50]). In this framework, the extender E is indexed at the cardinal successor of

[2] See [39], which also corrects some further errors in [25] and [33].

[3] The "proof" in [25] of the stronger initial segment condition goes wrong in the proof of Theorem 10.1, where on p. 98, in the "$\eta = \gamma$" case, the authors ignore the possibility that G might be of type Z. Schindler found this error. What the argument of [25] does prove is the weaker initial segment condition of Definition 2.4.

$i_E(\mathrm{crit}(E))$ in its ultrapower. For any fine extender sequence $\vec{E}$ there is a Friedman-Jensen sequence $\vec{F}$ such that $L[\vec{E}] = L[\vec{F}]$, and vice-versa, so both approaches lead to the same class of models. The Friedman-Jensen hierarchy grows more slowly than the one we are using, in that certain extenders are put on a Friedman-Jensen sequence which only appear on ultrapowers of its translation to a fine extender sequence. In particular, one can drop the counterpart of clause 3(b) of Definition 2.4 in the Friedman-Jensen approach.

2.6 Definition. A *potential premouse* (or ppm) is a structure of the form $(J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha, E_\alpha)$, where $\vec{E}$ is a fine extender sequence. We use $\mathcal{J}_\alpha^{\vec{E}}$ to denote this structure.

2.7 Definition. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm. We say $\mathcal{M}$ is *active* if $E_\alpha \neq \emptyset$, and passive otherwise. If $\mathcal{M}$ is active, then letting $\nu = \nu(E_\alpha)$ and $\kappa = \mathrm{crit}(E_\alpha)$, we say $\mathcal{M}$ is *type* I if $\nu = (\kappa^+)^{\mathcal{M}}$, $\mathcal{M}$ is *type* II if ν is a successor ordinal, and $\mathcal{M}$ is *type* III if ν is a limit ordinal $> (\kappa^+)^{\mathcal{M}}$.

The distinctions among potential premice introduced in Definition 2.7 are mostly important in the sort of details we shall suppress, but we need them in order to make certain definitions formally correct.

2.3. The Levy Hierarchy, Cores, and Soundness

Although it is possible to avoid fine structure theory entirely in the proofs of basic facts about smaller core models (for example, in the proof that $L[U] \models \mathsf{GCH}$), there is little one can show about larger core models (such as the minimal model satisfying "There is a Woodin cardinal") without fine structure theory.[4] It seems that one must marshall all one's forces in good order in order to advance; indeed, the very definition of the models requires fine structural notions. Therefore, in order to be able even to state precise definitions and theorems, we must lay out some of the fine structure theory of definability over potential premice.

We shall simplify matters by concentrating on the representative special case of Σ_1 definability, and indicating only briefly the appropriate notions at higher levels of the Levy hierarchy. In those few places where fine structural details crop up in proofs we give in later sections, the reader will lose little by considering only the special case $\Sigma_{n+1} = \Sigma_1$. The reader should see [38]

[4] Fine structure theory begins with Jensen's landmark paper [10]. Solovay (unpublished manuscript) extended Jensen's work to $L[U]$, and then Dodd and Jensen showed in [6–8], and [5] just how remarkably fruitful this extension could be. Dodd, Jensen, and Mitchell extended this older fine structure theory to still larger core models (in [23], and unpublished work), but the complexities became unmanageable just past core models with strong cardinals. The Baldwin-Mitchell idea of putting partial extenders on a coherent sequence cut through these difficulties. Ref. [25] was the first account to develop the Baldwin-Mitchell idea.

for an excellent full account of the fine structural underpinnings of the theory we present here.[5]

The subsets of $J_\alpha^{\vec{E}}$ belonging to $J_{\alpha+1}^{\vec{E}}$ are precisely those first-order definable over the ppm $\mathcal{J}_\alpha^{\vec{E}}$, but unfortunately, this structure is not amenable if $E_\alpha \neq \emptyset$.

2.8 Definition. A structure $(M, \in, A_1, A_2, \ldots)$ is *amenable* iff

$$\forall x \in M \forall i (A_i \cap x \in M).$$

Since amenability is important in basic ways,[6] we need an amenable structure with the same definable subsets as $(J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha, E_\alpha)$; that is, we need an amenable predicate coding E_α. The following lemma is the key.

2.9 Lemma. *Let $\vec{E}$ be a fine extender sequence, $E_\alpha \neq \emptyset$, $\kappa = \mathrm{crit}(E_\alpha)$, and $\nu = \nu(E_\alpha)$; then for any $\eta < \alpha$ and $\xi < (\kappa^+)^{J_\alpha^{\vec{E}}}$, $E_\alpha \cap ([\eta]^{<\omega} \times J_\xi^{\vec{E}}) \in J_\alpha^{\vec{E}}$. Moreover, if for $\xi < (\kappa^+)^{J_\alpha^{\vec{E}}}$ we set*

$$\gamma_\xi = \text{ least } \gamma < \alpha \text{ such that } E_\alpha \cap ([\nu]^{<\omega} \times J_\xi^{\vec{E}}) \in J_\gamma^{\vec{E}},$$

then

$$\sup(\{\gamma_\xi \mid \xi < (\kappa^+)^{J_\alpha^{\vec{E}}}\}) = \alpha.$$

Proof. Fix $\xi < (\kappa^+)^{J_\alpha^{\vec{E}}}$. Let $\langle A_\beta \mid \beta < \kappa\rangle$ be an enumeration of $\bigcup_{n<\omega}(P([\kappa]^n) \cap J_\xi^{\vec{E}})$ belonging to $J_\alpha^{\vec{E}}$. Let

$$i : J_\alpha^{\vec{E}} \to \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$$

be the canonical embedding, and notice that

$$\langle i(A_\beta) \mid \beta < \kappa\rangle \in \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha),$$

since $\langle i(A_\beta) \mid \beta < \kappa\rangle = i(\langle A_\beta \mid \beta < \kappa\rangle)\restriction\kappa$. But

$$E_\alpha \cap ([\eta]^{<\omega} \times J_\xi^{\vec{E}}) = \{(a, A_\beta) \mid a \in [\eta]^{<\omega} \wedge a \in i(A_\beta)\},$$

so $E_\alpha \cap ([\eta]^{<\omega} \times J_\xi^{\vec{E}}) \in \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$. Since α is a cardinal in this ultrapower, we have by acceptability that $E_\alpha \cap ([\eta]^{<\omega} \times J_\xi^{\vec{E}}) \in J_\alpha^{i(\vec{E}\restriction\alpha)}$. But $J_\alpha^{i(\vec{E}\restriction\alpha)} = J_\alpha^{\vec{E}}$ by coherence, so we are done with the first part of the lemma.

In order to show the γ_ξ are cofinal in α, it suffices to show that whenever $A \subseteq \nu$ and $A \in \mathrm{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$, then there is a ξ such that $A \in J_{\gamma_\xi+1}^{\vec{E}}$. So fix

[5] Jensen has developed a more general fine structure theory, using terminology somewhat different from that used here. See [47] or [50]. We shall not need this extra generality here.
[6] For example, in the proof that satisfaction for Σ_1 formulae is Σ_1, and in the proof of the Łoś's theorem for Σ_0 formulae. See [38, 1.12, 8.4].

such an A, and let $A = [a, f]$, where $a \subseteq \nu$ and, without loss of generality, $f \in J_\alpha^{\vec{E}}$ and $f : J_\kappa^{\vec{E}} \to J_\kappa^{\vec{E}}$. By acceptability, we have $\xi < (\kappa^+)^{J_\alpha^{\vec{E}}}$ such that $f \in J_\xi^{\vec{E}}$. Now for $\eta < \nu$, $\eta \in A$ iff for $(E_\alpha)_{a \cup \{\eta\}}$ a.e. u, $\mathrm{id}_{\{\eta\}, a \cup \{\eta\}}(u) \in f(u)$, and the set to be measured in answering this question about η is in $J_\xi^{\vec{E}}$. Thus A can be computed from $E_\alpha \cap ([\nu]^{<\omega} \times J_\xi^{\vec{E}})$, so $A \in J_{\gamma_\xi + 1}^{\vec{E}}$. ⊣

Given now a fine extender sequence $\vec{E}$ with $E_\alpha \neq \emptyset$, we can code E_α as follows: let E_α^c be the set of quadruples (γ, ξ, a, x) such that

$$\begin{aligned}&\big(\nu(E_\alpha) < \gamma < \alpha\big) \wedge (\mathrm{crit}(E_\alpha) < \xi < (\mathrm{crit}(E_\alpha)^+)^{J_\alpha^{\vec{E}}})\\ &\quad \wedge \big(E_\alpha \cap ([\nu(E_\alpha)]^{<\omega} \times J_\xi^{\vec{E}}) \in J_\gamma^{\vec{E}}\big) \wedge \big((a, x) \in (E_\alpha \cap ([\gamma]^{<\omega} \times J_\xi^{\vec{E}}))\big).\end{aligned}$$

It follows from Lemma 2.9 that $(J_\alpha^{\vec{E}}, \in, \vec{E}\restriction\alpha, E_\alpha^c)$ is amenable.

Certain ordinal parameters are important in the description of a ppm. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$. If $\mathcal{M}$ is active, then we set

$$\nu^{\mathcal{M}} = \nu(E_\alpha) \quad \text{and} \quad \mu^{\mathcal{M}} = \mathrm{crit}(E_\alpha).$$

If $\mathcal{M}$ is passive, set $\nu^{\mathcal{M}} = \mu^{\mathcal{M}} = 0$. If $\mathcal{M}$ is active of type II, then there is a longest non-type-Z proper initial segment F of E_α containing properly less information than E_α itself, and we let $\gamma^{\mathcal{M}}$ determine where F appears on $\vec{E}$ or an ultrapower of $\vec{E}$. More precisely, set

$$F = \begin{cases} (E_\alpha\restriction(\nu^{\mathcal{M}} - 1))^* & \text{if } (E_\alpha\restriction(\nu^{\mathcal{M}} - 1))^* \text{ is not type Z} \\ (E_\alpha\restriction\nu(E_\alpha\restriction(\nu^{\mathcal{M}} - 1)) - 1)^* & \text{otherwise.} \end{cases}$$

Then we let

$$\gamma^{\mathcal{M}} = \text{the unique } \xi \in \mathrm{dom}(\vec{E}) \quad \text{such that} \quad F = E_\xi,$$

if there is such a ξ.[7] If there is no such ξ, then setting $\eta = \nu(F)$, we have by 3(b) of Definition 2.4 that F is on the extender sequence of $\mathrm{Ult}(J_\eta^{\vec{E}}, E_\eta)$. We then let

$$\gamma^{\mathcal{M}} = (\eta, a, f), \quad \text{where } F = [a, f]_{E_\eta}^{J_\eta^{\vec{E}}},$$

and (a, f) is least in the order of construction on $J_\eta^{\vec{E}}$ with this property. Finally, if $\mathcal{M}$ is not active type II, then we set $\gamma^{\mathcal{M}} = 0$.

Since we shall put these parameters in all hulls we form, we might as well have names for them in our language.

2.10 Definition. $\mathcal{L}$ is the language of set theory with additional constant symbols $\dot{\mu}, \dot{\nu}, \dot{\gamma}$, and additional unary predicate symbols $\dot{E}$ and $\dot{F}$.

[7] $\gamma^{\mathcal{M}} = \mathrm{lh}(F)$ in this case.

2.11 Definition. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm; then the Σ_0 *code* of $\mathcal{M}$, or $\mathcal{C}_0(\mathcal{M})$, is the $\mathcal{L}$-structure $\mathcal{N}$ given by:

1. if $\mathcal{M}$ is passive, then $\mathcal{N}$ has universe $J_\alpha^{\vec{E}}$, $\dot{E}^{\mathcal{N}} = \vec{E}\restriction\alpha$, $\dot{F}^{\mathcal{N}} = \emptyset$, and $\dot{\mu}^{\mathcal{N}} = \dot{\nu}^{\mathcal{N}} = \dot{\gamma}^{\mathcal{N}} = 0$;

2. if $\mathcal{M}$ is active of types I or II, then $\mathcal{N}$ has universe $J_\alpha^{\vec{E}}$, $\dot{E}^{\mathcal{N}} = \vec{E}\restriction\alpha$, $\dot{F}^{\mathcal{N}} = E_\alpha^*$ (where E_α^* is the amenable coding of E_α), and $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{M}}$, $\dot{\nu}^{\mathcal{N}} = \nu^{\mathcal{M}}$, and $\dot{\gamma}^{\mathcal{N}} = \gamma^{\mathcal{M}}$;

3. if $\mathcal{M}$ is active type III, then letting $\nu = \nu(E_\alpha)$, $\mathcal{N}$ has universe $J_\nu^{\vec{E}}$, $\dot{E}^{\mathcal{N}} = \vec{E}\restriction\nu$, $\dot{F}^{\mathcal{N}} = E_\alpha\restriction\nu$, $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{M}}$, and $\dot{\nu}^{\mathcal{N}} = \dot{\gamma}^{\mathcal{N}} = 0$.

The Σ_0 code $\mathcal{C}_0(\mathcal{M})$ is amenable; this follows from our lemma unless $\mathcal{M}$ is active type III, in which case it follows at once from the initial segment condition of Definition 2.4. The reader may wonder why we treated the type III ppm differently in the definition above, but fortunately, the answer lies in fine structural details we shall avoid here.[8] The reader will lose nothing of importance if he pretends that all active premice are of type II. Notice that $\mathcal{M}$ is indeed coded into $\mathcal{C}_0(\mathcal{M})$; this is obvious unless $\mathcal{M}$ is active type III, and in that case we can recover $\mathcal{M}$ by forming $\mathrm{Ult}(\mathcal{C}_0(\mathcal{M}), \dot{F}^{\mathcal{C}_0(\mathcal{M})})$, then adding the trivial completion of $\dot{F}^{\mathcal{C}_0(\mathcal{M})}$ to its sequence at the proper place. There is little harm in identifying $\mathcal{M}$ with $\mathcal{C}_0(\mathcal{M})$.

We can now define the Σ_1 projectum, first standard parameter, and first core of a ppm $\mathcal{M}$.

2.12 Definition. Let $\mathcal{M}$ be a ppm; then the Σ_1 *projectum* of $\mathcal{M}$, or $\rho_1(\mathcal{M})$, is the least ordinal α such that for some boldface $\mathbf{\Sigma_1^{\mathcal{C}_0(\mathcal{M})}}$ set $A \subseteq \alpha$, $A \notin \mathcal{C}_0(\mathcal{M})$. (Thus $\rho_1(\mathcal{M}) \leq \mathrm{On} \cap \mathcal{C}_0(\mathcal{M})$.)

Notice that the new set A may not be (lightface) Σ_1-definable. Since there is a $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}$ map from the class of finite sets of ordinals onto $\mathcal{C}_0(\mathcal{M})$, we can take the parameter from which A is defined to be a finite set of ordinals. We standardize the parameter by minimizing it in a certain wellorder.

2.13 Definition. A *parameter* is a finite sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of ordinals such that $\alpha_0 > \cdots > \alpha_n$ (and could be empty). If $\mathcal{M}$ is a ppm, then the *first standard parameter of* $\mathcal{M}$, or $p_1(\mathcal{M})$, is the lexicographically least parameter p such that there is a $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{p\})$ set A such that $(A \cap \rho_1(\mathcal{M})) \notin \mathcal{C}_0(\mathcal{M})$.

2.14 Definition.

1. For any $\mathcal{L}$ structure $\mathcal{Q}$ and set $X \subseteq |\mathcal{Q}|$, $\mathcal{H}_1^{\mathcal{Q}}(X)$ is the transitive collapse of the substructure of $\mathcal{Q}$ whose universe consists of all $y \in |\mathcal{Q}|$ such that $\{y\}$ is $\Sigma_1^{\mathcal{Q}}$ definable from parameters in X.

[8] See [25, Sect. 3].

2. For any ppm $\mathcal{M}$, the *first core* of $\mathcal{M}$, $\mathcal{C}_1(\mathcal{M})$, is defined by: $\mathcal{C}_1(\mathcal{M}) = \mathcal{H}_1^{\mathcal{C}_0(\mathcal{M})}(\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\})$.

It is a routine matter to show that for any ppm $\mathcal{M}$, $\mathcal{C}_1(\mathcal{M})$ is the Σ_0 code of some ppm $\mathcal{N}$. One need only check that being a Σ_0 code can be expressed using Π_2 sentences of $\mathcal{L}$. (See [25, 2.5].)

We introduce two important ways in which the standard parameter $p_1(\mathcal{M})$ can behave well.

2.15 Definition. Let $\mathcal{M}$ be a ppm.

1. We say $p_1(\mathcal{M})$ is *1-universal* iff whenever $A \subseteq \rho_1(\mathcal{M})$ and $A \in \mathcal{C}_0(\mathcal{M})$, then $A \in \mathcal{C}_1(\mathcal{M})$.

2. Let $p_1(\mathcal{M}) = \langle \alpha_0, \ldots, \alpha_n \rangle$. We say $p_1(\mathcal{M})$ is *1-solid* iff whenever $i \leq n$ and A is $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{\alpha_0, \ldots, \alpha_{i-1}\})$, then $A \cap \alpha_i \in \mathcal{C}_0(\mathcal{M})$.

3. We say $\mathcal{M}$ is *1-solid* just in case $p_1(\mathcal{M})$ is 1-solid and 1-universal.

If $p_1(\mathcal{M})$ is 1-universal, then letting $\mathcal{C}_1(\mathcal{M}) = \mathcal{C}_0(\mathcal{N})$, we have $\rho_1(\mathcal{N}) = \rho_1(\mathcal{M})$, and $p_1(\mathcal{N})$ is the image of $p_1(\mathcal{M})$ under the transitive collapse.[9] The 1-solidity of $p_1(\mathcal{M})$ is important in showing that $i(p_1(\mathcal{M})) = p_1(\mathcal{Q})$ for certain ultrapower embeddings $i : \mathcal{M} \to \mathcal{Q}$.[10]

2.16 Definition. $\mathcal{M}$ is 1-*sound* iff $\mathcal{M}$ is 1-solid and $\mathcal{C}_1(\mathcal{M}) = \mathcal{C}_0(\mathcal{M})$.

Let $\mathcal{N}$ be the ppm whose Σ_0 code is $\mathcal{C}_1(\mathcal{M})$. It is easy to see that $\mathcal{C}_1(\mathcal{N}) = \mathcal{C}_1(\mathcal{M})$, so that if $\mathcal{N}$ is 1-solid, then $\mathcal{N}$ is 1-sound. We should now go on and define the nth projectum $\rho_n(\mathcal{M})$, the nth standard parameter $p_n(\mathcal{M})$, and the nth core $\mathcal{C}_n(\mathcal{M})$, as well as the notions of n-solidity and n-universality for $p_n(\mathcal{M})$ and n-soundness for $\mathcal{M}$, in the case $n > 1$. The definitions run parallel to those in the $n = 1$ case, but there are enough annoying details that we prefer to shirk our duty and refer the conscientious reader to [25, Sect. 2]. (Formally speaking, these objects and notions are defined by induction on n in such a way that $\rho_n(\mathcal{M})$, $p_n(\mathcal{M})$, etc., only make sense if $\mathcal{M}$ is $(n-1)$-solid.) There is one point worth mentioning here, namely, $\rho_n(\mathcal{M})$, $p_n(\mathcal{M})$, $\mathcal{C}_n(\mathcal{M})$, etc., are defined *from the viewpoint of* $\mathcal{C}_{n-1}(\mathcal{M})$. For example, $\rho_2(\mathcal{M})$ is the least ordinal α such that there is an $r\Sigma_2^{\mathcal{C}_1(\mathcal{M})}$-in-parameters set $A \subseteq \alpha$ such

[9] Let r be the image of $p_1(\mathcal{M})$ under the collapse. As the collapse is the identity on $\rho_1(\mathcal{M})$, r defines over $\mathcal{C}_0(\mathcal{N})$ a new Σ_1 subset of $\rho_1(\mathcal{M})$, so that $\rho_1(\mathcal{N}) \leq \rho_1(\mathcal{M})$ and $p_1(\mathcal{N}) \leq_{\text{lex}} r$. It is easy to see $\rho_1(\mathcal{N}) \geq \rho_1(\mathcal{M})$. Finally, if $s <_{\text{lex}} r$ and $A \subseteq \rho_1(\mathcal{M})$ is $\Sigma_1^{\mathcal{C}_0(\mathcal{N})}$ definable from s, then $A \in \mathcal{M}$ by the minimality of $p_1(\mathcal{M})$, so $A \in \mathcal{N}$ by the universality of $p_1(\mathcal{M})$. Thus $r \leq_{\text{lex}} p_1(\mathcal{N})$.

[10] For any parameter $s <_{\text{lex}} p_1(\mathcal{M})$, let T_s be the Σ_1 theory in $\mathcal{C}_0(\mathcal{M})$ of parameters from $\rho_1(\mathcal{M}) \cup \{s\}$; then $T_s \in \mathcal{M}$ by the definition of $p_1(\mathcal{M})$. The solidity of $p_1(\mathcal{M})$ is equivalent to the assertion that the map $s \mapsto T_s$ is a member of $\mathcal{M}$.

that $A \notin \mathcal{C}_1(\mathcal{M})$.[11] The class of $\Sigma_2^{\mathcal{C}_0(\mathcal{M})}$-definable relations is not relevant at this (or any) point, since random information can be coded into such relations by iterating some $\mathcal{C}_0(\mathcal{N})$ above $\rho_1(\mathcal{N})$.[12]

2.17 Definition. Let $\mathcal{M}$ be a ppm; then $\mathcal{M}$ is *ω-solid* iff $\mathcal{M}$ is n-solid for all $n < \omega$, and $\mathcal{M}$ is *ω-sound* iff $\mathcal{M}$ is n-sound for all $n < \omega$. If $\mathcal{M}$ is ω-solid, then we let $\rho_\omega(\mathcal{M})$ be the eventual value of $\rho_n(\mathcal{M})$ and $\mathcal{C}_\omega(\mathcal{M})$ the eventual value of $\mathcal{C}_n(\mathcal{M})$ as $n \to \omega$.

If $n < m$, then $\rho_n(\mathcal{M}) \geq \rho_m(\mathcal{M})$, so there is indeed an eventual value for $\rho_n(\mathcal{M})$, and hence $\mathcal{C}_n(\mathcal{M})$. Clearly, $\mathcal{M}$ is ω-sound iff $\mathcal{C}_0(\mathcal{M}) = \mathcal{C}_\omega(\mathcal{M})$. All levels of the core models we shall construct will be ω-sound. Nevertheless, we must study potential premice which are not ω-sound, since these can be produced from ω-sound potential premice by taking ultrapowers. (See Lemma 2.23 below.) However, all proper initial segments of such an ultrapower are ω-sound, so we can restrict ourselves to ppm all of whose proper initial segments are ω-sound.

2.18 Definition. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm, and let $\beta \leq \alpha$; then we write $\mathcal{J}_\beta^{\mathcal{M}}$ for $\mathcal{J}_\beta^{\vec{E}}$, and call $\mathcal{J}_\beta^{\mathcal{M}}$ an *initial segment* of $\mathcal{M}$. We write $\mathcal{N} \trianglelefteq \mathcal{M}$ ($\mathcal{N}$ is an initial segment of $\mathcal{M}$) iff $\exists\beta(\mathcal{N} = \mathcal{J}_\beta^{\mathcal{M}})$, and $\mathcal{N} \triangleleft \mathcal{M}$ ($\mathcal{N}$ is a proper initial segment of $\mathcal{M}$) iff $\exists\beta < \alpha(\mathcal{N} = \mathcal{J}_\beta^{\mathcal{M}})$.

2.19 Definition. A *premouse* is a potential premouse all of whose proper initial segments are ω-sound. A *coded premouse* is a structure of the form $\mathcal{C}_0(\mathcal{M})$, where $\mathcal{M}$ is a premouse.

It is easy to see that if $\vec{E}$ is an extender sequence with domain α such that all proper initial segments of $\mathcal{J}_\alpha^{\vec{E}}$ are ω-sound then $\vec{E}$ is acceptable at α. Indeed, soundness is simply a refinement of acceptability, in that we demand that whenever a new subset of κ appears in $J_{\tau+1}^{\vec{E}} - J_\tau^{\vec{E}}$, the surjection $f \in J_{\tau+1}^{\vec{E}}$ from κ onto $J_\tau^{\vec{E}}$ required by acceptability must actually be definable over $\mathcal{J}_\tau^{\vec{E}}$ at the same quantifier level that the new subset was. The acceptability of the fine extender sequences we shall construct will come from soundness in this way.

Perhaps the first substantial theorem in the fine structural analysis of L is Jensen's result that if $E_\beta = \emptyset$ for all $\beta \leq \alpha$, then $\mathcal{J}_\alpha^{\vec{E}}$ is ω-sound [10]. If μ is a normal ultrafilter on κ, then $(J_{\kappa+1}^\mu, \in, \mu)$ is not 1-sound (in the naturally

[11] The $r\Sigma_2$ relations are, roughly speaking, just those which are Σ_1-definable from the function T, where $T(\eta, q) = \Sigma_1$ theory of parameters in $\eta \cup \{q\}$, for $\eta < \rho_1$, and $T(\eta, q) = 0$ if $\eta \geq \rho_1$.

[12] The following example is due to Mitchell. Suppose that $\langle \kappa_i \mid i \in \omega \rangle$ is an increasing sequence of measurable cardinals of $\mathcal{N}$ with $\rho_1(\mathcal{N}) \leq \kappa_0$, and suppose that $\mathcal{N}$ is 1-sound and iterable. Let $a \subseteq \omega$ be arbitrary. Let $\mathcal{M}$ result from iterating $\mathcal{N}$ by hitting a normal measure with critical point κ_i iff $i \in a$. Then a is $\Sigma_2^{\mathcal{M}}$ since $i \in a$ iff κ_i is not $\Sigma_1^{\mathcal{M}}$-definable from parameters in $\kappa_i \cup \{p_i(\mathcal{M})\}$.

adapted meaning of the term). It is because we have followed the Baldwin-Mitchell approach in putting partial extenders on $\vec{E}$ that we have the very useful L-like fact that all levels of $L[\vec{E}]$ are ω-sound.

2.4. Fine Structure and Ultrapowers

If $\mathcal{M}$ is a premouse and E is an extender over $\mathcal{C}_0(\mathcal{M})$, then we can form $\mathrm{Ult}(\mathcal{C}_0(\mathcal{M}), E)$. One can show without too much difficulty that this structure is the Σ_0 code of a premouse. The key here is that the canonical embedding i into the ultrapower is not just Σ_1-elementary, but *cofinal*, in that both $i``(\mathrm{On} \cap \mathcal{C}_0(\mathcal{M}))$ is cofinal in $\mathrm{On} \cap \mathrm{Ult}(\mathcal{C}_0(\mathcal{M}), E)$, and $i``(\dot{\mu}^+)^{\mathcal{C}_0(\mathcal{M})}$ is cofinal in $i((\dot{\mu}^+)^{\mathcal{C}_0(\mathcal{M})})$. The second condition is of course only interesting if $\mathcal{M}$ is active.[13] If $\mathrm{crit}(E) < \rho_n(\mathcal{M})$, where $1 \leq n \leq \omega$, one can form a stronger ultrapower of $\mathcal{M}$, one for which Łoś's theorem holds for $r\Sigma_n$ formulae. Roughly speaking, instead of using only functions $f \in \mathcal{C}_0(\mathcal{M})$, one uses all functions f which are $r\Sigma_n$-definable from parameters over $\mathcal{C}_0(\mathcal{M})$. (See [25, Sect. 4] and [38] for details, and generally for the $r\Sigma_n$ hierarchy.) Since $\mathrm{crit}(E) < \rho_n(\mathcal{M})$, E measures enough sets that the construction makes sense, and Łoś's theorem holds for $r\Sigma_n$ formulae. We call this stronger ultrapower $\mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$, and sometimes call the earlier ultrapower $\mathrm{Ult}_0(\mathcal{C}_0(\mathcal{M}), E)$.

We shall only form $\mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$ in the case that $\mathcal{M}$ is n-sound. In this case, all of $\mathcal{C}_0(\mathcal{M})$ can be coded by the $r\Sigma_n$ theory of $\rho_n(\mathcal{M}) \cup \{p_n(\mathcal{M})\}$, which we can regard as a subset A_n of $\rho_n(\mathcal{M})$. The structure $(J^{\mathcal{M}}_{\rho_n}, A_n)$ is amenable. If one decodes $\mathrm{Ult}_0((J^{\mathcal{M}}_{\rho_n(\mathcal{M})}, A_n), E)$ in the natural way, one gets $\mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$. This is how Σ_n ultrapowers were treated by Dodd and Jensen [6], and the reader can find an exposition of their method in [38, §8]. The equivalence of the two approaches in the case that $\mathcal{M}$ is n-sound is proved in [25, §2].

We wish to record some basic facts concerning the elementarity of the canonical embedding associated to a Σ_n ultrapower. As a notational convenience, for any ppm $\mathcal{M}$ we let $\rho_0(\mathcal{M}) = \mathrm{On} \cap \mathcal{C}_0(\mathcal{M})$ and $p_0(\mathcal{M}) = \emptyset$, and we say $\mathcal{M}$ is 0-sound. Again, the concept of being $r\Sigma_n$ is treated in [25] and [38].

2.20 Definition. Let $\pi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$, and let $n < \omega$. We call π an *n-embedding* iff

1. $\mathcal{M}$ and $\mathcal{N}$ are n-sound,
2. π is $r\Sigma_{n+1}$-elementary,
3. $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leq n$, and
4. $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < n$ and $\sup(\pi``\rho_n(\mathcal{M})) = \rho_n(\mathcal{N})$.

[13] This is why we defined $\mathcal{C}_0(\mathcal{M})$ as we did in the case $\mathcal{M}$ is of type III. Had we defined it as in the type II case, the fact that i might not be continuous at $\nu^{\mathcal{M}}$ might lead to a failure of the initial segment condition for $\mathrm{Ult}(\mathcal{C}_0(\mathcal{M}), E)$. Having said this, we ask the reader to once again forget the type III case, and go back to identifying $\mathcal{C}_0(\mathcal{M})$ with $\mathcal{M}$.

We call π an *ω-embedding* iff π is fully elementary. Such an embedding preserves all projecta and standard parameters.

2.21 Lemma. *For any $n \leq \omega$, the canonical embedding associated to a Σ_n ultrapower is an n-embedding.*

We must also consider the behavior of $\rho_{n+1}(\mathcal{M})$ and $p_{n+1}(\mathcal{M})$ in Σ_n ultrapowers. Here we must impose an additional condition on the extender used to form the ultrapower.

2.22 Definition. Let E be a (κ,λ)-extender over $\mathcal{C}_0(\mathcal{M})$; then we say E *is close to* $\mathcal{C}_0(\mathcal{M})$ (or to $\mathcal{M}$ itself) iff for every $a \in [\lambda]^{<\omega}$

1. E_a is Σ_1-definable over $\mathcal{C}_0(\mathcal{M})$ from parameters, and
2. if $\mathcal{A} \in \mathcal{C}_0(\mathcal{M})$ and $\mathcal{C}_0(\mathcal{M}) \models |\mathcal{A}| \leq \kappa$, then $E_a \cap \mathcal{A} \in \mathcal{C}_0(\mathcal{M})$.

2.23 Lemma. *Let $\mathcal{M}$ be a premouse, and E a (κ,λ)-extender over $\mathcal{C}_0(\mathcal{M})$ which is close to $\mathcal{C}_0(\mathcal{M})$, with $\kappa < \rho_n(\mathcal{M})$ where $n \leq \omega$. Let $\mathcal{N}$ be such that $\mathcal{C}_0(\mathcal{N}) = \mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$. Then*

$$P(\kappa) \cap \mathcal{M} = P(\kappa) \cap \mathcal{N}.$$

If in addition $n < \omega$, $\mathcal{M}$ is n-sound and $(n+1)$-solid, and $\rho_{n+1}(\mathcal{M}) \leq \kappa$, then the canonical embedding $\pi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$ satisfies

$$\rho_{n+1}(\mathcal{M}) = \rho_{n+1}(\mathcal{N}) \quad \textit{and} \quad \pi(p_{n+1}(\mathcal{M})) = p_{n+1}(\mathcal{N}),$$

so that

$$\mathcal{C}_{n+1}(\mathcal{M}) = \mathcal{C}_{n+1}(\mathcal{N}),$$

and $\pi \restriction \mathcal{C}_{n+1}(\mathcal{M})$ is (an isomorphic copy of) the uncollapse map from $\mathcal{C}_{n+1}(\mathcal{N})$ to $\mathcal{C}_n(\mathcal{N})$. In particular, $\mathcal{N}$ is n-sound but not $(n+1)$-sound.

We omit the proof of Definition 2.23, which the reader can find in [25, 4.5, 4.6]. See also [38, 8.10]. It is a reasonable exercise to prove the lemma in the case $n = 0$. Here the only tricky part is showing that $\pi(p_1(\mathcal{M})) = p_1(\mathcal{N})$. At that point one uses heavily the solidity of $p_1(\mathcal{M})$. The prewellordering property for $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}$ relations is also used.[14]

Let $\mathcal{M}$ be a premouse, and E an extender over $\mathcal{C}_0(\mathcal{M})$ with $\mathrm{crit}(E) < \rho_n(\mathcal{M})$; then by $\mathrm{Ult}_n(\mathcal{M}, E)$ we shall mean the unique premouse $\mathcal{N}$ such that $\mathcal{C}_0(\mathcal{N}) = \mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$.[15]

[14] Let $p_1(\mathcal{M}) = \langle \alpha_0, \ldots, \alpha_k \rangle$, and let T be a universal $\Sigma_1^{\mathcal{M}}(\{\alpha_0, \ldots, \alpha_{i-1}\})$ subset of α_i. Let $\leq$ be the prewellorder of T given by the stages at which Σ_1 formulae are verified. Then the universal $\Sigma_1^{\mathcal{N}}(\{\pi(\alpha_0), \ldots, \pi(\alpha_{i-1})\})$ subset of $\pi(\alpha_i)$ is an initial segment of $\pi(T)$ under $\pi(\leq)$, and is therefore in $\mathcal{N}$. Thus $\pi(p_1(\mathcal{M}))$ is solid, and from this we easily see that $\pi(p_1(\mathcal{M})) = p_1(\mathcal{N})$.

[15] This gives us two definitions of $\mathrm{Ult}_0(\mathcal{M}, E)$, but they clearly agree with one another except possibly when $\mathcal{M}$ is active type III. In that case, we are now discarding the earlier definition.

3. Iteration Trees and Comparison

The key to Kunen's theory of $L[U]$ is the method of *iterated ultrapowers*. Given a structure $\mathcal{M}_0 = \langle L_\zeta[U], \in, U\rangle$ with appropriate ultrafilter U, one can form ultrapowers by U and its images under the canonical embeddings repeatedly, taking direct limits at limit ordinals. One obtains thereby structures $\mathcal{M}_\alpha$ and embeddings $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ for $\alpha < \beta$. We call the structures $\mathcal{M}_\alpha$ *iterates* of $\mathcal{M}_0$, and say that $\mathcal{M}_0$ is *iterable* just in case all its iterates are wellfounded. Kunen's key *comparison lemma* states that if $\mathcal{M}_0$ and $\mathcal{N}_0$ are two iterable structures of this form, then there are iterates $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ such that one of the two is an initial segment of the other.[16]

One can form iterated ultrapowers of an arbitrary premouse $\mathcal{M}_0$ similarly. In this case, the $\mathcal{M}_\alpha$-sequence may have more than one extender, and we are allowed to choose any one of them to continue. If E_α is the extender chosen, then we take $\mathcal{M}_{\alpha+1}$ to be $\mathrm{Ult}(\mathcal{M}_\alpha, E_\alpha)$.[17] At limit stages we form direct limits and continue. We call any such sequence $\langle(\mathcal{M}_\alpha, E_\alpha) : \alpha < \beta\rangle$ a *linear iteration* of $\mathcal{M}_0$, and the structures $\mathcal{M}_\alpha$ in it *linear iterates* of $\mathcal{M}_0$. We say $\mathcal{M}_0$ is *linearly iterable* just in case all its linear iterates are wellfounded.[18]

Given linearly iterable premice $\mathcal{M}_0$ and $\mathcal{N}_0$, there is a natural way to try to compare the two via linear iteration. Having reached $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$, and supposing neither is an initial segment of the other (as otherwise our work is finished), we pick extenders E and F representing the least disagreement between $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$, and use these to form $\mathcal{M}_{\alpha+1}$ and $\mathcal{N}_{\alpha+1}$.

If the extenders of the coherent sequence of $\mathcal{M}_0$ do not overlap one another too much, and similarly for $\mathcal{N}_0$, then this process must terminate with all disagreements between some $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ eliminated, so that one is an initial segment of the other. This is the key to core model theory at the level of strong cardinals. At bottom, the reason this comparison process must terminate is the following: if E and F are the extenders used at a typical stage α, then there will be a finite set a of generators and sets $\overline{X}$ and $\tilde{X}$ such that $X = i_{\eta,\alpha}(\overline{X}) = j_{\xi,\alpha}(\tilde{X})$, and X is measured differently by E_a and F_a.[19] But then $a \in i_{\alpha,\alpha+1}(X) \iff a \notin j_{\alpha,\alpha+1}(X)$, so $i_{\eta,\alpha+1}(\overline{X}) \neq j_{\xi,\alpha+1}(\tilde{X})$, and

[16] This means that there is a filter F such that $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ are of the form

$$\langle L_\xi[F], \in, F\rangle$$

and

$$\langle L_\eta[F], \in, F\rangle$$

for some ξ and η. (Here and elsewhere we identify wellfounded, extensional structures with their transitive isomorphs.) In fact, in this simple case we can take α to be $\sup(|\mathcal{M}_0|, |\mathcal{N}_0|)^+$ and F to be the club filter on α.

[17] This must be qualified, since if E_α does not measure all subsets of its critical point in $\mathcal{M}_\alpha$, then $\mathrm{Ult}(\mathcal{M}_\alpha, E_\alpha)$ makes no sense. In this case we take the "largest" E_α ultrapower of an initial segment of $\mathcal{M}_\alpha$ we can in order to form $\mathcal{M}_{\alpha+1}$. See below.

[18] In which case we identify these iterates with the premice to which they are isomorphic. Linear iterability should be taken to include the condition that no linear iteration of $\mathcal{M}_0$ drops to proper initial segments infinitely often.

[19] We use i for the embeddings in the $\mathcal{M}$-iteration, and j in the $\mathcal{N}$-iteration.

the images of $\overline{X}$ and $\tilde{X}$ do not participate in a disagreement at stage $\alpha+1$ the way they did at stage α. *If* all future extenders used in either iteration have critical point above $\sup(a)$, then $i_{\eta,\beta}(\overline{X}) \neq j_{\xi,\beta}(\tilde{X})$ for all β, so the images of $\overline{X}$ and $\tilde{X}$ never again participate in a disagreement, and we have made real progress at stage α. A simple reflection argument shows that if we never "move generators" in one of our iterations,[20] then eventually all disagreements are removed.[21] The lack of overlaps in the sequences of mice below a strong cardinal means that this process of iterating away the least disagreement does not move generators, and hence terminates in a successful comparison.

However, beyond a strong cardinal this linear comparison process definitely will lead to moving generators. There are tricks for making do with linear iterations a bit beyond strong cardinals, but the right solution is to give up linearity. If the extender E_α from the $\mathcal{M}_\alpha$-sequence we want to use has critical point less than $\nu(E_\beta)$ for some $\beta < \alpha$, then we apply E_α not to $\mathcal{M}_\alpha$, but to $\mathcal{M}_\beta$, for the least such β: i.e., we set $\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}_\beta, E_\alpha)$, where β is least such that $\mathrm{crit}(E_\alpha) < \nu(E_\beta)$.[22] We have an embedding $i_{\beta,\alpha+1} : \mathcal{M}_\beta \to \mathcal{M}_{\alpha+1}$. Thus this new iteration process gives rise to a *tree* of models, with embeddings along each branch of the tree. Along each branch the generators of the extenders used are not moved by later embeddings, and this is good enough to show that if a comparison process involving the formation of such "iteration trees" goes on long enough, it must eventually succeed.

What one needs to keep the construction of an iteration tree going past some limit ordinal λ is a branch of the tree which has been visited cofinally often before λ and is such that the direct limit of the premice along the branch is wellfounded. Thus the iterability we need for comparison amounts to the existence of some method for choosing such branches. We can formalize this as the existence of a winning strategy in a certain game. In giving the details of the necessary definitions, it is more convenient to introduce this "iteration game" first. We turn to this now.

3.1. Iteration Trees

Let $\mathcal{M}$ be a k-sound premouse, and let θ be an ordinal; we shall define the *iteration game* $\mathcal{G}_k(\mathcal{M}, \theta)$.

3.1 Definition. A *tree order on* α (for α an ordinal) is a strict partial order T of α with least element 0 such that for all $\gamma < \alpha$

1. $\beta \, T \, \gamma \implies \beta < \gamma$,
2. $\{\beta \mid \beta \, T \, \gamma\}$ is wellordered by T,

[20] That is, if $\nu(E) \leq \mathrm{crit}(E')$ whenever E is used before E' in the $\mathcal{M}$ iteration, and similarly on the $\mathcal{N}$ side.

[21] More precisely, there must be a stage $\alpha < \sup(|\mathcal{M}_0|, |\mathcal{N}_0|)^+$ at which $\mathcal{M}_\alpha$ is an initial segment of $\mathcal{N}_\alpha$, or *vice versa*.

[22] Again, if E_α fails to measure all sets in $\mathcal{M}_\beta$, we take the ultrapower of the longest possible initial segment of $\mathcal{M}_\beta$.

3. γ is a successor ordinal $\iff \gamma$ is a T-successor, and

4. γ is a limit ordinal $\implies \{\beta \mid \beta\, T\, \gamma\}$ is $\in$-cofinal in γ.

3.2 Definition. If T is a tree order then

$$[\beta, \gamma]_T = \{\eta \mid \eta = \beta \vee \beta\, T\, \eta\, T\, \gamma \vee \eta = \gamma\},$$

and similarly for $(\beta, \gamma]_T$, $[\beta, \gamma)_T$, and $(\beta, \gamma)_T$. Also, if γ is a successor ordinal, we let $\mathrm{pred}_T(\gamma)$ be the unique ordinal $\eta\, T\, \gamma$ such that $(\eta, \gamma)_T = \emptyset$.

3.3 Definition. Premice $\mathcal{M}$ and $\mathcal{N}$ *agree below* γ iff $\mathcal{J}_\beta^{\mathcal{M}} = \mathcal{J}_\beta^{\mathcal{N}}$ for all $\beta < \gamma$.

We now describe a typical run of $\mathcal{G}_k(\mathcal{M}, \theta)$. As play proceeds the players determine

- a tree order T on θ,
- premice $\mathcal{M}_\alpha$ for $\alpha < \theta$, with $\mathcal{M}_0 = \mathcal{M}$,
- an extender F_α from the $\mathcal{M}_\alpha$ sequence, for $\alpha < \theta$, and
- a set $D \subseteq \theta$, and embeddings $i_{\alpha,\beta} : \mathcal{C}_0(\mathcal{M}_\alpha) \to \mathcal{C}_0(\mathcal{M}_\beta)$ defined whenever $\alpha\, T\, \beta$ and $D \cap (\alpha, \beta]_T = \emptyset$.

The rules of the game guarantee the following agreement among the premice produced:

- $\alpha \le \beta \implies \mathcal{M}_\alpha$ agrees with $\mathcal{M}_\beta$ below $\mathrm{lh}(F_\alpha)$,
- $\alpha < \beta \implies \mathrm{lh}(F_\alpha)$ is a cardinal of $\mathcal{M}_\beta$.

Notice that the last condition implies that if $\alpha < \beta$, then $\mathcal{M}_\alpha$ does not agree with $\mathcal{M}_\beta$ below $\mathrm{lh}(F_\alpha) + 1$. This is because from F_α one can easily compute a map from $\nu(F_\alpha)$ onto $\mathrm{lh}(F_\alpha)$.

The game is played as follows. Suppose first that we are at move $\alpha + 1$, and have already defined F_ξ for $\xi < \alpha$, $\mathcal{M}_\xi$ for $\xi \le \alpha$, and T and D on $\alpha + 1$. (The first move is move 1, and in this case all we need is $\mathcal{M} = \mathcal{M}_0$ to get going.) At move $\alpha + 1$, I must pick an extender F_α from the $\mathcal{M}_\alpha$ sequence such that $\mathrm{lh}(F_\xi) < \mathrm{lh}(F_\alpha)$ for all $\xi < \alpha$. (If he does not, the game is over and he loses.) Now let $\beta \le \alpha$ be least such that $\mathrm{crit}(F_\alpha) < \nu(F_\beta)$. Let

$$\mathcal{M}^*_{\alpha+1} := \mathcal{J}_\gamma^{\mathcal{M}_\beta}, \quad \text{where } \gamma \text{ is the largest } \eta \text{ such that } F_\alpha \text{ is a pre-extender over } \mathcal{J}_\eta^{\mathcal{M}_\beta}.$$

Our agreement hypotheses imply that γ exists, $\mathrm{lh}(F_\beta) \le \gamma$, and F_α is a pre-extender over $\mathcal{C}_0(\mathcal{J}_\gamma^{\mathcal{M}_\beta})$. [Proof: this is clear if $\beta = \alpha$, so let $\beta < \alpha$. Let $\kappa = \mathrm{crit}(F_\alpha)$. Since $\mathrm{lh}(F_\beta) < \mathrm{lh}(F_\alpha)$ and $\mathrm{lh}(F_\beta)$ is a cardinal of $\mathcal{M}_\alpha$,

$$P(\kappa) \cap \mathcal{J}^{\mathcal{M}_\beta}_{\mathrm{lh}(F_\beta)} = P(\kappa) \cap \mathcal{M}_\alpha = P(\kappa) \cap \mathcal{J}^{\mathcal{M}_\alpha}_{\mathrm{lh}(F_\alpha)}.$$

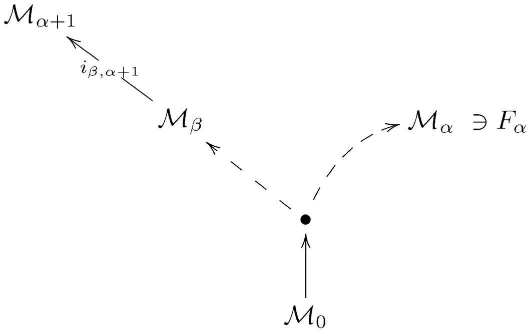
Thus F_α is a pre-extender over $\mathcal{J}^{\mathcal{M}_\beta}_{\mathrm{lh}(F_\beta)}$, so γ exists and $\mathrm{lh}(F_\beta) \le \gamma$. The last statement needs proof only in the case $\mathcal{J}^{\mathcal{M}_\beta}_\gamma$ is of type III. In this case, $\nu := \nu(\mathcal{J}^{\mathcal{M}_\beta}_\gamma)$ is the largest cardinal of $\mathcal{J}^{\mathcal{M}_\beta}_\gamma$. Thus if $\mathrm{lh}(F_\beta) < \gamma$, then $\mathrm{lh}(F_\beta) \le \nu$, so that $\kappa < \nu$, as desired. If $\mathrm{lh}(F_\beta) = \gamma$, then $\nu = \nu(F_\beta)$, so once again $\kappa < \nu$, as desired.] We put

$$\alpha + 1 \in D \iff \mathcal{M}^*_{\alpha+1} \text{ is a proper initial segment of } \mathcal{M}_\beta.$$

Let $n \le \omega$ be largest such that: (i) $\mathrm{crit}(F_\alpha) < \rho_n(\mathcal{M}^*_{\alpha+1})$ and (ii) if $D \cap [0, \alpha+1]_T = \emptyset$, then $n \le k$. Set

$$\mathcal{M}_{\alpha+1} := \mathrm{Ult}_n(\mathcal{M}^*_{\alpha+1}, F_\alpha),$$

if this ultrapower is wellfounded. (If the ultrapower is not wellfounded, then the game is over and II has lost.) Finally, we let $\beta\, T\, (\alpha+1)$, and if $\alpha + 1 \notin D$, then $i_{\beta,\alpha+1} : \mathcal{C}_0(\mathcal{M}_\beta) \to \mathcal{C}_0(\mathcal{M}_{\alpha+1})$ is the canonical ultrapower embedding, and $i_{\gamma,\alpha+1} = i_{\beta,\alpha+1} \circ i_{\gamma,\beta}$ whenever $\gamma\, T\, \beta$ and $D \cap (\gamma, \beta]_T = \emptyset$. If $\alpha + 1 \in D$, then we leave $i_{\beta,\alpha+1}$ undefined.



We must verify the agreement hypothesis we have carried along. For this, it suffices by induction to show that $\mathcal{M}_\alpha$ and $\mathcal{M}_{\alpha+1}$ have the necessary agreement. Let $\kappa = \mathrm{crit}(F_\alpha)$, and let $i : \mathcal{M}^*_{\alpha+1} \to \mathcal{M}_{\alpha+1}$, $j : \mathcal{M}^*_{\alpha+1} \to \mathrm{Ult}_0(\mathcal{M}^*_{\alpha+1}, F_\alpha) := \mathcal{P}$, and $h : \mathcal{J}^{\mathcal{M}_\alpha}_{\mathrm{lh}(F_\alpha)} \to \mathrm{Ult}_0(\mathcal{J}^{\mathcal{M}_\alpha}_{\mathrm{lh}(F_\alpha)}, F_\alpha) := \mathcal{Q}$ be the canonical embeddings. We have just shown, in effect, that $\mathcal{M}^*_{\alpha+1}$ and $\mathcal{J}^{\mathcal{M}_\alpha}_{\mathrm{lh}(F_\alpha)}$ agree below their common value λ for κ^+. It follows at once that $\mathcal{P}$ and $\mathcal{Q}$ agree below $j(\lambda) = h(\lambda)$. But $\mathcal{P}$ agrees below $i(\lambda) = j(\lambda)$ with $\mathcal{M}_{\alpha+1}$ because $\kappa < \rho_n(\mathcal{M}^*_{\alpha+1})$ (so that the $r\Sigma_n^{\mathcal{M}^*_{\alpha+1}}$ functions from κ to itself are all in $\mathcal{M}^*_{\alpha+1}$). Finally, $\mathcal{Q}$ agrees with $\mathcal{M}_\alpha$ below $\mathrm{lh}(F_\alpha)$, which is a cardinal of $\mathcal{Q}$, from the definition of fine extender sequences. Since $\mathrm{lh}(F_\alpha) < h(\lambda)$ we have the required agreement.

At a limit move λ, II picks a branch b of the tree T on λ determined by the play thus far. The branch b must be *cofinal* (i.e. $\in$-cofinal in λ), and *wellfounded*; otherwise II loses. (We say b is wellfounded iff $D \cap b$ is bounded below λ, and the direct limit of the $\mathcal{C}_0(\mathcal{M}_\beta)$ for $\beta \in (b \setminus \sup(D \cap \beta))$ under the embeddings $i_{\alpha,\beta}$ along b is wellfounded.) If II picks such a b, we set

$$\mathcal{M}_\lambda := \operatorname*{dirlim}_{\alpha \in b} \mathcal{M}_\alpha,$$

where we understand the direct limit here to be the premouse whose Σ_0 code is the direct limit of the $\mathcal{C}_0(\mathcal{M}_\alpha)$, for $\alpha \in b$ sufficiently large. We put $\alpha \, T \, \lambda$ for all $\alpha \in b$, and let $i_{\alpha,\lambda}$ be the canonical embedding into the direct limit for $\alpha \in b \setminus \sup(D \cap b)$.

This completes the rules of play for $\mathcal{G}_k(\mathcal{M}, \theta)$. If no one has lost after θ moves, then II wins.

3.4 Definition. A *k-maximal iteration tree on $\mathcal{M}$* is a partial play of the game $\mathcal{G}_k(\mathcal{M}, \theta)$ in which neither player has yet lost.[23]

We shall use calligraphic letters (e.g. $\mathcal{T}$) for iteration trees, and the corresponding roman letters (e.g. T) for their associated tree orders. ($\mathcal{T}$ is an *iteration tree* if it is a k-maximal iteration tree for some $k \leq \omega$.) We use $\mathcal{M}_\alpha^\mathcal{T}$ for the αth premouse of $\mathcal{T}$, $E_\alpha^\mathcal{T}$ for the αth extender used in $\mathcal{T}$, and $i_{\alpha,\beta}^\mathcal{T}$ for the canonical embeddings. (So $E_\alpha^\mathcal{T}$ is on the sequence of $\mathcal{M}_\alpha^\mathcal{T}$.) We use $D^\mathcal{T}$ for the set of all $\alpha+1$ such that $\mathcal{M}_{\alpha+1}^{*\mathcal{T}} \neq \mathcal{M}_{\mathrm{pred}_T(\alpha+1)}^\mathcal{T}$. In order to avoid a forest of superscripts, we shall often say "$\mathcal{T}$ is an iteration tree with models $\mathcal{N}_\alpha$, extenders F_α, and embeddings $j_{\alpha,\beta}$" when $\mathcal{N}_\alpha = \mathcal{M}_\alpha^\mathcal{T}$, $F_\alpha = E_\alpha^\mathcal{T}$, and $j_{\alpha,\beta} = i_{\alpha,\beta}^\mathcal{T}$. We will then write $\mathcal{N}_{\alpha+1}^*$ for $\mathcal{M}_{\alpha+1}^{*\mathcal{T}}$, and so forth. In general, we drop superscripts keeping track of an iteration tree whenever it seems like a good idea.

The *length* $\mathrm{lh}(\mathcal{T})$ of an iteration tree $\mathcal{T}$ is the domain of the associated tree order, so that $\mathrm{lh}(\mathcal{T}) = \alpha + 1$ iff $\mathcal{T}$ has last model $\mathcal{M}_\alpha^\mathcal{T}$.

In the course of describing $\mathcal{G}_k(\mathcal{M}, \theta)$ we proved the following lemma.

3.5 Lemma. *Let $\mathcal{T}$ be an iteration tree with models $\mathcal{M}_\alpha$ and extenders E_α, and let $\alpha < \beta < \mathrm{lh}(\mathcal{T})$; then*

1. *$\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ agree below $\mathrm{lh}(E_\alpha)$, and*
2. *$\mathrm{lh}(E_\alpha)$ is a cardinal of $\mathcal{M}_\beta$, so that $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ do not agree below $\mathrm{lh}(E_\alpha) + 1$.*

Here is another elementary fact:

3.6 Lemma. *Let $\mathcal{T}$ be an iteration tree, and let $\alpha + 1 < \mathrm{lh}(\mathcal{T})$; then E_α is close to $\mathcal{M}_{\alpha+1}^*$.*

The proof is a straightforward induction (see [25, 6.1.5]). This lemma puts the elementarity Lemma 2.23 at our disposal, and we can then describe the elementarity of the embeddings along the branches of an iteration tree as follows.

[23] More commonly now, such a tree is called *k-normal*. The word "maximal" is used for an entirely different descriptive set-theoretic property of iteration trees.

3.7 Definition. If $\mathcal{T}$ is an iteration tree with models $\mathcal{M}_\alpha$ and extenders E_α, and $\alpha+1 < \mathrm{lh}(\mathcal{T})$, then $\deg^{\mathcal{T}}(\alpha+1)$ is the largest $n \leq \omega$ such that $\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n(\mathcal{M}^*_{\alpha+1}, E_\alpha)$. Also, we use $i^{*\mathcal{T}}_{\alpha+1}$ for the canonical embedding from $\mathcal{M}^*_{\alpha+1}$ into this ultrapower.

3.8 Theorem. *Let $\mathcal{T}$ be a k-maximal iteration tree on a k-sound premouse, with models $\mathcal{M}_\alpha$ and embeddings $i_{\alpha,\beta}$, and let $(\alpha+1)\, T\, \beta$ and $D^{\mathcal{T}} \cap (\alpha+1, \beta]_T = \emptyset$; then*

1. $\deg^{\mathcal{T}}(\alpha+1) \geq \deg^{\mathcal{T}}(\xi+1)$ *for all* $\xi+1 \in (\alpha+1, \beta]_T$*, and*
2. *if* $\deg^{\mathcal{T}}(\alpha+1) = \deg^{\mathcal{T}}(\xi+1) = n$ *for all* $\xi+1 \in (\alpha+1,\beta]_T$*, then*

$$i_{\alpha+1,\beta} \circ i^*_{\alpha+1} \quad \textit{is an } n\textit{-embedding;}$$

moreover if $D^{\mathcal{T}} \cap [0, \alpha+1] \neq \emptyset$ *or* $n < k$*, then*

$$\rho_{n+1}(\mathcal{M}^*_{\alpha+1}) = \rho_{n+1}(\mathcal{M}_\beta) \leq \mathrm{crit}(i_{\alpha+1,\beta} \circ i^*_{\alpha+1}),$$

$$i_{\alpha+1,\beta} \circ i^*_{\alpha+1}(p_{n+1}(\mathcal{M}^*_{\alpha+1})) = p_{n+1}(\mathcal{M}_\beta),$$

and

$$\mathcal{C}_{n+1}(\mathcal{M}^*_{\alpha+1}) = \mathcal{C}_{n+1}(\mathcal{M}_\beta).$$

We omit the proof (see [25, 4.7]), which proceeds by induction on β, using the proof (not just the statement) of Lemma 2.23. Because of Theorem 3.8, we can for limit λ set $\deg^{\mathcal{T}}(\lambda)$ = eventual value of $\deg^{\mathcal{T}}(\alpha+1)$, for $(\alpha+1)\, T\, \lambda$ sufficiently large. When we are considering $\mathcal{T}$ as a play in $\mathcal{G}_k(\mathcal{M}, \theta)$, we set also $\deg^{\mathcal{T}}(0) = k$.[24] We then have that for any $\alpha < \mathrm{lh}(\mathcal{T})$, $\deg^{\mathcal{T}}(\alpha)$ is the largest $n \leq \omega$ such that $\mathcal{M}_\alpha$ is n-sound and $n \leq \deg^{\mathcal{T}}(0)$ if $D \cap [0, \alpha+1]_T = \emptyset$. If $\mathcal{M}^*_{\alpha+1}$ is $(n+1)$-sound, where $n+1 \leq \deg^{\mathcal{T}}(0)$ if $D \cap [0, \alpha+1]_T = \emptyset$, and $D \cap (\alpha+1, \beta]_T = \emptyset$ and $\deg^{\mathcal{T}}(\alpha+1) = \deg^{\mathcal{T}}(\beta) = n$, then by Theorem 3.8 the branch embedding $i_{\alpha+1,\beta} \circ i^*_{\alpha+1}$ is just the uncollapse map from $\mathcal{C}_{n+1}(\mathcal{M}_\beta)$ to $\mathcal{C}_n(\mathcal{M}_\beta)$.

3.9 Definition. A (k, θ)-*iteration strategy for* $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}_k(\mathcal{M}, \theta)$. We say $\mathcal{M}$ *is* (k, θ)-*iterable* iff there is such a strategy.

The iteration trees we have introduced have some special properties. If one drops the restriction on I in $\mathcal{G}_k(\mathcal{M}, \theta)$ that he pick extenders of increasing lengths, and allow him to apply the extender chosen to any initial segment of any earlier model over which it is an extender, one obtains a stronger notion of iterability which is perhaps more natural. We shall need an approximation to this stronger notion later.

It is customary to call an iterable premouse a *mouse*, and we shall follow this custom in informal discussion. We shall make no formal definition

[24] It is an awkward feature of our terminology that an iteration tree may be a play of $\mathcal{G}_k(\mathcal{M}, \theta)$ for more than one k.

of "mouse", however, as it is not clear what sort of iterability one should demand. The definition above captures only one variety of iterability. The question of iterability and its applications is of central importance and, at the same time, not very well understood. For this reason, we prefer to spell out in each instance how much iterability we can prove, or how much we need for a given purpose.

3.2. The Comparison Process

The most important use of iterability lies in the *comparison process* for mice. There are certainly mice $\mathcal{M}$ and $\mathcal{N}$ such that neither is an initial segment of the other, but if $\mathcal{M}$ and $\mathcal{N}$ are sufficiently iterable, then one can form iteration trees on $\mathcal{M}$ and $\mathcal{N}$ with last models $\mathcal{P}$ and $\mathcal{Q}$ respectively such that $\mathcal{P}$ is an initial segment of $\mathcal{Q}$ or vice-versa. Moreover, one can arrange that if, say, $\mathcal{P}$ is an initial segment of $\mathcal{Q}$, then the branch of the tree on $\mathcal{M}$ leading to $\mathcal{P}$ does not drop, and thus gives rise to an elementary embedding from $\mathcal{M}$ to $\mathcal{P}$. Intuitively, this means that $\mathcal{M}$ has been compared with $\mathcal{N}$, and found to be no stronger.

3.10 Definition. A branch b of the iteration tree $\mathcal{T}$ *drops (in model or degree)* iff $D^{\mathcal{T}} \cap b \neq \emptyset$ or $\deg^{\mathcal{T}}(b) < \deg^{\mathcal{T}}(0)$.

If b does not drop in model, then $i_{0,b}$ exists, and if in addition b does not drop in degree, then $i_{0,b}$ is a $\deg^{\mathcal{T}}(0)$-embedding. We shall also speak of "partial branches" of the form $[0,\alpha]_T$ dropping (in model or degree), with the obvious meaning. Again, if there is no such dropping, then $i_{0,\alpha}$ exists and is a $\deg^{\mathcal{T}}(0)$-embedding.

3.11 Theorem (The Comparison Lemma). *Let $\mathcal{M}$ and $\mathcal{N}$ be k-sound premice of size $\leq \theta$, and suppose that Σ and Γ are $(k, \theta^+ + 1)$-iteration strategies for $\mathcal{M}$ and $\mathcal{N}$ respectively; then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ played according to Σ and Γ respectively, and having last models $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{M}^{\mathcal{U}}_\eta$, such that either*

1. *$[0,\alpha]_T$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{T}}_\alpha$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_\eta$, or*
2. *$[0,\eta]_U$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{U}}_\eta$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_\alpha$.*

Proof. We build $\mathcal{T}$ and $\mathcal{U}$ by an inductive process known as "iterating away the least disagreement". Before step $\alpha + 1$ of the construction we have initial segments $\mathcal{T}_\alpha$ and $\mathcal{U}_\alpha$ of the trees we shall eventually construct, and these have last models $\mathcal{P}$ and $\mathcal{Q}$ respectively. ($\mathcal{T}_0$ and $\mathcal{U}_0$ are one-model trees with last models $\mathcal{P} = \mathcal{M}$ and $\mathcal{Q} = \mathcal{N}$.) If one of $\mathcal{P}$ and $\mathcal{Q}$ is an initial segment of the other, then the construction of $\mathcal{T}$ and $\mathcal{U}$ is finished. Otherwise, let

$$\lambda = \text{least } \gamma \text{ such that } \mathcal{J}^{\mathcal{P}}_\gamma \neq \mathcal{J}^{\mathcal{Q}}_\gamma.$$

This means that the predicates $\dot{F}^{\mathcal{J}^{\mathcal{P}}_\lambda}$ and $\dot{F}^{\mathcal{J}^{\mathcal{Q}}_\lambda}$ are different. If $\dot{F}^{\mathcal{J}^{\mathcal{P}}_\lambda} \neq \emptyset$, then letting $\mathrm{lh}(\mathcal{T}_\alpha) = \beta + 1$, we set

$$E^{\mathcal{T}_{\alpha+1}}_\beta := \text{pre-extender coded by } \dot{F}^{\mathcal{J}^{\mathcal{P}}_\lambda}$$

and let $\mathcal{T}_{\alpha+1}$ be the unique one-model extension of $\mathcal{T}_\alpha$ determined by this and the rules of $\mathcal{G}_k(\mathcal{M}, \theta^+ + 1)$. If $\dot{F}^{\mathcal{J}^{\mathcal{P}}_\lambda} = \emptyset$, then we just let $\mathcal{T}_{\alpha+1} = \mathcal{T}_\alpha$. Similarly, if $\dot{F}^{\mathcal{J}^{\mathcal{Q}}_\lambda} \neq \emptyset$, then letting $\mathrm{lh}(\mathcal{U}_\alpha) = \eta + 1$, we set

$$E^{\mathcal{U}_{\alpha+1}}_\eta := \text{pre-extender coded by } \dot{F}^{\mathcal{J}^{\mathcal{Q}}_\lambda}$$

and let $\mathcal{U}_{\alpha+1}$ be the one model extension of $\mathcal{U}_\alpha$ thereby determined; otherwise we let $\mathcal{U}_{\alpha+1} = \mathcal{U}_\alpha$. Notice that in any case, the last models of $\mathcal{T}_{\alpha+1}$ and $\mathcal{U}_{\alpha+1}$ agree below $\lambda + 1$. This means that future extenders used in the two trees will have length $> \lambda$, so that player I is not losing one of the iteration games by failing to play extenders increasing in length.

At limit steps λ in our construction, we set $\mathcal{T}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$ if this tree has a last model, that is, if $\mathcal{T}_\alpha$ is eventually constant as $\alpha \to \lambda$. Otherwise we let $\mathcal{T}_\lambda$ be the one-model extension of $\bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$ determined by the cofinal, wellfounded branch of this tree chosen by Σ. We define $\mathcal{U}_\lambda$ in parallel fashion.

The main thing we need to prove is that the inductive process just described stops at some step $\alpha < \theta^+$.

Claim. *There is an $\alpha < \theta^+$ such that the last model of $\mathcal{T}_\alpha$ is an initial segment of the last model of $\mathcal{U}_\alpha$, or vice-versa.*

Proof. If not, then we have trees $\mathcal{T} = \mathcal{T}_{\theta^+}$ and $\mathcal{U} = \mathcal{U}_{\theta^+}$. It is easy to see that, since $\mathcal{M}$ and $\mathcal{N}$ have size $\leq \theta$, both $\mathcal{T}$ and $\mathcal{U}$ have length $\theta^+ + 1$.

Let us say that extenders E and F are *compatible* iff for some η, E is the trivial completion of $F \restriction \eta$ or F is the trivial completion of $E \restriction \eta$. (This implies that the extenders have the same critical point, and measure the same subsets of that critical point.)

Subclaim. *For any $\alpha, \beta < \theta^+$, $E^{\mathcal{T}}_\alpha$ is incompatible with $E^{\mathcal{U}}_\beta$.*

Proof. Let $E = E^{\mathcal{T}}_\alpha$, $F = E^{\mathcal{U}}_\beta$, and suppose E is the trivial completion of $F \restriction \eta$, for some η. Let ξ be such that E is the extender used to go from $\mathcal{T}_\xi$ to $\mathcal{T}_{\xi+1}$, and let γ be such that F is used to go from $\mathcal{U}_\gamma$ to $\mathcal{U}_{\gamma+1}$. Since $\mathrm{lh}(E) \leq \mathrm{lh}(F)$, we have $\xi \leq \gamma$. But if $\xi = \gamma$, then E and F are used at the same stage in our process, so $\mathrm{lh}(E) = \mathrm{lh}(F)$, so $E = F$, contrary to the fact that we were iterating away disagreements. Thus $\xi < \gamma$, and hence $\mathrm{lh}(E) < \mathrm{lh}(F)$. Now let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}_\gamma$ and $\mathcal{U}_\gamma$ respectively. By Lemma 3.5, $\mathrm{lh}(E)$ is a cardinal of $\mathcal{P}$, and since $\mathcal{P}$ agrees with $\mathcal{Q}$ below $\mathrm{lh}(F)$, this means $\mathrm{lh}(E)$ is a cardinal of $\mathcal{J}^{\mathcal{Q}}_{\mathrm{lh}(F)}$. On the other hand, the initial segment condition of Definition 2.4 implies (in both its cases) that $E \in \mathcal{J}^{\mathcal{Q}}_{\mathrm{lh}(F)}$. Since E collapses its length in an easily computable way, this is a contradiction. $\dashv$

We now use a reflection argument to produce compatible extenders used on the branches $[0, \theta^+]_T$ and $[0, \theta^+]_U$, the desired contradiction. Let $X \prec V_\eta$ for some large η, with $\mathcal{T}, \mathcal{U} \in X$, $|X| = \theta$, and $X \cap \theta^+$ transitive. Let H be the transitive collapse of X, $\pi : H \to V_\eta$ the collapse map, and $\alpha = \mathrm{crit}(\pi) = X \cap \theta^+$. (Note that $\theta < \alpha$.) Let $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$ and $\bar{\mathcal{U}} = \pi^{-1}(\mathcal{U})$.

Since $\mathcal{M}$ and $\mathcal{N}$ have size $\leq \theta$, $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ are trees on $\mathcal{M}$ and $\mathcal{N}$ respectively. Similarly, $\bar{\mathcal{T}} \restriction \alpha = \mathcal{T} \restriction \alpha$ and $\bar{\mathcal{U}} \restriction \alpha = \mathcal{U} \restriction \alpha$. Also, $[0, \alpha]_{\bar{T}} = [0, \theta^+]_T \cap \alpha$ and $[0, \alpha]_{\bar{U}} = [0, \theta^+]_U \cap \alpha$. Since $[0, \alpha]_{\bar{T}}$ has limit order type, and any branch of an iteration tree must be closed below its sup (by clauses 3 and 4 of Definition 3.1), we have $\alpha \in [0, \theta^+]_T$, and thus $[0, \alpha]_{\bar{T}} = [0, \alpha]_T$. Similarly $\alpha \in [0, \theta^+]_U$ and $[0, \alpha]_{\bar{U}} = [0, \alpha]_U$. Since the direct limit construction is absolute to H, these facts imply that $\bar{\mathcal{T}} = \mathcal{T} \restriction (\alpha + 1)$ and $\bar{\mathcal{U}} = \mathcal{U} \restriction (\alpha + 1)$.

We can find a $\gamma \in [0, \alpha]_T$ such that $D^{\mathcal{T}} \cap [0, \alpha]_T \subseteq \gamma$, and using π we see that $D^{\mathcal{T}} \cap [0, \theta^+]_T \subseteq \gamma$. This means that $i^{\mathcal{T}}_{\alpha, \theta^+}$ is defined. In fact, if $x \in \mathcal{C}_0(\mathcal{M}^{\mathcal{T}}_\alpha)$, then letting

$$x = i^{\mathcal{T}}_{\gamma, \alpha}(\bar{x}) = i^{\bar{\mathcal{T}}}_{\gamma, \alpha}(\bar{x}),$$

we have

$$\pi(x) = i^{\mathcal{T}}_{\gamma, \theta^+}(\bar{x}) = i^{\mathcal{T}}_{\alpha, \theta^+}(i^{\mathcal{T}}_{\gamma, \alpha}(\bar{x})) = i^{\mathcal{T}}_{\alpha, \theta^+}(x).$$

In other words

$$i^{\mathcal{T}}_{\alpha, \theta^+} = \pi \restriction \mathcal{C}_0(\mathcal{M}^{\mathcal{T}}_\alpha).$$

Similarly, we get

$$i^{\mathcal{U}}_{\alpha, \theta^+} = \pi \restriction \mathcal{C}_0(\mathcal{M}^{\mathcal{U}}_\alpha).$$

Thus $i^{\mathcal{T}}_{\alpha, \theta^+}$ and $i^{\mathcal{U}}_{\alpha, \theta^+}$ agree wherever both are defined. Notice that they are defined on the same subsets of α, since

$$P(\alpha)^{\mathcal{M}^{\mathcal{T}}_\alpha} = P(\alpha)^{\mathcal{M}^{\mathcal{T}}_{\theta^+}} = P(\alpha)^{\mathcal{M}^{\mathcal{U}}_{\theta^+}} = P(\alpha)^{\mathcal{M}^{\mathcal{U}}_\alpha}.$$

Here the first and third identities hold because $\mathrm{crit}(i^{\mathcal{T}}_{\alpha, \theta^+}) = \mathrm{crit}(i^{\mathcal{U}}_{\alpha, \theta^+}) = \alpha$, and the second holds because $\mathcal{M}^{\mathcal{T}}_{\theta^+}$ agrees with $\mathcal{M}^{\mathcal{U}}_{\theta^+}$ below θ^+.

Now let $\xi + 1 \in [0, \theta^+]_T$ be such that $\mathrm{pred}_T(\xi + 1) = \alpha$, and $\gamma + 1 \in [0, \theta^+]_U$ be such that $\mathrm{pred}_U(\gamma + 1) = \alpha$. Let $\nu = \inf(\nu(E^{\mathcal{T}}_\xi), \nu(E^{\mathcal{U}}_\gamma))$. Then for any $a \in [\nu]^{<\omega}$ and $B \in \mathcal{C}_0(\mathcal{M}^{\mathcal{T}}_\alpha) \cap \mathcal{C}_0(\mathcal{M}^{\mathcal{U}}_\alpha)$,

$$\begin{aligned} B \in (E^{\mathcal{T}}_\xi)_a \quad &\Longleftrightarrow \quad a \in i^{\mathcal{T}}_{\alpha, \xi+1}(B) \\ &\Longleftrightarrow \quad a \in i^{\mathcal{T}}_{\alpha, \theta^+}(B) \\ &\Longleftrightarrow \quad a \in i^{\mathcal{U}}_{\alpha, \theta^+}(B) \\ &\Longleftrightarrow \quad a \in i^{\mathcal{U}}_{\alpha, \gamma+1}(B) \\ &\Longleftrightarrow \quad B \in (E^{\mathcal{U}}_\gamma)_a. \end{aligned}$$

The first and last equivalences displayed come from the relationship of an extender to its embedding, and the middle equivalence comes from the agreement between $i^{\mathcal{T}}_{\alpha, \theta^+}$ and $i^{\mathcal{U}}_{\alpha, \theta^+}$ our reflection argument produced. The second and fourth equivalences come from the fact that $\nu(E^{\mathcal{T}}_\xi) \leq \mathrm{crit}(i^{\mathcal{T}}_{\xi+1, \theta^+})$ and

$\nu(E_\gamma^{\mathcal{U}}) \leq \mathrm{crit}(i_{\gamma+1,\theta^+}^{\mathcal{U}})$. This is because *generators are not moved* along the branches of an iteration tree: if e.g. $(\xi+1)T(\eta+1)$, then $E_\eta^{\mathcal{T}}$ has been applied to a model with index $> \xi$, so $\nu(E_\xi^{\mathcal{T}}) \leq \mathrm{crit}(E_\eta^{\mathcal{T}})$.

This completes the proof of the claim. ⊣

Now let α be as in the claim, and set $\mathcal{T} = \mathcal{T}_\alpha$, $\mathcal{U} = \mathcal{U}_\alpha$, $\beta+1 = \mathrm{lh}(\mathcal{T})$, and $\gamma + 1 = \mathrm{lh}(\mathcal{U})$. In order to complete our proof, we must show that we have not dropped in model or degree in a way which would make our comparison meaningless. Now if $\mathcal{M}_\beta^{\mathcal{T}}$ is a proper initial segment of $\mathcal{M}_\gamma^{\mathcal{U}}$, then $\mathcal{M}_\beta^{\mathcal{T}}$ is ω-sound, and hence by the remarks following Theorem 3.8 there can have been no dropping in model or degree along $[0,\beta]_T$, so that $i_{0,\beta}^{\mathcal{T}}$ exists and is a k-embedding, as desired. Similarly, if $\mathcal{M}_\gamma^{\mathcal{U}}$ is a proper initial segment of $\mathcal{M}_\beta^{\mathcal{T}}$, then $i_{0,\gamma}^{\mathcal{U}}$ exists and is a k-embedding. Thus we may assume $\mathcal{M}_\beta^{\mathcal{T}} = \mathcal{M}_\gamma^{\mathcal{U}}$. If $D^{\mathcal{T}} \cap [0,\beta]_T = \emptyset$ and $\deg^{\mathcal{T}}(\beta) = k$, then we are done, so let us assume otherwise. Similarly, we may assume that $D^{\mathcal{U}} \cap [0,\gamma]_U \neq \emptyset$ or $\deg^{\mathcal{U}}(\gamma) < k$. It follows from these assumptions that $\deg^{\mathcal{T}}(\beta) = \deg^{\mathcal{U}}(\gamma) = n$, where n is largest such that $\mathcal{M}_\beta^{\mathcal{T}} = \mathcal{M}_\gamma^{\mathcal{U}}$ is n-sound. (See Theorem 3.8.) But then, from Theorem 3.8 and the remarks following it, we see that there are $\xi+1 \in [0,\beta]_T$ and $\eta + 1 \in [0,\gamma]_U$ such that

$$\begin{aligned} i_{\xi+1,\beta}^{\mathcal{T}} \circ i_{\xi+1}^{*\mathcal{T}} &= \text{uncollapse map from } \mathcal{C}_{n+1}(\mathcal{M}_\beta^{\mathcal{T}}) \text{ to } \mathcal{C}_n(\mathcal{M}_\beta^{\mathcal{T}}) \\ &= \text{uncollapse map from } \mathcal{C}_{n+1}(\mathcal{M}_\gamma^{\mathcal{U}}) \text{ to } \mathcal{C}_n(\mathcal{M}_\gamma^{\mathcal{U}}) \\ &= i_{\eta+1,\gamma}^{\mathcal{U}} \circ i_{\eta+1}^{*\mathcal{U}}. \end{aligned}$$

Because generators are not moved along the branches of an iteration tree, we get as in the proof of the claim that the extender $E_\xi^{\mathcal{T}}$ giving rise to $i_{\xi+1}^{*\mathcal{T}}$ is compatible with the extender $E_\eta^{\mathcal{U}}$ giving rise to $i_{\eta+1}^{*\mathcal{U}}$. This contradicts the subclaim, and thereby completes the proof of the comparison theorem. ⊣

We note that the conclusion of the Comparison Lemma can be strengthened a bit in the case that one is comparing ω-sound mice using ω-maximal trees, which is the case of greatest interest. In this case, if $\mathcal{T}$ drops in model or degree along the branch leading to its last model, then $\mathcal{U}$ does not, and the last model of $\mathcal{U}$ is a proper initial segment of the last model of $\mathcal{T}$. This follows at once from the proof of the Comparison Lemma 3.11 and the observation that the last model of $\mathcal{T}$ cannot be ω-sound in this case.

We can draw some simple corollaries concerning the definability of the reals belonging to mice.

3.12 Corollary. *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-sound (ω,ω_1+1)-iterable premice such that $\rho_\omega(\mathcal{M}) = \rho_\omega(\mathcal{N}) = \omega$; then $\mathcal{M}$ is an initial segment of $\mathcal{N}$, or vice-versa.*

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are ω-sound and project to ω, they are countable, and so we have enough iterability to compare them. Let $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$ be as in the conclusion of the Comparison Lemma 3.11, with last models $\mathcal{M}_\alpha$

and $\mathcal{N}_\eta$ respectively, and suppose without loss of generality that $\mathcal{M}_\alpha \trianglelefteq \mathcal{N}_\eta$ and $[0,\alpha]$ does not drop in model or degree. Since $\rho_\omega(\mathcal{M}) = \omega$, there are no extenders over $\mathcal{M}$ with critical point $< \rho_\omega(\mathcal{M})$, and therefore $\alpha > 0$ implies that $[0,\alpha]$ must drop in model or degree. So $\alpha = 0$. If $\eta = 0$ we are done, so assume $\eta > 0$. Since $\rho_\omega(\mathcal{N}) = \omega$, this implies $\mathcal{N}_\eta$ is not ω-sound. Thus $\mathcal{M}$ is a *proper* initial segment of $\mathcal{N}_\eta$, and $\mathcal{M}$ is countable in $\mathcal{N}_\eta$ because $\rho_\omega(\mathcal{M}) = \omega$. It is easy to see that this implies that $\mathcal{M}$ is an initial segment of $\mathcal{N}$, as desired. (One cannot gain reals by iterating, although one can lose them along some branch that drops.) ⊣

3.13 Corollary. *If $\mathcal{M}$ and $\mathcal{N}$ are (ω, ω_1+1)-iterable premice, then the $\mathcal{M}$-constructibility order on $\mathbb{R}\cap\mathcal{M}$ is an initial segment of the $\mathcal{N}$-constructibility order on $\mathbb{R}\cap\mathcal{N}$, or vice-versa.*

Proof. If $x \in \mathbb{R} \cap (\mathcal{J}^{\mathcal{M}}_{\alpha+1} \setminus \mathcal{J}^{\mathcal{M}}_{\alpha})$, then $\rho_\omega(\mathcal{J}^{\mathcal{M}}_{\alpha}) = \omega$. This observation and Corollary 3.12 easily yield the desired conclusion. ⊣

3.14 Corollary. *If $x \in \mathbb{R} \cap \mathcal{M}$ for some $(\omega, \omega_1 + 1)$-iterable premouse $\mathcal{M}$, then x is ordinal definable, and in fact x is Δ^2_2-definable from some countable ordinal.*

Proof. Say x is the αth real in the $\mathcal{M}$-constructibility order. By Corollary 3.13 we know that the formula "v is the αth real of some $(\omega, \omega_1 + 1)$-iterable premouse" characterizes x uniquely, so x is definable from α. In fact, by simply counting quantifiers one sees that $(\omega, \omega_1 + 1)$-iterability is Σ^2_3-definable, so x is Δ^2_3-definable from α. To see that x is Δ^2_2-definable, one uses the following equivalence:

$$\begin{aligned} y = x \quad \iff \quad & \exists\mathcal{M}\exists\Sigma(\mathcal{M} \text{ is a countable premouse and} \\ & \Sigma \text{ is an } (\omega,\omega_1)\text{-iteration strategy for } \mathcal{M} \text{ and} \\ & \forall\mathcal{N}\forall\Gamma(\text{if } \mathcal{N} \text{ is a countable premouse which} \\ & \text{has an } \alpha\text{th real } z \neq y, \text{ and} \\ & \Gamma \text{ is an } \omega_1\text{-iteration strategy for } \mathcal{N}, \text{ then} \\ & \text{if } (\mathcal{T},\mathcal{U}) \text{ is the } (\Sigma,\Gamma)\text{-coiteration of } \mathcal{M} \text{ with } \mathcal{N}, \\ & \text{then } \mathcal{U} \text{ has no cofinal branch})). \end{aligned}$$

Here by the (Σ, Γ)-coiteration we mean the pair of iteration trees determined by Σ and Γ through the process of iterating away the least disagreement, as in the Comparison Lemma 3.11. Since an ω_1-iteration strategy is essentially a set of reals, and the property of being an ω_1-iteration strategy is expressible using only real quantifiers, the formula displayed above is Σ^2_2, and hence x is Δ^2_2 in α. ⊣

We shall refine the proof of Corollary 3.14 later, and thereby obtain sharper upper bounds on the complexity of the reals in certain small mice. The refinement involves producing a logically simpler condition equivalent to $(\omega_1 + 1)$-iterability in the case of these small mice.

4. The Dodd-Jensen Lemma

The Dodd-Jensen Lemma on the minimality of iteration maps is a fundamental, often-used tool in inner model theory.

4.1. The Copying Construction

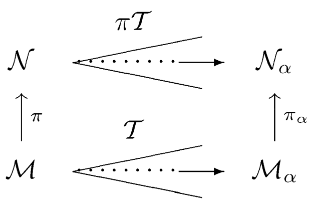
Given a k-embedding $\pi : \mathcal{M} \to \mathcal{N}$ and a k-maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$ with models $\mathcal{M}_\alpha$, we can lift $\mathcal{T}$ to a k-maximal iteration tree $\pi\mathcal{T}$ on $\mathcal{N}$ with models $\mathcal{N}_\alpha$. In fact, we need slightly less elementarity for π in order to construct $\pi\mathcal{T}$.

4.1 Definition. Let $\pi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$ and let $k < \omega$. We call π a *weak k-embedding* iff

1. $\mathcal{M}$ and $\mathcal{N}$ are k-sound,
2. π is $r\Sigma_k$-elementary, and $r\Sigma_{k+1}$-elementary on parameters from some set X cofinal in $\rho_k(\mathcal{M})$,
3. $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$, for all $i \leq k$, and
4. $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < k$, and $\sup(\pi``\rho_k(\mathcal{M})) \leq \rho_k(\mathcal{N})$.

A *weak ω-embedding* is just an ω-embedding, that is, a fully elementary map.

We shall construct $\pi\mathcal{T}$ by induction; at stage α we define its αth model $\mathcal{N}_\alpha$, together with an embedding π_α from $\mathcal{C}_0(\mathcal{M}_\alpha)$ to $\mathcal{C}_0(\mathcal{N}_\alpha)$, as in the following figure:



The next lemma describes the successor steps of this construction.

4.2 Lemma (Shift Lemma). *Let $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ be premice, let $\bar{\kappa} = \mathrm{crit}(\dot{F}^{\bar{\mathcal{N}}})$, and let*

$$\psi : \mathcal{C}_0(\bar{\mathcal{N}}) \to \mathcal{C}_0(\mathcal{N})$$

be a weak 0*-embedding, and*

$$\pi : \mathcal{C}_0(\bar{\mathcal{M}}) \to \mathcal{C}_0(\mathcal{M})$$

be a weak n-embedding. Suppose that $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ agree below $(\bar{\kappa}^+)^{\bar{\mathcal{M}}}$ and $(\bar{\kappa}^+)^{\bar{\mathcal{M}}} \le (\bar{\kappa}^+)^{\bar{\mathcal{N}}}$, while $\mathcal{M}$ and $\mathcal{N}$ agree below $(\kappa^+)^{\mathcal{M}}$ and $(\kappa^+)^{\mathcal{M}} \le (\kappa^+)^{\mathcal{N}}$, where $\kappa = \psi(\bar{\kappa})$. Suppose also

$$\pi\restriction(\bar{\kappa}^+)^{\bar{\mathcal{M}}} = \psi\restriction(\bar{\kappa}^+)^{\bar{\mathcal{M}}}.$$

Let $\bar{\kappa} < \rho_n(\bar{\mathcal{M}})$, so that $\mathrm{Ult}_n(\mathcal{C}_0(\bar{\mathcal{M}}), \dot{F}^{\bar{\mathcal{N}}})$ and $\mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), \dot{F}^{\mathcal{N}})$ make sense, and suppose the latter ultrapower is wellfounded. Then the former ultrapower is wellfounded; moreover, there is a unique embedding

$$\sigma : \mathrm{Ult}_n(\mathcal{C}_0(\bar{\mathcal{M}}), \dot{F}^{\bar{\mathcal{N}}}) \to \mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), \dot{F}^{\mathcal{N}})$$

satisfying the conditions:

1. *σ is a weak n-embedding,*
2. *$\mathrm{Ult}_n(\mathcal{C}_0(\bar{\mathcal{M}}), \dot{F}^{\bar{\mathcal{N}}})$ agrees with $\bar{\mathcal{N}}$ below $\rho_0(\bar{\mathcal{N}})$, and $\mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), \dot{F}^{\mathcal{N}})$ agrees with $\mathcal{N}$ below $\rho_0(\mathcal{N})$,*
3. *$\sigma\restriction(\rho_0(\bar{\mathcal{N}})) = \psi\restriction(\rho_0(\bar{\mathcal{N}}))$,*
4. *the diagram*

$$\begin{tikzcd} \mathrm{Ult}_n(\mathcal{C}_0(\bar{\mathcal{M}}), \dot{F}^{\bar{\mathcal{N}}}) \arrow[r, "\sigma"] & \mathrm{Ult}_n(\mathcal{C}_0(\mathcal{M}), \dot{F}^{\mathcal{N}}) \\ \mathcal{C}_0(\bar{\mathcal{M}}) \arrow[u, "i"] \arrow[r, "\pi"'] & \mathcal{C}_0(\mathcal{M}) \arrow[u, "j"'] \end{tikzcd}$$

commutes, where i and j are the canonical ultrapower embeddings.

The proof of the lemma is straightforward, so we omit it. In the representative special case $n = 0$, the desired map σ is defined by

$$\sigma([a, f]^{\bar{\mathcal{M}}}_{\dot{F}^{\bar{\mathcal{N}}}}) = [\psi(a), \pi(f)]^{\mathcal{M}}_{\dot{F}^{\mathcal{N}}}.$$

This is of course how it must be defined if we are to have conditions 3 and 4.

Now let $\pi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$ be a weak k-embedding, and let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$. We define the models of a k-maximal copied tree $\pi\mathcal{T}$ on $\mathcal{N}$ by induction. In order to avoid some fine structural details, we shall assume first that no model on $\mathcal{T}$ is a type III premouse. In that case, $\pi\mathcal{T}$ will be a tree with the same order and drop structure as $\mathcal{T}$, and we shall have embeddings

$$\pi_\alpha : \mathcal{C}_0(\mathcal{M}_\alpha) \to \mathcal{C}_0(\mathcal{N}_\alpha).$$

We shall have $\deg^{\mathcal{T}}(\alpha) \le \deg^{\pi\mathcal{T}}(\alpha)$, with perhaps strict inequality being forced on us by the desire that $\pi\mathcal{T}$ be k-maximal. We use E_β and $i_{\beta,\alpha}$ for the extenders and embeddings of $\mathcal{T}$, and F_β and $j_{\beta,\alpha}$ for the extenders and embeddings of $\pi\mathcal{T}$, and we maintain inductively:

- π_α is a weak $\deg^{\mathcal{T}}(\alpha)$-embedding,
- if $\beta < \alpha$ and E_β is the last extender of the initial segment $\mathcal{P}$ of $\mathcal{M}_\beta$, then $\pi_\beta \restriction \rho_0(\mathcal{P}) = \pi_\alpha \restriction \rho_0(\mathcal{P})$, and
- if $\beta\, T\, \alpha$ and $(\beta, \alpha]_T \cap D = \emptyset$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_0(\mathcal{N}_\beta) \arrow[r, "j_{\beta,\alpha}"] & \mathcal{C}_0(\mathcal{N}_\alpha) \\
\mathcal{C}_0(\mathcal{M}_\beta) \arrow[u, "\pi_\beta"] \arrow[r, "i_{\beta,\alpha}"'] & \mathcal{C}_0(\mathcal{M}_\alpha) \arrow[u, "\pi_\alpha"']
\end{tikzcd}$$

We define $\mathcal{N}_{\alpha+1}$ and $\pi_{\alpha+1}$ by applying the Shift Lemma. Following the notation of the lemma, we take $\bar{\mathcal{N}}$ to be the initial segment of $\mathcal{M}_\alpha$ whose last extender is E_α, and $\mathcal{N}$ to be $\pi_\alpha(\bar{\mathcal{N}})$ if $\bar{\mathcal{N}}$ is a proper initial segment of $\mathcal{M}_\alpha$, and $\mathcal{N} = \mathcal{N}_\alpha$ otherwise. (Because we have assumed $\mathcal{M}_\alpha$ is not of type III, $\mathcal{M}_\alpha$ is contained in the domain of π_α.) We take ψ to be the embedding with domain $\mathcal{C}_0(\bar{\mathcal{N}})$ induced by π_α. We let $F_\alpha = \dot{F}^{\mathcal{N}}$. Following further the Shift Lemma notation, $\bar{\mathcal{M}}$ is the initial segment $\mathcal{M}^*_{\alpha+1}$ of $\mathcal{M}_{\mathrm{pred}_T(\alpha+1)}$ to which E_α is applied, and $\pi : \mathcal{C}_0(\bar{\mathcal{M}}) \to \mathcal{C}_0(\mathcal{M})$ is the map induced by π_β, for $\beta = \mathrm{pred}_T(\alpha+1)$.) Let $n = \deg^{\mathcal{T}}(\alpha+1)$, and let $m = \deg^{\pi\mathcal{T}}(\alpha+1)$ be the degree dictated by F_α and our requirement that $\pi\mathcal{T}$ be k-maximal. One can check $n \le m$. If the ultrapower $\mathrm{Ult}_m(\mathcal{C}_0(\mathcal{N}), F_\alpha)$ giving rise to $\mathcal{N}_{\alpha+1}$ is illfounded, as may very well happen, then we stop the construction of $\pi\mathcal{T}$. Otherwise, let $\pi_{\alpha+1} = \tau \circ \sigma$, where σ is given by the Shift Lemma, and $\tau : \mathrm{Ult}_n(\mathcal{C}_0(\mathcal{N}), F_\alpha) \to \mathrm{Ult}_m(\mathcal{C}_0(\mathcal{N}), F_\alpha)$ is the natural map. It is easy to verify the induction hypotheses, and so we can continue.

At limit steps $\lambda < \mathrm{lh}(\mathcal{T})$ we let $\mathcal{N}_\lambda$ be the direct limit over all $\alpha \in [0, \lambda)_T$, α sufficiently large, of the $\mathcal{N}_\alpha$, provided that this limit is wellfounded. We let π_λ be the embedding given by our induction hypothesis (3): $\pi_\lambda(i_{\alpha,\lambda}(x)) = j_{\alpha,\lambda}(\pi_\alpha(x))$. It is easy to verify the induction hypotheses. If the direct limit is illfounded, as may very well happen, we stop the construction of $\pi\mathcal{T}$.

Suppose now that α is such that $\mathcal{M}_\alpha$ is type III. Letting $\bar{\mathcal{N}}$ be the initial segment of $\mathcal{M}_\alpha$ whose last extender is E_α, it is possible then that π_α does not act on $\mathcal{N}$, because the domain of π_α is only the squashed structure $\mathcal{C}_0(\mathcal{M}_\alpha)$. In the next paragraph, we include an outline of how to deal with this case, as a service to the scrupulous reader. We advise the unscrupulous reader to skip it.[25]

Let α be least such that $\mathcal{M}_\alpha$ is type III and let $\beta = \mathrm{pred}_T(\alpha+1)$. If $\bar{\mathcal{N}} = \mathcal{M}_\alpha$, then we can just take F_α to be the last extender of $\mathcal{N}_\alpha$, and everything works out. The problem comes when $\bar{\mathcal{N}}$ is a proper initial segment of $\mathcal{M}_\alpha$, but not in the domain of π_α. But notice then that "un-squashing" upstairs gives $\psi : \mathrm{Ult}(\mathcal{C}_0(\mathcal{M}_\alpha), \bar{F}) \to \mathrm{Ult}(\mathcal{C}_0(\mathcal{N}_\alpha), F)$ which extends π_α, where $\bar{F}$ and

[25] We ignored this problem in [25]. Schlutzenberg found that error, and its repair.

F are the last extenders of $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ respectively. Let $\mathcal{N} = \psi(\bar{\mathcal{N}})$. The problem is that $\mathcal{N}$ may not be an initial segment of $\mathcal{N}_\alpha$. So we extend $\pi\mathcal{T}$ by two steps: first apply F to the appropriate initial segment of the appropriate model (as dictated by maximality), forming $\mathcal{N}_{\alpha+1} = \mathrm{Ult}(\mathcal{Q}, F)$. It is easy to see that $\mathcal{N}$ is a proper initial segment of $\mathcal{P}$. We then take the last extender from $\mathcal{N}$ and apply it to the appropriate initial segment of $\mathcal{N}_\beta$ to get $\mathcal{N}_{\alpha+2}$. We have $\pi_{\alpha+1} : \mathcal{M}_{\alpha+1} \to \mathcal{N}_{\alpha+2}$ given by $\pi_{\alpha+1}([a, f]) = [\psi(a), \pi_\beta(f)]$. Again, everything works out. Thus in general, one step forward in $\mathcal{T}$ may correspond to two steps forward in $\pi\mathcal{T}$, and our copy maps π_γ map $\mathcal{M}^{\mathcal{T}}_\gamma$ to $\mathcal{N}^{\pi\mathcal{T}}_{\tau(\gamma)}$, where $\gamma < \tau(\gamma)$ is possible.

This completes the definition of $\pi\mathcal{T}$.

4.3 Remark. A *near k-embedding* is a weak k-embedding which is fully $r\Sigma_{k+1}$-elementary. If π_0 is a *near* k-embedding, then all π_α are near $\deg^{\mathcal{T}}$-embeddings, and moreover $\deg^{\mathcal{T}}(\alpha) = \deg^{\pi\mathcal{T}}(\alpha)$. See [35, 1.3]. There is an error in [25], where it is claimed that one can copy under weak embeddings, while maintaining both $\deg^{\mathcal{T}}(\alpha) = \deg^{\pi\mathcal{T}}(\alpha)$ and that $\pi\mathcal{T}$ is maximal.[26] See [35] for more on how various degrees of elementarity are propagated in the copying construction.

The Dodd-Jensen Lemma applies only to mice with a slightly stronger iterability property than the one we have introduced. In order to describe this property, we introduce an elaboration of the iteration game $\mathcal{G}_k(\mathcal{M}, \theta)$; a run of the new game is a linear composition of appropriately maximal iteration trees, rather than just a single such tree.

Let θ be an ordinal. In $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$, there are α rounds, the βth being played as follows: Let $\mathcal{Q}$ be the last model in the linear composition produced before round β; that is, let $\mathcal{Q} = \mathcal{M}$ if $\beta = 0$, $\mathcal{Q}$ be the last model of the tree played during round $\beta - 1$ if $\beta > 0$ is a successor, and $\mathcal{Q}$ be the direct limit along the unique cofinal branch in the linear composition of trees produced before β, if β is a limit ordinal. (I wins if this branch is illfounded.) We let q, the degree of $\mathcal{Q}$, be k if $\beta = 0$, the degree of $\mathcal{Q}$ as a model of the tree played during round $\beta - 1$ (see Definition 3.7) if $\beta > 0$ is a successor, and the eventual value of the degrees of previous rounds if β is a limit ordinal. I begins round β by choosing an initial segment $\mathcal{P}$ of $\mathcal{Q}$, and an $i \leq \omega$ such that if $\mathcal{P} = \mathcal{Q}$ then $i \leq q$, where q is the degree of $\mathcal{Q}$. The rest of round β is a run of $\mathcal{G}_i(\mathcal{P}, \theta)$,[27] except that we allow I to exit to round $\beta + 1$ before all θ moves have been played, and we require him to do so, on pain of losing, if θ is limit ordinal. (So if I has not lost, then when round β ends there will be in any case a last model to serve as $\mathcal{Q}$ for round $\beta + 1$.) II wins $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ just in case he does not lose any of the component games and, for $\beta \leq \alpha$ a limit ordinal, the unique cofinal branch in the composition of trees previously produced is wellfounded. A play of this game in which II has not yet lost is called a *k-bounded* iteration tree on $\mathcal{M}$. More commonly now,

[26] Schlutzenberg also found this error, and its repair.

[27] So by our earlier conventions, i is the degree of $\mathcal{P}$.

it is called a *stack of normal trees on $\mathcal{M}$.* Notice that any winning strategy Γ for II in $\mathcal{G}_k(\mathcal{M},\alpha,\theta)$ determines a winning strategy Σ for II in $\mathcal{G}_k(\mathcal{M},\theta)$ in an obvious way: Σ calls for II to play as if he were using Γ in the first round of $\mathcal{G}_k(\mathcal{M},\alpha,\theta)$, and I had not dropped to begin that round.

4.4 Definition. Let $\mathcal{M}$ be a k-sound premouse, where $k \le \omega$; then a (k,α,θ)-*iteration strategy* for $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}_k(\mathcal{M},\alpha,\theta)$, and $\mathcal{M}$ is (k,α,θ)-*iterable* just in case there is such a strategy.

The copying construction enables us to pull back iteration strategies for $\mathcal{N}$ to iteration strategies for premice embedded in $\mathcal{N}$.

4.5 Definition. Let $\pi : \mathcal{M} \to \mathcal{N}$ be a weak k-embedding, and Σ a strategy for II in $\mathcal{G}_k(\mathcal{N},\theta)$, or in $\mathcal{G}_i(\mathcal{P},\alpha,\theta)$ for some $\mathcal{P}$ such that $\mathcal{N}$ is an initial segment of $\mathcal{P}$ and i such that $i \le k$ if $\mathcal{N} = \mathcal{P}$; then the *pullback of* Σ *under* π is the strategy Σ^π in the corresponding game on $\mathcal{M}$ such that for any k-bounded $\mathcal{T}$ on $\mathcal{M}$,

$$\mathcal{T} \textit{ is by } \Sigma^\pi \iff \pi\mathcal{T} \textit{ is by } \Sigma.$$

Clearly, if Σ is a winning strategy for II an iteration game on $\mathcal{N}$, and $\pi : \mathcal{M} \to \mathcal{N}$ is sufficiently elementary, then Σ^π is a winning strategy for II in the corresponding game on $\mathcal{M}$. Thus

4.6 Theorem. *Suppose $\mathcal{N}$ is (k,θ)-iterable (respectively, (k,α,θ)-iterable), and there is a weak k-embedding from $\mathcal{M}$ into $\mathcal{N}$; then $\mathcal{M}$ is (k,θ)-iterable (respectively, (k,α,θ)-iterable).*

4.2. The Dodd-Jensen Lemma

The following definition enables us to state an abstract form of the Dodd-Jensen Lemma.

4.7 Definition. Let Σ be a (k,λ,θ)-iteration strategy for $\mathcal{M}$, where λ is additively closed, and let $\mathcal{T}$ be an iteration tree played according to Σ; then we say $\mathcal{T}$ *is* (k,λ,θ)-*unambiguous* iff whenever $\alpha < \mathrm{lh}(\mathcal{T})$ is a limit ordinal, then $[0,\alpha]_T$ is the unique cofinal branch b of $\mathcal{T}\restriction\alpha$ such that $\mathcal{M}^\mathcal{T}_b$ is $(\deg(b),\lambda,\theta)$-iterable.

So the unambiguous trees are just those which are played according to *every* (k,λ,θ)-iteration strategy for $\mathcal{M}$.

4.8 Theorem (The Dodd-Jensen Lemma). *Let λ be additively closed, let Σ be a (k,λ,θ)-iteration strategy for $\mathcal{M}$, and let $\mathcal{T}$ be an unambiguous iteration tree of length $\alpha+1$ played according to Σ. Suppose* $\deg^\mathcal{T}(\alpha) = k$, *and $\pi : \mathcal{M} \to \mathcal{N}$ is a weak k-embedding, where $\mathcal{N}$ is an initial segment of $\mathcal{M}^\mathcal{T}_\alpha$; then*

1. *$\mathcal{N} = \mathcal{M}^\mathcal{T}_\alpha$,*

2. $[0, \alpha]_T$ *does not drop (in model or degree), and*

3. *for all* $x \in \mathcal{M}$, $i^{\mathcal{T}}_{0,\alpha}(x) \leq_L \pi(x)$, *where* $\leq_L$ *is the order of construction.*

Proof. Assume first toward contradiction that $\mathcal{N}$ is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\alpha$. We shall construct a run r of $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$ which is a loss for Σ. The run r is divided into ω blocks, each consisting of a number of rounds of $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$ equal to the number of rounds in $\mathcal{T}$. We shall use $\mathcal{T}_n$ for the iteration tree played in the nth block of r, and $\mathcal{M}_n$ for the base model of $\mathcal{T}_n$. Thus $\mathcal{M}_{n+1}$ is the model player I drops to at the beginning of the first round in block $n+1$ of r; we have I drop to the degree k at the beginning of this round. We shall arrange that $\mathcal{M}_{n+1}$ is a proper initial segment of the last model of $\mathcal{T}_n$, so that the unique cofinal branch of the composition of the $\mathcal{T}_n$'s is illfounded, and r is indeed a loss for Σ. As an auxiliary we define maps $\pi_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ as we proceed.

Set $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{M}_1 = \mathcal{N}$, and $\pi_0 = \pi$.

Now suppose that $\mathcal{M}_n$, $\mathcal{T}_n$, $\mathcal{M}_{n+1}$, and π_n are given. Set $\mathcal{T}_{n+1} = \pi_n \mathcal{T}_n$. We shall check shortly that $\mathcal{T}_{n+1}$ is played according to Σ, so that $\mathrm{lh}(\mathcal{T}_{n+1}) = \mathrm{lh}(\mathcal{T}_n)$, and we have from the copying construction an embedding σ from the last model of $\mathcal{T}_n$ to the last model of $\mathcal{T}_{n+1}$. Now $\mathcal{M}_{n+1} \in \mathrm{dom}(\sigma)$, so we can set $\mathcal{M}_{n+2} = \sigma(\mathcal{M}_{n+1})$ and $\pi_{n+1} = \sigma \restriction \mathcal{M}_{n+1}$. This completes the construction of r, and thereby gives the desired contradiction.

We now show that $\mathcal{T}_{n+1}$ is a play according to Σ. Let us call a position u which is according to Σ *transitional* if $u = (s, (\mathcal{P}, i))$ where s represents some number $\beta < \lambda$ of complete rounds of play according to Σ in which I has not lost, and $(\mathcal{P}, i)$ is a way I might legally begin round β. Notice that in this situation, Σ determines an (i, λ, θ)-iteration strategy for $\mathcal{P}$. We call this strategy Σ_u. Now let u and v be the transitional initial segments of r ending with $(\mathcal{M}_n, k)$ and $(\mathcal{M}_{n+1}, k)$ respectively. Let $\psi = \pi_{n-1} \circ \cdots \circ \pi_0$ and $\tau = \pi_n \circ \cdots \circ \pi_0$, so that $\psi : \mathcal{M} \to \mathcal{M}_n$ and $\tau : \mathcal{M} \to \mathcal{M}_{n+1}$ are weak k embeddings. Since $(\Sigma_u)^\psi$ and $(\Sigma_v)^\tau$ are (k, λ, θ)-iteration strategies for $\mathcal{M}$ and $\mathcal{T}$ is unambiguous, $\mathcal{T}$ is a play by each of $(\Sigma_u)^\psi$ and $(\Sigma_v)^\tau$. Therefore $\psi\mathcal{T}$ and $\tau\mathcal{T}$ are plays according to Σ, and since $\tau\mathcal{T} = \pi_n \circ \psi\mathcal{T} = \pi_n \mathcal{T}_n = \mathcal{T}_{n+1}$, we are done.

The proofs of conclusions 2 and 3 of the Dodd-Jensen Lemma are similar. We construct $\mathcal{M}_n$, $\mathcal{T}_n$, and π_n as above, but now we have that $\mathcal{M}_{n+1}$ is the last model of $\mathcal{T}_n$. If the branch of $\mathcal{T}$ from $\mathcal{M}$ to $\mathcal{N} = \mathcal{M}_1$ drops, then the branch of $\mathcal{T}_n$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ drops for each n, and the unique cofinal branch of the composition of the $\mathcal{T}_n$'s is illfounded. Thus we may assume that the branch of $\mathcal{T}$ from $\mathcal{M}$ to $\mathcal{N}$ does not drop, so that 2 holds. This implies that for all n, the branch of $\mathcal{T}_n$ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ does not drop, so that we have an iteration map $i_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ given by $\mathcal{T}_n$. Assume that conclusion 3 fails, and fix $x_0 \in \mathcal{M}_0$ such that $\pi_0(x_0) <_L i_0(x_0)$. For any $n \geq 0$, define x_{n+1} by: $x_{n+1} = \pi_n(x_n)$. It is easy to check that $x_{n+1} <_L i_n(x_n)$ for all n. (This is true for $n = 0$ by hypothesis. But if $x_{n+1} <_L i_n(x_n)$,

then

$$x_{n+2} = \pi_{n+1}(x_{n+1}) <_L \pi_{n+1}(i_n(x_n)) = i_{n+1}(\pi_n(x_n)) = i_{n+1}(x_{n+1}),$$

because $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$ by the commutativity of the copy maps.) Thus again, the unique cofinal branch of the composition of the $\mathcal{T}_n$'s is illfounded, and we have a loss for Σ.

The following diagram illustrates the proof given for conclusion 3.

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "i_0=i^{\mathcal{T}}_{0\alpha}"] & \mathcal{M}_1 x \arrow[r, "i_1"] & \mathcal{M}_2 \arrow[r, "i_2"] & \mathcal{M}_3 \arrow[r, "i_3"] & \cdots \\
& \mathcal{M}_0 \arrow[u, "\pi_0=\pi"] \arrow[r, "i_0"] & \mathcal{M}_1 \arrow[u, "\pi_1"] \arrow[r, "i_1"] & \mathcal{M}_2 \arrow[u, "\pi_2"] \arrow[r, "i_2"] & \cdots \\
& & \mathcal{M}_0 \arrow[u, "\pi_0"] \arrow[r, "i_0"] & \mathcal{M}_1 \arrow[u, "\pi_1"] \arrow[r, "i_1"] & \cdots \\
& & & \mathcal{M}_0 \arrow[u, "\pi_0"] \arrow[r, "i_0"] & \cdots
\end{tikzcd}$$

$\dashv$

4.3. The Weak Dodd-Jensen Property

Unfortunately, there are important contexts in which one wants to use the Dodd-Jensen Lemma, but in which one does not know that the given iteration strategy is unambiguous. One such context is the proof of the key fine structural fact that the standard parameters of a sufficiently iterable mouse are solid and universal. (We shall prove this in the next section.) Fortunately, one can construct from any iteration strategy for a countable mouse another iteration strategy which satisfies a weak version of the Dodd-Jensen Lemma, and this weak version suffices for the proof of solidity and universality. Since the construction is simple and natural, we shall give it here.

The notions and results in this subsection come from [31].

Let $\mathcal{M}$ and $\mathcal{P}$ be premice; then we say that $\mathcal{P}$ is $(\mathcal{M}, k)$-*large* just in case there is a near k-embedding from $\mathcal{M}$ to an initial segment of $\mathcal{P}$. (A near k-embedding is a weak k-embedding which is $r\Sigma_{k+1}$ elementary. See [35, 1.2, 1.3], where it is shown that the copying construction gives rise to such embeddings. We could make do with weak k-embeddings here, but it would be a bit awkward at one point.) Let $\vec{e} = \langle e_i \mid i < \omega \rangle$ enumerate the universe of a countable premouse $\mathcal{M}$, and $\pi : \mathcal{M} \to \mathcal{P}$ be a near k-embedding; then we say π is $(k, \vec{e})$-*minimal* iff whenever σ is a near k-embedding from $\mathcal{M}$ to an initial segment $\mathcal{N}$ of $\mathcal{P}$, then $\mathcal{N} = \mathcal{P}$ and either $\sigma = \pi$, or $\sigma(e_i) >_L \pi(e_i)$ where i is least such that $\sigma(e_i) \neq \pi(e_i)$. Notice that if $\mathcal{P}$ is $(\mathcal{M}, k)$-large but no proper initial segment of $\mathcal{P}$ is $(\mathcal{M}, k)$-large, then there is a $(k, \vec{e})$-minimal embedding from $\mathcal{M}$ to $\mathcal{P}$. This embedding is just the leftmost branch through a certain tree.

4.9 Definition. Let Σ be a (k, α, θ)-iteration strategy for a countable premouse $\mathcal{M}$, and let $\vec{e} = \langle e_i \mid i < \omega \rangle$ enumerate the universe of $\mathcal{M}$ in order type ω; then we say Σ has the *weak Dodd-Jensen property* (relative to $\vec{e}$) iff whenever $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ played according to Σ, and $\beta < lh(\mathcal{T})$ is such that $\mathcal{M}_\beta^\mathcal{T}$ is $(\mathcal{M}, k)$-large, then $i_{0,\beta}^\mathcal{T}$ exists and is $(k, \vec{e})$-minimal.

4.10 Theorem (The Weak Dodd-Jensen Lemma). *Suppose $\mathcal{M}$ is (k, ω_1, θ)-iterable, and that $\vec{e}$ enumerates the universe of $\mathcal{M}$ in order type ω; then there is a (k, ω_1, θ)-iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$.*

Proof. Let Σ be any (k, ω_1, θ)-iteration strategy for $\mathcal{M}$. We shall construct a transitional position $u = (r, (\mathcal{P}, k))$ of Σ and a $(k, \vec{e})$-minimal embedding $\pi : \mathcal{M} \to \mathcal{P}$ such that π is strongly $(k, \vec{e})$ minimal, in the sense that whenever $\mathcal{R}$ is an $(\mathcal{M}, k)$-large Σ_u-iterate of $\mathcal{P}$, then there is no dropping in the iteration from $\mathcal{P}$ to $\mathcal{R}$, and if $i : \mathcal{P} \to \mathcal{R}$ is the iteration map, then $i \circ \pi$ is $(k, \vec{e})$ minimal. It is then easy to see that the π-pullback of Σ_u has the weak Dodd-Jensen property.

Let us call a pair $(r, \mathcal{Q})$ *suitable* if $(r, (\mathcal{Q}, k))$ is transitional, and $\mathcal{Q}$ is $(\mathcal{M}, k)$-large but no proper initial segment of $\mathcal{Q}$ is $(\mathcal{M}, k)$-large. In order to obtain the desired u and π, we define by induction on $n < \omega$ suitable pairs $(r_n, \mathcal{P}_n)$. We maintain inductively that r_{n+1} extends $(r_n, (\mathcal{P}_n, k))$. We begin by letting r_0 be the empty position, and $\mathcal{P}_0 = \mathcal{M}$. Now suppose that r_n and $\mathcal{P}_n$ have been defined.

Case 1. There is a suitable $(s, \mathcal{Q})$ such that s extends $(r_n, (\mathcal{P}_n, k))$ and the branch $\mathcal{P}_n$-to-$\mathcal{Q}$ in the iteration given by s has a drop.

In this case, we simply let $(r_{n+1}, \mathcal{P}_{n+1})$ be any such $(s, \mathcal{Q})$.

Case 2. Otherwise.

Let $\tau : \mathcal{M} \to \mathcal{P}_n$ be $(k, \vec{e})$-minimal.

Subcase 2a. There is a suitable $(s, \mathcal{Q})$ such that s extends $(r_n, (\mathcal{P}_n, k))$, and letting $i : \mathcal{P}_n \to \mathcal{Q}$ be the iteration map given by s, $i \circ \tau$ is not $(k, \vec{e})$-minimal.

In this case, let $m < \omega$ be least such that for some such s, $\mathcal{Q}$, and i we have, letting $\sigma : \mathcal{M} \to \mathcal{Q}$ be $(k, \vec{e})$-minimal, that $\sigma(e_m) \neq i \circ \tau(e_m)$ (and thus $\sigma(e_m) <_L i \circ \tau(e_m)$). We then let $(r_{n+1}, \mathcal{P}_{n+1})$ be a suitable pair $(s, \mathcal{Q})$ witnessing this property of m.

Subcase 2b. Otherwise.

In this case τ is strongly $(k, \vec{e})$-minimal in the sense advertised earlier, so we set $u = (r_n, (\mathcal{P}_n, k))$ and $\pi = \tau$, and stop the construction.

Now suppose that the construction never stops. Notice that case 1 can only apply finitely often, since otherwise we get an iteration tree played according to Σ whose unique cofinal branch has infinitely many drops. Suppose then that case 2 applies at all $n \geq n_0$, so that for all $n_0 \leq n \leq m$ we have a k-embedding $i_{n,m} : \mathcal{P}_n \to \mathcal{P}_m$ given by r_m. For $n \geq n_0$, let $\pi_n : \mathcal{M} \to \mathcal{P}_n$ be $(k, \vec{e})$-minimal; then if $n_0 \leq n < m$, π_m is "to the left" of $i_{n,m} \circ \pi_n$. It follows that for any j, $i_{n,m}(\pi_n(e_j)) = \pi_m(e_j)$ for all sufficiently large n, m (by induction on j). Let

$$r = \bigcup_{n<\omega} r_n, \qquad \mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n, \qquad u = (r, (\mathcal{P}, k)),$$

let $i_{n,\infty} : \mathcal{P}_n \to \mathcal{P}$ be the direct limit map (a k-embedding), and define $\pi : \mathcal{M} \to \mathcal{P}$ by

$$\pi(e_j) = \text{ eventual value of } i_{n,\infty}(\pi_n(e_j)), \quad \text{as } n \to \infty.$$

We claim that u and π are as advertised earlier.

Clearly π is a near k-embedding, and so $\mathcal{P}$ is $(\mathcal{M}, k)$-large. No proper initial segment $\mathcal{R}$ of $\mathcal{P}$ can be $(\mathcal{M}, k)$-large, as then $(u, \mathcal{R})$ could serve as the $(s, \mathcal{Q})$ witnessing the occurrence of case 1 at a stage $n > n_0$. Similarly, π is $(k, \vec{e})$-minimal. For if σ is a near k-embedding of $\mathcal{M}$ into $\mathcal{P}$ which is to the left of π, then take m_0 to be the least j such that $\sigma(e_j) \neq \pi(e_j)$, and let $l < \omega$ be so large that $n_0 < l$ and $\pi(e_j) = i_{l,\infty}(\pi_l(e_j))$ for all $j \leq m_0$ (and so $m > m_0$, where m is as in case 2a at stage l). Then r, $\mathcal{P}$, and σ could serve as the s, $\mathcal{Q}$, and σ witnessing $m \leq m_0$ at stage l, contradiction. Finally, let $\mathcal{R}$ be any $(\mathcal{M}, k)$-large iterate of $\mathcal{P}$ via Σ_u. Clearly, there is a transitional position $(v, (\mathcal{R}, k))$ such that v extends u. We can argue as above that there is no dropping in the iteration tree given by v from $\mathcal{P}$ to $\mathcal{R}$, and that if $i : \mathcal{P} \to \mathcal{R}$ is the iteration map, then $i \circ \pi$ is $(k, \vec{e})$-minimal. Thus u and π are as advertised.

We leave to the reader the easy verification that $(\Sigma_u)^\pi$ has the weak Dodd-Jensen property. ⊣

The weak Dodd-Jensen property isolates a unique iteration strategy, modulo the enumeration $\vec{e}$. Since the main ideas in the proof of this fact are used very often in inner model theory, we give it here.

4.11 Theorem. *Let $\vec{e}$ enumerate the universe of the k-sound premouse $\mathcal{M}$ in order type ω; then there is at most one $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$.*

Proof. Suppose that Σ and Γ are distinct such strategies. We can find a k-maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$ such that $\mathcal{T}$ has limit length $\lambda < \omega_1$, $\mathcal{T}$ is played according to both Σ and Γ, and $\Sigma(\mathcal{T}) \neq \Gamma(\mathcal{T})$. Let $\mathcal{U}^*$ and $\mathcal{V}^*$ be the iteration trees of length $\lambda + 1$ extending $\mathcal{T}$ produced by Σ and Γ respectively. We now proceed as if we had produced $\mathcal{U}^*$ and $\mathcal{V}^*$ on the two sides of a coiteration, and continue "iterating the least disagreement". We thereby extend $\mathcal{U}^*$ and $\mathcal{V}^*$ to k-maximal trees $\mathcal{U}$ and $\mathcal{V}$, played according to Σ and Γ respectively, in such a way that the last model of one is an initial segment of the last model of the other. We may as well assume that $\mathcal{M}^{\mathcal{U}}_\alpha$ is an initial segment of $\mathcal{M}^{\mathcal{V}}_\beta$. As in the Comparison Lemma 3.11, one of the two trees does not drop along the branch leading to its last model, so we can assume that $D^{\mathcal{U}} \cap [0,\alpha]_U = \emptyset$ and $\deg^{\mathcal{U}}(\alpha) = k$, and hence $i^{\mathcal{U}}_{0,\alpha}$ exists and is a k-embedding.

It follows that $\mathcal{M}^{\mathcal{V}}_\beta$ is $(\mathcal{M}, k)$-large. Since Γ has the weak Dodd-Jensen property relative to $\vec{e}$, $i^{\mathcal{V}}_{0,\beta}$ exists and is $(k,\vec{e})$-minimal. This implies that no proper initial segment of $\mathcal{M}^{\mathcal{V}}_\beta$ is $(\mathcal{M}, k)$-large, so $\mathcal{M}^{\mathcal{U}}_\alpha = \mathcal{M}^{\mathcal{V}}_\beta$. Because Σ also has the weak Dodd-Jensen property relative to $\vec{e}$, $i^{\mathcal{U}}_{0,\alpha}$ is also $(k,\vec{e})$-minimal. It follows that $i^{\mathcal{U}}_{0,\alpha} = i^{\mathcal{V}}_{0,\beta}$.

Notice that since $\Sigma(\mathcal{T}) \neq \Gamma(\mathcal{T})$, $[0,\alpha]_U \cap [0,\beta]_V$ is bounded in λ. As branches in an iteration tree are closed below their sups, we have a largest ordinal γ such that $\gamma \in [0,\alpha]_U \cap [0,\beta]_V \cap \lambda$. Let $\nu = \sup\{\nu(E^{\mathcal{T}}_\xi) \mid \xi\, T\, \gamma\}$. Every member of $\mathcal{M}^{\mathcal{T}}_\gamma$ is of the form $i^{\mathcal{T}}_{0,\gamma}(f)(a)$, for some $f \in \mathcal{M}$ and $a \in [\nu]^{<\omega}$. (We take $k = 0$ for notational simplicity; otherwise we have f $r\Sigma_k$ over $\mathcal{M}$.) Since $i^{\mathcal{U}}_{\gamma,\alpha}$ and $i^{\mathcal{V}}_{\gamma,\beta}$ have critical point at least ν, this representation of $\mathcal{M}^{\mathcal{T}}_\gamma$ and the fact that $i^{\mathcal{U}}_{0,\alpha} = i^{\mathcal{V}}_{0,\beta}$ yield that $i^{\mathcal{U}}_{\gamma,\alpha} = i^{\mathcal{V}}_{\gamma,\beta}$.

Let $\xi + 1 \in (\gamma,\alpha]_U$ be such that $\mathrm{pred}_U(\xi+1) = \gamma$. Let $\sigma + 1 \in (\gamma,\beta]_V$ be such that $\mathrm{pred}_V(\sigma+1) = \gamma$. Since $i^{\mathcal{U}}_{\gamma,\alpha} = i^{\mathcal{V}}_{\gamma,\beta}$, the extenders $E^{\mathcal{U}}_\xi$ and $E^{\mathcal{V}}_\sigma$ are compatible, that is, they agree up to the inf of the sups of their generators. If $\xi < \lambda$ or $\sigma < \lambda$, this is impossible as no extender used in an iteration tree is compatible with any extender used later in the same tree. (If $\alpha < \beta$ and E_α is compatible with E_β, then $E_\alpha \in \mathcal{M}_\beta$ by the initial segment condition. This implies that $\mathrm{lh}(E_\alpha)$ is not a cardinal in $\mathcal{M}_\beta$, contrary to Lemma 3.5.) If $\lambda \le \xi$ and $\lambda \le \sigma$, this is impossible as no two extenders used in a coiteration are compatible. (This was a subclaim in the proof of the Comparison Lemma 3.11.) This contradiction completes the proof. ⊣

5. Solidity and Condensation

In this section we shall sketch the proofs of two theorems which are central in the fine structural analysis of definability over mice. These results are much deeper than the fine structural results of Sect. 2. Their proofs involve comparison arguments, and hence require an iterability hypothesis. The proofs also use the weak Dodd-Jensen property, and they illustrate a very useful technique for insuring that in certain comparisons, the critical point of the embedding from the first to the last model in one of the trees is not too small.

Our first theorem is a condensation result.

5.1 Theorem. *Let $\mathcal{M}$ be ω-sound and $(\omega,\omega_1,\omega_1+1)$-iterable. Suppose that $\pi : \mathcal{H} \to \mathcal{M}$ is fully elementary, and* $\mathrm{crit}(\pi) = \rho^{\mathcal{H}}_\omega$*; then either*

1. *$\mathcal{H}$ is a proper initial segment of $\mathcal{M}$, or*

2. *there is an extender E on the $\mathcal{M}$-sequence such that $\mathrm{lh}(E) = \rho^{\mathcal{H}}_\omega$, and $\mathcal{H}$ is a proper initial segment of* $\mathrm{Ult}_0(\mathcal{M}, E)$.

5.2 Remarks. The complexities in the statement of Theorem 5.1 are necessary.

1. The hypothesis that $\mathrm{crit}(\pi) = \rho_\omega^{\mathcal{H}}$ is necessary in Theorem 5.1. For notice that $\mathrm{crit}(\pi) > \rho_\omega^{\mathcal{H}}$ is impossible since otherwise we would have $\rho_\omega^{\mathcal{H}} = \rho_\omega^{\mathcal{M}}$, and since $\mathcal{M}$ is ω-sound, this would imply that $\mathrm{crit}(\pi)$ is definable over $\mathcal{M}$ from points in the range of π. On the other hand, $\mathrm{crit}(\pi) < \rho_\omega^{\mathcal{H}}$ can occur while the conclusions of Theorem 5.1 fail: for example, let $\mathcal{M} = \mathrm{Ult}_\omega(\mathcal{H}, E)$, where E is on the $\mathcal{H}$-sequence and $\mathrm{crit}(E) < \rho_\omega^{\mathcal{H}}$, and let π be the canonical embedding.

2. The alternatives in the conclusion of Theorem 5.1 are mutually exclusive, since in the second case the extender E is on the $\mathcal{M}$-sequence, but not on the $\mathcal{H}$-sequence. The following example shows that the second alternative can occur. Suppose that $\mathcal{P}$ is an active, ω-sound mouse, and F is the last extender on the $\mathcal{P}$-sequence. Let $\kappa = \mathrm{crit}(F)$, and suppose $F{\restriction}\alpha$ is on the $\mathcal{P}$-sequence, where $\alpha > (\kappa^+)^{\mathcal{P}}$. (Under weak large cardinal hypotheses, there is such a $\mathcal{P}$.) Let

$$\sigma : \mathrm{Ult}_0(\mathcal{P}, F{\restriction}\alpha) \to \mathrm{Ult}_0(\mathcal{P}, F)$$

be the natural embedding. Since α is a cardinal in $\mathrm{Ult}_0(\mathcal{P}, F{\restriction}\alpha)$ by clause 1 of Definition 2.4, and not a cardinal in $\mathrm{Ult}_0(\mathcal{P}, F)$ because $F{\restriction}\alpha$ is in this model and collapses α, we have that $\alpha = \mathrm{crit}(\sigma)$. Let

$$\mathcal{H} = \mathcal{J}_{\alpha+1}^{\mathrm{Ult}_0(\mathcal{P}, F{\restriction}\alpha)}$$

and

$$\mathcal{M} = \sigma(\mathcal{H}), \pi = \sigma{\restriction}\mathcal{H}.$$

Clearly $\alpha = \mathrm{crit}(\pi) = \rho_\omega^{\mathcal{H}}$, π is fully elementary, and $\mathcal{H}$ is not an initial segment of $\mathcal{M}$.

Proof of Theorem 5.1. Let $\mathcal{H}$ and $\mathcal{M}$ constitute a counterexample. Let $X \prec V_\lambda$ for some limit ordinal λ, with X countable and $\mathcal{H}, \mathcal{M} \in X$, and let $\bar{\mathcal{H}}$ and $\bar{\mathcal{M}}$ be the images of $\mathcal{H}$ and $\mathcal{M}$ under the transitive collapse of X. It is easy to see that $\bar{\mathcal{H}}$ and $\bar{\mathcal{M}}$ still constitute a counterexample to Theorem 5.1. Thus we may assume without loss of generality that $\mathcal{M}$ is countable. We can therefore fix an enumeration $\vec{e}$ of $\mathcal{M}$ in order type ω, and an $(\omega, \omega_1, \omega_1 + 1)$-iteration strategy Σ for $\mathcal{M}$ having the weak Dodd-Jensen property relative to $\vec{e}$.

The natural plan is to compare $\mathcal{H}$ with $\mathcal{M}$, using Σ to iterate $\mathcal{M}$ and Σ^π to iterate $\mathcal{H}$. Suppose that $\mathcal{P}$ is the last model of the tree $\mathcal{T}$ on $\mathcal{H}$ and $\mathcal{Q}$ is the last model of the tree $\mathcal{U}$ on $\mathcal{M}$ in this comparison. We would like to see that $\mathcal{P} = \mathcal{H}$, for then it is clear that $\mathcal{H}$ is an initial segment of $\mathcal{Q}$, and a little further argument, given below, shows that $\mathcal{U}$ uses at most one extender, so that one of the alternatives in the conclusion of Theorem 5.1 must hold. Assume then that $\mathcal{P} \neq \mathcal{H}$.

If the branch $\mathcal{H}$-to-$\mathcal{P}$ of $\mathcal{T}$ drops in model or degree, then $\mathcal{M}$-to-$\mathcal{Q}$ does not drop in model or degree, and $\mathcal{Q}$ is a proper initial segment of $\mathcal{P}$. (Here we use that $\mathcal{T}$ and $\mathcal{U}$ are ω-maximal.) But then, letting $j : \mathcal{M} \to \mathcal{Q}$ be the iteration map, and $\tau : \mathcal{P} \to \mathcal{R}$ be the copy map from $\mathcal{P}$ to the last model of $\pi\mathcal{T}$, we have that $\tau \circ j$ maps $\mathcal{M}$ to a proper initial segment of $\mathcal{R}$, and $\mathcal{R}$ is a Σ-iterate of $\mathcal{M}$. This contradicts the weak Dodd-Jensen property of Σ. Thus $\mathcal{H}$-to-$\mathcal{P}$ does not drop in model or degree, and we have a fully elementary iteration map $i : \mathcal{H} \to \mathcal{P}$.

Since the branch $\mathcal{H}$-to-$\mathcal{P}$ does not drop in model or degree, we must have $\text{crit}(i) < \rho_\omega^{\mathcal{H}}$. Let $\rho = \rho_\omega^{\mathcal{H}}$. Since $\text{crit}(\pi) = \rho$, $\mathcal{H}$ and $\mathcal{M}$ agree below ρ, so that all extenders used in their comparison have length at least ρ. Nevertheless, it is possible that the first extender E used along $\mathcal{H}$-to-$\mathcal{P}$ is such that $\text{crit}(E) < \rho \leq \text{lh}(E)$. This possibility ruins our proof, so we must modify the construction of $\mathcal{T}$ so as to avoid it.

We modify the construction so that if E is an extender used in $\mathcal{T}$ and $\text{crit}(E) < \rho$, then E is used in $\mathcal{T}$ to take an ultrapower of $\mathcal{M}$, or rather the longest initial segment of $\mathcal{M}$ containing only subsets of $\text{crit}(E)$ measured by E, instead of being used to take an ultrapower of $\mathcal{H}$, as it would be in a tree on $\mathcal{H}$. This modification is possible because $\mathcal{M}$ and $\mathcal{H}$ agree below ρ. The system $\mathcal{T}$ we form in this way is not an ordinary iteration tree, but rather a "double-rooted" iteration tree whose base is the pair of models $(\mathcal{M}, \mathcal{H})$. We shall use $\mathcal{P}_\alpha$ for the αth model of $\mathcal{T}$, and E_α for the extender taken from the $\mathcal{P}_\alpha$-sequence and used to form $\mathcal{P}_{\alpha+1}$. Let

$$\mathcal{P}_0 = \mathcal{M}, \quad \text{and} \quad \mathcal{P}_1 = \mathcal{H}.$$

Let $E_0 = \emptyset$, and

$$\nu(E_0) = \rho.$$

For $\alpha \geq 1$, E_α is the extender on the $\mathcal{P}_\alpha$-sequence which participates in its least disagreement with the sequence of the current last model in $\mathcal{U}$. As in an ordinary iteration tree,

$$\text{pred}_T(\alpha + 1) = \text{least } \beta \text{ such that } \text{crit}(E_\alpha) < \nu(E_\beta),$$

and

$$\mathcal{P}_{\alpha+1} = \text{Ult}_n(\mathcal{P}^*_{\alpha+1}, E_\alpha),$$

where $\mathcal{P}^*_{\alpha+1}$ is the longest initial segment of $\mathcal{P}_\beta$ and n is the largest number $\leq \omega$ such that the ultrapower in question makes sense. (That is, we do so in all but one anomalous case, which we shall explain in the next paragraph.) Our convention on $\nu(E_0)$ and the fact that the $\nu(E_\alpha)$ are increasing then implies that if $\text{crit}(E_\alpha) < \rho$, then $\text{pred}_T(\alpha + 1) = 0$, so that E_α is applied in $\mathcal{T}$ to an initial segment of $\mathcal{M}$.

There is one anomalous case here.[28] Suppose that $\text{crit}(E_\alpha) := \kappa < \rho$, and let $\mathcal{P}^*_{\alpha+1}$ be the longest initial segment $\mathcal{Q}$ of $\mathcal{M}$ such that $P(\kappa) \cap \mathcal{Q} \subseteq \mathcal{P}_\alpha$. It

[28] This case was overlooked in [25]. It was discovered by Jensen. Our method of dealing with it is due to Schindler and Zeman; cf. [39].

can happen that $\mathcal{P}^*_{\alpha+1}$ is of type III, with $\nu(\mathcal{P}^*_{\alpha+1}) = \kappa$. (One can show easily then that $\rho = (\kappa^+)^{\mathcal{H}}$, and $\mathcal{P}^*_{\alpha+1} = \mathcal{J}^{\mathcal{M}}_\rho$.) In this case, $\mathrm{Ult}_0(\mathcal{C}_0(\mathcal{P}^*_{\alpha+1}), E_\alpha)$ does not make sense, because $\mathcal{C}_0(\mathcal{P}^*_{\alpha+1})$ has ordinal height $\mathrm{crit}(E_\alpha)$.[29] We must therefore return to our old, naïve meaning for $\mathrm{Ult}_0(\mathcal{P}^*_{\alpha+1}, E_\alpha)$. Let k be the canonical embedding associated to this ultrapower, and let F be the last extender of $\mathcal{P}^*_{\alpha+1}$. Then we set

$$\mathcal{P}_{\alpha+1} = \mathrm{Ult}_\omega(\mathcal{M}, k(F)).$$

Note here that $k(F)$ is indeed a total extender over $\mathcal{M}$ with critical point strictly less than $\rho_\omega(\mathcal{M})$.

Unfortunately, the extender $k(F)$ does not satisfy the initial segment condition, since $F{\restriction}\kappa$ is an initial segment of it not present in $\mathrm{Ult}_0(\mathcal{P}^*_{\alpha+1}, E_\alpha)$. This complicates the comparison argument to follow. We advise the reader who is going through this argument for the first time to simply ignore the anomalous case in the definition of $\mathcal{P}_{\alpha+1}$.

We can lift $\mathcal{T}$ to an ordinary iteration tree on $\mathcal{M}$ as follows. Let

$$\mathcal{R}_0 = \mathcal{R}_1 = \mathcal{M},$$

and let

$$\pi_0 : \mathcal{P}_0 \to \mathcal{R}_0 \quad \text{and} \quad \pi_1 : \mathcal{P}_1 \to \mathcal{R}_1$$

be given by: $\pi_0 =$ identity and $\pi_1 = \pi$. Note that π_0 and π_1 agree below $\nu(E_0)$. We can use (π_0, π_1) to lift $\mathcal{T}$ to a double-rooted tree $(\pi_0, \pi_1)\mathcal{T}$ on the pair $(\mathcal{R}_0, \mathcal{R}_1)$ just as we did in the copying construction for ordinary iteration trees. Since $\mathcal{R}_0 = \mathcal{R}_1 = \mathcal{M}$, the tree $(\pi_0, \pi_1)\mathcal{T}$, which we shall call $\mathcal{S}$, is nothing but an ordinary iteration tree on $\mathcal{M}$.[30]

We form $\mathcal{T}$ and $\mathcal{S}$ at limit stages as follows. Suppose that the initial segment $\mathcal{S}^*$ of $\mathcal{S}$ built so far is a play by Σ; then we can use Σ to obtain a cofinal wellfounded branch of $\mathcal{S}^*$, and as in the ordinary copying construction, the pullback of this branch is a cofinal wellfounded branch of the initial segment $\mathcal{T}^*$ of $\mathcal{T}$ built so far. We extend $\mathcal{S}^*$ and $\mathcal{T}^*$ by choosing these branches. Thus $\mathcal{S}$ is a play by Σ, and $\mathcal{T}$ is a play by its pullback $\Sigma^{(\pi_0,\pi_1)}$.

Since Σ is an $(\omega, \omega_1, \omega_1+1)$ iteration strategy, this inductive construction of $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$ can last as many as ω_1+1 steps. But $\mathcal{H}$ and $\mathcal{M}$ are countable, so as in the proof of the Comparison Lemma 3.11, the comparison represented by $\mathcal{T}$ and $\mathcal{U}$ actually terminates successfully at some countable stage. Let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}$ and $\mathcal{U}$ respectively. Let $\mathcal{R}$ be the last model of $\mathcal{S}$, and $\tau : \mathcal{P} \to \mathcal{R}$ the copy map. The key claim is:

Claim. *$\mathcal{P}$ is above $\mathcal{H}$ in $\mathcal{T}$.*

Proof. If not, then $\mathcal{P}$ is above $\mathcal{M}$ in $\mathcal{T}$. Suppose that the branch $\mathcal{M}$-to-$\mathcal{Q}$ of $\mathcal{U}$ drops in model or degree. Since $\mathcal{T}$ and $\mathcal{U}$ are ω-maximal trees on ω-sound mice, we then have that $\mathcal{P}$ is a proper initial segment of $\mathcal{Q}$, and the

[29] This problem cannot occur in the construction of an ordinary iteration tree, as we verified in the course of describing the successor steps in an iteration game.

[30] We are ignoring here some complications in the anomalous case.

branch $\mathcal{M}$-to-$\mathcal{P}$ of $\mathcal{T}$ does not drop in model or degree, so that there is a fully elementary iteration map $i : \mathcal{M} \to \mathcal{P}$. But then i maps $\mathcal{M}$ to a proper initial segment of a Σ-iterate of $\mathcal{M}$, which contradicts the weak Dodd-Jensen property of Σ. Thus $\mathcal{M}$-to-$\mathcal{Q}$ does not drop, and we have a fully elementary iteration map $j : \mathcal{M} \to \mathcal{Q}$ given by $\mathcal{U}$.

Suppose that the branch $\mathcal{M}$-to-$\mathcal{P}$ of $\mathcal{T}$ drops in model or degree. In this case $\mathcal{Q}$ must be a proper initial segment of $\mathcal{P}$. But then $\tau \circ j$ is a fully elementary map from $\mathcal{M}$ to a proper initial segment of $\mathcal{R}$, which is a Σ-iterate of $\mathcal{M}$. This contradicts the weak Dodd-Jensen property of Σ. Thus $\mathcal{M}$-to-$\mathcal{P}$ does not drop, and we have a fully elementary iteration map $i : \mathcal{M} \to \mathcal{P}$ given by $\mathcal{T}$.

These arguments also show that $\mathcal{P}$ is not a proper initial segment of $\mathcal{Q}$ and $\mathcal{Q}$ is not a proper initial segment of $\mathcal{P}$, so that $\mathcal{P} = \mathcal{Q}$. We claim that $i = j$ as well. For let x be first in the enumeration $\vec{e}$ of $\mathcal{M}$ such that $i(x) \neq j(x)$. If $i(x) <_L j(x)$, then j is an iteration map produced by Σ which is not $\vec{e}$-minimal, contrary to the weak Dodd-Jensen property of Σ. So $j(x) <_L i(x)$. But now, since $\mathcal{M}$-to-$\mathcal{P}$ did not drop in $\mathcal{T}$, the branch $\mathcal{M}$-to-$\mathcal{R}$ does not drop in the copied tree $\mathcal{S}$, and so we have an iteration map $k : \mathcal{M} \to \mathcal{R}$ given by $\mathcal{S}$. The copy maps commute with the tree embeddings, so we have $\tau \circ i = k \circ \pi_0 = k$. But then

$$\tau(j(x)) <_L \tau(i(x)) = k(x),$$

and $\tau \circ j$ witnesses that k is not $\vec{e}$-minimal, contrary to the fact that k is an iteration map produced by Σ. Thus $i = j$.

As in the proofs of Theorems 3.11 and 4.11, this implies that the first extenders used along the branches giving rise to i and j are compatible with each other. If these extenders satisfy the initial segment condition, then as in Theorems 3.11 and 4.11, that is a contradiction because they participated in disagreements when they were used.

We are left with the possibility that the first extender G used in i comes from our anomalous case. Here $G = k(F)$, where $k : \mathcal{J}_\rho^{\mathcal{M}} \to \mathrm{Ult}_0(\mathcal{J}_\rho^{\mathcal{M}}, E_\alpha)$ is the canonical embedding, and F is the last extender of $\mathcal{J}_\rho^{\mathcal{M}}$. We also have $\mathrm{crit}(k) = \nu(F)$, so that $F{\restriction}\nu(F)$ is an initial segment of G. It is in fact the first initial segment of G which is not in $\mathcal{P}$, and since it is compatible with the first extender used in j (which itself satisfies the initial segment condition), the trivial completion of $F{\restriction}\nu(F)$ *is* the first extender used in j. One can now show that the second extender used in j is compatible with E_α, and that is a contradiction because both of these extenders satisfy the initial segment condition. To prove the compatibility, one uses that for $A \subseteq \mathrm{crit}(G)$, $i_G(A) = k(i_F(A))$. The reader can find the remaining details in [39]. ⊣

So $\mathcal{P}$ is above $\mathcal{H}$ in $\mathcal{T}$. The branch $\mathcal{H}$-to-$\mathcal{P}$ cannot drop in model or degree, since otherwise $\mathcal{Q}$ is a proper initial segment of $\mathcal{P}$ and we have a fully elementary iteration map $j : \mathcal{M} \to \mathcal{Q}$, so that $\tau \circ j$ maps $\mathcal{M}$ into a proper initial segment of the Σ-iterate $\mathcal{R}$. Thus we have a fully elementary iteration map $i : \mathcal{H} \to \mathcal{P}$ given by $\mathcal{T}$. If i is not the identity, then the rules for $\mathcal{T}$ guarantee $\mathrm{crit}(i) \geq \rho$, so that $\mathcal{H}$-to-$\mathcal{P}$ would have to drop in model or degree at its first step. Therefore i is the identity; that is, $\mathcal{H} = \mathcal{P}$.

$\mathcal{Q}$ cannot be a proper initial segment of $\mathcal{H}$, for otherwise $\mathcal{M}$-to-$\mathcal{Q}$ does not drop, and letting j be the iteration map, $\tau \circ j$ maps $\mathcal{M}$ to a proper initial segment of itself. It cannot be that $\mathcal{H} = \mathcal{Q}$, for if so, then $\mathcal{M}$-to-$\mathcal{Q}$ does not drop, and letting j be the iteration map, $\rho_\omega^{\mathcal{H}} < \rho_\omega^{\mathcal{M}} \leq j(\rho_\omega^{\mathcal{M}}) = \rho_\omega^{\mathcal{Q}}$. Thus $\mathcal{H}$ is a proper initial segment of $\mathcal{Q}$.

We can now complete the proof of Theorem 5.1. Suppose that $\mathcal{H}$ is not an initial segment of $\mathcal{M}$, so that $\mathcal{U}$ uses at least one extender $E_0^{\mathcal{U}}$. Now $\rho \leq \mathrm{lh}(E_0^{\mathcal{U}})$ because $\mathcal{H}$ and $\mathcal{M}$ agree below ρ, while $\mathrm{lh}(E_0^{\mathcal{U}}) \leq \mathrm{On}^{\mathcal{H}}$ because $\mathcal{H}$ is not an initial segment of $\mathcal{M}$. But $\mathrm{lh}(E_0^{\mathcal{U}})$ is a cardinal of $\mathcal{Q}$, and $\mathcal{H}$ is a proper initial segment of $\mathcal{Q}$, so that $|\,\mathrm{On}^{\mathcal{H}}\,| \leq \rho_\omega^{\mathcal{H}}$ in $\mathcal{Q}$. It follows that $\mathrm{lh}(E_0^{\mathcal{U}}) = \rho$. Similarly, if $E_1^{\mathcal{U}}$ exists, then we must have $\mathrm{On}^{\mathcal{H}} < \mathrm{lh}(E_1^{\mathcal{U}})$, so in fact $E_1^{\mathcal{U}}$ does not exist. This means that $\mathcal{Q} = \mathrm{Ult}_k(\mathcal{M}, E_0^{\mathcal{U}})$ for some k. We can take $k = 0$ because $\mathrm{Ult}_0(\mathcal{M}, E_0^{\mathcal{U}})$ and $\mathrm{Ult}_k(\mathcal{M}, E_0^{\mathcal{U}})$ agree to their common value for ρ^+ and beyond. ⊣

One can prove a version of Theorem 5.1 in which $\rho_\omega^{\mathcal{H}}$ is replaced by $\rho_n^{\mathcal{H}}$, for some $n < \omega$. See [25, Sect. 8].

The technique by which Theorem 5.1 is proved is useful in many circumstances. One wants to compare two mice $\mathcal{H}$ and $\mathcal{M}$ in such a way that the iteration map on the $\mathcal{H}$ side has critical point at least ρ. An ordinary comparison might not have this property, but one finds models (such as $\mathcal{M}$ itself in the proof above) which agree with $\mathcal{H}$ to various extents below ρ, yet in some sense carry more information than $\mathcal{H}$. One then forms a many-rooted iteration tree on $\mathcal{H}$ "backed up" by these other models, and argues that the final model on this tree lies above the root $\mathcal{H}$. One can view the proof of Theorem 4.11 in this light.[31] Another important application of the technique lies in the proof of the following central fine-structural result concerning the good behavior of the standard parameter.

5.3 Theorem. *Let $k < \omega$, and let $\mathcal{M}$ be a k-sound, $(k, \omega_1, \omega_1 + 1)$-iterable premouse; then $\mathcal{C}_{k+1}(\mathcal{M})$ exists, and agrees with $\mathcal{M}$ below γ, for all γ of $\mathcal{M}$-cardinality $\rho_{k+1}(\mathcal{M})$.*

Sketch of Proof. We assume $k = 0$ for notational simplicity, and because only in that case have we given full definitions anyway. Let $r = p_1(\mathcal{M})$ be the first standard parameter of $\mathcal{M}$; we must show that r is 1-solid and 1-universal, so that $\mathcal{C}_1(\mathcal{M})$ exists, and that $\mathcal{C}_1(\mathcal{M})$ agrees with $\mathcal{M}$ as claimed. These properties of r and $\mathcal{M}$ are expressed by sentences in the first order theory of $\mathcal{M}$, so if they fail, they fail in some countable fully elementary

[31] In Theorem 4.11 one wanted to compare the last models of $\mathcal{U}^*$ and $\mathcal{V}^*$, but for the proof it was important to back them up with the earlier models of $\mathcal{T}$. Many-rooted iteration trees are also important in the inductive definition of K [43, Sect. 6], and in the proof of weak covering for K [26].

submodel of $\mathcal{M}$. Any countable elementary submodel of $\mathcal{M}$ inherits its $(0, \omega_1, \omega_1 + 1)$-iterability. Thus we may assume without loss of generality that $\mathcal{M}$ is countable.

We shall assume that r is solid, and briefly sketch the proof that r is universal and $\mathcal{C}_1(\mathcal{M})$ agrees with $\mathcal{M}$ below the cardinal successor of $\rho_1(\mathcal{M})$ in $\mathcal{M}$. So let $\rho = \rho_1(\mathcal{M})$, and let

$$\mathcal{H} = \mathcal{H}_1^{\mathcal{M}}(\rho \cup \{r\}).$$

We wish to show that $P(\rho) \cap \mathcal{M} \subseteq \mathcal{H}$, and for this the natural strategy is to compare $\mathcal{H}$ with $\mathcal{M}$. If the critical point of the embedding i from $\mathcal{H}$ to the last model $\mathcal{P}$ on the $\mathcal{H}$ side is at least ρ, then the $\Sigma_1^{\mathcal{H}}$ set $A \subseteq \rho$ which is not in $\mathcal{M}$ (witnessing that $\rho = \rho_1^{\mathcal{M}}$) is also $\Sigma_1^{\mathcal{P}}$. Since A is not in the last model $\mathcal{Q}$ on the $\mathcal{M}$ side, $\mathcal{Q}$ is an initial segment of $\mathcal{P}$, and one can then argue that

$$P(\rho)^{\mathcal{M}} = P(\rho)^{\mathcal{Q}} \subseteq P(\rho)^{\mathcal{P}} = P(\rho)^{\mathcal{H}},$$

as desired. In order to insure that $\mathrm{crit}(i) \geq \rho$, we once again form a double-rooted tree on the pair $(\mathcal{M}, \mathcal{H})$ on the $\mathcal{H}$ side of our comparison, going back to $\mathcal{M}$ whenever we use an extender with critical point $< \rho$.

Let $r = \langle \alpha_0, \dots, \alpha_n \rangle$, where the ordinals α_i are listed in decreasing order. Let $\vec{e}$ be an enumeration of the universe of $\mathcal{M}$ such that $e_i = \alpha_i$ for all $i \leq n$. Let Σ be a $(0, \omega_1, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ having the weak Dodd-Jensen property relative to $\vec{e}$. Let $\pi_0 =$ identity and $\pi_1 : \mathcal{H} \to \mathcal{M}$ be the collapse embedding. We form the double-rooted tree $\mathcal{T}$ on $(\mathcal{M}, \mathcal{H})$ using the pullback $\Sigma^{(\pi_0, \pi_1)}$ of Σ to choose branches at limit stages, and iterating the least disagreement with the last model of the tree $\mathcal{U}$ on $\mathcal{M}$ at successor stages. Let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{T}$ and $\mathcal{U}$.

As in the proof of Theorem 5.1, the weak Dodd-Jensen property of Σ implies that $\mathcal{P}$ is above $\mathcal{H}$, and not above $\mathcal{M}$, and that $\mathcal{H}$-to-$\mathcal{P}$ does not drop, and that $\mathcal{Q}$ is not a proper initial segment of $\mathcal{P}$. Thus we have a 0-embedding $i : \mathcal{H} \to \mathcal{P}$ given by $\mathcal{T}$. Since $\mathrm{crit}(i) \geq \rho$, A is $\Sigma_1^{\mathcal{P}}$, and since $A \notin \mathcal{Q}$, $\mathcal{P}$ is not a proper initial segment of $\mathcal{Q}$. Thus $\mathcal{P} = \mathcal{Q}$. We also get that $\mathcal{M}$-to-$\mathcal{Q}$ does not drop, so that $\mathcal{U}$ gives us an embedding $j : \mathcal{M} \to \mathcal{Q}$.

Let $\bar{\alpha}_e = \pi_1^{-1}(\alpha_e)$ be the image of α_e under collapse, for $e \leq n$. One can show by induction on e that

$$i(\bar{\alpha}_e) = j(\alpha_e),$$

using the solidity of $j(r)$ to show $i(\bar{\alpha}_e) \geq j(\alpha_e)$, and using the weak Dodd-Jensen property for the copied tree $(\pi_0, \pi_1)\mathcal{T}$ to show $i(\bar{\alpha}_e) \leq j(\alpha_e)$. (This is where we use the fact that $e_i = \alpha_i$ for all $i \leq n$.)

It follows that $\mathrm{crit}(j) \geq \rho$. For otherwise, letting $\kappa = \mathrm{crit}(j)$, and S be the Σ_1 theory in $\mathcal{M}$ of parameters from $\kappa \cup \{r\}$, then $S \in \mathcal{M}$. But then $j(S) \in \mathcal{Q}$, and from $j(S)$ one can compute the Σ_1 theory in $\mathcal{Q}$ of parameters from $j(\kappa) \cup \{j(r)\}$. (This is like the proof of Lemma 2.23 which we hinted at earlier.) Now $\rho < j(\kappa)$, $\mathcal{P} = \mathcal{Q}$, and $i(\bar{r}) = j(r)$, so this means the Σ_1 theory of $\rho \cup \{i(\bar{r})\}$ is in $\mathcal{P}$. This implies $A \in \mathcal{P}$, a contradiction.

Since i and j have critical point above ρ, $P(\rho)^{\mathcal{H}} = P(\rho)^{\mathcal{P}} = P(\rho)^{\mathcal{Q}} = P(\rho)^{\mathcal{M}}$, as desired. Also, $\mathcal{H} = \mathcal{C}_{k+1}(\mathcal{M})$ agrees with $\mathcal{P}$, hence $\mathcal{Q}$, hence $\mathcal{M}$, below any γ of $\mathcal{M}$-cardinality ρ, as desired. ⊣

One can use fine-structural condensation results such as Theorem 5.1 to show that iterable mice satisfy many of the useful combinatorial principles which Jensen has shown are true in L. For example

5.4 Theorem. *Let $\mathcal{M}$ be an $(\omega, \omega_1, \omega_1 + 1)$-iterable premouse satisfying the axioms of* ZF*, except perhaps Power Set; then the following are true in $\mathcal{M}$:*

1. *for all uncountable regular κ, $\diamondsuit_\kappa$,*
2. *for all uncountable regular κ ($\diamondsuit_\kappa^+ \iff \kappa$ is not ineffable),*
3. *for all infinite cardinals κ, $\square_\kappa$.*

Clause 1 follows immediately from Theorem 5.1 and Jensen's argument for L. Clause 2 is due to Ernest Schimmerling [33]. Clause 3 is work of Schimmerling and Zeman [37], building on the earlier work of Jensen, Solovay, Welch, Wylie, and Schimmerling. (See [10, 48, 49, 33], and [34].)

It follows immediately from Theorem 5.3 that if $\mathcal{M}$ is sufficiently iterable, then $\mathcal{C}_\omega(\mathcal{M})$ exists. We shall use this heavily in the construction of an iterable model, all of whose levels are ω-sound. We turn to that construction now.

6. Background-Certified Fine Extender Sequences

We have been studying mice in the abstract, but we have yet to produce any! In this section we shall describe a certain family of mouse constructions which we call, for obscure reasons, K^c*-constructions*. Such constructions are sufficiently cautious about adding extenders to the model that one gets an iterable model in the end,[32] yet can be sufficiently daring that they can capture the large cardinal strength present in the universe.[33]

6.1. K^c-Constructions

The natural idea is to construct a fine extender sequence $\vec{E}$ by induction. Given $\vec{E}\restriction\alpha$, we set $E_\alpha = \emptyset$ unless there is a certified[34] extender F such that $(\vec{E}\restriction\alpha)^\frown F$ is still a fine extender sequence; if there is such an F we may either set $E_\alpha = F$ or set $E_\alpha = \emptyset$. Here "certified" means roughly that F

[32] This is something between a conjecture and a theorem; see below.
[33] Again, there are qualifications to come.
[34] Whence the "c" in K^c.

is the restriction to $J_\alpha^{\vec{E}\restriction\alpha}$ of a "background extender" F^* which measures a broader collection of subsets of its critical point than does F, and whose ultrapower agrees with V a bit past $\nu(F)$. This background-certificate demand is necessary in order to insure that the premice we are constructing are iterable. Unfortunately, the background certificate demand conflicts with the demand that all levels of the model we are constructing be ω-sound.[35] K^c-constructions deal with this conflict by continually replacing the premouse $\mathcal{N}_\alpha$ currently approximating the model being built by its core $\mathcal{C}_\omega(\mathcal{N}_\alpha)$. Taking cores insures soundness, while the background extenders one can resurrect by going back into the history of the construction insure iterability.

This last claim must be qualified. We do not have a general proof of iterability for the premice $\mathcal{N}_\alpha$ produced in K^c-constructions. At the moment, in order to prove that such a premouse is appropriately iterable, we need to make an additional "smallness" assumption. One assumption that suffices, and which we shall spell out in more detail shortly, is that no initial segment of $\mathcal{N}_\alpha$ satisfies "There is an extender E on my sequence such that $\nu(E)$ is a Woodin cardinal". We shall call this property of $\mathcal{N}_\alpha$ *tameness*. Iterability is essential from the very beginning, for our proof that $\mathcal{C}_\omega(\mathcal{N}_\alpha)$ exists involves comparison arguments, and hence relies on the iterability of $\mathcal{N}_\alpha$. Thus, for all we know, a K^c-construction might simply break down by reaching a non-tame premouse $\mathcal{N}_\alpha$ such that $\mathcal{C}_\omega(\mathcal{N}_\alpha)$ does not exist.

The following definitions describe our background certificate condition. They come from [43, Sect. 1].

6.1 Definition. Let $\mathcal{M}$ be an active premouse, F the extender coded by $\dot{F}^{\mathcal{M}}$ (i.e. its last extender), $\kappa = \mathrm{crit}(F)$, and $\nu = \nu(F)$. Let $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$; then an *$\mathcal{A}$-certificate for $\mathcal{M}$* is a pair (N, G) such that

1. N is a transitive, power admissible set, $V_\kappa \cup \mathcal{A} \subseteq N$, N is closed under ω-sequences, and G is an extender over N,
2. $F \cap ([\nu]^{<\omega} \times \mathcal{A}) = G \cap ([\nu]^{<\omega} \times \mathcal{A})$,
3. $V_{\nu+1} \subseteq \mathrm{Ult}(N, G)$, and
4. $\forall\gamma(\omega\gamma < \mathrm{On}^{\mathcal{M}} \longrightarrow \mathcal{J}_\gamma^{\mathcal{M}} = \mathcal{J}_\gamma^{i(\mathcal{J}_\kappa^{\mathcal{M}})})$, where $i = i_G^N$ is the canonical embedding from N to $\mathrm{Ult}(N, G)$.

6.2 Definition. Let $\mathcal{M}$ be an active premouse, and κ the critical point of its last extender. We say $\mathcal{M}$ *is countably certified* iff for every countable $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{M}}$, there is an $\mathcal{A}$-certificate for $\mathcal{M}$.

In the situation described in Definition 6.1, we shall typically have $|N| = \kappa$, so that $\mathrm{On}^N < \mathrm{lh}(G)$. We are therefore not thinking of (N, G) as a structure

[35] Part of the requirement on F^* is that it be countably complete, and so $\mathrm{crit}(F^*)$ must be uncountable; on the other hand, if α is least so that $E_\alpha \neq \emptyset$, then $(J_\alpha^{\vec{E}\restriction\alpha}, \in, \vec{E}\restriction\alpha, E_\alpha)$ has Σ_1 projectum ω, so that $\mathrm{crit}(E_\alpha)$ must be countable if this structure is even 1-sound.

to be iterated; N simply provides a reasonably large collection of sets to be measured by G. The conditions $V_\kappa \subseteq N$ and $V_{\nu+1} \subseteq \mathrm{Ult}(N, G)$ are crucial (although the former can be weakened in a useful way; cf. [36, 2.1]). Power admissibility is simply a convenient fragment of ZFC; it can probably be weakened substantially.

6.3 Definition. A K^c*-construction* is a sequence $\langle \mathcal{N}_\alpha \mid \alpha < \theta \rangle$ of premice such that

1. $\mathcal{N}_0 = (V_\omega, \in, \emptyset, \emptyset)$;
2. if $\alpha + 1 < \theta$, then $\mathcal{N}_\alpha$ is ω-solid, and letting $\mathcal{M}$ be the unique ω-sound premouse such that $\mathcal{C}_\omega(\mathcal{N}_\alpha) = \mathcal{C}_\omega(\mathcal{M})$, either
   (a) $\mathcal{M}$ is passive, and $\mathcal{N}_{\alpha+1}$ is a countably certified premouse of the form $(|\mathcal{M}|, \in, \dot{E}^{\mathcal{M}}, F)$, for some F, or
   (b) letting $\omega\gamma = \mathrm{On}^{\mathcal{M}}$ and $\vec{E} = \dot{E}^{\mathcal{M}} \oplus \dot{F}^{\mathcal{M}}$, we have that $\mathcal{N}_{\alpha+1} = (J^{\vec{E}}_{\gamma+1}, \in, \vec{E}, \emptyset)$;
3. if $\lambda < \theta$ is a limit ordinal, then $\mathcal{N}_\lambda$ is the unique passive premouse $\mathcal{P}$ such that for all β, $\omega\beta < \mathrm{On}^{\mathcal{P}}$ iff $\mathcal{J}^{\mathcal{N}_\alpha}_\beta$ is defined and eventually constant as $\alpha \to \lambda$, and for all β such that $\omega\beta < \mathrm{On}^{\mathcal{P}}$, $\mathcal{J}^{\mathcal{P}}_\beta =$ eventual value of $\mathcal{J}^{\mathcal{N}_\alpha}_\beta$, as $\alpha \to \lambda$.

So at successor steps in a K^c-construction one replaces the previous model with its ωth core, and then either adds a countably certified extender to the resulting extender sequence or takes one step in its constructible closure. At limit steps one forms the natural "lim inf" of the previous premice.

Because we replace $\mathcal{N}_\alpha$ by its core at each step in a K^c-construction, the models of the construction may not grow by end-extension, and we need a little argument to show, for example, that a construction of proper class length converges to a premouse of proper class size. Our Theorem 5.3 on the agreement of $\mathcal{N}$ with $\mathcal{C}_\omega(\mathcal{N})$ is the key here.

6.4 Theorem. *Let κ be an uncountable regular cardinal or $\kappa = \mathrm{On}$, and let $\langle \mathcal{N}_\alpha \mid \alpha < \kappa \rangle$ be a K^c-construction; then there is a unique premouse $\mathcal{N}_\kappa$ of ordinal height κ such that $\langle \mathcal{N}_\alpha \mid \alpha \leq \kappa \rangle$ is a K^c-construction.*

Proof. For any limit ordinal κ and K^c-construction $\langle \mathcal{N}_\alpha \mid \alpha < \kappa \rangle$, there is a unique premouse $\mathcal{N}_\kappa$ satisfying the limit ordinal clause of Definition 6.3. We need only show that $\mathcal{N}_\kappa$ has ordinal height κ in the case κ is an uncountable cardinal or $\kappa = \mathrm{On}$. It is clear that $|\mathcal{N}_\alpha| < \kappa$ for all $\alpha < \kappa$, so $\mathcal{N}_\kappa$ has ordinal height $\leq \kappa$.

For $\nu < \kappa$, let

$$\vartheta_\nu = \inf\{\rho_\omega(\mathcal{N}_\alpha) \mid \nu \leq \alpha < \kappa\}.$$

So $\vartheta_0 = \omega$, and the ϑ's are nondecreasing. By Theorem 5.3, $\mathcal{N}_\nu$ agrees with all later $\mathcal{N}_\alpha$ below ϑ_ν, so if $\kappa = \sup(\{\vartheta_\nu \mid \nu < \kappa\})$, we are done. Since κ is regular, the alternative is that the ϑ's are eventually constant; say $\vartheta_\nu = \rho$ for all ν such that $\eta \le \nu < \kappa$. Now notice that if $\eta \le \nu < \kappa$ and $\rho_\omega(\mathcal{N}_\nu) = \rho$, then $\mathcal{C}_\omega(\mathcal{N}_\nu)$ is a proper initial segment of $\mathcal{N}_{\nu+1}$.[36] Moreover, $\mathcal{C}_\omega(\mathcal{N}_\nu)$ has cardinality ρ in $\mathcal{N}_{\nu+1}$ by soundness. It follows from Theorem 5.3 that $\mathcal{C}_\omega(\mathcal{N}_\nu)$ is an initial segment of $\mathcal{N}_\alpha$, for all $\alpha \ge \nu$. Since there are cofinally many $\nu < \kappa$ such that $\rho = \rho_\omega(\mathcal{N}_\nu)$, we again get that $\mathcal{N}_\kappa$ has height κ. ⊣

It is not hard to see that the ϑ_ν defined in the proof above are just the infinite cardinals of $\mathcal{N}_\kappa$.

6.2. The Iterability of K^c

It is clear by now that we have gotten nowhere unless we can prove that the premice we have constructed are sufficiently iterable. Here we encounter the central open problem of inner model theory. We formulate one instance of it as a conjecture:

6.5 Conjecture. Suppose $\mathcal{N}$ is a premouse occurring in a K^c-construction, that $k \le \omega$, and that $\mathcal{M}$ is a countable premouse such that there is a weak k-embedding from $\mathcal{M}$ into $\mathcal{C}_k(\mathcal{N})$; then $\mathcal{M}$ is $(k, \omega_1, \omega_1 + 1)$-iterable.

A proof of this conjecture would yield at once the basics of inner model theory at the level of models with superstrong cardinals.[37] At present we can prove the conjecture only for certain small mice.

In general, iterability proofs break up into an *existence* proof and a *uniqueness* proof for "sufficiently good" branches in iteration trees on the premice under consideration. The existence proof itself breaks into two parts, a direct existence argument in the countable case and a reflection argument in the uncountable case.

The direct existence argument applies to countable iteration trees on countable elementary submodels of the premice under consideration, and proceeds by using something like the countable completeness of the extenders involved in the iteration to transform an ill-behaved iteration into an infinite descending $\in$-chain. When coupled with the uniqueness proof, this shows that any countable elementary submodel of a premouse under consideration has an ω_1-iteration strategy, namely, the strategy of choosing the unique cofinal "sufficiently good" branch.[38]

The reflection argument extends this method of iterating to the uncountable: given an iteration tree $\mathcal{T}$ on $\mathcal{M}$, we go to $V[G]$ where G is $\mathrm{Col}(\omega, \kappa)$-generic over V and κ is large enough that $\mathcal{M}$ and $\mathcal{T}$ have become countable, and find a sufficiently good branch there. This branch is unique, and hence

[36] Assume the last extender predicate of $\mathcal{N}_\nu$ is empty here, as it obviously is for cofinally many such ν.

[37] New problems arise between superstrong and supercompact cardinals.

[38] Of course a sufficiently good branch must be wellfounded, but in general more is required, for we want to be able to find cofinal wellfounded branches later in the iteration game as well.

by the homogeneity of the collapse it is in V. In order to execute this argument one needs a certain level of absoluteness between V and $V[G]$. Once one gets past mice with Woodin cardinals, "sufficiently good" can no longer be taken simply to mean "wellfounded", and in fact "sufficiently good" is no longer a Σ^1_2 notion at all. Because of this, the generic absoluteness required by our reflection argument needs large cardinal/mouse existence principles that go beyond ZFC.[39]

The conjecture above overlaps slightly with the uncountable case because it is (ω_1+1)-iterability, rather than ω_1-iterability, which is at stake. One needs (ω_1+1)-iterability to guarantee the comparability of countable mice; the reflection argument that shows coiterations terminate requires a wellfounded branch of length ω_1. Nevertheless, we believe that the conjecture is provable in ZFC.[40]

At present, the strongest partial results on Conjecture 6.5 are those of [1] and [30], which show that it holds for levels $\mathcal{N}$ of K^c which are of limited complexity, in that they do not have too many extenders overlapping local Woodin cardinals. (Ref. [30] goes further than Conjecture 6.5, to levels with Woodin limits of Woodin cardinals, but it applies only to K^c constructions in which the background extenders measure all sets in V.) In this chapter we shall consider only premice having no extenders overlapping local Woodin cardinals. We call these special premice "tame". We shall outline a proof of Conjecture 6.5 for the tame levels of K^c. Our direct existence argument in the countable case seems perfectly general, but our uniqueness results are less definitive, and it is here that we resort to the tameness assumption. We begin by stating the existence theorem in the countable case.

We say that a branch b of an iteration tree $\mathcal{T}$ is *maximal* iff b has limit order type but is not continued in $\mathcal{T}$. Such a b must be $\in$-cofinal in some $\lambda \leq \mathrm{lh}(\mathcal{T})$, but different from $[0,\lambda]_T$ if $\lambda < \mathrm{lh}(\mathcal{T})$. Notice that any cofinal branch of $\mathcal{T}$ is maximal; the converse fails in general. Finally, a *putative* iteration tree is just like an ordinary iteration tree, except that we allow the last model, if there is one, to be illfounded.

6.6 Theorem (Branch Existence Theorem). *Let $\pi : \mathcal{M} \to \mathcal{C}_k(\mathcal{N}_\alpha)$ be a weak k-embedding, where $\mathcal{M}$ is countable and $\langle \mathcal{N}_\beta \mid \beta < \theta \rangle$ is a K^c-construction. Let $\mathcal{T}$ be a countable, k-maximal, putative iteration tree on $\mathcal{M}$; then either*

1. *There is a maximal branch b of $\mathcal{T}$ such that, letting $l = \deg^{\mathcal{T}}(b)$,*

    (a) *$D^{\mathcal{T}} \cap b = \emptyset$ and $l = k$, and there is a weak l-embedding $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{C}_l(\mathcal{N}_\alpha)$ such that*

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[39] For example, if it is consistent that there is a Woodin cardinal, then it is consistent that there is a premouse $\mathcal{N}$ occurring on a K^c-construction which is not θ-iterable for some θ.

[40] We suspect that if κ is strictly less than the infimum of the critical points of the background extenders, then the κ-iterability of the size κ elementary submodels of premice in a K^c-construction is provable in ZFC.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\pi"] \arrow[dr, "i_{0,b}"'] & \mathcal{C}_l(\mathcal{N}_\alpha) \\ & \mathcal{M}_b^{\mathcal{T}} \arrow[u, "\sigma"'] \end{tikzcd}$$

commutes, or

(b) $D^{\mathcal{T}} \cap b \neq \emptyset$ *or* $l < k$*, and there is a* $\beta \leq \alpha$ *and weak* l*-embedding* $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{C}_l(\mathcal{N}_\beta)$*, with* $\beta < \alpha$ *if* $D^{\mathcal{T}} \cap b \neq \emptyset$*, or*

2. $\mathcal{T}$ *has a last model* $\mathcal{M}_\gamma^{\mathcal{T}}$ *such that, letting* $l = \deg^{\mathcal{T}}(\gamma)$*,*

(a) $D^{\mathcal{T}} \cap [0, \gamma]_T = \emptyset$ *and* $l = k$*, and there is a weak* l*-embedding* $\sigma : \mathcal{M}_\gamma^{\mathcal{T}} \to \mathcal{C}_l(\mathcal{N}_\alpha)$ *such that*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\pi"] \arrow[dr, "i_{0,\gamma}"'] & \mathcal{C}_l(\mathcal{N}_\alpha) \\ & \mathcal{M}_\gamma^{\mathcal{T}} \arrow[u, "\sigma"'] \end{tikzcd}$$

commutes, or

(b) $D^{\mathcal{T}}[0, \gamma]_T \neq \emptyset$ *or* $l < k$*, and there is a* $\beta \leq \alpha$ *and weak* l*-embedding* $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{C}_l(\mathcal{N}_\beta)$*, with* $\beta < \alpha$ *if* $D^{\mathcal{T}} \cap [0, \gamma]_T \neq \emptyset$*.*

We shall not attempt to prove this theorem here. The reader can find a proof in [43, Sects. 2 and 9]. The theorem in the form stated here evolved from earlier results of [18] and [25].

If b is a branch satisfying clause 1 of the conclusion of the Branch Existence Theorem, then we say b (or $\mathcal{M}_b^{\mathcal{T}}$) is π-*realizable*, and call the map σ described in clause 1 a π-*realization* of b (or $\mathcal{M}_b^{\mathcal{T}}$). Similarly, if γ satisfies clause 2 of the conclusion, then we say γ (or $\mathcal{M}_\gamma^{\mathcal{T}}$) is π-realizable, and call the associated map σ a π-realization.

Given $\mathcal{M}$ and π as in the hypotheses of the Branch Existence Theorem, it is natural to attempt to iterate $\mathcal{M}$ using the following strategy: given $\mathcal{T}$ on $\mathcal{M}$ of countable limit length, pick the unique cofinal π-realizable branch of $\mathcal{T}$ with which to continue. Clause 2 in the conclusion of the Branch Existence Theorem guarantees that this strategy cannot break down at any countable successor stage. Clause 1 guarantees that if this strategy breaks down at some countable limit stage, then there are distinct cofinal π-realizable branches at that stage, since the uniqueness of the branches chosen at earlier stages implies that any maximal π-realizable branch of $\mathcal{T}$ must be cofinal. However, if we ever reach a stage at which our tree has distinct cofinal π-realizable branches (this is possible for some $\mathcal{M}$ and π; see [18, Sect. 5]), our troubles start. The best we can do, it seems, is to choose one such branch b and a π-realization σ of $\mathcal{M}_b^{\mathcal{T}}$. If our opponent in the iteration game is kind enough to continue by playing extenders which can be interpreted as forming a tree on $\mathcal{M}_b^{\mathcal{T}}$, then we can choose unique σ-realizable branches to continue, until we get distinct such branches and must pick one, realize it, and continue, etc. However, we are done for if our opponent applies an extender to a model from $\mathcal{T}$ (that is, a model with index $< \sup(b)$). Nothing in the Branch Existence Theorem even guarantees that the associated ultrapower will be wellfounded.[41]

Clearly, we need a uniqueness theorem to accompany our existence theorem. What we can show, roughly speaking, is that at a non-uniqueness stage in the process just described we pass a local Woodin cardinal.

6.7 Definition. Let $\kappa < \delta$ and $A \subseteq V_\delta$; then we say κ is *A-reflecting in δ* iff for all $\nu < \delta$ there is an extender E over V such that $\mathrm{crit}(E) = \kappa$, $i_E(\kappa) > \nu$, and $i_E(A) \cap V_\nu = A \cap V_\nu$.

6.8 Definition. A cardinal δ is a *Woodin cardinal* iff for all $A \subseteq \delta$ there is a $\kappa < \delta$ which is A-reflecting in δ.

It is perhaps no surprise to the reader that Woodin cardinals were discovered by Woodin. Woodin was inspired by the results of [9], and by earlier work of Saharon Shelah reducing the large cardinal hypotheses employed in [9]. The definition of Woodinness given above is different from Woodin's original one, but equivalent to it by an argument essentially due to Mitchell. (See [21, Theorem 4.1].) Mitchell's argument can also be used to show that if δ is Woodin, then δ is witnessed to be Woodin by extenders in V_δ.[42] It follows that the Woodinness of δ can be expressed by a Π_1 sentence about $(V_{\delta+1}, \in)$, so that the least Woodin cardinal is not weakly compact. It is easy to see that all Woodin cardinals are Mahlo.

The (local) Woodin cardinal we get from an iteration tree $\mathcal{T}$ having distinct good branches is the supremum of the lengths of the extenders used in $\mathcal{T}$.

6.9 Definition. Let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$ such that $\mathrm{lh}(\mathcal{T})$ is a limit ordinal; then we set

$$\delta(\mathcal{T}) = \sup\{\mathrm{lh}(E_\alpha^{\mathcal{T}}) \mid \alpha < \mathrm{lh}(\mathcal{T})\},$$

and

$$\begin{aligned}\mathcal{M}(\mathcal{T}) = {} & \text{unique passive } \mathcal{P} \text{ such that } \mathrm{On}^{\mathcal{P}} = \delta(\mathcal{T}) \text{ and} \\ & \forall \alpha < \delta(\mathcal{T})(\mathcal{M}(\mathcal{T}) \text{ agrees with } \mathcal{M}_\alpha^{\mathcal{T}} \text{ below } \mathrm{lh}(E_\alpha^{\mathcal{T}})).\end{aligned}$$

It is clear that if b is a cofinal branch of $\mathcal{T}$ such that $\delta(\mathcal{T}) \in \mathcal{M}_b^{\mathcal{T}}$, then $\delta(\mathcal{T})$ is a limit cardinal of $\mathcal{M}_b^{\mathcal{T}}$.

The main result connecting Woodin cardinals with the uniqueness of cofinal wellfounded branches in iteration trees is the following theorem of [18].

[41] We have described here how the Branch Existence Theorem yields a winning strategy for II in a game that requires less of him, the *weak iteration game.* We shall introduce this game formally in the next section.

[42] This observation is due to Woodin.

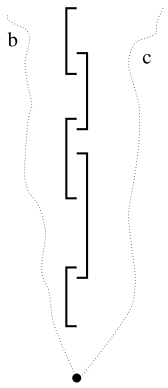

Figure 19.1: The overlapping pattern of two distinct wellfounded branches

6.10 Theorem (Branch Uniqueness Theorem). *Let b and c be distinct cofinal branches of the k-maximal iteration tree $\mathcal{T}$, let $\delta = \delta(\mathcal{T})$, and suppose that $A \subseteq \delta$ is such that $\delta, A \in \mathrm{wfp}(\mathcal{M}_b^{\mathcal{T}}) \cap \mathrm{wfp}(\mathcal{M}_c^{\mathcal{T}})$; then*

$$\mathcal{M}_b^{\mathcal{T}} \models \exists\kappa < \delta(\kappa \textit{ is } A\textit{-reflecting in } \delta).$$

Sketch of Proof. The extenders used on b and c have an overlapping pattern pictured in Fig. 19.1:

To see this, pick any successor ordinal

$$\alpha_0 + 1 \in b \setminus c,$$

and then let

$$\beta_n + 1 = \min\{\gamma \in c : \gamma > \alpha_n + 1\}$$

and

$$\alpha_{n+1} + 1 = \min\{\eta \in b : \eta > \beta_n + 1\},$$

for all $n < \omega$. Now for any n, the T-predecessor of β_n+1 is on c and $\leq \alpha_n+1$, hence $\leq \alpha_n$, so by the rules of the iteration game

$$\mathrm{crit}(F_{\beta_n}) < \nu(F_{\alpha_n}).$$

Similarly, for any n

$$\mathrm{crit}(F_{\alpha_{n+1}}) < \nu(F_{\beta_n}).$$

Now extenders used along the same branch of an iteration tree do not overlap (i.e., if E is used before F, then $\nu(E) \leq \mathrm{crit}(F)$), so we have

$$\begin{aligned}\mathrm{crit}(F_{\beta_n}) < \nu(F_{\alpha_n}) \leq \mathrm{crit}(F_{\alpha_{n+1}}) < \nu(F_{\beta_n})\\ \leq \mathrm{crit}(F_{\beta_{n+1}}) < \nu(F_{\alpha_{n+1}}) \leq \mathrm{crit}(F_{\alpha_{n+2}}),\end{aligned}$$

which is the overlapping pattern pictured.

Now $\sup(\{\alpha_n : n < \omega\}) = \sup(\{\beta_n : n < \omega\})$, and since branches of iteration trees are closed below their suprema in the order topology on On, the common supremum of the α_n and β_n is λ. Let us assume α_0 was chosen large enough that letting

$$\xi = \mathrm{pred}_T(\beta_0 + 1) \quad \text{and} \quad \eta = \mathrm{pred}_T(\alpha_1 + 1),$$

we have

$$A = i_{\xi,c}(A^*) = i_{\eta,b}(A^{**})$$

for some A^* and A^{**}. Let

$$\kappa = \mathrm{crit}(F_{\beta_0}) = \mathrm{crit}(i_{\xi,c});$$

we shall show that κ is A-reflecting in δ in the model $\mathcal{M}_b$.

Let $E_0 = F_{\beta_0} \restriction \mathrm{crit}(F_{\alpha_1})$. Because of the overlapping pattern, E_0 is a proper initial segment of F_{β_0}, and by initial segment condition on premice and the agreement of the models of an iteration tree, $E_0 \in \mathcal{M}_b$. Moreover, if $j : \mathcal{M}_b \to \mathrm{Ult}(\mathcal{M}_b, E_0)$ is the canonical embedding, then because A and A^* agree below κ, $j(A)$ and $i_{\xi,c}(A^*)$ agree below $\mathrm{crit}(F_{\alpha_1})$. That is, $j(A)$ agrees with A below $\mathrm{crit}(F_{\alpha_1})$, and hence E_0 witnesses that κ is A-reflecting up to $\mathrm{crit}(F_{\alpha_1})$ in $\mathcal{M}_b$.

To get A-reflection all the way up to δ, we set

$$E_{2n} = F_{\beta_n} \restriction \mathrm{crit}(F_{\alpha_{n+1}}) \quad \text{and} \quad E_{2n+1} = F_{\alpha_{n+1}} \restriction \mathrm{crit}(F_{\beta_{n+1}}),$$

for all n. Each of the E_n is in $\mathcal{M}_b$ for the same reason E_0 is in $\mathcal{M}_b$. Therefore the extender E which represents the embedding coming from "composing" the ultrapowers by the E_i for $0 \le i \le 2n$, is in $\mathcal{M}_b$. The argument above generalizes easily to show that E witnesses that κ is A-reflecting up to $\mathrm{crit}(F_{\alpha_{n+1}})$. Since $\mathrm{crit}(F_{\alpha_{n+1}}) \to \delta$ as $n \to \omega$, κ is A-reflecting in δ in the model $\mathcal{M}_b$. $\dashv$

We shall need a fine-structural refinement of Theorem 6.10. For this, we have to look closely at the first level of $\mathcal{M}_b^{\mathcal{T}}$ at which $\delta(\mathcal{T})$ is seen not to be Woodin, if there is one.

6.11 Definition. Let $\mathcal{T}$ be a k-maximal iteration tree on $\mathcal{M}$ of limit length, and let b be a cofinal wellfounded branch of $\mathcal{T}$. Let γ be the least ordinal, if there is one, such that either

$$\omega\gamma < \mathrm{On}^{\mathcal{M}_b} \quad \text{and} \quad \mathcal{J}_{\gamma+1}^{\mathcal{M}_b} \models \delta(\mathcal{T}) \text{ is not Woodin},$$

or

$$\omega\gamma = \mathrm{On}^{\mathcal{M}_b} \quad \text{and} \quad \rho_{n+1}(\mathcal{J}_\gamma^{\mathcal{M}_b}) < \delta(\mathcal{T})$$

for some $n < \omega$ such that $n + 1 \le k$ if $D^{\mathcal{T}} \cap b = \emptyset$. We set

$$\mathcal{Q}(b, \mathcal{T}) := \mathcal{J}_\gamma^{\mathcal{M}_b}$$

if there is such a γ, and let $\mathcal{Q}(b, \mathcal{T})$ be undefined otherwise.

Notice that if $\mathcal{Q}(b,\mathcal{T})$ exists and $\delta(\mathcal{T}) \in \mathcal{Q}(b,\mathcal{T})$, then $\mathcal{Q}(b,\mathcal{T})$ is just the longest initial segment $\mathcal{Q}$ of $\mathcal{M}_b^{\mathcal{T}}$ such that $\mathcal{Q} \models \delta(\mathcal{T})$ is Woodin. There is a failure of $\delta(\mathcal{T})$ to be Woodin definable over $\mathcal{Q}(b,\mathcal{T})$.[43] Notice also that if b drops in either model or degree, then $\rho_n(\mathcal{M}_b^{\mathcal{T}}) < \delta(\mathcal{T})$ for some appropriate n, and therefore $\mathcal{Q}(b,\mathcal{T})$ exists.[44]

Suppose that $\mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(c,\mathcal{T})$ (so both exist), and $\mathcal{Q}(b,\mathcal{T})$ is a proper initial segment of $\mathcal{M}_b^{\mathcal{T}}$ and $\mathcal{M}_c^{\mathcal{T}}$. Since $\mathcal{Q}(b,\mathcal{T})$ codes up a failure of Woodinness, Theorem 6.10 implies $b = c$. The following is a fine-structural strengthening of this fact.

6.12 Theorem. *Let $\mathcal{T}$ be k-maximal, and let b and c be distinct cofinal wellfounded branches of $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ exist; then neither is an initial segment of the other.*

Proof. If one is an initial segment of the other, then since they are minimal with respect to the same first-order property, $\mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(c,\mathcal{T})$. Since this property involves a failure of $\delta(\mathcal{T})$ to be Woodin, $\mathcal{Q}(b,\mathcal{T}) \notin \mathcal{M}_b$ and $\mathcal{Q}(c,\mathcal{T}) \notin \mathcal{M}_c$ by Theorem 6.10. Thus $\mathcal{M}_b = \mathcal{Q}(b,\mathcal{T}) = \mathcal{Q}(c,\mathcal{T}) = \mathcal{M}_c$.

It follows that $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ are defined by the second clause of Definition 6.11. If we let n be least such that $\rho_{n+1}(\mathcal{M}_b) < \delta(\mathcal{T})$, then there are $\eta \in b$ and $\xi \in c$ such that

$$\mathcal{M}_\eta^* = \mathcal{C}_{n+1}(\mathcal{M}_b) = \mathcal{C}_{n+1}(\mathcal{M}_c) = \mathcal{M}_\xi^*,$$

and $i_{\eta,b} \circ i_\eta^*$ and $i_{\xi,c} \circ i_\xi^*$ exist, and are n-embeddings with critical point at least $\rho_{n+1}(\mathcal{M}_\eta^*)$. But then, as in the fine structure argument at the end of the proof of the Comparison Lemma 3.11,

$$i_{\eta,b} \circ i_\eta^* = i_{\xi,c} \circ i_\xi^*,$$

since each is the core embedding from $\mathcal{C}_{n+1}(\mathcal{M}_b) = \mathcal{C}_{n+1}(\mathcal{M}_c)$ to $\mathcal{M}_b = \mathcal{M}_c$. Thus the extender applied to $\mathcal{M}_\eta^*$ in b is compatible with the extender applied to $\mathcal{M}_\xi^*$ in c, so that $\eta = \xi$.

Let α be the largest ordinal in $b \cap c$, so that $\alpha > \eta$ by the argument above. As usual, let us assume $n = 0$ to simplify matters a bit; the general case is essentially the same. Letting $\nu = \sup\{\nu(E_\beta) \mid \beta \, T \, \alpha\}$, we then have

$$\mathcal{M}_\alpha = \{i_{\eta,\alpha} \circ i_\eta^*(f)(a) \mid f \in \mathcal{M}_\eta^* \text{ and } a \in [\nu]^{<\omega}\}.$$

Since $i_{\alpha,b}$ and $i_{\alpha,c}$ are the identity on ν and agree on the range of $i_{\eta,\alpha} \circ i_\eta^*$, we have $i_{\alpha,b} = i_{\alpha,c}$. But this means the extender applied to $\mathcal{M}_\alpha$ in b is compatible with the extender applied to $\mathcal{M}_\alpha$ in c, so that α is not the largest element of $b \cap c$, a contradiction. $\dashv$

[43] The case $\rho_{n+1}(\mathcal{Q}(b,\mathcal{T})) < \delta(\mathcal{T})$ represents a failure of $\delta(\mathcal{T})$ to be a cardinal at all.

[44] Because $\mathcal{T}$ is maximal, b only drops when some extender used on b has critical point at least a projectum of the model to which it is applied. At the last drop, this projectum is preserved as a projectum of $\mathcal{M}_b^{\mathcal{T}}$.

6.13 Definition. We say η is a *cutpoint* of $\mathcal{M}$ iff for all extenders E on the $\mathcal{M}$-sequence, if $\text{crit}(E) < \eta$ then $\text{lh}(E) < \eta$.

6.14 Corollary. *Let $\mathcal{T}$ be k-maximal; then there is at most one cofinal, wellfounded branch b of $\mathcal{T}$ such that*

1. *$\mathcal{Q}(b, \mathcal{T})$ exists,*
2. *$\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}(b, \mathcal{T})$, and*
3. *$\mathcal{Q}(b, \mathcal{T})$ is $\delta(\mathcal{T})^+ + 1$-iterable.*

Proof. Suppose that b and c are distinct such branches. $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ have cardinality $\delta(\mathcal{T})$, so they are sufficiently iterable that their coiteration terminates successfully. Since $\delta(\mathcal{T})$ is a cutpoint of each model, and the two models agree below $\delta(\mathcal{T})$, all extenders used in this coiteration have critical point above $\delta(\mathcal{T})$. Also, each model is $\delta(\mathcal{T})$-sound and projects to $\delta(\mathcal{T})$, in the sense that there is an $n < \omega$ such that $\rho_{n+1}(\mathcal{Q}(b, \mathcal{T})) \leq \delta(\mathcal{T})$ and $\mathcal{Q}(b, \mathcal{T}) = \mathcal{H}_{n+1}^{\mathcal{Q}(b,\mathcal{T})}(\delta(\mathcal{T}) \cup \{p_{n+1}(\mathcal{Q}(b, \mathcal{T}))\})$, and similarly for $\mathcal{Q}(c, \mathcal{T})$. Just as in the proof of Corollary 3.12, this means that the side which comes out shorter does not move at all in the comparison, so that $\mathcal{Q}(b, \mathcal{T})$ is an initial segment of $\mathcal{Q}(c, \mathcal{T})$ or vice-versa. This contradicts Theorem 6.12. ⊣

Notice that all we needed in this argument was that $\mathcal{Q}(b, \mathcal{T})$ and $\mathcal{Q}(c, \mathcal{T})$ be iterable enough that we can compare them successfully. We can think of the structure $\mathcal{Q}(b, \mathcal{T})$ as a *branch oracle*, in that the fact that it is sufficiently iterable to be compared with other $\mathcal{Q}$-structures identifies b as the good branch of $\mathcal{T}$, the one any iteration strategy ought to choose. The sufficient-iterability-for-comparison of $\mathcal{Q}(b, \mathcal{T})$ only identifies b as the good branch, however, when $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}(b, \mathcal{T})$. This leads us to restrict our attention to mice all of whose Woodin cardinals are cutpoints.

6.15 Definition. A premouse $\mathcal{M}$ is *tame* iff whenever E is an extender on the $\mathcal{M}$-sequence, and $\lambda = \text{lh}(E)$, then

$$\mathcal{J}_\lambda^{\mathcal{M}} \models \forall \delta \geq \text{crit}(E)(\delta \text{ is not Woodin}).$$

In other words, tame mice cannot have extenders overlapping local Woodin cardinals. It is clear from the definition that any initial segment of a tame mouse is tame. Tame mice can satisfy large cardinal hypotheses as strong as "There is a strong cardinal which is a limit of Woodin cardinals". No tame mouse can satisfy "There is a Woodin cardinal which is a limit of Woodin cardinals".

The iterability conjecture above becomes a theorem when it is restricted to tame premice.

6.16 Theorem. *Let $\mathcal{N}$ be a tame premouse occurring on a K^c-construction, let $k \leq \omega$, and let $\mathcal{M}$ be countable and such that there is a weak k-embedding from $\mathcal{M}$ to $\mathcal{C}_k(\mathcal{N})$; then $\mathcal{M}$ is $(k, \omega_1, \omega_1 + 1)$-iterable.*

We shall not prove this theorem here, but in the next section we shall prove a fairly representative special case of it.

6.3. Large Cardinals in K^c

The iterability conjectures and theorems above show that K^c-constructions are sufficiently conservative about putting extenders on their sequences. We need also to know that they can be sufficiently liberal.

6.17 Definition. A K^c-construction $\langle \mathcal{N}_\alpha \mid \alpha < \theta \rangle$ is *maximal* iff $\mathcal{N}_{\alpha+1}$ is defined by case (2)(a) of Definition 6.3 whenever possible; that is, a new extender is added to the current sequence whenever there is one meeting all the requirements of (2)(a) in Definition 6.3.

One evidence of liberality is that large cardinal hypotheses true in V must also hold in K^c. Here is one such theorem.

6.18 Theorem. *Let δ be Woodin; then either*

1. *there is a maximal K^c-construction $\langle \mathcal{N}_\alpha \mid \alpha < \theta + 1 \rangle$ such that $\mathcal{N}_\theta$ is not tame, or*

2. *there is a maximal K^c-construction of length* $\mathrm{On} + 1$*, and for any such construction $\langle \mathcal{N}_\alpha \mid \alpha \leq \mathrm{On} \rangle$,*

$$\mathcal{N}_{\mathrm{On}} \models \delta \textit{ is Woodin.}$$

Sketch of Proof. If no maximal K^c-construction ever reaches a non-tame premouse, then by Theorems 6.16 and 5.3, every premouse occurring in a K^c-construction is ω-solid, and hence there are maximal K^c-constructions of length $\mathrm{On} + 1$.

Let $\langle \mathcal{N}_\alpha \mid \alpha \leq \mathrm{On} \rangle$ be such a construction, and let $\mathcal{N}_{\mathrm{On}} = (L[\vec{E}], \in, \vec{E})$. Let $A \subseteq \delta$ and $A \in L[\vec{E}]$; we must find a $\kappa < \delta$ which is satisfied by $L[\vec{E}]$ to be A-reflecting in δ.

Since δ is Woodin in V, we can find a $\kappa < \delta$ which is $(A, \vec{E} \restriction \delta)$-reflecting in δ. Now if F is an extender over V which witnesses this reflection up to η, where $\kappa < \eta < \delta$ and η is, say, inaccessible, then we can show that for any $\xi < \eta$,

$$G_\xi := F \restriction \xi \cap L[\vec{E}] \in L[\vec{E}].$$

This is enough, for the extenders G_ξ to witness that κ is A-reflecting in δ up to ξ in $L[\vec{E}]$.

To show that $G_\xi \in L[\vec{E}]$, we show by induction on ξ that if G_ξ is not of type Z, then the trivial completion of G_ξ is either on the sequence $\vec{E}$ or on an ultrapower of it, as in the initial segment condition in the definition of fine extender sequences. It is easy to see that G_ξ satisfies the requirements for being added to $\vec{E}$: coherence comes from the fact that F witnesses $\vec{E}\restriction\delta$-reflection,[45] the initial segment condition comes from our induction hypothesis, and F provides the necessary background certificates. However, there are some problems. First, there is a *timing* problem: the above shows that G_ξ could be added to the $L[\vec{E}]$ sequence somewhere, but we need to find an actual stage $\mathcal{N}_\alpha$ of the construction at which it can be added. Second, there is a *uniqueness of the next extender* problem: we need to conclude from the fact that G_ξ could be added to produce $\mathcal{N}_\alpha + 1$ that it *was* added to produce $\mathcal{N}_\alpha + 1$. For these arguments, we refer the reader to [25, Theorem 11.4]. $\dashv$

We note that the proof of Theorem 6.18 would have gone through if we had been even more conservative and required in Definition 6.3 that our background extenders measure all sets in V. This requirement simplifies the iterability proof for the resulting model, as it allows us to lift trees on it to trees on V.[46] It is important in some contexts, however, to allow partial background extenders. For example, in proving relative consistency results in which the theory assumed consistent does not imply the existence of measurable cardinals, we must construct core models satisfying large cardinal hypotheses without assuming there are any extenders which are total over V. What assures us that maximal K^c-constructions are sufficiently liberal in that situation is the following.

6.19 Theorem. *Suppose that μ is a normal measure on the measurable cardinal Ω, and that no K^c-construction reaches a non-tame premouse. Let $\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega\rangle$ be a maximal K^c-construction; then for μ-almost every $\alpha < \Omega$, $(\alpha^+)^{\mathcal{N}_\Omega} = \alpha^+$.*

This is essentially Theorem 1.4 of [43]. That is in turn an extension of earlier work of Jensen and Mitchell which in effect proved Theorem 6.19 under the hypothesis that no K^c-construction reaches the sharp for an inner model with a strong cardinal. Jensen and Mitchell did not require the measurable cardinal. Jensen and the author have recently proved in ZFC a version of Theorem 6.19 under the hypothesis that there is no proper class model with a Woodin cardinal, and used it to develop core model theory up to a Woodin cardinal in ZFC. See [32].

Our focus for the rest of this chapter will be on applications of core model theory in descriptive set theory, and so for simplicity we shall generally assume that there are Woodin cardinals in V. Therefore it will be Theorem 6.18 rather than Theorem 6.19 which is important for us. The reader should

[45] This is not actually as obvious as it might seem at first, because the G_ξ ultrapower of $L[\vec{E}]$ only obviously agrees with the F ultrapower (and hence $L[\vec{E}]$) out to $\nu(G_\xi)$, rather than to the successor of $\nu(G_\xi)$ in the G_ξ ultrapower, as required by coherence. The stronger agreement can be proved using the Condensation Theorem 5.1, applied to the natural embedding of the G_ξ ultrapower into the F ultrapower.

[46] This is the iterability proof given in [25, Sect. 12]. Of course, it only applies to tame mice; that is, it only proves a version of Theorem 6.16.

see [15] for an introductory article which turns at this point toward relative consistency results, results which make use of Theorem 6.19 rather than Theorem 6.18. See also [32] for a much more thorough survey of this area.

7. The Reals of M_ω

We shall show that the reals in the minimal iterable proper class model satisfying "There are ω Woodin cardinals" are precisely those reals which are ordinal definable over $L(\mathbb{R})$. Of course, in order to do this we must assume that there is such a model. It will simplify matters if we assume something a bit stronger, namely, that there are ω Woodin cardinals with a measurable cardinal above them all (in V). We shall do so throughout the rest of this chapter, sometimes without explicitly mentioning the assumption. One useful consequence of our assumption is $\mathsf{AD}^{L(\mathbb{R})}$, the axiom of determinacy restricted to sets of reals in $L(\mathbb{R})$.[47]

7.1 Definition. A premouse $\mathcal{M}$ is *ω-small* iff whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence, then

$$\mathcal{J}_\kappa^{\mathcal{M}} \not\models \text{There are } \omega \text{ Woodin cardinals.}$$

An ω-small mouse can satisfy "There are ω Woodin cardinals", but it cannot satisfy any significantly stronger large cardinal hypotheses.

7.2 Theorem. *If there are ω Woodin cardinals with a measurable cardinal above them all, then there is a $(\omega, \omega_1, \omega_1 + 1)$-iterable premouse which is not ω-small.*

Sketch of Proof. If a K^c construction reaches a nontame premouse, then it reaches a nontame premouse that is not ω-small, and we can apply Theorem 6.16. So, we may assume our maximal K^c construction reaches only tame mice. Let $j : V \to M$ witness the measurability of some κ below which there are ω Woodin cardinals. By Theorem 6.18, the Woodin cardinals of M are Woodin in $j(K^c)$, and hence there are ω Woodin cardinals of $j(K^c)$ below κ. Now for any $A \subseteq V_{\kappa+1}$ of cardinality κ, the fragment $E_j \cap (A \times [j(\kappa)]^{<\omega})$ of the extender determined by j is in M. These fragments provide sufficient background certificates to show that there is an extender on the K^c sequence whose critical point is above all the Woodin cardinals of $j(K^c)$ which are below κ. Thus our maximal K^c-construction reaches an $\mathcal{N}_\alpha$ which is not ω-small. By Theorem 6.16, any countable elementary submodel of $\mathcal{C}_\omega(\mathcal{N}_\alpha)$ witnesses the truth of the theorem. ⊣

7.3 Definition. $M_\omega^\#$ is the unique sound, $(\omega, \omega_1, \omega_1+1)$-iterable mouse which is not ω-small, but all of whose proper initial segments are ω-small.

[47] This is a result of Woodin, building on the work of [9] and [17]. See [28] for a proof.

It is easy to see that $\rho_1(M_\omega^\#) = \omega$, so that $M_\omega^\#$ is countable, and in fact every $x \in M_\omega^\#$ is Σ_1-definable over $M_\omega^\#$.[48] The uniqueness of $M_\omega^\#$ follows from Corollary 3.12. It is also clear that $M_\omega^\#$ is active; that is, it has a nonempty last extender predicate. We let M_ω be the proper class model left behind when the last extender of $M_\omega^\#$ is iterated out of the universe.

7.4 Definition. $M_\omega = \mathcal{J}^{\mathcal{P}}_{\mathrm{On}}$, where $\mathcal{P}$ is the Onth iterate of $M_\omega^\#$ by the last extender on its sequence.

It is clear that M_ω is an ω-small proper class model with ω Woodin cardinals, and that the Woodin cardinals of M_ω are countable in V. Their supremum is the supremum of the lengths of the extenders on the M_ω-sequence. The iterability of $M_\omega^\#$ easily implies that M_ω is $(\omega, \omega_1, \omega_1 + 1)$-iterable.

We shall show that the reals of M_ω are precisely the reals which are ordinal definable in $L(\mathbb{R})$.[49] We begin by showing that every real in M_ω is $\mathrm{OD}^{L(\mathbb{R})}$. Following the proof of Corollary 3.14, we see that for this it is enough to show that if $\alpha = \omega_1^{M_\omega}$, then $L(\mathbb{R})$ satisfies "$\mathcal{J}_\alpha^{\mathcal{M}_\omega}$ is $(\omega_1 + 1)$-iterable".[50]

7.1. Iteration Strategies in $L(\mathbb{R})$

Our task is complicated by the fact that M_ω is not itself $(\omega, \omega_1 + 1)$-iterable in $L(\mathbb{R})$, as we shall show later. We must drop to slightly smaller mice in order to find iteration strategies in $L(\mathbb{R})$.

7.5 Definition. A premouse $\mathcal{P}$ is *properly small* iff

1. $\mathcal{P}$ is ω-small,
2. $\mathcal{P} \models$ There are no Woodin cardinals, and
3. $\mathcal{P} \models$ There is a largest cardinal + ZF^-.

Here ZF^- is ZF without the Power Set Axiom. It is clear that if α is a successor cardinal of M_ω below its least Woodin cardinal, then $\mathcal{J}_\alpha^{M_\omega}$ is properly small. In particular, this is true when $\alpha = \omega_1^{M_\omega}$.

7.6 Lemma. *Let $\mathcal{T}$ be an ω-maximal iteration tree of limit length on a properly small premouse, and let b be a cofinal wellfounded branch of $\mathcal{T}$; then $\mathcal{Q}(b, \mathcal{T})$ exists.*

[48] Suppose that $\mathcal{M}$ is sufficiently iterable, not ω-small, and has only ω-small proper initial segments. The Σ_1 hull $\mathcal{H} := \mathcal{H}_1^{\mathcal{M}}(\emptyset)$ of $\mathcal{M}$ is sufficiently iterable that it can be compared with $\mathcal{J}_\alpha^{\mathcal{M}}$, for any $\alpha < \omega_1^{\mathcal{M}}$. Since $\mathcal{J}_\alpha^{\mathcal{M}}$ is ω-small, $\mathcal{H}$ must iterate past it, and it follows that for $\gamma = \omega_1^{\mathcal{M}}$, $\mathcal{J}_\gamma^{\mathcal{M}}$ is an initial segment of $\mathcal{H}$. Since we can easily compute a counting of $\mathcal{J}_\gamma^{\mathcal{H}}$ from the Σ_1 theory of $\mathcal{M}$, this theory is not a member of $\mathcal{M}$. Thus if $\mathcal{M}$ is 1-sound, $\mathcal{M} = \mathcal{H}$.

[49] Of course, M_ω and $M_\omega^\#$ have the same reals as members. $M_\omega^\#$ *is* (coded by) the simplest canonical real which is not $\mathrm{OD}^{L(\mathbb{R})}$; it is definable over $L(\mathbb{R} \cup \{\mathbb{R}^\#\})$ in a simple way.

[50] We are regarding this as a statement about the *parameter* $\mathcal{J}_\alpha^{M_\omega}$, which is in $L(\mathbb{R})$ because it is hereditarily countable. $L(\mathbb{R})$ need not believe that $\mathcal{J}_\alpha^{M_\omega}$ is obtained by implementing the definition of M_ω we gave in V.

Proof. We have already observed that if b drops in model or degree, then $\rho_{n+1}(\mathcal{M}_b) < \delta(\mathcal{T})$ for some n, so that $\mathcal{Q}(b, \mathcal{T})$ exists. Let $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$. The requirement that $\mathcal{M}$ satisfy ZF^- insures that $\rho_\omega(\mathcal{M}) = \mathrm{On}^{\mathcal{M}}$, so that any iteration map along a non-dropping branch of an ω-maximal tree on $\mathcal{M}$ is fully elementary. The requirement that there are no Woodin cardinals in $\mathcal{M}$ then implies that there are none in $\mathcal{M}_b$, so that if $\delta(\mathcal{T}) < \mathrm{On}^{\mathcal{M}_b}$ then $\mathcal{Q}(b, \mathcal{T})$ exists. But we must have $\delta(\mathcal{T}) < \mathrm{On}^{\mathcal{M}_b}$, since if $\delta(\mathcal{T}) = \mathrm{On}^{\mathcal{M}_b}$, then as $\mathrm{lh}(E_\alpha^{\mathcal{T}})$ is a cardinal of $\mathcal{M}_b$ for all $\alpha < \mathrm{lh}(\mathcal{T})$, there is no largest cardinal of $\mathcal{M}_b$. $\dashv$

This lemma will, together with Theorem 6.12, guarantee that there is at most one iteration strategy for a properly small $\mathcal{M}$, and ultimately the $L(\mathbb{R})$-definability of this strategy when it exists.

It is useful to introduce yet another iteration game, one which requires less of player II than $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$. We call this new game the *weak iteration game*. Suppose that $\mathcal{M}$ is a k-sound premouse; then the weak iteration game $\mathcal{W}_k(\mathcal{M}, \omega)$ is played in ω rounds as follows:

$$\begin{array}{c|cccccccc}
I & \mathcal{T}_0 & & \mathcal{P}_1, i_1, \mathcal{T}_1 & & \mathcal{P}_2, i_2, \mathcal{T}_2 & & \dots & \\
\hline
II & & b_0 & & b_1 & & b_2 & & \dots
\end{array}$$

Here I begins by playing a countable, k-maximal, putative iteration tree $\mathcal{T}_0$ on $\mathcal{M}$, after which II plays b_0, which may be either "accept" or a maximal wellfounded branch of $\mathcal{T}_0$, with the proviso that II cannot accept unless $\mathcal{T}_0$ has a last model, and this model is wellfounded. Let $\mathcal{Q}_1$ be this last model, if II accepts, and let $\mathcal{Q}_1 = \mathcal{M}_{b_0}^{\mathcal{T}_0}$ otherwise. Let k_1 be the degree of $\mathcal{Q}_1$. Play now goes into the next round as it did in $\mathcal{G}_k(\mathcal{M}, \lambda, \theta)$: I picks an initial segment $\mathcal{P}_1$ of $\mathcal{Q}_1$, and an $i_1 \leq \omega$ such that $i_1 \leq k_1$ if $\mathcal{P}_1 = \mathcal{Q}_1$, together with a countable, i_1-maximal, putative iteration tree on $\mathcal{P}_1$. Then II either accepts or plays a maximal wellfounded branch of $\mathcal{T}_1$, with the proviso that he can only accept if $\mathcal{T}_1$ has a last, wellfounded model. Etc.

If no one breaks any of these rules along the way, then we say II wins this run of $\mathcal{W}_k(\mathcal{M}, \omega)$ iff for all sufficiently large i, $\mathcal{P}_i = \mathcal{Q}_i$, the branch of $\mathcal{T}_i$ from $\mathcal{P}_i$ to $\mathcal{Q}_{i+1}$ does not drop, and the direct limit of the $\mathcal{P}_i$ under the iteration maps given by the $\mathcal{T}_i$ is wellfounded.

7.7 Definition. A *weak (k, ω)-iteration strategy* for $\mathcal{M}$ is a winning strategy for II in $\mathcal{W}_k(\mathcal{M}, \omega)$, and we say $\mathcal{M}$ is *weakly (k, ω)-iterable* (or $\Game^{\mathbb{R}}\Pi_1^1$*-iterable*) just in case there is such a strategy.

It is an immediate consequence of the Branch Existence Theorem 6.6 that every countable elementary submodel $\mathcal{M}$ of $\mathcal{C}_k(\mathcal{N}_\alpha)$, where $\mathcal{N}_\alpha$ occurs in a K^c-construction, is weakly (k, ω)-iterable. In fact, such mice are weakly (k, ω_1)-iterable, in the obvious sense.[51] Weak (k, ω_1)-iteration strategies suf-

[51] In $\mathcal{W}_k(\mathcal{M}, \omega_1)$, player I must play at limit $\lambda < \omega_1$ a tree $\mathcal{T}_\lambda$ on the direct limit of the models $\mathcal{P}_\eta$ for $\eta < \lambda$. Player II must insure that this direct limit is wellfounded.

fice for the comparison of tame mice, and this fact is what lies behind our iterability Theorem 6.16 for tame mice.[52]

If $\mathcal{M}$ is countable, and coded by the real x, then the weak iteration game $\mathcal{W}_k(\mathcal{M}, \omega)$ is (can be coded as) a game of length ω on $\mathbb{R}$ with $\Pi^1_1(x)$ payoff. Thus the set of reals coding weakly iterable premice is $\Game^{\mathbb{R}}\Pi^1_1$, which explains the alternate terminology. By [16], $\Game^{\mathbb{R}}\Pi^1_1$ statements are absolute between V and $L(\mathbb{R})$, so we have:

7.8 Theorem. *Let $\mathcal{M}$ be countable and weakly (k, ω)-iterable; then $L(\mathbb{R}) \models \mathcal{M}$ is weakly (k, ω)-iterable.*

It is also shown in [16] that $\Game^{\mathbb{R}}\Pi^1_1 = \Sigma_1^{L(\mathbb{R})}$, that is, that definitions in each form can be translated into the other.[53] We shall do our definability calculations below with Σ_1 formulae interpreted in $L(\mathbb{R})$. It is important here that we allow such formulae to contain a name $\dot{\mathbb{R}}$ for $\mathbb{R}$, so that quantification over $\mathbb{R}$ counts as bounded quantification. (Without this provision, we would have $\Sigma_1^{L(\mathbb{R})} = \Sigma^1_2$.) The sets whose definability we are calculating are generally subsets of HC, the class of hereditarily countable sets. Notice here that a set $A \subseteq \mathrm{HC}$ is $\Sigma_1^{L(\mathbb{R})}$ iff the set A^* of all reals coding (in some natural system) a member of A is $\Sigma_1^{L(\mathbb{R})}$. So we have:

7.9 Lemma. *The set of countable, weakly (k, ω)-iterable premice is $\Sigma_1^{L(\mathbb{R})}$.*

If we restrict our attention to properly small premice, weak (k, ω)-iterability suffices for comparison.

7.10 Theorem. *Assume $\mathsf{AD}^{L(\mathbb{R})}$, and let $\mathcal{M}$ be countable, properly small, and weakly (k, ω)-iterable; then*

$$L(\mathbb{R}) \models \mathcal{M} \textit{ is } (k, \omega_1 + 1)\textit{-iterable.}$$

Proof. We first note

7.11 Lemma. *In $L(\mathbb{R})$, every iteration tree of length ω_1 on a countable premouse has a cofinal, wellfounded branch.*

Proof. Let $\mathcal{T}$ be such a tree. Let j be the embedding coming from the club ultrafilter on ω_1. Now $\mathcal{T}$ can be coded by a subset of ω_1, so $\mathcal{T} \in L[\mathcal{T}]$. As $L[\mathcal{T}]$ is wellordered, $j \restriction L[\mathcal{T}]$ is elementary from $L[\mathcal{T}]$ to $L[j(\mathcal{T})]$. Thus $j(\mathcal{T})$ is an iteration tree of length $j(\omega_1) > \omega_1$, so that $j(\mathcal{T}) \restriction \omega_1$ has a cofinal, wellfounded branch. But $j(\mathcal{T}) \restriction \omega_1 = \mathcal{T}$. $\dashv$

Because of this, it is enough to show that $\mathcal{M}$ is (k, ω_1)-iterable in $L(\mathbb{R})$. We claim that the following is a (k, ω_1)-iteration strategy for $\mathcal{M}$: given that

[52] See [40, Theorem 1.1] for the comparison proof. The proof of our unique strategies result Theorem 4.11 is the other main ingredient in the proof of Theorem 6.16.

[53] We only need here that $\Game^{\mathbb{R}}\Pi^1_1 \subseteq \Sigma_1^{L(\mathbb{R})}$, and this is trivial.

you have reached $\mathcal{T}$ of countable limit length, pick the unique cofinal branch b of $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T})$ is weakly $(\deg^{\mathcal{T}}(b),\omega)$-iterable. Let us call this putative iteration strategy Γ.

Let $\mathcal{T}$ be played according to Γ, and of minimal length such that Γ breaks down at $\mathcal{T}$, either because $\mathcal{T}$ has limit length and there is no such unique branch to serve as $\Gamma(\mathcal{T})$, or because $\mathcal{T}$ has a last, illfounded model. Let Σ be a weak (k,ω)-iteration strategy for $\mathcal{M}$. If $\mathcal{T}$ has a last, illfounded model, then Σ cannot accept $\mathcal{T}$ as I's first move, so $\Sigma(\mathcal{T}) = b$ is a maximal branch of $\mathcal{T}$. Clearly, $\mathcal{Q}(b,\mathcal{T})$ is weakly $(\deg^{\mathcal{T}}(b),\omega)$-iterable, as witnessed by Σ. Letting $\lambda = \sup(b)$, we have from the definition of Γ that $b = \Gamma(\mathcal{T}\restriction\lambda)$, so $b = [0,\lambda]_T$, contrary to the maximality of b. Thus $\mathcal{T}$ has limit length. The argument just given shows that $b := \Sigma(\mathcal{T})$ is a cofinal branch of $\mathcal{T}$, and that $\mathcal{Q}(b,\mathcal{T})$ is weakly $(\deg^{\mathcal{T}}(b),\omega)$-iterable. Therefore there must be a second such branch; call it c. By Theorem 6.12 and the proof of Corollary 6.14, $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ cannot be compared. We shall use their weak iterability to compare them.

Let

$$\delta_0 = \sup\{\mathrm{lh}(E_\alpha^{\mathcal{T}}) \mid \alpha < \mathrm{lh}(\mathcal{T})\}.$$

Since δ_0 is Woodin in both $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$, it is a cutpoint of each model. Since $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ agree below δ_0, the comparison we are doing uses only extenders with critical point strictly greater than δ_0.

Let $\Sigma_0 = \Sigma$ and Σ_1 be any weak $(\deg(c),\omega)$-iteration strategy for $\mathcal{Q}(c,\mathcal{T})$. Let $\mathcal{T}_0^0 = \mathcal{T}$, $b_0^0 = b$, and $c_0 = c$. We coiterate $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ by iterating the least disagreement at successor steps, and choosing the unique cofinal branch with a weakly iterable $\mathcal{Q}$-structure at limit steps. This process is $L(\mathbb{R})$-definable, and must break down at some countable stage, as otherwise by Lemma 7.11 and the proof of the Comparison Lemma 3.11 we shall succeed in comparing $\mathcal{Q}(b,\mathcal{T})$ with $\mathcal{Q}(c,\mathcal{T})$. By the argument given above, the weak iterability of $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{Q}(c,\mathcal{T})$ implies that uniqueness is what breaks down. (It does not literally follow from Lemma 7.6 that cofinal branches always have $\mathcal{Q}$-structures, as the models we are comparing may no longer be properly small. But if, say, $\mathcal{Q}(b,\mathcal{T})$ is not properly small, then we have dropped along b getting to it, and this guarantees that in the tree on $\mathcal{Q}(b,\mathcal{T})$ we are now building, cofinal branches always have $\mathcal{Q}$-structures.) Let $\mathcal{T}_1^0$ on $\mathcal{Q}(b,\mathcal{T})$ and $\mathcal{T}_1^1$ on $\mathcal{Q}(c,\mathcal{T})$ be the trees produced by this process. Let

$$\begin{aligned}\delta_1 &= \sup\{\mathrm{lh}(E_\alpha^{\mathcal{T}_1^0}) \mid \alpha < \mathrm{lh}(\mathcal{T}_1^0)\}\\ &= \sup\{\mathrm{lh}(E_\alpha^{\mathcal{T}_1^1}) \mid \alpha < \mathrm{lh}(\mathcal{T}_1^1)\}.\end{aligned}$$

Let

$$b_1^0 = \Sigma_0\big(\langle \mathcal{T}_0^0, (\mathcal{Q}(b_0^0,\mathcal{T}_0^0), \deg(\mathcal{Q}(b_0^0,\mathcal{T}_0^0)), \mathcal{T}_1^0)\rangle\big)$$

and

$$b_1^1 = \Sigma_1(\mathcal{T}_1^1)$$

be the cofinal, weakly iterable branches of $\mathcal{T}_1^0$ and $\mathcal{T}_1^1$ chosen by Σ_0 and Σ_1. By hypothesis we have a third branch c_1 of some $\mathcal{T}_1^i$ (it does not matter which) such that $\mathcal{Q}(c_1, \mathcal{T}_1^i)$ is weakly $(\deg(c_1), \omega)$-iterable, say via the strategy Σ_2. It follows that the premice $\mathcal{Q}(b_1^0, \mathcal{T}_1^0)$, $\mathcal{Q}(b_1^1, \mathcal{T}_1^1)$, and $\mathcal{Q}(c_1, \mathcal{T}_1^i)$ cannot be compared.

We attempt to reach a contradiction by simultaneously comparing these three premice. (This means that we form three iteration trees simultaneously, iterating by the shortest extender on the sequence of any of the three last models which is not present on the sequences of both of the other two last models.) Again, we choose unique weakly iterable branches at limit ordinals, and again this process must break down due to non-uniqueness, giving trees $\mathcal{T}_2^0, \mathcal{T}_2^1$, and $\mathcal{T}_2^2$, with cofinal branches b_2^0, b_2^1, and b_2^2 chosen by Σ_0, Σ_1, and Σ_2. (It is because the $\mathcal{T}_2^i$ use only extenders with critical point above δ_1 that we can interpret them as played by the Σ_i.) We also have a new branch c_2 of some $\mathcal{T}_2^i$, and a weak iteration strategy Σ_3 for $\mathcal{Q}(c_2, \mathcal{T}_2^i)$. We let δ_2 be the sup of the lengths of the extenders used in the $\mathcal{T}_2^i$. And so on.

After ω steps in the process we have for each $i < \omega$ a weak iteration strategy Σ_i and a play by Σ_i in which the iteration trees played by I are the $\mathcal{T}_j^i$ for $j \geq i$ and the branches chosen by II are the b_j^i for $j \geq i$. Let $\mathcal{P}_i$ be the direct limit of the $\mathcal{M}_{b_j^i}^{\mathcal{T}_j^i}$. Since each Σ_i is winning, these direct limits are wellfounded. Clearly, all the δ_k are Woodin in each $\mathcal{P}_i$. Since $\mathcal{P}_i$ is ω-small, it has no extenders with index above the sup of the δ_k, and thus $\mathcal{P}_i$ is an initial segment of $\mathcal{P}_n$ or vice-versa, for all i and n. Since all $\mathcal{P}_i$ project below the sup of the δ_k, they must all be the same. Moreover, as in the proof of the Comparison Lemma 3.11, we can show that for no i does the composition of the trees $\mathcal{T}_j^i$ drop in model or degree on the branch leading to $\mathcal{P}_i$. But this means that $\mathcal{P}_0$ and $\mathcal{P}_1$ are the last models of a successful comparison of $\mathcal{Q}(b, \mathcal{T})$ with $\mathcal{Q}(c, \mathcal{T})$, a contradiction. ⊣

We have at once

7.12 Corollary. *Every real in M_ω is ordinal definable in $L(\mathbb{R})$.*

Proof. Let x be the αth real in the order of constructibility of M_ω; then

$$\begin{aligned} y = x \quad \iff \quad & L(\mathbb{R}) \models \exists \mathcal{M}(\mathcal{M} \text{ is countable, properly small,} \\ & (\omega, \omega_1 + 1)\text{-iterable, and } y \text{ is the } \alpha\text{th real} \\ & \text{in the constructibility order of } \mathcal{M}). \end{aligned}$$

⊣

The proof of Theorem 7.10 gives at once:

7.13 Corollary. *Assume $\mathsf{AD}^{L(\mathbb{R})}$, and let $\mathcal{M}$ be countable, properly small, and weakly (k, ω)-iterable; then in $L(\mathbb{R})$, $\mathcal{M}$ has a unique (k, ω_1)-iteration strategy Σ; moreover, Σ is $\Sigma_1^{L(\mathbb{R})}(\{\mathcal{M}\})$ definable, uniformly in $\mathcal{M}$, and Σ extends, in $L(\mathbb{R})$, to a $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$.*

7.2. Correctness and Genericity Iterations

We shall prove some correctness results for M_ω, and use them to show that every real ordinal definable over $L(\mathbb{R})$ is in M_ω. The key to these results is the following remarkable theorem of Woodin.

7.14 Theorem. *Let Σ be an (ω, ω_1+1)-iteration strategy for $\mathcal{M}$, and suppose that δ is a countable ordinal such that $\mathcal{M} \models \mathsf{ZF}^- +\ \delta$ is Woodin; then there is a $\mathbb{Q} \subseteq V_\delta^{\mathcal{M}}$ such that*

1. *$\mathcal{M} \models \mathbb{Q}$ is a δ-c.c. complete Boolean algebra, and*

2. *for any real x, there is a countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ played according to Σ with last model $\mathcal{M}_\alpha$ such that $i_{0,\alpha}$ exists and x is $i_{0,\alpha}(\mathbb{Q})$-generic over $\mathcal{M}_\alpha$.*

Proof. Working in $\mathcal{M}$, let $L_{\delta,0}$ be the infinitary language whose formulae are built up by means of conjunctions and disjunctions of size $< \delta$, and negation, from the propositional letters A_n, for $n < \omega$. (So all formulae are quantifier-free.) Any real x, regarded as a subset of ω, gives us an interpretation of $L_{\delta,0}$:

$$x \models A_n \iff n \in x.$$

We can then define $x \models \varphi$, for arbitrary formulae φ, by the obvious induction.

Still working in $\mathcal{M}$, consider the $L_{\delta,0}$ theory S which has the axioms

$$\bigvee\nolimits_{\alpha<\kappa} \varphi_\alpha \longleftrightarrow \bigvee\nolimits_{\alpha<\lambda} i_E(\langle \varphi_\xi \mid \xi < \kappa \rangle) \restriction \lambda$$

whenever E is on the $\mathcal{M}$-sequence, $\mathrm{crit}(E) = \kappa \leq \lambda$, and $\nu(E)$ is an $\mathcal{M}$-cardinal such that $i_E(\langle \varphi_\xi \mid \xi < \kappa \rangle) \restriction \lambda \in \mathcal{J}^{\mathcal{M}}_{\nu(E)}$.

We let $\mathbb{Q}$ be the Lindenbaum algebra of S. That is, we let

$$\varphi \sim \psi \quad \text{iff} \quad S \vdash \varphi \leftrightarrow \psi,$$

and

$$[\varphi] \leq [\psi] \quad \text{iff} \quad S \vdash \varphi \rightarrow \psi,$$

and we let

$$\mathbb{Q} := (\{[\varphi] \mid \varphi \in L_{\delta,0}\}, \leq).$$

Here provability in S means provability using the usual finitary rules together with the rule: from φ_α for all $\alpha < \kappa$ (where $\kappa < \delta$) infer $\bigwedge_{\alpha<\kappa} \varphi_\alpha$. Equivalently, $S \vdash \tau$ iff whenever x is a real in $\mathcal{M}[G]$ for some G generic over $\mathcal{M}$ and $x \models S$, then $x \models \tau$. (See [2]. Clearly, if $S \vdash \tau$, then *any* real which satisfies S satisfies τ.)

1 Claim. *$\mathbb{Q}$ is δ-c.c. in $\mathcal{M}$.*

Proof. We work in $\mathcal{M}$. Let $\langle[\varphi_\alpha] \mid \alpha < \delta\rangle$ be an antichain in $\mathbb{Q}$. Let $\kappa < \delta$ be $\langle\varphi_\alpha \mid \alpha < \delta\rangle$-reflecting. Let ν be a cardinal such that $\langle\varphi_\alpha : \alpha < \kappa+1\rangle \in \mathcal{J}_\nu^{\mathcal{M}}$, and let F on the $\mathcal{M}$-sequence witness the reflection of κ at this ν.[54] Let E be the trivial completion of $F{\restriction}\nu$. We then have

$$i_E\big(\textstyle\bigvee_{\alpha<\kappa}\varphi_\alpha\big){\restriction}(\kappa+1) = \textstyle\bigvee_{\alpha\leq\kappa}\varphi_\alpha,$$

so that

$$\textstyle\bigvee_{\alpha<\kappa}\varphi_\alpha \longleftrightarrow \textstyle\bigvee_{\alpha\leq\kappa}\varphi_\alpha$$

is provable in S. It follows that $[\varphi_\kappa] \leq \bigvee_{\alpha<\kappa}[\varphi_\alpha]$ in $\mathbb{Q}$, a contradiction. $\dashv$

2 Claim. *$\mathbb{Q}$ is a complete Boolean algebra in $\mathcal{M}$.*

Proof. $\mathbb{Q}$ is closed under sums of size $< \delta$ since $\bigvee_{\alpha<\kappa}[\varphi_\alpha] = [\bigvee_{\alpha<\kappa}\varphi_\alpha]$ for all $\kappa < \delta$. By Claim 1, $\mathbb{Q}$ is closed under arbitrary sums. $\dashv$

3 Claim. *If $x \models S$, then setting $G_x := \{[\varphi] \mid x \models \varphi\}$, we have that G_x is $\mathbb{Q}$-generic over $\mathcal{M}$ and $x \in \mathcal{M}[G_x]$.*

Proof. Since $x \models S$, G_x is well-defined on equivalence classes: if $S \vdash (\varphi \leftrightarrow \psi)$, then $x \models \varphi$ iff $x \models \psi$. It is also clear that G_x is an ultrafilter on $\mathbb{Q}$. To see that G_x is $\mathcal{M}$-generic, let $\langle[\varphi_\alpha] \mid \alpha < \nu\rangle$ be a maximal antichain. Since $[\bigvee_{\alpha<\nu}\varphi_\alpha] = 1$, we have $S \vdash \bigvee_{\alpha<\nu}\varphi_\alpha$. Since $x \models S$, we have $x \models \varphi_\alpha$ for some α; that is, $[\varphi_\alpha] \in G_x$ for some α. Finally, $n \in x$ iff $[A_n] \in G_x$, so $x \in \mathcal{M}[G_x]$. $\dashv$

An arbitrary real x may not satisfy S, but one can iterate $\mathcal{M}$ in such a way that x satisfies some image of S.

4 Claim. *For any real x, there is a countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ which is played according to Σ, has last model $\mathcal{M}_\alpha$, and is such that $[0,\alpha]_T$ does not drop and $x \models i_{0,\alpha}(S)$.*

Proof. We keep iterating away the first extender which induces an axiom not satisfied by x. More precisely, set $\mathcal{M}_0 = \mathcal{M}$, and now suppose that we have constructed the model $\mathcal{M}_\beta$ of $\mathcal{T}$, where $\beta < \omega_1$. Suppose also that $\mathcal{T}$ has not dropped anywhere yet; that is, $D^{\mathcal{T}} = \emptyset$ as of now. If $x \models i_{0,\beta}(S)$ we are done, so suppose not. Let E on the $\mathcal{M}_\beta$-sequence be such that E induces an axiom of $i_{0,\beta}(S)$ which is false of x, and $\mathrm{lh}(E)$ is minimal among all extenders on the $\mathcal{M}_\beta$-sequence with this property. We set $E_\beta^{\mathcal{T}} := E$, and use E according to the rules for ω-maximal iteration trees to extend $\mathcal{T}$ one more step.

We must check here that $\gamma < \beta \implies \mathrm{lh}(E_\gamma) < \mathrm{lh}(E_\beta)$. But if not, the agreement of models in an ω-maximal iteration tree implies that E_β is on

[54] We are using here the fact that the Woodinness of δ in $\mathcal{M}$ is witnessed by extenders on the $\mathcal{M}$-sequence. We might just have added this to the hypotheses of Theorem 7.14, but we need not do so because, by [36], it follows from the other hypotheses.

the sequence of $\mathcal{M}_\gamma$, and it is not hard to check that the false axiom of $i_{0,\beta}(S)$ it induces in $\mathcal{M}_\beta$ is also induced by it in $\mathcal{M}_\gamma$. (To see that $\nu(E_\beta)$ is a cardinal of $\mathcal{M}_\gamma$ in this situation, note that since $\nu(E_\gamma)$ is a cardinal of $\mathcal{M}_\gamma$, any cardinal of $\mathcal{M}_\beta$ which is $\leq \nu(E_\gamma)$ is a cardinal of $\mathcal{M}_\gamma$. But $\nu(E_\beta) < \mathrm{lh}(E_\beta) \leq \mathrm{lh}(E_\gamma)$ and there are no cardinals of $\mathcal{M}_\beta$ in the interval $(\nu(E_\gamma), \mathrm{lh}(E_\gamma))$, so $\nu(E_\beta) \leq \nu(E_\gamma)$.)

We must also check that $[0, \beta+1]$ does not drop; that is, that E_β measures all subsets of its critical point κ in the model $\mathcal{M}_\gamma$ to which it is applied. This is true because $\kappa < \nu(E_\gamma)$, $\nu(E_\gamma)$ is a cardinal of $\mathcal{M}_\gamma$, and $\mathcal{M}_\beta$ agrees with $\mathcal{M}_\gamma$ below $\nu(E_\gamma)$.

This finishes the successor step in the construction of $\mathcal{T}$. At limit ordinals $\lambda \leq \omega_1$ we use Σ to extend $\mathcal{T}$.

It is enough to show this process terminates at some countable ordinal, so suppose not. We reach a contradiction much as in the proof that the comparison process terminates. As in that argument, let

$$\pi : H \to V_\eta$$

be elementary, where H is a countable, transitive set, and V_η and the range of π are large enough to contain everything of interest. Let $\pi(\bar{\mathcal{T}}) = \mathcal{T}$, etc., and let $\alpha = \mathrm{crit}(\pi) = \omega_1^H$. We get as before, setting $\delta^* = i^{\mathcal{T}}_{0,\alpha}(\delta) = i^{\bar{\mathcal{T}}}_{0,\alpha}(\delta)$,

$$V_{\delta^*}^{\mathcal{M}_\alpha^{\bar{\mathcal{T}}}} = V_{\delta^*}^{\mathcal{M}_\alpha^{\mathcal{T}}}$$

and

$$\pi \restriction V_{\delta^*}^{\mathcal{M}_\alpha^{\bar{\mathcal{T}}}} = i^{\mathcal{T}}_{\alpha,\omega_1} \restriction V_{\delta^*}^{\mathcal{M}_\alpha^{\mathcal{T}}}.$$

Now let $\beta + 1$ be the T-successor of α on $[0, \omega_1]_T$. We have $\mathrm{crit}(E_\beta) = \mathrm{crit}(i_{\alpha,\omega_1}) = \alpha$, and we have an axiom

$$\bigvee\nolimits_{\gamma<\alpha} \varphi_\gamma \longleftrightarrow i_{E_\beta}\big(\bigvee\nolimits_{\gamma<\alpha} \varphi_\gamma\big) \restriction \lambda$$

of $i_{0,\beta}(S)$ induced by E_β and false of x. The falsity of this axiom means that the right hand side is true of x, but the left hand side is not. But now $\bigvee_{\gamma<\alpha} \varphi_\gamma$ is essentially a subset of α, and therefore is small enough that it is in $\mathcal{M}_\alpha$. Moreover, $\lambda < \nu(E_\beta)$, and since generators are not moved on $\mathcal{T}$

$$i_{E_\beta}\big(\bigvee\nolimits_{\gamma<\alpha} \varphi_\gamma\big) \restriction \lambda = i_{\alpha,\omega_1}\big(\bigvee\nolimits_{\gamma<\alpha} \varphi_\gamma\big) \restriction \lambda = \pi\big(\bigvee\nolimits_{\gamma<\alpha} \varphi_\gamma\big) \restriction \lambda.$$

But $x \in H$ and $\pi(x) = x$. Since $L_{\delta,0}$ satisfaction is sufficiently absolute and $x \not\models \bigvee_{\gamma<\alpha} \varphi_\gamma$, we have $x \not\models \pi(\bigvee_{\gamma<\alpha} \varphi_\gamma)$. This contradicts the fact that x satisfies the initial segment $i_{E_\beta}(\bigvee_{\gamma<\alpha} \varphi_\gamma) \restriction \lambda$ of this disjunction. ⊣

If $\mathcal{M}_\alpha$ is as in Claim 4, then we can replace $\mathcal{M}$ by $\mathcal{M}_\alpha$ in Claims 1, 2, and 3, and we see then that $\mathcal{T}$ and $\mathcal{M}_\alpha$ witness the conclusion of Theorem 7.14. ⊣

The complete Boolean algebra $\mathbb{Q}$ of Theorem 7.14 is known as the *extender algebra*.

We drop for a moment to smaller mice, and use the extender algebra to prove a correctness result for the minimal proper class model with one Woodin cardinal. (This was Woodin's original application of Theorem 7.14.) Let us call a premouse $\mathcal{M}$ *1-small* iff whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence, then $\mathcal{J}_\kappa^{\mathcal{M}} \models$ "There are no Woodin cardinals". Let $M_1^\#$ be the least mouse which is not 1-small, and M_1 the result of iterating the last extender of $M_1^\#$ out of the universe. (Granted that there is a Woodin cardinal with a measurable above it in V, $M_1^\#$ exists and is (ω, ω_1+1)-iterable.) Let $\mathbb{Q}$ be the extender algebra of M_1; then for any Σ_3^1 sentence φ, possibly involving real parameters from M_1, we have

$$\varphi \iff M_1 \models \exists p(p \Vdash \varphi).$$

The right-to-left direction comes from the fact that $P(\mathbb{Q}) \cap M_1$ is countable in V, so that any condition is extended by a generic filter in V. For the left-to-right direction: let x witness the outer existential quantifier of φ, and let $\mathcal{M}_\alpha$ be an iterate of M_1 over which x is $i_{0,\alpha}(\mathbb{Q})$-generic. Clearly, $\mathcal{M}_\alpha[G_x] \models \varphi$, so $\mathcal{M}_\alpha \models \exists p(p \Vdash \varphi)$, so by elementarity $M_1 \models \exists p(p \Vdash \varphi)$.

Thus M_1 can compute Σ_3^1 truth by asking what is forced in its extender algebra. (M_1 is not itself Σ_3^1-correct.) This easily implies that every real which is Δ_3^1 in a countable ordinal is in M_1. A careful look at the sort of iterability needed to compare "properly 1-small" mice (like $\mathcal{J}_\alpha^{M_1}$, for $\alpha = \omega_1^{M_1}$) shows every real in M_1 is Δ_3^1 in a countable ordinal, so we have a descriptive-set-theoretic characterization of the reals in M_1.[55]

$M_1^\#$ is essentially a real, and from this real we can recursively construct generic objects for the extender algebra of M_1 below any condition. It follows that every nonempty Σ_3^1 set of reals has a member recursive in $M_1^\#$. We can relativize the $M_1^\#$ construction to an arbitrary real x and obtain $M_1^\#(x)$; simply throw x into the model at the bottom. We get that any nonempty $\Sigma_3^1(x)$ set of reals has a member recursive in $M_1^\#(x)$, and therefore any premouse closed under the function $x \mapsto M_1^\#(x)$ is Σ_3^1-correct. In particular, M_ω is Σ_3^1-correct.

If we give M_n and $M_n^\#$ the obvious meaning, then we can show that the reals of M_n are precisely those which are Δ_{n+2}^1 in a countable ordinal, and that every nonempty Σ_{n+2}^1 set of reals has a member recursive in $M_n^\#$. (See [42].) Since M_ω is closed under $x \mapsto M_n^\#(x)$ for all $n < \omega$, M_ω is projectively correct. The following theorem gives us much more; it says that M_ω can compute $L(\mathbb{R})$ truth in much the same way that M_1 can compute Σ_3^1 truth.

We let $\mathrm{Col}(\omega, X)$ be the collapsing poset of all finite partial functions from ω into X. Notice that Theorem 7.14 implies that if $\mathcal{M}$, Σ, and δ satisfy its hypotheses, then for any real x there is a countable $\mathcal{T}$ played by Σ, with last model $\mathcal{M}_\alpha$, such that x is $\mathrm{Col}(\omega, i_{0,\alpha}(\delta))$-generic over $\mathcal{M}_\alpha$. This is true

[55] This set of reals is known in descriptive set theory as Q_3, and it has many other characterizations. $M_1^\#$ is also known from descriptive set theory; it is Turing equivalent to y_0. See [12].

because $\mathrm{Col}(\omega, \kappa)$ is universal for forcings of size κ. Unlike the extender algebra, $\mathrm{Col}(\omega, \delta)$ is not δ-c.c.; on the other hand, it is homogeneous.

By $\mathrm{Col}(\omega, <\lambda)$ we mean the finite support product of all $\mathrm{Col}(\omega, \alpha)$ such that $\alpha < \lambda$. If G is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, <\lambda)$, then we set

$$\mathbb{R}^*_G := \bigcup_{\alpha<\lambda} \mathbb{R} \cap \mathcal{M}[G \cap \mathrm{Col}(\omega, <\alpha)],$$

and say that $\mathbb{R}^*_G$ is the set of reals of a symmetric collapse of $\mathcal{M}$ below λ.

7.15 Theorem. *Suppose that $\mathcal{M} \models \lambda$ is a limit of Woodin cardinals, where λ is countable in V, and that Σ is an $(\omega, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$. Let H be $\mathrm{Col}(\omega, \mathbb{R})$-generic over V; then in $V[H]$ there is an iteration map $i : \mathcal{M} \to \mathcal{N}$ coming from an iteration tree all of whose proper initial segments are played by Σ, and a G which is $\mathrm{Col}(\omega, <i(\lambda))$-generic over $\mathcal{N}$, such that*

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

Proof. We shall need the following slight refinement of Theorem 7.14.

7.16 Lemma. *Let $\mathcal{M} \models \delta$ is Woodin, where δ is countable in V, and let Σ be an (ω, ω_1+1)-iteration strategy for $\mathcal{M}$. Let $\kappa < \delta$, and let G be $\mathcal{M}$-generic for a poset $\mathbb{P} \in V^{\mathcal{M}}_\kappa$. Then for any $x \subseteq \omega$, there is a countable iteration tree $\mathcal{T}$ played by Σ and having last model $\mathcal{M}_\alpha$ such that*

1. *$D^{\mathcal{T}} = \emptyset$ and $\mathrm{crit}(E^{\mathcal{T}}_\beta) > \kappa$ for all β, and*
2. *x is in some $\mathrm{Col}(\omega, \delta)$-generic extension of $\mathcal{M}_\alpha[G]$.*

Sketch of Proof. In $\mathcal{M}[G]$, δ is still Woodin via the extenders over $\mathcal{M}[G]$ which are "completions" of extenders on the $\mathcal{M}$-sequence with critical point $> \kappa$. So in $\mathcal{M}[G]$, the version of the extender algebra which uses only these extenders is a δ-c.c. complete Boolean algebra. The iteration $\mathcal{U}$ of $\mathcal{M}[G]$ we need to do to make x generic can be obtained from an iteration $\mathcal{T}$ of $\mathcal{M}$: $\mathcal{M}^{\mathcal{U}}_\beta = \mathcal{M}^{\mathcal{T}}_\beta[G]$ for all β. We omit further details. ⊣

We can now prove the theorem. Working in $V[H]$, let $\langle x_n \mid n < \omega\rangle$ be an enumeration of $\mathbb{R}^V$. Let $\langle \delta_n \mid n < \omega\rangle$ be an increasing sequence of Woodin cardinals of $\mathcal{M}$ which is cofinal in λ. We shall use Lemma 7.16 to successively absorb the x_n into the collapse of some image of δ_n in an iterate of $\mathcal{M}$.

More precisely, working in V we find a countable iteration tree $\mathcal{T}_0$ on $\mathcal{M}$ played by Σ with last model $\mathcal{P}_0$, and a G_0 which is $\mathcal{P}_0$-generic over $\mathrm{Col}(\omega, i_0(\delta_0))$, where $i_0 : \mathcal{M} \to \mathcal{P}_0$ is the iteration map, so that $x_0 \in \mathcal{P}_0[G_0]$. We then find an iteration tree $\mathcal{T}_1$ on $\mathcal{P}_0$ such that $\mathcal{T}_0 \oplus \mathcal{T}_1$ is according to Σ, and if $i_1 : \mathcal{P}_0 \to \mathcal{P}_1$ is the iteration map, then $\mathrm{crit}(i_1) > i_0(\delta_0)$, and there is a G_1 which is $\mathcal{P}_1[G_0]$-generic over $\mathrm{Col}(\omega, i_1 \circ i_0(\delta_1))$ such that $x_1 \in \mathcal{P}_1[G_0][G_1]$. And so on: given $\mathcal{P}_n$, we use Lemma 7.16 in V to obtain an iteration tree $\mathcal{T}_{n+1}$ on $\mathcal{P}_n$ such that $\mathcal{T}_0 \oplus \cdots \oplus \mathcal{T}_{n+1}$ is according to Σ, and if $i_{n+1} : \mathcal{P}_n \to \mathcal{P}_{n+1}$ is the iteration map, then $\mathrm{crit}(i_{n+1}) > i_n \circ \ldots \circ i_0(\delta_n)$, and there is a G_{n+1}

which is $\mathcal{P}_{n+1}[G_0, \ldots, G_n]$-generic over $\mathrm{Col}(\omega, i_{n+1} \circ \ldots \circ i_0(\delta_{n+1}))$ such that $x_{n+1} \in \mathcal{P}_{n+1}[G_0, \ldots, G_n][G_{n+1}]$.

Let $\mathcal{T} = \left(\bigoplus_n \mathcal{T}_n\right) \oplus b$, where b is the branch of $\bigoplus_n \mathcal{T}_n$ containing the $\mathcal{P}_n$. By construction, b is the unique cofinal branch of $\bigoplus_n \mathcal{T}_n$, and the $\mathcal{T}_n$ constitute a play by Σ. Let $\mathcal{N}$ be the last model of $\mathcal{T}$; clearly $\mathcal{N}$ is just the direct limit of the $\mathcal{P}_n$ under the i_n. A simple absoluteness argument shows that $\mathcal{N}$ is wellfounded: if not, then the tree of attempts to produce a sequence $\langle \mathcal{U}_n \mid n \in \omega \rangle$ which constitutes a play of ω rounds of $\mathcal{G}_\omega(\mathcal{M}, \omega, \omega_1+1)$ by Σ, together with a descending chain of ordinals in the direct limit along the unique cofinal branch, would have a branch in V. Let $i : \mathcal{M} \to \mathcal{N}$ be the direct limit map. By construction, each G_n is in V, so we have $x_n \in (\mathbb{R} \cap \mathcal{N}[G_0, \ldots, G_{n+1}]) \subseteq \mathbb{R}^V$, and therefore $\bigcup_n (\mathbb{R} \cap \mathcal{N}[G_0, \ldots, G_n]) = \mathbb{R}^V$. It is easy to see that $\bigcup_n (\mathbb{R} \cap \mathcal{N}[G_0, \ldots, G_n])$ is the set of reals of a symmetric collapse of $\mathcal{N}$ below $i(\lambda)$, so we are done. $\dashv$

7.17 Corollary. *Let $\mathcal{M}$ be a proper class premouse such that $\mathcal{M} \models \lambda$ is a limit of Woodin cardinals, where λ is countable in V, and suppose that $\mathcal{M}$ is (ω, ω_1+1)-iterable; then every real which is ordinal definable over $L(\mathbb{R})$ belongs to $\mathcal{M}$.*

Proof. Let $i : \mathcal{M} \to \mathcal{N}$ be as in Theorem 7.15, and let x be $\mathrm{OD}^{L(\mathbb{R})}$. We have, by the symmetry of $\mathrm{Col}(\omega, {<}i(\lambda))$ and the fact that $L(\mathbb{R})^V$ is realized as some $L(\mathbb{R}^*_G)$, that $x \in \mathcal{N}$. It follows that $x \in \mathcal{M}$. $\dashv$

The proof of Theorem 6.16 shows that if λ is a limit of Woodin cardinals and there is a measurable cardinal above λ, then $M_\omega^\#$ exists and is $(\omega, \omega_1, \omega_1+1)$-iterable, not just in V, but in $V^{\mathbb{P}}$, for any poset $\mathbb{P}$ of cardinality $< \lambda$. So we get at once:

7.18 Corollary. *If there are ω Woodin cardinals with a measurable above them all in V, then $\mathbb{R} \cap M_\omega = \{x \in \mathbb{R} \mid x$ is $\mathrm{OD}^{L(\mathbb{R})}\}$.*

We are in a position now to see that M_ω has no (ω, ω_1)-iteration strategy in $L(\mathbb{R})$. (We assume here that there are in V ω Woodin cardinals with a measurable above them all.) For if there were such a strategy in $L(\mathbb{R})$, then the set of reals which are not in M_ω would be a $\Sigma_1^{L(\mathbb{R})}$ set: $z \notin M_\omega$ iff $L(\mathbb{R}) \models$ (there is an (ω, ω_1)-iterable, ω-small premouse $\mathcal{N}$ of ordinal height ω_1 such that for some countable λ, $\mathcal{N} \models \lambda$ is a limit of Woodin cardinals, and such that $z \notin \mathcal{N}$). However, by [16], any nonempty $\Sigma_1^{L(\mathbb{R})}$ set of reals has an $\mathrm{OD}^{L(\mathbb{R})}$ member.[56] So there is an $\mathrm{OD}^{L(\mathbb{R})}$ real not in M_ω, contrary to Corollary 7.18.

The proof of Theorem 7.15 shows that any sufficiently iterable proper class model with ω Woodin cardinals can compute $L(\mathbb{R})$ truth by consulting its symmetric collapse; in fact

[56] We shall give a purely inner-model-theoretic proof of this result immediately after Theorem 7.20.

7.19 Theorem. *Let $\mathcal{M}$ be a proper class premouse such that $\mathcal{M} \models \lambda$ is a limit of Woodin cardinals, where λ is countable in V, and suppose that $\mathcal{M}$ is (ω, ω_1+1)-iterable. Let $\mathbb{R}^*$ be the set of reals of a symmetric collapse of $\mathcal{M}$ below λ; then in $V^{\mathrm{Col}(\omega,\mathbb{R})}$ there is an elementary $j : L(\mathbb{R}^*) \to L(\mathbb{R})^V$.*

Sketch of Proof. Let $\langle \delta_n \mid n < \omega \rangle$ be a sequence of Woodin cardinals with limit λ, and let G_n be $\mathrm{Col}(\omega, \delta_n)$-generic over $\mathcal{M}$ and such that

$$\mathbb{R}^* = \bigcup_n (\mathbb{R} \cap \mathcal{M}[G_n]).$$

Working in $V[H]$, where H is $\mathrm{Col}(\omega, \mathbb{R})$-generic over V, the proof of Theorem 7.15 gives for each n an iteration map

$$i_n : \mathcal{M} \to \mathcal{P}_n, \quad \text{with } \mathrm{crit}(i_n) > \delta_n,$$

such that $\mathbb{R}^V$ is the set of reals of a symmetric collapse of $\mathcal{P}_n$ below $i_n(\lambda)$. Let

$$\Gamma = \{\alpha \in \mathrm{On} \mid \forall n (i_n(\alpha) = \alpha)\},$$

and

$$X = \{x \mid x \text{ is definable over } L(\mathbb{R}) \text{ from elements of } \mathbb{R}^* \cup \Gamma\}.$$

Since the i_n are iteration maps, Γ is a proper class. Now i_n induces an elementary embedding $i_n^* : \mathcal{M}[G_n] \to \mathcal{P}_n[G_n]$, and by the homogeneity of the symmetric collapses we get, for all reals $\vec{x}$ in $\mathcal{M}[G_n]$, ordinals $\vec{\alpha}$, and formulae φ,

$$L(\mathbb{R}^*) \models \varphi[\vec{x}, \vec{\alpha}] \iff L(\mathbb{R})^V \models \varphi[\vec{x}, i_n(\vec{\alpha})].$$

It follows easily that

$$\mathbb{R} \cap X = \mathbb{R}^*.$$

Thus it suffices to show that $X \prec L(\mathbb{R})$, for then the inverse of the transitive collapse of X is the desired elementary embedding. So suppose

$$L(\mathbb{R}) \models \exists v \sigma[\vec{y}, \vec{\alpha}],$$

where $\vec{y} \in (\mathbb{R}^*)^{<\omega}$ and $\vec{\alpha} \in \Gamma^{<\omega}$. Pick n such that $\vec{y} \in M[G_n]$. Using the partial elementarity of i_n displayed above, we get

$$L(\mathbb{R}^*) \models \exists v \sigma[\vec{y}, \vec{\alpha}].$$

Since Γ is a proper class, we can take the witness v from $L(\mathbb{R}^*)$ to be definable over $L(\mathbb{R}^*)$ from z and $\vec{\beta}$, where $z \in \mathbb{R}^*$ and $\vec{\beta} \in \Gamma^{<\omega}$. Let $k \geq n$ be such that $z \in M[G_k]$; then the partial elementarity of i_k guarantees that there is a witness v to σ which is $L(\mathbb{R})$-definable from $z, \vec{y}, \vec{\beta}$, and $\vec{\alpha}$. This shows $X \prec L(\mathbb{R})$, as desired. ⊣

Although iterable class models with ω Woodin cardinals can compute $L(\mathbb{R})$ truth, they need not be correct for arbitrary statements about $L(\mathbb{R})$. We do have, however:

7.20 Theorem. *Let $\mathcal{M}$ be a proper class premouse such that $\mathcal{M} \models \eta$ is a limit of Woodin cardinals, for some $\eta < \omega_1^V$. Suppose that $\mathcal{M}$ is $(\omega, \omega_1 + 1)$-iterable; then for any real $x \in \mathcal{M}$ and Σ_1 formula φ, containing perhaps a name $\dot{\mathbb{R}}$ for $\mathbb{R}$,*

$$(L(\mathbb{R}) \models \varphi[x]) \quad \Longrightarrow \quad (L(\mathbb{R})^{\mathcal{M}} \models \varphi[x]).$$

Proof. We shall assume $x \in M_\omega$; the argument in general is only slightly more complicated.

Fix an ω-small proper class premouse $\mathcal{N}$ whose extender sequence is an initial segment of that of $\mathcal{M}$, and such that there is a $\lambda \leq \eta$ such that λ is a limit of Woodin cardinals in $\mathcal{N}$. To see that there is such an $\mathcal{N}$, note that either $\mathcal{M}$ is ω-small, in which case we can take $\mathcal{N} = \mathcal{M}$, or $M_\omega^\# = \mathcal{J}_\alpha^{\mathcal{M}}$ for some α, in which case we can take $\mathcal{N} = M_\omega$. The iterability of $\mathcal{M}$ implies that of $\mathcal{N}$. From Theorem 7.19 we get some α such that $J_\alpha^{\mathcal{N}} \models \mathsf{ZF}^- +$ "There is a λ which is a limit of Woodin cardinals, and $L(\mathbb{R}^*) \models \varphi[x]$, where $\mathbb{R}^*$ is the set of reals of a symmetric collapse below λ". By taking a Skolem hull inside $\mathcal{N}$ and comparing the result with $\mathcal{N}$, we see that if $\bar{\alpha}$ is the least such α, then $\bar{\alpha}$ is countable in $\mathcal{N}$. Define $\bar{\lambda}$ to be least λ such that $J_{\bar{\alpha}}^{\mathcal{N}} \models \lambda$ is a limit of Woodin cardinals.

We claim that $\mathcal{J}_{\bar{\alpha}}^{\mathcal{N}}$ is $(\omega, \omega_1 + 1)$-iterable in $\mathcal{M}$. (This is why we dropped from $\mathcal{M}$ to $\mathcal{N}$.) For $\mathcal{Q} := \mathcal{J}_{\bar{\alpha}+1}^{\mathcal{N}}$ is properly small, and therefore by Corollary 7.13, has a (ω, ω_1^V)-iteration strategy Σ which is $\Sigma_1^{L(\mathbb{R})}(\{\mathcal{Q}\})$. By Theorem 7.19, and the homogeneity of $\mathrm{Col}(\omega, {<}\eta)$, $V_\eta^{\mathcal{M}}$ is closed under Σ, and $\Sigma \restriction V_\eta^{\mathcal{M}} \in \mathcal{M}$.

We can now run the construction of Theorem 7.15 in $\mathcal{M}[H]$, where H is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mathbb{R})$. We obtain an iteration map

$$i : \mathcal{J}_{\bar{\alpha}}^{\mathcal{N}} \to \mathcal{P}$$

such that for some $\mathrm{Col}(\omega, {<}i(\bar{\lambda}))$-generic G over $\mathcal{P}$

$$\mathbb{R}^{\mathcal{M}} = \mathbb{R}^*_G.$$

Thus, for $\xi = \mathrm{On}^{\mathcal{P}}$, $L_\xi(\mathbb{R}^{\mathcal{M}}) \models \varphi[x]$, and hence $L(\mathbb{R}^{\mathcal{M}}) \models \varphi[x]$ since φ is Σ_1. $\dashv$

One can also prove this theorem using stationary tower forcing. (By Theorem 7.15 we have an iteration map $i : \mathcal{M} \to \mathcal{P}$ such that for some G which is $\mathrm{Col}(\omega, {<}i(\eta))$-generic over $\mathcal{P}$, $\mathbb{R}^V = \mathbb{R}^*_G$. Via stationary tower forcing over $\mathcal{P}$ one gets, for any α, a $\mathcal{P}$-generic elementary embedding $j : \mathcal{P} \to \mathcal{Q}$ with $\mathbb{R}^{\mathcal{Q}} = \mathbb{R}^*_G$ and $\alpha \in \mathrm{wfp}(\mathcal{Q})$. Then any Σ_1 fact true in $L(\mathbb{R})^V$ is true in some such $L(\mathbb{R})^{\mathcal{Q}}$, hence in $L(\mathbb{R})^{\mathcal{P}}$, and hence in $L(\mathbb{R})^{\mathcal{M}}$.) It is often the case that stationary tower forcing and genericity iterations can be made to do the same work.[57]

The argument of Theorem 7.20 yields another proof of the standard basis theorem for $\Sigma_1^{L(\mathbb{R})}$: every nonempty $\Sigma_1^{L(\mathbb{R})}$ set of reals has a $\Delta_1^{L(\mathbb{R})}$ member. For if φ defines our set over $L(\mathbb{R})$, then as in Theorem 7.20 we get an initial segment $\mathcal{Q}$ of M_ω of height $< \omega_1^{M_\omega}$ such that for some λ, $\mathcal{Q} \models \lambda$ is a limit of Woodin cardinals and it is forced in the symmetric collapse over $\mathcal{Q}$ below λ that $L(\mathbb{R}^*) \models \exists z \varphi(z)$. Working in M_ω, where λ is countable, we can find a generic object G for some $\mathrm{Col}(\omega, \delta)$, where $\delta < \lambda$, such that $\mathcal{Q}[G] \models \exists z(\mathrm{Col}(\omega, {<}\lambda) \Vdash \varphi(\check{z})^{L(\mathbb{R})})$. Picking such a $z \in \mathcal{Q}[G]$, we see from the iterability of $\mathcal{Q}$ in $V^{\mathrm{Col}(\omega,\mathbb{R})}$ that $L(\mathbb{R})^V \models \varphi[z]$. But z is in M_ω, hence z is $\mathrm{OD}^{L(\mathbb{R})}$. If we pick the least such z in the canonical wellorder of the reals of M_ω, we get that z is $\Delta_1^{L(\mathbb{R})}$.

The argument just given is closely related to the proof we gave that every nonempty Σ_3^1 set of reals has a member recursive in $M_1^\#$. One can extend the argument so as to show via inner model theory that the pointclass $\Sigma_1^{L(\mathbb{R})}$ has the scale property. (See [16] for the original proof, which used methods involving games and determinacy due to Yiannis Moschovakis.) In recent unpublished work, Itay Neeman has found a general method which uses definability over mice to produce many pointclasses with the scale property. Neeman's work gives a new proof that Σ_{2n}^1 and Π_{2n+1}^1 have the scale property, for any $n \geq 1$. Neeman's work builds on earlier ideas of Woodin (unpublished, but see [42]), who found a purely inner-model-theoretic proof of the weaker fact that Σ_{2n}^1 and Π_{2n+1}^1 have the prewellordering property, for all n.

7.21 Corollary. *Suppose that there are ω Woodin cardinals with a measurable above them all. Then $M_\omega \models \mathbb{R}$ has a $\Delta_1^{L(\mathbb{R})}$ wellorder.*

Proof. By the reflection theorem,

$$x \in \mathrm{OD}^{L(\mathbb{R})} \iff \exists \alpha (x \in \mathrm{OD}^{L_\alpha(\mathbb{R})}).$$

So being $\mathrm{OD}^{L(\mathbb{R})}$ is a $\Sigma_1^{L(\mathbb{R})}$ property. Thus, by Corollary 7.18 and Theorem 7.20,

$$M_\omega \models \forall x \in \mathbb{R}(x \text{ is } \mathrm{OD}^{L(\mathbb{R})}).$$

The reals can now be wellordered in M_ω via their definitions in $L(\mathbb{R})^{M_\omega}$. ⊣

One can also prove Corollary 7.21 by showing that the natural wellorder of $\mathbb{R} \cap M_\omega$ given by the stages of construction is Δ_1 over $L(\mathbb{R})^{M_\omega}$. The proof of this is implicit in the arguments just given.

[57] One can also show using the scale property for $\Sigma_1^{L(\mathbb{R})}$ that if $\mathcal{M}$ is any model of set theory such that $\mathbb{R}^{\mathcal{M}}$ is countable, and every $\mathrm{OD}^{L(\mathbb{R})}(\{\mathbb{R}^{\mathcal{M}}\})$ set $X \subseteq \mathbb{R}^{\mathcal{M}}$ is in $\mathcal{M}$, then the conclusion of Theorem 7.20 holds. Combining this with the natural extension of Corollary 7.17 to sets of reals, we get yet another proof of Theorem 7.20.

The author (unpublished) has shown that $M_\omega \models V = \mathrm{HOD}$. The proof builds on that of Corollary 7.21, but more is required.[58]

The correctness theorem 7.20 is best possible, in the sense that, if there are ω Woodin cardinals with a measurable cardinal above them all, then the statement "There is a wellorder of the reals" is a Σ_1 statement which is true in $L(\mathbb{R})^{M_\omega}$, but not true in $L(\mathbb{R})$. Another such statement is "Every real is ordinal definable over some $L_\alpha(\mathbb{R})$".

Iterations to make reals generic can be used to prove the generic absoluteness theorems one gets from stationary tower forcing. For example:

7.22 Theorem (Woodin). *Suppose that λ is a limit of Woodin cardinals, and there is a measurable cardinal above λ. Let G be $\mathbb{P}$-generic over V, where $|\mathbb{P}| < \lambda$, and let H be* $\mathrm{Col}(\omega, \mathbb{R})^{V[G]}$*-generic over $V[G]$; then in $V[G][H]$ there is an elementary*

$$j : L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}.$$

In particular, $L(\mathbb{R})^V$ is elementarily equivalent to $L(\mathbb{R})^{V[G]}$.

Proof. Let $\langle (i_n, \mathcal{P}_n) \mid n < \omega \rangle$ be a genericity iteration of M_ω such that setting $\mathcal{P} =$ dirlim $\mathcal{P}_n$, we have that $\mathbb{R}^V$ can be realized as the reals $\mathbb{R}^*_K$ of a symmetric collapse of $\mathcal{P}$ below the sup of its Woodin cardinals. We get such an iteration in $V[G][H]$ from the proof of Theorem 7.15, and we have from that proof that each $\mathcal{P}_n$ is countable in V, and $\mathbb{R}^*_K = \bigcup_n \mathbb{R} \cap \mathcal{P}_n[K_n]$, where K_n is in V and $\mathrm{Col}(\omega, i_n \circ \cdots \circ i_0(\delta_n))$-generic over $\mathcal{P}_n$. (Here δ_n is the nth Woodin cardinal of M_ω.) Applying Theorem 7.15 again, we have for each n an iteration map $j_n : \mathcal{P}_n \to \mathcal{Q}_n$ such that $\mathrm{crit}(j_n) > i_n \circ \cdots \circ i_0(\delta_n)$ and $\mathbb{R}^{V[G]}$ is the set of reals of a symmetric collapse of $\mathcal{Q}_n$. Note that j_n lifts to an elementary $\hat{j}_n$ from $\mathcal{P}_n[K_n]$ to $\mathcal{Q}_n[K_n]$. From the homogeneity of the two collapses it then follows that for any real $x \in \mathcal{P}_n[K_n]$, formula φ, and ordinal α, $L(\mathbb{R})^V \models \varphi[x, i_{n,\omega}(\alpha)]$ iff $L(\mathbb{R})^{V[G]} \models \varphi[x, j_n(\alpha)]$. As in the proof of Theorem 7.19, this means that if we let $X = \{\alpha \mid \forall n(j_n(\alpha) = \alpha = i_{n,\omega}(\alpha))\}$, and let j be the inverse of the transitive collapse of the hull in $L(\mathbb{R})^{V[G]}$ of $X \cup \mathbb{R}^V$, then $j : L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$ elementarily. $\dashv$

One can also use genericity iterations to eliminate stationary tower forcing from the proof of $\mathsf{AD}^{L(\mathbb{R})}$, and in fact this can be done in several different ways. See for example [29, 28], and [45].

The connection between correctness of mice and definability of their iteration strategies extends much further. How much further is one of the central open problems of inner model theory.

7.23 Definition. *Mouse capturing* is the following statement: for all $x, y \in \mathbb{R}$, x is ordinal definable from y if and only if for some (ω, ω_1)-iterable y-premouse $\mathcal{M}$, $x \in \mathcal{M}$.

[58] One shows that the inductive definition of K from [43] relativizes in such a way that one can define over M_ω its extender sequence in each interval between successive Woodin cardinals of M_ω.

Here a y-premouse is just like an ordinary premouse, except that we put y in at the bottom of its hierarchy. We have shown in this section that the existence of $M_\omega^\#$ implies that mouse capturing holds in $L(\mathbb{R})$. Results of Woodin show that $\mathsf{AD}^{L(\mathbb{R})}$ implies that mouse capturing holds in $L(\mathbb{R})$, and in fact, appropriately interpreted, it holds in every $J_\alpha(\mathbb{R})$. (See [13] and [46].) Woodin has also shown that mouse capturing holds in models of determinacy beyond $L(\mathbb{R})$: in any model of AD in which all ω_1-iterable mice are tame (see [46]), and even beyond that, in the minimal model of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$.[59]

In his PhD thesis (Berkeley 2009), Grigor Sargsyan has shown that in fact mouse capturing holds in the minimal model of $\mathsf{AD}_\mathbb{R}$ + "Θ is regular".[60] These results have local refinements: mouse capturing holds in any reasonably closed Wadge initial segment of the minimal model of $\mathsf{AD}_\mathbb{R}$ + "Θ is regular".[61] The capturing mice in the minimal model of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ can be nontame, but all capturing mice in the minimal model of $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" are below a Woodin limit of Woodin cardinals.

This leads us to the

Mouse Set Conjecture. Assume AD^+, and that there is no ω_1-iteration strategy for a premouse satisfying "There is a superstrong cardinal"; then mouse capturing holds.

AD^+ is a strengthening of AD which holds in all the models of AD we have constructed under large cardinal hypotheses. See for example [13, §8] for a precise definition. We might have stated the mouse set conjecture with AD as its hypothesis, but preferred to separate it from the open technical question as to whether AD implies AD^+.

It might be possible to drop the hypothesis that there is no ω_1-iteration strategy for a premouse satisfying "There is a superstrong cardinal" from the mouse set conjecture. One would presumably then have to enlarge the notion of mouse, so as to accommodate canonical models with supercompacts and more. The hypothesis that there is no ω_1-iteration strategy for a premouse satisfying "There is a superstrong cardinal" is a convenient way to say that we are in the initial segment of AD^+ models in which the capturing mice are premice in the sense of this chapter.

The author believes that it is unlikely that one can construct (ω_1+1)-iterable premice satisfying "There is a superstrong cardinal" under any hypothesis, even the hypothesis that there are superstrong cardinals, without proving the mouse set conjecture.

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59 $\mathsf{AD}_\mathbb{R}$ is the assertion that all games on $\mathbb{R}$ are determined.

60 This is a well-known, strong determinacy hypothesis. Θ is the least ordinal that is not the surjective image of $\mathbb{R}$. Θ for $L(\mathbb{R})$ is officially defined at the beginning of the next section.

61 Sargsyan also shows that if there is an iterable mouse with a Woodin limit of Woodin cardinals, then there is an inner model of $\mathsf{AD}_\mathbb{R}$ + "Θ is regular".

8. $\mathrm{HOD}^{L(\mathbb{R})}$ below Θ

Having characterized the reals in $\mathrm{HOD}^{L(\mathbb{R})}$ in terms of mice, it is natural to look for a similar characterization of the full model $\mathrm{HOD}^{L(\mathbb{R})}$. In this section we shall describe some work of the author [41] and Woodin (unpublished) which provides such a characterization.

The arguments of the last section give more in this direction than we stated there. Let $\mathcal{N}$ be the linear iterate of M_ω obtained by taking ultrapowers by the unique normal measure on the least measurable cardinal, and its images, ω_1^V times. Thus the least measurable cardinal of $\mathcal{N}$ is ω_1^V. One can show by the methods of the last section that $P(\omega_1^V) \cap \mathrm{HOD}^{L(\mathbb{R})} = P(\omega_1^V) \cap \mathcal{N}$. (See [40, Sect. 4].) This clearly suggests that the whole of $\mathrm{HOD}^{L(\mathbb{R})}$ might be an iterate of M_ω. We shall show in this section that is almost true.

8.1 Definition.

$$\Theta = \sup\{\alpha \mid \exists f \in L(\mathbb{R})(f : \mathbb{R} \to \alpha \text{ and } f \text{ is surjective})\}.$$

8.2 Definition.

$$\underset{\sim}{\delta}_1^2 = \sup\big\{\alpha \mid \exists f(f : \mathbb{R} \to \alpha \text{ and } f \text{ is surjective and } \boldsymbol{\Delta}_1^{L(\mathbb{R})}\big\}.$$

Standard notation would require that we write $\Theta^{L(\mathbb{R})}$ and $(\underset{\sim}{\delta}_1^2)^{L(\mathbb{R})}$ here, but since we shall only interpret the notions in question in $L(\mathbb{R})$, we have chosen to drop the superscripts. Similarly, we shall occasionally write HOD for $\mathrm{HOD}^{L(\mathbb{R})}$ in this section. We have nothing to say about HOD^V here.

We shall show that below $\underset{\sim}{\delta}_1^2$, HOD is the direct limit of a certain class $\mathcal{F}$ of countable, iterable mice, under the iteration maps given by the comparison process. (One gets a typical element of $\mathcal{F}$ by iterating M_ω, then cutting the iterate off at a successor cardinal below its bottom Woodin cardinal.) The mice in $\mathcal{F}$ are properly small, so that $L(\mathbb{R})$ knows how to iterate them correctly. They are as "full" as possible, given this smallness condition. Fullness guarantees that in the comparison of two mice in $\mathcal{F}$, neither side drops along the branch leading to the final model, and thus we have iteration maps on both sides. The Dodd-Jensen Lemma guarantees that these maps commute, so that we can indeed form a direct limit. The whole direct limit system is definable over $L(\mathbb{R})$ in a way that insures its direct limit M_∞ is included in $\mathrm{HOD} \cap V_{\underset{\sim}{\delta}_1^2}$. On the other hand, we shall see that in the bigger universe $V^{\mathrm{Col}(\omega_1,\mathbb{R})}$ there is an iterate N of $\mathcal{M}_\omega$ such that M_∞ is just N cut off at the least cardinal κ which is β-strong for all β below the bottom Woodin cardinal of N. The correctness properties of N can then be used to show that $\mathrm{HOD} \cap V_{\underset{\sim}{\delta}_1^2} \subseteq M_\infty$.

The maps in our direct limit system will come from compositions of iteration trees. In order to make the Dodd-Jensen Lemma applicable, we need to take care of some details regarding unique iterability. Let $\mathcal{M}$ be properly small. By $\mathcal{G}^*(\mathcal{M},\lambda,\theta)$ we mean the variant of the iteration game $\mathcal{G}_\omega(\mathcal{M},\lambda,\theta)$ in which player I is not allowed to drop at the beginning of a new round. That is, if $\mathcal{Q}$ is the model we get at the end of round α and q is its degree (with $\mathcal{Q}=\mathcal{M}$ and $q=\omega$ if $\alpha=0$), then round $\alpha+1$ of $\mathcal{G}^*(\mathcal{M},\lambda,\theta)$ must be a play of $\mathcal{G}_q(\mathcal{Q},\theta)$. Let us call a play of $\mathcal{G}^*(\mathcal{M},\lambda,\theta)$ in which II has not yet lost an *almost ω-maximal* iteration tree on $\mathcal{M}$; such a tree is just a linear composition of appropriately maximal trees, where "appropriately" means that the composition is itself maximal. Our proof of Corollary 7.13 gives

8.3 Lemma. *Let $\mathcal{M}$ be countable, properly small, and $\Game^{\mathbb{R}}\Pi^1_1$-iterable; then in $L(\mathbb{R})$, there is a unique winning strategy Σ for $\mathcal{G}^*(\mathcal{M},\omega_1,\omega_1)$; moreover, Σ is $\Sigma_1^{L(\mathbb{R})}(\{\mathcal{M}\})$ definable, uniformly in $\mathcal{M}$.*

8.4 Definition. Let $\mathcal{M}$ be countable, properly small, and $\Game^{\mathbb{R}}\Pi^1_1$-iterable. An almost ω-maximal iteration tree on $\mathcal{M}$ is *correct* just in case it is played according to the unique winning strategy for II in $\mathcal{G}^*(\mathcal{M},\omega_1,\omega_1)$. We say that $\mathcal{M}$ *iterates correctly* to $\mathcal{N}$ iff $\mathcal{N}$ is the last model of some correct $\mathcal{T}$ on $\mathcal{M}$ such that the branch $\mathcal{M}$-to-$\mathcal{N}$ of $\mathcal{T}$ has no drops.

From the last lemma we have at once:

8.5 Lemma. *The relations*

$$\{(\mathcal{M},\mathcal{T}) \mid \mathcal{T} \textit{ is a correct tree on } \mathcal{M}\}$$

and

$$\{(\mathcal{M},\mathcal{N}) \mid \mathcal{M} \textit{ iterates correctly to } \mathcal{N}\}$$

on HC *are Σ_1-definable over $L(\mathbb{R})$.*

There may in fact be more than one iteration tree witnessing that $\mathcal{M}$ iterates correctly to $\mathcal{N}$, but our proof of the Dodd-Jensen Lemma, together with the Uniqueness Lemma 8.3 above, easily implies that all such trees give rise to the same iteration map $\pi:\mathcal{M}\to\mathcal{N}$. Because properly small $\mathcal{M}$ satisfy ZF^-, π is fully elementary.

8.6 Definition. A properly small mouse $\mathcal{M}$ is *full* iff whenever $\mathcal{M}$ iterates correctly to $\mathcal{N}$, A is a bounded subset of $\mathrm{On}\cap\mathcal{N}$, and A is ordinal definable over $L(\mathbb{R})$ from the parameter $\mathcal{N}$, then $A\in\mathcal{N}$.

Fullness is clearly Π_1-definable over $L(\mathbb{R})$.[62] Since the $\mathrm{OD}^{L(\mathbb{R})}(\{\mathcal{N}\})$ sets are captured by mice, we can reformulate fullness in purely inner-model-theoretic terms.

8.7 Definition. We write $\mathcal{N}\trianglelefteq^*\mathcal{P}$ iff $\mathcal{N}=\mathcal{J}^{\mathcal{P}}_\eta$ for some cutpoint η of $\mathcal{P}$. In this case, we also call $\mathcal{N}$ a cutpoint of $\mathcal{P}$.

[62] Notice that a premouse which is not $\Game^{\mathbb{R}}\Pi^1_1$-iterable is vacuously full, since there are no correct trees on it. Of course, we are only interested in the full mice which *are* $\Game^{\mathbb{R}}\Pi^1_1$-iterable.

8.8 Lemma. *The following are equivalent:*

1. *$\mathcal{M}$ is full,*
2. *if $\mathcal{M}$ iterates correctly to $\mathcal{N}$, and $\mathcal{N} \trianglelefteq^* \mathcal{P}$, and $\mathcal{P}$ is $\Game^{\mathbb{R}}\Pi^1_1$-iterable above $\mathrm{On} \cap \mathcal{N}$,[63] then $\rho_\omega(\mathcal{P}) \geq \mathrm{On} \cap \mathcal{N}$.*

Proof. To see that $1 \Longrightarrow 2$, notice that the proof of Corollary 7.12 relativizes, and thus if $\mathcal{P}$ and $\mathcal{N}$ are as in clause 2, then $\mathcal{P}$ is $\mathrm{OD}^{L(\mathbb{R})}$ from $\mathcal{N}$ as a parameter.

For the converse, suppose that $\mathcal{N}$ is a correct iterate of $\mathcal{M}$, and let A be a bounded subset of $\lambda := \mathrm{On} \cap \mathcal{N}$ which is $\mathrm{OD}^{L(\mathbb{R})}$ from $\mathcal{N}$. We can modify the K^c construction by starting with $\mathcal{N}$ instead of $(V_\omega, \in, \emptyset, \emptyset)$ as our initial structure, and by adding only extenders with critical point strictly greater than λ. All ω-small structures we produce in such a construction are $\Game^{\mathbb{R}}\Pi^1_1$-iterable above λ, and so by clause 2 no such structure projects strictly below λ. It follows that $\mathcal{N}$ is an initial segment of all structures in the construction; indeed, λ is included in every core we take. Since $\mathcal{N}$ has a largest cardinal, λ is not the critical point of any extender in such a core, so that $\Game^{\mathbb{R}}\Pi^1_1$-iterability above λ is enough for comparison. We therefore get a proper class premouse $M_\omega(\mathcal{N})$ with ω Woodin cardinals which is iterable above λ and has $\mathcal{N}$ as a cutpoint. The proof of Corollary 7.18 relativizes so as to show that $A \in M_\omega(\mathcal{N})$. But by 2, no level of $M_\omega(\mathcal{N})$ projects strictly below λ, and therefore $A \in \mathcal{N}$. ⊣

We can now define our direct limit system. Set

$$\mathcal{F} := \{\mathcal{M} \mid \mathcal{M} \text{ is properly small, } \Game^{\mathbb{R}}\Pi^1_1\text{-iterable, and full}\},$$

and for $\mathcal{M}, \mathcal{N}$ in $\mathcal{F}$, let

$$\mathcal{M} \prec^* \mathcal{N} \iff \exists \mathcal{P}(\mathcal{M} \text{ iterates correctly to } \mathcal{P} \text{ and } \mathcal{P} \trianglelefteq^* \mathcal{N}).$$

The Dodd-Jensen Lemma implies that if $\mathcal{M} \prec^* \mathcal{N}$, then there is a unique $\mathcal{P} \trianglelefteq^* \mathcal{N}$ and a unique fully elementary $\pi : \mathcal{M} \to \mathcal{P}$ which is the iteration map given by some play of $\mathcal{G}^*(\mathcal{M}, \omega_1, \omega_1)$ according to the unique winning strategy for II. (There may be more than one such play giving rise to π.) We let

$$\pi_{\mathcal{M},\mathcal{N}} := \text{unique correct iteration map from } \mathcal{M} \text{ to some } \mathcal{P} \trianglelefteq^* \mathcal{N}.$$

It is clear that $\mathcal{F}$,$\prec^*$, and the function $(\mathcal{M}, \mathcal{N}) \mapsto \pi_{\mathcal{M},\mathcal{N}}$ are $\mathrm{OD}^{L(\mathbb{R})}$.

8.9 Lemma. *The relation $\prec^*$ is transitive; moreover, if $\mathcal{M} \prec^* \mathcal{N} \prec^* \mathcal{P}$, then $\pi_{\mathcal{M},\mathcal{P}} = \pi_{\mathcal{N},\mathcal{P}} \circ \pi_{\mathcal{M},\mathcal{N}}$.*

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[63] This means that II wins the variant of $\mathcal{W}_\omega(\mathcal{N}, \omega)$ in which I is constrained to play only extenders with critical point above $\mathrm{On} \cap \mathcal{N}$.

Proof. Let $\mathcal{T}$ and $\mathcal{U}$ be correct trees witnessing that $\mathcal{M} \prec^* \mathcal{N}$ and $\mathcal{N} \prec^* \mathcal{P}$ respectively. Let $\mathcal{Q}$ be the last model of $\mathcal{T}$. Since $\mathcal{Q}$ is a cutpoint in $\mathcal{N}$, we can re-arrange $\mathcal{U}$ as an iteration tree $\mathcal{R}$ on $\mathcal{N}$ which uses only extenders from the image of $\mathcal{Q}$, followed by an iteration tree $\mathcal{S}$ on the last model $\mathcal{M}_\alpha^{\mathcal{R}}$ of $\mathcal{R}$ which uses no extenders from $i_{0,\alpha}^{\mathcal{R}}(\mathcal{Q})$. (We leave the details to the reader.) But then $\mathcal{T} \oplus \mathcal{R}$ witnesses that $\mathcal{M} \prec^* \mathcal{P}$. Moreover, the embedding given by $\mathcal{T} \oplus \mathcal{R}$ from $\mathcal{M}$ to its last model is just $i_{0,\alpha}^{\mathcal{R}} \circ \pi_{\mathcal{M},\mathcal{N}}$. Since $i_{0,\alpha}^{\mathcal{R}} = \pi_{\mathcal{N},\mathcal{P}} \restriction \mathcal{Q}$ by construction, the embedding given by $\mathcal{T} \oplus \mathcal{R}$ is $\pi_{\mathcal{N},\mathcal{P}} \circ \pi_{\mathcal{M},\mathcal{N}}$, as desired. ⊣

The Comparison Lemma and fullness imply that $\prec^*$ is directed. For suppose that $\mathcal{M}, \mathcal{N} \in \mathcal{F}$, and let $\mathcal{T}$ and $\mathcal{U}$ be the correct trees on $\mathcal{M}$ and $\mathcal{N}$ constituting their coiteration. Let $\mathcal{P}$ and $\mathcal{Q}$ be their respective last models, and suppose for example that $\mathcal{P} \trianglelefteq^* \mathcal{Q}$. (We can always take one more ultrapower so as to guarantee that $\trianglelefteq^*$, rather than just $\trianglelefteq$, holds between the last models.) From the Comparison Lemma we get that $\mathcal{M}$-to-$\mathcal{P}$ has no drops, so that $\mathcal{M}$ iterates correctly to $\mathcal{P}$. But $\mathcal{M}$ is full, so $\rho_\omega(\mathcal{Q}) \geq \mathrm{On} \cap \mathcal{P}$. Now if $\mathcal{N}$-to-$\mathcal{Q}$ drops, then letting κ be the extender used at the last drop, we have $\rho_\omega(\mathcal{Q}) \leq \kappa < \mathrm{On} \cap \mathcal{P}$. Thus $\mathcal{N}$-to-$\mathcal{Q}$ has no drops, so that $\mathcal{N}$ iterates correctly to $\mathcal{Q}$, and we have $\mathcal{M} \prec^* \mathcal{Q}$ and $\mathcal{N} \prec^* \mathcal{Q}$.

We wish to show that $\prec^*$ is countably directed, and for this it is most convenient to first relate the system $(\mathcal{F}, \prec^*)$ to a natural system $(\mathcal{F}^+, \prec^+)$ of iterates of M_ω.

8.10 Definition. Let Σ_0 be the unique winning strategy for II in the game $\mathcal{G}^*(M_\omega, \omega_1, \omega_1 + 1)$.

We can extend Definition 8.4 from properly small mice to iterates of M_ω in the natural way. In general, let us say that $\mathcal{M}$ *iterates correctly* to $\mathcal{Q}$, or $\mathcal{Q}$ is a *correct iterate* of $\mathcal{M}$, iff there is a unique winning strategy for II in $\mathcal{G}^*(\mathcal{M}, \omega_1, \omega_1 + 1)$, and $\mathcal{Q}$ is the last model of a countable iteration tree $\mathcal{T}$ on $\mathcal{M}$ played according to this strategy such that the branch $\mathcal{M}$-to-$\mathcal{Q}$ of $\mathcal{T}$ does not drop.

8.11 Definition. We call an iteration tree on a premouse $\mathcal{M}$ which satisfies "There is a Woodin cardinal" δ_0*-bounded* if it uses only extenders from the image of $\mathcal{J}_\delta^{\mathcal{M}}$, where δ is the least Woodin cardinal of $\mathcal{M}$.

Thus a δ_0-bounded tree on $\mathcal{M}$ is just one which can be interpreted as a tree on $\mathcal{J}_\delta^{\mathcal{M}}$, where δ is the least Woodin cardinal of $\mathcal{M}$.

8.12 Definition. We set

$$\mathcal{F}^+ = \{\mathcal{Q} \mid M_\omega \text{ iterates correctly to } \mathcal{Q} \text{ via a } \delta_0\text{-bounded tree}\},$$

and for $\mathcal{P}, \mathcal{Q} \in \mathcal{F}^+$, put

$$\mathcal{P} \prec^+ \mathcal{Q} \iff \mathcal{P} \text{ iterates correctly via a } \delta_0\text{-bounded tree to } \mathcal{Q}.$$

In this case, we let

$$\pi^+_{\mathcal{P},\mathcal{Q}} := \text{unique iteration map from } \mathcal{P} \text{ to } \mathcal{Q}.$$

The uniqueness of the iteration map from $\mathcal{P}$ to $\mathcal{Q}$ follows from the Dodd-Jensen Lemma.

The pair $(\mathcal{F}^+, \prec^+)$ is not lightface definable over $L(\mathbb{R})$, since from it we can define M_ω. It does happen to be definable over $L(\mathbb{R})$ from M_ω as a parameter, but this is of no use to us now. The function $(\mathcal{P}, \mathcal{Q}) \mapsto \pi^+_{\mathcal{P},\mathcal{Q}}$ does not even belong to $L(\mathbb{R})$. One can regard the system $(\mathcal{F}, \prec^*)$, with its maps, as an $L(\mathbb{R})$-definable approximation to the direct limit system $(\mathcal{F}^+, \prec^+)$, with its maps. We shall spell this out in more detail momentarily, but first we should verify:

8.13 Lemma. *The relation $\prec^+$ is transitive and countably directed; moreover, if $\mathcal{M} \prec^+ \mathcal{N} \prec^+ \mathcal{Q}$, then $\pi^+_{\mathcal{M},\mathcal{Q}} = \pi^+_{\mathcal{N},\mathcal{Q}} \circ \pi^+_{\mathcal{M},\mathcal{N}}$.*

Proof. Transitivity is obvious because we can compose iterations. (The situation here is a little simpler than it was with $\prec^*$.) The commutativity of the maps is clear.

Let $\mathcal{P}_i \in \mathcal{F}^+$ for all $i \in \omega$. Let $\mathcal{Q}_0 = M_\omega$, and given $\mathcal{Q}_i$, let $\mathcal{Q}_{i+1}$ be the last model of the iteration tree $\mathcal{T}_i$ on $\mathcal{Q}_i$ which results from comparing $\mathcal{Q}_i$ with $\mathcal{P}_i$, using their unique iteration strategies in both cases. Let $\mathcal{U}_i$ be the tree on $\mathcal{P}_i$ in this comparison. Clearly, neither $\mathcal{T}_i$ nor $\mathcal{U}_i$ drops along the branch to its last model, so $\mathcal{Q}_{i+1}$ is a correct iterate of both $\mathcal{Q}_i$ and $\mathcal{P}_i$. Letting $\mathcal{Q}$ be the direct limit of the $\mathcal{Q}_i$, we have that for all i, $\mathcal{Q}$ is a correct iterate of $\mathcal{P}_i$. In order to show $\mathcal{P}_i \prec^+ \mathcal{Q}$ for all i, it is enough to show that all $\mathcal{T}_i$ and $\mathcal{U}_i$ are δ_0-bounded.

Suppose that this is true for all $j < i$. Now we can regard M_ω as an initial segment of $M_\omega^\#$, and the latter is ω-sound and has Σ_1 projectum ω. The iteration strategy Σ_0 is the restriction to M_ω of a winning strategy in $\mathcal{G}^*(M_\omega^\#, \omega_1, \omega_1 + 1)$. Thus $\mathcal{P}_i$ and $\mathcal{Q}_i$ are initial segments of Σ_0-iterates $\mathcal{P}_i^*$ and $\mathcal{Q}_i^*$ of $M_\omega^\#$, and since the iterations are δ_0-bounded, each of $\mathcal{P}_i^*$ and $\mathcal{Q}_i^*$ is Σ_1-generated by the ordinals below its bottom Woodin cardinal. Now let $\mathcal{T}$ and $\mathcal{U}$ be the longest δ_0-bounded initial segments of $\mathcal{T}_i$ and $\mathcal{U}_i$, let $\mathcal{R}$ and $\mathcal{S}$ be their last models, and let $\mathcal{R}^*$ and $\mathcal{S}^*$ be the corresponding iterates of $M_\omega^\#$. Then $\mathcal{R}^*$ and $\mathcal{S}^*$ agree below their common value δ for the least Woodin cardinal (because this least Woodin is a cutpoint in each, and the last models of $\mathcal{T}_i$ and $\mathcal{U}_i$ so agree). Moreover, each is Σ_1-generated by δ, and they have a common iterate $\mathcal{Q}_{i+1}^*$ obtained from the rest of $\mathcal{T}_i$ and $\mathcal{U}_i$, which is above δ. It follows that $\mathcal{R}^* = \mathcal{S}^*$, so that $\mathcal{R} = \mathcal{S} = \mathcal{Q}_{i+1}$, and $\mathcal{T}_i$ and $\mathcal{U}_i$ are δ_0-bounded. $\dashv$

We now relate our two direct limit systems.

8.14 Lemma.

1. *Let $\mathcal{T}$ be an iteration according to Σ_0 of M_ω with last model $\mathcal{Q}$, and suppose that M_ω-to-$\mathcal{Q}$ does not drop. If η is a successor cardinal of $\mathcal{Q}$ below its bottom Woodin cardinal, then $\mathcal{J}_\eta^{\mathcal{Q}}$ is full, and therefore in $\mathcal{F}$.*

2. *Let $\mathcal{P} \in \mathcal{F}$, and let $\mathcal{M}$ be a correct iterate of M_ω; then there is a correct iterate $\mathcal{Q}$ of $\mathcal{M}$, given by a δ_0-bounded iteration tree, such that $\mathcal{P} \prec^* \mathcal{J}_\eta^{\mathcal{Q}}$ for some successor cardinal cutpoint η of $\mathcal{Q}$ below its bottom Woodin cardinal.*

3. *If $\mathcal{P} \prec^+ \mathcal{Q}$, and $\mathcal{M}$ is a cutpoint of $\mathcal{P}$ at some successor cardinal below its bottom Woodin cardinal, and $\mathcal{N} = \pi^+_{\mathcal{P},\mathcal{Q}}(\mathcal{M})$, then $\mathcal{M} \prec^* \mathcal{N}$, and $\pi_{\mathcal{M},\mathcal{N}} = \pi^+_{\mathcal{P},\mathcal{Q}} \restriction \mathcal{M}$.*

Proof. Clause 1 follows easily from Lemma 8.8: suppose that $\mathcal{J}_\eta^{\mathcal{Q}}$ iterates correctly to $\mathcal{N}$, and $\mathcal{N} \trianglelefteq^* \mathcal{P}$, where $\mathcal{P}$ is ω-small and $\Game^{\mathbb{R}}\Pi^1_1$-iterable above $\lambda := \mathrm{On} \cap \mathcal{N}$. We must show $\rho_\omega(\mathcal{P}) \geq \lambda$. Now, since η is a successor cardinal cutpoint of $\mathcal{Q}$, our correct iteration $\mathcal{J}_\eta^{\mathcal{Q}}$-to-$\mathcal{N}$ lifts to an iteration $\mathcal{Q}$-to-$\mathcal{R}$ according to Σ_0; moreover λ is a successor cardinal cutpoint of $\mathcal{R}$. We can now compare $\mathcal{P}$ and $\mathcal{R}$, and the comparison is above λ since it is a cutpoint of each. If $\rho_\omega(\mathcal{P}) < \lambda$, then we must have $\mathcal{P} \trianglelefteq \mathcal{R}$, but this contradicts the fact that λ is a cardinal of $\mathcal{R}$.

For 2, we simply compare $\mathcal{P}$ with $\mathcal{M}$, forming iterations according to the unique (ω, ω_1+1)-iteration strategy on both sides. Since $\mathcal{P}$ is properly small, it must iterate into an initial segment $\mathcal{R}$ of the last model $\mathcal{Q}$ on the $\mathcal{M}$ side, with no dropping from $\mathcal{P}$ to $\mathcal{R}$. Since $\mathcal{P}$ is full, $\mathcal{M}$-to-$\mathcal{Q}$ does not drop. Since $\mathcal{R}$ is properly small and full, it must have the form described.

For 3, notice that the iteration from $\mathcal{P}$ to $\mathcal{Q}$ can be factored so as to give an iteration from $\mathcal{M}$ to $\mathcal{N}$ because $\mathcal{M}$ is a cutpoint in $\mathcal{P}$. The uniqueness of the iteration strategies gives the rest. ⊣

8.15 Definition. We let M_∞ be the direct limit of $(\mathcal{F}, \prec^*)$ under the $\pi_{\mathcal{M},\mathcal{N}}$, and M_∞^+ be the direct limit of $(\mathcal{F}^+, \prec^+)$ under the $\pi^+_{\mathcal{M},\mathcal{N}}$, transitively collapsed in each case.

Since $\prec^+$ is countably directed, M_∞^+ is wellfounded, so we can regard it as transitive. But Lemma 8.14 shows that M_∞ is an initial segment of M_∞^+, so it too is wellfounded. In fact

8.16 Corollary. *Let δ be the least Woodin cardinal of M_∞^+, and let $\kappa < \delta$ be the least cardinal of M_∞^+ which is $<\delta$-strong in M_∞^+; then $M_\infty = \mathcal{J}_\kappa^{M_\infty^+}$.*

Proof. By Lemma 8.14, the set of all $\mathcal{M}$ which are cutpoints of some $\mathcal{Q} \in \mathcal{F}^+$ at a successor cardinal below its bottom Woodin cardinal (and hence below the least cardinal strong to its bottom Woodin) are cofinal in $(\mathcal{F}, \prec^*)$; moreover, the π^+ maps act on these $\mathcal{M}$ the same way that the π maps act. Thus M_∞ is the direct limit of all such $\mathcal{M}$ under the π^+ maps. Clearly, this direct limit is M_∞^+ cut at the sup of all its successor cardinal cutpoints below δ. That sup is just κ. ⊣

We shall now show that the ordinal height of M_∞ is $\underset{\sim}{\delta}^2_1$.

8.17 Definition. Let $\mathcal{M}$ be a premouse, $\varphi(v)$ a Σ_1 formula, and $x \in \mathbb{R}$. We call $\mathcal{M}$ a (φ, x)-*witness* just in case $\mathcal{M}$ has ω Woodin cardinals with supremum λ, and for some set $\mathbb{R}^*$ of reals of a symmetric collapse below λ over $\mathcal{M}$, we have $x \in \mathbb{R}^*$ and $L_\alpha(\mathbb{R}^*) \models \varphi[x]$, where $\alpha = \mathrm{On} \cap \mathcal{M}$.

8.18 Lemma. *Let φ be Σ_1 and $x \in \mathbb{R}$. The following are equivalent:*

1. $L(\mathbb{R}) \models \varphi[x]$,
2. *There is an $(\omega, \omega_1 + 1)$-iterable (φ, x)-witness,*
3. $\exists \mathcal{M} \in \mathcal{F} \exists \beta (\mathcal{J}_\beta^{\mathcal{M}}$ *is a (φ, x)-witness*$)$.

Proof. For $3 \Longrightarrow 2$, notice that $\mathcal{J}_\beta^{\mathcal{M}}$ is $(\omega, \omega_1 + 1)$-iterable, because $\mathcal{M}$ is. For $2 \Longrightarrow 1$, we can easily adapt the proofs of Theorems 7.19 and 7.15 to mice of set size with ω Woodin cardinals. We get, in some generic extension of V, an iterate of our witness $\mathcal{P}$ which has a symmetric collapse of the form $L_\alpha(\mathbb{R}^V)$ such that $L_\alpha(\mathbb{R}^V) \models \varphi[x]$. Since φ is Σ_1, this implies that $L(\mathbb{R}^V) \models \varphi[x]$.

We now prove $1 \Longrightarrow 3$. Let $\mathcal{Q}$ be a correct iterate of M_ω such that x is generic over $\mathcal{Q}$ for the extender algebra at its least Woodin cardinal δ. Now $\mathcal{Q}$ is a (φ, x)-witness by Theorem 7.19, but it is not an initial segment of any $\mathcal{M} \in \mathcal{F}$. We must therefore take some Skolem hulls.

Since φ is Σ_1, we can fix α such that $\mathcal{J}_\alpha^{\mathcal{Q}}$ is a (φ, x)-witness. Let $G_x^{\mathcal{Q}}$ be the generic object on the extender algebra of $\mathcal{Q}$ at δ determined by x. (That is, $[\psi] \in G_x^{\mathcal{Q}}$ iff $x \models \psi$.) We then have some $p \in G_x^{\mathcal{Q}}$ such that

$$\mathcal{J}_\alpha^{\mathcal{Q}} \models \exists \lambda [\lambda \text{ is a limit of Woodins and } p \Vdash (1 \Vdash (L(\mathbb{R}^*) \models \varphi[\check{\dot{x}}]))],$$

where the first forcing is the extender algebra, the second is the symmetric collapse, and $\dot{x}$ is the canonical name for the real determined by the extender algebra generic. This is a Σ_1 fact about p and δ, so we may assume that $\mathcal{J}_\alpha^{\mathcal{Q}}$ is Σ_1-generated by $\delta \cup \{\delta\}$. (The Σ_1 hull of these parameters collapses to an initial segment of $\mathcal{Q}$ by a simple comparison argument. The extender algebra is definable over $\mathcal{J}_\delta^{\mathcal{Q}}$, hence contained in the hull, so that $G_x^{\mathcal{Q}}$ is still generic over the collapse of the hull.)

Now, working in $\mathcal{Q}[x]$, where δ is still a regular cardinal, we can find an η and an elementary submodel $Y \prec \mathcal{J}_\eta^{\mathcal{Q}}[x]$ such that $\delta, \alpha, p, x \in Y$ and $Y \cap \delta \in \delta$. Let N be the transitive collapse of Y, and $\mathcal{P}$ be the image of $\mathcal{J}_\alpha^{\mathcal{Q}}$ under the collapse. Letting $\bar{\delta} = Y \cap \delta$, we have that $\mathcal{P}$ is iterable, Σ_1 projects to $\bar{\delta}$, and agrees with $\mathcal{Q}$ below $\bar{\delta}$. It follows that $\mathcal{P}$ is an initial segment of $\mathcal{J}_\delta^{\mathcal{Q}}$, by comparison, and therefore $\mathcal{P}$ is an initial segment of some $\mathcal{M} \in \mathcal{F}$. Since the property of being a (φ, x)-witness is first-order over $\mathcal{J}_\eta^{\mathcal{Q}}[x]$, we have that $\mathcal{P}$ is a (φ, x)-witness, as desired. ⊣

8.19 Lemma. $\mathrm{On} \cap M_\infty = \underset{\sim}{\delta}_1^2$.

Proof. A direct computation shows that $\mathrm{On} \cap M_\infty \leq \underset{\sim}{\delta}_1^2$. For let $\alpha \in \mathrm{On} \cap M_\infty$, and fix $\mathcal{M} \in \mathcal{F}$ so that $\pi_{\mathcal{M},\infty}(\bar{\alpha}) = \alpha$ for some $\bar{\alpha}$. Let $\mathcal{G} := \{\mathcal{P} \mid \mathcal{M}$ iterates correctly to $\mathcal{P}\}$. Then $\mathcal{G} \subseteq \mathcal{F}$, and one can easily check that $\mathcal{G}$ is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$. Also, the relation R is $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$, where

$$\begin{aligned} &R(\langle \mathcal{P}, \bar{\beta}\rangle, \langle \mathcal{Q}, \bar{\gamma}\rangle) \\ &\quad\iff \quad (\mathcal{P}, \mathcal{Q} \in \mathcal{G} \wedge \bar{\beta} \in \mathrm{On}^{\mathcal{P}} \wedge \bar{\gamma} \in \mathrm{On}^{\mathcal{Q}} \wedge \pi_{\mathcal{P},\infty}(\bar{\beta}) \leq \pi_{\mathcal{Q},\infty}(\bar{\gamma})). \end{aligned}$$

This is because we can check whether $R(\langle \mathcal{P}, \bar{\beta}\rangle, \langle \mathcal{Q}, \bar{\gamma}\rangle)$ by comparing $\mathcal{P}$ with $\mathcal{Q}$, using their unique $\Sigma_1^{L(\mathbb{R})}(\{\mathcal{P}, \mathcal{Q}\})$ iteration strategies. Since every $\beta < \alpha$ is of the form $\pi_{\mathcal{P},\infty}(\bar{\beta})$ for some $\mathcal{P} \in \mathcal{G}$, there is a $\Delta_1^{L(\mathbb{R})}(\{\mathcal{M}\})$ prewellorder of H_{ω_1} of order type at least α. Thus $\alpha \leq \underset{\sim}{\delta}_1^2$.

Now suppose that $\mathrm{On} \cap M_\infty < \underset{\sim}{\delta}_1^2$. Since M_∞ can be coded simply by a subset of $\mathrm{On} \cap M_\infty$, we have by the Coding Lemma [27, Chap. 7] that for some real z, M_∞ is coded by a $\Delta_1^{L(\mathbb{R})}(\{z\})$ set of reals. But Lemma 8.18 implies that the universal $\Sigma_1^{L(\mathbb{R})}$ set of reals is projective in any set of reals coding M_∞, for we have, for all Σ_1 formulae φ and reals x:

$$L(\mathbb{R}) \models \varphi[x] \quad\iff\quad \exists \mathcal{M} \exists \beta \exists \pi (\mathcal{M} \text{ is a } (\varphi, x) \text{ witness and } \pi : \mathcal{M} \to \mathcal{J}_\beta^{M_\infty}).$$

(The left-to-right direction follows at once from $1 \Longrightarrow 3$ of Lemma 8.18, and the right-to-left direction follows from $2 \Longrightarrow 1$ of Lemma 8.18.) This implies that the universal $\Sigma_1^{L(\mathbb{R})}$ set of reals is $\Delta_1^{L(\mathbb{R})}(\{z\})$, a contradiction. ⊣

8.20 Theorem. $\mathrm{HOD} \cap V_{\underset{\sim}{\delta}_1^2} = M_\infty \cap V_{\underset{\sim}{\delta}_1^2}$.

Proof. We have shown that $\mathcal{F}, \prec^*$, and the function $(\mathcal{M}, \mathcal{N}) \mapsto \pi_{\mathcal{M},\mathcal{N}}$ are definable over $L(\mathbb{R})$. It follows that $M_\infty \in \mathrm{HOD}$. It is enough, then, to show that every bounded subset A of $\underset{\sim}{\delta}_1^2$ which is $\mathrm{OD}^{L(\mathbb{R})}$ is in M_∞. (Note here that $\underset{\sim}{\delta}_1^2$ is strongly inaccessible in HOD, by work of Harvey Friedman and Moschovakis.) So fix such an A. By the reflection theorem, we can fix a Σ_1 formula $\varphi(v_0, v_1)$ and an ordinal $\beta < \underset{\sim}{\delta}_1^2$ such that $A \subseteq \beta$, and for all $\alpha < \beta$

$$\alpha \in A \quad\iff\quad L(\mathbb{R}) \models \varphi[\alpha, \beta].$$

Since $M_\infty = \mathcal{J}_{\underset{\sim}{\delta}_1^2}^{M_\infty^+}$, and $\underset{\sim}{\delta}_1^2$ is a cardinal of M_∞^+ by Corollary 8.16, it will be enough to show that $A \in M_\infty^+$. Let λ be the sup of the Woodin cardinals of M_∞^+. By asking what is true in its own symmetric collapse below λ, M_∞^+ will be able to answer membership questions about A. More precisely, let $\bar{\varphi}(u)$ be the Σ_1 formula:

$$\text{“}u \in \mathbb{R} \text{ codes } (\mathcal{N}, \gamma, \delta) \text{ where } \mathcal{N} \in \mathcal{F} \text{ and } \varphi(\pi_{\mathcal{N},\infty}(\gamma), \pi_{\mathcal{N},\infty}(\delta))\text{”}.$$

Let η be a successor cardinal of M_∞ above β, and for each $\alpha < \beta$ let τ_α be a term for a real in the symmetric collapse below λ over M_∞^+ such that for all generic objects H for this collapse

$$\tau_\alpha^H \text{ codes } (\mathcal{J}_\eta^{M_\infty}, \alpha, \beta).$$

The map $\alpha \mapsto \tau_\alpha$, if chosen naturally, is definable over M_∞^+ from η and β. We claim that for all $\alpha < \beta$,

$$\alpha \in A \iff M_\infty^+ \models (1 \Vdash \bar{\varphi}(\tau_\alpha)^{L(\mathbb{R}^*)}).$$

It clearly suffices to prove this claim.

Fix $\alpha < \beta$. By Lemma 8.14, we can find $\mathcal{Q} \in \mathcal{F}^+$ and ordinals $\bar{\eta}$, $\bar{\beta}$, and $\bar{\alpha}$ in $\mathcal{Q}$ such that

$$\pi^+_{\mathcal{Q},\infty}(\langle \bar{\eta}, \bar{\beta}, \bar{\alpha}\rangle) = \langle \eta, \beta, \alpha\rangle.$$

Let $\bar{\tau}_{\bar{\alpha}}$ be definable over $\mathcal{Q}$ from $\bar{\eta}$, $\bar{\beta}$, and $\bar{\alpha}$ the way τ_α was from η, β, and α over M_∞^+, so that for any H generic over $\mathcal{Q}$ for the symmetric collapse below the sup $\bar{\lambda}$ of its Woodin cardinals, $\bar{\tau}^H_{\bar{\alpha}}$ is a real coding $(\mathcal{J}^{\mathcal{Q}}_{\bar{\eta}}, \bar{\alpha}, \bar{\beta})$. We have

$$\begin{aligned}
\alpha \in A \quad &\iff \quad L(\mathbb{R}) \models \varphi[\alpha, \beta]\\
&\iff \quad \forall H (H \text{ is } \mathrm{Col}(\omega, <\bar{\lambda}), \mathcal{Q}\text{-generic } \Rightarrow L(\mathbb{R}^*_H) \models \bar{\varphi}(\bar{\tau}^H_{\bar{\alpha}}))\\
&\iff \quad \mathcal{Q} \models (1 \Vdash \bar{\varphi}(\bar{\tau}_{\bar{\alpha}})^{L(\mathbb{R}^*)})\\
&\iff \quad M_\infty^+ \models (1 \Vdash \bar{\varphi}(\tau_\alpha)^{L(\mathbb{R}^*)}).
\end{aligned}$$

The second equivalence above follows from the correctness of $L(\mathbb{R}^*_H)$ and the fact that $\pi_{\mathcal{M},\infty}(\langle \bar{\alpha}, \bar{\beta}\rangle) = \langle \alpha, \beta\rangle$, for $\mathcal{M} = \mathcal{J}^{\mathcal{Q}}_{\bar{\eta}}$; this is true because the π and π^+ maps agree.

The displayed equivalences contain our claim. This completes the proof. $\dashv$

A different proof of Theorem 8.20 is sketched in [41]. One shows that in $L[M_\infty]$ there is a tree T on $\omega \times \underset{\sim}{\delta}^2_1$ projecting to the universal $\Sigma_1^{L(\mathbb{R})}$ set of reals, and that this tree is enough like the tree of a $\Sigma_1^{L(\mathbb{R})}$ scale that, by arguments of Martin, Becker, and Kechris [4], $\mathrm{HOD} \cap V_{\underset{\sim}{\delta}^2_1} \subseteq L[T]$. The tree T attempts to verify $\varphi(x)$ by building a (φ, x)-witness and embedding it into M_∞. In this version of the proof, the Dodd-Jensen Lemma corresponds nicely to the lower semi-continuity of a certain semi-scale.

Assuming sufficient determinacy, and given a pointclass Γ which resembles Π^1_1 in a certain technical sense, Moschovakis has defined a submodel of HOD corresponding to Γ-definability which he calls H_Γ. See [27, 8G]. Becker and Kechris show in [4] that $H_\Gamma = L[T]$, whenever T is the tree of a Γ-scale on a universal Γ set. The argument of the last paragraph actually shows that $L[M_\infty] = H_\Gamma$, where $\Gamma = \Sigma_1^{L(\mathbb{R})}$. The argument generalizes to many other Γ, with M_∞ replaced by a direct limit of mice whose iteration strategies and degree of correctness match Γ appropriately. This gives

8.21 Theorem. *Assume $\mathsf{AD}^{L(\mathbb{R})}$, and let Γ be either Π^1_n for n odd, or the pointclass $\Sigma_1^{L(\mathbb{R})}$; then H_Γ is an extender model.*

The theorem probably holds for all Γ resembling Π^1_1, but this has not been fully proved.

One immediate consequence of Theorem 8.20 is

8.22 Corollary. $\mathrm{HOD} \models \mathsf{GCH}$.

Proof. By Theorem 8.20, GCH holds in HOD at all $\alpha < \underset{\sim}{\delta}^2_1$. But Woodin [13] has shown that $\underset{\sim}{\delta}^2_1$ is $<\Theta$-strong in HOD, and thus GCH holds in HOD at all $\alpha < \Theta$. Since $\mathrm{HOD} = L(P)$ for some $P \subseteq \Theta$,[64] GCH holds in HOD at all α. $\dashv$

We emphasize that $\mathrm{HOD} = \mathrm{HOD}^{L(\mathbb{R})}$ in the statement of Corollary 8.22, and that $\mathsf{AD}^{L(\mathbb{R})}$ is a tacit hypothesis there.[65] Whether $\mathsf{AD}^{L(\mathbb{R})}$ implies that GCH holds in HOD was open for some time, and various partial results were obtained using the methods of "neo-classical" descriptive set theory, such as games and scales.[66] Our proof of Theorem 8.20 is evidence of what inner model theory can contribute to this mix. One gets not just GCH, of course, but the other consequences of fine structure theory, such as $\diamondsuit$ and $\square$.

It is natural to ask whether the full $\mathrm{HOD}^{L(\mathbb{R})}$ is a core model. Building on the proof of Theorem 8.20, Woodin has shown that this is essentially, but not literally, the case. We shall state Woodin's results, although it is beyond the scope of this chapter to prove them. The first is

8.23 Theorem (Woodin). *$M^+_\infty \subseteq \mathrm{HOD}$; moreover, the least Woodin cardinal of M^+_∞ is Θ, and $V_\Theta \cap \mathrm{HOD} = V_\Theta \cap M^+_\infty$.*

Since the full HOD is of the form $L(P)$ for some $P \subseteq \Theta$, M^+_∞ is not far from the full HOD. What is missing can be represented in inner-model-theoretic terms. Let X be the class of all δ_0-bounded iteration trees on M^+_∞ which belong to M^+_∞ and are satisfied to have cardinality strictly less than the sup of the Woodin cardinals in M^+_∞. There is a unique iteration strategy for M^+_∞; let us call it Σ.[67] Let $\Sigma^* := \{(\mathcal{T}, \alpha) \mid \mathcal{T} \in X$ and $\mathcal{T}$ is according to Σ and $\mathrm{lh}(\mathcal{T})$ is a limit ordinal, and $\alpha \in \Sigma(\mathcal{T})\}$. We then have

8.24 Theorem (Woodin). $\mathrm{HOD} = M^+_\infty[\Sigma^*]$.

Woodin has obtained results on HOD^M for M a model of AD larger than $L(\mathbb{R})$; for example, the Mouse Set Conjecture implies that $\mathrm{HOD}^M {\restriction} \Theta^M_0$ is an extender model. (Here Θ_0 is the supremum of the lengths of prewellorders of $\mathbb{R}$ which are ordinal definable from a real. If $V = L(\mathbb{R})$, then $\Theta_0 = \Theta$.) Woodin has also obtained an analysis of the full HOD^M analogous to that in

[64] This is another result of Woodin; P is a version of the Vopenka algebra which can add $\mathbb{R}$ to HOD.

[65] The proof we have given used a bit more, namely, that $M^\#_\omega$ exists and is $(\omega, \omega_1 + 1)$-iterable in $V^{\mathrm{Col}(\omega,\mathbb{R})}$. The proof can be made to work under the weaker hypothesis $\mathsf{AD}^{L(\mathbb{R})}$, however. The key is to prove the existence of mouse-witnesses, as stated in Lemma 8.18, assuming only $\mathsf{AD}^{L(\mathbb{R})}$. This is a result of Woodin. The method behind the original proof is described in [13]; there is another proof using the core model induction method.

[66] For example, Becker [3] showed that GCH holds in HOD at all $\alpha < \omega^V_1$.

[67] Granted ω Woodins plus a measurable above in V, Σ_0 prolongs uniquely to trees in $V^{\mathrm{Col}(\omega,\mathbb{R})}$.

Theorem 8.24. See [13, §8] for something on these results, on local forms of Theorem 8.24, and on open questions in the area.

We conclude with some applications of these results on HOD.

8.25 Lemma. *Let $\kappa < \Theta$ and suppose that* HOD $\models \kappa$ *is regular; then exactly one of the following holds:*

1. HOD $\models \kappa$ *is measurable,*

2. $\mathrm{cf}^{L(\mathbb{R})}(\kappa) = \omega$.

Proof. Let $\mathcal{Q} \in \mathcal{F}^+$ and $\bar{\kappa} \in \mathcal{Q}$ be such that $\pi^+_{\mathcal{Q},\infty}(\bar{\kappa}) = \kappa$. Thus $\mathcal{Q} \models$ $\bar{\kappa}$ is regular.

Suppose first that $\bar{\kappa}$ is not measurable in $\mathcal{Q}$. Now since $\pi^+_{\mathcal{Q},\infty}$ is essentially an iteration map, it is continuous at all regular, non-measurable cardinals of $\mathcal{Q}$. (In $V^{\mathrm{Col}(\omega,\mathbb{R})}$ we can find a $\prec^+$-increasing ω sequence starting with $\mathcal{Q}$ and cofinal in $\prec^+$. The map $\pi^+_{\mathcal{Q},\infty}$ is just the iteration map coming from composing iteration trees witnessing the $\prec^+$ relations along this sequence. So $\pi^+_{\mathcal{Q},\infty}$ is an iteration map in $V^{\mathrm{Col}(\omega,\mathbb{R})}$, which is good enough.) In particular, $\pi^+_{\mathcal{Q},\infty}$"$\bar{\kappa}$ is cofinal in κ. Since $\bar{\kappa}$ is below the least Woodin cardinal of $\mathcal{Q}$ by Theorem 8.23, and hence countable, $\mathrm{cf}^V(\kappa) = \omega$. But clearly, V and $L(\mathbb{R})$ have the same ω-sequences of ordinals $< \mu$, whenever $\mu < \Theta$. Thus $\mathrm{cf}^{L(\mathbb{R})}(\kappa) = \omega$. Note also that we have in this case that κ is not measurable in HOD.

Suppose next that $\bar{\kappa}$ is measurable in $\mathcal{Q}$. It is clear then that κ is measurable in HOD, and we need only show that $\mathrm{cf}^V(\kappa) > \omega$. Let X be a countable subset of κ. By the countable directedness of $\prec^+$, we can find an $\mathcal{R} \in \mathcal{F}^+$ such that $\mathcal{Q} \prec^+ \mathcal{R}$ and $X \subseteq \mathrm{dom}(\pi^+_{\mathcal{R},\infty})$. Let $\check{\kappa} = \pi^+_{\mathcal{Q},\mathcal{R}}(\bar{\kappa})$, and let $\mathcal{S}$ be the ultrapower of $\mathcal{R}$ by some normal measure on $\check{\kappa}$. Then $\mathcal{R} \prec^+ \mathcal{S}$, and it is easy to see that $X \subseteq \pi^+_{\mathcal{S},\infty}(\check{\kappa}) < \kappa$, so that X is bounded in κ, as desired. ⊣

We remark that the restriction of Lemma 8.25 to ordinals $\kappa < \underset{\sim}{\delta}^2_1$ requires only Theorem 8.20, rather than the full Theorem 8.23.

It follows from Lemma 8.25 that all successor cardinals of HOD below Θ have cofinality ω in $L(\mathbb{R})$, or equivalently, V. This is also true if we replace HOD by HOD_x, the sets hereditarily ordinal definable over $L(\mathbb{R})$ from x, for x a real. This is because our results relativize routinely to arbitrary reals x; we simply extend the notion of mouse by requiring that x be put in $\mathcal{J}^{\vec{E}}_0(x)$. The relativization of our dichotomy Lemma 8.25 gives the following result, known as the "boldface GCH" for $L(\mathbb{R})$.

8.26 Theorem. *Assume* AD *and* $V = L(\mathbb{R})$*; then for any* $\kappa < \Theta$*, every wellordered family of subsets of* κ *has cardinality at most* κ.

Proof. If not, we have some $A \subseteq \kappa^+$ which codes up a sequence of κ^+ distinct subsets of κ. Since $V = L(\mathbb{R})$, we can find a real x such that $A \in \mathrm{HOD}_x$. We have just observed that $(\kappa^+)^{\mathrm{HOD}_x} < \kappa^+$, by the relativization of our dichotomy Lemma 8.25 to x. But then A witnesses that GCH fails in HOD_x, contrary to the relativized version of Corollary 8.22. ⊣

Although we have quoted Corollary 8.22 in our proof of Theorem 8.26, we really only need Theorem 8.20. This is because "the boldface GCH fails at κ" is a $\Sigma_1^{L(\mathbb{R})}$ assertion about κ. Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ is a Σ_1 elementary substructure of $L(\mathbb{R})$, if the boldface GCH fails at some κ, it fails at some $\kappa < \underset{\sim}{\delta}^2_1$. But we can use Theorem 8.20 in the proof of Theorem 8.26 to see that this is not the case.

Finally, if $\kappa < \Theta$ is regular in $L(\mathbb{R})$, then by our dichotomy result, κ is measurable in HOD, and in fact, κ is measurable in HOD_x for all reals x. We can put the order zero measures on κ from the various HOD_x together, and we obtain:

8.27 Theorem. *Assume* AD *and* $V = L(\mathbb{R})$*; then for any regular* $\kappa < \Theta$*, the* ω*-closed unbounded filter on* κ *is a* κ*-complete, normal ultrafilter on* κ*. Thus all regular cardinals below* Θ *are measurable.*

Proof. For any real x, let μ_x be the order zero measure on κ of HOD_x, that is, the unique measure giving the set of measurable cardinals measure zero. There is such a measure by Lemma 8.25; it is unique because HOD_x is a core model. It will be enough to show that there is an ω-closed, unbounded set C which generates μ_x, in the sense that for all $A \subseteq \kappa$ such that $A \in \mathrm{HOD}_x$,

$$A \in \mu_x \quad \Longrightarrow \quad \exists \alpha < \kappa (C \setminus \alpha \subseteq A).$$

For this implies that the union over x of the μ_x is just the ω-closed unbounded filter on κ. Since every $A \subseteq \kappa$ is in some HOD_x, this union is an ultrafilter. Since every $f : \kappa \to \kappa$ is in some HOD_x, that ultrafilter is normal, and hence κ-complete.

We now construct the desired generating set for μ_x. Let us assume $x = 0$, so that we can use our earlier notation for the direct limit system giving $\mathrm{HOD}_x = \mathrm{HOD}$; the general case is only notationally different. Fix $\mathcal{Q} \in \mathcal{F}^+$ such that $\kappa \in \mathrm{ran}(\pi^+_{\mathcal{Q},\infty})$. Let

$$C := \{\alpha \mid \mathrm{cf}(\alpha) = \omega \text{ and } \mathrm{Hull}^{M^+_\infty}(\alpha \cup \mathrm{ran}(\pi^+_{\mathcal{Q},\infty})) \cap \kappa \subseteq \alpha\},$$

where the hull in question is the "uncollapsed" set of all points definable over M^+_∞ from parameters in $\mathrm{ran}(\pi^+_{\mathcal{Q},\infty})$ and ordinals $< \alpha$. Clearly, C is ω-closed and unbounded in κ. To see that C works, fix $A \in \mu_x = \mu_0$.

For any $\mathcal{S}$ such that $\mathcal{Q} \prec^+ \mathcal{S}$, let

$$\kappa(\mathcal{S}) := \text{ unique } \nu \in \mathcal{S} \text{ such that } \pi^+_{\mathcal{S},\infty}(\nu) = \kappa.$$

Fix $\mathcal{R}$ such that $\mathcal{Q} \prec^+ \mathcal{R}$ and $A \in \mathrm{ran}(\pi^+_{\mathcal{R},\infty})$, and for $\mathcal{S}$ such that $\mathcal{R} \prec^+ \mathcal{S}$ put

$$A(\mathcal{S}) := \text{ unique } B \in \mathcal{S} \text{ such that } \pi^+_{\mathcal{S},\infty}(B) = A.$$

We shall show that

$$C \setminus \sup(\mathrm{ran}(\pi^+_{\mathcal{R},\infty}) \cap \kappa) \subseteq A,$$

which will then finish the proof.

We need the following general fact about iterated ultrapower constructions.

1 Claim. *If $g \in \mathcal{R}$ and $g : [\kappa(\mathcal{R})]^{<\omega} \to \kappa(\mathcal{R})$, then there is a function $f \in \mathcal{Q}$ such that $g = \pi^+_{\mathcal{Q},\mathcal{R}}(f)(b)$ for some finite $b \subseteq \kappa(\mathcal{R})$.*

Proof. Let $\mathcal{T}$ be an iteration tree on $\mathcal{Q}$ with last model $\mathcal{R}$. One can show by an easy induction that if $\mathcal{R}^*$ is on the branch of $\mathcal{T}$ leading to $\mathcal{R}$, then the claim holds with $\mathcal{R}^*$ replacing $\mathcal{R}$. ⊣

Because our mice do not reach superstrong cardinals, we also have

2 Claim. *If $\mathcal{M}$ is a premouse, E is on the $\mathcal{M}$-sequence, $\mathrm{crit}(E) = \kappa$, and $i : \mathcal{M} \to \mathrm{Ult}_0(\mathcal{M}, E)$ is the canonical embedding, then $i(\kappa) = \sup\{i(f)(\kappa) \mid f : \kappa \to \kappa \wedge f \in \mathcal{M}\}$.*

Proof. Let λ be the sup in question. Clearly, $\lambda \leq i(\kappa)$, so suppose that $\lambda < i(\kappa)$ toward contradiction. Let $\nu = \nu(E)$.

Suppose that $\nu \leq \lambda$. Let $a \subseteq \nu$ and g be such that $\lambda = i(g)(a)$. Let h be such that $a \subseteq i(h)(\kappa)$. Now define $f : \kappa \to \kappa$ by

$$f(\alpha) := \sup\{g(u) \mid u \in [h(\alpha)]^{|a|}\}.$$

Then clearly, $\lambda \leq i(f)(\kappa)$, a contradiction. Therefore $\lambda < \nu$.

Arguing as in the last paragraph, we get that $i(g)(a) < \lambda$ for all finite $a \subseteq \lambda$ and $g : [\kappa]^{|a|} \to \kappa$. This means that $\lambda = j(\kappa)$, where $j : \mathcal{M} \to \mathrm{Ult}_0(\mathcal{M}, E{\restriction}\lambda)$ is the canonical embedding. But the initial segment condition on premice implies that the trivial completion E^* of $E{\restriction}\lambda$ is on the sequence of some premouse. Since $i_{E^*}(\kappa) < \mathrm{lh}(E^*)$, we do not allow such "long extenders" in a fine extender sequence, so this is a contradiction. ⊣

Now fix any $\alpha \in C \setminus \sup(\mathrm{ran}(\pi^+_{\mathcal{R},\infty}) \cap \kappa)$. Fix any $\mathcal{B}^* \in \mathcal{F}^+$ such that $\alpha \in \mathrm{ran}(\pi^+_{\mathcal{B}^*,\infty})$, and let $\mathcal{T}$ be the ω-maximal iteration tree on $\mathcal{R}$ which results from the coiteration of $\mathcal{B}^*$ with $\mathcal{R}$, using Σ_0 on both sides, and let $\mathcal{B}$ be the last model of $\mathcal{T}$. Since neither side drops, $\mathcal{B} \in \mathcal{F}^+$ and $\alpha \in \mathrm{ran}(\pi^+_{\mathcal{B},\infty})$; say

$$\alpha = \pi^+_{\mathcal{B},\infty}(\bar{\alpha}).$$

It will be enough to show that $\bar{\alpha} \in A(\mathcal{B})$.

Let us look closely at the tree $\mathcal{T}$ leading from $\mathcal{R}$ to $\mathcal{B}$. We use $\mathcal{M}_\xi, E_\xi$, and $i_{\xi,\gamma}$ for the models, extenders, and embeddings of $\mathcal{T}$. Let $\mathcal{B} = \mathcal{M}_\eta$. Now $i_{0,\eta}(\kappa(\mathcal{R})) = \kappa(\mathcal{B}) > \bar{\alpha}$, so we can set

$$\xi := \text{ least } \nu \in [0,\eta]_T \text{ such that } i_{0,\nu}(\kappa(\mathcal{R})) > \bar{\alpha}.$$

Note here that $\kappa(\mathcal{R}) \le \bar{\alpha}$, so that $\xi > 0$; this is because if $\gamma < \kappa(\mathcal{R})$, then $\pi^+_{\mathcal{R},\infty}(\gamma) < \alpha$, so $\pi^+_{\mathcal{R},\mathcal{B}}(\gamma) < \bar{\alpha}$, so $\gamma < \bar{\alpha}$.

Let $(\nu+1)\,T\,\xi$; we claim that $\mathrm{lh}(E_\nu) < \bar{\alpha}$. For letting $\beta = \mathrm{pred}_T(\nu+1)$, we have $\mathrm{crit}(E_\nu) = \mathrm{crit}(i_{\beta,\xi})$ because $\mathcal{T}$ is ω-maximal, and $\mathrm{crit}(i_{\beta,\xi}) \le \kappa(\mathcal{M}_\beta)$ by the minimality of ξ. But then $\mathrm{lh}(E_\nu) < i_{0,\nu+1}(\kappa(\mathcal{R})) \le \bar{\alpha}$ by the minimality of ξ.

It follows that ξ is a successor ordinal, for otherwise, since $\bar{\alpha} < i_{0,\xi}(\kappa(\mathcal{R}))$, we would get that $\bar{\alpha} = i_{0,\xi}(g)(a)$ for some $a \subseteq \mathrm{crit}(i_{\xi,\eta}) \cap \bar{\alpha}$ finite and $g : [\kappa(\mathcal{R})]^{|a|} \to \kappa(\mathcal{R})$. (We get $a \subseteq \mathrm{crit}(i_{\xi,\eta})$ because $\mathcal{T}$ is ω-maximal, and $a \subseteq \bar{\alpha}$ from the preceding paragraph and the assumption that ξ is a limit ordinal.) But by our first claim, we have $g = \pi^+_{\mathcal{Q},\mathcal{R}}(f)(b)$ for some $f \in \mathcal{Q}$ and $b \subseteq \kappa(\mathcal{R})$. We then have that

$$i_{\xi,\eta}(\bar{\alpha}) = i_{\xi,\eta}(i_{0,\xi}(g)(a)) = i_{0,\eta}(g)(a) = \pi^+_{\mathcal{Q},\mathcal{B}}(f)(\pi^+_{\mathcal{R},\mathcal{B}}(b))(a).$$

Since $\bar{\alpha} \le i_{\xi,\eta}(\bar{\alpha})$, we can apply $\pi^+_{\mathcal{B},\infty}$ to the identity above and obtain

$$\alpha \le \pi^+_{\mathcal{Q},\infty}(f)(\pi^+_{\mathcal{R},\infty}(b))(\pi^+_{\mathcal{B},\infty}(a)).$$

Now $\pi^+_{\mathcal{R},\infty}(b) \subseteq \alpha$ because we chose α as large as we did, and $\pi^+_{\mathcal{B},\infty}(a) \subseteq \alpha$ because $a \subseteq \bar{\alpha}$. Thus the ordinal named on the right side of the line just displayed witnesses that $\alpha \notin C$. This is a contradiction, and hence ξ is a successor ordinal.

Let $\xi = \gamma + 1$, $E = E_\gamma$, and $\beta = \mathrm{pred}_T(\xi)$. If $\nu(E) \le \bar{\alpha}$, then we get the same contradiction we got in the last paragraph, so we have $\nu(E) > \bar{\alpha}$. By the minimality of ξ, $\mathrm{crit}(E) \le \kappa(\mathcal{M}_\beta)$. We claim that $\mathrm{crit}(E) = \bar{\alpha}$. This is true because otherwise Claim 2 gives some $h : \kappa(\mathcal{M}_\beta) \to \kappa(\mathcal{M}_\beta)$ such that $\bar{\alpha} < i_{\beta,\xi}(h)(c)$, where $c = \{\mathrm{crit}(E)\} \subseteq \bar{\alpha}$. One can then proceed to a contradiction as in the last paragraph: represent h as $i_{0,\beta}(g)(d)$ where $d \subseteq \mathrm{crit}(E)$, so that $\bar{\alpha} = i_{0,\xi}(g)(a)$, where $a := c \cup d \subseteq \mathrm{crit}(i_{\xi,\eta}) \cap \bar{\alpha}$. Then let f, b be such that $\pi^+_{\mathcal{Q},\mathcal{R}}(f) = g$ and $b \subseteq \kappa(\mathcal{R})$, etc.

Since $\kappa(\mathcal{M}_\beta) \le \bar{\alpha}$ by the minimality of ξ, we have $\kappa(\mathcal{M}_\beta) = \mathrm{crit}(E) = \bar{\alpha}$. Now $\bar{\alpha}$ cannot be measurable in $\mathcal{M}_\xi = \mathrm{Ult}(\mathcal{M}_\beta, E)$, since then $\alpha = \pi^+_{\mathcal{B},\infty}(\bar{\alpha}) = \pi^+_{\mathcal{B},\infty}(i_{\xi,\eta}(\bar{\alpha}))$ is measurable in HOD. Since $\mathrm{cf}(\alpha) = \omega$, our dichotomy Lemma 8.25 rules this out. It follows that E is the order zero measure on $\kappa(\mathcal{M}_\gamma)$, and since using the order zero measure cannot move generators, that $\beta = \gamma$. We have then that $A(\mathcal{M}_\beta) \in E_a$, for $a = \{\kappa(\mathcal{M}_\beta)\}$, so $\bar{\alpha} = \kappa(\mathcal{M}_\beta) \in A(\mathcal{M}_\xi)$, so $\bar{\alpha} \in A(\mathcal{B})$, so $\alpha \in A$, as desired. ⊣

We remark that, once again, the negation of Theorem 8.27 is a Σ_1 statement about $L(\mathbb{R})$ by the Coding Lemma, so that if Theorem 8.27 fails, it fails below $\underset{\sim}{\delta}^2_1$. Therefore, we really needed only Theorem 8.20 for its proof. It is also worth noting that Theorems 8.26 and 8.27 make no mention of mice, or even HOD, in their statements.

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20. A Core Model Toolbox and Guide

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Contents

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1. Introduction

The subject of this chapter is core model theory at a level where it involves iteration trees. Our toolbox includes a list of fundamental theorems that set theorists can use off the shelf in applications; see Sect. 3. It also contains a catalog of applications of this sort of core model theory; see Sect. 5. The odd sections have no proofs and are basically independent of the even sections. For those interested in the nuts and bolts of core model theory, we offer a guide to the monograph *The Core Model Iterability Problem* [42] by John Steel in Sect. 2. We also provide an outline of the paper *The covering lemma up to a Woodin cardinal* by William Mitchell, Steel and the author [21] in Sect. 4.

What developed into the theory of core models began in earnest with theorems of Ronald Jensen L under the hypothesis that $0^{\#}$ does not exist. Jensen showed that if $0^{\#}$ does not exist, then L is the canonical core model, which is written $K = L$. He also showed that if $0^{\#}$ exists but $0^{\#\#}$ does not exist, then $K = L[0^{\#}]$ is the canonical core model. In general, K^V is the canonical core model (if there is one) whereas W is a core model if $W = K^M$ where M is a transitive class model of ZFC. Unfortunately, we must ask the reader to pay close attention to articles in the sense of grammar.

Whether or not it is possible to give a definition of K that allows us to make sense of K^M for all M is unknown. Up until recently, for those M for which K^M has been defined, K^M has turned out to be an extender model. Backing up slightly, recall that the existence of $0^{\#}$ is equivalent to the existence of an ordinal κ and an ultrafilter F over $\wp(\kappa) \cap L$ that gives rise to a non-trivial elementary embedding from L to itself. *Large cardinal axioms* such as the existence of $0^{\#}$ can all be phrased in terms of the existence of filters or systems of filters. Some of these systems are known as *extenders.* A *model* is a transitive set or proper class transitive model of ZFC. *Extender models* are models of the form J^E_Ω where $\Omega \le \mathrm{On}$, E is a sequence of length Ω and E_α is an extender for each $\alpha < \Omega$.[1]

Statements asserting that certain models with large cardinals do not exist are called *anti-large cardinal hypotheses.* Instead of making this precise, we list the four examples most relevant to our introduction.

- $0^{\#}$ does not exist.
- There is no proper class model with a measurable cardinal.
- There is no proper class model with a measurable cardinal κ with Mitchell order $o(\kappa) = \kappa^{++}$.
- There is no proper class model with a Woodin cardinal.

[1] Is every core model an extender model? Since we do not know how to define K in the abstract, it is impossible to answer this question. There are models that are not extender models that most likely will be accepted as core models but these are beyond the scope of this introduction.

For the last three examples, it would be equivalent to replace “model” by “extender model” although this is not obvious.

Much of core model theory deals with generalizations of Jensen’s theorems about L. The core model theorist adopts an anti-large cardinal hypothesis, possibly for the sake of obtaining a contradiction. Then he defines K and shows that K has many of the same useful properties that L has if $0^{\#}$ does not exist. With some exceptions, these properties fall into the following categories.

- Fine structure with the consequence, for example, that GCH and combinatorial principles such as $\diamondsuit$ and $\Box$ hold in K.

- Universality with the consequence, for example, that the existence of certain extender models is absolute to K.

- Maximality with the consequence, for example, that certain large cardinal properties of κ are downward absolute to K.

- Definability in a way that makes K absolute to set forcing extensions.

- Covering with the consequence, for example, that K computes successors of singular cardinals correctly.

Often, such properties of K are used in elaborate proofs by contradiction. In order to prove that a principle P implies the existence of a model with large cardinal C, one may assume that there is no model of C and use P to show that one of the basic properties of K fails. When this accomplished, it follows that the large cardinal consistency strength of P is at least C.

Dodd and Jensen developed the theory of K under the anti-large cardinal hypothesis that there is no proper class model with a measurable cardinal. Mitchell did this under the hypothesis that there is no proper class model with a measurable cardinal κ of order κ^{++}. Steel did this under the hypothesis that there is no proper class model with a Woodin cardinal except that he added a technical hypothesis, which we discuss momentarily.

It is important to emphasize that we do not know how to define K without an anti-large cardinal hypothesis. We do not refer to the Dodd-Jensen, Mitchell or Steel core model without the corresponding anti-large cardinal hypothesis. It is also important to know that the various definitions of K are consistent with each other. For example, if there is no transitive class model with a measurable cardinal, then the Dodd-Jensen, Mitchell and Steel definitions of K coincide. Quite reasonably, if $0^{\#}$ does not exist, then $K = L$ under all three definitions.

For all but the last section of this paper we assume:

Anti-Large Cardinal Hypothesis. *There is no proper class model with a Woodin cardinal.*

From what we said about the core model theories predating Steel's, the reader might expect that we could go straight into a discussion of K. But it is not known if the theory of K can be developed under this anti-large cardinal hypothesis alone. Following Steel, we add:

Technical Hypothesis. *Ω is a measurable cardinal and U is a normal measure over Ω.*

This means that U is a non-principal Ω-complete normal ultrafilter on $\wp(\Omega)$. Except in the last section, we also assume this Technical Hypothesis throughout this paper. Of course, by adding the Technical Hypothesis to ZFC we obtain a stronger theory. But, in this setting, it is not much stronger as measurable cardinals are much weaker than Woodin cardinals.[2]

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2. Basic Theory of K

2.1. Second-Order Definition of K

All of the results and proofs in Sects. 2.1 and 2.2 are due to Steel and come from [42]. But we only assume that the reader is familiar with [41] through the theory of countably certified construction.[3] Recall from [41, §6] that a countably certified construction is a sequence of premice $\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle$ where either

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha))$$

or

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha)^\frown \langle F \rangle),$$

where the second option (adding an extender) is permitted if

$$\mathfrak{C}(\mathcal{N}_\alpha)^\frown \langle F \rangle$$

is a countably certified mouse. When β is a limit ordinal, we define

$$\mathcal{N}_\beta = \liminf \langle \mathcal{N}_\alpha \mid \alpha < \beta \rangle.$$

The gist of [41, §6] as it applies to us is that the following statements hold for all $\gamma \leq \Omega$.

---

[2] In the Fall of 2007, Jensen and Steel found a way around the Technical Hypothesis. Their new idea does not supersede the core model theory described in this chapter; rather, it is an additional layer on top of what we are about to present.

[3] Countably certified constructions are called *K^c-constructions* in [41]. There, K^c-constructions are studied in generality before a particular maximal K^c-construction

$$\langle \mathcal{N}_\alpha \mid \alpha \leq \Omega \rangle$$

is fixed, at which point K^c is defined to be $\mathcal{N}_\Omega$. The terminology here is slightly different in this respect, and so is the definition of K^c.

1. $\mathcal{N}_\gamma$ is a 1-small premouse. In other words, no initial segment of $\mathcal{N}_\gamma$ has the first-order properties of a sharp for an inner model with one Woodin cardinal.

2. If $\mathcal{P}$ is a countable premouse that embeds into $\mathcal{N}_\gamma$, then $\mathcal{P}$ is ω_1+1 iterable.

3. Let $\alpha<\gamma$. Suppose that $\kappa \le \rho_\omega^{\mathcal{N}_\beta}$ for all β such that $\alpha<\beta<\gamma$. Then $\mathcal{N}_\alpha$ and $\mathcal{N}_\gamma$ agree below $(\kappa^+)^{\mathcal{N}_\alpha}$.

The proof of clause 1 uses our Anti-Large Cardinal Hypothesis. Countable certificates are used in the proof of clause 2. Clause 3 implies that $\mathcal{N}_\Omega$ has height Ω. Another important fact that we revisit in this paper is Theorem 6.19 of [41], which implies that if $\langle \mathcal{N}_\alpha \mid \alpha \le \Omega\rangle$ is a maximal countably certified construction, then $\mathcal{N}_\Omega$ computes κ^+ correctly for U almost all $\kappa<\Omega$. In this context, *maximal* means that at all successor stages of the construction, if it is possible to add an extender, then we do.

To define K^c, we consider a kind of countably certified construction that is not maximal but still computes the successors of U almost all cardinals correctly. The new condition is that we add an extender to form

$$\mathcal{N}_{\alpha+1} = \mathrm{rud}(\mathfrak{C}(\mathcal{N}_\alpha)^\frown\langle F\rangle)$$

whenever it is permitted so long as

$$\mathrm{crit}(F) \text{ is an inaccessible cardinal}$$

and, if

$$(\mathrm{crit}(F)^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \mathrm{crit}(F)^+$$

then

$$\{\kappa < \mathrm{crit}(F) \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \kappa^+\}$$

is stationary in $\mathrm{crit}(F)$. For the rest of this section, fix such a countably certified construction and let $K^c = \mathcal{N}_\Omega$.

2.1 Definition. A *weasel* is an $(\omega, \Omega+1)$ iterable premouse of height Ω.

This is slightly different from the notation in [42] where weasels are not required to be iterable at all.[4] The meaning of $(\omega,\Omega+1)$ *iterable* is given by Definition 4.4 of [41]. It says that there is a strategy for picking cofinal branches at limit stages that avoids illfounded models at all stages when building almost normal iteration trees. These are iteration trees obtained as follows.

- Build a normal iteration tree $\mathcal{T}_0$ of length $\le \Omega+1$.

[4] Following the convention on premice versus mice, a structure with the first-order properties of a weasel should have been called a *preweasel*.

- If $\mathcal{T}_n$ has successor length $\theta_n + 1 < \Omega + 1$, then build a normal iteration tree $\mathcal{T}_n$ on an initial segment of $\mathcal{M}^{\mathcal{T}_n}_{\theta_n}$.
- If $\mathcal{T}_n$ is defined for all $n < \omega$, then form the concatenation

$$\mathcal{T}_0{}^\frown\mathcal{T}_1{}^\frown\cdots{}^\frown\mathcal{T}_n{}^\frown\cdots.$$

In particular, the unique cofinal branch of the infinite concatenation may have only finitely many drops and its corresponding direct limit must be wellfounded. The original *raison d'être* for almost normal iteration trees is the Dodd-Jensen Lemma, Theorem 4.8 of [41]. The reader must forgive us for not saying whether we mean *normal* or *almost normal* when we write *iteration tree* in this basic account except at key places when the difference is most pronounced.

By the next theorem, the only way to iterate a weasel is to pick the unique cofinal wellfounded branch through an iteration tree of limit length $< \Omega$.

2.2 Theorem. *Let $\mathcal{P}$ be a premouse with no Woodin cardinals. Suppose that $\mathcal{T}$ is an iteration tree of limit length on $\mathcal{P}$. Assume that*

$$\delta(\mathcal{T}) < \mathrm{On} \cap \mathcal{P}.$$

Then $\mathcal{T}$ has at most one cofinal wellfounded branch.

Sketch. Let $\theta = \mathrm{lh}(\mathcal{T})$. Recall from Definition 6.9 of [41] that

$$\delta(\mathcal{T}) = \sup(\{\mathrm{lh}(E^{\mathcal{T}}_\eta) \mid \eta < \theta\})$$

and $\mathcal{M}(\mathcal{T})$ is the unique passive mouse of height $\delta(\mathcal{T})$ that agrees with $\mathcal{M}^{\mathcal{T}}_\eta$ below $\mathrm{lh}(E^{\mathcal{T}}_\eta)$ for all $\eta < \theta$. Our Anti-Large Cardinal Hypothesis implies that $\delta(\mathcal{T})$ is not a Woodin cardinal in $L[\mathcal{M}(\mathcal{T})]$. Let $\mathcal{Q}(\mathcal{M}(\mathcal{T}))$ be the premouse $\mathcal{R}$ of minimum height such that

$$\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{R} \triangleleft L[\mathcal{M}(\mathcal{T})]$$

and $\delta(\mathcal{T})$ is not a Woodin cardinal in $\mathrm{rud}(\mathcal{R})$. By Theorem 6.10 of [41], there is at most one cofinal branch b of $\mathcal{T}$ with the property that

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b).^{5}$$

Our assumptions about $\mathcal{P}$ and $\mathcal{T}$ imply that if b is a cofinal wellfounded branch of $\mathcal{T}$, then

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \trianglelefteq \mathcal{M}^{\mathcal{T}}_b.$$

⊣

[5] We define $\mathcal{M}^{\mathcal{T}}_b$ to be the Mostowski collapse of the direct limit of $\mathcal{M}^{\mathcal{T}}_\eta$ for $\eta \in b$ even if this direct limit is illfounded. By $\mathrm{wfp}(\mathcal{M}^{\mathcal{T}}_b)$ we mean the wellfounded part of $\mathcal{M}^{\mathcal{T}}_b$. In this case, the wellfounded part and the transitive part are the same because $\mathcal{M}^{\mathcal{T}}_b$ is its own Mostowski collapse.

2.3 Theorem. *K^c is a weasel.*

Sketch. We already know that K^c is a premouse of height Ω. It remains to see that K^c is $(\omega, \Omega+1)$ iterable. Here we show that it is Ω iterable. Our strategy is to pick the unique cofinal wellfounded branch through iteration trees of length $< \Omega$ and to use the fact that Ω is measurable to find a branch through iteration trees of length Ω.

First suppose that $\mathcal{T}$ is an iteration tree on K^c of length $\theta < \Omega$. Recalling that $H(\lambda)$ denotes the collection of sets hereditarily of cardinality $< \lambda$, let

$$\pi : N \to H(\Omega^+)$$

be an elementary embedding with N countable and transitive. Say $\pi(\mathcal{P}) = K^c$ and $\pi(\mathcal{S}) = \mathcal{T}$. By Theorem 6.16 of [41], $\mathcal{P}$ has an $\omega_1 + 1$ iteration strategy. Then $\mathcal{S}$ is consistent with this strategy because there is only one strategy: by Theorem 2.2, $[0, \eta)_S$ is the unique cofinal wellfounded branch of $\mathcal{S} \restriction \eta$ whenever η is a limit ordinal $< \mathrm{lh}(\mathcal{S})$.

Assume that $\theta = \eta + 1$ and F is an extender from the $\mathcal{M}^{\mathcal{T}}_\eta$ sequence such that $\mathrm{lh}(F) > \mathrm{lh}(E^{\mathcal{T}}_\zeta)$ for all $\zeta \leq \eta$. We claim that

$$\mathrm{Ult}(\mathcal{M}^*_{\zeta+1}, F)$$

is wellfounded where $\zeta \leq \eta$ is least so that $\mathrm{crit}(F) < \nu(E^{\mathcal{T}}_\zeta)$, $\mathcal{M}^*_{\zeta+1}$ is the maximal level of $\mathcal{M}^{\mathcal{T}}_\zeta$ that is measured by F, and the degree of the ultrapower is as large as possible. Otherwise, there exists such an F and a witness to illfoundedness in the range of π, so the corresponding extension of $\mathcal{S}$ using $\pi^{-1}(F)$ is also illfounded. This contradicts that $\mathcal{P}$ is $\mathrm{lh}(\mathcal{S}) + 1$ iterable.

Now assume that θ is a limit ordinal $< \Omega$. Let b be the unique cofinal wellfounded branch of $\mathcal{S}$. We know that b is the unique cofinal branch of $\mathcal{S}$ with the property that

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) \trianglelefteq \mathrm{wfp}(\mathcal{M}^{\mathcal{S}}_b).$$

By our Technical Hypothesis,

$$\mathcal{Q}(\mathcal{M}(\mathcal{T})) \triangleleft L_\Omega[\mathcal{M}(\mathcal{T})].$$

Therefore,

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) = \pi^{-1}(\mathcal{Q}(\mathcal{M}(\mathcal{T}))) \in N.$$

Let $\kappa < \pi^{-1}(\Omega)$ be a regular cardinal of N greater than the cardinality of $\mathcal{Q}(\mathcal{M}(\mathcal{S}))$ in N. For example, we may simply take

$$\kappa = (|\delta(\mathcal{S})|^+)^N.$$

Let $\mathcal{S}^*$ be $\mathcal{S}$ construed as an iteration tree on $\mathcal{J}^{\mathcal{P}}_\kappa$ and G be an N-generic filter over $\mathrm{Col}(\omega, \kappa)$.[6] Then $\mathcal{S}^*$ and $\mathcal{Q}(\mathcal{M}(\mathcal{S})) = \mathcal{Q}(\mathcal{M}(\mathcal{S}^*))$ are hereditarily

[6] We recall that $\mathrm{Col}\,(\omega, \kappa)$ is the collapsing poset consisting of the finite partial functions from ω to κ.

countable in $N[G]$. Moreover, in $N[G]$, there is a set Z and a subtree $\mathcal{U}$ of ${}^{<\omega}Z$ whose infinite branches correspond to picking an ordinal $\eta < \pi^{-1}(\theta)$ and a level $\mathcal{Q} \lhd \mathcal{M}_\eta^{\mathcal{S}^*}$, then, in infinitely many steps, picking a cofinal branch c of $\mathcal{S}^*$ and simultaneously defining an isomorphism

$$f : \mathcal{Q}(\mathcal{M}(\mathcal{S})) \simeq i_{\eta,c}^{\mathcal{S}^*}(\mathcal{Q}).$$

By being slightly more precise about the definition of $\mathcal{U}$, we guarantee that $\mathcal{U}$ has a unique branch, namely the one determined by b and the least ordinal $\eta \in b$ such that

$$\mathcal{Q}(\mathcal{M}(\mathcal{S})) \in \operatorname{ran}(i_{\eta,b}).$$

By the absoluteness of wellfoundedness, $b \in N[G]$. Then $b \in N$ by the uniqueness of b and the homogeneity of the poset $\operatorname{Col}(\omega, \kappa)$. The fact that b is a cofinal wellfounded branch of $\mathcal{S}$ is absolute to N. Therefore, $\pi(b)$ is a cofinal wellfounded branch of $\mathcal{T}$.

Finally, suppose that $\mathcal{T}$ is an iteration tree on K^c of length Ω. Let

$$b = [0, \Omega)_{j(T)}$$

where j is the ultrapower map corresponding to U. There is an elementary embedding from $\mathcal{M}_b^{\mathcal{T}}$ to $\mathcal{M}_\Omega^{j(\mathcal{T})}$. Since $\mathcal{M}_\Omega^{j(\mathcal{T})}$ is wellfounded, so is $\mathcal{M}_b^{\mathcal{T}}$. $\dashv$

2.4 Theorem. $\{\kappa < \Omega \mid (\kappa^+)^{K^c} = \kappa^+\} \in U$.

Sketch. Let $V' = \operatorname{Ult}(V, U)$ and $j : V \to V'$ be the ultrapower embedding. Then for all $\mathcal{A} \subseteq \wp(\Omega)$, if $|\mathcal{A}| \leq \Omega$, then $j \restriction \mathcal{A} \in V'$. Assume for contradiction that

$$(\Omega^+)^{j(K^c)} < \Omega^+.$$

Let F be the extender of length $j(\Omega)$ derived from $j \restriction j(K^c)$. Then $F \in V'$ and F is countably certified in V'. Now an elaborate induction similar to the proof of Theorem 6.18 of [41] shows that for all $\nu < j(\Omega)$, either the trivial completion of $F \restriction \nu$ is on the $j(K^c)$ sequence, or something close enough that still implies

$$F \restriction \nu \in j(K^c).$$

We could add F itself to $j(K^c)$ to get a model with a superstrong cardinal but it is enough to note that the initial segments of F witness that Ω is a Shelah cardinal in $j(K^c)$ for a contradiction. $\dashv$

2.5 Definition. Let

$$A_1 = \{\kappa < \Omega \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{K^c} = \kappa^+\}$$

and

$$A_0 = \{\lambda \in A_1 \mid A_1 \cap \lambda \text{ is not stationary in } \lambda\}.$$

2.6 Theorem. *The following hold.*

(1) $A_0 \notin U$.

(2) A_0 *is stationary.*

(3) If $\lambda \in A_0$*, then there are no total-on-*K^c *extenders on the* K^c *sequence with critical point* λ.

Sketch. From Theorem 2.4 it follows that $A_1 \in U$. Suppose for contradiction that $A_0 \in U$. Then $\Omega \in j(A_0)$. So $j(A_0) \cap \Omega$ is not stationary in Ω. But $A_0 = j(A_0) \cap \Omega$. Thus A_0 is not stationary in Ω. Since U is normal, $A_0 \notin U$, which is a contradiction.

Suppose for contradiction that A_0 is not stationary. Then there exists a C club in Ω such that for all $\lambda < \Omega$, if $\lambda \in A_1 \cap C$, then $A_1 \cap \lambda$ is stationary in λ. Let λ be the least element of $A_1 \cap \lim(C)$. Then $C \cap \lambda$ is club in λ and λ is an inaccessible cardinal, so $\lim(C) \cap \lambda$ is club in λ. Since $A_1 \cap \lambda$ is stationary in λ, there exists a $\kappa < \lambda$ such that $\kappa \in A_1 \cap \lim(C)$, which is a contradiction.

Let $\lambda \in A_0$. For all sufficiently large $\alpha < \Omega$,

$$(\lambda^+)^{\mathcal{N}_\alpha} = (\lambda^+)^{K^c}$$

and $\mathcal{N}_\alpha$ and K^c agree below their common λ^+. For such α,

$$(\lambda^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = (\lambda^+)^{\mathcal{N}_\alpha}$$

and $\mathfrak{C}(\mathcal{N}_\alpha)$ and $\mathcal{N}_\alpha$ agree below their common λ^+. Therefore,

$$(\lambda^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \lambda^+$$

and

$$\{\kappa < \lambda \mid \kappa \text{ is an inaccessible cardinal and } (\kappa^+)^{\mathfrak{C}(\mathcal{N}_\alpha)} = \kappa^+\}$$

is not stationary in λ. By the definition of K^c, we cannot add an extender with critical point λ to $\mathfrak{C}(\mathcal{N}_\alpha)$ in forming $\mathcal{N}_{\alpha+1}$. It follows that if $\lambda^+ < \xi < \Omega$, then $\mathrm{crit}(E_\xi^{K^c}) \neq \lambda$. Thus there are no total-on-K^c extenders on the K^c sequence with critical point λ. ⊣

Next we discuss some basic facts about coiteration. Suppose that $(\mathcal{P}, \mathcal{Q})$ is a pair of mice. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$ determined by their respective iteration strategies. Say $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. By Theorem 3.11 of [41], there are two possibly overlapping cases.

1. $\mathcal{P} \leq^* \mathcal{Q}$. That is, $[0, \eta]_S$ does not drop in model or degree and

$$\mathcal{M}_\eta^{\mathcal{S}} \trianglelefteq \mathcal{M}_\theta^{\mathcal{T}}.$$

2. $\mathcal{P} \geq^* \mathcal{Q}$. That is, $[0, \theta]_T$ does not drop in model or degree and

$$\mathcal{M}_\eta^{\mathcal{S}} \trianglerighteq \mathcal{M}_\theta^{\mathcal{T}}.$$

Moreover, by the proof of Theorem 3.11 of [41],

$$\eta, \theta < \max(|\mathcal{P}|, |\mathcal{Q}|)^+.$$

We continue the discussion above but assume instead that $\mathcal{P}$ and $\mathcal{Q}$ both have height $\leq \Omega$ and are $\Omega+1$ iterable. Using the fact that Ω is inaccessible, we can modify the proof of Theorem 3.11 of [41] to show that the coiteration of $(\mathcal{P}, \mathcal{Q})$ is successful. Moreover, with the same notation as above, $\eta, \theta \leq \Omega$ and, if

$$\max(\eta, \theta) = \Omega,$$

then at least one of the following holds.

1. $\mathcal{P} \leq^* \mathcal{Q}$, $\mathcal{P}$ is a weasel and $i^{\mathcal{S}}_{0,\eta}$"$\Omega \subseteq \Omega$.
2. $\mathcal{P} \geq^* \mathcal{Q}$, $\mathcal{Q}$ is a weasel and $i^{\mathcal{T}}_{0,\theta}$"$\Omega \subseteq \Omega$.

We leave it to the reader to fill in these details.

2.7 Definition. A weasel $\mathcal{Q}$ is *universal* iff $\mathcal{P} \leq^* \mathcal{Q}$ for all $\Omega+1$ iterable premice $\mathcal{P}$ of height $\leq \Omega$.

By the next theorem, K^c is a universal weasel.

2.8 Theorem. *If $\mathcal{Q}$ is a weasel and $\{\kappa < \Omega \mid (\kappa^+)^{\mathcal{Q}} = \kappa^+\}$ is stationary, then Q is universal.*

Sketch. Otherwise, there is an $\Omega+1$ iterable mouse $\mathcal{P}$ of height $\leq \Omega$ such that not $\mathcal{P} \leq^* \mathcal{Q}$. Therefore, $\mathcal{P} \geq^* \mathcal{Q}$ and, with notation as in our discussion on coiteration, $\eta = \Omega$. Moreover, for some $\xi \in [0, \Omega]_S$ and $\kappa < \Omega$,

$$i^{\mathcal{S}}_{\xi,\Omega}(\kappa) = \Omega.$$

Then the set

$$\{\lambda \in (\xi, \Omega)_S \mid i^{\mathcal{S}}_{\xi,\lambda}(\kappa) = \lambda\}$$

is club. Let us assume for simplicity that $\theta = \Omega$. Then also

$$\{\lambda \in (0, \Omega)_T \mid i^{\mathcal{T}}_{0,\lambda}\text{“}\lambda \subseteq \lambda\}$$

is club because $i^{\mathcal{T}}_{0,\Omega}$"$\Omega \subseteq \Omega$. Let λ be a regular cardinal in both these clubs with

$$(\lambda^+)^{\mathcal{Q}} = \lambda^+.$$

Then

$$i^{\mathcal{T}}_{0,\lambda}(\lambda) = \sup(i^{\mathcal{T}}_{0,\lambda}\text{“}\lambda) = \lambda$$

so

$$i^{\mathcal{T}}_{0,\lambda}(\lambda^+) = \lambda^+$$

and

$$(\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\lambda} = \lambda^+.$$

On the other hand,

$$(\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\lambda} = i^{\mathcal{S}}_{\xi,\lambda}((\kappa^+)^{\mathcal{M}^{\mathcal{S}}_\xi}) = \sup(i^{\mathcal{S}}_{\xi,\lambda}``(\kappa^+)^{\mathcal{M}^{\mathcal{S}}_\xi}) < \lambda^+.$$

Because $i^{\mathcal{S}}_{\lambda,\Omega}$ has critical point λ and $i^{\mathcal{T}}_{\lambda,\Omega}$ has critical point $\geq \lambda$,

$$(\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\Omega} = (\lambda^+)^{\mathcal{M}^{\mathcal{S}}_\lambda} < (\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\lambda} = (\lambda^+)^{\mathcal{M}^{\mathcal{T}}_\Omega}.$$

This contradicts that $\mathcal{M}^{\mathcal{T}}_\Omega \trianglelefteq \mathcal{M}^{\mathcal{S}}_\Omega$. ⊣

We are leading up to the definitions of the definability and hull properties for weasels. Historically, these derive from familiar properties of mice that have gone unnamed. Before dealing with weasels, we digress to discuss the analogous properties of mice as motivation. The fundamental intuition from fine-structure theory of mice is that cores and ultrapowers are inverse operations. Let us give an illustrative example. Suppose that $\mathcal{Q}$ is a 1-sound mouse, $E = \dot{F}^{\mathcal{Q}}$ and

$$\rho_1^{\mathcal{Q}} \leq \mathrm{crit}(E) = \kappa < \mathrm{On} \cap \mathcal{Q}.$$

Let

$$i : \mathcal{Q} \to \mathcal{R} = \mathrm{Ult}(\mathcal{Q}, E)$$

be the ultrapower map. Then i is a Σ_1-elementary embedding and cofinal in the sense that

$$\mathrm{On} \cap \mathcal{R} = \sup(i``(\mathrm{On} \cap \mathcal{Q})).$$

Moreover, $\rho_1^{\mathcal{R}} = \rho_1^{\mathcal{Q}}$ and $p_1^{\mathcal{R}} = i(p_1^{\mathcal{Q}})$. By the definition of 1-soundness,

$$\mathcal{Q} = \mathrm{Hull}_1^{\mathcal{Q}}(\rho_1^{\mathcal{Q}} \cup p_1^{\mathcal{Q}}).$$

By definition, $\mathrm{Hull}_1^{\mathcal{Q}}(X)$ has elements $\tau^{\mathcal{Q}}[c]$ where τ is a Σ_1-Skolem term and $c \in X^{<\omega}$. Therefore,

$$\mathrm{ran}(i) = \mathrm{Hull}_1^{\mathcal{R}}(\rho_1^{\mathcal{R}} \cup p_1^{\mathcal{R}}).$$

The moral is that by deriving an extender from the inverse of the Mostowski collapse of this hull, we recover E. We abstract two key notions from this example. Observe that κ is the least ordinal α such that

$$\alpha \notin \mathrm{Hull}_1^{\mathcal{R}}(\alpha \cup p_1^{\mathcal{R}}).$$

This says that κ is the least ordinal $\alpha \geq \rho_1^{\mathcal{R}}$ such that $\mathcal{R}$ fails to have a certain *definability property* at α. Observe also that

$$\wp(\kappa) \cap \mathcal{R} \subseteq \text{the Mostowski collapse of } \mathrm{Hull}_1^{\mathcal{R}}(\kappa \cup p_1^{\mathcal{R}}).$$

This says that $\mathcal{R}$ has a certain *hull property* at κ. The combination of the two observations above is the minimum required to derive an extender over $\mathcal{R}$ with critical point κ from the inverse of the Mostowski collapse of

$$\mathrm{Hull}_1^{\mathcal{R}}(\kappa \cup p_1^{\mathcal{R}}).$$

Of course, $\mathcal{Q}$ has the definability and hull properties at all $\alpha \geq \rho_1^{\mathcal{Q}}$ since we assumed that $\mathcal{Q}$ is 1-sound. We could go on to show that for all $\alpha \geq \rho_1^{\mathcal{R}}$, $\mathcal{R}$ fails to have the definability property at α iff α is a generator of E. And that $\mathcal{R}$ has the hull property at α iff $\alpha \leq \kappa$ or $\alpha \geq \nu(E)$ where

$$\nu(E) = \sup(\{(\kappa^+)^{\mathcal{Q}}\} \cup \{\xi + 1 \mid \xi \text{ is a generator of } E\}).$$

Taking our discussion to the next level, suppose instead that $\mathcal{Q}$ is a weasel. This is fundamentally different because

$$\rho_1^{\mathcal{Q}} = \mathrm{On} \cap \mathcal{Q} = \Omega.$$

Nevertheless, it is important to find an analogous way of undoing iterations of $\mathcal{Q}$. What we need are versions of the definability and hull properties that are appropriate for weasels. And we need a way to take hulls in K^c that produces weasels with these properties.

2.9 Definition. Let $\mathcal{Q}$ be a weasel and $\Gamma \subseteq \mathcal{Q}$. Then Γ *is thick in* $\mathcal{Q}$ iff there is a club C in Ω such that for all $\lambda \in A_0 \cap C$,

1. $(\lambda^+)^{\mathcal{Q}} = \lambda^+$,
2. λ is not the critical point of a total-on-$\mathcal{Q}$ extender on the $\mathcal{Q}$ sequence, and
3. there is a λ-club in $\Gamma \cap \lambda^+$.

2.10 Definition. $\mathcal{Q}$ is a *thick* weasel iff Ω is thick in $\mathcal{Q}$.

The reader will not find the expression *thick weasel* in the literature but the concept needed a name so we picked one. Clearly K^c is a thick weasel. The next three results are useful closure properties of thick sets.

2.11 Theorem. *Let $\mathcal{Q}$ be a thick weasel. Then*

$$\{\Gamma \subseteq \mathcal{Q} \mid \Gamma \text{ is thick in } \mathcal{Q}\}$$

is an Ω-complete filter.

2.12 Theorem. *Suppose that $\pi : \mathcal{P} \to \mathcal{Q}$ is an elementary embedding and* $\mathrm{ran}(\pi)$ *is thick in $\mathcal{Q}$. Let*

$$\Phi = \{\alpha < \Omega \mid \pi(\alpha) = \alpha\}.$$

Then Φ is thick in both $\mathcal{P}$ and $\mathcal{Q}$.

2.13 Theorem. *Let $\mathcal{T}$ be an iteration tree on a thick weasel $\mathcal{Q}$ with*

$$\mathrm{lh}(\mathcal{T}) = \theta + 1 \leq \Omega + 1.$$

Assume that there is no dropping along $[0, \theta]_T$ and $i^{\mathcal{T}}_{0,\theta}$“$\Omega \subseteq \Omega$. Let

$$\Phi = \{\alpha < \Omega \mid i^{\mathcal{T}}_{0,\theta}(\alpha) = \alpha\}.$$

Then Φ is thick in both $\mathcal{Q}$ and $\mathcal{M}^{\mathcal{T}}_\theta$.

The proofs of the previous three theorems are reasonable exercises for the reader. The $\theta = \Omega$ case of the Theorem 2.13 is why we used A_0 instead of A_1.

2.14 Definition. A thick weasel $\mathcal{Q}$ has the *definability property* at α iff

$$\alpha \in \mathrm{Hull}^{\mathcal{Q}}(\alpha \cup \Gamma)$$

whenever Γ is thick in $\mathcal{Q}$.

By definition, the elements of $\mathrm{Hull}^{\mathcal{Q}}(X)$ are those of the form $\tau^{\mathcal{Q}}[c]$ where τ is a Skolem term and $c \in X^{<\omega}$. Equivalently, $a \in \mathrm{Hull}^{\mathcal{Q}}(X)$ iff $\{a\}$ is first-order definable over $\mathcal{Q}$ with parameters from X.

2.15 Definition. A thick weasel $\mathcal{Q}$ has the *hull property* at α iff

$$\wp(\alpha) \cap \mathcal{Q} \subseteq \text{the Mostowski collapse of } \mathrm{Hull}^{\mathcal{Q}}(\alpha \cup \Gamma)$$

whenever Γ is thick in $\mathcal{Q}$.

2.16 Theorem. *Let $\beta < \Omega$ and $\mathcal{Q}$ be a thick weasel with the definability and hull properties for all $\alpha < \beta$. Suppose that $\mathcal{T}$ is an iteration tree on $\mathcal{Q}$ with*

$$\mathrm{lh}(\mathcal{T}) = \theta + 1 \leq \Omega + 1.$$

Assume that there is no dropping along $[0, \theta]_T$ and $i^{\mathcal{T}}_{0,\theta}$"$\Omega \subseteq \Omega$. Then the following hold for all $\alpha < \beta$.

(1) $\mathcal{M}^{\mathcal{T}}_{\theta}$ does not have the definability property at α iff there exists an

$$\eta + 1 \in [0, \theta]_T$$

such that α is a generator of $E^{\mathcal{T}}_{\eta}$.

(2) $\mathcal{M}^{\mathcal{T}}_{\theta}$ does not have the hull property at α iff there exists an $\eta + 1 \in [0, \theta]_T$ such that

$$(\mathrm{crit}(E^{\mathcal{T}}_{\eta})^{+})^{\mathcal{M}^{\mathcal{T}}_{\theta}} \leq \alpha < \nu(E^{\mathcal{T}}_{\eta}).$$

Sketch. For simplicity, we deal only with the case of a single ultrapower. In other words, $\theta = 2$. Let $E = E^{\mathcal{T}}_0$ and consider the following diagram.

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "j"] \arrow[dr, "i"'] & \mathrm{Ult}(\mathcal{Q}, E) \\
 & \mathrm{Ult}(\mathcal{Q}, E \restriction \alpha) \arrow[u, "k"']
\end{tikzcd}$$

Then $\mathrm{crit}(k) = \alpha$ iff α is a generator of E. Let $\Phi = \{\xi < \Omega \mid j(\xi) = \xi\}$. Then Φ is thick in all three models. Of course, $j(\xi) = \xi$ implies $k(\xi) = \xi$ and $i(\xi) = \xi$.

First we prove the *if* direction of (1). Assume that α is a generator of E. Equivalently, that $\alpha = \mathrm{crit}(k)$. Suppose for contradiction that $\mathrm{Ult}(\mathcal{Q}, E)$ has the definability property at α. Then there is a Skolem term τ and a parameter $c \in (\alpha \cup \Phi)^{<\omega}$ such that

$$\alpha = \tau^{\mathrm{Ult}(\mathcal{Q},E)}[c] = k(\tau^{\mathrm{Ult}(\mathcal{Q},E\restriction\alpha)}[c]\,).$$

This is a contradiction since $\alpha \notin \mathrm{ran}(k)$.

Second we prove the *if* direction of (2). Assume that

$$(\mathrm{crit}(E)^+)^{\mathcal{Q}} \leq \alpha < \nu(E).$$

The main point is that

$$E\restriction\alpha \in \mathrm{Ult}(\mathcal{Q}, E)$$

whereas

$$E\restriction\alpha \notin \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha).$$

We know this because E is on the $\mathcal{Q}$ sequence, which is a good extender sequence. Since

$$(\mathrm{crit}(E)^+)^{\mathcal{Q}} \leq \alpha,$$

it is possible to code $E\restriction\alpha$ by $A \subseteq \alpha$ with $A \in \mathrm{Ult}(\mathcal{Q}, E)$. Suppose for contradiction that $\mathrm{Ult}(\mathcal{Q}, E)$ has the hull property at α. Then there is a Skolem term τ and a parameter $c \in (\alpha \cup \Phi)^{<\omega}$ such that

$$A = \tau^{\mathrm{Ult}(\mathcal{Q},E)}[c] \cap \alpha.$$

Since $\mathrm{crit}(k) \geq \alpha$,

$$A = \tau^{\mathrm{Ult}(\mathcal{Q},E\restriction\alpha)}[c] \cap \alpha \in \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha),$$

so

$$E\restriction\alpha \in \mathrm{Ult}(\mathcal{Q}, E\restriction\alpha),$$

which is a contradiction.

Notice that the two *if* directions did not use the hypothesis that $\mathcal{Q}$ has the definability and hull properties at all ordinals $< \beta$. These are used for the two *only if* directions, which we leave to the reader. ⊣

The next theorem explains how the definability property and hull property are related, and its proof is a good example of how they are used.

2.17 Theorem. *For all $\beta < \Omega$, if $\mathcal{Q}$ has the definability property at all $\alpha < \beta$, then $\mathcal{Q}$ has the hull property at β.*

Sketch. By induction, we may assume that $\mathcal{Q}$ has the definability and hull properties at all $\alpha < \beta$. Suppose that Γ is thick in $\mathcal{Q}$. Let

$$\pi : \mathcal{P} \simeq \mathrm{Hull}^{\mathcal{Q}}(\beta \cup \Gamma)$$

be the inverse of the Mostowski collapse. We must show that

$$\wp(\beta) \cap \mathcal{P} = \wp(\beta) \cap \mathcal{Q}.$$

If Δ is thick in $\mathcal{P}$, then $\{\xi \in \Delta \mid \pi(\xi) = \xi\}$ is thick in $\mathcal{Q}$. From this it follows that $\mathcal{P}$ has the definability and hull properties at all $\alpha < \beta$. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Both $\mathcal{P}$ and $\mathcal{Q}$ are universal, so

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. Moreover, there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$. It is enough to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}), \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \beta.$$

For contradiction, suppose that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}) < \beta.$$

Apply Theorem 2.16 to $\mathcal{S}$ to see that $\mathrm{crit}(i^{\mathcal{S}}_{0,\eta})$ is equal to the least α such that $\mathcal{M}^{\mathcal{S}}_{\eta}$ does not have the definability property at α. And apply Theorem 2.16 to $\mathcal{T}$ to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}) = \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}).$$

Call this ordinal α and let

$$\alpha^* = \min(i^{\mathcal{S}}_{0,\eta}(\alpha), i^{\mathcal{T}}_{0,\theta}(\alpha)).$$

As $\alpha < \beta$, $\mathcal{Q}$ has the hull property at α, so

$$\wp(\alpha) \cap \mathcal{P} = \wp(\alpha) \cap \mathcal{Q}.$$

Next we use the fact that

$$\Phi = \{\xi < \Omega \mid i^{\mathcal{S}}_{0,\eta}(\xi) = \xi = i^{\mathcal{T}}_{0,\theta}(\xi)\}$$

is thick in both $\mathcal{P}$ and $\mathcal{Q}$ to show that if $X \subseteq \alpha$ with $X \in \mathcal{P}$, then

$$i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha^* = i^{\mathcal{T}}_{0,\theta}(X) \cap \alpha^*.$$

First note that $\alpha \subseteq \Phi$. Then, given $X \subseteq \alpha$ with $X \in \mathcal{P}$, choose a Skolem term τ and $c \in \Phi^{<\omega}$ such that

$$X = \tau^{\mathcal{P}}[c] \cap \alpha.$$

Let

$$Y = \tau^{\mathcal{Q}}[c] \cap \alpha.$$

Then

$$\begin{aligned}
i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha^* &= i^{\mathcal{S}}_{0,\eta}(\tau^{\mathcal{P}}[c] \cap \alpha) \cap \alpha^* \\
&= \tau^{\mathcal{M}^{\mathcal{S}}_{\eta}}[c] \cap \alpha^* \\
&= \tau^{\mathcal{M}^{\mathcal{T}}_{\theta}}[c] \cap \alpha^* \\
&= i^{\mathcal{T}}_{0,\theta}(\tau^{\mathcal{Q}}[c] \cap \alpha) \cap \alpha^* \\
&= i^{\mathcal{T}}_{0,\theta}(Y) \cap \alpha^*.
\end{aligned}$$

Also

$$X = i^{\mathcal{S}}_{0,\eta}(X) \cap \alpha = i^{\mathcal{T}}_{0,\theta}(Y) \cap \alpha = Y.$$

We have seen that the first extenders used along $[0,\eta]_S$ and $[0,\theta]_T$ are comparable, which is impossible in a coiteration. (E.g., see the subclaim in the proof of Theorem 3.11 of [41].)

The same contradiction is obtained similarly by assuming that

$$\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) < \beta.$$

$\dashv$

2.18 Definition. Let $\mathcal{P}$ be a mouse of height $< \Omega$. Then $\mathcal{P}$ is A_0-*sound* iff there exists a thick weasel $\mathcal{P}^*$ such that $\mathcal{P} \triangleleft \mathcal{P}^*$ and $\mathcal{P}^*$ has the definability property at all $\alpha \in \mathrm{On} \cap \mathcal{P}$.

The point of isolating A_0-sound mice is that they line up as the next theorem shows.

2.19 Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be A_0-sound mice. Then $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{P} \trianglerighteq \mathcal{Q}$.*

Sketch. Let $\mathcal{P}^*$ and $\mathcal{Q}^*$ be A_0-soundness witnesses for $\mathcal{P}$ and $\mathcal{Q}$ respectively. Let $(\mathcal{S},\mathcal{T})$ be the coiteration of $(\mathcal{P}^*,\mathcal{Q}^*)$. Then

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and there is no dropping along $[0,\eta]_S$ and $[0,\theta]_T$. It is enough to see that

$$\mathrm{crit}(i^{\mathcal{S}}_{0,\eta}), \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \min(\mathrm{On} \cap \mathcal{P}, \mathrm{On} \cap \mathcal{Q}).$$

This is done by contradiction exactly as in the proof of Theorem 2.17 using the hull property and definability property of $\mathcal{P}^*$ and $\mathcal{Q}^*$ at all

$$\alpha < \min(\mathrm{On} \cap \mathcal{P}, \mathrm{On} \cap \mathcal{Q}).$$

$\dashv$

2.20 Definition. K is the union of all the A_0-sound mice.

By Theorem 2.19, K is a premouse. But it is not immediate that K has height Ω.

2.21 Definition. Let $\mathcal{Q}$ be a thick weasel. Then

$$\mathrm{Def}(\mathcal{Q}) = \bigcap\{\mathrm{Hull}^{\mathcal{Q}}(\Gamma) \mid \Gamma \text{ is thick in } \mathcal{Q}\}$$

The plan for proving that K is a weasel is as follows. First we show that K is the Mostowski collapse of $\mathrm{Def}(K^c)$. Then we establish that K^c has the hull and definability properties at U-almost all $\alpha < \Omega$. The last step is to show that $\mathrm{Def}(K^c)$ is unbounded in Ω. The realization of this plan stretches over several theorems.

2.22 Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be thick weasels. Then* $\mathrm{Def}(\mathcal{P}) \simeq \mathrm{Def}(\mathcal{Q})$.

Sketch. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Then

$$\mathcal{M}^{\mathcal{S}}_{\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$. It is enough to see that

$$i^{\mathcal{S}}_{0,\eta}``\mathrm{Def}(\mathcal{P}) = \mathrm{Def}(\mathcal{M}^{\mathcal{S}}_{0,\eta})$$

and

$$i^{\mathcal{T}}_{0,\theta}``\mathrm{Def}(\mathcal{Q}) = \mathrm{Def}(\mathcal{M}^{\mathcal{T}}_{0,\theta}).$$

This is an easy exercise using the basic properties of thick sets. $\dashv$

2.23 Theorem. $K \simeq \mathrm{Def}(K^c)$.

Sketch. Let

$$\pi : K' \simeq \mathrm{Def}(K^c)$$

be the inverse of the Mostowski collapse. We must show that $K' = K$.

First let $\mathcal{P} \triangleleft K$ and $\mathcal{P}^*$ be a witness that $\mathcal{P}$ is A_0-sound. Since $\mathcal{P}^*$ has the definability property at all $\alpha < \mathrm{On} \cap \mathcal{P}$,

$$\mathcal{P} \subseteq \mathrm{Def}(\mathcal{P}^*).$$

But $\mathrm{Def}(K^c) \simeq \mathrm{Def}(\mathcal{P}^*) \simeq K'$ by Theorem 2.22. Therefore $\mathcal{P} \triangleleft K'$.

Now let $\mathcal{P} \triangleleft K'$. Let $\theta = \sup(\pi``(\mathrm{On} \cap \mathcal{P}))$. For each $\alpha \in \theta - \mathrm{Def}(K^c)$, pick an A_0-thick set Γ_α such that

$$\alpha \notin \mathrm{Hull}^{K^c}(\Gamma_\alpha).$$

Let

$$\Delta = \bigcap\{\Gamma_\alpha \mid \alpha \in \theta - \mathrm{Def}(K^c)\}$$

and $\mathcal{Q}$ be the Mostowski collapse of $\mathrm{Hull}^{K^c}(\Delta)$. It is an easy exercise to see that $\mathcal{Q}$ witnesses that $\mathcal{P}$ is A_0-sound. Therefore $\mathcal{P} \triangleleft K$. $\dashv$

By Theorems 2.22 and 2.23, $K \simeq \mathrm{Def}(\mathcal{P})$ whenever $\mathcal{P}$ is a thick weasel.

2.24 Theorem. *Let $\mathcal{Q}$ be a thick weasel. Then there exists a C club in Ω such that $\mathcal{Q}$ has the hull property at α for all inaccessible $\alpha \in C$. In particular,*

$$\{\alpha < \Omega \mid \mathcal{Q} \text{ has the hull property at } \alpha\} \in U.$$

Sketch. By recursion, define a continuous decreasing sequence

$$\langle X_\alpha \mid \alpha < \Omega \rangle$$

of thick elementary substructures of $\mathcal{Q}$ and an increasing sequence

$$\langle \lambda_\alpha \mid \alpha < \Omega \rangle$$

of cardinals of $\mathcal{Q}$. For all $\alpha < \Omega$, let $\pi_\alpha : \mathcal{P}_\alpha \simeq X_\alpha$ be the inverse of the Mostowski collapse and $\pi_\alpha(\kappa_\alpha) = \lambda_\alpha$. Arrange the construction so that $\langle \kappa_\alpha \mid \alpha \leq \beta \rangle$ is an initial segment of the infinite cardinals of $\mathcal{P}_\beta$ for all $\beta < \Omega$. Also arrange that for all $\alpha < \beta < \Omega$,

$$\pi_\alpha \restriction (\kappa_\alpha + 1) = \pi_\beta \restriction (\kappa_\alpha + 1)$$

and $\mathcal{P}_\beta$ has the hull property at all $\kappa \leq \kappa_\alpha$.

Start the construction with $\kappa_0 = \lambda_0 = \omega$, $X_0 = \mathcal{Q}$ and $\pi_0 = \mathrm{id} \restriction \mathcal{Q}$. If β is a limit ordinal, then $X_\beta = \bigcap_{\alpha<\beta} X_\alpha$ and this determines $\mathcal{P}_\beta$, π_β, κ_β and λ_β by what we said above. The successor step is more complicated. If $A \in \wp(\kappa_\alpha) \cap \mathcal{P}_\alpha$ and there exists a Γ thick in $\mathcal{P}_\alpha$ such that

$$A \notin \text{Mostowski collapse of } \mathrm{Hull}^{\mathcal{P}_\alpha}(\kappa_\alpha \cup \Gamma),$$

then pick such a Γ and call it Γ_A. Then let

$$\Gamma_\alpha = \bigcap(\{\mathcal{P}_\alpha\} \cup \{\Gamma_A \mid A \in \wp(\kappa_\alpha) \cap \mathcal{P}_\alpha \text{ and } \Gamma_A \text{ is defined}\})$$

and

$$X_{\alpha+1} = \mathrm{Hull}^{\mathcal{P}_\alpha}((\kappa_\alpha + 1) \cup \Gamma_\alpha).$$

This determines $\mathcal{P}_{\alpha+1}$, $\pi_{\alpha+1}$, $\kappa_{\alpha+1}$ and $\lambda_{\alpha+1}$ by what we said at the start.

2.25 Lemma. *If γ is a limit ordinal, then $\mathcal{P}_\gamma = \mathcal{P}_{\gamma+1}$.*

Sketch. Suppose not. Then Γ_A is defined for some $A \in \wp(\kappa_\gamma) \cap \mathcal{P}_\gamma$. Let $\mathcal{P}_A$ be the Mostowski collapse of

$$\mathrm{Hull}^{\mathcal{P}_\gamma}(\kappa_\gamma \cup \Gamma_A).$$

and $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}_A, \mathcal{P}_\gamma)$. Then $\mathcal{M}^{\mathcal{S}}_\eta = \mathcal{M}^{\mathcal{T}}_\theta$ where $\eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$, and there is no dropping along $[0, \eta]_S$ and $[0, \theta]_T$.

Suppose that $\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) \geq \kappa_\gamma$. Then

$$A \in \wp(\kappa_\gamma) \cap \mathcal{P}_\gamma = \wp(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{T}}_\theta = \wp(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{S}}_\eta \subseteq \wp(\kappa_\gamma) \cap \mathcal{P}_A$$

since all the extenders used on $\mathcal{S}$ have length at least κ_γ. But $A \notin \mathcal{P}_A$, contradiction!

Therefore, $\mathrm{crit}(i^{\mathcal{T}}_{0,\theta}) < \kappa_\gamma$. Let $\beta < \gamma$ be such that

$$\kappa_\beta = \mathrm{crit}(i^{\mathcal{T}}_{0,\theta}).$$

Then β is equal to the least ordinal $\alpha < \gamma$ such that $\mathcal{M}^{\mathcal{T}}_\theta$ does not have the hull property at $\kappa_{\alpha+1}$. This is not precisely what Theorem 2.16 says about $\mathcal{T}$ but the proof shows it. Applying a similar modification of Theorem 2.16 to $\mathcal{S}$ shows that $\kappa_\beta = \mathrm{crit}(i^{\mathcal{S}}_{0,\eta})$. Finally, use the hull property at κ_β in both $\mathcal{P}_A$ and $\mathcal{P}_\gamma$ to see that the first extenders used on $[0,\eta]_S$ and $[0,\theta]_T$ are compatible. This leads to a standard contradiction. ⊣

2.26 Lemma. *Let $X = \bigcap\{X_\alpha \mid \alpha < \Omega\}$. Then X is thick in $\mathcal{Q}$.*

Sketch. For each $\alpha < \Omega$, pick C_α club in Ω witnessing that X_α is thick in $\mathcal{Q}$. Let C be the diagonal intersection of $\langle C_\alpha \mid \alpha < \Omega\rangle$. We show that C witnesses that X is thick in $\mathcal{Q}$. Let $\beta \in A_0 \cap C$. Clearly $(\beta^+)^{\mathcal{Q}} = \beta^+$ and β is not the critical point of a total-on-$\mathcal{Q}$ extender on the $\mathcal{Q}$ sequence. For each $\alpha < \beta$, there exists a β-club $D_\alpha \subseteq X_\alpha \cap \beta^+$. Let $D = \bigcap\{D_\alpha \mid \alpha < \beta\}$. Then D is a β-club subset of

$$\textstyle\bigcap_{\alpha<\beta} X_\alpha \cap \beta^+ = X_\beta \cap \beta^+ = X_{\beta+1} \cap \beta^+ = X \cap \beta^+.$$

The first equation holds by the definition of X_β. The second holds by Lemma 2.25. The third holds because $\beta \le \lambda_\beta$, $\beta^+ \le \lambda_{\beta+1}$ and

$$X_{\beta+1} \cap (\lambda_{\beta+1} + 1) = X \cap (\lambda_{\beta+1} + 1).$$

In fact, by taking β closed under $\alpha \mapsto \lambda_\alpha$ we get that $\beta = \kappa_\beta = \lambda_\beta$ and $\beta^+ = \kappa_{\beta+1} = \lambda_{\beta+1}$. ⊣

2.27 Lemma. *Let $\mathcal{P}$ be the Mostowski collapse of X. Then $\mathcal{P}$ has the hull property at all $\alpha < \Omega$.*

Sketch. Lemma 2.26 implies that $\mathcal{P}$ is a thick weasel. By construction, $\langle \kappa_\alpha \mid \alpha < \Omega\rangle$ lists the infinite cardinals of $\mathcal{P}$ in increasing order and $\mathcal{P}$ has the hull property at κ_α for all $\alpha < \Omega$. ⊣

Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of $(\mathcal{P}, \mathcal{Q})$. Consider the case in which $\mathcal{S}$ and $\mathcal{T}$ both have length $\Omega + 1$, the other cases being similar. Then $\mathcal{M}^{\mathcal{S}}_\Omega = \mathcal{M}^{\mathcal{T}}_\Omega$ and there is no dropping along $[0,\Omega]_S$ and $[0,\Omega]_T$. Let C be the set of limit ordinals

$$\theta \in [0,\Omega]_S \cap [0,\Omega]_T$$

such that θ is the supremum of

$$\{\mathrm{lh}(E^{\mathcal{S}}_\eta) \mid \eta < \theta\} \cup \{\mathrm{lh}(E^{\mathcal{T}}_\eta) \mid \eta < \theta\}.$$

Then C is club in Ω. Consider an arbitrary $\theta \in C$. Then $\mathcal{M}^{\mathcal{S}}_{\theta}$ has the hull property at θ and, since

$$\operatorname{crit}(i^{\mathcal{S}}_{\theta,\Omega}) \geq \theta,$$

$\mathcal{M}^{\mathcal{S}}_{\Omega}$ has the hull property at θ. The fact that $\operatorname{crit}(i^{\mathcal{T}}_{\theta,\Omega}) \geq \theta$ can be used to see that $\mathcal{M}^{\mathcal{T}}_{\theta}$ has the hull property at θ. Now assume that θ is inaccessible. Then $i^{\mathcal{T}}_{0,\theta}(\theta) = \theta$. To finish the proof of the theorem, we show that $\mathcal{Q}$ has the hull property at θ. Suppose $A \in \wp(\theta) \cap \mathcal{Q}$ and Γ is thick in $\mathcal{Q}$. Let $B = i^{\mathcal{T}}_{0,\theta}(A)$. Then $B \in \wp(\theta) \cap \mathcal{M}^{\mathcal{T}}_{\theta}$ so there is a Skolem term τ and parameters $c \in \theta^{<\omega}$ and $d \in \Gamma^{<\omega}$ such that $d = i^{\mathcal{T}}_{0,\theta}(d)$ and

$$B = \tau^{\mathcal{M}^{\mathcal{T}}_{\theta}}[c, d] \cap \theta.$$

By minimizing c in this equation we find $b \in \theta^{<\omega}$ such that $c = i^{\mathcal{T}}_{0,\theta}(b)$. Thus

$$A = \tau^{\mathcal{Q}}[b, d] \cap \theta.$$

$\dashv$

We have used the Technical Hypothesis that Ω is measurable twice already. First, to see that the set of α such that $(\alpha^{+})^{K^c} = \alpha^{+}$ is stationary in Ω. Second, to show that K^c is $(\omega, \Omega+1)$ iterable starting from the fact that if $\mathcal{P}$ is countable and elementarily embeds into K^c, then $\mathcal{P}$ is (ω, ω_1+1) iterable. The third and final use of the Technical Hypothesis comes in the proof of the following theorem.

2.28 Theorem. $\{\alpha < \Omega \mid K^c$ *has the definability property at* $\alpha\} \in U$.

Sketch. Suppose not. Let

$$D = \{\alpha < \Omega \mid K^c \text{ does not have the definability property at } \alpha\}.$$

Then $D \in U$. For each $\alpha \in D$, pick a thick Γ_α such that

$$\alpha \notin \operatorname{Hull}^{K^c}(\alpha \cup \Gamma_\alpha).$$

We may assume $\Gamma_\beta \subseteq \Gamma_\alpha$ whenever $\alpha < \beta$ are elements of D. We write $\Gamma = \langle \Gamma_\alpha \mid \alpha \in D\rangle$. Form the iteration

$$V \xrightarrow[j]{} V' \xrightarrow[k]{} V''$$

with $V' = \operatorname{Ult}(V, U)$, $U' = j(U)$ and $V'' = \operatorname{Ult}(V', U')$. We will use the general fact that

$$j \circ j = k \circ j.$$

This equation holds because

$$j([\alpha \mapsto x]^V_U) = [\alpha \mapsto j(x)]^{V'}_{U'}.$$

Let $W = K^c$, $W' = j(W)$ and $W'' = k(W')$. By what we just said, $W'' = j(W')$. Consider the inverse of the Mostowski collapse

$$\pi : \mathcal{P} \simeq \mathrm{Hull}^{W'}(\Omega \cup \Gamma'_\Omega)$$

where $\Gamma'_\alpha = j(\Gamma)_\alpha$. Also let $\Gamma''_\alpha = k(\Gamma')_\alpha$. Then $\Gamma''_\alpha = j(\Gamma')_\alpha$. Since W' does not have the definability property at Ω, $\mathrm{crit}(\pi) = \Omega$. By Theorem 2.24,

$$(W' \text{ has the hull property at } \Omega)^{V'},$$

so

$$\wp(\Omega) \cap \mathcal{P} = \wp(\Omega) \cap W'.$$

Let $\Omega' = j(\Omega)$. Note that $\pi(\Omega) < \Omega'$ because Γ'_Ω is unbounded in Ω'. Let F be the extender of length $\pi(\Omega)$ derived from π. We claim that

$$\pi(A) = j(A) \cap \pi(\Omega)$$

for all $A \in \wp(\Omega) \cap W'$. From the claim, it follows that F is countably certified in V', which can be used to show that F witnesses that Ω is a superstrong cardinal in W'. To prove the claim, pick a Skolem term τ and parameters $c \in \Omega^{<\omega}$ and $d \in (\Gamma'_\Omega)^{<\omega}$ such that $A = \tau^{W'}[c, d] \cap \Omega$. Then

$$j(A) = \tau^{W''}[c, j(d)] \cap \Omega'$$

and

$$j(d) \in (\Gamma''_{\Omega'})^{<\omega} \subseteq (\Gamma''_\Omega)^{<\omega}$$

because Γ'' is a descending sequence and $\Omega' > \Omega$. In particular,

$$\tau^{W''}[c, j(d)] \in \mathrm{Hull}^{W''}(\Omega \cup \Gamma''_\Omega)$$

and

$$A = \tau^{W''}[c, j(d)] \cap \Omega.$$

By elementarity,

$$k(\pi) : k(\mathcal{P}) \simeq \mathrm{Hull}^{W''}(\Omega \cup \Gamma''_\Omega).$$

Finally, since $\mathrm{crit}(k) = \Omega' > \pi(\Omega)$ and $A \subseteq \Omega$,

$$\begin{aligned}
\pi(A) &= k(\pi(A)) = k(\pi)(k(A)) = k(\pi)(A) \\
&= k(\pi)(\tau^{W''}[c, j(d)] \cap \Omega) \\
&= \tau^{W''}[c, j(d)] \cap k(\pi(\Omega)) \\
&= j(A) \cap \pi(\Omega).
\end{aligned}$$

⊣

2.29 Theorem. *K is a weasel.*

Proof. Consider the following recursive construction. Let $\Gamma_0 = \Omega$. Assuming that Γ_α has been defined, if

$$\mathrm{Hull}^{K^c}(\Gamma_\alpha) = \mathrm{Def}(K^c),$$

then stop the construction. Otherwise, let

$$\gamma_\alpha = \min(\mathrm{Hull}^{K^c}(\Gamma_\alpha) - \mathrm{Def}(K^c))$$

and pick $\Gamma_{\alpha+1} \subseteq \Gamma_\alpha$ so that

$$\gamma_\alpha \notin \mathrm{Hull}^{K^c}(\Gamma_{\alpha+1}).$$

If β is a limit ordinal, then let

$$\Gamma_\beta = \bigcap\{\Gamma_\alpha \mid \alpha < \beta\}.$$

Suppose for contradiction that $\mathrm{Def}(K^c)$ is bounded in Ω. Then γ_α and Γ_α are defined for all $\alpha < \Omega$. And there exists an $\alpha < \Omega$ such that

$$\mathrm{Def}(K^c) \cap \Omega \subseteq \gamma_\alpha.$$

By Theorem 2.28, there exists a $\delta \in (\gamma_\alpha, \Omega)$ such that

$$\delta = \sup(\{\gamma_\beta \mid \beta < \delta\}) \leq \gamma_\delta$$

and K^c has the definability property at δ. This implies that there exist an ordinal $\beta \in (\alpha, \delta)$, parameters $c \in (\gamma_\beta)^{<\omega}$ and $d \in (\Gamma_{\delta+1})^{<\omega}$, and a Skolem term τ such that $\delta = \tau^{K^c}[c, d]$. Then c is a witness to the sentence:

There exists a $b \in (\gamma_\beta)^{<\omega}$ such that $\gamma_\beta < \tau^{K^c}[b, d] < \gamma_{\delta+1}$.

Since γ_β and $\gamma_{\delta+1}$ are elements of $\mathrm{Hull}^{K^c}(\Gamma_\beta)$ we may pick a witness b to this sentence with

$$b \in \mathrm{Hull}^{K^c}(\Gamma_\beta).$$

By the minimality of γ_β and the fact that $b \in (\gamma_\beta)^{<\omega}$,

$$b \in \mathrm{Def}(K^c).$$

Hence

$$\tau^{K^c}[b, d] \in \mathrm{Hull}^{K^c}(\Gamma_{\delta+1}).$$

By the choice of γ_α and the fact that $\gamma_\alpha < \gamma_\beta < \tau^{K^c}[b, d]$,

$$\tau^{K^c}[b, d] \notin \mathrm{Def}(K^c).$$

By the minimality of $\gamma_{\delta+1}$,

$$\tau^{K^c}[b, d] \geq \gamma_{\delta+1}.$$

But

$$\tau^{K^c}[b, d] < \gamma_{\delta+1},$$

which is a contradiction. $\dashv$

At the corresponding point in [42], Steel goes on to prove that

$$\{\alpha < \Omega \mid (\alpha^+)^K = \alpha^+\} \in U.$$

(Cf., Theorem 3.1 below.) The calculations involve combinatorics similar to the proof of Theorem 2.29 but we omit them here. From this and Theorem 2.8, it follows that K is universal. Also at this point, Steel shows that K is absolute under forcing in $H(\Omega)$. (Cf., Theorem 3.4 below.) The proof involves abstracting the properties of A_0 in the arguments we have given so far.

2.2. First-Order Definition of K

Now we head in a slightly different direction. Notice that the definition of K we have given is second-order over $H(\Omega)$. Moreover, there is no obvious sense in which the definition works locally. For example, it is not immediate from what we have said so far that $K \cap \mathrm{HC}$ is less complex than K.[7] Our next goal is to find an equivalent first-order definition of K that gives meaningful local bounds on complexity. For example, $K \cap \mathrm{HC}$ turns out to be Σ_1 definable over $L_{\omega_1}(\mathbb{R})$. (Cf., Theorem 3.5.) By results of Woodin, this is the best possible upper bound on the complexity of $K \cap \mathrm{HC}$. The ideas that go into the first-order definition of K are central to the proof of the weak covering theorem in Sect. 4.

Before launching into the details, let us motivate what is to come. It is not hard to see that all universal weasels have the same subsets of ω, namely those in

$$\mathcal{J}^K_{(\omega_1)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound mouse and } \rho^{\mathcal{Q}}_\omega = 1\}.$$

Nor is it hard to see that all universal weasels have the same subsets of $(\omega_1)^K$, namely those in

$$\mathcal{J}^K_{(\omega_2)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound mouse with } \rho^{\mathcal{Q}}_\omega = (\omega_1)^K \text{ and } \mathcal{J}^K_{(\omega_1)^K} \triangleleft \mathcal{Q}\}.$$

This points to a simultaneous definition of what it means for α to be a cardinal of K on one hand, and $\mathcal{J}^K_{(\alpha^+)^K}$ on the other, by induction on $\alpha < \Omega$. However, the general pattern is more complicated than we have indicated; it has to be by Woodin's result on the complexity of $K \cap \mathrm{HC}$. Instead, Steel wove together three definitions,

- α is a cardinal of K,

[7] By definition, $\mathrm{HC} = H(\aleph_1)$. The reader should be attentive here to the difference between

$$J^K_{(\omega_1)^K} = \mathrm{HC}^K$$

and

$$J^K_{(\omega_1)^V} = K \cap \mathrm{HC}.$$

- $\mathcal{Q}$ is an α-strong mouse, and
- $\mathcal{J}^K_{(\alpha^+)^K}$

by induction on $\alpha < \Omega$, where *α-strong* is a natural strengthening of iterable. In the end, if α is a cardinal of K, then

$$\mathcal{J}^K_{(\alpha^+)^K} = \bigcup\{\mathcal{Q} \mid \mathcal{Q} \text{ is a sound } \alpha\text{-strong mouse with } \omega\rho^{\mathcal{Q}}_\omega = \alpha \text{ and } \mathcal{J}^K_\alpha \lhd \mathcal{Q}\}.$$

The simpler pattern that leaves out α-strong holds for α less than the least measurable cardinal of K, as the reader familiar with the core model theory of Dodd and Jensen would expect. A remarkable fact due to Ralf Schindler is that the simpler pattern holds again if $\alpha \geq \aleph_2$. See Theorem 3.6.

Let us make the convention that if $\mathcal{P}$ is a mouse and $\mathcal{T}$ is an iteration tree on $\mathcal{P}$, then we have equipped $\mathcal{P}$ with an $(\omega, \Omega+1)$ iteration strategy $\Sigma^{\mathcal{P}}$ and $\mathcal{T}$ is consistent with $\Sigma^{\mathcal{P}}$. Unless, of course, we specify otherwise. This will save us some writing and make the main points clearer.

2.30 Definition. Suppose that $\mathcal{Q}$ is a premouse and $\alpha \leq \mathrm{On} \cap \mathcal{Q} \leq \Omega$. Let $\mathcal{P} = \mathcal{J}^{\mathcal{Q}}_\alpha$. Then $\mathcal{Q}$ is *α-strong* iff

1. $\mathcal{P}$ is A_0-sound (i.e., $\mathcal{P} \lhd K$) and
2. for each witness $\mathcal{P}^*$ that $\mathcal{P}$ is A_0-sound, there exist
   (a) an iteration tree $\mathcal{T}$ on $\mathcal{P}^*$ of successor length $\theta + 1 \leq \Omega + 1$ such that $\nu(E^{\mathcal{T}}_\eta) \geq \alpha$ for all $\eta < \theta$,
   (b) $\mathcal{R} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ and
   (c) an elementary embedding $\pi : \mathcal{Q} \to \mathcal{R}$ with $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$.

Definition 2.30 does not have the features advertised before in that it is not first-order over $H(\Omega)$ and it is not a natural strengthening of iterability. But there is a satisfactory equivalent formulation that we get to somewhat later. However, the following connection between K and α-strong tells us that we are on the right track.

2.31 Theorem. *Let α be a cardinal of K and $\mathcal{Q}$ be a sound premouse that agrees with K below α. Assume that $\rho^{\mathcal{Q}}_\omega = \alpha$. Then*

$$\mathcal{Q} \lhd K \quad \Longleftrightarrow \quad \mathcal{Q} \textit{ is } \alpha\textit{-strong.}$$

The proof of this and the following closely related basic result are left as reasonable exercises for the reader.

2.32 Theorem. *Let α be a cardinal of K and $\mathcal{P} = \mathcal{J}^K_\alpha$. Suppose that $\mathcal{P}^*$ is a witness that $\mathcal{P}$ is A_0-sound. Let $\beta = (\alpha^+)^K$. Then*

(1) $\beta = (\alpha^+)^{\mathcal{P}^*}$,

(2) $\mathcal{P}^*$ and K agree below β and

(3) $\mathcal{P}^*$ is α-strong.

Theorem 2.31 tells us how to formulate a recursive definition of K in terms of α-strong mice. But the definition of α-strong involves quantification over weasels that witness A_0-soundness, hence over subsets of $H(\Omega)$, so we are no better off than we started in terms of complexity. The first-order formulation we have in mind involves a generalization of the notion of an iteration tree on a mouse to an iteration tree on a phalanx, which is defined below. Such iteration trees also generalize the *double-rooted* iteration trees that appear in the proofs of condensation, Theorem 5.1 of [41], and solidity, Theorem 5.3 of [41], the difference being that we allow an arbitrary number of roots. (These condensation and solidity theorems originally appeared in [20] where double-rooted iteration trees are called *pseudo-iteration trees*.)

2.33 Definition. Suppose that $\vec{\lambda} = \langle \lambda_\alpha \mid \alpha < \gamma \rangle$ is an increasing sequence of ordinals, and $\vec{\mathcal{Q}} = \langle \mathcal{Q}_\alpha \mid \alpha \leq \gamma \rangle$ is a sequence of mice. Then $(\vec{\mathcal{Q}}, \vec{\lambda})$ is a *phalanx of length* $\gamma + 1$ iff $\mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$ agree below λ_α whenever $\alpha < \beta \leq \gamma$.

As an example, observe that if $\mathcal{S}$ is an iteration tree of successor length, then

$$(\langle \mathcal{M}^{\mathcal{S}}_\alpha \mid \alpha < \mathrm{lh}(\mathcal{S}) \rangle, \langle \mathrm{lh}(E^{\mathcal{S}}_\alpha) \mid \alpha < \mathrm{lh}(\mathcal{S}) - 1 \rangle)$$

is a phalanx of length $\mathrm{lh}(\mathcal{S})$. Notice that in passing from $\mathcal{S}$ to this phalanx we retain the models and record the relevant amount of agreement between the models but we lose all information about how the models were created and the tree order. Of course, not every phalanx comes from an iteration tree in this way.

2.34 Definition. Let $(\vec{\mathcal{Q}}, \vec{\lambda})$ be a phalanx of length $\gamma + 1$ and $\theta \geq \gamma + 1$. An *iteration tree $\mathcal{T}$ of length θ* on $(\vec{\mathcal{Q}}, \vec{\lambda})$ consists of

- a tree structure $<_T$ on θ for which each ordinal $\leq \gamma$ is a root,
- the corresponding root operation $\mathrm{root}^T : \theta \to \gamma + 1$,
- the corresponding predecessor operation pred^T that maps successor ordinals in the interval $[\gamma + 1, \theta)$ to ordinals $\leq \theta$,
- premice $\mathcal{M}^{\mathcal{T}}_\eta$ for $\eta < \theta$,
- extenders $E^{\mathcal{T}}_\eta$ whenever $\gamma < \eta + 1 < \theta$,
- a set of successor ordinals $D^{\mathcal{T}} \subseteq [\gamma + 1, \theta)$,
- a commutative system of embeddings

$$i^{\mathcal{T}}_{\zeta,\eta} : \mathcal{M}^{\mathcal{T}}_\zeta \to \mathcal{M}^{\mathcal{T}}_\eta$$

indexed by $\zeta <_T \eta$ for which

$$(\zeta, \eta]_T \cap D^{\mathcal{T}} = \emptyset$$

and

- an operation $\deg^{\mathcal{T}} : [\gamma + 1, \theta) \to \omega + 1$

with the following properties.

- If $\alpha \leq \gamma$, then $\mathcal{M}_\alpha^{\mathcal{T}} = \mathcal{Q}_\alpha$ and $\lambda_\alpha^{\mathcal{T}} = \lambda_\alpha$.
- If $\gamma < \eta + 1 < \theta$, then $E_\eta^{\mathcal{T}}$ is an extender from the $\mathcal{M}_\eta^{\mathcal{T}}$ sequence, $\mathrm{pred}^T(\eta + 1)$ is the least $\zeta \leq \eta$ such that

  $$\mathrm{crit}(E_\eta^{\mathcal{T}}) < \lambda_\zeta^{\mathcal{T}},$$

  and

  $$\mathcal{M}_{\eta+1}^{\mathcal{T}} = \mathrm{Ult}(\mathcal{N}, E_\eta^{\mathcal{T}})$$

  where $\mathcal{N}$ is the greatest initial segment of $\mathcal{M}_{\mathrm{pred}^T(\eta+1)}^{\mathcal{T}}$ such that $E_\eta^{\mathcal{T}}$ is an extender over $\mathcal{N}$. And

  $$\eta + 1 \in D^{\mathcal{T}} \quad \Longleftrightarrow \quad \mathcal{N} \neq \mathcal{M}_{\mathrm{pred}^T(\eta+1)}^{\mathcal{T}}.$$

  The degree of this ultrapower is $\deg^{\mathcal{T}}(\eta + 1)$, and this degree equals the largest $n \leq \omega$ such that

  $$\rho_n^{\mathcal{N}} > \mathrm{crit}(E_\eta^{\mathcal{T}})$$

  If $\eta + 1 \notin D^{\mathcal{T}}$, then

  $$i_{\mathrm{pred}^T(\eta+1), \eta+1}^{\mathcal{T}} : \mathcal{M}_{\mathrm{pred}^T(\eta+1)}^{\mathcal{T}} \to \mathcal{M}_{\eta+1}^{\mathcal{T}}$$

  is the ultrapower embedding. And

  $$\lambda_\eta^{\mathcal{T}} = \mathrm{lh}(E_\eta^{\mathcal{T}}).$$

- If $\gamma < \eta < \theta$ and η is a limit ordinal, then

  $$[\mathrm{root}^T(\eta), \eta)_T$$

  is a cofinal branch of $T \restriction \eta$. Moreover,

  $$D^{\mathcal{T}} \cap [\mathrm{root}^T(\eta), \eta)_T$$

  is finite and $\mathcal{M}_\eta^{\mathcal{T}}$ is the direct limit of the models $\mathcal{M}_\zeta^{\mathcal{T}}$ under the embeddings

  $$i_{\iota,\zeta}^{\mathcal{T}} : \mathcal{M}_\iota^{\mathcal{T}} \to \mathcal{M}_\zeta^{\mathcal{T}}$$

  for $\iota, \zeta \in [\mathrm{root}^T(\eta), \eta)_T - \max(D^{\mathcal{T}})$ with $\iota <_T \zeta$. In addition,

  $$\deg^{\mathcal{T}}(\eta) = \liminf_{\zeta <_T \eta} \deg^{\mathcal{T}}(\zeta).$$

Just like with iteration trees on a single mouse, in the literature one sees the ultrapower embedding $\mathcal{N} \to \mathcal{M}^{\mathcal{T}}_{\eta+1}$ above denoted

$$i^*_{\eta+1} : \mathcal{M}^*_{\eta+1} \to \mathcal{M}_{\eta+1}.$$

Adding a superscript $\mathcal{T}$ leads to admittedly unattractive notation but we do not break with tradition.

We remark that in most cases of interest, the degree is non-increasing between drops in model so the lim inf ends up being the eventual value. The phrase *drop in degree* has the obvious meaning.

The notion of an iteration strategy generalizes in the obvious way to phalanxes. An iteration strategy picks cofinal branches at limit stages and is responsible for wellfoundedness in both successor and limit stages. When we speak of an iteration tree on an iterable phalanx, the reader should assume that the iteration tree is compatible with a fixed iteration strategy on the phalanx.

The following theorem is the key step towards a recursive definition of α-strong. We write $<\beta$-strong to mean α-strong for all $\alpha < \beta$.

2.35 Theorem. *Suppose that α is a cardinal of K and $\mathcal{Q}$ is a premouse of height $\leq \Omega$ that agrees with K below α. Then the following are equivalent.*

(1) $\mathcal{Q}$ is α-strong.

(2) For all $<\alpha$-strong premice $\mathcal{P}$,

$$(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \alpha \rangle)$$

is an $\Omega + 1$ iterable phalanx.

Proof. That (2) implies (1) is an immediate consequence of Theorem 2.32(3) and the following result.

2.36 Lemma. *Suppose that α is a cardinal of K and $\mathcal{P} = \mathcal{J}^K_\alpha$. Let $\mathcal{P}^*$ be a witness that $\mathcal{P}$ is A_0-sound and $\mathcal{Q}$ be a premouse of height $\leq \Omega$. Suppose that $(\langle \mathcal{P}^*, \mathcal{Q} \rangle, \langle \alpha \rangle)$ is an $\Omega + 1$ iterable phalanx. Then clause 2 of Definition 2.30 holds for $\mathcal{Q}$ and $\mathcal{P}^*$.*

Sketch. Let $(\mathcal{S}, \mathcal{T})$ be the coiteration of the pair

$$((\langle \mathcal{P}^*, \mathcal{Q} \rangle, \langle \alpha \rangle), \mathcal{P}^*).$$

We have not discussed this sort of coiteration before but it is defined in the natural way, using comparison of extender sequences to decide which extenders to apply at successor stages. The proof of the comparison, Theorem 3.11 of [41], generalizes to show that this coiteration is successful, which means that either $\mathcal{M}^{\mathcal{S}}_{1+\eta} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\theta$ or vice-versa where $1 + \eta + 1 = \mathrm{lh}(\mathcal{S})$ and $\theta + 1 = \mathrm{lh}(\mathcal{T})$. And that $1 + \eta$, $\theta \leq \Omega$.

2.37 Claim. $\mathrm{root}^{\mathcal{S}}(1+\eta) = 1$.

Sketch. For contradiction, suppose that $\mathrm{root}^{\mathcal{S}}(1+\eta) = 0$. As in the proof of universality Theorem 2.8, the fact that $\mathcal{P}^*$ computes κ^+ correctly for stationary many $\kappa < \Omega$ can be used to see that

$$\mathcal{M}^{\mathcal{S}}_{1+\eta} = \mathcal{M}^{\mathcal{T}}_{\theta}$$

and there is no dropping along $[0, 1+\eta]_S$ and $[0,\theta]_T$. We have the embeddings

$$i^{\mathcal{S}}_{0,1+\eta} : \mathcal{P}^* \to \mathcal{M}^{\mathcal{S}}_{1+\eta}$$

and

$$i^{\mathcal{T}}_{0,\theta} : \mathcal{P}^* \to \mathcal{M}^{\mathcal{T}}_{1+\eta}$$

with

$$\mathrm{crit}(i^{\mathcal{S}}_{0,1+\eta}) < \alpha.$$

Theorems 2.13 and 2.16 generalize to iteration trees on phalanxes. Thus using the fact that $\mathcal{P}^*$ has the definability and hull properties at all ordinals $< \alpha$, we see that $i^{\mathcal{S}}_{0,1+\eta}$ and $i^{\mathcal{T}}_{0,\theta}$ have the same critical point and move subsets of their critical point the same way. In other words, the first extenders used along $[0, 1+\eta]_S$ and $[0,\theta]_T$ agree, which leads to a contradiction as in the proof of comparison, Theorem 3.11 of [41]. ⊣

Again as in the proof of Theorem 2.8,

$$\mathcal{M}^{\mathcal{S}}_{1+\eta} \trianglelefteq \mathcal{M}^{\mathcal{T}}_{\theta}$$

and there is no dropping along $[1, 1+\eta]_S$. So we have the embedding

$$i^{\mathcal{S}}_{1,1+\eta} : \mathcal{Q} \to \mathcal{M}^{\mathcal{S}}_{1+\eta}$$

with $\mathrm{crit}(i^{\mathcal{S}}_{1,1+\eta}) \geq \alpha$. Since $\mathcal{Q}$ and $\mathcal{P}^*$ agree below α, $\mathrm{lh}(E^{\mathcal{T}}_{\zeta}) \geq \alpha$ for all $\zeta < \theta$. Since α is a cardinal of K, it is a cardinal of $\mathcal{P}^*$. This can be used to see that $\nu(E^{\mathcal{T}}_{\zeta}) \geq \alpha$ for all $\zeta < \theta$. Thus $i^{\mathcal{S}}_{1,1+\eta}$ and $\mathcal{T}$ witness that $\mathcal{Q}$ is α-strong relative to $\mathcal{P}^*$ as desired. ⊣

We have seen that (2) implies (1). Let β be a cardinal of K. We show that (1) for β implies (2) for β. Suppose that $\mathcal{Q}$ is a β-strong premouse and $\mathcal{P}$ is a $<\beta$-strong premouse. We must show that $(\langle \mathcal{P}, \mathcal{Q}\rangle, \langle \beta \rangle)$ is $\Omega + 1$ iterable. By the proof of Theorem 2.29, there exist a witness W that $\mathcal{J}^K_\beta$ is A_0-sound and an elementary embedding

$$\sigma : W \to K^c.$$

We do not have a lower bound on the critical point of σ, nor is it relevant. By Definition 2.30, for each $\alpha \leq \beta$, we have an iteration tree $\mathcal{T}_\alpha$ of length $\theta_\alpha + 1 \leq \Omega + 1$ on W such that $\nu(F^{\mathcal{T}_\alpha}_\eta) \geq \alpha$ for all $\eta < \theta_\alpha$,

$$\mathcal{R}_\alpha \trianglelefteq \mathcal{M}^{\mathcal{T}_\alpha}_{\theta_\alpha}$$

and an elementary embedding π_α with $\pi_\alpha \restriction \alpha = \mathrm{id} \restriction \alpha$. If $\alpha < \beta$, then

$$\pi_\alpha : \mathcal{P} \to \mathcal{R}_\alpha$$

whereas

$$\pi_\beta : \mathcal{Q} \to \mathcal{R}_\beta.$$

Use σ to copy each $\mathcal{T}_\alpha$ to an iteration tree $\sigma\mathcal{T}_\alpha$ on K^c and let

$$\tau_\alpha : \mathcal{R}_\alpha \to \mathcal{S}_\alpha$$

be the restriction of the final copying map to $\mathcal{R}_\alpha$. Then

$$(\tau_\alpha \circ \pi_\alpha) \restriction \alpha = \tau_\alpha \restriction \alpha = \sigma \restriction \alpha$$

for all $\alpha \leq \beta$.

We wish to construe

$$(\langle \mathcal{S}_\alpha \mid \alpha \leq \beta \rangle, \langle \sigma(\alpha) \mid \alpha < \beta \rangle)$$

as a phalanx. Formally, for this we let $\langle \alpha_\eta \mid \eta \leq \theta \rangle$ enumerate the cardinals of K up to and including β and set

$$\mathfrak{F} = (\langle \mathcal{S}_{\alpha_\eta} \mid \eta \leq \theta \rangle, \langle \sigma(\alpha_\eta) \mid \eta < \theta \rangle).$$

Then $\mathfrak{F}$ is a phalanx. There are two basic elements to the remainder of the proof. Notice that all the models of $\mathfrak{F}$ are obtained by iterating K^c. We call such phalanxes K^c *based*. Steel proved that all K^c based phalanxes are $\Omega + 1$ iterable. The reader is referred to [42, §6] for the proof, which builds on Steel's proof that K^c is $(\omega, \Omega + 1)$ iterable. The second idea is that the sequence of embeddings

$$\psi_\eta = \tau_{\alpha_\eta} \circ \pi_{\alpha_\eta}$$

for $\eta \leq \theta$ can be used to pull back an iteration strategy on $\mathfrak{F}$ to an iteration strategy on $(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \beta \rangle)$. For this we use a generalization of the copying construction in Sect. 4.1 of [41]. The generalization is routine except for a few technical details. The main wrinkle comes in the case $\beta = (\alpha^+)^K$ when we apply the shift lemma to an ultrapower of $\mathcal{P}$ by an extender with critical point α. The difficulty is that $\psi_{\theta-1} = \tau_\alpha \circ \pi_\alpha$ and $\psi_\theta = \tau_\beta \circ \pi_\beta$ agree to α in this case whereas agreement to β would be needed to quote Lemma 4.2 of [41]. Nevertheless, a version of the shift lemma still goes through. We refer the reader to [42, pp. 49–50] for the details. This type of copying construction is used repeatedly in the proof of the weak covering property in Sect. 4. $\dashv$

We are about to arrive at the much promised definition of K that is first-order over $H(\Omega)$. Clause b of Theorem 2.35 quantifies over weasels so there is still something to do.

2.38 Definition. If $\mathcal{T}$ is an iteration tree of length θ, then $\mathcal{T}$ is called *bad* if it is a losing position for player II in the iteration game. In other words,

1. if $\theta = \eta + 1$, then there is an extender F on the $\mathcal{M}_\eta^\mathcal{T}$ sequence such that $\mathrm{lh}(F) > \mathrm{lh}(E_\zeta^\mathcal{T})$ for all $\zeta < \eta$ but if $\zeta \leq \eta$ is least such that $\mathrm{crit}(F) \geq \nu(E_\zeta^\mathcal{T})$ and $\mathcal{N}$ is the greatest initial segment of $\mathcal{M}_\zeta^\mathcal{T}$ over which F is an extender, then $\mathrm{Ult}(\mathcal{N}, F)$ is illfounded where the degree of the ultrapower is as large as possible, and

2. if θ is a limit ordinal, then all cofinal branches of $\mathcal{T}$ have infinitely many drops in model or are illfounded.

Because of our Technical Hypothesis, $\Omega + 1$ iterability is equivalent to Ω iterability. In light of our Anti-Large Cardinal Hypothesis, there are many cases in which $\Omega + 1$ iterability reduces further to the non-existence of a countable bad tree. For example, the proof of Theorem 2.3 can be extended to show that if a premouse $\mathcal{P}$ of height Ω is not $\Omega + 1$ iterable, then there is a countable bad tree on $\mathcal{P}$. We give another useful example.

2.39 Definition. A premouse $\mathcal{P}$ is defined to be *properly small* iff $\mathcal{P}$ has no Woodin cardinals and $\mathcal{P}$ has a largest cardinal.

Notice that if $\mathcal{P}$ is a weasel and $\mu < \Omega$, then $\mathcal{J}^\mathcal{P}_{(\mu^+)^\mathcal{P}}$ is properly small. It is also easy to see that the properly small levels of K that project to α are unbounded in $(\alpha^+)^K$. If each premouse of a phalanx is properly small, then the $\Omega + 1$ iterability of the phalanx reduces to the non-existence of a countable bad tree on the phalanx. Arguing along these lines we obtain the following characterization.

2.40 Theorem. *Suppose that α is a cardinal of K and $\mathcal{Q}$ is a properly small premouse of height $< \Omega$ that agrees with K below α. Then the following are equivalent.*

(1) $\mathcal{Q}$ is not α-strong.

(2) There is a properly small $<\alpha$-strong premouse $\mathcal{P}$ with the same cardinality as $\mathcal{Q}$ and a countable bad iteration tree on the phalanx

$$(\langle \mathcal{P}, \mathcal{Q} \rangle, \langle \alpha \rangle).$$

Sketch. Suppose that $\mathcal{Q}$ is not α-strong. Then there exist a weasel $\mathcal{P}^*$ that witnesses $\mathcal{J}^K_\alpha$ is A_0-sound and a bad iteration tree on

$$(\langle \mathcal{P}^*, \mathcal{Q} \rangle, \langle \alpha \rangle).$$

The same bad iteration tree can be construed as a bad iteration tree, call it $\mathcal{U}$, on

$$(\langle \mathcal{P}^{**}, \mathcal{Q} \rangle, \langle \alpha \rangle)$$

for some properly small $\mathcal{P}^{**} \lhd \mathcal{P}^*$. Let $Y \prec H(\Omega)$ with $\mathcal{U} \in Y$ such that Y has the same cardinality as $\mathcal{Q}$. Let $\tau : N \simeq Y$ with N transitive. Then

$\tau^{-1}(\mathcal{Q}) = \mathcal{Q}$. Let $\mathcal{P} = \tau^{-1}(\mathcal{P}^{**})$. Then $\mathcal{P}$ is $<\alpha$-strong, properly small and has the same cardinality as $\mathcal{Q}$. Let $X \prec N$ with X countable and $\tau^{-1}(\mathcal{U}) \in X$. Let $\sigma : M \simeq X$ with M transitive. Let $\mathcal{S} = (\tau \circ \sigma)^{-1}(\mathcal{U})$. An absoluteness argument like that used in the proof of Theorem 2.3 shows that $\mathcal{S}$ is bad. (Here is where the hypothesis that $\mathcal{Q}$ is properly small is used.) Let $\mathcal{T} = \sigma\mathcal{S}$. Then $\mathcal{T}$ is a countable bad iteration tree on $(\langle\mathcal{P}, \mathcal{Q}\rangle, \langle\alpha\rangle)$. ⊣

Finally, we reach Steel's recursive definition of K.

2.41 Theorem. *Let $\mathcal{M}$ be a premouse of height $< \Omega$. Then $\mathcal{M} \lhd K$ iff there exist $\theta < \Omega$, an increasing continuous sequence of ordinals*

$$\langle \alpha_\eta \mid \eta \leq \theta + 1\rangle$$

starting with $\alpha_0 = \omega$, an $\lhd$ increasing continuous sequence of premice

$$\langle \mathcal{R}_\eta \mid \eta \leq \theta + 1\rangle$$

with $\mathcal{M} \lhd \mathcal{R}_{\theta+1}$ and a double-indexed sequence of sets

$$\langle \mathcal{F}_{\zeta,\eta} \mid \zeta \leq \eta \leq \theta\rangle$$

that satisfy the following conditions.

(1) For all $\zeta \leq \eta \leq \theta$ and $\mathcal{Q}$,

$$\mathcal{Q} \in \mathcal{F}_{\zeta,\eta}$$

iff $\mathcal{Q}$ is a properly small premouse of cardinality $|\alpha_\eta|$ such that

$$\mathcal{R}_\zeta \lhd \mathcal{Q}$$

and, if

$$\mathcal{P} \in \bigcap\{\mathcal{F}_{\iota,\eta} \mid \iota < \zeta\},$$

then

$$(\langle\mathcal{P}, \mathcal{Q}\rangle, \langle\alpha_\zeta\rangle)$$

is a phalanx on which there is no countable bad iteration tree.

(2) For all $\eta \leq \theta$,

$$\{\mathcal{Q} \in \mathcal{F}_{\eta,\eta} \mid \mathcal{Q} \textit{ is sound and } \omega\rho_\omega^{\mathcal{Q}} = \alpha_\eta\}$$

is a family of premice that are pairwise comparable under $\trianglelefteq$. Moreover, the union of this family is $\mathcal{R}_{\eta+1}$, which is a premouse of height $\alpha_{\eta+1}$.

Sketch. The idea is that, for all $\zeta \leq \eta < \Omega$,

$$\alpha_\eta = (\aleph_\eta)^K$$

$$\mathcal{R}_\eta = \mathcal{J}^K_{(\aleph_\eta)^K}$$

and $\mathcal{F}_{\zeta,\eta}$ is the set of properly small $(\aleph_\zeta)^K$-strong premice of size $|(\aleph_\eta)^K|$. And θ is large enough so that

$$\mathcal{M} \lhd \mathcal{J}^K_{(\aleph_{\theta+1})^K}.$$

⊣

3. Core Model Tools

Throughout this section, we continue to assume the Anti-Large Cardinal Hypothesis,

there is no proper class model with a Woodin cardinal

and the Technical Hypothesis,

U is a normal measure over Ω.

Under these hypotheses, in the previous section, a certain transitive model of ZFC of ordinal height Ω is defined and named K. Here, we list properties of K that are useful in applications. For the most part, it is not necessary to read the previous section to make sense of these properties.

3.1. Covering Properties

Jensen showed that if $0^{\#}$ does not exist and A is an uncountable set of ordinals, then there exists a set $B \in L$ such that $A \subseteq B$ and $|A| = |B|$. Dodd and Jensen proved the same theorem for K under the hypothesis that there is no inner model with a measurable cardinal. If there is a measurable cardinal, then the Jensen covering property for K fails in any Prikry forcing extension. Mitchell proved that if there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, then K still satisfies several consequences of the Jensen covering property and that these weak covering properties are still useful in applications. Mitchell's work in this regard and the history behind it is the subject of the Handbook chapter [18].

The first result we list in this subsection, which is due to Steel, says that K computes the successor of almost every cardinal correctly.

3.1 Theorem.

$$\{\kappa < \Omega \mid (\kappa^+)^K = \kappa^+\} \in U.$$

The reader should cite Theorem 5.18(2) of [42] when applying Theorem 3.1. We mentioned this result in Sect. 2 just after the proof of Theorem 2.29.

Many people would identify the following result, which is due to Mitchell and the author, as the weak covering theorem for K. It implies that K computes successors of singular cardinals correctly but contains other applicable information.

3.2 Theorem. *Let κ be a cardinal of K such that*

$$\omega_2 \leq \kappa < \Omega.$$

and

$$\lambda = (\kappa^+)^K.$$

Then

$$\mathrm{cf}(\lambda) \geq |\kappa|.$$

Thus either $\lambda = |\kappa|^+$ or $\mathrm{cf}(\lambda) = |\kappa|$.

The reader should cite Theorem 0.1 of [19] when applying Theorem 3.2. The proof builds on that of Theorem 1.1 [21], which is the special case in which $|\kappa|$ is a countably closed cardinal. We outline the proof under this and further simplifying assumptions in Sect. 4.

The next result, which is due to Steel and the author, says that K computes successors of weakly compact cardinals correctly. The corresponding fact for L under the assumption that $0^{\#}$ does not exist was observed by Kunen in the 1970s.

3.3 Theorem. *Let κ be a weakly compact cardinal such that $\kappa < \Omega$. Then*

$$(\kappa^+)^K = \kappa^+.$$

The reader should cite Theorem 3.1 of [30] when applying Theorem 3.3.

3.2. Absoluteness, Complexity and Correctness

Steel proved the following theorem, which says that K is forcing absolute.

3.4 Theorem. *Let $\mathbb{P} \in H(\Omega)$ be a poset. Then*

$$\Vdash_{\mathbb{P}} K = K^V.$$

The reader should cite Theorem 5.18(3) of [42] when applying Theorem 3.4. We mentioned this result in Sect. 2 just after the proof of Theorem 2.29.

Using his first-order definition of K, Steel carried out the first part of the following computation of $K \cap \mathrm{HC}$. Think of this as the set of reals that code the countable levels of K, countable in V that is. The second part, a computation done by Schindler, shows that the complexity drops if only finitely many countable ordinals are strong cardinals in K.

3.5 Theorem. *There is a Σ_1 formula $\varphi(x)$ such that for all $a \in \mathrm{HC}$,*

$$a \in K \iff L_{\omega_1}(\mathbb{R}) \models \varphi[a].$$

Moreover, if $K \cap \mathrm{HC}$ has at most finitely many strong cardinals, then there is a formula $\psi(x)$ such that for all $a \in \mathrm{HC}$,

$$a \in K \iff \mathrm{HC} \models \psi[a].$$

The reader should cite Theorem 6.15 of [42] when applying the first part of Theorem 3.5. We mentioned this result in Sect. 2; it is a corollary to Theorem 2.41. The reader should cite Theorems 3.4 and 3.6 of [11] when applying the *moreover* part of Theorem 3.5.

Steel defined the levels of K by recursion on their ordinal height $< \Omega$. It turns out that iterability alone is not enough to guarantee that a mouse with all the right first-order properties to be a level of K is actually a level of K. So, simultaneous with his recursive definition of the levels of K, Steel defined increasingly strong forms of iterability. This is explained in detail in Sect. 2.2. The following theorem of Schindler shows that there is a tremendous simplification in the recursive definition for levels of K of height $\geq \aleph_2$.

3.6 Theorem. *Let κ be a cardinal of K such that $\aleph_2 \leq \kappa < \Omega$. Suppose that $\mathcal{M}$ is a mouse such that*

(1) $\mathcal{M}$ and K agree below κ,

(2) $\rho_\omega^{\mathcal{M}} \leq \kappa$ and

(3) $\mathcal{M}$ is sound above κ.

Then $\mathcal{M}$ is an initial segment of K.

One says that above $\aleph_2$, K is obtained by stacking mice. The reader should cite Lemma 3.5 of [10] when using Theorem 3.6 and should consult Lemma 2.2 of [34] as well. The proof of Theorem 3.6 builds on the proof of Theorem 3.2.

By definition, a class M is Σ^1_n correct iff $M \prec_{\Sigma^1_n} V$. In other words, for each Σ^1_n formula $\psi(x)$ and $a \in \mathbb{R} \cap M$,

$$\psi[a]^M \iff \psi[a].$$

Jensen proved that if $x^\#$ exists for all $x \subseteq \omega$ but there is no inner model with a measurable cardinal, then K is Σ^1_3 correct. The following result is due to Steel.

3.7 Theorem. *Suppose that there exists a measurable cardinal $< \Omega$. Then K is Σ^1_3 correct.*

The reader should cite Theorem 7.9 of [42] when applying Theorem 3.7. It is not known if the existence of a measurable cardinal $< \Omega$ is needed. There is also an attractive conjecture regarding Σ^1_4 correctness that has been open for about a decade.[8]

3.3. Embeddings of K

The first result in this subsection, which is due to Steel, says that K is rigid.

3.8 Theorem. *If $j : K \to K$ is an elementary embedding, then j is the identity.*

The reader should cite Theorem 8.8 of [42] when applying Theorem 3.8.

Steel proved the following result, which says that K is universal.

3.9 Theorem. *K is the unique universal weasel that elementarily embeds into all other universal weasels.*

The reader should cite Theorem 8.10 of [42] when applying Theorem 3.9. Universal weasels were defined in Sect. 2. See Definitions 2.1 and 2.7.

Now we turn to external embeddings and their actions on K. The question is whether the restriction to K of an embedding from an iteration of V is the embedding from an iteration of K.

[8] Assume that $M_1(x)$ exists for all sets x but that there is no model with two Woodin cardinals. Show that K is Σ^1_4 correct.

3.10 Theorem. *Suppose that $\mathcal{T}$ is an iteration tree on V with final model N and branch embedding*

$$\pi : V \to N.$$

Assume that

(1) $\mathcal{T}$ is finite and ${}^{\omega}N \subseteq N$, or

(2) $\mathcal{T}$ is countable and ρ-maximal in the sense of Neeman [23].

Then there is an iteration tree on K whose last model is K^N and whose branch embedding is $\pi \restriction K$.

Keep in mind that even if the external iteration tree $\mathcal{T}$ consists of a single ultrapower by a normal measure, the corresponding iteration tree on K may be infinite and quite complicated. Schindler proved Theorem 3.10 under assumption (1). The author observed that Schindler's proof goes through with assumption (2). The reader should cite Corollary 3.1 of [36] in case (1) and Corollary 3.2 [36] in case (2) when applying Theorem 3.10.

3.4. Maximality

Steel proved that K is maximal in the following sense.

3.11 Theorem. *Let F be an extender that coheres with the extender sequence of K. Suppose that (K, F) is countably certified. Then F is on the extender sequence of K.*

The reader should cite Theorem 8.6 of [42] when applying Theorem 3.11. This can be used to see that certain large cardinals reflect to K. For example, if $\kappa < \Omega$ and κ is a λ-strong cardinal for all $\lambda < \Omega$, then κ has the same property in K. The proof of a theorem slightly more general than Theorem 8.6 of [42], applications of maximality and other results along these lines by Steel and the author can be found in [30]. For example, Theorem 3.4 of [30] says that if κ is a cardinal such that $\aleph_2 \leq \kappa < \Omega$, then $H(\kappa) \cap K$ is universal for mice in $H(\kappa)$.

3.5. Combinatorial Principles

Jensen's results with the fine structure of L generalize to models of the form $L[E]$.

3.12 Theorem. *Let $\mathcal{Q}$ be a weasel. Then $\mathcal{Q}$ satisfies the following statements.*

(1) If κ is a cardinal, then $\diamondsuit^+_{\kappa^+}$ holds.

(2) If κ is an inaccessible cardinal, then

$$\diamondsuit^+_{\kappa} \text{ holds} \iff \kappa \text{ is not ineffable.}$$

(3) If κ is a cardinal, then

$$\square_\kappa \text{ holds} \iff \kappa \text{ is not subcompact.}$$

(4) If κ is a regular cardinal, then there is a κ^+ morass.

When applying Theorem 3.12, the reader should cite Theorem 1.2 of [27] for the clauses on diamond, which are due to the author. The reader should cite Theorem 2 of [31] for the existence of a $\square_\kappa$-sequence. (It is a theorem of ZFC due to Burke [4] that if κ is a subcompact cardinal, then $\square_\kappa$ fails.) Zeman and the author [32] proved the clause on morass.

Even though $\mathcal{Q} = K$ is its most interesting instance, Theorem 3.12 holds in situations in which we do not know how to define K. Neither the Anti-Large Cardinal Hypothesis nor the Technical Hypothesis is used in the proof of Theorem 3.12. This explains why we bothered to mention subcompact cardinals in the clause on square since subcompact cardinals are themselves Woodin cardinals, which do not exist under our Anti-Large Cardinal Hypothesis. We should add that only a weak form of iterability is needed for the proof of Theorem 3.12, much less than is assumed in the definition of weasel.

The next result gives conditions under which the $\square_\kappa$ sequence in K cannot be threaded in V. It is a result of the author.

3.13 Theorem. *Let κ be a cardinal such that*

$$\aleph_2 \leq \kappa < \Omega.$$

Suppose that κ is a limit cardinal of K. Let $\lambda = (\kappa^+)^K$. Then there exists a

$$\langle C_\alpha \mid \alpha < \lambda \rangle \in K$$

such that $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a $\square_\kappa$ sequence in K and $\langle C_\alpha \mid \alpha < \lambda \rangle$ has no thread. That is, there is no club D in λ such that

$$D \cap \alpha = C_\alpha$$

for all $\alpha \in \lim(D)$.

The reader should cite [29] when applying Theorem 3.13.

3.6. On the Technical Hypothesis

Schindler proved that below "zero hand grenade", the Technical Hypothesis can be avoided:

3.14 Theorem. *If there is no proper class model with a proper class of strong cardinals, then the Technical Hypothesis is not needed for the results in this chapter.*

The reader should consult [33] before applying Theorem 3.14. Not only is Theorem 3.14 loosely worded, it does not make sense, at least not literally, since there are results in this paper that explicitly refer to the normal measure U over Ω. For these, Ω should be replaced by On and statements about sets in U should be read as statements about stationary classes of ordinals.

Without going to a more restrictive anti-large cardinal hypothesis, it is not known how to get away without a technical hypothesis. But technical hypotheses weaker than a measurable cardinal are known to suffice. For example, Steel showed that the existence of $X^{\#}$ for all $X \in H(\Omega)$ is enough. Also, Steel and the author showed in Theorem 5.1 of [42] that an ω-Erdős cardinal is enough.

4. Proof of Weak Covering

In this section, we discuss elements of the proof of Theorem 3.2 under some simplifying assumptions. Earlier versions of this theorem due to Jensen, Dodd and Jensen, and Mitchell had no technical hypothesis and much stronger anti-large cardinal hypotheses. In particular, their proofs involved linear iterations at most whereas we deal with iteration trees and even some generalizations of iteration trees. To make our task manageable we assume that the reader is familiar with at least one of these earlier proofs, such as any proof in the Handbook chapter [18] or just the proof for L as presented in [28] or in the Handbook chapter [37]. Our emphasis here is on the new complications and how to overcome them, really just a segue into [21] for the reader.

4.1 Definition. A cardinal κ is *countably closed* iff $\mu^{\aleph_0} < \kappa$ for all cardinals $\mu < \kappa$.

For example, if $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega$ is countably closed. The following special case of Theorem 3.2 was proved in [21]. We continue to assume the same anti-large cardinal hypothesis and technical hypothesis as in all earlier sections, so that K exists.

4.2 Theorem. *Let κ be a cardinal of K such that $|\kappa|$ is countably closed and $\lambda = (\kappa^+)^K$. Then*

$$\mathrm{cf}(\lambda) \geq |\kappa|.$$

Thus either $\lambda = |\kappa|^+$ or $\mathrm{cf}(\lambda) = |\kappa|$.

For example, if $2^{\aleph_0} < \aleph_\omega$ and $\kappa = \aleph_\omega$, then $(\kappa^+)^K = \aleph_{\omega+1}$.

Outline. The proof begins pretty much as do earlier proofs of weak covering under stronger anti-large cardinal hypotheses. Let $\lambda = (\kappa^+)^K$ and assume for contradiction that

$$\mathrm{cf}(\lambda) < |\kappa|.$$

By taking the union of an elementary chain of length ω_1, we find

$$X \prec (V_{\Omega+1}, \in, U)$$

with

$$\sup(X \cap \lambda) = \lambda$$

and

$$^{\omega}X \subseteq X$$

such that

$$|X| < \mathrm{cf}(\lambda)^{\aleph_0} < |\kappa|.$$

Let $\pi : N \simeq X$ with N transitive and $\delta = \mathrm{crit}(\pi)$. Note that $\pi(\delta) \leq \kappa$. Let

$$\overline{\kappa} = \pi^{-1}(\kappa)$$

$$\overline{\lambda} = \pi^{-1}(\lambda)$$

and

$$\overline{\Omega} = \pi^{-1}(\Omega).$$

Consider an arbitrary $\mu \leq \Omega$. Let $E_\pi \restriction \mu$ be the extender of length μ derived from π. This means the following. For each $a \in [\mu]^{<\omega}$, let

$$\delta_a = \min(\{\gamma \in \overline{\Omega} \cap N \mid a \in [\pi(\gamma)]^{<\omega}\}).$$

Then let

$$(E_\pi)_a = \{X \subseteq [\delta_a]^{|a|} \mid a \in \pi(X)\}.$$

Notice that $(E_\pi)_a$ is an ultrafilter over

$$\wp([\delta_a]^{|a|}) \cap N.$$

And

$$E_\pi \restriction \mu = \{(a, X) \mid a \in [\mu]^{<\omega} \text{ and } X \in (E_\pi)_a\}.$$

The point of this extender is that if M is a transitive model and

$$\wp(\delta_a) \cap M \subseteq N$$

for all $a \in [\mu]^{<\omega}$, then it makes sense to talk about the ultrapower map

$$i_E^M : M \to \mathrm{Ult}(M, F)$$

where

$$F = E_\pi \cap ([\mu]^{<\omega} \times M).$$

Put another way, we may apply F to M iff

$$\wp(\gamma) \cap M \subseteq N$$

for all γ such that $\pi(\gamma) \geq \mu$. Define

$$\mathrm{Ult}(M, \pi, \mu) = \mathrm{Ult}(M, E_\pi \restriction \mu).$$

Here are a few more general remarks. If $a \in [\delta]^{<\omega}$, then $(E_\pi)_a$ is principal and therefore $E_\pi \restriction \delta$ is trivial in the sense that it gives rise to the identity embedding. Observe that $(E_\pi)_{\{\delta\}}$ is equivalent to the normal measure derived from π,

$$\{X \subseteq \delta \mid \delta \in \pi(X)\},$$

in the sense that they determine the same ultrapower. We call

$$E_\pi \restriction \pi(\delta)$$

the superstrong extender derived from π. And we call $E_\pi \restriction \mu$ a *long extender* whenever $\pi(\delta) < \mu$ or, equivalently, whenever $\delta_a > \delta$ for some $a \in [\mu]^{<\omega}$. Long extenders come up in the covering theorem for L in exactly the same way although the terminology had not been established when Jensen discovered the proof. The reader may refer to [28] for an account of Jensen's proof in these terms.

Instead of K we work with an A_0-soundness witness for a large enough initial segment of K. Large enough for us means height Ω_0 where

$$\Omega > \Omega_0 \geq |\lambda|^+ = |\kappa|^+.$$

But for convenience we assume that Ω_0 is an inaccessible cardinal. Let W be the witness that $\mathcal{J}^K_{\Omega_0}$ is A_0-sound that comes out of the proof of Theorem 2.24. There is an elementary embedding $\sigma : W \to K^c$ that is relevant later in the current proof. Let

$$\overline{W} = \pi^{-1}(W)$$

and $(\overline{\mathcal{T}}, \mathcal{T})$ be the coiteration of $(\overline{W}, W)$. Say

$$\theta + 1 = \mathrm{lh}(\mathcal{T})$$

and

$$\overline{\theta} + 1 = \mathrm{lh}(\overline{\mathcal{T}}).$$

Simplify the notation by setting

$$W_\eta = \mathcal{M}^{\mathcal{T}}_\eta$$

for $\eta \leq \theta$ and

$$\overline{W}_\eta = \mathcal{M}^{\overline{\mathcal{T}}}_\eta$$

for $\eta \leq \overline{\theta}$. Because W is universal,

$$D^{\overline{\mathcal{T}}} \cap [0, \overline{\theta}]_{\overline{\mathcal{T}}} = \emptyset$$

and

$$\overline{W}_{\overline{\theta}} \trianglelefteq W_\theta.$$

Nothing we have said so far differs significantly from earlier proofs of weak covering under stronger anti-large cardinal hypotheses except possibly that we are using W instead of K. Before continuing, let us review the main points of the earlier proofs and compare and contrast them with the current proof. In the earlier proofs, it is shown that $\overline{K} = \pi^{-1}(K)$ does not move in its coiteration with K. The current proof shows this too but in an indirect way.[9] Now let $\eta \leq \theta$ be least such that

$$\nu(E^{\mathcal{T}}_{\eta}) > \overline{\kappa}$$

if there is such an η; otherwise let $\eta = 0$. In the earlier proofs, it is shown that there exist $\mathcal{P} \trianglelefteq W_\eta$ and $n < \omega$ such that

$$\rho^{\mathcal{P}}_n \geq \overline{\lambda} = (\overline{\kappa}^+)^{\mathcal{P}}$$

and

$$\mathcal{P} = \mathrm{Hull}^{\mathcal{P}}_{n+1}(\overline{\kappa} \cup p^{\mathcal{P}}_{n+1}).$$

People refer to $\mathcal{P}$ as the *least mouse missing from N at $\overline{\kappa}$*. The current proof is different in that W_η might be a weasel and

$$(\overline{\kappa}^+)^{W_\eta} = \overline{\lambda}.$$

In this case, we set $\mathcal{P} = W_\eta$. Then $\mathcal{P}$ is a thick weasel. Moreover, because $\nu(E^{\mathcal{T}}_{\zeta}) \leq \overline{\kappa}$ for all $\zeta < \eta$, we conclude that $\mathcal{P}$ has the hull and definability properties at μ whenever $\overline{\kappa} \leq \mu < \Omega_0$. This collection of facts about $\mathcal{P}$ turns out to be an adequate substitute if $\mathcal{P}$ happens to be a weasel instead of a premouse of height $< \Omega$. Moving on with our discussion, in the earlier proofs, $E_\pi \restriction \lambda$ is an extender over $\mathcal{P}$ and one sets

$$\mathcal{R} = \mathrm{Ult}(\mathcal{P}, \pi, \lambda).$$

People refer to $\mathcal{R}$ as the *lift up of $\mathcal{P}$*. In the current proof, because iteration trees need not be linear, something along the lines of $\overline{W}$ not moving is needed just to make sense of the definition of $\mathcal{R}$. In the earlier proofs, a standard argument using the fact that ${}^\omega X \subseteq X$ shows that $\mathcal{R}$ is an iterable premouse. The current proof is different on this point. For although $\mathcal{R}$ is wellfounded, it can fail to be a premouse! This happens exactly when $\rho_1(\mathcal{P}) \leq \overline{\kappa}$ (that is, $n = 0$), $\mathcal{P}$ has a top extender with critical point $\mu < \overline{\kappa}$ and π is discontinuous at $(\mu^+)^{\mathcal{P}}$. For then the top extender of $\mathcal{R}$ is not total on $\mathcal{R}$ since its critical point is $\pi(\mu)$ but it only measures sets in

$$J^{\mathcal{R}}_{\sup(\pi``(\mu^+)^{\mathcal{P}})} \triangleleft J^{\mathcal{R}}_{(\pi(\mu)^+)^{\mathcal{R}}}.$$

We call $\mathcal{R}$ a *protomouse* and its top predicate $F^{\mathcal{R}}$ an *extender fragment.* Vaguely put, our answer to the possibility that $\mathcal{R}$ is not a premouse is to find an actual premouse that corresponds to $\mathcal{R}$. But let us set aside this

[9] The current proof is a complicated induction that shows no extender of length $< \overline{\Omega}_0$ is used on $\overline{\mathcal{T}}$.

complication until later and assume that $\mathcal{R}$ is a premouse. In the discussion so far, we have implicitly used some basic facts about the ultrapower embedding $\widetilde{\pi}:\mathcal{P}\to\mathcal{R}$, mainly that
$$\widetilde{\pi}\restriction\overline{\lambda}=\pi\restriction\overline{\lambda}.$$
It is also easy to see that
$$\mathcal{R}=\mathrm{Ult}_n(\mathcal{P},\pi,\kappa).$$
And that, if $\mathcal{P}$ is not a weasel, then
$$\mathcal{R}=\mathrm{Hull}^{\mathcal{R}}_{n+1}(\kappa\cup\widetilde{\pi}(p^{\mathcal{P}}_{n+1}))=\mathrm{Hull}^{\mathcal{R}}_{n+1}(\kappa\cup p^{\mathcal{R}}_{n+1}),$$
whereas if $\mathcal{P}$ is a weasel, then $\mathcal{R}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa\le\mu<\Omega_0$. The last step in the earlier proofs is to analyze the coiteration of $\mathcal{R}$ versus W to obtain the contradiction
$$\lambda=(\kappa^+)^K=(\kappa^+)^W>(\kappa^+)^{\mathcal{R}}=\sup(\widetilde{\pi}``\overline{\lambda})=\sup(\pi``\overline{\lambda})=\lambda.$$
At the analogous step in the current proof, we coiterate $(\langle W,\mathcal{R}\rangle,\langle\kappa\rangle)$ versus W. For this we need that the phalanx is iterable. Basically, we need to know that $\mathcal{R}$ is κ-strong whereas in the earlier proofs, iterability was enough. Our solution, which we make precise soon, is to work up to this phalanx by an induction that involves other phalanxes. In summary, the new complications are:

- how to show that $\overline{W}$ does not move,
- $\mathcal{P}$ and $\mathcal{R}$ could be weasels,
- $\mathcal{R}$ might be a protomouse but not a premouse and
- how to show that $\mathcal{R}$ is κ-strong.

We need more definitions to explain our strategy for dealing with these new complications. Let $\vec{\kappa}$ enumerate the infinite cardinals of $\overline{W}_{\overline{\theta}}$ up to $\overline{\Omega}_0=\pi^{-1}(\Omega_0)$. Thus
$$\kappa_\alpha=(\aleph_\alpha)^{\overline{W}_{\overline{\theta}}}$$
for all $\alpha<\overline{\Omega}_0$. Also let $\vec{\lambda}$ enumerate the infinite successor cardinals of $\overline{W}_{\overline{\theta}}$. Thus $\lambda_\alpha=\kappa_{\alpha+1}$ for all $\alpha<\overline{\Omega}_0$. The main idea for dealing with the first and last new complications involves an induction on $\gamma<\overline{\Omega}_0$ with six induction hypotheses. As they are introduced, we assume $(1)_\alpha$ through $(6)_\alpha$ for all $\alpha<\gamma$. Our obligation is to prove $(1)_\gamma$ through $(6)_\gamma$. The first induction hypothesis tells us that $\overline{W}$ has not moved yet. We use the notation $\overline{E}_\eta=E^{\mathcal{T}}_\eta$.

(1)$_\alpha$ *For all $\eta\le\overline{\theta}$, if $\overline{E}_\eta\neq\emptyset$, then* $\mathrm{lh}(\overline{E}_\eta)>\lambda_\alpha$.

The next step is to derive a phalanx from $\mathcal{T}$. Let $\eta(\alpha)$ be the least $\eta \leq \theta$ such that

$$\nu(E^{\mathcal{T}}_{\eta}) > \kappa_\alpha$$

if there is such an η; otherwise, let $\eta(\alpha) = 0$. Then let $\mathcal{P}_\alpha$ be the unique $\mathcal{P} \trianglelefteq W_{\eta(\alpha)}$ such that for some $n < \omega$

$$\rho^{\mathcal{P}}_n \geq \lambda_\alpha = (\kappa^+_\alpha)^{\mathcal{P}}$$

and $\rho^{\mathcal{P}}_{n+1} \leq \kappa_\alpha$ if it exists. In this case,

$$\mathcal{P} = \mathrm{Hull}^{\mathcal{P}}_{n+1}(\kappa_\alpha \cup p^{\mathcal{P}}_{n+1}).$$

Otherwise, let $\mathcal{P}_\alpha = W_{\eta(\alpha)}$. In this case, $\mathcal{P}_\alpha$ is a thick weasel with the hull and definability properties at μ whenever $\kappa_\alpha \leq \mu < \Omega_0$.

4.3 Lemma. *The phalanx* $(\vec{\mathcal{P}}\restriction(\gamma+1), \vec{\lambda}\restriction\gamma)$ *is iterable.*

Idea. We may construe an iteration tree on this phalanx as an iteration tree extending $\mathcal{T}\restriction(\eta(\gamma)+1)$. But W is iterable. ⊣

By our induction hypothesis $(1)_\alpha$, $E_\pi\restriction\pi(\kappa_\alpha)$ is an extender over $\mathcal{P}_\alpha$ for each $\alpha < \gamma$.[10] This allows us to define

$$\mathcal{R}_\alpha = \mathrm{Ult}(\mathcal{P}_\alpha, \pi, \pi(\kappa_\alpha))$$

and

$$\Lambda_\alpha = \sup(\pi``\lambda_\alpha) = (\pi(\kappa_\alpha)^+)^{\mathcal{R}_\alpha}.$$

A standard application of the fact that ${}^\omega X \subseteq X$ shows that $\mathcal{R}_\alpha$ is a transitive structure. Let $\pi_\alpha : \mathcal{P}_\alpha \to \mathcal{R}_\alpha$ be the ultrapower map. More standard calculations show that

$$\pi_\alpha\restriction\lambda_\alpha = \pi\restriction\lambda_\alpha$$

and

$$\pi_\alpha(\lambda_\alpha) = \Lambda_\alpha \leq \pi(\lambda_\alpha).$$

Also that

$$\mathcal{R}_\alpha = \mathrm{Ult}(\mathcal{P}_\alpha, \pi, \Lambda_\alpha).$$

And, if $\mathcal{P}_\alpha$ is not a weasel, then

$$\mathcal{R}_\alpha = \mathrm{Hull}^{\mathcal{R}_\alpha}_{n+1}(\pi(\kappa_\alpha) \cup \pi_\alpha(p^{\mathcal{P}_\alpha}_{n+1})) = \mathrm{Hull}^{\mathcal{R}_\alpha}_{n+1}(\pi(\kappa_\alpha) \cup p^{\mathcal{R}_\alpha}_{n+1})$$

for some $n < \omega$, whereas if $\mathcal{P}_\alpha$ is a weasel, then $\mathcal{R}_\alpha$ is a thick weasel with the hull and definability properties at μ whenever $\pi(\kappa_\alpha) \leq \mu < \Omega_0$. But notice

10 Models on a non-linear iteration tree are not necessarily contained in the starting model. In order to form $\mathrm{Ult}(\mathcal{P}_\alpha, \pi, \pi(\kappa_\alpha))$ we must know that $E_\pi\restriction\pi(\kappa_\alpha)$ measures all sets in $\mathcal{P}_\alpha$. The proof presented in [21] overlooks this detail but can be straightened out easily using the approach shown here.

that if $\rho_1(\mathcal{P}_\alpha) \leq \kappa_\alpha$, $\mathcal{P}_\alpha$ is an active premouse and π is discontinuous at the cardinal successor of $\mathrm{crit}(F^{\mathcal{P}_\alpha})$ in $\mathcal{P}_\alpha$, then $\mathcal{R}_\alpha$ is not a premouse.

Observe that

$$(\vec{\mathcal{R}}\restriction(\gamma+1), \vec{\Lambda}\restriction\gamma)$$

satisfies the agreement condition for being a phalanx. We call it a *phalanx of protomice*. Let us examine the situation in which $\beta \leq \gamma$ and $\mathcal{R}_\beta$ is not a premouse. Equivalently, there exist $\alpha < \beta$ with

$$\mathrm{crit}(F^{\mathcal{P}_\beta}) = \kappa_\alpha$$

and

$$\Lambda_\alpha < \pi(\lambda_\alpha).$$

In this case,

$$\mathrm{crit}(F^{\mathcal{R}_\beta}) = \pi(\kappa_\alpha).$$

And, although $F^{\mathcal{R}_\beta}$ is an extender fragment but not an extender over $\mathcal{R}_\beta$, it is an extender over $\mathcal{R}_\alpha$. More generally, if $\mathcal{U}$ is what would naturally be called an iteration tree on

$$(\vec{\mathcal{R}}\restriction(\gamma+1), \vec{\Lambda}\restriction\gamma)$$

and $\gamma < \beta' < \mathrm{lh}(\mathcal{U})$ with

$$\mathrm{root}^U(\beta') = \beta$$

and

$$D^{\mathcal{U}} \cap (\beta, \beta']_U = \emptyset,$$

then

$$\mathrm{crit}(F^{\mathcal{M}^{\mathcal{U}}_{\beta'}}) = \pi(\kappa_\alpha) = \mathrm{crit}(F^{\mathcal{R}_\beta})$$

and the two extender fragments are total over $\mathcal{R}_\alpha$. Thus $F^{\mathcal{M}^{\mathcal{U}}_{\beta'}}$ could legitimately be applied to $\mathcal{R}_\alpha$ to form an extension of $\mathcal{U}$. While the following result is not used in the current proof, others like it are.

4.4 Lemma. *The phalanx of protomice*

$$(\vec{\mathcal{R}}\restriction(\gamma+1), \vec{\Lambda}\restriction\gamma)$$

is iterable.

Idea. In the standard way, use the fact that ${}^\omega X \subseteq X$ to reduce the iterability of the above phalanx to that of

$$(\vec{\mathcal{P}}\restriction(\gamma+1), \vec{\lambda}\restriction\gamma).$$

The latter phalanx is iterable by Lemma 4.3. $\dashv$

Based on our discussion of earlier proofs of weak covering, we would expect to want to iterate

$$(\langle W, \mathcal{R}_\gamma\rangle, \langle\pi(\kappa_\gamma)\rangle).$$

We can make sense of what we mean by this even if $\mathcal{R}_\gamma$ is not a premouse, but iterating this phalanx of protomice does not seem to accomplish much in this case. Our solution to this problem is complicated. For each $\alpha \leq \gamma$, if $\mathcal{R}_\alpha$ is not a premouse, then we define a certain premouse $\mathcal{S}_\alpha$ that agrees with $\mathcal{R}_\alpha$ below Λ_α. We also find a premouse $\mathcal{Q}_\alpha$ that agrees with $\mathcal{P}_\alpha$ below λ_α such that

$$\mathcal{S}_\alpha = \mathrm{Ult}(\mathcal{Q}_\alpha, \pi, \pi(\kappa_\alpha)).$$

Only near the end of the current proof will we say exactly what $\mathcal{Q}_\alpha$ and $\mathcal{S}_\alpha$ are in this case. On the other hand, if $\mathcal{R}_\alpha$ is a premouse, then $\mathcal{Q}_\alpha = \mathcal{P}_\alpha$ and $\mathcal{S}_\alpha = \mathcal{R}_\alpha$. The reader is asked to consider this case only for the moment.

As we just indicated, the main thing we want to know besides $(1)_\gamma$ is that $\mathcal{S}_\gamma$ is $\pi(\kappa_\gamma)$-strong, so we make it an induction hypothesis in the following way.

(2)$_\alpha$ $(\langle W, \mathcal{S}_\alpha\rangle, \langle\pi(\kappa_\alpha)\rangle)$ *is an iterable phalanx.*

(3)$_\alpha$ $(\langle \overline{W}, \mathcal{Q}_\alpha\rangle, \langle\kappa_\alpha\rangle)$ *is an iterable phalanx.*

4.5 Lemma. $(3)_\gamma$ *implies* $(2)_\gamma$.

Idea. The proof uses the fact that $\mathcal{S}_\gamma = \mathrm{Ult}(\mathcal{Q}_\gamma, \pi, \lambda_\gamma)$ together with countable closure ${}^\omega X \subseteq X$. It is not as routine as Lemma 4.4 though. ⊣

The next hypothesis is the key to showing that $\overline{W}$ does not move. It also represents an interesting switch in that $\overline{W}$ appears as the starting model instead of the back-up model.

(4)$_\alpha$ $((\vec{\mathcal{P}}\restriction\alpha)^\frown\langle\overline{W}\rangle, \vec{\lambda}\restriction\alpha)$ *is an iterable phalanx.*

4.6 Lemma. $(4)_\gamma$ *implies* $(1)_\gamma$.

Idea. Let $(\mathcal{U}, \mathcal{V})$ be the coiteration of the phalanxes

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\overline{W}\rangle, \vec{\lambda}\restriction\gamma)$$

and

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\mathcal{P}_\gamma\rangle, \vec{\lambda}\restriction\gamma).$$

The former phalanx is iterable by $(4)_\gamma$. The latter phalanx is iterable by Lemma 4.3. In particular, $\mathcal{V}$ can be construed as an extension of $\mathcal{T}\restriction\eta(\gamma) + 1$. Let $\zeta + 1 = \mathrm{lh}(\mathcal{U})$. Standard arguments can be used to see that

$$\gamma = \mathrm{root}^U(\zeta)$$

and

$$D^{\mathcal{U}} \cap (\gamma, \zeta]_U = \emptyset.$$

These arguments use the hull and definability properties at κ_α when $\mathcal{P}_\alpha$ is a thick weasel and soundness at κ_α otherwise. Suppose for contradiction that $(1)_\gamma$ fails. Since $(1)_\alpha$ holds for all $\alpha < \gamma$,

$$\mathrm{lh}(\overline{E}_0) = \lambda_\gamma.$$

Consequently, the first extenders used on $\mathcal{U}$ and $\overline{\mathcal{T}}$ are the same, i.e.,

$$E^{\mathcal{U}}_\gamma = \overline{E}_0.$$

Hence

$$\mathrm{lh}(E^{\mathcal{U}}_\gamma) = \lambda_\gamma < ((\kappa_\gamma)^+)^{\overline{W}}.$$

This can be used to see that if

$$\gamma = \mathrm{pred}^U(\iota + 1)$$

then

$$\kappa_\gamma \leq \mathrm{crit}(E^{\mathcal{U}}_\iota)$$

so

$$\iota + 1 \in D^{\mathcal{U}},$$

which is a contradiction. $\dashv$

Here is a fact whose proof is like that of Lemma 4.6. Hypothesis $(4)_\alpha$ implies that there is an iteration tree $\mathcal{V}_\alpha$ on W that extends $\mathcal{T}\restriction(\eta(\alpha)+1)$, an initial segment $\mathcal{N}_\alpha$ of the last model of $\mathcal{V}_\alpha$ with

$$\wp(\kappa_\alpha) \cap \overline{W} = \wp(\kappa_\alpha) \cap \mathcal{N}_\alpha$$

and an elementary embedding $k_\alpha : \overline{W} \to \mathcal{N}_\alpha$ with $k_\alpha\restriction\kappa_\alpha = \mathrm{id}\restriction\kappa_\alpha$. This fact and the notation just established comes up again when we prove $(3)_\gamma$.

$(5)_\alpha$ $((\vec{\mathcal{R}}\restriction\alpha)^\frown\langle W\rangle, \vec{\Lambda}\restriction\alpha)$ *is an iterable phalanx of protomice.*

It makes sense to iterate this phalanx of protomice for reasons like those we gave before Lemma 4.4. The difference is that W is the starting model instead of $\mathcal{R}_\gamma$.

4.7 Lemma. $(5)_\gamma$ *implies* $(4)_\gamma$*.*

Idea. Consider the sequence of embeddings

$$\langle \pi_\alpha \mid \alpha < \gamma\rangle^\frown\langle\pi\rangle.$$

Since $\pi_\alpha\restriction\lambda_\alpha = \pi\restriction\lambda_\alpha$ for all $\alpha < \gamma$, this sequence can be used to reduce the iterability of

$$((\vec{\mathcal{P}}\restriction\gamma)^\frown\langle\overline{W}\rangle, \vec{\lambda}\restriction\gamma)$$

to that of

$$((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma).$$

The latter phalanx is iterable by $(5)_\gamma$. $\dashv$

It is worth noticing that the iteration trees on $((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$ that are relevant to the proof of Lemma 4.7 have a special form: whenever $\alpha < \gamma$ and an extender is applied to $\mathcal{R}_\alpha$, the critical point of the extender is exactly $\pi(\kappa_\alpha)$. Similarly, only special iteration trees are relevant to the proof of Lemma 4.4.

(6)$_\alpha$ *$((\vec{\mathcal{S}}\restriction\alpha)^\frown\langle W\rangle, \vec{\Lambda}\restriction\alpha)$ is an iterable phalanx.*

4.8 Lemma. $(6)_\gamma$ *implies* $(5)_\gamma$.

Since we have not defined $\vec{S}\restriction\gamma$ it would be meaningless to sketch the proof of Lemma 4.8, which is not easy. It is interesting, though, that the proof involves a variant of the usual copying constructions in which the tree structure changes. And an ultrapower by an extender fragment in an iteration tree on $((\vec{\mathcal{R}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$ corresponds to something like padding in the copied iteration tree on $((\vec{\mathcal{S}}\restriction\gamma)^\frown\langle W\rangle, \vec{\Lambda}\restriction\gamma)$.

Having seen that

$$(6)_\gamma \implies (5)_\gamma \implies (4)_\gamma \implies (1)_\gamma$$

and

$$(3)_\gamma \implies (2)_\gamma$$

it remains to prove $(3)_\gamma$ and $(6)_\gamma$, which we do next.

4.9 Lemma. $(3)_\gamma$ *holds.*

Idea. We must see that $(\langle\overline{W}, \mathcal{Q}_\gamma\rangle, \langle\kappa_\gamma\rangle)$ is iterable. Consider the sequence of embeddings

$$\langle k_\alpha \mid \alpha < \gamma\rangle^\frown\langle \mathrm{id}\restriction\mathcal{Q}_\gamma\rangle$$

where $k_\alpha : \overline{W} \to \mathcal{N}_\alpha$ was defined just after the proof of Lemma 4.6. Since $k_\alpha\restriction\kappa_\alpha = \mathrm{id}\restriction\kappa_\alpha$, this sequence of embeddings can be used to reduce the iterability of

$$(\langle\overline{W}, \mathcal{Q}_\gamma\rangle, \langle\kappa_\gamma\rangle)$$

to that of

$$(\langle\mathcal{N}_\alpha \mid \alpha < \gamma\rangle^\frown\langle\mathcal{Q}_\gamma\rangle, \vec{\kappa}\restriction\gamma).$$

There is a subtlety in the copying construction that also came up at the end of the proof of Theorem 2.35 but once again we omit this detail. The phalanx $(\langle\mathcal{N}_\alpha \mid \alpha < \gamma\rangle^\frown\langle\mathcal{Q}_\gamma\rangle, \vec{\kappa}\restriction\gamma)$ is what we call W *based* because each of its models appears on an iteration tree on W. For $\alpha < \gamma$, the iteration tree is $\mathcal{V}_\alpha$. And for $\mathcal{Q}_\gamma$ the iteration tree is $\mathcal{T}\restriction(\eta(\gamma)+1)$ because we are still assuming for simplicity that $\mathcal{Q}_\gamma = \mathcal{P}_\gamma$. (Otherwise a generalized notion of W based is used.) We chose W so that there is an elementary embedding from $\sigma : W \to K^c$. Copy each $\mathcal{V}_\alpha$ to $\sigma\mathcal{V}_\alpha$ and let

$$\sigma^*_\alpha : \mathcal{N}_\alpha \to \mathcal{N}^*_\alpha$$

be the final copy embedding restricted to $\mathcal{N}_\alpha$. Next, copy $\mathcal{T}\restriction(\eta(\gamma)+1)$ to $\sigma\mathcal{T}\restriction(\eta(\gamma)+1)$ and let

$$\sigma^*_\gamma : \mathcal{Q}_\gamma \to \mathcal{Q}^*_\gamma$$

be the final copy embedding restricted to $\mathcal{Q}_\gamma$. The sequence of embeddings

$$\langle \sigma^*_\alpha \mid \alpha \leq \gamma \rangle$$

can be used to reduce the iterability of $(\langle \mathcal{N}_\alpha \mid \alpha < \gamma \rangle ^\frown \langle \mathcal{Q}_\gamma \rangle, \vec{\kappa}\restriction\gamma)$ to that of

$$(\langle \mathcal{N}^*_\alpha \mid \alpha < \gamma \rangle ^\frown \langle \mathcal{Q}^*_\gamma \rangle, \langle \sigma(\kappa_\alpha) \mid \alpha < \gamma \rangle).$$

The latter phalanx is K^c based and hence iterable by §9 of [42]. ⊣

The following lemma is the last step in our induction.

4.10 Lemma. $(6)_\gamma$ *holds.*

Idea. Consider an arbitrary $\alpha < \gamma$. Freeing up earlier notation, let $(\mathcal{U}, \mathcal{V})$ be the coiteration of W versus $(\langle W, \mathcal{S}_\alpha \rangle, \langle \pi(\kappa_\alpha) \rangle)$. The latter phalanx is iterable by $(2)_\alpha$. Say $\mathrm{lh}(\mathcal{U}) = \zeta + 1$ and $\mathrm{lh}(\mathcal{V}) = \eta + 1$. Standard arguments as in Sect. 2 show that

$$\mathcal{M}^{\mathcal{U}}_\zeta \trianglerighteq \mathcal{M}^{\mathcal{V}}_\eta,$$

$1 = \mathrm{root}^V(\eta)$ and $D^{\mathcal{V}} \cap (1, \eta]_V = \emptyset$. Since $\mathcal{S}_\alpha$ and W agree below Λ_α, the extenders used on $\mathcal{U}$ and $\mathcal{V}$ all have length $\geq \Lambda_\alpha$. In the remainder of this proof, we refer to

$$j_\alpha = i^{\mathcal{V}}_{1,\eta}$$

and

$$\mathcal{M}_\alpha = \mathcal{M}^{\mathcal{V}}_\eta.$$

Note that $j_\alpha : \mathcal{S}_\alpha \to \mathcal{M}_\alpha$ is an elementary embedding and

$$j_\alpha\restriction\pi(\kappa_\alpha) = \mathrm{id}\restriction\pi(\kappa_\alpha).$$

To show that $(6)_\gamma$ holds we must see that the phalanx

$$((\vec{\mathcal{S}}\restriction\gamma)^\frown \langle W \rangle, \vec{\Lambda}\restriction\gamma)$$

is iterable. Using the sequence of embeddings

$$\langle j_\alpha \mid \alpha < \gamma \rangle ^\frown \langle \mathrm{id}\restriction W \rangle$$

this reduces to seeing that the phalanx

$$(\langle \mathcal{M}_\alpha \mid \alpha < \gamma \rangle ^\frown \langle W \rangle, \vec{\Lambda}\restriction\gamma)$$

is iterable. But the latter phalanx is W based. Since W embeds into K^c and all K^c based phalanxes are iterable, all W based phalanxes are iterable. ⊣

This concludes the proof by induction that $(1)_\alpha$ through $(6)_\alpha$ hold for all $\alpha < \overline{\Omega}_0$. Now fix $\overline{\alpha}$ so that $\kappa_{\overline{\alpha}} = \overline{\kappa}$. Then $\lambda_{\overline{\alpha}} = \overline{\lambda}$ and $\Lambda_{\overline{\alpha}} = \lambda$. Write $\mathcal{P} = \mathcal{P}_{\overline{\alpha}}$, $\mathcal{R} = \mathcal{R}_{\overline{\alpha}}$, $\mathcal{Q} = \mathcal{Q}_{\overline{\alpha}}$ and $\mathcal{S} = \mathcal{S}_{\overline{\alpha}}$. Also let $\widetilde{\pi} : \mathcal{Q} \to \mathcal{S}$ be the ultrapower embedding.

4.11 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is not a weasel.*

Idea. Assume otherwise. We build on the facts from the proof of Lemma 4.10 about the coiteration $(\mathcal{U}, \mathcal{V})$ of W versus $(\langle W, \mathcal{S}\rangle, \langle\kappa\rangle)$. We have that

$$\mathcal{S} = \mathrm{Hull}^{\mathcal{S}}_{n+1}(\kappa \cup p^{\mathcal{S}}_{n+1}).$$

Standard arguments show that either

$$\mathcal{S} \triangleleft \mathcal{M}^{\mathcal{V}}_0 = W$$

or

$$E^{\mathcal{U}}_0 = E^W_\lambda$$

and

$$\mathcal{S} = \mathcal{M}^{\mathcal{V}}_1 = \mathrm{Ult}((\mathcal{M}^*_1)^{\mathcal{V}}, E^{\mathcal{V}}_0).$$

Either way, we get the contradiction

$$\lambda = (\kappa^+)^{\mathcal{S}} < (\kappa^+)^W = (\kappa^+)^K.$$

⊣

4.12 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is a weasel.*

Idea. Assume otherwise. We build on facts about the coiteration $(\mathcal{U}, \mathcal{V})$ of W versus $(\langle W, \mathcal{S}\rangle, \langle\kappa\rangle)$ from the proof of Lemma 4.10. We have that $\mathcal{S}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa \leq \mu < \Omega_0$. By universality,

$$\mathcal{M}^{\mathcal{U}}_\zeta = \mathcal{M}^{\mathcal{V}}_\eta.$$

$\mathcal{M}^{\mathcal{V}}_\eta$ has the hull property at κ because $\mathrm{crit}(i^{\mathcal{V}}_{1,\eta}) \geq \kappa$. On the other hand, since $\mathcal{M}^{\mathcal{V}}_\eta$ also results from the iteration

$$W \xrightarrow[i^{\mathcal{T}}_{0,\eta(\overline{\alpha})}]{} \mathcal{Q} \xrightarrow[\widetilde{\pi}]{} \mathcal{S} \xrightarrow[i^{\mathcal{V}}_{1,\eta}]{} \mathcal{M}^{\mathcal{V}}_\eta$$

and $\mathrm{crit}(\widetilde{\pi}) = \mathrm{crit}(\pi) = \delta$, we conclude that $\mathcal{M}^{\mathcal{V}}_\eta$ does not have the definability property at δ. Here we are using that $W_{\eta(\overline{\alpha})} = \mathcal{P} = \mathcal{Q}$. This implies that $\mathcal{U}$ is not trivial. Let $E^{\mathcal{U}}_\iota$ be the first extender used along $[0, \zeta]_U$. That is,

$$0 = \mathrm{pred}^U(\iota + 1) \leq_U \zeta.$$

Since $\mathcal{M}^{\mathcal{U}}_\zeta$ does not have the definability property at δ,

$$\mathrm{crit}(E^{\mathcal{U}}_\iota) \leq \delta.$$

Recall that $\mathcal{S}$ and W agree below λ. But λ is a cardinal in both hence not the index of an extender on the sequence of either. Thus,

$$\mathrm{lh}(E_\iota^{\mathcal{U}}) > \lambda.$$

This implies that the generators of $E_\iota^{\mathcal{U}}$ are unbounded in λ. But then $\mathcal{M}_\zeta^{\mathcal{U}}$ does not have the hull property at κ. This is a contradiction. ⊣

To wrap things up for this section we give the definitions of $\mathcal{Q}_\beta$ and $\mathcal{S}_\beta$ and discuss how they fit with the outline of the proof of Theorem 4.2 given so far. Recall that if $\mathcal{R}_\beta$ is a premouse, then we already defined $\mathcal{Q}_\beta = \mathcal{P}_\beta$ and $\mathcal{S}_\beta = \mathcal{R}_\beta$. Suppose that $\mathcal{R}_\beta$ is not a premouse. Say $\alpha < \beta$ and $\mathrm{crit}(F_\beta^{\mathcal{R}}) = \pi(\kappa_\alpha)$. Recall that $F_\beta^{\mathcal{R}}$ is an extender over $\mathcal{R}_\alpha$. Suppose for the moment that $\mathcal{R}_\alpha$ is a premouse. Then what we would do is set

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{R}_\alpha, F^{\mathcal{R}_\beta}) = \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})$$

and

$$\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F^{\mathcal{P}_\beta}) = \mathrm{Ult}(\mathcal{Q}_\alpha, F^{\mathcal{P}_\beta}).$$

It is easy to see that, in this case, $\mathcal{S}_\beta$ is a premouse and

$$\begin{aligned} \mathcal{S}_\beta \text{ is a weasel} \quad &\iff \quad \mathcal{R}_\alpha \text{ is a weasel}\\ &\iff \quad \mathcal{P}_\alpha \text{ is a weasel}\\ &\iff \quad \mathcal{Q}_\beta \text{ is a weasel.}\end{aligned}$$

Of course, $\mathcal{R}_\beta$ is not a weasel. With a little more work, one sees that

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{Q}_\beta, \pi, \pi(\kappa_\alpha)) = \mathrm{Ult}(\mathcal{Q}_\beta, \pi, \Lambda_\alpha).$$

As for $\mathcal{Q}_\alpha$, it is a model on a finite extension of $\mathcal{T}\restriction(\eta(\alpha)+1)$ in this case but not so literally in others.

The general definition of $\mathcal{S}_\beta$ and $\mathcal{Q}_\beta$ is by induction. We set

$$\mathcal{S}_\beta = \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})$$

and

$$\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{Q}_\alpha, F^{\mathcal{P}_\beta})$$

whenever $\alpha < \beta$,

$$\mathrm{crit}(F_\beta^{\mathcal{R}}) = \pi(\kappa_\alpha)$$

and $\mathcal{R}_\beta$ is not a premouse. For example, we could have $\alpha < \beta < \gamma$,

$$\begin{aligned}\mathcal{S}_\alpha &= \mathcal{R}_\alpha\\ \mathcal{S}_\beta &= \mathrm{Ult}(\mathcal{S}_\alpha, F^{\mathcal{R}_\beta})\\ \mathcal{S}_\gamma &= \mathrm{Ult}(\mathcal{S}_\beta, F^{\mathcal{R}_\gamma})\end{aligned}$$

and the analogous equations for $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ and $\mathcal{Q}_\gamma$. Note that $\mathcal{Q}_\gamma$ is a model on a finite extension of $\mathcal{T}\restriction(\eta(\alpha)+1)$ but not in the conventional sense. What we mean by W based phalanxes and the theorems about them can be generalized accordingly though. This is needed to complete the proof of Lemma 4.9.

Beyond this, we do not attempt to explain how to incorporate this definition of $\mathcal{S}$ and $\mathcal{Q}$ into the proof by induction of $(1)_\alpha$ through $(6)_\alpha$. In particular, the proof of Lemma 4.8 is beyond the scope of this exposition. Instead, we finish this section by showing that it is still possible to obtain a contradiction assuming $(1)_\alpha$ through $(6)_\alpha$ hold for all $\alpha < \overline{\Omega}_0$ without assuming that $\mathcal{S} = \mathcal{R}$. The argument uses two additional concepts: the Dodd decomposition of an extender and fine structure for thick weasels. The simplest case in which $\mathcal{S} \neq \mathcal{R}$ already illustrates the main new ideas. First we look at the non-weasel subcase.

4.13 Lemma. *Let $\alpha < \beta$ and $F = F^{\mathcal{P}_\beta}$. Suppose that*

$$\rho_{n+1}^{\mathcal{P}_\alpha} \leq \kappa_\alpha < \lambda_\alpha \leq \rho_n^{\mathcal{P}_\alpha}$$

and $\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F)$. Then

$$\mathcal{Q}_\beta = \mathrm{Hull}_{n+1}^{\mathcal{Q}_\beta}(\kappa_\beta \cup p_{n+1}^{\mathcal{Q}_\beta}).$$

Idea. For simplicity, assume $n = 0$. (Otherwise, use the Σ_n mastercode structure for $\mathcal{P}_\alpha$.) Let $i : \mathcal{P}_\alpha \to \mathcal{Q}_\beta$ be the ultrapower map. The lemma is relatively easy to see if $\nu(F) = \kappa_\beta$ because then

$$p_1^{\mathcal{Q}_\beta} = i(p_1^{\mathcal{P}_\alpha}).$$

More generally, we show that

$$p_1^{\mathcal{Q}_\beta} - \kappa_\beta = i(p_1^{\mathcal{P}_\alpha}) \cup (s - \kappa_\beta)$$

for a certain $s \in [\mathrm{lh}(F)]^{<\omega}$ whose identity we are about to reveal.

The *Dodd projectum of* $\mathcal{P}_\beta$, written $\tau^{\mathcal{P}_\beta}$, is the least ordinal τ such that

$$\lambda_\alpha = (\mathrm{crit}(F)^+)^{\mathcal{P}_\beta} \leq \tau \leq \nu(F)$$

and there exists an $s \in [\nu(F)]^{<\omega}$ such that F and $F\restriction(\tau \cup s)$ have the same ultrapower. The *Dodd parameter of* $\mathcal{P}_\beta$, written $s^{\mathcal{P}_\beta}$, is the least parameter $s \in [\nu(F)]^{<\omega}$ such that F and $F\restriction(\tau^{\mathcal{P}_\beta} \cup s)$ have the same ultrapower.[11] In fact, $\tau = \max(\rho_1^{\mathcal{P}_\beta}, \lambda_\alpha)$. There is a relationship between the $s^{\mathcal{P}_\beta}$ and $p_1^{\mathcal{P}_\beta}$ that is slightly more complicated but not needed here. By a result of Steel in [38], if $\mathcal{P}_\beta$ is 1-sound, then for all $i < |s^{\mathcal{P}_\beta}|$,

$$F\restriction(s_i^{\mathcal{P}_\beta} \cup (s^{\mathcal{P}_\beta}\restriction i)) \in \mathcal{P}_\beta$$

[11] Recall that parameters, i.e., finite sets of ordinals, are often identified with descending sequences of ordinals, and that the ordering on parameters is lexicographic.

and for all $\xi < \tau^{\mathcal{P}_\beta}$,

$$F\restriction(\xi \cup s^{\mathcal{P}_\beta}) \in \mathcal{P}_\beta.$$

These properties are known as *Dodd solidity* and *Dodd amenability* respectively. Counterexamples for mice that are not 1-sound can be found in [27].

If $\mathcal{P}_\beta \lhd W_{\eta(\beta)}$, then certainly $\mathcal{P}_\beta$ is 1-sound and therefore Dodd solid and Dodd amenable. The fact that F and $F\restriction(\kappa_\beta \cup s^{\mathcal{P}_\beta})$ have the same ultrapower translates into

$$\mathcal{Q}_\beta = \mathrm{Hull}_1^{\mathcal{Q}_\beta}(\kappa_\beta \cup i(p_1^{\mathcal{P}_\alpha}) \cup s^{\mathcal{P}_\beta}).$$

The fact that

$$F\restriction(s_i^{\mathcal{P}_\beta} \cup (s^{\mathcal{P}_\beta}\restriction i)) \in \mathcal{P}_\beta$$

for all $i < |s^{\mathcal{P}_\beta} - \kappa_\beta|$ translates into

$$p_1^{\mathcal{Q}_\beta} - \kappa_\beta = i(p_1^{\mathcal{P}_\alpha}) \cup (s^{\mathcal{P}_\beta} - \kappa_\beta).$$

Suppose instead that $\mathcal{P}_\beta = W_{\eta(\beta)}$. Then $\mathcal{P}_\beta$ is not 1-sound. Let $\iota + 1$ be the last drop in model or degree along $[0, \eta(\beta)]_T$ and let

$$W^*_{\iota+1} \trianglelefteq W_{\mathrm{pred}^T(\iota+1)}$$

be the level to which we drop. Also let

$$i^*_{\iota+1} : W^*_{\iota+1} \to W_{\iota+1} = \mathrm{Ult}(W^*_{\iota+1}, F^{\mathcal{T}}_\iota)$$

be the ultrapower embedding. Since $W^*_{\iota+1}$ is 1-sound, it is Dodd solid and Dodd amenable by [38]. Now by induction on ζ such that

$$\iota + 1 \leq_T \zeta \leq_T \eta(\beta)$$

it is possible to show that if

$$s = i^{\mathcal{T}}_{\iota+1,\zeta}(i^*_{\iota+1}(s^{W^*_{\iota+1}})),$$

then F^{W_ζ} and $F^{W_\zeta}\restriction(\nu^{\mathcal{T}}_\zeta \cup s)$ have the same ultrapower,

$$F^{W_\zeta}\restriction(s_i \cup (s\restriction i)) \in W_\zeta$$

for all $i < |s - \nu^{\mathcal{T}}_\zeta|$ and

$$s - \nu^{\mathcal{T}}_\zeta = s^{W_\zeta} - \nu^{\mathcal{T}}_\zeta.$$

By definition, $\kappa_\beta \geq \nu^{\mathcal{T}}_{\eta(\beta)}$. So at the end of this induction we see that if $s = s^{\mathcal{P}_\beta}$, then F and $F\restriction(\kappa_\beta \cup s)$ have the same ultrapower and

$$F\restriction(s_i \cup (s\restriction i)) \in \mathcal{P}_\beta$$

for all $i < |s - \kappa_\beta|$. As before, these facts translate into the desired result. $\dashv$

The facts about Dodd solidity in the proof of Lemma 4.13 can be used to avoid a convoluted argument in [21].[12]

4.14 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is not a weasel.*

Idea. In the simplest case, which is the only one we discuss here, $\overline{\kappa} = \kappa_\beta$ satisfies the hypothesis of Lemma 4.13. Then

$$\mathcal{S} = \mathrm{Hull}_{n+1}^{\mathcal{S}}(\kappa \cup p_{n+1}^{\mathcal{S}}).$$

Now repeat the proof of Lemma 4.11 to obtain a contradiction. ⊣

4.15 Lemma. *It is not the case that $\mathcal{S} \neq \mathcal{R}$ and $\mathcal{S}$ is a weasel.*

Idea. There is an analog of Lemma 4.13 that is valid when $\mathcal{Q}_\beta$ is a weasel. With this analog, the proof of Lemma 4.12 can be adapted to give the proof of Lemma 4.15. The basic idea behind this analog is as follows.

Once again, we look only at the simplest instance of $\mathcal{Q}_\beta \neq \mathcal{P}_\beta$. That is, $\alpha < \beta$ and $F = F^{\mathcal{P}_\beta}$ and $\mathcal{Q}_\beta = \mathrm{Ult}(\mathcal{P}_\alpha, F)$. But this time suppose that $\mathcal{P}_\alpha = W_{\eta(\alpha)}$ is a thick weasel with the hull and definability properties at μ whenever $\kappa_\alpha \leq \mu < \Omega_0$. Then $\mathcal{Q}_\beta$ is also a thick weasel. If $\nu(F) = \kappa_\beta$, then we can show that $\mathcal{Q}_\beta$ has the hull and definability properties at μ whenever $\kappa_\beta \leq \mu < \Omega_0$, which is just what is needed to run the proof of Lemma 4.12 when $\kappa_\beta = \overline{\kappa}$. More generally, consider again the fact that F and its Dodd decomposition $F \restriction (\tau^{\mathcal{P}_\beta} \cup s^{\mathcal{P}_\beta})$ have the same ultrapower and $\tau^{\mathcal{P}_\beta} \leq \kappa_\beta$. There is a natural sense in which $\mathcal{Q}_\beta$ has the $s^{\mathcal{P}_\beta}$ definability property at μ whenever $\kappa_\beta \leq \mu < \Omega_0$. This fact motivates defining $\kappa^{\mathcal{Q}_\beta}$ to be the least ordinal μ_0 such that there exists a $c \in [\Omega_0]^{<\omega}$ such that $\mathcal{Q}_\beta$ has the c definability property at μ whenever $\mu_0 \leq \mu < \Omega_0$. This is the *class projectum*. We have that

$$\kappa^{\mathcal{Q}_\beta} \leq \kappa_\beta$$

as witnessed by $s^{\mathcal{P}_\beta}$. We also define $c^{\mathcal{Q}_\beta}$ to be the least parameter $c \in [\Omega_0]^{<\omega}$ such that $\mathcal{Q}_\beta$ has the c definability property at μ whenever $\kappa^{\mathcal{Q}_\beta} \leq \mu < \Omega_0$. This is the *class parameter*. The proof of Lemma 4.13 shows that F is Dodd solid above κ_β. This fact translates into

$$c^{\mathcal{Q}_\beta} - \kappa_\beta = s^{\mathcal{P}_\beta}.$$

The two displayed facts above are our version of Lemma 4.13 when $\mathcal{Q}_\beta$ is a weasel. They translate into

$$\kappa^{\mathcal{S}} \leq \kappa$$

and

$$c^{\mathcal{S}} - \kappa = s^{\mathcal{R}}$$

when $\kappa_\beta = \overline{\kappa}$. With some additional work we can adapt the proof of Lemma 4.12 to finish the proof of Lemma 4.15. ⊣

This concludes our outline of the proof of Theorem 4.2. ⊣

[12] Avoid Lemma 2.1.2 and Corollaries 2.1.3 and 2.1.6 of [21].

5. Applications of Core Models

In this section, we list some results whose proofs use core model theory at a level that involves iteration trees. These are stated in a way that minimizes core model prerequisites. We have also tried to avoid overly technical hypotheses. For example, in some theorems, the hypothesis that Ω is a measurable cardinal can be reduced to the existence of sharps for elements of $H(\Omega)$ or even less.

5.1. Determinacy

Some of the results in Sect. 5 are stated in terms of determinacy instead of large cardinals. Often it is easier to phrase things one way or the other but there are reasons to think that there is more to it than that. We begin this subsection by recalling some of the known equiconsistencies between large cardinals and determinacy.

5.1 Theorem. *The following are equiconsistent.*

(1) There exists a Woodin cardinal.

(2) Δ^1_2-determinacy.

5.2 Theorem. *The following are equiconsistent.*

(1) There exist infinitely many Woodin cardinals.

(2) $L(\mathbb{R})$-determinacy.

Theorems 5.1 and 5.2 are due to Woodin. A proof that (2) is consistent relative to (1) in Theorem 5.2 is given in the Handbook chapter [22]. The consistency of (1) relative to (2) in the two theorems is given in the Handbook chapter [14]. It would be reasonable for the reader to suspect that these parts of the proofs use core model theory. However, Woodin obtained these results in the 1980's before Steel developed the theory of K at the level of one Woodin cardinal. Woodin used HOD instead of K. In the proof of Theorem 5.1, Woodin showed that if Δ^1_2-determinacy holds, then there exists a real x such that $\omega_2^{L[y]}$ is a Woodin cardinal in $\mathrm{HOD}^{L[y]}$ whenever $x \in L[y]$. And his proof of Theorem 5.2 built on that of Theorem 5.1. More recently, Steel discovered alternate proofs that use core models.

Theorems 5.1 and 5.2 are equiconsistencies between determinacy and the existence of large cardinals. This is a good place to recall some of the known equivalences between determinacy and the existence of mice. For this, we must recall the definition of $M_n^\#(x)$, which can also be found in [41, §7].

The theory of mice generalizes to a theory of mice built over a real. If $n \leq \omega$ and $x \subseteq \omega$, then there is at most one structure

$$\mathcal{M} = \langle J_\beta^{E,x}, \in, E, F\rangle$$

such that $\mathcal{M}$ is a $\omega_1 + 1$ iterable sound premouse built over x,

$$J^{E,x}_{\mathrm{crit}(F)} \models \text{the number of Woodin cardinals} = n$$

and for all $\alpha < \beta$, if $E_\alpha \neq \emptyset$, then

$$J^{E,x}_{\mathrm{crit}(E_\alpha)} \models \text{the number of Woodin cardinals} < n.$$

If it exists, then this unique mouse built over x is called $M_n^\#(x)$. For $n = 0$, we have that $M_0^\#(x)$ is Turing equivalent to $x^\#$.

Let us point out some features of $M_n^\#$. Recall that the empty extender codes the identity embedding. The next weakest possibility is that the critical point of an extender is the only generator of the extender, in which case the extender codes the embedding from a normal measure. It follows from the definition that $F^{M_n^\#(x)}$ is a measure in this sense and that if δ is the supremum of the Woodin cardinals of $M_n^\#(x)$, then

$$\delta < \mathrm{crit}(F^{M_n^\#(x)})$$

and

$$E_\alpha^{M_n^\#(x)} = \emptyset$$

whenever $\delta \leq \alpha < \mathrm{crit}(F^{M_n^\#(x)})$. Regarding the projectum and standard parameter, it is easy to see that

$$\rho_1^{M_n^\#(x)} = 1$$

and

$$p_1^{M_n^\#(x)} = \emptyset.$$

In particular, $M_n^\#(x)$ is countable. We have enough iterability to guarantee that all (not just the first ω_1 many) iterates of $M_n^\#(x)$ by images of its top extender are wellfounded. By iterating away the top extender of $M_n^\#(x)$ in this way we obtain a proper class model that goes by the name $M_n(x)$. For $n = 0$ we have that $M_0(x) = L[x]$. Observe that $M_n(x)$ has the same Woodin cardinals as $M_n^\#(x)$ and that $M_n(x)$ is $\omega_1 + 1$ iterable. Moreover, the critical points of extenders used on this linear iteration form a club class of $M_n(x)$ indiscernibles. In the case $n = 0$, these are the $L[x]$ indiscernibles.

Let us call a structure that satisfies the first-order properties in the definition of $M_n^\#(x)$ but is λ iterable instead of $\omega_1 + 1$ iterable a λ *iterable* $M_n^\#(x)$.

5.3 Theorem. *Let $n < \omega$ and assume $\mathbf{\Pi^1_n}$-determinacy. Then the following are equivalent.*

(1) $\mathbf{\Pi^1_{n+1}}$-determinacy.

(2) For every $x \in \mathbb{R}$, there is an ω_1 iterable $M_n^\#(x)$.

(3) For every $x \in \mathbb{R}$, there is a unique ω_1 iterable $M_n^\#(x)$.

The case $n = 0$ boils down to the fact that

$$\boldsymbol{\Pi}^1_1\text{-determinacy} \iff \forall x \in \mathbb{R}\ (x^\# \text{ exists})$$

where the forward implication is due to Martin and the reverse is due to Harrington. The proof that (1) implies (3) is due to Woodin and uses core models. Parts of the proof can be found in the Handbook chapter [14] and Theorem 7.7 of [42]. The proof that (2) implies (1) is due to Woodin for odd n and Neeman for even n. See the Handbook chapter [22].

5.4 Corollary. *The following are equivalent.*

(1) Projective Determinacy.

(2) For all $n < \omega$ and $x \subseteq \omega$, there is an ω_1 iterable $M^\#_n(x)$.

(3) For all $n < \omega$ and $x \subseteq \omega$, there is a unique ω_1 iterable $M^\#_n(x)$.

(4) For all $n < \omega$ and $x \subseteq \omega$, there exists a Σ^1_n correct model M with n Woodin cardinals and $x \in M$.

This equivalence combines results of Martin, Steel and Woodin.

Woodin proved that if $M^\#_\omega(x)$ exists for all $x \subseteq \omega$, then $L(\mathbb{R})$-determinacy holds. See the Handbook chapter [22] for a proof due to Neeman. Steel and Woodin obtained the following optimal result.

5.5 Theorem. *The following are equivalent.*

(1) $L(\mathbb{R})$-determinacy.

(2) For all $x \subseteq \omega$ and every Σ_1 formula φ, if $\varphi[x, \mathbb{R}]$ holds in $L(\mathbb{R})$, then there is a countable, ω_1 iterable model M satisfying ZF$^-$ *plus there are ω Woodin cardinals such that $x \in M$, and $\varphi[x, \mathbb{R}^*]$ holds in the derived model of M.*

Next we state several theorems which show that some well-known consequences of determinacy are equivalent to determinacy.

5.6 Theorem. *Assume that for all $x \subseteq \omega$, $x^\#$ exists and the $\Sigma^1_3(x)$ separation property holds for subsets of ω. Then $\boldsymbol{\Delta}^1_2$-determinacy holds.*

Steel proved Theorem 5.6 by combining the Σ^1_3 correctness of K, Theorem 3.7, with ideas due to Alexander Kechris. See [42, Corollary 7.14].

Recall that if $A, B \subseteq {}^\omega\omega$, then $A \leq_w B$ iff there is a continuous function $f : {}^\omega\omega \to {}^\omega\omega$ such that $A = f^{-1}[B]$. This is Wadge reducibility, which can also be expressed in terms of games and winning strategies. By Γ Wadge determinacy we mean that for all $A, B \in \Gamma$, either $A \leq_w B$ or $B \leq_w {}^\omega\omega - A$. Under mild assumptions, Γ determinacy implies Γ Wadge determinacy. In the other direction, Harrington showed that $\Pi^1_1(x)$ Wadge determinacy implies $x^\#$ exists, hence $\Pi^1_1(x)$-determinacy by the result due to Martin mentioned earlier. One level up, Greg Hjorth proved the following.

5.7 Theorem. $\mathbf{\Pi^1_2}$ *Wadge determinacy implies* $\mathbf{\Pi^1_2}$*-determinacy.*

Theorem 5.7 is Theorem 3.15 of [12]. The proof uses the Σ^1_3 correctness of K, Theorem 3.7.

Projective Determinacy has the following well-known consequences: every projective subset of $\mathbb{R}$ is Lebesgue measurable and has the property of Baire, and every projective binary relation on $\mathbb{R}$ has a projective uniformization. Woodin once conjectured that the conjunction of these three consequences of PD implies PD, and he proved several theorems that provided evidence in favor of his conjecture. Eventually, Steel disproved Woodin's conjecture by showing that these three consequences of PD hold in $V^{\mathrm{Col}(\omega,\kappa)}$ if V is the minimal extender model with a cardinal λ such that the set of $\kappa < \lambda$ that are $<\lambda$-strong is unbounded in λ. This large cardinal axiom is weaker than the existence of a Woodin cardinal, hence weaker than the consistency strength of PD by Theorem 5.1. The reader is referred to Hauser and Schindler [11] where the history is reviewed more completely than here and Steel's theorem is reversed. While these consequences of determinacy do not match up with determinacy at the projective level, it turns out that they do match up at other levels. For example, Woodin proved the following theorem using his core model induction technique.

5.8 Theorem. *The following statements are equivalent.*

(1) $L(\mathbb{R})$*-determinacy.*

(2) For every $A \in L(\mathbb{R})$ *such that* $A \subseteq \mathbb{R} \times \mathbb{R}$ *and* A *is* Δ^2_1 *definable in* $L(\mathbb{R})$ *from real parameters,*

  (a) A *is Lebesgue measurable,*

  (b) A *has the property of Baire and*

  (c) A *can be uniformized by a function* $f \in L(\mathbb{R})$*. (By reflection,* f *can be chosen to be* Δ^2_1 *definable in* $L(\mathbb{R})$ *from real parameters.)*

(3) Same as (2) except instead of (c) we have

  (c′) *A can be uniformized by a function f such that every $B \subseteq \mathbb{R}$, if B is projective in f, then B is are Lebesgue measurable and has the property of Baire. (Note that f is not required to be in $L(\mathbb{R})$.)*

5.2. Tree Representations and Absoluteness

Shoenfield showed that all transitive proper class models of ZFC are Σ^1_2 correct. The proof involves a canonical recursive tree T that projects to a complete $\mathbf{\Sigma^1_1}$ subset of ${}^\omega\omega$ and a tree T^* on $\omega \times \mathrm{On}$ such that

$$\mathrm{proj}([T^*]) = {}^\omega\omega - \mathrm{proj}([T^*])$$

holds in every uncountable transitive model of ZFC. Forcing and Shoenfield absoluteness can be used to reprove the classical theorem that $\mathbf{\Sigma^1_1}$ sets are Lebesgue measurable; the argument is due to Solovay.

Suppose that κ be a measurable cardinal. Martin showed that all $\mathbf{\Pi^1_1}$ sets are κ-homogeneous and all κ-homogeneous sets are determined.[13] The projection of a κ-homogeneous set is called κ-weakly homogeneous and there is a corresponding notion of a κ-weakly homogeneous tree. Martin and Solovay showed that if T is a κ-weakly homogeneous tree, then there is a tree T^* on $\omega \times \mathrm{On}$ such that

$$\mathrm{proj}([T^*]) = {}^{\omega}\omega - \mathrm{proj}([T^*])$$

in $V^{\mathbb{P}}$ whenever $\mathbb{P} \in V_\kappa$. We say that T and T^* are $<\kappa$ absolutely complemented and that their projections are $<\kappa$ absolutely Suslin. (This property of the projections is also called $<\kappa$ universally Baire.) Martin and Solovay used this to show that $V^{\mathbb{P}}$ is Σ^1_3 correct in $V^{\mathbb{P}*\mathbb{Q}}$ for all $\mathbb{P}*\mathbb{Q} \in V_\kappa$. Forcing and Martin-Solovay absoluteness can be used to see that $\mathbf{\Sigma^1_2}$ sets are Lebesgue measurable.

The main theorem of Martin and Steel [16] is that if δ is a Woodin cardinal and $A \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ is δ^+ homogeneous, then ${}^{\omega}\omega - \mathrm{proj}(A)$ is $<\delta$ homogeneous. This has many important corollaries. For example, suppose that $\delta < \kappa$ where δ is a Woodin cardinal and κ is a measurable cardinal. Then $\mathbf{\Pi^1_2}$ sets are $<\delta$ homogeneous and $\mathbf{\Sigma^1_3}$ sets are $<\delta$ weakly homogeneous. (The latter was proved by Woodin before Martin and Steel obtained their result.) By Martin, $\mathbf{\Pi^1_2}$ sets are determined. By Martin-Solovay, $\mathbf{\Sigma^1_3}$ sets are $<\delta$ absolutely Suslin and $V^{\mathbb{P}}$ is Σ^1_4 correct in $V^{\mathbb{P}*\mathbb{Q}}$ whenever $\mathbb{P}*\mathbb{Q} \in V_\delta$. Forcing and Σ^1_4 absoluteness can be used to see that $\mathbf{\Sigma^1_3}$ sets are Lebesgue measurable.

Martin and Steel also combined their main theorem with an earlier theorem of Woodin to see that if there are $\delta < \kappa$ such that δ is a limit of Woodin cardinals and κ is a measurable cardinal, then all sets of reals in $L(\mathbb{R})$ are $<\delta$ homogeneous and hence determined. (Note that we are no longer assuming that δ is a Woodin cardinal.) From this it follows that all sets of reals in $L(\mathbb{R})$ are $< \delta$ weakly homogeneous, that the theory of $L(\mathbb{R})$ cannot be changed by forcing and that all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable. (These last three consequences were proved before Martin and Steel obtained their result; see Woodin-Shelah [40] and Woodin [50].)

Now we turn to lower bounds on the large cardinal consistency strength of the properties discussed above.

5.9 Theorem. *Let Ω be a measurable cardinal. Suppose that for all posets $\mathbb{P} \in H(\Omega)$,*

$$(L_{\omega_1}(\mathbb{R}))^{V^{\mathbb{P}}} \equiv (L_{\omega_1}(\mathbb{R}))^V.$$

Then

$$K^c \models \textit{there is a Woodin cardinal.}$$

[13] For these and other notions discussed below, we refer to the Handbook chapter [22].

Theorem 5.9 is due to Woodin and appears as Theorem 7.4 of [42]. The proof uses Theorem 3.5, that $K \cap \mathrm{HC}$ is Σ_1 definable over $L_{\omega_1}(\mathbb{R})$. It also uses almost everywhere weak covering, Theorem 3.1, which allows us to use forcing to change the truth value of the statement that ω_1 of the universe is a successor cardinal of K.

5.10 Theorem. *Let Ω be a measurable cardinal. Then the following are equivalent.*

(1) For all posets $\mathbb{P} \in V_\Omega$,

$$(L(\mathbb{R}))^{V^{\mathbb{P}}} \equiv (L(\mathbb{R}))^V.$$

(2) For all posets $\mathbb{P} \in V_\Omega$,

$$(L(\mathbb{R})\text{-}\mathit{determinacy})^{V^{\mathbb{P}}}.$$

(3) For all posets $\mathbb{P} \in V_\Omega$,

$$(L(\mathbb{R}) \text{ } \mathit{Lebesgue\ measurability})^{V^{\mathbb{P}}}.$$

(4) For all posets $\mathbb{P} \in V_\Omega$,

$$(\mathit{there\ is\ no}\ \omega_1\ \mathit{sequence\ of\ distinct\ reals\ in}\ L(\mathbb{R}))^{V^{\mathbb{P}}}.$$

(5) There exists an $\Omega + 1$ iterable model of height Ω with infinitely many Woodin cardinals.

Theorem 5.10 is due independently to Steel and Woodin and appears as Theorem 3.1 in [43]. The proof that the failure of (5) implies the failure of (4) uses core model theory. Instead of K^c and K, an "excellent" premouse $\mathcal{P}$ is found so that the maximal countably complete construction above $\mathcal{P}$ yields a relativized weasel $K^c(\mathcal{P})$ such that all the Woodin cardinals of $K^c(\mathcal{P})$ are in $\mathcal{P}$ and $K^c(\mathcal{P})$ is $\Omega + 1$ iterable above $\mathcal{P}$. The relativized core model $K(\mathcal{P})$ is extracted from $K^c(\mathcal{P})$ as in Sect. 2. One of the main tools is a version of the recursive definition of K, Theorem 3.5, that shows $K(\mathcal{P}) \cap \mathrm{HC} \in L(\mathbb{R})$ in this more general context.

5.3. Ideals and Generic Embeddings

Let κ be an uncountable cardinal and let I be a κ-complete ideal on $\mathcal{P}(\kappa)$. Assume that I is κ^+-saturated. In other words, $\mathcal{P}(\kappa)/I$ has the κ^+-chain condition. Suppose that G is V generic over $\mathcal{P}(\kappa)/I$. Let

$$j : V \to M = \mathrm{Ult}(V, G)$$

be the ultrapower map computed in $V[G]$. Then M is transitive, $\mathrm{crit}(j) = \kappa$ and

$$^{<j(\kappa)}M \subseteq M.$$

Such a j is called a *generic almost huge embedding*. The story of saturated ideals from the forcing side is far too rich to tell here but we do mention a couple of results. Shelah showed that if δ is a Woodin cardinal, then there is a semiproper poset $\mathbb{P}$ with the δ chain condition such that the nonstationary ideal over ω_1 is $\aleph_2$ saturated in $V^{\mathbb{P}}$. The following result in this subsection comes close to showing that one Woodin cardinal is the exact consistency strength.

5.11 Theorem. *Assume that Ω is a measurable cardinal and $\kappa < \Omega$. Let $\mathbb{P} \in V_\Omega$ be a poset. Suppose that forcing with $\mathbb{P}$ produces a generic almost huge embedding. Then there is a model of height Ω that satisfies "there is a Woodin cardinal".*

Theorem 5.11 is due to Steel and appears as Theorem 7.1 of [42]. The proof uses core model theory. In particular, it uses forcing absoluteness, Theorem 3.4 and the recursive definition of K, Theorem 3.5.

If I is a countably complete non-trivial ideal on $\mathcal{P}(\omega_1)$, then I is $\aleph_1$-dense iff $\mathcal{P}(\omega_1)/I$ has a dense subset of cardinality $\aleph_1$. This implies that forcing with $\mathcal{P}(\omega_1)/I$ is equivalent to forcing with $\mathrm{Col}(\omega, \omega_1)$, which in turn implies that I is $\aleph_2$ saturated. It also implies that $\mathcal{P}(\omega_1)/I$ is weakly homogeneous in the sense of forcing; we just say that I is homogeneous in this case. Woodin showed that the existence of an $\aleph_1$-dense ideal is consistent relative to $L(\mathbb{R})$-determinacy in [49]. Using core models, Steel proved that if there is a homogeneous ideal on ω_1 and CH holds, then PD holds. Building on this, Woodin showed his hypothesis was optimal.

5.12 Theorem. *The following are equiconsistent over* ZFC*.*

(1) There is an $\aleph_1$-dense ideal over ω_1.

(2) $L(\mathbb{R})$-determinacy.

The passage from (1) to (2) uses core models. In particular, it uses K and a method due to Woodin known as the *core model induction*. Woodin proves that if $A \subseteq \mathbb{R}$ and $A \in L(\mathbb{R})$, then A is determined. One could say that his proof is by induction on the least $(\alpha, n) \in \mathrm{On} \times \omega$ such that $A \in \Sigma_{n+1}(J_\alpha(\mathbb{R}))$. Steel uses a version of the core model induction in [44].

5.4. Square and Aronszajn Trees

This section is actually on the failure of square and the non-existence of Aronszajn trees, i.e., the tree property.

If λ is an ordinal and $C = \langle C_\alpha \mid \alpha < \lambda \rangle$, then C is a *coherent sequence* iff for all limit $\beta < \lambda$,

- C_β is club in β and
- if $\alpha \in \lim(C_\beta)$, then $C_\alpha = \alpha \cap C_\beta$.

If C is a coherent sequence, then D is a *thread* of C iff D is club in λ and $C_\alpha = \alpha \cap D$ for all $\alpha \in \lim(D)$. The principle $\square(\lambda)$ says that there is a coherent sequence of length λ with no thread. The principle $\square_\kappa$ says that there is a coherent sequence C of length κ^+ such that $\mathrm{type}(C_\alpha) \leq \kappa$ for all limit $\alpha < \lambda$. Coherent sequences are the topic of the Handbook chapter [46]. In this and the next section, it is convenient to set $\mathfrak{c} = 2^{\aleph_0}$.

5.13 Theorem. *Let $\kappa \geq \max(\aleph_2, \mathfrak{c})$. Suppose that both $\square_\kappa$ and $\square(\kappa)$ fail. Then $L(\mathbb{R})$-determinacy holds.*

See [29], which explains credit for Theorem 5.13 and related results, and has a proper introduction. Two basic elements of the proof are generalizations of Theorems 3.2 and 3.13. Theorem 3.6 is also used. The author derived PD from the hypothesis of Theorem 5.13. In fact, he showed that $M_n(X)$ exists for all $n < \omega$ and bounded $X \subseteq \kappa^+$. Steel observed that the author's proof meshed with techniques from [44] to give the result as stated.

Todorcevic proved that if $\square(\kappa)$ holds then there is an Aronszajn tree on κ. See [46]. From this and Theorem 5.13, one may conclude, for example, that if $\mathfrak{c} \leq \aleph_2$ and the tree property holds at $\aleph_2$ and $\aleph_3$, then $L(\mathbb{R})$-determinacy holds. Related theorems about the tree property were proved earlier without going through square; see [6] and its bibliography.

5.14 Theorem. *Suppose that κ is a singular strong limit cardinal and $\square_\kappa$ fails. Then $L(\mathbb{R})$-determinacy holds.*

Theorem 5.14 is Theorem 0.1 of [44], which includes an explanation of credit and related results. The proof uses Theorem 3.12, a generalization of Theorem 3.2 and a version of Woodin's core model induction due to Steel.

5.15 Theorem. *Suppose that κ is a weakly compact cardinal and $\square_\kappa$ fails. Then $L(\mathbb{R})$-determinacy holds.*

Theorem 5.15 is Corollary 8 of [31], which includes an explanation of credit. Two basic elements of the proof are generalizations of Theorems 3.3 and 3.12.

5.16 Theorem. *Suppose that κ is a measurable cardinal and $\square_\kappa$ fails. Then there is a model of height κ that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

See [2], which includes an explanation of credit. The proof uses a generalization of Theorem 3.12 for K^c and nothing about K. The hypothesis of Theorem 5.16 holds if κ is strongly compact by a well-known theorem of Solovay. Woodin has shown that the conclusion of Theorem 5.16 implies the consistency of $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$ where $\mathrm{AD}_\mathbb{R}$ asserts that all real games of length ω are determined.

The following is a very recent theorem due to Jensen, Schimmerling, Schindler and Steel [13].

5.17 Theorem. *Let $\kappa \geq \aleph_3$ be regular and countably closed. Suppose that both $\square_\kappa$ and $\square(\kappa)$ fail. Then there is a proper class model that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

5.5. Forcing Axioms

If $\mathcal{C}$ is a class of posets, then $\mathrm{FA}(\mathcal{C})$ says that for all $\mathbb{P} \in \mathcal{C}$ and $\mathcal{D}$ with $|\mathcal{D}| = \aleph_1$, there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$. By definition,

$$\mathrm{PFA} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is proper}\}).$$

This is the Proper Forcing Axiom. For any cardinal λ, we set

$$\mathrm{PFA}(\lambda) = \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is proper and } |\mathbb{P}| = \lambda\}).$$

Todorcevic and Velickovic showed that $\mathrm{PFA}(\mathfrak{c})$ implies that $\mathfrak{c} = \aleph_2$. See [48, Theorem 1.8] and [3, Theorem 3.16]. Todorcevic [47] showed that if λ is an ordinal such that $\mathrm{cf}(\lambda) \geq \aleph_2$, then $\mathrm{PFA}(\lambda^{\aleph_0})$ implies the failure of $\square(\lambda)$. Therefore $\mathrm{PFA}(\mathfrak{c}^+)$ implies the hypothesis of Theorem 5.13.

5.18 Corollary. $\mathrm{PFA}(\mathfrak{c}^+)$ *implies $L(\mathbb{R})$-determinacy.*

Note too that $\mathrm{PFA}(\mathfrak{c}^{++})$ implies the hypothesis of Theorem 5.17.

5.19 Corollary. $\mathrm{PFA}(\mathfrak{c}^{++})$ *implies that there is a proper class model that satisfies "there is a proper class of strong cardinals" and "there is a proper class of Woodin cardinals".*

Baumgartner and Shelah showed that PFA is consistent relative to the existence of a supercompact cardinal. The levels of the PFA hierarchy described above do not require a supercompact cardinal. For example, Neeman and Schimmerling [25] showed that the consistency strength of $\mathrm{PFA}(\mathfrak{c}^+)$ is strictly less than the existence of a cardinal κ that is κ^+-supercompact. More about this shortly.

By definition,

$$\mathrm{SPFA} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is semi-proper}\})$$

and

$$\mathrm{MM} \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves stationary subsets of } \omega_1\}).$$

These are the Semi-proper Forcing Axiom and Martin's Maximum respectively. It is straightforward to see that MM implies SPFA, which in turn implies PFA. Foreman, Magidor and Shelah showed that MM is consistent relative to a supercompact cardinal; see [7, Theorem 5]. Their proof used Shelah's revised countable support iteration. (Donder and Fuchs [5] is a good source for this.) Later, Shelah [39] proved that SPFA and MM are equivalent.

Recall that a poset $\mathbb{P} = (P, <_P)$ is λ-*linked* iff there is a function $\ell : P \to \lambda$ such that for all $p, q \in P$, if $\ell(p) = \ell(q)$, then p and q are compatible in $\mathbb{P}$. Here are two obvious comments. If $|P| = \lambda$, then $\mathbb{P}$ is λ-linked. If $\mathbb{P}$ is λ-linked, then $\mathbb{P}$ has the λ^+-chain condition. For any cardinal λ, we define

$$\mathrm{SPFA}(\lambda\text{-linked}) \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ is semi-proper and } \lambda\text{-linked}\})$$

and

$$\mathrm{MM}(\lambda) \equiv \mathrm{FA}(\{\mathbb{P} \mid \mathbb{P} \text{ preserves stationary subsets of } \omega_1 \text{ and } |P| = \lambda\}).$$

Shelah [39] showed that SPFA implies MM. In [25], this theorem is refined to SPFA($\mathfrak{c}^+$-linked) implies MM($\mathfrak{c}$). This is useful because Neeman and Schimmerling also show in [25] that SPFA($\mathfrak{c}^+$-linked) is consistent relative to the existence of a cardinal λ that is (λ, Σ^2_1)-subcompact. Without reproducing the definition, we remark that a witness that λ is (λ, Σ^2_1)-subcompact is a certain family of elementary embeddings of the form

$$\pi : H(\kappa^+) \to H(\lambda^+)$$

with $\mathrm{crit}(\pi) = \kappa$ and $\pi(\kappa) = \lambda$. Our point here is that each embedding of this sort comes from a superstrong extender. Consequently, (λ, Σ^2_1)-subcompactness is strictly weaker than κ^+-supercompactness in the large cardinal hierarchy. The consistency proof in [25] of SPFA($\mathfrak{c}^+$-linked) uses a revised countable support iteration of semi-proper posets of length λ as did Shelah's consistency proof of SPFA. Not surprisingly, if countable supports and proper posets are used instead, then one obtains a model of PFA($\mathfrak{c}^+$-linked) starting from the same large cardinal in the ground model. The theory of extender models can accommodate (λ, Σ^2_1)-subcompactness but core model techniques are not sufficiently developed to measure the consistency strength of PFA($\mathfrak{c}^+$-linked). However, there is evidence towards an equiconsistency: Neeman [24] showed that in order to force PFA($\mathfrak{c}^+$-linked) by proper forcing over an extender model, if λ is $\aleph_2$ of the generic extension, then λ is (λ, Σ^2_1)-subcompact in the ground model.

Taking a fundamentally different approach, Woodin [51] established that MM($\mathfrak{c}$) is consistent relative to the theory

$$\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \Theta \text{ is regular}$$

where

$$\Theta = \sup(\{\alpha \in \mathrm{On} \mid \text{ there is a surjection } f : \mathbb{R} \to \alpha\}).$$

The proof uses Woodin's $\mathbb{P}_{\max}$ theory; see the Handbook chapter [15] for an introduction to this technique.

Todorcevic showed that MM($\mathfrak{c}$) implies the Stationary Reflection Principle SRP(ω_2), which says that for every stationary $S \subseteq \mathcal{P}_{\omega_1}(\omega_2)$, if for all stationary $T \subseteq \omega_1$, $\{X \in S \mid X \cap \omega_1 \in T\}$ is stationary in $\mathcal{P}_{\omega_1}(\omega_2)$, then there exists an α such that $\omega_1 < \alpha < \omega_2$ and $S \cap \mathcal{P}_{\omega_1}(\alpha)$ contains a club in

$\mathcal{P}_{\omega_1}(\alpha)$. This version comes from Definition 9.74(3) and Lemma 9.75(1) of Woodin [51]. The reason we bring this up here is the following core model result of Steel and Zoble [45].

5.20 Theorem. SRP(ω_2) *implies* $L(\mathbb{R})$*-determinacy.*

The proof builds on that of Corollary 9.86 of Woodin [51], which says that SRP(ω_2) implies PD.

Another well-known variant of MM is Bounded Martin's Maximum or BMM, which says that if $\mathbb{P}$ preserves stationary subsets of ω_1, then

$$(H(\omega_2))^V \prec_{\Sigma_1} (H(\omega_2))^{V^{\mathbb{P}}}.$$

Woodin has shown that BMM is consistent relative to the existence of $\omega + 1$ many Woodin cardinals; see [51, Theorem 10.99]. The following lower bound by Schindler [35] uses core models.

5.21 Theorem. BMM *implies that for every set* X *there is a model with a strong cardinal containing* X*.*

5.6. The Failure of UBH

The theory of iteration trees was initiated by Martin and Steel in the context of inner models in [17] and determinacy in [16]. Three Hypotheses are isolated in Sect. 5 of the former paper: UBH (Unique Branches), CBH (Cofinal Branches), and SBH (Strategic Branches). These hypotheses have to do with iteration trees on V but their motivation is the construction of inner models with large cardinals. Results, both positive and negative, about the three hypotheses and their variants give useful information towards a solution to the inner model problem.

Woodin showed that UBH and CBH are false assuming sufficient large cardinals. This lead to the question of consistency strength and the following core model result of Steel [43].

5.22 Theorem. *Suppose that there is a non-overlapping iteration tree* $\mathcal{T}$ *on* V *with cofinal wellfounded branches* $b \neq c$*. Then there is an inner model with infinitely many Woodin cardinals. If, in addition,*

$$\delta(\mathcal{T}) \in \mathrm{ran}(i^{\mathcal{T}}_{0,b}) \cap \mathrm{ran}(i^{\mathcal{T}}_{0,c}),$$

then there is an inner model with a strong cardinal that is a limit of Woodin cardinals.

Woodin eventually reduced the large cardinal assumption in his refutations of UBH and CBH to a supercompact cardinal. Motivated by this, Neeman and Steel [26] constructed counterexamples starting from much less in the way of large cardinals. For example, under a large cardinal assumption slightly stronger than the one mentioned in Theorem 5.22, they constructed an iteration tree on V with distinct cofinal wellfounded branches. See [26] for a discussion on additional results on UBH and CBH and their failure.

5.7. Cardinality and Cofinality

Shelah famously showed that if $\aleph_\omega$ is a strong limit cardinal, then $(\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4}$. See the Handbook chapter [1]. An important conjecture is that the actual bound is $\aleph_{\omega_1}$. The following theorem appears as Theorem 1.1 of [10]. It provides valuable information about what it would take to obtain a counterexample to the conjecture.

5.23 Theorem. *Let α be a limit ordinal. Suppose that $2^{|\alpha|} < \aleph_\alpha$ and $2^{|\alpha|^+} < \aleph_{|\alpha|^+}$ but $(\aleph_\alpha)^{|\alpha|} > \aleph_{|\alpha|^+}$. Then $M_n(X)$ exists for all $n < \omega$ and bounded $X \subseteq \aleph_{|\alpha|^+}$.*

The following theorem appears as Theorem 1.4 of [10]. Recently, Gitik showed that its hypothesis is consistent relative to the existence of a supercompact cardinal. See [8].

5.24 Theorem. *Let λ be a cardinal such that $\omega < \mathrm{cf}(\lambda) < \lambda$. Suppose that $\{\kappa < \lambda \mid 2^\kappa = \kappa^+\}$ is stationary and co-stationary in λ. Then $M_n^\#(X)$ exists for all $X \subseteq \lambda$.*

In [9], Gitik showed that if there is a proper class of strongly compact cardinals, then there is a model of ZF in which all uncountable cardinals are singular. Towards measuring the consistency strength of this statement, Daniel Busche showed the following, which will appear in his PhD thesis.

5.25 Theorem. *Suppose that all uncountable cardinals are singular. Then* AD *holds in $L(\mathbb{R})^{\mathrm{HOD}^{\mathbb{P}}}$ for some $\mathbb{P} \in \mathrm{HOD}$.*

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21. Structural Consequences of AD

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Contents

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1. Introduction

In this chapter we survey recent advances in descriptive set theory, starting (roughly) from where Moschovakis' book [31] ends. Our survey is not intended to be complete, but focuses mainly on the structural consequences of determinacy for the model $L(\mathbb{R})$, including the important case of the projective sets. By "structural" we are referring to the combinatorial theory of the pointclasses (for example, the scale property which in some sense describes the structure of the set) as well as the cardinal structure up to the natural ordinal associated with these pointclasses. This might include determining their cofinalities, partition properties, and so forth.

The Axiom of Determinacy (AD), is the assertion that every two-player integer game is determined; we review the basic concepts below. The axiom was introduced by Mycielski and Steinhaus in the 1960s, and it soon became apparent that AD was a powerful tool for unlocking the combinatorial structure of sets of reals, and a program for doing this was begun. One of the central achievements during this period was the extension, assuming Projective Determinacy, to the general projective sets of the basic structural theory of the $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Sigma}^1_1$ sets developed by the "classical" descriptive set theorists from the 1920s through the 1940s. References [31] and [18] provide detailed accounts of the history of these developments. The results we discuss here can thus be viewed as extensions and refinements of the basic determinacy theory developed by the descriptive set theorists of the 1960s and 1970s as described in [31, 18].

We note that more recent work of Martin, Steel, and Woodin [28, 41] (described in a little more detail below) has pinpointed the connection between large cardinal axioms and various levels of determinacy, including $\mathrm{AD}^{L(\mathbb{R})}$.

We work throughout this chapter in the base theory $\mathrm{ZF} + \mathrm{DC}$ (except at a few points where we mention the use of DC). However, we will be dealing almost entirely in this chapter with the consequences of AD. In particular we assume AD throughout Sects. 3–5. In Sect. 2 we mention explicitly the hypotheses as needed. The reader should recall that AD contradicts the Axiom of Choice as well as many of its consequences. In particular, successor cardinals need not be regular and measurable cardinals need not be limit cardinals.

There were two ways in which the earlier theory was inadequate. First, the theory of the projective sets was described largely in terms of the so-called projective ordinals, the $\boldsymbol{\delta}^1_n$. The first four of these were computed, and several general results were proved (a good reference here is [16]). Kunen initiated a program for computing the $\boldsymbol{\delta}^1_n$. The idea of a homogeneous tree, which plays a central part in the program, arose independently in the work of Kunen and Martin. Kechris and Martin independently (see [17]) then formulated the general notion of a homogeneous tree. Despite this important progress, however, the values of $\boldsymbol{\delta}^1_n$ for $n \geq 5$, and the combinatorial structure of the intervening cardinals were left open. Secondly, the projective sets represent only the sets of reals occurring in the first level of the $L(\mathbb{R})$ hierarchy. It is generally believed, however, that AD should suffice to develop the structural theory (in some reasonable sense) of the entire model $L(\mathbb{R})$. Thus, two central goals emerge: first, to refine the arguments from the projective sets to permit a detailed analysis of the cardinal structure within the projective ordinals, and secondly, to extend this analysis as far as possible.

In the early 1980s, Martin and Steel [27] showed that $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})}$ was the largest pointclass in $L(\mathbb{R})$ having the scale property. Extending their methods, Steel [37] developed an analysis of the scale property in $L(\mathbb{R})$. This important work can be thought of as extending both the scale analysis of the projective sets and also the "fine structure" theory of L (developed by Jensen), and uses methods of both. This fine structure theory of $L(\mathbb{R})$, like the earlier theory of the projective sets, is not detailed enough to analyze the fine combinatorial structure of the cardinals, nor to answer many questions about the model $L(\mathbb{R})$ (though it suffices to answer many "scale type" questions, e.g., showing which pointclasses are the κ-Suslin sets for some κ, or showing every reliable cardinal is a Suslin cardinal; see [37]).

In the early 1980s, Martin [23] obtained a result on the ultrapowers of $\boldsymbol{\delta}^1_3$ by the normal measures on $\boldsymbol{\delta}^1_3$. Building on this and some joint work with Martin, the author computed $\boldsymbol{\delta}^1_5$. In the mid-1980s, this was extended to compute all the $\boldsymbol{\delta}^1_n$, and to develop the combinatorics of the cardinal structure of the cardinals up to that point. The analysis, naturally, proceeded by induction. The complete "first-step" of the induction appears in [11]. The analysis revealed a rich combinatorial structure to these cardinals. Indeed, even the answer $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$ hints at such a structure (in general $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\omega(2n-1)+1}$, where $\omega(0) = 1$ and $\omega(n+1) = \omega^{\omega(n)}$). A goal, then, is to extend some version of this "very-fine" structure theory to the entire model $L(\mathbb{R})$. In the late 1980s the author extended the analysis further, up to the least inaccessible cardinal in $L(\mathbb{R})$, although this lengthy analysis has never been written up. It was clear, however, that new, serious problems were being encountered shortly past the least inaccessible. In [10], for example, results were given that show that the theory fell far short of $\kappa^{\mathbb{R}}$, the ordinal of the inductive sets (the Wadge ordinal of the least non-selfdual pointclass closed under real quantification).

More recently, attempts have been made to isolate some of the "global" obstructions to extending the detailed $L(\mathbb{R})$ analysis. Some combinatorial principles were thus formulated which seem to be necessary for extending the theory sufficiently high in $L(\mathbb{R})$ and which seem not to be provable by an inductive "from below" argument. The most important principle along these lines is called the weak square property, $\boxminus_{\kappa,\lambda}$, for κ a Suslin cardinal and λ an ordinal $< \Theta$ (the supremum of the lengths of prewellorderings of the reals). Recently, the author has established this choice-like principle (to be defined in Sect. 6). This has a number of consequences. For example, it follows that every regular κ which is either a Suslin cardinal or the successor of a Suslin cardinal is $\boldsymbol{\delta}^2_1$-supercompact in $L(\mathbb{R})$. This and other global choice-like principles will be discussed in Sect. 6.

At the time of this writing, then, we may summarize the current situation as follows. A detailed structural analysis for an initial segment of $L(\mathbb{R})$ including the projective sets is known. Certain choice-like principles which seem to be important for further extensions have been established globally (that is, below Θ). What remains is to identify the further global principles required, and integrate them with the inductive analysis.

In the present chapter we will survey both approaches mentioned above. We will adopt a somewhat informal style at times in order to keep this chapter reasonably self-contained, still give proofs, and keep the discussion to a reasonable length. In Sect. 2 we will collect together and review various results from descriptive set theory and determinacy theory we will need. We will present proofs for some of these results of particular significance for us, and reference the others. In Sect. 3 we develop the AD theory of the Suslin cardinals, culminating in a classification theorem. In Sect. 4 we will develop a theory of "trivial" descriptions. This is not actually necessary, and in fact was omitted from [11]. Descriptions are the combinatorial ingredients that "describe" how to generate equivalence classes of functions with respect to certain measures. At the level of ω_1 (here the measures are simply the n-fold products of the normal measure), descriptions are trivial enough objects that they need not be introduced explicitly, but rather absorbed into the notation. Nevertheless, by introducing them explicitly the reader can see the general flavor of the arguments while the combinatorics is still trivial. Using this approach, we will show in this section the strong partition relation on ω_1, the weak partition relation on $\boldsymbol{\delta}^1_3$, and give a new proof of the Kechris-Martin Theorem for $\boldsymbol{\Pi}^1_3$. In all cases, our proofs will use only the theory of the trivial descriptions and techniques that will generalize to higher levels (in particular, no use is made of the theory of indiscernibles for L). In Sect. 5 we will introduce the (non-trivial) notion of a description of level 1. This corresponds to the analysis for computing $\boldsymbol{\delta}^1_5$ and proving the strong partition relation on $\boldsymbol{\delta}^1_3$, etc. Although we state complete definitions and theorems, we will illustrate proofs here frequently by considering examples which show the reader the ideas involved without getting lost in details. In Sect. 6 we introduce $\boxminus_{\kappa,\lambda}$ and other global choice-like principles. Some of the results presented here are of interest in their own right, and several are new.

We hope Sects. 4, 5 will provide a good introduction to, and an overview of, the modern theory of the projective sets (that is, the developments since [31]), and Sect. 6 will give some insight into the problems being faced in extending the theory and their possible solutions.

Although the focus of this chapter is on the consequences of AD, we mention briefly some connections with other hypotheses. Martin (see [25] and [26]) showed Borel Determinacy is a theorem of ZFC, although H. Friedman [4] showed that $\aleph_1$ iterations of the power set axiom applied to the reals is needed to prove it. Martin also showed $\boldsymbol{\Pi}^1_1$-determinacy followed from the existence of $x^\#$ for every $x \in \omega^\omega$, and Harrington proved the converse. Martin also showed the curious result that $\boldsymbol{\Delta}^1_{2n}$-determinacy implies $\boldsymbol{\Sigma}^1_{2n}$-

determinacy (see [19] for another proof). More recently, Martin and Steel [28] showed that the existence of n Woodin cardinals (see [28] for the definition) plus a larger measurable implies $\mathbf{\Delta}^1_{n+1}$-determinacy. Woodin showed that the existence of ω many Woodin cardinals plus a larger measurable implies $\mathrm{AD}^{L(\mathbb{R})}$. In fact, $\mathbf{\Delta}^1_{n+1}$-determinacy is equiconsistent with the existence of n Woodin cardinals, and AD is equiconsistent with the existence of ω many Woodin cardinals. Thus, the AD hypothesis now has the added measure of respect that $\mathrm{AD}^{L(\mathbb{R})}$ follows from more "conventional" large cardinal hypotheses. It should be noted, however, that the program of using AD to explore the structural theory of the projective sets and beyond was begun in the 1960s, well before this connection was known.

We review now some notation and terminology that we will use throughout the chapter. We generally work in the Polish space (complete, separable, metric space) ω^ω, the space of functions from ω to ω topologized with the product of the discrete topologies on ω. As a topological space this is homeomorphic to the space of irrationals, but any two uncountable Polish spaces are Borel isomorphic (in fact, isomorphic by a $\mathbf{\Delta}^0_3$-measurable function). We follow the usual convention of referring to the elements of ω^ω as "reals". By a *perfect product space* we mean a space of the form $X = X_1 \times \cdots \times X_n$, where each $X_i = \omega$ or ω^ω (ω always with the discrete topology), and at least one factor is ω^ω. All perfect product spaces are recursively homeomorphic to ω^ω by recursive coding and decoding maps, whose notation we now standardize.

For each $n \geq 2$, fix a recursive bijection

$$(m_0, \dots, m_{n-1}) \to \langle m_0, \dots, m_{n-1} \rangle_n$$

from ω^n to ω which is increasing in each argument, and let

$$m \to ((m)_0, \dots, (m)_{n-1})$$

denote the recursive inverse map. Let $(x_0, x_1, \dots) \to \langle x_0, x_1, \dots \rangle$ also denote the induced recursive bijection from $(\omega^\omega)^\omega$ to ω^ω defined by

$$\langle x_0, x_1, \dots \rangle(m) = x_{m_0}(m_1),$$

where (m_0, m_1) refers to the inverse of the coding map $\langle\ \rangle_2$. Similarly, for any perfect product space $X = X_1 \times \cdots \times X_{n-1}$, there is a recursive bijection between X and ω^ω, we will use the same notation $(x_0, \dots, x_{n-1}) \to \langle x_0, \dots, x_{n-1} \rangle_n$, and $x \to ((x)_0, \dots, (x)_{n-1})$, for the coding and decoding maps. Since the precise meaning is generally clear from the context, we will usually drop the subscripts and extra parentheses from the notation.

For X a perfect product space and $A \subseteq X$, we write A^c for $X - A$ (it will always be clear which X we are referring to). If $R \subseteq X \times Y$, the domain of R is defined by $\mathrm{dom}(R) = \{x : \exists y\ (x, y) \in R\}$. For any $x \in X$, we let R_x denote the section of R at x, that is, $R_x = \{y : (x, y) \in R\}$.

A (boldface) *pointclass* $\mathbf{\Gamma}$ is a collection of subsets of perfect product spaces X which is closed under continuous inverse image. That is, if $f : X \to Y$ is continuous and $A \subseteq Y$ is in $\mathbf{\Gamma}$, then $B = f^{-1}(A)$ is in $\mathbf{\Gamma}$. We also say A is *Wadge reducible* to B, written $A \leq_w B$. As is customary in descriptive set theory, we frequently use logical notation in describing sets, and thus we write $A(x)$ for $x \in A$. Thus we may rewrite the above as $B(x) \longleftrightarrow A(f(x))$, and for this reason pointclasses are referred to as being closed under continuous substitution (or Wadge reduction). Likewise $\neg A(x)$ means $x \notin A$. For $\mathbf{\Gamma}$ a pointclass, $\check{\mathbf{\Gamma}}$ denotes the *dual* pointclass, that is, $A \in \check{\mathbf{\Gamma}}$ iff $A^c \in \mathbf{\Gamma}$. We say $\mathbf{\Gamma}$ is *non-selfdual* if $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$, and otherwise say $\mathbf{\Gamma}$ is *selfdual.* We let $\exists^\omega$ and $\exists^{\omega^\omega}$ denote existential quantification over ω and ω^ω respectively, and likewise for $\forall^\omega$ and $\forall^{\omega^\omega}$. We apply this notation also to pointclasses. For example, $\exists^{\omega^\omega}\mathbf{\Gamma}$ denotes the $A \subseteq X$ for which there is a $B \subseteq X \times \omega^\omega$ such that $\forall x\ (A(x) \longleftrightarrow \exists y \in \omega^\omega\ B(x,y))$. For $\mathbf{\Gamma}$ a (usually non-selfdual) pointclass we let $\mathbf{\Delta}(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. When $\mathbf{\Gamma}$ is understood we frequently just write $\mathbf{\Delta}$.

Let X, Y be perfect product spaces, and $A \subseteq X$, $B \subseteq Y$. Assuming AD, Wadge's Lemma asserts that either $A \leq_w B$ or $B \leq_w A^c$. In fact, the proof shows something stronger. For example, suppose $X = Y = \omega^\omega$, and let $A, B \subseteq \omega^\omega$. Consider the integer game where I plays integers $a(i)$, and II plays $b(i)$, thereby producing reals $a, b \in \omega^\omega$, and II wins the run iff $(a \in A \longleftrightarrow b \in B)$. If II has a winning strategy τ, then τ defines a *Lipschitz continuous function* (which we also call τ) from ω^ω to ω^ω. By this we mean $\tau(a){\restriction}n$ depends only on $a{\restriction}n$. Also, $a \in A \longleftrightarrow \tau(a) \in B$. If I has a winning strategy σ, we likewise get a Lipschitz continuous function σ such that $b \in B \longleftrightarrow \sigma(b) \in A^c$. If we let $\leq_l$ denote reduction by a Lipschitz continuous function, we therefore have either $A \leq_l B$ or $B \leq_l A^c$.

For X a Polish space, we let $\mathbf{\Sigma}^0_1$ (respectively $\mathbf{\Pi}^0_1$) denote the collection of open (respectively closed) subsets of X. For $\alpha < \omega_1$, we recursively define $\mathbf{\Sigma}^0_\alpha$ to be the sets $A \subseteq X$ which are countable unions of sets A_n, each of which lies in $\mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$. Also, $\mathbf{\Pi}^0_\alpha = \check{\mathbf{\Sigma}}^0_\alpha$, and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$. The sets which are $\mathbf{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$ (equivalently $\mathbf{\Pi}^0_\alpha$ for some $\alpha < \omega_1$) are the *Borel* sets. The *projective hierarchy* is defined as follows. The $\mathbf{\Sigma}^1_1$ (or *analytic*) sets are the sets which are continuous images of closed sets in Polish spaces. An equivalent definition is $\mathbf{\Sigma}^1_1 = \exists^{\omega^\omega}\mathbf{\Pi}^0_1$. Also, we define $\mathbf{\Pi}^1_1 = \check{\mathbf{\Sigma}}^1_1$, and $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. Suslin's theorem says that the $\mathbf{\Delta}^1_1$ sets are exactly the Borel sets. In general, we define $\mathbf{\Sigma}^1_n = \exists^{\omega^\omega}\mathbf{\Pi}^1_{n-1}$, $\mathbf{\Pi}^1_n = \check{\mathbf{\Sigma}}^1_n$, and $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$. The *lightface* projective classes, Σ^1_n, Π^1_n, Δ^1_n are defined an analogous manner, except at the bottom we take Σ^0_1 to be the collection of sets of the form $\bigcup_{n \in \omega} V_{f(n)}$, where $f \colon \omega \to \omega$ is recursive, and $\{V_n\}_{n \in \omega}$ is a recursive presentation for the space X. See [31] for further details (we will always have X a perfect product space, in which case any reasonable enumeration of the usual basis for X will be a recursive presentation).

For s a finite sequence (i.e., a function with domain some $n \in \omega$), we let $\mathrm{lh}(s) = n$ denote the length of s. Thus, $s{\restriction}m$, for $m \leq n$, is the initial subsequence of s of length m.

By a tree on a set X we mean a set $T \subseteq X^{<\omega}$ closed under subsequence. Given sets X and Y, we frequently identify a tree T on the set $X \times Y$ with a set $T \subseteq \{(s,t) \in (X \times Y)^{<\omega} : \mathrm{lh}(s) = \mathrm{lh}(t)\}$ closed under initial segment, that is, if $(s,t) \in T$, then $(s\restriction m, t\restriction m) \in T$ for any $m < \mathrm{lh}(s)$. Similarly, we identify trees on $X_1 \times \cdots \times X_n$ with subsets of $X_1^{<\omega} \times \cdots \times X_n^{<\omega}$. $[T]$ denotes the set of paths or branches through T. We say a tree is *well-founded* iff it is well-founded under the extension relation, and otherwise say it is ill-founded. Thus, T is ill-founded iff $[T] \neq \emptyset$. If T is ill-founded, and X is equipped with a wellorder $<_X$, then T has a *left-most* branch $f \in [T]$. That is, for all $g \in [T]$, if $f \neq g$, then for the least n such that $f(n) \neq g(n)$ we have $f(n) <_X g(n)$. For T a tree on $X \times Y$, and $x \in X^\omega$, we let $T_x = \{s \in Y^{<\omega} : (x\restriction \mathrm{lh}(s), s) \in T\}$ denote the section of the tree T at x. For T a tree on $X \times Y$, we let $p[T]$ denote the projection of $[T]$, that is,

$$p[T] = \{x \in X^\omega : \exists y \in Y^\omega\ \forall n\ (x\restriction n, y\restriction n) \in T\}.$$

If T is a tree on $\lambda_1 \times \cdots \times \lambda_n$ and $\beta \in \mathrm{On}$, then we let

$$T\restriction\beta = \{(s_1, \ldots, s_n) \in T : \forall m < \mathrm{lh}(s_1)\ (s_1(m), \ldots, s_n(m) < \beta)\}.$$

If $(X, <_X)$ is a wellordered set, then for any $n \in \omega$ the set X^n is wellordered by the induced lexicographic ordering defined by:

$$s <_{\mathrm{lex}} t \longleftrightarrow \exists k(s(k) <_X t(k) \wedge \forall l < k\ s(l) = t(l)).$$

We let $|s|_{\mathrm{lex}}$ denote the rank of $s \in X^n$ in the lexicographic ordering. If T is a tree on X, then the wellorder $<_X$ also induces a linear order on T, known as the Kleene-Brouwer order, defined by:

$$\begin{aligned} s <_{\mathrm{KB}} t \longleftrightarrow{} & (s \text{ extends } t) \text{ or } \exists k < \min\{\mathrm{lh}(s), \mathrm{lh}(t)\} \\ & (s(k) <_X t(k) \wedge \forall l < k\ s(l) = t(l)). \end{aligned}$$

The Kleene-Brouwer order on T is a wellordering iff T is well-founded, that is, $[T] = \emptyset$. If T is a tree on X and $s \in T$, we let $T(s)$ denote $\{t \in T : t \text{ extends } s\}$. If T is a well-founded tree, we let $|T|$ denote the rank of T. In this case, we let $|T|_{\mathrm{KB}}$ denote the rank of T in the Kleene-Brouwer order (this also implicitly depends on the wellorder of X). We always have $|T| \leq |T|_{\mathrm{KB}}$. Note that $|T(s)|$ is the rank of s in T, and we also denote this by $|s|_T$. Also, $|(T\restriction\alpha)(s)|$ denotes the rank of s in the tree $T\restriction\alpha$, and likewise for $|(T\restriction\alpha)(s)|_{\mathrm{KB}}$. More generally, if $\prec$ is any well-founded relation on $\delta \in \mathrm{On}$, we let $\prec\restriction\alpha$ denote $\prec \cap \{(\beta,\gamma) : \beta, \gamma < \alpha\}$. Likewise, $\prec(\alpha) = \prec \cap \{(\beta,\gamma) : \beta, \gamma \prec \alpha\}$. We let $|\prec|$ denote the rank of $\prec$. Similarly, $|\prec(\alpha)| = |\alpha|_\prec$ both denote the rank of α in the well-founded relation $\prec$.

By a game on a set X we mean a two player game where players I and II alternate playing $x_0, x_1, \ldots \in X$ building $\vec{x} \in X^\omega$, and I wins the run iff $\vec{x} \in A$, where $A \subseteq X^\omega$ is the *payoff* set. Although the game is officially identified with its payoff set $A \subseteq X^\omega$, we sometimes write G_A to denote this game for conceptual clarity. A strategy for I (II) in a game on X is a function from the sequences from X of even (odd) length into X. The strategy is winning for I (II) if every run of the game where I (II) follows the strategy is a win for I (II). A game A is *determined* if one of the players has a winning strategy. AD is the assertion that every game on ω is determined. A *quasi-strategy* for I (II) is a relation $R \subseteq X^{<\omega} \times X$ such that $\forall s \in X^{<\omega}\ \exists x \in X\ R(s,x)$, and $\forall s \in X^{<\omega}$ of odd (even) length, $\forall x \in X\ R(s,x)$. We think of a quasi-strategy as a multi-valued strategy. A quasi-strategy is winning for I (II) if every run $\vec{x} \in X^\omega$ of the game such that $\forall n\ R(\vec{x}\restriction n, x(n))$ is a win for I (II). A game G_A is *quasi-determined* if one of the two players has a winning quasi-strategy. If X can be wellordered (e.g., $X = \omega$), then from a winning quasi-strategy for one of the players we easily get a winning strategy for the same player. If X is a perfect product space, Y is a set, and $A \subseteq X \times Y^\omega$, then $\Game^Y A$ is the set $B \subseteq X$ defined by $x \in B$ iff I has a winning strategy in the game $G_A(x)$ where I and II alternate playing $y_0, y_1, y_2, \ldots \in Y$ producing $\vec{y} \in Y^\omega$, and I wins the run if $(x, \vec{y}) \in A$. We abbreviate this by writing $x \in B \longleftrightarrow \exists y_0\ \forall y_1\ \exists y_2\ \forall y_3 \cdots (x, \vec{y}) \in A$.

By a *measure* on a set X we mean a countably additive ultrafilter on X. Recall that assuming AD, every ultrafilter on a set X is countably additive, that is, a measure. If ν is a measure on X, and f is a function with domain X, we let $[f]_\nu$ denote the equivalence class of f in the ultrapower, that is, $f \sim g \longleftrightarrow \nu(\{x \in X : f(x) = g(x)\}) = 1$. When considering ultrapowers by measures, we also let $[f]_\nu$ denote the image of $[f]_\nu$ in the transitive collapse of the ultrapower. We often say "A has measure one" in place of $\nu(A) = 1$. We also write $\forall^*_\nu x \in X$ to abbreviate "for ν measure one many $x \in X$".

We introduce a useful notational convention. Let $\nu_1, \ldots, \nu_n$ be measures. If $P \subseteq \mathrm{On}$ and $\delta \in \mathrm{On}$, we let $\forall^*_{\nu_1}\alpha_1\ \forall^*_{\nu_2}\alpha_2 \cdots \forall^*_{\nu_n}\alpha_n\ P(\delta(\alpha_1, \ldots, \alpha_n))$ abbreviate the statement: if $[f_1]_{\nu_1} = \delta$ then for ν_1 almost all α_1, if $[f_2]_{\nu_2} = f_1(\alpha_1)$, then for ν_2 almost all α_2, if $[f_3]_{\nu_3} = f_2(\alpha_2), \ldots,$ for ν_n almost all α_n, $P(f_n(\alpha_n))$. It is easily checked that this statement is well-defined. We also extend this convention to properties of pairs of ordinals, etc. For example, given measures $\nu_1, \ldots, \nu_n$ and ordinals δ, ϵ, we might write $\forall^*_{\nu_1}\alpha_1 \cdots \forall^*_{\nu_n}\alpha_n\ \mathrm{cf}(\delta(\alpha_1, \ldots, \alpha_n)) \leq \epsilon(\alpha_1, \ldots, \alpha_n)$. If the measures are understood, we simply write $\forall^*\alpha_1, \ldots, \alpha_n\ P(\delta(\alpha_1, \ldots, \alpha_n))$, etc.

If $f : X \to \mathrm{On}$ is a function and $\alpha \in \mathrm{On}$, we write $N_f(\alpha)$ for the least ordinal in the range of f greater than α. Likewise, if $A \subseteq \mathrm{On}$, we write $N_A(\alpha)$ for the least ordinal greater than α in A.

2. Survey of Basic Notions

Throughout Sect. 2 we work in the base theory ZF + DC, stating other hypotheses explicitly as needed.

2.1. Prewellordering, Scales, and Periodicity

We begin with a review of the basic concepts of scale and prewellordering. The definition of a scale was introduced by Moschovakis, and represents a distillation of the key ideas in the Novikov-Kondo proof of uniformization for $\mathbf{\Pi}^1_1$ sets.

Recall the basic notions of pointclass, Wadge reduction, etc., that were defined in the introduction. If $\mathbf{\Gamma}$ is a pointclass, we say $U \subseteq \omega^\omega \times X$ in $\mathbf{\Gamma}$ is *universal* for $\mathbf{\Gamma}\restriction X$ (where $X = X_0 \times \cdots \times X_n$ is a perfect product space) if for every $B \subseteq X$ in $\mathbf{\Gamma}$, there is a $y \in \omega^\omega$ such that $B = U_y = \{x : (y, x) \in U\}$. Assuming AD, Wadge's Lemma implies that every non-selfdual pointclass $\mathbf{\Gamma}$ has universal sets. For suppose $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$. For every product space $X = X_0 \times \cdots \times X_n$, define $U_X \subseteq \omega^\omega \times X$ by $U_X(y, x_0, \ldots, x_n) \longleftrightarrow f_y(\langle x_0, \ldots, x_n\rangle) \in A$ where we view every $y \in \omega^\omega$ as coding a Lipschitz continuous function $f_y : \omega^\omega \to \omega^\omega$ (say by $f_y(a_0, \ldots, a_n) = (y(\langle a_0\rangle), \ldots, y(\langle a_0, \ldots, a_n\rangle))$. Clearly $U_X \in \mathbf{\Gamma}$. If $B \subseteq X$ is in $\mathbf{\Gamma}$, then by Wadge $B \leq_l A$, so for some y we have $B = (U_X)_y$.

The usual diagonal argument shows that a universal $\mathbf{\Gamma}$ set $U \subseteq \omega^\omega \times \omega^\omega$ cannot lie in $\check{\mathbf{\Gamma}}$, and thus if $\mathbf{\Gamma}$ has a universal set, it is non-selfdual. Also, the *s-m-n* and Recursion Theorems go through at this level of generality. Specifically, we have:

2.1 Theorem. *Let $\mathbf{\Gamma}$ be a pointclass with a universal set. Then there are universal sets $U_X \subseteq \omega^\omega \times X$ for all product spaces X with the following properties:*

(1) (s-m-n Theorem) For every $X = X_1 \times \cdots \times X_n$, $Y = X_1 \times \cdots \times X_n \times \cdots \times X_m$ where $m > n$, there is a continuous $s_{Y,X} : \omega^\omega \times X \to \omega^\omega$ such that

$$U_Y(y, x_1, \ldots, x_n, \ldots, x_m) \longleftrightarrow U_{X'}(s_{Y,X}(y, x_1, \ldots, x_n), x_{n+1}, \ldots, x_m)$$

where $X' = X_{n+1} \times \cdots \times X_m$.

(2) (Recursion Theorem) For every product space $X = X_1 \times \cdots \times X_n$ and $\mathbf{\Gamma}$ set $A \subseteq \omega^\omega \times X$, there is a $y^* \in \omega^\omega$ such that for all $x \in X$, $U_X(y^*, x) \longleftrightarrow A(y^*, x)$.

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ in $\mathbf{\Gamma}$ be universal for $\mathbf{\Gamma}$ subsets of ω^ω. For $X = X_1 \times \cdots \times X_n$ define $U_X(y, (x_1, \ldots, x_n)) \longleftrightarrow U(y_0, \langle y_1, x_1, \ldots, x_n\rangle)$, where here $y \to (y_0, y_1)$ denotes our recursive bijection from ω^ω to $\omega^\omega \times \omega^\omega$. Suppose $Y = X_1 \times \cdots \times X_n \times \cdots \times X_m$. Then

$$U_Y(y, (x_1, \ldots, x_n, \ldots, x_m)) \longleftrightarrow U(y_0, \langle y_1, x_1, \ldots, x_n, \ldots, x_m\rangle)$$

and

$$U_{X'}(s, (x_{n+1}, \ldots, x_m)) \longleftrightarrow U(s_0, \langle s_1, x_{n+1}, \ldots, x_m\rangle).$$

Thus, it suffices to take $s_{Y,X}(y, x_1, \ldots, x_n) = \langle \epsilon, \langle y, x_1, \ldots, x_n\rangle\rangle$ where ϵ is such that $U(\epsilon, \langle\langle y, x_1, \ldots, x_n\rangle, x_{n+1}, \ldots, x_m\rangle) \longleftrightarrow U(y_0, \langle y_1, x_1, \ldots, x_m\rangle)$ for all $y, x_1, \ldots, x_m$. That is, choose ϵ so that for all z

$$U(\epsilon, z) \longleftrightarrow U(z_{0,0,0}, \langle z_{0,0,1}, z_{0,1}, \ldots, z_{0,n}, z_1, \ldots, z_{m-n}\rangle)$$

which is possible as U is universal (here $z_{0,0,0}$ abbreviates $(((z)_0)_0)_0$, etc., and these decoding functions refer to the obvious product spaces).

As for the Recursion Theorem, fix $X = X_1 \times \cdots \times X_n$, and let $A \subseteq \omega^\omega \times X$ be in $\mathbf{\Gamma}$. Let $\epsilon \in \omega^\omega$ be such that $U(\epsilon, y, x) \longleftrightarrow A(s(y,y), x)$, where s is the s-m-n function from $(\omega^\omega)^2$ to ω^ω corresponding to the spaces ω^ω and $\omega^\omega \times X$. Thus, $U(s(\epsilon, y), x) \longleftrightarrow U(\epsilon, y, x) \longleftrightarrow A(s(y,y), x)$ for all y, x, where we have dropped the cumbersome subscripts on the U. Let then $y = \epsilon$, and thus $y^* = s(\epsilon, \epsilon)$. ⊣

Following Moschovakis, we call sets U_X satisfying Theorem 2.1 *good* universal sets. We shall frequently implicitly assume (without loss of generality) that our universal sets are good. Note that the construction of the U_X is uniform in the universal sets A.

We review now some of the general theory of prewellorderings and scales.

2.2 Definition. A *(regular) norm* on a set $A \subseteq \omega^\omega$ is a map ϕ from A into (onto) some ordinal. A norm $\phi : A \to \mathrm{On}$ is said to be a $\mathbf{\Gamma}$-*norm* if there are $\mathbf{\Gamma}, \check{\mathbf{\Gamma}}$ binary relations $\leq_\phi^{\mathbf{\Gamma}}$, $\leq_\phi^{\check{\mathbf{\Gamma}}}$ on ω^ω such that for all $y \in A$, $\forall x\ [(x \in A \wedge \phi(x) \leq \phi(y)) \longleftrightarrow x \leq_\phi^{\mathbf{\Gamma}} y \longleftrightarrow x \leq_\phi^{\check{\mathbf{\Gamma}}} y]$. A pointclass $\mathbf{\Gamma}$ has the *prewellordering property*, written $\mathrm{pwo}(\mathbf{\Gamma})$, if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-norm.

Norms can be identified with *prewellorderings* of A (that is, transitive, reflexive, connected binary relations $\preceq$ on A). We let $\prec$ denote the strict part of a prewellordering $\preceq$ and vice versa (i.e., $x \prec y \longleftrightarrow x \preceq y \wedge \neg y \preceq x$).

The above definition generalizes immediately to any perfect product space X as well. A standard and straightforward lemma (Theorem 4B.1 of [31]) says that if $\mathbf{\Gamma}$ is closed under $\wedge, \vee$, then $\phi : A \to \mathrm{On}$ is a $\mathbf{\Gamma}$-norm on $A \in \mathbf{\Gamma}$ iff the following relations are in $\mathbf{\Gamma}$:

$$x \leq_\phi^* y \longleftrightarrow x \in A \wedge (y \notin A \vee \phi(x) \leq \phi(y)),$$
$$x <_\phi^* y \longleftrightarrow x \in A \wedge (y \notin A \vee \phi(x) < \phi(y)).$$

In fact, to show that the existence of the starred relations implies the prewellordering property requires no closure assumptions on $\mathbf{\Gamma}$.

If $\mathbf{\Gamma}$ has the prewellordering property and is closed under $\vee$, $\wedge$, then any two $\mathbf{\Gamma}$ sets A, B have comparable $\mathbf{\Gamma}$-norms. That is, there are $\mathbf{\Gamma}$-norms ϕ, ψ on A, B respectively and $\mathbf{\Gamma}$ relations $\leq_{\psi,\phi}^{\mathbf{\Gamma}}$, $\leq_{\phi,\psi}^{\mathbf{\Gamma}}$ and $\check{\mathbf{\Gamma}}$ relations $\leq_{\psi,\phi}^{\check{\mathbf{\Gamma}}}$, $\leq_{\phi,\psi}^{\check{\mathbf{\Gamma}}}$ such that $\forall y \in A\ \forall x\ [(x \in B \wedge \psi(x) \leq \phi(y)) \longleftrightarrow x \leq_{\psi,\phi}^{\mathbf{\Gamma}} y \longleftrightarrow x \leq_{\psi,\phi}^{\check{\mathbf{\Gamma}}} y]$, and likewise $\forall y \in B\ \forall x\ [(x \in A \wedge \phi(x) \leq \psi(y)) \longleftrightarrow x \leq_{\phi,\psi}^{\mathbf{\Gamma}} y \longleftrightarrow x \leq_{\phi,\psi}^{\check{\mathbf{\Gamma}}} y]$. To see this, let $E = \{\langle i, z\rangle : (i = 0 \wedge z \in A) \vee (i = 1 \wedge z \in B)\}$. Let ρ be a $\mathbf{\Gamma}$-norm on E, and let $\phi(x) = \rho(\langle 0, x\rangle)$ for $x \in A$, and $\psi(x) = \rho(\langle 1, x\rangle)$ for $x \in B$. We can take, for example $x \leq_{\phi,\psi}^{\mathbf{\Gamma}} y$ iff $\langle 0, x\rangle \leq_\rho^{\mathbf{\Gamma}} \langle 1, y\rangle$. Note, however, that these norms are not regular.

2.3 Definition. A set $A \subseteq \omega^\omega$ is κ-*Suslin* if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$. Let $S(\kappa)$ denote the pointclass of κ-Suslin sets. A cardinal κ is a *Suslin cardinal* if $S(\kappa) - \bigcup_{\kappa' < \kappa} S(\kappa') \neq \emptyset$.

A closely related concept (see Lemma 2.5) is that of a scale.

2.4 Definition. A *semi-scale* $\{\phi_n\}_{n\in\omega}$ on a set $A \subseteq X$ (X a perfect product space) is a collection of norms ϕ_n on A such that if $\{x_m\}_{m\in\omega} \subseteq A$ is a sequence of points in A converging to x, and for all n, $\phi_n(x_m)$ is eventually constant, then $x \in A$. We say $\{\phi_n\}$ is an α *semi-scale* if all norms map into α. We say $\{\phi_n\}$ is a *scale* if it in addition satisfies the lower semi-continuity property: $\forall n\ \phi_n(x) \leq \lambda_n \doteq \lim_{m\to\infty} \phi_n(x_m)$. Likewise we define α-scale.

A (semi)-scale $\{\phi_n\}$ on A is a *good* (semi)-scale if whenever $x_m \in A$ and for all n, $\phi_n(x_m)$ is eventually constant, then $x = \lim_{m\to\infty} x_m$ exists (and thus $x \in A$).

A (semi)-scale is called *very good* if it is good and whenever $x, y \in A$ and $\phi_n(x) \leq \phi_n(y)$, then $\phi_i(x) \leq \phi_i(y)$ for all $i < n$.

A (semi)-scale is called *excellent* if it is very good and whenever $x, y \in A$ and $\phi_n(x) = \phi_n(y)$ then $x\restriction n = y\restriction n$ (assuming now $X = \omega^\omega$).

The notions of good α-scale, etc., are defined in the same manner, requiring the norms to map into α.

The next lemma shows the essential equivalence of these concepts.

2.5 Lemma. *For every $A \subseteq \omega^\omega$ and every $\alpha \in \mathrm{On}$, A is α-Suslin iff A has an α-semiscale iff A has an α-scale iff A has an excellent α-scale.*

Proof. Clearly excellent scale $\to$ very good scale $\to$ good scale $\to$ scale $\to$ semi-scale, for any α. If $\{\phi_n\}$ is a semi-scale on A into α, define the tree of the semi-scale as follows:

$$\begin{aligned}&((a_0,\dots,a_{n-1}),(\beta_0,\dots,\beta_{n-1})) \in T_\phi\\ &\quad\longleftrightarrow \exists x \in A\ \big[x\restriction n = (a_0,\dots,a_{n-1}) \wedge \phi_0(x) = \beta_0,\dots,\phi_{n-1}(x) = \beta_{n-1}\big].\end{aligned}$$

Clearly $A \subseteq p[T]$. If $(x, f) \in [T]$, then $\exists x_m \in A$ such that $x_m \to x$ and $\phi_n(x_m) \to f(n)$ for all n. Thus, $x \in A$ by definition of a semi-scale. Hence, $A = p[T]$.

Thus, it suffices to show that A is α-Suslin implies A admits an excellent α-scale. First note that A is α-Suslin iff A is κ-Suslin, where $\kappa = |\alpha|$. Thus we may assume $\alpha = \kappa$ is a cardinal. Fix a tree T on $\omega \times \kappa$ such that $A = p[T]$. We consider two cases.

First assume $\mathrm{cf}(\kappa) > \omega$. Then $\forall x \in A\ \exists \beta < \kappa\ (x \in p[T\restriction\beta])$. Define a tree S by:

$$\begin{aligned}&((a_0,\dots,a_{n-1}),(\beta_0,\dots,\beta_{n-1})) \in S\\ &\quad\longleftrightarrow \beta_0 > \beta_1,\dots,\beta_{n-1} \wedge ((a_0,\dots,a_{n-2}),(\beta_1,\dots,\beta_{n-1})) \in T.\end{aligned}$$

Thus, $A = p[S]$ as well. For $x \in A$, define

$$\phi_n(x) = |(f(0), x(0),\dots, f(n-1), x(n-1)|^*_{\mathrm{lex}},$$

where $f : \omega \to \kappa$ is the leftmost branch through S_x, and $|\vec{s}|^*_{\text{lex}}$ denotes the rank of $\vec{s}$ in the lexicographic ordering restricted to $\vec{t} \in \kappa^{2n}$ such that $t(0) > t(1), \dots, t(2n-1)$. Thus, $\phi_n : A \to \kappa$. Suppose $\{x_m\} \subseteq A$ and $\phi_n(x_m) \to \lambda_n$ for all n. Let $\vec{s}_n$ be such that $|\vec{s}_n|^*_{\text{lex}} = \lambda_n$. Then $\vec{s}_{n+1}$ extends $\vec{s}_n$ for all n, so $x_m \to x \in \omega^\omega$ and the $\vec{s}_n$ define an $f : \omega \to \kappa$ for which $(x, f) \in [T]$. This shows $\{\phi_n\}$ is a semi-scale, and the lower semi-continuity and excellence are easily verified.

Suppose next that $\text{cf}(\kappa) = \omega$. Let $\kappa_i < \kappa$ with $\sup_{i\in\omega} \kappa_i = \kappa$. We may assume T is a tree on $\omega \times (\kappa - \omega)$. Define a tree S on $\omega \times \kappa$ by "padding" T as follows. An element of S will be of the form

$$((a_0, \dots, a_{n-1}), (k_0, 0, \dots, 0, \beta_0, \dots, k_i, 0, \dots, 0, \beta_i, \dots))$$

such that:

(1) Each $k_l \in \omega$, and after k_l occur k_l 0's. Also, $\beta_l < \kappa_{k_l}$.

(2) $((a_0, \dots, a_j), (\beta_0, \dots, \beta_j)) \in T$, where j is maximal so that $(k_0 + 2) + \dots + (k_j + 2) \leq n$.

Note that if $(\vec{a}, \vec{s}) \in S$, then $\forall i\ \vec{s}(i) < \kappa_i$. Easily, $A = p[S]$ as well. We now define ϕ_n as in the previous case (using lexicographic ordering on $(\kappa_n)^{2n}$). It is easily checked that $\{\phi_n\}$ is an excellent α-scale. $\dashv$

One standard consequence of scales is the Kunen-Martin Theorem (cf. [31, Theorem 2G.2]) which we now state.

2.6 Theorem (Kunen, Martin). *Every κ-Suslin well-founded relation on ω^ω has length less than κ^+.*

We next recall the fundamental notion of a $\mathbf{\Gamma}$-scale, a notion introduced by Moschovakis.

2.7 Definition. A scale $\{\phi_n\}$ on a set A is a $\mathbf{\Gamma}$-*scale* if all of the norms ϕ_n are $\mathbf{\Gamma}$-norms. We say $\mathbf{\Gamma}$ has *the scale property*, scale($\mathbf{\Gamma}$), if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-scale.

The prewellordering and scale properties are the basic structural ingredients in descriptive set theory, and have numerous applications there (this theory is developed in [31]). For example, if pwo($\mathbf{\Gamma}$) and $\mathbf{\Gamma}$ is closed under $\forall^\omega$ and $\wedge$, $\vee$, then $\mathbf{\Gamma}$ has the number uniformization property, that is, every $A \subseteq \omega^\omega \times \omega$ in $\mathbf{\Gamma}$ can be uniformized by a $\mathbf{\Gamma}$ relation $B \subseteq A$. Namely, set

$$B(x, n) \longleftrightarrow (\forall m\ (x, n) \leq^* (x, m)) \wedge (\forall m < n\ (x, n) <^* (x, m)),$$

where $\leq^*, <^*$ correspond to a $\mathbf{\Gamma}$-norm on A. [The number uniformization property can also be shown directly for pointclasses of the form $\exists^\omega\mathbf{\Gamma}$ where $\mathbf{\Gamma}$ is closed under $\forall^\omega$ but not $\exists^\omega$.] Likewise, if $\mathbf{\Gamma}$ has the scale property and is closed under $\forall^{\omega^\omega}$ and $\wedge, \vee$, then every $\mathbf{\Gamma}$ relation $A \subseteq \omega^\omega \times \omega^\omega$ has a $\mathbf{\Gamma}$

uniformization. To see this, note that if $\{\phi_n\}$ is a $\boldsymbol{\Gamma}$ scale on $A \subseteq \omega^\omega \times \omega^\omega$, and we define

$$\psi_n(x,y) = \big|(\phi_0(x,y), x(0), y(0), \ldots, \phi_{n-1}(x,y), x(n-1), y(n-1)\big|_{\text{lex}},$$

then $\{\psi_n\}$ is a very good $\boldsymbol{\Gamma}$-scale on A. For $x \in \text{dom}(A)$, $n \in \omega$, let $s_n = (\alpha_0, x(0), y(0), \ldots, \alpha_{n-1}, x(n-1), y(n-1))$ be lexicographically least such that for some $(x, y_n) \in A$, $\psi_n(x, y_n) = |s_n|_{\text{lex}}$. Note that s_{n+1} extends s_n. By the scale property, there is a $(x,y) \in A$ with $\psi_n(x,y) = |s_n|_{\text{lex}}$ for all n, and by very goodness this y is unique. Thus if we define $B(x,y) \longleftrightarrow \forall z \forall n\ (x,y) \leq^*_{\psi_n} (x,z)$, then B uniformizes A.

It is a relatively straightforward ZF result that the prewellordering and scale properties propagate from a pointclass $\boldsymbol{\Gamma}$ closed under $\forall^{\omega^\omega}$ to $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. The important periodicity theorems assert that, granted sufficient determinacy, they also propagate from a pointclass $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$ to $\forall^{\omega^\omega}\boldsymbol{\Gamma}$. We state without proof the first two of the three periodicity theorems (proofs may be found in [31]). These theorems are due to Martin-Moschovakis, Moschovakis, and Moschovakis respectively. We note that DC is not required for the following two theorems.

2.8 Theorem (First Periodicity). *Let $\boldsymbol{\Gamma}$ be a pointclass closed under $\exists^{\omega^\omega}$ with* pwo($\boldsymbol{\Gamma}$). *If $\boldsymbol{\Delta}$-determinacy holds, then* pwo($\forall^{\omega^\omega}\boldsymbol{\Gamma}$).

2.9 Theorem (Second Periodicity). *Let $\boldsymbol{\Gamma}$ be a pointclass closed under $\exists^{\omega^\omega}$ and $\wedge, \vee$ with the scale property. If $\boldsymbol{\Delta}$-determinacy holds, then $\forall^{\omega^\omega}\boldsymbol{\Gamma}$ has the scale property.*

2.10 Remark. The proof of the Second Periodicity Theorem also shows that if $A \subseteq \lambda^\omega \times \omega^\omega$ admits a scale (that is, is Suslin), then so does $B \subseteq \lambda^\omega$, where $B(\vec{\alpha}) \longleftrightarrow \forall x \in \omega^\omega\ A(\vec{\alpha}, x)$.

Thus, assuming Projective Determinacy the pointclasses amongst the $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ having the scale property are $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Pi}^1_3$, $\boldsymbol{\Sigma}^1_4, \ldots$, exhibiting a periodicity of order two.

We also recall a version of the Third Periodicity Theorem, due also to Moschovakis. Because we will have specific need for this result later, we give the proof. For the version we state, we require DC.

Let X be a set, and $A \subseteq X^\omega$. Recall G_A is the game where I, II alternate playing $x_0, x_1, \ldots \in X$, and I wins iff $(x_0, x_1, \ldots) \in A$. Assume I has a winning quasi-strategy in the game G_A, and A admits a very good semi-scale $\{\phi_n\}$ (defined in an obvious way using X^ω in place of ω^ω, where X is given the discrete topology). We define (assuming sufficient determinacy) a canonical winning quasi-strategy τ for I in G_A as follows. Suppose $s, t \in X^{<\omega}$ are winning positions for I in G_A of the same odd length (i.e., II's turn to move). For $n \in \omega$, consider the game $G^n_{s,t}$ played as follows:

s	F $a(0)$			S $a(1)$	F $a(2)$			S $a(3)$
t		S $b(0)$	F $b(1)$			S $b(2)$	F $b(3)$	

$$
\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s_3 & \mathrm{F}\, a_3(0) & a_3(1) & \mathrm{F}\, a_3(2) & a_3(3) & \cdots \\
\hline
s_3 & a_3(0) & \mathrm{F}\, a_3(1) & a_3(2) & \mathrm{F}\, a_3(3) & \cdots \\
 & & & & & \\
s_2 & \mathrm{F}\, a_2(0) & a_2(1) & \mathrm{F}\, a_2(2) & a_2(3) & \cdots \\
\hline
s_2 & a_2(0) & \mathrm{F}\, a_2(1) & a_2(2) & \mathrm{F}\, a_2(3) & \cdots \\
 & & & & & \\
s_1 & \mathrm{F}\, a_1(0) & a_1(1) & \mathrm{F}\, a_1(2) & a_1(3) & \cdots \\
\hline
s_1 & a_1(0) & \mathrm{F}\, a_1(1) & a_1(2) & \mathrm{F}\, a_1(3) & \cdots \\
 & & & & & \\
s_0 & \mathrm{F}\, a_0(0) & \mathrm{I}\ a_0(1) & \mathrm{F}\, a_0(2) & \mathrm{I}\ a_0(3) & \cdots
\end{array}
$$

Figure 21.1

The game consists of two players F and S (first and second), making moves from X as shown. Let $a, b \in X^\omega$ be the sequences they produce. We say S wins the run of the game iff $s^\frown a \leq^*_{\phi_n} t^\frown b$. Let W_m, for odd m, be the set of winning positions for I in G_A of length m (i.e., I has a winning quasi-strategy starting from that position). For $s, t \in W_m$ set $s \leq^m_n t$ iff S has a winning quasi-strategy in $G^n_{s,t}$. We assume here that the games $G^n_{s,t}$ are quasi-determined. We claim that each $\leq^m_n$ is a prewellordering on each W_m. First note that there cannot be an infinite sequence $s_0, s_1, \ldots \in W_m$ such that $\forall i\ s_i \not\leq^m_n s_{i+1}$. For if so, fix winning quasi-strategies for F in all of the games $G^n_{s_i,s_{i+1}}$, and fill in the sequence of games as shown in Fig. 21.1, using DC.

Here F follows the fixed winning quasi-strategies on all of the boards (moves made by following one of F's winning quasi-strategies are marked with an F), S's moves in the various boards are obtained by copying as shown, except in the bottom run where S follows a fixed winning quasi-strategy for the game G_A starting from s_0 (these moves are marked with a I). Let a_0, a_1, a_1, a_2, $a_2, \ldots,$ be the sequences they produce. Thus $s_0{}^\frown a_0 \in A$ and $\phi_n(s_0{}^\frown a_0) > \phi_n(s_1{}^\frown a_1) > \cdots$, a contradiction. It follows that $\leq^m_n$ is wellfounded, reflexive, and connected on each W_m. Transitivity of $\leq^m_n$ also easily follows, since if $s \leq^m_n t$, $t \leq^m_n u$, but $\neg(s \leq^m_n u)$, we could play quasi-strategies for S in the first two games against one for F in the third game to get a contradiction. Thus, each $\leq^m_n$ is a prewellordering on each W_m.

Define the quasi-strategy τ as follows. If $s = (s(0), \ldots, s(2n-1))$ is a winning position for I in G_A of even length, $s^\frown a \in \tau$ iff $s^\frown a$ is a winning position for I and $s^\frown a \leq^{2n+1}_n s^\frown b$ for all $b \in X$ such that $s^\frown b$ is a winning position for I.

To see this is a winning quasi-strategy, suppose $a = (a(0), a(1), \ldots)$ is a run following τ. Consider then the play of the games as shown in Fig. 21.2.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$a(0)$	$a(1)$	$a(2)$	$a(3)$	$a(4)$	$a(5)$	S $a_3(6)$	$\cdots$
$a(0)$	$a(1)$	$a(2)$	$a(3)$	$a_2(4)$	S $a_2(5)$	$a_2(6)$	$\cdots$
$a(0)$	$a(1)$	$a(2)$	$a(3)$	S $a_2(4)$	$a_2(5)$	S $a_2(6)$	$\cdots$
$a(0)$	$a(1)$	$a_1(2)$	S $a_1(3)$	$a_1(4)$	S $a_1(5)$	$a_1(6)$	$\cdots$
$a(0)$	$a(1)$	S $a_1(2)$	$a_1(3)$	S $a_1(4)$	$a_1(5)$	S $a_1(6)$	$\cdots$
I $a_0(0)$	S $a_0(1)$	I $a_0(2)$	S $a_0(3)$	I $a_0(4)$	S $a_0(5)$	I $a_0(6)$	$\cdots$

Figure 21.2

Here I is following a fixed winning quasi-strategy for G_A on the bottom run (these moves are marked with a I), and S is following winning quasi-strategies for the $G^n_{s,t}$, $s = (a(0), \ldots, a(2n))$, $t = (a(0), \ldots, a(2n-1), a_n(2n))$, on all the boards (these moves are marked with an S). If we let $a_n = (a(0), \ldots, a(2n-1), a_n(2n), a_n(2n+1), \ldots)$, then $a_n \in A$ and $\phi_n(a_n) \leq \phi_n(a_{n-1})$. Since $\{\phi_n\}$ is a very good scale, it follows that all the $\phi_n(a_m)$ are eventually constant and thus $a \in A$.

We state now our version of the Third Periodicity Theorem.

2.11 Theorem (Third Periodicity). *Let X be a set, $A \subseteq X^\omega$, and $\{\phi_n\}$ a very good semi-scale on A. Assume I has a winning quasi-strategy in G_A, and all of the games $G^n_{s,t}$ defined above are quasi-determined. Then the canonical quasi-strategy τ defined above is winning for I. If each of the games $G^n_{s,t}$ is determined, that is one of the players has a winning strategy, then each of the relations $\leq^{m*}_n, <^{m*}_n$ corresponding to the prewellordering $\leq^m_n$ on W_m is in $\Game^X \phi_n$. Specifically,*

$$s \leq^{m*}_n t \longleftrightarrow \forall a(0)\exists b(0)\forall b(1)\exists a(1)\cdots\ s^\frown a \leq^*_{\phi_n} t^\frown b,$$
$$s <^{m*}_n t \longleftrightarrow \exists b(0)\forall a(0)\exists a(1)\forall b(1)\cdots\ s^\frown a <^*_{\phi_n} t^\frown b.$$

Proof. Assuming all the games $G^n_{s,t}$ are quasi-determined, we have defined the quasi-strategy τ for I, and shown that it is winning for I. Assume now that all of the games $G^n_{s,t}$ are actually determined. For odd m, let $\leq^{m*}_n$, $<^{m*}_n$ be the starred relations corresponding to the prewellordering $\leq^m_n$ defined above; we must verify the equivalences stated in the theorem. Let $s, t \in X^m$, and suppose first that $s \leq^{m*}_n t$. In particular, $s \in W_m$. If $t \in W_m$ as well, so $s \leq^m_n t$, then the right hand side of the first equivalence holds, since it just asserts II has a winning strategy in the game $G^n_{s,t}$, which is the definition of $s \leq^m_n t$ in this case. If $t \notin W_m$, II easily wins $G^n_{s,t}$ by playing so the a, b produced in the run of $G^n_{s,t}$ satisfy $s^\frown a \in A$ and $t^\frown b \notin A$. Assume now the right-hand side of the first equivalence, that is, II wins $G^n_{s,t}$. We must

$$
\begin{array}{c|lllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s & \mathrm{S}\, a_3(0) & a_3(1) & \mathrm{S}\, a_3(2) & a_3(3) & \cdots \\
\hline
s & a_3(0) & \mathrm{F}\, a_3(1) & a_3(2) & \mathrm{F}\, a_3(3) & \cdots \\
 & & & & & \\
t & \mathrm{F}\, a_2(0) & a_2(1) & \mathrm{F}\, a_2(2) & a_2(3) & \cdots \\
\hline
t & a_2(0) & \mathrm{S}\, a_2(1) & a_2(2) & \mathrm{S}\, a_2(3) & \cdots \\
 & & & & & \\
s & \mathrm{S}\, a_1(0) & a_1(1) & \mathrm{S}\, a_1(2) & a_1(3) & \cdots \\
\hline
s & a_1(0) & \mathrm{F}\, a_1(1) & a_1(2) & \mathrm{F}\, a_1(3) & \cdots \\
 & & & & & \\
t & \mathrm{F}\, a_0(0) & \mathrm{I}\ a_0(1) & \mathrm{F}\, a_0(2) & \mathrm{I}\ a_0(3) & \cdots
\end{array}
$$

Figure 21.3

have $s \in W_m$, as otherwise I could easily win this game by playing to ensure $s^\frown a \notin A$. If $t \notin W_m$, $s \leq_n^{m*} t$ holds by definition, and if $t \in W_m$ then again by definition $s \leq_n^m t$ and so $s \leq_n^{m*} t$.

For the second equivalence, note first that the right-hand side is asserting that F has a winning strategy in the game $H_{s,t}^n$:

$$
\begin{array}{llllllll}
s & & \mathrm{S}\ a(0) & \mathrm{F}\ a(1) & & & \mathrm{S}\ a(2) & \mathrm{F}\ a(3) \\
t & \mathrm{F}\ b(0) & & & \mathrm{S}\ b(1) & \mathrm{F}\ b(2) & &
\end{array}
$$

where F wins the run iff $s^\frown a <_{\phi_n}^* t^\frown b$.

Assume first now that $s, t \in X^m$ and $s <_n^{m*} t$. In particular $s \in W_m$. If $t \notin W_m$, then easily F has a winning strategy in $H_{s,t}^n$ by playing to ensure that $s^\frown a \in A$ and $s^\frown b \notin A$. So, assume $t \in W_m$. Suppose, toward a contradiction, that S has a winning strategy ρ in $H_{s,t}^n$. Since $\neg(t \leq_n^m s)$, we may fix also a winning strategy σ for F in $G_{t,s}^n$. Using DC, fill in the runs of the games as in Fig. 21.3.

In the bottom run, the moves marked with a I are those following a winning quasi-strategy to produce a_0 with $t^\frown a_0 \in A$. In the boards with moves marked by F, those moves are made in accordance with σ. In the boards with moves marked by S, those moves are made in accordance with ρ. The other moves are made by copying as shown. Thus $t^\frown a_0 \in A$, and from the definitions of ρ, σ we have:

$$\phi_n(t^\frown a_0) > \phi_n(s^\frown a_1) \geq \phi_n(t^\frown a_2) > \phi_n(s^\frown a_3) \ldots,$$

a contradiction. Assume finally the right-hand side of the second equivalence, that is, F has a winning strategy in $H_{s,t}^n$. Easily this implies $s \in W_m$. If $t \notin W_m$, then the left-hand side is true by definition, so assume $t \in W_m$ as well. If $\neg(s <_n^{m*} t)$, then we would have $t \leq_n^m s$, and so S would have a winning strategy in $G^n_{t,s}$. These two strategies may be directly played against each other to get a contradiction. $\dashv$

A case of particular importance is when $X = \omega$, and $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^1_{2n}$. Assuming $\boldsymbol{\Delta}^1_{2n}$-determinacy ($\longleftrightarrow$ $\boldsymbol{\Sigma}^1_{2n}$-determinacy), Theorem 2.11 shows that if I wins G_A, then I has a $\boldsymbol{\Delta}^1_{2n+1}$ winning strategy. For we may define a canonical winning strategy τ for I in G_A by:

$$\tau(s) = k \longleftrightarrow \forall m\ (s^\frown k) \leq_l^{2l+1*} (s^\frown m) \wedge \forall m < k\ (s^\frown k) <_l^{2l+1*} (s^\frown m),$$

where $\mathrm{lh}(s) = 2l$. Thus the relation $\tau(s) = k$ is $\boldsymbol{\Pi}^1_{2n+1}$ from Theorem 2.11 since $\boldsymbol{\Sigma}^1_{2n}$ has the scale property and $\boldsymbol{\Pi}^1_{2n+1} = \Game^{\omega^\omega} \boldsymbol{\Sigma}^1_{2n}$. Since this relation does define a strategy, it follows that the relation is $\boldsymbol{\Delta}^1_{2n+1}$. Similarly, if I wins a $\boldsymbol{\Pi}^1_{2n+1}$ game, then, assuming $\boldsymbol{\Pi}^1_{2n+1}$-determinacy, I has a $\boldsymbol{\Delta}^1_{2n+2}$ winning strategy.

2.2. Projective Ordinals, Sets, and the Coding Lemma

The Moschovakis Coding Lemma is a basic tool in determinacy theory. We present the result in a general form.

2.12 Theorem (AD + DC; Coding Lemma). *Let $\boldsymbol{\Gamma}$ be a non-selfdual pointclass closed under $\exists^{\omega^\omega}$ and $\wedge$, and $\prec$ a $\boldsymbol{\Gamma}$ well-founded relation on ω^ω of rank $\theta \in \mathrm{On}$. Let $R \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ be such that $\forall x \in \mathrm{dom}(\prec)\ \exists y\ R(x, y)$. Then there is a $\boldsymbol{\Gamma}$ set $A \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ which is a choice set for R, that is:*

(1) $\forall \alpha < \theta\ \exists x \in \mathrm{dom}(\prec)\ \exists y\ [|x|_\prec = \alpha \wedge A(x, y)]$.

(2) $\forall x \forall y\ A(x, y) \to [x \in \mathrm{dom}(\prec) \wedge R(x, y)]$.

Proof. We may assume θ is minimal so that the theorem fails, and fix $\prec$, R, and a good universal set $U \subseteq (\omega^\omega)^3$ for the $\boldsymbol{\Gamma}$ subsets of $(\omega^\omega)^2$. Easily, θ is a limit ordinal. For $\delta < \theta$, say $u \in \omega^\omega$ codes a δ-choice set provided (1) holds for $\alpha \leq \delta$ using $A = U_u$, and (2) holds for $A = U_u$ where we replace $x \in \mathrm{dom}(\prec)$ with $x \in \mathrm{dom}(\prec) \wedge |x|_\prec \leq \delta$. By minimality of θ, for all $\delta < \theta$ there are δ-choice sets. Play the game where I, II play out $u, v \in \omega^\omega$, and II wins provided that if u codes a δ_1-choice set for some $\delta_1 < \theta$, then v codes a δ_2-choice set for some $\delta_2 > \delta_1$. If I has a winning strategy, we get a $\boldsymbol{\Sigma}^1_1$ set B of reals coding δ-choice sets for arbitrarily large $\delta < \theta$. Define then $A(x, y) \longleftrightarrow \exists w \in B\ U(w, x, y)$, which easily works.

Suppose now that τ is a winning strategy for II. From the *s-m-n* Theorem, let $s : (\omega^\omega)^2 \to \omega^\omega$ be continuous such that for all ϵ, x, t, w, $U(s(\epsilon, x), t, w) \longleftrightarrow \exists y\ \exists z\ [y \prec x \wedge U(\epsilon, y, z) \wedge U(z, t, w)]$. By the Recursion Theorem, let ϵ_0 be such that $U(\epsilon_0, x, z) \longleftrightarrow z = \tau(s(\epsilon_0, x))$. A straightforward induction on $|x|_\prec$ for $x \in \mathrm{dom}(\prec)$ shows that $\forall x \in \mathrm{dom}(\prec)\ \exists! z\ U(\epsilon_0, x, z)$, and $\forall x \in \mathrm{dom}(\prec)$ $\forall z\ [U(\epsilon_0, x, z) \to z$ codes a $\geq |x|_\prec$-choice set$]$. Let $A(x, y) \longleftrightarrow \exists z \in \mathrm{dom}(\prec)$ $\exists w\ [U(\epsilon_0, z, w) \wedge U(w, x, y)]$. $\dashv$

The Coding Lemma is frequently used where $\prec$ is the strict part of a prewellordering $\preceq$ which is also in $\mathbf{\Gamma}$ ($\mathbf{\Gamma}$ as in Theorem 2.12), and where the set R is invariant, that is, there is an $R' \subseteq \theta \times \omega^\omega$ such that $R(x,y) \longleftrightarrow R'(|x|_\prec, y)$. In this case, the relation A may be taken to have domain $\mathrm{dom}(\preceq)$. For we may define $A(x,y) \longleftrightarrow \exists x' \, [x' \preceq x \wedge x \preceq x' \wedge A'(x',y)]$, where A' is the $\mathbf{\Gamma}$ choice set from Theorem 2.12. A useful consequence of this is that if there is a $\mathbf{\Gamma}$ prewellordering $\preceq$ of length α whose strict part $\prec$ is also in $\mathbf{\Gamma}$, then every $S \subseteq \alpha$ is $\mathbf{\Delta}$ in the codes provided by $\prec$. That is, there are $\mathbf{\Gamma}, \check{\mathbf{\Gamma}}$ sets C, D such that for all $x \in \mathrm{dom}(\preceq)$, $S(|x|_\prec) \longleftrightarrow C(x) \longleftrightarrow D(x)$. To see this, apply the Coding Lemma to the relation $R(x,a) \longleftrightarrow (|x|_\prec \in S \wedge a = 1) \vee (|x|_\prec \notin S \wedge a = 0)$ (identifying $0, 1$ with two reals). Let A be an invariant choice set for R in $\mathbf{\Gamma}$, and set $C(x) \longleftrightarrow (x,1) \in A$, $D(x) \longleftrightarrow (x,0) \notin A$.

Finally, if $\mathbf{\Gamma}$ is a non-selfdual pointclass closed under $\forall^{\omega^\omega}$, $\vee$, and $\mathrm{pwo}(\mathbf{\Gamma})$, then we may improve the definability estimate. Namely, suppose $P \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$ and ϕ is a $\mathbf{\Gamma}$ norm on P mapping onto α. Then every $S \subseteq \alpha$ is $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ in the codes provided by ϕ (rather than $\mathbf{\Delta}(\exists^{\omega^\omega}\mathbf{\Gamma})$). To see this, let U be universal for $\check{\mathbf{\Gamma}} \restriction \omega^\omega \times \omega$. For $\beta < \alpha$ (we may assume α is a limit ordinal), say y codes $S \restriction \beta$ if for all (x,a),

$$\begin{aligned} U_y(x,a) \longleftrightarrow & (x \in P \wedge \phi(x) < \beta \wedge x \in S \wedge a = 1) \\ & \vee (x \in P \wedge \phi(x) < \beta \wedge x \notin S \wedge a = 0). \end{aligned}$$

From the Coding Lemma, for all $\beta < \alpha$ there is a y coding $S \restriction \beta$. Play the integer game where I plays x, II plays y, and II wins iff $[x \in P \rightarrow \exists \beta > \phi(x) \, (y \text{ codes } S \restriction \beta)]$. II wins by boundedness (a winning strategy for I would give a $\mathbf{\Sigma}^1_1$ set $S \subseteq P$ coding cofinally in α many ordinals, from which we would compute $P \in \check{\mathbf{\Gamma}}$ by $x \in P \longleftrightarrow \exists y \in S \; x \leq_{\check{\mathbf{\Gamma}}} y$), and if τ is winning for II, define $D(x) \longleftrightarrow U_{\tau(x)}(x,1)$ and $C(x) \longleftrightarrow \neg U_{\tau(x)}(x,0)$.

There is also a "uniform" version of the Coding Lemma. Roughly speaking, this asserts that A may be chosen so that $A \cap \{(x,w) : |x|_\preceq \leq \delta\}$ is $\mathbf{\Sigma}_1(\preceq_\delta)$, where $\preceq_\delta$ denotes the prewellordering restricted to reals of rank $\leq \delta$. We refer the reader to [22] for a precise statement. This is particularly useful for long prewellorderings, where the initial segments may be much simpler than the overall prewellordering (this also provides another proof of the result of the previous paragraph).

The following lemma, due to Moschovakis, is of frequent use. It can be proved using the Coding Lemma, or by a direct argument using the Recursion Theorem (cf. [31, 4C.14]). To illustrate the Coding Lemma and Recursion Theorem, we give both arguments.

2.13 Lemma (AD + DC). *Let $\mathbf{\Gamma}$ be non-selfdual and closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Then any $\check{\mathbf{\Gamma}}$ well-founded relation has length less than $\delta(\mathbf{\Gamma}) \doteq$ the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings (where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$).*

Proof. First we give the argument using the Coding Lemma. Let P be a $\mathbf{\Gamma}$-complete set, and ϕ a regular $\mathbf{\Gamma}$-norm on P. By definition of $\delta(\mathbf{\Gamma})$, ϕ maps into $\delta(\mathbf{\Gamma})$. Suppose that $\prec$ is a $\check{\mathbf{\Gamma}}$ well-founded relation of length $\geq |\phi|$ (the length of the norm ϕ). We may assume that $|\prec|$ is equal to $|\phi|$. Apply the Coding Lemma to the relation $R \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ given by $R(x, y) \longleftrightarrow (y \in P \wedge \phi(y) = |x|_\prec)$, where $|x|_\prec$ denotes the rank of x in the relation $\prec$. The Coding Lemma gives a $\check{\mathbf{\Gamma}}$ choice set $A \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ as in Theorem 2.12. Let $B(y) \longleftrightarrow \exists x\ A(x, y)$. Then $B \subseteq P$ is in $\check{\mathbf{\Gamma}}$, and for every $\alpha < |\phi|$ there is a $y \in B$ with $\phi(y) = \alpha$. This contradicts boundedness, namely, we could now compute $P \in \check{\mathbf{\Gamma}}$ by $P(z) \longleftrightarrow \exists y\ (B(y) \wedge z \leq_{\check{\mathbf{\Gamma}}} y)$, where $\leq_{\check{\mathbf{\Gamma}}}$ is the $\check{\mathbf{\Gamma}}$ relation corresponding to the norm ϕ.

Now we give the proof using the Recursion Theorem. Let now $U \subseteq \omega^\omega \times \omega^\omega$ be (good) universal for $\mathbf{\Gamma}$ (see Theorem 2.1), and let ϕ be a regular $\mathbf{\Gamma}$-norm on U. Again, $|\phi| \leq \delta(\mathbf{\Gamma})$. Let $\prec$ be a $\check{\mathbf{\Gamma}}$ well-founded relation, and we again show that $|\prec| < |\phi|$. It is enough to show that $|\prec| \leq |\phi|$, as it is easy, given any $\check{\mathbf{\Gamma}}$ relation $\prec$, to define a $\check{\mathbf{\Gamma}}$ relation $\prec'$ having length $|\prec|\ +1$. From the Recursion Theorem, let $x \in \omega^\omega$ be such that

$$U(x, y) \longleftrightarrow \forall z\ (z \prec y \rightarrow (x, z) <^*_\phi (x, y)),$$

holds for all y, where $<^*_\phi$ is the norm relation corresponding to ϕ. For every $y \in \mathrm{dom}(\prec)$, a straightforward induction on $|y|_\prec$ shows that $U(x, y)$ and $\phi(x, y) \geq |y|_\prec$. To see this, first prove by induction on $|y|_\prec$ that $U(x, y)$ holds. For if $|y|_\prec$ were a least violation, then for all $z \prec y$ we would have $U(x, z)$, and thus $(x, z) <^*_\phi (x, y)$ from the definition of $<^*_\phi$. But then $U(x, y)$ holds from the above equation defining U_x. So, for any $y \in \mathrm{dom}(\prec)$, $U(x, y)$ holds. From the definition of U_x it now follows that if $z \prec y$ then $\phi(x, z) < \phi(x, y)$. Thus, $|\phi| \geq |\prec|$. $\dashv$

2.14 Remark. The second proof given above has the advantage that it works in ZF provided $\mathbf{\Gamma}$ has a universal set.

The next lemma, due to Martin, is also frequently useful. It has the same hypotheses as the previous lemma.

2.15 Lemma (AD+DC). *Let $\mathbf{\Gamma}$ be non-selfdual and closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and assume* pwo$(\mathbf{\Gamma})$. *Then $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ is closed under $<\!\delta(\mathbf{\Gamma})$ length unions and intersections.*

Proof. It is enough to show $\mathbf{\Delta}$ is closed under $<\!\delta(\mathbf{\Gamma})$ unions. Suppose not, and let $\rho < \delta(\mathbf{\Gamma})$ be least such that $\mathbf{\Delta}$ is not closed under ρ-unions. Let $\{A_\alpha\}_{\alpha<\rho}$ be a sequence of $\mathbf{\Delta}$ sets with $A \doteq \bigcup_{\alpha<\rho} A_\alpha$ not in $\mathbf{\Delta}$. By minimality of ρ we may assume that the A_α form an increasing sequence. From the Coding Lemma and the fact that there is a $\mathbf{\Delta}$ prewellordering $\prec$ of length ρ it follows that $A \in \check{\mathbf{\Gamma}}$. [Apply the Coding Lemma to $R \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ given by $R(x, y) \longleftrightarrow (U_y = A_{|x|_\prec})$, where $U \subseteq \omega^\omega \times \omega^\omega$ is universal for $\check{\mathbf{\Gamma}}$. Let $S \in \check{\mathbf{\Gamma}}$ be the set produced by the Coding Lemma. Then $z \in A \longleftrightarrow \exists x\ \exists y\ (S(x, y) \wedge U(y, z))$.] So, A is $\check{\mathbf{\Gamma}}$-complete. Let ϕ be the norm on A corresponding to the union $A = \bigcup_{\alpha<\rho} A_\alpha$, that is, $\phi(x) =$ the least α such that $x \in A_\alpha$. Then ϕ

is a $\check{\boldsymbol{\Gamma}}$-norm on A. For example $x <^*_\phi y \longleftrightarrow \exists \alpha < \rho\ (x \in A_\alpha \wedge y \notin A_\alpha)$, which writes $<^*_\phi$ as a ρ-union of $\boldsymbol{\Delta}$ sets and thus shows $<^*_\phi \in \check{\boldsymbol{\Gamma}}$. This shows $\mathrm{pwo}(\check{\boldsymbol{\Gamma}})$. This, however, contradicts the ZF fact that for $\boldsymbol{\Gamma}$ having a universal set, $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$ cannot both have the prewellordering property. [We mention results along this line in Sect. 2.3, however we can also directly argue this last fact as follows. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\boldsymbol{\Gamma}$ set, and ϕ a $\boldsymbol{\Gamma}$ norm on A. Define $B(x,y) \longleftrightarrow (x_0,y) <^*_\phi (x_1,y)$, and $C(x,y) \longleftrightarrow (x_1,y) <^*_\phi (x_0,y)$, so $B, C \in \boldsymbol{\Gamma}$, and $B \cap C = \emptyset$ (here $x \mapsto (x_0,x_1)$ is our recursive bijection between ω^ω and $\omega^\omega \times \omega^\omega$). Assume toward a contradiction that $\mathrm{pwo}(\check{\boldsymbol{\Gamma}})$ also holds. Let ψ_1, ψ_2 be comparable $\check{\boldsymbol{\Gamma}}$-norms on B^c, C^c (see the discussion after Definition 2.2). Define $E(x,z) \longleftrightarrow (x,z) <^*_{\psi_1,\psi_2} (x,z)$. Note also $E(x,z) \longleftrightarrow \neg(x,z) \leq^*_{\psi_2,\psi_1} (x,z)$, since $B^c \cup C^c = \omega^\omega \times \omega^\omega$. So $E \in \boldsymbol{\Delta}$. However, E is also universal for $\boldsymbol{\Delta}$. For if $D \subseteq \omega^\omega$ is $\boldsymbol{\Delta}$, let x be such that $D^c = U_{x_0}$, $D = U_{x_1}$, and hence $D^c = B_x$, $D = C_x$. Then $E_x = D$. Being selfdual, however, the pointclass $\boldsymbol{\Delta}$ cannot have a universal set by the usual diagonal argument (the set $S(x) \longleftrightarrow \neg E(x,x)$ cannot be in $\boldsymbol{\Delta}$).] $\dashv$

We mention one more result of a general nature.

2.16 Lemma. *Let $\boldsymbol{\Gamma}$ be non-selfdual and closed under $\exists^{\omega^\omega}$, $\wedge$. Then the supremum of the lengths of the $\boldsymbol{\Gamma}$ well-founded relations is a regular cardinal.*

Proof. Let κ be the supremum of the lengths of the $\boldsymbol{\Gamma}$ well-founded relations. Clearly κ is a limit ordinal. Suppose $\rho \doteq \mathrm{cf}(\kappa) < \kappa$, and let $f : \rho \to \kappa$ be cofinal. Let $\prec$ be a $\boldsymbol{\Gamma}$ well-founded relation of length ρ. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\boldsymbol{\Gamma}$. Apply the Coding Lemma to $R \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ given by $R(x,y) \longleftrightarrow (x \in \mathrm{dom}(\prec) \wedge U_y$ is well-founded of length $\geq f(|x|_\prec))$. Let $A \subseteq \mathrm{dom}(\prec) \times \omega^\omega$ be as in the Coding Lemma, so $A \in \boldsymbol{\Gamma}$. Define then $(x,y,z) \ll (x',y',z') \longleftrightarrow ((x,y) \in A \wedge (x,y) = (x',y') \wedge U_y(z,z'))$. Then $\ll$ is a $\boldsymbol{\Gamma}$ well-founded relation of length κ, a contradiction. $\dashv$

Recall from the introduction the definitions of the projective pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$. Also, assuming Projective Determinacy, $\boldsymbol{\Pi}^1_{2n+1}$ and $\boldsymbol{\Sigma}^1_{2n+2}$ have the scale property for all n. We now define the projective ordinals and establish their basic properties.

2.17 Definition. $\boldsymbol{\delta}^1_n$ = the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of the reals.

2.18 Theorem (AD + DC). *For all n, $\boldsymbol{\delta}^1_n$ is the supremum of the lengths of the $\boldsymbol{\Sigma}^1_n$ well-founded relations. Each $\boldsymbol{\delta}^1_n$ is a regular cardinal, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$ for some Suslin cardinal λ_{2n+1} of cofinality ω. $\boldsymbol{\Sigma}^1_{2n+2} = \boldsymbol{\delta}^1_{2n+1}$-Suslin and $\boldsymbol{\Sigma}^1_{2n+1} = \lambda_{2n+1}$-Suslin. The Suslin cardinals within the projective ordinals are exactly the λ_{2n+1} and the $\boldsymbol{\delta}^1_{2n+1}$.*

Proof. If $\phi : A \to \delta$ is a regular $\boldsymbol{\Pi}^1_{2n+1}$ norm on a $\boldsymbol{\Pi}^1_{2n+1}$ set A, then by definition $\delta \leq \boldsymbol{\delta}^1_{2n+1}$ (as all initial segments of the prewellordering are in

$\boldsymbol{\Delta}^1_{2n+1}$). Thus, from the scale property for $\boldsymbol{\Pi}^1_{2n+1}$, every $\boldsymbol{\Pi}^1_{2n+1}$, and hence also every $\boldsymbol{\Sigma}^1_{2n+2}$, set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin. From the Coding Lemma it also follows that every $\boldsymbol{\delta}^1_{2n+1}$-Suslin set is $\boldsymbol{\Sigma}^1_{2n+2}$, so $S(\boldsymbol{\delta}^1_{2n+1}) = \boldsymbol{\Sigma}^1_{2n+2}$. If A is universal for $\boldsymbol{\Pi}^1_{2n+1}$, then from Lemma 2.13, $\delta = \boldsymbol{\delta}^1_{2n+1}$ and every $\boldsymbol{\Sigma}^1_{2n+1}$ well-founded relation has length less than δ (this also follows from the Kunen-Martin Theorem mentioned below, but the argument above does not need scales). So, $\boldsymbol{\delta}^1_{2n+1}$ is the supremum of the lengths of the $\boldsymbol{\Sigma}^1_{2n+1}$ well-founded relations. From Lemma 2.16, $\boldsymbol{\delta}^1_{2n+1}$ is regular.

From the Kunen-Martin Theorem 2.6, $\boldsymbol{\delta}^1_{2n+2} \leq$ the supremum of the lengths of the $\boldsymbol{\Sigma}^1_{2n+2}$ well-founded relations $\leq (\boldsymbol{\delta}^1_{2n+1})^+$. Conversely, let $\prec$ be a wellordering of $\boldsymbol{\delta}^1_{2n+1}$. The Coding Lemma implies that $\prec$ is $\boldsymbol{\Delta}^1_{2n+2}$ in the codes relative to a norm ϕ on a $\boldsymbol{\Pi}^1_{2n+1}$ universal set A, that is, the relation $(x, y \in A \wedge \phi(x) \prec \phi(y))$ is $\boldsymbol{\Delta}^1_{2n+2}$. Thus, $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$, and $\boldsymbol{\delta}^1_{2n+2} =$ the supremum of the $\boldsymbol{\Sigma}^1_{2n+2}$ well-founded relations. From Lemma 2.16, $\boldsymbol{\delta}^1_{2n+2}$ is also regular.

Note that $\boldsymbol{\delta}^1_n < \boldsymbol{\delta}^1_{n+1}$ as the $\boldsymbol{\Sigma}^1_n$ well-founded relations can be "put together" into a single $\boldsymbol{\Sigma}^1_n \wedge \boldsymbol{\Pi}^1_n$ well-founded relation via a universal $\boldsymbol{\Sigma}^1_n$ set.

Suppose A is a universal $\boldsymbol{\Sigma}^1_{2n+1}$ set, and write $A(x) \longleftrightarrow \exists y\, B(x, y)$ where $B \in \boldsymbol{\Pi}^1_{2n}$. Clearly if B is κ-Suslin, then so is A. Now B admits a $\boldsymbol{\Delta}^1_{2n+1}$ scale (using scale($\boldsymbol{\Pi}^1_{2n+1}$)), and thus A is λ-Suslin for some $\lambda < \boldsymbol{\delta}^1_{2n+1}$. Let λ_{2n+1} be the least such λ. It follows that every $\boldsymbol{\Sigma}^1_{2n+1}$ set is λ_{2n+1}-Suslin. From the Kunen-Martin Theorem and the definition of $\boldsymbol{\delta}^1_{2n+1}$, it follows that $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$. We claim that $\mathrm{cf}(\lambda_{2n+1}) = \omega$. For if not, then A could be written as a λ_{2n+1} union of sets A_α, each of which is $<\lambda_{2n+1}$-Suslin. Each A_α must be $\boldsymbol{\Sigma}^1_{2n}$, as otherwise by Wadge, some $\boldsymbol{\Pi}^1_{2n}$-complete set, and hence every $\boldsymbol{\Sigma}^1_{2n+1}$ set, would be $<\lambda_{2n+1}$-Suslin. By Lemma 2.15, $A \in \boldsymbol{\Delta}^1_{2n+1}$, a contradiction. So, $\mathrm{cf}(\lambda_{2n+1}) = \omega$. We noted above $\boldsymbol{\Sigma}^1_{2n+1} \subseteq S(\lambda_{2n+1})$. The reverse inclusion follows from the Coding Lemma. Thus, $S(\lambda_{2n+1}) = \boldsymbol{\Sigma}^1_{2n+1}$.

For any κ the pointclass of κ-Suslin sets is closed under $\exists^{\omega^\omega}$ (as well as $\wedge, \vee$). From Wadge's Lemma, the only pointclasses within the projective hierarchy that are closed under $\exists^{\omega^\omega}$ (and contain the closed sets) are the $\boldsymbol{\Sigma}^1_n$, and thus we have determined all the Suslin classes and cardinals within the projective hierarchy. ⊣

It follows also from our discussion above that $\boldsymbol{\delta}^1_1 = \omega_1$ and $\boldsymbol{\delta}^1_2 = \omega_2$. Martin and Solovay also computed $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, and $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$ (see also the next section). In Sect. 2.6 we will show (assuming AD) that each $\boldsymbol{\delta}^1_n$ is measurable, and in fact has the countable exponent partition property $\boldsymbol{\delta}^1_n \to (\boldsymbol{\delta}^1_n)^\lambda$ for all $\lambda < \omega_1$ (defined in Sect. 2.6).

2.3. Wadge Degrees and Abstract Pointclasses

We now recall some of the abstract theory of pointclasses. Additional background may be found in [35, 36], and [39]. We assume AD + DC throughout this section, though the determinacy required is "local", e.g., only Projective Determinacy is required within the projective sets, etc.

If $\mathbf{\Gamma}$ is a pointclass and κ a cardinal, let $\bigcup_\kappa \mathbf{\Gamma}$ denote those $A \subseteq \omega^\omega$ which can be written as $A = \bigcup_{\alpha<\kappa} A_a$ where each $A_\alpha \in \mathbf{\Gamma}$. We similarly define $\bigcap_\kappa \mathbf{\Gamma}$.

We note the simple observation that if $\mathbf{\Gamma}$ is a non-selfdual pointclass, then the closure of $\mathbf{\Gamma}$ under $\exists^\omega$ implies the closure of $\mathbf{\Gamma}$ under countable unions, and likewise the closure under $\forall^\omega$ implies the closure under countable intersections. For suppose $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$, and $A_n \in \mathbf{\Gamma}$ for $n \in \omega$. Thus, $A_n \leq_w A$ for all n, and it follows easily that $B \leq_w A$ as well, where $B(x) \longleftrightarrow \bar{x} \in A_{x(0)}$ and $\bar{x}(i) = x(i+1)$. Then $x \in \bigcup_n A_n \longleftrightarrow \exists i\ B(i^\frown x)$. Another simple but useful observation is that if $\mathbf{\Gamma}$ is non-selfdual and closed under countable intersections (respectively unions), then $\exists^{\omega^\omega}\mathbf{\Gamma}$ (respectively $\forall^{\omega^\omega}\mathbf{\Gamma}$) is closed under countable unions and intersections. As we already noted, $\exists^{\omega^\omega}\mathbf{\Gamma}$ is closed under countable unions (since $\mathbf{\Gamma}$ has a universal set, so does $\exists^{\omega^\omega}\mathbf{\Gamma}$, and hence it also is non-selfdual). To check countable intersections, let $A_n \in \exists^{\omega^\omega}\mathbf{\Gamma}$, say $A_n(x) \longleftrightarrow \exists y\ B_n(x,y)$ with $B_n \in \mathbf{\Gamma}$. Then $x \in \bigcap_n A_n \longleftrightarrow \exists y\ B(x,y)$, where $B = \bigcap_n B_n$ and $B_n(x,y) \longleftrightarrow A_n(x,(y)_n)$. Thus, $\bigcap_n A_n \in \exists^{\omega^\omega}\mathbf{\Gamma}$.

Recall the definitions of Lipschitz reduction $\leq_l$ and Wadge reduction $\leq_w$ from the introduction. We say a set $A \subseteq \omega^\omega$ is *selfdual* if $A \leq_l A^c$. A theorem of Steel [39] says that $A \leq_l A^c$ iff $A \leq_w A^c$. Thus, A is selfdual iff the pointclass generated by A, namely $\mathbf{\Gamma}_A = \{B : B \leq_w A\}$, is selfdual (i.e., closed under complements).

Consider now pairs of the form (A, A^c) (if A is selfdual, we may equivalently take just A in what follows). We extend $\leq_l$ to such pairs by setting $(A, A^c) \leq_l (B, B^c)$ iff one of A, A^c is $\leq_l$ to one of B, B^c. This is easily seen to be transitive, reflexive, and by Wadge's Lemma, connected. A Lipschitz degree, or l-degree, denotes an equivalence class of a pair under the relation $(A, A^c) \equiv_l (B, B^c)$ iff $(A, A^c) \leq_l (B, B^c)$ and $(B, B^c) \leq_l (A, A^c)$. An important basic result of Martin (cf. [39]) asserts that the strict part of $\leq_l$ is well-founded.

The Wadge degrees, or w-degrees, are defined analogously, using $\leq_w$ in place of $\leq_l$. Of course, a Wadge degree is an amalgamation of l-degrees, and it is immediate that the Wadge degrees are also wellordered. From the result of Steel mentioned above, it follows that only selfdual l-degrees are amalgamated in forming a w-degree. It is shown in [39] that for α a limit ordinal of cofinality ω, an l-degree of rank α must be selfdual, and for $\mathrm{cf}(\alpha) > \omega$ the pair is non-selfdual. Furthermore, following any non-selfdual l-degree (A, A^c), the next ω_1 l-degrees are all selfdual and of the same w-degree (that of the join of A and A^c, that is, $\{n^\frown x : (n \text{ is even } \wedge x \in A) \vee (n \text{ is odd } \wedge x \notin A)\}$). This gives a general picture of the w-degrees: the selfdual and non-selfdual w degrees alternate, and at limit ordinals α, a pair of Wadge degree α is selfdual iff $\mathrm{cf}(\alpha) = \omega$. For $A \subseteq \omega^\omega$, we let $o(A)$ denote the rank of (A, A^c) in $\leq_w$.

In [36, 35] some additional structural results for general pointclasses were obtained. Recall $\mathbf{\Gamma}$ has the reduction property, $\mathrm{red}(\mathbf{\Gamma})$, if for all $A, B \in \mathbf{\Gamma}$

$\exists A', B' \in \mathbf{\Gamma}$ such that $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$. $\mathbf{\Gamma}$ has the separation property, $\mathrm{sep}(\mathbf{\Gamma})$, if for all $A, B \in \mathbf{\Gamma}$ with $A \cap B = \emptyset$, $\exists C \in \mathbf{\Delta}$ $(A \subseteq C \subseteq B^c)$. A standard result in descriptive theory (see [31]) is that $\mathrm{pwo}(\mathbf{\Gamma}) \rightarrow \mathrm{red}(\mathbf{\Gamma})$ for $\mathbf{\Gamma}$ closed under $\wedge$, $\vee$, and $\mathrm{red}(\mathbf{\Gamma}) \rightarrow \mathrm{sep}(\check{\mathbf{\Gamma}})$. Reference [36] shows that for any non-selfdual pointclass $\mathbf{\Gamma}$, either $\mathrm{sep}(\mathbf{\Gamma})$ or $\mathrm{sep}(\check{\mathbf{\Gamma}})$, and from [39], both sides cannot have the separation property. Also, if $\mathbf{\Gamma}$ is closed under $\wedge, \vee$ then $\mathrm{red}(\mathbf{\Gamma})$ or $\mathrm{red}(\check{\mathbf{\Gamma}})$. More generally, Steel [35] shows that if $\mathbf{\Gamma}$ is closed under $\wedge$ and $\neg\,\mathrm{sep}(\mathbf{\Gamma})$, then $\mathrm{red}(\mathbf{\Gamma})$. Finally, [35] shows that if $\mathbf{\Gamma}$ is a *Levy* class, that is closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$, and if we make the technical assumption that $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ is not closed under wellordered unions (this is true in $L(\mathbb{R})$, for example, for all selfdual $\mathbf{\Delta} \neq \mathcal{P}(\omega^\omega)$) then either $\mathrm{pwo}(\mathbf{\Gamma})$ or $\mathrm{pwo}(\check{\mathbf{\Gamma}})$. It is also shown there that if $\mathbf{\Delta}$ is closed under real quantifiers, and $\mathrm{sep}(\mathbf{\Gamma})$, then $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$ (and thus $\check{\mathbf{\Gamma}}$ is closed under $\forall^{\omega^\omega}$).

2.19 Definition. Let $\mathbf{\Gamma}$ be a (possibly selfdual) pointclass. We let $o(\mathbf{\Gamma}) = \sup\{o(A) : A \in \mathbf{\Gamma}\}$. We let $\delta(\mathbf{\Gamma})$ = the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings of ω^ω (where $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$).

In [21] it is shown that for $\mathbf{\Delta}$ closed under real quantification, $\wedge$ and $\vee$, $o(\mathbf{\Delta}) = \delta(\mathbf{\Delta})$ = the supremum of the $\mathbf{\Delta}$ well-founded relations on ω^ω. We note that for $\mathbf{\Delta}$ closed under real quantification, closure under $\wedge$ and $\vee$ is almost automatic; it is needed only to rule out the case of a largest Wadge degree in $\mathbf{\Delta}$, which occurs only when $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ for some non-selfdual $\mathbf{\Gamma}$ closed under real quantification (by the hierarchy analysis below).

If $\mathbf{\Delta}$ is selfdual and closed under real quantifiers, $o(\mathbf{\Delta})$ has uncountable cofinality, and we again make the technical assumption that $\mathbf{\Delta}$ is not closed under wellordered unions, then Steel [35] shows there is a non-selfdual pointclass $\mathbf{\Gamma}$ closed under $\forall^{\omega^\omega}$ with $\mathrm{pwo}(\mathbf{\Gamma})$ such that $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. Steel establishes this by getting a useful representation for the $\mathbf{\Gamma}$ sets. Namely, if δ is the least ordinal such that $\mathbf{\Delta}$ is not closed under δ unions, then $\mathbf{\Gamma}$ is the collection of $\mathbf{\Sigma}^1_1$-bounded δ unions of $\mathbf{\Delta}$ sets. A union $A = \bigcup_{\alpha<\delta} A_\alpha$ is $\mathbf{\Sigma}^1_1$-*bounded* if for every $\mathbf{\Sigma}^1_1$ set $B \subseteq A$, $\exists \delta' < \delta$ $(B \subseteq \bigcup_{\alpha<\delta'} A_\alpha)$.

Steel shows in [35] (generalizing results of [21]) that these results suffice to place the prewellordering property within the Levy classes, as well as to classify the Levy classes within projective-like hierarchies. We summarize the conclusions. Suppose $\mathbf{\Gamma}$ is non-selfdual and closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$. Let α be the supremum of the limit ordinals β such that $\mathbf{\Delta}_\beta \doteq \{A : o(A) < \beta\}$ is closed under real quantifiers and $\mathbf{\Delta}_\beta \subseteq \mathbf{\Gamma}$. We have the following cases:

Type I Hierarchy $\mathrm{cf}(\alpha) = \omega$. The pointclass $\mathbf{\Lambda}$ of Wadge degree α is selfdual, consisting of ω-joins of sets of smaller degree. Let $\mathbf{\Gamma}_0$ be the class of countable unions of sets, each of degree $< \alpha$. Then $\mathbf{\Gamma}_0$ is the smallest class closed under $\exists^\omega$ containing $\mathbf{\Lambda}$, and we have $\mathrm{pwo}(\mathbf{\Gamma}_0)$. If we let $\mathbf{\Gamma}_1 = \forall^{\omega^\omega}\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_2 = \exists^{\omega^\omega}\mathbf{\Gamma}_1$, etc., then $\mathrm{pwo}(\mathbf{\Gamma}_n)$ for all n by first periodicity. $\mathbf{\Gamma}_0$ is closed under countable unions and finite intersections, and $\mathbf{\Gamma}_n$ for $n \geq 1$ is closed under countable unions and intersections. Also, $\mathbf{\Gamma} = \mathbf{\Gamma}_i$ or $\check{\mathbf{\Gamma}}_i$ for some i.

Type II, III Hierarchies $\mathrm{cf}(\alpha) > \omega$, so there is a non-selfdual pointclass $\mathbf{\Gamma}_0$ of degree α closed under $\forall^{\omega^\omega}$ and with $\mathrm{pwo}(\mathbf{\Gamma}_0)$. We assume in these cases that $\mathbf{\Gamma}_0$ is not closed under $\exists^{\omega^\omega}$. If we let $\mathbf{\Gamma}_1 = \exists^{\omega^\omega}\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_2 = \forall^{\omega^\omega}\mathbf{\Gamma}_1$, etc., then $\mathrm{pwo}(\mathbf{\Gamma}_n)$ for all n. For $n \geq 1$, $\mathbf{\Gamma}_n$ is closed under countable unions and intersections. If $\mathbf{\Gamma}_0$ is as well (by [35, Theorem 2.2] this is equivalent to $\mathbf{\Gamma}_0$ being closed under finite unions), this is referred to as a type III hierarchy, otherwise a type II hierarchy. Clearly, $\mathbf{\Gamma} = \mathbf{\Gamma}_i$ or $= \check{\mathbf{\Gamma}}_i$ for some i.

Type IV Hierarchy $\mathrm{cf}(\alpha) > \omega$, and for $\mathbf{\Gamma}_0$ as in the previous case, $\mathbf{\Gamma}_0$ is closed under real quantifiers. Let $\mathbf{\Gamma}_1 = \{A \cap B : A \in \mathbf{\Gamma}_0 \wedge B \in \check{\mathbf{\Gamma}}_0\}$. Let $\mathbf{\Gamma}_2 = \exists^{\omega^\omega}\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_3 = \forall^{\omega^\omega}\mathbf{\Gamma}_2$, etc. Then $\mathrm{pwo}(\mathbf{\Gamma}_n)$ for all n, and for $n \geq 2$ (or $n = 0$) $\mathbf{\Gamma}_n$ is closed under countable unions and intersections. Clearly, $\mathbf{\Gamma} = \mathbf{\Gamma}_i$ or $= \check{\mathbf{\Gamma}}_i$ for some i.

2.20 Remark. We refer to the pointclasses $\mathbf{\Gamma}_0$ as in the type II, III hierarchies above as *Steel pointclasses*.

We present one more result in the abstract theory of pointclasses which we will need later, and which illustrates the usefulness of the hierarchy classification.

2.21 Lemma. *If $\mathbf{\Gamma}$ is a non-selfdual pointclass closed under $\exists^{\omega^\omega}$ and $\mathrm{pwo}(\mathbf{\Gamma})$, then $\mathbf{\Gamma}$ is closed under wellordered unions.*

Proof. If $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$ as well, this is [14, Theorem 1.1], so assume $\forall^{\omega^\omega}\mathbf{\Gamma} \neq \mathbf{\Gamma}$. If $\mathbf{\Gamma}$ is closed under countable unions and intersections, the result follows from [21, Lemma 2.4.1]. Suppose now that $\mathbf{\Gamma}$ is not closed under countable intersections. The hierarchy analysis above shows that $\mathbf{\Gamma}$ is the base of a type I hierarchy, that is, $\mathbf{\Gamma} = \bigcup_\omega \mathbf{\Delta}$, and $\mathbf{\Delta}$ is closed under real quantifiers. Note that $\mathbf{\Gamma}$ is closed under $\wedge$. Towards a contradiction, let κ be the least cardinal so that $\bigcup_\kappa \mathbf{\Gamma} \nsubseteq \mathbf{\Gamma}$. Thus, κ is regular. Let $\mathbf{\Gamma}_1 = \exists^{\omega^\omega}\check{\mathbf{\Gamma}}$. By Wadge's Lemma, $\check{\mathbf{\Gamma}} \subseteq \bigcup_\kappa \mathbf{\Gamma}$, and thus $\mathbf{\Gamma}_1 \subseteq \bigcup_\kappa \mathbf{\Gamma}$. Using the regularity of κ, let $\langle A_\alpha \mid \alpha < \kappa\rangle$ be a strictly increasing κ sequence of sets in $\mathbf{\Gamma}$ whose union A is in $\check{\mathbf{\Gamma}} - \mathbf{\Gamma}$. Let $B = \{x : S_x \subseteq A\}$, where $S \subseteq (\omega^\omega)^2$ is universal $\mathbf{\Sigma}^1_1$. $B \in \check{\mathbf{\Gamma}}$ as $\check{\mathbf{\Gamma}}$ is closed under $\forall^{\omega^\omega}$ and $\vee$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$, where $B_\alpha \in \mathbf{\Gamma}$. If we replace A_α by $\{y : \exists x \in B_\alpha\ (y \in S_x)\}$, then the A_α form a $\mathbf{\Sigma}^1_1$-bounded sequence of $\mathbf{\Gamma}$ sets with union A. Let $U \subseteq (\omega^\omega)^2$ be a universal $\mathbf{\Gamma}$ set. Play the game where I plays x, II plays y, z, and II wins iff $x \in A \to \exists \alpha > |x|\ (U_y = A_\alpha \wedge z \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta)$, where $|x|$ is the least $\alpha < \kappa$ such that $x \in A_\alpha$. By $\mathbf{\Sigma}^1_1$-boundedness, II wins, say by τ. Define $x \prec y$ iff $x, y \in A \wedge \tau(y)_1 \notin U_{\tau(x)_0}$. Thus, $\prec$ is a $\check{\mathbf{\Gamma}}$ prewellordering of length κ. By the Coding Lemma, then, $\bigcup_\kappa \mathbf{\Gamma} \subseteq \mathbf{\Gamma}_1$, and hence $\bigcup_\kappa \mathbf{\Gamma} = \mathbf{\Gamma}_1$. Now, $\mathbf{\Delta}_1 = \mathbf{\Gamma}_1 \cap \check{\mathbf{\Gamma}}_1$ is clearly also not closed under κ unions, and any κ union of sets in $\mathbf{\Delta}_1$ is in $\mathbf{\Gamma}_1$. Thus, $\mathbf{\Gamma}_1 = \bigcup_{\kappa'} \mathbf{\Delta}_1$, where $\kappa' \leq \kappa$ is least such that $\mathbf{\Delta}_1$ is not closed under κ' unions. This, however, shows $\mathrm{pwo}(\mathbf{\Gamma}_1)$, a contradiction (this last part is Martin's argument from Theorem 2.15 again). ⊣

In the case of a type II, III, or IV hierarchy, the following observations are occasionally useful.

2.22 Lemma. *Let $\mathbf{\Gamma}$ be non-selfdual, closed under $\forall^{\omega^\omega}$, with $\mathrm{pwo}(\mathbf{\Gamma})$, and assume $\mathbf{\Delta}$ is closed under real quantification. Let $\kappa = o(\mathbf{\Delta})$. Then $\mathrm{cf}(\kappa)$ is the least ordinal ρ such that $\mathbf{\Delta}$ is not closed under ρ-length unions. Furthermore, there is a κ strictly increasing sequence of sets in $\mathbf{\Delta}$.*

Proof. From $\mathrm{pwo}(\mathbf{\Gamma})$ we have that $\mathbf{\Delta}$ is not closed under $\delta(\mathbf{\Delta}) = \delta(\mathbf{\Delta}) = \kappa$ length unions. Let ρ be least so that $\mathbf{\Delta}$ is not closed under ρ-length unions. Clearly $\rho \leq \kappa$ is a regular cardinal. Suppose $\rho > \mathrm{cf}(\kappa)$. Let $\{A_\alpha\}_{\alpha<\rho}$ be an increasing sequence of $\mathbf{\Delta}$ sets whose union A is not in $\mathbf{\Delta}$. Since $\rho > \mathrm{cf}(\kappa)$ is regular, there is a $\beta < \kappa$ such that for cofinally in ρ many α we have $o(A_\alpha) \leq \beta$. By Lemma 2.21 we may find a non-selfdual $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ which is closed under wellordered unions and with $o(\mathbf{\Gamma}_0) > \beta$. Then $A \in \mathbf{\Gamma}_0$, a contradiction. Suppose $\rho < \mathrm{cf}(\kappa)$ and again consider a sequence $\{A_\alpha\}_{\alpha<\rho}$ as above. Since $\rho < \mathrm{cf}(\kappa)$, there is a $\beta < \kappa$ such that for all $\alpha < \rho$, $o(A_\alpha) \leq \beta$, and we reach the same contradiction as before. So, $\rho = \mathrm{cf}(\kappa)$.

Fix now a sequence $\{A_\alpha\}_{\alpha<\mathrm{cf}(\kappa)}$ of sets in $\mathbf{\Delta}$ whose union A is not in $\mathbf{\Delta}$. Let $h(\alpha) = o(A_\alpha)$, so h is cofinal in κ. Without loss of generality we may assume that h is strictly increasing. Furthermore, we may assume that there is a prewellordering of length $h(\alpha)$ of Wadge degree less than $A_{\alpha+1}$. Let $\preceq$ be a $\mathbf{\Delta}$ prewellordering of length $\mathrm{cf}(\kappa)$ (we may assume $\mathrm{cf}(\kappa) < \kappa$ as otherwise there is nothing to show in the second claim). View every real y as coding a Lipschitz continuous function $f_y : \omega^\omega \to \omega^\omega$. By the Coding Lemma there is a $\mathbf{\Delta}$ relation $R \subseteq (\omega^\omega)^2$ with $\mathrm{dom}(R) = \mathrm{dom}(\preceq)$ and such that for all $(x, y) \in R$, $f_y^{-1}(A_{|x|+1})$ is a prewellordering of length $h(|x|)$, where $|x|$ denotes the rank of x in $\preceq$. For $\beta < \kappa$ define E_β by:

$$\begin{aligned}(x, y, z) \in E_\beta \longleftrightarrow{} & x \in \mathrm{dom}(\preceq) \wedge (\forall \gamma < |x|\ h(\gamma) \leq \beta) \\ & \wedge R(x, y) \wedge |z|_{f_y^{-1}(A_{|x|+1})} \leq \beta.\end{aligned}$$

Clearly the E_β form a κ-length strictly increasing sequence. To see that $E_\beta \in \mathbf{\Delta}$, let $\alpha_0 < \mathrm{cf}(\kappa)$ be least such that $h(\alpha_0) > \beta$. Then $E_\beta = \bigcup_{\alpha \leq \alpha_0} E_{\alpha,\beta}$ where:

$$(x, y, z) \in E_{\alpha,\beta} \longleftrightarrow x \in \mathrm{dom}(\preceq) \wedge (|x| = \alpha) \wedge R(x, y) \wedge |z|_{f_y^{-1}(A_{\alpha+1})} \leq \beta.$$

Since $\mathbf{\Delta}$ is closed under $<\mathrm{cf}(\kappa)$ unions, it is enough to show each $E_{\alpha,\beta} \in \mathbf{\Delta}$. The first three conjuncts are clearly in $\mathbf{\Delta}$. For the last, note that $P_y \doteq f_y^{-1}(A_{\alpha+1})$ is a $\mathbf{\Delta}$ prewellordering computed uniformly from y, that is, $(u, v) \in P_y \longleftrightarrow f_y(\langle u, v\rangle) \in A_{\alpha+1}$. From the Coding Lemma it is straightforward to compute that $\{(y, z) : |z|_{P_y} \leq \beta\}$ is projective in any pointclass of Wadge degree at least $o(A_{\alpha+1})$. ⊣

2.4. The Scale Theory of $L(\mathbb{R})$

The pointclass results of Sect. 2.3 can be considered to be a generalization of the "Spector" theory of the projective sets, that is, the theory which uses only the prewellordering property for the $\boldsymbol{\Pi}^1_{2n+1}$, $\boldsymbol{\Sigma}^1_{2n+2}$ sets. There is likewise a generalization of the scale theory of the projective sets to the sets of reals in $L(\mathbb{R})$. This theory is developed in [37]. We survey without proof the main results of this theory. We assume AD + $V = L(\mathbb{R})$. Recall that Θ is the supremum of the lengths of the prewellorderings of the reals.

Recall that the $J_\alpha(\mathbb{R})$ hierarchy building up $L(\mathbb{R})$ is defined similarly to the J_α hierarchy for L, except that we start with $J_1(\mathbb{R}) = V_{\omega+1}$. Thus, for limit α, $J_\alpha(\mathbb{R}) = \bigcup_{\alpha'<\alpha} J_{\alpha'}(\mathbb{R})$, and $J_{\alpha+1}(\mathbb{R})$ is the closure of $J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}$ under suitable rudimentary functions. $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ denotes the subsets of $J_\alpha(\mathbb{R})$ which are $\boldsymbol{\Sigma}_n$-definable over $J_\alpha(\mathbb{R})$ using parameters from $J_\alpha(\mathbb{R})$. We also let $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ denote the pointclass $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$. Note that $\boldsymbol{\Sigma}^1_n = \boldsymbol{\Sigma}_n(J_1(\mathbb{R}))$, so the $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$ hierarchy provides an extension of the projective hierarchy to all the pointclasses in $L(\mathbb{R})$. Recall that for $X = \omega^\omega$ or $X = \mathcal{P}(\omega^\omega)$, a relation $R \subseteq X$ is $\boldsymbol{\Sigma}^2_1$ if it can be written in the form $R(x) \longleftrightarrow \exists B \subseteq \omega^\omega \; P(x, B)$, where P is projective. That is, $P(x, B) \longleftrightarrow \exists z_1 \in \omega^\omega \; \forall z_2 \in \omega^\omega \cdots \exists(\forall) z_n \in \omega^\omega \; Q(x, B, z_1, \ldots, z_n)$, where Q is in the smallest collection containing any Borel relation on the real coordinates, the relations $z_i \in B$, $z_i \in x$ (if $X = \mathcal{P}(\omega^\omega)$), and closed under countable unions, intersections, and complements. Let ${\boldsymbol{\delta}^2_1}^{L(\mathbb{R})}$ be the supremum of the lengths of the $(\boldsymbol{\Delta}^2_1)^{L(\mathbb{R})}$ prewellorderings. We will henceforth just write $\boldsymbol{\delta}^2_1$ in place of ${\boldsymbol{\delta}^2_1}^{L(\mathbb{R})}$ (we will never consider $\boldsymbol{\delta}^2_1$ in a context outside of $L(\mathbb{R})$). $\boldsymbol{\delta}^2_1$ is the least ordinal δ such that $J_\delta(\mathbb{R}) \prec^{\mathbb{R}}_1 L(\mathbb{R})$, that is elementary for $\boldsymbol{\Sigma}_1$ formulas with real parameters. Also, $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})} = \boldsymbol{\Sigma}_1(J_{\boldsymbol{\delta}^2_1}(\mathbb{R})) \cap \mathcal{P}(\mathbb{R})$ and $(\boldsymbol{\Delta}^2_1)^{L(\mathbb{R})} = J_{\boldsymbol{\delta}^2_1}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$.

Martin and Steel [27] (using an idea of Moschovakis) show that $(\boldsymbol{\Sigma}^2_1)^{L(\mathbb{R})}$ has the scale property, and is the largest scaled pointclass in $L(\mathbb{R})$. Steel [37] refines this analysis as follows. Following Steel, call $[\alpha, \beta]$, where $\alpha \leq \beta$, a $\boldsymbol{\Sigma}_1$-*gap* if $J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 J_\beta(\mathbb{R})$ and the interval $[\alpha, \beta]$ is maximal with this property. The gaps thus partition the ordinals in $[1, \Theta]$. $[\boldsymbol{\delta}^2_1, \Theta]$ is the last gap, and for the first non-trivial gap, that is where $\beta > \alpha$, $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ already contains all the inductive sets (the smallest non-selfdual pointclass closed under real quantification). For α beginning a gap $[\alpha, \beta]$, $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ has the scale property, and $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R})) = \boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}); \mathbb{R})$, that is, every $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ set is $\boldsymbol{\Sigma}_1$-definable over $J_\alpha(\mathbb{R})$ using only parameters from $\mathbb{R}$. If $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ is not closed under real quantifiers, then periodicity propagates the scale property to $\boldsymbol{\Pi}_{2n}(J_\alpha(\mathbb{R}))$, $\boldsymbol{\Sigma}_{2n+1}(J_\alpha(\mathbb{R}))$. Otherwise (by a result of Martin), none of the $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$, $\boldsymbol{\Pi}_n(J_\alpha(\mathbb{R}))$ have the scale property for $n \geq 2$. If $\beta > \alpha$, then none of the classes $\boldsymbol{\Sigma}_n(J_\gamma(\mathbb{R}))$, $\boldsymbol{\Pi}_n(J_\gamma(\mathbb{R}))$ for $\alpha < \gamma < \beta$ have the scale property. The existence of scales at the end of a gap hinges on whether the gap satisfies a certain reflection property (a "strong" gap in the terminology of [37]). If so, there are no new scaled classes at the end of the gap. If not (a "weak" gap), then $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))$ has the scale property, where n is least such that $\boldsymbol{\Sigma}_n(J_\beta(\mathbb{R}))\cap\mathcal{P}(\mathbb{R}) \nsubseteq J_\beta(\mathbb{R})$. A periodicity argument then propagates the scale property to the $\boldsymbol{\Sigma}_{n+2k}(J_\beta(\mathbb{R}))$, $\boldsymbol{\Pi}_{n+2k+1}(J_\beta(\mathbb{R}))$. These results place exactly the scale property among the classes $\boldsymbol{\Sigma}_n(J_\alpha(\mathbb{R}))$, $\boldsymbol{\Pi}_n(J_\alpha(\mathbb{R}))$. With a little extra argument, this suffices to place exactly the scale property among the Levy classes in $L(\mathbb{R})$ (the only additional classes with the scale property are the Steel classes $\boldsymbol{\Gamma}_0$ such that $\exists^{\omega^\omega}\boldsymbol{\Gamma}_0 = \boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ for some α beginning a gap).

The analysis also shows that for α beginning a gap, a universal $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ set U_α and a $\boldsymbol{\Sigma}_1(J_\alpha(\mathbb{R}))$ scale $\{\phi_n^\alpha\}$ on U_α can be constructed uniformly in α. This uniformity, however, fails for the Steel pointclasses having the scale property; this presents an obstacle in some arguments.

2.5. Determinacy and Coding Results

We begin by recalling a useful ordinal determinacy result. If $\lambda_1, \dots, \lambda_n \in$ On and $A \subseteq \lambda_1^\omega \times \cdots \times \lambda_n^\omega$, we say A is *Suslin* if there is a tree T on $\lambda_1 \times \cdots \times \lambda_n \times \lambda_{n+1}$ for some $\lambda_{n+1} \in$ On such that $(\vec{\alpha}_1, \dots, \vec{\alpha}_n) \in A$ iff $\exists \vec{\alpha}_{n+1} \in \lambda_{n+1}^\omega\ (\vec{\alpha}_1, \dots, \vec{\alpha}_{n+1}) \in [T]$. The collection of Suslin sets contains the open and closed sets (${\lambda_i}^\omega$ endowed with the product of the discrete topology on λ_i), is closed under $\exists^{\omega^\omega}$, countable unions and intersections, continuous preimages, and assuming AD, $\forall^{\omega^\omega}$ (the only non-trivial part is closure under $\forall^{\omega^\omega}$ which follows from the proof of the Second Periodicity Theorem). We say A is *co-Suslin* if $(\lambda_1^\omega \times \cdots \times \lambda_n^\omega) - A$ is Suslin.

The following theorem is the basis for many ordinal determinacy results.

2.23 Theorem (AD). *Let $\lambda < \Theta$, and $A \subseteq \lambda^\omega$ be Suslin and co-Suslin. Then the ordinal game G_A is determined.*

The theorem is due originally to Moschovakis, and appears as Theorem 2.2 of [32] (though in a weaker form). The proof there is similar to that of the Third Periodicity Theorem, using also the Harrington-Kechris Theorem to ensure the determinacy of certain real games. A second proof appears in [22, Theorem 2.5], and is a more direct combinatorial proof.

An important tool in the theory of $L(\mathbb{R})$ is the Solovay Basis Theorem. This, along with Theorem 2.23, will provide the determinacy of some of the games we will consider later.

2.24 Theorem (ZF; Solovay Basis Theorem). *Let $P(A)$ be a $\boldsymbol{\Sigma}_1^2$ relation on sets $A \subseteq \omega^\omega$. If $L(\mathbb{R}) \models \exists A\ P(A)$ then $L(\mathbb{R}) \models \exists A \in \boldsymbol{\Delta}_1^2\ P(A)$.*

Proof. Write $P(A) \longleftrightarrow \exists B \subseteq \omega^\omega\ Q(A, B)$ where Q is projective. Work inside $L(\mathbb{R})$. Since every set in $L(\mathbb{R})$ is ordinal definable from a real, we may fix reals x_0, y_0 and formulas ϕ_1, ϕ_2 such that for some $\alpha, \beta \in$ On we have

$$L(\mathbb{R}) \models \exists! A\ \exists! B\ [\phi_1(x_0, \alpha, A) \wedge \phi_2(y_0, \beta, B) \wedge Q(A, B)].$$

Let $\psi(x_0, \alpha, y_0, \beta)$ denote the right-hand side and $\psi'(x_0, \alpha, y_0, \beta, A, B)$ denote the part inside the square brackets. Let $N \in \omega$ be large enough so that a transitive set model of $\mathrm{ZF}_N + V = L(\mathbb{R})$ containing the reals must be of the form $J_\delta(\mathbb{R})$. Let $(\delta_0, \alpha_0, \beta_0)$, where $\delta_0 > \alpha_0, \beta_0$, be the lexicographically least triple such that

$$J_{\delta_0}(\mathbb{R}) \models \mathrm{ZF}_N + \exists! A\ \exists! B[\phi_1(x_0, \alpha_0, A) \wedge \phi_2(y_0, \beta_0, B) \wedge Q(A, B)].$$

A hull argument shows that δ_0 exists, and there is a map from the reals onto $J_{\delta_0}(\mathbb{R})$, and thus $J_{\delta_0}(\mathbb{R})$ may be coded by a set of reals (that is, there is a structure $(\mathbb{R}, E)$ isomorphic to $J_{\delta_0}(\mathbb{R})$). Let A, B be the unique sets of reals in $J_{\delta_0}(\mathbb{R})$ such that $\phi_1(x_0, \alpha_0, A)$, $\phi_2(y_0, \beta_0, B)$, and $Q(A, B)$ hold in $J_{\delta_0}(\mathbb{R})$. Since Q is projective, $Q(A, B)$ holds in $L(\mathbb{R})$, and thus $P(A)$. We have:
$x \in A \longleftrightarrow \exists E \subseteq \omega^\omega \times \omega^\omega\ \exists x', x_0', y_0' \in \mathbb{R}$ satisfying the following:

(1) $(\mathbb{R}, E)$ is well-founded, $(\mathbb{R}, E) \models \mathrm{ZF}_N + V = L(\mathbb{R})$, and $\mathbb{R} \subseteq \pi(\mathbb{R}, E)$, where π is the transitive collapse map.

(2) $\pi(x') = x$, $\pi(x_0') = x_0$, $\pi(y_0') = y_0$.

(3) $(\mathbb{R}, E) \models \exists \alpha', \beta'\ [\psi(x_0', \alpha', y_0', \beta') \ \wedge\ \forall(\alpha'', \beta'') <_{\mathrm{lex}} (\alpha', \beta')\ \neg\psi(x_0', \alpha'', y_0', \beta'') \wedge \exists A' \exists B'\ \psi'(x_0', \alpha', y_0', \beta', A', B') \wedge x' \in A']$.

(4) $(\mathbb{R}, E) \models \forall \delta' \in \mathrm{On}\ J_{\delta'}(\mathbb{R}) \not\models \exists \alpha', \beta'\ \psi(x_0', \alpha', y_0', \beta')$.

Since (1)–(4) are projective statements about E, this shows that $A \in \boldsymbol{\Sigma}^2_1$, and a similar computation shows $A^c \in \boldsymbol{\Sigma}^2_1$. ⊣

From this, we get a useful determinacy result.

2.25 Theorem $(\mathrm{AD} + V = L(\mathbb{R}))$. *Let $\lambda < \Theta$ and $F : \lambda^\omega \times (\omega^\omega)^n \to \omega^\omega$ be continuous, and $A \subseteq \omega^\omega$. Consider the game $G_{\lambda,F,A}$ on λ where I, II play $\alpha_0, \alpha_1, \dots$ producing $\vec{\alpha} \in \lambda^\omega$, and I wins iff*

$$\exists x_1 \forall x_2 \cdots \exists(\forall) x_n\ F(\vec{\alpha}, x_1, x_2, \dots, x_n) \in A.$$

Then $G_{\lambda,F,A}$ is determined.

Proof. Suppose the theorem fails. By a hull argument, there is an $E \subseteq \omega^\omega \times \omega^\omega$ such that $(\mathbb{R}, E)$ is well-founded, $\mathbb{R} \subseteq \pi(M, E)$, where π denotes the transitive collapse map, and there are $\lambda', F', A' \in \mathbb{R}$ such that $(\mathbb{R}, E) \models \mathrm{ZF}_N +$ "(λ', F', A') witnesses the theorem fails". From Theorem 2.24, we may fix such an E which is $\boldsymbol{\Delta}^2_1$. Let $J_\delta(\mathbb{R}) = \pi(\mathbb{R}, E)$. Let $\lambda'' = \pi(\lambda')$, $A'' = \pi(A')$, $F'' = \pi(F')$. So, $J_\delta(\mathbb{R}) \models$ "(λ'', A'', F'') witness the theorem fails". Since $E \in \boldsymbol{\Delta}^2_1$, easily $A'' \in \boldsymbol{\Delta}^2_1$. Hence A'' is Suslin, co-Suslin in $L(\mathbb{R})$, and thus so is

$$\{(\vec{\alpha}, x_1, \dots, x_n) \in \lambda''^\omega \times (\omega^\omega)^n : F''(\vec{\alpha}, x_1, \dots, x_n,) \in A''\}.$$

By periodicity, $G_{\lambda'',A'',F''}$ is Suslin, co-Suslin in $L(\mathbb{R})$ and therefore determined. However, a winning strategy for this game can be identified with a subset of $\lambda'' < \Theta^{J_\delta(\mathbb{R})}$. By the Coding Lemma, the strategy must then lie in $J_\delta(\mathbb{R})$, a contradiction. ⊣

2.26 Corollary ($\mathrm{AD} + V = L(\mathbb{R})$). *Let $\lambda < \Theta$, $F : \lambda^\omega \to \omega^\omega$ be continuous, $A \subseteq \omega^\omega$, and G the game on λ with payoff $F^{-1}(A)$. Then G is determined.*

If $\mathbf{\Gamma}$ is a pointclass closed under $\forall^{\omega^\omega}, \vee$, and $\mathrm{pwo}(\mathbf{\Gamma})$, and if $\phi : A \xrightarrow{onto} \kappa$ is a $\mathbf{\Gamma}$ norm on the $\mathbf{\Gamma}$-complete set A, then the usual boundedness principle applies: every $B \subseteq A$ in $\check{\mathbf{\Gamma}}$ is bounded below κ with respect to ϕ. In this case, κ is the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings, and κ is regular. There is a useful generalization of this principle, due to Steel, which applies to all ordinals $\alpha < \Theta$. First, we recall one of the main results of [35]:

2.27 Theorem (AD; Steel). *Let $\mathbf{\Gamma}$ be non-selfdual, and $\exists^{\omega^\omega} \mathbf{\Delta} \subseteq \mathbf{\Delta}$. Then $\mathbf{\Gamma}$ is closed under intersections with κ-Suslin sets for $\kappa < \mathrm{cf}(o(\mathbf{\Delta}))$.*

The non-trivial case of Theorem 2.27 is when $\mathrm{sep}(\mathbf{\Gamma})$ holds, for if $\mathrm{sep}(\check{\mathbf{\Gamma}})$ then $\mathbf{\Gamma}$ is closed under $\wedge$ by [35].

Using Theorem 2.27 we now have the following general boundedness principle. We follow the proof in [12]. We say a norm $\phi : A \xrightarrow{onto} \alpha$ is *κ-Suslin bounded* if for every $B \subseteq A$ which is κ-Suslin we have $\sup\{\phi(x) : x \in B\} < \alpha$.

2.28 Theorem (AD; Steel). *Let $\alpha < \Theta$ be a limit ordinal. Then there is an $A \subseteq \omega^\omega$ and a norm $\phi : A \xrightarrow{onto} \alpha$ which is κ-Suslin bounded for all $\kappa < \mathrm{cf}(\alpha)$.*

Proof. First note that we may assume α is regular, since a norm of length $\mathrm{cf}(\alpha)$ which is κ-Suslin bounded for all $\kappa < \mathrm{cf}(\alpha)$ produces one of length α. For example, let $\delta = \mathrm{cf}(\alpha)$, $h : \delta \to \alpha$ be cofinal and increasing, and $\psi : B \to \delta$ a norm which is κ-Suslin bounded for all $\kappa < \delta$. Let $\rho : C \xrightarrow{onto} \alpha$ be a norm. Define for $\beta < \alpha$,

$$A_\beta(x) \longleftrightarrow [x_0 \in C \ \wedge \ \rho(x_0) = \beta \ \wedge x_1 \in B \ \wedge \ \beta < h(\psi(x_1))] \,.$$

Let $A = \bigcup_{\beta<\alpha} A_\beta$, and $\phi(x) = \rho(x_0)$ for $x \in A$. Suppose $S \subseteq A$ is κ-Suslin for some $\kappa < \delta$. Let $S_1 = \{x_1 : x \in S\}$. Then S_1 is κ-Suslin and $S_1 \subseteq B$, and so $\eta \doteq \sup\{\psi(x_1) : x \in S\} < \delta$. But clearly then $\sup\{\phi(x) : x \in S\} \leq h(\eta)$. So assume α is regular.

Similarly, it suffices to produce a norm $\psi : A \xrightarrow{onto} \rho$ which is κ-Suslin bounded for all $\kappa < \alpha$, for some ρ of cofinality α. For suppose $\psi : A \xrightarrow{onto} \rho$ is such a norm, and $\mathrm{cf}(\rho) = \alpha$. Let $h : \alpha \to \rho$ be cofinal. Define $\phi : A \xrightarrow{onto} \alpha$ by $\phi(x) =$ the least $\beta < \alpha$ such that $h(\beta) > \psi(x)$. Then easily ϕ is κ-Suslin bounded for all $\kappa < \alpha$.

Let $\rho > \alpha$ be a limit cardinal of cofinality α such that the collection $\mathbf{\Delta}$ of sets of Wadge degree $< \rho$ is closed under $\exists^{\omega^\omega}$. We produce an A and a norm $\phi : A \xrightarrow{onto} \rho$ which is κ-Suslin bounded for all $\kappa < \alpha$. Let $\mathbf{\Gamma}$ be the non-selfdual pointclass closed under $\forall^{\omega^\omega}$ with $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ (see [35], we assume $\alpha > \omega$ as otherwise the result is trivial). Let B be a $\mathbf{\Gamma}$ universal set. Define A by: $x \in A$ iff x_0, x_1 code continuous functions f_{x_0}, f_{x_1} with $f_{x_0}^{-1}(B) = \omega^\omega - f_{x_1}^{-1}(B)$. For $x \in A$, let $\phi(x)$ be the Wadge degree of the $\mathbf{\Delta}$ set $f_{x_0}^{-1}$. Clearly $\phi : A \xrightarrow{onto} \rho$. Suppose $S \subseteq A$ is κ-Suslin for some $\kappa < \alpha$, and assume towards a contradiction that $\sup\{\phi(x) : x \in S\} = \rho$. Define

$$C(x,y) \overset{\text{def}}{\longleftrightarrow} (x \in S \wedge f_{x_0}(y) \in B) \longleftrightarrow (x \in S \wedge f_{x_1}(y) \notin B)\,.$$

Thus, $C \in \mathbf{\Delta}$ by Theorem 2.27. This is a contradiction, since any $\mathbf{\Delta}$ set is Wadge reducible to C. For let $D \in \mathbf{\Delta}$, and take $x \in S$ so that $f_{x_0}^{-1}(B) = \omega^\omega - f_{x_1}^{-1}(B) = D' \geq_w D$. Then $y \in D' \longleftrightarrow (x,y) \in C$. ⊣

2.6. Partition Relations

We recall some facts and terminology associated with partition relations that we will be using frequently. We give the definitions working in our base theory ZF + DC, although to obtain non-trivial results we will have to assume AD.

If $f : \alpha \to \mathrm{On}$, we say f has *uniform cofinality* ω if there is a $f' : \alpha \times \omega \to \mathrm{On}$ such that $\forall \beta < \alpha\ f(\beta) = \sup\{f'(\beta,n) : n \in \omega\}$ and f' is increasing in the second argument, that is, $\forall \beta\ \forall n,m\ [n < m \to f'(\beta,n) < f'(\beta,m)]$.

2.29 Definition. We say $f : \alpha \to \mathrm{On}$ *is of the correct type* if f is increasing, everywhere discontinuous (i.e., for all $\beta < \alpha$, $f(\beta) > \sup\{f(\beta') : \beta' < \beta\}$), and of uniform cofinality ω.

Generalizing this, we define:

2.30 Definition. Let $f, S : \alpha \to \mathrm{On}$. We say f has *uniform cofinality* S if there is a function $l : \{(\beta,\gamma) : \beta < \alpha \wedge \gamma < S(\beta)\} \to \mathrm{On}$ which is increasing in the second argument and $\forall \beta < \alpha\ f(\beta) = \sup\{l(\beta,\gamma) : \gamma < S(\beta)\}$. We frequently just say $f(\beta)$ has uniform cofinality $S(\beta)$.

If μ is a measure (i.e., a countably additive ultrafilter) on α, we say f has *uniform cofinality* S *almost everywhere*, for S as above, if $\forall^*_\mu \beta < \alpha\ f(\beta) = f(\beta) = \sup\{l(\beta,\gamma) : \gamma < S(\beta)\}$. We usually just say $f(\beta)$ has uniform cofinality $S(\beta)$ almost everywhere with respect to μ.

Note that the statement "f has uniform cofinality S almost everywhere with respect to μ" depends only on $[f]_\mu$, $[S]_\mu$.

For κ a cardinal and $\lambda \leq \kappa$, we let $(\kappa)^\lambda$ denote the set of increasing functions from λ to κ. We write $\kappa \to (\kappa)^\lambda$ to mean: for every partition $\mathcal{P} : (\kappa)^\lambda \to \{0,1\}$ of the increasing functions from λ to κ into two pieces, there is a homogeneous $H \subseteq \kappa$ of size κ. That is, there is an $i \in \{0,1\}$ such that for all $f \in (H)^\lambda$ we have $\mathcal{P}(f) = i$. We define a variation on this as follows. We say $\kappa \xrightarrow{\text{c.u.b.}} (\kappa)^\lambda$ if for all partitions $\mathcal{P} : (\kappa)^\lambda \to \{0,1\}$ of the increasing functions from λ to κ into two pieces, there is a closed unbounded $C \subseteq \kappa$ such that for some $i \in \{0,1\}$ and all $f : \lambda \to C$ of the correct type, $\mathcal{P}(f) = i$.

The following well-known fact connects these two variations. The proof is straightforward, and left to the reader.

2.31 Fact. For all cardinals κ and ordinals $\lambda \leq \kappa$:

1. $\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda \implies \kappa \to (\kappa)^\lambda$.

2. $\kappa \to (\kappa)^{\omega \cdot \lambda} \implies \kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$.

The instances of the partition property of particular importance to us are expressed in the following definition.

2.32 Definition. We say a cardinal κ has the *strong partition property* if $\kappa \to (\kappa)^\kappa$. We say κ has the *weak partition property* if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.

From Fact 2.31 it follows that the notions of strong and weak partition property of κ do not depend on which of the two variations of the definition are used. In all of the determinacy arguments, it is the "c.u.b." version of the partition relation which is relevant. Since we will never need the other variation, we therefore adopt the convention that henceforth, $\kappa \to (\kappa)^\lambda$ means $\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$.

There are two slight generalizations of the strong partition property of κ which we will employ frequently. First, if $\prec$ is a wellordering of some set $\mathrm{dom}(\prec)$ of order-type κ, we have the strong partition property for partitions of functions $f : \mathrm{dom}(\prec) \to \kappa$ of the correct type (defined in the obvious manner). Second, instead of considering functions $f : \kappa \to \kappa$ or $f : \mathrm{dom}(\prec) \to \kappa$ of the correct type, we may consider f which are increasing, everywhere discontinuous, and of uniform cofinality S, for any fixed $S : \kappa \to \kappa$. Alternatively, we may consider partitions of functions f which are increasing, continuous at limit ordinals (or points of limit rank in $\prec$), and such that $f(\alpha)$ has uniform cofinality $S(\alpha)$ at points of successor rank. In either case, the generalized version of the strong partition property follows easily from the usual strong partition relation.

We present now an abstract form of Martin's proof of the strong partition relation on ω_1. We state it in the most general form for which we are able to prove it.

2.33 Definition. Let κ be a regular cardinal, $\lambda \in \mathrm{On}$, $\lambda \leq \kappa$. We say κ is λ-*reasonable* there is a non-selfdual pointclass $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$, and a map ϕ with domain ω^ω satisfying (where $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$):

(1) $\forall x\ \phi(x) \subseteq \lambda \times \kappa$.

(2) $\forall F : \lambda \to \kappa\ \exists x\ (\phi(x) = F)$.

(3) $\forall\beta<\lambda\ \forall\gamma<\kappa\ R_{\beta,\gamma}\in\mathbf{\Delta}$, where

$$x\in R_{\beta,\gamma}\longleftrightarrow\phi(x)(\beta,\gamma)\wedge\forall\gamma'<\kappa\ (\phi(x)(\beta,\gamma')\to\gamma'=\gamma).$$

(4) Suppose $\beta<\lambda$, $A\in\exists^{\omega^\omega}\mathbf{\Delta}$, and $A\subseteq R_\beta\doteq\{x:\exists\gamma<\kappa\ R_{\beta,\gamma}(x)\}$. Then $\exists\gamma_0<\kappa\ \forall x\in A\ \exists\gamma<\gamma_0\ R_{\beta,\gamma}(x)$.

We say κ is *reasonable* if it is κ-reasonable. If $\exists!\gamma\ \phi(x)(\beta,\gamma)$, then we write $\phi(x)(\beta)$ for this unique γ. Note that the pointclass hypotheses of the theorem are really just that $\mathbf{\Delta}=\mathbf{\Gamma}\cap\check{\mathbf{\Gamma}}$ for some Levy class $\mathbf{\Gamma}$ (i.e., $\mathbf{\Gamma}$ is non-selfdual and closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$) as the hypotheses are symmetric between $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$. Recall that from AD we have either pwo($\mathbf{\Gamma}$) or pwo($\check{\mathbf{\Gamma}}$).

2.34 Theorem (AD; Martin). *If κ is $\omega\cdot\lambda$-reasonable, then $\kappa\to(\kappa)^\lambda$.*

Proof. We will show below that $\mathbf{\Delta}$ is in fact closed under $<\kappa$ unions and intersections; we assume this for now. We refer below to the sets R_β, $R_{\beta,\gamma}$ of Definition 2.33.

Fix a partition $\mathcal{P}:(\kappa)^\lambda\to\{0,1\}$. Play the integer game where I plays out $x\in\omega^\omega$, II plays out $y\in\omega^\omega$. If there is a least ordinal $\beta<\omega\cdot\lambda$ such that $x\notin R_\beta$ or $y\notin R_\beta$, then II wins provided $x\notin R_\beta$. Otherwise, let f_x, $f_y:\omega\cdot\lambda\to\kappa$ be the functions they determine (e.g., $f_x(\beta)=\phi(x)(\beta)$). Define in this case $f_{x,y}:\lambda\to\kappa$ by

$$f_{x,y}(\beta)=\sup\{\max(f_x(\beta'),f_y(\beta')):\beta'<\omega\cdot(\beta+1)\}\,.$$

II then wins iff $\mathcal{P}(f_{x,y})=1$.

Assume without loss of generality that II has a winning strategy τ. For $\beta<\omega\cdot\lambda$ and $\gamma<\kappa$, define $x\in S_{\beta,\gamma}\longleftrightarrow\forall\beta'\le\beta\ \exists\gamma'\le\gamma\ x\in R_{\beta',\gamma'}$. Thus, $S_{\beta,\gamma}\in\mathbf{\Delta}$. Hence, for all $\beta<\omega\cdot\lambda$ and $\gamma<\kappa$, $\tau[S_{\beta,\gamma}]\in\exists^{\omega^\omega}\mathbf{\Delta}$ (note that for any Levy class $\mathbf{\Gamma}$ that $\exists^{\omega^\omega}\mathbf{\Delta}$ is closed under $\wedge$, $\vee$; an easy consequence of the hierarchy analysis of Sect. 2.3). Now, $\tau[S_{\beta,\gamma}]\subseteq R_\beta$. Thus, $\theta(\beta,\gamma)\doteq\sup\{\phi(x)(\beta):x\in\tau[S_{\beta,\gamma}]\}<\kappa$, from (4) of Definition 2.33. Let $C\subseteq\kappa$ be the set of points closed under θ, and $C'\subseteq C$ the set of limit points of C.

Suppose $F:\lambda\to C'$ is of the correct type; we show that $\mathcal{P}(F)=1$. Let x be such that $\phi(x)$ determines a function $f_x:\omega\cdot\lambda\to C$ such that $F(\beta)=\sup\{f_x(\beta'):\beta'<\omega\cdot(\beta+1)\}$. We may assume $f_x(\beta)\ge\beta$ for all β. Let $y=\tau(x)$. From the definition of C it follows that $\phi(y)$ determines a function $f_y:\omega\cdot\lambda\to\kappa$ such that $f_y(\beta)\le f_x(\beta+1)$ for all β. Thus, $F=f_{x,y}$, so $\mathcal{P}(F)=1$.

We show now that $\mathbf{\Delta}$ is closed under $<\kappa$ unions. Suppose not, and let $\delta<\kappa$ be least such that some union $A=\bigcup_{\alpha<\delta}A_\alpha$ is not in $\mathbf{\Delta}$. Note that $R_0=\bigcup_{\gamma<\kappa}R_{0,\gamma}$ is a κ union of $\mathbf{\Delta}$ sets, and $R_0\notin\exists^{\omega^\omega}\mathbf{\Delta}$. Suppose first pwo($\mathbf{\Gamma}$). Then $\mathbf{\Gamma}$ is closed under wellordered unions by Lemma 2.21. Thus $A\in\mathbf{\Gamma}$, and by Wadge's Lemma, $R_0=\bigcup_{\alpha<\delta}S_\alpha$ for some $S_\alpha\in\mathbf{\Delta}$. Since κ is regular, one of the $S_\alpha\subseteq R_0$ must be "unbounded" in κ, a contradiction to $\omega \cdot \lambda$-reasonableness. So assume pwo($\check{\boldsymbol{\Gamma}}$), and thus pwo($\boldsymbol{\Gamma}_1$), where $\boldsymbol{\Gamma}_1 = \exists^{\omega^\omega} \check{\boldsymbol{\Gamma}}$. Thus, $\boldsymbol{\Gamma}_1$ is closed under wellordered unions, and so $R_0 \in \boldsymbol{\Gamma}_1$. We cannot have $\bigcup_\delta \boldsymbol{\Delta} = \boldsymbol{\Gamma}$, as otherwise Martin's argument (Theorem 2.15) shows pwo($\boldsymbol{\Gamma}$). It follows that $\bigcup_\delta \boldsymbol{\Delta} \supseteq \check{\boldsymbol{\Gamma}}$, and so $\bigcup_\delta \exists^{\omega^\omega} \boldsymbol{\Delta} \supseteq \boldsymbol{\Gamma}_1$ (and hence actually $\boldsymbol{\Gamma}_1 = \bigcup_\delta \exists^{\omega^\omega} \boldsymbol{\Delta}$). Thus, $R_0 = \bigcup_{\alpha<\delta} S_\alpha$, with each $S_\alpha \in \exists^{\omega^\omega} \boldsymbol{\Delta}$. As before, this contradicts reasonableness. ⊣

2.35 Remark. The proof shows that if $\boldsymbol{\Gamma}$,ϕ witness the λ-reasonableness of κ, then $\boldsymbol{\Delta}$ is closed under $<\kappa$ unions. With a little extra work one can show $\boldsymbol{\Gamma}$ is closed under countable unions, intersections, and pwo($\check{\boldsymbol{\Gamma}}$).

The next lemma shows that all the $\boldsymbol{\delta}^1_{2n+1}$, and in particular ω_1, have the countable exponent partition relation. We will take this as the start of our analysis in the next section.

2.36 Theorem (AD). *Let $\boldsymbol{\Gamma}$ be a non-selfdual pointclass closed under $\forall^{\omega^\omega}$, $\wedge, \vee$ and assume* pwo($\boldsymbol{\Gamma}$). *Let $\delta = \delta(\boldsymbol{\Gamma})$ = the supremum of the lengths of the $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ prewellorderings. Then $\delta \to (\delta)^\lambda$ for all $\lambda < \omega_1$.*

Proof. Fix λ, and a bijection $\pi : \omega \to \lambda$. Fix also a $\boldsymbol{\Gamma}$ universal set P and a $\boldsymbol{\Gamma}$ norm ψ on P. We may assume ψ is onto an ordinal, in which case that ordinal is δ. We define the map ϕ so that $\check{\boldsymbol{\Gamma}}, \phi$ witness the λ-reasonableness of δ. Define $\phi(x)(\beta, \gamma)$ iff $x_n \in P \wedge \psi(x_n) = \gamma$, where $\pi(n) = \beta$. Items (1)–(3) are immediate, and (4) follows since a $\check{\boldsymbol{\Gamma}}$ subset of P is bounded. ⊣

Note that if we know directly that $\boldsymbol{\Gamma}$ is closed under countable unions and intersections, then the pointclass arguments in Theorem 2.34 are not necessary for the application to Theorem 2.36, as the sets $S(\alpha, \gamma)$ as defined there are in $\boldsymbol{\Delta}$ directly.

3. Suslin Cardinals

In this section we develop the basic theory of Suslin cardinals and scales assuming AD. The results presented here completely classify the Suslin cardinals κ and the corresponding Suslin classes $S(\kappa)$. They also suffice to completely determine the scaled pointclasses with one exception: if $\boldsymbol{\Gamma}$ is scaled and closed under quantifiers, then if λ denotes the next Suslin cardinal, we do not get the scale property at $\boldsymbol{\Sigma}_0$ or $\boldsymbol{\Pi}_1$ where $\boldsymbol{\Sigma}_0$ is the class of countable unions of sets of Wadge degree less than λ. We do, however, get the scale property at $\boldsymbol{\Sigma}_2$ (and by periodicity for the appropriate classes in the remainder of this projective-like hierarchy). As we show in this case, λ^+ is the next Suslin cardinal after λ, and $\boldsymbol{\Sigma}_2 = S(\lambda^+)$. Steel's analysis of scales in $L(\mathbb{R})$ (which we over-viewed in Sect. 2.4) provides a more detailed description if one assumes in addition $V = L(\mathbb{R})$. Namely, this analysis gives also the scale property at $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Pi}_1$.

The main result is Theorem 3.28. We assume AD throughout this section. The arguments of this section are mainly due to Martin and appear in [24], which we follow.

We assume in this section a basic knowledge of homogeneous and weakly homogeneous trees, though we only need here (aside from Theorem 3.2 below) the basic definitions and general properties of the homogeneous tree construction. The reader could skip ahead to Definition 5.1 and its following paragraphs for a discussion.

Throughout this section, ν will denote the Martin measure on the Turing degrees $\mathcal{D}$. Recall that $A \subseteq \mathcal{D}$ has ν measure one iff it contains a cone, that is, there is a degree d such that for all $d' \geq_T d$, $d' \in A$ (here $\leq_T$ denotes Turing reducibility). From AD, ν is a measure (i.e., a countably additive ultrafilter) on $\mathcal{D}$. Thus we will write "$\forall^*_\nu\, d$" to mean "for ν almost all degrees $d \in \mathcal{D}$". When we are clearly talking about degrees, we will frequently just write $x \leq d$ instead of $x \leq_T d$.

Recall that $S(\kappa)$ denotes the pointclass of κ-Suslin sets. Recall also the definitions of $o(\boldsymbol{\Gamma})$, $\delta(\boldsymbol{\Gamma})$ from Definition 2.19. We will use frequently the fact mentioned previously (from [21]) that for $\boldsymbol{\Delta}$ closed under real quantification, $\wedge$ and $\vee$, we have $o(\boldsymbol{\Delta}) = \delta(\boldsymbol{\Delta})$.

We state two theorems we will need for this analysis. The first, due to Steel and Woodin, is the following.

3.1 Theorem (Steel, Woodin). *The set of Suslin cardinals is closed below their supremum.*

Thus, assuming AD, the set of Suslin cardinals is closed below Θ except that the supremum of the Suslin cardinals, if less than Θ, may perhaps not be a Suslin cardinal. Woodin [42] has isolated a strengthening of AD called AD^+ which implies that the Suslin cardinals are closed below Θ. It is apparently unknown whether AD implies AD^+. We refer the reader to [42] for further discussion of AD^+.

We will also need the following theorem of Martin and Woodin on weak homogeneity. We refer the reader to [30] for a proof.

3.2 Theorem (Martin, Woodin). *Let κ be less than the supremum of the Suslin cardinals. Then every tree on $\omega \times \kappa$ is weakly homogeneous.*

3.1. Pointclass Arguments

We recall the following fact about the homogeneously Suslin sets.

3.3 Lemma. *Let κ be a cardinal. Let $\boldsymbol{\Gamma}$ be the collection of $A \subseteq \omega^\omega$ which can be written in the form $A = p[T]$ where T is a homogeneous tree on $\omega \times \kappa$. Then $\boldsymbol{\Gamma}$ is a pointclass and is closed under $\forall^{\omega^\omega}$.*

Proof. It is straightforward to check that $\boldsymbol{\Gamma}$ is a pointclass. We first show that $\forall^{\omega^\omega} \boldsymbol{\Gamma} \subseteq S(\kappa)$. Suppose $A(x) \longleftrightarrow \forall y\, B(x, y)$ where $B \in \boldsymbol{\Gamma}$, say $B = p[T]$

where T is a homogeneous tree on $\omega \times \omega \times \kappa$. Let $\{s_i\}_{i\in\omega}$ enumerate $\omega^{<\omega}$ in a reasonable manner. Define a tree U on $\omega \times \kappa$ by:

$$(t, \vec{\alpha}) \in U \longleftrightarrow \forall i < \mathrm{lh}(t)\ (t{\restriction}\,\mathrm{lh}(s_i), s_i, \vec{\beta}) \in T,$$

where $\vec{\beta} = (\alpha_{j(0)}, \alpha_{j(1)}, \ldots, \alpha_{j(\mathrm{lh}(s_i)-1)})$, and $j(a)$ is the integer such that $s_{j(a)} = s_i{\restriction}a$. Clearly $p[U] \subseteq A$. The inclusion $A \subseteq p[U]$ follows also if we have that for every $x \in A$ there is a Lipschitz continuous $f : \omega^\omega \to \kappa^\omega$ such that for all $y \in \omega^\omega$, $(x, y, f(y)) \in [T]$ (for this f will produce a branch through U_x). The existence of f follows from the homogeneity of T: play the (closed for II) game where I plays integers $y(i)$, II plays ordinals $\alpha(i) < \kappa$, and II wins the run iff for all n, $(x{\restriction}n, y{\restriction}n, \vec{\alpha}{\restriction}n) \in T$. Since T is homogeneous, II has a winning strategy in this game, and this gives the desired function f. So, $\forall^{\omega^\omega}\boldsymbol{\Gamma} \subseteq S(\kappa)$.

If $\boldsymbol{\Gamma} = S(\kappa)$, then this shows $\forall^{\omega^\omega}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}$ (and $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ as well). Suppose $\boldsymbol{\Gamma} \subsetneq S(\kappa)$. If κ is not the largest Suslin cardinal, then from Theorem 3.2 we have $S(\kappa) = \exists^{\omega^\omega}\boldsymbol{\Gamma}$. If $\forall^{\omega^\omega}\boldsymbol{\Gamma} \neq \boldsymbol{\Gamma}$, then by Wadge's Lemma $\forall^{\omega^\omega}\boldsymbol{\Gamma} \supseteq \forall^{\omega^\omega}\check{\boldsymbol{\Gamma}} = \check{S}(\kappa)$, a contradiction. If κ is the largest Suslin cardinal, then $S(\kappa)$ is closed under $\forall^{\omega^\omega}$ (as well as $\exists^{\omega^\omega}$) as otherwise periodicity would give a larger Suslin class. So, $\boldsymbol{\Delta}(S(\kappa))$ is closed under real quantification. Also, any $A \in S(\kappa) - \check{S}(\kappa)$ cannot be in $\boldsymbol{\Gamma}$ (or even be the projection of a weakly homogeneous tree), as otherwise $\omega^\omega - A$ would be Suslin from the homogeneous tree construction. Thus, in this case $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Delta}(S(\kappa))$. We borrow one fact from the upcoming Lemma 3.6, namely that $\mathrm{cf}(\kappa) > \omega$. Thus, every $S(\kappa)$ set is an increasing union of sets in $\bigcup_{\lambda<\kappa} S(\lambda)$. If $\bigcup_{\lambda<\kappa} S(\lambda)$ were properly contained in $\boldsymbol{\Delta}(S(\kappa))$, then we could find a pointclass $\boldsymbol{\Gamma}_0$ properly contained in $\boldsymbol{\Delta}(S(\kappa))$ which is closed under $\exists^{\omega^\omega}$, $\mathrm{pwo}(\boldsymbol{\Gamma}_0)$, and $\bigcup_{\lambda<\kappa} S(\lambda) \subseteq \boldsymbol{\Gamma}_0$. From Lemma 2.21 it follows that $S(\kappa) \subseteq \boldsymbol{\Gamma}_0$, a contradiction. So, every $\boldsymbol{\Delta}(S(\kappa))$ set is λ-Suslin for some $\lambda < \kappa$. From Theorem 3.2 it follows that every set in $\boldsymbol{\Delta}(S(\kappa))$ is the existential quantification of a set in $\boldsymbol{\Gamma}$, and so $\boldsymbol{\Gamma} = \boldsymbol{\Delta}(S(\kappa))$. Hence, $\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$. $\dashv$

We need the following simple lemma.

3.4 Lemma. *Let κ be a Suslin cardinal. Then there is a κ-length strictly increasing sequence of sets in $S(\kappa)$. If $\mathrm{cf}(\kappa) > \omega$, then there is a κ-length strictly increasing sequence of sets each of which is $<\kappa$-Suslin.*

Proof. The proof that every Suslin cardinal is reliable (cf. [37, Lemma 4.6] and Sect. 6.1) shows that for any Suslin cardinal κ there is an $A \in S(\kappa) - \check{S}(\kappa)$ and a scale $\{\phi_i\}$ on A with norms into κ and with ϕ_0 onto κ. We recall the argument. Let $B \in S(\kappa) - \check{S}(\kappa)$, and $\{\psi_i\}$ a regular scale on B with norms into κ. Let $A = \{x : x' \in B\}$, where $x'(n) = x(n + 1)$. Define for $x \in A$, $\phi_0(x) = \psi_{x(0)}(x')$, and $\phi_{i+1}(x) = \psi_i(x')$. Then ϕ_0 is onto κ as otherwise $A \in S(\lambda)$ for some $\lambda < \kappa$. For $\alpha < \kappa$ let $A_\alpha = \{x \in A : \phi_0(x) \leq \alpha\}$. Each A_α is in $S(\kappa)$. Moreover, the A_α form a strictly increasing sequence of $S(\kappa)$ sets of length κ.

Suppose now that $\mathrm{cf}(\kappa) > \omega$. We now use the argument of [14, Lemma 2.1]. Recall ϕ_0 maps onto κ. For $\alpha < \beta < \kappa$ define

$$A_{\alpha,\beta} = \{x : \forall i\ \phi_i(x) < \beta\} \cup \{x : \forall i\ \phi_i(x) \leq \beta \wedge \phi_0(x) \leq \alpha\}.$$

Each $A_{\alpha,\beta}$ is $<\kappa$-Suslin. If we view the increasing pairs (α, β) as ordered first by the second coordinate and then the first, clearly the $A_{\alpha,\beta}$ form a (not necessarily strictly) increasing sequence of order type κ. For each $\alpha < \kappa$, there is an x with $\phi_0(x) = \alpha$ and for this x there is a least $\beta > \alpha$ such that $\forall i\ \phi_i(x) \leq \beta$. It follows that for these α, β that $A_{\alpha,\beta} - \bigcup_{\alpha',\beta'} A_{\alpha',\beta'} \neq \emptyset$ where the union ranges over (α', β') less than (α, β) in the ordering described. Thus there is a κ length subsequence of the $A_{\alpha,\beta}$ which is strictly increasing. $\dashv$

We will need the following result, due to Chuang, in the theory of pointclasses. The methods used in the proof are similar to those of [14].

3.5 Theorem (Chuang). *Let* $\boldsymbol{\Gamma}$ *be non-selfdual and closed under* $\forall^{\omega^\omega}$, $\vee$, *and assume* $\mathrm{pwo}(\boldsymbol{\Gamma})$. *Then there is no strictly increasing or decreasing sequence of* $\boldsymbol{\Gamma}$ *sets of length* $(\delta(\boldsymbol{\Gamma}))^+$.

Proof. Let $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ as usual. Note that $\boldsymbol{\Gamma}$ is closed under countable intersections and $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ (which may be $\boldsymbol{\Gamma}$) is closed under countable unions and intersections. We fix a universal $\boldsymbol{\Gamma}$ set $U \subseteq \omega^\omega \times \omega^\omega$, so every real x codes a $\boldsymbol{\Gamma}$ set $U_x \subseteq \omega^\omega$. Let δ_0 be the supremum of those limit β such that $\{A : o(A) < \beta\}$ is closed under real quantification and is contained within $\boldsymbol{\Delta}$. Let $\boldsymbol{\Delta}_0 = \{A : o(A) < \delta_0\}$. Let $\delta = \delta(\boldsymbol{\Gamma}) =$ the supremum of the lengths of the $\boldsymbol{\Delta}$ prewellorderings. Suppose $\{A_\alpha\}_{\alpha<\delta^+}$ is a strictly increasing sequence of $\boldsymbol{\Gamma}$ sets. By thinning the sequence we may assume that for all $\alpha < \delta^+$ that $\bigcup_{\beta<\alpha} A_\beta \subsetneq A_\alpha$. Let $A = \bigcup_\alpha A_\alpha$. For $x \in A$ let $\phi(x) < \delta^+$ be the least ordinal α such that $x \in A_\alpha$. Let $\prec$ be the strict prewellordering defined by

$$x \prec y \longleftrightarrow x, y \in A \wedge \phi(x) < \phi(y).$$

Thus, $\prec$ has length δ^+.

We consider two cases, though the argument in each case is similar.

Case I. $\boldsymbol{\Gamma}$ is not closed under $\exists^{\omega^\omega}$.

By periodicity $\mathrm{pwo}(\exists^{\omega^\omega}\boldsymbol{\Gamma})$, and so by Lemma 2.21 $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ is closed under wellordered unions. It follows that $\prec\, \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$ since $x \prec y$ iff $\exists \alpha < \beta < \delta^+\ (x \in A_\alpha \wedge y \in A_\beta \wedge y \notin A_\alpha)$. We use here the fact that $\check{\boldsymbol{\Gamma}} \subseteq \exists^{\omega^\omega}\boldsymbol{\Gamma}$ as $\boldsymbol{\Gamma}$ is not closed under $\exists^{\omega^\omega}$. Let $C \subseteq (\omega^\omega)^3$ be defined by:

$$(x, y, z) \in C \longleftrightarrow \exists \alpha < \delta^+\ (U_x = A_\alpha \wedge y, z \in A \wedge \phi(y) = \alpha \wedge \phi(z) = \alpha + 1).$$

Applying the Coding Lemma to the $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ relation $\prec$ gives an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ set $S \subseteq C$ such that for all $\alpha < \delta^+$ there is an $(x, y, z) \in S$ with $\phi(y) = \alpha$. From $\mathrm{pwo}(\boldsymbol{\Gamma})$ and the closure of $\boldsymbol{\Gamma}$ under $\vee$ the usual boundedness argument shows that if ψ is a regular $\boldsymbol{\Gamma}$ norm on a $\boldsymbol{\Gamma}$ complete set B, then ψ maps onto δ and every $\check{\boldsymbol{\Gamma}}$ subset of B is bounded in the norm. In particular, every $\check{\boldsymbol{\Gamma}}$ well-founded relation has length less than δ (otherwise the Coding Lemma gives an unbounded $\check{\boldsymbol{\Gamma}}$ subset of B). Also from $\mathrm{pwo}(\boldsymbol{\Gamma})$, every $\boldsymbol{\Gamma}$ set is a δ union of $\boldsymbol{\Delta}$ sets. It follows that every $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ set is a δ union of $\exists^{\omega^\omega}\boldsymbol{\Delta} \subseteq \check{\boldsymbol{\Gamma}}$ sets. So write $S = \bigcup_{\beta<\delta} S_\beta$, where each $S_\beta \in \check{\boldsymbol{\Gamma}}$. For $\beta < \delta$ let $\leq_\beta$ be the prewellordering on S_β defined by

$$\begin{aligned}
&(x_1, y_1, z_1) \leq_\beta (x_2, y_2, z_2)\\
&\quad\longleftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge \phi(y_1) \leq \phi(y_2)\\
&\quad\longleftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge y_1 \in U_{x_2}\\
&\quad\longleftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge z_2 \notin U_{x_1}.
\end{aligned}$$

Thus, $\leq_\beta$ can be written as the intersection of $S_\beta \times S_\beta$ with a $\boldsymbol{\Gamma}$ set or with a $\check{\boldsymbol{\Gamma}}$ set. In particular, $\leq_\beta \in \check{\boldsymbol{\Gamma}}$, and has length less than δ. A similar computation shows the strict part $<_\beta$ of the prewellordering to be in $\check{\boldsymbol{\Gamma}}$ as well.

This however gives a one-to-one map of δ^+ into $\delta \times \delta$, a contradiction. Namely, given $\alpha < \delta^+$ let $\pi_0(\alpha)$ be the least ordinal $\beta < \delta$ such that there is an $(x, y, z) \in S_\beta$ with $\phi(y) = \alpha$. Let $\pi_1(\alpha)$ be the rank of any $(x, y, z) \in S_\beta$ with $\phi(y) = \alpha$ in the prewellordering $\leq_\beta$. It is easy to check that this is well-defined and that $\alpha \mapsto (\pi_0(\alpha), \pi_1(\alpha))$ is one-to-one.

Case II. $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$.

In this case $\boldsymbol{\Gamma}$ is closed under real quantification, countable unions and intersections. Define C as in case I. If there is a $\boldsymbol{\Gamma}$ well-founded relation of length δ^+, then using the coding as in case I gives a $\boldsymbol{\Gamma}$ set S as in that case. We still have that every $\check{\boldsymbol{\Gamma}}$ well-founded relation has length less than δ, and we thus get a contradiction exactly as in case I. So suppose every $\boldsymbol{\Gamma}$ well-founded relation has length less than δ^+. From the Coding Lemma there are $\boldsymbol{\Gamma}$ well-founded relations of any length less than δ^+, so δ^+ is the supremum of the lengths of the $\boldsymbol{\Gamma}$ well-founded relations. From this, the Coding Lemma easily implies that δ^+ is regular. Consider the integer game where I plays out $w \in \omega^\omega$ and II plays out $(x, y, z) \in (\omega^\omega)^3$. II wins the run iff (where U' is universal for $\boldsymbol{\Gamma}$ subsets of $(\omega^\omega)^2$):

$$U'_w \text{ is well-founded } \longrightarrow ((x, y, z) \in C \wedge \phi(y) > |U'_w|),$$

where $|U'_w|$ denotes the rank of the relation U'_w. I cannot have a winning strategy, as this would give a $\boldsymbol{\Sigma}^1_1$ set of codes of $\boldsymbol{\Gamma}$ well-founded relations whose lengths were unbounded in δ^+, and from this we would get a $\boldsymbol{\Gamma}$ well-founded relation of length δ^+ (in fact there can be no $\boldsymbol{\Gamma}$ set of codes of $\boldsymbol{\Gamma}$ well-founded relations having lengths unbounded in δ^+). Let τ be a winning strategy for II. Define the relation

$$w_1 \ll w_2 \longleftrightarrow (U'_{w_1}, U'_{w_2} \text{ are well-founded}) \wedge y_2 \notin U_{x_1},$$

where $\tau(w_1) = (x_1, y_1, z_1)$ and $\tau(w_2) = (x_2, y_2, z_2)$. From the closure of $\boldsymbol{\Gamma}$ under quantifiers it follows easily that $\ll \in \check{\boldsymbol{\Gamma}}$, and from the regularity of δ^+ an easy argument shows that $\ll$ has length δ^+ [For $\alpha < \delta^+$ let $f(\alpha) < \delta^+$ be least such that for some w with U'_w well-founded and $|U'_w| = \alpha$ we have $f(\alpha) = \phi(y)$ where $\tau(w) = (x, y, z)$. Let $C \subseteq \delta^+$ be closed unbounded and closed under f. By a straightforward induction check that for $\alpha \in C$ and w with $|U'_w| = \alpha$, we have $|w|_{\ll} \geq \gamma$, where α is the γth element of C.] Thus we have produced a $\check{\boldsymbol{\Gamma}}$ well-founded relation of length δ^+, a contradiction.

We have shown that there is no strictly increasing sequence of $\boldsymbol{\Gamma}$ sets of length $\delta(\boldsymbol{\Gamma})^+$. The argument for decreasing sequences is similar in each case (we use now for C the set of all (x, y, z) such that x codes some A_α, $y \in A_\alpha - A_{\alpha+1}$, and z codes $A_{\alpha+1}$). ⊣

By a *limit Suslin cardinal* we mean a Suslin cardinal which is the supremum of the smaller Suslin cardinals. A limit Suslin cardinal is necessarily a limit cardinal. By a *successor Suslin cardinal* we mean a Suslin cardinal κ which is the least Suslin cardinal greater than some Suslin cardinal λ. κ may or may not be a successor cardinal in this case. From Theorem 3.1 it follows that every Suslin cardinal is either a limit Suslin cardinal or a successor Suslin cardinal.

3.6 Lemma. *Let κ be a Suslin cardinal and assume $S(\kappa)$ is closed under $\forall^{\omega^\omega}$. Then κ is a regular limit of Suslin cardinals and* scale$(S(\kappa))$.

Proof. In this case $S(\kappa)$ is closed under real quantification and thus also countable unions and intersections. So $\boldsymbol{\Delta} = \boldsymbol{\Delta}(S(\kappa))$ is closed under real quantification, countable unions and intersections. Thus, $S(\kappa)$ is at the base of a type IV hierarchy. Let $\delta = o(\boldsymbol{\Delta})$. Since $S(\kappa)$ is closed under $\wedge$, $\vee$, an argument using the Coding Lemma shows that δ is regular. [If δ were singular, then the Coding Lemma would give an $S \in \boldsymbol{\Delta}$ consisting of pairs (x, y) coding Lipschitz continuous function f_x, f_y with $f_x^{-1}(A) = f_y^{-1}(B)$, where A is a $S(\kappa)$ complete set and B is $\check{S}(\kappa)$ complete. So, (x, y) codes a $\boldsymbol{\Delta}$ set of some Wadge rank $|(x, y)| < \delta$. Also, $\{|(x, y)| : (x, y) \in S\}$ will be cofinal in δ. Let

$$\begin{aligned} D &= \{(x, y, z) : (x, y) \in S \wedge f_x(z) \in A\} \\ &= \{(x, y, z) : (x, y) \in S \wedge f_y(z) \in B\}. \end{aligned}$$

But then $D \in \boldsymbol{\Delta}$ yet every set in $\boldsymbol{\Delta}$ is Wadge reducible to D, a contradiction.]

We cannot have $\delta > \kappa$ as then there would be a $\boldsymbol{\Delta}$ prewellordering of length κ, and by the Coding Lemma every subset of κ could be coded in $\boldsymbol{\Delta}$, and so $S(\kappa)$ would be contained in $\boldsymbol{\Delta}$. So, $\delta \leq \kappa$. On the other hand, from Lemma 3.4 there is a κ strictly increasing sequence of $S(\kappa)$ sets. Note that $\boldsymbol{\Delta}$ is not closed under wellordered unions, since if it were the standard tree computation would show that $S(\kappa)$ is contained in $\boldsymbol{\Delta}$. Thus (cf. the discussion before Definition 2.19) either pwo$(S(\kappa))$ or pwo$(\check{S}(\kappa))$. From Theorem 3.5

(applied to either $S(\kappa)$ or $\check{S}(\kappa)$, whichever has the prewellordering property) we have $\kappa < \delta^+$, so $\kappa \leq \delta$. Thus, $\kappa = \delta$. In particular, κ is regular.

Let λ be the supremum of the Suslin cardinals which are less than κ. We show that $\lambda = \kappa$. Suppose $\lambda < \kappa$. From Lemma 3.4 there is a κ strictly increasing sequence of sets in $S(\lambda)$. [Note: λ is actually a Suslin cardinal by Theorem 3.1, but we do not need this here. Note that $S(\lambda)$ is properly contained in $\boldsymbol{\Delta}$ from the regularity of δ, even if λ is not a Suslin cardinal.] Within the projective hierarchy over $S(\lambda)$ we may find a non-selfdual $\boldsymbol{\Gamma}_0$ closed under $\exists^{\omega^\omega}$ and $\mathrm{pwo}(\boldsymbol{\Gamma}_0)$, and so by Lemma 2.21, $\boldsymbol{\Gamma}_0$ is closed under wellordered unions. Then $S(\kappa) \subseteq \boldsymbol{\Gamma}_0$, a contradiction (as $\boldsymbol{\Gamma}_0 \subseteq \boldsymbol{\Delta}$). Thus, κ is a limit Suslin cardinal.

We cannot have $\mathrm{pwo}(\check{S}(\kappa))$ as then $\check{S}(\kappa)$ would be closed under wellordered unions and then $S(\kappa) \subseteq \check{S}(\kappa)$, a contradiction. So, $\mathrm{pwo}(S(\kappa))$. Thus, $S(\kappa)$ is closed under wellordered unions. Hence, $S(\kappa) = \bigcup_\kappa \boldsymbol{\Delta}$. To see now $\mathrm{scale}(S(\kappa))$, let $A \in S(\kappa) - \check{S}(\kappa)$ and let T be a tree on $\omega \times \kappa$ with $A = p[T]$. For $x \in A$, let $\phi_0(x)$ be the least $\alpha < \kappa$ such that $x \in p[T \restriction \alpha]$. For $i > 0$ let

$$\phi_i(x) = \langle \phi_0(x), \ell_0^{\phi_0(x)}, \ldots, \ell_i^{\phi_0(x)}(x) \rangle,$$

where $\ell_i^\beta(x)$ is the ith coordinate of the left-most branch of $(T\restriction\beta)_x$. Here $\langle \beta, \alpha_0, \ldots, \alpha_i \rangle$ denotes the rank of the tuple $(\beta, \alpha_0, \ldots, \alpha_i)$ in lexicographic ordering on those tuples satisfying $\beta \geq \max\{\alpha_0, \ldots, \alpha_i\}$. It is easy to check that $\{\phi_i\}$ is a scale on A with all norms into κ. Moreover, each of the norms ϕ_i is an $S(\kappa)$-norm as the norm relations $\leq_i^*$, $<_i^*$ are easily expressible as κ unions of $\boldsymbol{\Delta}$ sets. ⊣

We consider first Suslin cardinals of uncountable cofinality. First we consider the successor Suslin cardinals.

3.7 Lemma. *Let κ be a successor Suslin cardinal with* $\mathrm{cf}(\kappa) > \omega$. *Let λ be the largest Suslin cardinal less than κ. Then $\kappa = \lambda^+$ and* $\mathrm{cf}(\lambda) = \omega$. *Furthermore, $S(\kappa)$ has the scale property and $S(\kappa) = \exists^{\omega^\omega} \check{S}(\lambda)$. Also, κ is regular.*

Proof. Let $A \in S(\kappa) - \check{S}(\kappa)$, and let $\{\phi_i\}$ be a regular scale on A with norms into κ. From Lemma 3.4, there is a κ strictly increasing sequence of λ-Suslin sets. We cannot have $\mathrm{pwo}(S(\lambda))$ as then $S(\lambda)$ would be closed under wellordered unions by Lemma 2.21, and so A would be in $S(\lambda)$. So, $\mathrm{pwo}(\check{S}(\lambda))$. From Theorem 3.5 applied to $\check{S}(\lambda)$ it follows that $\kappa < \delta(\check{S}(\lambda))^+$, and thus $\kappa \leq \delta(\check{S}(\lambda))$. Every $\boldsymbol{\Delta}(\check{S}(\lambda))$ prewellordering is in $S(\lambda)$ and so by the Kunen-Martin Theorem has length $< \lambda^+$. So, $\kappa \leq \lambda^+$ and thus $\kappa = \lambda^+$.

Since $\kappa \leq \delta(\check{S}(\lambda))$, there is an $S(\lambda)$ well-founded relation of length λ. From the Coding Lemma it follows that $S(\lambda)$ is closed under λ unions. Suppose that $\mathrm{cf}(\lambda) > \omega$. Then every $S(\lambda)$ set is a λ union of $\boldsymbol{\Delta}(S(\lambda))$ sets. Thus $S(\lambda) = \bigcup_\lambda \boldsymbol{\Delta}(S(\lambda))$. By Martin's argument (cf. Lemma 2.15) this gives $\mathrm{pwo}(S(\lambda))$, a contradiction. So, $\mathrm{cf}(\lambda) = \omega$. Since $\mathrm{pwo}(\check{S}(\lambda))$ we also have

$\mathrm{pwo}(\exists^{\omega^\omega}\check{S}(\lambda))$ and so from Lemma 2.21, $\exists^{\omega^\omega}\check{S}(\lambda)$ is closed under wellordered unions. Note that $S(\lambda)$ is not closed under $\forall^{\omega^\omega}$ as otherwise by Lemma 3.6 λ would be regular. It follows that $S(\lambda) \subseteq \exists^{\omega^\omega}\check{S}(\lambda)$. Every $S(\kappa)$ set is a wellordered union of $S(\lambda)$ sets, and thus $S(\kappa) \subseteq \exists^{\omega^\omega}\check{S}(\lambda)$. To show the other inclusion it suffices to show that $\check{S}(\lambda) \subseteq S(\kappa)$, and this is immediate by Wadge's Lemma. So, $S(\kappa) = \exists^{\omega^\omega}\check{S}(\lambda)$.

Since $\mathrm{pwo}(\check{S}(\lambda))$ and $S(\kappa) = \exists^{\omega^\omega}\check{S}(\lambda)$ we have $\mathrm{pwo}(S(\kappa))$. Thus, $S(\kappa)$ is closed under wellordered unions by Lemma 2.21. It follows that $S(\kappa) = \bigcup_\kappa \mathbf{\Delta}(S(\kappa))$. Since $\mathrm{cf}(\kappa) > \omega$, the argument at the end of Lemma 3.6 shows that every $S(\kappa)$ set admits a scale all of whose norm relations can be written as κ unions of $\mathbf{\Delta}(S(\kappa))$ sets. Thus, $\mathrm{scale}(S(\kappa))$.

Finally, from $\kappa = \delta(S(\lambda))$ and the Kunen-Martin Theorem it follows that κ is the supremum of the lengths of the $S(\lambda)$ well-founded relations. From Lemma 2.16 it follows that κ is regular. ⊣

Next we consider the limit Suslin cardinals of uncountable cofinality.

3.8 Lemma. *Suppose that κ is a limit Suslin cardinal with $\mathrm{cf}(\kappa) > \omega$. Then $\mathrm{scale}(S(\kappa))$. Furthermore, if $\mathbf{\Delta} = \bigcup_{\lambda<\kappa} S(\lambda)$ and $\mathbf{\Gamma}$ is the corresponding Steel pointclass, then $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. Also, $\mathrm{scale}(\mathbf{\Gamma})$.*

Proof. Let $\mathbf{\Delta} = \bigcup_{\lambda<\kappa} S(\lambda)$. Thus $\mathbf{\Delta}$ is selfdual and closed under real quantification, countable unions and intersections (for countable unions and intersections we use $\mathrm{cf}(\kappa) > \omega$). Let $\delta = \delta(\mathbf{\Delta}) = o(\mathbf{\Delta})$. Let $\mathbf{\Gamma}$ be the corresponding Steel pointclass, that is, $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{\Gamma})$ and $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$. Similar to an earlier argument, we cannot have $\kappa < \delta$ as then the Coding Lemma would compute $S(\kappa) \subseteq \mathbf{\Delta}$. So, $\delta \leq \kappa$. Suppose $\delta < \kappa$. From Lemma 3.4 there is a strictly increasing sequence $\{A_\alpha\}_{\alpha<\kappa}$ of $\mathbf{\Delta}$ sets of length κ. For each $\beta < \delta$ let $S_\beta \subseteq \kappa$ consist of those α such that $o(A_\alpha) = \beta$. If all the S_β had size $< \delta$, then since $\kappa = \bigcup_{\beta<\delta} S_\beta$, we would have a map from $\delta \times \delta$ onto κ, a contradiction. Fix β so that $|S_\beta| \geq \delta$. Thus we have a δ-length strictly increasing sequence of sets $\{B_\gamma\}_{\gamma<\delta}$ of Wadge degree $\leq \beta$. Within $\mathbf{\Delta}$ we can find a non-selfdual pointclass $\mathbf{\Gamma}_0$ closed under $\wedge$, $\vee$ and closed under wellordered unions and properly containing the sets of Wadge degree $\leq \beta$ (from Lemma 2.21). The prewellordering associated to the B_γ (i.e., $x \prec y \longleftrightarrow \exists \eta_1 < \eta_2\ (x \in B_{\eta_1} \wedge y \in B_{\eta_2} \wedge y \notin B_{\eta_1})$) is a $\mathbf{\Gamma}_0$ prewellordering of length δ, a contradiction (recall δ is the supremum of the $\mathbf{\Delta}$ well-founded relations). So, $\kappa = \delta$.

Recall also $\mathrm{pwo}(\mathbf{\Gamma})$ (cf. [37]). Thus $\mathrm{pwo}(\exists^{\omega^\omega}\mathbf{\Gamma})$, and by Lemma 2.21 $\exists^{\omega^\omega}\mathbf{\Gamma}$ is closed under wellordered unions. Since $\mathrm{cf}(\kappa) > \omega$, every $S(\kappa)$ set is a union of $\mathbf{\Delta}$ sets, and so $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$. Since $S(\kappa)$ is closed under $\exists^{\omega^\omega}$, it follows from Wadge's Lemma that either $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$ or $S(\kappa) = \check{\mathbf{\Gamma}}$. We claim the first possibility holds. To see this, let $A \in S(\kappa) - \check{S}(\kappa)$, and let $A = p[T]$ with T a tree on $\omega \times \kappa$. From Theorem 3.2, T is weakly homogeneous. Thus, there is a homogeneous tree T' on $\omega \times \kappa$ such that $x \in A \longleftrightarrow \exists y\ \langle x, y\rangle \in p[T']$ (here $x, y \mapsto \langle x, y\rangle$ denotes our coding function). Let $\mathbf{\Gamma}'$ denote the pointclass of sets which are projections of homogeneous trees on $\omega \times \kappa$. Thus, $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}'$. From Lemma 3.3, $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$. If $\mathbf{\Gamma}$ is not closed under $\exists^{\omega^\omega}$, then the facts that $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$ and $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}'$ where $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$ imply that $\mathbf{\Gamma} = \mathbf{\Gamma}'$ and $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. If $\mathbf{\Gamma}$ is closed under real quantification, then $S(\kappa) \subseteq \exists^{\omega^\omega}\mathbf{\Gamma} = \mathbf{\Gamma}$ (and also $\mathbf{\Gamma} = \mathbf{\Gamma}'$) Thus in all cases $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$.

Since $\operatorname{pwo}(\mathbf{\Gamma})$ we also have $\operatorname{pwo}(S(\kappa))$, and so $S(\kappa)$ is closed under wellordered unions. Since $\operatorname{cf}(\kappa) > \omega$, the argument at the end of Lemma 3.6 shows that every $S(\kappa)$ sets admits a scale all of whose norm relations can be written as κ unions of $\mathbf{\Delta}$ sets, and thus are $S(\kappa)$ relations. Thus, $\operatorname{scale}(S(\kappa))$.

It remains to show $\operatorname{scale}(\mathbf{\Gamma})$. We use an argument similar to one of [35]. Let $A \in \mathbf{\Gamma}$. Let $U \subseteq (\omega^\omega)^2$ be universal $\mathbf{\Sigma}^1_1$. Define $B = \{y : U_y \subseteq A\}$. As $\check{\mathbf{\Gamma}}$ is closed under intersections with $\mathbf{\Sigma}^1_1$ sets (from [35, Theorem 2.1]), $B \in \mathbf{\Gamma}$. Let $B = p[T]$ where T is a tree on $\omega \times \kappa$. Let S be a tree on $(\omega)^3$ with $p[S] = \{(x, y) : U(y, x)\}$. Let V be the tree on $(\omega)^3 \times \kappa$ defined by:

$$(s, t, u, \vec{\alpha}) \in V \longleftrightarrow (s, t, u) \in S \wedge (t, \vec{\alpha}) \in T.$$

Clearly $A = p[V]$. We identify V with a tree V' on $\omega \times \kappa$ by ordering the triples $(a, b, \alpha) \in \omega \times \omega \times \kappa$ by reverse lexicographic order (i.e., order by α first). Let V'' be the slight modification of V' (using $\operatorname{cf}(\kappa) > \omega$) so that for $(s, \vec{\alpha}) \in V''$, $\alpha_0 \geq \max\{\alpha_1, \dots, \alpha_{\operatorname{lh}(s)}\}$. Let $\{\phi_i\}$ be the very good scale derived from V'', so each ϕ_i maps into κ. To show that $\leq^*_{\phi_i} \in \mathbf{\Gamma}$ (and similarly for $<^*_{\phi_i}$) it suffices to show that $\leq^*_{\phi_i}$ can be written as a $\mathbf{\Sigma}^1_1$-bounded κ-length union of $\mathbf{\Delta}$ sets (see the discussion after Definition 2.19). For $\alpha < \kappa$ let

$$C_\alpha = \{(x, y) : x \in A \wedge \phi_i(x) = \alpha \wedge \neg(y \in A \wedge \phi_i(y) < \alpha)\}.$$

Clearly $C_\alpha \in \mathbf{\Delta}$ (it is a Boolean combination of α-Suslin sets). Suppose $S \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$ and for all $(x, y) \in S$, $x \leq^*_{\phi_i} y$. In particular, $x \in A$. Let $S' = \{x : \exists y\ (x, y) \in S\}$. Let y be such that $U_y = S'$. Then $y \in B$. Fix $\vec{\alpha} \in \kappa^\omega$ so that $(y, \vec{\alpha}) \in p[T]$. Let $\alpha' \geq \max\{\vec{\alpha}_i\}$. Then $S' \subseteq p[V' \restriction (\omega)^3 \times \alpha']$. So for some $\alpha'' < \kappa$, $S' \subseteq \{x : \phi_i(x) \leq \alpha''\}$. It follows that $S \subseteq C_{\alpha''}$. Thus, $\{\phi_i\}$ is a $\mathbf{\Gamma}$-scale on A. $\dashv$

3.2. The Next Suslin Cardinal

The results presented so far suffice to give the theory of the Suslin cardinals up to the least κ so that $S(\kappa)$ is closed under real quantification, that is, at the base of a type IV hierarchy (we will give the details below). However, at the base of a type IV hierarchy a new method is needed since we cannot propagate the scale property upwards by periodicity. Martin [24] developed a method for analyzing the next Suslin cardinal which works not just in this case, but in general. Although we only need this construction in the case where we are at the base of a type IV hierarchy, we present Martin's method in general. The idea is to describe the pointclass where the next scale gets constructed as those sets which are "Wadge reducible to a measure on κ". We will make this statement precise later.

3.9 Definition (Martin). Let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ be a sequence of sets $A_\alpha \subseteq \omega^\omega$, for some $\kappa \in \mathrm{On}$. Then $\bar{\mathcal{A}}$ is the collection of $A \subseteq \omega^\omega$ such that for all countable $S \subseteq \omega^\omega$ there is an $\alpha < \kappa$ such that $S \cap A = S \cap A_\alpha$.

For $\boldsymbol{\Gamma}$ a pointclass and $\kappa \in \mathrm{On}$ we let

$$\Lambda(\boldsymbol{\Gamma}, \kappa) = \bigcup\{\bar{\mathcal{A}} : \mathcal{A} \subseteq \boldsymbol{\Gamma} \wedge \|\mathcal{A}\| \leq \kappa\},$$

where $\|\mathcal{A}\|$ denotes the cardinality of $\mathcal{A}$.

Note that $\bar{\mathcal{A}}$ is the closure of $\{A_\alpha : \alpha < \kappa\}$ under the topology on $\mathcal{P}(\omega^\omega)$ with basic open sets of the form $N_f = \{A : \forall x \in \mathrm{dom}(f)\ (x \in A \longleftrightarrow f(x) = 1)\}$, where $f : S \to \{0,1\}$ and $S \subseteq \omega^\omega$ is countable.

Following [24] we next prove a few basic facts about Λ.

3.10 Lemma. *Let $\boldsymbol{\Gamma}$ be non-selfdual and closed under $\forall^{\omega^\omega}$ and assume* $\mathrm{pwo}(\boldsymbol{\Gamma})$. *If $\boldsymbol{\Delta}$ is not closed under real quantification, then assume also that $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ has the scale property with norms into $\kappa \doteq \delta(\boldsymbol{\Gamma})$. Then there is an $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ such that for every $A \in \Lambda(\boldsymbol{\Gamma}, \kappa)$ there is a $B \in \bar{\mathcal{A}}$ with $A \leq_w B$.*

Proof. First consider the case where $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$ (so $\boldsymbol{\Gamma}$ is at the base of a type IV hierarchy), and so also countable unions and intersections. First we show in this case that for every $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Gamma}$ there is an $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ such that $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. Let W be a $\boldsymbol{\Gamma}$ complete set and ϕ a $\boldsymbol{\Gamma}$ norm on W. Let $C = \{(x,y) : x \in W \wedge y \in A_{\phi(x)}\}$. By the Coding Lemma $C \in \boldsymbol{\Gamma}$. Let $C = \bigcup_{\beta<\kappa} C_\beta$ where $C_\beta \in \boldsymbol{\Delta}$. For $\alpha, \beta < \kappa$ let $A'_{\alpha,\beta} = \{y : \exists x\ (x \in W \wedge \phi(x) = \alpha \wedge (x,y) \in C_\beta)\}$. Then $A'_{\alpha,\beta} \in \boldsymbol{\Delta}$ and easily $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. Next we show that there is a "universal" $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha \in \boldsymbol{\Gamma}$. That is, for all $A' \in \Lambda(\boldsymbol{\Gamma}, \kappa)$, $A' \leq_w A$ for some $A \in \bar{\mathcal{A}}$. Let $U_1 \subseteq (\omega^\omega)^2$, $U_2 \subseteq (\omega^\omega)^3$ in $\boldsymbol{\Gamma}$ be universal for $\boldsymbol{\Gamma}\restriction\omega^\omega$, $\boldsymbol{\Gamma}\restriction(\omega^\omega)^2$ respectively. For $\alpha < \kappa$ let $A_\alpha \subseteq \omega^\omega \times \omega^\omega$ be defined by

$$(x,y) \in A_\alpha \longleftrightarrow \exists z, u\ [z \in W \wedge \phi(z) = \alpha \wedge U_2(x,z,u) \wedge U_1(u,y)].$$

Each A_α lies in $\boldsymbol{\Gamma}$. Given $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with each $A'_\alpha \in \boldsymbol{\Gamma}$, by the Coding Lemma there is an x_0 such that for all $\alpha < \kappa$, $A'_\alpha = (A_\alpha)_{x_0}$. Suppose $A' \in \bar{\mathcal{A}}'$. For each Turing degree d, let $\beta(d)$ be the least $\beta < \kappa$ so that A' and A'_β agree on all $x \in d$ (i.e., $A' \cap d = A'_\beta \cap d$). Define $A \subseteq \omega^\omega \times \omega^\omega$ by:

$$(x,y) \in A \longleftrightarrow \forall^*_\nu\, d\ [(x,y) \in A_{\beta(d)}].$$

Clearly $A' = A_{x_0}$. Also, we easily have $A \in \bar{\mathcal{A}}$. [If $S \subseteq \omega^\omega \times \omega^\omega$ is countable, then for every $(x,y) \in S$ let $d_{x,y}$ be a large enough degree so that for all $d \geq_T d_{x,y}$ we have $(x,y) \in A_{\beta(d)}$ iff $(x,y) \in A$. Then for any d above all the $d_{x,y}$ for $(x,y)\in S$ we have that A and $A_{\beta(d)}$ agree on all $(x,y)\in S$.] So $A'=A_{x_0}\leq_w A\in\bar{\mathcal{A}}$.

Suppose now that we are in the case where $\boldsymbol{\Delta}$ is closed under real quantification, but $\boldsymbol{\Gamma}$ is not closed under $\exists^{\omega^\omega}$ (so $\boldsymbol{\Gamma}$ is at the base of a type II or III hierarchy). Again in this case $\kappa=o(\boldsymbol{\Delta})$. Suppose $\mathcal{A}=\{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$. From Lemma 2.22 there is an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ set W and an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ prewellordering of W of length κ with corresponding norm ϕ say, such that for each $\alpha<\kappa$, $W_\alpha=\{x\in W:\phi(x)=\alpha\}\in\boldsymbol{\Delta}$. Let C be defined as in the first case. Then $C\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$. We can write $C=\bigcup_{\beta<\rho}C_\beta$ where each $C_\beta\in\boldsymbol{\Delta}$ and $\rho\leq\kappa$ (in fact $\rho=\mathrm{cf}(\kappa)$ from Lemma 2.22). For $\alpha<\kappa$, $\beta<\rho$ let as in the first case $A'_{\alpha,\beta}=\{y:\exists x\ (x\in W\wedge\phi(x)=\alpha\wedge(x,y)\in C_\beta)\}$. Then $A'_{\alpha,\beta}\in\boldsymbol{\Delta}$ and $\bar{\mathcal{A}}\subseteq\bar{\mathcal{A}}'$. It suffices then to construct a universal $\mathcal{A}=\{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$. This is done as in the first case, using the W, ϕ just mentioned, universal sets U_1, U_2 for $\exists^{\omega^\omega}\boldsymbol{\Gamma}$, and the Coding Lemma with respect to the pointclass $\exists^{\omega^\omega}\boldsymbol{\Gamma}$.

Suppose finally that $\boldsymbol{\Delta}$ is not closed under real quantification. Inspecting the hierarchy analysis shows that $\boldsymbol{\Gamma}$ is not at the base of a projective hierarchy. [It cannot be at the base of a type I hierarchy because in that case the prewellordering property falls on the side closed under $\exists^{\omega^\omega}$, and it cannot be at the base of a type II, III, or IV hierarchy since then $\boldsymbol{\Delta}$ is closed under quantifiers.] It follows that $\boldsymbol{\Gamma}$ is closed under countable unions and intersections. From Lemma 2.13, a $\boldsymbol{\Gamma}$-norm on a $\boldsymbol{\Gamma}$-complete set has length κ, and every $\check{\boldsymbol{\Gamma}}$ well-founded relation has length less than κ. So, κ is the supremum of the lengths of the $\check{\boldsymbol{\Gamma}}$ well-founded relations, and from Lemma 2.16 it follows that κ is regular. We are assuming in this case that $\exists^{\omega^\omega}\boldsymbol{\Gamma}\subseteq S(\kappa)$. The reverse inclusion follows by the Coding Lemma, so $S(\kappa)=\exists^{\omega^\omega}\boldsymbol{\Gamma}$. The same argument as in the previous case also shows here that there is a single $\mathcal{A}=\{A_\alpha\}_{\alpha<\kappa}$ with all $A_\alpha\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$ which is universal for all such sequences. It remains to show that for such a $\mathcal{A}$ we can find a $\mathcal{A}'=\{A'_\alpha\}_{\alpha<\kappa}$ with $A'_\alpha\in\boldsymbol{\Delta}$ such that $\bar{\mathcal{A}}\subseteq\bar{\mathcal{A}}'$. Let $\{\phi_n\}$ be a regular $\exists^{\omega^\omega}\boldsymbol{\Gamma}$-scale on a $\exists^{\omega^\omega}\boldsymbol{\Gamma}$-universal set U. Let $T\subseteq(\omega\times\kappa)^{<\omega}$ be the tree of the scale $\{\phi_n\}$. From the Coding Lemma there is an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ relation $R\subseteq\omega^\omega\times\omega^\omega$ with $\mathrm{dom}(R)=W$ (a $\boldsymbol{\Gamma}$-complete set with a $\boldsymbol{\Gamma}$-norm ϕ onto κ) and such that $R(x,y)$ implies $U_y=A_{\phi(x)}$. For $\alpha<\kappa$ let $\beta(\alpha)$ be the least reliable ordinal $>\alpha$ with respect to $\{\phi_n\}$ ($\beta(\alpha)<\kappa$ since κ is regular). As in the Becker-Kechris Theorem (see [3]), there is an ordinal game $G_\alpha(z)$ (defined uniformly from α and $z\in\omega^\omega$) in which I plays ordinals $<\beta(\alpha)$, II plays ordinals $<\kappa$, and the game is closed for II satisfying: for all $z\in\omega^\omega$, $z\in A_\alpha$ iff II has a winning strategy in $G_\alpha(z)$. For $\gamma<\kappa$, let $G_{\alpha,\gamma}(z)$ be the game played as $G_\alpha(z)$ except now II's ordinal moves are restricted to be $<\gamma$. Let $A_{\alpha,\gamma}$ be the set of z such that II has a winning strategy in $G_{\alpha,\gamma}(z)$. Thus, $A_\alpha=\bigcup_\gamma A_{\alpha,\gamma}$. Finally, for $\delta<\kappa$ let $A_{\alpha,\gamma,\delta}$ be the set of z such that I does not win the open game $G_{\alpha,\gamma}(z)$ with ordinal $<\delta$ (that is, it is not the case that the empty node has rank $<\delta$ in the rank analysis of the open game). Clearly for any countable $S\subseteq\omega^\omega$ and any α, there are γ, $\delta<\kappa$ such that $A_\alpha\cap S=A_{\alpha,\gamma,\delta}\cap S$. Also, each $A_{\alpha,\gamma,\delta}\in\boldsymbol{\Delta}$ since $\boldsymbol{\Delta}$ is closed under $<\kappa$ unions and intersections by Martin's argument (cf. the proof of Theorem 2.15). We use here the fact that the games $G_\alpha(z)$ are "uniformly" closed for II, that is, the set of all $(\vec{\eta}, z)$ such that $\vec{\eta}$ is a winning run for II in $G_\alpha(z)$ is closed in $\kappa^\omega \times \omega^\omega$. If we let the A'_α enumerate the $A_{\alpha,\gamma,\delta}$ then $\bar{\mathcal{A}} \subseteq \bar{\mathcal{A}}'$. ⊣

Note that if $\boldsymbol{\Delta}$ is any selfdual class closed under $\wedge$, and κ is any cardinal, then $\Lambda(\boldsymbol{\Delta}, \kappa)$ is closed under $\wedge$, $\vee$, and $\neg$. For example, to see closure under $\wedge$, given $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, and $\mathcal{B} = \{B_\alpha\}_{\alpha<\kappa}$ and $A \in \bar{\mathcal{A}}$, $B \in \bar{\mathcal{B}}$, consider the sequence $\mathcal{C} = \{A_\alpha \cap B_\beta : \alpha, \beta < \kappa\}$. Easily $A \cap B \in \bar{\mathcal{C}}$. In particular, for any Levy class $\boldsymbol{\Gamma}$ and any κ, $\Lambda(\boldsymbol{\Delta}(\boldsymbol{\Gamma}), \kappa)$ is closed under $\wedge$, $\vee$, $\neg$. Thus as an immediate corollary to Lemma 3.10 we have the following.

3.11 Corollary. *Under the hypotheses of Lemma 3.10,* $\Lambda(\boldsymbol{\Gamma}, \kappa)$ *is closed under* $\wedge$, $\vee$, *and* $\neg$.

The next lemma shows that if $\boldsymbol{\Gamma}$ is closed under real quantification, then so will be Λ.

3.12 Lemma. *Let* $\boldsymbol{\Gamma}$ *be non-selfdual and closed under* $\forall^{\omega^\omega}$, $\exists^{\omega^\omega}$, *and assume* pwo$(\boldsymbol{\Gamma})$. *Let* $\kappa = \delta(\boldsymbol{\Delta}) = o(\boldsymbol{\Delta})$. *Then* $\Lambda(\boldsymbol{\Gamma}, \kappa)$ *is closed under* $\forall^{\omega^\omega}$, $\exists^{\omega^\omega}$.

Proof. Let $A \subseteq \omega^\omega \times \omega^\omega$ be in $\Lambda = \Lambda(\boldsymbol{\Gamma}, \kappa)$. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ and with $A \in \bar{\mathcal{A}}$ (here each $A_\alpha \subseteq \omega^\omega \times \omega^\omega$). Define $x \in B \longleftrightarrow \exists y\ (x, y) \in A$. View every real z as coding a countable $s_z \subseteq \omega^\omega$ and a $f_z : s_z \to \{0, 1\}$. For d a Turing degree, let $\alpha_z(d)$ be the least ordinal $<\kappa$, if one exists, such that $\forall x \in s_z\ (f_z(x) = 1 \longleftrightarrow \exists y \leq d\ (x, y) \in A_\alpha)$. Let $C = \{z : \forall^*_\nu d\ \alpha_z(d) \text{ is defined}\}$. A straightforward computation using the closure of $\boldsymbol{\Gamma}$ shows that $C \in \boldsymbol{\Gamma}$. Let ϕ be the norm on C corresponding to the prewellordering $z_1 \preceq z_2 \longleftrightarrow \forall^*_\nu d\ (\alpha_{z_1}(d) \leq \alpha_{z_2}(d))$. A straightforward computation shows that the norm relation $\leq^*_\phi$, $<^*_\phi$ are in $\boldsymbol{\Gamma}$, so ϕ is a $\boldsymbol{\Gamma}$-norm on C. For example,

$$\begin{aligned} z_1 \leq^*_\phi z_2 \longleftrightarrow \forall^*_\nu d\ \exists \alpha < \kappa\ \forall \beta < \alpha & \\ & \big[\forall x \in s_{z_1} (f_{z_1}(x) = 1 \longleftrightarrow \exists y \leq d\ (x, y) \in A_\alpha) \\ & \wedge \neg \forall x \in s_{z_2} (f_{z_2}(x) = 1 \longleftrightarrow \exists y \leq d\ (x, y) \in A_\beta)\big]. \end{aligned}$$

This is in $\boldsymbol{\Gamma}$ using the closure of $\boldsymbol{\Delta}$ under $<\kappa$ intersections. Thus ϕ has length κ.

For $z \in C$, define $B_{\phi(z)}$ by:

$$x \in B_{\phi(z)} \longleftrightarrow \forall^*_\nu d\ \exists y \leq d\ (x, y) \in A_{\alpha_z(d)}.$$

A straightforward computation as above shows $B_{\phi(z)} \in \boldsymbol{\Delta}$. It suffices to show that $B \in \bar{\mathcal{B}}$ where $\mathcal{B} = \{B_{\phi(z)} : z \in C\}$. Let $s \subseteq \omega^\omega$ be countable. Define $f : s \to \{0, 1\}$ by $f(x) = 1$ iff $x \in B$. Let z code s, f. Since $A \in \bar{\mathcal{A}}$ it follows that $z \in C$, so $\phi(z)$ is defined. Let d_0 be a large enough degree so that for all $x \in s$ we have $x \in B$ iff $\exists y \leq d_0\ (x, y) \in A$. For any $d \geq d_0$ and $x \in s$ we then have:

$$\begin{aligned} x \in B &\longleftrightarrow \exists y \leq d\ (x, y) \in A \\ &\longleftrightarrow \exists y \leq d\ (x, y) \in A_{\alpha_z(d)}, \end{aligned}$$

which shows that for $x \in s$, $x \in B$ iff $x \in B_{\phi(z)}$. $\dashv$

We next make precise the statement that $\Lambda(\mathbf{\Gamma}, \kappa)$ is the pointclass of sets "Wadge reducible to a measure on κ". Let $\mathbf{\Gamma}$, κ be as in Lemma 3.10. From the Coding Lemma, every subset of κ may be coded within the pointclass $\exists^{\omega^\omega}\mathbf{\Gamma}$. To make this precise, let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\exists^{\omega^\omega}\mathbf{\Gamma}$. Let ψ be an $\exists^{\omega^\omega}\mathbf{\Gamma}$-norm on a $\exists^{\omega^\omega}\mathbf{\Gamma}$ set W of length κ (the existence of ψ follows from Lemmas 2.21 and 2.22). For $z \in \omega^\omega$, define $B_z \subseteq \kappa$ by $\alpha \in B_z$ iff $\exists x \in W\ (\psi(x) = \alpha \wedge U(z, x))$. By the Coding Lemma, every subset of κ is of the form B_z for some z. We now define the code set C_μ of a measure μ on κ.

3.13 Definition. Let $\mathbf{\Gamma}$, κ be as in Lemma 3.10. For μ a measure on κ define $C_\mu = \{z : \mu(B_z) = 1\}$.

3.14 Lemma. *Let $\mathbf{\Gamma}$, κ be as in Lemma 3.10. Then $A \in \Lambda(\mathbf{\Gamma}, \kappa)$ iff there is a measure μ on κ such that $A \leq_w C_\mu$.*

Proof. First suppose μ is a measure on κ and we show $C_\mu \in \Lambda(\mathbf{\Gamma}, \kappa)$. For $\alpha < \kappa$ define $A_\alpha = \{z : \alpha \in B_z\}$. Clearly $A_\alpha \in \exists^{\omega^\omega}\mathbf{\Gamma}$. It suffices to observe that $C_\mu \in \bar{\mathcal{A}}$, where $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ (using the fact that $\Lambda(\exists^{\omega^\omega}\mathbf{\Gamma}, \kappa) = \Lambda(\mathbf{\Gamma}, \kappa)$ from the proof of Lemma 3.10). Let $s \subseteq \omega^\omega$ be countable. Let $\alpha(s)$ be the least element of $\bigcap_{z\in s} B'_z$, where $B'_z = B_z$ if $z \in C_\mu$ and otherwise $B'_z = \kappa - B_z$. Then for all $z \in s$ we have $z \in C_\mu$ iff $z \in A_{\alpha(s)}$. So, $C_\mu \in \Lambda(\mathbf{\Gamma}, \kappa)$.

Suppose next that $A \in \Lambda(\mathbf{\Gamma}, \kappa)$. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with $A \in \bar{\mathcal{A}}$. For d a Turing degree let $f(d)$ be the least ordinal less than κ such that for all $z \in d$, $z \in A$ iff $z \in A_{f(d)}$. Define $\mu = f(\nu)$ where again ν is the Martin measure on the Turing degrees. Consider the relation $R(y, x) \longleftrightarrow (x \in W \wedge y \in A_{\psi(x)})$, where W, ψ are defined just before Definition 3.13. From the Coding Lemma, $R \in \exists^{\omega^\omega}\mathbf{\Gamma}$. From the *s-m-n* Theorem there is a continuous function $h : \omega^\omega \to \omega^\omega$ such that $U_{h(y)} = \{x \in W : y \in A_{\psi(x)}\}$, and thus $B_{h(y)} = \{\alpha : y \in A_\alpha\}$. We then have that for all $y \in \omega^\omega$, $y \in A$ iff $\forall^*_\nu d\ (y \in A_{f(d)})$ iff $\forall^*_\mu \alpha\ (y \in A_\alpha)$ iff $h(y) \in C_\mu$. So, $A \leq_w C_\mu$. $\dashv$

We next show that the pointclass $\Lambda(\mathbf{\Gamma}, \kappa)$ is where the next semi-scale is constructed. We first show the upper bound.

3.15 Lemma. *Let $\mathbf{\Gamma}$ be non-selfdual and closed under $\forall^{\omega^\omega}$, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Assume also $\exists^{\omega^\omega}\mathbf{\Gamma}$ has the scale property with norms into $\kappa \doteq \delta(\mathbf{\Gamma})$. Assume also that there is a Suslin cardinal greater than κ. Then every set in $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ admits a semi-scale with each norm a $\Lambda(\mathbf{\Gamma}, \kappa)$-norm.*

Proof. In all cases we have $\mathrm{cf}(\kappa) > \omega$. [$\mathbf{\Gamma}$ cannot be at the base of a type I hierarchy as then prewellordering falls on the side closed under $\exists^{\omega^\omega}$. If $\mathbf{\Gamma}$ is at the base of a type II, III, or IV hierarchy, then $\mathrm{cf}(\kappa) = \mathrm{cf}(o(\mathbf{\Delta})) > \omega$. If $\mathbf{\Gamma}$ is not at the base of a hierarchy, then $\mathbf{\Gamma}$ is closed under countable unions and intersections, and it follows that $\delta(\mathbf{\Gamma})$ has uncountable cofinality.] If $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$ then there is a tree T on $\omega \times \kappa$ with $A = p[T]$ by hypothesis. Since $\mathrm{cf}(\kappa) > \omega$ we may assume T has the property that if $(s, \vec{\alpha}) \in T$ with $\vec{\alpha} = (\alpha_0, \dots, \alpha_{n-1})$, then $\alpha_0 > \max\{\alpha_1, \dots, \alpha_{n-1}\}$. From Theorem 3.2, T is weakly homogeneous. The homogeneous tree construction then produces a tree T' with $p[T'] = \omega^\omega - A \doteq B$. Let $\{\phi_n\}$ be the norms on B this construction gives. For $x \in B$, $\phi_n(x)$ is of the form $[f_x]_\mu$ where f_x is the rank function on T_x and μ is a measure on κ^j for some j. Our property of T gives that $f_x(\vec{\alpha}) < \kappa$ for every $\vec{\alpha} \in T_x$. The usual homogeneous tree argument shows that $\{\phi_n\}$ is a semi-scale on B [if $\{x_m\} \to x$ and for each n the norms $\phi_n(x_m)$ converge, then we get a sequence of measure one sets (with respect to the homogeneity measures μ_s for T) A_s, for $s \in \omega^{<\omega}$, and an order-preserving map from $T_x \restriction \{A_s\}$ to the ordinals. From the weak homogeneity of T this gives that T_x is well-founded.] It suffices to show that the norm relations $\leq^*_{\phi_n}$, $<^*_{\phi_n}$ are in $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$. We consider the case $\phi = \phi_0$ which is of the form $\phi(x) = [\alpha \mapsto f_x(\alpha)]_\mu$ (that is, $j = 1$). The general case is similar. For α, $\beta < \kappa$, let

$$A_{\alpha,\beta} = \{(x, y) : \exists \gamma \leq \beta\ (|\alpha|_{T_x} \leq \gamma \wedge \neg(|\alpha|_{T_y} \leq \gamma))\},$$

where $|\alpha|_{T_x} \leq \gamma$ means α is in the well-founded part of T_x and has rank $\leq \gamma$. We claim that each $A_{\alpha,\beta}$ is in $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{\Gamma})$. If $\mathbf{\Gamma}$ is closed under real quantification or $\mathbf{\Delta}$ is not closed under real quantification, then this follows from the fact that $\mathbf{\Delta}$ is closed under $<\kappa$ unions and intersections. The remaining case is when $\mathbf{\Gamma}$ is at the base of a type II or III hierarchy. Since $\mathbf{\Delta}$ is closed under real quantification, we may apply the Coding Lemma to a suitable non-selfdual pointclass $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ closed under $\exists^{\omega^\omega}$ to code functions $h : (\alpha + 1)^{<\omega} \to \beta$. From this and the closure of $\mathbf{\Delta}$ under quantifiers we easily compute $A_{\alpha,\beta} \in \mathbf{\Delta}$.

Suppose $s = \{(x_n, y_n) : n \in \omega\}$ is a countable set of pairs. For each n such that $x_n \leq^*_\phi y$, let B_n be a μ measure one set such that for all $\alpha \in B_n$, $f_x(\alpha) \leq f_y(\alpha)$ (we allow here the possibility that $\vec{\alpha}$ is in the illfounded part of f_y). For other n, let B_n be such that for all $\vec{\alpha} \in B_n$ we have either α is in the illfounded part of T_x or else $f_y(\alpha) < f_x(\alpha)$. Fix $\alpha \in \bigcap_n B_n$. Let β be large enough so that for all n, if $f_{x_n}(\alpha)$ is defined, then $f_{x_n}(\alpha) < \beta$. Then for all n we have $x_n \leq^*_\phi y_n$ iff $(x_n, y_n) \in A_{\alpha,\beta}$. This shows $\leq^*_\phi \in \bar{\mathcal{A}}$ where $\mathcal{A} = \{A_{\alpha,\beta}\}$. A similar argument works for $<^*_\phi$. ⊣

3.16 Remark. The $\{\phi_n\}$ produced in the above argument is only a semi-scale, not necessarily a scale. We could get a scale by using the left-most branch of the tree T', but then we seem to loose the definability estimate $\leq^*_{\phi_n} \in \Lambda$. This was an oversight in the original arguments, and was pointed out by Grigor Sargsyan. We also thank John Steel for some helpful correspondences.

We next get the lower bound.

3.17 Lemma. *Let* $\mathbf{\Gamma}$, κ *be as in Lemma 3.15. Let* A *be* $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$*-complete. Then* A *does not admit a scale all of whose norm relations are Wadge reducible to some* $B \in \Lambda(\mathbf{\Gamma}, \kappa)$*.*

Proof. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \mathbf{\Delta}$ and which is universal for $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$ (i.e., every $A \in \Lambda$ is Wadge reducible to some $B \in \bar{\mathcal{A}}$). This is possible from Lemma 3.10. View every real r as coding a continuous function $r : (\omega)^2 \times (\omega^\omega)^3 \to \omega^\omega$, and every real z as coding a Lipschitz continuous function $w \mapsto z(w)$. Define $D \subseteq \omega^\omega \times \omega^\omega$ by:

$$\begin{aligned} D(x,y) \longleftrightarrow & \exists r \leq_T x \; \forall^*_\nu d \; \exists \alpha < \kappa \; \forall n, m \in \omega \\ & (y(n) = m \longleftrightarrow \exists z \leq_T d \; \forall w \leq_T d \; r(n,m,x,z(w),w) \in A_\alpha). \end{aligned}$$

From the closure of $\exists^{\omega^\omega}\mathbf{\Gamma}$ under wellordered unions it follows easily that $D \in \forall^{\omega^\omega}\exists^{\omega^\omega}\mathbf{\Gamma}$. Also, each section D_x of D is countable since it can be wellordered (for fixed $r \leq_T x$, the wellordering is induced by the map $y \to [f]_\nu$ where ν is the Martin measure on the degrees and $f(d)$ is the least $\alpha < \kappa$ satisfying the definition above). Consider $D^c \in \exists^{\omega^\omega}\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$. Since we are assuming every $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ set has a scale all of whose norms are reducible to some $B \in \Lambda$, a standard argument shows that D^c has a uniformizing function $f : \omega^\omega \to \omega^\omega$ such that the relation $R(n,m,x) \longleftrightarrow f(x)(n) = m$ is in $\Game\Lambda$. To see this, write $\neg D(x,y) \longleftrightarrow \exists u \; E(x,y,u)$ with $E \in \forall^{\omega^\omega}\check{\mathbf{\Gamma}}$. Let $\{\phi_n\}$ be an excellent scale on E with all norm relations $\leq^*_{\phi_n}$ Wadge reducible to $B \in \Lambda$ (in particular $\phi_n(x,y,u)$ determines $x\restriction(n+1)$, $y\restriction(n+1)$, $z\restriction(n+1)$). This is possible since Λ is closed under $\wedge$, $\vee$. Let f be the canonical uniformizing function for D^c from this scale (that is, for some u, $(x, f(x), u)$ has minimal ϕ_n norm for all n). Then $f(x)(n) = m$ iff I wins the game $G^{n,m}_x$ where I and II play out $z = \langle y, u\rangle$, $w = \langle y', u'\rangle$ respectively and I wins the run iff $y(n) = m$ and $(x,y,u) \leq^*_{\phi_n} (x,y',u')$. Fix now a $B \in \Lambda$ and a real r coding a continuous function so that for all n, m, x, z, w,

$$\begin{aligned} f(x)(n) = m & \longleftrightarrow \text{I wins the game } G^{n,m}_x \\ & \longleftrightarrow \Game\langle z, w\rangle \; r(n,m,x,z,w) \in B \\ & \longleftrightarrow \forall^*_\nu d \; \exists z \leq_T d \; \forall w \leq_T d \; r(n,m,x,z(w),w) \in B. \end{aligned}$$

Fix now $x \geq_T r$ and let $y = f(x)$, so $\neg D(x,y)$. On the other hand, for d a Turing degree let $\alpha(d) < \kappa$ be least such that B and $A_{\alpha(d)}$ agree on reals in d. Then for any n, m we have

$$y(n) = m \;\longleftrightarrow\; \forall^* d \; \exists z \leq_T d \; \forall w \leq_T d \; r(n,m,x,z(w),w) \in A_{\alpha(d)}.$$

Intersecting countably many cones gives that

$$\forall^*_\nu d \; \forall n, m \; (y(n) = m \longleftrightarrow \exists z \leq_T d \; \forall w \leq_T d \; r(n,m,x,z(w),w) \in A_{\alpha(d)}).$$

Since $r \leq_T x$, this shows that $D(x,y)$, a contradiction. $\dashv$

Lemmas 3.15 and 3.17 show that $\Lambda(\mathbf{\Gamma}, \kappa)$ is precisely where the norms relations for the next semi-scale are constructed. We record this in the following theorem.

3.18 Theorem (Martin). *Let $\mathbf{\Gamma}$ be non-selfdual and closed under $\forall^{\omega^\omega}$, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Assume also $\exists^{\omega^\omega}\mathbf{\Gamma}$ has the scale property with norms into $\kappa \doteq \delta(\mathbf{\Gamma})$. Assume also that there is a Suslin cardinal greater than κ. Then every $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ set admits a semi-scale all of whose norm relations lie in the pointclass $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$. Furthermore, there is no scale on a $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$ complete set all of whose norm relations are Wadge reducible to some $B \in \Lambda$.*

In the case where $\mathbf{\Gamma}$ is closed under quantifiers, and hence so is Λ, the last sentence of the previous theorem is equivalent to saying that there is no semi-scale on a $\check{\mathbf{\Gamma}}$ complete set all of whose norm relations are Wadge reducible to some $B \in \Lambda$ (since a semi-scale can be converted to a scale within the next projective hierarchy). It follows that $\mathrm{cf}(o(\Lambda)) = \omega$. In fact, we can say a bit more in this case.

3.19 Lemma. *Let $\mathbf{\Gamma}$ be non-selfdual, closed under real quantification, and* $\mathrm{scale}(\mathbf{\Gamma})$. *Assume there is a Suslin cardinal greater than $\kappa = o(\mathbf{\Gamma})$. Then $\lambda \doteq o(\Lambda)$ (where $\Lambda = \Lambda(\mathbf{\Gamma}, \kappa)$) is the least Suslin cardinal greater than κ. Furthermore, every set in Λ is λ-Suslin.*

Proof. Recall in this case that Λ is a selfdual class closed under real quantification. If $A \in \check{\mathbf{\Gamma}}$ and $\{\phi_n\}$ is the semi-scale on A as in Lemma 3.15 (i.e., from the homogeneous tree construction) then each norm relation $\leq_n^*$ is in Λ and so has length less than $\delta(\Lambda) = o(\Lambda)$. Thus, A is λ-Suslin. On the other hand, if A were λ'-Suslin where $\lambda' < \lambda$, then from the Coding Lemma and the closure of Λ under real quantifiers a straightforward computation shows that for some $B \in \Lambda$, the scale corresponding to the λ' tree has norm relations $\leq_n^*$ Wadge reducible to B for all n. This contradicts Lemma 3.17. [Since $\delta(\mathbf{\Delta}) = o(\mathbf{\Delta})$ we can find a non-selfdual $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$, with $\mathrm{pwo}(\mathbf{\Gamma}_0)$ and such that there is a $\mathbf{\Gamma}_0$ prewellordering of length λ'. Then each $\leq_n^*$ is in $\Sigma_2(\mathbf{\Gamma}_0)$.] Thus, λ is the next Suslin cardinal after κ.

Suppose now that $B \in \Lambda$. Let $A \in \check{\mathbf{\Gamma}} - \mathbf{\Gamma}$, and let T be the tree of the semi-scale $\{\phi_n\}$ from Lemma 3.15. Let $\{\psi_n\}$ be the scale on A corresponding to T (we do not know that the norm relations $\leq_{\psi_n}^*$ are in Λ). Let $\mathbf{\Gamma}_0 \subseteq \Lambda$ be a non-selfdual pointclass containing B and $\omega^\omega - B$. Let $\mathbf{\Gamma}_1 \subseteq \Lambda$ be non-selfdual, closed under $\exists^{\omega^\omega}$, $\mathrm{pwo}(\mathbf{\Gamma}_1)$, and $\mathbf{\Gamma}_1$ properly contains $\mathbf{\Gamma}_0$. Let $\lambda_1 < \lambda$ be greater than the supremum of the lengths of the $\mathbf{\Gamma}_1$ prewellorders. Fix n so that ψ_n has length $> \lambda_1$ (by the previous paragraph). For α less than the length of ψ_n, let $A_\alpha = \{x \in A : \psi_n(x) \leq \alpha\}$. If all the A_α are in $\mathbf{\Gamma}_0$, then from Lemma 2.21 we would have a $\mathbf{\Gamma}_1$ prewellordering of length $\geq \lambda_1$, a contradiction. Fix α so that $A_\alpha \notin \mathbf{\Gamma}_0$. Then $B \leq_w A_\alpha$. The scale $\{\psi_n\}$ restricted to A_α is a scale on A_α with all the norm mapping into λ. So, A_α is λ-Suslin. Since $B \leq_w A_\alpha$, B also admits such a λ-scale. $\dashv$

3.3. More on Λ in the Type IV Case

In the case where $\boldsymbol{\Gamma}$ is closed under real quantification (the type IV case) we can say more about Λ. We will need these extra facts for the scale analysis to follow. However, these results are also of independent interest. These results were obtained after the oversight mentioned in Remark 3.16, and allow us to recover the full scale analysis except for the scaled pointclasses immediately after a type IV pointclass $\boldsymbol{\Gamma}$ (we state these results explicitly in the next section). We again thank Steel for some helpful correspondences.

Throughout this section $\boldsymbol{\Gamma}$ will denote a Suslin class $S(\kappa)$ which is closed under real quantification. We assume κ is not the largest Suslin cardinal. We let $\Lambda = \Lambda(\boldsymbol{\Gamma}, \kappa)$ be the corresponding Martin class. We let $\lambda = o(\Lambda) = \delta(\Lambda)$. We showed in Lemma 3.19 that λ is the next Suslin cardinal after κ. Also from that Lemma, $\Lambda \subseteq S(\lambda)$.

We let $\boldsymbol{\Sigma}_0 = \bigcup_\omega \Lambda$ denote the pointclass of countable unions of sets from Λ. Let $\boldsymbol{\Pi}_0$ denote the dual class to $\boldsymbol{\Sigma}_0$, and define $\boldsymbol{\Sigma}_1 = \exists^{\omega^\omega} \boldsymbol{\Pi}_0$, etc., as usual.

3.20 Lemma. $S(\lambda) = \boldsymbol{\Sigma}_1$.

Proof. Since $S(\lambda)$ contains Λ and is closed under countable unions, intersections, and $\exists^{\omega^\omega}$, it follows that $\boldsymbol{\Sigma}_1 \subseteq S(\lambda)$. Since there are clearly $\boldsymbol{\Sigma}_1$ prewellorderings of length λ, it follows from the Coding Lemma that every tree on $\omega \times \lambda$ can be coded in the pointclass $\boldsymbol{\Sigma}_1$. It follows easily that $S(\lambda) \subseteq \boldsymbol{\Sigma}_1$. ⊣

3.21 Lemma. pwo$(\boldsymbol{\Sigma}_0)$, pwo$(\boldsymbol{\Pi}_1)$, *and* $\delta_1 \doteq \delta(\boldsymbol{\Pi}_1) = \lambda^+$. *Also,* λ^+ *is regular.*

Proof. pwo$(\boldsymbol{\Sigma}_0)$ follows easily from $\boldsymbol{\Sigma}_0 = \bigcup_\omega \Lambda$ [if $A \in \boldsymbol{\Sigma}_0$, say $A = \bigcup_n A_n$ where $A_n \in \Lambda$, the norm on A given by $\phi(x) = \mu n\ (x \in A_n)$ is a $\boldsymbol{\Sigma}_0$-norm]. By periodicity we also have pwo$(\boldsymbol{\Pi}_1)$. Also, $\boldsymbol{\Pi}_1$ is closed under countable unions, intersections, and $\forall^{\omega^\omega}$. From Lemma 2.13 we have that δ_1 is the supremum of the lengths of the $\boldsymbol{\Sigma}_1$ wellfounded relations, and from the Coding Lemma it then follows that δ_1 is regular. Every $\boldsymbol{\Sigma}_1$ wellfounded relation has length $< \lambda^+$ by the Kunen-Martin Theorem. Since there are clearly $\boldsymbol{\Sigma}_1$ wellfounded relations of length greater than λ, we have $\delta_1 = \lambda^+$. ⊣

The next says, roughly, that the Martin class of the Martin class is still the Martin class.

3.22 Lemma. *Let* $B \in \Lambda$, $\rho < \lambda$, *and let* $\mathcal{A} = \{A_\alpha\}_{\alpha<\rho}$ *be a sequence where each* $A_\alpha \leq_w B$. *Then* $\bar{\mathcal{A}} \subseteq \Lambda$.

Proof. More generally, for ρ_0, $\rho_1 < \lambda$, let Λ_{ρ_0,ρ_1} be the pointclass of all sets in some $\bar{\mathcal{A}}$ where $\mathcal{A} = \{A_\alpha\}_{\alpha<\rho_0}$ and for each α we have $|A_\alpha|_w < \rho_1$. If the lemma fails, then for some ρ_0, $\rho_1 < \lambda$ we have $\boldsymbol{\Sigma}_0 \subseteq \Lambda_{\rho_0,\rho_1}$. We first show by following the proof of Lemma 3.12 that if $\boldsymbol{\Pi}_n \subseteq \Lambda_{\rho_0,\rho_1}$, then by increasing ρ_0, ρ_1 to ρ_0', $\rho_1' < \lambda$ we have $\boldsymbol{\Sigma}_{n+1} \subseteq \Lambda_{\rho_0',\rho_1'}$. Suppose $A(x) \longleftrightarrow \exists y\ B(x, y)$, where $B \in \mathbf{\Pi}_n \subseteq \Lambda_{\rho_0,\rho_1}$. Let $\mathcal{B} = \{B_\alpha\}_{\alpha<\rho_0}$ be such that $B \in \bar{\mathcal{B}}$, where each $|B_\alpha|_w < \rho_1$. View every real z as coding a countable set $s_z \subseteq \omega^\omega$ and a function $f_z : s_z \to \{0,1\}$. Define

$$z \in C \longleftrightarrow \forall^* d\ \exists \alpha < \rho_0\ \forall x \in s_z\ (f_z(x) = 1 \longleftrightarrow \exists y \leq_T d\ B_\alpha(x,y)).$$

Let $\alpha_z(d)$ be the least ordinal $\alpha < \rho_0$ in the above definition, if one exists. From the closure properties of Λ and the Coding Lemma it follows easily that $C \in \Lambda$. Define the prewellordering $\prec$ on C by:

$$z_1 \prec z_2 \longleftrightarrow \forall^* d\ (\alpha_{z_1}(d) < \alpha_{z_2}(d)).$$

Easily, $\prec$ is also in Λ. Let $\rho_0' = |\prec|$. For $\alpha < \rho_0'$, say $\alpha = |z|_\prec$ with $z \in C$, define:

$$x \in A'_\alpha \longleftrightarrow \forall^* d\ \exists y \leq_T d\ B_{\alpha_z(d)}(x,y)\,.$$

There is a bound ρ_1' on the Wadge degrees of the A'_α (all the A'_α lie in the projective hierarchy over a code set for the sequence of B_α). As in Lemma 3.12 we have that $A \in \bar{\mathcal{A}}'$, where $\mathcal{A} = \{A'_\alpha\}_{\alpha<\rho_0'}$.

So, by increasing ρ_0, ρ_1 we may assume that $\mathbf{\Sigma}_2 \subseteq \Lambda_{\rho_0,\rho_1}$. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\rho_0}$ such that $\bar{\mathcal{A}}$ contains a complete $\mathbf{\Sigma}_2$ set. We now proceed as in Lemma 3.17. Define $D \subseteq \omega^\omega \times \omega^\omega$ by:

$$\begin{aligned} D(x,y) \longleftrightarrow\ & \exists \tau \leq_T x\ \forall^* d\ \exists \alpha < \rho_0\ \forall m,n \in \omega \\ & (y(m) = n \longleftrightarrow \exists z \leq_T d\ \tau(x,z,n,m) \in A_\alpha), \end{aligned}$$

where we regard each real τ as coding a continuous function from $(\omega^\omega)^2 \times (\omega)^2 \to \omega^\omega$. Clearly $D \in \Lambda$ and all sections of D are countable. So, $\neg D$ is λ-Suslin and has a (total) uniformizing function f. Since the tree on $\omega \times \lambda$ projecting to $\neg D$ can be coded in the pointclass $\mathbf{\Sigma}_0$ (from the Coding Lemma), an easy computation shows the graph of f to be $\mathbf{\Sigma}_2$. Thus, there is a continuous function τ such that for all x we have:

$$f(x)(n) = m \longleftrightarrow \exists z\ \tau(x,z,n,m) \in A,$$

where $A \in \bar{\mathcal{A}}$. Let $x \geq_T \tau$. Then $D(x, f(x))$, a contradiction. $\dashv$

We say an ordinal δ is *closed under ultrapowers* if for every $\rho < \delta$ and measure μ on some $\eta < \delta$ we have $j_\mu(\rho) < \delta$. If $\mathrm{cf}(\delta) = \omega$, this is equivalent to saying that $j_\mu(\delta) = \delta$ for all measures μ on δ.

3.23 Lemma. *λ is closed under ultrapowers.*

Proof. Let μ be a measure on $\alpha < \lambda$, and let $\beta < \lambda$. Let $\prec$ be a prewellordering of length $\gamma > \max\{\alpha,\beta\}$ with $\prec$ in a pointclass $\mathbf{\Gamma}_0 \subseteq \Lambda$ which is non-selfdual, and closed under $\wedge$, $\vee$, $\exists^{\omega^\omega}$. Fix a universal set U for $\mathbf{\Gamma}_0$, and use U via the Coding Lemma to code subsets of γ. Define $\mathcal{A} = \{A_\alpha\}_{\alpha<\gamma}$ by:

$$x \in A_\alpha \longleftrightarrow \exists y\ (|y|_\prec = \alpha \wedge U(x,y)).$$

As in Lemma 3.14, the code set C_μ of the measure μ in $\bar{\mathcal{A}}$, and hence in Λ from the previous lemma. From this and the Coding Lemma it follows that the prewellordering corresponding to $j_\mu(\beta)$ lies in Λ, and so has length $<\lambda$. $\dashv$

3.24 Lemma. *δ_1 is closed under ultrapowers.*

Proof. This follows immediately from the previous lemma and the fact that $\delta_1 = \lambda^+$. That is, any measure μ on $\alpha < \delta_1$ is equivalent to a measure μ' on λ, and any $\beta < \delta_1$ is in bijection with λ. So, $j_\mu(\beta)$ has the same cardinality as $j_{\mu'}(\lambda) = \lambda$. $\dashv$

3.25 Lemma. *δ_1 is a Suslin cardinal and $S(\delta_1) = \boldsymbol{\Sigma}_2$.*

Proof. Let A be a complete $\boldsymbol{\Sigma}_1$ set, and write $A = p[T]$ where T is a tree on $\omega \times \lambda$. T is weakly homogeneous, and the homogeneous tree construction gives a tree T' with $p[T'] = B \doteq \omega^\omega - A$. Also. the ordinals in T' are bounded by an ordinal of the form $j_\mu(\beta)$ where $\beta < \lambda^+$ and μ is one of the homogeneity measures on λ. From Lemma 3.24 we have that T' is a tree on $\omega \times \delta_1$.

Hence, δ_1 is a Suslin cardinal and $\boldsymbol{\Pi}_1 \subseteq S(\delta_1)$. Since $S(\delta_1)$ is closed under $\exists^{\omega^\omega}$, we also have $\boldsymbol{\Sigma}_2 \subseteq S(\delta_1)$. From the Coding Lemma and $\mathrm{pwo}(\boldsymbol{\Pi}_1)$ (which gives $\boldsymbol{\Pi}_1$ prewellorderings of length δ_1) it follows that $S(\delta_1) \subseteq \boldsymbol{\Sigma}_2$. $\dashv$

3.26 Lemma. $\mathrm{scale}(\boldsymbol{\Sigma}_2)$ *with norms into δ_1.*

Proof. This follows from Lemma 3.7 since δ_1 is a regular successor Suslin cardinal. $\dashv$

We show one more result concerning measures on λ (which we do not need for the main result).

3.27 Lemma. *$\boldsymbol{\Delta}_1$, $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Pi}_1$ are closed under measure quantification for measures on λ. That is, if μ is a measure on λ and $\{A_\alpha\}_{\alpha<\lambda}$ is a sequence of sets in $\boldsymbol{\Delta}_1$ (or $\boldsymbol{\Sigma}_1$,* etc.*), then $A \in \boldsymbol{\Delta}_1$ where $A(x) \longleftrightarrow \forall^*_\mu \alpha\ (x \in A_\alpha)$.*

Proof. Consider first the case of $\boldsymbol{\Delta}_1$, Since $\mathrm{cf}(\lambda) = \omega$, we may assume μ is a measure on $\rho < \lambda$, and $\{A_\alpha\}_{\alpha<\rho}$ is a $<\lambda$ sequence of $\boldsymbol{\Delta}_1$ sets. Let $A(x) \longleftrightarrow \forall^*_\mu \alpha\ (x \in A_\alpha)$. It suffices to show that $A \in \boldsymbol{\Sigma}_1$. Let $\prec$ be a prewellordering in Λ of length greater than ρ. Let $\boldsymbol{\Gamma}_0 \subseteq \Lambda$ be non-selfdual, closed under $\wedge$, $\vee$, $\exists^{\omega^\omega}$, and such that $C_\mu \in \boldsymbol{\Delta}(\boldsymbol{\Gamma}_0)$, where C_μ denotes the code set of μ with respect to the prewellordering $\prec$. Recall this means the following. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\boldsymbol{\Gamma}_0$. We view every real x as coding a subset B_x of ρ by $\alpha \in B_x \longleftrightarrow \exists y\ (|y|_\prec = \alpha \wedge U(x, y))$. Then there is a $C \in \boldsymbol{\Delta}(\boldsymbol{\Gamma}_0)$ such that $C(z) \longleftrightarrow \mu(B_z) = 1$.

We have:

$$A(x) \longleftrightarrow \exists z\ (C(z) \wedge \forall \alpha \in B_z\ (x \in A_\alpha)).$$

It suffices to show that $D \in \boldsymbol{\Sigma}_1$ where $D(x, z) \longleftrightarrow \forall \alpha \in B_z\ (x \in A_\alpha)$. From Lemma 2.15, $\boldsymbol{\Delta}_1$ is closed under $<\delta_1$ length unions and intersections. So, it suffice to fix $\alpha < \rho$ and show that $D_\alpha \in \boldsymbol{\Delta}_1$, where:

$$D_\alpha(x, z) \longleftrightarrow (\alpha \notin B_z \vee x \in A_\alpha).$$

But, $\{z : \alpha \in B_z\} \in \boldsymbol{\Gamma}_0$ and $A_\alpha \in \boldsymbol{\Delta}_1$, so clearly $D_\alpha \in \boldsymbol{\Delta}_1$.

For the case of $\boldsymbol{\Sigma}_1$, the argument is similar, but we now need the fact that $\boldsymbol{\Sigma}_1$ is closed under $<\lambda$ intersections. This follows from the fact that $\boldsymbol{\Sigma}_1 = \exists^{\omega^\omega} \boldsymbol{\Pi}_0$ and the Coding Lemma. Briefly, let $\prec$ and $\boldsymbol{\Gamma}_0$ be as before. From the Coding Lemma there is a $\boldsymbol{\Gamma}_0$ relation $R \subseteq \omega^\omega \times \omega^\omega$ with $\mathrm{dom}(R) = \mathrm{dom}(\prec)$ and such that for all $(x, y) \in \mathrm{dom}(R)$, y is a $\boldsymbol{\Pi}_0$ code for a set B_y in $\omega^\omega \times \omega^\omega$ projecting to $A_{|x|_\prec}$. Let $\prec'$ be the relation on R given by $(x_1, y_1) \prec' (x_2, y_2)$ iff $(x_1 \prec x_2)$. If $w \in \bigcap_{\alpha<\rho} A_\alpha$, then by the Coding Lemma applied to $\prec'$, there is a $S \subseteq (\omega^\omega)^3$ in $\boldsymbol{\Gamma}_0$ such that (1) $(x, y, z) \in S \to (x, y) \in R$, (2) for every $\alpha < \rho$ there is an (x, y, z) in S with $|x|_\prec = \alpha$, and (3) for all (x, y, z) in S we have $B_y(w, z)$. It follows that

$$\begin{aligned} w \in \textstyle\bigcap_{\alpha<\rho} A_\alpha \longleftrightarrow\ & \exists S \in \boldsymbol{\Gamma}_0\ (S \text{ satisfies (1) and (2)} \\ & \wedge \forall (x, y, z) \in S \to B_y(w, z)) \end{aligned}$$

which shows that $\bigcap_{\alpha<\rho} A_\alpha$ is $\boldsymbol{\Sigma}_1$. ⊣

3.4. The Classification of the Suslin Cardinals

We are now ready to classify the Suslin cardinals and scale property assuming AD. This is the content of the next theorem. The reader should recall the definition of the Steel pointclass from Remark 2.20.

3.28 Theorem. *Let κ be a limit of Suslin cardinals, and suppose there is a Suslin cardinal greater than κ. Then κ is a Suslin cardinal and one of the following cases applies.*

Case I. $\mathrm{cf}(\kappa) = \omega$.

Let $\boldsymbol{\Sigma}_0 = \bigcup_\omega S_{<\kappa}$, where $S_{<\kappa} = \bigcup_{\rho<\kappa} S(\rho)$. Define $\boldsymbol{\Pi}_0 = \check{\boldsymbol{\Sigma}}_0$, and for $i > 0$, define as usual $\boldsymbol{\Sigma}_i = \exists^{\omega^\omega} \boldsymbol{\Pi}_{i-1}$ and $\boldsymbol{\Pi}_i = \forall^{\omega^\omega} \boldsymbol{\Sigma}_{i-1}$. Then $\boldsymbol{\Sigma}_0$ has the scale property with norms into κ. Also, $S(\kappa) = \boldsymbol{\Sigma}_1$. Let $\delta_{2i+1} = \delta(\boldsymbol{\Pi}_{2i+1})$. Then for all i, $\mathrm{scale}(\boldsymbol{\Pi}_{2i+1})$ and $\mathrm{scale}(\boldsymbol{\Sigma}_{2i+2})$ with scales into δ_{2i+1}. δ_{2i+1} is a Suslin cardinal and $S(\delta_{2i+1}) = \boldsymbol{\Sigma}_{2i+2}$. Also, $\delta_{2i+1} = (\lambda_{2i+1})^+$, where λ_{2i+1} is a Suslin cardinal of cofinality ω, and $\lambda_1 = \kappa$. $S(\lambda_{2i+1}) = \boldsymbol{\Sigma}_{2i+1}$. The sequence δ_1, λ_3, $\delta_3, \dots$ enumerates the first ω Suslin cardinals greater than κ.

Case II. $\mathrm{cf}(\kappa) > \omega$ *and the Steel pointclass $\boldsymbol{\Gamma}_0$ is not closed under $\exists^{\omega^\omega}$.*

Let $\boldsymbol{\Sigma}_0 = \exists^{\omega^\omega} \boldsymbol{\Gamma}_0$, $\boldsymbol{\Pi}_0 = \check{\boldsymbol{\Sigma}}_0$, and for $i > 0$ define the $\boldsymbol{\Sigma}_i$, $\boldsymbol{\Pi}_i$ as usual. Then $\mathrm{scale}(\boldsymbol{\Gamma}_0)$ and $\mathrm{scale}(\boldsymbol{\Sigma}_0)$ with norms into κ. Let $\delta_{2i+1} = \delta(\boldsymbol{\Pi}_{2i+1})$. Then for *all i,* $\mathrm{scale}(\boldsymbol{\Pi}_{2i+1})$ *and* $\mathrm{scale}(\boldsymbol{\Sigma}_{2i+2})$ *with scales into* δ_{2i+1}*.* δ_{2i+1} *is a Suslin cardinal and* $S(\delta_{2i+1}) = \boldsymbol{\Sigma}_{2i+2}$*. Also,* $\delta_{2i+1} = (\lambda_{2i+1})^+$*, where* λ_{2i+1} *is a Suslin cardinal of cofinality* ω*.* $S(\lambda_{2i+1}) = \boldsymbol{\Sigma}_{2i+1}$*. The sequence* λ_1*,* δ_1*,* λ_3*,* $\delta_3, \ldots$ *enumerates the first* ω *Suslin cardinals greater than* κ*.*

Case III. $\mathrm{cf}(\kappa) > \omega$ *and the Steel pointclass* $\boldsymbol{\Gamma}_0$ *is closed under* $\exists^{\omega^\omega}$*.*

$\boldsymbol{\Gamma}_0$ *has the scale property with norms into* κ*. Let* $\Lambda = \Lambda(\boldsymbol{\Gamma}_0, \kappa)$*. Let* $\boldsymbol{\Sigma}_0 = \bigcup_\omega \Lambda$*,* $\boldsymbol{\Pi}_0 = \check{\boldsymbol{\Sigma}}_0$*, and for* $i > 0$ *define* $\boldsymbol{\Sigma}_i$*,* $\boldsymbol{\Pi}_i$ *as usual. Let* $\delta_{2i+1} = \delta(\boldsymbol{\Pi}_{2i+1})$*. Then* $\mathrm{scale}(\boldsymbol{\Sigma}_2)$ *with norms into* $\delta_1 = \lambda_1^+$*, where* $\lambda_1 = o(\Lambda)$*. For all* $i > 0$*,* $\mathrm{scale}(\boldsymbol{\Pi}_{2i+1})$ *and* $\mathrm{scale}(\boldsymbol{\Sigma}_{2i+2})$ *with norms into* δ_{2i+1}*. For all* i*,* δ_{2i+1} *is a Suslin cardinal and* $S(\delta_{2i+1}) = \boldsymbol{\Sigma}_{2i+2}$*. Also,* $\delta_{2i+1} = (\lambda_{2i+1})^+$*, where* λ_{2i+1} *is a Suslin cardinal of cofinality* ω*.* $S(\lambda_{2i+1}) = \boldsymbol{\Sigma}_{2i+1}$*. The sequence* λ_1*,* δ_1*,* λ_3*,* $\delta_3, \ldots$ *enumerates the first* ω *Suslin cardinals greater than* κ*.*

Proof. Consider first case I, that is, $\mathrm{cf}(\kappa) = \omega$. Let $A \in \boldsymbol{\Sigma}_0$, and write $A = \bigcup_i A_i$ with $A_i \in S_{<\kappa}$. Let $\{\phi^i_n\}_{n\in\omega}$ be a scale on A_i of length $\gamma_i < \kappa$. From the Coding Lemma, all of the norm relations $\leq^*_{\phi^i_n}$, $<^*_{\phi^i_n}$ are in $S_{<\kappa}$. Define the norms ψ_n on A as follows. $\psi_0(x) = \mu i\ (x \in A_i)$, $\psi_{k+1}(x) = \langle \psi_0(x), \phi_k^{\psi_0(x)}(x)\rangle$, where $\langle \alpha, \beta\rangle$ denotes the rank of (α, β) in the lexicographic ordering on $(\gamma_i)^2$. This is easily checked to be a scale on A with norms into κ, and is clearly a $\boldsymbol{\Sigma}_0$-scale. Thus, $\mathrm{scale}(\boldsymbol{\Sigma}_0)$. From the Coding Lemma an easy computation show $S(\kappa) \subseteq \boldsymbol{\Sigma}_1$. Also, $\boldsymbol{\Sigma}_1 \subseteq S(\kappa)$ as $S(\kappa)$ is closed under $\exists^{\omega^\omega}$ and countable intersections. Thus, $S(\kappa) = \boldsymbol{\Sigma}_1$. By periodicity, $\mathrm{scale}(\boldsymbol{\Pi}_{2i+1})$, $\mathrm{scale}(\boldsymbol{\Sigma}_{2i})$. By definition of δ_{2i+1}, every regular $\boldsymbol{\Pi}_{2i+1}$ scale has length $\leq \delta_{2i+1}$. In particular, every $\boldsymbol{\Pi}_{2i+1}$, and hence every $\boldsymbol{\Sigma}_{2i+2}$ set is δ_{2i+1}-Suslin. A straightforward computation from the Coding Lemma shows $S(\delta_{2i+1}) \subseteq \boldsymbol{\Sigma}_{2i+2}$, so $S(\delta_{2i+1}) = \boldsymbol{\Sigma}_{2i+2}$. Every $\boldsymbol{\Pi}_{2i+1}$ set admits a $\boldsymbol{\Pi}_{2i+1}$-scale with norms into δ_{2i+1}, and the standard argument transferring the scale property to $\exists^{\omega^\omega}\boldsymbol{\Pi}_{2i+1}$ shows that every $\boldsymbol{\Sigma}_{2i+2}$ set admits a $\boldsymbol{\Sigma}_{2i+2}$-scale with norms into δ_{2i+1} (that is, the lengths of the norms do not increase with this transfer). Since $\boldsymbol{\Pi}_{2i+1}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$ and $\mathrm{pwo}(\boldsymbol{\Pi}_{2i+1})$, another standard argument (cf. the proof of Theorem 2.18) shows that δ_{2i+1} is the supremum of the lengths of the $\boldsymbol{\Sigma}_{2i+1}$ well-founded relations, and from the Coding Lemma this must be a regular cardinal. So, δ_{2i+1} is regular. From the Kunen-Martin Theorem $\delta_1 \leq \kappa^+$, and so $\delta_1 = \kappa^+$. Clearly δ_{2i+1} cannot be a limit Suslin cardinal (there are only finitely many Levy classes between $\boldsymbol{\Sigma}_o$ and $\boldsymbol{\Sigma}_{2i+1}$). From Lemma 3.7, $\delta_{2i+1} = \lambda^+_{2i+1}$ for some Suslin cardinal λ_{2i+1} with $\mathrm{cf}(\lambda_{2i+1}) = \omega$. Also from that lemma, $\boldsymbol{\Sigma}_{2i+2} = \exists^{\omega^\omega}\check{S}(\lambda_{2i+1})$, and thus $S(\lambda_{2i+1}) = \boldsymbol{\Sigma}_{2i+1}$ (this also follows from the fact there is only one Levy class closed under $\exists^{\omega^\omega}$ between $\boldsymbol{\Sigma}_{2i}$ and $\boldsymbol{\Sigma}_{2i+2}$). As we have now accounted for all the Levy classes closed under $\exists^{\omega^\omega}$, we must have that δ_1, λ_3, $\delta_3, \ldots$ enumerates the next ω Suslin cardinals after κ.

Consider next case II. From Lemma 3.8 we have $\mathrm{scale}(\boldsymbol{\Sigma}_0)$ with norms into κ. Since $\boldsymbol{\Sigma}_0$ is not closed under $\forall^{\omega^\omega}$ (as $\boldsymbol{\Pi}_0 \subseteq \forall^{\omega^\omega}\boldsymbol{\Sigma}_0$) we may propagate the scale property by periodicity to $\boldsymbol{\Pi}_{2i+1}$, $\boldsymbol{\Sigma}_{2i+2}$. As in the previous case, δ_{2i+1} is a Suslin cardinal, $S(\delta_{2i+1}) = \boldsymbol{\Sigma}_{2i+2}$, and $\delta_{2i+1} = \lambda^+_{2i+1}$, where λ_{2i+1}

is a Suslin cardinal of cofinality ω. In this case, since $\mathrm{cf}(\kappa) > \omega$, we have $\lambda_1 > \kappa$. From Lemma 3.7 again we have that $S(\lambda_{2i+1}) = \boldsymbol{\Sigma}_{2i+1}$. As we have again accounted for all the Levy classes closed under $\exists^{\omega^\omega}$, the sequence $\lambda_1, \delta_1, \lambda_3, \delta_3, \dots$ enumerates the next ω Suslin cardinals after κ. scale($\boldsymbol{\Gamma}_0$) follows from Lemma 3.8.

Consider now case III. From Lemma 3.8, $\boldsymbol{\Gamma}_0$ has the scale property with norms into κ, and also $S(\kappa) = \boldsymbol{\Gamma}_0$. Recall in this case that Λ is selfdual and closed under real quantification, and $\mathrm{cf}(\lambda) = \omega$, where $\lambda = o(\Lambda)$. From Lemma 3.19, λ is the next Suslin cardinal after κ, and $S(\lambda) = \boldsymbol{\Sigma}_1$ ($\boldsymbol{\Sigma}_1 \subseteq S(\lambda)$ follows from the closure of $S(\lambda)$ under $\exists^{\omega^\omega}$ and countable intersections, and $S(\lambda) \subseteq \boldsymbol{\Sigma}_1$ follows by a straightforward computation using the Coding Lemma and the closure of $\boldsymbol{\Sigma}_1$ under countable unions and intersections). From Lemma 3.26, $\boldsymbol{\Sigma}_2$ has the scale property with norms into $\delta_1 = \lambda^+$. In particular, $\lambda_1 = \lambda$. The remaining arguments are exactly as in the previous cases. $\dashv$

4. Trivial Descriptions: A Theory of $\boldsymbol{\omega_1}$

We assume AD + DC throughout Sect. 4.

Our goal in this section is to present a "theory of ω_1", using only techniques that will generalize to higher levels. Starting from the weak partition relation on ω_1 (proved in the last section), we prove the strong partition relation on ω_1, calculate $\boldsymbol{\delta}^1_3$, and prove the weak partition relation on $\boldsymbol{\delta}^1_3$. This represents the first step in the inductive analysis of the projective ordinals. We also use these techniques to present a proof of the Kechris-Martin Theorem on $\boldsymbol{\Pi}^1_3$. These results are not new (cf. [9, 33]), but our proofs do not rely on the theory of indiscernibles for L (as did the original proofs) but rather on a direct analysis on the measures on ω_1 (and the ω_n). The idea of using an analysis of measures to provide a good coding for sets was first used by Kunen (see [33]) in the original proof of the weak partition relation on $\boldsymbol{\delta}^1_3$.

As we mentioned in the introduction, one of our motivations in this section is to introduce and use the terminology of "descriptions", even though the concept at this level is rather trivial and could be dispensed with. This way, the arguments at the higher levels will have the same general form, though the concept of description will become non-trivial. This will free us, in the next section, to concentrate on this point.

Our first job is to analyze the measures on ω_1, from which the strong partition relation on ω_1 will follow.

Recall that if $T \subseteq \rho^{<\omega}$ is a tree on ρ and $\alpha < \rho$, then $T\restriction\alpha = T \cap \alpha^{<\omega}$ denotes the tree restricted to α, and if β is in the well-founded part of $T\restriction\alpha$, then $|(T\restriction\alpha)(\beta)|$ denotes the rank of β in $T\restriction\alpha$. Also, if T is a tree on $\alpha \times \rho$, then $T_x = \{\vec{s} \in \rho^{<\omega} : (x\restriction \mathrm{lh}(\vec{s}), \vec{s}) \in T\}$ is the section of the tree at x.

Let $\mathrm{WO} \subseteq \omega^\omega$ be the standard set of codes for wellorderings, that is $x \in \mathrm{WO}$ iff $\prec_x \doteq \{(n,m) : x(\langle n,m\rangle) = 1\}$ is a wellordering of ω. Let $\mathrm{WF} \supseteq \mathrm{WO}$ be the set of codes of well-founded, transitive relations on ω. That is, $x \in \mathrm{WF}$

iff $\prec_x$ is a well-founded and transitive relation. Both WO, WF are in $\Pi^1_1 - \boldsymbol{\Sigma}^1_1$. For $x \in \mathrm{WF}$, let $|x| = |\prec_x| = \sup\{|n|_{\prec_x} + 1 : n \in \mathrm{dom}(\prec_x)\}$.

The finite exponent partition relation on ω_1 easily implies that the closed unbounded filter on ω_1 is a normal measure, and is the unique normal measure on ω_1. We let W_1^m denote the m-fold product of this normal measure. Equivalently, $A \subseteq (\omega_1)^m$ has measure one with respect to W_1^m iff there is a closed unbounded $C \subseteq \omega_1$ such that $(C)^m \subseteq A$. Note that we regard W_1^m as a measure on the set of tuples $(\alpha_0, \dots, \alpha_{m-1})$ for which $\alpha_0 < \dots < \alpha_{m-1}$. With this convention, we may safely write the ordinals in the tuple in any order (which will be notationally convenient later).

Recall the construction of the Kunen tree on ω_1:

4.1 Lemma (Kunen). *There is a tree T on $\omega \times \omega_1$ such that for all $f : \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that T_x is well-founded and for all $\omega \leq \alpha < \omega_1$ we have $f(\alpha) \leq |T_x \restriction \alpha|$.*

Proof. Let S be a recursive tree on $\omega \times \omega$ with $p[S]$ a $\boldsymbol{\Sigma}^1_1$-complete set. Define the tree U on $\omega \times \omega_1$ by:

$$\begin{aligned} &((a_0, \dots, a_{n-1}), (\alpha_0, \dots, \alpha_{n-1})) \in U \\ &\quad \longleftrightarrow \forall i, j < n \ (a_{\langle i,j \rangle} = 1 \to \alpha_i < \alpha_j) \\ &\qquad\quad \wedge \forall i, j, k < n \ (a_{\langle i,j \rangle} = 1 \wedge a_{\langle j,k \rangle} = 1 \to a_{\langle i,k \rangle} = 1). \end{aligned}$$

Clearly, $p[U] = \mathrm{WF}$. Let V be the tree on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ defined by:

$$\begin{aligned} (\vec{s}, \vec{a}, \vec{\alpha}, \vec{b}, \vec{c}) \in V \longleftrightarrow &(\vec{a}, \vec{\alpha}) \in U \wedge (\vec{b}, \vec{c}) \in S \\ &\wedge \exists \sigma \text{ extending } \vec{s} \ (\sigma(\vec{a}) = \vec{b}), \end{aligned}$$

where we view every real σ as coding a strategy for II in some reasonable manner. Suppose $f : \omega_1 \to \omega_1$. Consider the game where I, II play out x, y, and II wins iff $[x \in \mathrm{WF} \to S_y$ is well-founded $\wedge\, |S_y| > \sup\{f(\beta) : \beta \leq |x|\}]$. II easily wins by boundedness, noting that

$$\sup\{|S_y| : S_y \text{ is well-founded}\} = \omega_1,$$

as otherwise $p[S]$ would be Borel. Let σ be winning for II. Note that for all $\alpha < \omega_1$, if $x \in \mathrm{WF}$ and $|x| = \alpha$, then $U_x \restriction \alpha$ is ill-founded. It follows that V_σ is well-founded and for all $\alpha \geq \omega$, we have $|V_\sigma \restriction \alpha| > f(\alpha)$. It is now easy to code the 2nd, 3rd, 4th, and 5th coordinates of V into the second coordinate of a tree T (say by weaving the values of the four components; this does not decrease rank, that is, $|T_\sigma \restriction \alpha| \geq |V_\sigma \restriction \alpha|$ for all σ, α). ⊣

We note that the Kunen tree T is Δ^1_1 in the codes. By this we mean that there are Σ^1_1, Π^1_1 relations $S(n, a, x)$, $R(n, a, x)$ such that for all x with $x_0, \dots, x_{n-1} \in \mathrm{WO}$, we have

$$\begin{aligned} S(n, a, x) &\longleftrightarrow R(n, a, x) \\ &\longleftrightarrow ((a_0, \dots, a_{n-1}), (|x_0|, \dots, |x_{n-1}|)) \in T. \end{aligned}$$

This follows immediately from the definition of T (see Lemma 4.1).

Although the difference is not large, it is sometimes to more convenient to deal with linear orderings rather than trees. The following theorem shows that we may modify the Kunen tree so as to make this possible. Recall that if $s \in \omega^n$, then $T_s = \{\vec{\alpha} \in \omega_1^n : (s, \vec{\alpha}) \in T\}$.

4.2 Theorem. *There is a function $s \to T(s)$ which assigns to each $s \in \omega^{<\omega}$ a wellordering of a subset of ω_1 with the following properties. If t extends s, then $T(t) \supseteq T(s)$. For $x \in \omega^\omega$, let $T(x) = \bigcup_n T(x \restriction n)$, so $T(x)$ is a linear order. Then for any $f : \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ such that $T(x)$ is a wellordering and for all $\alpha \geq \omega$, $f(\alpha) < |T(x) \restriction \alpha|$. Furthermore, the map $s \to T(s)$ is Δ^1_1 in the codes. That is, there are Σ^1_1, Π^1_1 relations $S(n, a, x, y)$, $R(n, a, x, y)$ such that for all $x, y \in \mathrm{WO}$, we have*

$$S(n, a, x, y) \longleftrightarrow R(n, a, x, y) \longleftrightarrow [(|x|, |y|) \in T(a_0, \ldots, a_{n-1})].$$

Proof. We modify the Kunen tree T as follows. Fix a bijection $\pi : (\omega_1)^{<\omega} \to \omega_1$ such that for all $\alpha_0, \ldots, \alpha_{n-1} < \omega$, we have $\pi(\alpha_0, \ldots, \alpha_{n-1}) < \omega$. For $s \in \omega^{<\omega}$, let $T(s)$ be the wellordering defined by

$$\alpha \ T(s) \ \beta \longleftrightarrow \pi^{-1}(\alpha), \pi^{-1}(\beta) \in T_s \wedge (\pi^{-1}(\alpha) <^{\mathrm{KB}}_{T_s} \pi^{-1}(\beta)),$$

where $<^{\mathrm{KB}}_{T_s}$ denotes the Kleene-Brouwer ordering on T_s. For $x \in \omega^\omega$, let $T(x) = \bigcup_n T(x \restriction n)$. Clearly $T(x)$ is a linear ordering, and is a wellordering iff T_x is well-founded. Suppose now that $f : \omega_1 \to \omega_1$. Let $C \subseteq \omega_1$ be the closed unbounded set of ordinals closed under π. Note that $\omega \in C$. For $\alpha \geq \omega$, let $l(\alpha)$ be the greatest element of C which is less than or equal to α. Define

$$f'(\alpha) = \sup\{f(\beta) : l(\beta) = l(\alpha)\}.$$

Let $x \in \omega^\omega$ be such that T_x is well-founded and for all $\alpha \geq \omega$, $|T_x \restriction \alpha| > f'(\alpha)$. We claim that for all $\alpha \geq \omega$, $|T(x) \restriction \alpha| > f(\alpha)$. Note that π^{-1} applied to $T(x) \restriction \alpha$ contains the entire tree $T_x \restriction l(\alpha)$. Thus, $|T(x) \restriction \alpha| \geq |T_x \restriction l(\alpha)| > f'(l(\alpha)) \geq f(\alpha)$. We may also choose the bijection π so that π is Δ^1_1 in the codes, and it is then straightforward to check that $s \to T(s)$ is Δ^1_1 in the codes. $\dashv$

In the future, we will use the notation T_s for the Kunen "tree", regardless of whether we are using the tree T_s or the linear ordering $T(s)$. The meaning will be clear from the context.

For the rest of Sect. 4, T will denote the Kunen tree of Lemma 4.1.

Note that for every $f : \omega_1 \to \omega_1$, the equivalence class $[f]_{W^1_1}$ may be coded by a pair (x, β) where $x \in \omega^\omega$, $\beta < \omega_1$. By this we mean $\forall^*_{W^1_1} \alpha \ f(\alpha) = |(T_x \restriction \alpha)(\beta)|$. To see that x and β exist, use normality and the fact that $\forall^* \alpha \ \exists \beta < \alpha \ f(\alpha) = |(T_x \restriction \alpha)(\beta)|$.

4.3 Definition. A level -1, or *trivial description*, d is simply a positive natural number $d \in \omega$. We let $\mathcal{D}^{-1} = \omega - \{0\}$ be the set of trivial descriptions. The interpretation function h assigns to each $d \in \mathcal{D}^{-1}$ an ordinal by: $h(d) = d$. We say a trivial description d is *well-defined*, or *satisfies condition D*, with respect to a measure W_1^m iff $d \leq m$. If d is well-defined with respect to W_1^m and $\beta_1 < \cdots < \beta_m < \omega_1$, we define $h(\beta_1, \ldots, \beta_m; d) = \beta_d$. If $g : \omega_1 \to \omega_1$, and d is well-defined with respect to W_1^m, we define an ordinal $(g; W_1^m; d)$. This is represented with respect to W_1^m by the function which assigns to $\beta_1, \ldots, \beta_m$ the ordinal $(g; \beta_1, \ldots, \beta_m; d) \doteq g(h(\beta_1, \ldots, \beta_m; d)) = g(\beta_d)$.

Clearly, $(g; W_1^m; d)$ only depends on $[g]_{W_1^1}$. In this way, the trivial descriptions, together with the equivalence classes of functions with respect to the normal measure on ω_1, generate equivalence classes with respect to the family of measures W_1^m. Note that for $g = \mathrm{id}$, the identity function on ω_1, we have $(\mathrm{id}; \vec{\beta}; d) = \beta_d$. We introduce a "lowering operator" $\mathcal{L}$ on $\mathcal{D}^{-1}$.

4.4 Definition. We define $\mathcal{L}(d) = d - 1$ for $d > 1$. If $d = 1$, we do not define $\mathcal{L}(d)$, and say $d = 1$ is *minimal* with respect to $\mathcal{L}$.

By definition, there is a unique $d \in \mathcal{D}^{-1}$ which is minimal with respect to $\mathcal{L}$.

As a warm-up, we use this terminology to recast one familiar proof of the computation $\boldsymbol{\delta}_3^1 = \omega_{\omega+1}$. In this context, our "main technical lemma" is the following (the reader will note that the lemma corresponds to a well-known property of indiscernibles for L).

4.5 Lemma. *Let $f : (\omega_1)^m \to \omega_1$, d be well-defined with respect to W_1^m, and assume $[f]_{W_1^m} < (\mathrm{id}; W_1^m; d)$. If d is non-minimal with respect to $\mathcal{L}$, then there is a $g : \omega_1 \to \omega_1$ such that $[f]_{W_1^m} < (g; W_1^m; \mathcal{L}(d))$. If d is minimal with respect to $\mathcal{L}$, then $\exists \alpha < \omega_1\ [f]_{W_1^m} < \alpha$.*

Proof. We have $\forall^*_{W_1^m} \beta_1, \ldots, \beta_m\ f(\vec{\beta}) < (\mathrm{id}; \beta_1, \ldots, \beta_m; d) = \beta_d$. Consider the case d non-minimal with respect to $\mathcal{L}$. Consider the partition $\mathcal{P}$, where we partition ordinals $\beta_1 < \cdots < \beta_m < \omega_1$ with the extra ordinal $\beta_{d-1} < \gamma < \beta_d$ according to whether $\gamma > f(\beta_1, \ldots, \beta_m)$. Clearly, on the homogeneous side the stated property holds. Let $C \subseteq \omega_1$ be closed unbounded and homogeneous for $\mathcal{P}$. Let $g(\alpha) =$ the least element of C greater than α. Then $\forall^*_{W_1^m} \beta_1, \ldots, \beta_m\ f(\vec{\beta}) < g(\beta_{d-1}) = (g; \beta_1, \ldots, \beta_m; \mathcal{L}(d))$, thus $[f]_{W_1^m} < (g; W_1^m; \mathcal{L}(d))$. The case where d is minimal with respect to $\mathcal{L}$ is similar. ⊣

We let $<$ be the transitive relation on $\mathcal{D}^{-1}$ generated by the relation $\mathcal{L}(d) < d$. Of course, $<$ is just the usual ordering on the positive integers. We let $|d|$ denote the rank of d in this ordering, so $|d| = d - 1$.

Lemma 4.5 and the analysis of functions $f : \omega_1 \to \omega_1$ with respect to the normal measure W_1^1 on ω_1 (i.e., the Kunen tree construction) suffice to compute upper bound for $j_{W_1^m}(\omega_1)$, where $j_{W_1^m}$ denotes the ultrapower embedding corresponding to the measure W_1^m. This is made explicit in the following theorem.

4.6 Theorem. *Let d be defined with respect to W_1^m. Then* $(\mathrm{id}; W_1^m; d) \leq \omega_{|d|+1}$.

Proof. By induction on $|d|$. If $|d| = 0$ (i.e., $d = 1$), then $(\mathrm{id}; W_1^m; d) \leq \omega_1$ from Lemma 4.5. Otherwise, let $\alpha < (\mathrm{id}; W_1^m; d)$. From Lemma 4.5, $\exists g : \omega_1 \to \omega_1$ such that $\alpha < (g; W_1^m; \mathcal{L}(d))$. Recall that T is the Kunen tree of Lemma 4.1. Let $x \in \omega^\omega$ be such that T_x is well-founded and $\forall^*_{W_1^m} \beta\, g(\beta) < |T_x \restriction \beta|$. Let $|T_x|$ also denote the function $\beta \to |T_x \restriction \beta|$. Thus, $\alpha < (|T_x|; W_1^m; \mathcal{L}(d))$. T_x also induces a bijection π between $(\mathrm{id}; W_1^m; \mathcal{L}(d))$ and $(|T_x|; W_1^m; \mathcal{L}(d))$. Namely, if $\delta < (\mathrm{id}; W_1^m; \mathcal{L}(d))$, then $\pi(\delta) < (|T_x|; W_1^m; \mathcal{L}(d))$ is defined by

$$\forall^*_{W_1^m} \beta_1, \ldots, \beta_m\ [\pi(\delta)(\vec{\beta}) = |(T_x \restriction (\mathrm{id}; \vec{\beta}; \mathcal{L}(d)))(\delta(\vec{\beta}))|].$$

Thus, $(g; W_1^m; d) < (\mathrm{id}; W_1^m; \mathcal{L}(d))^+$, and so $(\mathrm{id}; W_1^m; d) \leq (\mathrm{id}; W_1^m; \mathcal{L}(d))^+$. By induction, $(\mathrm{id}; W_1^m; \mathcal{L}(d)) \leq \omega_{|\mathcal{L}(d)|+1} = \omega_{|d|}$, so $(\mathrm{id}; W_1^m; d) \leq \omega_{|d|+1}$. ⊣

As a corollary, we have an upper bound for $\boldsymbol{\delta}^1_3$.

4.7 Corollary. $\boldsymbol{\delta}^1_3 \leq \omega_{\omega+1}$.

Proof. The homogeneous tree analysis, which we will not reproduce here, shows that every $\boldsymbol{\Pi}^1_2$ set is λ_3-Suslin, where $\lambda_3 \leq \sup j_\nu(\omega_1)$, the supremum ranging over measures ν occurring in the homogeneous tree on a $\boldsymbol{\Pi}^1_1$-complete set; that is, the measures W_1^m. [The homogeneous tree construction is described in detail in [17]. The definition of a homogeneous tree is given in Sect. 5, and more general arguments are presented there as well. In particular, the arguments immediately after Definition 5.1 suffice to prove the above claim.] From the Kunen-Martin Theorem, it follows that $\boldsymbol{\delta}^1_3 \leq [\sup_m j_{W_1^m}(\omega_1)]^+$. Now, a small variation in the proof of Lemma 4.5 shows that $j_{W_1^m}(\omega_1) \leq (\mathrm{id}; W_1^m; \tilde{d})^+$, where $\tilde{d} \doteq m$ is the *maximal* description defined with respect to W_1^m. Thus, $j_{W_1^m}(\omega_1) \leq \omega_{m+1}$, and so $\lambda_3 \leq \omega_\omega$. ⊣

We will get the lower bound for $\boldsymbol{\delta}^1_3$ as a consequence of a general result of Martin. However, we first need the strong partition relation on ω_1.

4.1. Analysis of Measures on $\boldsymbol{\delta}^1_1$

The next theorem is our analysis of an arbitrary measure ν on ω_1. The key idea is to exploit a pressing down argument with respect to ν.

4.8 Theorem. *Let ν be a measure on ω_1. Then there are finitely many reals $x_0, \ldots, x_n$ with T_{x_i} well-founded for $0 \leq i \leq n$, and an ordinal $\alpha < \omega_1$ such that for all $A \subseteq \omega_1$:*

$$\nu(A) = 1 \longleftrightarrow \forall^*_{W_1^{n+1}} \beta_0, \ldots, \beta_n\ (h^\alpha_{x_0,\ldots,x_n}(\beta_0, \ldots, \beta_n) \in A),$$

where for $0 \leq i \leq n$, $\delta_i \doteq h^\alpha_{x_0,\ldots,x_i}(\beta_0, \ldots, \beta_i)$ is defined inductively by: $\delta_i = |(T_{x_i} \restriction \beta_i)(\delta_{i-1})|$, and $\delta_{-1} = \alpha$. In particular, ν is equivalent to W_1^{n+1} for some $n \in \omega$.

Proof. We may assume that ν is non-principal. Let $g_0 : \omega_1 \to \omega_1$ satisfy:

(1) There is a ν measure one set A such that $g_0 \restriction A$ is monotonically increasing (i.e., if $\alpha < \beta$ are in A, then $g_0(\alpha) \leq g_0(\beta)$).

(2) There does not exist a ν measure one set A such that $g_0 \restriction A$ is constant.

(3) If $[g]_\nu < [g_0]_\nu$, then g does not satisfy (1) and (2).

g_0 exists, since the identity function satisfies (1) and (2). Consider the measure $g_0(\nu)$ (that is, $g_0(\nu)(A) = 1$ iff $\nu(g_0^{-1}(A)) = 1$). If $C \subseteq \omega_1$ is closed unbounded, then $g_0(\nu)(C) = 1$, as otherwise $p \circ g_0$ violates the minimality of g_0, where $p(\alpha) =$ the largest element of C less than or equal to α. Thus, $g_0(\nu) = W_1^1$. Fix A of ν measure one such that $g_0 \restriction A$ is monotonically increasing. Define $h(\alpha) = \sup\{\beta \in A : g_0(\beta) \leq \alpha\}$. Clearly $h : \omega_1 \to \omega_1$. Let x_0 be such that T_{x_0} is well-founded and $\forall^*_{W_1^1}\beta\ h(\beta) < |T_{x_0} \restriction \beta|$. Thus, $\forall^*_\nu \alpha\ \alpha < h(g_0(\alpha)) \leq |T_{x_0} \restriction g_0(\alpha)|$. Let $r_0 : \omega_1 \to \omega_1$ be such that $\forall^*_\nu \alpha\ r_0(\alpha) < g_0(\alpha)$ and $\forall^*_\nu \alpha\ [\alpha = |(T_{x_0} \restriction g_0(\alpha))(r_0(\alpha))|]$. Let $d_0 = d_1 = 1$, which are defined with respect to W_1^1. For α in the ν measure one set such that $r_0(\alpha) < g_0(\alpha)$ is defined, we define

$$\begin{aligned} h(g_0(\alpha); (W_1^1; x_0; d_0, d_1); r_0(\alpha)) &= |(T_{x_0} \restriction h(g_0(\alpha); d_0))(r_0(\alpha))| \\ &= |(T_{x_0} \restriction g_0(\alpha))(r_0(\alpha))|. \end{aligned}$$

We have thus produced a tuple $(\langle W_1^1; x_0; d_0, d_1 \rangle, g_0, r_0)$ satisfying the following:

(1) d_0, d_1 are defined with respect to W_1^1 and T_{x_0} is well-founded.

(2) $g_0(\nu) = W_1^1$.

(3) $\forall^*_\nu \alpha\ r_0(\alpha) < h(g_0(\alpha); d_1)$.

(4) $\forall^*_\nu \alpha\ [\alpha = h(g_0(\alpha); (W_1^1; x_0; d_0, d_1); r_0(\alpha))]$.

With this first step as motivation, we make the following definitions.

4.9 Definition. A *level*-1 *complex* is a tuple of the form

$$\mathcal{C} = \langle W_1^m; x_0, \ldots, x_{n-1}; d_0, \ldots, d_n \rangle$$

where $m \geq 1$, $n \in \omega$ (if $n = 0$, then no x_i appear). If $n \geq 1$, then each d_i is defined with respect to W_1^m for $0 \leq i \leq n$, $d_0 > d_1 > \cdots > d_{n-1} \geq d_n$, and each T_{x_i} is well-founded for $0 \leq i \leq n-1$.

If $\mathcal{C}$ as above is a complex with $n \geq 1$, $\gamma < \omega_1$, and $\beta_1 < \cdots < \beta_m < \omega_1$, define

$$h(\beta_1, \ldots, \beta_m; \mathcal{C}; \gamma) = |(T_{x_0} \restriction h(\vec{\beta}; d_0))(\alpha_1)|,$$

where $\alpha_1 = |(T_{x_1} \restriction h(\vec{\beta}; d_1))(\alpha_2)|, \ldots, \alpha_{n-1} = |(T_{x_{n-1}} \restriction h(\vec{\beta}; d_{n-1}))(\alpha_n)|$, and $\alpha_n = \gamma$. If $n = 0$, then we define $h(\vec{\beta}; \mathcal{C}; \gamma) = \gamma$.

Written out directly, the equation for h is:

$$h(\beta_1, \dots, \beta_m; \mathcal{C}; \gamma) = |T_{x_0} \restriction \beta_{d_0}(|T_{x_1} \restriction \beta_{d_1}(\dots |T_{x_{n-1}} \restriction \beta_{d_{n-1}}(\gamma)| \dots)).$$

The definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$ is actually only on the W_1^m measure one set of $\vec{\beta}$ such that $h(\vec{\beta}; d_0) > \alpha_1$, etc. Off this measure one set, we leave $h(\vec{\beta}; \mathcal{C}; \gamma)$ undefined. Note that the last description d_n is not used in the definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$; its role is to provide a bound for the r function in the following definition (the r stands for "remainder"—it represents, roughly speaking, the part of the measure that has not yet been analyzed).

We abstract the general step of the analysis into the following definition.

4.10 Definition. A *situation* for ν is a triple $(\mathcal{C}, g, r)$ satisfying the following:

(1) $\mathcal{C} = \langle W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ is a complex (defined immediately above).

(2) $g : \omega_1 \to (\omega_1)^m$, and $g(\nu) = W_1^m$.

(3) $r : \omega_1 \to \omega_1$, and $\forall^*_\nu \alpha \; r(\alpha) < h(g(\alpha); d_n)$.

(4) $\forall^*_\nu \alpha \; [\alpha = h(g(\alpha); \mathcal{C}; r(\alpha))]$.

Among all situations for ν, we now choose one with the minimal value for $[\alpha \to h(g(\alpha); d_n)]_\nu$ (i.e., minimizing the bound for the function r). We denote this situation by $(\mathcal{C}; g; r)$, where $\mathcal{C} = \langle W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$.

We claim that r is constant ν almost everywhere, that is, $\exists \gamma < \omega_1 \; \forall^*_\nu \alpha \; r(\alpha) = \gamma$. Granting this, it follows from (2) and (4) that $\nu(A) = 1$ iff $\exists$ c.u.b. $C \subseteq \omega_1 \; \forall \beta_1, \dots, \beta_m \in C \; h(\vec{\beta}; \mathcal{C}; \gamma) \in A$. This gives the theorem via a minor cosmetic change: if $m > n$ we replace W_1^m by W_1^n, and replace $d_0, \dots, d_{n-1}$ by $n, n-1, \dots, 1$, which gives the same measure (we eliminate the coordinates of W_1^m not used in the definition of $h(\vec{\beta}; \mathcal{C}; \gamma)$).

We consider the case where d_n is minimal, that is, $d_n = 1$, the other case being similar (in fact, as we remarked above, there is no loss of generality in assuming $m = n$ and $d_0, \dots, d_{n-1} = n, \dots, 1$, in which case $d_n = 1$). We consider two cases.

First assume that there is a ν measure one set A such that for $\alpha \in A$ there is a $\delta < h(g(\alpha); d_n)$ (that is, $\delta < \beta_1$, where $g(\alpha) = \beta_1, \dots, \beta_m$) such that for all $\alpha' \in A$ with $g(\alpha') = g(\alpha)$ we have $r(\alpha') < \delta$. Fix such a measure one set A, and by (3) we may assume $\forall \alpha \in A \; r(\alpha) < h(g(\alpha); d_n)$. By (2), let C be closed unbounded such that for all $\beta_1 < \dots < \beta_m \in C \; \exists \alpha \in A \; g(\alpha) = (\beta_1, \dots, \beta_m)$. Thus,

$$\forall \beta_1 < \dots < \beta_m \in C \; \exists \delta < \beta_1 \; \sup\{r(\alpha) : \alpha \in A \wedge g(\alpha) = (\beta_1, \dots, \beta_m)\} < \delta.$$

By normality, δ is constant on a W_1^m measure one set, and by countable additivity of ν, r is constant on a ν measure one set.

Suppose now that such a measure one set does not exist. It follows that for any A of ν measure one that $\forall^*_\nu \alpha \in A\ \forall \delta < \beta_1\ \exists \alpha' \in A\ [g(\alpha') = g(\alpha) \wedge r(\alpha') > \delta]$, where $g(\alpha) = (\beta_1, \ldots, \beta_m)$. Let $g' : \omega_1 \to \omega_1$ satisfy the following:

(1) $\forall^*_\nu \alpha\ g'(\alpha) < h(g(\alpha); d_n)$. In other words, $g'(\alpha) < \beta_1$, where $g(\alpha) = (\beta_1, \ldots, \beta_m)$.

(2) There is a ν measure one set A such that if $\alpha_1, \alpha_2 \in A$, $g(\alpha_1) = g(\alpha_2)$, and $r(\alpha_1) \leq r(\alpha_2)$, then $g'(\alpha_1) \leq g'(\alpha_2)$.

(3) For any A of ν measure one we have $\forall^*_\nu \alpha \in A\ \forall \delta < \beta_1\ \exists \alpha' \in A\ [g(\alpha') = g(\alpha) \wedge g'(\alpha') > \delta]$, where $g(\alpha) = (\beta_1, \ldots, \beta_m)$.

(4) If $[g'']_\nu < [g']_\nu$, then g'' does not satisfy (1)–(3).

g' exists, since r satisfies (1)–(3). Also, $g'(\nu) = W_1^1$, since if there were a closed unbounded C such that $\forall^*_\nu \alpha\ g'(\alpha) \notin C$, then $p \circ g'$ would violate the minimality of g', where $p(\alpha) =$ the largest element of C less than or equal to α. Let $\tilde{g}(\alpha) = g'(\alpha)^\frown g(\alpha)$, so $\tilde{g}(\nu) = W_1^{m+1}$. Fix a ν measure one set A witnessing (1) and (2) for g'. Consider the partition $\mathcal{P}$: we partition $\beta_0 < \delta < \beta_1 < \beta_2 < \cdots < \beta_m$ according to whether

$$\sup\{r(\alpha) : \alpha \in A \wedge g(\alpha) = (\beta_1, \ldots, \beta_m) \wedge g'(\alpha) \leq \beta_0\} < \delta.$$

Using (2) and (3) it follows that on the homogeneous side the stated property holds. Let $C \subseteq \omega_1$ be closed unbounded and homogeneous for $\mathcal{P}$, and let $N_C(\alpha) =$ the least element of C greater than α. Thus, $\forall^*_\nu \alpha\ r(\alpha) < N_C(g'(\alpha))$. Let x_n be such that T_{x_n} is well-founded and $\forall^*_{W_1^1} \beta\ N_C(\beta) < |T_{x_n} \restriction \beta|$. Define $r' : \omega_1 \to \omega_1$ so that $\forall^*_\nu \alpha\ r'(\alpha) < g'(\alpha)$ and $\forall^*_\nu \alpha\ [r(\alpha) = |(T_{x_n} \restriction g'(\alpha))(r'(\alpha))|$. Let $d'_0 = d_0 + 1, \ldots, d'_{n-1} = d_{n-1} + 1$ (to maintain the correspondence between the appropriate coordinates of W_1^m and W_1^{m+1}), and let $d'_n = d'_{n+1} = 1$. Let $\mathcal{C}' = \langle W_1^{m+1}; x_0, \ldots, x_n; d'_0, \ldots, d'_n, d'_{n+1}\rangle$. Note then that $\forall^*_\nu \alpha\ [\alpha = h(\tilde{g}(\alpha); \mathcal{C}'; r'(\alpha)]$. Since $\forall^*_\nu \alpha\ h(\tilde{g}(\alpha); d'_n) < h(g(\alpha); d_n)$, it follows that $(\mathcal{C}'; \tilde{g}; r')$ violates the minimality of $(\mathcal{C}; g; r)$, a contradiction. ⊣

4.2. The Strong Partition Relation on ω_1

We now convert this analysis of measures to a coding of the subsets of ω_1. As we mentioned before, this idea is due to Kunen. Throughout Sect. 4.2, T continues to denote the Kunen tree of Lemma 4.1.

First we code (enough) closed unbounded sets. If $\sigma \in \omega^\omega$, let

$$C_\sigma = \{\alpha < \omega_1 : \alpha > \omega \wedge \forall \beta < \alpha\ (T_\sigma \restriction \beta \text{ is well-founded } \wedge |T_\sigma \restriction \beta| < \alpha)\}.$$

For any σ, C_σ is a closed subset of ω_1, and if T_σ is well-founded then C_σ is also unbounded. Also, for all closed unbounded $C \subseteq \omega_1$, there is a $\sigma \in \omega^\omega$ such that T_σ is well-founded and $C_\sigma \subseteq C$. For we may choose σ so that for all infinite $\beta < \omega_1$, $|T_\sigma \restriction \beta| > N_C(\beta)$.

4.11 Definition. Suppose $\mathcal{C} = \langle W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ is a complex, T_σ is well-founded, and $\gamma < \omega_1$. If $n \geq 1$, we define (with h as in Definition 4.9)

$$S_{\sigma,\mathcal{C},\gamma} = \{h(\beta_1, \dots, \beta_m; \mathcal{C}; \gamma) : \vec{\beta} \in (C_\sigma)^m \wedge \gamma < \beta_1 \wedge h(\vec{\beta}; \mathcal{C}; \gamma) \geq \beta_m\}.$$

For $n = 0$ we define $S_{\sigma,\mathcal{C},\gamma} = \{\gamma\}$. We say $S \subseteq \omega_1$ is *simple* if it is of the form $S_{\sigma,\mathcal{C},\gamma}$ for some $\sigma, \mathcal{C}, \gamma$.

4.12 Theorem. *Every $A \subseteq \omega_1$ is a countable union of simple sets.*

Proof. Let $A \subseteq \omega_1$, and suppose the theorem fails for A. Let $\mathcal{I} \subseteq \mathcal{P}(A)$ denote the countably additive ideal of $I \subseteq A$ such that $I = \bigcup_i S_i$ is a countable union of simple sets $S_i \subseteq A$. Thus, $A \notin \mathcal{I}$. Also, $\mathcal{I}$ contains all singletons as every $\{\gamma\}$ is simple. By AD, every countably additive filter (dual to an ideal $\mathcal{I}$) on an ordinal $< \Theta$ (identified here with A) can be extended to a measure. [One way to see this: by the Coding Lemma, let $\pi : \omega^\omega \to \mathcal{I}$ be onto. For d a Turing degree, let $f(d) =$ least element of $A - \bigcup_{x \in d} \pi(x)$. If μ is the Martin measure on the degrees, then $f(\mu)$ is a measure on A with $f(\mu)(I) = 0$ for all $I \in \mathcal{I}$.]

Let ν be a measure on A extending the filter dual to $\mathcal{I}$. Since $\nu(A) = 1$, by Theorem 4.8 we have some simple set $S \subseteq A$ with $\nu(S) = 1$. This contradicts $S \in \mathcal{I}$. ⊣

We view each real z as coding countably many reals z_n, each of which codes reals σ_n, w_n, and a sequence $\mathcal{C}_n = \langle W_1^m; x_0, \dots, x_{t-1}; d_0, \dots, d_t \rangle$ which satisfies the definition of a complex except that we do not require the T_{x_i} to be well-founded (we call this a *partial complex*). To each z_n we associate a set A_{z_n} defined as follows. If $w_n \notin \mathrm{WO}$ (the set of codes for wellorderings of ω), we set $A_{z_n} = \emptyset$. Otherwise, set

$$\begin{aligned} \alpha \in A_{z_n} \longleftrightarrow\ & \exists \beta_1 < \cdots < \beta_m \leq \alpha \\ & [\beta_1 > |w_n| \wedge \{\forall i\ \beta_i \in C_{\sigma_n} \wedge\ \alpha = h(\vec{\beta}; \mathcal{C}_n; |w_n|)\}]. \end{aligned}$$

We define here $h(\vec{\beta}; \mathcal{C}_n; \gamma)$ for the partial complex $\mathcal{C}_n$ similarly to Definition 4.9: $h(\vec{\beta}; \mathcal{C}_n; \gamma) = |T_{x_0} \restriction h(\vec{\beta}; d_0)(\alpha_1)|$, where $\alpha_1 = |T_{x_1} \restriction h(\vec{\beta}; d_1)(\alpha_2)|$, etc., provided α_1 is in the well-founded part of $T_{x_0} \restriction h(\vec{\beta}; d_0)$, α_2 is in the well-founded part of $T_{x_1} \restriction h(\vec{\beta}; d_1)$, etc. If some α_{i+1} is not in the well-founded part of $T_{x_i} \restriction h(\vec{\beta}; d_i)$, we leave $h(\vec{\beta}; \mathcal{C}_n; \gamma)$ undefined. Thus, for all $z \in \omega^\omega$, the sets $A_{z_n} \subseteq \omega_1$ are defined. We set $A_z = \bigcup_{n \in \omega} A_{z_n}$.

The following theorem says that this coding is reasonable.

4.13 Theorem. *The coding $z \to A_z$ satisfies the following:*

(1) $\forall A \subseteq \omega_1\ \exists z\ A = A_z$.

(2) $\forall \alpha < \omega_1\ \{z : \alpha \in A_z\} \in \boldsymbol{\Delta}^1_1$.

Proof. (1) follows immediately from Theorem 4.12. (2) is a straightforward computation using the facts that $\forall \beta < \omega_1 \ \{\sigma : \beta \in C_\sigma\} \in \boldsymbol{\Delta}^1_1$, and $\forall \beta, \gamma, \delta < \omega_1 \ \{z : |(T_z \restriction \beta)(\gamma)| \leq \delta\} \in \boldsymbol{\Delta}^1_1$. ⊣

If we view functions $F : \omega_1 \to \omega_1$ as subsets of $(\omega_1)^2$, and use our coding above (identifying $(\omega_1)^2$ with ω_1), it is not quite good enough to witness that ω_1 is ω_1-reasonable. To get this, we must make a small modification to our coding, essentially modifying only the first step of the proof of Theorem 4.8. We sketch the changes that need to be made.

We code binary relations $F \subseteq \omega_1 \times \omega_1$ as follows. Every real z codes countably many reals z_n, each of which codes reals σ_n, w^1_n, w^2_n, and a partial complex $\mathcal{C}_n = \langle W^m_1; x_0, \dots, x_{t-1}; d_0, \dots, d_t \rangle$. Set

$$\begin{aligned}(\alpha, \beta) \in F_z \longleftrightarrow \exists n \ \big[& w^1_n, w^2_n \in \mathrm{WO} \wedge |w^1_n| < \alpha \wedge |w^2_n| < \alpha \\ & \wedge \exists \beta_1 < \cdots < \beta_m \leq \alpha \ \big[\beta_1 > \max\{|w^1_n|, |w^2_n\} \wedge \forall i \ \beta_i \in C_{\sigma_n} \\ & \wedge \alpha = h(\vec{\beta}; \mathcal{C}; |w^1_n|) \wedge \beta = h(\vec{\beta}; \mathcal{C}; |w^2_n|)\big]\big] \\ & \wedge \forall n' \in \omega \ \big[\{w^1_{n'}, w^2_{n'} \in \mathrm{WO} \wedge |w^1_{n'}| < \alpha \wedge |w^2_{n'}| < \alpha \wedge \\ & \exists \beta'_1 < \cdots < \beta'_m \leq \alpha \ \big[\beta'_1 > \max\{w^1_{n'}, w^2_{n'}\} \wedge \forall i \ \beta'_i \in C_{\sigma_{n'}} \\ & \wedge \alpha = h(\vec{\beta}'; \mathcal{C}; |w^1_{n'}|)\big]\big\} \to h(\vec{\beta}'; \mathcal{C}; |w^2_{n'}|) = \beta\big].\end{aligned}$$

The main difference now is that the β_i are required to be $\leq \alpha$ (rather than $\leq \max\{\alpha, \beta\}$). Note also that if $F_z(\alpha, \beta)$ and $F_z(\alpha, \beta')$, then $\beta = \beta'$.

The analog of Theorem 4.12 becomes:

4.14 Theorem. *For every function $F : \omega_1 \to \omega_1$, $F = F_z$ for some z.*

Proof. Fix $F : \omega_1 \to \omega_1$. Let $X = \{(\alpha, \beta) : \beta = F(\alpha)\}$. Let $\mathcal{I}$ be the countably additive ideal on X consisting of countable unions of sets I such that $I \subseteq S \subseteq F$ for some *simple* S, that is, $S = S_{\sigma, \mathcal{C}, \gamma_1, \gamma_2}$ for some complex $\mathcal{C}$, well-founded T_σ, and $\gamma_1, \gamma_2 < \omega_1$, where

$$\begin{aligned}&(\alpha, \beta) \in S_{\sigma, \mathcal{C}, \gamma_1, \gamma_2} \\ &\quad \longleftrightarrow \ \exists \beta_1 < \cdots < \beta_m \leq \alpha \ \big[\beta_1 > \max\{\gamma_1, \gamma_2\} \\ &\qquad \wedge \forall i (\beta_i \in C_{\sigma_n}) \wedge \alpha = h(\vec{\beta}; \mathcal{C}; \gamma_1) \wedge \beta = h(\vec{\beta}; \mathcal{C}; \gamma_2)\big].\end{aligned}$$

We finish as in Theorem 4.12 provided we show that every measure ν on X is generated by simple sets. To do this, we need only modify the first step in the proof of Theorem 4.8.

Let $\pi(\alpha, \beta) = \alpha$ be the projection onto the first coordinate. Let $\nu' = \pi(\nu)$. Define g_0 exactly as in Theorem 4.8, using ν'. Let A be a ν' measure one set on which g_0 is monotonically increasing, and define $h(\alpha) = \sup\{\max(\beta, \gamma) : \beta \in A \wedge \gamma = F(\beta) \wedge g_0(\beta) \leq \alpha\}$. Let T_{x_0} be well-founded and $\forall^*_{W^1_1} \gamma \ h(\gamma) < |T_{x_0} \restriction \gamma|$. Thus, $\forall^*_\nu (\alpha, \beta) \ \max(\alpha, \beta) < |T_{x_0} \restriction g_0(\alpha)|$. Let $r_0, s_0 : \omega_1 \to \omega_1$ be such that $\forall^*_\nu (\alpha, \beta) \ r_0(\alpha), s_0(\alpha) < g_0(\alpha)$ and

$$\forall^*_\nu (\alpha, \beta) \ [\alpha = |(T_{x_0} \restriction g_0(\alpha))(r_0(\alpha))| \wedge \beta = |(T_{x_0} \restriction g_0(\alpha))(s_0(\alpha))|].$$

The argument then proceeds as in Theorem 4.8. ⊣

This coding suffices for the strong partition relation on ω_1.

4.15 Theorem. *The coding $z \to F_z$ witnesses that ω_1 is reasonable relative to the pointclass $\mathbf{\Sigma}^1_1$.*

Proof. (1) and (2) in Definition 2.33 are immediate, and (3) is a straightforward computation using the definition of F_z. (4) follows from the fact that if $A \in \mathbf{\Sigma}^1_1$, $\alpha < \omega_1$ and $\forall z \in A\ \exists \beta\ F_z(\alpha, \beta)$, then there is a $\mathbf{\Sigma}^1_1$ relation $\prec$ such that for all $z_1, z_2 \in A$, $F_{z_1}(\alpha) < F_{z_2}(\alpha) \longleftrightarrow z_1 \prec z_2$. ⊣

4.16 Corollary. $\omega_1 \to (\omega_1)^{\omega_1}$.

We now obtain the lower bound for $\boldsymbol{\delta}^1_3$, again, using only techniques that will generalize. We use the following general theorem of Martin.

4.17 Theorem (Martin). *Assume $\kappa \to (\kappa)^\kappa$. Then for any measure ν on κ, the ultrapower $j_\nu(\kappa)$ is a cardinal.*

Proof. Toward a contradiction, fix ν such that $j_\nu(\kappa)$ is not a cardinal, and let $F : j_\nu(\kappa) \to \lambda$ be a bijection, where $\lambda < j_\nu(\kappa)$. Consider the partition $\mathcal{P}$: we partition $f, g : \kappa \to \kappa$ of the correct type with $f(\alpha) < g(\alpha) < f(\alpha + 1)$ according to whether $F([f]_\nu) < F([g]_\nu)$. It is clear by wellfoundedness that we cannot have a closed unbounded set homogeneous for the contrary side of the partition. Let $C \subseteq \kappa$ be closed unbounded and homogeneous for $\mathcal{P}$. Fix an ordinal θ with $\lambda < \theta < j_\nu(\kappa)$, and let $[h]_\nu = \theta$. Define $h' : \kappa \to C$ inductively by: $h'(\alpha) =$ the $\omega \cdot (h(\alpha) + 1)$st element in C greater than $\sup_{\beta<\alpha} h'(\beta)$. Since κ is regular, this is well defined. We then produce an order preserving map H from θ into λ, a contradiction. Namely, if $\delta < \theta$, let $[f_\delta]_\nu = \delta$, with $f_\delta < h$ everywhere. Define $f'_\delta : \kappa \to C$ by: $f'_\delta(\alpha) =$ the $\omega \cdot (f_\delta(\alpha) + 1)$st element of C greater than $\sup_{\beta<\alpha} h'(\beta)$. Then set $H(\delta) = F([f'_\delta]_\nu)$. It is easy to see that this is well-defined and order-preserving from θ into λ. ⊣

4.18 Corollary. $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.

Proof. Define $\pi : j_{W_1^m}(\omega_1) \to j_{W_1^{m+1}}(\omega_1)$ by: $\pi([f]_{W_1^m}) = [f']_{W_1^{m+1}}$ where $f'(\alpha_1, \ldots, \alpha_{m+1}) = f(\alpha_1, \ldots, \alpha_m)$. This gives an embedding from $j_{W_1^m}(\omega_1)$ into $[\mathrm{id}_{m+1}]_{W_1^{m+1}}$, where $\mathrm{id}_{m+1}(\alpha_1, \ldots, \alpha_{m+1}) = \alpha_{m+1}$. Thus, $j_{W_1^m}(\omega_1) < j_{W_1^{m+1}}(\omega_1)$. From Theorem 4.17 and the previous upper bound, it follows that $j_{W_1^n}(\omega_1) = \omega_{n+1}$. Using the Coding Lemma to code functions $f : (\omega_1)^m \to \omega_1$, and also our coding of closed unbounded sets, it is easy to compute the prewellordering corresponding to the ultrapower $j_{W_1^m}(\omega_1)$ as $\mathbf{\Delta}^1_3$ (in fact, a more careful computation shows that the prewellordering lies in the pointclass $\Game\, \omega \cdot m\text{–}\mathbf{\Pi}^1_1$). Thus, $\boldsymbol{\delta}^1_3 \geq \sup_m j_{W_1^m}(\omega_1) = \omega_\omega$. Since $\boldsymbol{\delta}^1_3$ is regular, $\boldsymbol{\delta}^1_3 \geq \omega_{\omega+1}$. ⊣

4.3. The Weak Partition Relation on $\boldsymbol{\delta}^1_3$

Starting from the weak partition relation on $\boldsymbol{\delta}^1_1 = \omega_1$, we have computed $\boldsymbol{\delta}^1_3$ and proved the strong partition relation on $\boldsymbol{\delta}^1_1$. To complete the cycle, we establish now the weak partition relation on $\boldsymbol{\delta}^1_3$. As we mentioned before, this is a result of Kunen (see [33]). Again, we wish to use only methods and terminology that will generalize. Nevertheless, the proof closely parallels Kunen's. An important concept which is introduced here is that of a *tree of uniform cofinalities*; this plays a central role in the general inductive analysis as well.

First, we analyze possible uniform cofinalities with respect to the measures W^m_1. This analysis is well known, but we emphasize that we use only the weak partition relation on ω_1 and the Kunen tree analysis.

4.19 Lemma. *Let $f : (\omega_1)^m \to \omega_1$, and assume $\forall^*_{W^m_1} \alpha_1, \ldots, \alpha_m\ f(\vec{\alpha})$ is a limit ordinal. Then either f has uniform cofinality ω almost everywhere with respect to W^m_1, or there is an i, $1 \leq i \leq m$, such that $f(\vec{\alpha})$ has uniform cofinality α_i almost everywhere. Also, each of these uniform cofinalities is distinct, that is, these cases are mutually exclusive.*

4.20 Remark. The uniform cofinalities other than ω are thus described by the descriptions d defined with respect to W^m_1. The lemma also holds for any $f : (\omega_1)^m \to \Theta$ assuming $\mathrm{AD} + V = L(\mathbb{R})$ (see Sect. 6).

Proof. Fix $f : (\omega_1)^m \to \omega_1$, and call a pair (S, l) a *liftup* to f provided $S : (\omega_1)^m \to \omega_1$, $l : \{(\alpha_1, \ldots, \alpha_m, \beta) : \beta < S(\vec{\alpha})\} \to \omega_1$ and $\forall^*_{W^m_1} \vec{\alpha}\ f(\vec{\alpha}) = \sup\{l(\vec{\alpha}, \beta) : \beta < S(\vec{\alpha})\}$. Fix a liftup (S, l) for which $[S]_{W^m_1}$ is minimal. If S is constant almost everywhere, then easily f has uniform cofinality ω. Let $1 \leq d \leq m$ be minimal so that $\forall^*_{W^m_1} \vec{\alpha}\ S(\vec{\alpha}) \leq h(\vec{\alpha}; d) = \alpha_d$. If equality holds almost everywhere we are done, as then $f(\vec{\alpha})$ has uniform cofinality α_d almost everywhere. Otherwise, there is an x with T_x well-founded such that $\forall^*_{W^m_1} \vec{\alpha}\ S(\vec{\alpha}) < |T_x \restriction \alpha_{d-1}|$ ($d > 1$ now). For almost all $\vec{\alpha}$, set $S'(\vec{\alpha}) = \alpha_{d-1}$, and for $\beta < \alpha_{d-1}$ define $l'(\vec{\alpha}, \beta) = \sup\{l(\vec{\alpha}, \gamma) : \gamma < |(T_x \restriction \alpha_{d-1})(\beta)|\}$ if $|(T_x \restriction \alpha_{d-1})(\beta)| < S(\vec{\alpha})$, and $= 0$ otherwise. Then (S', l') is a liftup to f with $[S']_{W^m_1} < [S]_{W^m_1}$, a contradiction. We leave the uniqueness proof to the reader. ⊣

We introduce some useful notation.

4.21 Definition. Suppose $\pi = (n, i_2, \ldots, i_n)$ is a permutation of $\{1, \ldots, n\}$ beginning with n. $<^\pi$ is the wellordering of $(\omega_1)^n$ defined by: $(\alpha_1, \ldots, \alpha_n)$ $<^\pi$ $(\beta_1, \ldots, \beta_n)$ iff $(\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_n}) <_{\mathrm{lex}} (\beta_n, \beta_{i_2}, \ldots, \beta_{i_n})$. We say an n-tuple of ordinals $(\gamma_1, \ldots, \gamma_n)$ has *type* π if it is order-isomorphic to π. By a *partial permutation* of n we mean a $\pi = (n, i_2, \ldots, i_m)$, $m \leq n$, which can be extended to a permutation. We likewise define the ordering $<^\pi$ on $(\omega_1)^n$ in this case by: $(\alpha_1, \ldots, \alpha_n) <^\pi (\beta_1, \ldots, \beta_n)$ iff $(\alpha_n, \alpha_{i_2}, \ldots, \alpha_{i_m}) <_{\mathrm{lex}}$ $(\beta_n, \beta_{i_2}, \ldots, \beta_{i_m})$. We identify $\vec{\alpha}, \vec{\beta} \in \mathrm{dom}(<^\pi)$ if $\alpha_n = \beta_n, \ldots, \alpha_{i_m} = \beta_{i_m}$.

Note that if $\pi = (n, i_2, \dots, i_m)$ is a partial permutation and $m < n$, then if $f : \mathrm{dom}(<^\pi) \to \omega_1$ is order-preserving, $f(\alpha_1, \dots, \alpha_n)$ depends only on $\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}$.

If $\pi = (n, i_2, \dots, i_k)$, $\pi' = (m, j_2, \dots, j_l)$ are partial permutations, we say π' *extends* π provided $m \geq n$, $l > k$, and $(m, j_2, \dots, j_k)$ is order-isomorphic to π. If, in addition, $l = k + 1$, we say π' is an *immediate extension* of π.

4.22 Definition. If $\pi = (n, i_2, \dots, i_n)$ is a permutation and $f : <^\pi \to \omega_1$ is order-preserving, we define the "mth invariant" $f(m) : (\omega_1)^m \to \omega_1$, for $m \leq n$ by:

$$f(m)(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}) \\ = \sup\{f(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}, \beta_{i_{m+1}}, \dots, \beta_{i_n}) : (\alpha_n, \dots, \beta_{i_n}) \text{ has type } \pi\}.$$

For example, if $\pi = (3, 1, 2)$, and $f : \mathrm{dom}(<_\pi) \to \omega_1$ is order-preserving, then $f(2)(\alpha_1, \alpha_3) = \sup\{f(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 < \alpha_2 < \alpha_3\}$. Recall our convention that for $g : (\omega_1)^2 \to \omega_1$ and $\alpha_1 < \alpha_3$, we write $g(\alpha_1, \alpha_3)$ interchangeably with $g(\alpha_3, \alpha_1)$, etc.

4.23 Lemma. *Let $f : (\omega_1)^n \to \omega_1$ with $\omega_n < [f]_{W_1^n}$. Then there is an $m \leq n$ and a partial permutation $\pi = (n, i_2, \dots, i_m)$ such that for some closed unbounded $C \subseteq \omega_1$ and any $\vec{\alpha}, \vec{\beta} \in (C)^n$, $f(\vec{\alpha}) < f(\vec{\beta})$ iff $(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_m}) <_{\mathrm{lex}} (\beta_n, \beta_{i_2}, \dots, \beta_{i_m})$.*

Proof. Suppose $(n, i_2, \dots, i_k)$ have been defined so that there is a closed unbounded $C \subseteq \omega_1$ satisfying

$$\forall \vec{\alpha}, \vec{\beta} \in (C)^n \ (\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_k}) <_{\mathrm{lex}} (\beta_n, \beta_{i_2}, \dots, \beta_{i_k}) \to f(\vec{\alpha}) < f(\vec{\beta}).$$

For each $i \in \{1, \dots, n\} - \{n, i_1, \dots, i_k\}$, consider the partition $\mathcal{P}_i$: we partition ordinals $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ with $\alpha_n = \beta_n, \dots, \alpha_{i_k} = \beta_{i_k}$, $\alpha_i < \beta_i < \alpha_{i+1}$, and $\alpha_{j-1} < \beta_j < \alpha_j$ for all other j. We partition according to whether $f(\vec{\alpha}) < f(\vec{\beta})$.

If all the partitions $\mathcal{P}_i$ are homogeneous for the contrary side, it is easy to see that on a closed unbounded set, if $(\alpha_n, \dots, \alpha_{i_k}) = (\beta_n, \dots, \beta_{i_k})$, then $f(\vec{\alpha}) = f(\vec{\beta})$, and we are done.

Otherwise, it is easy to see that there is a unique i such that $\mathcal{P}_i$ is homogeneous for the stated side. We may then extend $(n, i_2, \dots, i_k)$ to $(n, i_2, \dots, i_k, i_{k+1})$, setting $i_{k+1} = i$. Continuing, we establish the theorem. ⊣

Lemmas 4.19 and 4.23 completely analyze the "type" of a function $f : (\omega_1)^n \to \omega_1$. If $\pi = (n, i_2, \dots, i_m)$ is a partial permutation of n, we say $f : (\omega_1)^n \to \omega_1$ is of type π if f is order-preserving from $<^\pi$ to ω_1, of uniform cofinality ω, and is everywhere discontinuous, that is, for $\vec{\alpha} \in (\omega_1)^n$, $f(\vec{\alpha}) > \sup\{f(\vec{\beta}) : \vec{\beta} <^\pi \vec{\alpha}\}$. We say that f is of type (π, s) if f is order-preserving from $<^\pi$ and for $\vec{\alpha} \in (\omega_1)^n$ of limit rank in $<^\pi$, $f(\vec{\alpha}) = \sup\{f(\vec{\beta}) :$

$\vec{\beta} <^{\pi} \vec{\alpha}\}$ (and for $\vec{\alpha}$ of successor rank, $f(\vec{\alpha})$ has uniform cofinality ω). Finally, if $\pi' = (n+1, j_2, \ldots, j_{m+1})$ is a partial permutation extending π, we say that f is of type (π, π') if f is order-preserving with respect to $<^{\pi}$, everywhere discontinuous, and $f(\alpha_{n+1}, \alpha_{j_2}, \ldots, \alpha_{j_m})$ has uniform cofinality $\{\beta : (\alpha_{n+1}, \alpha_{j_2}, \ldots, \alpha_{j_m}, \beta)$ has type $\pi'\}$ if this set has limit order-type (and otherwise has uniform cofinality ω). We say that f has type π, (π, s) or (π, π') almost everywhere if there is a closed unbounded C such that $f\restriction(C)^n$ is order-preserving with respect to $<^{\pi}$, and $f\restriction(C)^n$ has the appropriate uniform cofinality and continuity properties.

Lemmas 4.19 and 4.23 consequently say that if $f : (\omega_1)^n \to \omega_1$, $[f] > \omega_n$, and $\forall^*_{W_1^n} \vec{\alpha}\ f(\vec{\alpha})$ is a limit ordinal, then f has type π, (π, s), or (π, π') almost everywhere, for some partial permutation(s) π, π'.

The next lemma is simple but important. It says we may change the values of functions on measure zero sets so that they are everywhere ordered correctly.

4.24 Lemma (Sliding Lemma). *Suppose $f : (\omega_1)^m \to \omega_1$, $g : (\omega_1)^n \to \omega_1$ have types $\pi_1 = (m, i_2, \ldots, i_k)$, $\pi_2 = (n, j_2, \ldots, j_l)$ almost everywhere respectively. Suppose $r \leq \min(k, l)$ is such that $[f(r-1)] = [g(r-1)]$, but $[f(r)] < [g(r)]$. Then there are f', g' of types π_1, π_2 with $[f'] = [f]$, $[g'] = [g]$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$, and such that for all $\vec{\alpha} \in (\omega_1)^m$, $\vec{\beta} \in (\omega_1)^n$, $g(\vec{\beta}) > f(\vec{\alpha})$ iff $(\beta_n, \beta_{j_2}, \ldots, \beta_{j_r}) \geq_{\mathrm{lex}} (\alpha_m, \alpha_{i_2}, \ldots, \alpha_{i_r})$.*

Proof. Note that $(m, i_2, \ldots, i_r)$, $(n, j_2, \ldots, j_r)$ are order-isomorphic, say to the permutation $\pi = (r, k_2, \ldots, k_r)$, by uniqueness of the uniform cofinality and the fact that $[f(r-1)] = [g(r-1)]$. Let $C_1 \subseteq \omega_1$ be closed unbounded such that $f\restriction(C_1)^m$ is order-preserving with respect to $<^{\pi_1}$ and of uniform cofinality ω, and similarly for g. Let $l : \omega_1 \to \omega_1$, and $C \subseteq C_1$ be closed unbounded and closed under l such that

$$\begin{aligned}\forall \vec{\alpha} \in (C)^r\ f(r)(\alpha_r, \alpha_{k_2}, \ldots, \alpha_{k_r}) &< g(r)(\alpha_r, \alpha_{k_2}, \ldots, \alpha_{k_r}) \\ &< f(r)(\alpha_r, \alpha_{k_2}, \ldots, l(\alpha_{k_r})).\end{aligned}$$

Let $p(\alpha) =$ the $\omega \cdot \alpha$th element of C. Define

$$f'(\alpha_1, \ldots, \alpha_m) = f(p(\alpha_1), \ldots, p(\alpha_m)),$$

and similarly

$$g'(\alpha_1, \ldots, \alpha_n) = g(p(\alpha_1), \ldots, p(\alpha_n)).$$

It is easy to check that f', g' have the desired properties. ⊣

A useful special case of the lemma is that if $f : (\omega_1)^n \to \omega_1$ has type π almost everywhere, then there is an f' of type π with $[f'] = [f]$ and $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$. A small variation of the argument shows this is also true for functions f of type (π, s) or (π, π') almost everywhere.

We now define the notion of a tree of uniform cofinalities, which plays an important role in the projective hierarchy analysis. The concept is similar to that of a homogeneous tree. Roughly speaking, for each node in this tree we allow as extensions all possible uniform cofinalities with respect to the measure which is associated to that node (these uniform cofinalities in turn define new measures).

In the following definition, we let $(s), (\omega)$ be two formal symbols (the first stands for "sup", and the second for "uniform cofinality ω").

4.25 Definition. A *type-1* tree of uniform cofinalities (of depth n) is a function $\mathcal{R}$ satisfying the following:

(1) $\langle p_1, i_1\rangle \in \mathrm{dom}(\mathcal{R})$ for $0 \le i_1 \le a$ for some integer a, and $p_1 =$ the unique permutation of length 1, namely $p_1 = (1)$. For $i_1 = 0$, $\mathcal{R}(\langle p_1, i_1\rangle) = (s)$, and for $i_1 > 0$, $\mathcal{R}(\langle p_1, i_1\rangle)$ is either (ω), or a permutation p_2 of length 2 (hence $p_2 = (2, 1)$). Also, $\langle p_1, i_1\rangle$ is maximal in $\mathrm{dom}(\mathcal{R})$ iff $\mathcal{R}(\langle p_1, i_1\rangle) = (s)$ or (ω).

(2) In general, $\mathrm{dom}(\mathcal{R})$ consists of tuples $\langle p_1, i_1, \dots, i_{m-1}, p_m, i_m\rangle$, $m \le n$, and such a tuple is maximal in $\mathrm{dom}(\mathcal{R})$ iff $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) = (s)$ or (ω) (these are the only values permitted, therefore, if $m = n$). $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) = (s)$ iff $i_m = 0$. If $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) \ne (s)$ or (ω), then $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle)$ is a permutation p_{m+1} immediately extending p_m. In this case, $(\langle p_1, i_1, \dots, p_m, i_m, p_{m+1}, i_{m+1}\rangle) \in \mathrm{dom}(\mathcal{R})$ for some integers $0 \le i_{m+1} \le a$ ($a \ge 0$ and depends on $\langle p_1, i_1, \dots, p_m, i_m, p_{m+1}\rangle$).

For $\mathcal{R}$ a type-1 tree of uniform cofinalities, we define $<^{\mathcal{R}}$ to be the lexicographic ordering on sequences $\langle \alpha_1, i_1, \dots, i_{m-1}, \alpha_m, i_m\rangle$, $m \le n$, satisfying:

(1) $\alpha_1, \dots, \alpha_m < \omega_1$.

(2) $(\alpha_1, \dots, \alpha_m)$ is of type p_m, where $(p_1, \dots, p_m)$ is the unique sequence such that $\langle p_1, i_1, \dots, p_m, i_m\rangle \in \mathrm{dom}(\mathcal{R})$.

We say a sequence $\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle$ with $(\alpha_1, \dots, \alpha_m)$ as in (2) is of type $\langle p_1, i_1, \dots, p_m, i_m\rangle$.

To each tree of uniform cofinalities $\mathcal{R}$ we associate a measure $M^{\mathcal{R}}$. First, say a function $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of *type* $\mathcal{R}$ if it is order-preserving, $f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle)$ has uniform cofinality ω if $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) = (\omega)$ or $\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle$ has successor rank in $<^{\mathcal{R}}$ (for $\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle$ having type $\langle p_1, i_1, \dots, p_m, i_m\rangle$), and otherwise

$$f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle) = \sup\{f(\vec{s}) : \vec{s} <^{\mathcal{R}} \langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle\}.$$

Define A to have measure one with respect to $M^{\mathcal{R}}$ iff $\exists$ c.u.b. $C \subseteq \omega_1$ $\forall f : \mathrm{dom}(<^{\mathcal{R}}) \to C$ of type $\mathcal{R}$ $(\dots, \alpha^{\langle p_1, i_1, \dots, p_m, i_m\rangle}, \dots) \in A$, where $\alpha^{\langle p_1, i_1, \dots, p_m, i_m\rangle}$ is represented with respect to W_1^m by:

$$f^{\langle p_1, i_1, \dots, p_m, i_m\rangle}(\alpha_1, \dots, \alpha_m) = f(\langle \alpha_1, i_1, \dots, \alpha_m, i_m\rangle).$$

(Recall our notational convention; the $\alpha_1, \dots, \alpha_m$ here are not written in increasing order).

Thus, each $M^{\mathcal{R}}$ is a measure on $(\omega_\omega)^{<\omega}$, and these measures generalize somewhat the measures occurring in the homogeneous tree construction on a $\boldsymbol{\Pi}^1_2$-complete set. We note that the requirement that $\langle p_1, i_1, \dots, p_m, i_m\rangle$ be non-maximal in $\mathrm{dom}(\mathcal{R})$ if $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) \neq (s)$ or (ω) causes no essential loss of generality as far as specifying functions of type $\mathcal{R}$ is concerned. For suppose $\mathcal{R}(\langle p_1, i_1, \dots, p_m, i_m\rangle) = p_{m+1}$, where p_{m+1} is the permutation $(m+1, j_2, \dots, j_m, j_{m+1})$. Thus, p_m is order-isomorphic to $(m+1, j_2, \dots, j_m)$. Suppose $f : \mathrm{dom}(<^{p_m}) \to \omega_1$ is order-preserving, f is discontinuous, that is, for $(\alpha_1, \dots, \alpha_m)$ of type p_m we have

$$f(\alpha_1, \dots, \alpha_m) > \sup\{f(\vec{\beta}) : \vec{\beta} <^{\pi} \vec{\alpha}\},$$

and for W^m_1 almost $(\alpha_1, \dots, \alpha_m)$ of type p_m we have $f(\vec{\alpha})$ has uniform cofinality $\{\beta : (\alpha_1, \dots, \alpha_m, \beta) \text{ has type } p_{m+1}\}$. By definition, there is an

$$f' : \{(\alpha_1, \dots, \alpha_m, \beta) : (\alpha_1, \dots, \alpha_m, \beta) \text{ has type } p_{m+1}\} \to \omega_1$$

order-preserving with respect to $<^{p_{m+1}}$, so that $f'(\vec{\alpha}, \beta) = \sup_{\beta' < \beta} f'(\vec{\alpha}, \beta')$ for limit β and f' induces f in that $f(\vec{\alpha}) = \sup_\beta f'(\vec{\alpha}, \beta)$ almost everywhere. Furthermore, it is not difficult to see that $[f']_{W_1^{m+1}}$ is uniquely determined from $[f]_{W_1^m}$. This shows that adding $\langle p_1, i_1, \dots, p_m, i_m, p_{m+1}, 0\rangle$ to $\mathrm{dom}(\mathcal{R})$ does no harm.

If $\mathcal{R}$ is a type-1 tree of uniform cofinalities, and $f_1, f_2 : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ are of type $\mathcal{R}$, we write $[f_1] = [f_2]$ to mean that for all $\langle p_1, i_1, \dots, p_k, i_k\rangle \in \mathrm{dom}(\mathcal{R})$ we have $[f_1^{\langle p_1, i_1, \dots, p_k, i_k\rangle}]_{W_1^k} = [f_2^{\langle p_1, i_1, \dots, p_k, i_k\rangle}]_{W_1^k}$.

We say the type-1 tree of uniform cofinalities $\mathcal{R}'$ *extends* $\mathcal{R}$ if there is a length preserving injection $\rho : \mathrm{dom}(\mathcal{R}) \to \mathrm{dom}(\mathcal{R}')$ such that $\mathcal{R}(\vec{s}) = \mathcal{R}'(\rho(\vec{s}))$ for all $\vec{s} \in \mathrm{dom}(\mathcal{R})$. We usually just say $\vec{s} \in \mathrm{dom}(\mathcal{R})$ is "identified" with $\rho(\vec{s}) \in \mathrm{dom}(\mathcal{R}')$. We say $\mathcal{R}'$ is an *immediate extension* of $\mathcal{R}$ is there is exactly one sequence $\vec{s}$ not ending in 0 in $\mathrm{dom}(\mathcal{R}') - \mathrm{dom}(\mathcal{R})$.

We say $\mathcal{R}'$ is a *partial extension* of $\mathcal{R}$ if $\mathcal{R}'$ has a unique extra sequence $\vec{s} = \langle q_1, j_1, \dots, q_l, j^*_l\rangle$ in $\mathrm{dom}(\mathcal{R}')$, and $j^*_l \neq 0$. More precisely, this means that $\langle q_1, j_1, \dots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R})$ for $0 \le j_l \le a$ and we have $\langle q_1, j_1, \dots, q_l, j_l\rangle \in \mathrm{dom}(\mathcal{R}')$ for $0 \le j_l \le a+1$. For $j_l < j^*_l$, we identify $\langle q_1, j_1, \dots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}})$ with $\langle q_1, j_1, \dots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}'})$, and for $j_l \ge j^*_l$, we identify $\langle q_1, j_1, \dots, q_l, j_l\rangle \in \mathrm{dom}(<^{\mathcal{R}})$ with $\langle q_1, j_1, \dots, q_l, j_l + 1\rangle \in \mathrm{dom}(<^{\mathcal{R}'})$. We leave $\mathcal{R}'(\vec{s})$ undefined (more formally, in order to still have $\vec{s} \in \mathrm{dom}(\mathcal{R}')$, we require $\mathcal{R}'(\vec{s})$ to be a formal symbol (u) for "undefined"). We define $<^{\mathcal{R}'}$ as for immediate extensions. Thus, after identifying $\mathrm{dom}(<^{\mathcal{R}})$ with a subset of $\mathrm{dom}(<^{\mathcal{R}'})$ we have

$$\mathrm{dom}(<^{\mathcal{R}'}) = \mathrm{dom}(<^{\mathcal{R}}) \cup \{\langle \alpha_1, j_1, \dots, \alpha_l, j^*_l\rangle : (\alpha_1, \dots, \alpha_l) \text{ has type } q_l\}.$$

Finally, we say that $g' : \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ is of *semi-type* $\mathcal{R}'$ if g' is order-preserving with respect to $<^{\mathcal{R}'}$, and the function $g : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ it induces by restriction is of type $\mathcal{R}$. Note that g' being of semi-type $\mathcal{R}'$ imposes no restriction on the uniform cofinality of the component function $g'^{\langle q_1, j_1, \dots, q_l, j_l^* \rangle}$.

Lemma 4.24 generalizes to level-1 trees of uniform cofinalities as follows.

4.26 Lemma. *Let $\mathcal{R}$ be a level-1 tree of uniform cofinalities, and suppose $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$. Suppose $\vec{s} = \langle p_1, i_1, \dots, p_k, i_k \rangle$ is non-maximal in $\mathrm{dom}(\mathcal{R})$, and $\mathcal{R}(\vec{s}) = p_{k+1}$. Suppose $\delta \in \mathrm{On}$ and for W_1^{k+1} almost all $(\alpha_1, \dots, \alpha_{k+1})$ of type p_{k+1} we have $\delta(\vec{\alpha}) < f(\langle \alpha_1, i_1, \dots, \alpha_k, i_k \rangle)$. Let i_{k+1}^* be maximal such that $\langle p_1, i_1, \dots, p_{k+1}, i_{k+1}^* \rangle \in \mathrm{dom}(\mathcal{R})$. Let $\mathcal{R}'$ be the partial extension of $\mathcal{R}$ with extra sequence $\langle p_1, i_1, \dots, p_{k+1}, i_{k+1}^* + 1 \rangle$ in its domain. Then there is an f' of semi-type $\mathcal{R}'$ such that:*

(1) $[f' \restriction \mathrm{dom}(<^{\mathcal{R}})] = [f]$. *That is, for each $\langle q_1, j_1, \dots, q_l, j_l \rangle \in \mathrm{dom}(\mathcal{R})$, $\forall^*_{W_1^l}(\alpha_1, \dots, \alpha_l)\ f'(\langle \alpha_1, q_1, \dots, \alpha_l, q_l \rangle) = f(\langle \alpha_1, q_1, \dots, \alpha_l, q_l \rangle)$.*

(2) $\mathrm{ran}(f') \subseteq \mathrm{ran}(f) \cup (\mathrm{ran}(f))'$.

(3) $\forall^*_{W_1^{k+1}}(\alpha_1, \dots, \alpha_{k+1})\ \delta(\vec{\alpha}) < f'(\langle \alpha_1, i_1 \dots, \alpha_{k+1}, i_{k+1}^* \rangle)$.

Proof. Fix a representing function $\vec{\alpha} \to \delta(\vec{\alpha})$ for δ, and fix $b : \omega_1 \to \omega_1$ and a closed unbounded $C \subseteq \omega_1$ closed under b such that for all $(\alpha_1, \dots, \alpha_{k+1})$ of type p_{k+1} in C, $\delta(\vec{\alpha}) < f(\langle \alpha_1, i_1, \dots, \alpha_k, i_k, b(\alpha_{k+1}), 0 \rangle)$. Let D be the set of closure points of C, and define $l(\alpha) = \alpha$th element of D. For $\langle \alpha_1, j_1, \dots, \alpha_l, j_l \rangle \in \mathrm{dom}(<^{\mathcal{R}})$, define

$$f'(\langle \alpha_1, j_1, \dots, \alpha_l, j_l \rangle) = f(\langle l(\alpha_1), j_1, \dots, l(\alpha_l), j_l \rangle).$$

Define also

$$f'(\langle \alpha_1, i_1, \dots, i_k, \alpha_{k+1}, i_{k+1}^* \rangle) = f(\langle l(\alpha_1), i_1, \dots, l(\alpha_k), i_k, \beta, 0 \rangle),$$

where β is the ωth element of C greater than $l(\alpha_{k+1})$. It is easy to check that f' satisfies (1)–(3) above, and also satisfies all of the requirements for being of type $\mathcal{R}$ except for the requirement that $f'(\vec{s})$ have uniform cofinality ω for $\vec{s}$ in certain measure zero sets. It is easy, however, to redefine $f'(\vec{s})$ at these points to guarantee this last requirement. ⊣

4.27 Definition. A *generalized trivial* (or *type-1*) *description* defined relative to a measure W_1^m and a type-1 tree of uniform cofinalities $\mathcal{R}$ is a sequence $d = \langle d_1, i_1, \dots, d_k, i_k \rangle$ where each d_i is a trivial description defined relative to W_1^m (i.e., $1 \leq d_i \leq m$), and for some (uniquely determined) $\langle p_1, i_1, \dots, p_k, i_k \rangle \in \mathrm{dom}(\mathcal{R})$ we have p_k is order-isomorphic to $(d_1, d_2, \dots, d_k)$. We order the generalized descriptions lexicographically.

We extend the interpretation function h to generalized descriptions. If $d = \langle d_1, i_1, \dots, d_k, i_k \rangle$ is defined relative to W_1^m and $\mathcal{R}$, $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ is of type $\mathcal{R}$, and $\beta_1 < \dots < \beta_m$, then we define

$$h(f; \vec{\beta}; d) = f(\langle h(\vec{\beta}; d_1), i_1, \dots, h(\vec{\beta}; d_k), i_k \rangle).$$

We extend also the lowering operation $\mathcal{L}$ to generalized descriptions as follows.

4.28 Definition. Suppose $d = \langle d_1, i_1, \ldots, d_k, i_k \rangle$ is defined relative to W_1^m and $\mathcal{R}$. Let $\langle p_1, i_1, \ldots, p_k, i_k \rangle \in \mathrm{dom}(\mathcal{R})$, where p_k is order-isomorphic to $(d_1, \ldots, d_k)$. We define $\mathcal{L}(d)$ through the following cases:

(1) If $\mathcal{R}(\langle p_1, i_1, \ldots, p_k, i_k \rangle) = (\omega)$, $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k - 1 \rangle$ (note here that $i_k > 0$).

(2) If $i_k = 0$, then $\mathcal{L}(d) = \langle d_1, i_1, \ldots, \mathcal{L}(d_k), i_k^* \rangle$ provided $\mathcal{L}(d_k)$ is defined (i.e., $d_k > 1$) and $(d_1, \ldots, \mathcal{L}(d_k))$ is order-isomorphic to p_k. Here i_k^* is maximal such that $\langle p_1, i_1, \ldots, p_k, i_k^* \rangle \in \mathrm{dom}(\mathcal{R})$. Otherwise, set $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_{k-1}, i_{k-1} - 1 \rangle$ (note that $i_{k-1} > 0$) provided $k > 1$. For $k = 1$ in this case (and thus $d = \langle d_1, 0 \rangle$, with $d_1 = 1$), we declare d to be $\mathcal{L}$-minimal, and do not define $\mathcal{L}(d)$.

(3) Suppose now that $i_k > 0$ and $\mathcal{R}(\langle p_1, i_1, \ldots, p_k, i_k \rangle) = p_{k+1}$. Let $1 \leq j \leq k$ be such that if $\alpha_{d_j - 1} < \beta < \alpha_{d_j}$, then $(\alpha_{d_1}, \ldots, \alpha_{d_k}, \beta)$ is order-isomorphic to p_{k+1}. If $d_{k+1}^* \doteq d_j - 1 \notin \{d_1, \ldots, d_k\}$, set $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k, d_{k+1}^*, i_{k+1}^* \rangle$, where i_{k+1}^* is maximal such that

$$\langle p_1, i_1, \ldots, p_k, i_k, p_{k+1}, i_{k+1}^* \rangle \in \mathrm{dom}(\mathcal{R}).$$

If $d_j - 1 = 0$ or $d_j - 1 \in \{d_1, \ldots, d_k\}$, set $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k - 1 \rangle$.

The significance of this definition is embodied in the following lemma.

4.29 Lemma. *Suppose $d = \langle d_1, i_1, \ldots, d_k, i_k \rangle$ is defined relative to $W_1^m, \mathcal{R}$, and $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ has type $\mathcal{R}$. Let $\delta \in \mathrm{On}$ be such that $\forall^*_{W_1^m} \vec{\beta}\ \delta(\vec{\beta}) < h(f; \vec{\beta}; d)$. Suppose d is non-minimal with respect to $\mathcal{L}$. Then there is an $f' : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f'] = [f]$ and $\forall^*_{W_1^m} \vec{\beta}\ \delta(\vec{\beta}) < N_{f'}(h(f'; \vec{\beta}; \mathcal{L}(d)))$.*

Proof. The result follows easily from Lemma 4.26 in all cases. For example, suppose $d = \langle d_1, i_1, \ldots, d_k, i_k \rangle$, and $\mathcal{L}(d) = \langle d_1, i_1, \ldots, d_k, i_k, d_{k+1}^*, i_{k+1}^* \rangle$, where $d_{k+1}^* = d_j - 1$. Thus,

$$\forall^*_{W_1^m} \vec{\alpha}\ \exists \beta < \alpha_{d_j}\ \ \delta(\vec{\alpha}) < f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, \beta, 0 \rangle).$$

Thus, there is an $l : \omega_1 \to \omega_1$ such that

$$\forall^*_{W_1^m} \vec{\alpha}\ \delta(\vec{\alpha}) < f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, l(\alpha_{d_j - 1}), 0 \rangle).$$

From Lemma 4.26 there is an f' of type $\mathcal{R}$ with $[f'] = [f]$ (and in fact with $\mathrm{ran}(f') \subseteq \mathrm{ran}(f) \cup \mathrm{ran}(f)'$), and

$$\begin{aligned} \forall^* \vec{\alpha}\ & N_{f'}(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, \alpha_{d_j j - 1}, i_{k+1}^* \rangle) \\ & > f(\langle h(\vec{\alpha}; d_1), i_1, \ldots, h(\vec{\alpha}; d_k), i_k, l(\alpha_{d_j - 1}), 0 \rangle). \end{aligned}$$

f' is as desired. ⊣

4.30 Definition. A *level-2 complex* $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ is a sequence where $\mathcal{R}$ is a type-1 tree of uniform cofinalities, $T_{x_0}, \dots, T_{x_{n-1}}$ are well-founded, and $d_0, \dots, d_n$ are generalized trivial descriptions defined relative to W_1^m.

For $\mathcal{C}$ a complex as above, $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, $\vec{\beta} \in (\omega_1)^m$, and $\gamma < \omega_1$, we define $h(f; \vec{\beta}; \mathcal{C}; \gamma) = |T_{x_0} \restriction h(f; \vec{\beta}; d_0)(\alpha_1)|$, where $\alpha_1 = |T_{x_1} \restriction h(f; \vec{\beta}; d_1)(\alpha_2)|, \dots, \alpha_{n-1} = |T_{x_{n-1}} \restriction h(f; \vec{\beta}; d_{n-1})(\alpha_n)|$, and $\alpha_n = \gamma$. For f of type $\mathcal{R}$, we let $h(f; W_1^m; \mathcal{C}; \gamma) < \omega_{m+1}$ be the ordinal represented with respect to W_1^m by the function $\vec{\beta} \to h(f; \vec{\beta}; \mathcal{C}; \gamma)$. Note that d_n is not used in the definition of $h(f; \vec{\beta}; \mathcal{C}; \gamma)$ (it plays a role in the definition of a situation below).

The following theorem analyzes the measures on ω_ω.

4.31 Theorem. *Let ν be a measure on ω_ω. Then there is a measure μ on ω_1 and a complex $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ such that for all $A \subseteq \omega_\omega$:*

$$\begin{aligned} \nu(A) = 1 \longleftrightarrow {} & \forall^*_\mu \gamma \; \exists \text{ c.u.b. } C \subseteq \omega_1 \\ & \forall f : \mathrm{dom}(<^{\mathcal{R}}) \to C \text{ of type } \mathcal{R} \; [h(f; W_1^m; \mathcal{C}; \gamma) \in A]. \end{aligned}$$

Proof. The proof is similar to that of Theorem 4.8. Fix the measure ν, and let m be least such that $\nu(\omega_{m+1}) = 1$. We may assume $m \geq 1$.

4.32 Definition. A *situation* for ν is a triple $(\mathcal{C}, g, r)$ consisting of a complex $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$ (same m as above), and functions g, r with domain ω_{m+1} satisfying:

(1) $g : \omega_{m+1} \to \omega_{m+1}^{<\omega}$ and $g(\nu) = M^{\mathcal{R}}$, the measure associated with $\mathcal{R}$.

(2) $r : \omega_{m+1} \to \omega_{m+1}$ and $\forall^*_\nu \alpha \; r(\alpha) < h(g(\alpha); W_1^m; d_n)$.

(3) $\forall^*_\nu \alpha \; [\alpha = h(g(\alpha); W_1^m; \mathcal{C}; r(\alpha))]$.

Note that in (2) and (3), when we write $h(g(\alpha); W_1^m; d_n)$, for example, we mean $h(g; W_1^m; d_n)$ where $g : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ represents $g(\alpha)$.

We fix now a situation $(\mathcal{C}; g; r)$ for ν with minimal value for

$$[\alpha \to h(g(\alpha); W_1^m; d_n)]_\nu,$$

where $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$.

We claim that $\forall^*_\nu \alpha \; r(\alpha) < \omega_1$. Granting this, let $\mu = r(\nu)$. Let ν' be the measure defined by:

$$\begin{aligned} \nu'(A) = 1 \longleftrightarrow {} & \forall^*_\mu \gamma \; \exists \text{ c.u.b. } C \subseteq \omega_1 \\ & \forall f : \mathrm{dom}(<^{\mathcal{R}}) \to C \text{ of type } \mathcal{R} \; [h(f; W_1^m; \mathcal{C}; \gamma) \in A]. \end{aligned}$$

We show that $\nu' = \nu$. Suppose not, and let $\nu'(A) = 1$, $\nu(A^c) = 1$. By the ω_2-additivity of $M^{\mathcal{R}}$ (which follows from an easy partition argument) fix a μ measure one set $D \subseteq \omega_1$, and a closed unbounded $C \subseteq \omega_1$ such that if $\gamma \in D$ and $f : \text{dom}(<^{\mathcal{R}}) \to C$ is of type $\mathcal{R}$, then $h(f; W_1^m; \mathcal{C}; \gamma) \in A$. Since $g(\nu) = M^{\mathcal{R}}$ and $r(\nu) = \mu$, there is an $\alpha \in A^c$ such that $r(\alpha) \in D$ and $g(\alpha)$ is representable by $f : \text{dom}(<^{\mathcal{R}}) \to C$ of type $\mathcal{R}$, and such that $\alpha = h(f; W_1^m; \mathcal{C}; r(\alpha))$. However, $h(f; W_1^m; \mathcal{C}; r(\alpha)) \in A$, a contradiction.

To fix notation, say $d_n = \langle d_1^n, i_1, \ldots, d_k^n, i_k\rangle$. We may assume d_n is non-minimal with respect to $\mathcal{L}$, since otherwise $\forall^*_\nu \alpha\, r(\alpha) < h(g(\alpha); W_1^m; d_n) = \omega_1$. We proceed to violate the minimality of $(\mathcal{C}; g; r)$. From Lemma 4.29 we have that $\forall^*_\nu \alpha\ \exists f : \text{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and $\forall^*_{W_1^m} \vec{\beta}\ r(\alpha)(\vec{\beta}) < N_f(h(f; \vec{\beta}; \mathcal{L}(d_n))$.

Define a partial extension $\mathcal{R}'$ of $\mathcal{R}$ as follows. If

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k - 1\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_k^n, i_k\rangle$ in its domain. We identify $\langle d_1^n, \ldots, i\rangle \in \text{dom}(\mathcal{R})$ with $\langle d_1^n, \ldots, i\rangle \in \text{dom}(\mathcal{R}')$ for $i < i_k$, and with $\langle d_1^n, \ldots, i+1\rangle \in \text{dom}(\mathcal{R}')$ for $i \geq i_k$. If $i_k = 0$ and

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, \mathcal{L}(d_k^n), i_k^*\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, \mathcal{L}(d_k^n), i_k^* + 1\rangle$ in its domain. If $i_k = 0$ and

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_{k-1}^n, i_{k-1} - 1\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_{k-1}^n, i_{k-1}\rangle$ in its domain. Finally, if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*, i_{k+1}^*\rangle,$$

then $\mathcal{R}'$ has the extra sequence $\langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*, i_{k+1}^*+1\rangle$ in its domain. Thus, in all cases the extra sequence is inserted according to Lemma 4.29. In all cases, let d_n^* be the generalized description corresponding to the extra sequence in the partial complex $\mathcal{R}'$. For example, if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k - 1\rangle,$$

then $d_n^* = d_n = \langle d_1^n, i_1, \ldots, d_k^n, i_k\rangle$, and if

$$\mathcal{L}(d_n) = \langle d_1^n, i_1, \ldots, d_k^n, i_k, d_{k+1}^*, i_{k+1}^*\rangle,$$

then $d_n^* = \langle d_1^n, i_1, \ldots, d_{k+1}^*, i_{k+1}^* + 1\rangle$.

To unify notation, let $\vec{s} = \langle q_1, j_1, \ldots, q_l, j_l\rangle$ be the extra sequence in $\text{dom}(\mathcal{R}') - \text{dom}(\mathcal{R})$. We therefore have: $\forall^*_\nu \alpha\ \exists f' : \text{dom}(\mathcal{R}') \to \omega_1$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and $\forall^*_{W_1^m} \vec{\beta}\ r(\alpha)(\vec{\beta}) < N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))$. As in Theorem 4.8 we consider two cases.

First suppose there is a ν measure one A such that for all $\alpha \in A$ there is a $f' : \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ of semi-type $\mathcal{R}'$ inducing (by restriction) the function $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ such that $[f] = g(\alpha)$ and

$$\forall \alpha' \in A \ \big[(g(\alpha') = g(\alpha)) \to \forall^*_{W_1^m} \vec{\beta} \ r(\alpha')(\vec{\beta}) \leq N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))\big].$$

Consider the partition $\mathcal{P}$ where we partition f' of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$, and where $f'(\langle \alpha_1, j_1, \ldots, \alpha_l, j_l \rangle)$ has uniform cofinality ω, according to whether

$$\forall \alpha' \in A \ \big[(g(\alpha') = [f]) \to \forall^*_{W_1^m} \vec{\beta} \ r(\alpha')(\vec{\beta}) \leq N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))\big].$$

We easily have, using Lemma 4.24, that on the homogeneous side the stated property holds. Fix $C \subseteq \omega_1$ homogeneous for $\mathcal{P}$. Let T_{x_n} be well-founded and $\forall^*_{W_1^m} \beta \ |T_{x_n} \restriction \beta| > N_C(\beta)$. Then $\forall^*_\nu \alpha \ \exists f$ of type $\mathcal{R}$ with $[f] = g(\alpha)$ and $\forall^*_{W_1^m} \vec{\beta} \ r(\alpha)(\vec{\beta}) < |T_{x_n} \restriction h(f; \vec{\beta}; \mathcal{L}(d_n))|$. Define r' by: $\forall^*_\nu \alpha$ if $[f] = g(\alpha)$, then

$$\forall^*_{W_1^m} \vec{\beta} \ r(\alpha)(\vec{\beta}) = |(T_{x_n} \restriction h(f; \vec{\beta}; \mathcal{L}(d_n)))(r'(\alpha)(\vec{\beta}))|.$$

Let $\mathcal{C}' = \langle \mathcal{R}; W_1^m; x_0, \ldots, x_{n-1}, x_n; d_0, \ldots, d_{n-1}, \mathcal{L}(d_n), \mathcal{L}(d_n) \rangle$. It follows that $(\mathcal{C}'; g; r')$ violates the minimality of the original situation.

Suppose next that such a measure one set does not exist. Let g' satisfy the following:

(1) $\forall^*_\nu \alpha \ g'(\alpha)$ is represented by an $f' : \mathrm{dom}(<^{\mathcal{R}'}) \to \omega_1$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ representing $g(\alpha)$.

(2) There is a ν measure one set A such that if $\alpha_1, \alpha_2 \in A$, $g(\alpha_1) = g(\alpha_2)$, and $r(\alpha_1) \leq r(\alpha_2)$, then $[f_1'^{\langle q_1, j_1, \ldots, q_l, j_l \rangle}]_{W_1^l} \leq [f_2'^{\langle q_1, j_1, \ldots, q_l, j_l \rangle}]_{W_1^l}$. Here f_1', f_2' represent $g'(\alpha_1)$, $g'(\alpha_2)$.

(3) There does not exist a ν measure one set A such that $\forall \alpha \in A \ \exists f'$ of semi-type $\mathcal{R}'$ inducing f of type $\mathcal{R}$ representing $g(\alpha)$ and

$$\forall \alpha' \in A \ [(g(\alpha') = g(\alpha)) \to [f_1'^{\langle q_1, j_1, \ldots, q_l, j_l \rangle}]_{W_1^l} \leq [f'^{\langle q_1, j_1, \ldots, q_l, j_l \rangle}]_{W_1^l}.$$

Here f_1' represents $g'(\alpha)$.

(4) For all $[g'']_\nu < [g']_\nu$, g'' does not satisfy (1)–(3).

Easily g' is well-defined [to satisfy (1)–(3), let $g'(\alpha)$ be least such that it is representable by an f' of semi-type $\mathcal{R}$ inducing f representing $g(\alpha)$ and such that $\forall^*_{W_1^m} \vec{\beta} \ r(\alpha)(\vec{\beta}) \leq N_{f'}(h(f; \vec{\beta}; \mathcal{L}(d_n)))$]. From (4) it follows by a pressing down argument that for any closed unbounded $C \subseteq \omega_1$, $\forall^*_\nu \alpha \ g'(\alpha)$ is represented by some $f' : \mathrm{dom}(<^{\mathcal{R}'}) \to C$ of semi-type $\mathcal{R}'$. By countable additivity of ν, there is an immediate extension $\mathcal{R}''$ of $\mathcal{R}$ extending $\mathcal{R}'$ (that is, $\mathcal{R}''(\vec{s})$ is now defined) such that $\forall^*_\nu \alpha \ g'(\alpha)$ is representable by f'

of type $\mathcal{R}''$. Fix a ν measure one set A on which (1) and (2) above hold. Consider the partition $\mathcal{P}$: we partition f' of type $\mathcal{R}''$ with the "extra values" $h(\langle \alpha_1, j_1, \dots, \alpha_l, j_l\rangle)$, for $(\alpha_1, \dots, \alpha_l)$ of type q_l, of uniform cofinality ω inserted between $f'(\langle \alpha_1, j_1, \dots, \alpha_l, j_l\rangle)$ and $N_{f'}(f'(\langle \alpha_1, j_1, \dots, \alpha_l, j_l\rangle))$ according to whether

$$\forall \alpha' \in A\ [(g'(\alpha') = [f']) \to \forall^*_{W_1^m} \vec{\beta}\ r(\alpha')(\vec{\beta}) \leq N_{f'}(f'; \vec{\beta}; d^*_n)].$$

From (2), (3) and the definition of A it follows that on the homogeneous side the stated property holds. Fix $C \subseteq \omega_1$ homogeneous for $\mathcal{P}$, and x_n with T_{x_n} well-founded such that $\forall^*_{W_1^m} \beta\ |T_{x_n} \restriction \beta| > N_C(\beta)$. Define r' so that

$$\forall^*_\nu \alpha\ r'(\alpha) < h(f'; W_1^m; d^*_n)$$

and

$$\forall^*_\nu \alpha\ \forall^*_{W_1^m} \vec{\beta}\ r(\alpha)(\vec{\beta}) = |(T_{x_n} \restriction h(f'; \vec{\beta}; d^*_n))(r'(\alpha)(\vec{\beta}))|,$$

where $[f'] = g'(\alpha)$. Let

$$\mathcal{C}' = \langle \mathcal{R}''; W_1^m; x_0, \dots, x_n; \pi(d_1), \dots, \pi(d_{n-1}), d^*_n, d^*_n \rangle\,,$$

where $\pi(d_i)$ is the generalized description defined relative to $\mathcal{R}''$ corresponding to d_i defined relative to $\mathcal{R}$. Then $(\mathcal{C}'; g'; r')$ violates the minimality of the original situation. $\dashv$

From Theorem 4.31, a suitable coding for the subsets of ω_ω follows, and thus the weak partition relation on $\boldsymbol{\delta}^1_3$. Since the details are now almost identical to those of Theorem 4.12, we merely sketch the results.

If $\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n\rangle$ is a level-2 complex, $B \subseteq \omega_1$, and $\sigma \in \omega^\omega$ codes the closed unbounded set $C_\sigma \subseteq \omega_1$, let $S_{\sigma,\mathcal{C},B} \subseteq \omega_{m+1}$ be the corresponding *simple* set defined by:

$$\alpha \in S_{\sigma,\mathcal{C},B} \longleftrightarrow \exists \gamma \in B\ \exists f : \mathrm{dom}(<^{\mathcal{R}}) \to C_\sigma \text{ of type } \mathcal{R}\ [\alpha = h(f; W_1^m; \mathcal{C}; \gamma)].$$

Theorem 4.31 then shows that every $A \subseteq \omega_{m+1}$ is a countable union of simple sets.

We define our coding $z \to A_z \subseteq \omega_\omega$ as follows. Every real z codes countably many reals z_n, each of which codes a real σ_n, a set $B_n \subseteq \omega_1$, and a sequence $\mathcal{C}_n = \langle \mathcal{R}_n; W_1^m; x_0, \dots, x_{t-1}; d_0, \dots, d_t\rangle$ (here m, t depend on n) satisfying the definition of a complex, except we do not require the T_{x_i} to be well-founded (the exact manner in which B_n is coded is not important; we could use the coding of Theorem 4.13, or simply use the Coding Lemma). For each $n \in \omega$, define $A_{z_n} \subseteq \omega_{m(n)+1}$ as follows. If σ_n does not code a closed unbounded set, or one of the T_{x_i} is ill-founded, set $A_{z_n} = \emptyset$. Otherwise, $A_{z_n} = S_{\sigma_n, \mathcal{C}_n, B_n}$. Then set $A_z = \bigcup_{n \in \omega} A_{z_n}$.

4.33 Theorem. *The coding $z \to A_z \subseteq \omega_\omega$ satisfies the following:*

(1) $\forall A \subseteq \omega_\omega\ \exists z\ A = A_z$.

(2) $\forall \alpha < \omega_\omega\ \{z : \alpha \in A_z\} \in \boldsymbol{\Delta}^1_3$.

Proof. The computation in (2) is straightforward using the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions and intersections, and the fact that if $A_{\beta_1,\dots,\beta_k}$ are $\mathbf{\Delta}^1_3$ sets for all $\vec{\beta} \in (\omega_1)^k$, then $\{x : \forall^*_{W_1^k} \vec{\beta}\ x \in A_{\vec{\beta}}\} \in \mathbf{\Delta}^1_3$ (this last fact is an easy computation using our coding of closed unbounded sets). ⊣

As a corollary we obtain the following result, due originally to Kunen.

4.34 Theorem. *For all* $\lambda < \boldsymbol{\delta}^1_3$, $\boldsymbol{\delta}^1_3 \to (\boldsymbol{\delta}^1_3)^\lambda$.

Proof. Fix $\lambda < \boldsymbol{\delta}^1_3$, and a bijection $\pi : \omega_\omega \to \lambda$. Fix the coding $z \to A_z \subseteq (\omega_\omega)^3$ satisfying (1), (2) above (identifying $(\omega_\omega)^3$ and ω_ω). Define $z \to R_z \subseteq \lambda \times \boldsymbol{\delta}^1_3$ as follows. If $\pi(\alpha) = \delta$, set $R_z(\delta, \epsilon) \longleftrightarrow \{(\beta,\gamma) : A_z(\alpha,\beta,\gamma)\}$ is a wellordering of length ϵ. From the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions and intersections, it is easy to see that the coding $z \to R_z$ satisfies (3) in the definition of reasonable, Definition 2.33. Theorem 4.33 also implies that there is a $\mathbf{\Delta}^1_3$ coding of the ordinals (i.e., singleton sets) less than ω_ω. That is, there is a map $x \to |x| < \omega_\omega$ from ω^ω onto ω_ω such that for all $\alpha < \omega_\omega$, $\{x : |x| = \alpha\} \in \mathbf{\Delta}^1_3$. [In Definition 4.36 below we define a better $\mathbf{\Delta}^1_3$ coding of ω_ω via code sets WO_m for the ordinals less than ω_m.] From the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions it follows that $\{(x,z) : |x| \in A_z\} \in \mathbf{\Delta}^1_3$. From this, it follows that (4) in the definition of reasonable is satisfied, since if $S \subseteq \{z : R_z$ is well-founded$\}$ is $\mathbf{\Sigma}^1_3$, then we get a $\mathbf{\Sigma}^1_3$ well-founded relation on ω^ω of length $\geq \sup\{|R_z| : z \in S\}$. ⊣

As another consequence of the analysis of measures we obtain the following result, also due originally to Kunen.

4.35 Theorem. *Let* $\alpha, \beta < \boldsymbol{\delta}^1_3$, *and* μ *a measure on* μ. *Then* $j_\mu(\beta) < \boldsymbol{\delta}^1_3$.

Proof (Sketch). We may assume $\alpha = \beta = \omega_\omega$. We use the coding of subsets of $\omega_\omega \times \omega_\omega$ given by Theorem 4.33. It is enough to show that the prewellordering $\preceq$ corresponding to the ultrapower relation, that is,

$$\begin{aligned} x \preceq y \longleftrightarrow\ & (x, y \text{ code functions } f_x, f_y : \omega_\omega \to \omega_\omega) \\ & \wedge\ \forall^*_\mu \alpha\ \exists \beta_1, \beta_2 < \omega_\omega\ (f_x(\alpha) = \beta_1 \wedge f_y(\alpha) = \beta_2 \wedge (\beta_1 \leq \beta_2)) \end{aligned}$$

is $\mathbf{\Delta}^1_3$. Using the closure of $\mathbf{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ length unions and intersections, we see that it suffices to show that if $\{B_\alpha\}_{\alpha < \boldsymbol{\delta}^1_3}$ is a sequence of $\mathbf{\Delta}^1_3$ sets, then $B \doteq \{x : \forall^*_\mu \alpha\ (x \in B_\alpha)\}$ is also $\mathbf{\Delta}^1_3$. It clearly suffices to show that B_μ is $\mathbf{\Sigma}^1_3$. Let

$$\mathcal{C} = \langle \mathcal{R}; W_1^m; x_0, \dots, x_{n-1}; d_0, \dots, d_n \rangle$$

be a level-2 complex, and μ_1 a measure on ω_1, which generate μ, as in Theorem 4.31. Let $y_0, \dots, y_n$ be reals with T_{y_j} well-founded for all j, and $\epsilon < \omega_1$ which generate μ_1 as in Theorem 4.8 (ϵ playing the role of α there). T is again the Kunen tree. Recall our coding of closed unbounded subsets of ω_1 from Sect. 4.2. We then have

$$\begin{aligned} x \in B \longleftrightarrow \exists \sigma_1, \sigma_2\ (&T_{\sigma_1}, T_{\sigma_2} \text{ are well-founded} \\ &\wedge \forall \gamma_1 < \cdots < \gamma_n \in C_{\sigma_1}\ \forall \vec{\delta} \in (\omega_\omega)^{<\omega}\ \forall \eta < \omega_\omega \text{ [If } \vec{\delta} \text{ is} \\ &\text{representable by an } f : \mathrm{dom}(<_{\mathcal{R}}) \to \omega_1 \text{ of the correct type,} \\ &\text{each } f^{\langle p_1, i_1, \ldots, p_k, i_k \rangle} \text{ has range in } C_{\sigma_2} \text{ almost everywhere,} \\ &\text{and } h(f; W_1^m; \mathcal{C}; \gamma') = \eta, \text{ then } x \in B_\eta]) \end{aligned}$$

where h is as in Theorem 4.31, and $\gamma' = h^{\epsilon}_{y_0,\ldots,y_n}(\gamma_1, \ldots, \gamma_n)$ as in Theorem 4.8. Note that $h(f; W_1^m; \mathcal{C}; \gamma')$ depends only on $\vec{\delta}$. From the closure properties of $\boldsymbol{\Delta}^1_3$ again, it is enough to show that if $C_{\alpha_1,\ldots,\alpha_l} \in \boldsymbol{\Delta}^1_3$ for all $\vec{\alpha} \in (\omega_1)^l$, then C defined by

$$z \in C \longleftrightarrow \forall^*_{W_1^l} \vec{\alpha}\ (z \in C_{\vec{\alpha}})$$

is also $\boldsymbol{\Delta}^1_3$. This special case now follows easily by the same type of computation, using just level-1 complexes. ⊣

Starting from the weak partition relation on $\boldsymbol{\delta}^1_1 = \omega_1$, we have obtained the strong partition relation on $\boldsymbol{\delta}^1_1$, calculated $\boldsymbol{\delta}^1_3$, and obtained the weak partition relation on $\boldsymbol{\delta}^1_3$. This completes the first step in the inductive projective hierarchy analysis. We have used only techniques that will generalize (when combined with a suitable notion of description at the higher levels). We will sketch how this generalization takes place in Sect. 5.

4.4. The Kechris-Martin Theorem Revisited

We finish this section by using our theory of "trivial descriptions" to give a proof of the Kechris-Martin Theorem for $\boldsymbol{\Pi}^1_3$ sets. This is an important result in descriptive set theory, although it and its higher level analogs are not needed for the inductive analysis of the projective sets. The proof we give follows closely the proof of Kechris and Martin, recast into the theory of trivial descriptions (their original proof appealed to the theory of indiscernibles for L). We assume AD throughout this section. We caution the reader that we will be using lightface notions in this section.

To state the theorem, we first introduce our coding for the ordinals $< \omega_\omega$. T continues to denote the Kunen tree from Lemma 4.1 (or Theorem 4.2).

4.36 Definition. $\mathrm{WO}_1 = \mathrm{WO} =$ the standard set of codes of wellorderings of ω. For $m \geq 1$,

$$\mathrm{WO}_{m+1} = \{\langle a, x_1, \ldots, x_m \rangle : a \in \mathrm{WO}_1 \wedge \forall i \leq m\ T_{x_i} \text{ is well-founded}\}.$$

For $y = \langle a, x_1, \dots, x_m\rangle \in \mathrm{WO}_{m+1}$, let $|y| = [f_y]_{W_1^m}$, where $f_y : (\omega_1)^m \to \omega_1$ is defined by:

$$\begin{aligned} f_y(\beta_1, \dots, \beta_m) &= |(T_{x_m} \restriction \beta_m)(\delta_{m-1})|, \\ &\text{where } \delta_{m-1} = |(T_{x_{m-1}} \restriction \beta_{m-1})(\delta_{m-2})|, \quad \dots, \\ &\delta_1 = |(T_{x_1} \restriction \beta_1)(\delta_0)|, \quad \text{and} \quad \delta_0 = |a|. \end{aligned}$$

Let $\mathrm{WO}_\omega = \bigcup_m \mathrm{WO}_m$.

Easily, $\mathrm{WO}_{m+1} \in \boldsymbol{\Pi}^1_2$ for all $m \geq 1$, and for all $\alpha < \omega_{m+1}$ there is a $y \in \mathrm{WO}_{m+1}$ with $|y| = \alpha$.

4.37 Definition. We say a relation $R \subseteq \omega^\omega \times \mathrm{WO}_{m+1}$, $m \geq 0$, is *invariant in the codes* if

$$\forall x, w_1, w_2\ [w_1, w_2 \in \mathrm{WO}_{m+1} \wedge |w_1| = |w_2| \wedge R(x, w_1) \to R(x, w_2)].$$

In this case, we write $R(x, \alpha)$, for $\alpha < \omega_{m+1}$, to denote $\exists w \in \mathrm{WO}_{m+1}\ [|w| = \alpha \wedge R(x, w)]$. We similarly define R being invariant in the codes for $R \subseteq \mathrm{WO}_m$, or $R \subseteq \omega^\omega \times \mathrm{WO}_m \times \mathrm{WO}_n$, etc.

4.38 Theorem (Kechris-Martin). *Let $R \subseteq \omega^\omega \times \mathrm{WO}_{m+1}$, $m \geq 0$ be Π^1_3 and invariant in the codes. Then $P(x) \longleftrightarrow \exists w \in \mathrm{WO}_{m+1}\ R(x, w)$ is also Π^1_3.*

For the sake of completeness we include first the $m = 0$ case, though the proof is unchanged here (cf. [15]). So, let $R \subseteq \omega^\omega \times \mathrm{WO}$ be Π^1_3 and invariant in the codes. We show that $P(x) \longleftrightarrow \exists w\ R(x, w) \longleftrightarrow \exists w \in \Delta^1_3(x)\ R(x, w)$, which suffices (cf. [31, Theorem 4D.3]). So fix x such that $P(x)$. Let $S(w) \longleftrightarrow w \in \mathrm{WO} \wedge \forall w' \in \Delta^1_3(w)\ [|w'| \leq |w| \to \neg R(x, w')]$. By "bounded quantification" [31, Theorem 4D.3] $S \in \Sigma^1_3(x)$. Clearly S is invariant in the codes and codes a proper initial segment of ω_1. Relativizing to x, it suffices to show the following claim.

4.39 Claim. *If $S \subseteq \mathrm{WO}$ is Σ^1_3, invariant in the codes, and* $\sup\{|w| : w \in S\} = \alpha_0 < \omega_1$, *then $\exists w^* \in \Delta^1_3 \cap \mathrm{WO}\ (|w^*| > \alpha_0)$.*

Proof. Let $S(w) \longleftrightarrow \exists z\ B(w, z)$, where $B \in \Pi^1_2$. Consider the integer game where I plays out w_1, z, and II plays out w_2. II wins iff $w_2 \in \mathrm{WO} \wedge [B(w_1, z) \to |w_2| > |w_1|]$. This is a Σ^1_2 game for II, and II clearly wins, so by third periodicity II has a Δ^1_3 winning strategy τ. Then $A \doteq \tau(\omega^\omega) \subseteq \mathrm{WO}$ is $\Sigma^1_1(\tau)$, so there is a $\Delta^1_1(\tau)$ real $w^* \in \mathrm{WO}$ with $|w^*| > \sup\{|w| : w \in A\} \geq \sup\{|w| : w \in S\}$. Since $\tau \in \Delta^1_3$, $w^* \in \Delta^1_3$. ⊣

A useful consequence of the $m = 0$ case which we shall need is the following lemma.

4.40 Lemma. *Let $R \subseteq \omega^\omega \times \mathrm{WO}$ be Σ^1_3 (Π^1_3, Δ^1_3) and invariant in the codes. Then $P(x) \longleftrightarrow \forall^*_{W^1_1} \alpha\ R(x, \alpha)$ is Σ^1_3 (Π^1_3, Δ^1_3).*

Proof. Suppose, for example, $R \in \Sigma^1_3$. We use a variation of our previous coding of closed unbounded subsets of ω_1. For any closed unbounded $C \subseteq \omega_1$, there is a strategy σ for II such that $\forall z \in \mathrm{WO}\ \sigma(z) \in \mathrm{WO}$ and $C(\sigma) \doteq \{\alpha < \omega_1 : \forall z \in \mathrm{WO}\ (|z| < \alpha \to |\sigma(z)| < \alpha)\}$ is a closed unbounded subset of C. This follows by playing a simple Solovay game (I plays x, II plays y, and II wins iff $(x \in \mathrm{WO} \to y \in \mathrm{WO}\ \wedge\ |y| > N_C(|x|))$.

Then

$$\begin{aligned} P(x) \longleftrightarrow \exists\sigma\ \big[&\forall w \in \mathrm{WO}\ (\sigma(w) \in \mathrm{WO}) \\ &\wedge \forall w \in \mathrm{WO}\ (\forall z(z \in \mathrm{WO} \wedge |z| < |w| \\ &\to |\sigma(z)| < |w|) \to R(x,w))\big]. \end{aligned}$$

From the $m = 0$ case, $P \in \Sigma^1_3$. From this, the result for Π^1_3, Δ^1_3 follows immediately. ⊣

We recall our coding $z \to F_z \subseteq (\omega_1)^2$ of Theorem 4.15, which we will need for the proof. Recall each real z codes countably many z_n, each of which codes reals σ_n, w^1_n, w^2_n, and a partial (level-1) complex

$$\mathcal{C}_n = \langle W^m_1; x_0, \ldots, x_{t-1}; d_0, \ldots, d_t \rangle.$$

As we noted previously, each F_z is a partial function.

The next lemma summarizes the properties of this coding we will need.

4.41 Lemma. *Consider the relations defined by:*

$$\begin{aligned} R_0(z) &\longleftrightarrow \forall\beta\ \exists\gamma\ F_z(\beta,\gamma) \\ R_1(z,y) &\longleftrightarrow y \in \mathrm{WO} \wedge \exists\gamma\ F_z(|y|,\gamma) \\ R_2(z,y) &\longleftrightarrow y \in \mathrm{WO} \wedge \forall\beta \le |y|\ \exists\gamma\ F_z(\beta,\gamma) \\ R_3(z,x,y) &\longleftrightarrow x,y \in \mathrm{WO} \wedge \forall\beta \le |x|\ \exists\gamma \le |y|\ F_z(\beta,\gamma). \end{aligned}$$

Then $R_0 \in \Pi^1_2$, and $R_1, R_2 \in \Pi^1_1$. Also, R_3 is Δ^1_1 in the codes for x, y, that is, there are Σ^1_1, Π^1_1 relations C, D such that for all z and $x, y \in \mathrm{WO}$, $R_3(z,x,y) \longleftrightarrow C(z,x,y) \longleftrightarrow D(z,x,y)$.

Proof. The computations are all straightforward, as in the proof of the strong partition relation. For example, (in this equation, t, x_i refer to $\mathcal{C}_n$, and t', x'_i refer to $\mathcal{C}_{n'}$)

$$\begin{aligned} R_1(z,y) \longleftrightarrow\ & y \in \mathrm{WO} \wedge \exists n\ \big\{w^1_n, w^2_n \in \mathrm{WO} \wedge |w^1_n|, |w^2_n| < |y| \\ & \wedge \exists\beta_{t-1} < \cdots < \beta_0 \le |y|\ \exists\gamma_{t-1}, \ldots, \gamma_1 < |y|\ \exists\delta_{t-1}, \ldots, \delta_1 < |y| \\ & \big[\beta_{t-1} > \max(|w^1_n|, |w^2_n|) \wedge \forall i\ \beta_i \in C_{\sigma_n} \\ & |(T_{x_{t-1}} \restriction \beta_{t-1})(|w^1_n|)| = \gamma_{t-1} \wedge |(T_{x_{t-2}} \restriction \beta_{t-2})(\gamma_{t-1})| = \gamma_{t-2} \\ & \wedge \cdots \wedge |(T_{x_0} \restriction \beta_0)(\gamma_1)| = |y| \wedge |(T_{x_{t-1}} \restriction \beta_{t-1})(|w^2_n|)| = \delta_{t-1} \end{aligned}$$

$$\begin{aligned}
&\wedge |(T_{x_{t-2}} \restriction \beta_{t-2})(\delta_{t-1})| = \delta_{t-2} \wedge \cdots \\
&\wedge \delta_1 \text{ is in the well-founded part of } (T_{x_0} \restriction \beta_0) \\
&\wedge \forall n' \in \omega \big[\big\{ w^1_{n'}, w^2_{n'} \in \mathrm{WO} \wedge |w^1_{n'}|, |w^2_{n'}| < |y| \\
&\wedge \exists \beta'_{t'-1} < \cdots < \beta'_0 \leq |y| \ \exists \gamma'_{t'-1}, \ldots, \gamma'_1 < |y| \\
&\exists \delta'_{t'-1}, \ldots, \delta'_1 < |y| (\forall i \ \beta'_i \in C_{\sigma_{n'}} \\
&|(T_{x'_{t'-1}} \restriction \beta'_{t'-1})(|w^1_{n'}|)| = \gamma'_{t'-1} \\
&\wedge |(T_{x'_{t'-2}} \restriction \beta'_{t'-2})(\gamma'_{t'-1})| = \gamma'_{t'-2} \wedge \cdots \wedge |(T_{x'_0} \restriction \beta'_0)(\gamma'_1)| = |y| \\
&\wedge |(T_{x'_{t'-1}} \restriction \beta'_{t'-1})(|w^2_{n'}|)| = \delta'_{t'-1} \\
&\wedge |(T_{x'_{t'-2}} \restriction \beta'_{t'-2})(\delta'_{t'-1})| = \delta'_{t'-2} \\
&\wedge \cdots \wedge |(T_{x'_1} \restriction \beta'_1)(\delta'_2)| = \delta'_1) \big\} \\
&\rightarrow |(T_{x'_0} \restriction \beta'_0)(\delta'_1)| = |(T_{x_0} \restriction \beta_0)(\delta_1)| \big] \big] \big\}.
\end{aligned}$$

In the last line of this formula, "$|(T_{x'_0} \restriction \beta'_0)(\delta'_1)| = |(T_{x_0} \restriction \beta_0)(\delta_1)|$" abbreviates "$\delta'_1$ is in the well-founded part of $T_{x'_0} \restriction \beta'_0$ and $|(T_{x'_0} \restriction \beta'_0)(\delta'_1)| = |(T_{x_0} \restriction \beta_0)(\delta_1)|$". It is straightforward to check that all clauses in this formula define Δ^1_1 relations except the clauses "$y \in \mathrm{WO}$" and "δ_1 is in the well-founded part of $T_{x_0} \restriction \beta_0$", which are Π^1_1. ⊣

We will write "F_z is a function" in place of $R_0(z)$, "$F_z(|y|)$ is defined" in place of $R_1(z, y)$.

4.42 Lemma. *The relation* $Q(x, z) \longleftrightarrow [x \in \mathrm{WO}_2 \wedge (F_z$ *is a function*$) \wedge |x| = [F_z]_{W^1_1}]$ *is* Δ^1_3.

Proof. $Q(x, z) \longleftrightarrow \exists \sigma \ [T_\sigma$ is well-founded $\wedge \forall w \in \mathrm{WO}(|w| \in C_\sigma \rightarrow \exists v \in \mathrm{WO} \ (f_x(|w|) = |v| \wedge F_z(|w|, |v|)))]$. The $m = 0$ case shows $Q \in \Sigma^1_3$. A similar computation shows that $Q^c \in \Sigma^1_3$. ⊣

If V is a tree on $\omega \times \omega_1$ and $x \in \omega^\omega$, let $\prec_x$ denote the Kleene-Brouwer ordering on V_x. Thus, $\prec_x$ is a linear ordering, and is a wellorder iff V_x is well-founded. We say $\alpha < \omega_1$ is *represented in the well-founded part of* $V_x \restriction \beta$ if there is an $s \in V_x \restriction \beta$ such that the initial segment determined by s in the Kleene-Brouwer ordering on $V_x \restriction \beta$ is order-isomorphic to α. For the trees used below this will be equivalent to saying that the initial segment of $\prec_x$ determined by s is order-isomorphic to α.

4.43 Lemma. *Let* $R \subseteq \omega^\omega \times \omega^\omega$ *be* Π^1_2. *Then there is a tree* V *on* $\omega \times \omega_1$ *such that:*

(1) V *is* Δ^1_1 *in the codes.*

(2) $\forall x, y \in \omega^\omega$ *[*$R(x, y) \longleftrightarrow V_{\langle x,y \rangle}$ *is well-founded* $\longleftrightarrow \forall \alpha < \omega_1$ *(*α *is represented in the well-founded part of* $V_{\langle x,y \rangle} \restriction \alpha$*)].*

(3) *The relation* $S(x,y,w) \longleftrightarrow [w \in \mathrm{WO} \wedge |w|$ *is represented in the well-founded part of* $V_{\langle x,y\rangle}\restriction|w|]$ *is* Δ^1_1 *in the codes for* w.

Proof. Let V' be the standard Shoenfield tree on $\omega \times \omega_1$ for R^c. Thus, V' is Δ^1_1 in the codes and $\forall x, y\ [R(x,y) \longleftrightarrow V'_{\langle x,y\rangle}$ is well-founded]. Let V be a minor modification of V' such that

$$\forall x\, \forall \alpha < \omega_1\ (V_x\restriction\alpha \text{ is well-founded } \rightarrow |V_x\restriction\alpha| > \alpha).$$

[For example, code into V_x all finite decreasing chains $\beta_0 > \beta_1 > \cdots > \beta_n$.] We may assume that if $((a_0,\ldots,a_n),(\alpha_0,\ldots,\alpha_n)) \in V$, then $\alpha_0 \geq \alpha_1,\ldots,\alpha_n$. Clearly V is Δ^1_1 in the codes.

If $R(x,y)$, then $V_{\langle x,y\rangle}$ is well-founded, and by construction $\forall\alpha\ |V_{\langle x,y\rangle}\restriction\alpha| > \alpha$, hence α is represented in the well-founded part of $V_{\langle x,y\rangle}\restriction\alpha$. If $(x,y) \notin R$, then $V_{\langle x,y\rangle}$ is ill-founded, say $(\langle x,y\rangle,(\beta_0,\beta_1,\ldots)) \in [V]$. If the initial segment $I^{\vec{\alpha}}_{x,y}$ of $\prec_{\langle x,y\rangle}$ determined by $\vec{\alpha} = (\alpha_0,\ldots,\alpha_{n-1}) \in V_{\langle x,y\rangle}$ is well-founded, we must have $\alpha_0,\ldots,\alpha_n \leq \beta_0$. Thus,

$$\gamma \doteq \sup\{|\vec{\alpha}|_{\prec_{\langle x,y\rangle}} : I^{\vec{\alpha}}_{x,y} \text{ is well-founded}\} < \omega_1.$$

For $\delta > \max\{\beta_0,\gamma\}$, δ is not represented in the well-founded part of $V_{\langle x,y\rangle}\restriction\delta$.

Finally, (3) is a standard computation using (1). ⊣

4.44 Lemma. *Let* $W \subseteq \mathrm{WO}_2$ *be* Σ^1_3, *invariant in the codes, and code a bounded initial segment of* ω_2. *Then there is a* Δ^1_3 *relation* $F \subseteq \mathrm{WO} \times \mathrm{WO}$ *which is invariant in the codes, and defines a total function* $F : \omega_1 \to \omega_1$ *(i.e.,* $\forall\alpha < \omega_1 \exists!\beta < \omega_1\ F(\alpha,\beta)$*) such that* $[F]_{W^1_1} > |x|$ *for all* $x \in W$.

Proof. Define $W'(w) \longleftrightarrow \exists x \in \mathrm{WO}_2\ [W(x) \wedge (w$ codes a function $F_w : \omega_1 \to \omega_1) \wedge (|x| = [F_w]_{W^1_1})]$. From Lemma 4.42, $W' \in \Sigma^1_3$. W' is also invariant in the sense that if w, w' code functions $F_w, F_{w'}$, $[F_w]_{W^1_1} = [F_{w'}]_{W^1_1}$, and $W'(w)$, then $W'(w')$. Let $W'(w) \longleftrightarrow \exists y\ R(w,y)$, where $R \in \Pi^1_2$. Let V be a tree on $\omega \times \omega_1$ as in Lemma 4.43, in particular $R(w,y) \longleftrightarrow V_{\langle w,y\rangle}$ is well-founded.

Say a real w is *α-good* if $F_w(\alpha)$ is defined, and say w is $\leq\alpha$-good if it is α' good for all $\alpha' \leq \alpha$. Say a pair (w,y) is *α-good* if w is α-good and α is represented in the well-founded part of $V_{\langle w,y\rangle}\restriction\alpha$.

Consider the integer game G where I plays out reals w_1, y, and II plays out w_2, and II wins the run iff there exists an $\eta_0 < \omega_1$ such that either:

(1) $\forall\eta < \eta_0\ (w_1,y), w_2$ are η-good, (w_1,y) is not η_0-good, and w_2 is η_0-good, or

(2) $\forall\eta \leq \eta_0\ (w_1,y), w_2$ are η-good, and $F_{w_1}(\eta_0) < F_{w_2}(\eta_0)$.

Using Lemmas 4.41 and 4.43, G is a Σ^1_2 game for II. II easily wins the game, by playing any w^* coding a function $F_{w^*} : \omega_1 \to \omega_1$ such that $[F_{w^*}]_{W^1_1} > \sup\{|x| : x \in W\}$. Thus, by third periodicity, II has a Δ^1_3 winning strategy τ.

Define $b : \omega_1 \to \omega_1$ inductively as follows. Let $b(\eta_0)$ be the maximum of $(\sup_{\eta<\eta_0} b(\eta)) + 1$ and

$$\sup\{F_{\tau(w_1,y)}(\eta_0) : \forall \eta < \eta_0 \ [(w_1, y) \text{ is } \eta\text{-good } \wedge F_{w_1}(\eta) = b(\eta)]\}.$$

We show by induction on η_0 that:

4.45 Claim.

(a) $b(\eta_0)$ is well-defined, that is, $b(\eta_0) < \omega_1$.

(b) If (w_1, y) is $\leq\eta_0$-good and $\forall \eta \leq \eta_0$ $F_{w_1}(\eta) = b(\eta)$, then $\forall \eta \leq \eta_0$ $F_{w_2}(\eta) \leq F_{w_1}(\eta)$, where $w_2 = \tau(w_1, y)$.

Proof. Suppose the claim holds for all $\eta < \eta_0$. Note that if (w_1, y) is η-good for all $\eta < \eta_0$ and $\forall \eta < \eta_0$ $F_{w_1}(\eta) = b(\eta)$, then by (b) and induction, $F_{w_2}(\eta_0)$ is defined, where $w_2 = \tau(w_1, y)$, as otherwise II would lose this run of the game. Now, $B_{\eta_0} \doteq \{(w_1, y) : \forall \eta < \eta_0 \ [(w_1, y) \text{ is } \eta\text{-good } \wedge F_{w_1}(\eta) = b(\eta)]\}$ is $\boldsymbol{\Delta}^1_1$, as it is Δ^1_1 in any real coding η_0 and $b\restriction\eta_0$. The reasonableness of the coding $z \to F_z$ now gives (a) (cf. (4) in Definition 2.33). (b) at η_0 is now immediate from the definition of $b(\eta_0)$. ⊣

Next we claim that $[b]_{W^1_1} > |x|$ for all $x \in W$. If not, then by the invariance and initial segment properties of W', there is a w_1 in W' with $F_{w_1} = b$. Let y be such that $R(w_1, y)$, and have I play (w_1, y) against τ, producing $w_2 = \tau(w_1, y)$. Since $\forall \eta_0 < \omega_1$ (w_1, y) is η_0-good, a straightforward induction using (b) shows that $\forall \eta_0 < \omega_1$ $F_{w_2}(\eta_0)$ is defined and $F_{w_2}(\eta_0) \leq F_{w_1}(\eta_0)$, a contradiction to II winning the game.

Finally, we show that the relation $F(z_1, z_2) \longleftrightarrow z_1, z_2 \in \mathrm{WO} \wedge b(|z_1|) = |z_2|$ is Δ^1_3. We have $F(z_1, z_2)$ iff the following hold:

(1) $z_1, z_2 \in \mathrm{WO}$.

(2) There is a $y \in \omega^\omega$ and a $z \in \mathrm{WO}$ with $|z| = |z_1| + 1$ and $|0|_{\prec_z} = |z_1|$ satisfying:

  (a) $\forall n \ y_n \in \mathrm{WO}$.

  (b) The map $n \to |y_n|$ defines an order-preserving map from $\prec_z$ to ω_1.

  (c) For any $n \in \mathrm{dom}(\prec_z)$, $|y_n|$ is the maximum of $(\sup\{|y_m| : m \prec_z n\}) + 1$ and

$$\sup\{F_{\tau(w_1,y)}(|n|_z) : \forall m \prec_z n \ ((w_1, y) \text{ is } |m|_{\prec_z}\text{-good} \wedge F_{w_1}(|m|_{\prec_z}) = |y_m|)\}.$$

  (d) $|y_0| = |z_2|$.

It follows easily that $F \in \Sigma^1_2(\tau)$, so $F \in \Delta^1_3$. ⊣

We prove now the $m = 1$ case of the Kechris-Martin Theorem. Let $R(x,w) \subseteq \omega^\omega \times \mathrm{WO}_2$ be Π^1_3 and invariant in the codes. Define $R'(x,w) \longleftrightarrow w \in \mathrm{WO}_2 \wedge \exists w' \in \mathrm{WO}_2\ [|w'| \leq |w| \wedge R(x,w')]$. Clearly R' is invariant in the codes, and we claim that $R' \in \Pi^1_3$. To see this, note that

$$R'(x,w) \longleftrightarrow w = \langle a,v\rangle \in \mathrm{WO}_2 \wedge \exists b \in \mathrm{WO}\ \big[\forall^* \beta < \omega_1\ |(T_v{\restriction}\beta)(|b|)| \\ \leq |(T_v{\restriction}\beta)(|a|)| \wedge \forall z \in \mathrm{WO}_2(|z| = |\langle b,v\rangle| \to R(x,z))\big].$$

From Lemma 4.40 and the $m = 0$ case of the theorem it follows that $R' \in \Pi^1_3$. Replacing now R with R', we may assume that $R(x,w)$ is also closed upwards in the codes w.

We employ a standard coding for the $\Delta^1_3(x)$ subsets of $\omega^\omega \times \omega^\omega$, uniformly in x. Thus, let $Q \subseteq (\omega^\omega)^3$ be Π^1_3 and such that for every $\Pi^1_3(x)$ set $A \subseteq (\omega^\omega)^2$, there is a real y recursive in x such that $A = Q_x$. Let $Q'_0(x,y,z) \longleftrightarrow Q(x_0,y,z)$, and $Q'_1(x,y,z) \longleftrightarrow Q(x_1,y,z)$. Let Q_0, Q_1 in Π^1_3 reduce Q'_0, Q'_1. We then say *x codes a Δ^1_3 set* if $\forall y,z\ [Q_0(x,y,z) \vee Q_1(x,y,z)]$, in which case x codes the $\Delta^1_3(x)$ set $D_x = \{(y,z) : Q_0(x,y,z)\}$.

Returning to the proof, let $P(x) \longleftrightarrow \exists w \in \mathrm{WO}_2\ R(x,w)$, where $R \in \Pi^1_3$ is invariant and closed upwards in the codes. From Lemma 4.44 we have:

$$\begin{aligned} P(x) \longleftrightarrow \exists y \in \Delta^1_3(x)\ \big[&(y \text{ codes a } \Delta^1_3 \text{ relation } D_y \subseteq (\omega^\omega)^2) \\ &\wedge (D_y \subseteq \mathrm{WO} \times \mathrm{WO} \wedge D_y \text{ is invariant in the codes}) \\ &\wedge (D_y \text{ defines a total function from } \omega_1 \text{ to } \omega_1) \\ &\wedge \forall w \in \mathrm{WO}_2\ \big[(\forall^*_{W^1_1} \alpha < \omega_1\ (\alpha, f_w(\alpha)) \in D_y) \to R(x,w)\big]\big]. \end{aligned}$$

For "D_y defines a total function from ω_1 to ω_1" we use:

$$\forall x, z_1, z_2 \in \mathrm{WO}\ [D_y(x,z_1) \wedge D_y(x,z_2) \to |z_1| = |z_2|] \\ \wedge \forall x \in \mathrm{WO}\ \exists z \in \mathrm{WO}\ [\forall z' \in \mathrm{WO}\ (|z'| = |z| \to D_y(x,z'))].$$

By the $m = 0$ case of the theorem, this expression defines a Π^1_3 set, so $P \in \Pi^1_3$, using Lemma 4.40. This completes the $m = 1$ case of the theorem.

We prove now the general case $m > 1$ of the theorem. So let $P(x) \longleftrightarrow \exists w \in \mathrm{WO}_{m+1}\ R(x,w)$, where $R \in \Pi^1_3$ is invariant in the codes w. Recall that for $w \in \mathrm{WO}_{m+1}$, f_w is the corresponding function from ω_1^m to ω_1 (defined W^m_1 almost everywhere) representing $|w|$. For any such f_w, there is a function $g : \omega_1 \to \omega_1$ such that $\forall^*_{W^m_1}(\alpha_1,\ldots,\alpha_m)\ f_w(\vec{\alpha}) < g(\alpha_m)$. We may take $g = f_y$ for some $y = \langle a,u\rangle \in \mathrm{WO}_2$. Thus we have:

$$\begin{aligned} P(x) \longleftrightarrow \exists y = \langle a,u\rangle \in \mathrm{WO}_2\ \exists z \in \mathrm{WO}_m \\ \big[&\forall^*_{W^{m-1}_1} \alpha_1,\ldots,\alpha_{m-1}\ f_z(\vec{\alpha}) \prec_{T_u} |a| \\ &\wedge \forall w \in \mathrm{WO}_{m+1}\ \big[(\forall^*_{W^m_1} \alpha_1,\ldots,\alpha_m\ f_w(\alpha_1,\ldots,\alpha_m) \\ &= |(T_u{\restriction}\alpha_m)(f_z(\alpha_1,\ldots,\alpha_{m-1}))|) \to R(x,w)\big]\big]. \end{aligned}$$

Note that the expression beginning with $\exists z \in \mathrm{WO}_m$ is invariant in the codes for y; it is equivalent to $\exists f : (\omega_1)^m \to \omega_1\ \forall^*_{W_1^m} \alpha_1, \dots, \alpha_m\ f(\alpha_1, \dots, \alpha_m) < f_y(\alpha_m)$ and $R(x, [f]_{W_1^m})$. By the $m = 1$ and $m - 1$ cases of the theorem, $P \in \Pi^1_3$. This completes the proof of Theorem 4.38. $\dashv$

5. Higher Descriptions

We assume AD throughout Sect. 5. We sketch in this section how the theory of Sect. 4 can be extended to higher levels. Indeed, the arguments here should extend to the general case of a successor Suslin cardinal in the hierarchy of $L(\mathbb{R})$. We will concentrate here, however, on the projective hierarchy, and in fact largely on the theory of $\boldsymbol{\delta}^1_5$, since all of the new ideas occur here. As we said in the introduction, our style here will be somewhat informal. We will concentrate on presenting the new ideas without getting lost in details; we will sometimes illustrate proofs by considering a representative example. The reader wishing to see the complete details for the next level of the analysis (the strong partition relation on $\boldsymbol{\delta}^1_3$, the computation of $\boldsymbol{\delta}^1_5$, and the weak partition relation on $\boldsymbol{\delta}^1_5$) can consult [11]. In Sect. 6 we will consider topics related to extending this theory further.

Reflecting on the arguments of the previous section, we see that there were two fundamental ingredients. First was the Kunen analysis, Lemma 4.1, which provided an analysis of the equivalence classes of function $f : \omega_1 \to \omega_1$ with respect to the normal measure W^1_1 on ω_1. Second was the analysis, embodied in Lemma 4.5, which showed how equivalence classes of functions with respect to the more general measures W^m_1 are generated from the normal measure analysis. The combinatorics of the process was described by the descriptions. Admittedly, the concept there was rather trivial (descriptions being just integers), and there was really no interesting combinatorics taking place. The situation changes as we move to the higher levels, though, and the concept of the description becomes a central point. Indeed, armed with the correct notion of description and proper generalization of the Kunen tree analysis (due to Martin, see below), the general step in the projective hierarchy analysis is quite similar to that of Sect. 4. Thus, we concentrate in this section on showing how these two key ingredients generalize.

5.1. Martin's Theorem on Normal Measures

From the weak partition relation on $\boldsymbol{\delta}^1_3$, Theorem 4.34, it follows that there are precisely three normal measures on $\boldsymbol{\delta}^1_3$, corresponding to the three regular cardinals $\omega, \omega_1, \omega_2$ below $\boldsymbol{\delta}^1_3$. [The weak partition relation shows that the closed unbounded filter restricted to points of one of these cofinalities is a normal measure. Conversely, any normal measure on $\boldsymbol{\delta}^1_3$ must give every closed unbounded set measure one, and by countable additivity must concentrate on one these cofinalities. Thus, it must coincide with one of these three normal measures.]

The ω-cofinal normal measure on $\boldsymbol{\delta}^1_3$ behaves just as the normal measure on ω_1; indeed, the Kunen tree analysis is quite general and holds for all the $\boldsymbol{\delta}^1_{2n+1}$. Thus, there is a tree T on $\omega \times \boldsymbol{\delta}^1_{2n+1}$ such that for all $f : \boldsymbol{\delta}^1_{2n+1} \to \boldsymbol{\delta}^1_{2n+1}$, there is an $x \in \omega^\omega$ with T_x well-founded such that $\forall^* \alpha < \boldsymbol{\delta}^1_{2n+1}\ f(\alpha) < |T_x \restriction \alpha|$ (where $\forall^*$ refers to the ω-cofinal normal measure). The proof is a small variation of Lemma 4.1. Instead of WF, one uses the set W defined by $W(x) \longleftrightarrow \forall n\ P(x_n)$, where P is a $\boldsymbol{\Pi}^1_{2n+1}$-complete set. Let $\{\phi_n\}$ be a $\boldsymbol{\Pi}^1_3$ scale on P. Using $\{\phi_n\}$, define a tree U on $\omega \times \boldsymbol{\delta}^1_3$ with $p[U] = W$ (a branch $\vec{\alpha}$ through U_x gives subsequences $\vec{\alpha}_1$, $\vec{\alpha}_2$, etc., such that for all m, $(x_m, \vec{\alpha}_m)$ is a branch through the tree of the scale $\{\phi_n\}$). We think of $x \in W$ as coding the ordinal $|x| \doteq \sup_n \phi_0(x_n)$. Note that for almost all $\alpha < \boldsymbol{\delta}^1_3$ with respect to the ω-cofinal normal measure, there is an $x \in W$ with $|x| = \alpha$ and $U_x \restriction \alpha$ is ill-founded. Let S be a complete $\boldsymbol{\Sigma}^1_{2n+1}$ set, which is $(\boldsymbol{\delta}^1_3)^-$-Suslin by Theorem 2.18. Say $S = p[V]$, V a tree on $\omega \times (\boldsymbol{\delta}^1_3)^-$. Since $S \notin \boldsymbol{\Delta}^1_3$, it follows from Theorem 2.15 that $\sup\{|V_x| : V_x \text{ is well-founded}\} = \boldsymbol{\delta}^1_3$. The Kunen tree T is then constructed as in Lemma 4.1.

For the other normal measures, however, the situation is different. To discuss this, we need to recall some facts from the homogeneous tree construction. A detailed account of this may be found in [17]; we summarize the main points.

Recall that if T is a tree on $\omega \times \kappa$, and $s \in \omega^n$, then $T_s = \{\vec{\alpha} \in \kappa^n : (s, \vec{\alpha}) \in T\}$. If t extends s, let $\pi_{s,t}$ denote the natural map from T_t to T_s defined by $\pi_{s,t}(\vec{\alpha}) = \vec{\alpha} \restriction \mathrm{lh}(s)$. If μ is a measure on T_t, then $\pi_{s,t}(\mu)$ is a measure on T_s (recall $\pi_{s,t}(\mu)(A) = \mu(\pi^{-1}_{s,t}(A))$).

5.1 Definition. A tree T on $\omega \times \kappa$ is *homogeneous* if there are measures μ_s, for $s \in \omega^{<\omega}$, with $\mu_s(T_s) = 1$ such that if t extends s then $\pi_{s,t}(\mu_t) = \mu_s$, and having the following homogeneity property: for all $x \in \omega^\omega$, if T_x is ill-founded and for each n a set $A_n \subseteq T_{x\restriction n}$ is given with $\mu_{x \restriction n}(A_n) = 1$, then there is a sequence $\vec{\alpha} \in \kappa^\omega$ such that for all n, $\alpha \restriction n \in A_n$.

The homogeneity property for T_x is equivalent to saying that the direct limit of the ultrapowers (of On) by the measures $\mu_{x\restriction n}$ is well-founded.

We extend the definition in the obvious way to trees T on $\omega \times \omega \times \kappa$, etc. (in this case, the measures $\mu_{s,t}$ are indexed by pairs of sequences of the same length). We say T is a homogeneous tree for $P \subseteq \omega^\omega$ if T is homogeneous and $p[T] = P$.

If $P \subseteq 2^\omega$ is $\boldsymbol{\Pi}^1_1$, the standard Shoenfield construction gives a tree T_1 on $2 \times \omega_1$ with $P = p[T_1]$. T_1 may be defined so that for each $s \in 2^\omega$, there is a permutation π_s of $\{1, \dots, n\}$, $n = \mathrm{lh}(s)$, with n occurring first, such that $(s, \vec{\alpha}) \in T_1$ iff $\vec{\alpha}$ is order-isomorphic to π_s. For π a permutation of $\{1, \dots, n\}$, let W^π_1 be the natural measure on n-tuples $\vec{\alpha}$ which are order-isomorphic to π (i.e., W^π_1 is equivalent to W^n_1 under the map which re-arranges $\vec{\alpha}$ into increasing order). Thus, the measures $W^{\pi_s}_1$ witness that T_1 is homogeneous.

If $S \subseteq 2^\omega$ is $\boldsymbol{\Sigma}^1_2$, then $S(x) \longleftrightarrow \exists y\ P(x, y)$, where $P \in \boldsymbol{\Pi}^1_1$, so $P = p[T_1]$ for some homogeneous tree T_1 on $2 \times 2 \times \omega_1$ (if we identify T_1 with a tree

T_1' on $2 \times \omega_1$ by identifying the second and third coordinates of T_1 with the second coordinate of T_1', then T_1' is said to be *weakly homogeneous*). For $s, t \in 2^{<\omega}$ of the same length, let $\pi_{s,t}$ and $W_1^{\pi_{s,t}}$ be the permutation and measure associated to s, t, and T_1. We may assume without loss of generality that for $s, t \in 2^{<\omega}$ of the same length, that $\pi_{s,t}$ depends only on $s \restriction (\mathrm{lh}(s) - 1)$, $t \restriction (\mathrm{lh}(t) - 1)$. For convenience we also assume (without loss of generality) that for any s, t of the same length, $(T_1)_{s,t} \neq \emptyset$. If $Q = S^c$, then the homogeneous tree construction shows how, using the strong partition relation on ω_1, to get a homogeneous tree T_2 for Q. One way of doing this is as follows. For $s \in 2^{<\omega}$, let $\prec_s$ be the Kleene-Brouwer ordering on $(T_1)_s \subseteq (2 \times \omega_1)^{\leq \mathrm{lh}(s)}$. In specifying the Kleene-Brouwer ordering, we order pairs $(n, \alpha) \in 2 \times \omega_1$ first by α. It is convenient here to adopt a minor variation of the definition of $\mathcal{R}$ being a type-1 tree of uniform cofinalities, Definition 4.25. First, we drop all sequences $\vec{p} = \langle p_1, i_1, \dots, p_m, i_m \rangle$ where $i_m = 0$ from the domain of $\mathcal{R}$. Thus, for any $\vec{p}$, $\mathcal{R}(\vec{p})$ is now either (ω) or a permutation p_{m+1} extending p_m. Second, we now allow either possibility for $\mathcal{R}(\vec{p})$ when $\vec{p}$ is maximal in $\mathrm{dom}(\mathcal{R})$. We define $<^{\mathcal{R}}$ exactly as before, and define $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ being of type $\mathcal{R}$ in the obvious way (for $\vec{p}$ maximal in $\mathrm{dom}(\mathcal{R})$ with $\mathcal{R}(\vec{p}) = p_{m+1}$, we require $f^{\vec{p}}(\alpha_1, \dots, \alpha_m)$ to almost everywhere have uniform cofinality $\{\beta : (\alpha_1, \dots, \alpha_m, \beta)$ is order-isomorphic to $p_{m+1}\}$). It is now easy to check that $\prec_s$ is of the form $<^{\mathcal{R}_s}$ for some type-1 tree of uniform cofinalities $\mathcal{R}_s$. In fact, we can define $\mathcal{R}_s$ as follows. Let $\vec{p} = \langle p_1, i_1, \dots, p_m, i_m \rangle \in \mathrm{dom}(\mathcal{R}_s)$ iff $m \leq \mathrm{lh}(s)$, each $i_k = 1$ or 2, and $p_m = \pi_{s \restriction m, t}$, where $t = (i_1 - 1, \dots, i_m - 1)$. We set $\mathcal{R}_s(\vec{p}) = \pi_{s', t'}$, where s', t' are any immediate extensions of s, t (by our assumption, this only depends on s and t).

We define $(s, \vec{\beta}) \in T_2$ iff there is an $f : \mathrm{dom}(<^{\mathcal{R}_s}) \to \omega_1$ of type $\mathcal{R}_s$ with $\vec{\beta} = [f] \restriction \mathrm{lh}(s)$. To say $\vec{\beta} = [f] \restriction \mathrm{lh}(s)$ means that for all $i < \mathrm{lh}(s)$, $\beta_i = [f^{\langle p_1, i_1, \dots, p_k, i_k \rangle}]_{W_1^k}$, where $(i_1, \dots, i_k)$ is the ith element of $2^{<\omega}$ in some reasonable enumeration, and $p_j = \pi_{(s \restriction j, (i_1, \dots, i_j))}$. From the strong partition relation on ω_1 it is not difficult to see that T_2 is homogeneous for Q, with measures $M^{\mathcal{R}_s}$. [For example, to show homogeneity, suppose $(T_2)_x$ is ill-founded, and each A_n has measure one with respect to $M^{\mathcal{R}_{x \restriction n}}$. Let C_n be a closed unbounded subset of ω_1 defining a $M^{\mathcal{R}_{x \restriction n}}$ measure one set contained in A_n. Let $C = \bigcap_n C_n$. Since $x \in Q = P^c$, $(T_1)_x$ is well-founded. Let f be order-preserving from the Kleene-Brouwer ordering on $(T_1)_x$ to C such that for all n, $f \restriction \mathrm{dom}(\prec_{x \restriction n})$ is of type $\mathcal{R}_{x \restriction n}$. Let $\beta_i = [f^{\langle p_1, i_1, \dots, p_k, i_k \rangle}]_{W_1^k}$, where again $(i_1, \dots, i_k)$ is the ith element of $2^{<\omega}$ in our enumeration. Then $(\beta_0, \dots, \beta_n) \in A_n$ for all n.]

Recall WO_2 is the $\mathbf{\Pi}^1_2$ set of codes for ordinals $< \omega_2$ (Definition 4.36), and for $x \in \mathrm{WO}_2$, $|x| = [f_x]_{W_1^1} < \omega_2$ is the ordinal coded by x. The next lemma shows that we may get a homogeneous tree for WO_2 with an additional property.

5.2 Lemma. *There is a homogeneous tree U on $\omega \times \omega_\omega$ with* $\mathrm{WO}_2 = p[U]$ *and with the following property:*

($\star$) *If* $\{\psi_n\}$ *is the scale on* WO_2 *corresponding to* U *(using left-most branches), then* $\forall x \in \mathrm{WO}_2 |x| \leq \psi_0(x)$.

Proof. Let T be the Kunen tree of Sect. 4, where we use the linear-order version of Theorem 4.2. Recall $x \in \mathrm{WO}_2$ if $x = \langle a, y\rangle$ where $a \in \mathrm{WO}$ and T_y is a wellordering. Since T is Δ^1_1 in the codes, there is a $\boldsymbol{\Sigma}^1_1$ relation E such that for all $b \in \mathrm{WO}$:

$$E(y, b, x_1, x_2) \longleftrightarrow \big[x_1, x_2 \in \mathrm{WO} \wedge |x_1|, |x_2| < |b| \wedge |x_2|\ T_y\ |x_1|\big].$$

Let W_1 be a tree on ω^5 projecting to E. Let W_2 be a homogeneous tree on $\omega \times \omega_1$ for WO. We may assume that for all $(s, (\alpha_0, \dots, \alpha_{n-1})) \in W_2$ that $\alpha_0 > \max\{\alpha_1, \dots, \alpha_{n-1}\}$. Define the tree V on $\omega^2 \times \omega_1 \times \omega^2$ as follows. Let $(p, s, \vec{\alpha}, v, w) \in V$ iff $(s, \vec{\alpha}) \in W_2$ and (p, s, v, w) satisfies: for all $i < \mathrm{lh}(p) - 1$, if j is maximal so that $\langle i, j\rangle$, $\langle i+1, j\rangle < \mathrm{lh}(p)$, then

$$\big(p{\restriction}j, s{\restriction}j, \big(v_i(0), \dots, v_i(j)\big), \big(v_{i+1}(0), \dots, v_{i+1}(j)\big), (w_i(0), \dots, w_i(j))\big) \in W_1,$$

where $v_i(k) = v(\langle i, k\rangle)$, and similarly for w. This last requirement is just building the Kunen-Martin tree for the relation E. In particular, V_y is well-founded iff for all $a \in \mathrm{WO}$, $T_y{\restriction}|a|$ is a wellorder, that is, iff T_y is a wellorder. Also, if T_y is well-founded, then for all $\beta < \omega_1$ the rank of $T_y{\restriction}\beta$ is less than or equal to the rank of $V_y^\beta \doteq$ those $(p, s, \vec{\alpha}, v, w) \in V_y$ with $\alpha_0 \leq \beta$ (as in the proof of the Kunen-Martin Theorem).

Let W_3 be the homogeneous tree on $\omega \times \omega_\omega$ constructed from V, so $p[W_3] = \{y : T_y \text{ is well-founded}\}$. For each $p \in \omega^n$, the homogeneity measure μ_p will be of the form $M^{\mathcal{R}_p}$ for some type-1 tree of uniform cofinalities $\mathcal{R}_p$. Thus, $(p, \vec{\beta}) \in W_3$ iff there is an f of type $\mathcal{R}_p$ such that for all $j < n$, $\beta_j = [f^{\langle p_1, i_1, \dots, p_k, i_k\rangle}]_{W_1^k}$, where $\langle p_1, i_1, \dots, p_k, i_k\rangle$ is the jth element of $\mathrm{dom}(\mathcal{R}_p)$ in some enumeration. We may assume that $\beta_0 = [f^{\langle p_1, i_1\rangle}]_{W_1^1}$, where i_1 is maximal so that $\langle p_1, i_1\rangle \in \mathrm{dom}(\mathcal{R}_p)$.

Finally, define U to be the tree which is the "conjunction" of W_2 and W_3:

$$\begin{aligned}&\big((y(0), a(0), \dots, y(n-1), a(n-1)), (\beta_0, \alpha_0, \dots, \beta_{n-1}, \alpha_{n-1})\big) \in U \\ &\quad\longleftrightarrow (a{\restriction}n, (\alpha_0, \dots, \alpha_{n-1})) \in W_2 \wedge (y{\restriction}n, (\beta_0, \dots, \beta_{n-1})) \in W_3.\end{aligned}$$

It is easy to see that U is homogeneous with measures of the form $W_1^\pi \times M^{\mathcal{R}}$. Clearly $p[U] = \mathrm{WO}_2$. To verify ($\star$), suppose that

$$x = \langle a, y\rangle = (y(0), a(0), y(1), a(1), \dots) \in \mathrm{WO}_2\,.$$

By definition $|x| \leq |T_y|$. It is enough to show that if $(y, \vec{\beta}) \in [W_3]$, then $\beta_0 \geq |T_y|$. For each n, let f_n be of type $\mathcal{R}_{y{\restriction}n}$ with $[f_n] = (\beta_0, \dots, \beta_{n-1})$. For $C \subseteq \omega_1$, let $V_y{\restriction}C$ denote those $(s, \vec{\alpha}, v, w) \in V_y$ with all $\alpha_j \in C$. From the f_n we get a function f and a closed unbounded $C \subseteq \omega_1$ such that:

(1) f is order-preserving from the Kleene-Brouwer ordering on $V_y \restriction C$ to On.

(2) For each n, f induces by restriction a function f'_n of type $\mathcal{R}_{y\restriction n}$ with $[f'_n] = [f_n] = (\beta_0, \dots, \beta_{n-1})$.

Thus, for all $\alpha \in C$ we have $f'_0(\alpha) \geq |V_y^\alpha \restriction C|$. However, for α a closure point of C (i.e., α is the α^{th} element of C), we easily have $|V_y^\alpha \restriction C| = |V_y^\alpha|$. Hence, $\beta_0 = [f'_0]_{W_1^1} \geq [\alpha \to |V_y^\alpha|]_{W_1^1} \geq [\alpha \to |T_y \restriction \alpha|]_{W_1^1} = |T_y|$. ⊣

The above homogeneous tree construction, without the ($\star$) argument, also shows that every $\boldsymbol{\Pi}^1_2$ set is the projection of a homogeneous tree on $\omega \times \omega_\omega$. This also shows that every $\boldsymbol{\Sigma}^1_3$ set is weakly homogeneous. Using the weak partition on $\boldsymbol{\delta}^1_3$ and the homogeneous tree construction again, one can then show that every $\boldsymbol{\Pi}^1_3$ set admits a homogeneous tree T on $\omega \times \boldsymbol{\delta}^1_3$. We wish, however, to modify this construction to obtain a homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set with some additional properties. Our argument is really just Martin's analysis of functions with respect to the normal measures. The form we present it here may be of use elsewhere. We sketch another version in Theorem 5.6.

5.3 Theorem. *There is a $\boldsymbol{\Pi}^1_3$ complete set P, a $\boldsymbol{\Pi}^1_3$-norm $z \to |z| < \boldsymbol{\delta}^1_3$ from P onto $\boldsymbol{\delta}^1_3$, and a homogeneous tree S on $\omega \times \boldsymbol{\delta}^1_3$ for P satisfying the following. There is a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$ there is a $z \in P$ with $|z| = \alpha$ and with $S_z \restriction (\sup_\nu j_\nu(\alpha))$ ill-founded, the supremum ranging over measures $\nu = M^{\mathcal{R}_s}$ occurring in the homogeneous tree U of Lemma 5.2. Furthermore, for any z and $\vec{\beta} = (\beta_0, \beta_1, \dots)$, if $(z, \vec{\beta}) \in [S]$, then $|z| \leq \beta_0$.*

Proof. Recall according to Theorem 4.33 our $\boldsymbol{\Delta}^1_3$ coding $z \to A_z$ of subsets of ω_ω (or $(\omega_\omega)^2$, etc.). We assume for this proof that for all z, $A_z \subseteq \omega_2 \times \omega_\omega \times \omega_\omega$. If $\alpha < \omega_2$, let $A_z^\alpha = \{(\beta, \gamma) : (\alpha, \beta, \gamma) \in A_z\}$. Define

$$P(z) \longleftrightarrow \forall \alpha < \omega_2 \; A_z^\alpha \text{ is well-founded.}$$

Note that the relation

$$C(z, x_1, x_2, x_3) \longleftrightarrow x_1 \in \mathrm{WO}_2 \wedge x_2, x_3 \in \mathrm{WO}_\omega \wedge (|x_1|, |x_2|, |x_3|) \in A_z$$

is $\boldsymbol{\Delta}^1_3$ by the closure of $\boldsymbol{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions (in fact, it is straightforward to show that C is Δ^1_3). Thus, $P \in \boldsymbol{\Pi}^1_3$. For $z \in P$, let $|z| = \sup\{|A_z^\alpha| : \alpha < \omega_2\}$. Using the closure of $\boldsymbol{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions and intersections, it is straightforward to check that this defines a $\boldsymbol{\Pi}^1_3$-norm onto $\boldsymbol{\delta}^1_3$.

Let $C^* \subseteq (\omega^\omega)^5$ be $\boldsymbol{\Pi}^1_2$ projecting to C. Let T_2^* be a homogeneous tree on $\omega^5 \times \omega_\omega$ for C^*. Let U be a homogeneous tree on $\omega \times \omega_\omega$ for WO_2 satisfying ($\star$). Define a tree V on $\omega^2 \times \omega_\omega \times \omega \times \omega_\omega$ as follows. Set $(p, s, \vec{\alpha}, u, \vec{\beta}) \in V$ iff $(s, \vec{\alpha}) \in U$ and if $u = (u(0), \dots, u(n-1))$, then the $u(i)$ code more and more of reals $u_0, u_1, \dots$ and reals $w_0, w_1, \dots$. Each $u(j+1)$ extends the array coded by $u(j)$ by adding one extra pair $(u_i(k), u_{i+1}(k))$ for some i, k, and one extra value $w_i(k)$. For each i,k, if the pairs $(u_i(0),u_{i+1}(0)),\dots,(u_i(k-1),u_{i+1}(k-1))$ are added at stages $q_0,\dots,q_{k-1}<n$, then we require that

$$(p{\restriction}k, s{\restriction}k, u_i{\restriction}k, u_{i+1}{\restriction}k, w_i{\restriction}k, (\beta_{q_0},\dots,\beta_{q_{k-1}}))\in T_2^*.$$

Again, V embodies the Kunen-Martin construction for the relation C. The tree V is also homogeneous, witnessed by measures $M_{p,s,u}=\nu_s\times\mu_{p,s,u}$ where ν_s are the homogeneity measures for U and $\mu_{p,s,u}$ are measures of the form $M^{\mathcal{R}_{p,s,u}}$ for some type-1 trees of uniform cofinalities $\mathcal{R}_{p,s,u}$ which are simply obtained from the type-1 trees giving the measures for T_2^*. (Note that for any two type-1 trees $\mathcal{R}_1$ and $\mathcal{R}_2$, there is a type-1 tree $\mathcal{R}$ whose measure $M^{\mathcal{R}}$ projects naturally to both $M^{\mathcal{R}_1}$ and $M^{\mathcal{R}_2}$.)

We now apply the homogeneous tree construction to V to produce a tree S on $\omega\times\boldsymbol{\delta}^1_3$ such that for all z, S_z is ill-founded iff V_z is well-founded. For any $p\in\omega^{<\omega}$, let $\prec_{V_p}$ denote the Kleene-Brouwer ordering on V_p. In specifying the Kleene-Brouwer order, we must say how the tuples $(s(n),\alpha_n,u(n),\beta_n)$ are ordered. In comparing two such tuples, it is important (at least for $n=0$) that we order first by α_n (the remaining order is unimportant). We define $(p,\vec{\gamma})\in S$ iff there exists an f which is order-preserving and of the correct type from V_p with the Kleene-Brouwer order to $\boldsymbol{\delta}^1_3$, and f represents $\vec{\gamma}$ in the following sense. First, we require $\gamma_0=\sup(f)$. Let (s_i,u_i), $i\geq 1$, enumerate the pairs of finite sequences of the same length, with $\mathrm{lh}(s_i)\leq i$. Then γ_i, for $i\geq 1$, is represented with respect to $M_{p\restriction \mathrm{lh}(s_i),s_i,u_i}$ by the function $f^{s_i,u_i}(\vec{\alpha},\vec{\beta})=f(s_i,\vec{\alpha},u_i,\vec{\beta})$. The weak partition relation on $\boldsymbol{\delta}^1_3$ shows that S is homogeneous (though we do not need this for the proof).

If $z\in P$, and hence V_z is well-founded, then easily S_z is ill-founded and in fact there is a $\vec{\gamma}$ with $(z,\vec{\gamma})\in[S]$ such that $\gamma_0\leq\omega\cdot|\prec_{V_z}|$, where $\prec_{V_z}$ is the Kleene-Brouwer ordering on V_z. That S_z being ill-founded implies $z\in P$ will be shown below.

We now define the closed unbounded set $C\subseteq\boldsymbol{\delta}^1_3$ as required. Fix for the moment $\alpha<\omega_2$, $\beta<\boldsymbol{\delta}^1_3$. Let

$$P_{\alpha,\beta}=\{z:\forall\alpha'\leq\alpha\ A_z^{\alpha'}\text{ is well-founded of rank }\leq\beta\}.$$

A standard computation, using the closure of $\boldsymbol{\Delta}^1_3$ under $<\boldsymbol{\delta}^1_3$ unions and intersections shows $P_{\alpha,\beta}\in\boldsymbol{\Delta}^1_3$. Note that if $z\in P_{\alpha,\beta}$ then V_z restricted to tuples $(s,\vec{\alpha},u,\vec{\beta})$ such that $\alpha_0\leq\alpha$ is well-founded; this uses property $(\star)$ of the tree U. Since $P_{\alpha,\beta}\in\boldsymbol{\Sigma}^1_3$ and is thus ω_ω-Suslin, an easy tree argument shows that $b(\alpha,\beta)\doteq\sup\{|\alpha|_{V_z}:z\in P_{\alpha,\beta}\}<\boldsymbol{\delta}^1_3$, where by $|\alpha|_{V_z}$ we mean the supremum of the ranks of $(s(0),\alpha,u(0),\gamma_0)$ in the Kleene-Brouwer ordering of V_z. This defines the function $b:\omega_2\times\boldsymbol{\delta}^1_3\to\boldsymbol{\delta}^1_3$. Let then $C\subseteq\boldsymbol{\delta}^1_3$ be a closed unbounded set consisting of limit ordinals and closed under b.

Suppose now that $\delta\in C$ and $\mathrm{cf}(\delta)=\omega_2$. Let $h:\omega_2\to\delta$ be increasing and cofinal. Let $z\in\omega^\omega$ be such that $\forall\alpha<\omega_2\ A_z^\alpha$ is well-founded of rank $h(\alpha)$. Thus $P(z)$, and for all $\alpha<\omega_2$, $z\in P_{\alpha,h(\alpha)}$. For all $\alpha<\omega_2$ we have $|\alpha|_{V_z}\leq b(\alpha,h(\alpha))<\delta$. Hence there is an order-preserving map f from the

Kleene-Brouwer ordering of V_z to δ. If f represents $\vec{\gamma}$, then $\gamma_0 \leq \delta$ and for all n, $\gamma_n \leq \sup_{p,s,u} j_{M_{p,s,u}}(\delta)$.

Suppose now that $(z, (\beta_0, \beta_1, \ldots)) \in [S]$. We show that $z \in P$, and if we let $h(\alpha) = |A_z^\alpha|$ for $\alpha < \omega_2$, then $\sup(h) \leq \beta_0$. For each j, let $f_j : V_{z\restriction j} \to \boldsymbol{\delta}^1_3$ be order-preserving representing $(\beta_1, \ldots, \beta_j)$, and with $\beta_0 = \sup(f_j)$. Recall that for $i \leq j$ and $i > 0$, β_i is represented with respect to $M_{z\restriction \mathrm{lh}(s_i), s_i, u_i} = \nu_{s_i} \times \mu_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ by the function $f_j^{s_i, u_i}$. For each $i > 0$, let E_i be a $M_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ measure one set such that for all $j_1, j_2 \geq i$ and all $(\vec{\alpha}, \vec{\beta}) \in E_i$ we have $f_{j_1}^{s_i,u_i}(\vec{\alpha}, \vec{\beta}) = f_{j_2}^{s_i,u_i}(\vec{\alpha}, \vec{\beta})$. For $(\vec{\alpha}, \vec{\beta}) \in E_i$, let $f^{s_i,u_i}(\vec{\alpha}, \vec{\beta})$ denote the common value of $f_j^{s_i,u_i}(\vec{\alpha}, \vec{\beta})$ for $j \geq i$. Let A_i be a ν_{s_i} measure one set such that for all $\vec{\alpha} \in A_i$ we have that for $\mu_{z\restriction \mathrm{lh}(s_i), s_i, u_i}$ almost all $\vec{\beta}$ that $(\vec{\alpha}, \vec{\beta}) \in E_i$. Consider now $\alpha < \omega_2$, and fix $x \in \mathrm{WO}_2$ with $|x| = \alpha$. By homogeneity of U, fix $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots)$ such that $(x, \vec{\alpha}) \in [U]$ and for all k, $\vec{\alpha}\restriction k \in \bigcap_i A_i$, where the intersection runs over the i such that $s_i = x\restriction k$.

Consider the tree

$$\begin{aligned} V_{z,x,\vec{\alpha}} \cap \vec{B} = &\big\{(u, \vec{\beta}) : (z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), \vec{\alpha}\restriction \mathrm{lh}(u), u, \vec{\beta}) \in V \\ &\wedge\ \vec{\beta} \in B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}\big\} \end{aligned}$$

where $B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}$ is a $\mu_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u}$ measure-one set satisfying $\{\vec{\alpha}\restriction \mathrm{lh}(u)\} \times B_{z\restriction \mathrm{lh}(u), x\restriction \mathrm{lh}(u), u} \subseteq E_i$, for that i with $(s_i, u_i) = (x\restriction \mathrm{lh}(u), u)$. The map $(u, \vec{\beta}) \to f^{x\restriction \mathrm{lh}(u), u}(\vec{\alpha}\restriction \mathrm{lh}(u), \vec{\beta})$ is order-preserving from the Kleene-Brouwer ordering of $V_{z,x,\vec{\alpha}} \cap \vec{B}$ to β_0. In particular, the tree $V_{z,x,\vec{\alpha}} \cap \vec{B}$ has rank at most β_0. On the other hand, the proof of the Kunen-Martin Theorem shows that the tree of finite sequences $(y_0, \ldots, y_n)$ of reals such that for all $i < n$, $C(z, x, y_{i+1}, y_i)$, embeds into $V_{z,x,\vec{\alpha}} \cap \vec{B}$ (we use here the homogeneity of T_2^*, which allows the "witness sequences" from the Kunen-Martin proof to be chosen in the B sets). Thus, A_z^α is well-founded of rank at most $|V_{z,x,\vec{\alpha}} \cap \vec{B}| \leq \beta_0$. Since $\alpha < \omega_2$ was arbitrary, we have $\sup(h) \leq \beta_0$.

We have proved the theorem for points of cofinality ω_2. A similar (but slightly easier) construction works for points of cofinality ω_1, using WO in place of WO_2. The cofinality ω case, we already observed, is the Kunen result. ⊣

5.4 Remark. If $\{\phi_n\}_{n\in\omega}$ is the semi-scale on P corresponding to S (using left-most branches), then one can show that $\{\phi_n\}$ is a (not necessarily regular) $\boldsymbol{\Pi}^1_3$-scale.

From Theorem 5.3, Martin's Theorem now follows quickly.

5.5 Theorem (Martin). *There is a tree T on $\omega \times \boldsymbol{\delta}^1_3$ such that for all $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ there is a $z \in \omega^\omega$ with T_z well-founded, and a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $f(\alpha) < |T_z \restriction \sup_\nu j_\nu(\alpha)|$, the supremum ranging over measures ν occurring in the homogeneous trees on $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Pi}^1_2$-complete sets.*

Proof. The argument is now almost identical to the Kunen case. Fix P, S as in Theorem 5.3. Let W be a tree on $\omega \times \omega_\omega$ such that $p[W]$ is $\mathbf{\Sigma}^1_3$-complete, and thus $\sup\{|W_w| : W_w \text{ is well-founded}\} = \boldsymbol{\delta}^1_3$. Let U be the tree on $\omega^2 \times \boldsymbol{\delta}^1_3 \times \omega \times \omega_\omega$ defined by:

$$\begin{aligned}&(p, s, \vec{\alpha}, u, \vec{\beta}) \in U\\ &\qquad \longleftrightarrow (s, \vec{\alpha}) \in S \wedge (u, \vec{\beta}) \in W \wedge\ \exists \sigma(\sigma \text{ extends } p \wedge \sigma(s) = u).\end{aligned}$$

If $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, then play the game where I plays out z, II plays out y, and II wins iff $(z \in P) \to (W_y \text{ is well-founded and } |W_y| > f(|z|))$. II wins by boundedness (a $\mathbf{\Sigma}^1_1$ subset of P codes a bounded below $\boldsymbol{\delta}^1_3$ set of ordinals). If σ is a winning strategy for II, and C is as in Theorem 5.3, then for all $\alpha \in C$, $|U_\sigma \restriction \sup_\nu j_\nu(\alpha)| > f(\alpha)$. Weaving the second, third, fourth, and fifth coordinates of U into a single coordinate produces a tree T as desired. ⊣

We refer to the tree T of Theorem 5.5 as the *Martin tree.*

In [11] a version of Martin's Theorem is presented which does not get the extra information about the homogeneous scale, but which refines slightly the inequality. Actually, the refined inequality can also be obtained by examining the proof of Theorem 5.3. Nevertheless, this second variation of the proof has, we feel, enough advantages to warrant presenting. Specifically, we will use this second variation of the proof in Theorem 5.7.

5.6 Theorem (Martin). *There is a tree T on $\omega \times \boldsymbol{\delta}^1_3$ such that for all $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$, there is a z with T_z well-founded and a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $f(\alpha) < |T_z \restriction \sup_\nu j_\nu(\alpha)|$, where if $\mathrm{cf}(\alpha) = \omega_1$ the supremum ranges over the measures W^m_1 occurring in the homogeneous tree on a $\mathbf{\Pi}^1_1$-complete set, and if $\mathrm{cf}(\alpha) = \omega_2$, the supremum ranges over the measures occurring in the homogeneous tree on a $\mathbf{\Pi}^1_2$-complete set (if $\mathrm{cf}(\alpha) = \omega$, we use $|T_z \restriction \alpha|$).*

Proof. The proof of this version is similar to that of the version presented above, so we give a sketch. Let P, U be as in Theorem 5.3, and let ν_s be the homogeneity measures for U. Define (where A^α_z is as in the proof of Theorem 5.3)

$$P'(z, x) \longleftrightarrow x \in \mathrm{WO}_2\ \wedge\ A^{|x|}_z \text{ is well-founded.}$$

Thus $P' \in \mathbf{\Pi}^1_3$ and $P(z) \longleftrightarrow \forall x\ [x \in \mathrm{WO}_2 \to P'(z, x)]$. Let V be a homogeneous tree on $\omega \times \omega \times \boldsymbol{\delta}^1_3$ for P' with $\boldsymbol{\delta}^1_3$-complete measures $\mu_{s,t}$ (which the usual homogeneous tree construction gives). Now define the tree W on $\omega \times \boldsymbol{\delta}^1_3$ as follows. Let $\{s_n\}_{n \in \omega}$ be an enumeration of $\omega^{<\omega}$ with any sequence preceding its extensions. Define $((z(0), \ldots, z(n-1)), (\beta_0, \ldots, \beta_{n-1})) \in W$ iff for all $j < n$, if $s_j = (s(0), \ldots, s(k-1))$ and $q_0 < q_1 < \cdots < q_{k-1} = j$ are such that $s_{q_l} = (s(0), \ldots, s(l))$, and $f_0, \ldots, f_{k-1}$ represent $\beta_{q_0}, \ldots, \beta_{q_{k-1}}$ with respect to $\nu_{s(0)}, \ldots, \nu_{(s(0), \ldots, s(k-1))}$, then

$$\forall^*_{\nu_{s_j}} \gamma_0, \ldots, \gamma_{k-1}\ (z \restriction k, s_j, (f_0(\gamma_0), \ldots, f_{k-1}(\gamma_0, \ldots, \gamma_{k-1}))) \in V.$$

We claim first that $P = p[W]$. $p[W] \subseteq P$ is easily checked, and uses only the homogeneity of U. If $z \in P$, consider the ordinal game G_z where I plays out $x \in \omega^\omega$, $\vec{\alpha} = (\alpha_0, \alpha_1, \dots) \in (\omega_\omega)^\omega$ with $\alpha_0 < \omega_2$, II plays out $\vec{\delta} \in (\boldsymbol{\delta}^1_3)^\omega$, and II wins iff $\forall n\ [(x \restriction n, \vec{\alpha} \restriction n) \in U \to (z \restriction n, x \restriction n, \vec{\delta} \restriction n) \in V]$. The game is closed for II, hence determined, and the usual homogeneity argument, using the $\boldsymbol{\delta}^1_3$ completeness of the measures for V, shows that I cannot have a winning strategy [otherwise, by $\boldsymbol{\delta}^1_3$-completeness of the measures $\mu_{s,t}$, we could stabilize I's moves on measure one sets, and then by the homogeneity of those measures, get a play for II which wins]. A winning strategy τ for II then gives functions f_s, and thus ordinals β_i, such that $(z, \vec{\beta}) \in [W]$. Namely, if $\mathrm{lh}(s_i) = k$, then β_i is represented with respect to ν_{s_i} by the function $f_{s_i}(\gamma_0, \dots, \gamma_{k-1}) = \tau(s_i, (\gamma_0, \dots, \gamma_{k-1}))$.

Fix for the moment $\alpha < \omega_2$, $\beta < \boldsymbol{\delta}^1_3$. Let $G_{z,\alpha}$ be the game defined just as G_z, except I's first ordinal move α_0 satisfies $\alpha_0 \le \alpha$. Let $P_{\alpha,\beta}$ be as in Theorem 5.3. For $z \in P_{\alpha,\beta}$, let $b(z,\alpha) < \boldsymbol{\delta}^1_3$ be least so that II has a winning strategy in $G_{z,\alpha}$ playing ordinals $< b(z,\alpha)$ (this exists since $\boldsymbol{\delta}^1_3$ is regular). We claim that $b(\alpha,\beta) \doteq \sup\{b(z,\alpha) : z \in P_{\alpha,\beta}\} < \boldsymbol{\delta}^1_3$. This follows from the fact that $P_{\alpha,\beta}$ is ω_ω-Suslin. To see this, consider the auxiliary game where I plays out $z, x \in \omega^\omega$, $\vec{\gamma} \in (\omega_\omega)^\omega$, $\vec{\alpha} \in (\omega_\omega)^\omega$ with $\alpha_0 \le \alpha$, and II plays out $\vec{\delta} \in (\boldsymbol{\delta}^1_3)^\omega$, and II wins iff

$$\forall n\ [((z \restriction n, \vec{\gamma} \restriction n) \in U_2 \wedge (x \restriction n, \vec{\alpha} \restriction n) \in U) \to (z \restriction n, x \restriction n, \vec{\delta} \restriction n) \in V],$$

where U_2 is a tree on $\omega \times \omega_\omega$ with $p[U_2] = P_{\alpha,\beta}$. II again wins, and $b(\alpha,\beta) \le$ any ordinal η large enough so that II can win this auxiliary game playing ordinals $< \eta$.

Let now $C \subseteq \boldsymbol{\delta}^1_3$ be a closed unbounded set consisting of limit ordinals and closed under b. Let $\delta \in C$ with $\mathrm{cf}(\delta) = \omega_2$. Let $h : \omega_2 \to \delta$ be increasing and cofinal, and $z \in P$ be such that $\forall \alpha < \omega_2\ |A^z_\alpha| = h(\alpha)$. Then we claim that $z \in p[W \restriction \sup_\nu j_\nu(\delta)]$, the supremum ranging over the measures ν_s for U. We must show that II can win G_z playing only ordinals $< \delta$. If I first moves $(x(0), \alpha_0)$, II picks the least ordinal $\eta \le b(\alpha_0, h(\alpha_0)) < \delta$ such that II can win $G_{z,\alpha}$ starting from that position playing only ordinals $< \eta$. II then follows the canonical winning strategy for the closed game $G_{z,\alpha} \restriction \eta$.

Thus, we have produced a tree W and a closed unbounded C such that $P = p[W]$ and for all $\alpha \in C$ of cofinality ω_2 there is a $z \in p[W \restriction \sup_\nu j_{\nu_s}(\alpha)]$ such that $|z| = \alpha$, that is, $\alpha = \sup\{|A^\beta_z| : \beta < \omega_2\}$. Again, the argument for cofinality ω_1 points is similar, using WO instead of WO_2. The argument now finishes exactly as before (i.e., Theorem 5.5). $\dashv$

Theorems 5.5 and 5.6 are the correct extension of the Kunen analysis on ω_1 to $\boldsymbol{\delta}^1_3$. The fact that the identity function of the Kunen analysis is replaced by $\alpha \to \sup_\nu j_\nu(\alpha)$ is, at bottom, the source of the combinatorial complications at the higher levels.

The proofs of Theorems 5.3, 5.5, and 5.6 are quite general. For example, the proof of Theorem 5.6 generalizes to the following (the proof is identical to that of Theorem 5.6 so we omit it).

5.7 Theorem. *Let $\lambda < \kappa$ be regular cardinals, and $\mathbf{\Gamma}$ a non-selfdual pointclass closed under $\forall^{\omega^\omega}, \wedge, \vee$. Assume:*

(1) There is a $\mathbf{\Delta}$ coding of the ordinals less than λ. That is, there is a $\mathbf{\Delta}$ set $C \subseteq \omega^\omega$ and a map $x \to |x|_C < \lambda$ for $x \in C$ such that the relations $(x_1, x_2 \in C \wedge |x_1|_C \leq |x_2|_C)$ and $(x_1, x_2 \in C \wedge |x_1|_C < |x_2|_C)$ are both in $\mathbf{\Delta}$.

(2) There is a homogeneous tree U on C such that for all $x \in C$, $|x| \leq \psi_0(x) < \lambda$, where $\{\psi_n\}$ is the semi-scale corresponding to U.

(3) There is a map $z \to A_z \subseteq \lambda \times \kappa$, for $z \in \omega^\omega$, satisfying:

(a) $\forall f : \lambda \to \kappa\ \exists z\ A_z = f$.

(b) The relation $P'(z, x) \longleftrightarrow [x \in C \wedge \exists! \beta\ A_z(|x|_C, \beta)]$ is in $\mathbf{\Gamma}$.

(c) For all $\alpha < \lambda$, $\beta < \kappa$, $P_{\alpha,\beta} \doteq \{z : \forall \alpha' \leq \alpha\ \exists \beta' \leq \beta\ [A_z(\alpha', \beta') \wedge \forall \beta''(A_z(\alpha', \beta'') \to \beta' = \beta'')]\}$ is in $\mathbf{\Delta}$.

(4) Every $\mathbf{\Gamma}$ set admits a homogeneous tree on κ with κ-complete measures.

(5) Every $\mathbf{\Delta}$ set is α-Suslin for some $\alpha < \kappa$. Also, if $A \subseteq P \doteq \{z : \forall x \in C\ P'(z, x)\}$ is in $\exists^{\omega^\omega} \mathbf{\Delta}$, then $\sup\{|z| : z \in A\} < \kappa$, *where for $z \in P$, $|z|$ is the supremum of the range of the function $A_z : \lambda \to \kappa$.*

Then there is a tree W on $\omega \times \kappa$ with $p[W] = P$ and a closed unbounded $D \subseteq \kappa$ such that for all $\alpha \in D$ with $\mathrm{cf}(\alpha) = \lambda$, *there is a $z \in P$ with $|z| = \alpha$ and $W_z \restriction (\sup_\nu j_\nu(\alpha))$ is ill-founded, the supremum ranging over the measures ν for the homogeneous tree U.*

5.8 Remark. The hypotheses imply that the prewellordering property falls on the $\mathbf{\Gamma}$ side. For if not, then $\check{\mathbf{\Gamma}}$ is closed under wellordered unions. From (3c) it follows that there is a κ increasing sequence of $\mathbf{\Delta}$ sets. Thus, there is a $\check{\mathbf{\Gamma}}$ prewellordering of length κ. If $\mathbf{\Delta}$ is not closed under $\exists^{\omega^\omega}$, then by (5.7) this prewellordering is α-Suslin for some $\alpha < \kappa$, a contradiction. If $\mathbf{\Delta}$ is closed under real quantification, then $\kappa = o(\mathbf{\Delta})$, and $\mathbf{\Delta}$ is closed under $<\kappa$ unions. Then P is a union of $\mathbf{\Delta}$ sets, so is in $\check{\mathbf{\Gamma}}$, and hence in $\mathbf{\Delta}$. This contradicts (5.7). Thus, pwo($\mathbf{\Gamma}$). Similarly, it can be shown that $\mathbf{\Delta}$ is closed under $<\kappa$ length unions and intersections.

This general form of Martin's Theorem is particularly useful when combined with another result of Martin and Steel (cf. [29]) which provides the existence of homogeneous trees in a general setting. We recall this theorem. Let $\{s_i\}_{i \in \omega}$ be an enumeration of $\omega^{<\omega}$ with each sequence preceding its proper extensions, and such that if $s_i = t^\frown a$, $s_j = t^\frown b$, then $a < b \to i < j$. View each real σ as a strategy for II via $\sigma(s_{i+1}) = \sigma(i)$ (note $s_0 = \emptyset$).

Let $\mathbf{\Gamma}$ be a Steel pointclass (i.e., $\mathbf{\Gamma}$ is non-selfdual, closed under $\forall^{\omega^\omega}$, pwo($\mathbf{\Gamma}$), and $\mathbf{\Delta}$ is closed under real quantification). By red($\mathbf{\Gamma}$), let $U, V \subseteq (\omega^\omega)^3$ be disjoint $\mathbf{\Gamma}$ sets such that for every disjoint $\mathbf{\Gamma}$ sets $A, B \subseteq (\omega^\omega)^2$ there is an $x \in \omega^\omega$ with $A = U_x$, $B = V_x$. We say $\mathbf{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$ if the relations

$$R(x,z) \longleftrightarrow \forall z, w\ [U_x(z,w) \vee V_x(z,w)] \wedge \exists w\ U_x(z,w),$$
$$S(x,z) \longleftrightarrow \forall z, w\ [U_x(z,w) \vee V_x(z,w)] \wedge \forall w\ U_x(z,w)$$

are in $\mathbf{\Gamma}$.

5.9 Theorem (AD; Martin, Steel). *Let $\mathbf{\Gamma}$ be a non-selfdual pointclass, $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$, and assume A and A^c are both Suslin. Let $B = \{\sigma : \forall y\ \sigma(y) \in A\}$. Then B is $\forall^{\omega^\omega}\mathbf{\Gamma}$-complete and B admits a scale $\{\psi_n\}$ whose corresponding tree T_ψ is homogeneous. If $\{\phi_n\}$ is a $\mathbf{\Gamma}$ very good scale on A and either $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$ or $\mathbf{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$, then $\{\psi_n\}$ is a $\forall^{\omega^\omega}\mathbf{\Gamma}$ scale. If $\mathbf{\Gamma}$ is closed under $\forall^\omega$, countable unions and intersections, then the measures in T_ψ will be κ-complete, where $\kappa =$ the supremum of the lengths of the $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ prewellorderings of the reals.*

Combining this with Theorem 5.7 we have the following.

5.10 Theorem (AD + $V = L(\mathbb{R})$). *Let $\kappa < \boldsymbol{\delta}^2_1$ be a regular limit Suslin cardinal, and $\lambda < \kappa$ be regular. Let $\mathbf{\Gamma}$ be the pointclass closed under $\forall^{\omega^\omega}$ such that $S(\kappa) = \exists^{\omega^\omega}\mathbf{\Gamma}$. Assume there is a $\mathbf{\Delta}$ coding of λ with homogeneous tree U as in (1) and (2) of Theorem 5.7. Then there is a tree W on $\omega \times \kappa$ with $p[W] = P$, a $\mathbf{\Gamma}$-complete set, and a map $z \to |z| < \kappa$ for $z \in P$ satisfying:*

(1) If $S \subseteq P$ is in $\mathbf{\Delta}$, then $\sup\{|z| : z \in S\} < \kappa$.

(2) There is a closed unbounded $C \subseteq \kappa$ such that for all $\alpha \in C$ of cofinality λ, there is a $z \in P$ such that $|z| = \alpha$ and $W_z \restriction \sup_\nu j_\nu(\alpha)$ is ill-founded, the supremum ranging over the homogeneity measures ν for U.

Proof (Sketch). $\mathbf{\Gamma}$ is closed under countable unions and intersections from [35], and also from [37] $\mathbf{\Gamma}$ has the scale property. The proof of Theorem 3.3 of [35] also shows that $\mathbf{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$. Note also that κ is the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings, and $\mathbf{\Delta}$ is closed under $<\kappa$ unions and intersections. By Theorem 5.9, some $\mathbf{\Gamma}$-complete set admits a homogeneous tree with κ-complete measures, and thus so does every $\mathbf{\Gamma}$ set. The coding $z \to A_z \subseteq \lambda \times \kappa$ is given simply from the Coding Lemma (relative to some $\mathbf{\Delta}$ prewellordering of length λ and some $\mathbf{\Gamma}$-norm on a $\mathbf{\Gamma}$-complete set). We define P, W as in Theorem 5.6, using a homogeneous tree V with κ-complete measures for P'. Items (1)–(5) of Theorem 5.7 are easily checked. ⊣

We note that if λ is a regular Suslin cardinal, the hypotheses of the previous theorem are automatically satisfied. For there is a non-selfdual pointclass $\mathbf{\Lambda}$

closed under $\forall^{\omega^\omega}$ with the scale property such that λ is the supremum of the lengths of the $\boldsymbol{\Delta} = \boldsymbol{\Delta}(\boldsymbol{\Lambda})$ prewellorderings. If λ is inaccessible, $\boldsymbol{\Delta}$ is uniformly closed under $\exists^{\omega^\omega}$. If λ is a successor Suslin, then from Theorem 4.3 of [37], $\boldsymbol{\Lambda} = \forall^{\omega^\omega}\boldsymbol{\Lambda}^-$ with $\boldsymbol{\Lambda}^-$ closed under $\exists^{\omega^\omega}$ with scale($\boldsymbol{\Lambda}^-$). In either case, by Theorem 5.9, there is a $\boldsymbol{\Lambda}$-scale $\{\psi_n\}$ on a $\boldsymbol{\Lambda}$-complete set C whose tree U is homogeneous. We must have ψ_0 onto λ in this case (assuming without loss of generality that the scale is regular, i.e., all norms map onto an initial segment on the ordinals). Letting our coding of λ be given by $|x| = \psi_0(x)$ for $x \in C$, this verifies (1) and (2) of Theorem 5.7.

We make some comments on the possible significance of Theorem 5.10 to extending the inductive analysis of the projective sets to higher levels of $L(\mathbb{R})$. In analyzing the measures on κ, the first step of the proof of Theorem 4.8 shows it is necessary to have at hand an analysis of the functions on κ with respect to the semi-normal measures (the measures that give every closed unbounded set measure one). At successor Suslin cardinals, Theorem 5.7 and the analysis below κ should give the analog of Theorem 5.6. If κ is singular, there should be no analog of Martin's Theorem required, though other methods become necessary (we will discuss some of these in the next section). For κ inaccessible Suslin, Theorem 5.10 is a step towards providing the necessary result, but is not complete as it handles only the normal measures on κ corresponding to fixed cofinalities $\lambda < \kappa$.

5.2. Some Canonical Measures

For the remainder of this section we return to the projective hierarchy, and discuss the other main ingredient; the theory of descriptions. As we said in the introduction, our purpose here is not to present complete details, but rather to exposit the main ideas. We will frequently illustrate a proof by considering a case which shows the central idea.

According to Theorem 5.6, in analyzing functions $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the ω_1 cofinal normal measure, we need to consider ultrapowers by the measures W^m_1. There is nothing more to say in this case. With respect to the ω_2-cofinal normal measure, we need consider ultrapowers by the measures $M^{\mathcal{R}}$ occurring in the homogeneous tree on a $\boldsymbol{\Pi}^1_2$ set. One suspects that there is a combinatorially simpler family which "dominates" these measures. Indeed, it simplifies considerably the resulting theory to have such a family at hand.

Let $<_m$ be the ordering on $(\omega_1)^m$ corresponding to the permutation $\pi = (m, 1, 2, \ldots, m-1)$. Thus,

$$\begin{aligned}&(\alpha_1, \ldots, \alpha_m) <_m (\beta_1, \ldots, \beta_m)\\ &\quad\longleftrightarrow (\alpha_m, \alpha_1, \ldots, \alpha_{m-1}) <_{\text{lex}} (\beta_m, \beta_1, \ldots, \beta_{m-1}).\end{aligned}$$

Recall that a function f from the domain of a wellordering $\prec$ to the ordinals is of the correct type if it order-preserving with respect to $\prec$, of uniform cofinality ω, and everywhere discontinuous.

5.11 Definition. S_1^m is the measure on ω_{m+1} induced by the strong partition relation on ω_1 and functions $h : \mathrm{dom}(<_m) \to \omega_1$ of the correct type. That is, A has measure one if there is a closed unbounded $C \subseteq \omega_1$ such that $[f]_{W_1^m} \in A$ for all $f : \mathrm{dom}(<_m) \to C$ of the correct type.

Thus, S_1^1 is the ω cofinal normal measure on ω_2. We let W_1 denote the collection of measures W_1^m, and S_1 the collection S_1^m.

According to the next theorem, we need only consider ultrapowers by the measures S_1^m in Theorem 5.6.

5.12 Theorem. *There is a closed unbounded $C \subseteq \boldsymbol{\delta}_3^1$ such that for all $\theta \in C$ of cofinality ω_2, $\sup_\nu j_\nu(\theta) = \sup_m j_{S_1^m}(\theta)$, the first supremum ranging over measures ν of the form $M^{\mathcal{R}}$ for some type-1 tree of uniform cofinalities $\mathcal{R}$.*

We let C be the set of θ closed under the embeddings j_ν, for $\nu = M^{\mathcal{R}}$ (recall Theorem 4.35). The theorem follows easily from the following lemma:

5.13 Lemma. *Let $\nu = M^{\mathcal{R}}$. There is an $m \in \omega$, a measure μ on ω_ω, and a function $h : \omega_{m+1} \to (\omega_\omega)^{<\omega}$ satisfying the following:*

(1) $\forall^*_{S_1^m}\alpha\ \exists\vec{\gamma} \in \mathrm{dom}(\nu)\ \forall^*_\mu\beta\ h(\alpha)(\beta) < \vec{\gamma}$. By $h(\alpha)(\beta) < \vec{\gamma}$ we mean that if $\vec{\gamma}$ is represented by $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, and $h(\alpha)(\beta)$ by $g : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$, then $[g^{\langle p_1,i_1\rangle}]_{W_1^1} < [f^{\langle p_1,i_1\rangle}]_{W_1^1}$ for all $\langle p_1, i_1\rangle \in \mathrm{dom}\,\mathcal{R}$.

(2) If $A \subseteq \mathrm{dom}(\nu)$ has ν measure one, then $\forall^*_{S_1^m}\alpha\ \forall^*_\mu\beta\ h(\alpha)(\beta) \in A$.

To see this proves the theorem, fix $\delta = [F]_\nu < j_\nu(\theta)$, where $\theta \in C$. Define ϵ by: $\forall^*_{S_1^m}\alpha\ \forall^*_\mu\beta\ \epsilon(\alpha)(\beta) = F(h(\alpha)(\beta))$. From (5.13) this depends only on $[F]_\nu$, and from (5.13) it follows easily that $\epsilon < j_{S_1^m}(\theta)$. [We use here the fact that if $F : \mathrm{dom}(\nu) \to \mathrm{On}$, then there is a ν measure one set A such that if $f, g : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of type $\mathcal{R}$ represent $[f], [g] \in A$ and $[g^{\langle p_1,i_1\rangle}] < [f^{\langle p_1,i_1\rangle}]$ for all $\langle p_1, i_1\rangle \in \mathrm{dom}(\mathcal{R})$, then $F([g]) \le F([f])$. This follows by an easy partition argument.] From (5.13), the map $\pi(\delta) = \epsilon$ is an embedding of $j_\nu(\theta)$ into $j_{S_1^m}(\theta)$.

The lemma is proved by a direct construction of the measure μ. We illustrate with a case. Suppose that $<^{\mathcal{R}}$ is the lexicographic ordering on tuples $\langle \gamma_1, i_1, \gamma_2, \gamma_3\rangle$ where $\gamma_3 < \gamma_2 < \gamma_1 < \omega_1$, and $i_1 \in \{0, 1\}$ (we have removed irrelevant indices from our notation now). Also, ν is induced by functions $f : \mathrm{dom}(<^{\mathcal{R}}) \to \omega_1$ of the correct type (ν is actually the two-fold product of the measure on ω_4 corresponding to the permutation $(3, 2, 1)$). In this case, take $m = 4$ and $\mu = W_1^2 \times S_1^2$. We define the function h as required. Let $\alpha < \omega_5$ be represented by $f_\alpha : \mathrm{dom}(<_4) \to \omega_1$ of the correct type. Define $h(\alpha)$ so that for almost all $\vec{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathrm{dom}\,\mu$, so $\beta_1 < \beta_2 < \omega_1$ and $\beta_3 < \omega_3$ is represented by $f_{\beta_3} : \mathrm{dom}(<_2) \to \omega_1$ of the correct type, $h(\alpha)(\vec{\beta}) \in \mathrm{dom}(\nu)$ is represented by $h(\alpha)(\vec{\beta})(\langle \gamma_1, i_1, \gamma_2, \gamma_3\rangle) = f_\alpha(\beta_{i_1}, \gamma_2, f_{\beta_3}(\gamma_3, \gamma_2), \gamma_1)$. It is easy to check that this is well-defined and satisfies (1) and (2). $\dashv$

We state one more embedding theorem which helps to simplify the analysis. By Theorem 2.18, $\boldsymbol{\delta}^1_5 = \lambda_5^+$, where λ_5 is least such that every $\boldsymbol{\Pi}^1_4$ set is λ_5-Suslin. The weak partition relation on $\boldsymbol{\delta}^1_3$ and the homogeneous tree construction shows every $\boldsymbol{\Pi}^1_3$ set admits a homogeneous tree with measures ν_s on $\boldsymbol{\delta}^1_3$. Thus, every $\boldsymbol{\Sigma}^1_4$ set is weakly homogeneous with the same family of measures. The homogeneous tree construction again shows every $\boldsymbol{\Pi}^1_4$ set is λ-Suslin, where $\lambda = \sup_\nu j_\nu(\boldsymbol{\delta}^1_3)$, ν ranging over the measures in a homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set (granting the strong partition relation on $\boldsymbol{\delta}^1_3$, the resulting tree on $\omega \times \lambda$ is also homogeneous). Thus, $\lambda_5 \leq \lambda$. Computing the upper bound for $\boldsymbol{\delta}^1_5$ is thus reduced to bounding the ultrapowers $j_\nu(\boldsymbol{\delta}^1_3)$.

Again, one suspects that a simpler family of measures will suffice here.

5.14 Definition. W^m_3 is the measure on $\boldsymbol{\delta}^1_3$ induced by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type, and the measure S^m_1 on ω_{m+1}. That is, $W^m_3(A) = 1$ iff there is a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $f : \omega_{m+1} \to C$ of the correct type, $[f]_{S^m_1} \in A$.

5.15 Theorem. *Let ν be a measure on $\boldsymbol{\delta}^1_3$ occurring in the homogeneous tree on a $\boldsymbol{\Pi}^1_3$-complete set. Then for some $m \in \omega$, $j_\nu(\boldsymbol{\delta}^1_3) \leq j_{W^m_3}(\boldsymbol{\delta}^1_3)$.*

5.16 Remark. The theorem actually holds for any measure ν on $\boldsymbol{\delta}^1_3$, although this requires the analysis of measures on $\boldsymbol{\delta}^1_3$ to show. The proof of Theorem 5.15 is similar to that of Theorem 5.12. The reader can find the details in [11].

5.3. The Higher Descriptions

In view of Theorem 5.15, the basic problem in computing the upper bound for $\boldsymbol{\delta}^1_5$ is to analyze equivalence classes of functions $F : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to the measures W^m_3. This leads us to the notion of a level 1 description. It is helpful to consider some examples first.

Let us construct first an equivalence class of a function $F : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to W^1_3. We must define $F([f]_{S^1_1})$ for $f : \omega_2 \to \boldsymbol{\delta}^1_3$ of the correct type. We will define $F(f)$ for any such f, and note that our definition only depends on $[f]_{S^1_1}$. $F(f)$ is defined to be the ordinal represented with respect to $K_1 = S^1_1$ by the function which assigns to $[h_1]_{W^1_1}$ (here $h_1 : \omega_1 \to \omega_1$ is of the correct type) the value $F(f, [h_1])$. We will define $F(f, h_1)$ for any such h_1, and note that this depends only on $[h_1]_{W^1_1}$, so we set $F(f, [h_1]) = F(f, h_1)$. Finally, $F(f, h_1)$ is the ordinal represented with respect to $K_2 = S^1_1$ by the function which assigns to $[h_2]_{W^1_1}$ the value $F(f, h_1, h_2) \doteq f([h_1 \circ h_2]_{W^1_1})$. Extending our earlier notational convention, we abbreviate this definition by saying $\forall^* f\ \forall^* h_1\ \forall^* h_2\ F(f, h_1, h_2) = f([h_1 \circ h_2])$. We could also write $\forall^* f\ \forall^* h_1\ \forall^* h_2\ F(f, h_1, h_2) = f([h])$, where $\forall^*_{W^1_1} \alpha\ h(\alpha) = h_1(h_2(\alpha))$.

It is easy to see that this definition is well-defined. Note, however, the map $(h_1, h_2) \to f([h_1 \circ h_2]_{W^1_1})$ is not well-defined with respect to $S^1_1 \times S^1_1$, that is, it does not just depend on $[h_1]_{W^1_1}, [h_2]_{W^1_1}$. Note also that it is important that we compose the functions h_1, h_2 in the order shown; the other way does not lead to a well-defined definition.

This simple example shows that the basic operation of composition leads to well-defined definitions of equivalence classes of functions $F : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. In the general definition of a level 1 description, we generalize by allowing finitely many functions $h_1, \dots, h_t$, where each h_i is either a function $h_i : \mathrm{dom}\,(<_m) \to \omega_1$ of the correct type, or a finite tuple of ordinals $\beta_1 < \cdots < \beta_m < \omega_1$. That is, instead of S^1_1 in the example above, we allow measures S^m_1, W^m_1.

We first consider one more example. We now define the equivalence class of three functions F_1, F_2, F_3 with respect to the measure W^2_3. Thus, we must define $F_i([f]_{S^2_1}) = F_i(f)$ where $f : \omega_3 \to \boldsymbol{\delta}^1_3$ is of the correct type. In all three case we will use the sequence of measures $K_1 = K_2 = S^2_1$, $K_3 = W^2_1$. We define F_1 by:

$$\forall^* f\ \forall^* h_1\ \forall^* h_2 \forall^* h_3 = (\beta_1, \beta_2)\ F_1(f, h_1, h_2, h_3) = \sup\{f(\delta) : \delta < [h]_{W^2_1}\},$$

where $\forall^*_{W^2_1} \alpha_1, \alpha_2\ h(\alpha_1, \alpha_2) = h_1(1)(\alpha_2)$ (recall that $h_1(1)(\alpha_2) = \sup\{h_1(\gamma, \alpha_2) : \gamma < \alpha_2\}$). We define $F_2(f, h_1, h_2, h_3) = f([h])$, where

$$h(\alpha_1, \alpha_2) = h_1(h_2(1)(\alpha_1), \alpha_2).$$

We define $F_3(f, h_1, h_2, h_3) = f([h])$, where

$$h(\alpha_1, \alpha_2) = h_1(h_2(\beta_2, \alpha_1), \alpha_2).$$

It is easy to check that all three functions are well-defined, and that $\forall^*_{W^2_3}[f]$ $F_3(f) < F_2(f) < F_1(f)$. We will return to this example in a moment.

In general, a description d will be a finitary object defined relative to a sequence $K_1, \dots, K_t$ of measures, each of the form W^r_1 or S^r_1. It will describe how, given the functions $h_1, \dots, h_t$, to generate the function h as in the above examples. In the first example, $h : \omega_1 \to \omega_1$, and in the second example, we had $h : (\omega_1)^2 \to \omega_1$ in all three cases. In general we will have $h : (\omega_1)^m \to \omega_1$ for some $m \in \omega$. The set $\mathcal{D}$ of descriptions will be partitioned accordingly as $\mathcal{D} = \bigcup_m \mathcal{D}_m$. The $d \in \mathcal{D}_m$, which we call the m-descriptions, will thus generate $h : (\omega_1)^m \to \omega_1$ given the functions $h_1, \dots, h_t$. We write also $\mathcal{D}_m(K_1, \dots, K_t)$ to denote those m-descriptions defined relative to $K_1, \dots, K_t$.

Thus, in the first example the underlying description (which we haven't defined yet) lies in $\mathcal{D}_1(S^1_1, S^1_1)$, and in the second, the descriptions lie in $\mathcal{D}_2(S^2_1, S^2_1, W^2_1)$.

Given a description $d \in \mathcal{D}_m(K_1, \dots, K_t)$, we will write $h(d; h_1, \dots, h_t)$ to denote the function $h : (\omega_1)^m \to \omega_1$ generated according to d from the $\vec{h}$.

Fix now $m \in \omega$ and the sequence $K_1, \dots, K_t$. The primitive descriptions in $\mathcal{D}_m(K_1, \dots, K_t)$ are those which do not involve composing functions. We refer to these as the *basic* descriptions, and the others as non-basic. As we define the descriptions, we simultaneously define $k(d) \in \{1, \dots, t\} \cup \{\infty\}$ for each description which gives the least k so that the function h_k is involved in the definition of $h(d;\vec{h})$. As we define the descriptions, we also define how to interpret them, i.e. we define $h(d;\vec{h})(\alpha_1,\dots,\alpha_m)$. For $h_k : \mathrm{dom}(<_r) \to \omega_1$ order-preserving, and $l \le r$, recall the definition of $h_k(l) : (\omega_1)^l \to \omega_1$ from Definition 4.22. In this case it reduces to (for $l = r$, $h_k(l) = h_k$):

$$\begin{aligned} &h_k(l)(\alpha_1,\dots,\alpha_l)\\ &\quad = \sup\{h_k(\alpha_1,\dots,\alpha_{l-1},\beta_1,\dots,\beta_{r-l},\alpha_l) : \alpha_{l-1} < \beta_1 < \cdots < \beta_{r-l} < \alpha_l\}. \end{aligned}$$

5.17 Definition (Descriptions).

(1) (Basic) We allow:

(a) $d = (p)$ where p is an integer $1 \le p \le m$ (which we put in parentheses to distinguish from a level-1 description). We define $h(d;\vec{h})(\alpha_1,\dots,\alpha_m) = \alpha_p$. We set $k(d) = \infty$.

(b) $d = (k;p)$ where $1 \le k \le t$, K_k is of the form $K_k = W_1^r$, and $1 \le p \le r$. In this case, we define $h(d;\vec{h})(\alpha_1,\dots,\alpha_m) = \beta_p$, where $h_k = (\beta_1,\dots,\beta_r)$. We set $k(d) = k$.

(2) (Non-Basic) Suppose $1 \le k \le t$ and $K_k = S_1^r$. We allow:

(a) $d = (k; d_r, d_1,\dots,d_l)$, where $d_1,\dots,d_l,d_r \in \mathcal{D}_m(K_1,\dots,K_t)$, $l < r$, and $k(d_1),\dots,k(d_l),k(d_r) > k$. We set $k(d) = k$ and define:

$$h(d;\vec{h})(\vec{\alpha}) = h_k(l+1)(h(d_1;\vec{h})(\vec{\alpha}),\dots,h(d_l;\vec{h})(\vec{\alpha}),h(d_r;\vec{h})(\vec{\alpha}))$$

(b) $d = (k; d_r, d_1,\dots,d_l)^s$, where $d_1,\dots,d_l,d_r \in \mathcal{D}_m(K_1,\dots,K_t)$, $l < r$, $k(d_1),\dots,k(d_l),k(d_r) > k$, and s is a formal symbol (which stands for "sup"). We require in this case that $r \ge 2$, $l \ge 1$. We set $k(d) = k$ and define:

$$\begin{aligned} h(d;\vec{h})(\vec{\alpha}) = &\sup\bigl\{h_k(l+1)(h(d_1;\vec{h})(\vec{\alpha}),\dots,h(d_{l-1};\vec{h}),\beta,\\ &h(d_r;\vec{h})(\vec{\alpha})) : \beta < h(d_l;\vec{h})\bigr\}. \end{aligned}$$

We write $(k; d_r, d_1,\dots,d_l)^{(s)}$ to indicate the symbol s may or may not appear.

This completes the definition of $\mathcal{D}$ and the "interpretation function" h. We will write d^m when we wish to emphasize $d \in \mathcal{D}_m$. In the first example above, the description is given by $d^1 = (1;(2;(1)))$. In the second example, the three descriptions are given by $d_a^2 = (1;(2))$, $d_b^2 = (1;(2);(2;(1)))$, and $d_c^2 = (1;(2);(2;(1),(3;2)))$.

If $d_1, d_2 \in \mathcal{D}_m(K_1, \ldots, K_t)$, define $d_1 < d_2$ iff $\forall^* h_1 \cdots \forall^* h_t\ h(d_1; \vec{h}) < h(d_2; \vec{h})$ almost everywhere. Here $\forall^*$ refers to the measure on functions of the correct type induced by the strong partition relation. Note that in a non-basic description $d = (k; d_r, d_1, \ldots, d_l)$ the component descriptions are listed in their order of significance in determining $h(d; \vec{h})$. It is not difficult to reformulate the $<$ relation on $\mathcal{D}_m(K_1, \ldots, K_t)$ in a purely "syntactical" manner, we leave this to the reader.

The requirement that $k(d_r), k(d_1), \ldots, k(d_l) > k = k(d)$ for non-basic descriptions d is one of two necessary to ensure the equivalence class of $h(d; \vec{h})$ is well-defined with respect to the measures $K_1, \ldots, K_t$. This guarantees that for almost all $h_{k+1}, \ldots, h_t$, the values $h(d_r; \vec{h})(\vec{\alpha})$, etc. that we are putting into h_k will lie in a closed unbounded set on which two functions h_k, h'_k (with $[h_k] = [h'_k]$) agree. The other requirement is that these values be in the correct order. We can now state this requirement, which is referred to as "condition C".

5.18 Definition (Definition of C). We say $d \in \mathcal{D}$ *satisfies condition* C if either d is basic or else d is non-basic, say of the form $d = (k; d_r, d_1, \ldots, d_l)^{(s)}$, all $d_r, d_1, \ldots, d_l$ satisfy C, and $d_1 < d_2 < \cdots < d_l < d_r$.

It is now easy to check that if d satisfies this condition, then the equivalence class of $h(d; \vec{h})$ is well-defined in the following precise sense:

5.19 Lemma. *Suppose $d \in \mathcal{D}_m(K_1, \ldots, K_t)$ satisfies condition* C. *Then for almost all h_1, if $[h'_1] = [h_1]$, then for almost all h_2, if $[h'_2] = [h_2], \ldots$ for almost all h_t if $[h'_t] = [h_t]$, then $[h(d; h_1, \ldots, h_t)]_{W_1^m} = [h(d; h'_1, \ldots, h'_t)]_{W_1^m}$.*

All of the descriptions in the examples above satisfy C.

From now on, we officially let $h(d; \vec{h})$ be the ordinal $< \omega_{m+1}$ (where $d \in \mathcal{D}_m$) represented by the function h.

We expand a little our notational convention mentioned at the end of the introduction. Suppose that $K_1, \ldots, K_t$ is a sequence of measures, $d \in \mathcal{D}(K_1, \ldots, K_t)$ satisfies C, and $P \subseteq \mathrm{On}$. When we write

$$\forall^* h_1, \ldots, h_t\ P(h(d; h_1, \ldots, h_t)),$$

we mean: $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2, \ldots, \forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $P(h(d; h_1, \ldots, h_t))$. If $\theta \in \mathrm{On}$, we write

$$\forall^* h_1, \ldots, h_t\ P(\theta(h_1, \ldots, h_t))$$

to mean: if we fix a representing function $\eta_1 \to \theta(\eta_1)$ for θ with respect to K_1 then $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then if we fix a representing function $\eta_2 \to \theta([h_1], \eta_2)$ for $\theta([h_1])$ with respect to K_2, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2$, $\ldots$, if we fix a representing function $\eta_t \to \theta([h_1], \ldots, [h_{t-1}], \eta_t)$ for $\theta([h_1], \ldots, [h_{t-1}])$ with respect to K_t, then $\forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $P(\theta([h_1], \ldots, [h_t]))$.

We can use these conventions simultaneously. For example, for d satisfying C we may write

$$\forall^* h_1, \dots, h_t \;\; \mathrm{cf}(h(d; h_1, \dots, h_t)) < \theta(h_1, \dots, h_t).$$

This abbreviates: $\forall^*_{K_1} \eta_1$ if $[h_1] = \eta_1$, then $\forall^*_{K_2} \eta_2$ if $[h_2] = \eta_2, \dots, \forall^*_{K_t} \eta_t$ if $[h_t] = \eta_t$, then $\mathrm{cf}(h(d; h_1, \dots, h_t)) < \theta([h_1], \dots, [h_t])$. Written out in full, this becomes: if $\eta_1 \to \theta(\eta_1)$ represents θ with respect to K_1, then $\forall^*_{K_1} \eta_1$, if $[h_1] = \eta_1$ and $\eta_2 \to \theta([h_1], \eta_2)$ represents $\theta([h_1])$ with respect to K_2, then $\forall^*_{K_2} \eta_2$, if $[h_2] = \eta_2$ and $\eta_3 \to \theta([h_1], [h_2], \eta_3)$ represents $\theta([h_1], [h_2])$ with respect to $K_3, \dots, \forall^*_{K_t} \eta_t$, if $[h_t] = \eta_t$, then $\mathrm{cf}(h(d; h_1, \dots, h_t)) < \theta([h_1], \dots, [h_t])$. Such a statement is well-defined by Lemma 5.19.

Recall the purpose of a description $d \in \mathcal{D}_m$ is to generate an ordinal $\alpha < \omega_{m+1}$ which we can plug into a function $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ in our attempt to generate an equivalence class $[F]_{W^m_3}$. By "plug in" we mean either take $f(\alpha)$ or $\sup\{f(\alpha') : \alpha' < \alpha\}$. Condition C guarantees the ordinal α is well-defined in an appropriate sense. However, it does not guarantee α will be representable by a function $h : \mathrm{dom}(<_m) \to \omega_1$ of the correct type, or that it is a limit of such ordinals. Thus we introduce another condition, condition D, below.

First we extend slightly the notion of a description. Let $\bar{\mathcal{D}}_m$ be the set of objects ("extended descriptions") of the form (d) or $(d)^s$ where $d \in \mathcal{D}_m$ satisfies C, together with one new object $()^s$. We write $(d)^{(s)}$ to denote either (d) or $(d)^s$.

5.20 Definition (Definition of D). Suppose $d \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$. Then:

1. (d) *satisfies condition* D if $\forall^* h_1, \dots, h_t \; h(d; \vec{h}) : (\omega_1)^m \to \omega_1$ is of the correct type almost everywhere (i.e. restricted to a measure one set with respect to W^m_1).

2. $(d)^s$ *satisfies condition* D if $\forall h_1, \dots, h_t \; h(d; \vec{h})$ is the supremum of ordinals representable by functions $h : (\omega_1)^m \to \omega_1$ of the correct type almost everywhere. We also define $()^s$ to satisfy D.

We can now describe our generation of equivalence classes.

5.21 Definition. Let $m \in \omega$, $K_1, \dots, K_t \in W_1 \cup S_1$, let $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$ satisfy condition D, and let $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. We define an ordinal which we denote by $(g; (d)^{(s)}; K_1, \dots, K_t)$. We represent this ordinal with respect to W^m_3 by the function which assigns to $[f]_{S^m_1}$ the ordinal $(g; f; (d)^{(s)}; K_1, \dots, K_t)$, for $f : \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type. We represent $(g; f; (d)^{(s)}; K_1, \dots, K_t)$ with respect to K_1 by the function which assigns to $[h_1]$ the ordinal $(g; f; (d)^{(s)}; h_1, K_2, \dots, K_t)$, and in general, represent $(g; f; (d)^{(s)}; h_1, \dots, h_{i-1}\ K_i, \dots, K_t)$ with respect to K_i by the function which assigns to $[h_i]$ the ordinal $(g; f; (d)^{(s)}; h_1, \dots,\ h_i,\ K_{i+1}, \dots, K_t)$. Finally, we define $(g; f; (d)^{(s)}, h_1, \dots, h_t)$ by cases as follows:

(1) If s does not appear, then $(g; f; (d); h_1, \dots, h_t) = g(f(h(d; \bar{h})))$.

(2) If s appears, then $(g; f; (d)^s; h_1, \dots, h_t) = g(\sup\{f(\beta) : \beta < h(d; \vec{h})\})$.

(3) For the object $()^s$, $(g; f; ()^s; h_1, \dots, h_t) = g(\sup\{f(\beta) : \beta < \omega_{m+1}\})$.

Of particular importance is the case $g = \mathrm{id}$, the identity function. In the first example considered previously, the equivalence class $[F]_{W_3^1}$ constructed was $(\mathrm{id}; (d); K_1, K_2)$, where $d = (1; (2; (1)))$, and $K_1 = K_2 = S_1^1$. The three functions of the second example represent the ordinals $(\mathrm{id}; (d_a)^s; K_1, K_2, K_3)$, $(\mathrm{id}; (d_b); K_1, K_2, K_3)$, and $(\mathrm{id}; (d_c); K_1, K_2, K_3)$ respectively.

It turns out, though we will not prove this fully here, that the cardinals $\boldsymbol{\delta}_3^1 < \kappa < (\boldsymbol{\delta}_5^1)^-$ are precisely the ordinals of the form $(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t)$.

We have thus seen how descriptions generate equivalence classes of functions $F : \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ with respect to the measures W_3^m, just as the trivial descriptions did for functions $F : \omega_1 \to \omega_1$ with respect to the measures W_1^m, Definition 4.3. The main task remaining is to formulate and prove a result analogous to Lemma 4.5, the "main lemma" in the theory of trivial descriptions. To do that we need the correct analog of the lowering operator $\mathcal{L}$ (which we will still call $\mathcal{L}$).

If we fix $m \in \omega$ and measures $K_1, \dots, K_t \in W_1 \cup S_1$, then the relation $<$ on $\mathcal{D}_m(K_1, \dots, K_t)$ is a linear ordering. For $d \in \mathcal{D}_m(K_1, \dots, K_t)$, $\mathcal{L}(d)$ will again give the description preceding d in this ordering, except for a unique minimal description. For the sake of completeness we will give the complete definition of $\mathcal{L}$, though will be content to illustrate the proof of the main lemma through one of our examples. The $\mathcal{L}$ operation is defined by defining a series of approximations $\mathcal{L}^k$ to it. $\mathcal{L}^k(d)$ will only be defined for d with $k(d) \geq k$. Roughly speaking, $\mathcal{L}^k$ is the result of holding $h_1, \dots, h_{k-1}$ constant and lowering with respect to $h_k, \dots, h_t$ only. We will thus take $\mathcal{L}(d) = \mathcal{L}^1(d)$. Following [11], we define $\mathcal{L}^k$ as follows.

5.22 Definition (Definition of $\mathcal{L}^k$). Let $m \in \omega$ and $K_1, \dots, K_t \in W_1 \cup S_1$. Let $k \in \{1, \dots, t\} \cup \{\infty\}$, and assume $d \in \mathcal{D}_m(K_1, \dots, K_t)$ with $k(d) \geq k$. Then $\mathcal{L}^k(d)$ is defined by reverse induction on k through the following cases:

I $k = \infty$. So, $d = (i)$ where $1 \leq i \leq m$. If $i > 1$, then $\mathcal{L}^\infty(d) = (i-1)$. For $i = 1$, d is minimal with respect to $\mathcal{L}^\infty$.

II $1 \leq k \leq t$.

(1) $k = k(d)$.

(a) $d = (k; p)$ is basic. For $p > 1$, we set $\mathcal{L}^k(d) = (k; p-1)$, and for $p = 1$ we define d to be minimal.

(b) $d = (k; d_r, d_1, \dots, d_l)$, where $K_k = S_1^r$ and $l = r - 1$. We set $\mathcal{L}^k(d) = (k; d_r, d_1, \dots, d_l)^s$ if $l \geq 1$, and if $r = 1$ and $d = (k; d_r)$, we set $\mathcal{L}^k(d) = d_r$.

(c) $d = (k; d_r, d_1, \ldots, d_l)$, where $K_k = S_1^r$ and $l < r - 1$. First assume $l \geq 1$. If $\mathcal{L}^{k+1}(d_r)$ is defined and $> d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_l, \mathcal{L}^{k+1}(d_r))$. If $\mathcal{L}^{k+1}(d_r)$ is not defined or is $\leq d_l$, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_l)^s$. If $l = 0$ (so $d = (k; d_r)$), we set $\mathcal{L}^k(d) = (d_r; \mathcal{L}^{k+1}(d_r))$ if $\mathcal{L}^{k+1}(d_r)$ is defined, and otherwise $\mathcal{L}^k(d) = d_r$.

(d) $d = (k; d_r, d_1, \ldots, d_l)^s$, where $K_k = S_1^r$ (so $l \geq 1$). We set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l))$ if $\mathcal{L}^{k+1}(d_l)$ is defined and if $l \geq 2$ also satisfies $\mathcal{L}^{k+1}(d_l) > d_{l-1}$. Otherwise, we set $\mathcal{L}^k(d) = (k; d_r, d_1, \ldots, d_{l-1})^s$ if $l \geq 2$ and for $l = 1$, $\mathcal{L}^k(d) = d_r$.

(2) $k < k(d)$, $K_k = W_1^r$.

(a) d not minimal with respect to $\mathcal{L}^{k+1}$. We then set $\mathcal{L}^k(d) = \mathcal{L}^{k+1}(d)$.

(b) d is minimal with respect to $\mathcal{L}^{k+1}(d)$. We then set $\mathcal{L}^k(d) = (k; r)$.

(3) $k < k(d)$, $K_k = S_1^r$.

(a) d not minimal with respect to $\mathcal{L}^{k+1}$. We then set $\mathcal{L}^k(d) = (k; \mathcal{L}^{k+1}(d))$.

(b) d minimal with respect to $\mathcal{L}^{k+1}$. Then d is minimal with respect to $\mathcal{L}^k$.

Recall our previous example where $m = 2$, $K_1 = K_2 = S_1^2$, $K_3 = W_1^2$, and we had the three descriptions $d_a = (1; (2))$, $d_b = (1; (2); (2; (1)))$, and $d_c = (1; (2); (2; (1), (3; 2)))$. The reader can check now that $\mathcal{L}(d_a) = d_b$, $\mathcal{L}(d_b) = (1; (2); (2; (1)))^s$, and $\mathcal{L}((1; (2); (2; (1)))^s) = d_c$.

We also extend the $\mathcal{L}$ operation to $\bar{\mathcal{D}}_m(K_1, \ldots, K_t)$ as follows. We set $\mathcal{L}((d)) = ((d)^s)$, and $\mathcal{L}((d)^s) = (\mathcal{L}^{(p)}(d))$, where $\mathcal{L}^{(p)}(d)$ denotes the pth iterate of $\mathcal{L}$, and p is least so that $\mathcal{L}^{(p)}(d)$ satisfies condition D. If such a p does not exist, we say $(d)^s$ is minimal with respect to $\mathcal{L}$. Finally, for the distinguished object $()^s$, we define $\mathcal{L}(()^s) = (\tilde{d})^{(s)}$, where $\tilde{d}$ is the maximal description in $\mathcal{D}_m(K_1, \ldots, K_t)$ such that $(\tilde{d})$ or $(\tilde{d})^s$ satisfies D (in the first case, s does not appear, and in the second it does). If there are no descriptions satisfying D, then $()^s$ is declared minimal.

To illustrate, let $m = 2$, and consider the sequence of measures K_1, K_2, K_3 where $K_1 = K_2 = S_1^2$, and $K_3 = W_1^2$. Applying the $\mathcal{L}$ operation repeatedly to $()^s \in \bar{\mathcal{D}}_2(K_1, K_2, K_3)$ results in a sequence whose first few terms are: $(d_0)^s$, (d_1), $(d_1)^s$, (d_2), $(d_2)^s$, (d_3), $(d_3)^s$, $(d_4)^s$, (d_5), $(d_5)^s$, (d_6), $(d_6)^s$, (d_7), $(d_7)^s$,

(d_8), $(d_8)^s$, (d_9), where:

$$\begin{aligned}
d_0 &= (1;(2;(2))) \\
d_1 &= (1;(2;(2));(2;(2);(1))) \\
d_2 &= (1;(2;(2));(2;(2);(1)))^s \\
d_3 &= (1;(2;(2));(2;(2);(1))^s) \\
d_4 &= (1;(2;(2));(2;(2);(1))^s)^s \\
d_5 &= (1;(2;(2));(2;(2);(3;2))) \\
d_6 &= (1;(2;(2));(2;(2);(3;2)))^s \\
d_7 &= (1;(2;(2));(2;(2);(3;2))^s) \\
d_8 &= (1;(2;(2));(2;(2);(3;2))^s)^s \\
d_9 &= (1;(2;(2));(2;(2);(3;1)))
\end{aligned}$$

Note that (d_0), and (d_4) do not satisfy Condition D.

We now state our "main lemma", the analog of Lemma 4.5.

5.23 Theorem (Main Lemma). *Let $(d)^{(s)} \in \bar{\mathcal{D}}(K_1,\dots,K_t)$ satisfy Condition* D. *Suppose $\theta < (\mathrm{id};(d)^{(s)};K_1,\dots,K_t)$. Then:*

(1) If $(d)^{(s)}$ is not minimal with respect to $\bar{\mathcal{L}}$, then there is a $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that $\theta < (g;\mathcal{L}((d)^{(s)});\ K_1,\dots,K_t)$.

(2) If $(d)^{(s)}$ is minimal with respect to $\mathcal{L}$, then $\theta < \boldsymbol{\delta}^1_3$.

We will illustrate the proof of the main lemma by considering the example $(d_a)^s$ above, where $d_a = (1;(2))$. So, fix $\theta < (\mathrm{id};(d_a)^s;K_1,K_2,K_3)$. Thus,

$$\forall^*_{W^2_3} f\ \forall^* h_1,h_2,h_3\ [\theta(f,\vec{h}) < (\mathrm{id};f;(d_a)^s;\vec{h}) = \sup\{f(\gamma) : \gamma < h(d_a;\vec{h})\}].$$

Hence,

$$\forall^* f\ \exists\delta\ \forall^* h_1,h_2,h_3\ [\delta(\vec{h}) < h(d_a;\vec{h}) \wedge \theta(f,\vec{h}) < f(\delta(\vec{h}))].$$

Suppose now $\delta \in \mathrm{On}$ is such that $\forall^* h_1,h_2,h_3\ \delta(\vec{h}) < h(d_a;\vec{h})$. In other words,

$$\forall^* h_1,h_2,h_3\ \forall^*_{W^2_1} \alpha_1,\alpha_2\ \delta(\vec{h})(\alpha_1,\alpha_2) < h(d_a;\vec{h})(\alpha_1,\alpha_2).$$

Recall that $h(d_a;\vec{h})(\alpha_1,\alpha_2) = h_1(1)(\alpha_2) = \sup\{h_1(\eta,\alpha_2) : \eta < \alpha_2\}$. Thus,

$$\forall^* h_1,h_2,h_3\ \forall^* \alpha_1,\alpha_2\ \exists\eta < \alpha_2\ [\delta(\vec{h})(\alpha_1,\alpha_2) < h_1(\eta,\alpha_2)].$$

It follows that

$$\forall^* h_1,h_2,h_3\ \exists g : \omega_1 \to \omega_1\ \forall^* \alpha_1,\alpha_2\ [\delta(\vec{h})(\alpha_1,\alpha_2) < h_1(g(\alpha_1),\alpha_2)].$$

In this expression, only the equivalence class of g with respect to W^1_1 matters, and we may assume the g is of the correct type. Using the ω_2-additivity of the measure S^1_1, it follows that

$$\forall^* h_1, h_2\ \exists g : \omega_1 \to \omega_1\ \forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(g(\alpha_1), \alpha_2)].$$

For fixed h_1 of the correct type, and fixed $\delta(h_1) \in \mathrm{On}$ such that

$$\forall^* h_2\ \exists g : \omega_1 \to \omega_1\ \forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < h_1(g(\alpha_1), \alpha_2)],$$

we consider the partition $\mathcal{P}$: we partition $h_2 : \mathrm{dom}(<_2) \to \omega_1$ of the correct type with the extra value $g(\alpha)$ inserted between $h_2(1)(\alpha)$ and $N_{h_2}(h_2(1)(\alpha))$ (with $g(\alpha)$ of uniform cofinality ω) according to whether

$$\forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < \ h_1(g(\alpha_1), \alpha_2)].$$

From Lemma 4.24 (with $m = 2$, $n = 1$, $r = 1$) it follows that a closed unbounded set cannot be homogeneous for the contrary side of the partition. Let C be homogeneous for $\mathcal{P}$, and $g(\alpha) =$ the ωth element of C greater that α. We then have that for any $h_2 : \mathrm{dom}(<_2) \to C'$ of the correct type

$$\forall^* h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(h_1)(h_2, h_3)(\alpha_1, \alpha_2) < h_1(g(h_2(1)(\alpha_1)), \alpha_2)].$$

Since for almost all h_1, $\delta(h_1)$ satisfies the hypothesis of the partition, we have:

$$\begin{aligned}&\forall^* h_1\ \exists g : \omega_1 \to \omega_1\ \forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\\ &\quad [\delta(h_1, h_2, h_3)(\alpha_1, \alpha_2) < h_1(g(h_2(1)(\alpha_1)), \alpha_2)].\end{aligned}$$

Fix a representing function $h_1 \to \delta(h_1)$ for δ, and consider the partition $\mathcal{P}$: we partition $h_1 : \mathrm{dom}(<_2) \to \omega_1$ of the correct type with the extra values $g(\gamma_1, \gamma_1)$ (of uniform cofinality ω) inserted between $h_1(\gamma_1, \gamma_2)$ and $N_{h_1}(h_1(\gamma_1, \gamma_2))$ according to whether

$$\begin{aligned}&\forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\\ &\quad [\delta(h_1, h_2, h_3)(\alpha_1, \alpha_2) < N_g(h_1(h_2(1)(\alpha_1), \alpha_2)) = g(h_2(1)(\alpha_1), \alpha_2)].\end{aligned}$$

It follows from Lemma 4.24 that we cannot have a closed unbounded set C homogeneous for the contrary side of the partition. For if so, fix a representing function $h_1 \to \delta(h_1)$ for δ, and fix then a $h_1 : \mathrm{dom}(<_2) \to C$ of the correct type such that for some $\bar{g} : \omega_1 \to \omega_1$ of the correct type, which we fix, we have:

$$\forall^* h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < h_1(\bar{g}(h_2(1)(\alpha_1)), \alpha_2)].$$

Define $\tilde{g}(\gamma_1, \gamma_2) = h_1(\bar{g}(\gamma_1), \gamma_2)$. Apply then Lemma 4.24 to $f = h_1$ and $\tilde{g}$ (and $r = 2$). This produces h_1', g' which are ordered as in $\mathcal{P}$, have range in C, and for which the property stated in $\mathcal{P}$ holds, a contradiction. If we fix now a closed unbounded C homogeneous for $\mathcal{P}$, and define $g(\alpha)=$ the ωth element of C greater than α, then we have:

$$\forall^* h_1, h_2, h_3\ \forall^* \alpha_1, \alpha_2\ [\delta(\vec{h})(\alpha_1, \alpha_2) < g(h_1(h_2(1)(\alpha_1), \alpha_2))].$$

We now have that for almost all $f : \omega_3 \to \boldsymbol{\delta}^1_3$ of the correct type, there is a $g : \omega_1 \to \omega_1$ such that

$$\forall^* h_1, h_2, h_3\ \theta(f, \vec{h}) < f(\gamma) \tag{21.1}$$

where:

$$\forall^* \alpha_1, \alpha_2\ \gamma(\alpha_1, \alpha_2) = g(h_1(h_2(1)(\alpha_1), \alpha_2)) = g(h(d_b; \vec{h})(\alpha_1, \alpha_2)), \tag{21.2}$$

and $d_b = (1; (2); (2; (1)))$ as before.

Fix a representing function $f \to \theta(f)$ for θ with respect to W^2_3, and consider the partition $\mathcal{P}$: we partition $f : \omega_3 \to \boldsymbol{\delta}^1_3$ of the correct type with the extra values $f_2(\alpha)$ of uniform cofinality ω inserted between $f(\alpha)$ and $f(\alpha+1)$ according to whether $\forall^* h_1, h_2, h_3\ \theta(f)(\vec{h}) < f_2(\gamma)$, where $\gamma = h(d_b; \vec{h})$. There cannot be a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ homogeneous for the contrary side, for if so, fix $f : \omega_3 \to C$ of the correct type such that there is a $g : \omega_1 \to \omega_1$ as in (21.1), (21.2), and fix such a g. Define $f_2 : \omega_3 \to C$ by: $f_2(\gamma) = f(\delta)$, where $\forall^* \alpha_1, \alpha_2\ \delta(\alpha_1, \alpha_2) = g(\gamma(\alpha_1, \alpha_2))$. A variation of Lemma 4.24 shows that there are $f', f'_2 : \omega_3 \to C$ of the correct type and ordered as in $\mathcal{P}$ such that $[f']_{S^2_1} = [f]_{S^2_1}$, $[f'_2]_{S^2_1} = [f_2]_{S^2_1}$. This contradicts the homogeneity of C for the contrary side. Let $C \subseteq \boldsymbol{\delta}^1_3$ now be a closed unbounded set homogeneous for $\mathcal{P}$. Define $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ by $g(\alpha) =$ the ω^{th} element of C greater than α. We then have that $\forall^* f\ \forall^* h_1, h_2, h_3\ \theta(f, \vec{h}) < g(f(\gamma))$, where $\gamma = h(d_b; \vec{h})$. In other words, $\theta < (g; (d_b); K_1, K_2, K_3)$. ⊣

From Lemma 5.23 and Theorem 5.5 our main result analyzing equivalence classes with respect to the measures W^m_3 on $\boldsymbol{\delta}^1_3$ now follows.

5.24 Theorem. *Suppose $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \dots, K_t)$ satisfies Condition* D. *If $(d)^{(s)}$ is not minimal with respect to $\mathcal{L}$ then*

$$(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t) \le \Big[\sup_{K_{t+1} \in W_1 \cup S_1} (\mathrm{id}; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t, K_{t+1}) \Big]^+.$$

Here $\mathcal{L}((d)^{(s)})$ is computed relative to the sequence $K_1, \dots, K_t$. If $(d)^{(s)}$ is minimal with respect to $\mathcal{L}$, then $(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t) = \boldsymbol{\delta}^1_3$.

Proof. We consider the first case, and suppose $\theta < (\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t)$. By Lemma 5.23, there is a $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ such that

$$\theta < (g; \mathcal{L}((d)^{(s)}); K_1, \dots, K_t).$$

Let T be as in Theorem 5.5, and fix a real x and a closed unbounded $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$, $g(\alpha) < |T_x \restriction \sup_{K_{t+1}} j_{K_{t+1}}(\alpha)|$, the supremum ranging over measures $K_{t+1} \in W_1 \cup S_1$. For $\alpha < \boldsymbol{\delta}^1_3$, let

$$l(\alpha) = \sup_{K_{t+1}} j_{K_{t+1}}(\alpha) \quad \text{and} \quad l'(\alpha) = |T_x \restriction l(\alpha)|.$$

We define a well-founded relation $\prec$ on $\lambda \doteq (l; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t)$ by: $\rho_1 \prec \rho_2$ iff

$$\forall^*_{W_3^m} f \ \forall^* h_1, \ldots, h_t \ |T_x \restriction \lambda(f, \vec{h})(\rho_1(f, \vec{h}))| < |T_x \restriction \lambda(f, \vec{h})(\rho_2(f, \vec{h}))|.$$

Easily, $|\prec| \geq (l'; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t) \geq \theta$. It follows that

$$(\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t) \leq \Big[\Big(\sup_{K_{t+1}} j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t\Big)\Big]^+.$$

By countable additivity of the measures $W_3^m, K_1, \ldots, K_t$, we have that if $\alpha < (\sup_{K_{t+1}} j_{K_{t+1}};\ \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t)$ then there is a K_{t+1} such that $\alpha < (j_{K_{t+1}};\mathcal{L}((d)^{(s)}); K_1, \ldots, K_t)$. Also, from the definitions of these ordinals it is immediate that

$$(j_{K_{t+1}}; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t) = (\mathrm{id}; \mathcal{L}((d)^{(s)}); K_1, \ldots, K_t, K_{t+1}).$$

The result now follows. ⊣

To compute the upper bound for $\boldsymbol{\delta}^1_5$, it suffices to compute the rank of the $\mathcal{L}$ operation, in a suitable sense. Namely, fix m and consider the set of all tuples $((d)^{(s)}; K_1, \ldots, K_t)$ where $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \ldots, K_t)$ satisfies Condition D relative to $K_1, \ldots, K_t$. Let $\prec_m$ be the transitive relation on this set generated by the relation $((d)^{(s)}; K_1, \ldots, K_t) \prec_m (\mathcal{L}((d)^{(s)}); K_1, \ldots, K_t, K_{t+1})$ for all K_{t+1}, where $\mathcal{L}((d)^{(s)})$ is again computed relative to the sequence $K_1, \ldots, K_t$. The relation $\prec_m$ is easily well-founded. Let $|\vec{s}\rceil_m$ denote the rank of the tuple $\vec{s}$, computed in the slightly non-standard manner by: $|\vec{s}\rceil_m = (\sup\{|\vec{t}\rceil_m : \vec{t} \prec_m \vec{s}\}) + 1$; by convention if $\vec{s}$ is minimal, then $|\vec{s}\rceil = 0$ (thus, at limit ranks, this is one more than the usual rank).

An immediate induction on the $\prec_m$ rank using Theorem 5.24 then shows:

5.25 Theorem. *For all $m \in \omega$, $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \ldots, K_t)$ satisfying Condition* D*, we have* $(\mathrm{id}; (d)^{(s)}; K_1, \ldots, K_t) \leq \aleph_{\omega+1+|((d)^{(s)}; K_1, \ldots, K_t)|_m}$. ⊣

Let θ_m be the supremum of the $|\ |_m$ ranks of the tuples $((d)^{(s)}; K_1, \ldots, K_t)$ where $(d)^{(s)} \in \bar{\mathcal{D}}_m(K_1, \ldots, K_t)$. From Theorem 5.25 and the homogeneous tree analysis (cf. two paragraphs before Definition 5.14) we thus have $\boldsymbol{\delta}^1_5 \leq [\aleph_{\sup_m \theta_m}]^+$.

The computation of the θ_m is a purely combinatorial problem, and is relatively straightforward. We omit the proof, and simply state the result that $\theta_m = \omega^{\omega^m}$ (ordinal exponentiation). As an immediate corollary we have:

5.26 Corollary (Jackson). $\boldsymbol{\delta}^1_5 \le \aleph_{\omega^{\omega^\omega}+1}$.

At this point we have extended the basic ingredients in the theory, Martin's theorem and the description analysis, from the $\boldsymbol{\delta}^1_1$ level to the $\boldsymbol{\delta}^1_3$ level, and we have used this to do one step in the next level of the inductive analysis, namely the upper bound for $\boldsymbol{\delta}^1_5$. To finish the next level analysis, it remains to prove the strong partition relation on $\boldsymbol{\delta}^1_3$, the lower bound for $\boldsymbol{\delta}^1_5$, and the weak partition relation for $\boldsymbol{\delta}^1_5$. The proofs in all cases follow in outline those of Sect. 4, using the description analysis, Theorem 5.24. Since we have now illustrated all of the ideas which go into these arguments, we will content ourselves with this. The complete details of these arguments can be found in [11]. We mention only that the analysis of measures on $\boldsymbol{\delta}^1_3$ and on λ_5 require the notions of type-2 and type-3 trees of uniform cofinalities respectively (roughly corresponding to the measures occurring in homogeneous trees on $\boldsymbol{\Pi}^1_3, \boldsymbol{\Pi}^1_4$-complete sets).

5.4. Some Further Results

We close this section with some remarks on generalizations and refinements of the results discussed. All the ordinals $(\mathrm{id}; (d)^{(s)}; K_1, \dots, K_t)$, it can be shown, are actually cardinals (Theorem 5.24 shows that all cardinals between $\boldsymbol{\delta}^1_3$ and λ_5 must be of this form). This is proved in [13]. It is also not difficult to show that if μ is a semi-normal measure on a cardinal κ having the strong partition relation, then $j_\mu(\kappa)$ is a regular cardinal. For the three normal measures μ_ω, μ_{ω_1}, μ_{ω_2} on $\boldsymbol{\delta}^1_3$, these ultrapowers are computed to be $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$, $\aleph_{\omega\cdot 2+1}$, and $\aleph_{\omega^\omega+1}$ respectively. These three regular cardinals κ all satisfy $\kappa \to (\kappa)^\lambda$ for all $\lambda < \boldsymbol{\delta}^1_4$, but $\kappa \not\to (\kappa)^{\boldsymbol{\delta}^1_4}$. One can also compute the cofinalities of all successor cardinals between $\boldsymbol{\delta}^1_4$ and λ_5. The result, from [13] is:

5.27 Theorem. *Suppose $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1} < \aleph_{\alpha+1} < \aleph_{\omega^{\omega^\omega}+1} = \boldsymbol{\delta}^1_5$. Let $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, where $\omega^\omega > \beta_1 \ge \cdots \ge \beta_n$ be the normal form for α. Then:*

(1) If $\beta_n = 0$, then $\mathrm{cf}(\kappa) = \boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$.

(2) If $\beta_n > 0$, and is a successor ordinal, then $\mathrm{cf}(\kappa) = \aleph_{\omega\cdot 2+1}$.

(3) If $\beta_n > 0$ and is a limit ordinal, then $\mathrm{cf}(\kappa) = \aleph_{\omega^\omega+1}$.

Finally, one can extend the results of this section to all levels of the projective hierarchy. One first defines the measures W^m_{2n+1}, $S^{l,m}_{2n+1}$, for $m \in \omega$, $1 \le l \le 2^{n+1} - 1$, assuming the weak and strong partition relations on $\boldsymbol{\delta}^1_{2n+1}$ respectively (for $n = 0$, we set $S^m_1 = S^{1,m}_1$ to agree with our previous notation). Order these (to be defined) families of measures as: W^m_1, $S^{1,m}_1$, W^m_3, $S^{1,m}_3$, $S^{2,m}_3$, $S^{3,m}_3$, W^m_5, etc. W^m_{2n+1} is defined to be the measure on $\boldsymbol{\delta}^1_{2n+1}$ induced from the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f : \mathrm{dom}(S^{l_0,m}_{2n-1}) \to \boldsymbol{\delta}^1_{2n+1}$, where $l_0 = 2^n - 1$, and the measure $S^{l_0,m}_{2n-1}$ on $\mathrm{dom}(S^{l_0,m}_{2n-1}) < \lambda_{2n+1}$. $S^{1,m}_{2n+1}$ is the measure on $(\boldsymbol{\delta}^1_{2n+1})^{+m}$ defined just as

S_1^m was, using $\boldsymbol{\delta}^1_{2n+1}$ in place of ω_1. For $l > 1$, $S^{l,m}_{2n+1}$ is the measure on $\mathrm{dom}(S^{l,m}_{2n+1}) < \lambda_{2n+3}$ induced by the strong partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f : \boldsymbol{\delta}^1_{2n+1} \to \boldsymbol{\delta}^1_{2n+1}$ of the correct type, and the measure ν on $\boldsymbol{\delta}^1_{2n+1}$. Here ν is the measure on $\boldsymbol{\delta}^1_{2n+1}$ induced by the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $h : \mathrm{dom}(\mu) \to \boldsymbol{\delta}^1_{2n+1}$, and the measure μ, where μ is the mth measure in the $(l-1)$st family. In [8] it is shown that these measures dominate the general measures in the homogeneous trees on $\boldsymbol{\Pi}^1_{2n+1}, \boldsymbol{\Pi}^1_{2n+2}$-complete sets, in the sense of Theorems 5.12, 5.15. The notion of a level n description is introduced there, and the analog of the main theorem, Theorem 5.24, is proved (with a suitable generalization of the $\mathcal{L}$ operator). The ranks of these generalized $\mathcal{L}$ operations are also computed, giving the upper bounds for the $\boldsymbol{\delta}^1_{2n+1}$. As mentioned in the introduction, the result is $\boldsymbol{\delta}^1_{2n+1} \leq \aleph_{\omega(2n-1)+1}$, where $\omega(0) = 0$ and $\omega(n+1) = \omega^{\omega(n)}$. With the main theorem, the inductive step is then similar to that of Sect. 4 or [11] (see the forthcoming [7]). In particular, $\sup_n \boldsymbol{\delta}^1_n = \aleph_{\epsilon(0)}$, where $\epsilon(0) = \sup_m \omega(m)$. Also, the regular cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ are given by the ultrapowers of $\boldsymbol{\delta}^1_{2n+1}$ by the normal measures on $\boldsymbol{\delta}^1_{2n+1}$, corresponding to the regular cardinals below $\boldsymbol{\delta}^1_{2n+1}$. Thus, there are $2^{n+1} - 1$ regular cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$.

In fact, these arguments extend with little modification up to $\boldsymbol{\delta}^1_{\omega_1} = \aleph_{\omega_1}$. Here $\boldsymbol{\delta}^1_\alpha$ is the supremum of the lengths of the $\boldsymbol{\Delta}^1_\alpha$ prewellorderings, where $\boldsymbol{\Sigma}^1_\alpha$ is the αth pointclass closed under $\exists^{\omega^\omega}$ (so $\boldsymbol{\Sigma}^1_0 = \boldsymbol{\Sigma}^0_1$, and for limit α, $\boldsymbol{\delta}^1_\alpha = \sup_{\beta<\alpha} \boldsymbol{\delta}^1_\beta$). For there are no new measures on $\boldsymbol{\delta}^1_\alpha$ for α limit $< \omega_1$, and a coding of $\mathcal{P}(\boldsymbol{\delta}^1_\alpha)$ may be constructed trivially from codings of $\mathcal{P}(\boldsymbol{\delta}^1_\beta)$, $\beta < \alpha$. Also, the only normal measures on the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ for limit α correspond to the fixed cofinalities below $\boldsymbol{\delta}^1_{\alpha+2n+1}$ (since there will be only countably many regular cardinals below $\boldsymbol{\delta}^1_{\alpha+2n+1}$). Again, the ultrapowers of the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ by these normal measures, together with the $\boldsymbol{\delta}^1_{\alpha+2n+1}$ precisely constitute the regular cardinals below $\boldsymbol{\delta}^1_{\omega_1}$.

At $\aleph_{\omega_1}$, or at any limit Suslin cardinal δ of cofinality $> \omega$, the situation changes as there are, of course, new measures on δ. One can show directly here that the next Suslin cardinal after $\boldsymbol{\delta}^1_{\omega_1}$ has cardinality the supremum of the ultrapowers of $\boldsymbol{\delta}^1_{\omega_1}$ by the measures on $\boldsymbol{\delta}^1_{\omega_1}$. Actually, Martin has proved a general result which shows the same fact for any limit Suslin cardinal of cofinality $> \omega$ (or any successor Suslin cardinal as well). In unpublished work, the author has analyzed these measures and shown that the supremum of their ultrapowers is $\aleph_{\aleph_\omega}$. We will consider some of the problems associated with further extending the theory in the next section.

6. Global Results

We consider in this section some results and problems of a "global" nature, that is, related to the attempt to push the structural theory of $L(\mathbb{R})$ up through Θ. The problems we consider here (and in some cases solve) seem to be necessary for further extensions of the theory, but are almost certainly not sufficient. Identifying the remaining obstructions remains a central goal of this subject. Nevertheless, some of the results we mention in this section are of independent interest. We will not require any of the results of Sects. 4, 5 for this section. We assume AD throughout this section, occasionally assuming $V = L(\mathbb{R})$ as well.

6.1. Generic Codes

Kechris and Woodin [20] have developed a theory of generic codes for uncountable ordinals which we will use in several of the arguments of this section. We will only need, however, the most basic lemma of their theory, the one asserting the existence of a generic coding function. For the sake of completeness, we give their proof of this result.

We say an ordinal α is *reliable* if there is a $P \subseteq \omega^\omega$ and a scale $\{\phi_n\}_{n\in\omega}$ with $\phi_n : P \to \alpha$ with ϕ_0 onto α. Every Suslin cardinal is easily reliable (cf. [37, Lemma 4.6]), and in [37] it is shown from $\mathrm{AD} + V = L(\mathbb{R})$ that every reliable cardinal is a Suslin cardinal. Actually, using some additional arguments this can be shown to follow from just AD. There are, however, many reliable ordinals which are not cardinals, as, for example, the set of reliable ordinals is closed unbounded in every $\boldsymbol{\delta}^1_{2n+1}$. For the purposes of generic codes, it is convenient to slightly strengthen the definition of reliable to include the requirement that the scale relations $\leq^*_n$, $<^*_n$ are both Suslin and co-Suslin. This only has the effect of removing the largest Suslin cardinal, if there is one, from consideration. We henceforth officially adopt this stronger form of the definition.

If α is reliable, $S \in \mathcal{P}_{\omega_1}(\alpha)$, and $\beta \in S$, we say S is β-*honest* if there is an $x \in P$ with $\phi_0(x) = \beta$ and $\forall n\ \phi_n(x) \in S$ (this notion is defined relative to the choice of P and $\{\phi_n\}$). We say S is *honest* if it is β-honest for all $\beta \in S$. For $x \in P$ we frequently write $|x|$ for $\phi_0(x)$. If S is a countable set of ordinals, then S^ω (S having the discrete topology) is homeomorphic to ω^ω and so carries a natural notion of category. When we speak of meager or comeager, we are always referring to this topology. If $p \in S^{<\omega}$, we write $\forall^*_p s \in S^\omega$ to mean for comeager many s in the neighborhood $N_p = \{s \in S^\omega : s{\restriction}\mathrm{lh}(p) = p\}$. We just write $\forall^* s \in S^\omega$ to mean for comeager many s in the space S^ω. Recall from Definition 2.3 that $S(\kappa)$ denotes the pointclass of κ-Suslin sets.

Recall that from AD, every set $A \subseteq S^\omega$ has the Baire property. In particular, A is either meager or else comeager on some neighborhood N_p. Also from AD, a wellordered union of meager sets is meager ("additivity of category").

6.1 Theorem (Kechris-Woodin). *Let α be reliable, as witnessed by P, $\{\phi_n\}$. Then there is Lipschitz continuous function $G : \alpha^\omega \to \omega^\omega$ satisfying:*

(1) $\forall \vec{s} = (\alpha_0, \alpha_1, \dots) \in \alpha^\omega\ \forall n \in \omega\ [G(\vec{s})_n \in P \wedge \phi_0(G(\vec{s})_n) \leq \alpha_n]$.

(2) If $\vec{s} = (\alpha_0, \alpha_1, \dots) \in \alpha^\omega$ enumerates an honest set S, then for all $n \in \omega$, $\phi_0(G(\vec{s})_n) = \alpha_n$.

Proof. Let T be the tree of the scale $\{\phi_n\}$, so T is a tree on $\omega \times \alpha$. For $\gamma < \alpha$, let

$$T_\gamma = \{(s, \vec{\alpha}) \in T : \alpha_0 = \gamma\}.$$

Consider the following ordinal game:

I	α_0		α_1		α_2		α_3		$\dots$	
II		β_0		β_1		β_2		β_3		$\dots$
		$x(0)$		$x(1)$		$x(2)$		$x(3)$		$\dots$

Here $\alpha_i, \beta_i < \alpha$, $x(i) \in \omega$, so I plays out $\vec{\alpha} \in \alpha^\omega$, and II plays out $\vec{\beta} \in \alpha^\omega$ and $x \in \omega^\omega$. Let $S = \{\alpha_i : i \in \omega\}$. x codes reals x_i and $\vec{\beta}$ codes sequences $(\vec{\beta})_i \in \alpha^\omega$ in the usual manner, so $x_i(j) = x(\langle i, j\rangle)$, and $(\vec{\beta})_i(j) = \beta_{\langle i,j\rangle}$ (we assume $\langle i, j\rangle \geq i$ for all j, so II does not have to play any of the $x_i(j)$ or $(\vec{\beta})_i(j)$ until I has played α_i). II wins the run of the game iff

$$\forall i\ (x_i, \alpha_i{}^\frown(\vec{\beta})_i) \in [T] \wedge \forall i\, \forall y\ [y \in p[T_{\alpha_i}\restriction S] \to \phi_0(y) \leq \phi_0(x_i)].$$

Now I cannot have a winning strategy, for as soon as I plays α_i, II can pick some $x_i \in P$ with $\phi_0(x_i) = \alpha_i$, and pick $(\vec{\beta})_i$ with $(x_i, \alpha_i{}^\frown(\vec{\beta})_i) \in [T]$ and proceed to play these to defeat I's strategy. Thus, if the game is determined, then II has a winning strategy τ. Ignoring the ordinal moves of τ gives the function G as desired.

To show the game is determined, it is enough to observe that it is Suslin, co-Suslin by Theorem 2.23. The first conjunct in the payoff definition trivially defines a Suslin, co-Suslin set (in fact, a closed set). For the second conjunct, note that

$$y \in p[T_{\alpha_i}\restriction S] \longleftrightarrow \exists \pi \in \omega^\omega\ \forall n\ (y\restriction n, (\alpha_i, \alpha_{\pi(1)}, \dots, \alpha_{\pi(n-1)})) \in T.$$

From the closure of the Suslin sets under $\vee^\omega, \wedge^\omega, \exists^{\omega^\omega}$, and $\forall^{\omega^\omega}$ (the latter by the Second Periodicity Theorem; see Remark 2.10), it follows that this relation, and thus the second conjunct, is Suslin, co-Suslin. ⊣

If there is a largest Suslin cardinal Ξ, then Theorem 6.1 does not immediately give a generic coding function $G : \Xi^\omega \to \omega^\omega$ at Ξ. In this case $\boldsymbol{\Gamma} = S(\Xi)$ will be a non-selfdual pointclass closed under real quantification, $\wedge$, $\vee$, and scale($\boldsymbol{\Gamma}$). The scale relations $\leq_n^*$, $<_n^*$ will not be co-Suslin, however. Nevertheless, we can argue that a generic coding function G still exists. To see this, fix a $\boldsymbol{\Gamma}$-scale $\{\phi_n\}_{n\in\omega}$ on a $\boldsymbol{\Gamma}$-complete set A. Without loss of generality, all of the norms ϕ_n are onto Ξ. Recall that Ξ is regular and a limit of Suslin cardinals. Also, the Suslin cardinals are closed unbounded in Ξ. By boundedness, there is a closed unbounded $C \subseteq \Xi$ such that for all $\alpha \in C$ and

$\beta < \alpha$, if $B^i_\beta = \{x \in A : \phi_i(x) < \beta\}$, then $\sup\{\phi_j(x) : j \in \omega \wedge x \in B^i_\beta\} < \alpha$. Note that every Suslin cardinal in C is reliable with respect $\{\phi_n\}$. For every Suslin cardinal $\alpha \in C$, there is a generic coding function at α with respect to $A_\alpha = \{x \in A : \forall i\ \phi_i(x) < \alpha\}$ and the norms $\phi_i \restriction A_\alpha$ (from Theorem 6.1). It suffices to show that we can get a function which to each such α assigns a generic coding function G_α with respect to A_α and the $\phi_i \restriction A_\alpha$. For then we can define $G : \Xi^\omega \to \omega^\omega$ by: $G(\alpha_0, \alpha_1, \dots) = z$ where for all j we have $(z)_j = (G_{\alpha'_j}(\beta_0, \beta_1, \dots))_j$ where α'_j is the least Suslin cardinal in C greater than α_j and $\beta_k = \alpha_k$ if $\alpha_k < \alpha'_j$ and otherwise $\beta_k = \alpha_j$. Using the definition of C it is not hard to check that G is a generic coding function (the point is that if S is α_j-honest then $S \cap \alpha'_j$ is α_j-honest). It remains to show that we can uniformly define the G_α for $\alpha \in C$ a Suslin cardinal. Let $\mathcal{G}_\alpha$ denote the generic coding game as in Theorem 6.1 using A_α and the $\phi_i \restriction A_\alpha$. The payoff set for II is Suslin and co-Suslin, and is uniformly Suslin (but not uniformly co-Suslin). For all Suslin cardinals α in C, II has a winning strategy in $\mathcal{G}_\alpha$. From a Suslin representation for II's payoff set and the fact that II has a winning strategy, we can uniformly in α get a winning strategy for II in $\mathcal{G}_\alpha$, which then gives us the generic coding function G_α. In fact, either of the two proofs that Suslin, co-Suslin ordinal games are determined shows this (cf. [32, Theorem 2.2] or [22, Theorem 2.5], also [27, Theorem 2]).

We are frequently only concerned with getting a real which codes the ordinal α_0. Thus, let $G_0 : \alpha^\omega \to \omega^\omega$ be a Lipschitz continuous function so that $\forall \vec{s} \in \alpha^\omega\ G_0(\vec{s}) = G(\vec{s})_0$. The functions G_0, G are referred to as generic coding functions, and we fix them for the remainder of this section. Of course, these functions depend on the choice of the set P and the scale $\{\phi_n\}$, but we suppress writing this. Frequently, α will be a Suslin cardinal.

Recall Theorem 2.25, according to which any game on α whose payoff depends only on $G(\vec{s})$ is determined (where $\vec{s}$ is the sequence I and II build, assuming now $V = L(\mathbb{R})$). Thus, for any game on α with payoff set $R \subseteq \alpha^\omega$, there is a determined game $R' \subseteq \alpha^\omega$ approximating R. Namely, define $R'(\vec{s}) \longleftrightarrow R(|G(\vec{s})_0|, |G(\vec{s})_1|, \dots)$. R' is always determined, and if $\vec{s}$ enumerates an honest set, then $R'(\vec{s}) \longleftrightarrow R(\vec{s})$.

The generic coding functions are particularly useful when combined with the existence of supercompactness measures. Recall that from $\mathrm{AD} + V = L(\mathbb{R})$ there is a supercompactness measure (i.e., a fine, normal, countably additive ultrafilter) ν on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, which in turn induces one on $\mathcal{P}_{\omega_1}(\delta)$ for any $\delta \leq \boldsymbol{\delta}^2_1$. Woodin has shown [40] that the supercompactness measure ν on $\mathcal{P}_{\omega_1}(\delta)$, for any $\delta \leq \boldsymbol{\delta}^2_1$ is unique. Woodin has also shown that there is a supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ for any $\lambda < \Theta$ assuming $\mathrm{AD} + V = L(\mathbb{R})$.

In fact, the existence of generic coding functions can be used to give a quick proof of the existence of the supercompactness measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, assuming $\mathrm{AD} + V = L(\mathbb{R})$. Recall $\boldsymbol{\delta}^2_1$ is a Suslin cardinal, and $S(\boldsymbol{\delta}^2_1) = \boldsymbol{\Sigma}^2_1$ has the scale property. Let $G : (\boldsymbol{\delta}^2_1)^\omega \to \omega^\omega$ be a generic coding function at $\boldsymbol{\delta}^2_1$ (see the remarks after the proof of Theorem 6.1). If $A \subseteq \mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, define $\nu(A) = 1$

iff II has a winning strategy in the game G_A: I and II alternate playing $\alpha_0, \alpha_1, \ldots$ building $\vec{s} \in (\boldsymbol{\delta}^2_1)^\omega$, and II wins iff $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\} \in A$. It is not hard to check that this defines a fine, normal measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$, using standard dovetailing arguments and the fact that either player can play to ensure S is honest. Alternatively, one can argue just using Theorem 6.1 as follows. Let μ be a normal measure on $\boldsymbol{\delta}^2_1$ (the proof that the $\boldsymbol{\delta}^1_{2n+1}$ are measurable works for $\boldsymbol{\delta}^2_1$). For every reliable $\lambda < \boldsymbol{\delta}^2_1$ there is a generic coding function $G : \lambda^\omega \to \omega^\omega$ from Theorem 6.1, and this gives a supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ as above. By Woodin's theorem, this supercompactness measure on $\mathcal{P}_{\omega_1}(\lambda)$ is unique, call it ν_λ. A supercompactness measure on $\mathcal{P}_{\omega_1}(\boldsymbol{\delta}^2_1)$ can then be defined by $\nu(A) = 1$ iff $\forall^*_\mu \lambda\ \forall^*_{\nu_\lambda} S \in \mathcal{P}_{\omega_1}(\lambda)\ S \in A$.

As an example of using generic coding arguments, we prove, following [12], the following theorem.

6.2 Theorem. *Let κ be a regular Suslin cardinal less than the supremum of the Suslin cardinals. Let $\lambda < \Theta$ be a cardinal with $j_{\nu_\alpha}(\lambda) = \lambda$ for all $\alpha < \kappa$, where ν_α is the supercompactness measure on $\mathcal{P}_{\omega_1}(\alpha)$. Then* $\mathrm{cf}(\lambda^+) > \kappa$.

6.3 Corollary. *If $\omega_1 < \lambda^+ < \Theta$, then* $\mathrm{cf}(\lambda^+) > \omega_1$.

Proof. Fix κ, λ as above, and assume $f : \kappa \to \lambda^+$ is cofinal. For $\alpha < \kappa$, let $\alpha' < \kappa$ denote the least reliable ordinal $> \alpha$ relative to the scale used in constructing the generic coding function G_0 for κ (it is not hard to see that $\alpha' < \kappa$ using the regularity of κ). Consider the game where I, II play $\alpha_0, \alpha_1, \ldots$ building $\vec{s}$, and II plays also $x(0), x(1), \ldots \in \omega$ building $x \in \omega^\omega$. II wins iff x codes a wellordering of λ of length $\geq f(\phi_0(G_0(\vec{s})))$. Here we code subsets of λ by reals in some manner which is not important, say by the Coding Lemma. I cannot have a winning strategy, for as soon as I plays α_0, II can enumerate an honest set containing α_0 and closed under I's winning strategy, and play some x coding a wellordering of λ of length $\geq f(\alpha_0)$.

A winning strategy τ for II gives (ignoring II's ordinal moves) a Lipschitz continuous $\mathcal{F} : \kappa^\omega \to \omega^\omega$ such that

(1) For all $\vec{s} \in \kappa^\omega$, $\mathcal{F}(\vec{s})$ codes a wellordering of λ.

(2) For all $\alpha < \kappa$ and all $\vec{s}$ enumerating an honest S containing α, $\mathcal{F}(\alpha{}^\frown\vec{s})$ codes a wellordering of λ of length $\geq f(\alpha)$.

Let $|\beta|_x$ denote the rank of β in the wellordering coded by x.

Fix for the moment $\alpha < \kappa$ and an honest set $S \in \mathcal{P}_{\omega_1}(\alpha')$ containing α. Define a tree T on $S^{<\omega} \times \lambda$ as follows. Put $((p_0, \beta_0), \ldots, (p_n, \beta_n))$ in T iff

(1) For all $i < n$, p_{i+1} extends p_i.

(2) $\forall^*_{p_{i+1}} \vec{s} \in S^\omega\ |\beta_{i+1}|_{\mathcal{F}(\alpha{}^\frown\vec{s})} < |\beta_i|_{\mathcal{F}(\alpha{}^\frown\vec{s})}$.

(3) $\forall i \leq n\ \exists \eta_i \in \mathrm{On}\ \forall^*_{p_i} \vec{s} \in S^\omega\ |\beta_i|_{\mathcal{F}(\alpha{}^\frown\vec{s})} = \eta_i$.

The last clause guarantees that T is well-founded, as the η_i are decreasing along any branch. We define an order-preserving map π from the tree of the ϵ relation on $f(\alpha)$ into T. Suppose

$$\pi(\gamma_0, \gamma_1, \ldots, \gamma_n) = ((p_0, \beta_0), \ldots, (p_n, \beta_n))$$

has been defined, and assume inductively that

$$\forall^*_{p_n} \vec{s} \in S^\omega \ |\beta_n|_{\mathcal{F}(\alpha^\frown \vec{s})} = \eta_n \geq \gamma_n.$$

If $\gamma_{n+1} < \gamma_n$, then let $\pi(\gamma_0, \ldots, \gamma_{n+1})$ be the least sequence

$$((p_0, \beta_0), \ldots, (p_{n+1}, \beta_{n+1}))$$

extending $((p_0, \beta_0), \ldots, (p_n, \beta_n))$ such that

$$\forall^*_{p_{n+1}} \vec{s} \in S^\omega \ |\beta_{n+1}|_{\mathcal{F}(\alpha^\frown \vec{s})} = \eta_{n+1} \geq \gamma_{n+1} \text{ and } \eta_{n+1} < \eta_n.$$

The additivity of category shows that $p_{n+1}, \beta_{n+1}, \eta_{n+1}$ exist.

The definition of the tree $T = T^\alpha_S$ is uniform in α, S. Let now $F(\alpha) = [S \to T^\alpha_S]_{\nu_{\alpha'}}$. $F(\alpha)$ may be viewed as a wellordering of $j_{\nu_{\alpha'}}(\lambda) = \lambda$, and clearly $|F(\alpha)| \geq f(\alpha)$. This gives a wellordering of λ of length λ^+, a contradiction. $\dashv$

The analog of Corollary 6.3 with $\kappa = \boldsymbol{\delta}^1_{2n+1}$ replacing ω_1 was shown by Kechris and Woodin to hold for λ below the supremum of the projective ordinals, i.e., $\lambda < \aleph_{\epsilon(0)}$ (this also follows from the projective hierarchy analysis, cf. Theorem 5.27). Along with Corollary 6.3, this suggests the following conjecture:

6.4 Conjecture. If κ is a regular Suslin cardinal and $\kappa < \lambda^+ < \Theta$, then $\mathrm{cf}(\lambda^+) > \kappa$.

6.2. Weak Square and Uniform Cofinalities

One of the important ingredients in the projective hierarchy analysis is the analysis of uniform cofinalities. In the general step, one has the notion of a type-n tree of uniform cofinalities $\mathcal{R}$ with associated measure $M^{\mathcal{R}}$, and it is necessary to analyze the possible uniform cofinalities with respect to these measures. Consider then the general question: given a measure ν on κ, what are the possible uniform cofinalities of a function $f : \kappa \to \lambda \in \mathrm{On}$ with respect to ν? One possibility is that for some function g from κ to the regular cardinals we have $\forall^*_\nu \alpha \ f(\alpha)$ has uniform cofinality $g(\alpha)$. We call these the trivial uniform cofinalities. What are the possible non-trivial uniform cofinalities? Suppose, for example, $\kappa = \omega_2$, $\nu = S^1_1$, and $\lambda \in \mathrm{On}$. Are the possible non-trivial uniform cofinalities for $f : \omega_2 \to \lambda$ the same as for $f : \omega_2 \to \boldsymbol{\delta}^1_3$? Intuitively, it seems as though the possible non-trivial uniform cofinalities should depend only on κ (and ν), and not on λ (a phenomenon reminiscent of the Coding Lemma). For small λ, one can extend the projective hierarchy arguments directly to answer this question, but for large λ, such an inductive approach does not seem to help.

From these considerations the author formulated a combinatorial principle called $\boxminus_{\kappa,\lambda}$. In fact, this principle also arose independently from attempts to extend some joint work with Becker [2, 6]. The statement of the principle follows.

6.5 Definition. Let κ, $\lambda < \Theta$. $\boxminus_{\kappa,\lambda}$ is the assertion that for all $f : \kappa \to \lambda$ such that $\mathrm{cf}(f(\alpha)) \leq \kappa$ for all $\alpha < \kappa$, there is an $A \subseteq \lambda$ of size $\leq \kappa$ such that for all $\alpha < \kappa$, $A \cap f(\alpha)$ is cofinal in $f(\alpha)$.

Thus, $\boxminus_{\kappa,\lambda}$ can be viewed as a choice principle. The principle can be stated for any cardinal κ, but for non-Suslin κ can fail. For example, it is not difficult to see that $\boxminus_{\omega_2,\omega_3}$ fails (using just that $\mathrm{cf}(\omega_3) = \omega_2$). In [12], the following theorem is proved.

6.6 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$). *$\boxminus_{\kappa,\lambda}$ holds for any Suslin cardinal κ and any $\lambda < \Theta$.*

This theorem provides a positive answer to the question on uniform cofinalities asked above. Specifically, we have the following.

6.7 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$). *Let κ be a Suslin cardinal, μ a measure on κ, $\lambda < \Theta$, and $f : \kappa \to \lambda$. Then one of the following holds.*

(1) $\forall^*_\mu \alpha\ \mathrm{cf}(f(\alpha)) \leq \kappa$. Then the uniform cofinality of f with respect to μ is realized by a function $f' : \kappa \to \kappa$.

(2) $\forall^*_\mu \alpha\ \mathrm{cf}(f(\alpha)) > \kappa$. Let $g(\alpha) = \mathrm{cf}(f(\alpha))$. Then there is an h with domain $\{(\alpha, \beta) : \alpha < \kappa \wedge \beta < g(\alpha)\}$ such that $h(\alpha, \beta) < f(\alpha)$ and $\forall^*_\mu \alpha\ f(\alpha) = \sup\{h(\alpha, \beta) : \beta < g(\alpha)\}$.

We require a preliminary lemma. Throughout, ν denotes the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$.

6.8 Lemma. *Let κ be a Suslin cardinal and $\lambda \in \mathrm{On}$ with $\mathrm{cf}(\lambda) > \kappa$. Suppose $F : \mathcal{P}_{\omega_1}(\kappa) \to \lambda$. Then $\exists \delta < \lambda\ \forall^*_\nu S\ F(S) < \delta$.*

Proof. Fix an $S(\kappa)$-bounded prewellordering (C, ψ) of length λ according to Theorem 2.28. Play the game where I plays $\alpha_0, \alpha_2, \ldots$, II plays $\alpha_1, \alpha_3, \ldots$ and $x(0), x(1), \ldots \in \omega$, and II wins iff $x \in C$ and $\psi(x) \geq F(S)$, where $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\}$ and $\vec{s} = (\alpha_0, \alpha_1, \ldots)$. The game is determined and easily I cannot win. A winning strategy for II gives a Lipschitz continuous function $\mathcal{F} : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that $\forall \vec{s} \in \kappa^\omega\ \mathcal{F}(\vec{s}) \in C$ (ignoring II's ordinal moves in computing $\mathcal{F}(\vec{s})$), and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $\psi(\mathcal{F}(\vec{s})) \geq F(S)$. Let $w \in B \longleftrightarrow \exists \vec{s} \in \kappa^\omega\ w = \mathcal{F}(\vec{s})$. $B \subseteq C$ and is κ-Suslin, so $\delta = \sup\{\phi(w) : w \in B\} < \lambda$ (to see B is κ-Suslin, note that the Lipschitz continuous $\mathcal{F}$ can be coded by the Coding Lemma with the pointclass $S(\kappa)$). Thus, $\forall^*_\nu S\ F(S) < \delta$. ⊣

Proof of Theorem 6.7. Assume first $\forall^*_\mu\alpha\ \mathrm{cf}(f(\alpha)) \leq \kappa$. By $\boxminus_{\kappa,\lambda}$, let $A \subseteq \lambda$ have size κ such that $\forall^*_\mu\alpha\ (A \cap f(\alpha)$ is cofinal in $f(\alpha))$. Taking the transitive collapse of A, we may assume that $\lambda < \kappa^+$. Let $\prec$ be a wellordering of κ of length $> \lambda$. For $\alpha < \kappa$, let $R(\alpha) \leq \kappa$ be least such that $\sup\{|\beta|_\prec : \beta < R(\alpha) \wedge |\beta|_\prec < f(\alpha)\} = f(\alpha)$. For $\beta < R(\alpha)$, let $l(\alpha, \beta) = |\beta|_\prec$ if $|\beta|_\prec < f(\alpha)$, and 0 otherwise. R, l provide a liftup to f, as in the proof of Lemma 4.19. The uniform cofinality of f with respect to μ is the same as that of R.

Assume now $\forall^*_\mu\alpha\ \mathrm{cf}(f(\alpha)) > \kappa$. Let $g(\alpha) = \mathrm{cf}(f(\alpha))$. The game argument above produces a Lipschitz continuous $\mathcal{F} : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all $\alpha < \kappa$, and all $\vec{s} \in \kappa^\omega$ enumerating an honest set containing α and closed under $\mathcal{F}$, $\mathcal{F}(\alpha^\frown s)$ codes (ignoring II's ordinal moves) an increasing $g(\alpha)$ sequence cofinal in $f(\alpha)$. The exact manner in which reals code $g(\alpha)$ sequences below λ is not important, say by the Coding Lemma with respect to a suitably large pointclass.

Fix for the moment $\alpha < \kappa$ and an honest S containing α and closed under $\mathcal{F}$. For $p \in S^{<\omega}$ and $\beta < g(\alpha)$ define $h(\alpha, \beta, S, p)$ to be the least $\gamma < f(\alpha)$ such that $\forall^*_p s \in S^\omega\ \mathcal{F}(\alpha^\frown s)(\beta) < \gamma$ if one exists, and 0 otherwise. Define $h(\alpha, \beta, S) = \sup\{h(\alpha, \beta, S, p) : p \in S^{<\omega}\}$. Clearly $h(\alpha, \beta, S) < f(\alpha)$. If $\gamma < f(\alpha)$, then by additivity of category there is a $p \in S^{<\omega}$, a $\beta < g(\alpha)$, and a $\eta < f(\alpha)$ such that $\eta > \gamma$ and

$$\forall^*_p s \in S^\omega\ \mathcal{F}(\alpha^\frown s)(\beta) = \eta.$$

Thus, $h(\alpha, \beta, S, p) > \gamma$. Hence, $f(\alpha) = \sup\{h(\alpha, \beta, S) : \beta < g(\alpha)\}$. Also, an easy argument shows that $h(\alpha, \beta, S)$ is monotonically increasing in β. Define

$$h(\alpha, \beta) = \text{ the least } \delta < f(\alpha) \text{ such that } \forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)\ h(\alpha, \beta, S) < \delta.$$

By Lemma 6.8, this is well-defined. Fix now $\alpha < \kappa$, and suppose towards a contradiction that $\rho \doteq \sup\{h(\alpha, \beta) : \beta < g(\alpha)\} < f(\alpha)$. We have

$$\forall^*_\nu S\ \exists \beta < g(\alpha)\ [h(\alpha, \beta, S) > \rho].$$

By Lemma 6.8 and monotonicity, $\exists \beta_0 < g(\alpha)\ \forall^*_\nu S\ [h(\alpha, \beta_0, S) > \rho]$. However, $\forall^*_\nu S\ h(\alpha, \beta_0, S) \leq h(\alpha, \beta_0) \leq \rho$. ⊣

Theorem 6.6 has other applications as well. For example, in [12] it is used to show the following.

6.9 Theorem ($\mathrm{AD} + V = L(\mathbb{R})$). *Let κ be a regular cardinal which is either a Suslin cardinal or the successor of a Suslin cardinal. Then κ is $\boldsymbol{\delta}^2_1$-supercompact.*

6.10 Corollary ($\mathrm{AD} + V = L(\mathbb{R})$). *All the projective ordinals $\boldsymbol{\delta}^1_n$ are $\boldsymbol{\delta}^2_1$-supercompact.*

Solovay [34] first showed, assuming $\mathrm{AD}^{\mathbb{R}}$, that $\boldsymbol{\delta}^1_1 = \omega_1$ is λ-supercompact for all $\lambda < \Theta$. The work of Martin-Steel [27] and Harrington-Kechris [5] showed that $\boldsymbol{\delta}^1_1$ is $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$-supercompact from just AD. Woodin later showed from AD that ω_1 is λ-supercompact for all $\lambda < \Theta$. The $\kappa = \boldsymbol{\delta}^1_2$ case of Corollary 6.10 is due to Becker [1].

One of the main ideas in the proof of Theorem 6.6 involves combining certain category methods with the generic coding arguments. We will not prove Theorem 6.6 in detail here. Rather, we present a result whose proof uses the same idea.

We fix some notation for the remainder of this section. We assume AD + $V = L(\mathbb{R})$. κ will henceforth denote a Suslin cardinal, and ν the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$. From the scale analysis in $L(\mathbb{R})$ (see [37]) there is a κ-Suslin set P and a κ-Suslin scale $\{\phi_n\}$ on P with ϕ_0 onto κ (for κ below the supremum of the projective ordinals, that is, $\kappa = \boldsymbol{\delta}^1_{2n+1}$ or $\kappa = (\boldsymbol{\delta}^1_{2n+1})^-$, this is immediate). We write $|x|$ for $\phi_0(x)$, for $x \in P$. The generic coding functions G_0, G are defined relative to $P, \{\phi_n\}$, and are henceforth fixed. The pointclass $S(\kappa)$ of κ-Suslin sets is closed under $\exists^{\omega^\omega}$, so by the Coding Lemma we may code subsets of κ within the pointclass $S(\kappa)$. In particular, strategies (Lipschitz continuous functions) may be coded within $S(\kappa)$. If $\tau \in \omega^\omega$ codes a strategy, we also write τ for the strategy it codes. Thus, if τ codes a strategy $\tau : \kappa^\omega \to \kappa^\omega \times \omega^\omega$, the relation $\tau(\alpha_1, \dots, \alpha_n) = ((\beta_0, \dots, \beta_n), (a_0, \dots, a_n))$ is κ-Suslin in the codes (with respect to ϕ_0). For any other object we need to code by reals, the exact manner in which we do so is not important, say by using the Coding Lemma with respect to some sufficiently large pointclass.

Suppose $\tau : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ is a strategy, and $\vec{s} = (\alpha_0, \alpha_2, \dots) \in \kappa^\omega$. Let $(\alpha_1, \alpha_3, \dots)$ be the ordinal part of τ's response, and $x = (x(0), x(1), \dots)$ the integer part. Let $S = \{\alpha_0, \alpha_1, \dots\} \in \mathcal{P}_{\omega_1}(\kappa)$. We say x codes a comeager set $A \subseteq S^\omega$ and a continuous function $f : A \to \omega^\omega$ provided x_0 codes the comeager set A, and x_1 the continuous function f as follows. x_0 codes A by having each $(x_0)_n$ code a dense open $D_n \subseteq S^\omega$ such that $A = \bigcap_n D_n$. To say y codes the dense open set $D \subseteq S^\omega$ means each $y(k)$ codes a sequence $u_k \in \omega^{<\omega}$, and $D = \bigcup_k N_{u_k^*}$, where if $u_k = (a_0, \dots, a_l)$, then $N_{u_k^*}$ is the basic open set in S^ω determined by the sequence $u_k^* = (\alpha_{a_0}, \dots, \alpha_{a_l})$. Likewise, x_1 codes f by coding a sequence of tuples of integers $(a_0, \dots, a_l, b_0, \dots, b_m)$, where for $u = (a_0, \dots, a_l)$ coding a basic open set in D_k we have $m \geq k$ and for $s \in A \cap N_{u^*}$, $f(s)$ extends $(b_0, \dots, b_m)$. It is easy to see that for a fixed enumeration $\alpha_0, \alpha_1, \dots$ of a set S, the set of x coding a comeager set and a continuous function on S^ω is $\boldsymbol{\Pi}^0_2$.

If $\tau : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ is a strategy and $\vec{s} \in \kappa^\omega$, we usually write $\tau(\vec{s})$ to denote the real obtained as the integer moves of τ against $\vec{s}$.

6.11 Theorem (AD + $V = L(\mathbb{R})$). *Let κ be a Suslin cardinal, and ν the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$. Suppose $F : \mathcal{P}_{\omega_1}(\kappa) \to \lambda < \Theta$. Then* $\mathrm{cf}([F]_\nu) > \kappa$ *iff* $\forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)$ $[\mathrm{cf}(F(S)) > \omega]$.

The full proof of this theorem can be found in [12]. We will prove here a somewhat weaker version which still suffices to illustrate the main idea used in the proof of Theorem 6.6. Specifically, we show here that

(1) If $\forall^*_\nu S$ $[\mathrm{cf}(F(S)) \leq \omega]$ then $\mathrm{cf}([F]_\nu) \leq \kappa$.

(2) If $\forall^*_\nu S$ $[\mathrm{cf}(F(S)) > \kappa]$, then $\mathrm{cf}([F]_\nu) > \kappa$.

The proof of the full Theorem in [12] uses these ideas plus also the Becker-Kechris method used in proving the invariance of $L[T_{2n+1}]$ (see [3]).

Proof. Suppose first that $\forall^*_\nu S$ $\mathrm{cf}(F(S)) \leq \omega$. Play the game where I plays $\alpha_0, \alpha_2, \ldots$, II plays $\alpha_1, \alpha_3, \ldots$ and $x(0), x(1), \in \omega$, and II wins iff x codes an ω sequence of ordinals cofinal in $F(S')$, where $S' = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\}$, and $\vec{s} = (\alpha_0, \alpha_1, \ldots)$. II has a winning strategy, since II can defeat any strategy for I by enumerating an honest set S closed under I's strategy, and playing an x coding an ω sequence cofinal in $F(S)$. A winning strategy for II gives a Lipschitz continuous function $\mathcal{F} : \kappa^\omega \to \omega^\omega$ (ignoring the ordinal moves) such that for all $\vec{s} \in \kappa^\omega$ $\mathcal{F}(\vec{s})$ codes an ω sequence of ordinals $|\mathcal{F}(\vec{s})_0|, |\mathcal{F}(\vec{s})_1|, \ldots$, and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $\sup_i |\mathcal{F}(\vec{s})_i| = F(S)$.

For $S \in \mathcal{P}_{\omega_1}(\kappa)$ honest and closed under $\mathcal{F}$, define $G(S) \subseteq F(S)$ by: $G(S) = \{G_{n,p}(S) : n \in \omega, p \in S^{<\omega}\}$, where $G_{n,p}(S) \doteq$ the least ordinal $\beta < F(S)$ such that $\forall^*_p \vec{s} \in S^\omega$ $|\mathcal{F}(\vec{s})_n| = \beta$ if such an ordinal exists, and $G_{n,p}(S) = 0$ otherwise. The additivity of category shows that $G(S)$ is cofinal in $F(S)$. Thus, $[G]_\nu$ is cofinal in $[F]$, and by normality, $[G]_\nu$ is a set of size $\leq \kappa$. This shows the first claim.

Suppose now that $\forall^*_\nu S$ $\mathrm{cf}(F(S)) > \kappa$. Suppose towards a contradiction that $\mathrm{cf}([F]_\nu) \leq \kappa$, and let $B \subseteq [F]_\nu$ be cofinal with $|B| \leq \kappa$. By the usual game argument as above and Theorem 2.28, there is a Lipschitz continuous $\mathcal{F} : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that for all $\vec{s} \in \kappa^\omega$, $u \doteq \mathcal{F}(\vec{s})$ codes a $S(\kappa)$-bounded prewellordering (C_u, ψ_u) of some limit length which we denote by $|u|$, and for all $\vec{s}$ enumerating an honest set S closed under $\mathcal{F}$, $|u| = F(S)$.

If $\beta < [F]_\nu$, we say a real z is *β-good* if:

(1) z codes a Lipschitz continuous function $z : \kappa^\omega \to \kappa^\omega \times \omega^\omega$ such that if $s_0 \in \kappa^\omega$ enumerates an honest set S closed under z and $\mathcal{F}$, then $z(s_0)$ codes a comeager set $A_{z(s_0)} \subseteq S^\omega$, and a continuous function $z(s_0, -) : A_{z(s_0)} \to \omega^\omega$ such that for all $s_1 \in A_{z(s_0)}$, $w \doteq z(s_0, s_1)$ is in the $S(\kappa)$-bounded union (C_u, ψ_u) coded by $u \doteq \mathcal{F}(s_1)$.

(2) $\forall^*_\nu S \in \mathcal{P}_{\omega_1}(\kappa)$ $\forall^* s_0$ $\forall^* s_1 \in S^\omega$ the rank of $w = z(s_0, s_1)$ in the $S(\kappa)$-bounded union coded by $u = \mathcal{F}(s_1)$ is greater than $\beta(S)$. (Recall $S \to \beta(S)$ represents β.)

We first claim that for all $\beta < [F]_\nu$ there is a $z \in \omega^\omega$ such that z is β-good. To see this, fix a function $S \to \beta(S)$ representing β with respect to ν, and play the game where I plays $\alpha_0, \alpha_2, \ldots$, II plays $\alpha_1, \alpha_3, \ldots$, and $x(0), x(1), \ldots$, and II wins iff x codes a comeager set $A_x \subseteq S^\omega$, where $S = \{|G(\vec{s})_0|, |G(\vec{s})_1|, \ldots\}$, $\vec{s} = (\alpha_0, \alpha_1, \ldots)$, and a continuous function $x(-) : A_x \to \omega^\omega$ such that for all $s_1 \in A_x$, $x(s_1) = \overline{0}$ if $\mathcal{F}(s_1) = u$ does not code $F(S)$, and otherwise the rank of $x(s_1)$ in the $S(\kappa)$-bounded union coded by u is $> \beta(S)$. This game is again determined. Suppose I won by σ. Let S be an honest set closed under σ and $\mathcal{F}$, and such that $\mathrm{cf}(F(S)) > \kappa$, and $F(S) > \beta(S)$. II will enumerate S in the $\alpha_1, \alpha_3, \ldots$. Let $R \subseteq S^\omega \times \omega^\omega$ be defined by: $R(s_1, w) \longleftrightarrow [s_1$ enumerates $S \wedge w \in C_u$ where $u = \mathcal{F}(s_1) \wedge \psi_u(w) \geq \beta(S)]$. From AD, we may uniformize R by R' on a comeager set. Also, every function defined on a comeager subset of S^ω is continuous restricted to a comeager set. Let $x \in \omega^\omega$ code such a comeager set $A_x \subseteq S^\omega$ (coding neighborhoods using only the $\alpha_1, \alpha_3, \ldots$) and continuous function $x(-) : A_x \to \omega^\omega$. If II plays this x, then II defeats I. A winning strategy τ for II then gives a β-good real.

By the Coding Lemma, there is an $S(\kappa)$ set $C \subseteq \omega^\omega$ such that $\forall \tau \in C$ $\exists \beta < [F]_\nu$ τ is β-good, and $\forall \beta \in B$ $\exists \tau \in C$ τ is β-good. We now define $G : \mathcal{P}_{\omega_1}(\kappa) \to \mathrm{On}$ such that $[G]_\nu < [F]_\nu$ but $\forall \beta \in B$ $\forall^*_\nu S$ $G(S) > \beta(S)$, a contradiction.

Let S be honest and closed under $\mathcal{F}$. Let $G(S)$ be the least $\alpha \in \mathrm{On}$ such that $\forall^* s_1 \in S^\omega$ $G(S, s_1) < \alpha$, where $G(S, s_1)$ is defined as follows. Let $u = \mathcal{F}(s_1)$, so (C_u, ψ_u) is a $S(\kappa)$-bounded prewellordering of length $F(S)$. Set $G(S, s_1) = \sup\{\psi_u(w) : w \in B_{s_1}\}$, where

$$w \in B_{s_1} \longleftrightarrow \exists s_0 \text{ enumerating } S \; \exists \tau \in C \; \big[(S \text{ is closed under } \tau) \wedge (s_1 \in A_{\tau(s_0)}) \wedge (w = \tau(s_0, s_1))\big].$$

Easily, $B_{s_1} \in S(\kappa)$. Also, $B_{s_1} \subseteq C_u$, and so by boundedness, $G(S, s_1) < F(S)$ for all $s_1 \in S^\omega$. By additivity of category, $G(S) < F(S)$. Thus, $[G]_\nu < [F]_\nu$.

Fix now $\beta \in B$ and a function $S \to \beta(S)$ representing β, and let $\tau \in C$ be β-good. Let S be honest, closed under $\mathcal{F}$ and τ, and such that (6.2) in the definition of β-good above holds for S for this τ. We show that $G(S) > \beta(S)$, a contradiction. It is enough to show that $\forall^* s_1 \in S^\omega$ $G(S, s_1) > \beta(S)$. Fix s_0 enumerating S so that the remaining clause in (6.2) of β-good is satisfied. If $s_1 \in A_{\tau(s_0)}$ and $w = \tau(s_0, s_1)$, then $w \in B_{s_1}$ (using τ and s_0 as witnesses) and so $G(S, s_1) \geq \psi_u(w)$, where $u = \mathcal{F}(s_1)$. On the other hand, from the choice of S, s_0 and (6.2) of β-good we have $\forall^* s_1$ $\psi_u(w) > \beta(S)$. Thus, $\forall^* s_1$ $G(S, s_1) > \beta(S)$. ⊣

6.3. Some Final Remarks

We close this chapter with some final (somewhat tentative) thoughts on extending the structural theory throughout $L(\mathbb{R})$. The analysis, of course, is inductive, and proceeds by induction on the Suslin cardinals. As we remarked earlier, the arguments of Sects. 4, 5 should provide the necessary ingredients at successor Suslin cardinals. At singular Suslin cardinals δ, Theorems 6.6, 6.7, and similar results should provide a basis for the analysis. Aside from providing an analysis of the uniform cofinalities (Theorem 6.7), these techniques should provide a method for “gluing together” the description analyses at the lower Suslin cardinals to obtain one at δ (Theorem 6.7 may be viewed as a simple case of this; it shows how to glue together the pointwise cofinalities below δ to obtain a uniform cofinality).

At inaccessible Suslin cardinals δ, one gets some facts for "free", such as the strong partition relation on δ (see [22]). However, it still seems necessary to analyze the measures on δ to permit analysis at the next Suslin cardinal. One of the main problems here, as we mentioned earlier, is analyzing the semi-normal measures on δ (which is where the first step in the "pressing down" analysis of the measures leaves one; see the proof of Theorem 4.8). The normal measures on δ corresponding to fixed cofinalities $\kappa < \delta$ seem well-behaved (for example, we get Theorem 5.10), but there are, in general, many more semi-normal measures on δ.

Let $S \subseteq \delta$ be a "thin" stationary set. By thin we mean for all $\alpha \in S$, $S \cap \alpha$ is not stationary in α. Then the closed unbounded filter restricted to S defines a normal measure μ_S on δ. We refer to this as the *atomic normal measure* corresponding to S. This is shown using the strong partition relation on δ. For example, to see this defines an ultrafilter, for $A \subseteq \delta$ consider the partition of $f : \delta \to \delta$ of the correct type according to whether $\alpha(f, S) \in A$, where $\alpha(f, S) =$ the least limit point of $\mathrm{ran}(f)$ in S.

Given thin stationary sets S_1, S_2, define $S_1 \prec S_2$ iff there is a closed unbounded $C \subseteq \delta$ such that for all $f : \delta \to C$ of the correct type, $\alpha(f, S_1) < \alpha(f, S_2)$. The strong partition relation on δ shows that $\prec$ is a wellordering on equivalence classes $[S]$, where $S \sim T$ iff there is a closed unbounded $C \subseteq \delta$ such that $S \cap C = T \cap C$. Equivalently, $S_1 \prec S_2$ iff there is a closed unbounded $C \subseteq \delta$ such that for all $\alpha \in C \cap S_2$, S_1 is stationary in α. Let $o(\delta)$ denote the rank of this prewellordering, and $o(S)$ the rank of S in $\prec$ (this forms a generalized notion of Mahlo rank; δ is inaccessible if $o(\delta) \geq \delta$, Mahlo if $o(\delta) \geq \delta + 1$, etc.). Note that the atomic normal measure corresponding to S depends only on $[S]$. For $\beta < o(\delta)$, let $[S_\beta]$ denote the βth equivalence class in the stationary set ordering.

If $o(\delta)$ is fairly small compared with δ, we can transfer a semi-normal measure μ on δ onto a smaller ordinal, and thereby begin to analyze μ. Suppose, for example, $o(\delta) = \delta + \omega_1$, and μ concentrates on inaccessible cardinals. A generic coding argument, which we omit, shows that we may pick thin stationary sets S_α for $\alpha < \omega_1$ which are pairwise disjoint and $o(S_\alpha) = \delta + \alpha$. For μ almost all $\alpha < \delta$, let $\alpha \in S_{f(\alpha)}$. Then $\nu = f(\mu)$ is a measure on ω_1. Let μ' be the measure obtained by integrating the μ_S with respect to ν, that is, $\mu'(A) = 1$ iff $\forall^*_\nu \alpha < \omega_1\ \forall^*_{\mu_{S_\alpha}} \beta\ (\beta \in A)$. It is then not hard to see that $\mu = \mu'$. Thus we have analyzed the semi-normal measures on δ.

One can attempt to extend these arguments to larger values of $o(\delta)$. Suppose, for example, that $o(\delta) = j_{\nu_\omega}(\delta)$, where ν_ω denotes the ω-cofinal normal measure on δ. Using the strong partition relation on δ, ν_ω induces a measure $\mathcal{V}$ on $j_{\nu_\omega}(\delta)$ (using functions of the correct type, say). Integrating the $\nu_{[S_\alpha]}$ using $\mathcal{V}$ produces a semi-normal measure ν on δ (there is a problem now in trying to pick representatives S_α for the equivalence classes; the measure $\nu_{[S_\alpha]}$, however, is still well-defined). In fact, if $o(\delta)$ is large enough we may lift an arbitrary measure μ on δ to a new measure ν on δ in this manner (using μ in place of ν_ω). It seems reasonable (but not clear) that one might reverse the above process, and thus reduce the semi-normal measure ν on δ to a "smaller" measure μ on δ and proceed inductively. The measure is smaller in the following sense.

6.12 Lemma. *Let μ be a measure on δ, where δ is an inaccessible Suslin cardinal (and so has the strong partition property). Let $\mathcal{V}$ be the measure on $j_\mu(\delta)$ induced by the strong partition property, functions $F : \delta \to \delta$ of the correct type, and the measure μ on δ. Assume $o(\delta) \geq j_\mu(\delta)$, and let ν be the measure on δ defined by: $\nu(A) = 1$ iff $\forall^*_{\mathcal{V}} \beta < j_\mu(\delta)\ \forall^*_{[S_\beta]} \alpha < \delta\ \alpha \in A$. Then $j_\mu(\delta) < j_\nu(\delta)$.*

The proof of the lemma is not difficult, we omit it (the basic fact is that if $S_1 \prec S_2$ then $j_{S_1}(\delta)$ embeds into $[f]_{S_2}$, where $f(\alpha) =$ the next Suslin cardinal after α).

Unfortunately, the main definite result along these lines at the moment is a negative one; it asserts that for Suslin cardinals δ where $S(\delta)$ is closed under real quantification, $o(\delta)$ is closed under the above ultrapower operation in the following precise sense (cf. [10]).

6.13 Theorem. *Let δ be a Suslin cardinal with $S(\delta)$ closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$. Let $\beta < o(\delta)$, and ν_β the corresponding atomic normal measure. Then $j_{\nu_\beta}(\delta) < o(\delta)$. Furthermore, $\mathrm{cf}(o(\delta)) > \delta$, and $\mathrm{cf}(o(\delta)) \neq j_{\nu_\beta}(\delta)$ for all $\beta < o(\kappa)$.*

It is conjectured in [10] that $o(\delta)$ is regular for such δ; Theorem 6.13 seems a step towards showing that. A recent theorem of Steel [38] shows that in $L(\mathbb{R})$ every regular cardinal below Θ is measurable (using techniques of inner model theory). Thus, granting the above conjecture, new (normal) measures appear on δ which seem not to be approachable "from below" in the previous sense. Undoubtedly, new techniques will be necessary.

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22. Determinacy in $L(\mathbb{R})$

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Contents

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$$\begin{array}{c|ccccccc}
\mathrm{I} & x(0) & & x(2) & & \ldots\ldots & & \\
\hline
\mathrm{II} & & x(1) & & x(3) & & & \ldots\ldots
\end{array}$$

Diagram 22.1: The game $G_\omega(C)$

Given a set $C \subseteq \omega^\omega$ define $G_\omega(C)$, the length ω game with payoff set C, to be played as follows: Players I and II collaborate to produce an infinite sequence $x = \langle x(i) \mid i < \omega \rangle$ of natural numbers. They take turns as in Diagram 22.1, I picking $x(i)$ for even i and II picking $x(i)$ for odd i. If at the end the sequence x they produce belongs to C then player I wins; and otherwise player II wins. $G_\omega(C)$, or any other game for that matter, is *determined* if one of the two players has a *winning strategy*, namely a strategy for the game that wins against all possible plays by the opponent. The set C is said to be determined if the corresponding game $G_\omega(C)$ is determined. Determinacy is said to hold for a pointclass $\mathbf{\Gamma}$ if all sets of reals in $\mathbf{\Gamma}$ are determined. (Following standard abuse of notation we identify $\mathbb{R}$ with ω^ω.)

Perhaps surprisingly, determinacy has turned out to have a crucial and central role in the study of definable sets of reals. This role resulted from two lines of discoveries. On the one hand it was seen that determinacy for definable sets of reals, taken as an axiom, can be used to prove many desirable results about these sets, and indeed to obtain a rich and powerful structure theory. On the other hand it was seen that determinacy can be proved for definable sets of reals, from large cardinal axioms.

The earliest work on consequences of determinacy, by Banach, Mazur, and Ulam [23] at the famous Scottish Café in the 1930's, Oxtoby [35], Davis [3], and Mycielski-Swierczkowski [27], established that determinacy for a pointclass Γ implies that all sets of reals in Γ have the Baire property, have the perfect set property, and are Lebesgue measurable. Later on Blackwell [2] used the determinacy of open sets to prove Kuratowski's reduction theorem. (In modern terminology this theorem states that for any $\mathbf{\Pi}^1_1$ sets A, B, there are $A^* \subseteq A$ and $B^* \subseteq B$ so that $A^* \cup B^* = A \cup B$ and $A^* \cap B^* = \emptyset$.) Inspired by his methods, Martin [15] and Addison-Moschovakis [1] used determinacy for projective sets to prove reduction for each of the pointclasses $\mathbf{\Pi}^1_n$, $n > 1$ odd, and indeed prove for these pointclasses some of the structural properties that hold for $\mathbf{\Pi}^1_1$. Their results initiated a wider study of consequences of the Axiom of Determinacy (AD), the assertion that all sets of reals are determined, proposed initially by Mycielski-Steinhaus [26]. Over time this line of research, which the reader may find in Moschovakis [25], Jackson [7], and of course the Cabal volumes [9–12], established determinacy axioms as natural assumptions in the study of definable sets of reals.

It should be emphasized that AD was not studied as an assumption about V. (It contradicts the axiom of choice.) Rather, it was studied as an assumption about more restrictive models, models which contain all the reals but have only definable sets of reals. A prime example was the model $L(\mathbb{R})$, consisting of all sets which are constructible from $\{\mathbb{R}\} \cup \mathbb{R}$. It was known by work of Solovay [37] that this model need not satisfy the axiom of choice, and that in fact it is consistent that all sets of reals in this model are Lebesgue measurable. The extra assumption of AD allowed for a very careful analysis of $L(\mathbb{R})$, in terms that combined descriptive set theory, fine structure, and infinitary combinatorics. It seemed plausible that if there were a model of AD, $L(\mathbb{R})$ would be it.

Research into the consequences of determinacy was to some extent done on faith. The established hierarchy of strength in set theory involved large cardinals axioms, axioms asserting the existence of elementary embeddings from the universe of sets into transitive subclasses, not determinacy axioms. A great deal of work has been done in set theory on large cardinal axioms, with Kanamori [8] a good reference, and large cardinals have come to be regarded as the backbone of the universe of sets, providing a hierarchy of consistency strengths against which all other statements are measured. From $\mathsf{AD}^{L(\mathbb{R})}$ one could obtain objects in $L(\mathbb{R})$ which are very strongly reminiscent of large cardinal axioms in V, suggesting a connection between the two. Perhaps the most well-known of the early results in this direction is Solovay's proof that ω_1 is measurable under AD. Further justification for the use of $\mathsf{AD}^{L(\mathbb{R})}$ was provided by proofs of determinacy for simply definable sets: for open sets in Gale-Stewart [6], for Borel sets in Martin [18, 17], and for $\mathbf{\Pi}^1_1$ sets from a measurable cardinal in Martin [16], to name the most well-known. Additional results, inspired by Solovay's proof that ω_1 is measurable under AD and Martin's proof of $\mathbf{\Pi}^1_1$ determinacy from a measurable cardinal, identified detailed and systematic correspondences of strength, relating models for many measurable cardinals to determinacy for pointclasses just above $\mathbf{\Pi}^1_1$. These levels are well below the pointclass of all sets in $L(\mathbb{R})$, but still the accumulated evidence of the results suggested that there should be a proof of $\mathsf{AD}^{L(\mathbb{R})}$ from large cardinals, and conversely a construction of inner models with these large cardinals from $\mathsf{AD}^{L(\mathbb{R})}$. In 1985 the faith in this connection was fully vindicated. A sequence of results of Foreman, Magidor, Martin, Shelah, Steel, and Woodin (see [5, 36, 21, 22, 43] for the papers involved and the introduction in [30] for an overview) brought the identification of a new class of large cardinals, known now as Woodin cardinals, new structures of iterated ultrapowers, known now as iteration trees, and new proofs of determinacy, including a proof of $\mathsf{AD}^{L(\mathbb{R})}$. Additional results later on produced Woodin cardinals from determinacy axioms, and indeed established a deep and intricate connection between the descriptive set theory of $L(\mathbb{R})$ under AD, and inner models for Woodin cardinals.

In this chapter we prove $\mathsf{AD}^{L(\mathbb{R})}$ from Woodin cardinals. Our exposition is complete and self-contained: the necessary large cardinals are introduced in Sect. 1, and every result about them which is needed in the course of proving $\mathsf{AD}^{L(\mathbb{R})}$ is included in the chapter, mostly in Sects. 2 and 3. The climb to $\mathsf{AD}^{L(\mathbb{R})}$ is carried out progressively in the remaining sections. In Sect. 4 we introduce homogeneously Suslin sets and present a proof of determinacy for $\mathbf{\Pi}^1_1$ sets from a measurable cardinal. In Sect. 5 we move up and present a proof of projective determinacy from Woodin cardinals. The proof in essence converts the quantifiers over reals appearing in the definition of a projective set to quantifiers over iteration trees and branches through the trees, and these quantifiers in turn are tamed by the iterability results in Sect. 2. In Sect. 6 we improve on the results in Sect. 5 by reducing the large cardinal assumption needed for the determinacy of universally Baire sets. The section also lays the ground for Sect. 7, where we show that models with Woodin cardinals can be iterated to absorb an arbitrary given real into a generic extension. Finally, in Sect. 8 we derive $\mathsf{AD}^{L(\mathbb{R})}$.

There is much more to be said about proofs of determinacy that cannot be fitted within the scope of this chapter. Martin [18, 19] and Neeman [32] for example prove weaker forms of determinacy (from weaker assumptions) using completely different methods, which handle increments of payoff complexity corresponding to countable unions, rather than real quantifiers. Perhaps more importantly there are strengthenings of $\mathsf{AD}^{L(\mathbb{R})}$ in two directions, one involving stronger payoff sets, and the other involving longer games. In the former direction the reader should consult Steel [39], which contains a proof of Woodin's derived model theorem, a fundamental theorem connecting models of AD to symmetric extensions of models of choice with Woodin cardinals, and uses this theorem to establish AD in models substantially stronger than $L(\mathbb{R})$. In the latter direction the reader should consult Neeman [30, 33], which contain proofs of determinacy for games of fixed countable lengths, variable countable lengths, and length ω_1.

Historical Remarks. With some exceptions, noted individually inside the various sections, the following remarks summarize credits for the material in the chapter. Extenders were introduced by Mitchell [24], then simplified to their present forms by Jensen. The related material on ultrapowers in Sect. 1 is by now folklore within set theory. Its history can be found in Kanamori [8]. The material on iteration trees in Sect. 1 is due to Martin-Steel [22] and so is all the material in Sect. 2. The material in Sect. 3 is due to Martin-Steel [21], and follows the exposition of Neeman [30]. The material in Sect. 4 is due to Martin. The material in Sect. 5 is due to Martin-Steel [21]. (The exposition here is specifically geared to easing the transition to the next section.) The material in Sects. 6 and 7 is due to Neeman. $\mathsf{AD}^{L(\mathbb{R})}$ from infinitely many Woodin cardinals and a measurable cardinal above them is due to Woodin, proved using the methods of stationary tower forcing (see Larson [14]) and an appeal to the main theorem, Theorem 5.11, in Martin-Steel [21]. A proof using Woodin's genericity iterations [38, 4.3] and fine structure instead of stationary tower forcing is due to Steel, and the proof reached in this chapter (using a second form of genericity iterations and no fine structure) is due to Neeman.

1. Extenders and Iteration Trees

Throughout this chapter we shall deal with elementary embeddings of the universe into transitive classes. Here we develop tools for the study of such embeddings. Most basic among them is the ultrapower construction, which allows the creation of an embedding $\pi : V \to M$ from the restriction of such an embedding to a set. We begin by characterizing the restrictions.

1.1 Remark. By embedding we always mean elementary embedding, even when this is not said explicitly. As a matter of convention when we say a wellfounded model of set theory we mean a *transitive* model equipped with the standard membership relation $\in$. More generally we always take the wellfounded parts of our models to be transitive.

Let $(*, *)$ denote the Gödel pairing operation on ordinals. Given sets of ordinals A and B define $A \times B$ to be $\{(\alpha, \beta) \mid \alpha \in A \wedge \beta \in B\}$. Note that $A \times B$ is then a set of ordinals too. We refer to it as the *product* of A and B. In general define finite products of sets of ordinals as follows: For $n = 0$ set $\prod_{i \leq n} A_i$ equal to A_0; for $n > 0$ set $\prod_{i \leq n} A_i$ equal to $(\prod_{i \leq n-1} A_i) \times A_n$. Define finite sequences of ordinals similarly by setting the empty sequence equal to 0, setting (α) equal to α, and setting $(\alpha_0, \ldots, \alpha_n)$ equal to $((\alpha_0, \ldots, \alpha_{n-1}), \alpha_n)$ for $n > 0$.

If A is a set of ordinal sequences of length n, and $\sigma : n \to n$ is a permutation of n, then define σA by setting

$$(\alpha_0, \ldots, \alpha_{n-1}) \in \sigma A \iff (\alpha_{\sigma^{-1}(0)}, \ldots, \alpha_{\sigma^{-1}(n-1)}) \in A.$$

If A is a set of ordinal sequences of length $n + 1$, then define $\mathrm{bp}(A)$ to be the set $\{(\alpha_0, \ldots, \alpha_{n-1}) \mid (\exists \xi \in \alpha_0)(\alpha_0, \ldots, \alpha_{n-1}, \xi) \in A\}$. $\mathrm{bp}(A)$ is the *bounded projection* of A.

By a *fiber* of sets $\langle A_i \mid i < \omega \rangle$ we mean a sequence $\langle \alpha_i \mid i < \omega \rangle$ so that $(\alpha_0, \ldots, \alpha_{i-1}) \in A_i$ for every $i < \omega$.

1.2 Definition. A (*short*) *extender* is a function E that satisfies the following conditions:

(1) The domain of E is equal to $\mathcal{P}(\kappa)$ for an ordinal κ closed under Gödel pairing.

(2) E sends ordinals to ordinals and sets of ordinals to sets of ordinals.

(3) $E(\alpha) = \alpha$ for $\alpha < \kappa$, and $E(\kappa) \neq \kappa$.

(4) E respects products, intersections, set differences, membership, the predicates of equality and membership, permutations, and bounded projections. More precisely this means that for all $A, B \in \mathrm{dom}(E)$, all ordinals $\alpha \in \mathrm{dom}(E)$, and all permutations σ of the appropriate format:

(a) $E(A \times B) = E(A) \times E(B)$, $E(A \cap B) = E(A) \cap E(B)$, and $E(A - B) = E(A) - E(B)$.

(b) $\alpha \in A \Longrightarrow E(\alpha) \in E(A)$.

(c) $E(\{(\alpha, \beta) \in A \times A \mid \alpha = \beta\})$ is equal to $\{(\alpha, \beta) \in E(A) \times E(A) \mid \alpha = \beta\}$, and similarly with $\alpha \in \beta$ replacing $\alpha = \beta$.

(d) $E(\sigma A) = \sigma E(A)$.

(e) $E(\mathrm{bp}(A)) = \mathrm{bp}(E(A))$.

(5) E is *countably complete*. Precisely, this means that for any sequence $\langle A_i \mid i < \omega \rangle$ of sets which are each in the domain of E, if there exists a fiber through $\langle E(A_i) \mid i < \omega \rangle$ then there exists also a fiber through $\langle A_i \mid i < \omega \rangle$.

The first ordinal moved by E is called the *critical point* of E, denoted $\mathrm{crit}(E)$. By condition (3), this critical point is precisely equal to the ordinal κ of condition (1). The set $\bigcup_{A \in \mathrm{dom}(E)} E(A)$ is called the *support* of E, denoted $\mathrm{spt}(E)$. Using condition (2) it is easy to see that the support of E is an ordinal.

1.3 Remark. Condition (3) limits our definition to extenders with domains consisting of just the subsets of the extenders' critical points. It is this condition that makes our extenders "short". We shall see later that it has the effect of limiting the strength of embeddings generated by our (short) extenders to a level known as superstrong. This level is more than adequate for our needs. We shall therefore deal exclusively with short extenders in this chapter, and refer to them simply as extenders. For a more general definition see Neeman [31].

1.4 Definition. A (two-valued) *measure* over a set U is a function μ from $\mathcal{P}(U)$ into $\{0, 1\}$ with the properties that $\mu(\emptyset) = 0$, $\mu(U) = 1$, and $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for any disjoint $X, Y \subseteq U$.

1.5 Remark. Given $a \in \mathrm{spt}(E)$ define $E_a : \mathcal{P}(\kappa) \to \{0, 1\}$ to be the function given by $E_a(X) = 1$ if $a \in E(X)$, and 0 otherwise. E_a is then a measure over κ. It has been customary to define extenders by specifying properties of the sequence $\langle E_a \mid a \in \mathrm{spt}(E) \rangle$ equivalent to the properties of E specified in Definition 1.2. For a definition of extender through properties of $\langle E_a \mid a \in \mathrm{spt}(E) \rangle$ see Martin-Steel [21, §1A] (short extenders) and Kanamori [8, §26] (the general case).

By a *pre-extender* over a model Q we mean an object E that satisfies conditions (1)–(4) in Definition 1.2, with $\mathcal{P}(\kappa)$ in condition (1) replaced by $\mathcal{P}^Q(\kappa)$, but not necessarily condition (5). The point of this distinction is that condition (5) involves second-order quantification over E, whereas conditions (1)–(4) involve only E, the power set of κ, and bounded quantifiers over the transitive closure of E. By removing condition (5) we obtain a notion that is absolute in the sense given by Claim 1.7:

1.6 Definition. Two models Q and N *agree to* an ordinal ρ if (ρ is contained in the wellfounded part of both models, and) $\mathcal{P}^Q(\xi) = \mathcal{P}^N(\xi)$ for each $\xi < \rho$. Q and N *agree past* an ordinal κ if they agree to $\kappa + 1$.

1.7 Claim. *Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree past κ. Then E is a pre-extender over Q.*

Extenders are naturally induced by elementary embeddings. Let $\pi : V \to M$ be a non-trivial elementary embedding of V into some wellfounded class model M. Let κ be the critical point of π, namely the first ordinal moved by π. Let $\lambda \leq \pi(\kappa)$ be an ordinal closed under Gödel pairing. Define the λ-*restriction* of π to be the map E given by:

(R1) $\mathrm{dom}(E) = \mathcal{P}(\kappa)$.

(R2) $E(X) = \pi(X) \cap \lambda$ for each $X \in \mathrm{dom}(E)$.

It is then easy to check that E is an extender. The items in condition (4) of Definition 1.2 follow directly from the elementarity of π and, in the case of condition (4e), the absoluteness between M and V of formulae with only bounded quantifiers. Condition (5) follows from the elementarity of π and the *wellfoundedness* of M. If a fiber through $\langle E(A_i) \mid i < \omega\rangle$ exists in V then using the wellfoundedness of M such a fiber must also exist in M. Its existence can then be pulled back via π to yield a fiber through $\langle A_i \mid i < \omega\rangle$.

1.8 Remark. The λ-restriction makes sense also in the case of an embedding into an illfounded model M, so long as the wellfounded part of M contains λ. But countable completeness may fail in this case, and the λ-restriction need only be a pre-extender.

The description above shows how extenders are induced by elementary embeddings into wellfounded models. Extenders also give rise to such elementary embeddings, through the ultrapower construction, which we describe next.

Let ZFC^- consist of the standard axioms of ZFC excluding the Power Set Axiom. Fix a model Q of ZFC^- and a pre-extender E over Q. Let $\kappa = \mathrm{crit}(E)$. Let $\mathcal{F}$ be the class of functions $f \in Q$ so that $\mathrm{dom}(f) \subseteq \kappa$. Let $\mathcal{D} = \{\langle f, a\rangle \mid f \in \mathcal{F} \wedge a \in E(\mathrm{dom}(f))\}$.

For two functions $f, g \in \mathcal{F}$ set $Z^{=}_{f,g} = \{(\alpha, \beta) \mid f(\alpha) = g(\beta)\}$ and $Z^{\in}_{f,g} = \{(\alpha, \beta) \mid f(\alpha) \in g(\beta)\}$. Both $Z^{=}_{f,g}$ and $Z^{\in}_{f,g}$ are then subsets of κ in Q, and therefore elements of the domain of E.

Define a relation $\sim$ on $\mathcal{D}$ by setting $\langle f, a\rangle \sim \langle g, b\rangle$ iff $(a, b) \in E(Z^{=}_{f,g})$. One can check using condition (4) in Definition 1.2 that $\sim$ is an equivalence relation. Let $[f, a]$ denote the equivalence class of $\langle f, a\rangle$. Let $\mathcal{D}^*$ denote $\mathcal{D}/{\sim}$. Define a relation R on $\mathcal{D}^*$ by setting $[f, a]\ R\ [g, b]$ iff $(a, b) \in E(Z^{\in}_{f,g})$. Again using condition (4) in Definition 1.2 one can check that R is well defined.

The following property, known as Łoś's Theorem, can be proved from the various definitions, by induction on the complexity of φ:

1.9 Theorem (Łoś). *Let $[f_1, a_1], \ldots, [f_n, a_n]$ be elements of $\mathcal{D}^*$. Let $\varphi = \varphi(v_1, \ldots, v_n)$ be a formula. Let Z be the set*

$$\{(\alpha_1, \ldots, \alpha_n) \mid Q \models \varphi[f_1(\alpha_1), \ldots, f_n(\alpha_n)]\}.$$

Then $(\mathcal{D}^*, R) \models \varphi[[f_1, a_1], \ldots, [f_n, a_n]]$ iff $(a_1, \ldots, a_n)$ belongs to $E(Z)$.

For each set x let c_x be the function with domain $\{0\}$ and value $c_x(0) = x$. From Łoś's Theorem it follows that the map $x \mapsto [c_x, 0]$ is elementary, from Q into $(\mathcal{D}^*, R)$. (In particular, $(\mathcal{D}^*, R)$ satisfies ZFC^-.)

1.10 Definition. The *ultrapower* of Q by E, denoted $\mathrm{Ult}(Q, E)$, is the structure $(\mathcal{D}^*; R)$. The *ultrapower embedding* is the map $j : Q \to \mathrm{Ult}(Q, E)$ defined by $j(x) = [c_x, 0]$.

In general $\mathrm{Ult}(Q, E)$ need not be wellfounded. (If it is then we of course identify it with its transitive collapse, and identify R with $\in$.) But notice that wellfoundedness is a consequence of countable completeness: if $\langle [f_i, a_i] \mid i < \omega \rangle$ is an infinite descending sequence in R, then the sequence of sets $A_i = \{(\alpha_0, \ldots, \alpha_{i-1}) \mid f_0(\alpha_0) \ni f_1(\alpha_1) \ni \cdots \ni f_{i-1}(\alpha_{i-1})\}$ violates countable completeness. Ultrapowers by extenders, as opposed to mere pre-extenders, are therefore wellfounded.

Let $\lambda = \mathrm{spt}(E)$. Using the various definitions one can prove the following two properties of the ultrapower. The first relates the ultrapower embedding back to the extender E, and the second describes a certain minimality of the ultrapower:

(U1) The λ-restriction of j is precisely equal to E.

(U2) Every element of $\mathrm{Ult}(Q, E)$ has the form $j(f)(a)$ for some function $f \in \mathcal{F}$ and some $a \in \lambda$.

These properties determine the ultrapower and the embedding completely.

The following lemma relates an embedding $\pi : V \to M$ to the ultrapower embedding by the extender over V derived from π. It shows that the ultrapower by the λ-restriction of π captures π up to λ.

1.11 Lemma. *Let $\pi : V \to M$ be an elementary embedding of V into a wellfounded model M, and let $\kappa = \mathrm{crit}(\pi)$. Let $\lambda \leq \pi(\kappa)$ be an ordinal closed under Gödel pairing. Let E be the λ-restriction of π. Let $N = \mathrm{Ult}(V, E)$ and let $j : V \to N$ be the ultrapower embedding.*

Then there is an elementary embedding $k : N \to M$ with $\pi = k \circ j$ (see Diagram 22.2) and $\mathrm{crit}(k) \geq \lambda$.

1.12 Exercise. Let μ be a two-valued measure over a cardinal κ. Let $\mathcal{F}$ be the class of functions from κ into V. For $f, g \in \mathcal{F}$ set $f \sim g$ iff $\{\xi < \kappa \mid f(\xi) = g(\xi)\}$ has measure one. Show that $\sim$ is an equivalence relation. Let $\mathcal{F}^* = \mathcal{F}/{\sim}$. For $f \in \mathcal{F}$ let $[f]$ denote the equivalence class of f. Define

$$\begin{tikzcd} & M \\ V \arrow[ur, "\pi"] \arrow[r, "j"'] & N = \mathrm{Ult}(V, E) \arrow[u, "k"'] \end{tikzcd}$$

Diagram 22.2: The original map π and the ultrapower map j

a relation R on $\mathcal{F}^*$ by $[f]\ R\ [g]$ iff $\{\xi < \kappa \mid f(\xi) \in g(\xi)\}$ has measure one. Show that R is well defined.

Define $\mathrm{Ult}(V, \mu)$, the *ultrapower* of V by μ, to be the structure $(\mathcal{F}^*; R)$, and define the *ultrapower embedding* $j : V \to \mathrm{Ult}(V, \mu)$ by $j(x) = [c_x]$ where $c_x : \kappa \to V$ is the constant function which takes the value x.

Show that ultrapower embedding is elementary. Show that if μ is countably complete, meaning that $\mu(\bigcap_{n<\omega} X_n) = 1$ whenever $\langle X_n \mid n < \omega\rangle$ is a sequence of sets of measure one, then the ultrapower is wellfounded.

1.13 Exercise. The *seed* of a measure μ is the element [id] of the ultrapower, where $\mathrm{id} : \kappa \to V$ is the identity function. Let s be the seed of μ. Prove that every element of $\mathrm{Ult}(V, \mu)$ has the form $j(f)(s)$, where $j : V \to \mathrm{Ult}(V, \mu)$ is the ultrapower embedding.

1.14 Exercise. A (two-valued) measure μ over a set U is called *non-principal* just in case that $\mu(\{\xi\}) = 0$ for each singleton $\{\xi\}$. μ is *κ-complete* if $\mu(\bigcap_{\alpha<\tau} X_\alpha) = 1$ whenever $\tau < \kappa$ and $X_\alpha \subseteq U$ $(\alpha < \tau)$ are all sets of measure one. A cardinal κ is called *measurable* if there is a two-valued, non-principal, κ-complete measure over κ. Let κ be measurable, let μ witness this, and let $j : V \to \mathrm{Ult}(V, \mu)$ be the ultrapower embedding. Show that $\mathrm{crit}(j) = \kappa$.

1.15 Exercise. Let κ be measurable and let μ witness this. Let $M = \mathrm{Ult}(V, \mu)$. Prove that $\mathcal{P}(\kappa) \subseteq M$, and that $\mathcal{P}(\mathcal{P}(\kappa)) \not\subseteq M$.

Hint. To see that $\mathcal{P}(\kappa) \subseteq M$, note that $j(X) \cap \kappa = X$ for each $X \subseteq \kappa$ (where $j : V \to M$ is the ultrapower embedding).

To see that $\mathcal{P}(\mathcal{P}(\kappa)) \not\subseteq M$, prove that $\mu \notin M$: Suppose for contradiction that $\mu \in \mathrm{Ult}(V, \mu)$. Without loss of generality you may assume that κ is the smallest cardinal carrying a measure μ with $\mu \in \mathrm{Ult}(V, \mu)$. Derive a contradiction to the analogous minimality of $j(\kappa)$ in M by showing that $\mu \in \mathrm{Ult}(M, \mu)$. $\dashv$

1.16 Definition. An embedding $\pi : V \to M$ is *α-strong* just in case that $\mathcal{P}(\xi) \subseteq M$ for all $\xi < \alpha$. An extender E is *α-strong* just in case that $\mathcal{P}(\xi) \subseteq \mathrm{Ult}(V, E)$ for all $\xi < \alpha$. The *strength* of $\pi : V \to M$ is defined to be the largest α so that π is α-strong. The strength of an extender E is defined similarly, using the ultrapower, and is denoted $\mathrm{Strength}(E)$. (Notice that the strength of an embedding is always a cardinal.) An embedding π with critical point κ is *superstrong* if it is $\pi(\kappa)$-strong. A cardinal κ is *α-strong* if it is the critical point of an α-strong embedding, and *superstrong* if it is the critical point of a superstrong embedding.

Measurable cardinals lie at the low end of the hierarchy of strength: assuming GCH, an ultrapower embedding by a measure on κ is κ^+-strong and no more. Superstrong embeddings lie much higher in the hierarchy. These embedding are the most we can hope to capture using (short) extenders:

1.17 Lemma. *Let E be a (short) extender with critical point κ. Let j be the ultrapower embedding by E. Then E is at most $j(\kappa)$-strong.*

Proof. Using the ultrapower construction and the elementarity of j, one can see that every element x of $j(\kappa^+)$ has the form $j(f)(a)$ for a function $f : \kappa \to \kappa^+$ and an $a \in \mathrm{dom}(j(f)) = j(\kappa)$. (The fact that f can be taken to have domain κ traces back to the fact that the domain of E consists precisely of the subsets of its critical point, in other words to the fact that E is a *short* extender.) It follows that $j(\kappa^+)$ has cardinality at most $\theta = (\kappa^+)^\kappa \cdot j(\kappa)$. If j is $j(\kappa)$-strong then $j(\kappa)$ is a strong limit cardinal in V, and a quick calculation shows that $\theta = j(\kappa)$. Thus $j(\kappa^+) = (j(\kappa)^+)^{\mathrm{Ult}(V,E)}$ has cardinality $j(\kappa)$ in V, and from this it follows that $\mathrm{Ult}(V, E)$ must be missing some subsets of $j(\kappa)$. So E is not $j(\kappa) + 1$-strong. $\dashv$

1.18 Lemma. *Let $\pi : V \to M$ with critical point κ. Suppose that π is α-strong where $\alpha \leq \pi(\kappa)$. Let $\lambda \leq \pi(\kappa)$ be an ordinal closed under Gödel pairing and such that $\lambda \geq (2^{<\alpha})^M$. Then the λ-restriction of π is an α-strong extender.*

Proof. Immediate from Lemma 1.11. $\dashv$

Lemma 1.18 shows that (short) extenders are adequate means for capturing the strength of embeddings at or below the level of superstrong. On the other hand Lemma 1.17 shows that (short) extenders cannot capture embeddings beyond superstrong. Such stronger embeddings can be captured using the general extenders mentioned in Remark 1.3, but for our purpose in this chapter the greater generality is not necessary.

1.19 Definition. We write $Q\|\alpha$ to denote V^Q_α. We say that Q and N *agree well beyond* κ if the first inaccessible above κ is the same in both Q and N, and, letting $\alpha > \kappa$ be this inaccessible, $Q\|\alpha = N\|\alpha$. Given further embeddings $i : Q \to Q^*$ and $j : N \to N^*$ we say that i and j *agree well beyond* κ if $i{\restriction}(Q\|\alpha \cup \{Q\|\alpha\}) = j{\restriction}(N\|\alpha \cup \{N\|\alpha\})$.

We shall use the notion of Definition 1.19 as an all-purpose security blanket, giving us (more than) enough room in several arguments below.

$$\begin{tikzcd}
Q^* \arrow[rr, bend left, "i^*"] & E^* \in N^* & \mathrm{Ult}(Q^*, E^*) \\
Q \arrow[u, "\pi"] \arrow[rr, bend right, "i"'] & E \in N \arrow[u, "\sigma"] & \mathrm{Ult}(Q, E) \arrow[u, "\tau"]
\end{tikzcd}$$

Diagram 22.3: Copying the ultrapower of Q by E to an ultrapower of Q^* by E^*

1.20 Claim. *Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree well beyond κ, so that (in particular) E is a pre-extender over Q. Let i be the ultrapower embedding of Q by E, and let j be the ultrapower embedding of N by E. Then i and j agree well beyond κ, and $\mathrm{Ult}(Q, E)$ and $\mathrm{Ult}(N, E)$ agree well beyond $i(\kappa) = j(\kappa)$.*

Let Q and N be models of set theory. Suppose that E is an extender in N, and let $\kappa = \mathrm{crit}(E)$. Suppose that Q and N agree well beyond κ, so that in particular E is a pre-extender over Q.

Let $\pi : Q \to Q^*$ and $\sigma : N \to N^*$ be elementary. Let $E^* = \sigma(E)$. Suppose that π and σ agree well beyond κ. Hence in particular Q^* and N^* agree well beyond $\pi(\kappa) = \sigma(\kappa)$, and E^* is therefore a pre-extender over Q^*. The models and embeddings are presented in Diagram 22.3.

For an element $x = [f, a]$ of $\mathrm{Ult}(Q, E)$ define $\tau(x)$ to be the element $[\pi(f), \sigma(a)]$ of $\mathrm{Ult}(Q^*, E^*)$.

Then τ is a well defined (meaning invariant under the choice of representatives for $x \in \mathrm{Ult}(Q, E)$) elementary embedding from $\mathrm{Ult}(Q, E)$ into $\mathrm{Ult}(Q^*, E^*)$; $\tau \restriction \mathrm{spt}(E) = \sigma \restriction \mathrm{spt}(E)$; and τ makes Diagram 22.3, with i and i^* being the relevant ultrapower embeddings, commute.

The ultrapower of Q^* by E^* is called the *copy*, via the pair $\langle \pi, \sigma \rangle$, of the ultrapower of Q by E. τ is called the *copy embedding*. Note that the definition of τ involves both π and σ, and the agreement between these two embeddings is important for the proof that τ is well defined.

1.21 Remark. Recall that every element of $\mathrm{Ult}(Q, E)$ has the form $i(f)(a)$ for a function $f \in Q$ and an ordinal $a \in \mathrm{spt}(E)$. The copy embedding τ is characterized completely by the condition $\tau(i(f)(a)) = (i^* \circ \pi)(f)(\sigma(a))$ for all f and a.

Next we describe how to repeatedly form ultrapowers by extenders, to obtain a chain, or a tree, of models. For the record let us start by defining direct limits.

1.22 Definition. Let $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \alpha\rangle$ be a system of models M_ξ and elementary embeddings $j_{\zeta,\xi} : M_\zeta \to M_\xi$, commuting in the natural way. Let $\mathcal{D} = \{\langle \xi, x\rangle \mid \xi < \alpha \wedge x \in M_\xi\}$.

Define an equivalence relation $\sim$ on $\mathcal{D}$ by setting $\langle \xi, x\rangle \sim \langle \xi', x'\rangle$ iff $j_{\xi,\nu}(x) = j_{\xi',\nu}(x)$ where $\nu = \max\{\xi, \xi'\}$. Let $\mathcal{D}^* = \mathcal{D}/{\sim}$.

Define a relation R on $\mathcal{D}^*$ by setting $[\xi, x]\ R\ [\xi', x']$ iff $j_{\xi,\nu}(x) \in j_{\xi',\nu}(x)$ where again $\nu = \max\{\xi, \xi'\}$. It is easy to check that R is well defined.

The structure $M^* = (\mathcal{D}^*; R)$ is called the *direct limit* of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \alpha\rangle$. The embeddings $j_{\xi,*} : M_\xi \to M^*$ determined by $j_\xi(x) = [\xi, x]$ are called the *direct limit embeddings*. It is easy to check that these embeddings commute with the embeddings $j_{\zeta,\xi}$ in the natural way.

1.23 Remark. If $(\mathcal{D}^*; R)$ is wellfounded then we identify it with its transitive collapse, and identify R with $\in$.

We pass now to the matter of iterated ultrapowers.

1.24 Definition. A *tree order* is an order T on an ordinal α so that:

(1) T is a suborder of $<\restriction(\alpha \times \alpha)$.

(2) For each $\eta < \alpha$, the set $\{\xi \mid \xi\ T\ \eta\}$ is linearly ordered by T.

(3) For each ξ so that $\xi + 1 < \alpha$, the ordinal $\xi + 1$ is a successor in T.

(4) For each limit ordinal $\gamma < \alpha$, the set $\{\xi \mid \xi\ T\ \gamma\}$ is cofinal in γ.

1.25 Definition. An *iteration tree* $\mathcal{T}$ of length α on a model M consists of a tree order T on α and a sequence $\langle E_\xi \mid \xi + 1 < \alpha\rangle$, so that the following conditions hold with an additional sequence $\langle M_\xi, j_{\zeta,\xi} \mid \zeta\ T\ \xi < \alpha\rangle$ which is determined completely by the conditions:

(1) $M_0 = M$.

(2) For each ξ so that $\xi + 1 < \alpha$, E_ξ is an extender of M_ξ, or $E_\xi =$ "pad".

(3) (a) If $E_\xi =$ "pad" then $M_{\xi+1} = M_\xi$, the T-predecessor of $\xi + 1$ is ξ, and $j_{\xi,\xi+1}$ is the identity.
(b) If $E_\xi \neq$ "pad" then $M_{\xi+1} = \mathrm{Ult}(M_\zeta, E_\xi)$ and $j_{\zeta,\xi+1} : M_\zeta \to M_{\xi+1}$ is the ultrapower embedding, where ζ is the T-predecessor of $\xi + 1$. It is implicit in this condition that M_ζ must agree with M_ξ past $\mathrm{crit}(E_\xi)$, so that E_ξ is a pre-extender over M_ζ by Claim 1.7.

(4) For limit $\lambda < \alpha$, M_λ is the direct limit of the system $\langle M_\zeta, j_{\zeta,\xi} \mid \zeta\ T\ \xi\ T\ \lambda\rangle$, and $j_{\zeta,\lambda} : M_\zeta \to M_\lambda$ for $\zeta\ T\ \lambda$ are the direct limit embeddings.

(5) The remaining embeddings $j_{\zeta,\xi}$ for $\zeta\ T\ \xi < \alpha$ are obtained through composition.

M_ξ and $j_{\zeta,\xi}$ for $\zeta\ T\ \xi < \alpha$ are the *models* and *embeddings of* $\mathcal{T}$. We view them as part of $\mathcal{T}$, though formally they are not.

$$\begin{tikzcd} M_{n+1} & \\ & E_n \in M_n \\ M_k \arrow[uu, "j_{k,n+1}"] & \end{tikzcd}$$

Diagram 22.4: Forming M_{n+1}

1.26 Remark. The inclusion of pads in iteration tree is convenient for purposes of indexing in various constructions, and we shall use it later on. But for much of the discussion below we make the implicit assumption that the iteration tree considered has no pads. This assumption poses no loss of generality.

We shall only need iteration trees of length ω in this chapter. We shall construct these trees recursively. In stage n of the construction we shall have the models $M_0, \ldots, M_n$. During the stage we shall pick an extender E_n in M_n, and pick further some $k \leq n$ so that M_k and M_n agree past $\mathrm{crit}(E_n)$. We shall then set k to be the T-predecessor of $n+1$ and set $M_{n+1} = \mathrm{Ult}(M_k, E_n)$. This is illustrated in Diagram 22.4. After ω stages of a construction of this kind we obtain an iteration tree of length ω.

A *branch* through an iteration tree $\mathcal{T}$ is a set b which is linearly ordered by T. The branch is *cofinal* if $\sup(b) = \mathrm{lh}(\mathcal{T})$. By the *direct limit* along b, denoted $M_b^{\mathcal{T}}$ or simply M_b, we mean the direct limit of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta \; T \; \xi \in b\rangle$. We use $j_{\zeta,b}^{\mathcal{T}}$, or simply $j_{\zeta,b}$, to denote the direct limit embeddings of this system. The branch b is called *wellfounded* just in case that the model M_b is wellfounded.

2. Iterability

The existence of wellfounded cofinal branches through certain iteration trees is crucial to proofs of determinacy. This existence is part of the general topic of iterability. In this section we briefly describe the topic, point out its most important open problem, and sketch a proof of the specific iterability necessary for the determinacy results in this chapter.

Let M be a model of ZFC^-. In the (*full*) *iteration game* on M players "good" and "bad" collaborate to construct an iteration tree $\mathcal{T}$ of length ω_1^V+1 on M. "bad" plays all the extenders, and determines the T-predecessor of $\xi+1$ for each ξ. "good" plays the branches $\{\zeta \mid \zeta \; T \; \lambda\}$ for limit λ, thereby determining the T-predecessors of λ and the direct limit model M_λ. Note that "good" is also responsible for the final move, which determines $M_{\omega_1^V}$.

If ever a model along the tree is reached which is illfounded then "bad" wins. Otherwise "good" wins. M is (*fully*) *iterable* if "good" has a winning strategy in this game. An *iteration strategy* for M is a strategy for the good player in the iteration game on M. The *Strategic Branches Hypothesis* (SBH) asserts that every countable model which embeds into a rank initial segment of V is iterable.

As stated the hypothesis is more general than necessary. The iteration trees that come up in applications follow a specific format, and only the restriction of SBH to trees of such format is needed.

Call an iteration tree $\mathcal{T}$ on M *nice* if:

(1) The extenders used in $\mathcal{T}$ have increasing strengths. More precisely, $\langle \mathrm{Strength}^{M_\xi}(E_\xi) \mid \xi + 1 < \mathrm{lh}(\mathcal{T})\rangle$ is strictly increasing.

(2) For each ξ, $\mathrm{Strength}^{M_\xi}(E_\xi)$ is inaccessible in M_ξ.

(3) For each ξ, $\mathrm{spt}(E_\xi) = \mathrm{Strength}^{M_\xi}(E_\xi)$.

2.1 Remark. Throughout this chapter, whenever a result claims the existence of an iteration tree, the iteration tree is nice. In the later sections we often neglect to mention this explicitly.

A model N is λ-*closed* if every subset of N of size λ in V belongs to N.

2.2 Exercise. Let $\mathcal{T}$ be a nice, finite iteration tree on V. Prove that each of the models in $\mathcal{T}$ is countably closed, and conclude from this that each of the models in $\mathcal{T}$ is wellfounded. Prove further that each of the models in $\mathcal{T}$ is $2^{\aleph_0}$-closed.

Hint. Prove the general fact that if $Q \models$"E is an extender with inaccessible support", N agrees with Q past the critical point of E, and both N and Q are countably (respectively $2^{\aleph_0}$) closed, then $\mathrm{Ult}(N, E)$ is countably (respectively $2^{\aleph_0}$) closed. Wellfoundedness follows from countable closure, since by elementarity each of the models in $\mathcal{T}$ satisfies internally that "there are no infinite descending sequences of ordinals". $\dashv$

Call M *iterable for nice trees* if "good" has a winning strategy in the iteration game on M when "bad" is restricted to extenders which give rise to nice trees. Let nSBH be the assertion that every countable model which embeds elementarily into a rank initial segment of V is iterable for nice trees. nSBH is a technical weakening of SBH, sufficient for all known applications. A proof of nSBH would constitute a substantial breakthrough in the study of large cardinals, particularly in inner model theory.

For the sake of the determinacy proofs in this chapter we need only a weak form of iterability, involving linear compositions of trees of length ω. This iterability was proved by Martin-Steel [22]. We now proceed to state the iterability precisely, and give its proof.

A *weak iteration* of M of length α consists of objects M_ξ, $\mathcal{T}_\xi$, b_ξ for $\xi < \alpha$ and embeddings $j_{\zeta,\xi} : M_\zeta \to M_\xi$ for $\zeta < \xi < \alpha$, so that:

$$\begin{tikzcd} M \arrow[r, "\mathcal{T}_0"', "b_0"] & M_1 \arrow[r, dotted] & M_\xi \arrow[r, "\mathcal{T}_\xi"', "b_\xi"] & M_{\xi+1} \arrow[r, dotted] & {} \end{tikzcd}$$

Diagram 22.5: A weak iteration of M

$$\begin{tikzcd} V\|\theta & \\ M \arrow[u, "\pi"] \arrow[r, "\mathcal{T}"', "j_b"] & M_b \arrow[ul, "\sigma"'] \end{tikzcd}$$

Diagram 22.6: Theorem 2.3

(1) $M_0 = M$.

(2) For each $\xi < \alpha$, $\mathcal{T}_\xi$ is a nice iteration tree of length ω on M_ξ; b_ξ is a cofinal branch through $\mathcal{T}_\xi$; $M_{\xi+1}$ is the direct limit along b_ξ; and $j_{\xi,\xi+1} : M_\xi \to M_{\xi+1}$ is the direct limit embedding along b_ξ.

(3) For limit $\lambda < \alpha$, M_λ is the direct limit of the system $\langle M_\xi, j_{\zeta,\xi} \mid \zeta < \xi < \lambda\rangle$ and $j_{\zeta,\lambda} : M_\zeta \to M_\lambda$ are the direct limit embeddings.

(4) The remaining embeddings $j_{\zeta,\xi}$ are obtained by composition.

A weak iteration is thus a linear composition of length ω iteration trees.

In the *weak iteration game* on M players "good" and "bad" collaborate to produce a weak iteration of M, of length ω_1^V. "Bad" plays the iteration trees $\mathcal{T}_\xi$ and "good" plays the branches b_ξ. (These moves determine the iteration completely.) If ever a model M_ξ, $\xi < \omega_1$, is reached which is illfounded, then "bad" wins. Otherwise "good" wins. M is *weakly iterable* if "good" has a winning strategy in the weak iteration game on M.

2.3 Theorem. *Let $\pi : M \to V\|\theta$ be elementary with M countable. Let $\mathcal{T}$ be a nice iteration tree of length ω on M. Then there is a cofinal branch b through $\mathcal{T}$, and an embedding $\sigma : M_b \to V\|\theta$, so that $\sigma \circ j_b = \pi$. (Note that b is then a wellfounded branch, since M_b embeds into $V\|\theta$.)*

2.4 Corollary. *Let $\pi : M \to V\|\theta$ be elementary with M countable. Then "good" has a winning strategy in the weak iteration game on M.*

Proof. Immediate through iterated applications of Theorem 2.3. "Good" should simply keep choosing branches given by the theorem, successively embedding each $M_{\xi+1}$ into $V\|\theta$, and preserving commutativity which is needed for the limits.

The idea of proving iterability by embedding back into V, simple only in retrospect, was first used by Jensen in the context of linear iterations. ⊣

Theorem 2.3 and Corollary 2.4 provide the iterability necessary for the determinacy proofs in this chapter. In the remainder of this section we give the proof of the theorem.

2.5 Definition. Let $\mathcal{T}$ be a nice iteration tree of length ω on a model M, giving rise to models and embeddings $\langle M_m, j_{m,n} \mid m \; T \; n < \omega\rangle$. $\mathcal{T}$ is *continuously illfounded* if there exists a sequence of ordinals $\alpha_n \in M_n$ $(n < \omega)$ so that $j_{m,n}(\alpha_m) > \alpha_n$ whenever $m \; T \; n$.

Note that a continuously illfounded iteration tree has *no* wellfounded cofinal branches. Indeed, for any cofinal branch b, the sequence $j_{n,b}(\alpha_n)$ for $n \in b$ witnesses that M_b is illfounded. Continuously illfounded iteration trees, on countable models M which embed into rank initial segments of V, thus contradict Theorem 2.3 in a very strong way. We begin by showing that in fact any counterexample to Theorem 2.3 gives rise to a continuously illfounded iteration tree.

2.6 Lemma. *Let $\pi : M \to V\|\theta$ be elementary with M countable. Let $\mathcal{T}$ be a nice iteration tree of length ω on M, and suppose that the conclusion of Theorem 2.3 fails for $\mathcal{T}$. Then there is a continuously illfounded nice iteration tree on V.*

Proof. Let E_n, M_n, and $j_{m,n}$ $(m \; T \; n < \omega)$ denote the extenders, models, and embeddings of $\mathcal{T}$. Working recursively define a length ω iteration tree $\mathcal{T}^*$ on V, and embeddings $\pi_n : M_n \to M_n^*$ through the conditions:

- $M_0^* = V$ and $\pi_0 = \pi$.
- $E_n^* = \pi_n(E_n)$.
- The T^*-predecessor of $n+1$ is the same as the T-predecessor of $n+1$.
- $M_{n+1}^* = \mathrm{Ult}(M_k^*, E_n^*)$ where k is the T-predecessor of $n+1$, and π_{n+1} is the copy embedding via the pair $\langle \pi_k, \pi_n\rangle$.

It is easy to check that this definition goes through, giving rise to a nice iteration tree $\mathcal{T}^*$ and the commuting diagram presented in Diagram 22.7. We will show that $\mathcal{T}^*$ is continuously illfounded.

2.7 Definition. The tree $\mathcal{T}^*$ defined through the conditions above is the *copy* of $\mathcal{T}$ via $\pi : M \to V$. It is denoted $\pi\mathcal{T}$.

From the fact that M is countable it follows that each M_n is countable. Let $\vec{e}^n = \langle e_l^n \mid l < \omega\rangle$ enumerate M_n. Given an embedding σ with domain M_n, we use $\sigma \restriction l$ to denote the restriction of σ to $\{e_0^n, \dots, e_{l-1}^n\}$, and we write $M_n \restriction l$ to denote $\{e_0^n, \dots, e_{l-1}^n\}$.

Working in V let R be the tree of attempts to create a cofinal branch b through $\mathcal{T}$ and a commuting system of embeddings realizing the models along b into V. More precisely, a node in R consists of a finite branch a through T, and of partial embeddings $\sigma_i : M_i \to V$, $i \in a$, satisfying the following conditions (where l is the length of a):

$$\begin{tikzcd}
\mathcal{T}^* & V \arrow[r, dotted, dash] & M_k^* \arrow[rr, bend left, "j^*_{k,n_1}"] & M_n^* & M_{n+1}^* \arrow[r, dotted, dash] & {} \\
\mathcal{T} & M_0 \arrow[u, "\pi"'] \arrow[r, dotted, dash] & M_k \arrow[u, "\pi_k"'] \arrow[rr, bend right, "j_{k,n+1}"'] & M_n \arrow[u, "\pi_n"'] & M_{n+1} \arrow[u, "\pi_{n+1}"'] \arrow[r, dotted, dash] & {}
\end{tikzcd}$$

Diagram 22.7: $\mathcal{T}$ and $\mathcal{T}^*$

- For each i the domain of σ_i is precisely $M_i{\restriction}l$.
- (Commutativity) If $i \ T \ i' \in a$, $x \in M_i{\restriction}l$, $x' \in M_{i'}{\restriction}l$, and $x' = j_{i,i'}(x)$, then $\sigma_{i'}(x') = \sigma_i(x)$.
- σ_0 is equal to $\pi{\restriction}l$.

The tree R consists of these nodes, ordered naturally by extension for each component.

An infinite branch through R gives rise to a corresponding infinite branch $b = \{n_0, n_1, \ldots\}$ through T and an embedding σ_∞ of the direct limit along b into V, with the commutativity $\sigma_\infty \circ j_b = \pi$. Thus, an infinite branch through R produces precisely the objects b and σ necessary for the conclusion of Theorem 2.3.

The assumption of the current lemma is that $\mathcal{T}$ witnesses the failure of Theorem 2.3. The tree R must therefore have *no* infinite branches. Let $\varphi : R \to \mathrm{On}$ be a rank function, that is a function assigning to each node in R an ordinal, in such a way that if a node s' extends a node s then $\varphi(s') < \varphi(s)$. The existence of such a function follows from the fact that R has no infinite branches.

For each finite branch $a = \langle 0 = n_0 \ T \ n_1 \ldots T \ n_{l-1} \rangle$ through T, let s_a consist of a itself and the embeddings $(\pi_{n_{l-1}} \circ j_{n_i, n_{l-1}}){\restriction}l$ for each $i < l$. Using the commutativity of Diagram 22.7 it is easy to check that s_a is a node in $j^*_{0,n_{l-1}}(R)$.

For $k < \omega$ let s_k be the node s_a where a is the branch of T ending at k. s_k is then a node in $j^*_{0,k}(R)$. For $k \ T \ k'$ it is easy to check, again using the commutativity of Diagram 22.7, that $s_{k'}$ extends $j^*_{k,k'}(s_k)$.

Let $\alpha_k = j^*_{0,k}(\varphi)(s_k)$. This is the rank of the node s_k of $j^*_{0,k}(R)$ given by the shift of the rank function φ to M_k^*. From the fact that $s_{k'}$ extends $j^*_{k,k'}(s_k)$ for $k \ T \ k'$ it follows that $\alpha_{k'} < j_{k,k'}(\alpha_k)$. The ordinals $\langle \alpha_k \mid k < \omega \rangle$ therefore witness that $\mathcal{T}^*$ is continuously illfounded. ⊣

2.8 Lemma. *Let $\mathcal{U}$ be a nice length ω iteration tree on V. Then $\mathcal{U}$ is* not *continuously illfounded.*

Proof. Suppose for contradiction that $\mathcal{U}$ is a nice, length ω, continuously illfounded iteration tree on V, and let $\langle \beta_n \mid n < \omega \rangle$ witness this. Let η be large enough that $\mathcal{U}$ belongs to $V \| \eta$. By replacing each β_n with the β_nth regular cardinal of $M_n^{\mathcal{U}}$ above $j_{0,n}^{\mathcal{U}}(\eta)$ we may assume that β_n is regular in $M_n^{\mathcal{U}}$ for each n, and larger than $j_{0,n}^{\mathcal{U}}(\eta)$.

Let θ be large enough that both $\mathcal{U}$ and $\langle \beta_n \mid n < \omega \rangle$ belong to $V \| \theta$. Let H be a countable Skolem hull of $V \| \theta$ with $\mathcal{U}$ and $\langle \beta_n \mid n < \omega \rangle$ elements of H. Let M be the transitive collapse of H and let $\pi : M \to V \| \theta$ be the anticollapse embedding. Let $\mathcal{T} = \pi^{-1}(\mathcal{U})$ and let $\langle \alpha_n \mid n < \omega \rangle = \pi^{-1}(\langle \beta_n \mid n < \omega \rangle)$. Then $\mathcal{T}$ is a nice, length ω, continuously illfounded iteration tree on M; $\langle \alpha_n \mid n < \omega \rangle$ witnesses this; for each n, α_n is regular in $M_n = M_n^{\mathcal{T}}$; and, for each n, $E_n = E_n^{\mathcal{T}}$ belongs to $M_n \| \alpha_n$. (The last clause follows from the fact that β_n is greater than $j_{0,n}^{\mathcal{U}}(\eta)$, obtained in the previous paragraph, and the fact that η was chosen large enough that $E_n^{\mathcal{U}} \in V \| j_{0,n}^{\mathcal{U}}(\eta)$.)

Let M_n, E_n, and $j_{m,n}$ ($m \; T \; n < \omega$) be the models and embeddings of $\mathcal{T}$. Let ρ_n be the strength of E_n in M_n. The sequence $\langle \rho_n \mid n < \omega \rangle$ is increasing, and for each $n < n^*$, M_n and M_{n^*} agree to ρ_n.

Let $P_0 = V \| \beta_0$ and let $\sigma_0 = \pi \restriction (M \| \alpha_0)$. We work by recursion to produce models P_n and embeddings σ_n satisfying the following conditions:

(1) σ_n is elementary from $M_n \| \alpha_n$ into P_n.

(2) σ_n belongs to P_n and is countable in P_n.

(3) For $\bar{n} < n$, $\sigma_{\bar{n}}$ and σ_n agree on $M_{\bar{n}} \| \rho_{\bar{n}}$.

We shall construct so that:

(i) For each n, $P_{n+1} \in P_n$.

At the end of the construction we shall thus have an infinite $\in$-decreasing sequence, a contradiction.

We already have conditions (1) and (2) for $n = 0$, and condition (3) is vacuous for $n = 0$. Suppose inductively that we have conditions (1)–(3) for n. We describe how to construct P_{n+1} and σ_{n+1}.

Let k be the T-predecessor of $n+1$, so that M_{n+1} is the ultrapower of M_k by E_n. We wish to copy this ultrapower to an ultrapower of P_k via the pair $\langle \sigma_k, \sigma_n \rangle$. We cannot quite manage this, since the domain of σ_k is $M_k \restriction \alpha_k$ rather than M_k. We adjust our wishes as follows: Let $\gamma = j_{k,n+1}(\alpha_k)$. $M_{n+1} \| \gamma$ is then the ultrapower of $M_k \| \alpha_k$ by E_n. Now let P_n^* be the copy of this ultrapower via the pair $\langle \sigma_k, \sigma_n \rangle$, and let $\sigma_n^* : M_{n+1} \| \gamma \to P_n^*$ be the copy embedding.

We would have liked to simply set $P_{n+1} = P_n^*$ and σ_{n+1} equal to the restriction of σ_n^* to $M_{n+1} \| \alpha_{n+1}$. There are two problems with this. First, P_n^* does not belong to P_n, so we lose condition (i), the crucial condition in our scheme for a contradiction. Second, σ_n^* does not belong to P_n^*, so we lose condition (2). We handle the second problem first.

2.9 Claim. *Let τ denote the restriction of σ_n^* to $M_{n+1}\|\rho_n$. Then τ belongs to P_n^*.*

Proof. Let φ_n denote $\sigma_n(\rho_n)$. Let F_n denote $\sigma_n(E_n)$.

P_n^* is the ultrapower of P_k by F_n. F_n is φ_n-strong in P_n. It follows that P_n^* and P_n agree to φ_n.

The definition of copy embedding requires that σ_n^* and σ_n agree on the support of E_n. This support must contain ρ_n, since otherwise E_n could not be ρ_n-strong. σ_n^* and σ_n thus agree on ρ_n. By condition (2) and the inaccessibility of φ_n in P_n, $\sigma_n\restriction\rho_n$ belongs to $P_n\restriction\varphi_n$. Since P_n and P_n^* agree to φ_n, $\sigma_n\restriction\rho_n$ belongs to P_n^*. Now σ_n^* is the same as σ_n up to ρ_n, so $\sigma_n^*\restriction\rho_n$ belongs to P_n^*. From this, using the inaccessibility of ρ_n in M_{n+1}, one can argue that $\sigma_n^*\restriction(M_{n+1}\|\rho_n)$ belongs to P_n^*. $\dashv$

Let $\alpha_n^* = \sigma_n^*(\alpha_{n+1})$. Notice that the definition makes sense, as α_{n+1} is smaller than $\gamma = j_{k,n+1}(\alpha_k)$, and therefore belongs to the domain of σ_n^*.

2.10 Claim. *There is an elementary embedding $\sigma_n^{**} : M_{n+1}\|\alpha_{n+1} \to P_n^*\|\alpha_n^*$ so that:*

- *The restriction of σ_n^{**} to $M_{n+1}\|\rho_n$ is equal to τ.*
- *$\sigma_n^{**}(\rho_n) = \varphi_n$.*
- *σ_n^{**} belongs to P_n^* and is countable in P_n^*.*

Notice that σ_n^*, restricted to $M_{n+1}\|\alpha_{n+1}$, already satisfies the first two demands of the claim. Replacing it by an embedding σ_n^{**} that also satisfies the third demand solves our "second problem" mentioned above.

Proof of Claim 2.10. This is a simple matter of absoluteness. Using the fact that τ belongs to P_n^* we can put together, inside P_n^*, the tree of attempts to construct an embedding σ_n^{**} satisfying the demands of the claim. This tree of attempts has an infinite branch in V, given by the restriction of σ_n^* to $M_{n+1}\|\alpha_{n+1}$. By absoluteness then it has an infinite branch inside P_n^*. $\dashv$

Let $P_n^{**} = P_n^*\|\alpha_n^*$. Note that P_n^{**} is then a *strict* rank initial segment of P_n^*, ultimately because $\alpha_{n+1} < j_{k,n+1}(\alpha_n)$.

Taking $P_{n+1} = P_n^{**}$ and $\sigma_{n+1} = \sigma_n^{**}$ would satisfy conditions (1)–(3). But we need one more adjustment to obtain condition (i), the crucial condition in our scheme for a contradiction. This final adjustment hinges on the fact that P_n^{**} is a strict initial segment of P_n^*, and therefore an element of P_n^*. Let H be the Skolem hull of $P_n^{**}\|\varphi_n \cup \{\varphi_n, \sigma_n^{**}\}$ inside P_n^{**}. Let P_{n+1} be the transitive collapse of H, and let $j : P_{n+1} \to H$ be the anticollapse embedding. Let $\sigma_{n+1} = j^{-1} \circ \sigma_n^{**}$. It is easy to check that conditions (1)–(3) hold with these assignments.

Since P_n^{**} and σ_n^{**} belong to P_n^*, the Skolem hull H taken above has cardinality φ_n *inside* P_n^*. It follows that P_{n+1} can be coded by a subset of φ_n

inside P_n^*. Now P_n^* is equal to $\mathrm{Ult}(P_k, F_n)$. Since P_k and P_n agree well beyond the critical point of F_n, the ultrapowers $\mathrm{Ult}(P_k, F_n)$ and $\mathrm{Ult}(P_n, F_n)$ agree well beyond the image of this critical point (Claim 1.20). This image in turn is at least φ_n, that is the strength of F_n, since F_n is a short extender. (See Lemma 1.17.) It follows that all subsets of φ_n in $P_n^* = \mathrm{Ult}(P_k, F_n)$ belong also to $\mathrm{Ult}(P_n, F_n)$. Now $\mathrm{Ult}(P_n, F_n)$ can be computed over P_n (as $F_n \in P_n$). So all subsets of φ_n in P_n^* belong to P_n. We noted at the start of this paragraph that P_{n+1} can be coded by such a subset. So P_{n+1} belongs to P_n, and we have condition (i), as required. ⊣

Lemmas 2.6 and 2.8 combine to prove Theorem 2.3.

2.11 Remark. The contradiction in Lemma 2.8 is obtained through the very last adjustment in the proof, replacing P_n^{**} by a Skolem hull which belongs to P_n. It is crucial for that final adjustment that P_n^{**} is a strict rank initial segment of P_n^*, and this is where the continuous illfoundedness of $\mathcal{T}$ is used. The ordinals witnessing the continuous illfoundedness provide the necessary drops in rank.

2.12 Lemma. *Let $\mathcal{T}$ be a nice iteration tree of length ω on V. Then $\mathcal{T}$ has a cofinal branch leading to a wellfounded direct limit.*

Proof. Suppose not. For each cofinal branch b through $\mathcal{T}$ fix a sequence $\langle \alpha_n^b \mid n \in b\rangle$ witnessing that the direct limit along b is illfounded, more precisely satisfying $j_{m,n}(\alpha_m^b) > \alpha_n^b$ for all $m < n$ both in b. Let θ be large enough that all the ordinals α_n^b are smaller than θ.

For each $n < \omega$ let B_n be the set of cofinal branches b through $\mathcal{T}$ with $n \in b$. Let F_n be the set of functions from B_n into θ. Let $\prec$ be the following relation: $\langle n, f\rangle \prec \langle m, g\rangle$ iff $f \in F_n$, $g \in F_m$, $m\ T\ n$, and $f(b) < g(b)$ for every $b \in F_n$. The relation $\prec$ is wellfounded: if $\langle\langle n_i, f_i\rangle \mid i < \omega\rangle$ were an infinite descending chain in $\prec$, then $\langle f_i(b) \mid i < \omega\rangle$, where b is the cofinal branch through $\mathcal{T}$ generated by $\{n_i \mid i < \omega\}$, would be an infinite descending sequence of ordinals.

For each $n < \omega$ let φ_n be the function $b \mapsto \alpha_n^b$, defined on $b \in B_n$, that is on branches b so that $n \in b$. By Exercise 2.2, each of the models M_n of $\mathcal{T}$ is $2^{\aleph_0}$-closed, and it follows that for each $n < \omega$, φ_n belongs to M_n. Let $\prec_n$ denote the relation $j_{0,n}(\prec)$. Using the fact that $j_{m,n}(\alpha_m^b) > \alpha_n^b$ for all b and all $m < n$ both in b, it is easy to check that $\langle n, \varphi_n\rangle \prec_n \langle m, j_{m,n}(\varphi_m)\rangle$ whenever $m\ T\ n$. Letting γ_n be the rank of φ_n in $\prec_n$ it follows that $\gamma_n < j_{m,n}(\gamma_m)$ whenever $m\ T\ n$. But then the sequence $\langle \gamma_n \mid n < \omega\rangle$ is a witness that $\mathcal{T}$ is continuously illfounded, contradicting Lemma 2.8. ⊣

3. Creating Iteration Trees

The creation of iteration trees with non-linear tree orders is not a simple matter. Recall that the model M_{n+1} in an iteration tree $\mathcal{T}$ is created by picking an extender $E_n \in M_n$, picking $k \leq n$ so that M_k and M_n agree past $\text{crit}(E_n)$, and setting $M_{n+1} = \text{Ult}(M_k, E_n)$. The agreement between M_k and M_n is necessary for the ultrapower to make sense. The agreement can be obtained trivially by taking $k = n$. But doing this repeatedly would generate a linear iteration, that is an iteration with the simple tree order $0\ T\ 1\ T\ 2 \ldots$. For the creation of iteration trees with more complicated orders we need a way of ensuring that M_n has extenders with critical points within the level of agreement between M_n and previous models in the tree.

This section introduces the large cardinals and machinery that will allow us to create iteration trees with as complicated a tree order as we wish. The results here are due to Martin-Steel [21]. The terminology follows Neeman [30, §1A(1)].

3.1 Definition. u is called a *(κ, n)-type*, where κ is a limit ordinal and $n < \omega$, if u is a set of formulae involving n free variables $v_0, \ldots, v_{n-1}$, a constant $\widetilde{\delta}$, and additional constants $\widetilde{c}$ for each $c \in V\|\kappa \cup \{\kappa\}$.

A (κ, n)-type can be coded by a subset of $(V\|\kappa)^{<\omega}$. Since κ is assumed to be a limit ordinal, $(V\|\kappa)^{<\omega} \subseteq V\|\kappa$. We may therefore view (κ, n)-types as subsets of $V\|\kappa$.

We refer to κ as the *domain* of u, denoted $\text{dom}(u)$. For $\tau \leq \kappa$ and $m \leq n$, we let

$$\begin{aligned}\text{proj}^m_\tau(u) = \big\{&\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{m-1}) \mid k \in \mathbb{N},\ c_0, \ldots, c_k \in V\|\tau \cup \{\tau\},\\ &\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \in u,\ \text{and } \phi \text{ makes no mention of}\\ &v_m, \ldots, v_{n-1}\big\}.\end{aligned}$$

We use $\text{proj}_\tau(u)$ to denote $\text{proj}^n_\tau(u)$, and $\text{proj}^m(u)$ to denote $\text{proj}^m_\kappa(u)$.

3.2 Definition. We say that a (κ, n)-type u is *realized* (*relative to* δ) *by* $x_0, \ldots, x_{n-1}$ *in* $V\|\eta$ just in case that:

- $x_0, \ldots, x_{n-1}$ and δ are elements of $V\|\eta$.
- For any $k < \omega$, any $c_0, \ldots, c_k \in V\|\kappa \cup \{\kappa\}$, and any formula $\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1})$, $\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \in u$ if and only if $V\|\eta \models \phi[\delta, c_0, \ldots, c_k, x_0, \ldots, x_{n-1}]$. (Implicitly we must have $\eta > \kappa$ and $\eta > \delta$.)

We call u the *κ-type of* $x_0, \ldots, x_{n-1}$ *in* $V\|\eta$ (relative to δ) if u is the unique (κ, n)-type which is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$. A (κ, n)-type u is *realizable* (relative to δ) if it is realized by some $x_0, \ldots, x_{n-1}$ in some $V\|\eta$.

We often neglect to mention the set δ involved in the realization. In applications δ is usually fixed, and clear from the context.

3.3 Note. If u is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$, then $\text{proj}^m_\tau(u)$ is realized by $x_0, \ldots, x_{m-1}$ in $V\|\eta$.

3.4 Definition. If the formula "there exists a largest ordinal" and the formula "$\widetilde{\kappa}, \widetilde{\delta}, v_0, \ldots, v_{n-1} \in V\|\nu$, where ν is the largest ordinal" are both elements of the (κ, n)-type u we define

$$\begin{aligned} u^- = \bigl\{\phi(\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}) \mid\, & k \in \mathbb{N},\ c_0, \ldots, c_k \in V\|\kappa \cup \{\kappa\}, \\ & \text{and the formula "}V\|\nu \models \phi[\widetilde{\delta}, \widetilde{c}_0, \ldots, \widetilde{c}_k, v_0, \ldots, v_{n-1}] \\ & \text{where } \nu \text{ is the largest ordinal" is an element of } u\bigr\}. \end{aligned}$$

3.5 Note. If $\kappa, \delta, x_0, \ldots, x_{n-1} \in V\|\eta$ and u is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta + 1$ then u^- is defined and is realized by the same $x_0, \ldots, x_{n-1}$ in $V\|\eta$.

3.6 Definition. Let u be a (κ, n)-type, and let w be a (τ, m)-type. We say that w is a *subtype* of u (and write $w < u$) if:

- $\tau < \kappa$.
- $m \geq n$.
- The formula "there is an ordinal ν and $v_n, \ldots, v_{m-1} \in V\|\nu$ such that $\widetilde{w}$ is realized by some permutation of $v_0, \ldots, v_{m-1}$ in $V\|\nu$" is an element of the type u.

3.7 Note. Let u be the κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta$. Then w is a subtype of u iff there are $\tau < \kappa$, $\nu < \eta$, $m \geq n$, and sets $x_n, \ldots, x_{m-1}$ so that w is the τ-type of some permutation of $x_0, \ldots, x_{m-1}$ in $V\|\nu$.

3.8 Remark. Definition 3.6 makes no mention of realizability but only stipulates that one particular formula belongs to u. It is immediate then that the property $w < u$ is absolute for any two models of set theory which have w and u as elements.

3.9 Definition. We say that a (τ, m)-type w *exceeds* the (κ, n)-type u, if:

- $\tau > \kappa$.
- $m \geq n$.
- There exist ordinals ν, η, and sets $x_0, \ldots, x_{m-1} \in V\|\nu$ such that
  - u is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$,
  - w is realized by some permutation of $x_0, \ldots, x_{m-1}$ in $V\|\nu$, and
  - $\nu + 1 < \eta$.

ν, η, and $x_0, \ldots, x_{m-1}$ are said to witness the fact that w exceeds u.

3.10 Remark. The definition here is slightly more liberal than the corresponding definition in Neeman [30], where it is required that w be realized by $x_0, \ldots, x_{m-1}$ in their original order, not by a permutation of $x_0, \ldots, x_{m-1}$. A similar comment applies to Definition 3.6.

3.11 Note. Let u be the κ-type of $x_0, \dots, x_{n-1}$ in $V\|\eta$. Suppose there are $\tau > \kappa$, ν with $\nu + 1 < \eta$, $m \geq n$, and $x_n, \dots, x_{m-1}$ so that w is the τ-type of a permutation of $x_0, \dots, x_{m-1}$ in $V\|\nu$. Then w exceeds u. This should be compared with Note 3.7. There τ is smaller than κ, and here τ must be larger than κ.

3.12 Definition. Let $\kappa < \lambda$, E a λ-strong extender with $\operatorname{crit}(E) = \kappa$, and u a type with $\operatorname{dom}(u) = \kappa$. Let $i_E : V \to \operatorname{Ult}(V, E)$ be the ultrapower embedding. We define $\operatorname{Stretch}^E_\lambda(u)$ to be equal to $\operatorname{proj}_\lambda(i_E(u))$.

$i_E(u)$ in Definition 3.12 is a type in $\operatorname{Ult}(V, E)$ with domain $i_E(\kappa)$. $i_E(\kappa)$ is at least as large as λ by Lemma 1.17, since E is λ-strong. So the projection to λ in Definition 3.12 makes sense.

3.13 Definition. A (κ, n)-type u is called *elastic* just in case that u^- is defined and u contains the following formulae:

- "$\widetilde{\delta}$ is an inaccessible cardinal."
- "Let ν be the largest ordinal. Then for all $\lambda < \widetilde{\delta}$ there exists an extender $E \in V\|\widetilde{\delta}$ such that
  - $\operatorname{crit}(E) = \widetilde{\kappa}$, $\operatorname{spt}(E) = \operatorname{Strength}(E)$, $\operatorname{Strength}(E)$ is an inaccessible cardinal greater than λ, and
  - $\operatorname{Stretch}^E_\lambda(u^-)$ is realized (relative to $\widetilde{\delta}$) by $v_0, \dots, v_{n-1}$ in $V\|\nu$."

Formally the last clause should begin with "$\operatorname{Stretch}^E_\lambda(w)$, where w is the type of $v_0, \dots, v_{n-1}$ in $V\|\nu$" instead of $\operatorname{Stretch}^E_\lambda(u^-)$, as u^- is not a parameter in formulae in u.

3.14 Remark. The requirements on support and of inaccessible strength in Definition 3.13 are not part of the definition in Neeman [30]. They are added in this chapter so as to make sure, later on, that our iteration trees are nice.

3.15 Remark. Definition 3.13 makes no mention of realizability but only stipulates that certain formulae belong to u. It is immediate then that the property of being elastic is absolute between models of set theory.

Ordinarily if u is realized by $x_0, \dots, x_{n-1}$ in $V\|\nu + 1$ then $\operatorname{Stretch}^E_\lambda(u^-)$ is realized by $i_E(x_0), \dots, i_E(x_{n-1})$ in $\operatorname{Ult}(V, E)\|i_E(\nu)$, and relative to $i_E(\delta)$. The demand in Definition 3.13 that it must also be realized by $x_0, \dots, x_{n-1}$ in $V\|\nu$, and relative to δ, places a requirement of certain strength on the extender E. The existence of realizable elastic types is dependent on the existence of enough extenders with such strength.

3.16 Definition. Let H be a set. Let E be an extender and let $\kappa = \operatorname{crit}(E)$. Let j be the ultrapower embedding by E. Let $\alpha \leq j(\kappa)$. E is said to be *α-strong with respect to H* if (a) it is α-strong; and (b) $j(H \cap \kappa)$ and H agree to α, i.e., $j(H \cap \kappa) \cap \alpha = H \cap \alpha$.

A cardinal κ is said to be *α-strong with respect to H* if it is the critical point of an extender which is α-strong with respect to H.

A cardinal κ is said to be *$<\alpha$-strong with respect to H* if it is β-strong with respect to H for each $\beta < \alpha$.

3.17 Lemma. *Let τ be the critical point of a superstrong extender. Let $H \subseteq \tau$. Then there is a $\kappa < \tau$ which is $<\tau$-strong with respect to H.*

Proof. Let E be a superstrong extender with critical point τ, let $M = \mathrm{Ult}(V, E)$, and let $\pi : V \to M$ be the ultrapower embedding. Let $\tau^* = \pi(\tau)$. For each $\alpha < \tau^*$ let F_α be the λ-restriction of π, where $\lambda < \tau^*$ is the least ordinal satisfying the requirements in Lemma 1.18 relative to α. Notice that F_α is then an element of $V \| \tau^*$, and therefore, through of the agreement between V and M, an element of M. Notice further that, by Lemma 1.18, F_α is α-strong. Let j_α be the ultrapower embedding by F_α, and notice finally that $j_\alpha(H)$ and $\pi(H)$ agree up to λ, meaning that $j_\alpha(H) \cap \lambda = \pi(H) \cap \lambda$. Since $H = \pi(H) \cap \kappa$ it follows that F_α is α-strong with respect to $\pi(H)$.

The extenders F_α, $\alpha < \tau^*$, thus witness that τ is $<\tau^*$-strong in M with respect to $\pi(H)$. So M is a model of the statement "there is a $\kappa < \tau^*$ which is $<\tau^*$-strong with respect to $\pi(H)$." Using the elementarity of π to pull this statement back to V it follows that there is a $\kappa < \tau$ which is $<\tau$-strong with respect to H. $\dashv$

3.18 Definition. A cardinal δ is called a *Woodin cardinal* if for every $H \subseteq \delta$, there exists a $\kappa < \delta$ which is $<\delta$-strong with respect to H.

Lemma 3.17 shows that the critical point of a superstrong extender is Woodin. The next exercise shows that there are Woodin cardinals below the critical point. In fact Woodin cardinals sit *well* below such critical points in the large cardinal hierarchy, and there are many large cardinal axioms strictly between the existence of Woodin cardinals and the existence of superstrong extenders.

3.19 Exercise. Let E be a superstrong extender. Show that there are Woodin cardinals below the critical point of E. In fact, show that the critical point of E is a limit of Woodin cardinals.

3.20 Exercise. Let δ be a Woodin cardinal. Show that δ is a limit of (strongly) inaccessible cardinals, and that it is (strongly) inaccessible itself.

3.21 Exercise. Let δ be a Woodin cardinal. Let $H \subseteq \delta$ and let κ be $<\delta$-strong with respect to H. Let $\alpha < \delta$ be given. Prove that there is an extender E with critical point κ so that E is α-strong with respect to H, and so that $\mathrm{spt}(E) = \mathrm{Strength}(E)$ and $\mathrm{Strength}(E)$ is an inaccessible cardinal greater than α.

Hint. Let $\lambda < \delta$ be the first inaccessible cardinal above α. Using the fact that κ is $<\delta$-strong with respect to H, get an extender F with critical point

κ so that F is λ-strong with respect to H. In particular then F is α-strong with respect to H, and $\mathrm{Strength}(F) \geq \lambda$. Let π be the ultrapower embedding by F, and let E be the λ-restriction of π. Show that the strength of E is precisely λ, and that E is α-strong with respect to H. ⊣

3.22 Lemma. *Let δ be a Woodin cardinal. Let $\eta > \delta$, and let $x_0, \ldots, x_{n-1}$ be elements of $V\|\eta$. Then there exist unboundedly many $\kappa < \delta$ such that the κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta + 1$ relative to δ is elastic.*

Proof. For each (strongly) inaccessible $\gamma < \delta$, let A_γ be the γ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta$ relative to δ, viewed as a subset of γ. Let $H = \{(\xi, \gamma) \mid \xi \in A_\gamma\}$, where $(*, *)$ is the Gödel pairing.

Let κ be $<\delta$-strong with respect to H. Let u be the κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta + 1$.

It is easy to check that if λ^* is closed under Gödel pairing and E is λ^*-strong with respect to H, then for every $\lambda < \lambda^*$, $\mathrm{Stretch}^E_\lambda(u^-)$ is realized by $x_0, \ldots, x_{n-1}$ in $V\|\eta$. Using Exercise 3.21 it follows that the formula in the second clause of Definition 3.13 holds for $x_0, \ldots, x_{n-1}$ in $V\|\eta + 1$, and is therefore an element of u. By Exercise 3.20, δ is inaccessible, and so the formula in the first clause of Definition 3.13 belongs to u. This shows that u is elastic.

We have so far obtained one cardinal $\kappa < \delta$ such that the κ-type of $x_0, \ldots, x_{n-1}$ in $V\|\eta$ is elastic. We leave it to the reader to show that there are unboundedly many. ⊣

We now know that Woodin cardinals provide the strength necessary for the existence of many elastic types. The usefulness of elastic types appears through the following lemma. The lemma essentially says that an elastic type u which is exceeded by a type w can be stretched to a supertype of w.

3.23 Lemma (One-Step Lemma). *Assume that u is an elastic type, and that w exceeds u (with all realizations relative to δ). Let $\tau = \mathrm{dom}(w)$ and let $\kappa = \mathrm{dom}(u)$. Suppose that $\tau < \delta$. Then there exists an extender $E \in V\|\delta$ so that*

- $\mathrm{crit}(E) = \kappa$, $\mathrm{spt}(E) = \mathrm{Strength}(E)$, *the strength of E is an inaccessible cardinal greater than τ, and*
- $w < \mathrm{Stretch}^E_{\tau+\omega}(u)$.

Proof. Let $\nu, \eta, x_0, \ldots, x_{m-1}$ witness that w exceeds u. Since u^- exists, η is a successor ordinal. Say $\eta = \bar{\eta} + 1$. Pick $E \in V\|\delta$ so that $\mathrm{crit}(E) = \kappa$, E has inaccessible strength greater than τ, and $\mathrm{Stretch}^E_{\tau+\omega}(u^-)$ is realized by $x_0, \ldots, x_{n-1}$ in $V\|\bar{\eta}$ relative to δ. This is possible since u is elastic.

Then w is a subtype of $\mathrm{Stretch}^E_{\tau+\omega}(u^-)$, as it is realized by a permutation of $x_0, \ldots, x_{n-1}, x_n, \ldots, x_m$ in $V\|\nu$ and $\nu < \bar{\eta}$. Simple properties of realizable types now imply that w is a subtype of $\mathrm{Stretch}^E_{\tau+\omega}(u)$. ⊣

We now have the tools necessary for the creation of iteration trees. We work for the rest of the section under the assumption that δ is a Woodin cardinal.

3.24 Lemma. *Let $M_0 = V$. There is an iteration tree with the structure of models presented in the following diagram:*

$$\begin{tikzcd} M_0 \arrow[rrr, bend left] & = & M_1 \arrow[rr, bend right] & M_2 & M_3 \end{tikzcd}$$

Proof. Let η be an ordinal greater than δ. Let $\kappa_0 < \delta$ be such that the κ_0-type of η in $V\|\eta + 5$ is elastic. Let u_0 be this type.

Let $\kappa_1 > \kappa_0$ be such that the κ_1-type of η in $V\|\eta + 3$ is elastic. Let u_1 be this type.

Notice that u_1 exceeds u_0. Using the One-Step Lemma pick a $\kappa_1 + 1$-strong extender $E_1 \in M_1\|\delta$ so that $\mathrm{crit}(E_1) = \kappa_0$, and u_1 is a subtype of $\mathrm{Stretch}^{E_1}_{\kappa_1+\omega}(u_0)$.

Set $M_2 = \mathrm{Ult}(M_0, E_1)$, and let $j_{0,2} : M_0 \to M_2$ be the ultrapower embedding. Then u_1 is a subtype of $j_{0,2}(u_0)$. By the elementarity of $j_{0,2}$, $j_{0,2}(u_0)$ is realized by $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta) + 5$. It follows from this and from the fact that u_1 is a subtype of $j_{0,2}(u_0)$, that u_1 is also realized in M_2, specifically it must be realized by $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta) + 3$. The level $j_{0,2}(\eta) + 3$ is reached by observing that u_1 contains the formula "$v_0 + 2$ is the largest ordinal".

Working now in M_2, let $\kappa_2 > \kappa_1$ be such that the κ_2-type of $j_{0,2}(\eta)$ in $M_2\|j_{0,2}(\eta) + 1$ is elastic. Let u_2 be this type. Notice that u_2 then exceeds u_1, inside M_2. This uses the realization of u_1 in M_2, reached in the previous paragraph. Applying the One-Step Lemma pick an extender $E_2 \in M_2$ which stretches u_1 to a supertype of u_2. E_2 has critical point κ_1, and κ_1 is within the level of agreement between M_2 and M_1. E_2 can therefore be applied to M_1. Set $M_3 = \mathrm{Ult}(M_1, E_2)$. ⊣

3.25 Exercise. Construct an iteration tree with the structure presented in the following diagram:

$$\begin{tikzcd} M_0 \arrow[rrr, bend left] & = & M_1 \arrow[rr, bend right] & M_2 \arrow[rr, bend left] & M_3 & M_4 \end{tikzcd}$$

3.26 Exercise. Construct a length ω iteration tree with the tree order presented in the following diagram:

$$\begin{tikzcd} M_0 \arrow[rrr, bend left] & = & M_1 \arrow[rr, bend right] & M_2 \arrow[rr, bend left] & M_3 \arrow[rr, bend right] & M_4 & M_5 & \cdots \end{tikzcd}$$

Hint. The following definition is useful:

3.27 Definition. Let $\nu_{\mathrm{L}} < \nu_{\mathrm{H}}$ be ordinals greater than δ. We say that $\langle \nu_{\mathrm{L}}, \nu_{\mathrm{H}} \rangle$ is a pair of *local indiscernibles* relative to δ just in case that:

$$(V\|\nu_{\mathrm{L}} + \omega) \models \phi[\nu_{\mathrm{L}}, c_0, \ldots, c_{k-1}] \quad \Longleftrightarrow \quad (V\|\nu_{\mathrm{H}} + \omega) \models \phi[\nu_{\mathrm{H}}, c_0, \ldots, c_{k-1}]$$

for any $k < \omega$, any formula ϕ with $k+1$ free variables, and any $c_0, \ldots, c_{k-1} \in V\|\delta + \omega$.

Given local indiscernibles $\nu_{\mathrm{L}} < \nu_{\mathrm{H}}$, note that a type u is realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 1$ iff it is realized by ν_{H} in $V\|\nu_{\mathrm{H}} + 1$. Notice further that if u is realized by ν_{H} in $V\|\nu_{\mathrm{H}} + 1$, then any type of larger domain, which is realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 3$, exceeds $\mathrm{proj}^0(u)$, because $\nu_{\mathrm{L}} + 3 < \nu_{\mathrm{H}} + 1$. (It should be pointed out that the use of the projection is necessary here, to pass to a type which does not involve ν_{H} as a parameter.) In sum then you have:

3.28 Claim. *Let u be κ-type realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. Let $\tau > \kappa$ and let w be a τ-type realized by ν_{L} in $V\|\nu_{\mathrm{L}} + 3$. Then w exceeds* $\mathrm{proj}^0(u)$.

You have also the following claim, directly from the definitions:

3.29 Claim. *Let α be an ordinal greater than δ. Let u be a κ-type realized by α in $V\|\alpha + 3$. Let $\tau > \kappa$ and let w be a τ-type realized by α in $V\|\alpha + 1$. Then w exceeds u.*

Use the two claims alternately, to construct the iteration tree required for the exercise, types $u_n \in M_n$, and ordinals α_n for $n < \omega$ odd, with $\alpha_1 = \nu_{\mathrm{L}}$, so that:

(1) For even $n < \omega$, u_n is realized by $j_{0,n}(\nu_{\mathrm{L}})$ in $M_n\|j_{0,n}(\nu_{\mathrm{L}}) + 3$.

(2) For odd $n < \omega$, u_{n-1} is realized by α_n in $M_n\|\alpha_n + 3$, and u_n is realized by α_n in $M_n\|\alpha_n + 1$.

The construction is similar to that of the previous exercise, except that the use of the projection introduces some changes. The ordinals α_n for $n > 1$ odd are chosen using the third clause of Definition 3.6, applied to the fact that u_{n-1} is a subtype of $j_{n-2,n}(\mathrm{proj}^0(u_{n-2}))$. If you get $\alpha_n < j_{n-2,n}(\alpha_{n-2})$ for $n > 1$ odd, then you are on the right track. $\dashv$

3.30 Exercise. Go back to the last exercise, and make sure that the tree you construct is nice.

4. Homogeneously Suslin Sets

By a *tree* on a set X we mean a set $T \subseteq X^{<\omega}$, closed under initial segments. We use $[T]$ to denote the set of infinite branches through T, that is the set $\{x \in X^\omega \mid (\forall n) x \restriction n \in T\}$. Given a tree T on $X \times Y$ we often think of T as a subset of $X^{<\omega} \times Y^{<\omega}$ rather than $(X \times Y)^{<\omega}$, and similarly we think of $[T]$ as a subset of $X^\omega \times Y^\omega$. For T a tree on $X \times Y$ we use $p[T]$ to denote the *projection* of $[T]$ to X^ω, namely the set $\{x \in X^\omega \mid (\exists y)\langle x, y\rangle \in [T]\}$. We use T_s (for $s \in X^{<\omega}$) to denote the set $\{t \in Y^{<\omega} \mid \langle s, t\rangle \in T\}$, and use T_x (for $x \in X^\omega$) to denote the tree $\bigcup_{n<\omega} T_{x\restriction n}$. x is an element of $p[T]$ iff $[T_x]$ is non-empty. We sometimes apply similar terminology in the case that T is a tree on a product of more than two sets, for example $p[T] = \{x \mid (\exists y)(\exists z)\langle x, y, z\rangle \in [T]\}$ in the case that T is a tree on $X \times Y \times Z$.

Recall that a set $A \subseteq X^\omega$ is $\mathbf{\Sigma}^1_1$ iff there is a tree R on $X \times \omega$ so that $A = p[R]$. A set is $\mathbf{\Pi}^1_n$ if its complement is $\mathbf{\Sigma}^1_n$; and a set $A \subseteq X^\omega$ is $\mathbf{\Sigma}^1_{n+1}$ (for $n \geq 1$) if there is a $\mathbf{\Pi}^1_n$ set $B \subseteq X^\omega \times \omega^\omega$ so that $x \in A \iff (\exists y)\langle x, y\rangle \in B$. A set is *projective* if it is $\mathbf{\Pi}^1_n$ for some $n < \omega$. The projective sets are thus obtained from closed sets using complementations and projections along the real line.

The sets of reals in the very first level $L_1(\mathbb{R})$ (Definition 8.3) are precisely the projective sets, and our climb to $\mathsf{AD}^{L(\mathbb{R})}$ begins at the low end of the projective hierarchy. We prove determinacy for $\mathbf{\Pi}^1_1$ sets assuming measurable cardinals. The proof, due to Martin [16], can with hindsight be divided into two parts: a proof, using a measurable cardinal κ, that all $\mathbf{\Pi}^1_1$ sets are κ-homogeneously Suslin (see below for the definition); and a proof that all homogeneously Suslin sets are determined.

Let γ be an ordinal and let $m < n < \omega$. For $Z \subseteq \gamma^m$ let $Z^* = \{f \in \gamma^n \mid f\restriction m \in Z\}$. A measure ν over γ^n is an *extension* of a measure μ over γ^m just in case that for every $Z \subseteq \gamma^m$, $\mu(Z) = 1 \rightarrow \nu(Z^*) = 1$.

A *tower of measures* over γ is a sequence $\langle \mu_n \mid n < \omega\rangle$ so that:

(i) μ_n is a measure over γ^n for each n.

(ii) μ_n is an extension of μ_m for all $m < n < \omega$.

The tower is *countably complete* just in case that:

(iii) If $\mu_n(Z_n) = 1$ for each n then there is a fiber through $\langle Z_n \mid n < \omega\rangle$, namely a sequence $\langle \alpha_i \mid i < \omega\rangle$ so that $\langle \alpha_0, \ldots, \alpha_{n-1}\rangle \in Z_n$ for each n.

For sequences s and t we write $s \leq t$ to mean that s is an initial segment of t, and $s < t$ to mean that s is a proper initial segment of t.

4.1 Definition. A tree T on $X \times \gamma$ is *homogeneous* if there is a sequence of measures $\langle \mu_s \mid s \in X^{<\omega}\rangle$ so that:

(1) For each $s \in X^{<\omega}$, μ_s is a measure over T_s (equivalently, over $\gamma^{\mathrm{lh}(s)}$ with $\mu_s(T_s) = 1$), and μ_s is $\mathrm{card}(X)^+$-complete.

(2) If $s \leq t$ then μ_t is an extension of μ_s.

It follows from condition (2) that for every $x \in X^\omega$, the sequence $\langle \mu_{x\restriction n} \mid n < \omega\rangle$ is a tower.

(3) If $x \in p[T]$ then the tower $\langle \mu_{x\restriction n} \mid n < \omega\rangle$ is countably complete.

T is *κ-homogeneous* if in addition each of the measures μ_s is κ-complete.

4.2 Exercise. Let T be a homogeneous tree on $X \times \gamma$. Prove that there is a system $\langle M_s, f_s, j_{s,t} \mid s \le t \in X^{<\omega}\rangle$ of (wellfounded) models M_s, nodes f_s, and embeddings $j_{s,t}$ satisfying the following conditions:

(1) $j_{s,t} : M_s \to M_t$ for each $s \le t$, $\mathrm{crit}(j_{s,t})$ is larger than $\mathrm{card}(X)$, $M_\emptyset = V$, and the system $\langle M_s, j_{s,t} \mid s \le t \in X^{<\omega}\rangle$ commutes in the natural way.

(2) $f_s \in j_{\emptyset,s}(T_s)$ for each $s \in X^{<\omega}$, and the nodes $\langle f_s \mid s \in X^{<\omega}\rangle$ cohere in the natural way, meaning that $s < t \Longrightarrow j_{s,t}(f_s) < f_t$.

(3) If $x \in p[T]$ then the system $\langle M_s, j_{s,t} \mid s \le t < x\rangle$ has a wellfounded direct limit.

Hint. Let $M_s = \mathrm{Ult}(V, \mu_s)$ and let $j_s : V \to M_s$ be the ultrapower embedding. Let f_s be the seed of the measure μ_s. Notice that f_s is an element of $j_s(T_s)$.

Recall that each element of M_s has the form $j_s(g)(f_s)$ for some function $g : \gamma^{\mathrm{lh}(s)} \to V$. For $s \le t \in X^{<\omega}$ define an embedding $j_{s,t} : M_s \to M_t$ by letting $j_{s,t}$ send $j_s(g)(f_s)$ to $j_t(g)(f_t \restriction \mathrm{lh}(f_s))$.

Prove that the resulting system satisfies conditions (1)–(3). $\dashv$

4.3 Exercise. Let T be a tree on $X \times \gamma$ and suppose that there is a system $\langle M_s, f_s, j_{s,t} \mid s \le t \in X^{<\omega}\rangle$ satisfying the conditions in Exercise 4.2. Prove that T is homogeneous.

Suppose in addition that the embeddings $j_{s,t}$ all have critical points at least κ. Show that T is κ-homogeneous.

Hint. Set $\mu_s(Z) = 1$ iff $f_s \in j_{\emptyset,s}(Z)$. Prove that the resulting system of measures $\langle \mu_s \mid s \in X^{<\omega}\rangle$ satisfies the conditions in Definition 4.1. $\dashv$

The existence of a system satisfying the conditions in Definition 4.1 is thus equivalent to the existence of a system satisfying the conditions in Exercise 4.2. We use the two systems alternately, and refer to both of them as *homogeneity systems* for the tree T.

4.4 Exercise. Show that the converse of condition (3) in Exercise 4.2 follows from conditions (1) and (2) in the exercise. Condition (3) can therefore be strengthened to an equivalence, and so can condition (3) in Definition 4.1.

Hint. Fix x. Let M_x be the direct limit of the system $\langle M_s, j_{s,t} \mid s \le t < x\rangle$, and let $j_{s,x} : M_s \to M_x$ for $s < x$ be the direct limit embeddings. Let $f_x = \bigcup_{s<x} j_{s,x}(f_s)$, and notice that using condition (2), f_x is an infinite branch through $j_{\emptyset,x}(T_x)$. Use the wellfoundedness of M_x to find some infinite branch f through $j_{\emptyset,x}(T_x)$ with $f \in M_x$, and then using the elementarity of $j_{\emptyset,x}$ argue that $x \in p[T]$. $\dashv$

A set $A \subseteq X^\omega$ is *Suslin* if there is an ordinal γ and a tree T on $X \times \gamma$ so that $p[T] = A$. $A \subseteq X^\omega$ is *homogeneously Suslin* if in addition T can be taken to be homogeneous, and κ*-homogeneously Suslin* if T can be taken to be κ-homogeneous. These definitions are due independently to Kechris and Martin. In the context of the axiom of choice, which we employ throughout the chapter, every $A \subseteq X^\omega$ is Suslin. But of course not every set is homogeneously Suslin.

Let κ be a measurable cardinal. Fix a set $X \in V \| \kappa$ and a $\mathbf{\Pi}^1_1$ set $A \subseteq X^\omega$. We aim to show that A is κ-homogeneously Suslin.

4.5 Exercise. Let $R \subseteq \omega^{<\omega}$ be a tree. The *Kleene-Brouwer order* on R is the strict order $\prec$ defined by the condition: $s \prec t$ iff s extends t or $s(n) < t(n)$ where n is least so that $s(n) \neq t(n)$. Prove that $\prec$ is illfounded iff R has an infinite branch.

4.6 Exercise. Show that there is a map $s \mapsto \prec_s$, defined on $s \in X^{<\omega}$, so that:

- $\prec_s$ is a linear order on $\mathrm{lh}(s)$.
- If $s \leq t$ then $\prec_s \subseteq \prec_t$.
- $x \in A$ iff $\prec_x$ is wellfounded, where $\prec_x = \bigcup_{n<\omega} \prec_{x \restriction n}$.

The last condition is the most important one. The first two conditions are needed to make sense of $\prec_x$.

Hint to Exercise 4.6. Let $R \subseteq (X \times \omega)^{<\omega}$ be a tree so that $p[R]$ is precisely equal to the complement of A. Define the map $s \mapsto \prec_s$ in such a way that for each $x \in X^\omega$, $\prec_x$ is isomorphic to the Kleene-Brouwer order on R_x. $\dashv$

Let $T \subseteq X \times \kappa$ be the tree consisting of nodes $\langle s, f \rangle$ so that f has the form $\langle \alpha_0, \ldots, \alpha_{\mathrm{lh}(s)-1} \rangle$ with $\alpha_i < \kappa$ for each i, and $\alpha_i < \alpha_j$ iff $i \prec_s j$.

4.7 Exercise. Show that $p[T] = A$.

Since κ is measurable, there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. Let μ be the measure over κ defined by $\mu(Z) = 1$ iff $\kappa \in j(Z)$.

4.8 Exercise. Prove that μ is a κ-complete, non-principal measure on κ.

4.9 Exercise. A function $f : \kappa \to \kappa$ is *pressing down* if $f(\alpha) < \alpha$ for all $\alpha < \kappa$. A measure over κ is called *normal* if every pressing down function on κ is constant on a set of measure one. Prove that the measure μ defined above is normal.

4.10 Exercise. The *diagonal intersection* of the sets Z_α $(\alpha < \kappa)$ is defined to be the set $\{\xi \in \kappa \mid (\forall \alpha < \xi) \xi \in Z_\alpha\}$. Prove, for the measure μ defined above, that the diagonal intersection of sets of measure one has measure one.

For each $s \in X^{<\omega}$ and each $C \subseteq \kappa$ define C^s to be the set of tuples $\langle \alpha_0, \dots, \alpha_{\mathrm{lh}(s)-1} \rangle$ with $\alpha_i \in C$ for each i, and $\alpha_i < \alpha_j$ iff $i \prec_s j$. Define a filter $\mathcal{F}_s$ over $\kappa^{\mathrm{lh}(s)}$ by setting $Z \in \mathcal{F}_s$ iff there exists a set $C \subseteq \kappa$ so that $Z \supseteq C^s$ and $\mu(C) = 1$.

4.11 Exercise. Prove that $\mathcal{F}_s$ is an ultrafilter over $\kappa^{\mathrm{lh}(s)}$, meaning that for every $Z \subseteq \kappa^{\mathrm{lh}(s)}$, either $Z \in \mathcal{F}_s$ or else $\kappa^s - Z \in \mathcal{F}_s$.

Hint. Work by induction on the length of s. The inductive step makes several uses of Exercises 4.9 and 4.10. $\dashv$

Define a two-valued measure μ_s on κ^s by setting $\mu_s(Z) = 1$ iff $Z \in \mathcal{F}_s$.

4.12 Exercise. Prove that μ_s is κ-complete.

4.13 Exercise. Let $s \leq t \in X^{<\omega}$. Prove that μ_t extends μ_s.

4.14 Exercise. Let $x \in X^\omega$, and suppose that x belongs to A, so that $\prec_x$ is wellfounded. Prove that the tower $\langle \mu_{x \restriction n} \mid n < \omega \rangle$ is countably complete.

Hint. Suppose that $\mu_{x \restriction n}(Z_n) = 1$ for each $n < \omega$. Fix C_n so that $\mu(C_n) = 1$ and $C_n^s \subseteq Z_n$. Let $C = \bigcap_{n<\omega} C_n$. Then $C^s \subseteq Z_n$ for each n, and $\mu(C) = 1$ by countable completeness. Since $x \in A$, $\prec_x$ is wellfounded. The order $\prec_x$ can therefore be embedded into the ordinals, and in fact into C since C is uncountable. Pick then a sequence $\langle \alpha_i \mid i < \omega \rangle$ of ordinals in C so that $i \prec_x j$ iff $\alpha_i < \alpha_j$. The sequence $\langle \alpha_i \mid i < \omega \rangle$ is a fiber through $\langle Z_n \mid n < \omega \rangle$. $\dashv$

4.15 Theorem. *Let κ be a measurable cardinal. Let X belong to $V \| \kappa$ and let $A \subseteq X^\omega$ be $\mathbf{\Pi}^1_1$. Then A is κ-homogeneously Suslin.*

Proof. Let $T \subseteq (X \times \kappa)^{<\omega}$ be the tree defined above and let μ_s be the measures defined above. Exercises 4.12 through 4.14 establish that $\langle \mu_s \mid s \in X^{<\omega} \rangle$ is a κ-homogeneity system for T. $\dashv$

Next we prove that homogeneously Suslin sets are determined. We work for the rest of the section with some set X and a homogeneously Suslin set $A \subseteq X^\omega$. Let T and $\langle \mu_s \mid s \in X^{<\omega} \rangle$ witness that A is homogeneously Suslin.

Define G^* to be the game played according to Diagram 22.8 and the following rules:

- $x_n \in X$ for each $n < \omega$.
- $\langle x_0, \alpha_0, \dots, x_{n-1}, \alpha_{n-1} \rangle \in T$ for each $n < \omega$.

The first rule is a requirement on player I if n is even, and on player II if n is odd. The second rule is a requirement on player I. A player who violates a rule loses. Infinite runs of G^* are won by player I.

4.16 Exercise. Prove that G^* is determined.

$$\begin{array}{c|ccccccccc}
\text{I} & x_0 & \alpha_0 & & \alpha_1 & x_2 & \alpha_2 & & \alpha_3 & \dots & \\
\hline
\text{II} & & & x_1 & & & & x_3 & & & \dots
\end{array}$$

Diagram 22.8: The game G^*

Hint. You are asked to prove the famous theorem of Gale-Stewart [6] that infinite games with closed payoff are determined. Let S be the set of positions in G^* from which player II has a winning strategy. If the initial position belongs to S, then player II has a winning strategy in G^*. Suppose that the initial position does not belong to S, and prove that there is a strategy for player I which stays on positions outside S, and that this strategy is winning. $\dashv$

4.17 Exercise. Suppose that player I has a winning strategy in G^*. Prove that player I has a winning strategy in $G_\omega(A)$.

Hint. Let σ^* be a winning strategy for I in G^*. Say that a position $p = \langle x_0, \dots, x_{n-1}\rangle$ in $G_\omega(A)$ is nice if it can be expanded to a position $p^* = \langle x_0, \alpha_0, \dots, x_{n-1}, \alpha_{n-1}\rangle$ in G^* so that p^* is according to σ^*. Note that if such an expansion exists, then it is unique. Define a strategy σ for I in $G_\omega(A)$ by setting $\sigma(p) = \sigma^*(p^*)$. Show that every infinite run according to σ belongs to $p[T]$, and is therefore won by player I in $G_\omega(A)$. $\dashv$

4.18 Lemma. *Suppose that player* II *has a winning strategy in* G^*. *Then player* II *has a winning strategy in* $G_\omega(A)$.

Proof. Let σ^* be a winning strategy for II in G^*.

Let $s = \langle x_0, \dots, x_{i-1}\rangle$ be a position of odd length in $G_\omega(A)$. For each $\varphi = \langle \alpha_0, \dots, \alpha_{i-1}\rangle \in T_s$, let $h_s(\varphi)$ be σ^*'s move following the position $\langle x_0, \alpha_0, \dots, x_{i-1}, \alpha_{i-1}\rangle$ in G^*. h_s is then a function from T_s into X. By the completeness of μ_s there is a specific move x_i so that:

($*$) $\{\varphi \mid h_s(\varphi) = x_i\}$ has μ_s-measure one.

Define $\sigma(s)$ to be equal to this x_i.

Suppose now that $x = \langle x_i \mid i < \omega\rangle$ is an infinite run of $G_\omega(A)$, played according to σ. We have to show that x is won by player II.

Using condition ($*$) fix for each odd $n < \omega$ a set $Z_n \subseteq T_{x\restriction n}$ so that $h_{x\restriction n}(\varphi) = x_n$ for every $\varphi \in Z_n$ and $\mu_{x\restriction n}(Z_n) = 1$. For even $n < \omega$ let $Z_n = T_{x\restriction n}$.

Suppose for contradiction that $x \in A$. Then $\langle \mu_{x\restriction n} \mid n < \omega\rangle$ is countably complete and so there is a fiber $\langle \alpha_i \mid i < \omega\rangle$ for the sequence $\langle Z_n \mid n < \omega\rangle$. In other words there is a sequence $\langle \alpha_i \mid i < \omega\rangle$ in $[T_x]$ so that $h_{x\restriction n}(\langle \alpha_0, \dots, \alpha_{n-1}\rangle) = x_n$ for each odd $n < \omega$. But then $\langle x_i, \alpha_i \mid i < \omega\rangle$ is a run of G^* and is consistent with σ^*. This is a contradiction, since σ^* is a winning strategy for player II, and infinite runs of G^* are won by player I. $\dashv$

4.19 Corollary. *Let $A \subseteq X^\omega$ be homogeneously Suslin. Then $G_\omega(A)$ is determined.*

Proof. By Exercise 4.16, G^* is determined. By Exercise 4.17 and Lemma 4.18, the player who has a winning strategy in G^* has a winning strategy in $G_\omega(A)$. ⊣

Theorem 4.15 and Corollary 4.19 establish the determinacy of $\mathbf{\Pi}^1_1$ subsets of ω^ω, assuming the existence of a measurable cardinal. In the next section we deal with $\mathbf{\Pi}^1_2$ sets.

5. Projections and Complementations

Martin and Steel [21] use Woodin cardinals to propagate the property of being homogeneously Suslin under complementation and existential real quantification, proving in this manner that all projective sets are determined. In this section we present their results. We begin by proving that if δ is a Woodin cardinal, and $A \subseteq X^\omega \times \omega^\omega$ is δ^+-homogeneously Suslin, then the set $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$ is determined. We then go on to show that B is κ-homogeneously Suslin for all $\kappa < \delta$. Together with the results in Sect. 4 this shows that all $\mathbf{\Pi}^1_{n+1}$ sets are determined, assuming that there are n Woodin cardinals and a measurable cardinal above them.

Let δ be a Woodin cardinal. Let X be a set in $V \| \delta$, and let $A \subseteq X^\omega \times \omega^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+-homogeneously Suslin, and let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ (for some ordinal γ) and $\langle \mu_{s,t} \mid \langle s, t\rangle \in (X \times \omega)^{<\omega}\rangle$ witness this.

5.1 Remark. The objects in the homogeneity system are given for pairs $\langle s, t\rangle \in X^{<\omega} \times \omega^{<\omega}$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$. We sometimes write $\mu_{s,t}$ or $S_{s,t}$ also when s and t are of different length. We mean $\mu_{s\restriction n, t\restriction n}$ where $n = \min\{\mathrm{lh}(s), \mathrm{lh}(t)\}$, and similarly with $S_{s,t}$. We also write $\mu_{x,t}$ for $x \in X^\omega$ to mean $\mu_{x\restriction n,t}$ where $n = \mathrm{lh}(t)$, and similarly with $S_{x,t}$.

5.2 Exercise (Martin-Solovay [20]). Let t_i ($i < \omega$) enumerate $\omega^{<\omega}$. The *Martin-Solovay tree* for the complement of $p[A]$, where $A \subseteq X^\omega \times \omega^\omega$ is the projection of a tree S with homogeneity system $\langle \mu_{s,t} \mid \langle s, t\rangle \in (X \times \omega)^{<\omega}\rangle$, is the tree of attempts to create $x \in X^\omega$ and a sequence $\langle \rho_i \mid i < \omega\rangle$ so that:

(i) ρ_i is a partial function from $S_{x\restriction \mathrm{lh}(t_i), t_i}$ into $|S|^+$, and the domain of ρ_i has $\mu_{x\restriction \mathrm{lh}(t_i), t_i}$-measure one.

(ii) If $t_i < t_j$ then $\rho_i(f \restriction \mathrm{lh}(t_i)) > \rho_j(f)$ for every $f \in \mathrm{dom}(\rho_j)$.

Prove that this tree projects to $X^\omega - p[A]$.

Definitions 5.3 and 5.6 below essentially code a subset of the Martin-Solovay tree for B by a relation on types. We will use this coding to prove that

$G_\omega(B)$ is determined, and that in fact B is homogeneously Suslin. Martin-Steel [21] proved that the Martin-Solovay tree itself is homogeneous. We work with types, rather than the Martin-Solovay tree of functions, in preparation for Sect. 6.

The constructions below use the definitions of Sect. 3. By type here we always mean a type with domain less than δ and greater than $\mathrm{rank}(X)$. All realizations in V are relative to the fixed Woodin cardinal δ. The variable v_0 in each type will always be realized by S. (Realizations in iterates M of V are made relative to the appropriate image of δ, and with the first variable realized by the image of S.)

5.3 Definition. Let $\langle s,t\rangle \in X^{<\omega}\times\omega^{<\omega}$, with $\mathrm{lh}(s)=\mathrm{lh}(t)=k$ say. Let w be a $(k+2)$-type. Define $Z_{s,t}$ to be the set of $f \in S_{s,t}$ for which $(\exists\eta\in\mathrm{On})(\exists\alpha > \max\{\delta,\mathrm{rank}(S)\})$ so that w is realized by S, $\langle 0,f(0)\rangle$, ..., $\langle k-1,f(k-1)\rangle$, and α in $V\|\eta$. Define $\rho_{s,t} : Z_{s,t} \to \mathrm{On}$ by setting $\rho_{s,t}(f)$ equal to the least η witnessing the existential statement above.

5.4 Remark. Both $Z_{s,t}$ and $\rho_{s,t}$ depend on w. When we wish to emphasize the dependence we write $Z_{s,t}(w)$ and $\rho_{s,t}(w)$.

Definition 5.3 lets us view types as defining partial functions $\rho_{s,t}$ from $S_{s,t}$ into the ordinals. The domain of the partial function $\rho_{s,t}$ is $Z_{s,t}$. Connecting the definition to the homogeneity system, let us say that w is $\langle s,t\rangle$-*nice* if $Z_{s,t}$ has $\mu_{s,t}$-measure one.

5.5 Claim. *Let w be a $(k+2)$-type and suppose that w is $\langle s,t\rangle$-nice. Then w contains the formula "$\{v_1,\dots,v_k\}$ is a node in the tree $(v_0)_{\widetilde{s},\widetilde{t}}$".*

Note that both s and t belong to the domain of w, since they are elements of $X^{<\omega}$, and the domain of w is greater than $\mathrm{rank}(X)$ (see the comment following Remark 5.1). The reference to $\widetilde{s}$ and $\widetilde{t}$ in a formula which may potentially belong to w therefore makes sense. $(v_0)_{\widetilde{s},\widetilde{t}}$ in the formula stands for the tree of nodes g so that $\langle s,t,g\rangle$ belongs to the interpretation of v_0.

Proof of Claim 5.5. Let f be any element of $Z_{s,t}(w)$. ($Z_{s,t}$ has $\mu_{s,t}$-measure one, and so certainly it is not empty.) Then

(1) w is realized by S, $\langle 0,f(0)\rangle$, ..., $\langle k-1,f(k-1)\rangle$, α in $V\|\eta$ for some α and η.

(2) $\langle s,t,f\rangle$ belongs to S, meaning that f, which is formally equal to the set $\{\langle 0,f(0)\rangle,\dots,\langle k-1,f(k-1)\rangle\}$, belongs to $S_{s,t}$.

It follows that the formula in the claim belongs to w. ⊣

5.6 Definition. Let s' and t' extend s and t (perhaps not strictly), with $\mathrm{lh}(s') = \mathrm{lh}(t') = k'$. Let w be $\langle s,t\rangle$-nice and let w' be $\langle s',t'\rangle$-nice. We write $w' \prec w$ to mean that the set $\{f' \in S_{s',t'} \mid \rho_{s',t'}(w')(f') < \rho_{s,t}(w)(f'{\restriction}k)\}$ has $\mu_{s',t'}$-measure one.

5.7 Claim. *The relation $\prec$ is transitive.*

5.8 Definition. Given a $k{+}2$-type w we use $\mathrm{dcp}(w)$ (pronounced "decap w") to denote $\mathrm{proj}^{k+1}(w)$. If w is realized by S, $\langle 0, f(0)\rangle$, $\dots$, $\langle k-1, f(k-1)\rangle$, and α, then $\mathrm{dcp}(w)$ is realized by S, $\langle 0, f(0)\rangle$, $\dots$, and $\langle k-1, f(k-1)\rangle$.

5.9 Claim. *Let w be $\langle s,t\rangle$-nice, and suppose that w contains the formula "$\mathrm{v}_{k+1}+2$ exists" (where $k = \mathrm{lh}(s) = \mathrm{lh}(t)$, and w is a $(k+2)$-type). Let s' and t' extend s and t, with $\mathrm{lh}(s') = \mathrm{lh}(t') = k'$. Then there is a $(k'+2)$-type u so that:*

(1) u is $\langle s', t'\rangle$-nice.

(2) u contains the formula "$\mathrm{v}_{k'+1}$ is the largest ordinal".

(3) $\mathrm{dcp}(u)$ is elastic.

(4) u exceeds w.

(5) $u \prec w$.

Proof. Fix for a moment some $f' \in S_{s',t'}$, and suppose that $f'{\restriction}k \in Z_{s,t}(w)$. Let $\eta = \rho_{s,t}(f'{\restriction}k)$, so that w is realized by S, $\langle 0, f'(0)\rangle, \dots, \langle k-1, f'(k-1)\rangle$, and some $\alpha > \max\{\delta, \mathrm{rank}(S)\}$ in $V\|\eta$. Since w contains the formula "$v_{k+1}+2$ exists", it must be that $\eta > \alpha+2$.

Let $\tau < \delta$ be such that the τ-type of S, $\langle 0, f'(0)\rangle, \dots,$ and $\langle k'-1, f'(k'-1)\rangle$ in $V\|\alpha+1$ is elastic, and such that $\tau > \mathrm{dom}(w)$. Such a τ exists by Lemma 3.22. Let u be the τ-type of S, $\langle 0, f'(0)\rangle$, $\dots$, $\langle k'-1, f'(k'-1)\rangle$, and α in $V\|\alpha+1$. Then u contains the formula "$v_{k'+1}$ is the largest ordinal", u exceeds w, and $\mathrm{dcp}(u)$ is elastic.

The type u defined above depends on the node $f' \in S_{s',t'}$ used. To emphasize the dependence let us from now on write $u(f')$ to denote this type. Let us similarly write $\alpha(f')$ and $\eta(f')$ to emphasize the dependence of α and η on f'.

The function $f' \mapsto u(f')$ maps $\{f' \in S_{s',t'} \mid f'{\restriction}k \in Z_{s,t}\}$ into $V\|\delta$. Using the fact that $Z_{s,t}$ has $\mu_{s,t}$-measure one it is easy to check that the domain of this function has $\mu_{s',t'}$-measure one. From this and the δ^+-completeness of the measures it follows that the function is fixed on a set of $\mu_{s',t'}$-measure one. Thus, there exists a particular type u, and a set $Z \subseteq S_{s',t'}$, so that $u(f') = u$ for each $f' \in Z$, and Z has $\mu_{s',t'}$-measure one.

Clearly $Z_{s',t'}(u) \supseteq Z$, and it follows from this that u is $\langle s',t'\rangle$-nice. It is also clear that $\rho_{s',t'}(u)(f') \le \alpha(f')+1 < \eta(f')$ for each $f' \in Z$, and it follows from this that $u \prec w$. $\dashv$

5.10 Claim. *Let u be $\langle s,t\rangle$-nice, where $\mathrm{lh}(s) = \mathrm{lh}(t) = k$. Let w be a $(k+2)$-type, containing the formula "$\mathrm{v}_{k+1} > \max\{\widetilde{\delta}, \mathrm{rank}(\mathrm{v}_0)\}$". Suppose that w is a subtype of $\mathrm{dcp}(u)$. Then w is $\langle s,t\rangle$-nice, and $w \prec u$.*

Proof. Fix for a moment some $f \in Z_{s,t}(u)$. Let $\eta = \rho_{s,t}(u)(f)$, so that u is realized by S, $\langle 0, f(0)\rangle, \ldots, \langle k-1, f(k-1)\rangle$, and some α in $V\|\eta$.

Since w is a subtype of $\text{dcp}(u)$, there must be some β and some ν so that w is realized by S, $\langle 0, f(0)\rangle, \ldots, \langle k-1, f(k-1)\rangle$, and β in $V\|\nu$, and so that $\nu < \eta$. Since w contains the formula "$v_{k+1} > \max\{\widetilde{\delta}, \text{rank}(v_0)\}$", β is greater than $\max\{\delta, \text{rank}(S)\}$.

It follows from the argument of the previous paragraph that, for each $f \in Z_{s,t}(u)$, there exist ν and $\beta > \max\{\delta, \text{rank}(S)\}$ so that w is realized by S, $\langle 0, f(0)\rangle, \ldots, \langle k-1, f(k-1)\rangle$, and β in $V\|\nu$, and that the least ν witnessing this is smaller than $\rho_{s,t}(u)(f)$. In other words $f \in Z_{s,t}(w)$ and $\rho_{s,t}(w)(f) < \rho_{s,t}(u)(f)$, for each $f \in Z_{s,t}(u)$. Since $Z_{s,t}(u)$ has $\mu_{s,t}$-measure one this implies that w is $\langle s,t\rangle$-nice and that $w \prec u$. ⊣

5.11 Claim. *Let $x \in X^\omega$. Suppose that there are types $\langle w_t \mid t \in \omega^{<\omega}\rangle$ so that:*

(1) Each w_t is $\langle x,t\rangle$-nice.

(2) For each $t < t^* \in \omega^{<\omega}$, $w_{t^*} \prec w_t$.

Then $x \in B$.

Proof. We have to show that $(\forall y \in \omega^\omega)\langle x,y\rangle \notin A$. Fix $y \in \omega^\omega$. For each $n < \omega$ let μ_n denote $\mu_{x\restriction n, y\restriction n}$. Let ρ_n denote $\rho_{x,y\restriction n}(w_{y\restriction n})$. ρ_n is a partial function with domain a μ_n-measure one subset of $S_{x\restriction n, y\restriction n}$.

Set $Z_0 = S_{\emptyset,\emptyset}$ and for each $n < \omega$ set $Z_{n+1} = \{f \in S_{x\restriction n+1, y\restriction n+1} \mid \rho_{n+1}(f) < \rho_n(f\restriction n)\}$. By assumption $w_{y\restriction n+1} \prec w_{y\restriction n}$ so Z_{n+1} has μ_{n+1}-measure one.

Suppose for contradiction that $\langle x,y\rangle \in A$. The tower $\langle \mu_n \mid n < \omega\rangle$ is then countably complete by Definition 4.1, so the sequence $\langle Z_n \mid n < \omega\rangle$ has a fiber, $f = \langle \alpha_i \mid i < \omega\rangle$ say. Then $f\restriction n+1 \in Z_{n+1}$ for each $n < \omega$, meaning that $\rho_{n+1}(f\restriction n+1) < \rho_n(f\restriction n)$, so that $\langle \rho_n(f\restriction n) \mid n < \omega\rangle$ is an infinite descending sequence of ordinals, contradiction. ⊣

Let $\langle \nu_{\text{L}}, \nu_{\text{H}}\rangle$ be the lexicographically least pair of local indiscernibles (Definition 3.27) of V relative to $\max\{\delta, \text{rank}(S)\}$, minimizing first over the second coordinate.

5.12 Claim. *For each $\kappa < \delta$, the κ-type of S and ν_{L} in $V\|\nu_{\text{L}} + 1$ is $\langle\emptyset,\emptyset\rangle$-nice.*

For $t \in \omega^{<\omega}$ let $\text{pred}(t)$ denote $t\restriction(\text{lh}(t) - 1)$.

5.13 Definition. Define G^*, illustrated in Diagram 22.9, to be played according to the following rules:

(1) $x_n \in X$.

(2) $t_n \in \omega^{<\omega}$.

I		w_0	x_0		w_1			w_2	x_2	$\ldots$
II	l_0, t_0, u_0			l_1, t_1, u_1		x_1	l_2, t_2, u_2			$\ldots$

Diagram 22.9: The game G^*

(3) u_n is a (k_n+2)-type, $\mathrm{dcp}(u_n)$ is elastic, and u_n contains the formula "$\{v_1, \ldots, v_{k_n}\}$ is a node in the tree $(v_0)_{\tilde{s}_n, \tilde{t}_n}$", where $k_n = \mathrm{lh}(t_n)$ and $s_n = x \restriction k_n$.

(4) If $n > 0$ then $\mathrm{dom}(u_n) > \mathrm{dom}(u_{n-1})$. (And $\mathrm{dom}(u_0) > \mathrm{rank}(X)$, see the comment following Remark 5.1.)

(5) If $t_n = \emptyset$ then u_n is realized by S and ν_{L} in $V \| \nu_{\mathrm{L}} + 1$.

(6) If $t_n \neq \emptyset$ then $l_n < n$ is such that $t_{l_n} = \mathrm{pred}(t_n)$, and u_n exceeds w_{l_n}.

(7) w_n too is a (k_n+2)-type, w_n is a subtype of $\mathrm{dcp}(u_n)$, and w_n contains the formulae "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$" and "$v_{k_n+1}+2$ exists and is the largest ordinal".

The first player to violate any of the rules loses. Infinite runs where all rules have been followed are won by player I.

5.14 Lemma. *Suppose that player* I *has a winning strategy in G^*. Then player* I *has a winning strategy in $G_\omega(B)$.*

Proof. Let σ^* be a winning strategy for player I in G^*. Let $\langle t^*_n \mid n < \omega \rangle$ enumerate $\omega^{<\omega}$ in such a way that $(\forall t \in \omega^{<\omega})$ $\mathrm{pred}(t)$ is enumerated before t. In particular $t^*_0 = \emptyset$. For $n > 0$ let $l^*_n < n$ be such that $\mathrm{pred}(t^*_n) = t^*_{l^*_n}$. Let $l^*_0 = 0$.

Fix an opponent willing to play for II in $G_\omega(B)$. We describe how to play against the opponent, and win. Our description takes the form of a construction of a run of G^*. σ^* supplies moves for I. The opponent supplies the moves $x_1, x_3, x_5, \ldots$ for II. It is up to us to come up with the remaining moves, l_n, t_n, u_n for $n < \omega$. We make sure as we play that:

(1) $t_n = t^*_n$ and $l_n = l^*_n$.

(2) u_n contains the formula "v_{k_n+1} is the largest ordinal" where $k_n = \mathrm{lh}(t_n)$.

(3) u_n is $\langle x, t_n \rangle$-nice.

(We write $\langle x, t_n \rangle$-nice, but notice that only $x \restriction \mathrm{lh}(t_n)$ is relevant to the condition.)

w_n, by the rules of G^*, is a (k_n+2)-type, is a subtype of $\mathrm{dcp}(u_n)$, and contains the formula "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$". It follows by Claim 5.10 that:

(i) w_n is $\langle x, t_n\rangle$-nice.

(ii) $w_n \prec u_n$.

Let us now describe how to play l_n, t_n, and u_n. We begin with the case $n = 0$. Set $t_0 = \emptyset$ and $l_0 = 0$. Using Lemma 3.22 let $\kappa_0 < \delta$ be such that the κ_0-type of S in $V\|\nu_{\text{L}} + 1$ is elastic. Set u_0 to be the κ_0-type of S and ν_{L} in $V\|\nu_{\text{L}} + 1$. These assignments determine the moves l_0, t_0, and u_0. It is easy to check that they satisfy the relevant rules of G^*, and conditions (1)–(3) above for $n = 0$.

Suppose next that rounds 0 through $n-1$ have been played, subject to the relevant rules and to conditions (1)–(3) above. Set $t_n = t_n^*$ and $l_n = l_n^*$. Note that by condition (i), w_{l_n} is $\langle x, t_{l_n}\rangle$-nice. Let $k_n = \text{lh}(t_n)$. Using Claim 5.9, set u_n to be a (k_n+2)-type so that: u_n is $\langle x, t_n\rangle$-nice; u_n contains the formula "v_{k_n+1} is the largest ordinal"; $\text{dcp}(u_n)$ is elastic; u_n exceeds w_{l_n}; and $u_n \prec w_{l_n}$. These assignments determine the moves l_n, t_n, and u_n. It is again easy to check that they satisfy the relevant rules of G^*, and conditions (1)–(3) above. For the record let us note that we have also the following condition:

(iii) $u_n \prec w_{l_n}$.

The assignments made above, together with moves supplied by σ^* and by the opponent, determine an infinite run $\langle l_n, t_n, w_n, u_n, x_n \mid n < \omega\rangle$ of G^*. It remains to check that the real $x = \langle x_n \mid n < \omega\rangle$ constructed as part of this run is won by player I in $G_\omega(B)$.

By conditions (ii) and (iii), $w_n \prec w_{l_n}$ for each $n > 0$. It follows from this that $w_n \prec w_m$ whenever $t_n > t_m$. By Claim 5.11, $x \in B$. So x is won by player I in $G_\omega(B)$, as required. ⊣

5.15 Lemma. *Suppose that player* II *has a winning strategy in G^*. Then player* II *has a winning strategy in $G_\omega(B)$.*

Proof. Let σ^* be a winning strategy for player II in G^*. Fix an opponent willing to play for I in $G_\omega(B)$. We describe how to play against the opponent, and win. Again our description takes the form of a construction. But this time we do not construct a run of G^*. Rather we construct an iteration tree $\mathcal{T}$ with an even branch consisting of $\{0, 2, 4, \ldots\}$, and a run of $j_{\text{even}}(G^*)$, played according to $j_{\text{even}}(\sigma^*)$.

Precisely, we construct:

(A) l_n, t_n, u_n, w_n, and x_n for $n < \omega$.

(B) An iteration tree $\mathcal{T}$ giving rise to models M_k for $k < \omega$ and embeddings $j_{l,k}$ for $l\ T\ k < \omega$.

(C) Nodes $g_n \in j_{0,2n+1}(S)_{x,t_n}$ for $n < \omega$.

x in the last condition is the sequence $\langle x_n \mid n < \omega\rangle$, although of course only $x\restriction \mathrm{lh}(t_n)$ is relevant to the condition.

We construct so that:

- $0\, T\, 2\, T\, 4 \dots$.
- If $t_n \neq \emptyset$ then the T-predecessor of $2n+1$ is $2l_n+1$.
- If $t_n = \emptyset$ then the T-predecessor of $2n+1$ is $2n$.

Note that these conditions determine the tree order T completely.

Let $p_0 = \emptyset$ and recursively define

$$p_{n+1} = j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n, x_n\rangle.$$

We construct so that p_n is a position in $j_{0,2n}(G^*)$, played according to $j_{0,2n}(\sigma^*)$. This amounts to maintaining the following conditions:

(1) l_n, t_n, and u_n are the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n.

(2) w_n is a legal move for player I following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n)\rangle$.

(3) If n is odd then x_n is the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$.

Notice that conditions (1) and (3) determine l_n, t_n, and u_n for each n, and x_n for odd n.

Let k_n denote $\mathrm{lh}(t_n)$. Condition (C) above already places some restriction on the nature of g_n. It must be a sequence of length k_n, and $\langle x\restriction k_n, t_n, g_n\rangle$ must belong to $j_{0,2n+1}(S)$. We maintain the following additional condition during the construction:

(4) w_n is realized by $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle$, $\dots$, $\langle k_n-1, g_n(k_n-1)\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$.

Notice that from this it automatically follows that w_n is a (k_n+2)-type and that it contains the formulae "$v_{k_n+1} > \max\{\widetilde{\delta}, \mathrm{rank}(v_0)\}$" and "$v_{k_n+1}+2$ exists and is the largest ordinal" as demanded by rule (7) of G^*.

Finally, we maintain the conditions:

(5) w_n is elastic.

(6) M_{2n+1} agrees with all later models of $\mathcal{T}$, that is all models M_i for $i > 2n+1$, past $\mathrm{dom}(w_n)$. w_n belongs to M_i for each $i > 2n+1$.

(7) All the extenders used in $\mathcal{T}$ have critical points above $\mathrm{rank}(X)$. For each $m > n$, the critical point of $j_{2n+2,2m+2}$ is greater than the domain of w_n. In particular $j_{2n+2,2m+2}(w_n) = w_n$ for each $m \geq n$.

5.16 Remark. It follows directly from the last condition that p_n has the form $\langle l_i, t_i, j_{2i,2n}(u_i), w_i, x_i \mid i < n\rangle$.

Let us now describe the construction in round n, assuming inductively that we have already constructed the objects corresponding to rounds 0 through $n-1$, and that we maintained conditions (1)–(7) for these rounds.

Set l_n, t_n, and u_n to be the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n, in line with condition (1). The construction continues subject to one of the following cases:

Case 1. $t_n = \emptyset$. The rules of G^* are such that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $M_{2n}\|j_{0,2n}(\nu_{\mathrm{L}})+1$. From the local indiscernibility of ν_{L} and ν_{H} it follows that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{H}})$ in $M_{2n}\|j_{0,2n}(\nu_{\mathrm{H}})+1$. Working in M_{2n} using Lemma 3.22, let $\tau < j_{0,2n}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $j_{0,2n}(\nu_{\mathrm{L}})+3$ is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\mathrm{dcp}(u_n)$ in M_{2n}.

Set E_{2n} = "pad" so that $M_{2n+1} = M_{2n}$ and $j_{2n,2n+1}$ is the identity. Using the One-Step Lemma 3.23, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$. It is easy now to check that w_n satisfies the conditions of rule (7) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, t_n, u_n\rangle)$.

Finally, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n){}^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in $G_\omega(B)$ following $\langle x_0, \dots, x_{n-1}\rangle$ if n is even. This completes the round. ⊣ (Case 1)

Case 2. $t_n \neq \emptyset$. The rules of $j_{0,2n}(G^*)$ following the position p_n are such that u_n exceeds w_{l_n} in M_{2n}. (We are making an implicit use of Remark 5.16 here.) Let κ denote the domain of u_n. Using the One-Step Lemma in M_{2n} find an extender E_{2n} with critical point $\mathrm{dom}(w_{l_n})$, so that u_n is a subtype of $\mathrm{Stretch}^{E_{2n}}_{\kappa+\omega}(w_{l_n})$. Set $M_{2n+1} = \mathrm{Ult}(M_{2l_n+1}, E_{2n})$, and set $j_{2l_n+1,2n+1}$ to be the ultrapower embedding, so that u_n is a subtype of $j_{2l_n+1,2n+1}(w_{l_n})$.

5.17 Exercise. Complete the precise details of this construction, verifying that there is enough agreement between the various models to make sense of the ultrapower taken.

Let k denote $\mathrm{lh}(t_n)$. Note that $t_{l_n} = \mathrm{pred}(t_n)$, so $\mathrm{lh}(t_{l_n}) = k-1$. Let $\bar{k}$ denote $k-1$. Let $\bar{g}$ denote g_{l_n}, and let $\bar{g}' = j_{2l_n+1,2n+1}(\bar{g})$.

Now w_{l_n} is realized by $j_{0,2l_n+1}(S)$, $\langle 0, \bar{g}(0)\rangle$, $\dots$, $\langle \bar{k}-1, \bar{g}(\bar{k}-1)\rangle$ and $j_{0,2l_n+1}(\nu_{\mathrm{L}})$ in $M_{2l_n+1}\|j_{0,2l_n+1}(\nu_{\mathrm{L}})+3$. Using the elementarity of the embedding $j_{2l_n+1,2n+1}$ it follows that $j_{2l_n+1,2n+1}(w_{l_n})$ is realized by $j_{0,2n+1}(S)$, $\langle 0, \bar{g}'(0)\rangle$, $\dots$, $\langle \bar{k}-1, \bar{g}'(\bar{k}-1)\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$. Since u_n is a subtype of $j_{2ln+1,2n+1}(w_{l_n})$ it must be realized, by the same objects and one more object, at a lower rank. Combining this with the fact that u_n is a $(k{+}2)$-type which contains the formula in rule (3) of the definition of G^* (Definition 5.13), we see that there must exist some set z so that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, \bar{g}'(0)\rangle$, ..., $\langle \bar{k}-1, \bar{g}'(\bar{k}-1)\rangle$, $\langle \bar{k}, z\rangle$ and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}}){+}1$, and that moreover the function $g = \bar{g}' \cup \{\langle \bar{k}, z\rangle\}$ is a node in $j_{0,2n+1}(S)_{x,t_n}$. Set g_n equal to this function g, securing the demands of condition (C) above. For the record let us note that:

(i) g_n extends $j_{2l_n+1,2n+1}(g_{l_n})$.

We now continue very much as we did in case 1. Using the local indiscernibility of ν_{L} and ν_{H}, we see that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle$, ..., $\langle k-1, g_n(k-1)\rangle$, and $j_{0,2n+1}(\nu_{\mathrm{H}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{H}})+1$. Working in M_{2n+1} using Lemma 3.22, let $\tau < j_{0,2n+1}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n+1}(S)$, $\langle 0, g_n(0)\rangle$, ..., $\langle k-1, g_n(k-1)\rangle$, $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1}\|j_{0,2n+1}(\nu_{\mathrm{L}})+3$ is elastic. Let w_n be this type. w_n then exceeds $\mathrm{dcp}(u_n)$ in M_{2n+1}.

Using the One-Step Lemma in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$, with critical point equal to the domain of u_n, so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$, and this secures the main requirement on w_n posed by rule (7) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, t_n, u_n\rangle)$.

Finally, as in case 1, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, t_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in $G_\omega(B)$ following $\langle x_0, \dots, x_{n-1}\rangle$ if n is even. This completes the round. $\dashv$ (Case 2)

5.18 Exercise. Verify that the construction described above maintains conditions (1)–(7).

It remains now to check that every sequence $x = \langle x_n \mid n < \omega\rangle \in X^\omega$ that can be obtained by following the construction described above (with moves x_n for even n supplied by the opponent) is won by player II in $G_\omega(B)$.

Let x, $\mathcal{T}$, $\langle l_n, t_n, u_n, w_n \mid n < \omega\rangle$, and $\langle g_n \mid n < \omega\rangle$ be obtained through the construction above. We work through a series of claims to show that $x \notin B$.

5.19 Claim. *The even branch of $\mathcal{T}$ has an illfounded direct limit.*

Proof. Suppose for contradiction that M_{even} is wellfounded. Let R be the tree of attempts to construct an infinite run of G^*, played according to σ^*. Note that $j_{\mathrm{even}}(R)$ has an infinite branch, consisting of $\bigcup_{n<\omega} j_{2n,\mathrm{even}}(p_n)$. Since M_{even} is wellfounded, the existence of an infinite branch through $j_{\mathrm{even}}(R)$ reflects to M_{even}. Thus, $M_{\mathrm{even}} \models$ "there is an infinite run of $j_{\mathrm{even}}(G^*)$, played according to $j_{\mathrm{even}}(\sigma^*)$". Using the elementarity of j_{even} it follows that $V \models$ "there is an infinite run of G^* played according to σ^*". But this contradicts the fact that σ^* is a winning strategy for player II, the player who loses infinite runs. $\dashv$

Let θ be a regular cardinal, large enough that all the objects involved in the construction belong to $V\|\theta$. Let H be a countable elementary substructure of $V\|\theta$, with x, $\mathcal{T}$, $\langle l_n, t_n, u_n, w_n \mid n < \omega\rangle$ and $\langle g_n \mid n < \omega\rangle$ in H. Let P be the transitive collapse of H, and let $\pi : P \to H$ be the anti-collapse embedding. Let $\mathcal{U} = \pi^{-1}(\mathcal{T})$ and let $h_n = \pi^{-1}(g_n)$. Let P_i and $\bar{j}_{i,i'}$ denote the models and embeddings of $\mathcal{U}$. Let $\bar{S}$ denote $\pi^{-1}(S)$. Let $\bar{x}_i = \pi^{-1}(x_i)$ and let $\bar{x} = \langle \bar{x}_i \mid i < \omega\rangle$.

Using Theorem 2.3 find an infinite branch b through $\mathcal{U}$ so that there is an embedding $\sigma : \bar{P}_b \to V\|\theta$ with $\sigma \circ \bar{j}_b = \pi$.

5.20 Claim. *b is not the even branch.*

Proof. The fact that $\bar{P}_b$ embeds into $V\|\theta$ implies that it is wellfounded. $\bar{P}_{\text{even}}$ is not wellfounded, by Claim 5.19. ⊣

Let $m_0, m_1, \dots$ be such that $2m_0 + 1, 2m_1 + 1, 2m_2 + 2, \dots$ lists, in increasing order, all the odd models in b. The tree structure of $\mathcal{T}$, and hence of $\mathcal{U}$, is such that:

- $t_{m_0} = \emptyset$.
- $\text{pred}(t_{m_{i+1}}) = t_{m_i}$.

From the last condition and from condition (i) of the construction it follows that:

- $h_{m_{i+1}}$ extends $\bar{j}_{2m_i+1,2m_{i+1}+1}(h_{m_i})$.

Letting $h^*_i = \bar{j}_{2m_i+1,b}(h_{m_i})$ it follows that:

- h^*_{i+1} extends h^*_i for each i.

Let $y = \bigcup_{i<\omega} t_{m_i}$ and let $h^* = \bigcup_{i<\omega} h^*_i$. Condition (C) of the construction implies that $\langle \bar{x}\restriction i, y\restriction i, h^*\restriction i\rangle$ is a node in $\bar{j}_b(\bar{S})$. Applying the embedding $\sigma : P_b \to V\|\theta$ to this statement, and using the fact that $\sigma \circ \bar{j}_b = \pi$, it follows that $\langle x\restriction i, y\restriction i, \sigma(h^*\restriction i)\rangle$ is a node in $\pi(\bar{S}) = S$. This is true for each i, and hence:

5.21 Claim. *$\langle x, y\rangle \in p[S]$.*

Proof. Let $h^{**} = \bigcup_{i<\omega} \sigma(h^*\restriction i)$. The argument of the previous paragraph shows that $\langle x, y, h^{**}\rangle$ is an infinite branch through S. ⊣

Recall that $A = p[S]$ and that $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. From the last claim it follows that $x \notin B$, and therefore x is won by player II in $G_\omega(B)$, as required. ⊣

5.22 Definition. Let M be a model of ZFC^-. Let X belong to M and let $S \in M$ be a tree on $X \times U$ for some set $U \in M$. Define $\text{gp}(S)$, the *generalized projection* of S, by setting $x \in \text{gp}(S)$ iff there exists a length ω iteration tree $\mathcal{T}$

on M, using only extenders with critical points above $\text{rank}(X)$, so that for every wellfounded cofinal branch b of $\mathcal{T}$, $x \in p[j_b^{\mathcal{T}}(S)]$. An iteration tree $\mathcal{T}$ witnessing that $x \in \text{gp}(S)$ is said to put x in a *shifted projection* of S. Notice that the tree must be such that $x \in p[j_b^{\mathcal{T}}(S)]$ for *all* wellfounded cofinal branches of $\mathcal{T}$.

5.23 Exercise. Let M be a model of ZFC and let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$ and let $S \in M$ be a tree on $X \times \omega \times \gamma$ for some ordinal γ. Let G^* be the game of Definition 5.13 but relativized to M. Suppose $M \models$ "player II has a winning strategy in G^*". Prove that there is a strategy σ for player II in the game on X so that, in V, every infinite play according to σ belongs to $\text{gp}(S)$.

Hint. Let $\sigma^* \in M$ be a winning strategy for player II in G^*. Imitate the construction in the proof of Lemma 5.15 to define a strategy σ for II in the game on X. Show that if $x \in X^\omega$ and $\mathcal{T}$ are produced by the construction in the proof of Lemma 5.15, then $\mathcal{T}$ witnesses that x belongs to a shifted projection of S: Claim 5.19 shows that the even branch of $\mathcal{T}$ is illfounded, and the argument following Claim 5.20 can be modified to produce, for each cofinal branch b other than the even branch, some y and f so that $\langle x, y, f\rangle \in [j_b(S)]$. ⊣

Lemmas 5.14 and 5.15 combine to show that $G_\omega(B)$ is determined: G^* is determined since it is a closed game, and by Lemmas 5.14 and 5.15 the player who has a winning strategy in G^* has a winning strategy in $G_\omega(B)$. We thus obtained the following theorem:

5.24 Theorem. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$ and let $A \subseteq (X \times \omega)^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+-homogeneously Suslin. Then B is determined.*

In the next section we weaken the assumption, from homogeneously Suslin to universally Baire. But first we continue toward a proof that B is homogeneously Suslin.

Let Γ be the map that assigns to each position $q^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ in the game G^* the move $\langle l_n, t_n, u_n\rangle$ described in the proof of Lemma 5.14. By this we mean the move that the construction there would produce for round n, assuming that the moves of the previous rounds were $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$. (The construction appears between conditions (ii) and (iii) in the proof of Lemma 5.14. Notice that this part does not depend on the strategy σ^*.) If the moves in $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ do not satisfy the inductive conditions in the proof of Lemma 5.14, then leave $\Gamma(q^*)$ undefined.

Given a sequence $q = \langle x_i, w_i \mid i < n\rangle$ define q^* to be the sequence $\langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ where for each $m < n$, $\langle l_m, t_m, u_m\rangle$ is equal to $\Gamma(q^* \restriction m)$. If for some $m < n$, $q^* \restriction m$ is not a legal position in G^* or $\Gamma(q^* \restriction m)$ is undefined, then leave q^* undefined.

Let $R \subseteq (X \times V\|\delta)^{<\omega}$ be the tree of sequences $q = \langle x_i, w_i \mid i < n\rangle$ so that q^* is defined.

5.25 Exercise. Suppose that $x \in p[R]$. Prove that $x \in B$.

Hint. Let $\langle w_i \mid i < \omega \rangle$ be such that $\langle x_i, w_i \mid i < n \rangle \in R$ for each $n < \omega$. Let q_n denote $\langle x_i, w_i \mid i < n \rangle$. Note that for each $n < \omega$, q_n^* is defined. Let $q^* = \bigcup_{n<\omega} q_n^*$. Check that q^* is an infinite run of G^*, satisfying all the conditions in the proof of Lemma 5.14. Use the final argument in that proof to conclude that $x \in B$. $\dashv$

Given $z \in X^\omega$ let $\langle l_n^z, t_n^z, u_n^z, w_n^z, x_n^z \mid n < \omega \rangle$, $\mathcal{T}^z$, and $\langle g_n^z \mid n < \omega \rangle$ be the objects obtained by constructing subject to the conditions in the proof of Lemma 5.15, with condition (1) replaced by the condition "$\langle l_n, t_n, u_n \rangle = j_{0,2n}(\Gamma)(p_n)$", and condition (3) replaced by the condition "$x_n = z_n$ for all n". These two replacements remove the use of the opponent and of σ^* in the construction. The use of σ^* is replaced by a use of Γ and of the odd half of z. The use of the opponent is replaced by a use of the even half of z.

Notice that the dependence of the construction on z is continuous, in the sense that knowledge of $z \restriction n$ suffices to determine the construction in rounds 0 through $n-1$. These rounds construct, among other things, $\mathcal{T}^z \restriction 2n+1$, and $\langle w_0, \ldots, w_{n-1} \rangle$. We have therefore maps $s \mapsto \mathcal{T}^s$, $s \mapsto \langle l_i^s, t_i^s, u_i^s, w_i^s, x_i^s \mid i < \mathrm{lh}(s) \rangle$, and $s \mapsto \langle g_i^s \mid i < \mathrm{lh}(s) \rangle$, defined on $s \in X^{<\omega}$, with the properties:

- $\mathcal{T}^s$ is an iteration tree of length $2\,\mathrm{lh}(s) + 1$, leading to a final model indexed $2\,\mathrm{lh}(s)$.
- $\mathcal{T}^z = \bigcup_{n<\omega} \mathcal{T}^{z \restriction n}$.
- $l_i^z = l_i^s$ whenever z extends s and $i < \mathrm{lh}(s)$, and similarly with t_i^z, u_i^z, w_i^z, x_i^z, and g_i^z.

Let M_i^s, for $i \leq 2\,\mathrm{lh}(s)$, be the models of the tree $\mathcal{T}^s$. Let $j_{i,i'}^s$ be the embeddings of the tree.

5.26 Exercise. Show that $\langle x_i^s, w_i^s \mid i < \mathrm{lh}(s) \rangle$ belongs to $j_{0,2\,\mathrm{lh}(s)}^s(R)$.

Hint. Let $q = \langle x_i^s, w_i^s \mid i < \mathrm{lh}(s) \rangle$. Let $p = \langle l_i^s, t_i^s, j_{2i,2\,\mathrm{lh}(s)}^s(u_i^s), w_i^s, x_i^s \mid i < \mathrm{lh}(s) \rangle$. Use the fact that $\langle l_i^s, t_i^s, u_i^s \rangle = j_{0,2i}(\Gamma)(p \restriction i)$ to show that q^* (in the sense of $M_{2\,\mathrm{lh}(s)}^s$) is equal to p. $\dashv$

Define M_s to be the last model of the tree $\mathcal{T}^s$, namely the model $M_{2\,\mathrm{lh}(s)}^s$. Define $j_{s,s^*} : M_s \to M_{s^*}$ to be the embedding $j_{2\,\mathrm{lh}(s),2\,\mathrm{lh}(s^*)}^{s^*}$. Define φ_s to be the sequence $\langle w_i^s \mid i < \mathrm{lh}(s) \rangle$.

5.27 Exercise. Prove that R is homogeneous by showing that the system $\langle M_s, \varphi_s, j_{s,s^*} \mid s < s^* \in X^{<\omega} \rangle$ satisfies the conditions in Exercise 4.2. Conclude that B is homogeneously Suslin.

Hint. Condition (2) of Exercise 4.2 follows from the previous exercise. For condition (3): The direct limit of $\langle M_s, j_{s,s^*} \mid s < s^* < x \rangle$ is simply the direct limit along the even branch of $\mathcal{T}^x$. You can use its illfoundedness as a replacement for Claim 5.19, and proceed from there as in the proof of Lemma 5.15, to show that $x \notin B$, and hence by Exercise 5.25, $x \notin p[R]$. To conclude that B is homogeneously Suslin you now only need the converse to Exercise 5.25. To prove it use the fact that illfoundedness of the direct limit of $\langle M_s, j_{s,s^*} \mid s < s^* < x \rangle$ implies not only $x \notin p[R]$, but $x \notin B$. $\dashv$

5.28 Exercise. Prove that the Martin-Solovay tree for B (see Exercise 5.2) is homogeneous.

Hint. Embed R into the Martin-Solovay tree for B, and use the embedding to transfer the homogeneity measures on R to the Martin-Solovay tree. $\dashv$

The exercises above establish that B is homogeneously Suslin. With a small additional adjustment we obtain the following:

5.29 Exercise. Let δ be a Woodin cardinal. Let X belong to $V \| \delta$ and let $A \subseteq (X \times \omega)^\omega$. Let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. Suppose that A is δ^+-homogeneously Suslin. Then B is κ-homogeneously Suslin for each $\kappa < \delta$.

Hint. Fix $\kappa < \delta$. Revise the construction in the proof of Lemma 5.14 to make sure that $\mathrm{dom}(u_0) > \kappa$. Show that if Γ is defined using this revised construction, then the embeddings j_{s,s^*} obtained above all have critical points above κ. $\dashv$

5.30 Corollary. *Suppose that there are n Woodin cardinals and a measurable cardinal above them. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_{n+1}$. Then A is homogeneously Suslin.*

Proof. Let $\delta_1 < \cdots < \delta_n$ be the Woodin cardinals, and let $\kappa > \delta_n$ be the measurable cardinal. Let $\delta_0 = \aleph_0$.

Let $A_k \subseteq (\omega^\omega)^k$ be such that A_{n+1} is $\mathbf{\Pi}^1_1$, $A_k = \{\langle x, y_1, \ldots, y_{k-1}\rangle \mid (\forall y_k)\langle x, y_1, \ldots, y_k\rangle \notin A_{k+1}\}$ for each $k \leq n$, and $A_1 = A$.

By Theorem 4.15, A_{n+1} is $(\delta_n)^+$-homogeneously Suslin. Successive applications of Exercise 5.29, starting from $k = n$ and working down to $k = 1$, show that A_k is $(\delta_{k-1})^+$-homogeneously Suslin. Finally then $A = A_1$ is homogeneously Suslin. $\dashv$

5.31 Corollary. *Suppose that there are n Woodin cardinals and a measurable cardinal above them. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_{n+1}$. Then $G_\omega(A)$ is determined.*

6. Universally Baire Sets

Let δ be a Woodin cardinal. Let X belong to $V \| \delta$. Let S be a tree on $X \times \omega \times \gamma$ for some ordinal γ, let $A = p[S] \subseteq X^\omega \times \omega^\omega$, and let $B = \{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin A\}$. In the previous section we showed that if S is δ^+-homogeneous then $G_\omega(B)$ is determined. Here we work without the assumption of homogeneity, and try to salvage as much determinacy as we can. We cannot hope for actual determinacy since every set is Suslin under the axiom of choice, but not every set is determined. The approximation for determinacy that we salvage is the following lemma. Recalling a standard notation, $\mathrm{Col}(\omega, \delta)$ is the poset that adjoins a map from ω onto δ using finite partial maps as conditions.

6.1 Lemma. *Let g be $\mathrm{Col}(\omega, \delta)$-generic over V. In $V[g]$ define B^* to be the set $\{x \in X^\omega \mid (\forall y)\langle x, y\rangle \notin p[S]\}$, where X^ω, the quantifier $(\forall y)$, and the projection $p[S]$ are all computed in $V[g]$. Then at least one of the following cases hold:*

(1) In V, player II *has a winning strategy in the game $G_\omega(B)$.*

(2) In $V[g]$, player I *has a winning strategy in $G_\omega(B^*)$.*

With a sufficiently absolute set B the lemma can be used to obtain actual determinacy, as we shall see later on.

Proof of Lemma 6.1. Let G^* be the game defined in the previous section, specifically in Definition 5.13. Notice that the game is defined with no reference to the homogeneity system of the previous section, and so we may use it in the current context. Notice further that Lemma 5.15 is proved without use of the homogeneity system. It too applies in the current context, showing that if player II has a winning strategy in G^* then player II has a winning strategy in $G_\omega(B)$. To complete the proof of Lemma 6.1 it thus suffices to show that if player I has a winning strategy in G^*, then condition (2) of Lemma 6.1 holds true.

Let σ^* be a winning strategy for player I in G^*. Let $\rho : \delta \to V\|\delta$ be a bijection. To be precise we emphasize that both σ^* and ρ are taken in V. Working now in $V[g]$, notice that $\rho \circ g$ is a bijection of ω and $V\|\delta$.

In Lemma 5.14 we used the homogeneity measures for S to ascribe auxiliary moves for player II in G^* while playing against σ^*. We cannot do the same here since T is not assumed to be homogeneous. Instead, we plan to ascribe to player II the $\rho \circ g$-first legal move in each round.

6.2 Claim. *Let $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ be a legal position in G^*. Then there is a move $\langle l_n, t_n, u_n\rangle$ which is legal for player* II *in G^* following p^*.*

Proof. Let $\zeta < \delta$ be large enough that all the moves made in p belong to $V\|\zeta$. Using Lemma 3.22 let $\kappa < \delta$ be such that the κ-type of S in $V\|\nu_{\mathrm{L}} + 1$ is elastic, and such that $\kappa > \zeta$. Set u to be the κ-type of S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. (ν_{L} here is taken from the lexicographically least pair of local indiscernibles relative to $\max\{\delta, \mathrm{rank}(S)\}$.) It is easy to check that the triple $\langle 0, \emptyset, u\rangle$ is legal for II in G^* following p^*. It falls under the case of rule (5) in Definition 5.13. $\dashv$

Call a number $e < \omega$ *valid* at a position $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle$ in G^* just in case that $(\rho \circ g)(e)$ is a legal move for player II in G^* following p^*.

By this we mean that $(\rho \circ g)(e)$ is equal to a tuple $\langle l_n, t_n, u_n\rangle \in V\|\delta$ that satisfies the relevant rules in Definition 5.13. By the last claim there is always a number which is valid at p^*.

6.3 Definition. A position $\langle x_0, \dots, x_{n-1}\rangle$ in $G_\omega(B^*)$ is *nice* if it can be expanded to a position $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < n\rangle^\frown\langle l_n, t_n, u_n, w_n\rangle$ in G^* so that:

(1) p^* is according to σ^*.

(2) For each $m \le n$, $\langle l_m, t_m, u_m\rangle$ is equal to $(\rho\circ g)(e)$ for the *least* number e which is valid at $p^*\restriction m$.

Notice that if p is nice then the expansion p^* is unique: condition (1) uniquely determines w_m for each $m \le n$, and condition (2) uniquely determines l_m, t_m, and u_m for each $m \le n$. Define a strategy σ for player I in $G_\omega(B^*)$ by setting $\sigma(p) = \sigma(p^*)$ in the case that p is a nice position of even length. (It is easy to check that all finite plays by σ lead to nice positions. So there is no need to define σ on positions which are not nice.)

The generic g comes in to the definition of σ through condition (2) in Definition 6.3. σ is thus not an element of V, but of $V[g]$. We now aim to show that, in $V[g]$, σ is winning for I in $G_\omega(B^*)$.

Let $x \in V[g]$ be an infinite run, played according to σ. Suppose for contradiction that $x \notin B$, and let $y \in V[g]$ and $f \in V[g]$ be such that $\langle x, y, f\rangle$ is an infinite branch through S.

For each $n < \omega$ let p^*_n be the unique expansion of $x\restriction n$ that satisfies the conditions of Definition 6.3. Let $p^* = \bigcup_{n<\omega} p^*_n$. Let l_i, t_i, u_i, and w_i be such that $p^* = \langle l_i, t_i, u_i, w_i, x_i \mid i < \omega\rangle$. Let e_n be the least number valid at $p^*\restriction n$, so that $\langle l_i, t_i, u_i\rangle = (\rho \circ g)(e_i)$.

We work recursively to construct sequences $n_0 < n_1 < \cdots$ and $\alpha_0, \alpha_1, \dots$ so that for each i:

(1) $t_{n_i} = y\restriction i$.

(2) u_{n_i} is realized by S, $\langle 0, f(0)\rangle$, …, $\langle i-1, f(i-1)\rangle$, and α_i in $V\|\alpha_i + 1$.

Set to begin with $n_0 = 0$ and $\alpha_0 = \nu_{\mathrm{L}}$. The rules of G^* are such that $t_0 = \emptyset$ and u_0 is the type of S and ν_{L} in $V\|\nu_{\mathrm{L}} + 1$. Conditions (1) and (2) for $i = 0$ therefore hold with these settings.

Suppose now that n_i and α_i have been defined and that conditions (1) and (2) hold for i. The rules of G^* are such that w_{n_i} is a subtype of $\mathrm{dcp}(u_{n_i})$, and must therefore be realized at a lower level. In fact, using the realization of u_{n_i} given by condition (2) above, the specific requirements in rule (7) in Definition 5.13 are such that there must exist some ordinal $\beta < \alpha_i$ so that w_{n_i} is realized by S, $\langle 0, f(0)\rangle$, …, $\langle i-1, f(i-1)\rangle$, and β in $V\|\beta + 3$, and so that $\beta > \max\{\delta, \mathrm{rank}(S)\}$.

Let α_{i+1} be this ordinal β. For the record we note that:

(i) $\alpha_{i+1} < \alpha_i$.

(ii) w_{n_i} is realized by S, $\langle 0, f(0)\rangle$, …, $\langle i-1, f(i-1)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+3$.

It remains to define n_{i+1} in such a way that conditions (1) and (2) hold for $i+1$.

6.4 Claim. *Let $E = \max\{e_0, \ldots, e_{n_i}\}$. Then there exist $e < \omega$ and $\kappa < \delta$ so that:*

(a) $(\rho \circ g)(e)$ has the form $\langle l, t, u\rangle$ with $l = n_i$, $t = y{\restriction}i+1$, and u equal to the κ-type of S, $\langle 0, f(0)\rangle$, …, $\langle i, f(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+1$.

(b) $\mathrm{dcp}(u)$ is elastic.

(c) $e > E$.

(d) κ is large enough that $(\rho\circ g)(0)$, …, $(\rho\circ g)(e-1)$ all belong to $V\|\kappa$.

Proof. Let $D \subseteq \mathrm{Col}(\omega, \delta)$ be the set of conditions q so that conditions (a)–(d) hold for some $e < \mathrm{dom}(q)$ and $\kappa < \delta$, with $(\rho\circ g)$ replaced by $(\rho \circ q)$ in conditions (a) and (d). Notice that D is defined in V: it only makes reference to $f{\restriction}i+1$ and $y{\restriction}i+1$. Using Lemma 3.22 it is easy to check that D is dense. Thus $g \cap D$ is non-empty and the claim follows. ⊣

Let e be given by the last claim. Let $\langle l, t, u\rangle = (\rho\circ g)(e)$, and let $\kappa = \mathrm{dom}(u)$.

6.5 Claim. *$\langle l, t, u\rangle$ is a legal move for player II in G^* following $p^*{\restriction}n$, for every n such that:*

(1) $n > n_i$.

(2) $\mathrm{dom}(u_{n-1}) < \kappa$.

Proof. This is easy to verify, using conditions (1), (ii), (a), and (b) above, and the fact that $\langle x{\restriction}i+1, y{\restriction}i+1, f{\restriction}i+1\rangle$ is a node in S. ⊣

6.6 Claim. *There exists an $n < \omega$ so that $e_n = e$.*

Proof. Let n be least so that $e_n \geq e$. Since $e > E = \max\{e_0, \ldots, e_{n_i}\}$, certainly $n > n_i$. Note that $e_{n-1} < e$ and so from condition (d) it follows that $(\rho\circ g)(e_{n-1})$ belongs to $V\|\kappa$. In particular then u_{n-1} belongs to $V\|\kappa$, so certainly $\mathrm{dom}(u_{n-1}) < \kappa$. Applying Claim 6.5 it follows that $\langle l, t, u\rangle$ is legal for II in G^* following $p^*{\restriction}n$, and hence e is valid at $p^*{\restriction}n$. Since e_n is the least number which is valid at $p^*{\restriction}n$, it must be that $e_n \leq e$. We have $e_n \geq e$ by the initial choice of n. Thus $e_n = e$. ⊣

Set n_{i+1} equal to the number n given by the last claim. By condition (a) of Claim 6.4 then, $l_{n_{i+1}}$, $t_{n_{i+1}}$, and $u_{n_{i+1}}$ are such that $l_{n_{i+1}} = n_i$, $t_{n_{i+1}} = y{\restriction}i+1$, and $u_{n_{i+1}}$ is equal to the κ-type of $S, \langle 0, f(0)\rangle, \ldots, \langle i, f(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1}+1$. In particular conditions (1) and (2) hold for $i+1$.

Working by recursion we completed the construction of the sequences $\langle n_i \mid i < \omega\rangle$ and $\langle \alpha_i \mid i < \omega\rangle$. By condition (i) above the sequence $\langle \alpha_i \mid i < \omega\rangle$ is descending. The construction of this infinite descending sequence was based on the assumption that $\langle x, y, f\rangle$ is an infinite branch through S. (This assumption was used in the proof of Claim 6.5.) The assumption must therefore be false, and this shows that x, an arbitrary play according to σ^* in $V[g]$, must belong to B^*. This completes the proof of Lemma 6.1. ⊣

6.7 Corollary. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be* $\mathrm{Col}(\omega, \delta)$*-generic over V. Then at least one of the following holds:*

(1) $V \models$ "player II *has a winning strategy in the game $G_\omega(\neg p[T])$".*

(2) $V[g] \models$ "player I *has a winning strategy in the game $G_\omega(\neg p[T])$".*

($\neg p[T]$ here is the complement of the projection of T. Notice that $\neg p[T]$ need not be the same in $V[g]$ and in V.)

Proof. Immediate from Lemma 6.1 by introducing a vacuous coordinate, more precisely by using the tree $S = \{\langle s, t, f\rangle \in (X \times \omega \times \gamma)^{<\omega} \mid \langle s, f\rangle \in T\}$. ⊣

6.8 Exercise. It may seem that we are losing ground in passing from the lemma to the corollary, but in fact we are not. Prove that Lemma 6.1 is a consequence of Corollary 6.7.

Hint. Let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ be given. Let $\varphi : \omega \times \gamma \to \gamma'$ be a bijection of $\omega \times \gamma$ onto an ordinal γ'. Define a tree T on $X \times \gamma'$ in such a way that $\langle x, y, f\rangle \in [S]$ iff $\langle x, g\rangle \in [T]$ where $g(n) = \varphi(\langle y_n, f(n)\rangle)$. Use Corollary 6.7 with T. ⊣

6.9 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Let $T \in M$ be a tree on $X \times \gamma$ for some ordinal γ. Let g be $\mathrm{Col}(\omega, \delta)$-generic over M. Prove that at least one of the following holds:

(1) There is a strategy σ for player II in the game on X so that, in V, every infinite play according to σ belongs to $\mathrm{gp}(T)$.

(2) There is a strategy $\sigma \in M[g]$ for player I in the game on X so that, in $M[g]$, every infinite play according to σ avoids $p[T]$.

Hint. Relativize the proof of Corollary 6.7 to M, but replace the use of Lemma 5.15, which ultimately leads to the case of condition (1) in Corollary 6.7, with a use of Exercise 5.23. ⊣

6.10 Corollary. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be $\mathrm{Col}(\omega, \delta)$-generic over V. Then at least one of the following holds:*

(1) $V \models$ "player I *has a winning strategy in the game $G_\omega(p[T])$".*

(2) $V[g] \models$ "player II *has a winning strategy in the game $G_\omega(p[T])$".*

(Notice that $p[T]$ need not be the same in $V[g]$ and in V.)

Proof. Immediate from Corollary 6.7, using continuous substitution to reverse the roles of the players. Let us just point out that both here and in Corollary 6.7, the player who has a winning strategy in V is the player who wants to get into $p[T]$, and the player who has a winning strategy in $V[g]$ is the player who wants to avoid $p[T]$. ⊣

We can use various forms of absoluteness to obtain actual determinacy, either in V or in $V[g]$, from Corollary 6.10:

6.11 Lemma. *Let δ be a Woodin cardinal. Let X belong to $V\|\delta$. Let T be a tree on $X \times \gamma$ for some ordinal γ. Let g be $\mathrm{Col}(\omega, \delta)$-generic over V. Suppose that there is a tree S in V so that $V[g] \models$ "$p[S] = \neg p[T]$". Then $V[g] \models$ "$G_\omega(p[T])$ is determined".*

Proof. It is enough to show that if case 1 of Corollary 6.10 holds, then player I wins $G_\omega(p[T])$ also in $V[g]$.

Suppose then that player I wins $G_\omega(p[T])$ in V, and let σ witness this. Let R be the tree of attempts to construct a pair $\langle x, f\rangle$ so that $x \in X^\omega$ is a play according to σ, and $\langle x, f\rangle \in [S]$.

The tree R belongs to V. An infinite branch in V through R would produce an x which belongs to both $p[T]$ and $p[S]$. But then the same x, taken in $V[g]$, would exhibit a contradiction to the assumption of the lemma that $(p[S])^{V[g]}$ and $(p[T])^{V[g]}$ are complementary.

Thus R has no infinite branches in V. By absoluteness R has no infinite branches in $V[g]$ either. It follows that all plays according to σ in $V[g]$ belong to the complement of $(p[S])^{V[g]}$, which by assumption is $(p[T])^{V[g]}$. So σ witnesses that player I wins $G_\omega(p[T])$ in $V[g]$. ⊣

6.12 Corollary (Woodin). *Let δ be a Woodin cardinal and g a $\mathrm{Col}(\omega, \delta)$-generic filter over V. Then $V[g]$ is a model of Δ^1_2 (lightface) determinacy.*

Let X be hereditarily countable. A set $C \subseteq X^\omega$ is λ*-universally Baire* if all its continuous preimages, to topological spaces with regular open bases of cardinality $\leq \lambda$, have the property of Baire. C is ∞-universally Baire if it is λ-universally Baire for all cardinals λ. Feng-Magidor-Woodin [4] provides the following convenient characterization of universally Baire sets, and the basic results in Exercises 6.15 and 6.16:

6.13 Definition. A pair of trees T and T^* on $X \times \gamma$ and $X \times \gamma^*$ respectively is *exhaustive* for a poset $\mathbb{P}$ if the statement "$p[T] \cup p[T^*] = X^\omega$" is forced to hold in all generic extensions of V by $\mathbb{P}$.

6.14 Fact (Feng-Magidor-Woodin [4]). Let X be hereditarily countable, let $C \subseteq X^\omega$, and let λ be an infinite cardinal. C is λ-universally Baire iff there are trees T and T^* so that:

(1) $p[T] = C$ and $p[T^*] = X^\omega - C$.

(2) The pair $\langle T, T^* \rangle$ is exhaustive for all posets of size $\leq \lambda$.

6.15 Exercise. Suppose T and T^* are trees so that:

(1) $p[T] \cap p[T^*]$ is empty.

(2) $\langle T, T^* \rangle$ is exhaustive for $\mathrm{Col}(\omega, \lambda)$.

Prove that $p[T^*] = \mathbb{R} - p[T]$, and that $p[T]$ is λ-universally Baire.

Hint. Use condition (2) and simple absoluteness to argue that $p[T] \cup p[T^*] = \mathbb{R}$. This establishes that $p[T^*] = \mathbb{R} - p[T]$. Basic forcing arguments show that condition (2) here is equivalent to the corresponding condition in Fact 6.14. $\dashv$

6.16 Exercise (Feng-Magidor-Woodin [4]). A set $C \subseteq X^\omega$ is *weakly homogeneously Suslin* (respectively, *weakly λ-homogeneously Suslin*) if it is the projection to X^ω of a homogeneously Suslin (respectively, λ-homogeneously Suslin) subset of $X^\omega \times \omega^\omega$. Prove that if C is weakly λ^+-homogeneously Suslin then it is λ-universally Baire.

Hint. Let $A \subseteq X^\omega \times \omega^\omega$ be λ^+-homogeneously Suslin with $p[A] = C$. Let $S \subseteq (X \times \omega \times \gamma)^{<\omega}$ be a λ^+-homogeneous tree projecting to A, and let $\langle \mu_{s,t} \mid \langle s, t \rangle \in (X \times \omega)^{<\omega} \rangle$ be a λ^+-homogeneity system for S.

Let T be equal to S, viewed as a tree on $X \times (\omega \times \gamma)$, so that T projects to $p[A] = C$. Let T^* be the Martin-Solovay tree for the complement of $p[A]$, defined in Exercise 5.2. Prove that $\langle T, T^* \rangle$ is exhaustive for every poset $\mathbb{P}$ of size $\leq \lambda$. You will need the following claim, which follows from the completeness of the measures $\mu_{s,t}$: Let $\dot{\rho} \in V^{\mathbb{P}}$ be a function from $S_{s,t}$ into the ordinals. Then there is a $\mu_{s,t}$-measure one set Z so that $\dot{\rho} \restriction \check{Z}$ is forced to belong to V. $\dashv$

Using the characterization in Fact 6.14 we can prove, from a Woodin cardinal δ, that δ-universally Baire sets are determined. In light of Exercise 6.16 this is a strengthening of Theorem 5.24:

6.17 Theorem. *Suppose that C is δ-universally Baire and δ is a Woodin cardinal. Then $G_\omega(C)$ is determined.*

Proof. Let T and T^* witness that C is δ-universally Baire. Apply Corollary 6.10 with T and Corollary 6.7 with T^*.

If case 1 of Corollary 6.10 with T holds, then player I wins $G_\omega(C)$ in V. If case 1 of Corollary 6.7 with T^* holds, then player II wins $G_\omega(C)$ in V. Thus it suffices to show that it cannot be that case 2 holds in both applications.

Suppose for contradiction that case 2 holds in both applications. Then in $V[g]$ player II wins $G_\omega(p[T])$ and player I wins $G_\omega(\neg p[T^*])$. Pitting I's winning strategy against II's winning strategy we obtain a real $x \in V[g]$ which does not belong to $(p[T])^{V[g]}$ and does belong to $(\neg p[T^*])^{V[g]}$. In other words x belongs to neither $(p[T])^{V[g]}$ nor $(p[T^*])^{V[g]}$. But this contradicts the fact that $\langle T, T^*\rangle$ is exhaustive for $\mathrm{Col}(\omega, \delta)$. $\dashv$

Our plan for the future is to prove $\mathsf{AD}^{L(\mathbb{R})}$ by proving, from large cardinals, that the least non-determined set in $L(\mathbb{R})$, if it exists, is universally Baire, and then appealing to Theorem 6.17 to conclude that in fact the set is determined.

7. Genericity Iterations

Given a tree S on $X \times U_1 \times U_2$, define $\mathrm{dp}(S)$, the *demanding projection* of S, by putting $x \in \mathrm{dp}(S)$ iff there exist $f_1 : \omega \to U_1$ and $f_2 : \omega \to U_2$ so that $\langle x, f_1, f_2\rangle \in [S]$ and so that f_1 is onto U_1. It is the final clause, that f_1 must be onto U_1, that makes the demanding projection more demanding than the standard projection $p[S]$.

Let M be a model of ZFC and let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$ and let $S \in M$ be a tree on $X \times U_1 \times U_2$ for some sets $U_1, U_2 \in M$. For convenience suppose that $U_1 \cap U_2 = \emptyset$. For further convenience suppose that U_1 and U_2 are the smallest (meaning $\subseteq$-minimal) sets so that S is a tree on $X \times U_1 \times U_2$. U_1 and U_2 are then definable from S.

Define $\mathrm{gdp}(S)$, the *generalized demanding projection* of S, by setting $x \in \mathrm{gdp}(S)$ iff there exists a length ω iteration tree $\mathcal{T}$ on M, using only extenders with critical points above $\mathrm{rank}(X)$, so that for every wellfounded cofinal branch b of $\mathcal{T}$, $x \in \mathrm{dp}(j_b^{\mathcal{T}}(S))$.

An iteration tree $\mathcal{T}$ witnessing that $x \in \mathrm{gdp}(S)$ is said to put x in a *shifted demanding projection* of S. Note that the tree must be such that $x \in \mathrm{dp}(j_b^{\mathcal{T}}(S))$ for *every* cofinal wellfounded branch of $\mathcal{T}$.

The generalized projection here is similar to the one in Definition 5.22, only using the demanding projection instead of the standard projection. We work next to obtain some parallel to the result in Exercise 6.9, for the generalized demanding projection. We work with the objects M, X, δ, and S fixed. We assume throughout that U_1 and $\mathcal{P}^M(\delta)$ are countable in V, so that in V there are surjections onto U_1, and there are $\mathrm{Col}(\omega, \delta)$ filters which are generic over M.

7.1 Definition. Working inside M, define G^* to be played according to Diagram 22.10 and the following rules:

I		w_0	x_0		w_1			w_2	x_2	$\dots$
II	l_0, u_0			l_1, u_1		x_1	l_2, u_2			$\dots$

Diagram 22.10: The game G^*

I	$f_1(0)$		$f_1(1)$		$\dots$	
II		$f_2(0)$		$f_2(1)$		$\dots$

Diagram 22.11: The game $G(\neg S_x)$

(1) $x_n \in X$.

(2) u_n is a $(2k_n + 2)$-type for some number k_n, $\text{dcp}(u_n)$ is elastic, and, setting $s_n = x{\restriction}k_n$, u_n contains the formula "$\langle \widetilde{s}_n, a, b\rangle$ is a node in v_0 where $a = \{v_1, v_3, \dots, v_{2k_n-1}\}$ and $b = \{v_2, v_4, \dots, v_{2k_n}\}$".

(3) If $n > 0$ then $\text{dom}(u_n) > \text{dom}(u_{n-1})$. $\text{dom}(u_0) > \text{rank}(X)$.

(4) If $k_n = 0$ then u_n is realized by S and ν_{L} in $V\|\nu_{\text{L}} + 1$.

(5) If $k_n \neq 0$ then $l_n < n$ is such that $k_{l_n} = k_n - 1$, and u_n exceeds w_{l_n}.

(6) w_n is a $2k_n + 3$-type, w_n is a subtype of $\text{dcp}(u_n)$, and w_n contains the formulae "$v_{2k_n+2} > \max\{\widetilde{\delta}, \text{rank}(v_0)\}$", "$v_{2k_n+2} + 2$ exists and is the largest ordinal", and "v_{2k_n+1} has the form $\langle k_n, z\rangle$ with $z \in A_1$, where A_1, A_2 are the smallest sets so that v_0 is a tree on $\widetilde{X} \times A_1 \times A_2$".

The first player to violate any of the rules loses. Infinite runs where all rules have been followed are won by player I.

7.2 Remark. The key difference between the definition here and that in Sect. 5 is the addition of variables to the types. The use of these variables is governed by rules (2) and (6). Rule (2) is such that the sets realizing $v_1, \dots, v_{2k}$ must form a node $\langle a, b\rangle$ of S_x. Rule (6) is such that v_{2k+1} must be realized by a pair $\langle k, z\rangle$ with $z \in U_1$.

A smaller difference is the elimination here of the moves t_n of Sect. 5. These moves correspond to the vacuous coordinate in the derivation of Corollary 6.7 from Lemma 6.1, and are not needed in a direct proof.

For $x \in X^\omega$ define $G(\neg S_x)$ to be the following game: players I and II alternate moves as in Diagram 22.11 to construct sequences $f_1 = \langle f_1(n) \mid n < \omega\rangle \in (U_1)^\omega$ and $f_2 = \langle f_2(n) \mid n < \omega\rangle \in (U_2)^\omega$. If at any point $\langle x{\restriction}n, f_1{\restriction}n, f_2{\restriction}n\rangle \notin S$ then player I wins. Otherwise player II wins.

Define $\Game(\neg S)$ by setting $x \in \Game(\neg S)$ iff I has a winning strategy in $G(\neg S_x)$.

7.3 Exercise. Suppose $x \notin \text{dp}(S)$. Prove that $x \in \Game(\neg S)$.

7.4 Lemma. *Let g be* $\mathrm{Col}(\omega,\delta)$*-generic over M, and let $B = \Game(\neg S)$ in the sense of $M[g]$. Suppose that $M \models$ "player* I *has a winning strategy in G^*". Then $M[g] \models$ "player* I *has a winning strategy in $G_\omega(B)$".*

Proof. We adapt the construction in the proof of Lemma 6.1.

Let $\sigma \in M$ be a winning strategy for player I in G^*. Let $\rho \in M$ be a bijection of δ onto $V\|\delta$. Call a number $e < \omega$ *valid* at a position $p^* = \langle l_i, u_i, w_i, x_i \mid i < n\rangle$ in G^* just in case that $(\rho \circ g)(e)$ is a legal move for player II in G^* following p^*. Adapting the proof of Claim 6.2, it is easy to see that player II always has a legal move in G^*, so that there is always a number which is valid at p^*.

7.5 Definition. Call a position $\langle x_0, \ldots, x_{n-1}\rangle$ in $G_\omega(B)$ *nice* if it can be expanded to a position $p^* = \langle l_i, u_i, w_i, x_i \mid i < n\rangle^\frown\langle l_n, u_n, w_n\rangle$ in G^* so that:

(1) p^* is according to σ^*.

(2) For each $m \leq n$, $\langle l_m, u_m\rangle$ is equal to $(\rho \circ g)(e)$ for the *least* number e which is valid at $p^* \restriction m$.

Notice that if p is nice then the expansion p^* is unique. Define a strategy σ for player I in $G_\omega(B)$ by setting $\sigma(p) = \sigma(p^*)$ in the case that p is a nice position of even length. (All finite plays by σ lead to nice positions, so there is no need to define σ on positions which are not nice.)

We now aim to show that, in $M[g]$, σ is winning for I in $G_\omega(B)$. Again we adapt the argument in the proof of Lemma 6.1.

Let $x \in M[g]$ be an infinite run of $G_\omega(B)$, played according to σ. Suppose for contradiction that $x \notin B$. This implies that there is a strategy $\tau \in M[g]$ which is winning for II in $G(\neg S_x)$. We intend to use τ and the nature of rule (6) in Definition 7.1 as replacements for the infinite branch through S_x used in the proof of Lemma 6.1.

For each $n < \omega$ let p_n^* be the unique expansion of $x \restriction n$ that satisfies the conditions of Definition 7.5. Let $p^* = \bigcup_{n<\omega} p_n^*$. Let l_i, u_i, and w_i be such that $p^* = \langle l_i, u_i, w_i, x_i \mid i < \omega\rangle$. Let e_i be the least number valid at $p^* \restriction n$, so that $\langle l_i, u_i\rangle = (\rho \circ g)(e_i)$.

We work recursively to construct $f_1 \in (U_1)^\omega$, $f_2 \in (U_2)^\omega$, and sequences $n_0 < n_1 < \cdots$ and $\alpha_0, \alpha_1, \ldots$ so that for each i:

(1) $k_{n_i} = i$ (see Definition 7.1 for the definition of k_n).

(2) $\langle x \restriction i, f_1 \restriction i, f_2 \restriction i\rangle \in S$.

(3) $\langle f_1 \restriction i, f_2 \restriction i\rangle$, viewed as a position in $G(\neg S_x)$, is according to τ.

(4) u_{n_i} is realized by S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\ldots$, $\langle i-1, f_1(i-1)\rangle$, $\langle i-1, f_2(i-1)\rangle$ and α_i in $V\|\alpha_i + 1$.

As in the proof of Lemma 6.1, we shall have $\alpha_{i+1} < \alpha_i$, leading to a contradiction.

Set to begin with $n_0 = 0$, $\alpha_0 = \nu_{\text{L}}$, $f_1\restriction 0 = \emptyset$, and $f_2\restriction 0 = \emptyset$. It is easy to check that these assignments satisfy conditions (1)–(4). In the case of condition (4) note that condition (5) in Definition 7.1 implies that $k_0 = 0$, whence by condition (4) of the definition, u_0 is realized by S and ν_{L} in $V\|\nu_{\text{L}} + 1$.

Suppose now that n_i, α_i, $f_1\restriction i$, and $f_2\restriction i$ have been defined and that conditions (1)–(4) hold for i. The rules of G^* are such that w_{n_i} is a subtype of u_{n_i}. Using the realization of u_{n_i} given by condition (4) and the conditions placed on w_{n_i} by rule (6) in Definition 7.1, it follows that there are $\beta < \alpha_i$ and $z \in U_1$ so that w_{n_i} is realized by S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i-1, f_1(i-1)\rangle$, $\langle i-1, f_2(i-1)\rangle$, $\langle i, z\rangle$, and β, in $V\|\beta + 3$, and so that $\beta > \max\{\delta, \text{rank}(S)\}$.

Let $\alpha_{i+1} = \beta$ and let $f_1(i) = z$. Let $f_2(i)$ be τ's reply to the move $f_1(i) = z$ following the position $\langle f_1\restriction i, f_2\restriction i\rangle$ in $G(\neg S_x)$. Since τ is a winning strategy for II in $G(\neg S_x)$, $\langle x\restriction i+1, f_1\restriction i+1, f_2\restriction i+1\rangle$ is a node in S.

7.6 Remark. The use of rule (6) in Definition 7.1 to obtain $f_1(i)$, and the use of τ to obtain $f_2(i)$, together replace the use of the infinite branch through S in the proof of Lemma 6.1.

We have so far determined α_{i+1}, $f_1\restriction i+1$, and $f_2\restriction i+1$. It remains to determine n_{i+1}.

7.7 Claim. *Let $E = \max\{e_0, \dots, e_{n_i}\}$. Then there exist $e < \omega$ and $\kappa < \delta$ so that:*

(a) $(\rho \circ g)(e)$ has the form $\langle l, u\rangle$ with $l = n_i$, and u equal to the κ-type of S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i, f_1(i)\rangle$, $\langle i, f_2(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1} + 1$.

(b) $\text{dcp}(u)$ is elastic, $e > E$, and κ is large enough that $(\rho \circ q)(0)$, $\dots$, $(\rho \circ q)(e-1)$ all belong to $V\|\kappa$.

Proof. Similar to the proof of Claim 6.4. ⊣

Let e be given by the last claim. Let $\langle l, u\rangle = (\rho \circ g)(e)$, and let $\kappa = \text{dom}(u)$. An argument similar to that in the proof of Claim 6.5, using the fact that $\langle x\restriction i+1, f_1\restriction i+1, f_2\restriction i+1\rangle$ is a node in S, shows that $\langle l, u\rangle$ is a legal move for player I following $p^*\restriction n$. An argument similar to that in the proof of Claim 6.6 produces $n < \omega$ so that $e_n = e$. Set n_{i+1} equal to this n. By condition (a) then, $l_{n_{i+1}} = n_i$ and $u_{n_{i+1}}$ is equal to the κ-type of S, $\langle 0, f_1(0)\rangle$, $\langle 0, f_2(0)\rangle$, $\dots$, $\langle i, f_1(i)\rangle$, $\langle i, f_2(i)\rangle$, and α_{i+1} in $V\|\alpha_{i+1} + 1$. It is easy now to check that conditions (1)–(4) hold for $i + 1$.

The recursive construction above is such that $\alpha_{i+1} < \alpha_i$ for each $i < \omega$. This contradiction, similar to the one obtained in the proof of Lemma 6.1, completes the proof of Lemma 7.4. ⊣

7.8 Lemma. *Suppose that player* II *has a winning strategy in G^*. Then there is a strategy σ for player* II *in the game on X so that, in V, every infinite play according to σ belongs to* $\text{gdp}(S)$.

Proof. We adapt the solution for Exercise 5.23 to the current setting.

Let $\sigma \in M$ be a winning strategy for player II in G^*. Fix an opponent, willing to play for I in the game on X. We describe how to play against the opponent, making sure that each infinite play according to our description ends up in $\mathrm{gdp}(S)$. As usual our description takes the form of a construction. Precisely, we construct:

(A) l_n, u_n, w_n, and x_n for $n < \omega$.

(B) An iteration tree $\mathcal{T}$ on M giving rise to models M_k for $k < \omega$ and embeddings $j_{l,k}$ for $l\ T\ k < \omega$.

(C) Nodes $\langle a_n, b_n \rangle \in j_{0,2n+1}(S)_x$ for $n < \omega$.

(D) $z_n \in j_{0,2n+1}(U_1)$ for $n < \omega$.

This list of objects is similar to the one in the proof of Lemma 5.15, and our construction too will be similar to the one in that proof.

As in Lemma 5.15 we construct so that: $0\ T\ 2\ T\ 4\ldots$; if $k_n \neq 0$ then the T-predecessor of $2n+1$ is $2l_n+1$; and if $k_n = 0$ then the T-predecessor of $2n+1$ is $2n$. k_n here is such that u_n is a $(2k_n+2)$-type, see Definition 7.1.

Let $p_0 = \emptyset$ and recursively define

$$p_{n+1} = j_{2n,2n+2}(p_n)^\frown \langle l_n, j_{2n,2n+2}(u_n), w_n, x_n \rangle.$$

We construct so that p_n is a position in $j_{0,2n}(G^*)$, played according to $j_{0,2n}(\sigma^*)$. In addition we maintain the conditions:

(1) w_n is realized by the objects $j_{0,2n+1}(S)$, $\langle 0, a_n(0) \rangle$, $\langle 0, b_n(0) \rangle, \ldots, \langle k_n - 1, a_n(k_n-1) \rangle$, $\langle k_n - 1, b_n(k_n-1) \rangle$, $\langle k_n, z_n \rangle$, and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\mathrm{L}}) + 3$.

(2) w_n is elastic.

(3) M_{2n+1} agrees with all later models of $\mathcal{T}$, that is all models M_i for $i > 2n+1$, past $\mathrm{dom}(w_n)$. w_n belongs to M_i for each $i > 2n+1$.

(4) All the extenders used in $\mathcal{T}$ have critical points above $\mathrm{rank}(X)$. For each $m > n$, the critical point of $j_{2n+2,2m+2}$ is greater than the domain of w_n. In particular $j_{2n+2,2m+2}(w_n) = w_n$ for each $m \geq n$.

Notice that from condition (1) and the fact that $z_n \in j_{0,2n+1}(U_1)$ it automatically follows that w_n is a $(2k_n+3)$-type and that it contains the formulae required by rule (6) of G^*.

To begin round n of the construction set l_n, and u_n to be the moves played by $j_{0,2n}(\sigma^*)$ following the position p_n. Let k_n be such that u_n is a $(2k_n+2)$-type. The construction in round n continues subject to one of the following cases:

Case 1. $k_n = 0$. The rules of G^* are such that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{L}})$ in $M_{2n} \| j_{0,2n}(\nu_{\mathrm{L}}) + 1$. From the local indiscernibility of ν_{L} and ν_{H} it follows that u_n is realized by $j_{0,2n}(S)$ and $j_{0,2n}(\nu_{\mathrm{H}})$ in $M_{2n} \| j_{0,2n}(\nu_{\mathrm{H}}) + 1$. Pick a set $z_n \in j_{0,2n}(U_1)$. We shall say more on how this set should be picked, later on. Working in M_{2n} using Lemma 3.22, let $\tau < j_{0,2n}(\delta)$ be such that $\tau > \mathrm{dom}(u_n)$ and such that the τ-type of $j_{0,2n}(S)$, $\langle 0, z_n\rangle$, and $j_{0,2n}(\nu_{\mathrm{L}})$ in $j_{0,2n}(\nu_{\mathrm{L}}) + 3$ is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\mathrm{dcp}(u_n)$ in M_{2n}.

Set E_{2n} = "pad" so that $M_{2n+1} = M_{2n}$ and $j_{2n,2n+1}$ is the identity. Applying the One-Step Lemma 3.23 in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\mathrm{Stretch}^{E_{2n+1}}_{\tau+\omega}(\mathrm{dcp}(u_n))$. Set $M_{2n+2} = \mathrm{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. Note that these settings are such that w_n is a subtype of $j_{2n,2n+2}(\mathrm{dcp}(u_n))$. It is easy now to check that w_n satisfies the conditions of rule (6) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n{}^\frown\langle l_n, u_n\rangle)$.

Finally, set x_n to be the move played $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in the game on X following $\langle x_0, \ldots, x_{n-1}\rangle$ if n is even. This completes the round. ⊣ (Case 1)

Case 2. $k_n \neq 0$. The rules of $j_{0,2n}(G^*)$ following the position p_n are such that u_n exceeds w_{l_n} in M_{2n}. Let κ denote the domain of u_n. Using the One-Step Lemma in M_{2n} find an extender E_{2n} with critical point $\mathrm{dom}(w_{l_n})$, so that u_n is a subtype of $\mathrm{Stretch}^{E_{2n}}_{\kappa+\omega}(w_{l_n})$. Set $M_{2n+1} = \mathrm{Ult}(M_{2l_n+1}, E_{2n})$, and set $j_{2l_n+1,2n+1}$ to be the ultrapower embedding, so that u_n is a subtype of $j_{2l_n+1,2n+1}(w_{l_n})$.

Let $k = k_n$ be such that u_n is a $(2k_n + 2)$-type. Let $\bar{k}$ denote $k - 1$. The rules of G^* are such that w_{l_n} is a $(2\bar{k} + 3)$-type. Let $\bar{a}$, $\bar{b}$, and $\bar{z}$ denote a_{l_n}, b_{l_n}, and z_{l_n}. Let $a = j_{2l_n+1,2n+1}(\bar{a})$ and similarly with b and z.

Our construction is such that w_{l_n} is realized by $j_{0,2l_n+1}(S)$, $\langle 0, \bar{a}(0)\rangle$, $\langle 0, \bar{b}(0)\rangle$, $\ldots$, $\langle \bar{k} - 1, \bar{a}(\bar{k} - 1)\rangle$, $\langle \bar{k} - 1, \bar{b}(\bar{k} - 1)\rangle$, $\langle \bar{k}, z\rangle$, and $j_{0,2l_n+1}(\nu_{\mathrm{L}})$ in $M_{2l_n+1} \| j_{0,2l_n+1}(\nu_{\mathrm{L}}) + 3$. Using the elementarity of $j_{2l_n+1,2n+1}$, the fact that u_n is a subtype of $j_{2l_n+1,2n+1}$, and the conditions placed on u_n by rule (2) of Definition 7.1, it follows that there must exist some set z' so that u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a(0)\rangle$, $\langle 0, b(0)\rangle$, $\ldots$, $\langle \bar{k} - 1, a(\bar{k} - 1)\rangle$, $\langle \bar{k} - 1, b(\bar{k} - 1)\rangle$, $\langle \bar{k}, z\rangle$, $\langle \bar{k}, z'\rangle$, and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\mathrm{L}}) + 1$, and that moreover $\langle a^\frown\langle z\rangle, b^\frown\langle z'\rangle\rangle$ is a node in $j_{0,2n+1}(S)_x$. Set $a_n = a^\frown\langle z\rangle$ and set $b_n = b^\frown\langle z'\rangle$. Then $\langle a_n, b_n\rangle$ is a node in $j_{0,2n+1}(S)$, and u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, $\ldots$, $\langle k - 1, a_n(k - 1)\rangle$, $\langle k - 1, b_n(k - 1)\rangle$, and $j_{0,2n+1}(\nu_{\mathrm{L}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\mathrm{L}}) + 1$. For the record we note that:

(i) a_n extends $j_{2l_n+1,2n+1}(a_{l_n})$, and similarly with b_n.

(ii) $j_{2l_n+1,2n+1}(z_{l_n})$ belongs to the range of a_n.

From here we continue as in case 1.

By the local indiscernibility of ν_{L} and ν_{H}, u_n is realized by $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, ..., $\langle k-1, a_n(k-1)\rangle$, $\langle k-1, b_n(k-1)\rangle$, and $j_{0,2n+1}(\nu_{\text{H}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\text{H}}) + 1$.

Pick some set $z_n \in j_{0,2n+1}(U_1)$. We shall say more on how this set should be picked, later on. Working in M_{2n+1} using Lemma 3.22, let $\tau < j_{0,2n+1}(\delta)$ be such that $\tau > \text{dom}(u_n)$ and such that the τ-type of $j_{0,2n+1}(S)$, $\langle 0, a_n(0)\rangle$, $\langle 0, b_n(0)\rangle$, ..., $\langle k-1, a_n(k-1)\rangle$, $\langle k-1, b_n(k-1)\rangle$, $\langle k, z_n\rangle$, and $j_{0,2n+1}(\nu_{\text{L}})$ in $M_{2n+1} \| j_{0,2n+1}(\nu_{\text{L}}) + 3$ is elastic. Let w_n be this type. It is easy to check that w_n exceeds $\text{dcp}(u_n)$ in M_{2n+1}.

Using the One-Step Lemma 3.23, in M_{2n+1}, find an extender $E_{2n+1} \in M_{2n+1}$ so that w_n is a subtype of $\text{Stretch}^{E_{2n+1}}_{\tau+\omega}(\text{dcp}(u_n))$. Set $M_{2n+2} = \text{Ult}(M_{2n}, E_{2n+1})$, and set $j_{2n,2n+2}$ to be the ultrapower embedding. As in case 1, w_n satisfies the conditions of rule (6) of G^*, shifted to M_{2n+2}, following the position $j_{2n,2n+2}(p_n {}^\frown \langle l_n, u_n\rangle)$.

Finally, set x_n to be the move played by $j_{0,2n+2}(\sigma^*)$ following the position $j_{2n,2n+2}(p_n)^\frown\langle l_n, j_{2n,2n+2}(u_n), w_n\rangle$ if n is odd, and the move played by the opponent in the game on X following $\langle x_0, \ldots, x_{n-1}\rangle$ if n is even. This completes the round. ⊣ (Case 2)

The description above completes the construction, except that we have yet to specify how the sets z_n are picked. Note that the structure of the iteration tree $\mathcal{T}$ is such that cofinal branches other than the even branch have the form $0, 2, \ldots, 2m_0, 2m_0+1, 2m_1+1, \ldots$ for some increasing sequence $\{m_i\}$. The sets z_n should be picked during the construction in such a way that:

(iii) For every cofinal branch b other than the even branch, for every odd node $2m+1 \in b$, and for every set $y \in j_{0,2m+1}(U_1)$, there exists a node $2m^*+1 \in b$, with $m^* > m$, so that z_{m^*} is equal to $j_{2m+1,2m^*+1}(y)$.

Securing this through some condition on the way z_n is chosen is a simple matter of book-keeping, using the fact that U_1 is countable in V. Let us just note that this book-keeping cannot in general be phrased inside M, since U_1 is only assumed to be countable in V. Thus the strategy σ which our construction describes need not be an element of M.

With the construction complete, it remains to check that every sequence $x = \langle x_n \mid n < \omega\rangle \in X^\omega$ that can be obtained by following the construction, with moves x_n for even n supplied by the opponent, belongs to $\text{gdp}(S)$.

Let x, $\mathcal{T}$, $\langle l_n, u_n, w_n \mid n < \omega\rangle$, $\langle a_n \mid n < \omega\rangle$, $\langle b_n \mid n < \omega\rangle$ and $\langle z_n \mid n < \omega\rangle$ be obtained through the construction above. We work through a series of claims to show that x belongs to $\text{gdp}(S)$.

7.9 Claim. *The even branch of $\mathcal{T}$ has an illfounded direct limit.*

Proof. Identical to the proof of Claim 5.19. ⊣

7.10 Claim. *Let b be a branch of $\mathcal{T}$ other than the even branch. Let $\{m_i\}$ be such that $b = \{0, 2, \ldots, 2m_0, 2m_0+1, \ldots, 2m_i+1, \ldots\}$. Let $a^*_i = j_{2m_i+1,b}(a_i)$ and let $b^*_i = j_{2m_i+1,b}(b_i)$. Let $a^* = \bigcup_{i<\omega} a^*_i$ and let $b^* = \bigcup_{i<\omega} b^*_i$. Then:*

(1) $\langle x, a^*, b^* \rangle \in [j_{0,b}(S)]$.

(2) a^* *is onto* $j_{0,b}(U_1)$.

Proof. Note first that by condition (i), $\bigcup_{i<\omega} a_i^*$ and $\bigcup_{i<\omega} b_i^*$ are both increasing unions giving rise to infinite sequences. By condition (C), below Lemma 7.8, $\langle x \restriction i, a_i^*, b_i^* \rangle$ is a node in $j_{0,b}(S)$ for each i. Thus $\langle x, a^*, b^* \rangle$ is an infinite branch through $j_{0,b}(S)$.

By conditions (ii), $j_{2m_i+1,b}(z_{m_i})$ belongs to the range of a^* for each i. From this and condition (iii) it follows that a^* is onto $j_{0,b}(U_1)$. ⊣

Claims 7.9 and 7.10 together combine to show that $x \in \mathrm{dp}(j_{0,b}(S))$ for every wellfounded cofinal branch b of $\mathcal{T}$. $\mathcal{T}$ therefore witnesses that $x \in \mathrm{gdp}(S)$. ⊣

7.11 Corollary. *Let M be a model of* ZFC*. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$.*

Let $S \in M$ be a tree. Suppose that both S and $\mathcal{P}^M(\delta)$ are countable in V. Let g be $\mathrm{Col}(\omega, \delta)$*-generic over M.*

Then at least one of the following conditions holds:

(1) There is a strategy σ for player II *in the game on X so that, in V, every infinite run according to σ belongs to* $\mathrm{gdp}(S)$.

(2) There is a strategy $\sigma \in M[g]$ for player I *in the game on X so that, in $M[g]$, every infinite run according to σ belongs to $\supset(\neg S)$.*

Proof. Immediate from Lemma 7.4, Lemma 7.8, and the fact that the game G^* is closed and therefore determined in M. ⊣

Sometimes we want to restrict players on X to some specific subtree of $X^{<\omega}$. The next exercise is useful in such circumstances.

7.12 Exercise. Work in the setting of Corollary 7.11, and in addition to the objects there let $R \in M$ be a tree on X with no terminal nodes. Show that at least one of the cases in the corollary holds, with “game on X” replaced by “game on R” in both cases.

Hint. Define $\pi : X^{<\omega} \to R$ so that $\mathrm{lh}(\pi(s)) = \mathrm{lh}(s)$ for each $s \in X^{<\omega}$, $s < t \Longrightarrow \pi(s) < \pi(t)$ for all $s, t \in X^{<\omega}$, and so that π is onto R. Let $\hat{S} = \{\langle s, u_1, u_2 \rangle \mid \langle \pi(s), u_1, u_2 \rangle \in S\}$. Use Corollary 7.11 on $\hat{S}$. ⊣

One can use Corollary 7.11 to directly obtain determinacy results. Here instead we use the corollary to obtain a genericity result, and then use the genericity result in conjunction with Theorem 6.17 to obtain determinacy.

7.13 Definition. Let $\mathbb{P} \in M$ be a poset. An iteration tree $\mathcal{T}$ on M is said to *absorb* x to an extension by an image of $\mathbb{P}$ just in case that for every wellfounded cofinal branch b through $\mathcal{T}$, there is a generic extension $M_b^{\mathcal{T}}[g]$ of $M_b^{\mathcal{T}}$ by the poset $j_{0,b}^{\mathcal{T}}(\mathbb{P})$, so that $x \in M_b^{\mathcal{T}}[g]$.

7.14 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.

Let U_1 be the set of dense sets in $\mathrm{Col}(\omega,\delta)$. Let A be the set of canonical names in M for functions from ω into X. Let U_2 be the union of A with the set of conditions in $\mathrm{Col}(\omega,\delta)$. Working in M let $S \subseteq (X \times U_1 \times U_2)^{<\omega}$ be the tree of attempts to construct sequences $x = \langle x_0, x_1, \dots\rangle \in X^\omega$, $\langle D_0, D_1, \dots\rangle \in (U_1)^\omega$, and $\langle \dot{x}, p_1, p_2, \dots\rangle \in (U_2)^\omega$ so that:

(1) $\dot{x} \in A$ and $p_n \in \mathrm{Col}(\omega,\delta)$ for each n.

(2) $p_{n+1} < p_n$ and $p_{n+1} \in D_n$ for each n.

(3) $p_n \Vdash$"$\dot{x}(\check{n}) = \check{x}_n$" for each n.

Prove that $x \in \mathrm{dp}(S)$ iff there is a g which is $\mathrm{Col}(\omega,\delta)$-generic over M with $x \in M[g]$.

7.15 Exercise. Continuing to work with the tree of the previous exercise, prove that $x \in \eth(\neg S)$ iff there is *no* g which is $\mathrm{Col}(\omega,\delta)$-generic over M with $x \in M[g]$.

7.16 Theorem. *Let M be a model of* ZFC*. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.*

Then for every $x \in X^\omega$ there is a length ω iteration tree $\mathcal{T}$ on M which absorbs x into an extension by an image of $\mathrm{Col}(\omega,\delta)$.

Note that in particular any real number in V can be absorbed into a generic extension of an iterate of M.

Proof of Theorem 7.16. Let g be $\mathrm{Col}(\omega,\delta)$-generic over M, and apply Corollary 7.11 to the tree S of Exercise 7.14. Notice that condition (2) of the corollary cannot hold: the strategy σ in that condition belongs to $M[g]$, and certainly then there are plays $x \in X^\omega$ which are according to σ, and which belong to $M[g]$. But from Exercise 7.15 and the fact that x belongs to $M[g]$ it follows that $x \notin \eth(\neg S)$, while from condition (2) of the corollary and the fact that x is according to σ it follows that $x \in \eth(\neg S)$.

Thus condition (1) of the corollary must hold, and this immediately implies that for every sequence $\langle x_0, x_2, \dots\rangle \in X^\omega$, there is a sequence $\langle x_1, x_3, \dots\rangle \in X^\omega$ and a length ω iteration tree $\mathcal{T}$ on M, so that the combined sequence $x = \langle x_0, x_1, \dots\rangle$ belongs to $\mathrm{dp}(j_{0,b})(S)$ for every cofinal wellfounded branch b of $\mathcal{T}$. By Exercise 7.14 then, x belongs to a generic extension of $M_b^{\mathcal{T}}$ by $j_{0,b}(\mathrm{Col}(\omega,\delta))$. So $\mathcal{T}$ absorbs x, and therefore certainly $\langle x_0, x_2, \dots\rangle$, into an extension by an image of $\mathrm{Col}(\omega,\delta)$. ⊣

Theorem 7.16 was proved in Neeman [28, 29]. It is the second of two genericity results. The first is due to Woodin [42]. Woodin's theorem uses a forcing notion which has the δ chain condition, and it does not require any assumption on the size of δ or its power set in V. These properties often make it more useful than Theorem 7.16, see for example Neeman-Zapletal [34]. On the other hand Woodin's theorem requires full iterability for trees of lengths up to ω_1, and in our setting this is a disadvantage.

7.17 Definition. Let M be a model of ZFC, let δ be a cardinal of M, let $X \in M\|\delta$, and let $\dot{A} \in M$ be a $\mathrm{Col}(\omega, \delta)$-name for a subset of X^ω.

$x \in X^\omega$ belongs to the *generalized interpretation* of $\dot{A}$ if there exists a length ω iteration tree $\mathcal{T}$ on M using only extenders with critical points above $\mathrm{rank}(X)$, and a map $h : \omega \to \mathrm{On}^{<\omega}$, so that for every wellfounded cofinal branch b of $\mathcal{T}$:

(1) $h_b = \bigcup_{n \in b} h(n)$ is $\mathrm{Col}(\omega, j_b^{\mathcal{T}}(\delta))$-generic over $M_b^{\mathcal{T}}$.

(2) x belongs to $j_b^{\mathcal{T}}(\dot{A})[h_b]$.

7.18 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V. Let g be $\mathrm{Col}(\omega, \delta)$-generic over M.

Let $\dot{A} \in M$ be a $\mathrm{Col}(\omega, \delta)$-name for a subset of X^ω. Prove that at least one of the following conditions holds:

(1) In V, player I has a winning strategy in $G_\omega(A^*)$, where A^* is the generalized interpretation of $\dot{A}$.

(2) In $M[g]$, player II has a winning strategy in $G_\omega(\dot{A}[g])$.

Hint. First note that by changing the roles of the players (and modifying the name $\dot{A}$ accordingly) the exercise can be reduced to proving that at least one of the following conditions holds:

(1) There is a strategy σ for player II so that, in V, every play according to σ belongs to the generalized interpretation of $\dot{A}$.

(2) There is a strategy $\sigma \in M[g]$ for player I so that, in $M[g]$, every play according to σ belongs to the complement of $\dot{A}[g]$.

Were it not for the need for continuity of the map $b \mapsto h_b$ in Definition 7.17, this could be derived from Corollary 7.11, using a tree S similar to the one defined in Exercise 7.14, but replacing the set A used in that exercise with the set of names forced to belong to $\dot{A}$. The continuity of the map $b \mapsto h_b$ is a consequence of the proof of Corollary 7.11, tracing back to the way b^* is constructed in Claim 7.10. $\dashv$

Exercise 7.18 appeared in Neeman [28]. When applied with an iterable model M and a name $\dot{A}$ for a set defined by an absolute condition, the exercise leads to determinacy, and Neeman [28] uses it to prove projective determinacy and indeed $\mathsf{AD}^{L(\mathbb{R})}$.

Tracing through the construction leading to the exercise, the reader can check that in condition (1), the tree $\mathcal{T}$ and the function h witnessing that x

belongs to the generalized interpretation of $\dot{A}$ depend on x continuously. This element of continuity is expressed more explicitly in Lemma 1.7 of Neeman [28]. It is crucial for proofs of determinacy of long games, but we shall not get into this here. The interested reader may find more in Neeman [30].

7.19 Exercise (Windßus [41], see [13, Lemma 4.5, Theorem 5.2]). Let $\pi : P \to V\|\theta$ be elementary, with P countable. Let $\bar{\kappa} \in P$. Let A be the set of sequences $\langle u_i \mid i < \omega\rangle \in P^\omega$ so that:

(i) u_i is a (nice) finite iteration tree on P. If $i < j$ then u_j extends u_i, so that $\mathcal{U} = \bigcup_{i<\omega} u_i$ is a (nice) iteration tree of length ω. The trees use only extenders with critical points above $\bar{\kappa}$.

(ii) Let $n_i + 1 = \mathrm{lh}(u_i)$. Then $b = \{n_i \mid i < \omega\}$ is a branch through U.

(iii) The direct limit of the models of $\pi\mathcal{U}$ along b is wellfounded. (Recall that $\pi\mathcal{U}$ is the copy of $\mathcal{U}$ via π, see Definition 2.7. It is an iteration tree on V.)

Prove that A is $\pi(\bar{\kappa})$-homogeneously Suslin.

Proof. The proof builds on that of Lemma 2.12. Let B be the set of sequences $\langle u_i \mid i < \omega\rangle$ satisfying conditions (i) and (ii), but such that the direct limit of $\pi\mathcal{U}$ along b is illfounded. For each $x = \langle u_i \mid i < \omega\rangle$ in B fix a sequence $\langle \alpha_i^x \mid i < \omega\rangle$ witnessing the illfoundedness, more precisely a sequence so that:

(1) for all $i < \omega$, $j^{\pi u_{i+1}}_{n_i,n_{i+1}}(\alpha_i^x) > \alpha_{i+1}^x$.

Let θ be larger than all the ordinals α_i^x.

For $s = \langle u_0, \dots, u_{i-1}\rangle$ let B_s be the set of $x \in B$ which extend s. Let T be the tree of attempts to construct sequences $x = \langle u_i \mid i < \omega\rangle$ and $\langle \sigma_i \mid i < \omega\rangle$ so that:

(2) x satisfies conditions (i) and (ii).

(3) $\sigma_i : B_{\langle u_0,\dots,u_i\rangle} \to \theta$.

(4) For all i and all $y \in B_{\langle u_0,\dots,u_{i+1}\rangle}$, $\sigma_i(y) > \sigma_{i+1}(y)$.

Prove that $x \in B \implies x \notin p[T]$, and hence $p[T] \subseteq A$. You will prove that $A \subseteq p[T]$ later on.

Let $M_\emptyset = V$. For $s = \langle u_0, \dots, u_i\rangle$ let M_s be the final model $M^{\pi u_i}_{n_i}$ of the copied tree πu_i. Let φ_s be the function $x \mapsto \alpha_i^x$, defined for $x \in B_s$, where α_i^x are the ordinals witnessing condition (1) above. The models of πu_i are $2^{\aleph_0}$-closed by Exercise 2.2, and hence $\varphi_s \in M_s$.

For $t = \langle u_0, \dots, u_{i^*}\rangle$ extending $s = \langle u_0, \dots, u_i\rangle$ let $j_{s,t} : M_s \to M_t$ be the embedding $j^{\pi u_{i^*}}_{n_i,n_{i^*}}$. Let $j_{\emptyset,t} : V \to M_t$ be the embedding $j^{\pi u_{i^*}}_{0,i^*}$. Notice that all these embeddings have critical points above $\pi(\bar{\kappa})$.

Show using condition (1) that $f_s = \langle j_{s\restriction 1,s}(\varphi_{s\restriction 1}), j_{s\restriction 2,s}(\varphi_{s\restriction 2}), \ldots, \varphi_s\rangle$ is a node in $j_{\emptyset,s}(T_s)$, and use the models M_s, embeddings $j_{s,t}$, and nodes f_s to assemble a homogeneity system for T along the conditions of Exercise 4.2. Finally use the converse of condition (3) of Exercise 4.2, given by Exercise 4.4, to show that $A \subseteq p[T]$. ⊣

7.20 Exercise (Woodin, see [14, Theorem 3.3.8]). Let δ be Woodin in V and let $A \subseteq \omega^\omega$ be δ-universally Baire. Prove that A is weakly κ-homogeneously Suslin for each $\kappa < \delta$.

Hint. Fix κ. Let $\langle T, T^*\rangle$ witness that A is δ-universally Baire. Let θ be large enough that δ, T, and T^* belong to $V\|\theta$. Let $\pi : P \to V\|\theta$ be elementary, with P countable and κ, δ, T, and T^* in the range of π. Let $\bar{\kappa}$ be such that $\pi(\bar{\kappa}) = \kappa$, and similarly with $\bar{\delta}$, $\bar{T}$, and $\bar{T}^*$.

Let B be the set of tuples $\langle x, \mathcal{U}, b, n, \dot{x}, g\rangle$ so that: $x \in \omega^\omega$; $\mathcal{U}$ is a (nice) length ω iteration tree on P using only extenders with critical points above $\bar{\kappa}$; b is a cofinal branch through U, leading to a wellfounded direct limit in the copy tree $\pi\mathcal{U}$ on V; $n \in b$; $\dot{x} \in P_n$ is a name in $\mathrm{Col}(\omega, j_{0,n}(\bar{\delta}))$, forced by the empty condition to be a real belonging to $p[j_{0,n}(\bar{T})]$; g is $\mathrm{Col}(\omega, j_b(\bar{\delta}))$-generic over P_b; and $j_{n,b}(\dot{x})[g] = x$.

Show using Exercise 7.19 that B is κ-homogeneously Suslin. Then show using Theorem 7.16 and Lemma 2.12 that $x \in A$ iff $(\exists\mathcal{U})(\exists b)(\exists n)(\exists\dot{x})(\exists g)$ $\langle x, \mathcal{U}, b, n, \dot{x}, g\rangle \in B$. The quantifiers all involve elements of P and P^ω, which are isomorphic to ω and ω^ω. Use this to present A as the projection of a κ-homogeneously Suslin subset of $\omega^\omega \times \omega^\omega$. ⊣

7.21 Remark. If κ is a limit of Woodin cardinals, then for any $A \subseteq \omega^\omega$, Exercises 5.29, 6.16, and 7.20 together imply that A is $<\kappa$-universally Baire iff A is $<\kappa$-homogeneously Suslin iff A is weakly $<\kappa$-homogeneously Suslin.

7.22 Exercise. Let $j : M \to N$ be elementary. Let h be $\mathrm{Col}(\omega, \kappa)$-generic over M. Suppose that $\mathrm{crit}(j) > \kappa$. Prove that j can be extended to an embedding $j^* : M[h] \to N[h]$.

Hint. Define j^* by setting $j^*(\dot{a}[h]) = (j(\dot{a}))[h]$. Show that j^* is well defined and elementary. ⊣

7.23 Exercise. Let M be a model of ZFC. Let δ be a Woodin cardinal of M. Let X belong to $M\|\delta$. Suppose that $\mathcal{P}^M(\delta)$ is countable in V.

Let $\kappa < \delta$. Let h be $\mathrm{Col}(\omega, \kappa)$-generic over M.

Let $x \in X^\omega$. Then there is a length ω iteration tree $\mathcal{T}$ on M so that:

(1) All the extenders used in $\mathcal{T}$ have critical points above κ. (In particular then the embeddings along branches of $\mathcal{T}$ extend to act on $M[h]$.)

(2) For every cofinal wellfounded branch b of $\mathcal{T}$, there is a g $\mathrm{Col}(\omega, j_b(\delta))$-generic over $M_b[h]$ so that x belongs to $M_b[h][g]$.

Note that in particular any real in V can be absorbed into a generic extension of $M_b[h]$ for an iterate M_b of M.

Hint to Exercise 7.23. Let $\widehat{X} = M \| \kappa + \omega$. Let $R \subseteq \widehat{X}^{<\omega}$ be the tree of attempts to construct a sequence $\langle\langle x_0, q_0\rangle, E_0, \langle x_1, q_1\rangle, E_1, \dots\rangle$ so that:

(1) $x_n \in X$ for each n, and q_n is a condition in $\mathrm{Col}(\omega, \kappa)$.

(2) E_n is a dense subset of $\mathrm{Col}(\omega, \kappa)$ for each n.

(3) $q_{n+1} < q_n$ and $q_{n+1} \in E_n$ for each n.

For clarity let us point out that in games on R, player I plays the objects $\langle x_n, q_n\rangle$, and player II plays the objects E_n.

Working in M let U_1 be the set of $\mathrm{Col}(\omega, \kappa)$-names for dense subsets of $\mathrm{Col}(\omega, \delta)$, let A be the set of canonical $\mathrm{Col}(\omega, \kappa) \times \mathrm{Col}(\omega, \delta)$-names for functions from ω into X, and let U_2 be the union of A with the set of conditions in $\mathrm{Col}(\omega, \delta)$.

Let $S \subseteq (\widehat{X} \times U_1 \times U_2)^{<\omega}$ be the tree of attempts to construct a sequence $\langle\langle x_0, q_0\rangle, E_0, \langle x_1, q_1\rangle, E_1, \dots\rangle \in [R]$, a sequence $\langle \dot{D}_0, \dot{D}_1, \dots\rangle \in (U_1)^\omega$, and a sequence $\langle \dot{x}, p_1, p_2, \dots\rangle \in (U_2)^\omega$ so that:

(1) $\dot{x} \in A$ and $p_n \in \mathrm{Col}(\omega, \delta)$ for each n.

(2) For each n and each $i \leq n$, $p_{n+1} < p_n$ and $q_{n+1} \not\Vdash^{\mathrm{Col}(\omega,\kappa)}$ "$\check{p}_{i+1} \notin \dot{D}_i$."

(3) For each n and each $i \leq n$, $\langle q_n, p_n\rangle \not\Vdash^{\mathrm{Col}(\omega,\kappa)\times\mathrm{Col}(\omega,\delta)}$ "$\dot{x}(\check{i}) \neq \check{x}_i$."

Apply Exercise 7.12 to $\widehat{X}$, R, and S as defined above. Argue first that case (2) cannot hold. (For this you will need the following forcing claim: Let g be $\mathrm{Col}(\omega, \delta)$-generic over M. Let h^* belong to $M[g]$ and suppose that h^* is $\mathrm{Col}(\omega, \kappa)$-generic over M. Then there exists a g^* which is $\mathrm{Col}(\omega, \delta)$-generic over $M[h^*]$ and so that $M[h^*][g^*] = M[g]$.) Then use case (1) of Exercise 7.12 to reach the conclusion of the current exercise. ⊣

7.24 Remark. Let $\kappa_1 < \kappa_2 < \cdots < \kappa_i = \kappa$. $\mathrm{Col}(\omega, \kappa)$ is then isomorphic to $\mathrm{Col}(\omega, \kappa_1) \times \cdots \times \mathrm{Col}(\omega, \kappa_i)$. Exercise 7.23 can therefore be rephrased to replace h by a generic $h_1 \times \cdots \times h_i$ for $\mathrm{Col}(\omega, \kappa_1) \times \cdots \times \mathrm{Col}(\omega, \kappa_i)$. This sets the stage for an iterated use of the exercise, assuming an increasing sequence of Woodin cardinals. We shall make such a use in the next section.

8. Determinacy in $L(\mathbb{R})$

Let M be a model of ZFC and let $\delta_0 < \delta_1 < \cdots$ be ω Woodin cardinals in M. Let $\delta_\infty = \sup_{n<\omega} \delta_n$. Suppose that $\mathcal{P}^M(\delta_\infty)$ is countable in V.

Let $\mathbb{P}$ be the finite support product $\mathrm{Col}(\omega, \delta_0) \times \mathrm{Col}(\omega, \delta_1) \times \cdots$.

Given a filter $G = \langle g_i \mid i < \omega\rangle$ which is $\mathbb{P}$-generic over M define $R^*[G]$ to be $\bigcup_{n<\omega} \mathbb{R}^{M[G\restriction n]}$. We refer to $R^*[G]$ as the reals in the *symmetric collapse* of M induced by G. We refer to $L_{M\cap\mathrm{On}}(R^*[G])$ as the *derived model* of M induced by G. (This is $L(R^*[G])$ if M is a class model.)

8.1 Remark. Suppose that $v_1, \dots, v_k \in M[G\restriction n]$. Let $\mathbb{P}_{\mathrm{L}} = \mathrm{Col}(\omega, \delta_0) \times \cdots \times \mathrm{Col}(\omega, \delta_{n-1})$, so that $G\restriction n$ is $\mathbb{P}_{\mathrm{L}}$-generic over M, and let $\mathbb{P}_{\mathrm{H}} = \mathrm{Col}(\omega, \delta_n) \times \cdots$. Because of the symmetry of $\mathbb{P}_{\mathrm{H}}$, any statement $\varphi[v_1, \dots, v_k]$ which holds in $M(R^*[G])$ must be forced to hold in $M(R^*[G])$ by the *empty condition* in $\mathbb{P}_{\mathrm{H}}$ over $M[G\restriction n]$.

8.2 Exercise. Let R^* denote the reals of the symmetric collapse of M induced by G, and let W denote the derived model of M induced by G. Prove that $\mathbb{R}^W = R^*$.

Hint. The inclusion $\mathbb{R}^W \supseteq R^*$ is clear. For the reverse inclusion: let $b \in \mathbb{R}^W$. b is definable in W from some parameters in $R^* \cup (\mathrm{On} \cap M)$. Thus there is some $n < \omega$ so that the parameters defining b belong to $M[G\restriction n]$. Use this and the symmetry given by Remark 8.1 to argue that b belongs to $M[G\restriction n]$, and therefore $b \in R^*$. $\dashv$

Exercise 8.2 makes no use of the assumption that δ_∞ is a limit of Woodin cardinals in M. But without this assumption the derived model need not even satisfy the axiom of dependent choice for reals, and in such circumstances the conclusion of the exercise is less meaningful than it appears.

8.3 Definition. By a $\Sigma_1(\mathbb{R})$ *statement over* $L(\mathbb{R})$, $\Sigma_1(\mathbb{R})$ for short, we mean a statement of the form $(\exists Q \supseteq \mathbb{R})\, Q \models \psi[x_1, \dots, x_n]$, where $x_1, \dots, x_n \in \mathbb{R}$.

We say that $L_\alpha(R)$ is an *initial segment* of $L_\beta(R)$ if: (1) $\alpha \le \beta$; and (2) $\mathbb{R}^{L_\alpha(R)} = \mathbb{R}^{L_\beta(R)} = R$.

8.4 Claim. *Suppose that $L_\alpha(R)$ is an initial segment of $L_\beta(R)$. Then any $\Sigma_1(R)$ statement true in $L_\alpha(R)$ is also true in $L_\beta(R)$.*

The failure of $\mathsf{AD}^{L(\mathbb{R})}$ is $\Sigma_1(\mathbb{R})$, and so is the failure of dependent choice for reals in $L(\mathbb{R})$.

8.5 Lemma. *Let $\varphi[x_1, \dots, x_k]$ be $\Sigma_1(\mathbb{R})$ over $L(\mathbb{R})$. Suppose that $x_1, \dots, x_k$ belong to the symmetric collapse of M induced by G. Suppose that M is countable and embeds into a rank initial segment of V. Then if $\varphi[x_1, \dots, x_k]$ holds in the derived model of M induced by G, it must hold also in (the true) $L(\mathbb{R})$.*

Proof. Let Σ be the weak iteration strategy for M given by Corollary 2.4. Let θ be a cardinal large enough that M, Σ, G, and $\mathbb{R}$ all belong to $V\|\theta$, and so that $V\|\theta$ satisfies enough of ZFC for the argument below. Let X be a countable elementary substructure of $V\|\theta$ containing these objects. Let P be the transitive collapse of X and let $\tau : P \to V\|\theta$ be the anti-collapse embedding. Notice that M, being countable, is not moved by the collapse. So $\tau(M) = M$. Notice further that $\tau^{-1}(\Sigma)$ is simply equal to $\Sigma \cap P$. This is because the iteration trees which come up in weak iteration games on M are countable, and not moved by τ.

Let $\langle a_i \mid n \leq i < \omega\rangle$ be an enumeration of the reals of P, which is $\mathrm{Col}(\omega, \mathbb{R}^P)$-generic over P. Let $M_0 = M_1 = \cdots = M_n = M$ and let $j_{i,i'}$ for $i \leq i' \leq n$ be the identity. For $i < n$ let $h_i = g_i$. Below let h^i denote $h_0 \times h_1 \times \cdots \times h_{i-1}$. Using repeated applications of Exercise 7.23 and Remark 7.24 construct $\mathcal{T}_i$, b_i, M_i, and h_i for $i \geq n$, and a commuting system of embeddings $j_{i,i'} : M_i \to M_{i'}$ for $i \leq i' < \omega$ so that:

(1) $\mathcal{T}_i$ is a length ω iteration tree on M_i, using only extenders with critical points above $j_{0,i}(\delta_{i-1})$.

(2) b_i is the cofinal branch through $\mathcal{T}_i$ given by Σ (equivalently by $\bar{\Sigma}$).

(3) M_{i+1} is the direct limit of the models of $\mathcal{T}_i$ along b_i. $j_{i,i+1} : M_i \to M_{i+1}$ is the direct limit embedding.

(4) h_i is $\mathrm{Col}(\omega, j_{0,i+1}(\delta_i))$-generic over $M_{i+1}[h^i]$.

(5) a_i belongs to $M_{i+1}[h^i \times h_i]$.

The key point in the construction is the last condition, condition (5). It is obtained through an application of Exercise 7.23, inside P, on the model $M_i[h^i]$, to absorb the real a_i into a generic extension of an iterate. $\mathcal{T}_i$ is the iteration tree given by the exercise.

The construction is dependent on the sequence $\langle a_i \mid n \leq i < \omega\rangle$ which does not belong to P. Thus the sequence $\langle M_i, \mathcal{T}_i, b_i, h_i \mid i < \omega\rangle$ does not belong to P. But notice that every stage of the construction is done inside P. Each of the individual objects in the sequence is therefore an element of P (and countable in P, since M is countable in P). Using this and some book-keeping it is easy to arrange that:

(i) For every $i < \omega$, and every $D \in M_i$ which is dense in $j_{0,i}(\mathbb{P})$, there exists some $i^* > i$ so that the filter $h_0 \times \cdots \times h_{i^*-1}$ meets $j_{i,i^*}(D)$.

The book-keeping requires an enumeration of $\bigcup_{i<\omega} M_i$. Notice that there are such enumerations in $P[a_i \mid n \leq i < \omega]$ since each M_i is countable in P, and therefore coded by a real.

Let M_∞ be the direct limit of the system $\langle M_i, j_{i,i'} \mid i \leq i' < \omega\rangle$, and let $j_{i,\infty}$ be the direct limit maps. M_∞ is wellfounded since it is obtained in a play of the weak iteration game according to Σ.

From condition (1) it follows that $\mathrm{crit}(j_{i^*,\infty}) \geq j_{0,i^*}(\delta_{i-1})$ for every $i^* < \omega$. Conditions in h^{i^*} are therefore not moved by $j_{i^*,\infty}$. From this and condition (i) it follows that $H = \langle h_i \mid i < \omega\rangle$ is $j_{0,\infty}(\mathbb{P})$-generic over M_∞.

8.6 Claim. *$\varphi[x_1, \ldots, x_k]$ holds in the derived model of M_∞ induced by H.*

Proof. We know that $\varphi[x_1,\dots,x_k]$ holds in the derived model of M induced by G. By Remark 8.1 this statement, let us denote it $(*)$, is forced, over $M[g_0\times\cdots\times g_{n-1}] = M[h_0\times\cdots\times h_{n-1}]$ by the empty condition in $\mathbb{P}_{\text{H}}$. $j_{0,\infty}$ has critical point above δ_{n-1} and therefore extends to an elementary embedding of $M[h_0 \times\cdots\times h_{n-1}]$ into $M_\infty[h_0\times\cdots\times h_{n-1}]$. $x_1,\dots,x_k$, being reals, are not moved by the embedding. From this and elementarity if follows that the statement $(*)$ is forced to hold also over $M_\infty[h_0\times\cdots\times h_{n-1}]$. It follows that $\varphi[x_1,\dots,x_k]$ holds in the derived model of M_∞ induced by H. ⊣

8.7 Claim. $R^*(H) = \mathbb{R}^P$.

Proof. From the restriction on the critical points in condition (1) it follows that $\mathbb{R}\cap M_\infty[H{\restriction}i] = \mathbb{R}\cap M_i[H{\restriction}i]$. Since M_i and $H{\restriction}i$ belong to P it follows that $\mathbb{R}\cap M_\infty[H{\restriction}i]\subseteq P$, and hence $R^*(H)\subseteq\mathbb{R}^P$.

Conversely, every real in P belongs to $\{a_i \mid n\le i<\omega\}$, and is, by construction, an element of $M_{i+1}[h^i][h_i] = M_{i+1}[H{\restriction}i+1]$ for some i. Using the restriction on the critical points in condition (1), $\mathbb{R}\cap M_{i+1}[H{\restriction}i+1] = \mathbb{R}\cap M_\infty[H{\restriction}i+1]$. So $\mathbb{R}^P\subseteq R^*(H)$. ⊣

8.8 Claim. *$\varphi[x_1,\dots,x_k]$ holds in $(L(\mathbb{R}))^P$.*

Proof. Notice that the ordinals of M_∞ are contained in the ordinals of P. (This is because M_∞ belongs to $P[a_i \mid n\le i<\omega]$.) From this and the last claim it follows that the derived model of M_∞ induced by H is an initial segment of the model $(L(\mathbb{R}))^P$. By Claim 8.6, $\varphi[x_1,\dots,x_k]$ holds in the former model. From this and the fact that φ is $\Sigma_1(\mathbb{R})$ it follows that $\varphi[x_1,\dots,x_k]$ holds also in the latter. ⊣

We showed so far that $\varphi[x_1,\dots,x_k]$ holds in $(L(\mathbb{R}))^P$, where P is the transitive collapse of a Skolem hull of a rank initial segment of V. Using the elementarity of the anti-collapse embedding it follows that $\varphi[x_1,\dots,x_k]$ holds in $(L(\mathbb{R}))^{V\|\theta}$, and since $\varphi[x_1,\dots,x_k]$ is $\Sigma_1(\mathbb{R})$ this implies that it holds in $(L(\mathbb{R}))^V$. ⊣

8.9 Lemma. *Suppose that $\langle\eta_i \mid i<\omega\rangle$ is an increasing sequence of Woodin cardinals of V. Let $\mathbb{Q}$ be the finite support product $\mathrm{Col}(\omega,\eta_1)\times\mathrm{Col}(\omega,\eta_2)\times\cdots$. Let $H=\langle h_i \mid i<\omega\rangle$ be $\mathbb{Q}$-generic over V.*

Then the derived model of V induced by H satisfies the axiom of dependent choice for reals (and hence the full axiom of dependent choice).

Proof. Suppose not. Let θ be a cardinal large enough that $\mathbb{Q}\in V\|\theta$ and so that $V\|\theta$ satisfies the fragment of ZFC that must be assumed in a model M for Lemma 8.5 to hold for the model. Let $\pi: M\to V\|\theta$ be elementary, with M countable and $\mathbb{Q}\in\mathrm{range}(\pi)$. By elementarity, dependent choice for reals fails in the derived models of M. The failure of dependent choice for reals is $\Sigma_1(\mathbb{R})$. Thus by Lemma 8.5 dependent choice for reals must fail also in the true $L(\mathbb{R})$. But this is a contradiction. Dependent choice for reals in the true $L(\mathbb{R})$ follows from the axiom of choice in V and the fact that countable sequences of reals can be coded by reals. $\dashv$

8.10 Theorem. *Suppose that $\langle \eta_i \mid i < \omega \rangle$ is an increasing sequence of Woodin cardinals of V. Let $\mathbb{Q}$ be the finite support product* $\mathrm{Col}(\omega, \eta_1) \times \mathrm{Col}(\omega, \eta_2) \times \cdots$. *Let $H = \langle h_i \mid i < \omega \rangle$ be $\mathbb{Q}$-generic over V.*

Then the derived model of V induced by H satisfies AD.

Proof. Let R^* denote $R^*[H]$, and suppose for contradiction that there is a set $A \in L(R^*)$ so that $A \subseteq R^*$ and $G_\omega(A)$ is not determined in $L(R^*)$.

Since every set in $L(R^*)$ is definable from real and ordinal parameters in a level of $L(R^*)$, there must be a parameter $a \in R^*$, a formula φ, and ordinals γ, ζ so that

$$x \in A \iff L_\gamma(R^*) \models \varphi[x, a, \zeta].$$

Without loss of generality we may assume that $a \in \mathbb{R}^V$. Otherwise we may simply replace V by $V[h_0 \times \cdots \times h_i]$ for i large enough that $a \in \mathbb{R}^{V[h_0 \times \cdots \times h_i]}$.

Again without loss of generality we may assume that $\langle \gamma, \zeta \rangle$ is the lexicographically least pair of ordinals for which the set $\{x \mid L_\gamma(R^*) \models \varphi[x, a, \zeta]\}$ is not determined. By the symmetry of the collapse, this minimality of $\langle \gamma, \zeta \rangle$ is forced by the empty condition in $\mathbb{Q}$ over V.

8.11 Remark. We refer to A as the *least* non-determined set definable from a and ordinal parameters in $L(R^*)$.

Let θ be a cardinal larger than $\sup_{i<\omega} \eta_i$, larger than γ, and so that $V\|\theta$ satisfies the fragment of ZFC that must be assumed in a model M for Lemma 8.5 to hold for the model. Let $\dot{R}^* \in V$ be the canonical name for $R^*[H]$.

8.12 Definition. Working in V let $T_{\text{in}} \subseteq \omega \times V\|\theta$ be the tree of attempts to construct a real x, and a sequence $\langle \langle e_i, f_i \rangle \mid i < \omega \rangle \in (V\|\theta)^\omega$ so that:

(1) $\{e_i \mid i < \omega\}$ is an elementary substructure of $V\|\theta$.

Let M be the transitive collapse of $\{e_i \mid i < \omega\}$, and let $\pi : M \to V\|\theta$ be the anticollapse embedding.

(2) $e_0 = a$, e_1 is equal to $\langle \eta_i \mid i < \omega \rangle$, $e_2 = \mathbb{Q}$, $e_3 = \dot{R}^*$, $e_4 = \gamma$, $e_5 = \zeta$, and e_6 is a name for a real in the symmetric collapse of V by $\mathbb{Q}$.

(3) It is forced by the empty condition in $\mathbb{Q}$ that $L_{\check{\gamma}}(\dot{R}^*) \models \varphi[e_6, \check{a}, \check{\zeta}]$.

Let $\dot{x}$ denote $\pi^{-1}(e_6)$. Let $\mathbb{P}$ denote $\pi^{-1}(\mathbb{Q})$.

(4) The set $G = \{\pi^{-1}(e_{f_i}) \mid i < \omega\}$ forms a $\mathbb{P}$-generic filter over M.

(5) $\dot{x}[G]$ is equal to x.

Let $T_{\text{out}} \in V$ be defined similarly, only changing "$\models$" in condition (3) to "$\not\models$".

8.13 Remark. We emphasize that both T_{in} and T_{out} are defined in V, that is with no reference to H.

8.14 Remark. Let $x \in p[T_{\text{in}}]$ and let $\langle\langle e_i, f_i\rangle \mid i < \omega\rangle$ witness this. Let M, π, and G be as in Definition 8.12. Note in this case that the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinal parameters, and x belongs to the least such set". This follows from the minimality of $\langle\gamma, \zeta\rangle$, the elementarity of π, condition (3) of Definition 8.12, and condition (5) of the definition.

Similarly, if $x \in p[T_{\text{out}}]$, then the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinal parameters, and x belongs to the *complement* of the least such set."

8.15 Claim. *The pair* $\langle T_{\text{in}}, T_{\text{out}}\rangle$ *is exhaustive for* $\mathrm{Col}(\omega, \eta_0)$.

Proof. Let x be a real in $V[h_0]$. Recall that $A = \{x \mid L_\gamma(R^*) \models \varphi[x, a, \zeta]\}$. If $x \in A$ then a Skolem hull argument in $V[H]$ easily shows that $x \in (p[T_{\text{in}}])^{V[H]}$, and from this by absoluteness it follows that $x \in (p[T_{\text{in}}])^{V[h_0]}$. If $x \notin A$ then a similar argument shows that $x \in (p[T_{\text{out}}])^{V[h_0]}$. ⊣

8.16 Claim. *Let x be a real in V. Suppose that $x \in p[T_{\text{in}}]$. Then, in $L(\mathbb{R})$, there is a non-determined set definable from a and ordinal parameters, and x belongs to the least such set.*

Proof. Let $\langle\langle e_i, f_i\rangle \mid i < \omega\rangle$ witness that $x \in p[T_{\text{in}}]$. Let M, π, $\dot{x}$, and G be as in Definition 8.12. By Remark 8.14, the derived model of M induced by G satisfies the statement "there is a non-determined set definable from a and ordinals parameters, and x belongs to the least such set". This statement is $\Sigma_1(\mathbb{R})$. By Lemma 8.5 the statement must hold of x and a in the true $L(\mathbb{R})$. ⊣

8.17 Claim. *Let x be a real in V. Suppose that $x \in p[T_{\text{out}}]$. Then, in $L(\mathbb{R})$, there is a non-determined set definable from a and ordinal parameters, and x belongs to the* complement *of the least such set.*

Proof. Similar to the proof of the previous claim. ⊣

8.18 Claim. $V \models$ *"$p[T_{\text{in}}] \cap p[T_{\text{out}}] = \emptyset$".*

Proof. This follows immediately from the last two claims: x cannot belong to both the least non-determined set and its complement. ⊣

From Claims 8.15 and 8.18, and Exercise 6.15, it follows that, in V, $p[T_{\text{out}}]$ is precisely equal to the complement of $p[T_{\text{in}}]$. In particular this means that, in the true $L(\mathbb{R})$, there is a non-determined set definable from a and ordinal parameter, for otherwise both $p[T_{\text{in}}]$ and $p[T_{\text{out}}]$ would be empty by Claims 8.16 and 8.17. $p[T_{\text{in}}]$ is equal to the least such set.

Again from Exercise 6.15, $p[T_{\text{in}}]$ is η_0-universally Baire. By Theorem 6.17, $G_\omega(p[T_{\text{in}}])$ must be determined. But this is a contradiction since $p[T_{\text{in}}]$ is the least *non*-determined set. The contradiction completes the proof of Theorem 8.10. $\dashv$

8.19 Definition. Let $A \subseteq \mathbb{R}$ in V be $<\eta$-universally Baire, i.e. κ-universally Baire for each $\kappa < \eta$. Let H be $\mathrm{Col}(\omega, <\eta)$-generic over V and let $\mathbb{R}^* = R^*[H] = \bigcup_{\alpha<\eta} \mathbb{R}^{V[H\restriction\alpha]}$. The set A has a *canonical extension* to a set $A^* \subseteq \mathbb{R}^*$, defined as follows: $x \in \mathbb{R}^{V[H\restriction\alpha]}$ belongs to A^* iff $x \in p[T]$ for some, and equivalently any, pair $\langle T, T^* \rangle \in V$ witnessing that A is α-universally Baire. (The equivalence is easy to prove using the conditions in Fact 6.14, and makes the canonical extension useful.)

8.20 Exercise. Let η be a limit of Woodin cardinals and H a $\mathrm{Col}(\omega, <\eta)$-generic filter over V. Let $A \subseteq \mathbb{R}$ in V be $<\eta$-universally Baire (equivalently, by Remark 7.21, $<\eta$-homogeneously Suslin, or weakly $<\eta$-homogeneously Suslin). Let A^* be the canonical extension of A to a subset of $\mathbb{R}^* = R^*[H]$. Prove that $L(\mathbb{R}^*, A^*)$ satisfies AD.

Exercise 8.20 is a first step towards Woodin's derived model theorem, which the reader can find in Steel [40]. Assuming enough large cardinals, it can be shown that there are universally Baire sets which do not belong to $L(\mathbb{R})$, and in that case Exercise 8.20 is a proper strengthening of Theorem 8.10, taking determinacy to sets outside $L(\mathbb{R}^*)$.

Hint to Exercise 8.20. Adapt the proof of Theorem 8.10, replacing $L(\mathbb{R})$ by $L(\mathbb{R}, A)$ and, for countable N and $\sigma : N \to V\|\theta$, replacing derived models of N by models of the form $L_{N\cap\mathrm{On}}(\bar{\mathbb{R}}^*, \bar{A}^*)$ where $\bar{\mathbb{R}}^*$ is the set of reals of the derived model and $\bar{A}^*$ is the canonical extension of $\bar{A} = \sigma^{-1}(A)$ to a subset of $\bar{\mathbb{R}}^*$. (Notice that all the countable models which come up during the proof of Theorem 8.10 embed into rank initial segments of V, either directly by construction or because they are obtained through uses of Theorem 2.3.) You will need the following observation, which is easily verified, to connect $L(\bar{\mathbb{R}}^*, \bar{A}^*)$ with $L(\mathbb{R}, A)$: Let $\sigma : N \to V\|\theta$ be elementary, with N countable and $\sigma(\bar{A}) = A$, $\sigma(\bar{\eta}) = \eta$. Let $\bar{H} \in V$ be $\mathrm{Col}(\omega, \bar{\eta})$-generic over N. Let $\bar{\mathbb{R}}^* = \bigcup_{\alpha<\bar{\eta}} \mathbb{R}^{N[\bar{H}\restriction\alpha]}$ and let $\bar{A}^*$ be the canonical extension of $\bar{A}$ to a subset of $\bar{\mathbb{R}}^*$, as defined inside $N[\bar{H}]$. Then for every $x \in \bar{\mathbb{R}}^*$, $x \in \bar{A}^* \iff x \in A$. $\dashv$

8.21 Theorem. *Suppose that there is a model M of* ZFC *so that:*

- *M has ω Woodin cardinals and a measurable cardinal above them.*
- *M is countable in V.*
- *M is weakly iterable.*

Then the true $L(\mathbb{R})$ satisfies AD.

Proof. Let Σ be a weak iteration strategy for M. Let θ be a cardinal large enough that $\Sigma \in V\|\theta$, and so that $V\|\theta$ satisfies enough of ZFC for the argument below. Let X be a countable elementary substructure of $V\|\theta$ with $M, \Sigma \in X$. Let P be the transitive collapse of X and let $\tau : P \to V\|\theta$ be the anti-collapse embedding. We intend to show that $(L(\mathbb{R}))^P$ satisfies AD, and then use the elementarity of τ.

Let $\langle \delta_i \mid i < \omega \rangle \in M$ be an increasing sequence of Woodin cardinals of M, and let ρ be a measurable cardinal of M above these Woodin cardinals. Let $\mathbb{P}$ denote the finite support product $\mathrm{Col}(\omega, \delta_0) \times \mathrm{Col}(\omega, \delta_1) \times \cdots$.

Using iterated applications of Exercise 7.23 construct a weak iteration $\langle M_i, j_{i,i'} \mid i \leq i' \leq \omega \rangle$ of $M_0 = M$, and a filter H, so that: the iteration is according to Σ, H is $j_{0,\omega}(\mathbb{P})$-generic over M_ω, and $R^*[H]$ is precisely equal to $\mathbb{R} \cap P$. The construction is similar to the main construction in the proof of Lemma 8.5.

By Theorem 8.10, the derived model of M_ω induced by H satisfies AD. This model is an initial segment of $(L(\mathbb{R}))^P$: it has the reals that P has, but it does not have all the ordinals P has. We now add ordinals by passing from M_ω to an iterate of M_ω obtained through ultrapowers by a measure on ρ and its images.

Let μ witness that ρ is measurable in M. Extend the iteration $\langle M_i, j_{i,i'} \mid i \leq i' \leq \omega \rangle$ of M to a weak iteration of length ω_1 by setting $M_{\xi+1} = \mathrm{Ult}(M_\xi, j_{0,\xi}(\mu))$ for each $\xi \geq \omega$ and setting $j_{\xi,\xi+1}$ to be the ultrapower embedding. This completely determines the iteration.

Let η_α denote the ordinal height of M_α, that is $\mathrm{On} \cap M_\alpha$.

8.22 Exercise. Show that $\eta_\alpha \geq \alpha$.

Hint. The map $\xi \mapsto j_{0,\xi}(\rho)$ embeds $\alpha - \omega$ into the ordinals of M_α. ⊣

Note that, for $\alpha \geq \omega$, $j_{\omega,\alpha}$ has critical point $j_{0,\omega}(\rho)$, and this is larger than $j_{0,\omega}(\sup_{i<\omega} \delta_i)$. It follows that H is generic also over M_α, and that the reals of the symmetric collapse induced by H over M_α are the same as the reals of the symmetric collapse induced by H over M_ω, which in turn are the same as the reals of P. Thus, for each $\alpha \geq \omega$:

(i) The derived model of M_α induced by H is equal to $L_{\eta_\alpha}(\mathbb{R}^P)$.

From this and Theorem 8.10 it follows that:

(ii) $L_{\eta_\alpha}(\mathbb{R}^P)$ satisfies AD.

Using (i) and Exercise 8.2:

(iii) $\mathbb{R}^{L_{\eta_\alpha}(\mathbb{R}^P)}$ is equal to $\mathbb{R}^P$.

Using Exercise 8.22 fix some $\alpha < \omega_1$ so that $\eta_\alpha > \mathrm{On} \cap P$. By condition (iii) then, $(L(\mathbb{R}))^P$ is an initial segment of $L_{\eta_\alpha}(\mathbb{R}^P)$. From this and condition (ii) it follows that $(L(\mathbb{R}))^P$ satisfies AD. Using the elementarity of τ it follows that $(L(\mathbb{R}))^{V\|\theta} = L_\theta(\mathbb{R})$ satisfies AD. Since θ could be chosen arbitrarily large, it follows finally that $L(\mathbb{R})$ satisfies AD. ⊣

8.23 Remark. Readers familiar with sharps can verify, by adapting the proof given above, that the assumption in Theorem 8.21 can be weakened, from demanding that M has ω Woodin cardinals and a measurable cardinal above them, to demanding that M is a sharp for ω Woodin cardinals.

8.24 Theorem. *Suppose that in V there are ω Woodin cardinals and a measurable cardinal above them. Then $L(\mathbb{R})$ satisfies* AD.

Proof. Let θ be a cardinal large enough that $V\|\theta \models$ "there are ω Woodin cardinals and a measurable cardinal above them", and so that $V\|\theta$ satisfies the fragment of ZFC necessary in a model M for Theorem 8.21 to hold for the model. Let X be a countable elementary substructure of $V\|\theta$ and let M be the transitive collapse of X. Then $M \models$ "there are ω Woodin cardinals and a measurable cardinal above them", M is countable in V, and, by Corollary 2.4, M is weakly iterable. Applying Theorem 8.21 it follows that $L(\mathbb{R})$ satisfies AD. $\dashv$

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23. Large Cardinals from Determinacy

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Contents

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1. Introduction

In this chapter we give an account of Woodin's technique for deriving large cardinal strength from determinacy hypotheses. These results appear here for the first time and for this reason we have gone into somewhat more detail than is customary in a handbook. All unattributed results that follow are either folklore or due to Woodin.

1.1. Determinacy and Large Cardinals

In the era of set theory following the discovery of independence a major concern has been the discovery of new axioms that settle the statements left undecided by the standard axioms (ZFC). One interesting feature that has emerged is that there are often deep connections between axioms that spring from entirely different sources. In this chapter we will be concerned with one instance of this phenomenon, namely, the connection between axioms of definable determinacy and large cardinal axioms.

In this introduction we will give a brief overview of axioms of definable determinacy and large cardinal axioms (in Sects. A and B), discuss their interconnections (in Sects. C and D), and give an overview of the chapter (in Sect. E). At some points we will draw on notation and basic notions that are explained in fuller detail in Sects. 1.2 and 2.1.

A. Determinacy

For a set of reals $A \subseteq \omega^\omega$ consider the game where two players take turns playing natural numbers:

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(2) & & x(4) & & \ldots \\ \mathrm{II} & & x(1) & & x(3) & & \ldots & \end{array}$$

At the end of a round of this game the two players will have produced a real x, obtained through "interleaving" their plays. We say that Player I wins the round if $x \in A$; otherwise Player II wins the round. The set A is said to be *determined* if one of the players has a "winning strategy" in the associated game, that is, a strategy which ensures that the player wins a round regardless of how the other player plays. The *Axiom of Determinacy* (AD) is the statement that *every* set of reals is determined.

It is straightforward to see that very simple sets are determined. For example, if A is the set of all reals then clearly I has a winning strategy; if A is empty then clearly II has a winning strategy; and if A is countable then II has a winning strategy (by "diagonalizing"). A more substantive result is that if A is closed then one player must have a winning strategy. This might lead one to expect that *all* sets of reals are determined. However, it is straightforward to use the Axiom of Choice (AC) to construct a non-determined set (by listing all winning strategies and "diagonalizing" across them). For this reason AD was never really considered as a serious candidate for a new axiom. However, there is an interesting class of related axioms that are consistent with AC, namely, the axioms of *definable determinacy*. These axioms extend the above pattern by asserting that all sets of reals at a given level of complexity are determined, notable examples being, $\utilde{\Delta}^1_1$-determinacy (all Borel sets of reals are determined), PD (all projective sets of reals are determined) and $\mathrm{AD}^{L(\mathbb{R})}$ (all sets of reals in $L(\mathbb{R})$ are determined).

One issue is whether these are really new axioms or whether they follow from ZFC. In the early development of the subject the result on the determinacy of closed sets was extended to higher levels of definability. These developments culminated in Martin's proof of $\utilde{\Delta}^1_1$-determinacy in ZFC. It turns out that this result is close to optimal—as one climbs the hierarchy of definability, shortly after $\utilde{\Delta}^1_1$ one arrives at axioms that fall outside the provenance of ZFC. For example, this is true of PD and $\mathrm{AD}^{L(\mathbb{R})}$. Thus, we have here a hierarchy of axioms (including PD and $\mathrm{AD}^{L(\mathbb{R})}$) which are genuine candidates for new axioms.

There are actually two hierarchies of axioms of definable determinacy, one involving *lightface* notions of definability (by which we mean notions (such as Δ^1_2) that do not involve real numbers as parameters) and the other involving *boldface* notions of definability (by which we mean notions (such as $\utilde{\Delta}^1_2$) that do involve real numbers as parameters). (See Jackson's chapter in this Handbook for details concerning the various grades of definability and the relevant notation.) Each hierarchy is, of course, ordered in terms of increasing complexity. Moreover, each hierarchy has a natural limit: the natural limit of the lightface hierarchy is OD-determinacy (all OD sets of reals are determined) and the natural limit of the boldface hierarchy is $\mathrm{OD}(\mathbb{R})$-determinacy (all $\mathrm{OD}(\mathbb{R})$ sets of reals are determined). The reason these are natural limits is that the notions of lightface and boldface ordinal definability are candidates for the richest lightface and boldface notions of definability. To see this (for the lightface case) notice first that any notion of definability which does not render all of the ordinals definable can be transcended (as can be seen by considering the least ordinal which is not definable according to the notion) and second that the notion of ordinal definability cannot be so transcended (since by reflection OD is ordinal definable). It is for this reason that Gödel proposed the notion of ordinal definability as a candidate for an "absolute" notion of definability. Our limiting cases may thus be regarded as two forms of absolute definable determinacy.

So we have two hierarchies of increasingly strong candidates for new axioms and each has a natural limit. There are two fundamental questions concerning such new axioms. First, are they *consistent*? Second, are they *true*? In the most straightforward sense these questions are asked in an *absolute* sense and not relative to a particular theory such as ZFC. But since we are dealing with new axioms, the traditional means of answering such questions—namely, by establishing their consistency or provability relative to the standard axioms (ZFC)—is not available. Nevertheless, one can hope to establish results—such as relative consistency and logical connections with respect to other plausible axioms—that collectively shed light on the original, absolute question. Indeed, there are a number of results that one can bring to bear in favour of PD and $\mathrm{AD}^{L(\mathbb{R})}$. For example, these axioms lift the structure theory that can be established in ZFC to their respective domains, namely, second-order arithmetic and $L(\mathbb{R})$. Moreover, they do so in a fashion which settles a remarkable number of statements that are independent of ZFC. In fact, there is no "natural" statement concerning their respective domains that is known to be independent of these axioms. (For more on the structure theory provided by determinacy and the traditional considerations in their favour see [9] and for more recent work see Jackson's chapter in this Handbook.) The results of this chapter figure in the case for PD and $\mathrm{AD}^{L(\mathbb{R})}$. However, our concern will be with the question of relative consistency; more precisely, we wish to calibrate the consistency strength of axioms of definable determinacy—in particular, the ultimate axioms of lightface and boldface determinacy—in terms of the large cardinal hierarchy.

There are some reductions that we can state at the outset. In terms of consistency strength the two hierarchies collapse at a certain stage: Kechris and Solovay showed that $\mathrm{ZF}+\mathrm{DC}$ implies that in the context of $L[x]$ for $x \in \omega^\omega$, OD-determinacy and Δ^1_2-determinacy are equivalent (see Theorem 6.6). And it is a folklore result that $\mathrm{ZFC}+\mathrm{OD}(\mathbb{R})$-determinacy and $\mathrm{ZFC}+\mathrm{AD}^{L(\mathbb{R})}$ are equiconsistent. Thus, in terms of consistency strength, the lightface hierarchy collapses at Δ^1_2-determinacy and the boldface hierarchy collapses at $\mathrm{AD}^{L(\mathbb{R})}$. So if one wishes to gauge the consistency strength of lightface and boldface determinacy it suffices to concentrate on Δ^1_2-determinacy and $\mathrm{AD}^{L(\mathbb{R})}$.

Now, it is straightforward to see that if Δ^1_2-determinacy holds then it holds in $L[x]$ for some real x and likewise if $\mathrm{AD}^{L(\mathbb{R})}$ (or AD) holds then it holds in $L(\mathbb{R})$. Thus, the natural place to study the consistency strength of lightface definable determinacy is $L[x]$ for some real x and the natural place to study the consistency strength of boldface definable determinacy (or full determinacy) is $L(\mathbb{R})$. For this reason these two models will be central in what follows.

To summarize: We shall be investigating the consistency strength of lightface and boldface determinacy. This reduces to Δ^1_2-determinacy and $\mathrm{AD}^{L(\mathbb{R})}$. The settings $L[x]$ and $L(\mathbb{R})$ will play a central role. Consistency strength will be measured in terms of the large cardinal hierarchy. Before turning to a discussion of the large cardinal hierarchy let us first briefly discuss stronger forms of determinacy.

Our concern in this chapter is with axioms of determinacy of the above form, where the games have length ω and the moves are natural numbers. However, it is worthwhile to note that there are two directions in which one can generalize these axioms.

First, one can consider games of length greater than ω (where the moves are still natural numbers). A simple argument shows that one cannot have the determinacy of all games of length ω_1 but there is a great deal of room below this upper bound and much work has been done in this area. For more on this subject see [10].

Second, one can consider games where the moves are more complex than natural numbers (and where the length of the game is still ω). One alternative is to consider games where the moves are real numbers. The axiom $\mathrm{AD}_{\mathbb{R}}$ states that all such games are determined. One might try to continue in this direction and consider the axiom $\mathrm{AD}_{\mathscr{P}(\mathbb{R})}$ asserting the determinacy of all games where the moves are sets of real numbers. It is straightforward to see that this axiom is inconsistent. In fact, even the definable version asserting that all OD subsets of $\mathscr{P}(\mathbb{R})^\omega$ is inconsistent. Another alternative is to consider games where the moves are ordinal numbers. Again, a simple argument shows that one cannot have the determinacy of all subsets of ${\omega_1}^\omega$. However, a result of Harrington and Kechris shows that in this case if one adds a definability constraint then one can have determinacy at this level. In fact, OD-determinacy implies that every OD set $A \subseteq {\omega_1}^\omega$ is determined. It is natural then to extend this to large ordinals. The ultimate axiom in this direction would simply assert that every OD set $A \subseteq \mathrm{On}^\omega$ is determined. Perhaps surprisingly, at this stage a certain tension arises since recent work in inner model theory provides evidence that this axiom is in fact inconsistent. See [12] for more on this subject.

B. Large Cardinals

Our aim is to calibrate the consistency strength of lightface and boldface determinacy in terms of the large cardinal hierarchy. The importance of the large cardinal hierarchy in this connection is that it provides a canonical means of climbing the hierarchy of consistency strength. To show, for a given hypothesis φ and a given large cardinal axiom L, that the theories $\mathrm{ZFC} + \varphi$ and $\mathrm{ZFC} + L$ are equiconsistent one typically uses the dual methods of *inner model theory* and *outer model theory* (also known as *forcing*). Very roughly, given a model of $\mathrm{ZFC} + L$ one forces to obtain a model of $\mathrm{ZFC} + \varphi$ and given a model of $\mathrm{ZFC} + \varphi$ one uses the method of inner model theory to construct a model of $\mathrm{ZFC} + L$. The large cardinal hierarchy is (for the most part) naturally well-ordered and it is a remarkable phenomenon that given any two "natural" theories extending ZFC one can compare them in terms of consistency strength (equivalently, interpretability) by lining them up with the large cardinal hierarchy.

In a very rough sense large cardinal axioms assert that there are "large" levels of the universe. A template for formulating a broad class of large cardinal axioms is in terms of elementary embeddings. The basic format of the template is as follows: There is a transitive class M and a non-trivial elementary

$$j : V \to M.$$

To say that the embedding is non-trivial is simply to say that it is not the identity, in which case one can show that there is a least ordinal moved. This ordinal is denoted $\mathrm{crit}(j)$ and called the *critical point* of j. A cardinal is said to be a *measurable cardinal* if and only if it is the critical point of such an embedding.

It is easy to see that for any such elementary embedding there is necessarily a certain degree of agreement between V and M. In particular, it follows that $V_{\kappa+1} \subseteq M$, where $\kappa = \mathrm{crit}(j)$. This degree of agreement in conjunction with the elementarity of j can be used to show that κ has strong reflection properties, in particular, κ is strongly inaccessible, Mahlo, weakly compact, etc.

One way to strengthen a large cardinal axiom of the above form is to demand greater agreement between M and V. For example, if one demands that $V_{\kappa+2} \subseteq M$ then the fact that κ is measurable is recognized within M and hence it follows that M satisfies that there is a measurable cardinal below $j(\kappa)$, namely, κ. Thus, by the elementarity of the embedding, V satisfies that there is a measurable cardinal below κ. The same argument shows that there are arbitrarily large measurable cardinals below κ.

This leads to a natural progression of increasingly strong large cardinal axioms. It will be useful to discuss some of the major axioms in this hierarchy: If κ is a cardinal and $\eta > \kappa$ is an ordinal then κ is *η-strong* if there is a transitive class M and a non-trivial elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \eta$ and $V_\eta \subseteq M$. A cardinal κ is *strong* iff it is η-strong for all η. As we saw above if κ is $(\kappa+2)$-strong then κ is measurable and there are arbitrarily large measurable cardinals below κ. Next, one can demand that the embedding preserve certain classes: If A is a class, κ is a cardinal, and $\eta > \kappa$ is an ordinal then κ is *η-A-strong* if there exists a $j : V \to M$ which witnesses that κ is η-strong and which has the additional feature that $j(A \cap V_\kappa) \cap V_\eta = A \cap V_\eta$. The following large cardinal notion will play a central role in this chapter.

1.1 Definition. A cardinal κ is a *Woodin cardinal* if κ is strongly inaccessible and for all $A \subseteq V_\kappa$ there is a cardinal $\kappa_A < \kappa$ such that

$$\kappa_A \text{ is } \eta\text{-}A\text{-strong},$$

for each η such that $\kappa_A < \eta < \kappa$.

It should be noted that in contrast to measurable and strong cardinals, Woodin cardinals are not characterized as the critical point of an embedding or collection of embeddings. In fact, a Woodin cardinal need not be measurable. However, if κ is a Woodin cardinal, then V_κ is a model of ZFC and from the point of view of V_κ there is a proper class of strong cardinals.

Going further, a cardinal κ is *superstrong* if there is a transitive class M and a non-trivial elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. If κ is superstrong then κ is a Woodin cardinal and there are arbitrarily large Woodin cardinals below κ.

One can continue in this vein, demanding greater agreement between M and V. The ultimate axiom in this direction would, of course, demand that $M = V$. This axiom was proposed by Reinhardt. But shortly after its introduction Kunen showed that it is inconsistent with ZFC. In fact, Kunen showed that assuming ZFC, there can be no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. (An interesting open question is whether these axioms are inconsistent with ZF or whether there is a hierarchy of "choiceless" large cardinal axioms that climb the hierarchy of consistency strength far beyond what can be reached with ZFC.)

There is a lot of room below the above upper bound. For example, a very strong axiom is the statement that there is a non-trivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$. The strongest large cardinal axiom in the current literature is the axiom asserting that there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\mathrm{crit}(j) < \lambda$. Surprisingly, this axiom yields a structure theory of $L(V_{\lambda+1})$ which is closely analogous to the structure theory of $L(\mathbb{R})$ under the axiom $\mathrm{AD}^{L(\mathbb{R})}$. This parallel between axioms of determinacy and large cardinal axioms suggests seeking stronger large cardinal axioms by following the guide of the strong axioms of determinacy discussed at the close of the previous section. In fact, there is evidence that the parallel extends. For example, there is a new large cardinal axiom that is the analogue of $\mathrm{AD}_{\mathbb{R}}$. See [12] for more on these recent developments.

C. Determinacy from Large Cardinals

Let us return to the questions of the truth and the consistency of axioms of definable determinacy, granting that of large cardinal axioms. In the late 1960s Solovay conjectured that $\mathrm{AD}^{L(\mathbb{R})}$ is provable from large cardinal axioms (and hence that ZF+AD is consistent relative to large cardinal axioms). This conjecture was realized in stages.

In 1970 Martin showed that if there is a measurable cardinal then all $\undertilde{\Sigma}^1_1$ sets of reals are determined. Later, in 1978, he showed that under the much stronger assumption of a non-trivial iterable elementary embedding $j : V_\lambda \to V_\lambda$ all $\undertilde{\Sigma}^1_2$ sets of reals are determined. It appeared that there would be a long march up the hierarchy of axioms of definable determinacy. However, in 1984 Woodin showed that if there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$, then $\mathrm{AD}^{L(\mathbb{R})}$ holds.

The next major advances concerned reducing the large cardinal hypothesis used to obtain $\mathrm{AD}^{L(\mathbb{R})}$. The first step in this direction was made shortly after, in 1985, when Martin and Steel proved the following remarkable result, using a completely different technique:

1.2 Theorem (Martin and Steel). *Assume* ZFC. *Suppose that there are n Woodin cardinals with a measurable cardinal above them all. Then $\undertilde{\Sigma}^1_{n+1}$-determinacy holds.*

It follows that if there is a Woodin cardinal with a measurable cardinal above, then Δ^1_2-determinacy holds and if there are infinitely many Woodin cardinals then PD holds. Finally, the combination of Martin and Steel's work and Woodin's work on the stationary tower (see [8]) led to a significant reduction in the hypothesis required to obtain $\mathrm{AD}^{L(\mathbb{R})}$.

1.3 Theorem. *Assume* ZFC. *Suppose there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then* $\mathrm{AD}^{L(\mathbb{R})}$.

A more recent development is that, in addition to being implied by large cardinal axioms, $\mathrm{AD}^{L(\mathbb{R})}$ is implied by a broad array of other strong axioms, which have nothing to do with one another—in fact, there is reason to believe that $\mathrm{AD}^{L(\mathbb{R})}$ is implied by *all* sufficiently strong "natural" theories. For further discussion of this subject and other more recent results that contribute to the case for certain axioms of definable determinacy see [7, 11, 13].

Each of the above results concerns the *truth* of axioms of definable determinacy, granting large cardinal axioms. A closely related question concerns the *consistency* of axioms of definable determinacy, granting that of large cardinal axioms. For this one can get by with slightly weaker large cardinal assumptions.

1.4 Theorem. *Assume* ZFC. *Suppose δ is a Woodin cardinal. Suppose $G \subseteq \mathrm{Col}(\omega, \delta)$ is V-generic. Then $V[G] \models \Delta^1_2$-determinacy.*

1.5 Theorem. *Assume* ZFC. *Suppose that λ is a limit of Woodin cardinals. Suppose $G \subseteq \mathrm{Col}(\omega, {<}\lambda)$ is V-generic and let $\mathbb{R}^* = \bigcup\{\mathbb{R}^{V[G\restriction\alpha]} \mid \alpha < \lambda\}$. Then $L(\mathbb{R}^*) \models \mathrm{AD}$.*

For more on the topic of this section see Neeman's chapter in this Handbook.

D. Large Cardinals from Determinacy

The above results lead to the question of whether the large cardinal assumptions are "necessary". Of course, large cardinal assumptions (in the traditional sense of the term) cannot be necessary in the strict sense since axioms of definable determinacy (which concern sets of reals) do not outright imply the existence of large cardinals (which are much larger objects). The issue is whether they are necessary in the sense that one cannot prove the axioms of definable determinacy with weaker large cardinal assumptions. To establish this one must show that the consistency of the axioms of definable determinacy implies that of the large cardinal axioms and one way to do this is to show that axioms of definable determinacy imply that there are *inner models* of the large cardinal axioms.

There are two approaches to inner model theory, each originating in the work of Gödel. These approaches have complementary advantages and disadvantages. The first approach is based on L, the universe of constructible sets. The advantage of this approach is that L is very well understood; in fact, it is fair to say that within ZFC one can carry out a "full analysis" of this model. As a consequence of this one can show, for example, that under ZF + AD, ω_1^V is inaccessible in L. The disadvantage is that L is of limited applicability since it cannot accommodate strong large cardinal axioms such as the statement that there is a measurable cardinal. So if the large cardinal assumptions in Theorems 1.4 and 1.5 are close to optimal then L is of no use in establishing this.

The second approach is based on HOD, the universe of hereditarily ordinal definable sets. This inner model can accommodate virtually all large cardinal axioms that have been investigated. But it has a complementary defect in that one cannot carry out a full analysis of this structure within ZFC.

A major program in set theory—*the inner model program*—aims to combine the advantages of the two approaches by building inner models that resemble L in having a highly ordered inner structure but which resemble HOD in that they can accommodate strong large cardinal axioms.

"L-like" inner models at the level of Woodin cardinals were developed in stages beginning with work of Martin and Steel, and continuing with work of Mitchell and Steel. The Mitchell-Steel inner models are true analogs of L. Martin and Steel used their models to show that the large cardinal hypotheses in their proofs of determinacy were essentially optimal. For example, they showed that if there is a Woodin cardinal then there is a canonical inner model M that contains a Woodin cardinal and has a Δ^1_3 well-ordering of the reals. It follows that one cannot prove $\underset{\sim}{\Sigma}^1_2$-determinacy from the assumption of a Woodin cardinal alone.

However, this still left open a number of questions. First, does the consistency of ZFC + "There is a Woodin cardinal" follow from that of ZFC + Δ^1_2-determinacy? Second, can one build an inner model of a Woodin cardinal directly from ZFC + Δ^1_2-determinacy? Third, what is the strength of ZFC + $\mathrm{AD}^{L(\mathbb{R})}$? To approach these questions it would seem that one would need *fine-structural* inner model theory. However, at the time when the central results of this chapter were proved, fine-structural inner model theory had not yet reached the level of Woodin cardinals. One option was to proceed with HOD.

In contrast to L the structure of HOD is closely tied to the universe in which it is constructed. In the general setting, where one works in ZF and constructs HOD in V, the structure theory of HOD is almost as intractable as that of V. Surprisingly if one strengthens the background theory then the structure theory of HOD becomes tractable. For example, Solovay showed that under ZF + AD, HOD satisfies that ω_1^V is a measurable cardinal. It turns out that both lightface and boldface definable determinacy are able to illuminate the structure of HOD (when constructed in the natural inner models of these axioms—$L[x]$ and $L(\mathbb{R})$) to the point where one can recover the large cardinals that are necessary to establish their consistency.

In the case of lightface definable determinacy the result is the following:

1.6 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone of* x*,*

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

Thus, the consistency strength of ZFC + OD-determinacy is precisely that of ZFC + "There is a Woodin cardinal". For the case of boldface determinacy let us first state a preliminary result of which the above result is a localization. First we need a definition. Let

$$\Theta = \sup\{\alpha \mid \text{there is a surjection } \pi : \omega^\omega \to \alpha\}.$$

1.7 Theorem. *Assume* ZF + AD*. Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

In fact, both of these results are special instances of a general theorem on the generation of Woodin cardinals—the Generation Theorem. In addition to giving the above results, the Generation Theorem will also be used to establish the optimal large cardinal lower bound for boldface determinacy:

1.8 Theorem. *Assume* ZF + AD*. Suppose* Y *is a set. There is a generalized Prikry forcing* $\mathbb{P}_Y$ *through the* Y*-degrees such that if* $G \subseteq \mathbb{P}_Y$ *is* V*-generic and* $\langle [x_i]_Y \mid i<\omega\rangle$ *is the associated sequence, then*

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle,V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$ *is the* Y*-degree of* x*.*

Thus, the consistency strength of ZFC + OD($\mathbb{R}$)-determinacy and of ZF + AD is precisely that of ZFC + "There are ω-many Woodin cardinals".

The main results of this chapter have applications beyond equiconsistency; in particular, the theorems play an important role in the structure theory of AD^+ (a potential strengthening of AD that we will define and discuss in Sect. 8). For example, Steel showed that under AD, in $L(\mathbb{R})$ every uncountable regular cardinal below Θ is a measurable cardinal. (See Steel's chapter in this Handbook for a proof.) This theorem generalizes to a theorem of AD^+ and the theorems of this chapter are an important part of the proof. We will discuss some other applications in the final section of this chapter.

E. Overview

The results on the strength of lightface and boldface determinacy were established in the late 1980s. However, the current presentation and many of the results that follow are quite recent. One of the key new ingredients is the following abstract theorem on the generation of Woodin cardinals, which lies at the heart of this chapter:

1.9 Theorem (Generation Theorem). *Assume* ZF. *Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$,

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$,

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α,

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models$ *Strategic determinacy with respect to* B.

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

Here T_0 is the theory $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) - \text{Power Set} + \text{“}\mathscr{P}(\omega)\text{exists”}$ and the notion of “strategic determinacy” is a technical notion that we will state precisely later.

The Generation Theorem provides a template for generating models containing Woodin cardinals. One simply has to show that in a particular setting the various conditions can be met, though this is often a non-trivial task. The theorem is also quite flexible in that it is a result of ZF that does not presuppose DC and has applications in both lightface and boldface settings. It will play a central role in the calibration of the strength of both lightface and boldface determinacy.

We shall approach the proof of the Generation Theorem by proving a series of increasingly complex approximations.

In Sect. 2 we take the initial step by proving Solovay's theorem that under $\mathrm{ZF} + \mathrm{AD}$, ω_1^V is a measurable cardinal in HOD and we show that the associated measure is normal. The proof that we give is slightly more complicated than the standard proof but has the virtue of illustrating in a simple setting some of the key components that appear in the more complex variations. We illustrate this at the end of the section by showing that the proof of Solovay's theorem generalizes to show that under $\mathrm{ZF} + \mathrm{AD}$, the ordinal $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. Our main aim in this section is to illustrate the manner in which "boundedness" and "coding" combine to yield normal ultrafilters. In subsequent sections stronger forms of boundedness (more precisely, "reflection") and stronger forms of coding will be used to establish stronger forms of normality.

In Sect. 3 we prove the strong forms of coding that will be central throughout.

In Sect. 4, as a precursor to the proof of the Generation Theorem, we prove the following theorem:

1.10 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

The assumption of DC is merely provisional—it will ultimately be eliminated when we prove the Generation Theorem. Toward the proof of the above theorem, we begin in Sect. 4.1 by establishing the reflection phenomenon that will play the role played by boundedness in Sect. 2. We will then use this reflection phenomenon in $L(\mathbb{R})$ to define for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$ that is intended to witness that $\underset{\sim}{\delta}^2_1$ is λ-strong. In Sect. 4.2 we shall introduce and motivate the notion of *strong normality* by showing that the strong normality of μ_λ ensures that $\underset{\sim}{\delta}^2_1$ is λ-strong. We will then show how reflection and uniform coding combine to secure strong normality. In Sect. 4.3 we will prove the above theorem by relativizing the construction of Sect. 4.2 to subsets of $\Theta^{L(\mathbb{R})}$.

In Sect. 5 we extract the essential components of the above construction and prove two abstract theorems on Woodin cardinals in a general setting, one that involves DC and one that does not. The first theorem is proved in Sect. 5.1. The importance of this theorem is that it can be used to show that in certain strong determinacy settings HOD can contain many Woodin cardinals. The second theorem is the Generation Theorem, the proof of which will occupy the remainder of the section. The aim of the Generation Theorem is to show that the construction of Sect. 4 can be driven by lightface determinacy alone. The difficulty is that the construction of Sect. 4 involves games that are defined in terms of real parameters. To handle this we introduce the notion of "strategic determinacy", a notion that resembles boldface determinacy in that it involves real parameters but which can nonetheless hold in settings where one has AC. To motivate the notion of "strategic determinacy" we shall begin in Sect. 5.2 by examining one such setting, namely, $L[S, x]$ where S is a class of ordinals and x is a real. Once we show that "strategic determinacy" can hold in this setting we shall return in Sect. 5.3 to the general setting and prove the Generation Theorem. In the final subsection, we prove a number of special cases, many of which are new. Although some of these applications involve lightface settings, they all either involve assuming full AD or explicitly involve "strategic determinacy".

In Sect. 6 we use two of the special cases of the Generation Theorem to calibrate the consistency strength of lightface and boldface definable determinacy in terms of the large cardinal hierarchy. In the case of the first result the main task is to show that Δ^1_2-determinacy suffices to establish that "strategic determinacy" can hold. In the case of the second result the main task is to show that the Generation Theorem can be iteratively applied to generate ω-many Woodin cardinals.

In Sect. 7 we show that the Generation Theorem can itself be localized in two respects. In the first localization we show that Δ^1_2-determinacy implies that for a Turing cone of x, $\omega_1^{L[x]}$ is a Woodin cardinal in an inner model of $L[x]$. In the second localization we show that the proof can in fact be carried out in second-order arithmetic.

In Sect. 8 we survey some further results. First, we discuss results concerning the actual equivalence of axioms of definable determinacy and axioms asserting the existence of inner models with Woodin cardinals. Second, we revisit the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ and $\mathrm{HOD}^{L[x][g]}$, for certain generic extensions $L[x][g]$, in light of the advances that have been made in fine-structural inner model theory. Remarkably, it turns out that not only are these models well-behaved in the context of definable determinacy—they are actually fine-structural inner models, but of a kind that falls outside of the traditional hierarchy.

We have tried to keep the account self-contained, presupposing only acquaintance with the constructible universe, the basics of forcing, and the basics of large cardinal theory. In particular, we have tried to minimize appeal to fine structure and descriptive set theory. Fine structure enters only in Sect. 8 where we survey more recent developments, but even there one should be able to get a sense of the lay of the land without following the details. For the relevant background and historical development of the subject see [1, 2, 9].

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1.2. Notation

For the most part our notational conventions are standard. Nevertheless, some comments are in order.

(1) We use $\mu\alpha\,\varphi(\alpha)$ to indicate the least ordinal α such that $\varphi(\alpha)$.

(2) In writing OD_X and HOD_X we always mean that X itself (as opposed to its elements) is allowed as a parameter. The notation $\mathrm{OD}_{\{X\}}$ is

sometimes used for this, for example, in contexts where one would like to speak of both $\mathrm{OD}_{\{X\}}$ and OD_X. However, in this chapter we will have no occasion to speak of the latter and so we have dropped the curly brackets on the ground that they would only serve to clutter the text. We also use OD_X as both a name (for the class of sets which are ordinal definable from X) and as an adjective (for example when we say that a particular class is OD_X.) We use $<_{\mathrm{OD}_X}$ for a fixed canonical OD_X-well-ordering of OD_X sets. The notation $\mathrm{OD}(\mathbb{R})$ is used in analogy with $L(\mathbb{R})$.

(3) A *strategy* for Player I is a function $\sigma : \bigcup_{i<\omega} \omega^{2i} \to \omega$. Letting $\sigma * y$ be the real produced when Player I follows σ and Player II plays y, we say that σ is a *winning strategy* for Player I in the game with payoff $A \subseteq \omega^\omega$ if for all $y \in \omega^\omega$, $\sigma * y \in A$. The corresponding notions for Player II are defined similarly. We typically reserve σ for strategies for Player I and τ for strategies for Player II. The play that results from having II play y against σ is denoted $\sigma * y$ and likewise the play that results from having I play x against τ is denoted $x * \tau$. We write $x * y$ for the real that results from having Player I play x and Player II play y and in this case we let $(x * y)_I = x$ and $(x * y)_{II} = y$. For example, $(\sigma * y)_I$ is the real that Player I plays when following the strategy σ against II's play of y. If σ is a strategy for Player I and τ is a strategy for Player II we write $\sigma * \tau$ for the real produced by playing the strategies against one another. Occasionally, when $z = x * y$ we write z_{even} to indicate x and z_{odd} to indicate y.

(4) If X is a subset of the plane $\omega^\omega \times \omega^\omega$ we use $\mathrm{proj}_1(X)$ for the "projection to the first coordinate" and $\mathrm{proj}_2(X)$ for the "projection to the second coordinate".

(5) For $n_0, \ldots, n_{k-1} \in \omega$, we use $\langle n_0, \ldots, n_{k-1} \rangle$ to denote the natural number encoding $(n_0, \ldots, n_{k-1})$ via a recursive bijection between ω^k and ω (which we fix throughout) and we let $(n)_i$ be the associated projection functions. For $x \in \omega^\omega$ and $i \in \omega$ we also use $(x)_i$ for the projection function associated to a recursive bijection between $(\omega^\omega)^\omega$ and ω^ω. See [9, Chap. 3] for further details on such recursive coding and decoding functions.

There is a slight conflict in notation in that angle brackets are also traditionally used for sequences and n-tuples. We have lapsed into this usage at points but the context resolves the ambiguity; for example, when we write $\langle x_\alpha \mid \alpha < \lambda \rangle$ it is clear that we are referring to a sequence.

(6) In this chapter by the "reals" we mean ω^ω, which, under the standard topology, is homeomorphic to the irrationals as normally construed. However, we continue to use the symbol '$\mathbb{R}$' in contexts where it is traditional, for example, in $L(\mathbb{R})$.

(7) We use $\mathrm{tc}(x)$ for the transitive closure of x.

(8) A base theory that will play a central role throughout is

$$\mathrm{T}_0 = \mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) - \text{Power Set} + \text{“}\mathscr{P}(\omega) \text{ exists”}.$$

2. Basic Results

The central result of this section is Solovay's theorem to the effect that under ZF + AD, ω_1 is a measurable cardinal. The proof that we will give is slightly more involved than the standard proof but it has the advantage of illustrating some of the key components in the more general theorems to be proved in later sections. One thing we would like to illustrate is the manner in which "boundedness" and "coding" combine to yield normal ultrafilters. In subsequent sections stronger forms of boundedness (more precisely, "reflection") and stronger forms of coding will be used to establish stronger forms of normality. This will culminate in the production of Woodin cardinals.

In Sect. 2.1 we review some basic consequences of ZF + AD. In Sect. 2.2 we prove $\utilde{\Sigma}^1_1$-boundedness and use it to prove the Basic Coding Lemma, a simple case of the more general coding lemmas to be proved in Sect. 3. In Sect. 2.3 we use $\utilde{\Sigma}^1_1$-boundedness to show that the club filter on ω_1 witnesses that ω_1 is a measurable cardinal and we use $\utilde{\Sigma}^1_1$-boundedness and the Basic Coding Lemma to show that this ultrafilter is normal. In Sect. 2.4 we introduce $\utilde{\delta}^2_1$ and establish its basic properties. Finally, in Sect. 2.5 we draw on the Coding Lemma of Sect. 3 to show that the proof of Solovay's theorem generalizes to show that assuming ZF + DC + AD then in the restricted setting of $L(\mathbb{R})$ the ordinal $(\utilde{\delta}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal. Later, in Sect. 4, we will dispense with DC and reprove this theorem in ZF + AD.

2.1. Preliminaries

In order to keep this account self-contained, in this subsection we shall collect together some of the basic features of the theory of determinacy. These concern (1) the connection between determinacy and choice, (2) the implications of determinacy for regularity properties, and (3) the implications of determinacy for the Turing degrees. See [2, 9], and Jackson's chapter in this Handbook for further details and references.

Let us begin with the axiom of choice. A straightforward diagonalization argument shows that AD contradicts the full axiom of choice, AC. However, certain fragments of AC are consistent with AD and, in fact, certain fragments of AC follow from AD.

2.1 Definition. The *Countable Axiom of Choice*, AC_ω, is the statement that every countable set consisting of non-empty sets has a choice function.

The *Countable Axiom of Choice for Sets of Reals*, $\mathrm{AC}_\omega(\mathbb{R})$, is the statement that every countable set consisting of non-empty sets of reals has a choice function.

2.2 Theorem. *Assume* ZF + AD. *Then* $\mathrm{AC}_\omega(\mathbb{R})$.

Proof. Let $\{X_n \mid n < \omega\}$ be a countable collection of non-empty sets of reals. Consider the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

where I wins if and only if $y \notin X_{x(0)}$. (Notice that we are leaving the definition of the payoff set of reals A implicit. In this case the payoff set is $\{x \in \omega^\omega \mid x_{\text{odd}} \notin X_{x(0)}\}$. In the sequel we shall leave such routine transformations to the reader.) Thus, Player I is to be thought of as playing an element X_n of the countable collection and Player II must play a real which is not in this element. Of course, Player I cannot win. So there must be a winning strategy τ for Player II. The function

$$\begin{aligned} f : \omega &\to \omega^\omega \\ n &\mapsto (\langle n, 0, 0, \dots \rangle * \tau)_{II} \end{aligned}$$

is a choice function for $\{X_n \mid n < \omega\}$. $\dashv$

2.3 Corollary. *Assume* ZF + AD. *Then* ω_1 *is regular.*

2.4 Definition. The *Principle of Dependent Choices*, DC, is the statement that for every non-empty set X and for every relation $R \subseteq X \times X$ such that for all $x \in X$ there is a $y \in X$ such that $(x, y) \in R$, there is a function $f : \omega \to X$ such that for all $n < \omega$, $(f(n), f(n+1)) \in R$. The *Principle of Dependent Choices for Sets of Reals*, $\mathrm{DC}_\mathbb{R}$, is simply the restricted version of DC where X is $\mathbb{R}$.

It is straightforward to show that DC implies AC_ω and Jensen showed that this implication cannot be reversed. Solovay showed that $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD}_\mathbb{R})$ implies $\mathrm{Con}(\mathrm{AD} + \neg\mathrm{DC})$ and this was improved by Woodin.

2.5 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AD} + V = L(\mathbb{R})$. *Then in a forcing extension there is an inner model of* $\mathrm{AD} + \neg\mathrm{AC}_\omega$.

2.6 Theorem (Kechris). *Assume* ZF + AD. *Then* $L(\mathbb{R}) \models \mathrm{DC}$.

1 Open Question. Does AD imply $\mathrm{DC}_\mathbb{R}$?

Thus, of the above fragments of AC, $\mathrm{AC}_\omega(\mathbb{R})$ is known to be within the reach of AD, $\mathrm{DC}_\mathbb{R}$ could be within the reach of AD, and the stronger principles AC_ω and DC are known to be consistent with but independent of AD (assuming consistency of course). For this reason, to minimize our assumptions, in what follows we shall work with $\mathrm{AC}_\omega(\mathbb{R})$ as far as this is possible. There are two places where we invoke DC, namely, in Kunen's theorem (Theorem 3.11) and in Lemma 4.8 concerning the well-foundedness of certain ultrapowers. However, in our applications, DC will reduce to $\mathrm{DC}_\mathbb{R}$ and so if the above open question has a positive answer then these appeals to DC can also be avoided.

We now turn to regularity properties. The axiom of determinacy has profound consequences for the structure theory of sets of real numbers. See [9] and Jackson's chapter in this Handbook for more on this. Here we mention only one central consequence that we shall need below.

2.7 Theorem (Mycielski-Swierczkowski; Mazur, Banach; Davis). *Assume* $\mathrm{ZF}+\mathrm{AD}$*. Then all sets are Lebesgue measurable, have the property of Baire, and have the perfect set property.*

Proof. See [2, Sect. 27]. ⊣

Another important consequence we shall need is the following:

2.8 Theorem. *Assume* $\mathrm{ZF}+\mathrm{AD}$*. Then every ultrafilter is* ω_1*-complete.*

Proof. Suppose $\mathscr{U} \subseteq \mathscr{P}(X)$ is an ultrafilter. If $\mathscr{U}$ is not ω_1-complete then there exists $\{X_i \mid i<\omega\}$ such that

(1) for all $i<\omega$, $X_i \in \mathscr{U}$ and

(2) $\bigcap_{i<\omega} X_i \notin \mathscr{U}$.

Without loss of generality we can suppose that $\bigcap_{i<\omega} X_i = \varnothing$. So this gives a partition $\{Y_i \mid i<\omega\}$ of X into disjoint non-empty sets each of which is not in $\mathscr{U}$. Define $\mathscr{U}^* \subseteq \mathscr{P}(\omega)$ as follows:

$$\sigma \in \mathscr{U}^* \quad \text{iff} \quad \bigcup\{Y_i \mid i \in \sigma\} \in \mathscr{U}.$$

This is an ultrafilter on ω which is not principal since by assumption $Y_i \notin \mathscr{U}$ for each $i<\omega$. However, as Sierpiński showed, a non-principal ultrafilter over ω (construed as a set of reals) is not Lebesgue measurable. ⊣

Finally, we turn to the implications of determinacy for the Turing degrees. For $x, y \in \omega^\omega$, we say that x is *Turing reducible* to y, $x \leqslant_T y$, if x is recursive in y and we say that x is *Turing equivalent* to y, $x \equiv_T y$, if $x \leqslant_T y$ and $y \leqslant_T x$. The *Turing degrees* are the corresponding equivalence classes $[x]_T = \{y \in \omega^\omega \mid y \equiv_T x\}$. Letting

$$\mathscr{D}_T = \{[x]_T \mid x \in \omega^\omega\}$$

the relation $\leqslant_T$ lifts to a partial ordering on $\mathscr{D}_T$. A *cone of Turing degrees* is a set of the form

$$\{[y]_T \mid y \geqslant_T x_0\}$$

for some $x_0 \in \omega^\omega$. A *Turing cone of reals* is a set of the form

$$\{y \in \omega^\omega \mid y \geqslant_T x_0\}$$

for some $x_0 \in \omega^\omega$. In each case x_0 is said to be the *base* of the cone. In later sections we will discuss different degree notions. However, when we speak of a "cone of x" without qualification we always mean a "Turing cone of x". The *cone filter on* $\mathscr{D}_T$ is the filter consisting of sets of Turing degrees that contain a cone of Turing degrees.

2.9 Theorem (Cone Theorem; Martin). *Assume* ZF + AD. *The cone filter on* $\mathscr{D}_T$ *is an ultrafilter.*

Proof. For $A \subseteq \mathscr{D}_T$ consider the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where I wins iff $[x * y]_T \in A$. If I has a winning strategy σ_0 then σ_0 witnesses that A is in the cone filter since for $y \geqslant_T \sigma_0$, $[y]_T = [\sigma_0 * y]_T \in A$. If II has a winning strategy τ_0 then τ_0 witnesses that $\mathscr{D}_T \smallsetminus A$ is in the cone filter since for $x \geqslant_T \tau_0$, $[x]_T = [x * \tau_0]_T \in \mathscr{D}_T \smallsetminus A$. $\dashv$

It follows that under ZF + AD each statement $\varphi(x)$ either holds for a Turing cone or reals x or fails for a Turing cone of reals x.

The proof of the Cone Theorem easily relativizes to fragments of definable determinacy. For example, assuming Σ^1_2-determinacy every Σ^1_2 set which is invariant under Turing equivalence either contains or is disjoint from a Turing cone of reals.

It is of interest to note that when Martin proved the Cone Theorem he thought that he would be able to refute AD by finding a property that "toggles". He started with Borel sets and, when no counterexample arose, moved on to more complicated sets. We now know (assuming there are infinitely many Woodin cardinals with a measurable above) that no counterexamples are to be found in $L(\mathbb{R})$. Moreover, the statement that there are no counterexamples in $L(\mathbb{R})$ (i.e. the statement that *Turing determinacy* holds in $L(\mathbb{R})$) actually implies $\mathrm{AD}^{L(\mathbb{R})}$ (over ZF + DC). Thus, the basic intuition that the Cone Theorem is strong is correct—it is just not as strong as $0 = 1$.

2.2. Boundedness and Basic Coding

We begin with some definitions. For $x \in \omega^\omega$, let E_x be the binary relation on ω such that mE_xn iff $x(\langle m, n\rangle) = 0$, where recall that $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ is a recursive bijection. The real x is said to *code* the relation E_x. Let $\mathrm{WO} = \{x \in \omega^\omega \mid E_x$ is a well-ordering$\}$. For $x \in \mathrm{WO}$, let α_x be the ordertype of E_x and, for $\alpha < \beta < \omega_1$ let $\mathrm{WO}_\alpha = \{x \in \mathrm{WO} \mid \alpha_x = \alpha\}$, $\mathrm{WO}_{<\alpha} = \{x \in$

$\mathrm{WO} \mid \alpha_x < \alpha\}$, $\mathrm{WO}_{(\alpha,\beta]} = \{x \in \mathrm{WO} \mid \alpha < \alpha_x \leqslant \beta\}$ and likewise for other intervals of countable ordinals. For $x \in \mathrm{WO}$, let $\mathrm{WO}_x = \mathrm{WO}_{\alpha_x}$. It is straightforward to see that these sets are Borel and that WO is a complete $\underset{\sim}{\Pi}^1_1$ set. (See [1, Chap. 25] for details.)

In addition to the topological and recursion-theoretic characterizations of $\underset{\sim}{\Sigma}^1_1$ there is a model-theoretic characterization which is helpful in simplifying complexity calculations. A model (M, E) satisfying T_0 (or some sufficiently strong fragment of ZF) is an *ω-model* if $(\omega^M, E{\restriction}\omega^M) \cong (\omega, \in{\restriction}\omega)$, where recall that T_0 is the theory $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) - \text{Power Set} + \text{“}\mathscr{P}(\omega)\text{ exists”}$. Notice that ω-models are correct about arithmetical statements and hence Π^1_1 statements are downward absolute to ω-models. Moreover, the statement “There exists a real coding an ω-model of T_0” is Σ^1_1, in contrast to the statement “There exists a real coding a well-founded model of T_0”, which is Σ^1_2. Thus we have the following characterization of the pointclass $\underset{\sim}{\Sigma}^1_1$: $A \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1$ iff there is a formula φ and there exists a $z \in \omega^\omega$ such that

$$\begin{aligned} A = \{y \in \omega^\omega \mid{} & \text{there is a real coding an } \omega\text{-model } M \text{ with } z \in M \\ & \text{such that } y \in M \text{ and } M \models \mathrm{T}_0 + \varphi[y, z]\}. \end{aligned}$$

The lightface version Σ^1_1 is defined similarly by omitting the parameter z, as are the Σ^1_1 subsets of $(\omega^\omega)^n$ and the Σ^1_1 statements, etc. Theories much weaker than T_0 yield an equivalent definition. For example, one can use the finite theory ZF_N of the first N axioms of ZF for some sufficiently large N.

As an illustration of the utility of this model-theoretic characterization of $\underset{\sim}{\Sigma}^1_1$ we shall use it to show that for each $x \in \mathrm{WO}$, $\mathrm{WO}_{<x}$ is $\underset{\sim}{\Delta}^1_1$: Notice that ω-models of T_0 correctly compute “$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$” in the following sense: If $x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$ and M is an ω-model of T_0 which contains x and y, then $M \models$ “$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$”. (By downward absoluteness M satisfies that $x, y \in \mathrm{WO}$ and hence that α_y and α_x are defined. Furthermore, since M is an ω-model it correctly computes the ordering of α_x and α_y.) If $x \in \mathrm{WO}$ and M is an ω-model of T_0 which satisfies “$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$” then $y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$. (The point is that M satisfies that there is an order-preserving map $f : E_y \to E_x$ and, since ω-models are correct about such maps and since E_x is truly well-founded, it follows that $y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$). So, for $x \in \mathrm{WO}$,

$$\begin{aligned} \mathrm{WO}_{<x} = \{y \in \omega^\omega \mid{} & \text{there is a real coding an } \omega\text{-model } M \text{ such that} \\ & x, y \in M \text{ and } M \models \mathrm{T}_0 + \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}\} \\ = \{y \in \omega^\omega \mid{} & \text{for all reals coding } \omega\text{-models } M \text{ if } x, y \in M \\ & \text{and } M \models \mathrm{T}_0 \text{ then } M \models \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}\}. \end{aligned}$$

Thus, for $x \in \mathrm{WO}$, $\mathrm{WO}_{<x}$ is $\underset{\sim}{\Delta}^1_1$ and hence Borel.

2.10 Lemma ($\underset{\sim}{\Sigma}^1_1$-Boundedness; Luzin-Sierpiński). *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. *Suppose* $X \subseteq \mathrm{WO}$ *and* X *is* $\underset{\sim}{\Sigma}^1_1$. *Then there exists an* $\alpha < \omega_1$ *such that* $X \subseteq \mathrm{WO}_{<\alpha}$.

Proof. Assume toward a contradiction that X is unbounded. Then

$$y \in \mathrm{WO} \quad \text{iff} \quad \text{there is a } x \in X \text{ such that } \alpha_y < \alpha_x$$

since for $x \in X \subseteq \mathrm{WO}$, ω-models of T_0 correctly compute $\alpha_y < \alpha_x$. By the above remark, we can rewrite this as

$$\begin{aligned} y \in \mathrm{WO} \quad \text{iff} \quad & \text{there is an } x \in X \text{ and there is an } \omega\text{-model } M \text{ such that} \\ & x, y \in M \text{ and } M \models \mathrm{T}_0 + \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}. \end{aligned}$$

Thus, WO is $\underset{\sim}{\Sigma}^1_1$, which contradicts the fact that WO is a complete $\underset{\sim}{\Pi}^1_1$ set. (Without appealing to the fact that WO is a complete $\underset{\sim}{\Pi}^1_1$ set we can arrive at a contradiction (making free use of AC) as follows: Let $z \in \omega^\omega$ be such that both X and WO are $\Sigma^1_1(z)$. Let α be such that $V_\alpha \models \mathrm{T}_0$ and choose $Y \prec V_\alpha$ such that Y is countable and $z \in Y$. Let N be the transitive collapse of Y. By correctness, $X \cap N = X^N$. Choose a uniform ultrafilter $U \subseteq \mathscr{P}(\omega_1)^N$ such that if

$$j : N \to \mathrm{Ult}(N, U)$$

is the associated embedding then $\mathrm{crit}(j) = \omega_1^N$ and $j(\omega_1^N)$ is not well-founded. (To obtain such an ultrafilter build a generic for $(\mathscr{P}(\omega_1)/\text{countable})^N$. See Lemma 22.20 of [1].) Since $\mathrm{Ult}(N, U)$ is an ω-model of T_0 it correctly computes WO. It follows that $(\mathrm{WO})^{\mathrm{Ult}(N,U)} \subseteq \mathrm{WO}$, which in turn contradicts the fact that $\omega_1^{\mathrm{Ult}(N,U)}$ is not well-founded.) ⊣

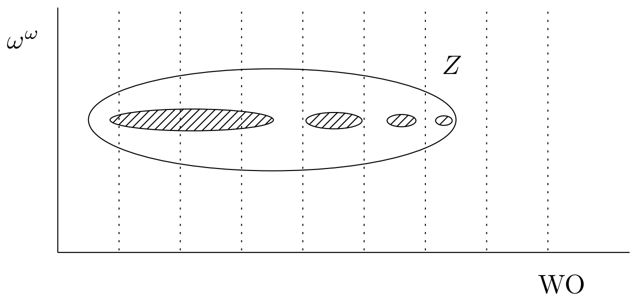
2.11 Lemma (Basic Coding; Solovay). *Assume* ZF + AD. *Suppose* $Z \subseteq \mathrm{WO} \times \omega^\omega$. *Then there exists a* $\underset{\sim}{\Sigma}^1_2$ *set* Z^* *such that*

(1) $Z^* \subseteq Z$ *and*

(2) for all $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ *iff* $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

Moreover, there is such a Z^* *which is of the form* $X \cap (\mathrm{WO} \times \omega^\omega)$ *where* $X \subseteq \omega^\omega \times \omega^\omega$ *is* $\underset{\sim}{\Sigma}^1_1$.

Proof. Here is the picture:



The space $\mathrm{WO} \times \omega^\omega$ is sliced into sections $\mathrm{WO}_\alpha \times \omega^\omega$ for $\alpha < \omega_1$. Z is represented by the unshaded ellipse and Z^* is represented by the shaded region. Basic Coding says that whenever Z meets one of the sections so does Z^*. In such a situation we say that Z^* is a *selector* for Z.

To see that Z^* exists, consider the game

$$\begin{array}{cccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

where II wins iff whenever $x \in \mathrm{WO}$ then y codes a countable set Y such that

(1) $Y \subseteq Z$ and

(2) for all $\alpha \leqslant \alpha_x$, $Y \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

The idea is that Player I challenges by playing a countable ordinal α_x and Player II meets this challenge provided he can play (a code for a) a selector Y for $Z \cap (\mathrm{WO}_{\leqslant \alpha_x} \times \omega^\omega)$.

Claim. *There can be no winning strategy for Player I in this game.*

Proof. Suppose σ is a winning strategy for I. As the play unfolds, Player I can attempt to increase α_x as Player II's play is revealed. However, Player II can anticipate all such attempts as follows: The set

$$X = \{(\sigma * y)_I \mid y \in \omega^\omega\}$$

is $\Sigma^1_1(\sigma)$ and, since σ is winning for I, $X \subseteq \mathrm{WO}$. So, by $\underset{\sim}{\Sigma}^1_1$-boundedness, there is a $\beta < \omega_1$ such that $X \subseteq \mathrm{WO}_{<\beta}$. Since we have $\mathrm{AC}_\omega(\mathbb{R})$ (by Theorem 2.2), we can choose a countable set $Y \subseteq Z$ such that for all $\alpha < \beta$, $Y \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$. Let y code Y and play y against σ. The resulting play $\sigma * y$ is a win for II, which is a contradiction. ⊣

Thus II has a winning strategy τ. For $x \in \mathrm{WO}$, let Y^x be the countable subset of Z coded by $(x * \tau)_{II}$. Then

$$Z^* = \bigcup \{Y^x \mid x \in \mathrm{WO}\}$$

is $\Sigma^1_2(\tau)$ and such that

(1) $Z^* \subseteq Z$,

(2) for all $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

Hence Z^* is as desired.

To see that we can choose Z^* to be of the form $X \cap (\mathrm{WO} \times \omega^\omega)$ where $X \subseteq \omega^\omega \times \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1$, let

$$\begin{aligned} X = \{(a, b) \mid {} & \text{there is an } \omega\text{-model } M \\ & \text{such that } a, b, \tau \in M \text{ and } M \models \mathrm{T}_0 + (a, b) \in Z^{**}\} \end{aligned}$$

where

$$Z^{**} = \bigcup \{Y^x \cap (\mathrm{WO}_{\alpha_x} \times \omega^\omega) \mid x \in \mathrm{WO}\}.$$

This set is $\Sigma^1_1(\tau)$. The trouble is that although for $a \in \mathrm{WO}$ such models M are correct about $(a, b) \in Z^{**}$, M might think $a \in \mathrm{WO}$ when $a \notin \mathrm{WO}$. To overcome this difficulty we pare down, letting $Z^* = X \cap (\mathrm{WO} \times \omega^\omega)$. ⊣

2.3. Measurability

2.12 Theorem (Solovay). *Assume* ZF + AD. *Then the club filter is an ω_1-complete ultrafilter on ω_1.*

Proof. The ultrafilter on ω_1 will be extracted from a game. As motivation, for the moment work in ZFC. For $S \subseteq \omega_1$, consider the game

$$\begin{array}{lccccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \alpha_2 & & \dots \\ \mathrm{II} & & \beta_0 & & \beta_1 & & \dots & \end{array}$$

where we demand that $\alpha_0 < \beta_0 < \alpha_1 < \cdots < \omega_1$ and where the first player that fails to meet this demand loses and if both players meet the demand then I wins provided $\sup_{i<\omega} \alpha_i \in S$.

We claim that I wins this game for S if and only if S contains a club in ω_1. Suppose first that S contains a club C. Let σ be a strategy for I which chooses an element of C larger than the last ordinal played by II. This is a winning strategy for I. For if II meets the first condition then the ordinals played form an increasing sequence. The even elements of this sequence are in C and hence the supremum of the sequence is in C (since C is club) and hence in S. Thus σ is a winning strategy for I. Suppose next that I have a winning strategy σ. Let C be the set of limit ordinals $\gamma < \omega_1$ with the feature that for every $i < \omega$ and for every increasing sequence $\xi_0, \dots, \xi_{2i}$ of ordinals less than γ, the response $\sigma(\langle \xi_0, \dots, \xi_{2i} \rangle)$ is also less than γ. Let C' be the limit points of C. Since ω_1 is regular it follows that C and C' are club in ω_1. Now suppose $\gamma \in C'$. Let $\langle \gamma_i \mid i < \omega \rangle$ be an increasing sequence of ordinals in C which is cofinal in γ and such that γ_0 is greater than I's first move via σ. The key point is that this sequence is a legal play for II. Player II has "taken control" of the game. Now, since σ is a winning strategy for I it follows that the supremum, γ, is in S. Thus, S contains the club C'. So, if we had determinacy (which of course is impossible in ZFC) we would have an ultrafilter on ω_1.

Now return to ZF + AD. We want to mimic the above game via a game where each player plays natural numbers. This can be done since in an integer game each player ultimately plays a real x that can be regarded as coding ω-many reals $(x)_i$ each of which (potentially) codes a countable ordinal. More precisely, for $S \subseteq \omega_1$, let $G(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

with the following rules: Rule 1: For all $i < \omega$, $(x)_i, (y)_i \in \mathrm{WO}$. If Rule 1 is violated then, letting i be least such that either $(x)_i \notin \mathrm{WO}$ or $(y)_i \notin \mathrm{WO}$, I wins if $(x)_i \in \mathrm{WO}$; otherwise II wins. Now suppose Rule 1 is satisfied. Rule 2: $\alpha_{(x)_0} < \alpha_{(y)_0} < \alpha_{(x)_1} < \alpha_{(y)_1} < \cdots$. The first failure defines who wins as above. If both rules are satisfied then I wins if and only if $\sup_{i<\omega} \alpha_{(x)_i} \in S$.

Now let

$$\mu = \{S \subseteq \omega_1 \mid \mathrm{I} \text{ wins } G(S)\}.$$

We claim that if I has a winning strategy in $G(S)$ then S contains a club: Let σ be a winning strategy for I. For $\alpha < \omega_1$, let

$$X_\alpha = \big\{((\sigma * y)_I)_n \mid n < \omega,\ y \in \omega^\omega,\ \text{and} \\ \forall i < n \big((y)_i \in \mathrm{WO} \text{ and } \alpha_{(y)_i} < \alpha\big)\big\}.$$

Notice that each $X_\alpha \subseteq \mathrm{WO}$ (since X_α is $\underset{\sim}{\Sigma}^1_1$ (in σ and the code for α) and σ is a winning strategy) and so by $\underset{\sim}{\Sigma}^1_1$-boundedness, there exists an α' such that $X_\alpha \subseteq \mathrm{WO}_{<\alpha'}$. Let $f : \omega_1 \to \omega_1$ be the function which given α chooses the least α' such that $X_\alpha \subseteq \mathrm{WO}_{<\alpha'}$. As before let C be the set of limit ordinals $\gamma < \omega_1$ with the feature that for every $\xi < \gamma$, $f(\xi) < \gamma$. Let C' be the limit points of C. Since ω_1 is regular (by Corollary 2.3) it follows that C and C' are club in ω_1. Now suppose $\gamma \in C'$. Let $\langle \gamma_i \mid i < \omega\rangle$ be an increasing sequences of ordinals in C which is cofinal in γ. Let $y \in \omega^\omega$ be such that for all $i < \omega$, $\alpha_{(y)_i} = \gamma_i$. We claim that playing y against σ witnesses that $\gamma \in S$. It suffices to show that y is legal with respect to the two rules. For then the supremum, γ, must be in S since σ is a winning strategy for I. Now the first rule is trivially satisfied since we chose y such that for all $i < \omega$, $(y)_i \in \mathrm{WO}$. To see that the second rule is satisfied we need to see that for each $i < \omega$, $\alpha_{((\sigma * y)_I)_i} < \gamma_i$. This follows from the fact that $X_{\gamma_i} \subseteq \mathrm{WO}_{<\gamma_i}$. Again, Player II has "taken control" of the game.

A similar argument shows that if II has a winning strategy in $G(S)$ then $\omega_1 \smallsetminus S$ contains a club. Thus the club filter on ω_1 is an ultrafilter and so μ is that ultrafilter. Finally, the fact that μ is ω_1-complete follows from Theorem 2.8. ⊣

We now wish to show that under AD the club filter is normal. This was proved by Solovay, using DC. We shall give a proof that avoids appeal to DC and generalizes to larger ordinals.

2.13 Theorem. *Assume* ZF + AD. *Then the club filter is an ω_1-complete normal ultrafilter on ω_1.*

Proof. For $S \subseteq \omega_1$ let $G(S)$ be the game from the previous proof and let μ be as defined there. We know that μ is the club filter. To motivate the proof of normality we give a proof of ω_1-completeness that will generalize to produce normal ultrafilters on ordinals larger than ω_1. This is merely for illustration—the proof uses DC but this will be eliminated in Claim 2.

Claim 1. *μ is ω_1-complete.*

Proof. Suppose $S_j \in \mu$ for $j < \omega$. We have to show that $S = \bigcap_{j<\omega} S_j \in \mu$. Let σ_j be a winning strategy for I in $G(S_j)$. Assume toward a contradiction that $S \notin \mu$—that is, that I does not win $G(S)$—and let σ be a winning strategy for I in $G(\omega_1 \smallsetminus S)$. Our strategy is to build a play y that is legal for II against each σ_j and against σ. This will give us our contradiction by implying that $\sup_{i<\omega} \alpha_{(y)_i}$ is in each S_j but not in S.

We build $z_n = (y)_n$ by recursion on n using the foresight provided by $\underset{\sim}{\Sigma}^1_1$-boundedness. For the initial step we use $\underset{\sim}{\Sigma}^1_1$-boundedness to get $\beta_0 < \omega_1$ such that for all $j < \omega$ and for all $y \in \omega^\omega$

$$\alpha_{((\sigma_j * y)_I)_0} < \beta_0 \quad \text{and} \quad \alpha_{((\sigma * y)_I)_0} < \beta_0.$$

Choose $z_0 \in \mathrm{WO}_{\beta_0}$. For the $(n+1)$st step we use $\underset{\sim}{\Sigma}^1_1$-boundedness to get $\beta_{n+1} < \omega_1$ such that $\beta_n < \beta_{n+1}$ and for all $j < \omega$ and for all $y \in \omega^\omega$, if $(y)_i = z_i$ for all $i \leqslant n$, then

$$\alpha_{((\sigma_j * y)_I)_{n+1}} < \beta_{n+1} \quad \text{and} \quad \alpha_{((\sigma * y)_I)_{n+1}} < \beta_{n+1}.$$

Choose $z_{n+1} \in \mathrm{WO}_{\beta_{n+1}}$. Finally, let y be such that for all $n < \omega$, $(y)_n = z_n$. The play y is legal for II against each σ_j and σ, which is a contradiction. $\dashv$

Claim 2. *μ is normal.*

Proof. Assume toward a contradiction that $f : \omega_1 \to \omega_1$ is regressive and that there is no $\alpha < \omega_1$ such that $\{\xi < \omega_1 \mid f(\xi) = \alpha\} \in \mu$ or, equivalently (by AD), that for all $\alpha < \omega_1$,

$$S_\alpha = \{\xi < \omega_1 \mid f(\xi) \neq \alpha\} \in \mu.$$

Our strategy is to recursively define

(1.1) an increasing sequence $\langle \eta_i \mid i < \omega \rangle$ of countable ordinals with supremum η,

(1.2) a sequence of collections of strategies $\langle X_i \mid i < \omega \rangle$ where X_i contains winning strategies for I in games $G(S_\alpha)$ for $\alpha \in [\eta_{i-1}, \eta_i)$, where $\eta_{-1} = 0$, and

(1.3) a sequence $\langle y_i \mid i < \omega \rangle$ of plays such that y_i is legal for II against any $\sigma \in X_i$ and $\sup_{j<\omega} \alpha_{(y_i)_j} = \eta$.

Since each $\sigma \in X_i$ is a winning strategy for I, y_i will witness that $\eta \in S_\alpha$ for each $\alpha \in [\eta_{i-1}, \eta_i)$, i.e. y_i will witness that $f(\eta) \neq \alpha$ for each $\alpha \in [\eta_{i-1}, \eta_i)$. Thus collectively the y_i will guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts our assumption that $f(\eta) < \eta$.

We begin by letting

$$Z = \{(x, \sigma) \mid x \in \mathrm{WO} \text{ and } \sigma \text{ is a winning strategy for I in } G(S_{\alpha_x})\}.$$

By the Basic Coding Lemma, there is a $Z^* \subseteq Z$ such that

(2.1) for all $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$,

(2.2) $Z^* = X \cap (\mathrm{WO} \times \omega^\omega)$ where X is $\underset{\sim}{\Sigma}^1_1$.

The key point is that for each $\alpha < \omega_1$,

$$X \cap (\mathrm{WO}_{\leqslant\alpha} \times \omega^\omega)$$

is $\underset{\sim}{\Sigma}^1_1$ since $\mathrm{WO}_{\leqslant\alpha}$ is Borel. Thus, we can apply $\underset{\sim}{\Sigma}^1_1$-boundedness to these sets.

The difficulty is that to construct the sequence $\langle y_i \mid i < \omega\rangle$ we shall need DC. For this reason we drop down to a context where we have DC and run the argument there.

Let t be a real such that X is $\Sigma^1_1(t)$. By absoluteness, for each $\alpha < \omega_1^{L[t,f]}$, there exists an $(x,\sigma) \in Z^* \cap L[t,f]$ such that $\alpha = \alpha_x$ and σ is a winning strategy for Player I in $G(S_\alpha^{L[t,f]})$ where

$$S_\alpha^{L[t,f]} = \{\eta < \omega_1^{L[t,f]} \mid f(\eta) \neq \alpha\}.$$

For the remainder of the proof we work in $L[t,f]$ and interpret S_α and X via their definitions, simply writing S_α and X.

For the first step let

$$\begin{aligned}
\eta_0 &= \text{some ordinal } \eta \text{ such that } \eta < \omega_1\\
X_0 &= \mathrm{proj}_2\left(X \cap (\mathrm{WO}_{[0,\eta_0)} \times \omega^\omega)\right)\\
Y_0 &= \left\{((\sigma * y)_I)_0 \mid \sigma \in X_0 \wedge y \in \omega^\omega\right\}\\
z_0 &= \text{some real } z \text{ such that } Y_0 \subseteq \mathrm{WO}_{<\alpha_z}.
\end{aligned}$$

So X_0 is a collection of strategies for games $G(S_\alpha)$ where $\alpha < \eta_0$. Since these strategies are winning for I the set Y_0 is contained in WO. Furthermore, Y_0 is $\underset{\sim}{\Sigma}^1_1$ and hence has a bound α_{z_0}. For the next step let

$$\begin{aligned}
\eta_1 &= \text{some ordinal } \eta \text{ such that } \eta_0, \alpha_{z_0} < \eta < \omega_1\\
X_1 &= \mathrm{proj}_2\left(X \cap (\mathrm{WO}_{[\eta_0,\eta_1)} \times \omega^\omega)\right)\\
Y_1 &= \left\{((\sigma * y)_I)_1 \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } (y)_0 = z_0\right\}\\
&\quad \cup \left\{((\sigma * y)_I)_0 \mid \sigma \in X_1,\ y \in \omega^\omega\right\}\\
z_1 &= \text{some real } z \text{ such that } Y_1 \subseteq \mathrm{WO}_{<\alpha_z}.
\end{aligned}$$

For the $(n+1)$st step let

$$\begin{aligned}
\eta_{n+1} &= \text{some ordinal } \eta \text{ such that } \eta_n, \alpha_{z_n} < \eta < \omega_1\\
X_{n+1} &= \mathrm{proj}_2\left(X \cap (\mathrm{WO}_{[\eta_n,\eta_{n+1})} \times \omega^\omega)\right)\\
Y_{n+1} &= \left\{((\sigma * y)_I)_{n+1} \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n\ (y)_i = z_i\right\}\\
&\quad \cup \left\{((\sigma * y)_I)_n \mid \sigma \in X_1,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n-1\ (y)_i = z_{i+1}\right\}\\
&\quad\quad \vdots\\
&\quad \cup \left\{((\sigma * y)_I)_0 \mid \sigma \in X_{n+1},\ y \in \omega^\omega\right\}\\
z_{n+1} &= \text{some real } z \text{ such that } Y_{n+1} \subseteq \mathrm{WO}_{<\alpha_z}.
\end{aligned}$$

Finally, for each $k < \omega$, let y_k be such that $(y_k)_i = z_{i+k}$ for all $i < \omega$. Since each of these reals contains a tail of the z_i's, if $\eta = \sup_{n<\omega} \eta_n$, then

$$\sup_{i<\omega}(\alpha_{(y_k)_i}) = \eta$$

for all $k < \omega$. Furthermore, y_k is a legal play for II against any $\sigma \in X_k$, as witnessed by the $(k+1)$st components of Y_n for $n \geqslant k$. Since each $\sigma \in X_k$ is a winning strategy for I, y_k witnesses that $\eta \in S_\alpha$ for $\alpha \in [\eta_{k-1}, \eta_k)$, i.e. that $f(\eta) \neq \alpha$ for any $\alpha \in [\eta_{k-1}, \eta_k)$. Thus, collectively the y_k guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts the fact that $f(\eta) < \eta$. ⊣

This completes the proof of the theorem. ⊣

It should be noted that using DC normality can be proved without using Basic Coding since in place of the sequence $\langle X_i \mid i \in \omega \rangle$ one can use DC to construct a sequence $\langle \sigma_\alpha \mid \alpha < \eta \rangle$ of strategies. This, however, relies on the fact that η is countable. Our reason for giving the proof in terms of Basic Coding is that it illustrates in miniature how we will obtain normal ultrafilters on ordinals much larger than ω_1.

2.14 Corollary (Solovay). *Assume* ZF + AD*. Then*

$$\text{HOD} \models \omega_1^V \textit{ is a measurable cardinal.}$$

Proof. We have that

$$\text{HOD} \models \mu \cap \text{HOD is a normal ultrafilter on } \omega_1^V,$$

since $\mu \cap \text{HOD} \in \text{HOD}$ (as μ is OD and OD is OD). ⊣

Thus, if ZF + AD is consistent, then ZFC + "There is a measurable cardinal" is consistent.

There is also an *effective* version of Solovay's theorem, which we shall need.

2.15 Theorem. *Assume* ZFC + OD*-determinacy. Then*

$$\text{HOD} \models \omega_1^V \textit{ is a measurable cardinal.}$$

Proof. If S is OD then the game $G(S)$ is OD and hence determined. It follows (by the above proof) that if I has a winning strategy in $G(S)$ then S contains a club and if II has a winning strategy in $G(S)$ then $\omega_1 \setminus S$ contains a club. Thus,

$$V \models \mu \cap \text{HOD is an ultrafilter on HOD}$$

and so

$$\text{HOD} \models \mu \cap \text{HOD is an ultrafilter.}$$

Similarly the proof of Claim 1 in Theorem 2.13 shows that

$$V \models \mu \cap \text{HOD is } \omega_1\text{-complete}$$

and so

$$\text{HOD} \models \mu \cap \text{HOD is } \omega_1\text{-complete},$$

which completes the proof. ⊣

2.4. The Least Stable

We now take the next step in generalizing the above result. For this purpose it is useful to think of ω_1 in slightly different terms: Recall the following definition:

$$\underset{\sim}{\delta}{}^1_1 = \sup\{\alpha \mid \text{there is a } \underset{\sim}{\Delta}{}^1_1\text{-surjection } \pi : \omega^\omega \to \alpha\}.$$

It is a classical result that $\omega_1 = \underset{\sim}{\delta}{}^1_1$. Now consider the following higher-order analogue of $\underset{\sim}{\delta}{}^1_1$:

$$\underset{\sim}{\delta}{}^2_1 = \sup\{\alpha \mid \text{there is a } \underset{\sim}{\Delta}{}^2_1\text{-surjection } \pi : \omega^\omega \to \alpha\}.$$

In this section we will work without determinacy and establish the basic features of this ordinal in the context of $L(\mathbb{R})$. In the next section we will solve for U in the equation

$$\frac{\underset{\sim}{\delta}{}^1_1}{\mathrm{WO}} = \frac{\underset{\sim}{\delta}{}^2_1}{U}$$

in such a way that U is accompanied by the appropriate boundedness and coding theorems required to generalize Solovay's proof to show that $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}^{L(\mathbb{R})}$ implies that $(\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$.

The following model-theoretic characterization of the pointclass $\underset{\sim}{\Sigma}{}^2_1$ will be useful in complexity calculations: $A \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}{}^2_1$ iff for some formula φ and some real $z \in \omega^\omega$,

$$\begin{aligned} A = \{y \in \omega^\omega \mid \ & \text{there is a transitive set } M \text{ such that} \\ & \text{(a) } \omega^\omega \subseteq M, \\ & \text{(b) there is a surjection } \pi : \omega^\omega \to M, \text{ and} \\ & \text{(c) } M \models \mathrm{T}_0 + \varphi[y, z]\}. \end{aligned}$$

As before, theories much weaker than T_0 yield an equivalent definition and our choice of T_0 is simply one of convenience. The lightface version Σ^2_1 is defined similarly by omitting the parameter z.

We wish to study $\underset{\sim}{\delta}{}^2_1$ in the context of $L(\mathbb{R})$. In the interest of keeping our account self-contained and free of fine structure we will give a brief introduction to the basic features of $L(\mathbb{R})$ under the stratification $L_\alpha(\mathbb{R})$ for $\alpha \in \mathrm{On}$. For credits and references see [2].

Definability issues will be central. Officially our language is the language of set theory with an additional constant $\dot{\mathbb{R}}$ which is always to be interpreted as $\mathbb{R}$. For a set M such that $X \cup \{\mathbb{R}\} \subseteq M$, let $\Sigma_n(M, X)$ be the collection of sets definable over M via a Σ_n formula with parameters in $X \cup \{\mathbb{R}\}$. For example, x is $\Sigma_1(L(\mathbb{R}), X)$ iff x is Σ_1-definable over $L(\mathbb{R})$ with parameters from $X \cup \{\mathbb{R}\}$. It is important to note that the parameter $\mathbb{R}$ is always allowed in our definability calculations. To emphasize this we will usually make it explicit.

The basic features of L carry over to $L(\mathbb{R})$, one minor difference being that $\mathbb{R}$ is allowed as a parameter in all definability calculations. For example, for each limit ordinal λ, the sequence $\langle L_\alpha(\mathbb{R}) \mid \alpha < \lambda\rangle$ is $\Sigma_1(L_\lambda(\mathbb{R}), \{\mathbb{R}\})$.

For $X \cup \{\mathbb{R}\} \subseteq M \subseteq N$, let $M \prec_n^X N$ mean that for all parameter sequences $\vec{a} \in (X \cup \{\mathbb{R}\})^{<\omega}$ and for all Σ_n formulas φ, $M \models \varphi[\vec{a}]$ iff $N \models \varphi[\vec{a}]$. Let $M \prec_n N$ be short for $M \prec_n^M N$.

2.16 Definition. The *least stable in* $L(\mathbb{R})$, $\delta_{\mathbb{R}}$, is the least ordinal δ such that

$$L_\delta(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}).$$

A related ordinal of particular importance is δ_F, the least ordinal δ such that

$$L_\delta(\mathbb{R}) \prec_1 L(\mathbb{R}).$$

We aim to show that $(\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})} = \delta_{\mathbb{R}} = \delta_F$. For notational convenience we write $\underset{\sim}{\delta}{}^2_1$ for $(\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})}$ and Θ for $\Theta^{L(\mathbb{R})}$.

The definability notions involved in the previous definition also have useful model-theoretic characterizations, which we will routinely employ. For example, $A \subseteq \omega^\omega$ is Σ_1-definable over $L(\mathbb{R})$ with parameters from $\mathbb{R} \cup \{\mathbb{R}\}$ iff there is a formula φ and a $z \in \omega^\omega$,

$$\begin{aligned} A = \{y \in \omega^\omega \mid \exists\alpha \in \mathrm{On} \text{ such that } & \\ &\text{(a) } L_\alpha(\mathbb{R}) \models \mathrm{T}_0 \text{ and} \\ &\text{(b) } L_\alpha(\mathbb{R}) \models \varphi[y, z, \mathbb{R}]\}. \end{aligned}$$

Again, theories weaker than T_0 (such as ZF_N for sufficiently large N) suffice. The existence of arbitrarily large levels $L_\alpha(\mathbb{R})$ satisfying T_0 will be proved below in Lemma 2.22.

2.17 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Suppose*

$$\begin{aligned} X = \big\{x \in L_\lambda(\mathbb{R}) \mid x \textit{ is definable over } L_\lambda(\mathbb{R}) & \\ \textit{from parameters in } \mathbb{R} \cup \{\mathbb{R}\}\big\}&, \end{aligned}$$

where λ is a limit ordinal. Then $X \prec L_\lambda(\mathbb{R})$.

Proof. It suffices (by the Tarski-Vaught criterion) to show that if A is a non-empty set which is definable over $L_\lambda(\mathbb{R})$ from parameters in $\mathbb{R} \cup \{\mathbb{R}\}$, then $A \cap X \neq \varnothing$. Let A be such a non-empty set and choose $x_0 \in A$. Since every set in $L_\lambda(\mathbb{R})$ is definable over $L_\lambda(\mathbb{R})$ from a real and an ordinal parameter,

$$\{x_0\} = \{x \in L_\lambda(\mathbb{R}) \mid L_\lambda(\mathbb{R}) \models \varphi_0[x, c_0, \alpha_0, \mathbb{R}]\}$$

for some formula φ_0, and parameters $c_0 \in \omega^\omega$ and $\alpha_0 \in \mathrm{On}$. Let α_1 be least such that there is exactly one element x such that $L_\lambda(\mathbb{R}) \models \varphi_0[x, c_0, \alpha_1, \mathbb{R}]$ and $x \in A$. Notice that α_1 is definable in $L_\lambda(\mathbb{R})$ from c_0 and the real parameter used in the definition of A. Thus, letting x_1 be the unique element such that $L_\lambda(\mathbb{R}) \models \varphi_0[x_1, c_0, \alpha_1, \mathbb{R}]$ we have a set which is in A (by the definition of x_1) and in X (since it is definable in $L_\lambda(\mathbb{R})$ from c_0 and the real parameter used in the definition of A.) $\dashv$

2.18 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$*. For each* $\alpha < \Theta$*, there is an* OD *surjection* $\pi : \omega^\omega \to \alpha$*.*

Proof. Fix $\alpha < \Theta$. Since every set in $L(\mathbb{R})$ is OD_x for some $x \in \omega^\omega$ there is an OD_x surjection $\pi : \omega^\omega \to \alpha$. For each $x \in \omega^\omega$, let π_x be the $<_{\mathrm{OD}_x}$-least such surjection if one exists and let it be undefined otherwise. We can now "average over the reals" to eliminate the dependence on real parameters, letting

$$\begin{aligned} \pi : \omega^\omega &\to \alpha \\ x &\mapsto \begin{cases} \pi_{(x)_0}((x)_1) & \text{if } \pi_{(x)_0} \text{ is defined} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is an OD surjection. ⊣

2.19 Lemma (Solovay). *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$*. Then* Θ *is regular in* $L(\mathbb{R})$*.*

Proof. By the proof of the previous lemma, there is an OD sequence

$$\langle \pi_\alpha \mid \alpha < \Theta \rangle$$

such that each $\pi_\alpha : \omega^\omega \to \alpha$ is an OD surjection. Assume for contradiction that Θ is singular. Let

$$f : \alpha \to \Theta$$

be a cofinal map witnessing the singularity of Θ. Let $g : \omega^\omega \to \alpha$ be a surjection. It follows that the map

$$\begin{aligned} \pi : \omega^\omega &\to \Theta \\ x &\mapsto \pi_{f \circ g((x)_0)}((x)_1) \end{aligned}$$

is a surjection, which contradicts the definition of Θ. ⊣

2.20 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$*. Then*

$$L_\Theta(\mathbb{R}) = \{x \in L(\mathbb{R}) \mid \textit{there is a surjection } \pi : \omega^\omega \to \mathrm{tc}(x)\}.$$

Thus, $\mathscr{P}(\mathbb{R}) \subseteq L_\Theta(\mathbb{R})$*.*

Proof. For the first direction suppose $x \in L_\Theta(\mathbb{R})$. Let $\lambda < \Theta$ be a limit ordinal such that $x \in L_\lambda(\mathbb{R})$. Thus $\mathrm{tc}(x) \subseteq L_\lambda(\mathbb{R})$. Moreover, there is a surjection $\pi : \omega^\omega \to L_\lambda(\mathbb{R})$, since every element of $L_\lambda(\mathbb{R})$ is definable from an ordinal and real parameters.

For the second direction suppose $x \in L(\mathbb{R})$ and that there is a surjection $\pi : \omega^\omega \to \mathrm{tc}(x)$. We wish to show that $x \in L_\Theta(\mathbb{R})$. Let λ be a limit ordinal such that $x \in L_\lambda(\mathbb{R})$. Thus $\mathrm{tc}(x) \subseteq L_\lambda(\mathbb{R})$. Let

$$\begin{aligned} X = \big\{ y \in L_\lambda(\mathbb{R}) \mid y &\text{ is definable over } L_\lambda(\mathbb{R}) \\ &\text{from parameters in } \mathrm{tc}(x) \cup \mathbb{R} \cup \{\mathbb{R}\} \big\}, \end{aligned}$$

By the proof of Lemma 2.17, $X \prec L_\lambda(\mathbb{R})$ and $\mathrm{tc}(x) \subseteq X$. By Condensation, the transitive collapse of X is $L_{\bar\lambda}(\mathbb{R})$ for some $\bar\lambda$. Since there is a surjection $\pi : \omega^\omega \to \mathrm{tc}(x)$ and since all members of $L_{\bar\lambda}(\mathbb{R})$ are definable from parameters in $\mathrm{tc}(x) \cup \mathbb{R} \cup \{\mathbb{R}\}$, there is a surjection $\rho : \omega^\omega \to L_{\bar\lambda}(\mathbb{R})$. So $\bar\lambda < \Theta$ and since $x \in L_{\bar\lambda}(\mathbb{R})$ this completes the proof. ⊣

2.21 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then*

$$L_\Theta(\mathbb{R}) \models \mathrm{T}_0.$$

Proof. It is straightforward to see that $L_\Theta(\mathbb{R})$ satisfies T_0 − Separation − Replacement.

To see that $L_\Theta(\mathbb{R}) \models$ Separation note that if $S \subseteq x \in L_\Theta(\mathbb{R})$ then $S \in L_\Theta(\mathbb{R})$, by Lemma 2.20. To see that $L_\Theta(\mathbb{R}) \models$ Replacement we verify Collection, which is equivalent to Replacement, over the other axioms. Suppose

$$L_\Theta(\mathbb{R}) \models \forall x \in a\, \exists y\, \varphi(x, y),$$

where $a \in L_\Theta(\mathbb{R})$. Let

$$\begin{aligned} f &: a \mapsto \Theta \\ &\quad x \to \mu\alpha\,(\exists y \in L_\alpha(\mathbb{R}) \text{ such that } L_\Theta(\mathbb{R}) \models \varphi(x, y)). \end{aligned}$$

The ordertype of $\mathrm{ran}(f)$ is less that Θ since otherwise there would be a surjection $\pi : \omega^\omega \to \Theta$ (since there is a surjection $\pi : \omega^\omega \to a$). Moreover, since Θ is regular, it follows that $\mathrm{ran}(f)$ is bounded by some $\lambda < \Theta$. Thus,

$$L_\Theta(\mathbb{R}) \models \forall x \in a\, \exists y \in L_\lambda(\mathbb{R})\, \varphi(x, y),$$

which completes the proof. ⊣

2.22 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *There are arbitrarily large α such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$.*

Proof. The proof is similar to the previous proof. Let us say that α is an $\mathbb{R}$-*cardinal* if for every $\gamma < \alpha$ there does not exist a surjection $\pi : \mathbb{R} \times \gamma \to \alpha$. For each limit ordinal $\gamma \in \mathrm{On}$, letting

$$\Theta(\gamma) = \sup\{\alpha \mid \text{there is a surjection } \pi : \mathbb{R} \times \gamma \to \alpha\}$$

we have that $\Theta(\gamma)$ is an $\mathbb{R}$-cardinal. For each γ which is closed under the Gödel pairing function, the argument of Lemma 2.19 shows that $\Theta(\gamma)$ is regular. The proof of the previous lemma generalizes to show that for every regular $\Theta(\gamma)$, $L_{\Theta(\gamma)}(\mathbb{R}) \models \mathrm{T}_0$. ⊣

2.23 Lemma (Solovay). *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. $L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R})$.

Proof. Suppose
$$L(\mathbb{R}) \models \varphi[a],$$
where $a \in L_\Theta(\mathbb{R})$ and φ is Σ_1. By Reflection, let λ be a limit ordinal such that
$$L_\lambda(\mathbb{R}) \models \varphi[a].$$
Let
$$X = \big\{y \in L_\lambda(\mathbb{R}) \mid y \text{ is definable over } L_\lambda(\mathbb{R}) \\ \text{from parameters in } \mathrm{tc}(a) \cup \mathbb{R} \cup \{\mathbb{R}\}\big\},$$
By Condensation and Lemma 2.20, the transitive collapse of X is $L_{\bar\lambda}(\mathbb{R})$ for some $\bar\lambda < \Theta$. Thus, by upward absoluteness,
$$L_\Theta(\mathbb{R}) \models \varphi[a].$$
⊣

2.24 Lemma. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$ + $V = L(\mathbb{R})$. *There are arbitrarily large* $\alpha < \delta_F$ *such that* $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$

Proof. Suppose $\xi < \delta_F$. Since $L_\Theta(\mathbb{R}) \models \mathrm{T}_0$,
$$L(\mathbb{R}) \models \exists \alpha > \xi\, (L_\alpha(\mathbb{R}) \models \mathrm{T}_0).$$
The formula is readily seen to be Σ_1 with parameters in $\{\mathbb{R}, \xi\}$ by our model-theoretic characterization. Thus, by the definition of δ_F,
$$L_{\delta_F}(\mathbb{R}) \models \exists \alpha > \xi\, (L_\alpha(\mathbb{R}) \models \mathrm{T}_0),$$
which completes the proof. ⊣

2.25 Lemma. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Suppose* φ *is a formula and* $a \in \omega^\omega$. *Suppose* λ *is least such that* $L_\lambda(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a]$. *Let*
$$X = \big\{x \in L_\lambda(\mathbb{R}) \mid y \textit{ is definable over } L_\lambda(\mathbb{R}) \\ \textit{from parameters in } \mathbb{R} \cup \{\mathbb{R}\}\big\}.$$
Then $X = L_\lambda(\mathbb{R})$. *Moreover, there is a surjection* $\pi : \omega^\omega \to L_\lambda(\mathbb{R})$ *such that* π *is definable over* $L_{\lambda+1}(\mathbb{R})$ *from* $\mathbb{R}$ *and* a.

Proof. By Lemma 2.17 we have that $X \prec L_\lambda(\mathbb{R})$. By condensation the transitive collapse of X is some $L_{\bar\lambda}(\mathbb{R})$. So $L_{\bar\lambda}(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a]$ and thus by the minimality of λ we have $\bar\lambda = \lambda$. Since every $x \in X$ is definable from a real parameter and since $L_\lambda(\mathbb{R}) \cong X$, we have that every $x \in L_\lambda(\mathbb{R})$ is definable from a real parameter, in other words, $X = L_\lambda(\mathbb{R})$. The desired map $\pi : \omega^\omega \to L_\lambda(\mathbb{R})$ is the map which takes a real coding the Gödel number of φ and a real parameter a to the set $\{x \in L_\lambda(\mathbb{R}) \mid L_\lambda(\mathbb{R}) \models \varphi[x, a]\}$. This map is definable over $L_{\lambda+1}(\mathbb{R})$. ⊣

2.26 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Suppose* $0 < \alpha < \delta_{\mathbb{R}}$. *Then there is a surjection* $\pi : \omega^\omega \to L_\alpha(\mathbb{R})$ *such that* $\{(x, y) \mid \pi(x) \in \pi(y)\}$ *is* $\underset{\sim}{\Delta}{}^2_1$. *Thus,* $\delta_{\mathbb{R}} \leqslant \underset{\sim}{\delta}{}^2_1$.

Proof. Fix α such that $0 < \alpha < \delta_{\mathbb{R}}$. By the minimality of $\delta_{\mathbb{R}}$,

$$L_\alpha(\mathbb{R}) \not\prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}).$$

So there is an $a \in \omega^\omega$ and a Σ_1 formula φ such that if β is the least ordinal such that $L_\beta(\mathbb{R}) \models \varphi[a]$ then $\beta > \alpha$. Let γ be least such that $\gamma > \beta > \alpha$ and $L_\gamma(\mathbb{R}) \models \mathrm{T}_0$ (which exists by Lemma 2.22). So γ is least such that $L_\gamma(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a]$ and, by Lemma 2.25, there is a surjection $\pi : \omega^\omega \to L_\gamma(\mathbb{R})$ which is definable over $L_{\gamma+1}(\mathbb{R})$ with the parameters $\mathbb{R}$ and a. Let $A = \{(x, y) \mid \pi(x) \in \pi(y)\}$. Let ψ_1 and ψ_2 be the formulas defining A and $(\omega^\omega)^2 \smallsetminus A$ over $L_{\gamma+1}(\mathbb{R})$, respectively. By absoluteness,

$$\begin{array}{lll} (x,y) \in A & \text{iff} & \text{there is a transitive set } M \text{ such that} \\ & & \omega^\omega \subseteq M, \\ & & \text{there is a surjection } \pi : \omega^\omega \to M, \text{ and} \\ & & M \models \mathrm{T}_0 + \exists\gamma\,(L_\gamma(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a] \text{ and} \\ & & L_{\gamma+1}(\mathbb{R}) \models \psi_1[x,y]). \end{array}$$

This shows, by our model-theoretic characterization of $\underset{\sim}{\Sigma}{}^2_1$ that A is $\underset{\sim}{\Sigma}{}^2_1$. A similar argument shows that $(\omega^\omega)^2 \smallsetminus A$ is $\underset{\sim}{\Sigma}{}^2_1$. Finally, the desired map can be extracted from π. ⊣

We now use a universal $\underset{\sim}{\Sigma}{}^2_1$ set to knit together all of these "$\underset{\sim}{\Delta}{}^2_1$ projection maps".

2.27 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then there is a partial surjection* $\rho : \omega^\omega \to L_{\delta_{\mathbb{R}}}(\mathbb{R})$ *such that* $\mathrm{dom}(\rho)$ *and* ρ *are both* Σ_1*-definable over* $L_{\delta_{\mathbb{R}}}(\mathbb{R})$ *with the parameter* $\mathbb{R}$. *Thus,* $L_{\delta_{\mathbb{R}}}(\mathbb{R}) \prec_1 L(\mathbb{R})$ *and hence* $\delta_F \leqslant \delta_R$.

Proof. Let U be a Σ^2_1 subset of $\omega^\omega \times \omega^\omega \times \omega^\omega$ that is universal for $\underset{\sim}{\Sigma}{}^2_1$ subsets of $\omega^\omega \times \omega^\omega$, that is, such that for each $\underset{\sim}{\Sigma}{}^2_1$ subset $A \subseteq \omega^\omega \times \omega^\omega$ there is an $x \in \omega^\omega$ such that $A = U_x$ where by definition

$$U_x = \{(y, z) \in \omega^\omega \times \omega^\omega \mid (x, y, z) \in U\}.$$

We define ρ using U. For the domain of ρ we take

$$\begin{aligned} \mathrm{dom}(\rho) = \{x \in \omega^\omega \mid \exists\alpha \in \mathrm{On}\,(&L_\alpha(\mathbb{R}) \models \mathrm{T}_0 \text{ and} \\ &L_\alpha(\mathbb{R}) \models U_{(x)_0} = (\omega^\omega \times \omega^\omega) \smallsetminus U_{(x)_1})\}. \end{aligned}$$

Notice that $\mathrm{dom}(\rho)$ is $\Sigma_1(L(\mathbb{R}), \{\mathbb{R}\})$ and hence $\Sigma_1(L_{\delta_{\mathbb{R}}}(\mathbb{R}), \{\mathbb{R}\})$. Notice also that in general if $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ then

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} \subseteq U_{(x)_0}$$

and thus, if in addition,

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = (\omega^\omega \times \omega^\omega) \smallsetminus (U_{(x)_1})^{L_\alpha(\mathbb{R})},$$

then,

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = U_{(x)_0}.$$

We can now define ρ as follows: Suppose $x \in \mathrm{dom}(\rho)$. Let $\alpha(x)$ be the least α as in the definition of $\mathrm{dom}(\rho)$. If there is an ordinal η and a surjection $\pi : \omega^\omega \to L_\eta(\mathbb{R})$ such that

$$\{(t_1, t_2) \mid \pi(t_1) \in \pi(t_2)\} = (U_{(x)_0})^{L_{\alpha(x)}(\mathbb{R})}$$

then let $\rho(x) = \pi((x)_2)$; otherwise let $\rho(x) = \varnothing$. Notice that the map ρ is $\Sigma_1(L(\mathbb{R}), \{\mathbb{R}\})$ and hence $\Sigma_1(L_{\delta_\mathbb{R}}(\mathbb{R}), \{\mathbb{R}\})$. By Lemma 2.26, $\rho : \mathrm{dom}(\rho) \to L_{\delta_\mathbb{R}}(\mathbb{R})$ is a surjection.

For the last part of the proof recall that by definition $L_{\delta_\mathbb{R}}(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$. The partial surjection $\rho : \omega^\omega \to L_{\delta_\mathbb{R}}(\mathbb{R})$ allows us to reduce arbitrary parameters in $L_{\delta_\mathbb{R}}(\mathbb{R})$ to parameters in ω^ω. ⊣

2.28 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. $\underset{\sim}{\delta}^2_1 = \delta_\mathbb{R} = \delta_F$.

Proof. We have $\delta_\mathbb{R} \leqslant \underset{\sim}{\delta}^2_1$ (by Lemma 2.26), $\delta_\mathbb{R} \leqslant \delta_F$ (by definition), and $\delta_F \leqslant \delta_\mathbb{R}$ (by Lemma 2.27). It remains to show $\underset{\sim}{\delta}^2_1 \leqslant \delta_\mathbb{R}$.

Suppose $\gamma < \underset{\sim}{\delta}^2_1$. We wish to show that $\gamma < \delta_\mathbb{R}$. Let $\pi : \omega^\omega \to \alpha$ be a surjection such that $A = \{(x, y) \mid \pi(x) < \pi(y)\}$ is $\underset{\sim}{\Delta}^2_1$. Using the notation from the previous proof let x be such that

$$U_{(x)_0} = A \quad \text{and} \quad U_{(x)_1} = (\omega^\omega \times \omega^\omega) \smallsetminus A.$$

There is an ordinal α such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = (\omega^\omega \times \omega^\omega) \smallsetminus (U_{(x)_1})^{L_\alpha(\mathbb{R})}.$$

Since

$$L_{\delta_\mathbb{R}}(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}),$$

the least such ordinal, $\alpha(x)$, is less than $\delta_\mathbb{R}$. Thus,

$$(U_{(x)_0})^{L_{\alpha(x)}(\mathbb{R})} = A.$$

Finally, since $L_{\alpha(x)}(\mathbb{R}) \models \mathrm{T}_0$, this model can compute the ordertype, γ, of A. Thus, $\gamma < \alpha(x) < \delta_\mathbb{R}$. ⊣

2.29 Remark. Although we will not need these facts it is worthwhile to note that the above proofs show

(1) $(\underset{\sim}{\Sigma}^2_1)^{L(\mathbb{R})} = \underset{\sim}{\Sigma}_1(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})) \cap \mathscr{P}(\omega^\omega)$,

(2) $(\underset{\sim}{\Delta}^2_1)^{L(\mathbb{R})} = L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \cap \mathscr{P}(\omega^\omega)$, and

(3) (Solovay's Basis Theorem) if $L(\mathbb{R}) \models \exists X\, \varphi(X)$ where φ is $\underset{\sim}{\Sigma}^2_1$ then $L(\mathbb{R}) \models \exists X \in \underset{\sim}{\Delta}^2_1\ \varphi(X)$.

2.5. Measurability of the Least Stable

We are now in a position to show that under ZF + DC + AD,

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \text{ is a measurable cardinal.}$$

This serves as a warm-up to Sect. 4, where we will show that under ZF + DC + AD,

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \text{ is } \lambda\text{-strong,}$$

for each $\lambda < \Theta^{L(\mathbb{R})}$, and, in fact, that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \text{ is a Woodin cardinal.}$$

The proof that we give in Sect. 4 will show that DC can be eliminated from the result of the present section.

First we need an analogue U of WO that enables us to encode (unboundedly many) ordinals below $\underset{\sim}{\delta}^2_1$ and is accompanied by the boundedness and coding theorems required to push the above proof through for $\underset{\sim}{\delta}^2_1$. The following works: Let U be a Σ^2_1 subset of $\omega^\omega \times \omega^\omega$ that is universal for $\underset{\sim}{\Sigma}^2_1$ subsets of ω^ω. For $y \in \omega^\omega$ we let $U_y = \{z \in \omega^\omega \mid (y,z) \in U\}$. For $(y,z) \in U$, let $\Theta_{(y,z)}$ be least such that

$$L_{\Theta_{(y,z)}}(\mathbb{R}) \models \mathrm{T}_0 \quad \text{and} \quad (y,z) \in U^{L_{\Theta_{(y,z)}}(\mathbb{R})}.$$

Let $\delta_{(y,z)} = (\underset{\sim}{\delta}^2_1)^{L_{\Theta_{(y,z)}}(\mathbb{R})}$. These ordinals are the analogues of α_x from the proof that ω_1 is measurable. For notational convenience we will routinely use our recursive bijection from $\omega^\omega \times \omega^\omega$ to ω^ω to identify pairs of reals (y,z) with single reals $x = \langle y,z\rangle$. Thus we will write Θ_x and δ_x instead of $\Theta_{(y,z)}$ and $\delta_{(y,z)}$.

2.30 Lemma. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$ + $V = L(\mathbb{R})$. $\{\delta_x \mid x \in U\}$ *is unbounded in* $\underset{\sim}{\delta}^2_1$.

Proof. Let $\alpha < \underset{\sim}{\delta}^2_1$. Let A be (the set of reals coding) a $\underset{\sim}{\Delta}^2_1$ prewellordering of length greater than α. Let $y, z \in \omega^\omega$ be such that $U_y = A$ and $U_z = \omega^\omega \smallsetminus A$. So $L(\mathbb{R}) \models$ "$U_y = \omega^\omega \smallsetminus U_z$". Since $\underset{\sim}{\delta}^2_1$ is the least stable, there is a $\beta < \underset{\sim}{\delta}^2_1$ such that $L_\beta(\mathbb{R}) \models$ "$U_y = \omega^\omega \smallsetminus U_z$" and since $(U_y)^{L_\beta(\mathbb{R})} \subseteq A$ and $(U_z)^{L_\beta(\mathbb{R})} \subseteq \omega^\omega \smallsetminus A$ we have that $A = (U_y)^{L_\beta(\mathbb{R})}$. Now, letting $x \in U \smallsetminus U^{L_\beta(\mathbb{R})}$ and $\gamma < \underset{\sim}{\delta}^2_1$ be such that $L_\gamma(\mathbb{R}) \models \mathrm{T}_0$ + "$x \in U$", we have that $\alpha < \delta_x$ since $A \in L_\gamma(\mathbb{R})$ and $L_\gamma(\mathbb{R})$ can compute the ordertype of A. ⊣

In analogy with WO, for $x \in U$ let $U_{\delta_x} = \{y \in U \mid \delta_y = \delta_x\}$, $U_{<\delta_x} = \{y \in U \mid \delta_y < \delta_x\}$ and so on.

2.31 Lemma ($\underset{\sim}{\Delta}^2_1$-Boundedness, Moschovakis). *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$ + $V = L(\mathbb{R})$. *Suppose* $X \subseteq U$ *and* X *is* $\underset{\sim}{\Delta}^2_1$. *Then there exists an* $x \in U$ *such that such that* $X \subseteq U_{<\delta_x}$.

Proof. Let $y, z \in \omega^\omega$ be such that $U_y = X$ and $U_z = \omega^\omega \setminus X$. (Notice that we are identifying X with the set of reals that recursively encodes it.) As above, there is a $\beta_0 < \underset{\sim}{\delta}^2_1$ such that $X = (U_y)^{L_{\beta_0}(\mathbb{R})}$. Choose γ such that $\beta_0 < \gamma < \underset{\sim}{\delta}^2_1$ and $L_\gamma(\mathbb{R})$ satisfies T_0. Then for all $z \in X$, $\delta_z < \gamma$. Now choose $x \in U$ such that $\delta_x > \gamma$. ⊣

2.32 Lemma (Coding; Moschovakis). *Assume* ZF + AD. *Suppose* $Z \subseteq U \times \omega^\omega$. *Then there exists a* $Z^* \subseteq Z$ *such that for all* $x \in U$

(i) $Z^* \cap (U_{\delta_x} \times \omega^\omega) \neq \varnothing$ *iff* $Z \cap (U_{\delta_x} \times \omega^\omega) \neq \varnothing$,

(ii) $Z^* \cap (U_{\leqslant \delta_x} \times \omega^\omega)$ *is* $\underset{\sim}{\Delta}^2_1$.

This lemma will follow from the more general coding lemmas of the next section. See Remark 3.6.

2.33 Theorem (Moschovakis). *Assume* ZF + DC + AD. *Then*

$$L(\mathbb{R}) \models \textit{There is a normal ultrafilter on } \underset{\sim}{\delta}^2_1.$$

Proof. Work in $L(\mathbb{R})$. The proof is virtually a carbon copy of the proof for ω_1. One just replaces $\underset{\sim}{\delta}^1_1$, WO, α_x, and $\underset{\sim}{\Sigma}^1_1$ with $\underset{\sim}{\delta}^2_1$, U, δ_x, and $\underset{\sim}{\Delta}^2_1$, respectively. For completeness we include some of the details, noting the main changes.

For $S \subseteq \underset{\sim}{\delta}^2_1$, let $G(S)$ be the game

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

with the following rules: Rule 1: For all $i < \omega$, $(x)_i, (y)_i \in U$. If Rule 1 is violated then, letting i be least such that either $(x)_i \notin U$ or $(y)_i \notin U$, I wins if $(x)_i \in U$; otherwise II wins. Now suppose Rule 1 is satisfied. Rule 2: $\delta_{(x)_0} < \delta_{(y)_0} < \delta_{(x)_1} < \delta_{(y)_1} < \cdots$. The first failure defines who wins as above. If both rules are satisfied then I wins iff $\sup_{i \in \omega} \delta_{(x)_i} \in S$.

Now let

$$\mu = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \text{I wins } G(S)\}.$$

Notice that as before (using $\underset{\sim}{\Delta}^2_1$-boundedness) if I has a winning strategy in $G(S)$ then S contains a set C which is unbounded and closed under ω-sequences. The proof that U is an ultrafilter is exactly as before. To see that it is $\underset{\sim}{\delta}^2_1$-complete and normal one uses the new versions of Boundedness and Coding. We note the minor changes in the proof of normality.

Assume for contradiction that $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ and that there is no $\alpha < \underset{\sim}{\delta}^2_1$ such that $\{\xi \mid f(\xi) = \alpha\} \in \mu$ or, equivalently (by AD) that for all $\alpha < \underset{\sim}{\delta}^2_1$,

$$S_\alpha = \{\xi \mid f(\xi) \neq \alpha\} \in \mu.$$

Let $\langle \delta_\alpha \mid \alpha < \underset{\sim}{\delta}^2_1 \rangle$ enumerate $\langle \delta_x \mid x \in U \rangle$. Here we are appealing to the fact that $\underset{\sim}{\delta}^2_1$ is regular, which can be shown using the Coding Lemma (see [9, p. 433]). In analogy with WO, for $\alpha < \beta < \underset{\sim}{\delta}^2_1$, let $U_\alpha = \{x \in U \mid \delta_x = \delta_\alpha\}$,

$U_{(\alpha,\beta]} = \{x \in U \mid \delta_\alpha < \delta_x \leqslant \delta_\beta\}$ and likewise for other intervals. Let $\leqslant_U$ be the associated prewellordering.

As before, our strategy is to inductively define

(1.1) an increasing sequence $\langle \eta_i \mid i < \omega\rangle$ of ordinals with supremum η,

(1.2) a sequence of collections of strategies $\langle X_i \mid i < \omega\rangle$ where X_i contains winning strategies for I in games $G(S_\alpha)$ for $\alpha \in [\eta_{i-1}, \eta_i)$, where $\eta_{-1} = 0$, and

(1.3) a sequence $\langle y_i \mid i < \omega\rangle$ of plays such that y_i is legal for II against any $\sigma \in X_i$ and $\sup_{j<\omega} \delta_{(y_i)_j} = \eta$.

Thus the y_i will collectively witness that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts our assumption that $f(\eta) < \eta$. The key difference is that in our present case we need the Coding Lemma since there are too many games. Let

$$\begin{aligned} Z = \{(x,\sigma) \mid x \in U &\text{ and } \sigma \text{ is a winning strategy for I} \\ &\text{in } G(S_\alpha) \text{ where } \alpha \text{ is such that } \delta_\alpha = \delta_x\} \end{aligned}$$

and, by our new Coding Lemma, let $Z^* \subseteq Z$ be such that for all $\alpha < \utilde{\delta}^2_1$,

(2.1) $Z^* \cap (U_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (U_\alpha \times \omega^\omega) \neq \varnothing$

(2.2) $Z^* \cap (U_{\leqslant\alpha} \times \omega^\omega)$ is $\utilde{\Delta}^2_1$.

This puts us in a position to apply $\utilde{\Delta}^2_1$-boundedness.

For the first step let

$$\begin{aligned} \eta_0 &= \text{some ordinal } \eta \text{ such that } \eta < \utilde{\delta}^2_1 \\ X_0 &= \operatorname{proj}_2\left(Z^* \cap (U_{[0,\eta_0)} \times \omega^\omega)\right) \\ Y_0 &= \left\{((\sigma * y)_I)_0 \mid \sigma \in X_0 \wedge y \in \omega^\omega\right\} \\ z_0 &= \text{some real } z \text{ such that } Y_0 \subseteq U_{<\delta_z}. \end{aligned}$$

So X_0 is a collection of strategies for games $G(S_\alpha)$ where $\alpha < \eta_0$. Since these strategies are winning for I the set Y_0 is contained in U. Furthermore, Y_0 is $\utilde{\Delta}^2_1$ and hence has a bound δ_{z_0}. For the induction step let

$$\begin{aligned} \eta_{n+1} &= \text{some ordinal } \eta \text{ such that } \eta_n, \delta_{z_n} < \eta < \utilde{\delta}^2_1 \\ X_{n+1} &= \operatorname{proj}_2\left(Z^* \cap (U_{[\eta_n,\eta_{n+1})} \times \omega^\omega)\right) \\ Y_{n+1} &= \left\{((\sigma * y)_I)_{n+1} \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n\ (y)_i = z_i\right\} \\ &\quad \cup \left\{((\sigma * y)_I)_n \mid \sigma \in X_1,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n-1\ (y)_i = z_{i+1}\right\} \\ &\quad\ \ \vdots \\ &\quad \cup \left\{((\sigma * y)_I)_0 \mid \sigma \in X_{n+1},\ y \in \omega^\omega\right\} \\ z_{n+1} &= \text{some real } z \text{ such that } Y_{n+1} \subseteq U_{<\delta_z}. \end{aligned}$$

Finally, for $k < \omega$, let y_k be such that $(y_k)_i = z_{i+k}$ for all $i < \omega$. Since each of these reals contains a tail of the z_i's, if $\eta = \sup_{n<\omega} \eta_n$, then

$$\sup_{i<\omega} \left(\delta_{(y_k)_i}\right) = \eta$$

for all $k < \omega$. Furthermore, y_k is a legal play for II against any $\sigma \in X_k$, as witnessed by the $(k+1)$st components of Y_n with $n \geqslant k$. Since each $\sigma \in X_k$ is a winning strategy for I, y_k witnesses that $\eta \in S_\alpha$ for $\alpha \in [\eta_{k-1}, \eta_k)$, i.e. that $f(\eta) \neq \alpha$ for any $\alpha \in [\eta_{k-1}, \eta_k)$. So collectively the y_k guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts the fact that $f(\eta) < \eta$. ⊣

2.34 Corollary. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \textit{ is a measurable cardinal.}$$

The above proof uses DC. However, as we shall see in Sect. 4.1 the theorem can be proved in $\mathrm{ZF} + \mathrm{AD}$. See Lemma 4.7.

The Coding Lemma was used to enable II to "collect together" the relevant strategies and then the $\underset{\sim}{\Delta}^2_1$-boundedness lemma was used to enable II to "take control of the ordinal played" in all such games by devising a play that is legal against all of the relevant strategies and (in each case) has the same fixed ordinal as output. This technique is central in what follows. It is important to note, however, that the above ultrafilter (and, more generally, ultrafilters obtained by such a "sup" game) concentrates on points of cofinality ω. Later we will use a slightly different game, where the role of the $\underset{\sim}{\Delta}^2_1$-boundedness lemma will be played by a certain reflection phenomenon. Before turning to this we prove the coding lemmas we shall need.

3. Coding

In the Basic Coding Lemma we constructed selectors relative to WO; we now do so relative to more general prewellorderings.

3.1. Coding Lemma

We begin by fixing some notation. For $P \subseteq \omega^\omega$, the notion of a $\underset{\sim}{\Sigma}^1_1(P)$ set is defined exactly like that of a $\underset{\sim}{\Sigma}^1_1$ set only now we allow reference to P and to $\omega^\omega \smallsetminus P$. In model-theoretic terms, $X \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1(P)$ iff there is a formula φ and a real z such that

$$\begin{aligned} X = \big\{ y \in \omega^\omega \mid {} & \text{there is an } \omega\text{-model } M \text{ such that} \\ & y, z, P \cap M \in M \text{ and } M \models \mathrm{T}_0 + \varphi[y, z, P \cap M] \big\}. \end{aligned}$$

The notion of a $\underset{\sim}{\Sigma}^1_1(P, P')$ set is defined in the same way, only now reference to both P and P' and their complements is allowed. The lightface versions of these notions and the versions involving $P \subseteq (\omega^\omega)^n$ are all defined in the obvious way.

Let $U^{(n)}(P)$ be a $\Sigma^1_1(P)$ subset of $(\omega^\omega)^{n+1}$ that is universal for $\utilde{\Sigma}^1_1(P)$ subsets of $(\omega^\omega)^n$, that is, such that for each $\utilde{\Sigma}^1_1(P)$ set $A \subseteq (\omega^\omega)^n$ there is an $e \in \omega^\omega$ such that $A = U^{(n)}_e(P) = \{y \in (\omega^\omega)^n \mid (e,y) \in U^{(n)}(P)\}$. We do this in such a way that the same formula is used, so that the definition is uniform in P. Likewise, for $U^{(n)}(P,P')$ etc. (The existence of such a universal set $U^{(n)}(P)$ is guaranteed by the fact that the pointclass in question, namely, $\utilde{\Sigma}^1_1(P)$, is ω-parameterized and closed under recursive substitution. See [9], 3E.4 on p. 160 and especially 3H.1 on p. 183. We further assume that the universal sets are "good" in the sense of [9], p. 185 and we are justified in doing so by [9], 3H.1. A particular component of this assumption is that our universal sets satisfy the *s-m-n*-theorem (uniformly in P (or P and P')). See Jackson's chapter in this Handbook for further details.)

3.1 Theorem (Recursion Theorem; Kleene). *Suppose $f : \omega^\omega \to \omega^\omega$ is $\utilde{\Sigma}^1_1(P)$. Then there is an $e \in \omega^\omega$ such that*

$$U^{(2)}_e(P) = U^{(2)}_{f(e)}(P).$$

Proof. For $a \in \omega^\omega$, let

$$T_a = \{(b,c) \mid (a,a,b,c) \in U^{(3)}(P)\}.$$

Let $d : \omega^\omega \to \omega^\omega$ be Σ^1_1 such that $T_a = U^{(2)}_{d(a)}(P)$. (The function d comes from the *s-m-n*-theorem. In fact, $d(a) = s(a,a)$ (in the notation of Jackson's chapter) and d is continuous.) Let

$$Y = \{(a,b,c) \mid (b,c) \in U^{(2)}_{f(d(a))}(P)\}$$

and let a_0 be such that $Y = U^{(3)}_{a_0}(P)$. Notice that Y is $\utilde{\Sigma}^1_1(P)$ using the parameter for Y (as can readily be checked using the model-theoretic characterization of $\utilde{\Sigma}^1_1(P)$). We have

$$\begin{aligned} (b,c) \in U^{(2)}_{d(a_0)}(P) \quad &\text{iff} \quad (a_0,a_0,b,c) \in U^{(3)}(P) \\ &\text{iff} \quad (a_0,b,c) \in U^{(3)}_{a_0}(P) = Y \\ &\text{iff} \quad (b,c) \in U^{(2)}_{f(d(a_0))}(P) \end{aligned}$$

and so $d(a_0)$ is as desired. ⊣

3.2 Theorem (Coding lemma; Moschovakis). *Assume* ZF + AD. *Suppose $X \subseteq \omega^\omega$ and $\pi : X \to$* On. *Suppose $Z \subseteq X \times \omega^\omega$. Then there is an $e \in \omega^\omega$ such that*

(1) $U^{(2)}_e(Q) \subseteq Z$ and

(2) for all $a \in X$, $U^{(2)}_e(Q) \cap (Q_a \times \omega^\omega) \neq \varnothing$ iff $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$,

where $Q = \{\langle a,b\rangle \mid \pi(a) \leqslant \pi(b)\}$.

Proof. Assume toward a contradiction that there is no such e. Consider the set G of reals e for which (1) in the statement of the theorem is satisfied:

$$G = \{e \in \omega^\omega \mid U_e^{(2)}(Q) \subseteq Z\}.$$

So, for each $e \in G$, (2) in the statement of the theorem fails for some $a \in X$. Let α_e be the least α such that (2) fails at the αth-section:

$$\alpha_e = \min\{\alpha \mid \exists a \in X \ (\pi(a) = \alpha \wedge U_e^{(2)}(Q) \cap (Q_a \times \omega^\omega) = \varnothing \\ \wedge Z \cap (Q_a \times \omega^\omega) \neq \varnothing)\}.$$

Now play the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where I wins if $x \in G$ and either $y \notin G$ or $\alpha_x \geqslant \alpha_y$. Notice that by our assumption that there is no index e as in the statement of the theorem, neither I nor II can win a round of this game by playing a selector. The best they can do is play "partial" selectors. For a play $e \in G$, let us call $U_e^{(2)}(Q) \cap (Q_{<\alpha_e} \times \omega^\omega)$ the *partial* selector played. Using this terminology we can restate the winning conditions by saying that II wins either by ensuring that I does not play a subset of Z or, failing this, by playing a partial selector which is *longer* than that played by I.

We will arrive at a contradiction by showing that neither player can win this game.

Claim 1. *Player I does not have a winning strategy.*

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As in the proof of the Basic Coding Lemma our strategy will be to "bound" all of I's plays and then use this bound to construct a play e^* which defeats σ.

Since σ is a winning strategy,

$$U^{(2)}_{(\sigma * y)_I}(Q) \subseteq Z$$

for all $y \in \omega^\omega$. Let e_σ be such that

$$U^{(2)}_{e_\sigma}(Q) = \bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(Q).$$

By assumption, $U^{(2)}_{e_\sigma}(Q)$ is not a selector. So α_{e_σ} exists. Since for all $y \in \omega^\omega$, $\alpha_{e_\sigma} \geqslant \alpha_{(\sigma * y)_I}$, we can take α_{e_σ} as our bound. Choose $a \in X$ such that $\pi(a) = \alpha_{e_\sigma}$. Pick $(x_1, x_2) \in Z \cap (Q_a \times \omega^\omega)$. Let e^* be such that

$$U^{(2)}_{e^*}(Q) = U^{(2)}_{e_\sigma} \cup \{(x_1, x_2)\}.$$

So $e^* \in G$. But $\alpha_{e_\sigma} < \alpha_{e^*}$. In summary, we have that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma} < \alpha_{e^*}$. Thus, by playing e^*, II defeats σ. ⊣

Claim 2. *Player II does not have a winning strategy.*

Proof. Assume toward a contradiction that τ is a winning strategy for II. We shall show that τ yields a selector for Z; in other words, it yields an e^* such that

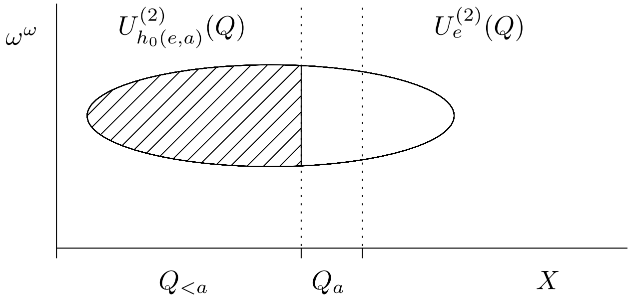
(1) $U^{(2)}_{e^*}(Q) \subseteq Z$ and

(2) for all $a \in X$, $U^{(2)}_{e^*}(Q) \cap (Q_a \times \omega^\omega) \neq \varnothing$ iff $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$.

Choose $h_0 : \omega^\omega \times X \to \omega^\omega$ such that h_0 is $\Sigma^1_1(Q)$ and for all $e, a \in \omega^\omega$,

$$U^{(2)}_{h_0(e,a)}(Q) = U^{(2)}_e(Q) \cap (Q_{<a} \times \omega^\omega).$$

Thus, the set coded by $h_0(e,a)$ is the result of taking the initial segment given by a of the set coded by e.



Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is $\underset{\sim}{\Sigma}^1_1(Q)$ and for all $e \in \omega^\omega$,

$$U^{(2)}_{h_1(e)}(Q) = \bigcup_{a \in X} \big(U^{(2)}_{(h_0(e,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega)\big).$$

Thus, the set coded by $h_1(e)$ is the union of all "a-sections" of sets played by II in response to "$<a$-initial segments" of the set coded by e.

By the recursion theorem there is a fixed point for h_1; that is, there is an e^* such that

$$U^{(2)}_{e^*}(Q) = U^{(2)}_{h_1(e^*)}(Q).$$

This set has the following closure property: if I plays an initial segment of it then II responds with a subset of it. We shall see that $e^* \in G$. Moreover, if $U^{(2)}_{e^*}(Q)$ is *not* a selector then having I play the largest initial segment which is a partial selector, II responds with a *larger* selector, which is a contradiction. Thus, e^* codes a selector. Here are the details.

Subclaim 1. $e^* \in G$.

Proof. Suppose for contradiction that $U^{(2)}_{e^*}(Q) \setminus Z \neq \varnothing$. Choose $(x_1, x_2) \in U^{(2)}_{e^*}(Q) \setminus Z$ with $\pi(x_1)$ minimal. So

$$\begin{aligned}(x_1, x_2) \in U^{(2)}_{e^*}(Q) &= U^{(2)}_{h_1(e^*)}(Q) \\ &= \bigcup_{a \in X} \big(U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega)\big).\end{aligned}$$

Fix $a \in X$ such that

$$(x_1, x_2) \in U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega).$$

The key point is that $h_0(e^*, a) \in G$ since we chose (x_1, x_2) with $\pi(x_1) = \pi(a)$ minimal. Thus, since τ is a winning strategy, $(h_0(e^*, a) * \tau)_{II} \in G$, and so $(x_1, x_2) \in Z$, which is a contradiction. ⊣

Subclaim 2. *α_{e^*} does not exist.*

Proof. Suppose for contradiction that α_{e^*} exists. Let $a \in X$ be such that $\pi(a) = \alpha_{e^*}$. Thus $h_0(e^*, a) \in G$ and $\alpha_{h_0(e^*,a)} = \alpha_{e^*}$. Since τ is a winning strategy for II,

$$\alpha_{(h_0(e^*,a)*\tau)_{II}} > \alpha_{h_0(e^*,a)} = \alpha_{e^*},$$

which is impossible since

$$U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \subseteq U^{(2)}_{e^*}(Q).$$

Thus α_{e^*} does not exist. ⊣

Hence e^* is the code for a selector. ⊣

This completes the proof of the Coding Lemma. ⊣

3.2. Uniform Coding Lemma

We shall need a uniform version of the above theorem. The version we prove is different than that which appears in the literature [6]. We shall need the following uniform version of the recursion theorem.

3.3 Theorem (Uniform Recursion Theorem; Kleene). *Suppose $f : \omega^\omega \to \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1$. Then there is an $e \in \omega^\omega$ such that for all $P, P' \subseteq \omega^\omega$,*

$$U^{(2)}_e(P, P') = U^{(2)}_{f(e)}(P, P').$$

Proof. The proof is the same as before. The key point is that the definition of the fixed point $d(a_0)$ depends only on f and, of course, d, which is uniform in P, P'. ⊣

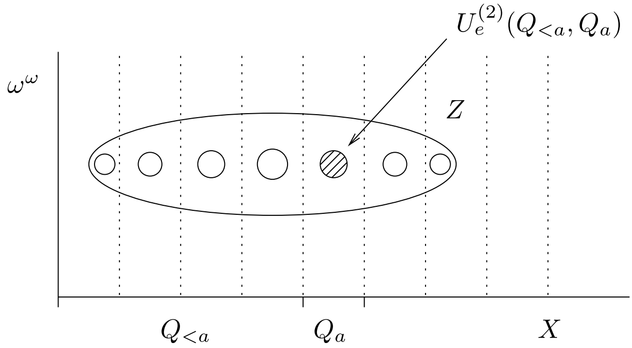
3.4 Theorem (Uniform Coding Lemma). *Assume* ZF + AD. *Suppose $X \subseteq \omega^\omega$ and $\pi : X \to \mathrm{On}$. Suppose $Z \subseteq X \times \omega^\omega$. Then there exists an $e \in \omega^\omega$ such that for all $a \in X$,*

(1) $U^{(2)}_e(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega)$ and

(2) $U^{(2)}_e(Q_{<a}, Q_a) \neq \varnothing$ iff $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$,

where $Q_{<a} = \{b \in X \mid \pi(b) < \pi(a)\}$ and $Q_a = \{b \in X \mid \pi(b) = \pi(a)\}$.

Proof. Here is the picture:



Think of e as providing a "rolling selector". The unshaded ellipse, Z, is sliced into sections $Z \cap (Q_a \times \omega^\omega)$. The Uniform Coding Lemma tells us that there is a simple selector $U_e^{(2)}(Q_{<a}, Q_a)$ for each of these sections which is uniform in the parameters $Q_{<a}$, Q_a; that is, there is a fixed e such that $U_e^{(2)}(Q_{<a}, Q_a)$ selects from $Z \cap (Q_a \times \omega^\omega)$, for all parameters $Q_{<a}$, Q_a.

Assume toward a contradiction that there is no such e. Consider the set G of reals e for which (1) in the statement of the theorem is satisfied:

$$G = \left\{e \in \omega^\omega \mid \forall a \in X \ \left(U_e^{(2)}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega)\right)\right\}.$$

So, for each $e \in G$, (2) in the statement of the theorem fails for some $a \in X$. Let α_e be least such that (2) fails at the α_eth-section:

$$\alpha_e = \min\left\{\alpha \mid \exists a \in X \ (\pi(a) = \alpha \wedge U_e^{(2)}(Q_{<a}, Q_a) = \varnothing \\ \wedge Z \cap (Q_a \times \omega^\omega) \neq \varnothing)\right\}.$$

Now play the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where I wins if $x \in G$ and either $y \notin G$ or $\alpha_x \geqslant \alpha_y$.

Claim 1. *Player I does not have a winning strategy.*

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As before our strategy is to "bound" all of I's plays and then use this bound to construct a play e^* for II which defeats σ.

The proof is as before except that we have to take care to choose a parameter e_σ that works uniformly for all parameters $Q_{<a}$, Q_a: Choose e_σ such that for all $P, P' \subseteq \omega^\omega$,

$$U_{e_\sigma}^{(2)}(P, P') = \bigcup\nolimits_{y \in \omega^\omega} U_{(\sigma * y)_I}^{(2)}(P, P').$$

In particular, e_σ is such that for all $a \in X$,

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) = \bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(Q_{<a}, Q_a).$$

Since σ is a winning strategy for I, $(\sigma * y)_I \in G$ for all $y \in \omega^\omega$. Thus,

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega),$$

that is, $e_\sigma \in G$. Notice that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma}$. We have thus "bounded" all of I's plays. It remains to construct a defeating play e^* for II.

Choose $a \in X$ such that $\pi(a) = \alpha_{e_\sigma}$. So

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) = \varnothing$$

and

$$Z \cap (Q_a \times \omega^\omega) \neq \varnothing.$$

Pick $(x_1, x_2) \in Z \cap (Q_a \times \omega^\omega)$. Choose e^* such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e^*}(P, P') = \begin{cases} U^{(2)}_{e_\sigma}(P, P') & \text{if } x_1 \notin P' \\ U^{(2)}_{e_\sigma}(P, P') \cup \{(x_1, x_2)\} & \text{if } x_1 \in P'. \end{cases}$$

In particular, e^* is such that for all $a' \in X$,

$$U^{(2)}_{e^*}(Q_{<a'}, Q_{a'}) = \begin{cases} U^{(2)}_{e_\sigma}(Q_{<a'}, Q_{a'}) & \text{if } x_1 \notin Q_{a'} \\ U^{(2)}_{e_\sigma}(Q_{<a'}, Q_{a'}) \cup \{(x_1, x_2)\} & \text{if } x_1 \in Q_{a'}. \end{cases}$$

So $e^* \in G$. But $\alpha_{e_\sigma} < \alpha_{e^*}$. In summary, we have that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma} < \alpha_{e^*}$. Thus, by playing e^*, II defeats σ. ⊣

Claim 2. *Player II does not have a winning strategy.*

Proof. Assume toward a contradiction that τ is a winning strategy for II. We seek e^* such that

$$\begin{aligned} &U^{(2)}_{e^*}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega) \quad \text{and} \\ &U^{(2)}_{e^*}(Q_{<a}, Q_a) \neq \varnothing \quad \text{iff} \quad Z \cap (Q_a \times \omega^\omega) \neq \varnothing. \end{aligned}$$

Choose $h_0 : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that h_0 is Σ^1_1 and for all $e, z \in \omega^\omega$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_0(e,z)}(P, P') = \begin{cases} U^{(2)}_e(P, P') & \text{if } z \notin P \cup P' \\ \varnothing & \text{if } z \in P \cup P'. \end{cases}$$

In particular, for all $a \in X$,

$$U^{(2)}_{h_0(e,z)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_e(Q_{<a}, Q_a) & \text{if } z \notin Q_{<a} \cup Q_a \\ \varnothing & \text{if } z \in Q_{<a} \cup Q_a. \end{cases}$$

Notice that for $e \in \omega^\omega$ and $z \in X$, the set $U^{(2)}_{h_0(e,z)}(\cdot,\cdot)$ is such that it agrees with $U^{(2)}_e(\cdot,\cdot)$ for parameters $Q_{<a}, Q_a$ where $\pi(a) < \pi(z)$ and is empty for parameters $Q_{<a}, Q_a$ where $\pi(a) \geqslant \pi(z)$.

Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is $\Sigma^1_1(\tau)$ and for all $e \in \omega^\omega$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_1(e)}(P,P') = \bigcup\nolimits_{z \in P'} U^{(2)}_{(h_0(e,z)*\tau)_{II}}(P,P').$$

In particular, for all $a \in X$,

$$U^{(2)}_{h_1(e)}(Q_{<a},Q_a) = \bigcup\nolimits_{z \in Q_a} U^{(2)}_{(h_0(e,z)*\tau)_{II}}(Q_{<a},Q_a).$$

The idea is roughly this: Fix $e \in \omega^\omega$ and $z \in Q_a$. $U^{(2)}_{h_0(e,z)}(\cdot,\cdot)$ is such that it agrees with $U^{(2)}_e(\cdot,\cdot)$ for parameters $Q_{<\bar a}, Q_{\bar a}$ where $\pi(\bar a) < \pi(a)$ and is empty for parameters $Q_{<\bar a}, Q_{\bar a}$ where $\pi(\bar a) \geqslant \pi(a)$. Think of this as a play for I. In the case of interest, this play will be in G. And since τ is a winning strategy, II's response will be in G and when provided with parameters $Q_{<a}$,Q_a it will select from the a-component. $U_{h_1(e)}(Q_{<a},Q_a)$ is the union of these over $z \in Q_a$.

Let e^* be a fixed point for h_1, by Theorem 3.3.

Subclaim 1. *$e^* \in G$.*

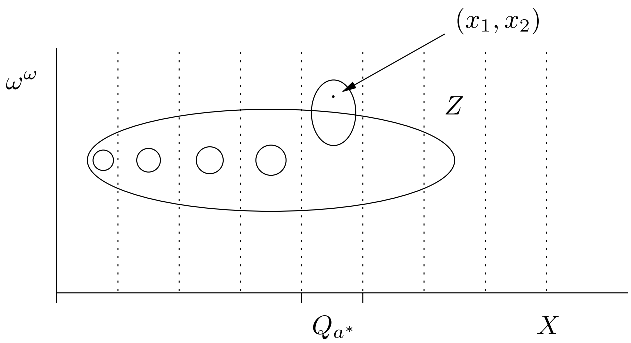
Proof. Suppose for contradiction that for some $a \in X$,

$$U^{(2)}_{e^*}(Q_{<a},Q_a) \smallsetminus (Z \cap (Q_a \times \omega^\omega)) \neq \varnothing.$$

Let a^* be an a where $\pi(a)$ is least such that

$$U^{(2)}_{e^*}(Q_{<a},Q_a) \smallsetminus (Z \cap (Q_a \times \omega^\omega)) \neq \varnothing.$$

Choose $(x_1,x_2) \in U^{(2)}_{e^*}(Q_{<a^*},Q_{a^*}) \smallsetminus (Z \cap (Q_{a^*} \times \omega^\omega))$.



So

$$\begin{aligned}(x_1,x_2) \in U^{(2)}_{e^*}(Q_{<a^*},Q_{a^*}) &= U^{(2)}_{h_1(e^*)}(Q_{<a^*},Q_{a^*}) \\ &= \bigcup\nolimits_{z \in Q_{a^*}} U^{(2)}_{(h_0(e^*,z)*\tau)_{II}}(Q_{<a^*},Q_{a^*}).\end{aligned}$$

Fix $z^* \in Q_{a^*}$ such that

$$(x_1, x_2) \in U^{(2)}_{(h_0(e^*,z^*)*\tau)_{II}}(Q_{<a^*}, Q_{a^*}).$$

The key point is that $h_0(e^*, z^*) \in G$: By the definition of h_0, for all $a \in X$ and for all $z \in \omega^\omega$,

$$U^{(2)}_{h_0(e^*,z)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_{e^*}(Q_{<a}, Q_a) & \text{if } z \notin Q_{<a} \cup Q_a \\ \varnothing & \text{if } z \in Q_{<a} \cup Q_a. \end{cases}$$

We have fixed $z^* \in Q_{a^*}$. For this fixed value, allowing a to vary, we have (i) $z^* \notin Q_{<a} \cup Q_a$ iff $\pi(a) < \pi(a^*)$ and (ii) $z^* \in Q_{<a} \cup Q_a$ iff $\pi(a) \geqslant \pi(a^*)$. So

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = U^{(2)}_{e^*}(Q_{<a}, Q_a),$$

for all a such that $\pi(a) < \pi(a^*)$ and

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = \varnothing,$$

for all a such that $\pi(a) \geqslant \pi(a^*)$. Thus,

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega),$$

for *all* $a \in X$, i.e. $h_0(e^*, z^*) \in G$.

Now since τ is a winning strategy for II, $(h_0(e^*, z^*) * \tau)_{II} \in G$, which means that $(x_1, x_2) \in Z$, a contradiction. ⊣

Subclaim 2. *α_{e^*} does not exist.*

Proof. Suppose not. Let $a^* \in X$ be such that $\pi(a^*) = \alpha_{e^*}$, and choose $z^* \in Q_{a^*}$. Thus, $h_0(e^*, z^*) \in G$, since $e^* \in G$ by Subclaim 1, and $h_0(e^*, z^*)$ is defined such that for all $a \in X$,

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_{e^*}(Q_{<a}, Q_a) & \text{if } \pi(a) < \pi(a^*) \\ \varnothing & \text{if } \pi(a) \geqslant \pi(a^*). \end{cases}$$

So, $\alpha_{h_0(e^*,z^*)} = \alpha_{e^*}$. Since τ is a winning strategy for II,

$$\alpha_{(h_0(e^*,z^*)*\tau)_{II}} > \alpha_{h_0(e^*,z^*)} = \alpha_{e^*},$$

which is impossible since

$$U^{(2)}_{(h_0(e^*,z^*)*\tau)_{II}}(Q_{<a}, Q_a) \subseteq U^{(2)}_{e^*}(Q_{<a}, Q_a)$$

for all $a \in X$. ⊣

Thus, e^* is the code for a uniform selector. ⊣

This completes the proof of the Uniform Coding Lemma. ⊣

3.5 Remark. The game in the above proof is definable from X, π, and Z and no choice is required to show that it works. Thus, if these parameters are OD, then ZF + OD-determinacy suffices for the proof.

3.6 Remark. The version of the Coding Lemma stated in Lemma 2.32 follows from the Uniform Coding Lemma: Take $X = U$ and $\pi : U \to \mathrm{On}$ given by $\pi(x) = \delta_x$. Then

$$Z^* = \bigcup_{x \in U} U_e^{(2)}(Q_{<\delta_x}, Q_{\delta_x}).$$

This gives (i). For (ii) note that

$$Z^* \cap (U_{\leqslant \delta_x} \times \omega^\omega) = \bigcup_{y \in U_{\leqslant \delta_x}} U_e^{(2)}(Q_{<\delta_y}, Q_{\delta_y}),$$

which is $\underset{\sim}{\Delta}^2_1$.

2 Open Question (Strong Coding Lemma). Suppose $X \subseteq \omega^\omega$ and $\pi : X \to \mathrm{On}$. Let $\leqslant_X$ be the prewellordering associated with π. Suppose $Z \subseteq X^{<\omega}$ is a tree. Then there exists a subtree $Z^* \subseteq Z$ such that

(1) Z^* is $\Sigma^1_1(\leqslant_X)$ and

(2) for all $\vec{s} \in (Z^*)^{<\omega}$ and for all $a \in X$, if there exists a $t \in Q_a$ such that $\vec{s}^\frown t \in Z$ then there exists a $t \in Q_a$ such that $\vec{s}^\frown t \in Z^*$,

where $Q_a = \{b \in X \mid \pi(b) = \pi(a)\}$.

3.3. Applications

In this section we will bring together some basic results and key applications of the above coding lemmas that will be of use later. It will be useful to do things in a slightly more general fashion than is customary.

For a set X, let

$$\Theta_X = \sup\{\alpha \mid \text{there is an } \mathrm{OD}_X \text{ surjection } \pi : \omega^\omega \to \alpha\}.$$

3.7 Lemma. *Assume* ZF *and suppose X is a set. Then there is an* OD_X *sequence $A = \langle A_\alpha \mid \alpha < \Theta_X\rangle$ such that A_α is a prewellordering of the reals of length α.*

Proof. Let A_α be the $<_{\mathrm{OD}_X}$-least prewellordering of the reals of length α, where $<_{\mathrm{OD}_X}$ is the canonical OD_X well-ordering of the OD_X sets. ⊣

3.8 Lemma. *Assume* ZF *and suppose X is a set. Suppose that every set is* $\mathrm{OD}_{X,y}$ *for some real y. Then $\Theta = \Theta_X$.*

Proof. Fix $\alpha < \Theta$. We have to show that there is an OD_X surjection $\pi : \omega^\omega \to \alpha$. There is certainly an $\mathrm{OD}_{X,y}$ surjection for some y. For each $y \in \omega^\omega$, let π_y be the $<_{\mathrm{OD}_{X,y}}$-least such surjection if one exists and let it be undefined otherwise. We can now "average over the reals" to eliminate the dependence on real parameters, letting

$$\begin{aligned} \pi : \omega^\omega &\to \alpha \\ y &\mapsto \begin{cases} \pi_{(y)_0}((y)_1) & \text{if } \pi_{(y)_0} \text{ is defined} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is an OD_X surjection. ⊣

The following theorem is essentially due to Moschovakis. We are just replacing AD with OD_X-determinacy and the changes are straightforward.

3.9 Theorem. *Assume* $\mathrm{ZF} + \mathrm{OD}_X$*-determinacy, where* X *is a set. Then*

$$\mathrm{HOD}_X \models \Theta_X \textit{ is strongly inaccessible.}$$

Proof. First we show that Θ_X is regular in HOD_X. By Lemma 3.7 there is an OD_X sequence

$$\langle \pi_\alpha \mid \alpha < \Theta_X \rangle$$

where each $\pi_\alpha : \omega^\omega \to \alpha$ is a surjection. Assume for contradiction that Θ_X is singular in HOD_X and let

$$f : \eta \to \Theta_X$$

be an OD_X cofinal map. Let g be an OD_X surjection from ω^ω onto η. Then the map

$$\begin{aligned} \pi : \omega^\omega &\to \Theta_X \\ x &\mapsto \pi_{f \circ g((x)_0)}((x)_1) \end{aligned}$$

is an OD_X surjection, which contradicts the definition of Θ_X.

We now show that Θ_X is a strong limit in HOD_X. For each $\eta < \Theta_X$, we have to show that $|\mathscr{P}(\eta)|^{\mathrm{HOD}_X} < \Theta_X$. For this it suffices to show that there is an OD_X surjection

$$\pi : \omega^\omega \to \mathscr{P}(\eta)^{\mathrm{HOD}_X},$$

since if $|\mathscr{P}(\eta)|^{\mathrm{HOD}_X} \geqslant \Theta_X$ then there would be an OD_X surjection $\rho : \mathscr{P}(\eta) \to \Theta_X$ and so $\rho \circ \pi : \omega^\omega \to \Theta_X$ would be an OD_X surjection, which contradicts the definition of Θ_X.

Let $\pi_\eta : \omega^\omega \to \eta$ be an OD_X surjection and, for $\alpha < \eta$, let $Q_{<\alpha}$ and Q_α be the usual objects defined relative to π_η. For $e \in \omega^\omega$, let

$$S_e = \{\beta < \eta \mid U_e^{(2)}(Q_{<\beta}, Q_\beta) \neq \varnothing\}.$$

The key point is that since π_η is OD_X the game for the Uniform Coding Lemma for $Z = \bigcup\{Q_\alpha \times \omega^\omega \mid \alpha \in S\}$ is determined for each $S \in \mathscr{P}(\eta)^{\mathrm{HOD}_X}$.

(See Remark 3.5.) Thus, every $S \in \mathscr{P}(\eta)^{\mathrm{HOD}_X}$ has the form S_e for some $e \in \omega^\omega$ and hence

$$\begin{aligned}\pi : \omega^\omega &\to \mathscr{P}(\eta)^{\mathrm{HOD}_X} \\ e &\mapsto S_e\end{aligned}$$

is a surjection. Moreover, π is OD_X (since π_η is OD_X), which completes the proof. ⊣

The above theorem has the following corollary. The first part also follows from early work of Friedman and Solovay. The second part is a simple application of the Coding Lemma and Solovay's Lemma 2.23.

3.10 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AD} + V = L(\mathbb{R})$. *Then*

(1) $\mathrm{HOD}^{L(\mathbb{R})} \models \Theta$ *is strongly inaccessible and*

(2) $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$.

Proof. (1) This follows immediately from Theorem 3.9 and Lemma 3.8.

(2) Since $\mathrm{HOD}^{L(\mathbb{R})}$ is Σ_1-definable over $L(\mathbb{R})$ (with the parameter $\mathbb{R}$) and since $L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R})$ (by Lemma 2.23),

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

Thus, it suffices to show

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

The right-to-left inclusion is immediate. For the left-to-right inclusion suppose $x \in \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$. We have to show that $x \in L_\Theta(\mathbb{R})$. Since Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$, x is coded by a set of ordinals $A \subseteq \alpha$ where $\alpha < \Theta$. However, by the proof of Theorem 3.9, $\mathscr{P}(\alpha) \in L_\Theta(\mathbb{R})$, for each $\alpha < \Theta$. Thus, $x \in L_\Theta(\mathbb{R})$, which completes the proof. ⊣

3 Open Question. Assume $\mathrm{ZF} + \mathrm{DC} + V = L(\mathbb{R})$.

(1) Suppose that for every $\alpha < \Theta$ there is a surjection $\pi : \omega^\omega \to \mathscr{P}(\alpha)$. Must AD hold in $L(\mathbb{R})$?

(2) Suppose Θ is inaccessible. Must AD hold in $L(\mathbb{R})$?

3.11 Theorem (Kunen). *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Suppose* $\lambda < \Theta$ *and* μ *is an ultrafilter on* λ. *Then* μ *is* OD.

Proof. Let $\leqslant$ be a prewellordering of ω^ω of length λ. Let $\pi : \omega^\omega \to \mathscr{P}(\lambda)$ be the surjection derived from $\leqslant$ as in the above proof. For $x \in \omega^\omega$, let

$$A_x = \bigcap\{\pi(y) \mid \pi(y) \in \mu \wedge y \leqslant_T x\}.$$

Since there are only countably many such y and AD implies that all ultrafilters are countably complete (Theorem 2.8), A_x is non-empty. Let

$$f(x) = \bigcap A_x.$$

Notice that A_x and $f(x)$ depend only on the Turing degree of x. In particular, we can regard f as a function from the Turing degrees $\mathscr{D}_T$ into the ordinals. Notice also that

$$A \in \mu \quad \text{iff} \quad \text{for a cone of } x, f(x) \in A$$

since if $B \in \mu$ then, for any $x \geqslant_T x_0$ we have $f(x) \in B$, where x_0 is such that $\pi(x_0) = B$. We can now "erase" reference to the prewellordering by taking the ultrapower. Let μ_T be the cone ultrafilter on the Turing degrees (see Theorem 2.9) and consider the ultrapower $V^{\mathscr{D}_T}/\mu_T$. By DC the ultrapower is well-founded. So we can let M be the transitive collapse of $V^{\mathscr{D}_T}/\mu_T$ and let

$$j : V \to M$$

be the canonical map. Letting γ be the ordinal represented by f, we have

$$B \in \mu \quad \text{iff} \quad \gamma \in j(B)$$

and so μ is OD. ⊣

4. A Woodin Cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$

Our main aim in this section is to prove the following theorem:

4.1 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

This will serve as a warm-up for the proof of the Generation Theorem in the next section. The proof that we give appeals to DC at only one point (Lemma 4.8) and as we shall see in the next section one can avoid this appeal and prove the result in $\mathrm{ZF} + \mathrm{AD}$. See Theorem 5.36.

In Sect. 4.1 we will establish the reflection phenomenon that will play the role played by boundedness in Sect. 2 and we will define for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$ that is intended to witness that $\underset{\sim}{\delta}^2_1$ is λ-strong. In Sect. 4.2 we shall introduce and motivate the notion of *strong normality* by showing that the strong normality of μ_λ ensures that $\underset{\sim}{\delta}^2_1$ is λ-strong. We will then show how reflection and uniform coding combine to secure strong normality. In Sect. 4.3 we will prove the main theorem by relativizing the construction to subsets of Θ. Throughout this section we work in $L(\mathbb{R})$ and so when we write $\underset{\sim}{\delta}^2_1$ and Θ we will always be referring to these notions as interpreted in $L(\mathbb{R})$.

4.1. Reflection

We have seen that ZF + AD implies that Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$. Our next task is to show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \underset{\sim}{\delta}^2_1 \text{ is } \lambda\text{-strong},$$

for all $\lambda < \Theta$. The proof will then relativize to subsets of Θ that are in $\mathrm{HOD}^{L(\mathbb{R})}$ and thereby establish the main theorem.

The ultrafilters that witness strength cannot come from the "sup" game of Sect. 2 since the ultrafilters produced by this game concentrate on ω-club sets, whereas to witness strength we will need ultrafilters according to which there are measure-one many measurable cardinals below $\underset{\sim}{\delta}^2_1$. For this reason we will have to use a variant of the "sup" game. In this variant the role of boundedness will be played by a certain reflection phenomenon.

The reflection phenomenon we have in mind does not presuppose any determinacy assumptions. For the time being work in ZF + $\mathrm{AC}_\omega(\mathbb{R})$. The main claim is that there is a function $F : \underset{\sim}{\delta}^2_1 \to L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ which is Δ_1-definable over $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ and for which the following *reflection phenomenon* holds:

> For all $X \in L(\mathbb{R}) \cap \mathrm{OD}^{L(\mathbb{R})}$, for all Σ_1 formulas φ, and for all $z \in \omega^\omega$, if
>
> $$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$
>
> then there exists a $\delta < \underset{\sim}{\delta}^2_1$ such that
>
> $$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

One should think of F as a sequence that contains "proxies" or "generic witnesses" for each $\mathrm{OD}^{L(\mathbb{R})}$ set X: Given *any* Σ_1 fact (with a real parameter) about *any* $\mathrm{OD}^{L(\mathbb{R})}$ set X there is a "proxy" $F(\delta)$ in our fixed sequence that witnesses the same fact.

The function F is defined (much like $\diamondsuit$) in terms of the least counterexample. To describe this in more detail let us first recall some basic facts from Sect. 2.4 concerning $L(\mathbb{R})$ and the theory T_0: There are arbitrarily large α such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. In particular,

$$L_\Theta(\mathbb{R}) \models \mathrm{T}_0.$$

Moreover, since

$$L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R}),$$

there are arbitrarily large $\alpha < \Theta$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. Similarly, there are arbitrarily large $\alpha < \underset{\sim}{\delta}^2_1$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. However, notice that it is *not* the case $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \models \mathrm{T}_0$ (by Lemma 2.27).

Because of the greater maneuvering room provided by levels $L_\alpha(\mathbb{R})$ that satisfy T_0 we will concentrate (for example, in reflection arguments) on such levels. For example, we can use these levels to give a first-order definition of

$\mathrm{OD}^{L(\mathbb{R})}$ and the natural well-ordering $<_{\mathrm{OD}^{L(\mathbb{R})}}$ on the $\mathrm{OD}^{L(\mathbb{R})}$ sets. For the latter, given $X \in \mathrm{OD}^{L(\mathbb{R})}$, let

$$\begin{aligned}\alpha_X = {} & \text{the least } \alpha \text{ such that} \\ & (1)\ L_\alpha(\mathbb{R}) \models \mathrm{T}_0, \\ & (2)\ X \in L_\alpha(\mathbb{R}), \text{and} \\ & (3)\ X \text{ is definable in } L_\alpha(\mathbb{R}) \text{ from ordinal parameters;}\end{aligned}$$

let φ_X be the least formula that defines X from ordinal parameters in $L_\alpha(\mathbb{R})$; and let $\vec{\xi}_X$ be the lexicographically least sequence of ordinal parameters used to define X in $L_\alpha(\mathbb{R})$ via φ_X. Given X and Y in $\mathrm{OD}^{L(\mathbb{R})}$, working in $L(\mathbb{R})$ set

$$\begin{aligned} X <_{\mathrm{OD}} Y \quad \text{iff} \quad & \alpha_X < \alpha_Y \text{ or} \\ & \alpha_X = \alpha_Y \text{ and } \varphi_X < \varphi_Y \text{ or} \\ & \alpha_X = \alpha_Y \text{ and } \varphi_X = \varphi_Y \text{ and } \vec{\xi}_X <_{\mathrm{lex}} \vec{\xi}_Y. \end{aligned}$$

Since the $L_\alpha(\mathbb{R})$ hierarchy is Σ_1-definable in $L(\mathbb{R})$, it follows that $\mathrm{OD}^{L(\mathbb{R})}$ and $(<_{\mathrm{OD}})^{L(\mathbb{R})}$ are Σ_1-definable in $L(\mathbb{R})$. (This is in contrast to the usual definitions of these notions, which are Σ_2 since they involve existential quantification over the V_α hierarchy, which is Π_1.) Notice that if $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$, then

$$(<_{\mathrm{OD}})^{L_\alpha(\mathbb{R})} \trianglelefteq (<_{\mathrm{OD}})^{L(\mathbb{R})}.$$

Furthermore, if $L_\alpha(\mathbb{R}) \prec_1 L(\mathbb{R})$, then

$$\mathrm{OD}^{L_\alpha(\mathbb{R})} = \mathrm{OD}^{L(\mathbb{R})} \cap L_\alpha(\mathbb{R}) \quad \text{and} \quad (<_{\mathrm{OD}})^{L_\alpha(\mathbb{R})} = (<_{\mathrm{OD}})^{L(\mathbb{R})} \restriction L_\alpha(\mathbb{R}).$$

For example,

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

(For this it is crucial that we use the Σ_1 definition given above since the Σ_2 definition involves quantification over the V_α hierarchy and yet in $L_{\utilde{\delta}^2_1}(\mathbb{R})$ even the level $V_{\omega+2}$ does not exist.) Our goal can thus be rephrased as that of showing

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} \models \utilde{\delta}^2_1 \text{ is a strong cardinal.}$$

We are now in a position to define the reflection function F. If the reflection phenomenon fails in $L(\mathbb{R})$ with respect to $F \restriction \utilde{\delta}^2_1$ then (by Replacement) there is some level $L_\alpha(\mathbb{R})$ which satisfies T_0 over which the reflection phenomenon fails with respect to $F \restriction \utilde{\delta}^2_1$. This motivates the following definition:

4.2 Definition. Assume T_0. Suppose that $F \restriction \delta$ is defined. Let $\vartheta(\delta)$ be least such that

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \mathrm{T}_0 \text{ and there is an } X \in L_{\vartheta(\delta)}(\mathbb{R}) \cap \mathrm{OD}^{L_{\vartheta(\delta)}(\mathbb{R})} \text{ such that}$$

($\star$) there is a Σ_1 formula φ and a real z such that

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \varphi[z, X, \delta, \mathbb{R}]$$

and for all $\bar{\delta} < \delta$,

$$L_{\vartheta(\delta)}(\mathbb{R}) \not\models \varphi[z, F(\bar{\delta}), \bar{\delta}, \mathbb{R}]$$

(if such an ordinal exists) and then set $F(\delta) = X$ where X is $(<_{\mathrm{OD}})^{L_{\vartheta(\delta)}}$-least such that ($\star$) holds.

We have to establish two things: First, $F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$. Second, $F(\underset{\sim}{\delta}^2_1)$ is not defined. This implies that the reflection phenomenon holds with respect to F.

4.3 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. *Then*

(1) if $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$, *then* $(F)^{L_\alpha(\mathbb{R})} = F \restriction \gamma$ *for some* γ,

(2) $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})} = F \restriction \underset{\sim}{\delta}^2_1$, *and*

(3) $F(\delta)$ *is defined for all* $\delta < \underset{\sim}{\delta}^2_1$.

Proof. For (1) suppose that $(F \restriction \delta)^{L_\alpha(\mathbb{R})} = F \restriction \delta$ with the aim of showing that $(F(\delta))^{L_\alpha(\mathbb{R})} = (F(\delta))^{L(\mathbb{R})}$. The point is that

$$L_\alpha(\mathbb{R}) \models \vartheta(\delta) \text{ exists}$$

if and only if

$$(\vartheta(\delta))^{L(\mathbb{R})} < \alpha,$$

in which case

$$(\vartheta(\delta))^{L_\alpha(\mathbb{R})} = (\vartheta(\delta))^{L(\mathbb{R})} \quad \text{and} \quad (F(\delta))^{L_\alpha(\mathbb{R})} = (F(\delta))^{L(\mathbb{R})},$$

by the locality of the definition of F and the assumption that $(F \restriction \delta)^{L_\alpha(\mathbb{R})} = F \restriction \delta$.

For (2) first notice that we can make sense of F as defined over levels (such as $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$) that do not satisfy T_0 by letting, for an arbitrary ordinal ξ,

$$F^{L_\xi(\mathbb{R})} = \bigcup \{ F^{L_\alpha(\mathbb{R})} \mid \alpha < \xi \text{ and } L_\alpha(\mathbb{R}) \models \mathrm{T}_0 \}.$$

Thus, $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})} = F \restriction \gamma$ for some γ, by (1). Assume for contradiction that (2) fails, that is, for some $\gamma < \underset{\sim}{\delta}^2_1$, $F(\gamma)$ is defined and yet $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}(\gamma)$ is not defined. Since in $L(\mathbb{R})$, $\vartheta(\gamma)$ and $F(\gamma)$ are defined, the following is a true Σ_1 statement about γ:

$$\exists \alpha > \gamma \, (L_\alpha(\mathbb{R}) \models \mathrm{T}_0 + \vartheta(\gamma) \text{ exists.})$$

Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \prec_1 L(\mathbb{R})$, this statement holds in $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ and so $F^{L_\alpha(\mathbb{R})}(\gamma)$ is defined and hence $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}(\gamma)$ is defined, which is a contradiction.

For (3) assume for contradiction that $\gamma < \underset{\sim}{\delta}^2_1$, where $\gamma = \mathrm{dom}(F)$. By (2) (and the definition of $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}$) there is an $\alpha < \underset{\sim}{\delta}^2_1$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and $F^{L_\alpha(\mathbb{R})} = F\restriction\gamma = F$. We claim that this implies that

$$L_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}),$$

which is a contradiction (by Theorem 2.28). Suppose

$$L(\mathbb{R}) \models \psi[z, \mathbb{R}]$$

where ψ is a Σ_1 formula and $z \in \omega^\omega$. We have to show that $L_\alpha(\mathbb{R}) \models \psi[z, \mathbb{R}]$. By Replacement there is an ordinal β such that

$$L_\beta(\mathbb{R}) \models \psi[z, \mathbb{R}].$$

Consider the Σ_1 statement $\varphi[z, X, \mathbb{R}]$ expressing "There exists ξ such that $X = L_\xi(\mathbb{R})$ and $X \models \psi[z, \mathbb{R}]$". Letting $\vartheta > \beta$ be such that $L_\vartheta(\mathbb{R}) \models \mathrm{T}_0$ we have: there exists an $X \in L_\vartheta(\mathbb{R}) \cap \mathrm{OD}^{L_\vartheta(\mathbb{R})}$ (namely, $X = L_\beta(\mathbb{R})$) such that

$$L_\vartheta(\mathbb{R}) \models \mathrm{T}_0 + \varphi[z, X, \mathbb{R}].$$

Moreover, since $\vartheta(\gamma)$ does not exist, it follows (by the definition of $\vartheta(\gamma)$) that there exists a $\bar{\delta} < \gamma$ such that

$$L_\vartheta(\mathbb{R}) \models \varphi[z, F(\bar{\delta}), \mathbb{R}].$$

Thus (unpacking $\varphi[z, X, \mathbb{R}]$) there exists a ξ such that $F(\bar{\delta}) = L_\xi(\mathbb{R})$ and $L_\xi(\mathbb{R}) \models \psi[z, \mathbb{R}]$. Since $F \subseteq L_\alpha(\mathbb{R})$, $\xi < \alpha$ and so, by upward absoluteness,

$$L_\alpha(\mathbb{R}) \models \psi[z, \mathbb{R}],$$

which completes the proof. $\dashv$

It follows that $F\restriction\underset{\sim}{\delta}^2_1 : \underset{\sim}{\delta}^2_1 \to L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ is total and Δ_1-definable over $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$. It remains to see that $F(\underset{\sim}{\delta}^2_1)$ is not defined.

4.4 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. *For all* $X \in \mathrm{OD}^{L(\mathbb{R})}$, *for all* Σ_1 *formulas* φ, *and for all* $z \in \omega^\omega$ *if*

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

then there exists a $\delta < \underset{\sim}{\delta}^2_1$ *such that*

$$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Proof. The idea of the proof is straightforward but the details are somewhat involved.

Assume for contradiction that there is an $X \in \mathrm{OD}^{L(\mathbb{R})}$, a Σ_1 formula φ, and $z \in \omega^\omega$ such that

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

and for all $\delta < \underset{\sim}{\delta}^2_1$,

$$L(\mathbb{R}) \not\models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Step 1. By Replacement, let $\vartheta_0 > \underset{\sim}{\delta}^2_1$ be least such that

(1.1) $L_{\vartheta_0}(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_{\vartheta_0}(\mathbb{R}) \cap \mathrm{OD}^{L_{\vartheta_0}(\mathbb{R})}$ and

($\star$) there is a Σ_1 formula φ and a real z such that

$$L_{\vartheta_0}(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

and for all $\delta < \underset{\sim}{\delta}^2_1$

$$L_{\vartheta_0}(\mathbb{R}) \not\models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Let X_0 be least (in the order of definability) such that (1.1) and for this choice pick φ_0 and z_0 such that ($\star$). (Thus we have let $\vartheta_0 = \vartheta(\underset{\sim}{\delta}^2_1)$, $X_0 = F(\underset{\sim}{\delta}^2_1)$, and we have picked witnesses φ_0 and z_0 to the failure of reflection with respect to $F(\underset{\sim}{\delta}^2_1)$.)

Notice that $L_{\vartheta_0}(\mathbb{R}) \models \underset{\sim}{\delta}^2_1$ exists $+ F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$. Since

$$F^{L_{\vartheta_0}(\mathbb{R})} \restriction \underset{\sim}{\delta}^2_1 = F \restriction \underset{\sim}{\delta}^2_1,$$

by Lemma 4.3, (1.1) is equivalent to the internal statement $L_{\vartheta_0}(\mathbb{R}) \models \mathrm{T}_0 +$ "reflection fails with respect to $F \restriction \underset{\sim}{\delta}^2_1$". It is this internal statement that we will reflect to get a contradiction. We have that for all $\delta < \underset{\sim}{\delta}^2_1$,

(1.2) $L_{\vartheta_0}(\mathbb{R}) \not\models \varphi_0[z_0, F(\delta), \delta, \mathbb{R}]$.

Our strategy is to reflect to get $\bar{\vartheta} < \underset{\sim}{\delta}^2_1$ such that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}].$$

By upward absoluteness, this will contradict (1.2). To implement this strategy we need the appropriate Σ_1 fact (in a real) to reflect.

Step 2. The following is a true Σ_1 statement about φ_0 and z_0 (as witnessed by taking α to be ϑ_0 from Step 1): There is an α such that

(2.1) $L_\alpha(\mathbb{R}) \models \underset{\sim}{\delta}^2_1$ exists $+$ $F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$,

(2.2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_\alpha(\mathbb{R}) \cap \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and

($\star$) there is a Σ_1 formula φ and a real z such that

$$L_\alpha(\mathbb{R}) \models \varphi[z, X, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\alpha(\mathbb{R}) \not\models \varphi[z, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}],$$

(2.3) if $\beta < \alpha$ then it is not the case that $L_\beta(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_\beta(\mathbb{R}) \cap \mathrm{OD}^{L_\beta(\mathbb{R})}$ and

($\star$) there is a Σ_1 formula φ and a real z such that

$$L_\beta(\mathbb{R}) \models \varphi[z, X, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\beta(\mathbb{R}) \not\models \varphi[z, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}],$$

and

(2.4) if $\bar{X}$ is least (in the order of definability) such that (2.2) then

$$L_\alpha(\mathbb{R}) \models \varphi_0[z_0, \bar{X}, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\alpha(\mathbb{R}) \not\models \varphi_0[z_0, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}].$$

(Notice that in (2.3) the ordinal $\underset{\sim}{\delta}^2_1$ and the function F are computed in $L_\alpha(\mathbb{R})$ while the formulas are evaluated in $L_\beta(\mathbb{R})$.) Thus (2.1) ensures (by Lemma 4.3) that $F^{L_\alpha(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})} = F\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$, (2.2) says that $L_\alpha(\mathbb{R})$ satisfies "reflection is failing with respect to $F^{L_\alpha(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$" and, because of (2.1), this ensures that $\vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$ exists, (2.3) ensures in addition that $\alpha = \vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$, and (2.4) says that φ_0 and z_0 (as chosen in Step 1) witness the existence of $\vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$.

Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \prec_1^{\mathbb{R}} L(\mathbb{R})$ and φ_0 and z_0 can be coded by a single real, the least ordinal α witnessing the existential of the above statement must be less than $\underset{\sim}{\delta}^2_1$. Let $\bar{\vartheta}$ be this ordinal.

Step 3. We claim that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}],$$

which finishes the proof since by upward absoluteness this contradicts (1.2).

The ordinal $\bar{\vartheta}$ has the Σ_1-properties listed in (2.1)–(2.4) for α. So we have: (4.1) $L_{\bar{\vartheta}}(\mathbb{R}) \models$ "$\underset{\sim}{\delta}^2_1$ exists" + "$F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$" and so (by Lemma 4.3) $F^{L_{\bar{\vartheta}}(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})} = F\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}$, (4.2) $L_{\bar{\vartheta}}(\mathbb{R})$ satisfies "reflection is failing with respect to $F^{L_{\bar{\vartheta}}(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}$" and, because of (4.1), this ensures that $\vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$ exists, (4.3) $\bar{\vartheta} = \vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$, and (4.4) φ_0 and z_0 (as chosen in Step 1) witness the existence of $\vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$. Therefore, by the definition of F, (4.4) implies that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}],$$

which contradicts (1.2). ⊣

We will need a slight strengthening of the above theorem. This involves the notion of the *reflection filter*, which in turn involves various universal sets.

Let U_X be a good universal $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$ set of reals. So U_X is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$ subset of $\omega^\omega \times \omega^\omega$ such that each $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R} \cup \{\mathbb{R}\}\})$ subset of ω^ω is of the form $(U_X)_t$ for some $t \in \omega^\omega$. For each $\delta < \underset{\sim}{\delta}^2_1$, let U_δ be the universal $\Sigma_1(L(\mathbb{R}), \{F(\delta), \delta, \mathbb{R}\})$ set obtained using the same definition used for U_X except with X and $\underset{\sim}{\delta}^2_1$ replaced by the reflected proxies $F(\delta)$ and δ. As before, we shall identify each of U_X and U_δ with a set of reals using our recursive bijection between $\omega^\omega \times \omega^\omega$ and ω^ω.

For each Σ_1 formula φ and for each real y, there exists a $z_{\varphi,y} \in \omega^\omega$ such that

$$z_{\varphi,y} \in U_X \ \text{ iff } \ L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}^2_1, \mathbb{R}].$$

In such a situation we say that $z_{\varphi,y}$ *certifies* the Σ_1 fact φ about y. The key property is, of course, that if $z_{\varphi,y} \in U_\delta$ then $L(\mathbb{R}) \models \varphi[y, F(\delta), \delta, \mathbb{R}]$. Notice that the real $z_{\varphi,y}$ is recursive in y (uniformly).

In what follows we will drop reference to φ and y and simply write $z \in U_X$, it being understood that the formula and parameter are encoded in z. In these terms Theorem 4.4 can be recast as stating that if $z \in U_X$ then there is an ordinal $\delta < \underset{\sim}{\delta}^2_1$ such that $z \in U_\delta$, in other words, $U_X \subseteq \bigcup_{\delta < \underset{\sim}{\delta}^2_1} U_\delta$. But notice that equality fails since different X can have radically different "reflection points".

For $z \in U_X$, let

$$S_z = \{\delta < \underset{\sim}{\delta}^2_1 \mid z \in U_\delta\}$$

and set

$$\mathscr{F}_X = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \exists z \in U_X\, (S_z \subseteq S)\}.$$

Equivalently, for a Σ_1 formula φ and a real y such that

$$L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

let

$$S_{\varphi,y} = \{\delta < \underset{\sim}{\delta}^2_1 \mid L(\mathbb{R}) \models \varphi[y, F(\delta), \delta, \mathbb{R}]\}$$

and set

$$\mathscr{F}_X = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \text{there is a } \Sigma_1 \text{ formula } \varphi \text{ and a } y \in \omega^\omega \text{ such that } L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}^2_1, \mathbb{R}] \text{ and } S_{\varphi,y} \subseteq S\}.$$

Notice that we can reflect to points δ where the proxies $F(\delta), \delta$ resemble $X, \underset{\sim}{\delta}^2_1$ as much as we like. For example, suppose

$$L(\mathbb{R}) \models \psi[y, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

and consider the following Σ_1 statement:

There is an α such that

$$\begin{aligned}
&L_\alpha(\mathbb{R}) \models \mathrm{T}_0,\\
&\delta = (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \quad F\restriction\delta = (F)^{L_\alpha(\mathbb{R})}, \quad \text{and} \quad F(\delta) \in \mathrm{OD}^{L_\alpha(\mathbb{R})}, \quad \text{and}\\
&L_\alpha(\mathbb{R}) \models \psi[y, F(\delta), \delta, \mathbb{R}].
\end{aligned}$$

If we replace δ by $\underset{\sim}{\delta}^2_1$ and $F(\delta)$ by X then this statement is true. It follows that the statement holds for $\mathscr{F}_X$-almost all δ. The second clause ensures that each such δ is a "local $\underset{\sim}{\delta}^2_1$" and that the "local computation of F up to δ" coincides with F. By altering ψ and y we can increase the degree to which the proxies $F(\delta), \delta$ resemble $X, \underset{\sim}{\delta}^2_1$.

4.5 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. *Then* $L(\mathbb{R}) \models \mathscr{F}_X$ *is a countably complete filter.*

Proof. Upward closure and the non-triviality condition are immediate. It remains to prove countable completeness. Suppose $\{S_n \mid n < \omega\} \subseteq \mathscr{F}_X$. For $n < \omega$, let $z_n \in U_X$ be such that $S_{z_n} \subseteq S_n$. Let $z \in \omega^\omega$ be such that $(z)_n = z_n$ for all $n < \omega$. The following is a true Σ_1 statement about z, X, $\underset{\sim}{\delta}^2_1$, and $\mathbb{R}$:

There is an α such that

(1) $\underset{\sim}{\delta}^2_1 < \alpha$,
(2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$,
(3) $X \in \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and
(4) for all $n < \omega, (z)_n \in (U_X)^{L_\alpha(\mathbb{R})}$.

Let $z^* \in U_X$ certify this statement. It follows that for each $\delta < \underset{\sim}{\delta}^2_1$ such that $z^* \in U_\delta$ the following holds:

There is an α such that

(1) $\delta < \alpha$,
(2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$,
(3) $F(\delta) \in \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and
(4) for all $n < \omega, (z)_n \in (U_\delta)^{L_\alpha(\mathbb{R})}$.

But then, by upward absoluteness, $\delta \in \bigcap\{S_{z_n} \mid n < \omega\}$ and so $S_{z^*} \subseteq \bigcap\{S_{z_n} \mid n < \omega\} \subseteq \bigcap\{S_n \mid n < \omega\}$. ⊣

We shall call $\mathscr{F}_X$ the *reflection filter* since, by definition, there are $\mathscr{F}_X$-many reflecting points in the Reflection Theorem.

We wish now to extend the Reflection Theorem by allowing various parameters $S \subseteq \underset{\sim}{\delta}^2_1$ and their "reflections" $S \cap \delta$. For this we bring in AD.

4.6 Theorem (Reflection Theorem). *Assume* ZF + AD. *Suppose* $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ *and* $S \subseteq \underset{\sim}{\delta}^2_1$ *are in* $L(\mathbb{R})$. *For all* $X \in \mathrm{OD}^{L(\mathbb{R})}$, *for all* Σ_1 *formulas* φ, *and for all* $z \in \omega^\omega$, *if*

$$L(\mathbb{R}) \models \varphi[z, X, f, S, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

then for $\mathscr{F}_X$*-many* $\delta < \underset{\sim}{\delta}^2_1$,

$$L(\mathbb{R}) \models \varphi[z, F(\delta), f \restriction \delta, S \cap \delta, \delta, \mathbb{R}],$$

where here f *and* S *occur as predicates.*

Proof. First we show that the theorem holds for $S \subseteq \underset{\sim}{\delta}^2_1$. For each $\delta < \underset{\sim}{\delta}^2_1$, let

$$Q_\delta = U_\delta \setminus \bigcup \{U_\gamma \mid \gamma < \delta\}.$$

The sequence

$$\langle Q_\delta \mid \delta < \underset{\sim}{\delta}^2_1 \rangle$$

gives rise to a prewellordering of length $\underset{\sim}{\delta}^2_1$. By the Uniform Coding Lemma, there is an $e(S) \in \omega^\omega$ such that

$$U^{(2)}_{e(S)}(Q_{<\delta}, Q_\delta) \neq \varnothing \quad \text{iff} \quad \delta \in S.$$

The key point is that for $\mathscr{F}_X$-almost all δ

$$F^{L_{\vartheta(\delta)}(\mathbb{R})} = F \restriction \delta.$$

To see this let $z \in U_X$ be such that if $z \in U_\delta$ then

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \delta = \underset{\sim}{\delta}^2_1 \quad \text{and} \quad F \restriction \delta \text{ is defined.}$$

Thus, if $\delta \in S_z$, then

$$\delta = (\underset{\sim}{\delta}^2_1)^{L_{\vartheta(\delta)}(\mathbb{R})} \quad \text{and} \quad F \restriction \delta = F^{L_{\vartheta(\delta)}(\mathbb{R})},$$

which implies

$$\langle Q_\gamma \mid \gamma < \delta \rangle = \langle Q_\gamma \mid \gamma < \delta \rangle^{L_{\vartheta(\delta)}(\mathbb{R})}.$$

It follows that for $\delta \in S_z$, $e(S)$ codes $S \cap \delta$.

This enables us to associate with each Σ_1 sentence φ involving the predicate S, a Σ_1 sentence φ^* involving instead the real $e(S)$ in such a way that

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, S, \mathbb{R}]$$

if and only if

$$L(\mathbb{R}) \models \varphi^*[z, X, \underset{\sim}{\delta}^2_1, e(S), \mathbb{R}]$$

and, for $\delta \in S_z \in \mathscr{F}_X$,

$$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, S \cap \delta, \mathbb{R}]$$

if and only if

$$L(\mathbb{R}) \models \varphi^*[z, F(\delta), \delta, e(S), \mathbb{R}].$$

In this fashion, the predicate S can be eliminated in favour of the real $e(S)$, thereby reducing the present version of the reflection theorem to the original version (Theorem 4.4).

To see that we can also include parameters of the form $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ simply note that $\mathscr{F}_X$-almost all δ are closed under the Gödel pairing function and so we can include functions $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ by coding them as subsets of $\underset{\sim}{\delta}^2_1$. $\dashv$

We are now in a position to define, for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$. For the remainder of this section fix an ordinal $\lambda < \Theta$ and (by the results of Sect. 3.3) an OD-prewellordering $\leqslant_\lambda$ of ω^ω of length λ. Our interest is in applying the Reflection Theorem to

$$X = (\leqslant_\lambda, \lambda).$$

For each $S \subseteq \underset{\sim}{\delta}^2_1$, let $G^X(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X$. If the rule is violated then, letting i be the least such that either $(x)_i \notin U_X$ or $(y)_i \notin U_X$, I wins if $(x)_i \in U_X$; otherwise II wins. If the rule is satisfied then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta$, (which exists by reflection since (as in Lemma 4.5) we can regard this as a Σ_1 statement about a single real) I wins iff $\delta \in S$. Thus, I is picking δ by steering into the δth-approximation U_δ. (Note that the winning condition is not Σ_1.)

Now set

$$\mu_X = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \mathrm{I} \text{ wins } G^X(S)\}.$$

We let $\mu_\lambda = \mu_X$ but shall typically write μ_X to emphasize the dependence on the prewellorder. For $z \in U_X$, Player I can win $G^X(S_z)$ by playing x such that $(x)_i \in U_X$ for all $i < \omega$ and, for some $i < \omega$, $(x)_i = z$. Thus,

$$\mathscr{F}_X \subseteq \mu_X.$$

It is easy to see that μ_X is upward closed and contains either S or $\underset{\sim}{\delta}^2_1 \setminus S$ for each $S \subseteq \underset{\sim}{\delta}^2_1$.

4.7 Lemma. *Assume* ZF+AD. *Then* $L(\mathbb{R}) \models \mu_X$ *is a* $\underset{\sim}{\delta}^2_1$*-complete ultrafilter.*

Proof. The proof is similar to the proof of Theorem 2.33 (which traces back to the proof of Theorem 2.13). Consider $\{S_\alpha \mid \alpha < \gamma\}$ where $S_\alpha \in \mu_X$ and $\gamma < \underset{\sim}{\delta}^2_1$. Let $S = \bigcap_{\alpha<\gamma} S_\alpha$ and assume for contradiction that $S \notin \mu_X$. Let σ' be a winning strategy for I in $G^X(\underset{\sim}{\delta}^2_1 \smallsetminus S)$. Let

$$Z = \{(x,\sigma) \mid \text{for some } \alpha < \gamma,\ x \in Q_\alpha \text{ and } \sigma \text{ is a winning strategy for I in } G^X(S_\alpha)\}$$

where $Q_\alpha = \{x \in \omega^\omega \mid |x|_{\leqslant_U} = \alpha\}$ and $\leqslant_U$ is the prewellordering of length $\underset{\sim}{\delta}^2_1$ from Theorem 2.33. (One can also use the prewellordering from Theorem 4.6.)

By the Uniform Coding Lemma, let $e_0 \in \omega^\omega$ be such that for all $\alpha < \gamma$,

$$U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha) \subseteq Z \cap (Q_\alpha \times \omega^\omega) \quad \text{and} \quad U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha) \neq \varnothing.$$

Let

$$\Sigma = \mathrm{proj}_2\left(\bigcup\nolimits_{\alpha<\gamma} U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha)\right).$$

Notice that Σ is $\underset{\sim}{\Delta}^2_1$ since $\leqslant_U \restriction \gamma$ is $\underset{\sim}{\Delta}^2_1$. The key point is that (as in Lemma 2.27) we can choose a real that ensures that in a reflection argument we reflect to a level that correctly computes $\leqslant_U \restriction \gamma$ and hence Σ. We assume that all reals below have this feature.

Now we can "take control" of the output ordinal δ_0 with respect to σ' and all $\tau \in \Sigma$:

Base Case. We have

(1.1) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_X$ and

(1.2) $\forall y \in \omega^\omega\, \forall \sigma \in \Sigma\, ((\sigma * y)_I)_0 \in U_X$

since σ' and σ (as in (1.2)) are winning strategies for I. Since Σ is $\underset{\sim}{\Delta}^2_1$ this is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\} \cup \mathbb{R})$ fact about σ' and hence certified by a real $z_0 \in U_X$ such that $z_0 \leqslant_T \sigma'$; more precisely, $z_0 \leqslant_T \sigma'$ is such that for all δ if $z_0 \in U_\delta$ then

(1.3) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_\delta$ and

(1.4) $\forall y \in \omega^\omega\, \forall \sigma \in \Sigma\, ((\sigma * y)_I)_0 \in U_\delta$.

$(n+1)$st *Step.* Assume we have defined $z_0, \ldots, z_n$ in such a way that $z_n \leqslant_T \cdots \leqslant_T z_0$ and

(2.1) $\forall y \in \omega^\omega\, \bigl(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_X\bigr)$ and

(2.2) $\forall y \in \omega^\omega\, \forall \sigma \in \Sigma\, \bigl(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma * y)_I)_{n+1} \in U_X\bigr)$.

Let $z_{n+1} \in U_X$ be such that $z_{n+1} \leqslant_T z_n$ and for all δ, if $z_{n+1} \in U_\delta$ then

(2.3) $\forall y \in \omega^\omega\, \bigl(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_\delta\bigr)$ and

(2.4) $\forall y \in \omega^\omega\, \forall \sigma \in \Sigma\, \big(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma * y)_I)_{n+1} \in U_\delta\big)$.

Finally, let $z \in \omega^\omega$ be such that $(z)_i = z_i$ for all $i < \omega$ and let δ_0 be least such that $(z)_i \in U_{\delta_0}$ for all $i \in \omega$. Notice that by our choice of z_n DC is not required to define z. Then, for all $i < \omega$,

(3.1) $((\sigma' * z)_I)_i \in U_{\delta_0}$ by (1.3) and (2.3) and

(3.2) $((\sigma * z)_I)_i \in U_{\delta_0}$ for all $\sigma \in \Sigma$ by (1.4) and (2.4).

So

(4.1) δ_0 is the ordinal produced by $\sigma' * z$, i.e. $\delta_0 \in \underset{\sim}{\delta}^2_1 \smallsetminus S$ and

(4.2) δ_0 is the ordinal produced by $\sigma_\alpha * z$ where $\sigma_\alpha \in \Sigma$ is a winning strategy for I in $G^X(S_\alpha)$, i.e. $\delta_0 \in S_\alpha$ for all $\alpha < \gamma$.

This is a contradiction. ⊣

4.2. Strong Normality

Assuming ZF + AD, in $L(\mathbb{R})$ we have defined, for cofinally many $\lambda < \Theta$, an $\mathrm{OD}^{L(\mathbb{R})}$ ultrafilter on $\underset{\sim}{\delta}^2_1$ and shown that these ultrafilters are $\underset{\sim}{\delta}^2_1$-complete. We now wish to take the ultrapower of $\mathrm{HOD}^{L(\mathbb{R})}$ with these ultrafilters and show that collectively they witness that for each $\lambda < \Theta$, $\underset{\sim}{\delta}^2_1$ is λ-strong in $\mathrm{HOD}^{L(\mathbb{R})}$. This will be achieved by showing that reflection and uniform coding combine to show that μ_λ is *strongly normal.*

We begin with the following basic lemma on the ultrapower construction, which we shall prove in greater generality than we need at the moment.

4.8 Lemma. *Assume* ZF+DC*. Suppose μ is a countably complete ultrafilter on δ and that μ is* OD*. Suppose T is a set. Let $(\mathrm{HOD}_T)^\delta$ be the class of all functions $f : \delta \to \mathrm{HOD}_T$. Then the transitive collapse M of $(\mathrm{HOD}_T)^\delta/\mu$ exists, the associated embedding*

$$j : \mathrm{HOD}_T \to M$$

is OD_T*, and*

$$M \subseteq \mathrm{HOD}_T.$$

Proof. For $f, g : \delta \to \mathrm{HOD}_T$, let $f \sim_\mu g$ iff $\{\alpha < \delta \mid f(\alpha) = g(\alpha)\} \in \mu$ and let $[f]_\mu$ be the set consisting of the members of the equivalence class of f which have minimal rank. The structure $(\mathrm{HOD}_T)^\delta/\mu$ is the class consisting of all such equivalence classes. Let E be the associated membership relation. So $[f]_\mu\ E\ [g]_\mu$ if and only if $\{\alpha < \delta \mid f(\alpha) \in g(\alpha)\} \in \mu$. Notice that both $(\mathrm{HOD}_T)^\delta/\mu$ and E are OD_T.

The map

$$\begin{aligned} j_\mu : \mathrm{HOD}_T &\to (\mathrm{HOD}_T)^\delta/\mu \\ a &\mapsto [c_a]_\mu, \end{aligned}$$

where $c_a \in (\mathrm{HOD}_T)^\delta$ is the constant function with value a, is an elementary embedding, since Łoś's theorem holds, as HOD_T can be well-ordered. Notice that j_μ is OD_T.

Claim 1. $((\mathrm{HOD}_T)^\delta/\mu, E)$ *is well-founded.*

Proof. Suppose for contradiction that $((\mathrm{HOD}_T)^\delta/\mu, E)$ is not well-founded. Then, by DC, there is a sequence

$$\langle [f_n]_\mu \mid n < \omega \rangle$$

such that $[f_{n+1}]_\mu \; E \; [f_n]_\mu$ for all $n < \omega$. For each $n < \omega$, let

$$A_n = \{\alpha < \delta \mid f_{n+1}(\alpha) \in f_n(\alpha)\}.$$

For all $n < \omega$, $A_n \in \mu$ and since μ is countable complete,

$$\bigcap \{A_n \mid n < \omega\} \in \mu.$$

This is a contradiction since for each α in this intersection, $f_{n+1}(\alpha) \in f_n(\alpha)$ for all $n < \omega$. ⊣

Claim 2. $((\mathrm{HOD}_T)^\delta/\mu, E)$ *is isomorphic to a transitive class* $(M, \in)$*.*

Proof. We have established well-foundedness and extensionality is immediate. It remains to show that for each $a \in (\mathrm{HOD}_T)^\delta/\mu$,

$$\{b \in (\mathrm{HOD}_T)^\delta/\mu \mid b \; E \; a\}$$

is a set. Fix $a \in (\mathrm{HOD}_T)^\delta/\mu$ and choose $f \in (\mathrm{HOD}_T)^\delta$ such that $a = [f]_\mu$. Let α be such that $f \in V_\alpha$. Then for each $b \in (\mathrm{HOD}_T)^\delta/\mu$ such that bEa, letting $g \in (\mathrm{HOD}_T)^\delta$ be such that $b = [g]_\mu$,

$$\{\beta < \delta \mid g(\beta) \in V_\alpha\} \in \mu.$$

Thus,

$$\{b \in (\mathrm{HOD}_T)^\delta/\mu \mid b \; E \; a\} = \{[g]_\mu \mid [g]_\mu \; E \; [f]_\mu \text{ and } g \in V_\alpha\},$$

which completes the proof. ⊣

Let

$$\pi : ((\mathrm{HOD}_T)^\delta/\mu, E) \to (M, \in)$$

be the transitive collapse map and let

$$j : \mathrm{HOD}_T \to M$$

be the composition map $\pi \circ j_\mu$. Since π and j_μ are OD_T, j and M are OD_T.

It remains to see that $M \subseteq \mathrm{HOD}_T$. For this it suffices to show that for all α, $M \cap V_{j(\alpha)} \subseteq \mathrm{HOD}_T$. We have

$$M \cap V_{j(\alpha)} = j(\mathrm{HOD}_T \cap V_\alpha).$$

Let $A \in \mathrm{HOD}_T \cap \mathscr{P}(\gamma)$ be such that

$$\mathrm{HOD}_T \cap V_\alpha \subseteq L[A]$$

for some γ. We have

$$M \cap V_{j(\alpha)} = j(\mathrm{HOD}_T \cap V_\alpha) \subseteq L[j(A)].$$

But j and A are OD_T. Thus, $j(A) \in \mathrm{HOD}_T$ and hence $L[j(A)] \subseteq \mathrm{HOD}_T$, which completes the proof. $\dashv$

4.9 Remark. The use of DC in this lemma is essential in that assuming mild large cardinal axioms (such as the existence of a strong cardinal) there are models of $\mathrm{ZF} + \mathrm{AC}_\omega$ in which the lemma is false. In these models the club filter on ω_1 is an ultrafilter and the ultrapower of On by the club filter is not well-founded.

The ultrafilter μ_X defined in Sect. 4.1 is $\mathrm{OD}^{L(\mathbb{R})}$. Thus, by Lemma 4.8 (with $T = \varnothing$), letting

$$\pi : (\mathrm{HOD}^{L(\mathbb{R})})^{\utilde{\delta}^2_1}/\mu_X \to M_X$$

be the transitive collapse map and letting

$$j_X : \mathrm{HOD}^{L(\mathbb{R})} \to M_X$$

be the induced elementary embedding we have that $M_X \subseteq \mathrm{HOD}^{L(\mathbb{R})}$ and the fragments of j_X are in $\mathrm{HOD}^{L(\mathbb{R})}$ (in other words, j_X is amenable to $\mathrm{HOD}^{L(\mathbb{R})}$). Moreover, since μ_X is $\utilde{\delta}^2_1$-complete, the critical point of j_X is $\utilde{\delta}^2_1$.

Our next aim is to show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \utilde{\delta}^2_1 \text{ is } \lambda\text{-strong}$$

and for this it remains to show that

$$j_X(\utilde{\delta}^2_1) > \lambda \quad \text{and} \quad \mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda \subseteq M_X.$$

From now on we will also assume that λ is such that $L_\lambda(\mathbb{R}) \prec L_\Theta(\mathbb{R})$ and $\utilde{\delta}^2_1 < \lambda$. There are arbitrarily large $\lambda < \Theta$ with this feature (by the proof of Lemma 2.20). Since

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$$

(by Theorem 3.10), it follows that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda = \mathrm{HOD}^{L_\lambda(\mathbb{R})}.$$

Thus, letting $A \subseteq \lambda$ be an $\mathrm{OD}^{L(\mathbb{R})}$ set coding $\mathrm{HOD}^{L_\lambda(\mathbb{R})}$, we have

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda = L_\lambda[A].$$

Thus, it remains to show that $A \in M_X$. In fact, we will show that

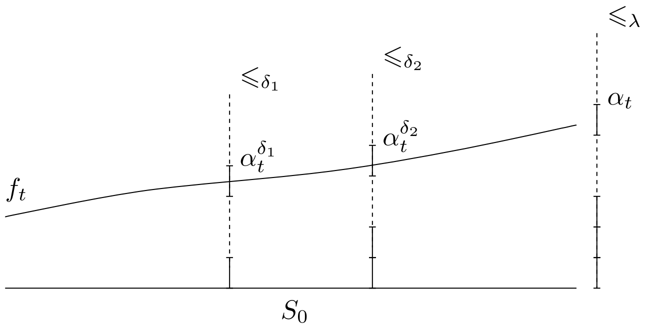
$$\mathscr{P}(\lambda) \cap \mathrm{HOD}^{L(\mathbb{R})} \subseteq M_X.$$

Let

$$S_0 = \{\delta < \underset{\sim}{\delta}^2_1 \mid F(\delta) = (\leqslant_\delta, \lambda_\delta) \text{ where } \leqslant_\delta \text{ is a prewellordering of length } \lambda_\delta \text{ and } L_{\lambda_\delta}(\mathbb{R}) \models \mathrm{T}_0\}.$$

Note that $S_0 \in \mathscr{F}_X$. For $\alpha < \lambda$, let $Q^{\underset{\sim}{\delta}^2_1}_\alpha$ be the αth-component of $\leqslant_\lambda$ and, for $\delta \in S_0$ and $\alpha < \lambda_\delta$, let Q^δ_α be the αth-component of $\leqslant_\delta$. Each $t \in \omega^\omega$ determines a *canonical function* f_t as follows: For $\delta \in S_0$, let α^δ_t be the unique ordinal α such that $t \in Q^\delta_\alpha$ and then set

$$\begin{aligned} f_t : S_0 &\to \underset{\sim}{\delta}^2_1 \\ \delta &\mapsto \alpha^\delta_t. \end{aligned}$$



For $t \in \omega^\omega$, let $\alpha_t = |t|_{\leqslant_\lambda}$ be the rank of t according to $\leqslant_\lambda$, that is, $\alpha_t = |t|_{\leqslant_\lambda} = \mu\alpha\,(t \in Q^{\underset{\sim}{\delta}^2_1}_\alpha)$.

4.10 Lemma. *Assume* $\mathrm{ZF} + \mathrm{AD}$. $j_X(\underset{\sim}{\delta}^2_1) > \lambda$.

Proof. Suppose $t_1, t_2 \in \omega^\omega$ and $|t_1|_{\leqslant_\lambda} = |t_2|_{\leqslant_\lambda}$. This is a true Σ_1 statement in $L(\mathbb{R})$ about t_1, t_2, X and $\mathbb{R}$. Thus, by reflection (Theorem 4.6), it follows that for $\mathscr{F}_X$-almost all $\delta < \underset{\sim}{\delta}^2_1$, $|t_1|_{\leqslant_\delta} = |t_2|_{\leqslant_\delta}$ and so the ordinal $[f_t]_{\mu_X}$ represented by f_t only depends on $|t|_{\leqslant_\lambda}$. Likewise, if $|t_1|_{\leqslant_\lambda} < |t_2|_{\leqslant_\lambda}$ then $[f_{t_1}]_{\mu_X} < [f_{t_2}]_{\mu_X}$. Therefore, the map

$$\begin{aligned} \rho : \lambda &\to \prod \lambda_\delta / \mu_X \\ |t|_{\leqslant_\lambda} &\mapsto [f_t]_{\mu_X} \end{aligned}$$

is well-defined and order-preserving and it follows that $\lambda \leqslant \prod \lambda_\delta / \mu_X < j_X(\underset{\sim}{\delta}^2_1)$. ⊣

We now turn to showing $\mathscr{P}(\lambda) \cap \mathrm{HOD}^{L(\mathbb{R})} \subseteq M_X$. Fix $A \subseteq \lambda$ such that $A \in \mathrm{HOD}^{L(\mathbb{R})}$. By the Uniform Coding Lemma there is an index $e(A) \in \omega^\omega$ such that for all $\alpha < \lambda$,

$$U^{(2)}_{e(A)}(Q^{\underset{\sim}{\delta}^2_1}_{<\alpha}, Q^{\underset{\sim}{\delta}^2_1}_{\alpha}) \neq \varnothing \quad \text{iff} \quad \alpha \in A.$$

For all $\delta \in S_0$, let

$$A^\delta = \{\alpha < \lambda_\delta \mid U^{(2)}_{e(A)}(Q^{\delta}_{<\alpha}, Q^{\delta}_{\alpha}) \neq \varnothing\}$$

be the "reflection of A". Since the statement

$$\{\alpha < \lambda \mid U^{(2)}_{e(A)}(Q^{\underset{\sim}{\delta}^2_1}_{<\alpha}, Q^{\underset{\sim}{\delta}^2_1}_{\alpha}) \neq \varnothing\} \in \mathrm{HOD}^{L(\mathbb{R})}$$

is a true Σ_1 statement about X, $\mathbb{R}$ and $e(A)$, there is a set $S \in \mathscr{F}_X$ such that for all $\delta \in S$, $A^\delta \in \mathrm{HOD}^{L(\mathbb{R})}$.

We wish to show that

$$\begin{aligned} h_A : S &\to \mathrm{HOD}^{L(\mathbb{R})} \\ \delta &\mapsto A^\delta \end{aligned}$$

represents A in the ultrapower. Notice that

$$\begin{aligned} |t|_{\leqslant\lambda} \in A \quad &\text{iff} \quad \{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) \in A^\delta\} \in \mu_X \\ &\text{iff} \quad [f_t]_{\mu_X} \in [h_A]_{\mu_X}. \end{aligned}$$

The last equivalence holds by definition. For the first equivalence note that if $|t|_{\leqslant\lambda} \in A$ then since this is a true Σ_1 statement about $e(A)$, t and X, for μ_X-almost all δ, $|t|_{\leqslant\delta} \in A^\delta$, that is, $\{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) \in A^\delta\} \in \mu_X$. Likewise, if $|t|_{\leqslant\lambda} \notin A$ then since this is a true Σ_1 statement about $e(A)$, t and X, for μ_X-almost all δ, $|t|_{\leqslant\delta} \notin A^\delta$.

So it suffices to show that the map

$$\begin{aligned} \rho : \lambda &\to \prod \lambda_\delta/\mu_X \\ |t|_{\leqslant\lambda} &\mapsto [f_t]_{\mu_X} \end{aligned}$$

is an isomorphism since then $\pi([h_A]_{\mu_X}) = A \in M_X$, where recall that $\pi : (\mathrm{HOD}^{L(\mathbb{R})})^{\underset{\sim}{\delta}^2_1}/\mu_X \cong M_X$ is the transitive collapse map. We already know that ρ is well-defined and order-preserving (by Lemma 4.10). It remains to show that ρ is onto, that is, that every function $f \in \prod \lambda_\delta/\mu_X$ is equivalent (modulo μ_X) to a canonical function f_t. To say that this is true is to say that μ_X is *strongly normal*:

4.11 Definition (Strong normality). μ_X is *strongly normal* iff whenever $f : S_0 \to \underset{\sim}{\delta}^2_1$ is such that

$$\{\delta \in S_0 \mid f(\delta) < \lambda_\delta\} \in \mu_X$$

then there exists a $t \in \omega^\omega$ such that

$$\{\delta \in S_0 \mid f(\delta) = f_t(\delta)\} \in \mu_X.$$

Notice that normality is a special case of strong normality since if

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) < \delta\} \in \mu_X$$

then (since for $\mathscr{F}_X$-almost all δ, $\lambda_\delta > \delta$), by strong normality there is a $t \in \omega^\omega$ such that

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) = f(\delta)\} \in \mu_X.$$

So if β is such that $t \in Q^{\delta^2_1}_\beta$ then $\beta < \underset{\sim}{\delta}^2_1$, since otherwise by reflection this would contradict the assumption that

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) < \delta\} \in \mu_X.$$

Thus,

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) = \beta\} \in \mu_X.$$

4.12 Theorem. *Assume* ZF + AD. $L(\mathbb{R}) \models \mu_X$ *is strongly normal.*

Proof. Assume toward a contradiction that f is a counterexample to strong normality. So, for each $t \in \omega^\omega$,

$$\{\delta \in S_0 \mid f(\delta) \neq f_t(\delta)\} \in \mu_X.$$

Let

$$\eta = \min\left\{\beta < \lambda \mid \forall t \in Q^{\delta^2_1}_\beta \ \{\delta \in S_0 \mid f(\delta) < f_t(\delta)\} \in \mu_X\right\}$$

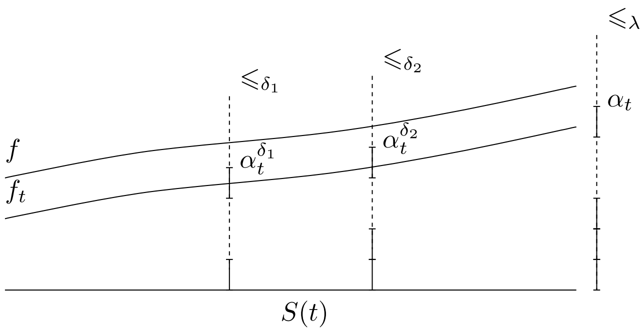
if such β exist; otherwise, let $\eta = \lambda$. Fix $y_\eta \in Q^{\delta^2_1}_\eta$ (unless $\eta = \lambda$, in which case we ignore this parameter) and, for $\delta \in S_0$, let $\eta_\delta = f_{y_\eta}(\delta)$ and for $\delta = \underset{\sim}{\delta}^2_1$, let $\eta_\delta = \eta$. Note that $f_{y_\eta}(\delta) > f(\delta)$ for μ_X-almost all δ. In the proof we will be working on this set and so we modify S_0 by intersecting it with this set if necessary. For convenience let

$$S(t) = \{\delta \in S_0 \mid f_t(\delta) < f(\delta)\}.$$

Notice that by the definition of η and our assumption that f is a counterexample to strong normality, we have that

$$S(t) \in \mu_X$$

for all $t \in Q^{\delta^2_1}_{<\eta}$.



Our aim is to compute f from a real parameter by coding relative to the various prewellorderings. Our computation will give us "$f(\underset{\sim}{\delta}^2_1)$". Then, picking a real $y_f \in Q^{\underset{\sim}{\delta}^2_1}_{\text{“}f(\underset{\sim}{\delta}^2_1)\text{”}}$ we shall have by reflection that for μ_X-almost every δ, $f(\delta) = f_{y_f}(\delta)$, which is a contradiction.

The proof involves a number of parameters which we list here. We will also give a brief description which will not make complete sense at this point but will serve as a useful reference to consult as the proof proceeds.

e_0 is the index of the universal set that selects the Z^δ_α's (represented in the diagrams as ellipses) from Z' (represented in the diagrams as chimneys).

e_1 is the index of the universal set that selects *subsets* of the Z^δ_α's (represented in the diagrams as black dots inside the ellipses).

y_η is the real in $Q^{\underset{\sim}{\delta}^2_1}_\eta$ that determines η_δ for $\delta \in S_0$.

y_f is the real in $Q^{\underset{\sim}{\delta}^2_1}_{\text{“}f(\underset{\sim}{\delta}^2_1)\text{”}}$ that determines $f(\delta)$ for $\delta \in S_3$.

We will successively shrink S_0 to S_1, S_2, and finally S_3. All four of these sets will be members of μ_X. We now proceed with the proof.

Let

$$Z' = \big\{(t, \sigma) \mid t \in Q^{\underset{\sim}{\delta}^2_1}_{<\eta} \text{ and } \sigma \text{ is a winning strategy for I in } G^X(S(t))\big\}.$$

Thus, by our assumption that f is a counterexample to strong normality and by our choice of η we have, for all $\beta < \eta$,

$$Z' \cap (Q^{\underset{\sim}{\delta}^2_1}_\beta \times \omega^\omega) \neq \varnothing,$$

since for all $t \in Q^{\underset{\sim}{\delta}^2_1}_{<\eta}$, I wins $G^X(S(t))$. By the Uniform Coding Lemma, let $e_0 \in \omega^\omega$ be such that for all $\beta < \eta$,

(1.1) $U^{(2)}_{e_0}(Q^{\underset{\sim}{\delta}^2_1}_{<\beta}, Q^{\underset{\sim}{\delta}^2_1}_\beta) \subseteq Z' \cap (Q^{\underset{\sim}{\delta}^2_1}_\beta \times \omega^\omega)$ and

(1.2) $U^{(2)}_{e_0}(Q^{\underset{\sim}{\delta}^2_1}_{<\beta}, Q^{\underset{\sim}{\delta}^2_1}_\beta) \neq \varnothing$.

By reflection, we have that for $\mathscr{F}_X$-almost all δ, for all $\beta < \eta_\delta$,

(2.1) $U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta) \subseteq Q^\delta_\beta \times \omega^\omega$ and

(2.2) $U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta) \neq \varnothing$.

Notice that in the reflected statement we have had to drop reference to Z' since we cannot reflect Z' as the games involved in its definition are not Σ_1. Let S'_1 be the set of such δ and let $S_1 = S'_1 \cap S_0$. Notice that S_1 is $\Sigma_1(L(\mathbb{R}), \{e_0, y_\eta, f, X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$.

For $\delta \in S_1 \cup \{\underset{\sim}{\delta}^2_1\}$ and $\beta < \eta_\delta$ let

$$Z^\delta_\beta = U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta)$$

and for $\delta \in S_1 \cup \{\underset{\sim}{\delta}^2_1\}$ let

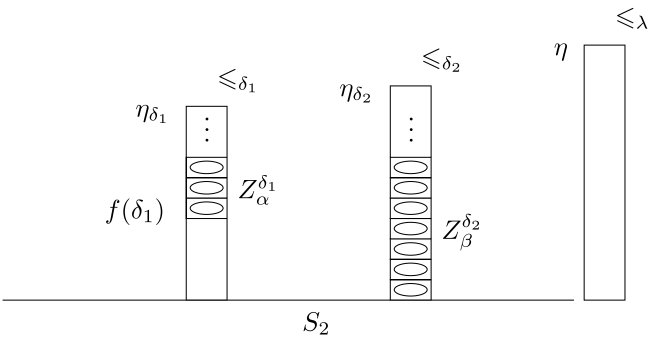
$$Z^\delta = \bigcup_{\beta<\eta_\delta} Z^\delta_\beta.$$

Claim A (Disjointness Property). *There is an $S_2 \subseteq S_1$, $S_2 \in \mu_X$ such that for $\delta_1, \delta_2 \in S_2 \cup \{\underset{\sim}{\delta}^2_1\}$ with $\delta_1 < \delta_2 \leqslant \underset{\sim}{\delta}^2_1$,*

$$Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing$$

for all $\alpha \in [f(\delta_1), \eta_{\delta_1})$ and $\beta \in [0, \eta_{\delta_2})$.

Proof. Here is the picture:



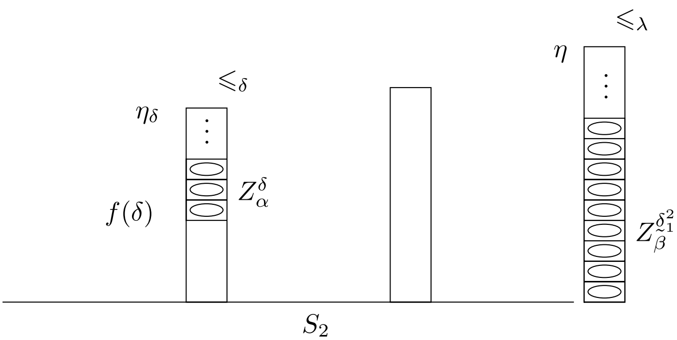
We begin by establishing a special case.

Subclaim. *For μ_X-almost all δ,*

$$Z^\delta_\alpha \cap Z^{\underset{\sim}{\delta}^2_1}_\beta = \varnothing$$

for all $\alpha \in [f(\delta), \eta_\delta)$ and $\beta \in [0, \eta)$.

Proof. The picture is similar:



Let

$$T = \{\delta \in S_1 \mid Z^\delta_\alpha \cap Z^{\underset{\sim}{\delta}^2_1}_\beta = \varnothing \text{ for all } \alpha \in [f(\delta), \eta_\delta) \text{ and } \beta \in [0, \eta)\}$$

and assume, toward a contradiction, that $T \notin \mu_X$. So $(\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1 \in \mu_X$. Let σ' be a winning strategy for I in $G^X((\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1)$.

Let us first motivate the main idea: Suppose z is a legal play for II against σ' (by which we mean a play for II that satisfies the Main Rule) and suppose that the ordinal associated with this play is δ_0. So $\delta_0 \in (\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1$ and (by the definition of T) there exists an $\alpha_0 \in [f(\delta_0), \eta_\delta)$ and $\beta_0 \in [0, \eta)$ such that $Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0} \neq \varnothing$. Pick $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. In virtue of the fact that $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0}$ we have

(3.1) $f_{t_0}(\delta_0) = \alpha_0 \geqslant f(\delta_0)$

and in virtue of the fact that $(t_0, \sigma_0) \in Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ we have

(3.2) σ_0 is a winning strategy for I in $G^X(S(t_0))$, where

$$S(t_0) = \{\delta \in S_0 \mid f_{t_0}(\delta) < f(\delta)\}.$$

So we get a contradiction if δ_0 happens to be in $S(t_0)$ (since then $f_{t_0}(\delta_0) < f(\delta_0)$, contradicting (3.1)). Notice that this will occur if we can arrange the play z to be such that in addition to being a legal play against σ' with associated ordinal δ_0 it is *also* a legal play against σ_0 (in the game $G^X(S(t_0))$) with associated ordinal δ_0. We can construct such a play z recursively as in the proof of completeness.

Base Case. We have

(4.1) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_X$ and

(4.2) $\forall y \in \omega^\omega\, \forall (t, \sigma) \in Z^{\underset{\sim}{\delta}^2_1}\, ((\sigma * y)_I)_0 \in U_X$

since σ' and σ (as in (4.2)) are winning strategies for I. Now all of this is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$ fact about σ' and e_0 (the index for $Z^{\underset{\sim}{\delta}^2_1}$) and so it is certified by a real $z_0 \in U_X$ such that $z_0 \leqslant_T \langle \sigma', e_0 \rangle$; so z_0 is such that if $z_0 \in U_\delta$ then

(4.3) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_\delta$ and

(4.4) $\forall y \in \omega^\omega\, \forall (t, \sigma) \in Z^\delta\, ((\sigma * y)_I)_0 \in U_\delta$.

$(n+1)$st *Step.* Assume we have defined $z_0, \dots, z_n$ in such a way that $z_n \leqslant_T \cdots \leqslant_T z_0$ and

(5.1) $\forall y \in \omega^\omega\, \big(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_X\big)$ and

(5.2) $\forall y \in \omega^\omega\, \forall (t, \sigma) \in Z^{\underset{\sim}{\delta}^2_1}\, \big(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma * y)_I)_{n+1} \in U_X\big)$.

Again, all of this is a $\Sigma_1(L(\mathbb{R}),\{X,\underset{\sim}{\delta}^2_1,\mathbb{R}\})$ fact about $\sigma', e_0, z_0,\dots,z_n$ and so it is certified by a real $z_{n+1}\in U_X$ such that $z_{n+1}\leqslant_T z_n$; so z_{n+1} is such that if $z_{n+1}\in U_\delta$ then

(5.3) $\forall y\in\omega^\omega\,\big(\forall i\leqslant n\,(y)_i=z_i\rightarrow((\sigma'*y)_I)_{n+1}\in U_\delta\big)$ and

(5.4) $\forall y\in\omega^\omega\,\forall(t,\sigma)\in Z^\delta\,\big(\forall i\leqslant n\,(y)_i=z_i\rightarrow((\sigma*y)_I)_{n+1}\in U_\delta\big)$.

Finally, let $z\in\omega^\omega$ be such that $(z)_i=z_i$ for all $i<\omega$ and let δ_0 be least such that $(z)_i\in U_{\delta_0}$ for all $i\in\omega$. Notice that since we chose z_{n+1} to be recursive in z_n DC is not required to form z. Since $(z)_i\in U_X$ for all $i\in\omega$, z is a legal play for II in any of the games $G^X(S)$ relevant to the argument. Moreover, for all $i\in\omega$,

(6.1) $((\sigma'*z)_I)_i\in U_{\delta_0}$ by (4.3) and (5.3) and

(6.2) $((\sigma*z)_I)_i\in U_{\delta_0}$ for all $\sigma\in\mathrm{proj}_2(Z^{\delta_0})$ by (4.4) and (5.4)

and so

(7.1) δ_0 is the ordinal produced by $\sigma'*z$, i.e. $\delta_0\in(\underset{\sim}{\delta}^2_1\smallsetminus T)\cap S_1$ and

(7.2) δ_0 is the ordinal produced by $\sigma*z$ for any $\sigma\in\mathrm{proj}_2(Z^{\delta_0})$.

Since $\delta_0\in(\underset{\sim}{\delta}^2_1\smallsetminus T)\cap S_1$, by the definition of T there exists an $\alpha_0\in[f(\delta_0),\eta_{\delta_0})$ and $\beta_0\in[0,\eta)$ such that $Z^{\delta_0}_{\alpha_0}\cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}\neq\varnothing$. Pick $(t_0,\sigma_0)\in Z^{\delta_0}_{\alpha_0}\cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. In virtue of the fact that $(t_0,\sigma_0)\in Z^{\delta_0}_{\alpha_0}$ we have

(8.1) $f_{t_0}(\delta_0)=\alpha_0\geqslant f(\delta_0)$

and in virtue of the fact that $(t_0,\sigma_0)\in Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ we have

(8.2) σ_0 is a winning strategy for I in $G^X(S(t_0))$, where

$$S(t_0)=\{\delta\in S_0\mid f_{t_0}(\delta)<f(\delta)\}.$$

Combined with (7.2) this implies $\delta_0\in S(t_0)$, in other words, $f_{t_0}(\delta_0)<f(\delta_0)$, which contradicts (8.1). ⊣

Thus, $T\in\mu_X$ and we have

(9.1) $\forall\delta\in T\,\forall\beta\in[0,\eta_{\underset{\sim}{\delta}^2_1})\,\forall\alpha\in[f(\delta),\eta_\delta)\,(Z^\delta_\alpha\cap Z^{\underset{\sim}{\delta}^2_1}_\beta=\varnothing)$.

This is a true Σ_1 statement in $L(\mathbb{R})$ about e_0, y_η, f, X, $\mathbb{R}$, $\underset{\sim}{\delta}^2_1$, and T. Since T is $\Sigma_1(L(\mathbb{R}),\{e_0,y_\eta,f,X,\underset{\sim}{\delta}^2_1,\mathbb{R}\})$, the above statement is $\Sigma_1(L(\mathbb{R}),\{e_0,y_\eta,f,X,\underset{\sim}{\delta}^2_1,\mathbb{R}\})$. Thus by the Reflection Theorem (Theorem 4.6) there exists an $S_2\subseteq S_1$, $S_2\in\mu_X$ such that for all $\delta_2\in S_2$,

(9.2) $\forall\delta_1\in T\cap\delta_2\,\forall\beta\in[0,\eta_{\delta_2})\,\forall\alpha\in[f(\delta_1),\eta_{\delta_1})\,(Z^{\delta_1}_\alpha\cap Z^{\delta_2}_\beta=\varnothing)$.

Notice that S_2 is $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$ in e_0, y_η and the parameters for coding. This completes the proof of Claim A. $\dashv$

Claim B (Tail Computation). *There exists an index* $e_1 \in \omega^\omega$ *such that for all* $\delta \in S_2$,

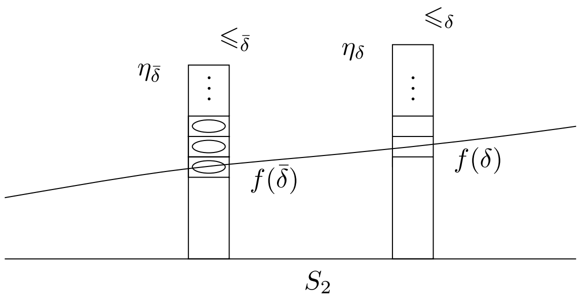
(1) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \subseteq Z^\delta_\beta$ *for all* $\beta < \eta_\delta$,

(2) $U^{(2)}_{e_1}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing$, *and*

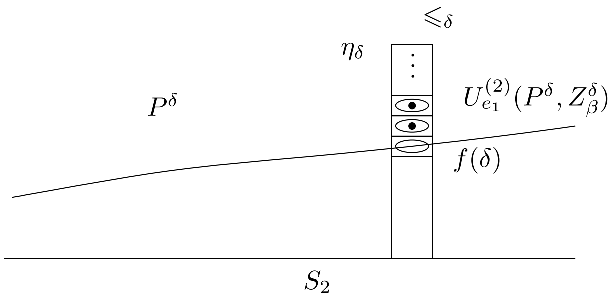
(3) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \neq \varnothing$ *for* β *such that* $f(\delta) < \beta < \eta_\delta$,

where $P^\delta = \bigcup\{Z^{\bar{\delta}}_\alpha \mid \bar{\delta} \in S_2 \cap \delta$ *and* $\alpha \in [f(\bar{\delta}), \eta_{\bar{\delta}})\}$ *and* S_2 *is from the end of the proof of Claim A.*

Proof. Here is the picture of the "tail parameter" P^δ:



Here is the picture of the statement of Claim B:



Assume toward a contradiction that there is no such e_1. We follow the proof of the Uniform Coding Lemma. To begin with, notice that it suffices to find an $e_1 \in \omega^\omega$ satisfying (2) and

(3′) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta \neq \varnothing$ for β such that $f(\delta) < \beta < \eta_\delta$

since given the parameter Z^δ_β we can easily ensure (1).

Consider the set of reals such that (2) of the (revised) claim holds, that is,

$$G = \left\{e \in \omega^\omega \mid \forall \delta \in S_2 \left(U^{(2)}_e(P^\delta, Z^\delta_{f(\delta)}) = \varnothing\right)\right\}.$$

So, for each $e \in G$, ($3'$) in the claim fails for some $\delta \in S_2$ and $\beta \in (f(\delta), \eta_\delta)$. For each $e \in G$, let

$$\begin{aligned} \alpha_e = {} & \text{lexicographically least pair } (\delta, \beta) \text{ such that} \\ & (1)\ \delta \in S_2, \\ & (2)\ f(\delta) < \beta < \eta_\delta, \text{ and} \\ & (3)\ U^{(2)}_e(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta = \varnothing. \end{aligned}$$

Now play the game

$$\begin{array}{lccccccc} \text{I} & x(0) & & x(1) & & x(2) & & \dots \\ \text{II} & & y(0) & & y(1) & & \dots & \end{array}$$

where II wins iff $(x \in G \to (y \in G \wedge \alpha_y >_{\text{lex}} \alpha_x))$

Claim 1. *Player I does not have a winning strategy.*

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As in the proof of the Uniform Coding Lemma, we aim to "bound" all of I's plays and then use this bound to construct a play e^* for II which defeats σ. We will make key use of the Disjointness Property.

Choose $e_\sigma \in \omega^\omega$ such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e_\sigma}(P, P') = \bigcup\nolimits_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(P, P').$$

In particular, for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) = \bigcup\nolimits_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(P^\delta, Z^\delta_\beta).$$

Note two things: First, since σ is a winning strategy for I, $(\sigma * y)_I \in G$ for all $y \in \omega^\omega$; so $e_\sigma \in G$. Second, for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant_{\text{lex}} \alpha_{e_\sigma}$. So e_σ is "at least as good" as any $(\sigma * y)_I$. We have to do "better".

Pick $x_0 \in Z^{\delta_0}_{\beta_0}$ where $(\delta_0, \beta_0) = \alpha_{e_\sigma}$. Choose e^* such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e^*}(P, P') = \begin{cases} U^{(2)}_{e_\sigma}(P, P') & \text{if } x_0 \notin P' \\ U^{(2)}_{e_\sigma}(P, P') \cup \{x_0\} & \text{if } x_0 \in P'. \end{cases}$$

In particular, for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) & \text{if } x_0 \notin Z^\delta_\beta \\ U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) \cup \{x_0\} & \text{if } x_0 \in Z^\delta_\beta. \end{cases}$$

Since we chose $x_0 \in Z^{\delta_0}_{\beta_0}$, by the Disjointness Property (and the fact that for fixed δ, $Z^\delta_\alpha \cap Z^\delta_\beta = \varnothing$ for $\alpha < \beta < \eta_\delta$) we have

(10.1) $x_0 \notin Z^\delta_\beta$ for $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$,

(10.2) $x_0 \notin Z^{\delta_0}_\beta$ for $\beta \in [0, \eta_{\delta_0}) \setminus \{\beta_0\}$, and

(10.3) $x_0 \notin Z^\delta_\beta$ for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ and $\beta \in [0, \eta_\delta)$.

Thus, by the definition of e^*, we have, by (10.1–3),

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_{f(\delta)}) = U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_{f(\delta)})$$

for all $\delta \in S_2$. Since $e_\sigma \in G$, this means $e^* \in G$. So α_{e^*} exists. Similarly, by the definition of e^*, we have, by (10.1) and (10.2),

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta)$$

for all $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$ and for $\delta = \delta_0$ and $\beta \in [f(\delta_0), \beta_0)$. So e^* is "at least as good" as e_σ. But since $x_0 \in Z^{\delta_0}_{\beta_0}$, we have that $x_0 \in U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0})$, by the definition of e^*; that is, e^* is "better" than e_σ. In other words, $\alpha_{e^*} >_{\text{lex}} \alpha_{e_\sigma} \geqslant_{\text{lex}} \alpha_{(\sigma * y)_I}$ for all $y \in \omega^\omega$ and so, by playing e^*, II defeats σ. ⊣

Claim 2. *Player II does not have a winning strategy.*

Proof. Suppose toward a contradiction that τ is a winning strategy for II. We shall find an e^* such that $e^* \in G$ (Subclaim 1) and α_{e^*} does not exist (Subclaim 2), which is a contradiction.

Choose $h_0 : \omega^\omega \times (\omega^\omega \times \omega^\omega) \to \omega^\omega$ such that h_0 is Σ^1_1 and for all $(e, x) \in \omega^\omega \times (\omega^\omega \times \omega^\omega)$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_0(e,x)}(P, P') = \begin{cases} U^{(2)}_e(P, P') & \text{if } x \notin P \cup P' \\ \varnothing & \text{if } x \in P \cup P'. \end{cases}$$

In particular, for $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{h_0(e,x)}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_e(P^\delta, Z^\delta_\beta) & \text{if } x \notin P^\delta \cup Z^\delta_\beta \\ \varnothing & \text{if } x \in P^\delta \cup Z^\delta_\beta. \end{cases}$$

Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is Σ^1_1 and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_1(e)}(P, P') = \bigcup\nolimits_{x \in P'} U^{(2)}_{(h_0(e,x) * \tau)_{II}}(P, P').$$

In particular, for $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{h_1(e)}(P^\delta, Z^\delta_\beta) = \bigcup\nolimits_{x \in Z^\delta_\beta} U^{(2)}_{(h_0(e,x) * \tau)_{II}}(P^\delta, Z^\delta_\beta).$$

By the Recursion Theorem, there is an $e^* \in \omega^\omega$ such that for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = U^{(2)}_{h_1(e^*)}(P^\delta, Z^\delta_\beta).$$

Subclaim 1. *$e^* \in G$.*

Proof. Suppose for contradiction that $e^* \notin G$. Let $\delta_0 \in S_2$ be least such that

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing.$$

Now

$$\begin{aligned} U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) &= U^{(2)}_{h_1(e^*)}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \\ &= \textstyle\bigcup_{x \in Z^{\delta_0}_{f(\delta_0)}} U^{(2)}_{(h_0(e^*,x)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}). \end{aligned}$$

So choose $x_0 \in Z^{\delta_0}_{f(\delta_0)}$ such that

$$U^{(2)}_{(h_0(e^*,x_0)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing.$$

If we can show $h_0(e^*, x_0) \in G$ then we are done since this implies that $(h_0(e^*, x_0) * \tau)_{II} \in G$ (as τ is a winning strategy for II), which contradicts the previous statement.

Subsubclaim. *$h_0(e^*, x_0) \in G$, that is, for all $\delta \in S_2$,*

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing.$$

Proof. By the definition of h_0, for all $\delta \in S_2$,

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \begin{cases} U^{(2)}_{e^*}(P^{\delta}, Z^{\delta}_{f(\delta)}) & \text{if } x_0 \notin P^{\delta} \cup Z^{\delta}_{f(\delta)} \\ \varnothing & \text{if } x_0 \in P^{\delta} \cup Z^{\delta}_{f(\delta)}. \end{cases}$$

Since $x_0 \in Z^{\delta_0}_{f(\delta_0)}$, by the Disjointness Property, this definition yields the following: For $\delta \in S_2 \cap [0, \delta_0)$ we have $x_0 \notin P^{\delta} \cup Z^{\delta}_{f(\delta)}$ and so,

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = U^{(2)}_{e^*}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing,$$

where the latter holds since we chose δ_0 to be least such that

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing;$$

for $\delta = \delta_0$ we have $x_0 \in Z^{\delta}_{f(\delta)}$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing;$$

and for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ we have $x_0 \in P^{\delta}$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing.$$

Thus, $h_0(e^*, x_0) \in G$. ⊣

This completes the proof of Subclaim 1. ⊣

Subclaim 2. *α_{e^*} does not exist.*

Proof. Suppose for contradiction that α_{e^*} exists. Recall that

$$\begin{aligned}\alpha_{e^*} = {} & \text{lexicographically least pair } (\delta, \beta) \text{ such that} \\ & (1)\ \delta \in S_2, \\ & (2)\ f(\delta) < \beta < \eta_\delta, \text{ and} \\ & (3)\ U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta = \varnothing.\end{aligned}$$

Let $(\delta_0, \beta_0) = \alpha_{e^*}$. We shall show $U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing$, which is a contradiction. By the definition of h_1,

$$\begin{aligned}U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) &= U^{(2)}_{h_1(e^*)}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \\ &= \textstyle\bigcup_{x \in Z^{\delta_0}_{\beta_0}} U^{(2)}_{(h_0(e^*, x) * \tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}).\end{aligned}$$

Fix $x_0 \in Z^{\delta_0}_{\beta_0}$. Since $e^* \in G$, $h_0(e^*, x_0) \in G$, by the Disjointness Property. (This is because for $\delta \in S_2 \cap [0, \delta_0)$ we have $x_0 \notin P^\delta \cup Z^\delta_{f(\delta)}$ and so

$$U^{(2)}_{h_0(e^*, x_0)}(P^\delta, Z^\delta_{f(\delta)}) = U^{(2)}_{e^*}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing,$$

where the latter holds since $e^* \in G$; for $\delta = \delta_0$ we have $x_0 \notin P^\delta \cup Z^\delta_{f(\delta)}$ and since $e^* \in G$ this implies

$$U^{(2)}_{h_0(e^*, x_0)}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing;$$

and for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ we have $x_0 \in P^\delta$ and so

$$U^{(2)}_{h_0(e^*, x_0)}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing.)$$

So $\alpha_{h_0(e^*, x_0)}$ exists.

Subsubclaim. $\alpha_{h_0(e^*, x_0)} = \alpha_{e^*}$.

Proof. By the definition of h_0,

$$U^{(2)}_{h_0(e^*, x_0)}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) & \text{if } x_0 \notin P^\delta \cup Z^\delta_\beta \\ \varnothing & \text{if } x_0 \in P^\delta \cup Z^\delta_\beta \end{cases}$$

for $\delta \in S_2$ and $\beta < \eta_\delta$. So

$$U^{(2)}_{h_0(e^*, x_0)}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} = \varnothing,$$

since $x_0 \in P^{\delta_0} \cup Z^{\delta_0}_{\beta_0}$. And, when either $\delta = \delta_0$ and $\beta \in (f(\delta_0), \beta_0)$ or $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$, we have, by the Disjointness Property, $x_0 \notin P^\delta \cup Z^\delta_\beta$, hence

$$U^{(2)}_{h_0(e^*, x_0)}(P^\delta, Z^\delta_\beta) = U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta).$$

Thus, $\alpha_{h_0(e^*, x_0)} = \alpha_{e^*}$. ⊣

Since τ is winning for II,

$$(h_0(e^*, x_0) * \tau)_{II} \in G$$

and

$$\alpha_{(h_0(e^*,x_0)*\tau)_{II}} >_{\text{lex}} \alpha_{h_0(e^*,x_0)} = \alpha_{e^*}.$$

So

$$U^{(2)}_{(h_0(e^*,x_0)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing.$$

Since

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) = \bigcup_{x \in Z^{\delta_0}_{\beta_0}} U^{(2)}_{(h_0(e^*,x)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0})$$

we have

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing,$$

which is a contradiction. ⊣

This completes the proof of Claim 2. ⊣

We have a contradiction and therefore there is an e_1 is as desired. ⊣

Notice that $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha)$, for variable α, allows us to pick out $f(\delta)$. Now we can consider the ordinal "$f(\underset{\sim}{\delta}^2_1)$" picked out in this fashion.

Claim C. *There exists a $\beta_0 < \eta$ such that*

(1) $U^{(2)}_{e_1}(P^{\underset{\sim}{\delta}^2_1}, Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^{\underset{\sim}{\delta}^2_1}, Z^{\underset{\sim}{\delta}^2_1}_{\beta}) \neq \varnothing$ for all $\beta \in (\beta_0, \eta)$, where

$$P^{\underset{\sim}{\delta}^2_1} = \bigcup \{ Z^\delta_\alpha \mid \delta \in S_2 \textit{ and } \alpha \in [f(\delta), \eta_\delta) \}.$$

Proof. Suppose for contradiction that the claim is false. The statement that the claim fails is a true Σ_1 statement about e_0, e_1, y_η, X, $\mathbb{R}$, f and S_2. But then by the Reflection Theorem (Theorem 4.6) this fact reflects to $\mathscr{F}_X$-almost all δ, which contradicts Claim B. ⊣

Pick $y_f \in Q^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. Now the statement that $y_f \in Q^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ where β_0 is such that (1) and (2) of Claim C hold is a true Σ_1 statement about e_0, e_1, y_η, y_f, f, X, $\mathbb{R}$, and $\underset{\sim}{\delta}^2_1$. Thus, by Theorem 4.6, for $\mathscr{F}_X$-almost every $\delta < \underset{\sim}{\delta}^2_1$ this statement reflects. Let $S_3 \subseteq S_2$ be in μ_X and such that the above statement reflects to each point in S_3. Now by Claim B, for $\delta \in S_3$, the least β_0 such that $y_f \in Q^\delta_{\beta_0}$ is $f(\delta)$. Thus,

$$\{\delta \in S_0 \mid f_{y_f}(\delta) = f(\delta)\} \in \mu_X$$

and hence μ_X is strongly normal. ⊣

This completes the proof of the following:

4.13 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$. *Then, for each* $\lambda<\Theta^{L(\mathbb{R})}$,

$$\mathrm{HOD}^{L(\mathbb{R})}\models \mathrm{ZFC}+(\undertilde{\delta}^2_1)^{L(\mathbb{R})} \textit{ is } \lambda\textit{-strong.}$$

4.14 Remark. For simplicity we proved Lemma 4.7 and Claim A of Theorem 4.12 using a proof by contradiction. This involves an appeal to determinacy. However, one can prove each result more directly, without appealing to determinacy.

Call a real y *suitable* if $(y)_i\in U_X$ for all $i<\omega$. Call a strategy σ a *proto-winning strategy* if σ is a winning strategy for I in $G^X(\undertilde{\delta}^2_1)$. Thus if y is suitable and σ is a proto-winning strategy then

$$\{((\sigma*y)_I)_i,(y)_i\mid i<\omega\}\subseteq U_X$$

and so we can let

$$\delta_{(\sigma,y)}=\text{the least }\delta\text{ such that }\{((\sigma*y)_I)_i\mid i<\omega\}\cup\{(y)_i\mid i<\omega\}\subseteq U_\delta.$$

Let κ be least such that $X\in L_\kappa(\mathbb{R})$ and $L_\kappa(\mathbb{R})\prec_1 L(\mathbb{R})$. This is the "least stable over X". It is easy to see that

$$\mathscr{P}(\mathbb{R})\cap L_\kappa(\mathbb{R})=\Delta_1(L(\mathbb{R}),\mathbb{R}\cup\{X,\undertilde{\delta}^2_1,\mathbb{R}\})$$

and so if $\Sigma\in\mathscr{P}(\mathbb{R})\cap L_\kappa(\mathbb{R})$ then for $\mathscr{F}_X$-almost all δ there is a reflected version Σ_δ of Σ. We can now state the relevant result:

Suppose $\Sigma\in\mathscr{P}(\mathbb{R})\cap L_\kappa(\mathbb{R})$ is a set of proto-winning strategies for I. Then there is a proto-winning strategy σ such that for all suitable reals y, for all $\tau\in\Sigma_{\delta_{(\sigma,y)}}\cap\Sigma$, there is a suitable real y_τ such that

$$\delta_{(\sigma,y)}=\delta_{(\tau,y_\tau)}.$$

The proof of this is a variant of the above proofs and it provides a more direct proof of completeness and strong normality.

4.3. A Woodin Cardinal

We now wish to show that $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. In general, in inner model theory there is a long march up from strong cardinals to Woodin cardinals. However, in our present context, where we have the power of AD and are working with the special inner model $\mathrm{HOD}^{L(\mathbb{R})}$, this next step comes almost for free.

4.15 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})}\models\Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

Proof. For notational convenience let $\Theta = \Theta^{L(\mathbb{R})}$. To show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta \text{ is a Woodin cardinal}$$

it suffices to show that for each $T \in \mathscr{P}(\Theta) \cap \mathrm{OD}^{L(\mathbb{R})}$, there is an ordinal δ_T such that

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC} + \delta_T \text{ is } \lambda\text{-}T\text{-strong},$$

for each $\lambda < \Theta$. Since Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies that $V_\Theta^{\mathrm{HOD}^{L(\mathbb{R})}}$ is a model of second-order ZFC. Thus, since $T \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ and $V_\Theta^{\mathrm{HOD}^{L(\mathbb{R})}} = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$,

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC}.$$

It remains to establish strength. Since this is almost exactly as before we will just note the basic changes.

The model $L_\Theta(\mathbb{R})[T]$ comes with a natural Σ_1 stratification, namely,

$$\big\langle L_\alpha(\mathbb{R})[T \cap \alpha] \;\big|\; \alpha < \Theta \big\rangle.$$

Since Θ is regular in $L(\mathbb{R})$ and $L_\Theta(\mathbb{R}) \models \mathrm{T}_0$, the set

$$\big\{\alpha < \Theta \mid L_\alpha(\mathbb{R})[T \cap \alpha] \prec L_\Theta(\mathbb{R})[T]\big\}$$

contains a club in Θ. To see this is note that for each $n < \omega$,

$$C_n = \big\{\alpha < \Theta \mid L_\alpha(\mathbb{R})[T \cap \alpha] \prec_n L_\Theta(\mathbb{R})[T]\big\}$$

is club (by Replacement) and, since Θ is regular, $\bigcap\{C_n \mid n < \omega\}$ is club. Thus, there are arbitrarily large $\alpha < \Theta$ such that

$$L_\alpha(\mathbb{R})[T \cap \alpha] \models \mathrm{T}_0.$$

For this reason OD_T, $<_{\mathrm{OD}_T}$ and HOD_T are Σ_1-definable in $L_\Theta(\mathbb{R})[T]$ exactly as before. (Here, as usual, we are working in the language of set theory supplemented with a predicate for T, which is assumed to be allowed in all of our definability calculations.)

Let

$$\delta_T = \text{ the least } \lambda \text{ such that } L_\lambda(\mathbb{R})[T \cap \lambda] \prec_1 L_\Theta(\mathbb{R})[T].$$

As will be evident, the relevant facts concerning $\underset{\sim}{\delta}^2_1$ carry over to the present context. For example, δ_T is the least ordinal λ such that

$$L_\lambda(\mathbb{R})[T \cap \lambda] \prec_1^{\mathbb{R} \cup \{\mathbb{R}\}} L_\Theta(\mathbb{R})[T].$$

The function $F_T : \delta_T \to L_{\delta_T}(\mathbb{R})[T \cap \delta_T]$ is defined as before as follows: Work in T_0. Suppose that $F_T \restriction \delta$ is defined. Let $\vartheta(\delta)$ be least such that

$L_{\vartheta(\delta)}(\mathbb{R})[T \cap \vartheta(\delta)] \models \mathrm{T}_0$ and there is an $X \in L_{\vartheta(\delta)}(\mathbb{R})[T \cap \vartheta(\delta)] \cap \mathrm{OD}_T^{L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]}$ and

($\star$) there is a Σ_1 formula φ and a real z such that

$$L_{\vartheta(\delta)}(\mathbb{R})[T \cap \vartheta(\delta)] \models \varphi[z, X, \delta_T, T \cap \vartheta(\delta), \mathbb{R}]$$

and for all $\bar{\delta} < \delta$,

$$L_{\vartheta(\delta)}(\mathbb{R})[T \cap \vartheta(\delta)] \not\models \varphi[z, F(\bar{\delta}), \bar{\delta}, T \cap \vartheta(\delta), \mathbb{R}]$$

(if such an ordinal exists) and then let $F_T(\delta)$ be the $(<_{\mathrm{OD}_T})^{L_{\vartheta(\delta)(\mathbb{R})}[T\cap\vartheta(\delta)]}$-least X such that ($\star$).

The proof of the Reflection Theorem carries over exactly as before to establish the following: For all $X \in \mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$, for all Σ_1 formulas φ, and for all $z \in \omega^\omega$ if

$$L_\Theta(\mathbb{R})[T] \models \varphi[z, X, \delta_T, T, \mathbb{R}]$$

then there exists a $\delta < \delta_T$ such that

$$L_\Theta(\mathbb{R})[T] \models \varphi[z, F_T(\delta), \delta, T \cap \delta, \mathbb{R}].$$

Let U_X^T be a universal $\Sigma_1(L_\Theta(\mathbb{R})[T], \{X, \delta_T, T, \mathbb{R}\})$ set of reals and, for $\delta < \delta_T$, let U_δ^T be the universal $\Sigma_1(L_\Theta(\mathbb{R})[T], \{F_T(\delta), \delta, T\cap\delta, \mathbb{R}\})$ set obtained by using the same definition. For $z \in U_X^T$, let $S_z^T = \{\delta < \delta_T \mid z \in U_\delta^T\}$ and set

$$\mathscr{F}_X^T = \{S \subseteq \delta_T \mid \exists z \in U_X^T\, (S_z^T \subseteq S)\}.$$

As before, $\mathscr{F}_X^T$ is a countably complete filter and in the Reflection Theorem we can reflect to $\mathscr{F}_X^T$-many points $\delta < \delta_T$ and allow parameters $A \subseteq \delta_T$ and $f : \delta_T \to \delta_T$.

Fix an ordinal $\lambda < \Theta$. By the results of Sect. 3.3, there is an $\mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$-prewellordering $\leqslant_\lambda$ of ω^ω of length λ. Our interest is in applying the Reflection Theorem to

$$X = (\leqslant_\lambda, \lambda).$$

Working in $L_\Theta(\mathbb{R})[T]$, for each $S \subseteq \delta_T$, let $G_T^X(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X^T$. If the rule is violated then, letting i be the least such that either $(x)_i \notin U_X^T$ or $(y)_i \notin U_X^T$ I wins if $(x)_i \in U_X^T$; otherwise II wins. If the rule is satisfied then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta^T$, I wins iff $\delta \in S$.

Now set

$$\mu_X^T = \{S \subseteq \delta_T \mid \mathrm{I} \text{ wins } G_T^X(S)\}.$$

Notice that $\mu_X^T \in \mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$. As before $\mathscr{F}_X^T \subseteq \mu_X^T$ and μ_X^T is a δ_T-complete ultrafilter.

Let

$$S_0 = \{\delta < \delta_T \mid F_T(\delta) = (\leqslant_\delta, \lambda_\delta) \text{ where } \leqslant_\delta \text{ is a prewellordering of length } \lambda_\delta \text{ and } L_{\lambda_\delta}(\mathbb{R})[T \cap \lambda_\delta] \models \mathrm{T}_0\}.$$

By reflection, $S_0 \in \mathscr{F}_X^T$.

As before we say that μ_X^T is *strongly normal* iff whenever $f : S_0 \to \delta_T$ is such that

$$\{\delta \in S_0 \mid f(\delta) < \lambda_\delta\} \in \mu_X^T$$

then there exists a $t \in \omega^\omega$ such that

$$\{\delta \in S_0 \mid f(\delta) = f_t(\delta)\} \in \mu_X^T.$$

The proof that μ_X^T is strongly normal is exactly as before. As in the proof of Lemma 4.8 we can use μ_X^T to take the ultrapower of $\mathrm{HOD}^{L_\Theta(\mathbb{R})[T]}$. In $L_\Theta(\mathbb{R})[T]$ form

$$\left(\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]}\right)^{\delta_T} / \mu_X^T.$$

As before we get an elementary embedding

$$j_\lambda : \mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \to M,$$

where M is the transitive collapse of the ultrapower. By completeness, this embedding has critical point δ_T and as in Lemma 4.10 the canonical functions witness that $j_\lambda(\delta_T) > \lambda$. Assuming further that λ is such that

$$L_\lambda(\mathbb{R})[T \cap \lambda] \prec_1 L_\Theta(\mathbb{R})[T]$$

we have that

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \subseteq M_\lambda.$$

As before, strong normality implies that

$$\begin{aligned} \rho : \lambda &\to \textstyle\prod \lambda_\delta / \mu_X^T \\ |t|_{\leqslant_\lambda} &\mapsto [f_t]_{\mu_X^T} \end{aligned}$$

is an isomorphism. It remains to establish T-strength, that is,

$$|t|_{\leqslant_\lambda} \in T \cap \lambda \quad \text{iff} \quad \{\delta < \delta_T \mid f_t(\delta) \in T \cap \lambda_\delta\} \in \mu_X^T.$$

The point is that both

$$|t|_{\leqslant_\lambda} \in T \cap \lambda$$

and

$$|t|_{\leqslant_\lambda} \notin T \cap \lambda$$

are $\Sigma_1(L_\Theta(\mathbb{R})[T], \{X, \delta_T, T, \mathbb{R}\})$ and so the result follows by the Reflection Theorem (Theorem 4.6) and the fact that $\mathscr{F}^T_X \subseteq \mu^T_X$.

Thus,

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC} + \delta_T \text{ is } \lambda\text{-}T\text{-strong},$$

which completes the proof. ⊣

In the above proof DC was only used in one place—to show that the ultrapowers were well-founded (Lemma 4.8). This was necessary since although the ultrapowers were ultrapowers of HOD and HOD satisfies AC, the ultrapowers were "external" (in that the associated ultrafilters were not in HOD) and so we had to assume DC in V to establish well-foundedness. However, this use of DC can be eliminated by using the extender formulation of being a Woodin cardinal. In this way one obtains strength through a network of "internal" ultrapowers (that is, via ultrafilters that live in HOD) and this enables one to bypass the need to assume DC in V. We will take this route in the next section.

5. Woodin Cardinals in General Settings

Our aim in this section is to abstract the essential ingredients from the previous construction and prove two abstract theorems on Woodin cardinals in general settings, one that requires DC and one that does not.

The first abstract theorem will be the subject of Sect. 5.1:

5.1 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$*. Suppose X and Y are sets. Let*

$$\Theta_{X,Y} = \sup\{\alpha \mid \textit{there is an } \mathrm{OD}_{X,Y} \textit{ surjection } \pi : \omega^\omega \to \alpha\}.$$

Then

$$\mathrm{HOD}_X \models \mathrm{ZFC} + \Theta_{X,Y} \textit{ is a Woodin cardinal.}$$

There is a variant of this theorem (which we will prove in Sect. 5.4) where one can drop DC and assume less determinacy, the result being that Θ_X is a Woodin cardinal in HOD_X. The importance of the version involving $\Theta_{X,Y}$ is that it enables one to show that in certain settings HOD_X can have many Woodin cardinals. To describe one such key application we introduce the following notion due to Solovay. Assume $\mathrm{ZF} + \mathrm{DC}_\mathbb{R} + \mathrm{AD} + V = L(\mathscr{P}(\mathbb{R}))$ and work in $V = L(\mathscr{P}(\mathbb{R}))$. The sequence $\langle \Theta_\alpha \mid \alpha \leqslant \Omega\rangle$ is defined to be the shortest sequence such that Θ_0 is the supremum of all ordinals γ for which there is an OD surjection of ω^ω onto γ, $\Theta_{\alpha+1}$ is the supremum of all ordinals γ for which there is an OD surjection of $\mathscr{P}(\Theta_\alpha)$ onto γ, $\Theta_\lambda = \sup_{\alpha<\lambda} \Theta_\alpha$ for nonzero limit ordinals $\lambda \leqslant \Omega$, and $\Theta_\Omega = \Theta$.

5.2 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC}_\mathbb{R} + \mathrm{AD} + V = L(\mathscr{P}(\mathbb{R}))$*. Then for each $\alpha < \Omega$,*

$$\mathrm{HOD} \models \mathrm{ZFC} + \Theta_{\alpha+1} \textit{ is a Woodin cardinal.}$$

The second abstract theorem provides a template that one can use in various contexts to generate inner models containing Woodin cardinals.

5.3 Theorem (Generation Theorem). *Assume* ZF*. Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$*,*

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$*,*

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α*,*

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models$ *Strategic determinacy with respect to* B*.*

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

The motivation for the statement of the theorem—in particular, the notion of "strategic determinacy"—comes from the attempt to run the construction of Sect. 4.2 using lightface determinacy alone. In doing this one must simulate enough boldface determinacy to handle the real parameters that arise in that construction. To fix ideas we begin in Sect. 5.2 by examining a particular lightface setting, namely, $L[S, x]$ where S is a class of ordinals. Since $(\mathrm{OD}_{S,x})^{L[S,x]} = L[S, x]$ and $L[S, x]$ satisfies AC one cannot have boldface determinacy in $L[S, x]$. However, by assuming full determinacy in the background universe, strong forms of lightface determinacy hold in $L[S, x]$, for an S-cone of x. (The notion of an S-cone will be defined in Sect. 5.2). We will extract stronger and stronger forms of lightface determinacy until ultimately we reach the notion of "strategic determinacy", which is sufficiently rich to simulate boldface determinacy and drive the construction. With this motivation in place we will return to the general setting in Sect. 5.3 and prove the Generation Theorem. Finally, in Sect. 5.4 we will use the Generation Theorem as a template reprove the theorem of the previous section in ZF + AD and to deduce a number of special cases, two of which are worth mentioning here:

5.4 Theorem. *Assume* ZF + AD*. Then for an* S*-cone of* x*,*

$$\mathrm{HOD}_S^{L[S,x]} \models \mathrm{ZFC} + \omega_2^{L[S,x]} \textit{ is a Woodin cardinal.}$$

5.5 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AD}$*. Suppose* Y *is a set and* $a \in H(\omega_1)$*. Then for a* Y*-cone of* x*,*

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \mathrm{ZFC} + \omega_2^{\mathrm{HOD}_{Y,a,x}} \textit{ is a Woodin cardinal,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$.

(The notion of a Y-cone will be defined in Sect. 5.4.) In Sect. 6 these two results will be used as the basis of a calibration of the consistency strength of lightface and boldface definable determinacy in terms of the large cardinal hierarchy. The second result will also be used to reprove and generalize Kechris' classical result that $\mathrm{ZF} + \mathrm{AD}$ implies that DC holds in $L(\mathbb{R})$. For this reason it is important to note that the theorem does not presuppose DC.

5.1. First Abstraction

5.6 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$*. Suppose* X *and* Y *are sets. Then*

$$\mathrm{HOD}_X \models \mathrm{ZFC} + \Theta_{X,Y} \textit{ is a Woodin cardinal.}$$

Proof. By Theorem 3.9,

$$\mathrm{HOD}_{X,Y} \models \Theta_{X,Y} \text{ is strongly inaccessible}$$

and so

$$\mathrm{HOD}_X \models \Theta_{X,Y} \text{ is strongly inaccessible.}$$

A direct approach to showing that in addition

$$\mathrm{HOD}_X \models \Theta_{X,Y} \text{ is a Woodin cardinal}$$

would be to follow Sect. 4.3 by showing that for each $T \in \mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_X$ there is an ordinal δ_T such that

$$\mathrm{HOD}_X \cap V_{\Theta_{X,Y}} \models \delta_T \text{ is } \lambda\text{-}T\text{-strong}$$

for each $\lambda < \Theta_{X,Y}$. However, such an approach requires that for each $\lambda < \Theta_{X,Y}$, there is a prewellordering of ω^ω of length λ which is OD in $L_{\Theta_{X,Y}}(\mathbb{R})[T]$ and in our present, more general setting we have no guarantee that this is true. So our strategy is to work with a larger model (where such prewellorderings exist), get the ultrafilters we need, and then pull them back down to $L_{\Theta_{X,Y}}(\mathbb{R})[T]$ by Kunen's theorem (Theorem 3.11).

We will actually first show that

$$\mathrm{HOD}_{X,Y} \models \Theta_{X,Y} \text{ is a Woodin cardinal.}$$

Let T be an element of $\mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_{X,Y}$ and let (by Lemma 3.7)

$$A = \langle A_\alpha \mid \alpha < \Theta_{X,Y} \rangle$$

be an $\mathrm{OD}_{X,Y}$ sequence such that each A_α is a prewellordering of ω^ω of length α. We will work with the structure

$$L_{\Theta_{X,Y}}(\mathbb{R})[T,A]$$

and the natural hierarchy of structures that it provides.

To begin with we note some basic facts. First, notice that

$$\Theta_{X,Y} = (\Theta_{T,A})^{L(\mathbb{R})[T,A]} = \Theta^{L(\mathbb{R})[T][A]}.$$

(For the first equivalence we have

$$(\Theta_{T,A})^{L(\mathbb{R})[T,A]} \geqslant \Theta_{X,Y}$$

because of A and we have

$$(\Theta_{T,A})^{L(\mathbb{R})[T,A]} \leqslant \Theta_{X,Y}$$

because $L(\mathbb{R})[T,A]$ is $\mathrm{OD}_{X,Y}$. The second equivalence holds since every element in $L(\mathbb{R})[T,A]$ is $\mathrm{OD}_{T,A,y}^{L(\mathbb{R})[T][A]}$ for some $y \in \omega^\omega$. So the "averaging over reals" argument of Lemma 3.8 applies.) It follows that our earlier arguments generalize. For example, by the proof of Theorem 3.10,

$$\Theta_{X,Y} \text{ is strongly inaccessible in } \mathrm{HOD}^{L(\mathbb{R})[T,A]}$$

and

$$\Theta_{X,Y} \text{ is regular in } L(\mathbb{R})[T,A].$$

(Note that $\Theta_{X,Y}$ need not be regular in V. For example, assuming $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_\mathbb{R}$, Θ_0 has cofinality ω in V.) Moreover, the proof of Lemma 2.21 shows that

$$L_{\Theta_{X,Y}}(\mathbb{R})[T,A] \models \mathrm{T}_0$$

and the proof of Lemma 2.23 shows that

$$L_{\Theta_{X,Y}}(\mathbb{R})[T,A] \prec_1 L(\mathbb{R})[T,A].$$

This implies (in conjunction with the fact that $\Theta_{X,Y}$ is regular in $L(\mathbb{R})[T,A]$) that

$$\{\alpha < \Theta_{X,Y} \mid L_\alpha(\mathbb{R})[T{\restriction}\alpha, A{\restriction}\alpha] \prec L_{\Theta_{X,Y}}(\mathbb{R})[T,A]\}$$

is club in $\Theta_{X,Y}$ and hence that each such level satisfies T_0.

So we are in exactly the situation of Sect. 4.3 except that now the prewellorderings are explicitly part of the structure. The proof of Theorem 4.15 thus shows that: For each $T \in \mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_{X,Y}$ there is an ordinal $\delta_{T,A}$ such that

$$\mathrm{HOD}_{T,A}^{L_{\Theta_{X,Y}}(\mathbb{R})[T,A]} \models \delta_{T,A} \text{ is } \lambda\text{-}T\text{-strong}$$

for each $\lambda < \Theta_{X,Y}$, as witnessed by an ultrafilter μ_λ^T on $\delta_{T,A}$. These ultrafilters are $\mathrm{OD}_{T,A}^{L_{\Theta_{X,Y}}(\mathbb{R})[T,A]}$.

The key point is that all of these ultrafilters μ_λ^T are actually OD by Kunen's theorem (Theorem 3.11). This is where DC is used.

Now we return to the smaller model $L_{\Theta_{X,Y}}(\mathbb{R})[T]$. Since $\Theta_{X,Y}$ is strongly inaccessible in HOD_X there is a set $H \in \mathscr{P}(\Theta_{X,Y})^{L_{\Theta_{X,Y}}(\mathbb{R})[T]}$ such that

$$\mathrm{HOD}_X \cap V_{\Theta_{X,Y}} = L_{\Theta_{X,Y}}[H].$$

We may assume without loss of generality that H is folded into T. Thus

$$\mathrm{HOD}_T^{L_{\Theta_{X,Y}}(\mathbb{R})[T]} = \mathrm{HOD}_X \cap V_{\Theta_{X,Y}}$$

and this structure contains all of the ultrafilters μ_λ^T. These ultrafilters can now be used (as in the proof of Lemma 4.8) to take the ultrapower and so we have

$$\mathrm{HOD}_T^{L_{\Theta_{X,Y}}(\mathbb{R})[T]} \models \delta_{T,A} \text{ is } \lambda\text{-}T\text{-strong},$$

which completes the proof. ⊣

5.2. Strategic Determinacy

Let us now turn to the Generation Theorem. We shall begin by motivating the notion of "strategic determinacy" by examining the special case of $L[S,x]$ where S is a class of ordinals.

For $x \in \omega^\omega$, the *S-degree of x* is $[x]_S = \{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$. The *$S$-degrees* are the sets of the form $[x]_S$ for some $x \in \omega^\omega$. Let $\mathscr{D}_S = \{[x]_S \mid x \in \omega^\omega\}$. Define $x \leqslant_S y$ to hold iff $x \in L[S,y]$ and define the notions $x \equiv_S y$, $x <_S y$, $x \geqslant_S y$, $[x]_S \leqslant_S [y]_S$ in the obvious way. A *cone of S-degrees* is a set of the form $\{[y]_S \mid y \geqslant_S x_0\}$ for some $x_0 \in \omega^\omega$. An *S-cone of reals* is a set of form $\{y \in \omega^\omega \mid y \geqslant_S x_0\}$ for some $x_0 \in \omega^\omega$. The *cone filter on $\mathscr{D}_S$* is the filter consisting of sets of S-degrees that contain a cone of S-degrees. Given a formula $\varphi(x)$ we say that *φ holds for an S-cone of x* if there is a real x_0 such that for all $y \geqslant_S x_0$, $L[S,y] \models \varphi(y)$. The proof of the Cone Theorem (Theorem 2.9) generalizes.

5.7 Theorem (Martin). *Assume* ZF + AD*. The cone filter on $\mathscr{D}_S$ is an ultrafilter.*

Proof. For $A \subseteq \mathscr{D}_S$ consider the game

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

where I wins iff $[x * y]_S \in A$. If I has a winning strategy σ_0 then σ_0 witnesses that A is in the S-cone filter since for $y \geqslant_S \sigma_0$, $[y]_S = [\sigma_0 * y]_S \in A$. If II has a winning strategy τ_0 then τ_0 witnesses that $\mathscr{D}_S \smallsetminus A$ is in the S-cone filter since for $x \geqslant_S \tau_0$, $[x]_S = [x * \tau_0]_S \in \mathscr{D}_S \smallsetminus A$. ⊣

It follows that each statement φ either holds on an S-cone or fails on an S-cone. In fact, the entire theory stabilizes. However, in order to fully articulate this fact one needs to invoke second-order assumptions (like the existence of a satisfaction relation). Without invoking second-order assumptions one has the following:

5.8 Corollary. *Assume* ZF + AD. *For each $n < \omega$, there is an x_n such that for all $x \geqslant_S x_n$,*

$$L[S,x] \models \varphi \quad \textit{iff} \quad L[S,x_n] \models \varphi,$$

for all Σ^1_n sentences φ.

Proof. Let $\langle \varphi_i \mid i < \omega\rangle$ enumerate the Σ^1_n sentences of the language of set theory and, for each i, let y_i be the base of an S-cone settling φ_i. Now using $\mathrm{AC}_\omega(\mathbb{R})$ (which is provable in ZF + AD) let x_n encode $\langle y_i \mid i < \omega\rangle$. ⊣

A natural question then is: "What is the stable theory?"

5.9 Theorem. *Assume* ZF + AD. *Then for an S-cone of x,*

$$L[S,x] \models \mathrm{CH}.$$

Proof. Suppose for contradiction (by Theorem 5.7) that ¬CH holds for an S-cone of x. Let x_0 be the base of this cone.

We will arrive at a contradiction by producing an $x \geqslant_S x_0$ such that $L[S,x] \models \mathrm{CH}$. This will be done by forcing over $L[S,x_0]$ in two stages, first to get CH and then to get a real coding this generic (while preserving CH). It will be crucial that the generics actually exist.

Claim. *ω_1^V is strongly inaccessible in any inner model M of* AC.

Proof. We first claim that there is no ω_1^V-sequence of distinct reals: Let μ be the club filter on ω_1^V. By Solovay's theorem (Theorem 2.12, which doesn't require DC) μ is a countably complete ultrafilter on ω_1^V. Suppose $\langle a_\alpha \mid \alpha < \omega_1^V\rangle$ is a sequence of characteristic functions for distinct reals. By countable completeness there is a μ-measure one set X_n of elements of this sequence that agree on their nth-coordinate. Thus, $\bigcap_{n<\omega} X_n$ has μ-measure one, which is impossible since it only has one member.

It follows that for each $\gamma < \omega_1^V$, $(2^\gamma)^M < \omega_1^V$ since otherwise (γ being countable) there would be an ω_1^V sequence of distinct reals. Since ω_1^V is clearly regular in M the result follows. ⊣

Step 1. Let G be $L[S,x_0]$-generic for $\mathrm{Col}(\omega_1^{L[S,x_0]}, \mathbb{R}^{L[S,x_0]})$. (By the Claim this generic exists in V). So

$$L[S,x_0][G] \models \mathrm{CH} \quad \text{and} \quad \mathbb{R}^{L[S,x_0][G]} = \mathbb{R}^{L[S,x_0]}.$$

The trouble is that $L[S,x_0][G]$ is not of the form $L[S,x]$ for $x \in \mathbb{R}$. (We could code G via a real by brute force but doing so might destroy CH. A more delicate approach is needed.)

Step 2. Code G using almost disjoint forcing: First, view G as a subset of $\omega_1^{L[S,x_0]}$ by letting $A \subseteq \omega_1^{L[S,x_0]}$ be such that

$$L[S,x_0][G] = L[S,x_0,A].$$

Now let

$$\langle \sigma_\alpha \mid \alpha < \omega_1^{L[S,x_0]} \rangle \in L[S,x_0]$$

be a sequence of infinite almost disjoint subsets of ω (that is, such that if $\alpha \neq \beta$ then $\sigma_\alpha \cap \sigma_\beta$ is finite). By almost disjoint forcing, in $L[S,x_0,A]$ there is a c.c.c. forcing $\mathbb{P}_A$ of size $\omega_1^{L[S,x_0,A]}$ such that if $H \subseteq \mathbb{P}_A$ is $L[S,x_0,A]$-generic then there is a $c(A) \subseteq \omega$ such that

$$\alpha \in A \quad \text{iff} \quad c(A) \cap \sigma_\alpha \text{ is infinite.}$$

(See [1, pp. 267–268] for details concerning this forcing notion.) Also

$$L[S,x_0,A][H] = L[S,x_0,A][c(A)] = L[S,x_0,c(A)].$$

Finally,

$$L[S,x_0,c(A)] \models \mathrm{CH}$$

as $\mathbb{P}_A$ is c.c.c., $|\mathbb{P}_A| = \omega_1^{L[S,x_0,A]}$, and $L[S,x_0,A] \models \mathrm{CH}$, and so there are, up to equivalence, only $\omega_1^{L[S,x_0,A]}$-many names for reals. ⊣

5.10 Corollary. *Assume* $\mathrm{ZF} + \mathrm{AD}$. *For an* S*-cone of* x*,*

$$L[S,x] \models \mathrm{GCH} \textit{ below } \omega_1^V.$$

Proof. Let x_0 be such that for all $x \geqslant_S x_0$,

$$L[S,x] \models \mathrm{CH}.$$

Fix $x \geqslant_S x_0$. We claim that $L[S,x] \models \mathrm{GCH}$ below ω_1^V: Suppose for contradiction that there is a $\lambda < \omega_1^V$ such that $L[S,x] \models 2^\lambda > \lambda^+$. Let $G \subseteq \mathrm{Col}(\omega,\lambda)$ be $L[S,x]$-generic. Thus $L[S,x][G] \models \neg\mathrm{CH}$. But $L[S,x][G] = L[S,y]$ for some real y and so $L[S,x][G] \models \mathrm{CH}$. ⊣

A similar proof shows that $\diamondsuit$ holds for an S-cone of x, the point being that adding a Cohen subset of ω_1 forces $\diamondsuit$ and this forcing is c.c.c. and of size ω_1. See [1], Exercises 15.23 and 15.24.

5.11 Conjecture. Assume $\mathrm{ZF} + \mathrm{AD}$. For an S-cone of x,

$$L[S,x] \cap V_{\omega_1^V}$$

is an "L-like" model in that it satisfies Condensation, $\square$, Morasses, etc.

Corollary 5.10 tells us that for an S-cone of x,

$$\Theta^{L[S,x]} = (\mathfrak{c}^+)^{L[S,x]} = \omega_2^{L[S,x]}.$$

Thus, to prove that for an S-cone of x,

$$L[S,x] \models \omega_2 \text{ is a Woodin cardinal in } \mathrm{HOD}_S,$$

we can apply our previous construction concerning Θ provided we have enough determinacy in $L[S,x]$.

5.12 Theorem (Kechris and Solovay). *Assume* ZF + AD. *For an S-cone of x,*

$$L[S,x] \models \mathrm{OD}_S\textit{-determinacy.}$$

Proof. Play the following game

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a,b,c,d\rangle$, I wins if $L[S,p] \not\models \mathrm{OD}_S$-determinacy and $L[S,p] \models$ "$a * d \in A^p$", where A^p is the least (in the canonical ordering) undetermined $\mathrm{OD}_S^{L[S,p]}$ set in $L[S,p]$. In such a game the reals are played so as to be "interleaved" in the pattern $(a(0), c(0), b(0), d(0), \dots)$. Here the two players are to be thought of as cooperating to determine the playing field $L[S,p]$ in which they will simultaneously play (via a and d) an auxiliary round of the game on the least undetermined OD_S set A^p (assuming, of course, that such a set exists, as I is trying to ensure).

Case 1: I has a winning strategy σ_0.

We claim that for all $x \geqslant_S \sigma_0$, $L[S,x] \models \mathrm{OD}_S$-determinacy, which contradicts the assumption that σ_0 is a winning strategy for I. For consider such a real x and suppose for contradiction that $L[S,x] \not\models \mathrm{OD}_S$-determinacy. As above let $A^x \in \mathrm{OD}_S^{L[S,x]}$ be least such that A^x is not determined. We will arrive at a contradiction by deriving a winning strategy σ for I in A^x from the strategy σ_0. Run the game according to σ_0 while having Player II feed in x for c and playing some auxiliary play $d \in L[S,x]$. This ensures that the resulting model $L[S,p]$ that the two players jointly determine is just $L[S,x]$ and so $A^p = A^x$. We can now derive a winning strategy σ for I in A^x from σ_0 as follows: For $d \in L[S,x]$, let σ be the strategy such that $(\sigma * d)_I$ is the a such that $(\sigma_0 * \langle x,d\rangle)_I = \langle a,b\rangle$.

(It is crucial that we have II play $c = x$ and $d \in L[S,x]$ since otherwise we would get $a * d \in A^p$ for varying p. By having II play $c = x$ and $d \in L[S,x]$, II has "steered into the right model", namely $L[S,x]$, and we have "fixed" the set A^x. This issue will become central later on when we refine this proof.)

Case 2: II has a winning strategy τ_0.

We claim that for $x \geqslant_S \tau_0$, $L[S,x] \models \mathrm{OD}_S$-determinacy. This is as above except that now we run the game according to τ_0, having I steer into $L[S,x]$ by playing x for b and some $a \in L[S,x]$. Then, as above, we derive a winning strategy for I in $\omega^\omega \smallsetminus A^x$ and hence a winning strategy τ for II in A^x. ⊣

To drive the construction of a model containing a Woodin cardinal we need more than OD_S-determinacy since some of the games in the construction are definable in a real parameter. Unfortunately, we cannot hope to get

$$L[S,x] \models \mathrm{OD}_{S,y}\text{-determinacy}$$

for each y since $(\mathrm{OD}_{S,x})^{L[S,x]} = L[S,x]$ and $L[S,x]$ is a model of AC. Nevertheless, it is possible to have $\mathrm{OD}_{S,y}$-determinacy in $L[S,x]$ for certain specially chosen reals y. There is therefore hope of approximating a sufficient amount of boldface definable determinacy to drive the construction. To make precise the approximation we need, we introduce the notion of a "prestrategy".

Let A and B be sets of reals. A *prestrategy for* I (*respectively* II) *in* A is a continuous function f such that for all $x \in \omega^\omega$, $f(x)$ is a strategy for I (respectively II) in A. A prestrategy f in A (for either I or II) is *winning with respect to the basis* B if, in addition, for all $x \in B$, $f(x)$ is a winning strategy in A. The *strategic game with respect to the predicates* $P_1, \ldots, P_k$ *and the basis* B is the game $SG^B_{P_0,\ldots,P_k}$

$$\begin{array}{llllll} \mathrm{I} & A_0 & \cdots & A_n & \cdots & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{P_0,\ldots,P_k}$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{P_0,\ldots,P_k,f_0,\ldots,f_n}$ and

(2) f_n is a prestrategy for A_n that is winning with respect to B,

and II wins iff II can play all ω rounds. We say that *strategic definable determinacy holds with respect to the predicates* $P_0, \ldots, P_k$ *and the basis* B ($\mathrm{ST}_{P_0,\ldots,P_k}$-*determinacy*) if II wins $SG^B_{P_0,\ldots,P_k}$ and we say that *strategic definable determinacy for* n *moves holds with respect to the predicates* $P_0, \ldots, P_k$ *and the basis* B ($\mathrm{ST}_{P_0,\ldots,P_k}$-*determinacy for* n *moves*) if II can play n rounds of $SG^B_{P_0,\ldots,P_k}$. When these parameters are clear from context we shall often simply refer to SG and ST-determinacy.

In the context of $L[S,x]$ the predicate will be S and the basis B will be the S-degree of x. Thus to say that $L[S,x]$ satisfies ST_S-determinacy (or ST-determinacy for short) is to say that II can play all rounds of the game

$$\begin{array}{llllll} \mathrm{I} & A_0 & \cdots & A_n & \cdots & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_S^{L[S,x]}$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{S,f_0,\ldots,f_n}^{L[S,x]}$, and

(2) $f_n \in L[S,x]$ is a prestrategy for A_n that is winning with respect to $[x]_S$.

The ability to survive a single round of this game implies that $L[S,x]$ satisfies OD_S-determinacy. So this notion is indeed a generalization of OD_S-determinacy.

Before turning to the main theorems, some remarks are in order. First, notice that the games $\mathrm{ST}_{P_0,\ldots,P_k}$ are closed for Player II, hence determined. The only issue is whether II wins.

Second, notice also that if I wins then I has a *canonical* strategy. This can be seen as follows: Player I can rank partial plays, assigning rank 0 to partial plays in which he wins; Player I can then play by reducing rank. The result is a quasi-strategy that is definable in terms of the tree of partial plays which in turn is ordinal definable. Since I is essentially playing ordinals this quasi-strategy can be converted into a strategy in a definable fashion. We take this to be I's canonical strategy.

Third, notice that each prestrategy can be coded by a real number in a canonical manner. We assume that such a coding has been fixed and, for notational convenience, we will identify a prestrategy with its code.

Fourth, it is important to note that if II is to have a hope of winning then we must allow II to play prestrategies and not strategies. To see this, work in $L[S,x]$ and consider the variant of SG_S^B where we have II play strategies τ_0, τ_1, ... instead of prestrategies. The set $A_0 = \{y \in \omega^\omega \mid L[S, y_{\mathrm{even}}] = L[S,x]\}$ is $\mathrm{OD}_S^{L[S,x]}$ and hence a legitimate first move for I. But then II's response must be a winning strategy for I in A_0 since I can win a play of A_0 by playing x. However, $\mathrm{OD}_{S,\tau_0}^{L[S,x]} = L[S,x]$ and so in the next round I is allowed to play any $A_1 \in L[S,x]$. But then II cannot hope to always respond with a winning strategy since $L[S,x] \not\models \mathrm{AD}$. The upshot is that if II is to have a hope of winning a game of this form then we must allow II to be less committal.

Fifth, although one can use a base B which is slightly larger than $[x]_S$, the previous example motivates the choice of $B = [x]_S$. Let A_0 be as in the previous paragraph and let f_0 be II's response. By the above argument, it follows that for all $z \in B$, $\langle f_0, z\rangle \in [x]_S$ and so in a sense we are "one step away" from showing that one must have $B \subseteq [x]_S$.

Finally, as we shall show in the next section, for every OD basis $B \subseteq \omega^\omega$ there is an OD set $A \subseteq \omega^\omega$ such that there is no OD prestrategy which is winning for A with respect to B (Theorem 6.11). Thus, for each basis B, ST^B-determinacy does not trivially reduce to OD-determinacy.

5.13 Theorem. *Assume* ZF + AD. *Then for an S-cone of x, for each n,*

$$L[S,x] \models \mathrm{ST}_S\text{-}\textit{determinacy for } n \textit{ moves},$$

where $B = [x]_S$.

5.14 Theorem. *Assume* ZF + DC$_{\mathbb{R}}$ + AD. *Then for an S-cone of x,*

$$L[S,x] \models \mathrm{ST}_S\text{-}determinacy,$$

where $B = [x]_S$.

Proofs of Theorems 5.13 and 5.14. Assume toward a contradiction that the statement of Theorem 5.14 is false. By Theorem 5.7, there is a real x_0 such that if $x \geqslant_T x_0$,

$$L[S,x] \models \text{I wins } SG,$$

(where here and below we drop reference to S and B since these are fixed throughout). For $x \geqslant_T x_0$, let σ^x be I's canonical winning strategy in $SG^{L[S,x]}$. Note that the strategy depends only on the model, that is, if $y \equiv_S x$ then $\sigma^y = \sigma^x$.

Our aim is to construct a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that the winning strategies (for whichever player wins) enable us to define, for an S-cone of x, prestrategies $f_0^x, f_1^x, \ldots, f_n^x, \ldots$ which constitute a non-losing play against σ^x in $SG^{L[S,x]}$.

Step 0. Consider (in V) the game G_0

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d, x_0\rangle$ and $A_0^p = \sigma^p(\varnothing)$, I wins iff $a * d \in A_0^p$. Notice that by including x_0 in p we have ensured that σ^p is defined and hence that the winning condition makes sense. In this game I and II are cooperating to steer into the model $L[S,p]$ and they are simultaneously playing (via a and d) an auxiliary round of the game A_0^p, where A_0^p is I's first move according to the canonical strategy in the strategic game $SG^{L[S,p]}$. I wins a round iff I wins the auxiliary round of this auxiliary game.

Claim 1. *There is a real x_1 such that for all $x \geqslant_S x_1$ there is a prestrategy f_0^x that is a non-losing first move for II against σ^x in $SG^{L[S,x]}$.*

Proof. Case 1: I has a winning strategy σ_0 in G_0.

For $x \geqslant_T \sigma_0$, let f_0^x be the prestrategy derived from σ_0 by extracting the response in the auxiliary game where we have II feed in y for c, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f_0^x(y)$ be such that $f_0^x(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle a, b\rangle$. Note that $f_0^x \in L[S,x]$ as it is definable from σ_0. Let $x_1 = \langle \sigma_0, x_0\rangle$ and for $x \geqslant_S x_1$ let $A_0^x = \sigma^x(\varnothing)$. We claim that for $x \geqslant_S x_1$, f_0^x is a prestrategy for I in A_0^x that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f_0^x is a non-losing first move for II against σ^x in $SG^{L[S,x]}$. To see this fix $x \geqslant_S x_1$ and y such that $L[S,y] = L[S,x]$ and consider $d \in L[S,x]$. The value $f_0^x(y)$ of the prestrategy was defined by running G_0, having II feed in y for c:

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & y, d \end{array}$$

By our choice of y and d, we have solved the "steering problem", that is, we have $L[S,p] = L[S,x]$ and $A_0^p = A_0^x$ where $p = \langle a, b, y, d, x_0 \rangle$. Now, f_0^x is such that $f_0^x(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d \rangle)_I = \langle a, b \rangle$. Since σ_0 is winning for I, we have $f_0^x(y) * d = a * d \in A_0^p = A_0^x$.

Case 2: II has a winning strategy τ_0 in G_0.

Let f_0^x be the prestrategy derived from τ_0 by extracting the response in the auxiliary game where we have I feed in y for b, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f_0^x(y)$ be such that $a * f_0^x(y) = a * d$ where d is such that $(\langle a, y \rangle * \tau_0)_{II} = \langle c, d \rangle$. Let $x_1 = \langle \tau_0, x_0 \rangle$ and for $x \geqslant_S x_1$ let $A_0^x = \sigma^x(\varnothing)$. As before, we have that for $x \geqslant_S x_1$, f_0^x is a prestrategy for II in A_0^x that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f_0^x is a non-losing first move for II against σ^x in $SG^{L[S,x]}$.

Let x_1 be as described in whichever case holds. ⊣

Step $n+1$. Assume that we have defined games $G_0, \ldots, G_n$, reals $x_0, \ldots, x_{n+1}$ such that $x_0 \leqslant_S x_1 \leqslant_S \cdots \leqslant_S x_{n+1}$, and prestrategies $f_0^x, \ldots, f_n^x$ which depend only on the degree of x and such that for all $x \geqslant_S x_{n+1}$,

$$f_0^x, \ldots, f_n^x$$

is a non-losing partial play for II against σ^x in $SG^{L[S,x]}$.

Consider (in V) the game G_{n+1}

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d, x_{n+1} \rangle$ and A_{n+1}^p be I's response via σ^p to the partial play $f_0^p, \ldots, f_n^p$, I wins iff $a * d \in A_{n+1}^p$. Notice that we have included x_{n+1} in p to ensure that $\sigma^p, f_0^p, \ldots, f_n^p$ are defined and hence that the winning condition makes sense. In this game I and II are cooperating to steer into the model $L[S,p]$ and they are simultaneously playing an auxiliary round (via a and d) on A_{n+1}^p, where A_{n+1}^p is I's response via σ^p to II's non-losing partial play $f_0^p, \ldots, f_n^p$ in the strategic game $SG^{L[S,p]}$. I wins a round iff he wins the auxiliary round of this auxiliary game.

Claim 2. *There is a real x_{n+2} such that for all $x \geqslant_S x_{n+2}$ there is a prestrategy f_{n+1}^x such that $f_0^x, \ldots, f_n^x, f_{n+1}^x$ is a non-losing partial play for II against σ^x in $SG^{L[S,x]}$.*

Proof. Case 1: I has a winning strategy σ_{n+1} in G_{n+1}.

Let f_{n+1}^x be the prestrategy derived from σ_{n+1} by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f_{n+1}^x(y)$ be such that $f_{n+1}^x(y) * d = a * d$ where a is such that $(\sigma_{n+1} * \langle y, d \rangle)_I = \langle a, b \rangle$. Let $x_{n+2} = \langle \sigma_{n+1}, x_{n+1} \rangle$ and for $x \geqslant_S x_{n+2}$ let $A_{n+1}^x = \sigma^x(\langle f_0^x, \ldots, f_n^x \rangle)$, i.e. A_{n+1}^x is the

$(n+2)$nd move of I in $SG^{L[S,x]}$ following σ^x against II's play of $f_0^x, \ldots, f_n^x$. As in Claim 1, f_{n+1}^x is a prestrategy for I in A_{n+1}^x that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f_{n+1}^x is a non-losing $(n+2)$nd move for II against σ^x in $SG^{L[S,x]}$.

Case 2: II has a winning strategy τ_{n+1} in G_{n+1}.

Let f_{n+1}^x be the prestrategy derived from τ_{n+1} by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f_{n+1}^x(y)$ be such that $a * f_{n+1}^x(y) = a * d$ where d is such that $(\langle a, y\rangle * \tau_{n+1})_{II} = \langle c, d\rangle$. Let $x_{n+2} = \langle \tau_{n+1}, x_{n+1}\rangle$ and for $x \geqslant_S x_{n+2}$ let $A_{n+1}^x = \sigma^x(\langle f_0^x, \ldots, f_n^x\rangle)$, as above. As before, we have that for $x \geqslant_S x_{n+2}$, f_{n+1}^x is a prestrategy for II in A_{n+1}^x that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f_{n+1}^x is a non-losing $(n+2)$nd move for II against σ^x in $SG^{L[S,x]}$.

Let x_{n+2} be as described in whichever case holds. ⊣

Finally, using $\mathrm{DC}_{\mathbb{R}}$, we get a sequence of reals $x_0, \ldots, x_n, \ldots$ and prestrategies $f_0^x, \ldots, f_n^x, \ldots$ as in each of the steps. Letting $x^\infty \geqslant_S x_n$, for all n, we have that for all $x \geqslant_S x^\infty$, $f_0^x, \ldots, f_n^x, \ldots$ is a non-losing play for II against σ^x in $SG^{L[S,x]}$, which is a contradiction. This completes the proof of Theorem 5.14.

For Theorem 5.13 simply note that $\mathrm{DC}_{\mathbb{R}}$ is not needed to define the finite sequences $x_0, \ldots, x_n, x_{n+1}$ and $f_0^x, \ldots, f_n^x$ for $x \geqslant_S x_{n+1}$ (as these prestrategies are definable from $x_0, \ldots, x_n, x_{n+1}$). ⊣

5.3. Generation Theorem

In the previous section we showed (assuming ZF + AD) that for an S-cone of x,

$$L[S,x] \models \mathrm{OD}_S\text{-determinacy},$$

and (even more) that for each n,

$$L[S,x] \models \mathrm{ST}_S\text{-determinacy for } n \text{ moves},$$

where $B = [x]_S$. It turns out that for a sufficiently large choice of n this degree of determinacy is sufficient to implement the previous arguments and show that

$$L[S,x] \models \omega_2 \text{ is a Woodin cardinal in } \mathrm{HOD}_S.$$

At this stage we could proceed directly to this result but instead, with this motivation behind us, we return to the more general setting. The main theorem to be proved is the Generation Theorem:

5.15 Theorem (Generation Theorem). *Assume* ZF. *Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$,

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$,

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α,

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models \mathrm{ST}_{T,A,B}$*-determinacy for four moves.*

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

The importance of the restriction to strategic determinacy for four moves is that in a number of applications of this theorem strategic determinacy for n moves (for each n) can be established without any appeal to DC (as for example in Theorem 5.13) in contrast to full strategic determinacy which (just as in Theorem 5.14) uses $\mathrm{DC}_{\mathbb{R}}$.

The external assumption that Θ_M is a regular cardinal is merely for convenience—it ensures that there are cofinally many stages in the stratification of M where T_0 holds. The dedicated reader can verify that this assumption can be dropped by working instead with the theory $\mathrm{ZF}_N + \mathrm{AC}_\omega(\mathbb{R})$ for some sufficiently large N.

The remainder of this section is devoted to a proof of the Generation Theorem.

Proof. Let us start by showing that $\mathrm{HOD}^M_{T,A,B}$ satisfies ZFC. When working with structures of the form $L_{\Theta_M}(\mathbb{R})[T, A, B]$ it is to be understood that we are working in the language of ZFC augmented with constant symbols for T, A, B, and $\mathbb{R}$. The first step is to show that $\mathrm{HOD}^M_{T,A,B}$ is first-order over M. For $\gamma < \Theta_M$, let

$$M_\gamma = L_\gamma(\mathbb{R})[T\restriction\gamma, A\restriction\gamma, B],$$

it being understood that the displayed predicates are part of the structure. Since Θ_M is regular and $M \models \mathrm{T}_0$ there are cofinally many $\gamma < \Theta_M$ such that $M_\gamma \models \mathrm{T}_0$. So a set $x \in M$ is $\mathrm{OD}^M_{T,A,B}$ if and only if there is a $\gamma < \Theta_M$ such that $M_\gamma \models \mathrm{T}_0$ and x is definable in M_γ from ordinal parameters (and the constant symbols for the parameters). It follows that $\mathrm{OD}^M_{T,A,B}$ and $\mathrm{HOD}^M_{T,A,B}$ are Σ_1-definable over M (in the expanded language).

With this first-order characterization of $\mathrm{HOD}^M_{T,A,B}$ all of the standard results carry over to our present setting. For example, since $M \models \mathrm{ZF} -$ Power Set we have that $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} -$ Power Set. (The proofs that AC holds in $\mathrm{HOD}^M_{T,A,B}$ and that for all $\alpha < \Theta_M$, $V_\alpha \cap \mathrm{HOD}^M_{T,A,B} \in \mathrm{HOD}^M_{T,A,B}$ require that $\mathrm{OD}^M_{T,A,B}$ be ordinal definable.)

5.16 Lemma. $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC}$.

Proof. We have seen that $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC}-\text{Power Set}$. Since $\mathrm{HOD}^M_{T,A,B} \models$ AC it remains to show that for all $\lambda < \Theta_M$,

$$\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in \mathrm{HOD}^M_{T,A,B}.$$

The point is that since $M \models \mathrm{OD}_{T,A,B}$-determinacy, for each $S \in \mathrm{OD}^M_{T,A,B} \cap \mathscr{P}(\lambda)$ the game for coding S relative to the prewellordering A_λ is determined: Without loss of generality, we may assume A_λ has length λ. For $\alpha < \lambda$, let $Q^\kappa_{<\alpha}$ and Q^κ_α be the usual objects defined relative to A_λ. For $e \in \omega^\omega$, let

$$S_e = \{\alpha < \lambda \mid U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing\}.$$

Since A_λ is trivially $\mathrm{OD}^M_{T,A,B}$ the game for the Uniform Coding Lemma for $Z = \bigcup\{Q^\kappa_\alpha \times \omega^\omega \mid \alpha \in S\}$ is determined for each $S \in \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}}$. Thus, every $S \in \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}}$ has the form S_e for some $e \in \omega^\omega$ and hence

$$\begin{aligned} \pi : \omega^\omega &\longrightarrow \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \\ e &\mapsto S_e \end{aligned}$$

is an $\mathrm{OD}^M_{T,A,B}$ surjection. Thus, $\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in M$ and so, by our first-order characterization, $\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in \mathrm{HOD}^M_{T,A,B}$. $\dashv$

The ordinal κ that we will show to be T-strong in $\mathrm{HOD}^M_{T,A,B}$ is "the least stable in M":

5.17 Definition. Let κ be least such that

$$M_\kappa \prec_1 M.$$

As before the $\Diamond$-like function $F : \kappa \to M_\kappa$ is defined inductively in terms of the least counterexample: Given $F\restriction\delta$ let $\vartheta(\delta)$ be least such that

$$M_{\vartheta(\delta)} \models \mathrm{T}_0 \text{ and there is an } X \in M_{\vartheta(\delta)} \cap \mathrm{OD}^{M_{\vartheta(\delta)}}_{T\restriction\vartheta(\delta),A\restriction\vartheta(\delta),B} \text{ and}$$

($\star$) there is a Σ_1 formula φ and a $t \in \omega^\omega$ such that

$$M_{\vartheta(\delta)} \models \varphi[t, X, \delta, \mathbb{R}]$$

and for all $\bar{\delta} < \delta$

$$M_{\vartheta(\delta)} \not\models \varphi[t, F(\bar{\delta}), \bar{\delta}, \mathbb{R}]$$

(if such an ordinal exists) and then let $F(\delta)$ be the $<^{M_{\vartheta(\delta)}}_{\mathrm{OD}_{T\restriction\vartheta(\delta),A\restriction\vartheta(\delta),B}}$-least X such that ($\star$) holds.

5.18 Theorem. *For all $X \in \mathrm{OD}^M_{T,A,B}$, for all Σ_1 formulas φ, and for all $t \in \omega^\omega$, if*

$$M \models \varphi[t, X, \kappa, \mathbb{R}]$$

then there exists a $\delta < \kappa$ such that

$$M \models \varphi[t, F(\delta), \delta, \mathbb{R}].$$

Proof. Same as the proof of Theorem 4.6. ⊣

Our interest is in applying Theorem 5.18 to

$$X = (\leqslant_\lambda, \lambda)$$

where $\leqslant_\lambda = A_\lambda$ is the prewellordering of length λ, for $\lambda < \Theta_M$. Clearly X is $\mathrm{OD}^M_{T,A,B}$.

Let U_X be a universal $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$ set of reals and, for $\delta < \kappa$, let U_δ be the reflected version (using the same definition used for U except with $F(\delta)$ and δ in place of X and κ). For $z \in U_X$, let $S_z = \{\delta < \kappa \mid z \in U_\delta\}$ and set

$$\mathscr{F}_X = \{S \subseteq \kappa \mid \exists z \in U_X\,(S_z \subseteq S)\}.$$

As before, $\mathscr{F}_X$ is a countably complete filter and in Theorem 5.18 we can reflect to $\mathscr{F}_X$-many points $\delta < \kappa$. Let

$$S_0 = \{\delta < \kappa \mid F(\delta) = (A_{\lambda_\delta}, \lambda_\delta) \text{ for some } \lambda_\delta > \delta\}.$$

Notice that $S_0 \in \mathscr{F}_X$. For notational conformity let $\leqslant_\delta$ be A_{λ_δ}. For $\alpha < \lambda$, let Q^κ_α be the αth-component of $\leqslant_\lambda$ and, for $\delta \in S_0$ and $\alpha < \lambda_\delta$, let Q^δ_α be the αth-component of $\leqslant_\delta$ (where without loss of generality we may assume that each A_α has length exactly α).

In our previous settings we went on to do two things. First, using the Uniform Coding Lemma we showed that one can allow parameters of the form $A \subseteq \kappa$ and $f : \kappa \to \kappa$ in the Reflection Theorem. Second, for $S \subseteq \kappa$, we defined the games $G^X(S)$ that gave rise to the ultrafilter extending the reflection filter, an ultrafilter that was either explicitly OD in the background universe (as in Sect. 4.3) or shown to be OD by appeal to Kunen's theorem (as in Sect. 5.1). In our present setting (where we have a limited amount of determinacy at our disposal) we will have to manage our resources more carefully. The following notion will play a central role.

5.19 Definition. A set $x \in M$ is *n-good* if and only if II can play n rounds of $(SG^B_{T,A,B,x})^M$. For $y \in M$, a set $x \in M$ is *n-y-good* if and only if (x, y) is n-good.

Notice that if M satisfies $\mathrm{ST}_{T,A,B,y}$-determinacy for $n+1$ moves then II's first move f_0 is n-y-good. Notice also that if x is 1-y-good then every $\mathrm{OD}^M_{T,A,B,x,y}$ set of reals is determined. For example, if $S \subseteq \kappa$ is 1-good then the game for coding S relative to A_κ using the Uniform Coding Lemma is determined. Thus we have the following version of the Reflection Theorem.

5.20 Theorem. *Suppose $f : \kappa \to \kappa$, $G \subseteq \kappa$, $S \subseteq \kappa$ and (f, G, S) is 1-good. For all $X \in M \cap \mathrm{OD}^M_{T,A,B}$, for all Σ_1 formulas φ, and for all $t \in \omega^\omega$, if*

$$M \models \varphi[t, X, \kappa, \mathbb{R}, f, G, S]$$

then for $\mathscr{F}_X$-many $\delta < \kappa$,

$$M \models \varphi[t, F(\delta), \delta, \mathbb{R}, f{\restriction}\delta, G \cap \delta, S \cap \delta].$$

For each $S \subseteq \kappa$, let $G^X(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X$. If the rule is violated, then, letting i be the least such that either $(x)_i \notin U_X$ or $(y)_i \notin U_X$, I wins if $(x)_i \in U_X$; otherwise II wins. If the rule is satisfied, then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta$, I wins iff $\delta \in S$.

As before, if $S \in \mathscr{F}_X$ then I wins $G^X(S)$ by playing any x such that for all $i < \omega$, $(x)_i \in U_X$ and for some $i < \omega$, $(x)_i = z$, where z is such that $S_z \subseteq S$. But we cannot set

$$\mu_X = \{S \subseteq \kappa \mid \mathrm{I} \text{ wins } G^X(S)\}$$

since we have no guarantee that $G^X(S)$ is determined for an arbitrary $S \subseteq \kappa$.

However, if S is 1-good then $G^X(S)$ is determined. In particular, $G^X(S)$ is determined for each $S \in \mathscr{P}(\kappa) \cap \mathrm{HOD}^M_{T,A,B}$. Thus, setting

$$\mu = \{S \in \mathscr{P}(\kappa) \cap \mathrm{HOD}^M_{T,A,B} \mid \mathrm{I} \text{ wins } G^X(S)\}$$

we have directly shown that κ is measurable in $\mathrm{HOD}^M_{T,A,B}$.

It is useful at this point to stand back and contrast the present approach with the two earlier approaches. In both of the earlier approaches (namely, that of Sect. 4.3 and that of Sect. 5.1) the ultrafilters were ultrafilters in V and seen to be complete and normal in V and the ultrafilters were OD^V, the only difference being that in the first case the ultrafilters were *directly* seen to be OD^V, while in the second case they were *indirectly* seen to be OD^V by appeal to Kunen's theorem (Theorem 3.11). Now, in our present setting, there is no hope of getting such ultrafilters in V since we do not have enough determinacy. Instead we will get ultrafilters in $\mathrm{HOD}^M_{T,A,B}$. However, the construction will still be "external" in some sense since we will be defining the ultrafilters in V.

We also have to take care to ensure that the ultrafilters "fit together" in such a way that they witness that κ is T-strong. In short, we will define a (κ, λ)-pre-extender $E_X \in \mathrm{HOD}^M_{T,A,B}$, a notion we now introduce.

For $n \in \omega$ and $z \in [\mathrm{On}]^n$, we write $z = \{z_1, \ldots, z_n\}$, where $z_1 < \cdots < z_n$. Suppose $b \in [\mathrm{On}]^n$ and $a \subseteq b$ is such that $a = \{b_{i_1}, \ldots b_{i_k}\}$, where $i_1 < \cdots < i_k$. For $z \in [\mathrm{On}]^n$, set

$$z_{a,b} = \{z_{i_1}, \ldots, z_{i_k}\}.$$

Thus the elements of $z_{a,b}$ sit in z in the same manner in which the elements of a sit in b. For $\alpha \in \mathrm{On}$ and $X \subseteq [\alpha]^k$, let

$$X^{a,b} = \{z \in [\alpha]^n \mid z_{a,b} \in X\}.$$

For $\alpha \in \mathrm{On}$ and $f : [\alpha]^k \to V$, let $f^{a,b} : [\alpha]^n \to V$ be such that

$$f^{a,b}(z) = f(z_{a,b}).$$

Thus we use 'a, b' as a subscript to indicate that $z_{a,b}$ is the "drop of z from b to a" and we use 'a, b' as a superscript to indicate that $X^{a,b}$ is the "lift of X from a to b".

5.21 Definition. Let κ be an uncountable cardinal and let $\lambda > \kappa$ be an ordinal. The sequence

$$E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$$

is a (κ, λ)-*extender* provided:

(1) For each $a \in [\lambda]^{<\omega}$,

$$E_a \text{ is a } \kappa\text{-complete ultrafilter on } [\kappa]^{|a|}$$

that is principal if and only if $a \subseteq \kappa$.

(2) (Coherence) If $a \subseteq b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X^{a,b} \in E_b$.

(3) (Countable Completeness) If $X_i \in E_{a_i}$ where $a_i \in [\lambda]^{<\omega}$ for each $i < \omega$, then there is an order-preserving map

$$h : \textstyle\bigcup_{i<\omega} a_i \to \kappa$$

such that $h``a_i \in X_i$ for all $i < \omega$.

(4) (Normality) If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$ is such that

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a$$

for some $i \leqslant |a|$, then there is a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|a \cup \{\beta\}|} \mid f(z_{a, a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$$

where k is such that β is the kth element of $a \cup \{\beta\}$.

If conditions (1) and (2) alone are satisfied then we say that E is a (κ, λ)-*pre-extender*.

We need to ensure that the ultrafilter E_a on $[\kappa]^{|a|}$ depends on a in such a way that guarantees coherence and the other properties. The most natural way to define an ultrafilter E_a on $[\kappa]^{|a|}$ that depends on a is as follows:

(1) For $\mathscr{F}_X$-almost all δ define a “reflected version” $a^\delta \in [\lambda_\delta]^{<\omega}$ of the “generator” a.

(2) For $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$, let

$$S(a,Y) = \{\delta < \kappa \mid a^\delta \in Y\}$$

and set

$$E_a = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a,Y))\}.$$

In other words, we regard Y as “E_a-large” if and only if it contains the “reflected generators” on a set which is large from the point of view of the game.

The trouble is that we have not guaranteed that $S(a,Y)$ is determined. This set will be determined if it is 1-good but we have not ensured this. So we need to “reflect” a in such a way that $S(a,Y)$ is 1-good. Now the most natural way to reflect $a \in [\lambda]^k$ is as follows: Choose

$$(y_1,\dots,y_k) \in Q^\kappa_{a_1} \times \cdots \times Q^\kappa_{a_k}$$

and, for $\delta \in S_0$, let $a^\delta = \{a^\delta_1, a^\delta_2, \dots, a^\delta_k\}$ be such that

$$(y_1,\dots,y_k) \in Q^\delta_{a^\delta_1} \times \cdots \times Q^\delta_{a^\delta_k}.$$

There is both a minor difficulty and a major difficulty with this approach. The minor difficulty is that we have to ensure that there is no essential dependence on our particular choice of $(y_1,\dots,y_k)$. The major difficulty is that unless $(y_1,\dots,y_k)$ is 1-good we still have no guarantee that $S(a,Y)$ is 1-good. The trouble is that there is in general no way of choosing such 1-good reals. However, assuming that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves, there *is* a way of generating 1-good prestrategies which (for all $x \in B$) hand us the reals we want. We will prove something slightly more general.

5.22 Lemma. *Assume z is $(n+1)$-good. Then for each $a \in [\lambda]^{<\omega}$ there is a function $f_a : \omega^\omega \to (\omega^\omega)^k$ such that*

(1) f_a is n-z-good and

(2) for all $x \in B$,

$$f_a(x) \in Q^\kappa_{a_1} \times \cdots \times Q^\kappa_{a_k},$$

where $k = |a|$.

Proof. The set

$$A_0 = \{x \in \omega^\omega \mid (x_{\mathrm{even}})_i \in Q^\kappa_{a_{i+1}} \text{ for all } i < k\}$$

is $\mathrm{OD}^M_{T,A,B}$ and clearly I wins A_0. Let A_0 be I's first move in $(SG^B_{T,A,B,z})^M$ and let f_0 be II's response. Notice that f_0 is n-z-good. We have

$$\forall x \in B\, \forall y \in \omega^\omega \,\big(f_0(x) * y \in A_0\big)$$

and hence

$$\forall x \in B\, \forall y \in \omega^\omega \,\big(((f_0(x) * y)_{\mathrm{even}})_i \in Q^\kappa_{a_{i+1}} \text{ for all } i < k\big).$$

Thus the function

$$\begin{aligned} f_a : \omega^\omega &\to (\omega^\omega)^k \\ x &\mapsto \begin{cases} \big(((f_0(x) * 0)_{\mathrm{even}})_0, \ldots, ((f_0(x) * 0)_{\mathrm{even}})_{k-1}\big) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is n-z-good (since it is definable from the n-z-good object f_0) and has the desired property. ⊣

5.23 Definition. Assume M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. For $a \in [\lambda]^{<\omega}$, we call a 1-good function $f_a : \omega^\omega \to (\omega^\omega)^{|a|}$ given by Lemma 5.22, a 1-*good code* for a.

The importance of a 1-good code f_a is twofold. First, any game defined in terms of f_a is determined. Second, for $\mathscr{F}_X$-almost all δ a 1-good code f_a selects a reflected version a^δ of a in a manner that is independent of $x \in B$; moreover, we can demand that a^δ inherits any $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$-property that a has. To see this, consider a statement such as the following: For all $x, x' \in B$, if $\alpha_1, \ldots, \alpha_k$ are such that

$$f_a(x) \in Q^\kappa_{\alpha_1} \times \cdots \times Q^\kappa_{\alpha_k}$$

then

$$f_a(x') \in Q^\kappa_{\alpha_1} \times \cdots \times Q^\kappa_{\alpha_k}$$

and

$$\alpha_1 < \cdots < \alpha_k.$$

This is a true $\Sigma_1(M, \{X, \mathbb{R}, \kappa\})$ statement. Thus, for $\mathscr{F}_X$-almost all δ, the statement reflects.

5.24 Definition. Suppose $a \in [\lambda]^{<\omega}$ and f_a is a 1-good code for a. Let

$$\begin{aligned} S_0(f_a) = \big\{\delta < \kappa \mid \forall x, x' \in B\, \forall \alpha_1, \ldots, \alpha_k\, (f_a(x) \in Q^\delta_{\alpha_1} \times \cdots \times Q^\delta_{\alpha_k} \\ \to f_a(x') \in Q^\delta_{\alpha_1} \times \cdots \times Q^\delta_{\alpha_k} \wedge \alpha_1 < \cdots < \alpha_k)\big\}. \end{aligned}$$

Notice that $S_0(f_a) \in \mathscr{F}_X$ and $S_0(f_a)$ is $\mathrm{OD}^M_{T,A,B,f_a}$.

5.25 Definition. Suppose $a \in [\lambda]^{<\omega}$ and f_a is a 1-good code for a. For $\delta \in S_0(f_a)$ and some (any) $x \in B$, let

$$a^{\delta}_{f_a} = \{|(f_a(x))_1|_{\leqslant\delta}, \ldots, |(f_a(x))_{|a|}|_{\leqslant\delta}\}$$

be the *reflected generator* of a.

5.26 Definition. For $a \in [\lambda]^{<\omega}$, f_a a 1-good code for a, and $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$, let

$$S(a, f_a, Y) = \{\delta \in S_0(f_a) \mid a^{\delta}_{f_a} \in Y\}.$$

Since f_a is 1-good and $S(a, f_a, Y)$ is $\mathrm{OD}^M_{T,A,B,f_a}$ it follows that $S(a, f_a, Y)$ is 1-good and hence $G^X(S(a, f_a, Y))$ is determined.

For $a \in [\lambda]^{<\omega}$ and f_a a 1-good code for a, let

$$E_a(f_a) = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a, f_a, Y))\}$$

and let

$$\begin{aligned} E_X(f_a) : [\lambda]^{<\omega} &\to \mathrm{HOD}^M_{T,A,B} \\ a &\mapsto E_a(f_a). \end{aligned}$$

The only trouble with this definition is that there is no guarantee that $E_X(f_a)$ is in $\mathrm{HOD}^M_{T,A,B}$ because there is no guarantee that $E_a(f_a)$ is in $\mathrm{HOD}^M_{T,A,B}$. We have to "erase" the dependence on the choice of f_a in the definition of E_a.

5.27 Lemma. *Suppose $a \in [\lambda]^{<\omega}$ and f_a and $\hat{f}_a$ are 1-good codes for a. Suppose $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$. Then*

(1) I wins $G^X(\{\delta \in S_0(f_a) \cap S_0(\hat{f}_a) \mid a^{\delta}_{f_a} = a^{\delta}_{\hat{f}_a}\})$.

(2) I wins $G^X(S(a, f_a, Y))$ iff I wins $G^X(S(a, \hat{f}_a, Y))$, and

(3) $E_a(f_a) = E_a(\hat{f}_a)$.

Proof. (1) The statement

$$\forall x \in B\, \forall i \leqslant |a| \left((f_a(x))_i =_\lambda (\hat{f}_a(x))_i\right)$$

is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$ statement about f_a and $\hat{f}_a$. So, by reflection, the set $\{\delta \in S_0(f_a) \cap S_0(\hat{f}_a) \mid a^{\delta}_{f_a} = a^{\delta}_{\hat{f}_a}\}$ is in $\mathscr{F}_X$ and hence in μ_X.

(2) Assume I wins $G^X(S(a, f_a, Y))$. We have that I wins the game in (1). Let $G^X(S_0(f_a, \hat{f}_a))$ abbreviate this game. So I wins $G^X(S(a, f_a, Y) \cap S_0(f_a, \hat{f}_a))$. But

$$S(a, f_a, Y) \cap S_0(f_a, \hat{f}_a) \subseteq S(a, \hat{f}_a, Y).$$

So I wins $G^X(S(a, \hat{f}_a, Y))$. Likewise if I wins $G^X(S(a, \hat{f}_a, Y))$ then I wins $G^X(S(a, f_a, Y))$.

(3) This follows immediately from (2). $\dashv$

Thus, we may wash out reference to f_a by setting

$$\begin{aligned} E_a &= \bigcap\{E_a(f_a) \mid f_a \text{ is a 1-good code of } a\} \\ &= E_a(f_a) \quad \text{for some (any) 1-good code } f_a \text{ of } a. \end{aligned}$$

Let

$$\begin{gathered} E_X : [\lambda]^{<\omega} \to \mathrm{HOD}^M_{T,A,B} \\ a \mapsto E_a \end{gathered}$$

Note that $E_a \in \mathrm{OD}^M_{T,A,B}$ and $E_a \subseteq \mathrm{HOD}^M_{T,A,B}$. Thus, $E_a \in \mathrm{HOD}^M_{T,A,B}$ and $E_X \in \mathrm{HOD}^M_{T,A,B}$.

Our definition of the extender E_X presupposes that for each $a \in [\lambda]^{<\omega}$ there is a 1-good code f_a of a and the existence of such codes is guaranteed by the assumption that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. Thus we have proved the following:

5.28 Lemma. *Assume that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. Then E_X is well-defined and $E_X \in \mathrm{HOD}^M_{T,A,B}$.*

It is important to stress that although the extender E_X is in $\mathrm{HOD}^M_{T,A,B}$ it is defined in M. For example, the certification that a certain set Y is in E_a depends on the existence of a winning strategy for a game in M. In general both the strategy and the game will not be in $\mathrm{HOD}^M_{T,A,B}$. So in establishing properties of E_X that hold in $\mathrm{HOD}^M_{T,A,B}$ we nevertheless have to consult the parent universe M.

5.29 Lemma. *Assume that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. Then*

$$\mathrm{HOD}^M_{T,A,B} \models E_X \textit{ is a pre-extender},$$

that is, $\mathrm{HOD}^M_{T,A,B}$ satisfies

(1) for each $a \in [\lambda]^{<\omega}$,

(a) E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$ and

(b) E_a is principal iff $a \subseteq \kappa$, and

(2) if $a \subseteq b \in [\lambda]^{<\omega}$ and $Y \in E_a$ then $Y^{a,b} \in E_b$.

Proof. (1)(a) It is easy to see that E_a is an ultrafilter in $\mathrm{HOD}^M_{T,A,B}$. It remains to see that E_a is κ-complete in $\mathrm{HOD}^M_{T,A,B}$. The proof is similar to that of Lemma 4.7. Let f_a be a 1-good code of a such that $E_a = E_a(f_a)$ and recall that

$$E_a(f_a) = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a, f_a, Y))\}.$$

Consider $\{Y_\alpha \mid \alpha < \gamma\} \in \mathrm{HOD}^M_{T,A,B}$ such that $\gamma < \kappa$ and for each $\alpha < \gamma$, $Y_\alpha \in E_\alpha(f_a)$. We have to show that

$$Y = \bigcap\{Y_\alpha \mid \alpha < \gamma\} \in E_a(f_a).$$

The key point is that

$$S(a, f_a, Y) = \bigcap\{S(a, f_a, Y_\alpha) \mid \alpha < \gamma\}$$

and so we are in almost exactly the situation as Lemma 4.7, only now we have to carry along the parameter f_a.

Since $Y \in \mathrm{HOD}^M_{T,A,B}$, $S(a, f_a, Y) \in \mathrm{OD}^M_{T,A,B,f_a}$. Since f_a is 1-good it follows that $G^X(S(a, f_a, Y))$ is determined. Assume for contradiction that I does not win $G^X(S(a, f_a, Y))$ and let σ' be a winning strategy for I in $G^X(\kappa \smallsetminus S(a, f_a, Y))$. We will derive a contradiction by finding a play that is legal against σ' and against winning strategies for I in each game $G^X(S(a, f_a, Y_\alpha))$, for $\alpha < \gamma$.

As in the case of Lemma 4.7, for the purposes of coding the winning strategies (in the games $G^X(S(a, f_a, Y_\alpha))$ for $\alpha < \gamma$) we need a prewellordering of length γ which is such that in a reflection argument we can ensure that it reflects to itself. For this purpose, for $\delta < \kappa$, let

$$Q_\delta = U_\delta \smallsetminus \bigcup\{U_\xi \mid \xi < \delta\}.$$

The sequence

$$\langle Q_\xi \mid \xi < \kappa\rangle$$

gives rise to an $\mathrm{OD}^M_{T,A,B}$ prewellordering with the feature that for $\mathscr{F}_X$-almost all δ,

$$\langle Q_\xi \mid \xi < \delta\rangle = \langle Q_\xi \mid \xi < \delta\rangle^{M_{\vartheta(\delta)}}$$

and, by choosing a real, we can ensure that we always reflect to some such point $\delta > \gamma$.

Now set

$$\begin{aligned} Z = \{(x, \sigma) \mid{} & \text{for some } \alpha < \gamma,\ x \in Q_\alpha \text{ and} \\ & \sigma \text{ is a winning strategy for I in } G^X(S(a, f_a, Y_\alpha))\}. \end{aligned}$$

This set is $\mathrm{OD}^M_{T,A,B,f_a}$, hence determined (as f_a is 1-good). So the game in the Uniform Coding Lemma is determined. The rest of the proof is exactly as before.

(b) By κ-completeness, E_a is principal if and only if there exists $b \in [\kappa]^{|a|}$ such that

$$E_a = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid b \in Y\}.$$

Suppose that $a \in [\kappa]^{|a|}$. We claim that $b = a$ witnesses that E_a is principal. Let f_a be a 1-good code of a. For $\mathscr{F}_X$-almost all δ, $a^\delta_{f_a} = a$. So, for $Y \in$

$\mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$,

$$\begin{aligned} Y \in E_a &\leftrightarrow \text{I wins } G^X(S(a, f_a, Y)) \\ &\leftrightarrow \text{I wins } G^X(\{\delta \in S_0(f_a) \mid a^\delta_{f_a} = a \in Y\}) \\ &\leftrightarrow a \in Y. \end{aligned}$$

Suppose that $a \notin [\kappa]^{|a|}$. We claim that no $\beta \in [\kappa]^{|a|}$ witnesses that E_a is principal. Consider $b \in [\kappa]^{|a|}$ and let f_b be a 1-good code for b and let f_a be a 1-good code for a. For $\mathscr{F}_X$-almost all δ, $a^\delta_{f_a} \neq b^\delta_{f_b} = b$. Let S be the set of such δ and let $Y = \{a^\delta_{f_a} \mid \delta \in S\}$. Then $Y \in E_a$ and $b \notin Y$. Hence E_a is not principal.

(2) Suppose $a \subseteq b \in [\lambda]^{<\omega}$ and $Y \in E_a$. So I wins $G^X(S(a, f_a, Y))$ for some (any) 1-good code f_a of a. We must show that I wins $G^X(S(b, f_b, Y^{a,b}))$ for some (any) 1-good code f_b of b. Let f_b be a 1-good code of b and consider the statement describing the manner in which a sits inside b. This is a $\Sigma_1(M, \{X, \mathbb{R}, \kappa\})$ statement about f_a and f_b. So, by reflection, there exists an $S_0(f_a, f_b) \in \mathscr{F}_X$ such that for all $\delta \in S_0(f_a, f_b)$,

$$\langle a^\delta_{f_a}, b^\delta_{f_b}, \in\rangle \cong \langle a, b, \in\rangle.$$

We claim that $S(a, f_a, Y) \cap S_0(f_a, f_b) \subseteq S(b, f_b, Y^{a,b})$. Let δ be an ordinal in $S(a, f_a, Y) \cap S_0(f_a, f_b)$. We have $a^\delta_{f_a} \in Y$ and $\langle a^\delta_{f_a}, b^\delta_{f_b}, \in\rangle \cong \langle a, b, \in\rangle$. Since, by definition,

$$Y^{a,b} = \{z \in [\kappa]^{|b|} \mid z_{a,b} \in Y\},$$

this means that $b^\delta_{f_b} \in Y^{a,b}$ (as $(b^\delta_{f_b})_{a,b} = a^\delta_{f_a}$), that is, $\delta \in S(b, f_b, Y^{a,b})$. Finally, since I wins $G^X\big(S(a, f_a, Y) \cap S_0(f_a, f_b)\big)$, I wins $G^X(S(b, f_b, Y^{a,b}))$. $\dashv$

5.30 Lemma. *Assume that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. Then*

$$\mathrm{HOD}^M_{T,A,B} \models E_X \text{ is countably complete.}$$

Proof. Let $\{a_i \mid i < \omega\} \in \mathrm{HOD}^M_{T,A,B}$ and suppose that for each $i < \omega$, $X_i \in E_{a_i}$, that is, I wins $G^X(S(a_i, f_{a_i}, X_i))$ for some (any) 1-good code f_{a_i} of a_i. Let $S = \bigcap_{i<\omega} S(a_i, f_{a_i}, X_i)$. We need to ensure that $G^X(S)$ is determined. The point is that since $\{a_i \mid i < \omega\} \in \mathrm{HOD}^M_{T,A,B}$, a slight modification of the proof of Lemma 5.22 shows that there are f_{a_i} such that $\langle f_{a_i} \mid i < \omega\rangle$ is 1-good. So $G^X(S)$ is determined. As in the proof of the completeness of E_a we have that I wins $G^X(S)$.

As in the proof of coherence there is a set $S_0(f_{a_1}, \ldots, f_{a_n}, \ldots) \in \mathscr{F}_X$ such that for all $\delta \in S_0(f_{a_1}, \ldots, f_{a_n}, \ldots)$,

$$\langle a^\delta_1, \ldots, a^\delta_i, \ldots\rangle \cong \langle a_1, \ldots, a_i, \ldots\rangle.$$

Fix $\delta \in S \cap S_0(f_{a_1}, \dots, f_{a_n}, \dots)$. Set

$$\begin{aligned} h_i : a_i &\to \kappa \\ (a_i)_j &\mapsto ((a_i)^\delta_{f_{a_i}})_j. \end{aligned}$$

Since $\delta \in S_0(f_{a_1}, \dots, f_{a_n}, \dots)$, the function

$$h = \bigcup_{i<\omega} h_i : \bigcup_{i<\omega} a_i \to \kappa$$

is well-defined. Since $\delta \in S(a_i, f_{a_i}, X_i)$, $h``a_i = (a_i)^\delta_{f_{a_i}} \in X_i$. However, h may not belong to $\mathrm{HOD}^M_{T,A,B}$. To see that there is such an h in $\mathrm{HOD}^M_{T,A,B}$ consider the tree $\mathscr{T}$ of attempts to build such a function. (The nth level of T consists of approximations $h^* : \bigcup_{i<n} a_i \to \kappa$ and the order is by inclusion.) Thus $\mathscr{T} \in \mathrm{HOD}^M_{T,A,B}$ and the existence of h in V shows that $\mathscr{T}$ is ill-founded in V. But well-foundedness is absolute, so some such h must belong to $\mathrm{HOD}^M_{T,A,B}$. ⊣

It remains to establish that

$$\mathrm{HOD}^M_{T,A,B} \models E_X \text{ is normal.}$$

This will follow from an analogue of the earlier strong normality theorems.

5.31 Definition. Assume M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for two moves. For $\alpha < \lambda$, let $f_\alpha : \omega^\omega \to \omega^\omega$ be a 1-good code of $\{\alpha\}$ (as in Lemma 5.22) and (as in Definition 5.25), for $\delta \in S_0(f_\alpha)$, let $\alpha^\delta_{f_\alpha}$ be the "reflected version" of α. We call the function

$$\begin{aligned} g_{f_\alpha} : S_0(f_\alpha) &\to \kappa \\ \delta &\mapsto \alpha^\delta_{f_\alpha} \end{aligned}$$

the *canonical function* associated to f_α.

Notice that the manner in which the ordinal $\alpha^\delta_{f_\alpha}$ is determined is different than in Sect. 4. In Sect. 4 we just chose $t \in Q_\alpha$ and let α^δ_t be unique such that $t \in Q^\delta_{\alpha^\delta_t}$. Notice also that g_{f_α} is 1-good since it is $\mathrm{OD}^M_{T,A,B,f_a}$.

The statement and proof of strong normality are similar to before, only now we have to ensure that the objects are sufficiently good to guarantee the determinacy of the games defined in terms of them. The real parameters that arise in the proof of strong normality will now have to be generated using the technique of Lemma 5.22 and every time we use this technique we will sacrifice one degree of goodness. There will be finitely many such sacrifices and so it suffices to assume that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for n moves for some sufficiently large n. Furthermore, there is no loss in generality in making this assumption since in all of the applications of the Generation Theorem, one will be able to show without DC that M satisfies

$\mathrm{ST}_{T,A,B}$-determinacy for $n < \omega$. As we shall see there will in fact only be two sacrifices of goodness. Thus, since we want our final object to be 1-good (to ensure that the games defined in terms of it are determined) it suffices to start with an object which is 3-good.

5.32 Theorem (Strong Normality). *Suppose $g : \kappa \to \kappa$ is such that*

(1) g is 3-good and

(2) I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$.

Then there exists an $\alpha < \lambda$ such that

$$I \text{ wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\}),$$

where f_α is any 1-g-good code of α.

Proof. We begin with a few comments. First, note that since g is 3-good, by Lemma 5.22 we have that for each $\alpha < \lambda$ there is a 1-g-good code f_α of α (in fact, there is a 2-g-good code) and hence each game $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$ is determined. The only issue is whether I wins some such game.

Second, notice that α is uniquely specified. For suppose $f_{\hat{\alpha}}$ is a 1-g-good code of $\hat{\alpha}$ such that I wins the corresponding game. If $\alpha < \hat{\alpha}$, then

$$\{\delta \in S_0(f_\alpha) \cap S_0(f_{\hat{\alpha}}) \mid g_{f_\alpha}(\delta) < g_{f_{\hat{\alpha}}}(\delta)\} \in \mathscr{F}_X$$

and I wins $G^X(S)$ where S is this set. But then I cannot win both

$$G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$$

and

$$G^X(\{\delta \in S_0(f_{\hat{\alpha}}) \mid g(\delta) = g_{f_{\hat{\alpha}}}(\delta)\}).$$

Third, it will be useful at this point to both list the parameters that will arise in the proof and motivate the need for assuming that g is 3-good. In outline the proof will follow that of Theorem 4.12. The final game in the present proof (the one involving e_1) will be defined in terms of three parameters: g, f_η and e_0, corresponding respectively to f, y_η, and e_0 from Theorem 4.12. To ensure the determinacy of the final game we will need to take steps to ensure that (g, f_η, e_0) is 1-good. Now, the parameter e_0 will be obtained by applying the technique of Lemma 5.22 to the parameter (g, f_η) and so we will need to take steps to ensure that this parameter is 2-good. And the parameter f_η will in turn be obtained by applying the technique of Lemma 5.22 to the parameter g and so we have had to assume from the start that g is 3-good.

We now turn to the proof proper. Suppose $g : \kappa \to \kappa$ is such that

(1.1) g is 3-good and

(1.2) I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$.

Assume for contradiction that for each $\alpha < \lambda$ and for each 1-g-good code f_α of α,

(2.1) I does not win $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$,

and hence (since each such game is determined, as f_α is 1-g-good)

(2.2) I wins $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) \neq g_{f_\alpha}(\delta)\})$.

Step 1. Let

$$\eta = \min\big(\{\beta < \lambda \mid \text{I wins } G^X(\{\delta \in S_0(f_\beta) \mid g(\delta) < g_{f_\beta}(\delta)\}) \text{ for each 1-}g\text{-good code } f_\beta \text{ of } \beta\}\big)$$

if such β exist; otherwise let $\eta = \lambda$. (So if there are such β then η is a limit ordinal.) Notice that

(3.1) whenever $\alpha < \eta$ and f_α is a 1-g-good code of α,

$$\text{I wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\}),$$

which is the desired situation. By Lemma 5.22, let

$$f_\eta \text{ be a 2-}g\text{-good code of } \eta.$$

For notational convenience, for $\delta \in S_0(f_\eta)$, let η_δ be $\eta^\delta_{f_\eta}$. By the definition of η, I wins $G^X(\{\delta \in S_0(f_\eta) \mid g(\delta) < g_{f_\eta}(\delta)\})$. Now update S_0 to be $S_0 \cap \{\delta \in S_0(f_\eta) \mid g(\delta) < g_{f_\eta}(\delta)\}$. We will work on this "large" set. Notice that S_0 is $\mathrm{OD}^M_{T,A,B,g,f_\eta}$. If $\eta = \lambda$ then $\eta_\delta = \delta$ and we may omit mention of f_η in what follows.

For convenience let us write "$S \in \mu_X$" as shorthand for "I wins $G^X(S)$". To summarize:

(4.1) g is 3-good,

(4.2) (g, f_η) is 2-good (First Drop in Goodness),

(4.3) S_0 is $\mathrm{OD}^M_{T,A,B,g,f_\eta}$,

(4.4) $S_0 \in \mu_X$ and for all $\delta \in S_0$, $g(\delta) < g_{f_\eta}(\delta)$, and

(4.5) for all $\alpha < \eta$ and for all 1-g-good codes f_α of α,

$$\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\} \in \mu_X.$$

Step 2. We now establish the "disjointness property".

Let

$$Z' = \big\{(x, \langle y, \sigma\rangle) \mid x \in Q^\kappa_\alpha \text{ for some } \alpha < \eta, \\ y \text{ codes a 1-}g\text{-good code } f_\alpha \text{ of } \alpha \\ \text{such that } x \in \operatorname{ran}(f_\alpha \restriction B), \text{ and} \\ \sigma \text{ is a winning strategy for I in} \\ G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\})\big\}.$$

We have

(5.1) Z' is $\mathrm{OD}^M_{T,A,B,g,f_\eta}$ and $Z' \subseteq Q^\kappa_{<\eta} \times \omega^\omega$, and

(5.2) for all $\alpha < \eta$,

$$Z' \cap (Q^\kappa_\alpha \times \omega^\omega) \neq \varnothing,$$

by (3.1).

Since (g, f_η) is 2-good the game in the proof of the Uniform Coding Lemma (Theorem 3.4) is determined. So *there is* an index $e \in \omega^\omega$ such that for all $\alpha < \eta$,

(6.1) $U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega)$ and

(6.2) $U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing$.

The trouble is that we have no guarantee that such an index e has any degree of (g, f_η)-goodness; yet this is essential for the present proof since we shall go on to define games in terms of this index and we need some guarantee that these games are determined. As usual, we retreat from the reals we want to the good functions that capture them and this will lead to the second (and final) drop in goodness. Let

$$A_0 = \big\{x \in \omega^\omega \mid x_{\text{even}} \text{ is such that for all } \alpha < \eta \\ (1)\ U^{(2)}_{x_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega) \text{ and} \\ (2)\ U^{(2)}_{x_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing\big\}.$$

So $A_0 \in \mathrm{OD}^M_{T,A,B,g,f_\eta}$. Now have I play A_0 in $(SG^B_{T,A,B,g,f_\eta})^M$ and let f_0 be II's response. Since (g, f_η) is 2-good, II's response f_0 is 1-(g, f_η)-good. Furthermore,

(7.1) $\forall x \in B\, \forall y \in \omega^\omega\, (f_0(x) * y \in A_0)$, which is to say,

(7.2) $\forall x \in B\, \forall y \in \omega^\omega\, (f_0(x) * y)_{\text{even}}$ is an index as in (6.1) and (6.2), hence

(7.3) $\forall \alpha < \eta$, $\bigcup_{x \in B} U^{(2)}_{(f_0(x)*0)_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha)$ is as in (6.1) and (6.2).

The union in (7.3) is itself $\underset{\sim}{\Sigma}^1_1(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha)$ and so there is an $e_0 \in \omega^\omega$ which is definable from f_0 (and hence inherits the 1-(g, f_η)-goodness of f_0) such that

(8.1) (g, f_η, e_0) is 1-good (Second Drop in Goodness) and

(8.2) for all $\alpha < \eta$,

(1) $U^{(2)}_{e_0}(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega)$ and

(2) $U^{(2)}_{e_0}(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing$.

Omitting Z', (8.2) is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_\eta, e_0\})$. So, for $\mathscr{F}_X$-almost all δ,

(8.3) for all $\alpha < \eta_\delta$,

(1) $U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \subseteq (Q^\delta_\alpha \times \omega^\omega)$ and

(2) $U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \neq \varnothing$.

The set S_1' of such δ is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_\eta, e_0\})$. Let $S_1 = S_1' \cap S_0$. Since $S_1' \in \mu_X$ and $S_0 \in \mu_X$, it follows that $S_1 \in \mu_X$. Notice also that S_1 is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, g, f_\eta, e_0\})$. For $\delta \in S_1 \cup \{\kappa\}$ and $\alpha < \eta_\delta$, let

$$\begin{aligned} Z^\delta_\alpha &= U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \quad \text{and} \\ Z^\delta &= \textstyle\bigcup_{\alpha<\eta_\delta} Z^\delta_\alpha. \end{aligned}$$

Claim A (Disjointness Property). *There is an $S_2 \subseteq S_1$ such that $S_2 \in \mu_X$ and for $\delta_1, \delta_2 \in S_2 \cup \{\kappa\}$ with $\delta_1 < \delta_2 \leqslant \kappa$,*

$$Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing$$

for all $\alpha \in [g(\delta_1), \eta_{\delta_1})$ and $\beta \in [0, \eta_{\delta_2})$.

Proof. We begin by establishing a special case.

Subclaim. *For μ_X-almost all δ,*

$$Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing$$

for all $\alpha \in [g(\delta), \eta_\delta)$ and $\beta \in [0, \eta)$.

Proof. Let

$$G = \big\{\delta \in S_1 \mid Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing \text{ for all } \alpha \in [g(\delta), \eta_\delta) \text{ and } \beta \in [0, \eta)\big\}$$

be the set of "good points". Our aim is to show that $G \in \mu_X$. Note that G is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$. Since (g, f_η, e_0) is 1-good, $G^X(G)$ is determined. Assume for contradiction that $G \notin \mu_X$. Then, by determinacy, $\kappa \smallsetminus G \in \mu_X$. Since $S_1 \in \mu_X$, we have $(\kappa \smallsetminus G) \cap S_1 \in \mu_X$. Let σ' be a winning strategy for I in $G^X((\kappa \smallsetminus G) \cap S_1)$.

We get a contradiction much as before: We can "take control" of the games to produce a play z and an ordinal δ_0 such that

(9.1) z is a legal play for II against σ' and δ_0 is the associated ordinal and

(9.2) z is a legal play for II against each $\sigma \in (\mathrm{proj}_2(Z^{\delta_0}))_1$ and in each case δ_0 is the associated ordinal.

This will finish the proof: By (9.1) and the definition of G, there is an $\alpha_0 \in [g(\delta_0), \eta_{\delta_0})$ and a $\beta_0 \in [0, \eta)$ such that $Z^{\delta_0}_{\alpha_0} \cap Z^{\kappa}_{\beta_0} \neq \varnothing$. Fix $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0} \cap Z^{\kappa}_{\beta_0}$. Since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\kappa}_{\beta_0} \subseteq Z' \cap (Q^{\kappa}_{\beta_0} \times \omega^\omega)$ we have, by the definition of Z', $x_0 \in Q^{\kappa}_{\beta_0}$, y_0 codes a 1-g-good code f_{β_0} of β_0, $x_0 \in \mathrm{ran}(f_{\beta_0} \restriction B)$, and σ_0 is a winning strategy for I in $S(\{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\})$. Since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0}$, $\sigma_0 \in (\mathrm{proj}_2(Z^{\delta_0}))_1$. Now, by (9.2), z is a legal play for II against σ_0 with associated ordinal δ_0, and since σ_0 is a winning strategy for I in $S(\{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\})$, this implies

(10.1) $\delta_0 \in \{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\}$,

that is, $g(\delta_0) > g_{f_{\beta_0}}(\delta_0)$. We now argue

(10.2) $g_{f_{\beta_0}}(\delta_0) = \alpha_0$,

which is a contradiction since $\alpha_0 \geqslant g(\delta_0)$. Recall that by definition $g_{f_{\beta_0}}(\delta_0) = |f_{\beta_0}(x)|_{\leqslant_{\delta_0}}$, where x is any element of B. Since we arranged $x_0 \in \mathrm{ran}(f_{\beta_0} \restriction B)$ and since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0}$, this implies that $g_{f_{\beta_0}}(\delta_0) = |f_{\beta_0}(x)|_{\leqslant_{\delta_0}} = \alpha_0$, where x is any element of B. Thus, a play z as in (9.1) and (9.2) will finish the proof.

The play z is constructed as before:

Base Case. We have

(11.1) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_X$ and

(11.2) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\kappa))_1\, ((\sigma * y)_I)_0 \in U_X$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \sigma', e_0, f_\eta\})$ statement. So there is a $z_0 \in U_X$ such that $z_0 \leqslant_T \langle \sigma', e_0, f_\eta\rangle$ and for all δ if $z_0 \in U_\delta$ then

(11.3) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_\delta$ and

(11.4) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\delta))_1\, ((\sigma * y)_I)_0 \in U_\delta$.

$(n+1)$st Step. Assume we have defined $z_0, \ldots, z_n$ in such a way that

(12.1) $\forall y \in \omega^\omega\, \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma' * y)_I)_{n+1} \in U_X\big)$ and

(12.2) $\forall y \in \omega^\omega\, \sigma \in (\mathrm{proj}_2(Z^\kappa))_1, \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_X\big)$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \sigma', e_0, f_\eta, z_0, \ldots, z_n\})$ statement. So there is a $z_{n+1} \in U_X$ such that $z_{n+1} \leqslant_T z_n$ and for all if $z_{n+1} \in U_\delta$ then

(12.3) $\forall y \in \omega^\omega\, \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma' * y)_I)_{n+1} \in U_\delta\big)$ and

(12.4) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\delta))_1\, \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_\delta\big)$.

Finally, let $z \in \omega^\omega$ be such that $(z)_i = z_i$ for all $i \in \omega$ and let δ_0 be least such that $(z)_i \in U_{\delta_0}$ for all $i \in \omega$. Notice that by our choice of z_n, for $n < \omega$, no DC is required to construct z. We have that for all $i \in \omega$,

(13.1) $((\sigma' * z)_I)_i \in U_{\delta_0}$ by (11.3) and (12.3) and

(13.2) $((\sigma * z)_I)_i \in U_{\delta_0}$ for all $\sigma \in (\mathrm{proj}_2(Z^{\delta_0}))_1$ by (11.4) and (12.4).

So we have (9.1) and (9.2), which is a contradiction. ⊣

By the subclaim,

(14.1) $\forall \delta \in G\, \forall \alpha \in [g(\delta), \eta_\delta)\, \forall \beta \in [0, \eta)\, (Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing)$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, e_0\rangle, g, G\})$ statement φ. Notice that since G is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$ and (g, f_η, e_0) is 1-good, it follows that (G, g, f_η, e_0) is 1-good. In particular, (G, g) is 1-good, and so Theorem 5.20 applies (taking $\langle f_\eta, e_0\rangle$ for the real t in the statement of that theorem) and we have that

(14.3) for $\mathscr{F}_X$-almost all δ_2,

- (1) $M \models \varphi[\langle f_\eta, e_0\rangle, F(\delta_2), \delta_2, g\restriction\delta_2, G \cap \delta_2]$, that is,
- (2) $\forall \delta_1 \in G \cap \delta_2\, \forall \alpha \in [g(\delta_1), \eta_{\delta_1})\, \forall \beta \in [0, \eta_{\delta_2})\, (Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing)$.

Let S'_2 be the set of such δ_2 in (14.3) and let $S_2 = S'_2 \cap G$. Since $S'_2 \in \mathscr{F}_X \subseteq \mu_X$ and $G \in \mu_X$, we have that $S_2 \in \mu_X$. Hence S_2 is as desired in Claim A. Also, S_2 is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$. ⊣

Notice that two additional parameters have emerged, namely, G and S_2, but these do not lead to a drop in goodness since

(15.1) $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0,G,S_2} = \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0,G} = \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, and so

(15.2) (g, f_η, e_0, G, S_2) is 1-good.

Step 3. We are now in a position to "compute g".
For $\delta \in S_2$, let

$$P^\delta = \bigcup\{Z^{\bar\delta}_\alpha \mid \bar\delta \in S_2 \cap \delta \wedge \alpha \in [g(\bar\delta), \eta_{\bar\delta})\}.$$

By (15.1), $P^\delta \in \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$.

Claim B (Tail Computation). *There exists an index $e_1 \in \omega^\omega$ such that for all $\delta \in S_2$,*

(1) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \subseteq Z^\delta_\alpha$ for all $\alpha \in [0, \eta_\delta)$,

(2) $U^{(2)}_{e_1}(P^\delta, Z^\delta_{g(\delta)}) = \varnothing$, and

(3) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \neq \varnothing$ for $\alpha \in (g(\delta), \eta_\delta)$.

Proof. As before it suffices to show (2) and (3′) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \cap Z^\delta_\alpha \neq \varnothing$ for $\alpha \in (g(\delta), \eta_\delta)$.

Let

$$G = \{e \in \omega^\omega \mid \forall \delta \in S_2\, (U^{(2)}_e(P^\delta, Z^\delta_{g(\delta)}) = \varnothing)\}.$$

Toward a contradiction assume that for each $e \in G$, (3′) in the claim fails for some δ and α. For each $e \in G$, let

$$\begin{aligned} \alpha_e = {} & \text{lexicographically least pair } (\delta, \alpha) \text{ such that} \\ & (1)\ \delta \in S_2, \\ & (2)\ g(\delta) < \alpha < \eta_\delta, \text{ and} \\ & (3)\ U^{(2)}_e(P^\delta, Z^\delta_\alpha) \cap Z^\delta_\alpha = \varnothing. \end{aligned}$$

Now play the game

$$\begin{array}{lccccccc} \text{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \text{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where II wins iff $(x \in G \to (y \in G \wedge \alpha_y >_{\text{lex}} \alpha_x))$.

The key point is that this payoff condition is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, by (15.1), and hence, the game is determined, since (g, f_η, e_0) is 1-good.

The rest of the proof is exactly as before. ⊣

From this point on there are no uses of determinacy that require further "joint goodness".

Claim C. *There exists an $\alpha_0 < \eta$ such that*

(1) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_{\alpha_0}) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_\alpha) \neq \varnothing$ for all $\alpha \in (\alpha_0, \eta)$, where

$$P^\kappa = \bigcup \{Z^\delta_\alpha \mid \delta \in S_2 \wedge \alpha \in [g(\delta), \eta_\delta)\}.$$

Proof. The statement that there is not a largest ordinal α_0 which is "empty" is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, e_0, e_1\rangle, g, G, S_2\})$. Since (g, f_η, e_0) is 1-good and G and S_2 are $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, it follows that (g, G, S_2) is 1-good. Thus, the Reflection Theorem (Theorem 5.20) applies and we have that for $\mathscr{F}_X$-many δ, the statement reflects, which contradicts Claim B. ⊣

Let α_0 be the unique ordinal as above and let f_{α_0} be a 1-g-good code of α_0 (which exists by Lemma 5.22). The statement

(16.2) $\forall x \in B\, f_{\alpha_0}(x) \in Q^\kappa_\alpha$ where α is such that

(1) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_\alpha) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_\beta) \neq \varnothing$ for $\beta \in (\alpha, \eta_\delta)$.

is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, f_{\alpha_0}, e_0, e_1\rangle, g, G, S_2\})$. Since (g, G, S_2) is 1-good, the Reflection Theorem (Theorem 5.20) applies and hence for $\mathscr{F}_X$-almost all δ the statement reflects. Let $S_3' \in \mathscr{F}_X$ be this set. Let $S_3 = S_3' \cap S_2$. So $S_3 \in \mu_X$. By Claim B and Claim C, for $\delta \in S_3$ the ordinal α in question is $g(\delta)$. So I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \cap S_3 \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$ and hence I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$. This game is determined since f_{α_0} is 1-g-good.

To summarize:

(17.1) f_{α_0} is 1-g-good and

(17.2) I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$,

which completes the proof of strong normality. ⊣

Since every $g : \kappa \to \kappa$ in $\mathrm{HOD}^M_{T,A,B}$ is 3-good and since in the context of the main theorem we assume that M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for four moves, we have shown:

5.33 Corollary. *Suppose $g : \kappa \to \kappa$ is in $\mathrm{HOD}^M_{T,A,B}$ and such that I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$. Then there exists an $\alpha < \lambda$ and a 1-g-good code f_α of α such that*

$$I \text{ wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\}).$$

5.34 Lemma (Normality). *In $\mathrm{HOD}^M_{T,A,B}$: If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$ is such that*

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a$$

for some $i \leqslant |a|$, then there is a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|a \cup \{\beta\}|} \mid f(z_{a, a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$$

where k is such that β is the kth element of $a \cup \{\beta\}$.

Proof. The proof just involves chasing through the definitions: Suppose $f : \kappa^{|a|} \to \kappa$ is a function in $\mathrm{HOD}^M_{T,A,B}$ such that for some $i \leqslant |a|$,

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a.$$

Since M satisfies $\mathrm{ST}_{T,A,B}$-determinacy for four moves, f is 4-good. So, by Lemma 5.22, there is a 3-good code f_a of a. Hence

(1.1) I wins $G^X(\{\delta \in S_0(f_a) \mid f(a^\delta_{f_a}) < (a^\delta_{f_a})_i\})$.

Let

$$\begin{aligned} f^* &: \kappa \to \kappa \\ \delta &\mapsto \begin{cases} f(a^\delta_{f_a}) & \text{if } \delta \in S_0(f_a) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So $f^* \in \mathrm{HOD}^M_{T,A,B,f_a}$ and hence f^* is 3-good. By Theorem 5.32, (1.2) there is an f_β such that

(1) f_β is a 1-f^*-good code of β,
(2) I wins $G^X(\{\delta \in S_0(f_\beta) \mid f^*(\delta) = g_{f_\beta}(\delta)\})$, and
(3) I wins $G^X(\{\delta \in S_0(f_\beta) \cap S_0(f_a) \mid f(a^\delta_{f_a}) = g_{f_\beta}(\delta)\})$.

Note that $\beta < a_i$ since if $\beta \geqslant a_i$ then for $\mathscr{F}_X$-almost all δ, $g_{f_\beta}(\delta) \geqslant (a^\delta_{f_a})_i$ and we get that I wins $G^X(\{\delta \in S_0(f_\beta) \cap S_0(f_a) \mid f(a^\delta_{f_a}) \geqslant (a^\delta_{f_a})_i\})$, which contradicts (1.1).

Let k be such that $\beta = (a \cup \{\beta\})_k$. Let $f_{a\cup\{\beta\}}$ be a 1-good code of $a \cup \{\beta\}$. Note that

(2.1) for $\mathscr{F}_X$-almost all δ,

$$\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_k = g_{f_\beta}(\delta)$$

and

(2.2) for $\mathscr{F}_X$-almost all δ,

$$\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_{a,a\cup\{\beta\}} = a^\delta_{f_a}$$

and, moreover, I wins on these sets (since the parameters in the definitions are 1-good). So (1.2)(3) yields

(3.1) I wins $G^X\big(\big\{\delta \in S_0(f_{a\cup\{\beta\}}) \mid f\big(\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_{a,a\cup\{\beta\}}\big) = \big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_k\big\}\big)$, that is,

(3.2) $\{z \in [\kappa]^{|a\cup\{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$,

as desired. ⊣

We are now in a position to take the "ultrapower" of $\mathrm{HOD}^M_{T,A,B}$ by E_X. It will be useful to recall this construction and record some basic facts concerning it. For further details see Steel's chapter in this Handbook.

Let

$$D = \{\langle a, f\rangle \in \mathrm{HOD}^M_{T,A,B} \mid a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to \mathrm{HOD}^M_{T,A,B}\}.$$

We get an equivalence relation on D by letting

$$\langle a, f\rangle \sim_E \langle b, g\rangle \in D \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) = g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

Let $[a, f]$ be the elements of minimal rank of the equivalence class of $\langle a, f\rangle$. Let Ult be the structure with domain

$$\{[a, f] \mid \langle a, f\rangle \in D\}$$

and membership relation defined by

$$[a, f] \in_{E_X} [b, g] \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) \in g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

Since $\mathrm{HOD}^M_{T,A,B}$ satisfies AC, Łoś's theorem holds in the following form: For all formulas $\varphi(x_1, \ldots, x_n)$ and all elements $[a_1, f_1], \ldots, [a_n, f_n] \in \mathrm{Ult}$,

$$\begin{aligned} &\mathrm{Ult} \models \varphi\big[[a_1, f_1], \ldots, [a_n, f_n]\big] \\ &\quad \leftrightarrow \quad \{z \in [\kappa]^{|b|} \mid \mathrm{HOD}^M_{T,A,B} \models \varphi[f_1(z_{a_1,b}), \ldots, f_n(z_{a_n,b})]\} \in E_b, \end{aligned}$$

where $b = \bigcup_{1\leqslant i\leqslant n} a_i$. It follows that

$$\begin{aligned} j'_E : \mathrm{HOD}^M_{T,A,B} &\to \mathrm{Ult} \\ x &\mapsto [\varnothing, c_x], \end{aligned}$$

where c_x is the constant function with value x, is an elementary embedding. The countable completeness of E_X ensures that Ult is well-founded and it is straightforward to see that it is extensional and set-like. So we can take the transitive collapse. Let

$$\pi : \mathrm{Ult} \to M_X$$

be the transitive collapse map and let

$$j_E : \mathrm{HOD}^M_{T,A,B} \to M_X$$

be the elementary embedding obtained by letting $j_E = \pi \circ j'_E$. The κ-completeness of each E_a, for $a \in [\lambda]^{<\omega}$, implies that j_E is the identity on $\mathrm{HOD}^M_{T,A,B} \cap V_\kappa$ and that κ is the critical point of j_E. Normality implies that for each $a \in [\lambda]^{<\omega}$, $\pi([a, z \mapsto z_i]) = a_i$, for each i such that $1 \leqslant i \leqslant |a|$. In particular, if $\alpha < \lambda$ then $\alpha = \pi([\{\alpha\}, z \mapsto \cup z])$. It follows that $\lambda \leqslant j_E(\kappa)$.

5.35 Lemma (T-STRENGTH).

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

Proof. We already have that

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC},$$

by Lemma 5.16. It follows that there are arbitrarily large $\lambda < \Theta_M$ such that

$$\mathrm{HOD}^M_{T,A,B} \cap V_\lambda^{\mathrm{HOD}^M_{T,A,B}} = L_\lambda[A],$$

where $A \subseteq \lambda$ and $A \in \mathrm{HOD}^M_{T,A,B}$. Let λ be such an ordinal and let κ, j_E, etc. be as above. We have that $j_E(\kappa) \geqslant \lambda$ and it remains to show that

$$V_\lambda^{\mathrm{HOD}^M_{T,A,B}} \subseteq M_X$$

and

$$j_E(T \cap \kappa) \cap \lambda = T \cap \lambda.$$

The proof of each is the same. Let us start with the latter. We have to show that for all $\alpha < \lambda$,

$$\alpha \in j_E(T \cap \kappa) \leftrightarrow \alpha \in T.$$

We have

$$\begin{aligned} \alpha \in j_E(T \cap \kappa) \quad &\leftrightarrow \quad \pi([\{\alpha\}, z \mapsto \cup z]) \in \pi([\varnothing, c_{T \cap \kappa}]) \\ &\leftrightarrow \quad [\{\alpha\}, z \mapsto \cup z] \in_{E_X} [\varnothing, c_{T \cap \kappa}] \\ &\leftrightarrow \quad \{z \in [\kappa]^1 \mid \cup z \in T \cap \kappa\} \in E_{\{\alpha\}}. \end{aligned}$$

So we have to show that

$$\alpha \in T \leftrightarrow \{\{z\} \mid z \in T \cap \kappa\} \in E_{\{\alpha\}}.$$

Let $f_{\{\alpha\}}$ be a 1-good code of $\{\alpha\}$.

Assume $\alpha \in T$. We have to show that

$$\text{I wins } G^X\big(S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\})\big).$$

The key point is that the statement "for all $x \in B$, $|f_{\{\alpha\}}(x)|_{\leqslant \lambda} \in T$" is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_{\{\alpha\}}\})$ statement. So the set S of δ to which this statement reflects is in $\mathscr{F}_X$. Since $S \in \mathrm{OD}^M_{T,A,B,f_{\{\alpha\}}}$ and $f_{\{\alpha\}}$ is 1-good, $G^X(S)$ is determined and I wins. But

$$S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\}) = S_0(f_{\{\alpha\}}) \cap S$$

and so I wins this game as well.

Assume $\alpha \notin T$. We have to show that

$$\text{I does not win } G^X\big(S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\})\big).$$

Again, the point is that the statement "for all $x \in B$, $|f_{\{\alpha\}}(x)|_{\leqslant \lambda} \notin T$" is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_{\{\alpha\}}\})$ statement. So this statement reflects to $\mathscr{F}_X$-almost all δ, which implies that I cannot win the above game.

Exactly the same argument with 'A' in place of 'T' shows that

$$j_E(A \cap \kappa) \cap \lambda = A \cap \lambda,$$

and hence that

$$V_\lambda^{\mathrm{HOD}^M_{T,A,B}} = L_\lambda[A] \subseteq M_X,$$

which completes the proof. ⊣

This completes the proof of the Generation Theorem. ⊣

5.4. Special Cases

We now consider a number of special instances of the Generation Theorem. In each case all we have to do is find appropriate values for the parameters Θ_M, T, A, and B. We begin by recovering the main result of Sect. 4.

5.36 Theorem. *Assume* $\mathrm{ZF}+\mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

Proof. For notational convenience let $\Theta = \Theta^{L(\mathbb{R})}$. Our strategy is to meet the conditions of the Generation Theorem while at the same time arranging that $M = L_\Theta(\mathbb{R})[T, A, B]$ is such that

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta.$$

We will do this by taking care to ensure that the ingredients T, A, and B are in $\mathrm{HOD}^{L(\mathbb{R})}$ while at the same time packaging $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$ as part of T. It will then follow from the Generation Theorem that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal},$$

and by varying T the result follows.

To begin with let $\Theta_M = \Theta^{L(\mathbb{R})}$ and, for notational convenience, we continue to abbreviate this as Θ. By Theorem 3.9, Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$. Also,

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})},$$

by Theorem 3.10. So we can let $H \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ code

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta.$$

Fix $T' \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ and let $T \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ code T' and H. By Lemmas 3.7 and 3.8, there is an $\mathrm{OD}^{L(\mathbb{R})}$ sequence $A = \langle A_\alpha \mid \alpha < \Theta \rangle$ such that each A_α is a prewellordering of reals of length α. Let $B = \mathbb{R}$.

Let

$$M = L_\Theta(\mathbb{R})[T, A, B]$$

where Θ, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L(\mathbb{R})$ satisfies AD and M contains all reals. Moreover, since we have arranged that all of the ingredients T, A, and B are in $\mathrm{OD}^{L(\mathbb{R})}$ and also that T codes $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$, we have

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$$

and, since T' was arbitrary, the result follows as noted above. ⊣

We can also recover the following approximation to Theorem 5.6.

5.37 Theorem. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$*. Suppose* ST_X*-determinacy holds, where* X *is a set and* B *is non-empty and* OD_X*. Then*

$$\mathrm{HOD}_X \models \Theta_X \textit{ is a Woodin cardinal.}$$

Proof. Let $\Theta_M = \Theta_X$. Let $A = \langle A_\alpha \mid \alpha < \Theta_X\rangle$ be such that A_α codes the OD_X-least prewellordering of reals of length α. By Theorem 3.9, Θ_X is strongly inaccessible in HOD_X and so there exists an $H \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$ coding $\mathrm{HOD}_X \cap V_{\Theta_X}$. Let $T \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$ code H and some arbitrary $T' \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$.

Let

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

where Θ_M, T, A, and B are as above. Work in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Conditions (3)–(5) of the Generation Theorem are clearly met. For condition (2) note that by Lemma 3.7, $\Theta_X = \Theta^{\mathrm{HOD}_{\{X\}\cup\mathbb{R}}}$ and that by the arguments of Lemma 3.8 and Lemma 3.9, $\Theta^{\mathrm{HOD}_{\{X\}\cup\mathbb{R}}}$ is regular in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Thus, Θ_M is regular in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Condition (6) follows from the fact that M is OD_X and M contains all of the reals. It remains to see that condition (1) can be met. For this we just have to see that Replacement holds in M. If Replacement failed in M then there would be a cofinal map $\pi : \omega^\omega \to \Theta_X$ that is definable from parameters in M, which in conjunction with A would lead to an OD_X surjection from ω^ω onto Θ_X, which is a contradiction. ⊣

5.38 Remark. Work in ZF+DC. For μ a δ-complete ultrafilter on δ let E_μ be the (δ, λ)-extender derived from μ where $\lambda = j(\delta)$ (or $\lambda = \delta^\delta/\mu$) and j is the ultrapower map. We have the following corollary: Assume $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}$. Then Θ_X is Woodin in HOD_X and this is witnessed by the collection of $E_\mu \cap \mathrm{HOD}_X$ where μ is a normal ultrafilter on some $\delta < \Theta_X$.

5.39 Remark. Theorem 5.6 cannot be directly recovered from the Generation Theorem and this is why we have singled it out for special treatment. However, it follows from the proof of the Generation Theorem, as can be seen by noting that in the case where one has full boldface determinacy the ultrafilters are actually in HOD_X by Kunen's theorem (Theorem 3.11).

4 Open Question. There are some interesting questions related to Theorem 5.37.

(1) Suppose $\Theta_X = \Theta_0$. Suppose ST_X-determinacy, where B is non-empty and OD_X. Is Θ_0 a Woodin cardinal in HOD?

(2) Suppose ST_X-determinacy, where X is a set and B is non-empty and OD_X. Is Θ_X a Woodin cardinal in HOD?

(3) In the AD^+ setting, every Θ_X is of the form Θ_α and there are constraints on this sequence. For example, each Θ_X must be of the form $\Theta_{\alpha+1}$. Does this constraint apply in the lightface setting?

5.40 Theorem. *Assume* ZF + AD. *Let S be a class of ordinals. Then for an S-cone of x,*

$$\mathrm{HOD}_S^{L[S,x]} \models \omega_2^{L[S,x]} \textit{ is a Woodin cardinal.}$$

Proof. For an S-cone of x,

$$L[S,x] \models \mathrm{ZFC} + \mathrm{GCH} \text{ below } \omega_1^V,$$

by Corollary 5.10, and, for all $n < \omega$,

$$L[S,x] \models \mathrm{ST}_S\text{-determinacy for } n \text{ moves},$$

where $B = [x]_S$, by Theorem 5.13. Let x be in this S-cone.

Let $\Theta_M = \omega_2^{L[S,x]}$. Since $L[S,x]$ satisfies GCH below ω_1^V and $L[S,x] = \mathrm{OD}_{S,x}^{L[S,x]}$, by Lemma 3.8 we have that

$$\omega_2^{L[S,x]} = \sup\{\alpha \mid \text{there is an } \mathrm{OD}_S^{L[S,x]} \text{ prewellordering of length } \alpha\},$$

in other words, $\omega_2^{L[S,x]} = (\Theta_S)^{L[S,x]}$. Let $A = \langle A_\alpha \mid \alpha < \omega_2^{L[S,x]}\rangle$ be such that A_α is the $\mathrm{OD}_S^{L[S,x]}$-least prewellordering of length α. Since $L[S,x] \models \mathrm{OD}_S$-determinacy, it follows (by Theorem 3.9) that $\omega_2^{L[S,x]}$ is strongly inaccessible in $\mathrm{HOD}_S^{L[S,x]}$. So there is a set $H \subseteq \omega_2^{L[S,x]}$ coding $\mathrm{HOD}_S^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}$. Let T' be in $\mathscr{P}(\omega_2^{L[S,x]}) \cap \mathrm{OD}_S^{L[S,x]}$ and let $T \in \mathscr{P}(\omega_2^{L[S,x]}) \cap \mathrm{OD}_S^{L[S,x]}$ code T and H. Let $B = [x]_S$.

Let

$$M = L_{\Theta_M}(\mathbb{R}^{L[S,x]})[T, A, B],$$

where Θ_M, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L[S,x]$ satisfies ST_S-determinacy for four moves, M is OD_S in $L[S,x]$ and M contains the reals of $L[S,x]$. Thus,

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \text{ There is a } T\text{-strong cardinal.}$$

Since we have arranged that all of the ingredients T, A, and B are in $\mathrm{OD}^{L[S,x]}$ and also that T codes $\mathrm{HOD}^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}$, we have

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}.$$

Since T' was arbitrary, the result follows. ⊣

5.41 Theorem. *Assume* ZF + AD. *Then for an S-cone of x,*

$$\mathrm{HOD}_{S,\mathrm{HOD}_{S,x}} \models \omega_2^{\mathrm{HOD}_{S,x}} \textit{ is a Woodin cardinal.}$$

Proof. This will follow from the next theorem which is more general. ⊣

The next two theorems require some notation. Suppose Y is a set and $a \in H(\omega_1)$. For $x \in \omega^\omega$, the (Y,a)-*degree* of x is the set

$$[x]_{Y,a} = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,a,z} = \mathrm{HOD}_{Y,a,x}\}.$$

The (Y,a)-*degrees* are the sets of the form $[x]_{Y,a}$ for some $x \in \omega^\omega$. Define $x \leqslant_{Y,a} y$ to hold iff $x \in \mathrm{HOD}_{Y,a,y}$. A *cone of* (Y,a)-*degrees* is a set of the form $\{[y]_{Y,a} \mid y \geqslant_{Y,a} x_0\}$ for some $x_0 \in \omega^\omega$ and a (Y,a)-*cone of reals* is a set of the form $\{y \in \omega^\omega \mid y \geqslant_{Y,a} x_0\}$ for some $x_0 \in \omega^\omega$. The proof of the Cone Theorem (Theorem 2.9) generalizes to the present context. In the case where $a = \varnothing$ we speak of Y-*degrees*, etc.

5.42 Theorem. *Assume* ZF+AD. *Suppose Y is a set and $a \in H(\omega_1)$. Then for a (Y,a)-cone of x,*

$$\mathrm{HOD}_{Y,a,[x]_{Y,a}} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \textit{ is a Woodin cardinal,}$$

where $[x]_{Y,a} = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,a,z} = \mathrm{HOD}_{Y,a,x}\}$.

Proof. By determinacy it suffices to show that the above statement holds for a Turing cone of x, which is what we shall do. The key issues in this case are getting a sufficient amount of GCH and strategic determinacy. To establish the first we need two preliminary claims. Recall that a set $A \subseteq \omega^\omega$ is *comeager* if and only if $\omega^\omega \setminus A$ is meager.

Claim 1. *Assume* ZF + AD. *Suppose that $\langle A_\alpha \mid \alpha < \gamma\rangle$ is a sequence of sets which are comeager in the space ω^ω, where either $\gamma \in \mathrm{On}$ or $\gamma = \mathrm{On}$, in which case the sequence is a definable proper class. Then $\bigcap_{\alpha<\gamma} A_\alpha$ is comeager.*

Proof. Assume for contradiction that the claim fails and let γ be least such that there is a sequence $\langle A_\alpha \mid \alpha < \gamma\rangle$ the intersection of which is not comeager. By AD, $\bigcap_{\alpha<\gamma} A_\alpha$ has the property of Baire and so we may assume without loss of generality that $\bigcap_{\alpha<\gamma} A_\alpha$ is meager. So, every proper initial segment has comeager intersection while the whole sequence has meager intersection. We can now violate the Kuratowski-Ulam Theorem. (This is the analogue for category of Fubini's theorem. See [9, 5A.9].) Define f on the complement of $\bigcap_{\alpha<\gamma} A_\alpha$ as follows:

$$f(x) = \min(\{\alpha < \gamma \mid x \notin A_\alpha\}).$$

So if $y \in \bigcap_{\xi<\alpha} A_\xi$ then $f(y) > \alpha$. Since $\bigcap_{\alpha<\gamma} A_\alpha$ is meager, $\mathrm{dom}(f)$ is comeager. Consider the subset of the plane

$$Z = \{(x,y) \in \mathrm{dom}(f) \times \mathrm{dom}(f) \mid f(x) < f(y)\}.$$

For each $x \in \mathrm{dom}(f)$ the vertical section

$$Z_x = \{y \in \mathrm{dom}(f) \mid f(y) > f(x)\}$$

is comeager since it includes $\bigcap_{\alpha \leqslant f(x)} A_\alpha$ and for each $y \in \mathrm{dom}(f)$ the horizontal section

$$Z^y = \{x \in \mathrm{dom}(f) \mid f(x) < f(y)\}$$

is meager since its complement contains the comeager set $\bigcap_{\alpha < f(y)} A_\alpha$. Since Z has the property of Baire, this contradicts the Kuratowski-Ulam Theorem, the proof of which requires only $\mathrm{AC}_\omega(\mathbb{R})$, which follows from AD (Theorem 2.2). ⊣

Claim 2. *Assume* ZF + AD. *Suppose Y is a set, $a \in H(\omega_1)$ and $\mathbb{P} \in \mathrm{HOD}_{Y,a} \cap H(\omega_1)$ is a partial order. Then for comeager many $\mathrm{HOD}_{Y,a}$-generic $G \subseteq \mathbb{P}$,*

$$\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G].$$

Proof. For each G we clearly have $\mathrm{HOD}_{Y,a}[G] \subseteq \mathrm{HOD}_{Y,a,G}$. We seek a set A that is comeager in the Stone space of $\mathbb{P}$ and such that for all $G \in A$, $\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G]$. We will do this by showing that for each $G \in A$ the latter model can compute the "ordinal theory" of the former model.

For every Σ_2 statement φ and finite sequence of ordinals $\vec{\xi}$ consider the statement $\varphi[\vec{\xi}, Y, a, G]$ about a generic G. Let $B^{\varphi,\vec{\xi},Y,a}$ be the associated collection of filters on $\mathbb{P}$ and let

$$P^{\varphi,\vec{\xi}} = \{p \in \mathbb{P} \mid B^{\varphi,\vec{\xi},Y,a} \text{ is comeager in } O_p\} \quad \text{and}$$
$$N^{\varphi,\vec{\xi}} = \{p \in \mathbb{P} \mid B^{\neg\varphi,\vec{\xi},Y,a} \text{ is comeager in } O_p\},$$

where O_p is the open set of generics containing p. These are the sets of conditions which "positively" and "negatively" decide $\varphi[\vec{\xi}, Y, a, G]$, respectively. So $P^{\varphi,\vec{\xi}} \cup N^{\varphi,\vec{\xi}}$ is predense. Now let

$$\begin{aligned} A_{\varphi,\vec{\xi}} = &\{G \subseteq \mathbb{P} \mid \varphi[\vec{\xi}, Y, a, G] \leftrightarrow G \cap P^{\varphi,\vec{\xi}} \neq \varnothing\} \\ &\cup \{G \subseteq \mathbb{P} \mid \neg\varphi[\vec{\xi}, Y, a, G] \leftrightarrow G \cap N^{\varphi,\vec{\xi}} \neq \varnothing\}. \end{aligned}$$

Each such set is comeager. We thus have a class size well-order of comeager sets and so, by the previous lemma,

$$A = \bigcap\{A_{\varphi,\vec{\xi}} \mid \varphi \text{ is a } \Sigma_2 \text{ formula and } \vec{\xi} \in \mathrm{On}^{<\omega}\}$$

is comeager. But now we have that for all $G \in A$

$$\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G]$$

since the latter can compute all answers to questions involving the former—that is, questions of the form $\varphi[\vec{\xi}, Y, a, G]$ where φ is Σ_2—by checking whether G hits $P^{\varphi,\vec{\xi}}$ or $N^{\varphi,\vec{\xi}}$. (Notice that the restriction to Σ_2 formulas suffices (by reflection) since any statement about an initial segment of $\mathrm{HOD}_{Y,a,G}$ is Σ_2.) ⊣

Claim 3. *Assume* ZF + AD. *Suppose Y is a set and $a \in H(\omega_1)$. Then for a Turing cone of x,*

$$\mathrm{HOD}_{Y,a,x} \models \mathrm{GCH} \textit{ below } \omega_1^V.$$

Proof. It suffices to show that CH holds on a cone since given this the proof that GCH below ω_1^V holds on a cone goes through exactly as before.

Suppose for contradiction (by the Cone Theorem (Theorem 2.9)) that there is a real x_0 such that for all $x \geqslant_T x_0$,

$$\mathrm{HOD}_{Y,a,x} \models \neg\mathrm{CH}.$$

We will arrive at a contradiction by producing an $x \geqslant_T x_0$ with the feature that $\mathrm{HOD}_{Y,a,x} \models \mathrm{CH}$. As before x is obtained by forcing (in two steps) over HOD_{Y,a,x_0}. First, we get a HOD_{Y,a,x_0}-generic

$$G \subseteq \mathrm{Col}(\omega_1^{\mathrm{HOD}_{Y,a,x_0}}, \mathbb{R}^{\mathrm{HOD}_{Y,a,x_0}})$$

and then we use almost disjoint forcing to code G with a real. Viewing the generic g as a real, by the previous claim we have that for comeager many g,

$$\mathrm{HOD}_{Y,a,x_0,g} = \mathrm{HOD}_{Y,a,x_0}[g] \models \mathrm{CH},$$

and hence

$$\mathrm{HOD}_{Y,a,\langle x_0,g\rangle} \models \mathrm{CH},$$

which is a contradiction. ⊣

Claim 4. *Suppose Y is a set and $a \in H(\omega_1)$. Then for a Turing cone of x, for each $n < \omega$, II can play n moves of $SG^B_{Y,a,[x]_{Y,a}}$, where $B = [x]_{Y,a}$, and we demand in addition that II's moves belong to $\mathrm{HOD}_{Y,a,x}$, in other words, II can play n moves of the game*

$$\begin{array}{lccccc} \mathrm{I} & A_0 & & \cdots & A_{n-1} & \\ \mathrm{II} & & f_0 & \cdots & & f_{n-1} \end{array}$$

where we require, for $i+1 < n$,

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^V_{Y,a,[x]_{Y,a}}$, $A_{i+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^V_{Y,a,[x]_{Y,a},f_0,\ldots,f_i}$ and

(2) f_{i+1} is prestrategy for A_{i+1} that belongs to $\mathrm{HOD}_{Y,a,x}$ and is winning with respect to $[x]_{Y,a}$.

Proof. The proof of Theorem 5.13 actually establishes this stronger result. ⊣

We are now in a position to meet the conditions of the Generation Theorem. For a Turing cone of x,

$$\mathrm{HOD}_{Y,a,x} \models \mathrm{ZFC} + \mathrm{GCH} \text{ below } \omega_1^V,$$

by Claim 3, and for all $n<\omega$,

$$\mathrm{ST}_{Y,a,[x]_{Y,a}}\text{-determinacy for } n \text{ moves}$$

holds in V where $B=[x]_{Y,a}$, by Claim 4. Let x be in this cone.

Let $\Theta_M=\omega_2^{\mathrm{HOD}_{Y,a,x}}$. Since $\mathrm{HOD}_{Y,a,x}\models \mathrm{ZFC}+\mathrm{GCH}$ below ω_1^V,

$$\Theta_M=\Theta.$$

Since every set is $\mathrm{OD}_{Y,a,x}$, and hence $\mathrm{OD}_{Y,a,[x]_Y,x}$,

$$\Theta=\Theta_{Y,a,[x]_Y},$$

by Lemma 3.8. Thus,

$$\omega_2^{\mathrm{HOD}_{Y,a,x}}=\Theta^{\mathrm{HOD}_{Y,a,x}}_{Y,a,[x]_{Y,a}}.$$

Letting $A=\langle A_\alpha\mid\alpha<\omega_2^{\mathrm{HOD}_{Y,a,x}}\rangle$ be such that A_α is the $\mathrm{OD}_{Y,a,[x]_{Y,a}}$-least prewellordering of length α we have that A is $\mathrm{OD}_{Y,a,[x]_{Y,a}}$. We also have that $\omega_2^{\mathrm{HOD}_{Y,a,x}}$ is strongly inaccessible in $\mathrm{HOD}_{Y,a,[x]_{Y,a}}$, by Theorem 3.9. So there is a set $H\subseteq\omega_2^{\mathrm{HOD}_{Y,a,x}}$ coding $\mathrm{HOD}_{Y,a,[x]_{Y,a}}\cap V_{\omega_2^{\mathrm{HOD}_{Y,a,x}}}$. Let T' be in $\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,a,x}})\cap\mathrm{OD}_{Y,a,[x]_{Y,a}}$ and let $T\in\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,a,x}})\cap\mathrm{OD}_{Y,a,[x]_{Y,a}}$ code T' and H. Let $B=[x]_{Y,a}$.

Let

$$M=L_{\Theta_M}(\mathbb{R}^{\mathrm{HOD}_{Y,a,x}})[T,A,B],$$

where Θ_M, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met. Condition (6) follows from the fact that M is $\mathrm{OD}_{Y,a,[x]_{Y,a}}$ and we have arranged (in Claim 4) that all of II's moves in $SG^B_{Y,a,[x]_{Y,a}}$ are in M.

Thus,

$$\mathrm{HOD}^M_{T,A,B}\models\mathrm{ZFC}+\text{ There is a } T\text{-strong cardinal},$$

and since we have arranged that

$$\mathrm{HOD}^M_{T,A,B}=\mathrm{HOD}_{Y,a,[x]_{Y,a}}\cap V_{\omega_2^{\mathrm{HOD}_{Y,a,x}}},$$

and T was arbitrary, the result follows. $\dashv$

In the above theorem the degree notion $[x]_{Y,a}$ depends on the initial choice of a. However, later (in Sect. 6.2) we will want to construct models with many Woodin cardinals. A natural approach to doing this is to iteratively apply the previous theorem, starting off with $a=\varnothing$, increasing the degree of x to get that $\omega_2^{\mathrm{HOD}_{Y,x}}$ is a Woodin cardinal in $\mathrm{HOD}_{Y,[x]_Y}$, and then taking $a=[x]_Y$, increasing the degree of x yet again to get that $\omega_2^{\mathrm{HOD}_{Y,[x]_Y,x}}$ is a Woodin cardinal in $\mathrm{HOD}_{Y,[x]_Y,[x]_{Y,[x]_Y}}$, etc. This leads to serious difficulties since the degree notion is changing. We would like to keep the degree notion fixed as we supplement a and for this reason we need the following variant of the previous theorem.

5.43 Theorem. *Assume* ZF + AD. *Suppose Y is a set and $a \in H(\omega_1)$. Then for a Y-cone of x,*

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \textit{ is a Woodin cardinal,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$.

Proof. The proof is essentially the same as that of the previous theorem. Claims 1 to 3 are exactly as before. The only difference is that now in Claim 4 we have $[x]_Y$ in place of $[x]_{Y,a}$. The proof of this version of the claim is the same, as is that of the rest of the theorem. ⊣

6. Definable Determinacy

We now use the Generation Theorem to derive the optimal amount of large cardinal strength from both lightface and boldface definable determinacy.

The main result concerning lightface definable determinacy is the following:

6.1 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone of x,*

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

When combined with the results mentioned in the introduction this has the consequence that the theories ZFC + OD-determinacy and ZFC + "There is a Woodin cardinal" are equiconsistent. In order to prove this theorem we will have to get into the situation of the Generation Theorem. The issue here is that Δ^1_2-determinacy does not imply that for a cone of x strategic determinacy holds in $L[x]$ with respect to the constructibility degree of x. Instead we will use a different basis set B, one for which we can establish ST^B-determinacy for four moves, using Δ^1_2-determinacy alone.

The main result concerning boldface definable determinacy is the following:

6.2 Theorem. *Assume* ZF + AD. *Suppose Y is a set. There is a generalized Prikry forcing $\mathbb{P}_Y$ through the Y-degrees such that if $G \subseteq \mathbb{P}_Y$ is V-generic and $\langle [x_i]_Y \mid i < \omega\rangle$ is the associated sequence, then*

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle, V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals.}$$

When combined with the results mentioned in the introduction this has the consequence that the theories ZFC + OD($\mathbb{R}$)-determinacy and ZFC + "There are ω-many Woodin cardinals" are equiconsistent. As an application we show that when conjoined with the Derived Model Theorem (Theorem 1.5 or, more generally, Theorem 8.12) this result enables one to reprove and generalize Kechris' theorem (Theorem 2.6).

6.1. Lightface Definable Determinacy

In this subsection we will work in the theory $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$-determinacy and examine the features of the model $L[x]$ for a Turing cone of reals x. Our aim is to show that for a Turing cone of x,

$$\mathrm{HOD}^{L[x]} \models \omega_2^{L[x]} \text{ is a Woodin cardinal.}$$

This will be done by showing that the conditions of the Generation Theorem can be met. We already know that this is true assuming full boldface determinacy in the background universe. But now we are working with a weak form of lightface definable determinacy and this presents new obstacles. The main difficulty is in showing that for a Turing cone of x,

$$L[x] \models \mathrm{ST}^B\text{-determinacy}$$

for an appropriate basis B. In the boldface setting we took our basis B to be the constructibility degree of x. But as we shall see (in Theorem 6.12) in our present setting one cannot secure this version of strategic determinacy. Nevertheless, it turns out that strategic determinacy holds for a different, smaller basis. This leads to the notion of *restricted strategic determinacy.*

We shall successively extract stronger and stronger forms of determinacy until we ultimately reach the version we need. The subsection closes with a series of limitative results, including results that motivate the need for strategic and restricted strategic determinacy.

6.3 Theorem (Martin). *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then* Σ^1_2*-determinacy.*

Proof. Consider $A=\{x\in\omega^\omega \mid \varphi(x)\}$ where φ is Σ^1_2. We have to show that A is determined. Our strategy is to show that if II (the Π^1_2 player) does not have a winning strategy for A then I (the Σ^1_2 player) has a winning strategy for A.

Assume that II does not have a winning strategy for A. First, we have to shift to a "local" setting where we can apply Δ^1_2-determinacy. For each $x\in\omega^\omega$,

$$L[x] \models \text{II does not have a winning strategy in } \{y\in\omega^\omega \mid \varphi(y)\}$$

(since otherwise, by Σ^1_3 upward absoluteness, II would have a winning strategy in V, contradicting our initial assumption) and so, by the Löwenheim-Skolem theorem, there is a countable ordinal λ such that

$$L_\lambda[x] \models \mathrm{T} + \text{II does not have a winning strategy in } \{y\in\omega^\omega \mid \varphi(y)\},$$

where T is some fixed sufficiently strong fragment of ZFC (such as $\mathrm{ZFC_N}$ where N is large or $\mathrm{ZFC}-\mathrm{Replacement}+\Sigma_2$-Replacement). For $x\in\omega^\omega$, let

$$\lambda(x)=\mu\lambda\,(L_\lambda[x]\models \mathrm{T}+\text{II does not have a winning strategy in } \{y\in\omega^\omega \mid \varphi(y)\}).$$

For convenience let $A^x = \{y \in \omega^\omega \mid \varphi(y)\}^{L_{\lambda(x)}[x]}$.

Consider the game G

$$\begin{array}{ll} \mathrm{I} & a, x \\ \mathrm{II} & b \end{array}$$

where I wins iff $a * b \in L_{\lambda(x)}[x]$ and $L_{\lambda(x)}[x] \models \varphi(a * b)$. Here Player I is to be thought of as choosing the playing field $L_{\lambda(x)}[x]$ in which the two players are to play an auxiliary round (via a and b) of the localized game A^x. The key point is that since $\lambda(x)$ is always defined this game is Δ^1_2 and hence determined.

We claim that I has a winning strategy in G (and so I wins each round of the localized games A^x) and, furthermore, that (by ranging over these rounds and applying upward Σ^1_2-absoluteness) this winning strategy yields a winning strategy for I in A.

Assume for contradiction (by Δ^1_2-determinacy) that II has a winning strategy τ_0 in G. For each $x \geqslant_T \tau_0$, in $L_{\lambda(x)}[x]$ we can derive a winning strategy τ^x for II in A^x as follows: For $a \in (\omega^\omega)^{L_{\lambda(x)}[x]}$, let $(a * \tau^x)_{II} = b$ where b is such that $(\langle a, x\rangle * \tau_0)_{II} = b$. Since τ_0 is a winning strategy for II in G and we have arranged that $a * b \in L_{\lambda(x)}[x]$, II must win in virtue of the second clause, which means that $a * b \notin A^x$. Thus, $L_{\lambda(x)}[x] \models$ "τ^x is a winning strategy for II in A^x", which is a contradiction.

Thus I has a winning strategy σ_0 in G. Consider the derived strategy σ such that for $b \in \omega^\omega$, $(\sigma * b)_I = a$ where a is such that $(\sigma_0 * b)_I = \langle a, x\rangle$. Since σ_0 is a winning strategy for I in G, $\sigma * b \in L_{\lambda(x)}[x]$ and $L_{\lambda(x)}[x] \models \varphi(\sigma * b)$ and so, by upward Σ^1_2-absoluteness, $V \models \varphi(\sigma * b)$. Thus, σ is a winning strategy for I in A. ⊣

6.4 Remark.

(1) The above proof relativizes to a real parameter to show that $\Delta^1_2(x)$-determinacy implies $\Sigma^1_2(x)$-determinacy.

(2) A similar but more elaborate argument shows that if Δ^1_2-determinacy holds and for every real x, $x^\#$ exists, then $\mathrm{Th}(L[x])$ is constant for a Turing cone of x. See [4].

6.5 Theorem (Martin). *Assume* ZF + DC + Δ^1_2*-determinacy. If I has a winning strategy in a* Σ^1_2 *game then I has a* Δ^1_3 *strategy.*

Proof. Consider $A = \{x \in \omega^\omega \mid \varphi(x)\}$ where φ is Σ^1_2. Our strategy is to show that if II (the Π^1_2 player) does not win A then I (the Σ^1_2 player) wins A via a Δ^1_3 strategy.

Assume that II does not have a winning strategy in A. For $x \in \omega^\omega$, let $\lambda(x)$, A^x, G, and σ_0 be as in the previous proof. Since σ_0 is a winning strategy for I in G, for $x \geqslant_T \sigma_0$, in $L_{\lambda(x)}[x]$ we can derive a winning strategy σ^x for I in A^x as follows: For $x \geqslant_T \sigma_0$ and $b \in (\omega^\omega)^{L_{\lambda(x)}[x]}$ let $(\sigma * b)_I = a$ where a is such that $(\sigma_0 * b)_I = \langle a, x\rangle$.

Next we show that there is an $x_0 \geqslant_T \sigma_0$ such that for all $x \geqslant_T x_0$, $L_{\lambda(x)}[x] \models \Delta^1_2$-determinacy. Let $\langle \varphi_n, \psi_n \rangle$ enumerate the pairs of Σ^1_2 formulas and let $A_\varphi = \{x \in \omega^\omega \mid \varphi(x)\}$. Using DC let z_n be such that z_n codes a winning strategy for A_{φ_n} if $A_{\varphi_n} = \omega^\omega \smallsetminus A_{\psi_n}$ (i.e. A_{φ_n} is Δ^1_2); otherwise let $z_n = \langle 0, 0, \ldots \rangle$. Finally, let x_0 code $\langle z_n \mid n < \omega \rangle$. Thus, for $x \geqslant_T x_0$,

$$L_{\lambda(x)}[x] \models \text{I has a } \Sigma^1_4 \text{ strategy in } A^x$$

by the Third Periodicity Theorem of Moschovakis.

(For a proof of Third Periodicity see Jackson's chapter in this Handbook. The statement of Third Periodicity typically involves boldface determinacy. However, the proof shows that lightface Δ^1_2 determinacy suffices to get Σ^1_4 winning strategies for Σ^1_2 games that I wins. To see this note that $\text{Scale}(\Sigma^1_2)$ holds in $\text{ZF} + \text{DC}$. Furthermore, we also have the determinacy of the Σ^1_2 games (denoted $G^n_{s,t}$ in Jackson's chapter) that are used to define the prewellorderings and ultimately the definable strategies. It follows that these prewellorderings and strategies are $\Game\Sigma^1_2 \subseteq \Sigma^1_4$. (Notice that if we had $\underset{\sim}{\Delta}^1_2$-determinacy then we could flip the quantifiers and conclude that $\Game\Sigma^1_2 = \Pi^1_3$ and hence get Δ^1_3 strategies. However, in our present lightface setting some more work is required.))

For $x \geqslant_T x_0$, let $\hat{\sigma}^x$ be the Σ^1_4-strategy for I in A^x. For a Turing cone of x the formula $\varphi(y, z)$ defining this strategy is constant. We can now "freeze out" the value of $\hat{\sigma}^x$ on a Turing cone of x. The key point is that the function $x \mapsto L_{\lambda(x)}[x]$ is Δ^1_2. So, for each $s \in \omega^{2n}$ and $m \in \omega$ the statement

$$L_{\lambda(x)}[x] \models \varphi(s, m)$$

is Δ^1_2. Thus, for each $s \in \omega^{2n}$, the m such that $L_{\lambda(x)}[x] \models \varphi(s, m)$ is fixed for a Turing cone of x. Since there are only countably many $s \in \omega^{2n}$ this means that the value of $\hat{\sigma}^x$ is fixed on a Turing cone of x. Finally, letting

$$\begin{aligned} \sigma(s) = m &\leftrightarrow \exists x_0 \forall x \geqslant_T x_0 \, (L_{\lambda(x)}[x] \models \varphi(s, m)) \\ &\leftrightarrow \forall x_0 \exists x \geqslant_T x_0 \, (L_{\lambda(x)}[x] \models \varphi(s, m)) \end{aligned}$$

(where we have used Δ^1_2-determinacy to flip the quantifiers) we have that σ is a Δ^1_3 winning strategy for I in A. $\dashv$

Kechris and Solovay showed (in [3]) that under $\text{ZF} + \text{DC} + \Delta^1_2$-determinacy there is a real x_0 that "enforces" OD-determinacy in the following sense: For all $x \geqslant_T x_0$, $L[x] \models$ OD-determinacy. We will need the following strengthening of this result, which involves a stronger notion of "enforcement". We need the following definition: An ordinal λ is *additively closed* (*a.c.*) iff for all $\alpha, \beta < \lambda$, $\alpha + \beta < \lambda$.

6.6 Theorem. *Assume* $\text{ZF} + \text{DC} + \Delta^1_2$-*determinacy. Then there is a real* x_0 *such that for all additively closed* $\lambda > \omega$, *and for all reals* x, *if* $x_0 \in L_\lambda[x]$, *then* $L_\lambda[x] \models$ OD-*determinacy.*

Proof. Some preliminary remarks are in order. First, for λ additively closed, $L_\lambda[x]$ might satisfy only a very weak fragment of ZFC; so the statement "$L_\lambda[x] \models$ OD-determinacy" is to be taken in the following external sense: For each $\xi < \lambda$ and for each formula φ, $L_\lambda[x] \models$"Either I or II has a winning strategy for $\{z \in \omega^\omega \mid \varphi(z, \xi)\}$". The point is that this statement makes sense even when $\{z \in \omega^\omega \mid \varphi(z, \xi)\}$ is a proper class from the point of view of $L_\lambda[x]$. Second, the key feature of additively closed $\lambda > \omega$, is that if $y \in L_\lambda[x]$ then $L_\lambda[y] \subseteq L_\lambda[x]$. This is true since additively closed ordinals $\lambda > \omega$ are such that $\alpha + \lambda = \lambda$ for all $\alpha < \lambda$ and so if y is constructed at stage α, then $L_\lambda[x]$ still has λ-many remaining stages in which to "catch up" and construct everything in $L_\lambda[y]$. Third, the proof of the theorem is a "localization" of the proof of Theorem 5.12.

Assume for contradiction that for every real x_0 there is an additively closed $\lambda > \omega$ and a real x such that $x_0 \in L_\lambda[x]$ and $L_\lambda[x] \not\models$ OD-determinacy. So, for every real x_0 there is an additively closed $\lambda > \omega$ and a real $x' \geqslant_T x_0$ such that $L_\lambda[x'] \not\models$ OD-determinacy (since we can take $x' = \langle x, x_0\rangle$ where x and x_0 are as in the first statement) and hence, by the Löwenheim-Skolem theorem,

$$\begin{aligned}&\forall x_0 \in \omega^\omega\, \exists x \geqslant_T x_0 \exists\lambda(\lambda \text{ is a.c.} \wedge \omega < \lambda < \omega_1)\\ &\quad \wedge L_\lambda[x] \not\models \text{OD-determinacy},\end{aligned}$$

where 'a.c.' abbreviates 'additively closed'. Since the condition on x in this statement is Σ^1_2 and since we have Σ^1_2-determinacy (by Theorem 6.3)

$$\begin{aligned}&\exists x_0 \in \omega^\omega \forall x \geqslant_T x_0 \exists\lambda(\lambda \text{ is a.c.} \wedge \omega < \lambda < \omega_1)\\ &\quad \wedge L_\lambda[x] \not\models \text{OD-determinacy}\end{aligned}$$

by the Cone Theorem (Theorem 2.9). Let

$$\lambda(x) = \begin{cases} \mu\lambda\,(\omega < \lambda < \omega_1 \wedge \lambda \text{ is a.c.} \wedge L_\lambda[x] \not\models \text{OD-det}) & \text{if such a } \lambda \text{ exists}\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Notice that for a Turing cone of x

$$\lambda(x) \text{ is defined}$$

and that there are x_0 of arbitrarily large Turing degree such that for all $x \geqslant_T x_0$

$$\lambda(x) \geqslant \lambda(x_0).$$

To see this last point it suffices to observe that otherwise (by Σ^1_2-determinacy and the Cone Theorem (Theorem 2.9)) there would be an infinite descending sequence of ordinals. This point will be instrumental below in ensuring that Player II can "steer into the right model".

For each x such that $\lambda(x)$ is defined let (φ_x, ξ_x) be lexicographically least such that

$$L_{\lambda(x)}[x] \models \{z \in \omega^\omega \mid \varphi_x(z, \xi_x)\} \text{ is not determined}$$

and let $A^x = \{z \in \omega^\omega \mid \varphi_x(z, \xi_x)\}$. (However, note that since A^x might be a proper class from the point of view of $L_{\lambda(x)}[x]$, when we write '$L_{\lambda(x)}[x] \models a \in A^x$' we really mean '$L_{\lambda(x)}[x] \models \varphi_x(a, \xi_x)$'.)

Consider the game

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d\rangle$, I wins iff $\lambda(p)$ is defined and $L_{\lambda(p)}[p] \models$ "$a * d \in A^p$". This game is Σ^1_2, hence determined.

(Notice that in contrast to the proof of Theorem 5.12 we cannot include x_0 in p since we need our game to be lightface definable. However, in the plays of interest we will have one player fold in x_0. This will ensure that the first clause of the winning condition is satisfied and so the players are to be thought of as cooperating to determine the model $L_{\lambda(p)}[p]$ and simultaneously playing an auxiliary game (via a and d) on the least non-determined OD set of this model, namely, A^x.)

We will arrive at a contradiction by showing that neither player can win.

Case 1: I has a winning strategy σ_0.

Let $x_0 \geqslant_T \sigma_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. We claim that $L_{\lambda(x_0)}[x_0] \models$ "I has a winning strategy σ in A^{x_0}", which is a contradiction. The strategy σ is the strategy derived by playing the main game according to σ_0 while having II feed in x_0 for c and playing some auxiliary play $d \in L_{\lambda(x_0)}[x_0]$; that is, $(\sigma * d)_I = a$ where a is such that $(\sigma_0 * \langle x_0, d\rangle)_I = \langle a, b\rangle$:

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & x_0, d. \end{array}$$

Let $p = \langle a, b, x_0, d\rangle$. Since we have ensured that $p \geqslant_T x_0$ we know that $\lambda(p)$ is defined and, since σ_0 is winning for I, I must win in virtue of the first clause and so $L_{\lambda(p)}[p] \models$ "$a * d \in A^p$". It remains to see that II has managed to "steer into the right model", that is, that

$$L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0]$$

and hence

$$A^p = A^{x_0}.$$

Since $x_0 \geqslant_T \sigma_0$ and $d \in L_{\lambda(x_0)}[x_0]$ we have that $p \in L_{\lambda(x_0)}[x_0]$ and

$$L_{\lambda(x_0)}[p] = L_{\lambda(x_0)}[x_0]$$

(where for the left to right inclusion we have used that $\lambda(x_0)$ is additively closed). Furthermore, by arrangement, $\lambda(p) \geqslant \lambda(x_0)$ since $p \geqslant_T x_0$. But $\lambda(p)$ is the *least* additively closed λ such that $\omega < \lambda < \omega_1$ and $L_\lambda[p] \not\models$ OD-determinacy. Thus, $\lambda(p) = \lambda(x_0)$ and

$$L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0].$$

So $L_{\lambda(x_0)}[x_0] \models$ "$\sigma * d \in A^{x_0}$". Since this is true for any $d \in L_{\lambda(x_0)}[x_0]$, this means that $L_{\lambda(x_0)}[x_0] \models$ "σ is a winning strategy for I in A^{x_0}", which is a contradiction.

Case 2: II has a winning strategy τ_0.

Let $x_0 \geqslant_T \tau_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. For $a \in L_{\lambda(x_0)}[x_0]$ let $(a * \tau)_{II}$ where d is such that $(\langle a, x_0\rangle * \tau_0)_{II} = \langle c, d\rangle$. Since $p \geqslant_T x_0$, II must win in virtue of the second clause. The rest of the argument is exactly as above. So we have that $L_{\lambda(x_0)}[x_0] \models$ "τ is a winning strategy for II in A^{x_0}", which is a contradiction. ⊣

6.7 Remark. The proof relativizes to a real parameter to show $\mathrm{ZF} + \mathrm{DC} + \Sigma^1_2(x)$-determinacy implies that there is a real enforcing (in the strong sense of Theorem 6.6) OD_x-determinacy.

6.8 Corollary (Kechris and Solovay). *Assume* ZF. *Suppose* $L[x] \models \Delta^1_2$-*determinacy, where* $x \in \omega^\omega$. *Then* $L[x] \models$ OD-*determinacy.*

Proof. This follows by reflection. ⊣

We will now extract an even stronger form of determinacy from Δ^1_2-determinacy. We begin by recalling some definitions. The *strategic game with respect to the basis* B is the game SG^B

$$\begin{array}{lccccccc} \mathrm{I} & A_0 & & \cdots & A_n & & \cdots & \\ \mathrm{II} & & f_0 & & \cdots & f_n & & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{f_0,\ldots,f_n}$ and

(2) f_n is a prestrategy for A_n that is winning with respect to B,

and II wins iff he can play all ω rounds. We say that *strategic determinacy holds with respect to the basis* B (ST^B-*determinacy*) if II wins SG^B.

In the context of $L[S, x]$ we dropped reference to the basis B since it was always understood to be $\{y \in \omega^\omega \mid L[S, y] = L[S, x]\}$. In our present lightface setting we will have to pay more careful attention to B since (as we will see in Theorem 6.12) Δ^1_2-determinacy is insufficient to ensure that for a Turing cone of x, $L[x] \models \mathrm{ST}^B$-determinacy, where $B = \{y \in \omega^\omega \mid L[y] = L[x]\}$. We will now be interpreting strategic determinacy in the local setting of models $L_\lambda[x]$ where $x \in \omega^\omega$ and λ is a countable ordinal and the relevant basis will be of the form $C \cap \{y \in \omega^\omega \mid L_\lambda[y] = L_\lambda[x]\}$ where C is a Π^1_2 set of $L_\lambda[x]$. It is in the attempt to "localize" the proof of Theorem 5.14 that the need for the Π^1_2 set becomes manifest. The issue is one of "steering into the right model" and can be seen to first arise in the proof of Claim 3 below.

Let RST-determinacy abbreviate the statement "There is a Π^1_2 set C such that C contains a Turing cone and ST^B-determinacy holds where $B =$

$C \cap \{y \in \omega^\omega \mid L[y] = V\}$". Here 'R' stands for 'restrictive'. We will be interpreting this notion over models $L_\lambda[x]$ that do not satisfy full Replacement. In such a case it is to be understood that the statement involves the Σ_1 definition of ordinal definability.

6.9 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone of* x*,*

$$L[x] \models RST\text{-}determinacy.$$

Proof. We will actually prove something stronger: Assume $\mathrm{ZF} + V = L[x] + \Delta^1_2$-determinacy for some $x \in \omega^\omega$. Let T be the theory $\mathrm{ZFC} - \mathrm{Replacement} + \Sigma_2$-Replacement. Then there is a real z_0 such that if $L_\lambda[z]$ is such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models$ RST-determinacy. The theorem follows by reflection.

Assume for contradiction that for every real z_0 there is a real $z \geqslant_T z_0$ and an ordinal λ such that $L_\lambda[z] \models \mathrm{T} + \neg$RST-determinacy. The preliminary step is to reduce to a local setting where we can apply Δ^1_2-determinacy. By the Löwenheim-Skolem theorem

$$\forall z_0 \in \omega^\omega\, \exists z \geqslant_T z_0\, \exists \lambda < \omega_1\, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}).$$

Since the condition on z in this statement is Σ^1_2 and since we have OD-determinacy (by Corollary 6.8) it follows (by the Cone Theorem (Theorem 2.9)) that

$$\exists z_0 \in \omega^\omega\, \forall z \geqslant_T z_0\, \exists \lambda < \omega_1\, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}).$$

For $z \in \omega^\omega$, let

$$\lambda(z) = \begin{cases} \mu\lambda\, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Thus, if $\lambda(z)$ is defined, then for every $(\Pi^1_2)^{L_{\lambda(z)}[z]}$ set C that contains a Turing cone, I wins the game

$$\begin{array}{lcccccc} \mathrm{I} & A_0 & \cdots & A_n & \cdots & & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathrm{OD}^{L_{\lambda(z)}[z]}$, $A_{n+1} \in \mathrm{OD}^{L_{\lambda(z)}[z]}_{f_0,\ldots,f_n}$, and

(2) f_n is a prestrategy for A_n that is winning with respect to $C \cap \{y \in \omega^\omega \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$.

We now need to specify a particular $(\Pi^1_2)^{L_{\lambda(z)}[z]}$ set since (i) we want to get our hands on a canonical winning strategy σ^z for I and (ii) we need to solve the "steering problem". The naïve approach would be to forget about the Π^1_2 sets and just work with $\{y \in \omega^\omega \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$. The trouble is that for an element y of this set we might have $\lambda(y) < \lambda(z)$ and yet (when we implement the proof of Theorem 5.13) we will need to ensure that $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and thus $A^y = A^z$ and for this we require that $\lambda(y) = \lambda(z)$. So we will need to intersect with a set C that "holds up the value of $\lambda(y)$". A good candidate is the following: For each z such that $\lambda(z)$ is defined let

$$C_z = \{y \in \omega^\omega \mid \lambda(y) \text{ is undefined}\}^{L_{\lambda(z)}[z]}.$$

This is a $(\Pi^1_2)^{L_{\lambda(z)}[z]}$-set. It contains z (since in $L_{\lambda(z)}[z]$ the ordinal $\lambda(z)$ is certainly undefined). We would like to ensure that it contains the cone above z.

Claim 1. *For a Turing cone of z,*

(1) $\lambda(z)$ is defined,

(2) for all reals $y \in L_{\lambda(z)}[z]$, if $y \geqslant_T z$ then $\lambda(y) = \lambda(z)$.

Proof. We have already proved (1). Assume for contradiction that (2) does not hold on a Turing cone. Then (by OD-determinacy) for every real z there is a real $z' \geqslant_T z$ such that $\lambda(z')$ is defined and in $L_{\lambda(z')}[z']$ there is a real z'' such that $z'' \geqslant_T x'$ and $\lambda(z'') < \lambda(z)$. But then, for each $n < \omega$, we can successively choose $z_{n+1} \geqslant_T z_n$ such that $\lambda(z_{n+1}) < \lambda(z_n)$, which is a contradiction. ⊣

For each z as in Claim 1 we now have that C_z contains the Turing cone above z (since, by (2) of Claim 1, $\lambda(y) = \lambda(z)$ and so $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and again in $L_{\lambda(z)}[z]$ the ordinal $\lambda(y) = \lambda(z)$ is undefined). Letting

$$B_z = \{y \in C_z \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$$

we have that

$$\text{I wins } (SG^{B_z})^{L_{\lambda(z)}[z]}.$$

Moreover, since we have arranged that $L_{\lambda(z)}[z] \models \mathrm{T}$, Player I has a canonical strategy $\sigma^z \in \mathrm{HOD}^{L_{\lambda(z)}[z]}$. (This is because, since $L_{\lambda(z)}[z] \models \mathrm{T}$, the $\mathrm{OD}^{L_{\lambda(z)}[z]}$ sets of reals are sets (and not proper classes) in $L_{\lambda(z)}[z]$. So the tree on which $(SG^{B_z})^{L_{\lambda(z)}[z]}$ is played is an element of $\mathrm{HOD}^{L_{\lambda(z)}[z]}$.) Notice also that σ^z depends only on the model $L_{\lambda(z)}[z]$, in the sense that if $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ then $\sigma^y = \sigma^z$.

Our aim is to obtain a contradiction by defeating σ^z for some z in the Turing cone of Claim 1. We will do this by constructing a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that I must win via $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ and, for a Turing cone of z, the winning strategies give rise to prestrategies $f_0^z, f_1^z, \ldots, f_n^z, \ldots$ that constitute a non-losing play against σ^z in the game $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Step 0. Consider (in $L[x]$) the game G_0

$$\begin{array}{lll} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is either 1 or 2 and, letting $p = \langle a, b, c, d\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A_0^p$", where $A_0^p = \sigma^p(\varnothing)$.

In the plays of interest we will ensure that p is in the cone of Claim 1. So clause (1) of the winning condition will be automatically satisfied and the decisive factor will be whether in $L_{\lambda(p)}[p]$ Player ϵ wins the auxiliary round (via a and d) of A_0^p. This game is Σ_2^1 (for Player I), hence determined.

Claim 2. *I has a winning strategy σ_0 in G_0.*

Proof. Assume for contradiction that I does not have a winning strategy in G_0. Then, by Σ_2^1-determinacy, II has a winning strategy τ_0 in G_0. Let $z_0 \geqslant_T \tau_0$ be such that for all $z \geqslant_T z_0$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models$ OD-determinacy (by Theorem 6.6).

Consider $A_0^{z_0} = \sigma^{z_0}(\varnothing)$. Since $L_{\lambda(z_0)}[z_0] \models$ OD-determinacy, $L_{\lambda(z_0)}[z_0] \models$ "$A_0^{z_0}$ is determined". We will use τ_0 to show that neither player can win this game. Suppose for contradiction that $L_{\lambda(z_0)}[z_0] \models$ "σ is a winning strategy for I in $A_0^{z_0}$". Run G_0 according to τ_0, having Player I (falsely) predict that Player I wins the auxiliary game, while steering into $L_{\lambda(z_0)}[z_0]$ by playing $b = z_0$ and using σ to respond to τ_0 on the auxiliary play:

$$\begin{array}{lll} \mathrm{I} & 1 & (\sigma * d)_I, z_0 \\ \mathrm{II} & & c, d \end{array}$$

We have to see that Player I has indeed managed to steer into $L_{\lambda(z_0)}[z_0]$, that is, we have to see that $L_{\lambda(p)}[p] = L_{\lambda(z_0)}[z_0]$, where $p = \langle(\sigma * d)_I, z_0, c, d\rangle$. Since $\sigma, z_0, \tau_0 \in L_{\lambda(z_0)}[z_0]$ and $\lambda(z_0)$ is additively closed, we have $L_{\lambda(z_0)}[p] = L_{\lambda(z_0)}[z_0]$. But $\lambda(p) = \lambda(z_0)$ since z_0 satisfies Claim 1. Thus, $L_{\lambda(p)}[p] = L_{\lambda(z_0)}[z_0]$ and hence $A_0^p = A_0^{z_0}$. Finally, since τ_0 is a winning strategy for II in G_0 and $\epsilon = 1$, we have that $L_{\lambda(p)}[p] \models$ "$\sigma * d \notin A_0^p$", and hence $L_{\lambda(z_0)}[z_0] \models$ "$\sigma * d \notin A_0^{z_0}$", which contradicts the assumption that σ is a winning strategy for I. Similarly, we can use τ_0 to defeat any strategy τ for II in $A_0^{z_0}$. ⊣

Since the game is Σ_2^1 for Player I, Player I has a Δ_3^1-strategy σ_0, by Theorem 6.5.

Claim 3. *For every real $z \geqslant_T \sigma_0$ in the Turing cone of Claim 1, there is a prestrategy f_0^z such that f_0^z is definable in $L_{\lambda(z)}[z]$ from σ_0 and f_0^z is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.*

Proof. Fix $z \geqslant_T \sigma_0$ as in Claim 1. Consider $A_0^z = \sigma^z(\varnothing)$. Let f_0^z be the prestrategy derived from σ_0 in $L_{\lambda(z)}[z]$ by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L_{\lambda(z)}[z]}$ let $f_0^z(y)$ be such that for $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$, $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. f_0^z is clearly definable in $L_{\lambda(z)}[z]$ from σ_0. We claim that f_0^z is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

To motivate the need for the Π^1_2 set, let us first see why f_0^z need not be a prestrategy for II in A_0^z that is winning with respect to $\{y \in (\omega^\omega)^{L_{\lambda(z)}[z]} \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$. Consider such a real y and an auxiliary play $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$. By definition $f_0^z(y)$ is such that $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. Assume first that $\epsilon = 1$. Since σ_0 is a winning strategy for I in G_0, $f_0^z(y) * d = a * d \in A_0^p$ where $p = \langle a, b, y, d\rangle$. What we need, however, is that $f_0^z(y) * d = a * d \in A_0^z$. The trouble is that we may have $L_{\lambda(p)}[p] = L_{\lambda(y)}[y] \subsetneq L_{\lambda(z)}[z]$ because although $L_{\lambda(z)}[y] = L_{\lambda(z)}[z]$ we might have $\lambda(y) < \lambda(z)$. And if this is indeed the case then we cannot conclude that $A_0^p = A_0^z$. If $\epsilon = 0$ then $f_0^z(y) * d = a * d \notin A_0^p$ but again what we need is that $f_0^z(y) * d = a * d \notin A_0^z$ and the same problem arises.

The above problem is solved by demanding in addition that $y \in C_z$, since then $\lambda(y) = \lambda(z)$ and so $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f_0^z(y) * d = a * d \in A_0^p = A_0^z$" as desired. Thus f_0^z is a non-losing first move for II against σ_z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. $\dashv$

Step $n+1$. Assume that we have defined (in $L[x]$) games $G_0, \ldots, G_n$ with winning strategies $\sigma_0, \ldots, \sigma_n \in \mathrm{HOD}$ such that for all $z \geqslant_T \langle \sigma_0, \ldots, \sigma_n\rangle$ in the Turing cone of Claim 1 there are prestrategies $f_0^z, \ldots, f_n^z$ such that f_i^z is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \ldots, \sigma_i$ (for all $i \leqslant n$) and $f_0^z, \ldots, f_n^z$ is a non-losing partial play for II in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Consider (in $L[x]$) the game G_{n+1}

$$\begin{array}{lcc} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is 1 or 2 and, letting $p = \langle a, b, c, d, \sigma_0, \ldots, \sigma_n\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A_{n+1}^p$", where A_{n+1}^p is I's response via σ^p to II's partial play $f_0^p, \ldots, f_n^p$.

If p satisfies condition (1) then, since $p \geqslant_T \langle \sigma_0, \ldots, \sigma_n\rangle$, we have, by the induction hypothesis, prestrategies $f_0^p, \ldots, f_n^p$ such that f_i^p is definable in $L_{\lambda(p)}[p]$ from $\sigma_0, \ldots, \sigma_i$ (for all $i \leqslant n$) and $f_0^p, \ldots, f_n^p$ is a non-losing partial play for II in $(SG^{B_p})^{L_{\lambda(p)}[p]}$. Thus, condition (2) in the definition of the game makes sense.

This game is $\Sigma^1_2(\sigma_0, \dots, \sigma_n)$ (for Player I) and hence determined (since $\sigma_0, \dots, \sigma_n \in \mathrm{HOD}$ and we have OD-determinacy, by Theorem 6.6).

Claim 4. *I has a winning strategy σ_{n+1} in G_{n+1}.*

Proof. Assume for contradiction that I does not have a winning strategy. Then, by OD-determinacy, II has a winning strategy τ_{n+1}. Let $z_{n+1} \geqslant_T \langle \tau_{n+1}, \sigma_0, \dots, \sigma_n \rangle$ be such that for all $z \geqslant_T z_{n+1}$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_{n+1} \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models \mathrm{OD}_{\sigma_0,\dots,\sigma_n}$-determinacy (by the relativized version of Theorem 6.6).

It follows that

$$L_{\lambda(z_{n+1})}[z_{n+1}] \models A^{z_{n+1}}_{n+1} \text{ is determined},$$

where $A^{z_{n+1}}_{n+1} = \sigma^{z_{n+1}}(\langle f^{z_{n+1}}_0, \dots, f^{z_{n+1}}_n \rangle)$. This is because $A^{z_{n+1}}_{n+1}$ is an element of $\mathrm{HOD}^{L_{\lambda(z_{n+1})}[z_{n+1}]}(\sigma_0, \dots, \sigma_n)$ (as all of the ingredients $\sigma^{z_{n+1}}$, $f^{z_{n+1}}_0$, $\dots$, $f^{z_{n+1}}_n$ used to define $A^{z_{n+1}}_{n+1}$ are in this model) and we arranged that $L_{\lambda(z_{n+1})}[z_{n+1}]$ satisfies $\mathrm{OD}_{\sigma_0,\dots,\sigma_n}$-determinacy.

[The enforcement of the parameterized version of OD-determinacy in (2) appears to be necessary. The point is that even though, in Step 1 for example, σ_0 is Δ^1_3 and $\sigma_0 \in L_{\lambda(z)}[z]$ we have no guarantee that in $L_{\lambda(z_1)}[z_1]$, σ_0 satisfies this definition. If we did then we would have that A^z_1 is in $\mathrm{HOD}^{L_{\lambda(z_1)}[z_1]}$ and hence just enforce OD-determinacy.]

We will use τ_{n+1} to show that neither player can win this game. The argument is exactly as in Step 0 except with the subscripts '0' replaced by '$n+1$': Suppose for contradiction that $L_{\lambda(z_{n+1})}[z_{n+1}] \models$ "σ is a winning strategy for I in $A^{z_{n+1}}_{n+1}$". Run G_{n+1} according to τ_{n+1}, having Player I (falsely) predict that Player I wins the auxiliary game, while steering into $L_{\lambda(z_{n+1})}[z_{n+1}]$ by playing $b = z_{n+1}$ and using σ to respond to τ_{n+1} on the auxiliary play:

$$\begin{array}{llll} \mathrm{I} & 1 & (\sigma * d)_I, z_{n+1} \\ \mathrm{II} & & c, d \end{array}$$

We have to see that Player I has indeed managed to steer into $L_{\lambda(z_{n+1})}[z_{n+1}]$, that is, we have to see that $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$, where p is the set $\langle (\sigma * d)_I, z_{n+1}, c, d \rangle$. Since $\sigma, z_{n+1}, \tau_{n+1} \in L_{\lambda(z_{n+1})}[z_{n+1}]$ and $\lambda(z_{n+1})$ is additively closed, we have $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$. Since $p \geqslant_T z_{n+1}$ and z_{n+1} satisfies the condition of Claim 1, $\lambda(p) = \lambda(z)$, and so $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$ and hence $A^p_{n+1} = A^{z_{n+1}}_{n+1}$. Finally, since τ_{n+1} is a winning strategy for II in G_{n+1} and $\epsilon = 1$, we have that $L_{\lambda(p)}[p] \models$ "$\sigma * d \notin A^p_{n+1}$", and hence $L_{\lambda(z_{n+1})}[z_{n+1}] \models$ "$\sigma * d \notin A^{z_{n+1}}_{n+1}$", which contradicts the assumption that σ is a winning strategy for I. Similarly, we can use τ_{n+1} to defeat any strategy τ for II in $A^{z_{n+1}}_{n+1}$. ⊣

Since the game is $\Sigma^1_2(\sigma_0, \dots, \sigma_n)$ for Player I, Player I has a $\Delta^1_3(\sigma_0, \dots, \sigma_n)$ strategy σ_{n+1}, by the relativized version of Theorem 6.5.

Claim 5. *For every real $z \geqslant_T \langle \sigma_0, \dots, \sigma_n \rangle$ as in Claim 1, there is a prestrategy f^z_{n+1} that is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \dots, \sigma_{n+1}$ and such that $f^z_0, \dots, f^z_{n+1}$ is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.*

Proof. The proof is just like the proof of Claim 3. Fix $z \geqslant_T \langle \sigma_0, \dots, \sigma_n \rangle$ as in Claim 1 and consider $A^z_{n+1} = \sigma^z(\langle f^z_0, \dots, f^z_n \rangle)$. Let f^z_{n+1} be the prestrategy derived from σ_{n+1} in $L_{\lambda(z)}[z]$ by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L_{\lambda(z)}[z]}$ let $f^z_{n+1}(y)$ be such that for $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$, $f^z_{n+1}(y) * d = a * d$ where a is such that $(\sigma_{n+1} * \langle y, d \rangle)_I = \langle \epsilon, a, b \rangle$. Clearly, f^z_{n+1} is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \dots, \sigma_{n+1}$. We claim that f^z_{n+1} is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. Again the point is that for $y \in B_z$, $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$, hence $A^y_{n+1} = A^z_{n+1}$. Thus, $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f^z_{n+1}(y) * d = a * d \in A^p_{n+1} = A^z_{n+1}$" as desired. Hence $\langle f^z_0, \dots, f^z_{n+1} \rangle$ is a non-losing play for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. ⊣

Finally, letting z^∞ be such that $z^\infty \geqslant_T z_n$ for all n and z^∞ is as in Claim 1, we have that $f^{z^\infty}_0, \dots, f^{z^\infty}_n, \dots$ defeats σ^{z^∞} in $(SG^{B_{z^\infty}})^{L_{\lambda(z^\infty)}[z^\infty]}$, which is a contradiction. ⊣

6.10 Theorem. *Assume* ZF + DC + Δ^1_2*-determinacy. Then for a Turing cone of x,*

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

Proof. For a Turing cone of x, $L[x] \models$ RST-determinacy, by Theorem 6.9. Let x be in this cone. We have to meet the conditions of the Generation Theorem. Let $\Theta_M = \omega_2^{L[x]}$. Since $L[x]$ satisfies GCH and $L[x] = \mathrm{OD}^{L[x]}_x$,

$$\omega_2^{L[x]} = \sup\{\alpha \mid \text{there is an } \mathrm{OD}^{L[x]} \text{ prewellordering of length } \alpha\},$$

in other words, $\omega_2^{L[x]} = (\Theta_0)^{L[x]}$. Let $A = \langle A_\alpha \mid \alpha < \omega_2^{L[x]} \rangle$ be such that A_α is the $\mathrm{OD}^{L[x]}$-least prewellordering of length α. Since $L[x] \models$ OD-determinacy, it follows (by Theorem 3.9) that $\omega_2^{L[x]}$ is strongly inaccessible in $\mathrm{HOD}^{L[x]}$. So there is a set $H \subseteq \omega_2^{L[x]}$ coding $\mathrm{HOD}^{L[x]} \cap V_{\omega_2^{L[x]}}$. Let T' be in $\mathscr{P}(\omega_2^{L[x]}) \cap \mathrm{OD}^{L[x]}$ and let $T \in \mathscr{P}(\omega_2^{L[x]}) \cap \mathrm{OD}^{L[x]}$ code T' and H. Let B be as in the statement of RST-determinacy.

Let

$$M = (L_{\Theta_M}(\mathbb{R})[T, A, B])^{L[x]},$$

where Θ_M, T, A, B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L[x]$ satisfies RST-determinacy, M is OD in $L[x]$ and M contains the reals of $L[x]$. Thus,

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal.}$$

Since, by arrangement, $\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L[x]} \cap V_{\omega_2^{L[x]}}$, it follows that

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal.}$$

Since T' was arbitrary, the theorem follows. ⊣

We close with four limitative results. The first result motivates the need for the notion of strategic determinacy by showing that strategic determinacy does not follow trivially from OD-determinacy in the sense that for some OD basis there are OD prestrategies.

6.11 Theorem. *Assume* ZF*. Then for each non-empty* OD *set $B \subseteq \omega^\omega$, there is an* OD *set $A \subseteq \omega^\omega$ such that there is no* OD *prestrategy in A which is winning with respect to the basis B.*

Proof. Assume for contradiction that there is a set $B \subseteq \omega^\omega$ which is OD and such that for all OD sets $A \subseteq \omega^\omega$ there is an OD prestrategy f_A in A which is winning with respect to B. We may assume OD-determinacy since if OD-determinacy fails then the theorem trivially holds (as clearly one cannot have a prestrategy which is winning with respect to a non-empty basis for a non-determined game).

We shall need to establish three claims.

Claim 1. *Assume* ZF*. Then*

$$\bigcap\{A \subseteq \omega^\omega \mid A \in \mathrm{OD},\ A \textit{ is Turing invariant,}\\ \textit{and } A \textit{ contains a Turing cone}\} = \varnothing.$$

Proof. For each $\alpha < \omega_1$, let

$$A_\alpha = \{z \in \omega^\omega \mid \exists x, y \in \omega^\omega \text{ such that}\\ x \equiv_T y \leqslant_T z \text{ and } x \text{ codes } \alpha\}.$$

Notice that A_α is OD, Turing invariant, and contains a Turing cone. But clearly

$$\textstyle\bigcap_{\alpha<\omega_1} A_\alpha = \varnothing$$

since otherwise there would be a real z which recursively encodes all countable ordinals. ⊣

Claim 2. *Assume* $\mathrm{ZF} + \mathrm{OD}$*-determinacy. Then*

$$\mathrm{HOD} \models \textit{There is a countably complete ultrafilter on } \omega_1^V.$$

Proof. Since we are not assuming $\mathrm{AC}_\omega(\mathbb{R})$, the proof of Theorem 2.12 does not directly apply. To see this note that ω_1^V may not be regular in V—in fact, we do not even know whether ω_1^V is regular in HOD. Nevertheless, we will be able to implement some of the previous arguments by dropping into an appropriate model of $\mathrm{AC}_\omega(\mathbb{R})$. In the case of countable completeness an additional change will be required since without $\mathrm{AC}_\omega(\mathbb{R})$ we cannot choose countably many strategies as we did in the earlier proof. Let

$$\mu = \{S \subseteq \omega_1^V \mid S \in \mathrm{HOD} \text{ and I has a winning strategy in } G(S)\},$$

where $G(S)$ is the game from Theorem 2.12.

Subclaim 1. $\mathrm{HOD} \models \mu \cap \mathrm{HOD}$ *is an ultrafilter.*

Proof. It is clear that $\omega_1^V \in \mu$ and $\varnothing \notin \mu$. It is also clear that if $S \in \mu$ and $S' \in \mathrm{HOD} \cap \mathscr{P}(\omega_1^V)$ and $S \subseteq S'$ then $S' \in \mu$.

Suppose that $S \in \mathrm{HOD} \cap \mathscr{P}(\omega_1^V)$ and that II has a winning σ strategy in $G(S)$. We claim that I has a winning strategy in $G(\omega_1^V \smallsetminus S)$. Suppose for contradiction that I does not have a winning strategy. Then, by OD-determinacy, II has a winning strategy σ'. Now work in $L[\sigma, \sigma']$. Using $\underset{\sim}{\Sigma}^1_1$-boundedness, by the usual arguments, one can construct a play x for I which is legal against both σ and σ' and in each case has the same associated ordinal $\alpha < \omega_1^{L[\sigma,\sigma']}$. This is a contradiction.

We now show that if $S_1, S_2 \in \mu$ then $S_1 \cap S_2 \in \mu$. Let σ_1 be a winning strategy for I in $G(S_1)$ and let σ_2 be a winning strategy for I in $G(S_2)$. Suppose for contradiction that $S_1 \cap S_2 \notin \mu$. Since $S_1 \cap S_2$ is OD, $G(S_1 \cap S_2)$ is determined and so II has a winning strategy in $G(S_1 \cap S_2)$, which implies that I has a winning strategy σ in $G(\omega_1^V \smallsetminus (S_1 \cap S_2))$. Work in $L[\sigma_1, \sigma_2, \sigma]$. The strategy σ_1 witnesses (by the usual argument using $\underset{\sim}{\Sigma}^1_1$-boundedness) that $S_1 \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club. Likewise, σ_2 witnesses that $S_2 \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club and σ witnesses that $(\omega_1^V \smallsetminus (S_1 \cap S_2)) \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club. This contradiction completes the proof of Subclaim 1. ⊣

Subclaim 2. $\mathrm{HOD} \models \mu \cap \mathrm{HOD}$ *is countably complete.*

Proof. Suppose for contradiction that the subclaim fails. Let $\langle S_i \mid i < \omega \rangle \in \mathrm{HOD}$ be such that for each $i < \omega$, $S_i \in \mu$ and $\bigcap_{i<\omega} S_i = \varnothing$. Consider the game

$$\begin{array}{lccccccc} \mathrm{I} & i & & y(0) & & y(1) & & \dots \\ \mathrm{II} & & x(0) & & x(1) & & \dots & \end{array}$$

where II wins if and only if $x * y$ is a winning play for I in $G(S_i)$. The idea is that Player I begins by specifying a set S_i in our fixed sequence and then the two players play an auxiliary round of $G(S_i)$, with Player I playing as Player II and Player II playing as Player I.

Notice that this game is OD, hence determined. We claim that II has a winning strategy. Suppose for contradiction that I has a winning strategy σ. In the first move the strategy σ produces a fixed k. Since σ is winning for I, for each $x \in \omega^\omega$, $x * \sigma$ is a win for II in $G(S_k)$. But this is impossible since $S_k \in \mu$ and so I has a winning strategy τ_k in $G(S_k)$; thus, by following τ_k in the auxiliary game, II (playing as I) can defeat σ.

Let τ be a winning strategy for II. Work in $L[\tau]$. We claim that in $L[\tau]$, τ witnesses that for all $i < \omega$, $S_i \cap \omega_1^{L[\tau]}$ contains a club. For our purposes we just need a single $\alpha \in \bigcap_{i<\omega} S_i$. The point is that Player I can play any i as the first move and then use $\underset{\sim}{\Sigma}^1_1$-boundedness to produce a real y such that for all $i < \omega$, $i^\frown y$ is a legal play and in each case the ordinal produced in the auxiliary game is some fixed $\alpha < \omega_1^{L[\tau]}$. This contradiction completes the proof of Subclaim 2. $\dashv$

Thus,

$$\mathrm{HOD} \models \mu \cap \mathrm{HOD} \text{ is a countably complete ultrafilter on } \omega_1^V,$$

which completes the proof. $\dashv$

It follows that ZF + OD-determinacy proves that

$$\mathrm{HOD} \models \exists\kappa \leqslant \omega_1^V \ (\kappa \text{ is a measurable cardinal})$$

(as witnessed by letting κ be the completeness of the ultrafilter), and hence that $\mathbb{R} \cap \mathrm{HOD}$ is countable. Let $\alpha < \omega_1$ be the length of the canonical well-ordering of $\mathbb{R} \cap \mathrm{HOD}$. Let t code α. Then in HOD_t there is a real y^* such that for all $z \in \mathbb{R} \cap \mathrm{HOD}$, $z \leqslant_T y^*$. Let y^* be such a real.

Claim 3. *Suppose $z \in B$. Suppose A is* OD*, A is Turing invariant, and A contains a Turing cone. Then A contains the Turing cone above $\langle y^*, z\rangle$.*

Proof. By our original supposition for contradiction recall that we let f_A be an OD prestrategy which is winning with respect to B. Since A contains a Turing cone f_A must be winning for Player I. This means that for all $z \in B$, for all $y \in \omega^\omega$, $f_A(z) * y \in A$. Now let $y \geqslant_T \langle y^*, z\rangle$. We wish to show that $y \in A$. The point is that

$$y \equiv_T f_A(z) * y \in A$$

and since A is Turing invariant, this implies that $y \in A$. $\dashv$

Claim 3 contradicts Claim 1, which completes the proof. $\dashv$

The second result motivates the need for restricted strategic determinacy by showing that $V = L[x]$ + Δ^1_2-determinacy does not imply ST^B-determinacy, where B is the constructibility degree of x. Thus, in Theorem 6.9 it was necessary to drop down to a restricted form of strategic determinacy. It also follows from the theorem that something close to $\underset{\sim}{\Delta}^1_2$-determinacy is required to establish that ST^B-determinacy holds with respect to the constructibility degree of x since the statement "$\underset{\sim}{\Delta}^1_2$-determinacy" is equivalent to the statement "for every $y \in \omega^\omega$ there is an inner model M such that $y \in M$ and $M \models \mathrm{ZFC} +$ There is a Woodin cardinal".

6.12 Theorem. *Assume* $\mathrm{ZF} + V = L[x]$ *for some* $x \in \omega^\omega$. *Suppose* ST^B-*determinacy, where* $B = \{y \in \omega^\omega \mid L[y] = L[x]\}$. *Suppose there exists an* $\alpha > \omega_1^{L[x]}$ *such that* $L_\alpha[x] \models \mathrm{ZFC}$. *Then for every* $y \in \omega^\omega$ *there is a transitive model* M *such that* $y \in M$ *and* $M \models \mathrm{ZFC} +$ *"There is a Woodin cardinal".*

Proof. Let

$$\begin{aligned} A_0 = \{y \in \omega^\omega \mid \neg\exists M\, (&M \text{ is transitive} \wedge y \in M \\ &\wedge M \models \mathrm{ZFC} + \text{There is a Woodin cardinal})\}. \end{aligned}$$

Suppose for contradiction that $A_0 \neq \varnothing$. Let $t \in A_0$. It follows that A_0 contains a Turing cone of reals. Let Player I play A_0 in SG^B and let f_0 be II's response. Since Player I can win a round of A_0 by playing t, f_0 is winning for I with respect to B, that is, for all $y \in B$, $f_0(y) \in A_0$. We will arrive at a contradiction by constructing a real $y \in B$ such that $f_0(y) \notin A_0$.

We claim that

$$\mathrm{HOD}_{f_0}^{L_\alpha[x]} \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

First note that

$$L[x] \models \mathrm{ST}_{f_0}\text{-determinacy.}$$

Since $L_\alpha[x]$ is ordinal definable in $L[x]$ (as $\alpha > \omega_1^{L[x]}$ and so $L_\alpha[x] = L_\alpha[x']$ for any real x' such that $V = L[x']$) it follows that

$$L_\alpha[x] \models \mathrm{ST}_{f_0}\text{-determinacy.}$$

Thus, by the relativized version of Theorem 6.10,

$$\mathrm{HOD}_{f_0}^{L_\alpha[x]} \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

Therefore $f_0 \notin A_0$.

By $\Sigma_2^1(f_0)$-absoluteness, $L[f_0]$ satisfies that there is a countable transitive model M such that $f_0 \in M$ and

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

Since $L_\alpha[x] \models$ "$f_0^\#$ exists" (by the effective version of Solovay's Theorem (Theorem 2.15) there is a countable ordinal λ such that $L_\lambda[f_0]$ satisfies ZFC+ "M is countable". In $L_\lambda[f_0]$ let P be a perfect set of reals that are Cohen generic over M. Since P is perfect in $L_\lambda[f_0]$ there is a path $c \in [P]$ which codes x in the sense that $c \geqslant_T x$.

Our desired real y is $\langle f_0, c\rangle$. To see that $\langle f_0, c\rangle \in B$ note that $L_\lambda(\langle f_0, c\rangle)$ can compute x and hence $L_{\omega_1}[\langle f_0, c\rangle] = L_{\omega_1}[x]$. To see that $f_0(\langle f_0, c\rangle) \notin A_0$ note that since c is Cohen generic over M, the model $M[c]$ is a transitive model containing $f_0(\langle f_0, c\rangle)$ satisfying ZFC + "There is a Woodin cardinal". This is a contradiction. ⊣

The third result shows that Martin's "lightface form" of Third Periodicity (Theorem 6.3) does not generalize to higher levels. In fact, the result shows that even ZFC + OD-determinacy (assuming consistency of course) does not imply that for every Σ^1_4 game which Player I wins, Player I has a Δ^1_5 strategy (or even an OD strategy). The reason that the "lightface form" of Third Periodicity holds at the level of Σ^1_2 but not beyond is that in Third Periodicity boldface determinacy is used to get scales but in ZF + DC we get Scale($\utilde{\Sigma}^1_2$) for free.

6.13 Theorem. *Assume* $\mathrm{ZF}+V=L[x]+\mathrm{OD}$*-determinacy for some* $x\in\omega^\omega$*. There is a* Π^1_2 *set of reals which contains a Turing cone but which does not contain a member in* HOD*.*

Proof. Consider the set

$$A=\{y\in\omega^\omega\mid \text{for all additively closed } \lambda<\omega_1,\\ \text{for all } z\geqslant_T y,\ \text{if } x\in\mathrm{OD}^{L_\lambda[z]} \text{ then } x\leqslant_T y\}.$$

This is a Π^1_2 set. Notice that each $y\in A$ witnesses that $\mathbb{R}\cap\mathrm{HOD}^{L[z]}$ is countable for each $z\geqslant_T y$.

Claim 1. *A contains a Turing cone.*

Proof. For $y\in\omega^\omega$ and α such that $\omega<\alpha<\omega_1$, let $R_{\alpha,y}$ be the set of reals which are ordinal definable in $L_\alpha[y]$ and let $<_{\alpha,y}$ the canonical well-ordering of $R_{\alpha,y}$, where we arrange that $<_{\alpha,y}$ is an initial segment of $<_{\alpha',y}$ when $\alpha<\alpha'$. For $y\in\omega^\omega$, let $R_y=\bigcup\{R_{\alpha,y}\mid\omega<\alpha<\omega_1\}$ and let $<_y$ be the induced order on R_y (where we order first by α and then by $<_{\alpha,y}$). Let z^y_α be the αth real in $<_y$ and let ϑ_y be the ordertype of $<_y$. Notice that $R_{\alpha,y}$, R_y, $<_{\alpha,y}$, $<_y$, z^y_α and ϑ_y depend only on the Turing degree of y.

Our strategy is to "freeze out" the values of R_y and $<_y$ on a Turing cone of y. For $\alpha<\omega_1$, the set

$$A_\alpha=\{y\in\omega^\omega\mid\vartheta_y>\alpha\}$$

is OD and hence, by OD-determinacy, either it or its complement contains a Turing cone. Moreover, if A_α contains a Turing cone and $\bar\alpha<\alpha$ then $A_{\bar\alpha}$ contains a Turing cone. Thus,

$$A'=\{\alpha<\omega_1\mid A_\alpha\text{ contains a Turing cone}\}$$

is an initial segment of ω_1. For each $\alpha\in A'$, and for each $y\in\omega^\omega$, the statement "$\vartheta_y>\alpha$ and $z^y_\alpha(n)=m$" is an OD-statement about y. So, by OD-determinacy, the value of z^y_α is fixed for a Turing cone of y. We write z_α for this stable value. It follows that $\langle z_\alpha\mid\alpha\in A'\rangle$ is a definable well-ordering of reals and hence, by OD-determinacy, A' must be countable (by the effective version of Solovay's theorem (Theorem 2.15) and the argument in the Claim in Theorem 5.9). Let $\vartheta = \sup\{\alpha+1 \mid \alpha \in A'\}$. Finally, let $R_\infty = \{z_\alpha \mid \alpha < \vartheta\}$ and $<_\infty = \{(z_\alpha, z_\beta) \mid \alpha < \beta < \vartheta\}$. We claim that for a Turing cone of y, $R_y = R_\infty$. To see this let $y \in L[x]$ be such that $x \leqslant_T y$ (so, in particular $L[x] = L[y]$) and y belongs to all of the (countably many) cones fixing z_α for $\alpha \in A'$. Then $\vartheta_y = \vartheta$ and $R_y = R_\infty$. (In fact, $R_y = \mathbb{R}^{\mathrm{HOD}}$.)

Let z_0 be such that for all $z \geqslant_T z_0$, $R_z = R_{z_0} = R_\infty$. Since R_∞ is countable, we can choose $y_0 \geqslant_T z_0$ such that $R_\infty \leqslant_T y_0$. Then for all $z \geqslant_T y_0$, $R_z = R_\infty \leqslant_T y_0$, that is, $y_0 \in A$. Likewise, if $y \geqslant_T y_0$, then $y \in A$. ⊣

Claim 2. *$A \cap \mathrm{HOD}^{L[x]} = \varnothing$.*

Proof. Suppose for contradiction that $y \in A \cap \mathrm{HOD}^{L[x]}$. Since $y \in A$, y witnesses that R_z is countable for all $z \geqslant_T y$. Let z be such that

$$R_x = R_z.$$

Then since $y \in \mathrm{HOD}^{L[x]}$,

$$\mathrm{HOD}^{L[x]} \models \mathbb{R} \text{ is countable},$$

which is impossible. ⊣

This completes the proof. ⊣

The final result is a refinement of a theorem of Martin [5, Theorem 13.1]. It shows that $\mathrm{ZF} + \mathrm{DC} + \Delta^1_2$-determinacy implies that for a Turing cone of x, $\mathrm{HOD}^{L[x]}$ has a Δ^1_3 well-ordering of reals and hence that for a Turing cone of x, Δ^1_2-determinacy fails in $\mathrm{HOD}^{L[x]}$.

6.14 Theorem. *Assume $\mathrm{ZF} + V = L[x] + \Delta^1_2$-determinacy, for some $x \in \omega^\omega$. Then in HOD there is a Δ^1_3-well-ordering of the reals.*

Proof. For $y \in \omega^\omega$ and α such that $\omega < \alpha < \omega_1$, let $R_{\alpha,y}$, $<_{\alpha,y}$, z^y_α, R_y, $<_y$, and ϑ_y be as in the proof of Theorem 6.13. Let A' and R_∞ be as in the proof of Theorem 6.13. The argument of Claim 1 of Theorem 6.13 shows that $R_\infty = \mathbb{R}^{\mathrm{HOD}}$: To see this let $x' \in L[x]$ be such that $x \leqslant_T x'$ (so, in particular $L[x] = L[x']$) and x' belongs to all of the (countably many) cones fixing z_α for $\alpha \in A'$. Then $\vartheta_{x'} = \vartheta$ and $R_\infty = R_{x'} = \mathbb{R} \cap \mathrm{HOD}^{L[x']} = \mathbb{R}^{\mathrm{HOD}}$.

Notice that

$$\exists y_0 \forall y \geqslant_T y_0 \forall \omega < \alpha < \omega_1 \, (R_{\alpha,y} \subseteq R_\infty \wedge <_{\alpha,y} \trianglelefteq <_\infty),$$

where $\trianglelefteq$ denotes ordering by initial segment; x' as above is such a y_0. Since R_∞ and $<_\infty$ are countable they can be coded by a real. Let y_0 be the base of the above cone and let a be a real coding $\langle y_0, R_\infty, <_\infty \rangle$. The statement "$a$ codes $\langle y_0, R_\infty, <_\infty \rangle$ and for all $y \geqslant_T y_0$, for all $\alpha < \omega_1$, $R_{\alpha,y} \subseteq R_\infty$ and

$<_{\alpha,y} \trianglelefteq <_\infty$" is a Π^1_2 truth about a. Writing $\psi(a)$ for this statement we have the following Π^1_3 definitions (in $L[x]$) of $\omega^\omega \cap \mathrm{HOD}$ and $<_\infty$:

$$z \in R_\infty \leftrightarrow \forall a\,[\,a \text{ codes } (z, R, <) \wedge \psi(a) \rightarrow z \in R\,]$$

and

$$z_0 <_\infty z_1 \leftrightarrow \forall a\,[\,a \text{ codes } (z, R, <) \wedge \psi(a) \rightarrow z_0 < z_1\,].$$

We now look at things from the point of view of HOD. Fix $\xi < \vartheta$. We claim that ξ is countable in HOD. Consider the game

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c \end{array}$$

where I wins iff there is an $\alpha < \omega_1$ such that $z_\xi \in R_{\alpha,\langle b,c\rangle}$ and a codes the ordertype of $<_{\alpha,\langle b,c\rangle}\restriction z_\xi$. This game is $\Sigma^1_2(z_\xi)$ (for Player I) and since $z_\xi \in \mathrm{HOD}$ the game is determined. Moreover, I must win (since I can play $b = y_0$ and an a coding ξ). By (the relativized version of) Theorem 6.5, Player I has a winning strategy $\sigma \in \mathrm{HOD}$. It follows that ξ is less than the least admissible relative to σ, which in turn is countable in HOD.

Thus, we can let z be a real in HOD coding $<_\infty\restriction z_\xi$. Consider the game $G(z, z_\xi)$

$$\begin{array}{ll} \mathrm{I} & a \\ \mathrm{II} & b \end{array}$$

where I wins iff there exists an α such that $z_\xi \in R_{\alpha,\langle a,b\rangle}$ and $<_{\alpha,\langle a,b\rangle}\restriction z_\xi = z$. This game is $\Sigma^1_2(\langle z, z_\xi\rangle)$, hence determined. Moreover, I must win. So I has a winning strategy $\sigma_\xi \in \mathrm{HOD}$.

Finally, notice the following: If $y \geqslant_T \sigma_\xi$ then $y \equiv_T \sigma_\xi * y$ and

$$\forall\alpha\,(\omega < \alpha < \omega_1 \wedge z_\xi \in R_{\alpha,y} \rightarrow <_{\alpha,y}\restriction z_\xi = <_\infty\restriction z_\xi).$$

So the following is a Σ^1_3 calculation of $<_\infty$ in HOD:

$$\begin{aligned} x <_\infty y \;\leftrightarrow\; & \exists a \in \omega^\omega \text{ coding } (y_0, <, z) \text{ such that} \\ & < \text{ is a linear ordering of its domain, } \mathrm{dom}(<), \\ & x, y, z \in \mathrm{dom}(<), \\ & x < y \text{ and } y < z, \text{ and} \\ & \forall y' \geqslant_T y_0\, \forall\alpha\,(\omega < \alpha < \alpha_1 \\ & \wedge z \in R_{\alpha,y'} \rightarrow <_{\alpha,y'}\restriction z = <\restriction z). \end{aligned}$$

This completes the proof, since clearly a Σ^1_3 total ordering is also Π^1_3. ⊣

Putting everything together we have that ZF + DC + Δ^1_2-determinacy implies that for a Turing cone of x, $\mathrm{HOD}^{L[x]}$ is an inner model with a Woodin cardinal and a Δ^1_3 well-ordering of reals. It follows that Δ^1_2-determinacy fails in $\mathrm{HOD}^{L[x]}$ for a Turing cone of x.

Some interesting questions remain. For example: Does $\mathrm{HOD}^{L[x]}$ satisfy GCH, for a Turing cone of x? Is $\mathrm{HOD}^{L[x]}$ a fine-structural model, for a Turing cone of x? We will return to this topic in Sect. 8.

6.2. Boldface Definable Determinacy

In this section we will work in ZF + AD. Our aim is to extract the optimal amount of large cardinal strength from boldface determinacy by constructing a model of ZFC that contains ω-many Woodin cardinals.

We shall prove a very general theorem along these lines. Our strategy is to iteratively apply Theorem 5.43. Recall that this theorem states that under ZF + AD, for a Y-cone of x,

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \text{ is a Woodin cardinal,}$$

where

$$[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}.$$

We start by taking a to be the empty set. By Theorem 5.43, there exists an x_0 such that for all $x \geqslant_Y x_0$,

$$\mathrm{HOD}_{Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x}} \text{ is a Woodin cardinal.}$$

To generate a model with two Woodin cardinals we would like to apply Theorem 5.43 again, this time taking a to be $[x_0]_Y$. This gives us an $x_1 \geqslant_Y x_0$ such that for all $x \geqslant_Y x_1$,

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ is a Woodin cardinal}$$

and we would like to argue that

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ are Woodin cardinals.}$$

But there are two difficulties in doing this. First, in the very least, we need to ensure that

$$\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}}$$

and this is not immediate. Second, in moving to the larger model we need to ensure that we have not collapsed the first Woodin cardinal; a sufficient condition for this is that

$$\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} = \mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y},$$

but again this is not immediate. It turns out that both difficulties can be overcome by taking x to be of sufficiently high "Y-degree". This will be the content of an elementary observation and a "preservation" lemma. Once these two hurdles are overcome we will be able to generate models with n Woodin cardinals for each $n < \omega$. We shall then have to take extra steps to ensure that we can preserve ω-many Woodin cardinals. This will be achieved by shooting a Prikry sequence through the Y-degrees and proving an associated "generic preservation" lemma.

6.15 Remark. It is important to note that in contrast to Theorem 5.42 here the degree notion in Theorem 5.43 does not depend on a and this is instrumental in iteratively applying the theorem to generate several Woodin cardinals. In contexts such as $L(\mathbb{R})$ where HOD "relativizes" in the sense that $\mathrm{HOD}_a = \mathrm{HOD}[a]$, one could also appeal to Theorem 5.42, since in such a case $\mathrm{HOD}_{Y,a,[x]_{Y,a}} = \mathrm{HOD}_{Y,a,[x]_Y}$. Our reason for not taking this approach is twofold. First, it would take us too far afield to give the argument that $\mathrm{HOD}_a = \mathrm{HOD}[a]$ in, for example, $L(\mathbb{R})$. Second, it is of independent interest to work in a more general setting.

We shall be working with the "Y-degrees"

$$\mathscr{D}_Y = \{[x]_Y \mid x \in \omega^\omega\}.$$

Let μ_Y be the cone filter over $\mathscr{D}_Y$. As noted earlier, the argument of Theorem 2.9 shows that μ_Y is an ultrafilter. Also, by Theorem 2.8 we know that μ_Y is countably complete.

6.16 Lemma (Preservation Lemma). *Assume* $\mathrm{ZF} + \mathrm{AD}$. *Suppose Y is a set, $a \in H(\omega_1)$, and $\alpha < \omega_1$. Then for a Y-cone of x,*

$$\mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a,[x]_Y} = \mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a}.$$

Proof. The right-to-left direction is immediate. Suppose for contradiction that the left-to-right direction fails. For sufficiently large $[x]_Y$, let

$$f([x]_Y) = \text{ least } Z \in \mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a,[x]_Y} \smallsetminus \mathrm{HOD}_{Y,a},$$

where the ordering is the canonical ordering of $\mathrm{OD}_{Y,a,[x]_Y}$. This function is defined for a Y-cone of x and it is $\mathrm{OD}_{Y,a}$. Let $Z_0 \in \mathscr{P}(\alpha)$ be such that

$$\xi \in Z_0 \quad \text{iff} \quad \xi \in f([x]_Y) \text{ for a } Y\text{-cone of } x.$$

Since α is countable and since μ_Y is countably complete $Z_0 = f([x]_Y)$ for sufficiently large x. Thus, $Z_0 \in \mathrm{HOD}_{Y,a}$, which is a contradiction. ⊣

We are now in a position to iteratively apply Theorem 5.43 to generate a model with n Woodin cardinals.

Step 0. By Theorem 5.43, let x_0 be such that for all $x \geqslant_Y x_0$,

$$\mathrm{HOD}_{Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x}} \text{ is a Woodin cardinal.}$$

Step 1. Recall that ω_1^V is strongly inaccessible in any inner model of ZFC, by the Claim of Theorem 5.9. It follows that $\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_1^V$ and so when we choose $x_1 \geqslant_Y x_0$ we may assume that x_1 codes $\omega_2^{\mathrm{HOD}_{Y,x_0}}$. Thus, there exists an $x_1 \geqslant_Y x_0$ such that for all $x \geqslant_Y x_1$,

$$\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}},$$

and, by the Preservation Lemma (taking a to be $[x_0]_Y$),

$$\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} = \mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y},$$

and, by Theorem 5.9 (taking a to be $[x_0]_Y$),

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ is a Woodin cardinal.}$$

It follows that

$$\mathrm{HOD}_{Y,[x_0]_Y,[x_1]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x_1}} \text{ are Woodin cardinals.}$$

Step $n+1$. It is useful at this stage to introduce a piece of notation: For $x_0 \leqslant_Y \cdots \leqslant_Y x_{n+1}$, let

$$\delta_0(x_0) = \omega_2^{\mathrm{HOD}_{Y,x_0}}$$

and

$$\delta_{n+1}(x_0, \ldots, x_{n+1}) = \omega_2^{\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle, x_{n+1}}}.$$

Suppose that we have chosen $x_0 \leqslant_Y x_1 \leqslant_Y \cdots \leqslant_Y x_n$ such that

$$\begin{aligned}&\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle} \models \delta_0(x_0) < \cdots < \delta_n(x_0, \ldots, x_n)\\ &\quad \text{are Woodin cardinals.}\end{aligned}$$

Again, since ω_1^V is strongly inaccessible in any inner model of ZFC, it follows that each of these ordinals is countable in V and so when we choose $x_{n+1} \geqslant_Y x_n$ we may assume that x_{n+1} collapses these ordinals. Thus, there exists an $x_{n+1} \geqslant_Y x_n$ such that for all $x \geqslant_Y x_{n+1}$,

$$\delta_n(x_0, \ldots, x_n) < \delta_{n+1}(x_0, \ldots, x_n, x)$$

and, by the Preservation Lemma (taking a to be $\langle [x_0]_Y, \ldots, [x_n]_Y\rangle$),

$$\begin{aligned}&\mathscr{P}(\delta_n(x_0, \ldots, x_n)) \cap \mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y}\\ &\quad = \mathscr{P}(\delta_n(x_0, \ldots, x_n)) \cap \mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle}\end{aligned}$$

and, by Theorem 5.43 (taking a to be $\langle [x_0]_Y, \ldots, [x_n]_Y\rangle$),

$$\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y} \models \delta_{n+1}(x_0, \ldots, x_n, x) \text{ is a Woodin cardinal.}$$

It follows that

$$\begin{aligned}&\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y} \models \delta_0(x_0) < \cdots < \delta_{n+1}(x_0, \ldots, x_n, x)\\ &\quad \text{are Woodin cardinals.}\end{aligned}$$

We now need to ensure that when we do the above stacking for ω-many stages, the Woodin cardinals $\delta_n(x_0, \ldots, x_n)$ are preserved in the final model. This is not immediate since, for example, if we are not careful then the reals $x_0, x_1, \ldots$ might code up a real that collapses $\sup_{n<\omega} \delta_n(x_0, \ldots, x_n)$. To circumvent this difficulty we implement the construction relative to a "Prikry sequence" of degrees $[x_0]_Y, [x_1]_Y, \ldots$.

6.17 Definition (The forcing $\mathbb{P}_Y$). Assume ZF + AD. Suppose Y is a set. Let $\mathscr{D}_Y$ and μ_Y be as above. The conditions of $\mathbb{P}_Y$ are of the form $\langle [x_0]_Y, \ldots, [x_n]_Y, F\rangle$ where $F : \mathscr{D}_Y^{<\omega} \to \mu_Y$. The ordering on $\mathbb{P}_Y$ is:

$$\langle [x_0]_Y, \ldots, [x_n]_Y, [x_{n+1}]_Y, \ldots, [x_m]_Y, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle [x_0]_Y, \ldots, [x_n]_Y, F\rangle$$

if and only if

(1) $[x_{i+1}]_Y \in F(\langle [x_0]_Y, \ldots, [x_i]_Y\rangle)$ for all $i \geqslant n$ and

(2) $F^*(p) \subseteq F(p)$ for all $p \in \mathscr{D}_Y^{<\omega}$.

The point of the following lemma is to avoid appeal to DC.

6.18 Lemma. *Assume* ZF + AD. *Suppose φ is a formula in the forcing language and $\langle p, F\rangle \in \mathbb{P}_Y$. Then there is an F^* such that $\langle p, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle$ and $\langle p, F^*\rangle$ decides φ. Moreover, F^* is uniformly definable from $\langle p, F\rangle$ and φ.*

Proof. Fix φ a formula and $\langle p, F\rangle \in \mathbb{P}_Y$. Let us use '$p$' and '$q$' for "lower parts" of conditions—that is, finite sequences of $\mathscr{D}_Y$—'F' and 'G' for the corresponding "upper parts", and 'a' for elements of $\mathscr{D}_Y$. Write $q \trianglerighteq p$ to indicate that p is an initial segment of q. Set

$$\begin{aligned} Z_0 &= \{\, q \mid q \trianglerighteq p \text{ and } \exists G\ \langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle \text{ and } \langle q, G\rangle \Vdash \varphi \,\},\\ Z_{\alpha+1} &= \{\, q \mid \{\, a \mid q^\frown a \in Z_\alpha \,\} \in \mu_Y \,\}, \quad \text{and}\\ Z_\lambda &= \textstyle\bigcup_{\alpha<\lambda} Z_\alpha \quad \text{for } \lambda \text{ a limit.} \end{aligned}$$

Let $D_q^\alpha = \{a \mid q^\frown a \in Z_\alpha\}$. So $Z_{\alpha+1} = \{q \mid D_q^\alpha \in \mu_Y\}$. We claim that for each α,

(1) if $q \in Z_\alpha$, then $D_q^\alpha \in \mu_Y$, and hence

(2) $Z_a \subseteq Z_{\alpha+1}$.

The proof is by induction on α: For $\alpha = 0$ suppose $q \in Z_0$ and let G witness this. So $G(q) \in \mu_Y$. Notice that for each $a \in G(q)$,

$$\langle q^\frown a, G\rangle \leqslant_{\mathbb{P}_Y} \langle q, G\rangle$$

and so $q^\frown a \in Z_0$, i.e. $G(q) \subseteq \{a \mid q^\frown a \in Z_0\} = D_q^0$ and so $D_q^0 \in \mu_Y$ and $Z_0 \subseteq Z_1$. Assume (1) holds for $\alpha + 1$. It follows that $Z_{\alpha+1} \subseteq Z_{\alpha+2}$. Suppose $q \in Z_{\alpha+2}$. Then, by the definition of $Z_{\alpha+2}$, $D_q^{\alpha+1} \in \mu_Y$. However, since $Z_{\alpha+1} \subseteq Z_{\alpha+2}$, it follows that $D_q^{\alpha+1} \subseteq D_q^{\alpha+2}$. So $D_q^{\alpha+2} \in \mu_Y$. For λ a limit ordinal suppose $q \in Z_\lambda$. Then $q \in Z_\alpha$ for some $\alpha < \lambda$. So, by the induction hypothesis, $D_q^\alpha \in \mu_Y$. Since $Z_\alpha \subseteq Z_\lambda$, $D_q^\alpha \subseteq D_q^\lambda$ and so $D_q^\lambda \in \mu_Y$.

Now define a ranking function $\rho : \mathscr{D}_Y^{<\omega} \to \mathrm{On} \cup \{\infty\}$ by

$$\rho(q) = \begin{cases} \text{least } \alpha \text{ such that } q \in Z_\alpha & \text{if there is such an } \alpha\\ \infty & \text{otherwise.} \end{cases}$$

We begin by noting the following three persistence properties which will aid us in shrinking F so as to decide φ. First, if $\rho(q) = \infty$ then set

$$A_q = \{a \mid \rho(q^\frown a) = \infty\}$$

and notice that $A_q \in \mu_Y$ since otherwise we would have $\{a \mid \rho(q^\frown a) \in \mathrm{On}\} \in \mu_Y$ (as μ_Y is an ultrafilter) and letting $\beta = \sup\{\rho(q^\frown a) \mid \rho(q^\frown a) \in \mathrm{On}\}$ we would have $q \in Z_{\beta+1}$, a contradiction. Second, if $\rho(q) \in \mathrm{On} \smallsetminus \{0\}$ then set

$$B_q = \{a \mid \rho(q^\frown a) < \rho(q)\}$$

and notice that $B_q \in \mu_Y$ since $\rho(q)$ is clearly a successor, say $\alpha + 1$, and $q \in Z_\alpha$ and so (by our claim) $D_q^\alpha \in \mu_Y$; but $D_q^\alpha \subseteq B_q$. Third, if $\rho(q) = 0$ then set

$$C_q = \{a \mid \rho(q^\frown a) = 0\}$$

and notice that $C_q \in \mu_Y$ since clearly $q \in Z_0$ and so, by the claim, $D_q^0 \in \mu_Y$; but $D_q^0 \subseteq C_q$.

Now either $\rho(p) = \infty$ or $\rho(p) \in \mathrm{On}$.

Claim 1. *If $\rho(p) = \infty$ then there is an F^* such that $\langle p, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle$ and $\langle p, F^*\rangle \Vdash \neg\varphi$.*

Proof. Define F^* as follows:

$$F^*(q) = \begin{cases} F(q) \cap A_q & \text{if } \rho(q) = \infty \\ F(q) & \text{otherwise.} \end{cases}$$

Suppose that it is not the case that $\langle p, F^*\rangle \Vdash \neg\varphi$. Then $\exists \langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F^*\rangle$ such that $\langle q, G\rangle \Vdash \varphi$. But then q is such that $\rho(q) = 0$. However, F^* witnesses that in fact $\rho(q) = \infty$: Suppose $q = p^\frown a_0{}^\frown \cdots {}^\frown a_k$. Since $\rho(p) = \infty$ and $a_0 \in F^*(p)$, we have that $a_0 \in A_p$ and so $\rho(p^\frown a_0) = \infty$. Continuing in this manner, we get that $\rho(q) = \infty$. This is a contradiction. ⊣

Claim 2. *If $\rho(p) \in \mathrm{On}$ then there is an F^* such that $\langle p, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle$ and $\langle p, F^*\rangle \Vdash \varphi$.*

Proof. Define F^* as follows:

$$F^*(q) = \begin{cases} F(q) \cap B_q & \text{if } \rho(q) \in \mathrm{On} \smallsetminus \{0\} \\ F(q) \cap C_q & \text{if } \rho(q) = 0 \\ F(q) & \text{otherwise.} \end{cases}$$

We claim that $\langle p, F^*\rangle \Vdash \varphi$. Assume not. Then $\exists \langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F^*\rangle$ such that $\langle q, G\rangle \Vdash \neg\varphi$. Since $\langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F^*\rangle$ and $\rho(p) \in \mathrm{On}$ we have that $\rho(q) \in \mathrm{On}$ (by an easy induction using the definition of F^*). We may assume that $\langle q, G\rangle$ is chosen so that $\rho(q)$ is as small as possible. But then $\rho(q) = 0$ as otherwise there is an a such $\langle q^\frown a, G\rangle \leqslant_{\mathbb{P}_Y} \langle q, G\rangle$, $\langle q^\frown a, G\rangle \Vdash \neg\varphi$ and $\rho(q^\frown a) < \rho(q)$, contradicting the minimality of $\rho(q)$. Now since $\rho(q) = 0$, $q \in Z_0$ and so $\exists G' \langle q, G'\rangle \Vdash \varphi$. But this is a contradiction since $\langle q, G'\rangle$ is compatible with $\langle q, G\rangle$ and $\langle q, G\rangle \Vdash \neg\varphi$. ⊣

This completes the proof of the lemma. $\dashv$

We can now obtain the following "generic preservation" lemma.

6.19 Lemma (Generic Preservation Lemma). *There exists an F such that if $G \subseteq \mathbb{P}_Y$ is V-generic and $\langle \varnothing, F\rangle \in G$ and $\langle [x_i]_Y \mid i < \omega\rangle$ is the generic sequence associated to G, then, for all $i < \omega$,*

$$\begin{aligned}\mathscr{P}(\delta_i(x_0, \ldots, x_i))^{V[G]} &\cap \mathrm{HOD}^{V[G]}_{Y,\langle [x_j]_Y \mid j<\omega\rangle, V} \\ &= \mathscr{P}(\delta_i(x_0, \ldots, x_i)) \cap \mathrm{HOD}^{V}_{Y,\langle [x_0]_Y, \ldots, [x_i]_Y\rangle}.\end{aligned}$$

Proof. We need the following extension of Lemma 6.18: Suppose $\langle \varphi_\xi \mid \xi < \alpha\rangle$ is a countable sequence of formulas in the forcing language (evaluated in a rank initial segment) and $\langle p, F\rangle \in \mathbb{P}_Y$ is a condition. Then there is an F^* such that $\langle p, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle$ and $\langle p, F^*\rangle$ decides φ_ξ, for each $\xi < \alpha$, and F^* is uniformly definable from $\langle \varphi_\xi \mid \xi < \alpha\rangle$ and $\langle p, F\rangle$. For each $\xi < \alpha$, let F_ξ be as in Lemma 6.18 (where it is denoted F^*). Letting F^* be the "intersection" of the F_α—i.e., such that $F^*(q) = \bigcap_{\alpha<\beta} F_\alpha(q)$ for each $q \in D^{<\omega}_Y$—we have that $\langle p, F^*\rangle$ decides φ_ξ for each $\xi < \alpha$ and that F^* is uniformly definable from $\langle \varphi_\xi \mid \xi < \alpha\rangle$ and $\langle p, F\rangle$.

Suppose $a \in H(\omega_1)$ and $\alpha < \omega_1$. We claim that we can definably associate with a and α a function $F_{a,\alpha} : \mathscr{D}^{<\omega}_Y \to \mu_Y$ such that $\langle \varnothing, F_{a,\alpha}\rangle$ forces

$$\mathscr{P}(\alpha)^{V[G]} \cap \mathrm{HOD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle, V} = \mathscr{P}(\alpha) \cap \mathrm{HOD}^{V}_{Y,a}.$$

Let φ be the formula in the forcing language that expresses the displayed statement. By Lemma 6.18 there is an $\mathrm{OD}_{Y,a}$ condition $\langle \varnothing, G\rangle$ deciding φ. Suppose for contradiction that this condition forces $\neg\varphi$. Since right-to-left inclusion holds trivially (as we are including V as a parameter) it must be that the left-to-right inclusion fails. Let $A \subseteq \alpha$ be $<\mathrm{OD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle, V}$-least such that

$$A \in \mathrm{HOD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle, V} \smallsetminus \mathrm{HOD}^{V}_{Y,a}.$$

Now, for each $\xi < \alpha$ let φ_ξ be the statement expressing "$\xi \in A$". In an $\mathrm{OD}_{Y,a}$ fashion we can successively shrink $\langle \varnothing, G\rangle$ to decide each φ_ξ. But then A is $\mathrm{OD}_{Y,a}$ and hence in $\mathrm{HOD}^V_{Y,a}$, which is a contradiction.

We now define a "master function" $F : \mathscr{D}^{<\omega}_Y \to \mu_Y$ such that for all $\langle [x_0]_Y, \ldots, [x_n]_Y\rangle \in \mathscr{D}^{<\omega}_Y$,

$$F(\langle [x_0]_Y, \ldots, [x_n]_Y\rangle) = F_{\langle [x_0]_Y\rangle, \delta_0(x_0)} \cap \cdots \cap F_{\langle [x_0]_Y, \ldots, [x_n]_Y\rangle, \delta_n(x_0, \ldots, x_n)}.$$

Suppose $\langle \varnothing, F\rangle \in G$. Suppose $\langle p, H\rangle \in G$ and $p{\restriction} i+1 = \langle [x_0]_Y, \ldots, [x_i]_Y\rangle$. It follows that $\langle p, H \wedge F\rangle \in G$, where, by definition, $H \wedge F$ is such that $(H \wedge F)(q) = H(q) \cap F(q)$ for each q. But

$$\begin{aligned}\langle p, H \wedge F\rangle &\leqslant_{\mathbb{P}_Y} \langle p{\restriction} i+1, H \wedge F\rangle \\ &\leqslant_{\mathbb{P}_Y} \big\langle p{\restriction} i+1, H \wedge F_{p\restriction i+1, \delta_i(x_0, \ldots, x_i)}\big\rangle \\ &\leqslant_{\mathbb{P}_Y} \big\langle \varnothing, F_{p\restriction i+1, \delta_i(x_0, \ldots, x_i)}\big\rangle,\end{aligned}$$

(using the definition of F for the second line) and so $\langle \varnothing, F_{p\restriction i+1,\delta_i(x_0,\ldots,x_i)}\rangle \in G$. Finally, by the definition of $F_{p\restriction i+1,\delta_i(x_0,\ldots,x_i)}$, this condition forces

$$\begin{aligned}\mathscr{P}(\delta_i(x_0,\ldots,x_i))^{V[G]} &\cap \mathrm{HOD}^{V[G]}_{Y,\langle [x_0]_Y,\ldots,[x_i]_Y\rangle,\langle [x_i]_Y \mid i<\omega\rangle,V}\\ &= \mathscr{P}(\delta_i(x_0,\ldots,x_i)) \cap \mathrm{HOD}^{V}_{Y,\langle [x_0]_Y,\ldots,[x_i]_Y\rangle},\end{aligned}$$

which completes the proof. ⊣

6.20 Theorem. *Assume* ZF + AD. *Then there is a condition* $\langle \varnothing, F\rangle \in \mathbb{P}_Y$ *such that if* $G \subseteq \mathbb{P}_Y$ *is* V*-generic and* $\langle \varnothing, F\rangle \in G$*, then*

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle,V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals,}$$

where $\langle [x_i]_Y \mid i<\omega\rangle$ *is the sequence associated with* G.

Proof. Let $\langle \varnothing, F, \rangle$ be the condition from the Generic Preservation Lemma (Lemma 6.19). We claim that

$$\begin{aligned}\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle,V} &\models \delta_0(x_0) < \cdots < \delta_n(x_0,\ldots,x_n) < \cdots\\ &\text{are Woodin cardinals.}\end{aligned}$$

By the Generic Preservation Lemma it suffices to show that for each $n < \omega$

$$\begin{aligned}\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle} &\models \delta_0(x_0) < \cdots < \delta_n(x_0,\ldots,x_n)\\ &\text{are Woodin cardinals,}\end{aligned}$$

which follows by genericity and the argument for the finite case. ⊣

As an interesting application of this theorem in conjunction with the Derived Model Theorem (Theorem 8.12), we obtain Kechris' theorem that under ZF + AD, DC holds in $L(\mathbb{R})$. This alternate proof is of interest since it is entirely free of fine structure and it easily generalizes.

6.21 Theorem (Kechris). *Assume* ZF + AD. *Then* $L(\mathbb{R}) \models \mathrm{DC}$.

Proof Sketch. Work in ZF + AD + $V = L(\mathbb{R})$. Let $Y = \varnothing$ and let $N = \mathrm{HOD}^{V[G]}_{\langle [x_i]_Y \mid i<\omega\rangle,V}$ where G and $[x_i]_Y$ are as in the above theorem. By genericity, the Woodin cardinals δ_i of N have ω_1^V as their supremum. By Vopěnka's theorem (see the proof of Theorem 7.8 below for the statement and a sketch of the proof), each $x \in \mathbb{R}^V$ is N-generic for some $\mathbb{P} \in N \cap V_{\omega_1^V}$. Thus, $N(\mathbb{R}^V)$ is a symmetric extension of N. The derived model of $N(\mathbb{R}^V)$ (see Theorem 8.12 below) satisfies $\mathrm{DC}_{\mathbb{R}}$ and therefore DC since $N \models \mathrm{AC}$. Furthermore $N(\mathbb{R}^V)$ contains $L(\mathbb{R})$ and cannot contain more since then $L(\mathbb{R})$ would have forced its own sharp. (This follows from AD^+ theory: Assume $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD} + V = L(\mathscr{P}(\mathbb{R}))$. Suppose $A \subseteq \mathbb{R}$. Then either $V = L(A,\mathbb{R})$ or $A^{\#}$ exists. See Definition 8.10 below.) Thus, $L(\mathbb{R}) \models \mathrm{AD} + \mathrm{DC}$. ⊣

7. Second-Order Arithmetic

The statement that all Δ^1_2 sets are determined is really a statement of second-order arithmetic. So a natural question is whether the construction culminating in Sect. 6.1 can be implemented in this more limited setting. In this section we show that a variant of the construction can be carried out in this context. We break the construction into two steps. First, we show that a variant of the above construction can be carried out with respect to an object smaller than $\omega_2^{L[x]}$, one that is within the reach of second-order arithmetic. Second, we show that this version of the construction can be carried out in the weaker theory of second-order arithmetic.

The need to alter the previous construction is made manifest in the following result:

7.1 Theorem. *Assume* $\mathrm{ZF} + V = L[x] + \Delta^1_2$*-determinacy, for some* $x \in \omega^\omega$*. Suppose* N *is such that*

(1) $\mathrm{On} \subseteq N \subseteq \mathrm{HOD}^{L[x]}$ *and*

(2) $N \models \delta$ *is a Woodin cardinal.*

Then $\delta \geqslant \omega_2^{L[x]}$.

However, it turns out that $\omega_1^{L[x]}$ can be a Woodin cardinal in an inner model that overspills $\mathrm{HOD}^{L[x]}$.

7.2 Theorem. *Assume* $\mathrm{ZF} + V = L[x] + \Delta^1_2$*-determinacy, for some* $x \in \omega^\omega$*. Then there exists an* $N \subseteq L[x]$ *such that*

$$N \models \mathrm{ZFC} + \omega_1^{L[x]} \textit{ is a Woodin cardinal.}$$

Moreover, this result is optimal.

7.3 Theorem. *Assume* $\mathrm{ZF} + \Delta^1_2$*-determinacy. Then there is a real* x *such that*

(1) $L[x] \models \Delta^1_2$*-determinacy, and*

(2) for all $\alpha < \omega_1^{L[x]}$*,* α *is not a Woodin cardinal in any inner model* N *such that* $\mathrm{On} \subseteq N$*.*

In Sect. 7.1 we prove Theorem 7.2. More precisely, we prove the following:

7.4 Theorem. *Assume* $\mathrm{ZF} + \mathrm{DC} + \Delta^1_2$*-determinacy. Then for a Turing cone of* x*,*

$$\mathrm{HOD}^{L[x]}_{[x]_T} \models \mathrm{ZFC} + \omega_1^{L[x]} \textit{ is a Woodin cardinal.}$$

This involves relativizing the previous construction to the Turing degree of x, replacing the notions that concerned reals (for example, winning strategies) with relativized analogues that concern only those reals in the Turing degree of x.

In Sect. 7.2 we show that the relativized construction goes through in the setting of second-order arithmetic.

7.5 Theorem. *Assume that* $\mathrm{PA}_2 + \Delta^1_2$*-determinacy is consistent. Then* $\mathrm{ZFC} +$ *"*On *is Woodin" is consistent.*

Here PA_2 is the standard axiomatization of second-order arithmetic (without AC). The statement that On is Woodin is to be understood schematically. Alternatively, one could work with the conservative extension GBC of ZFC and the analogous conservative extension of PA_2. This would enable one to fuse the schema expressing that On is Woodin into a single statement.

7.1. First Localization

To prove Theorem 7.4 we have to prove an analogue of the Generation Theorem where ω_2 is replaced by ω_1. The two main steps are (1) getting a suitable notion of strategic determinacy and (2) getting definable prewellorderings for all ordinals less than ω_1.

For $x \in \omega^\omega$ we "relativize" our previous notions to the Turing degree of x. The *relativized reals* are $R_x = \{y \in \omega^\omega \mid y \leqslant_T x\}$. Fix $A \subseteq R_x$. A *relativized strategy for I* is a function $\sigma : \bigcup_{n<\omega} \omega^{2n} \to \omega$ such that $\sigma \in R_x$. A relativized strategy σ for I is *winning in A* iff for all $y \in R_x$, $\sigma * y \in A$. The corresponding notions for II are defined similarly. A *relativized prestrategy* is a continuous function f such that (the code for) f is in R_x and for all $y \in R_x$, $f(y)$ is a relativized strategy for either I or II. We say that a relativized prestrategy f is *winning in A for I (II) with respect to $B \subseteq R_x$* if in addition we have that for all $y \in B$, $f(y)$ is a relativized winning strategy for I (II) in A. (In our present setting our basis B will always be $[x]_T$.) We say that a set $A \subseteq R_x$ is *determined in the relativized sense* if either I or II has a relativized winning strategy for A. Let OD-$[x]_T$-determinacy be the statement that for every $\mathrm{OD}_{[x]_T}$ subset of R_x either Player I or Player II has a relativized winning strategy.

The *strategic game relativized to* $[x]_T$ is the game SG-$[x]_T$

$$
\begin{array}{lccccccc}
\mathrm{I} & A_0 & & \cdots & & A_n & & \cdots \\
\mathrm{II} & & f_0 & & \cdots & & f_n & & \cdots
\end{array}
$$

where we require

(1) $A_0 \in \mathscr{P}(R_x) \cap \mathrm{OD}_{[x]_T}$, $A_{n+1} \in \mathscr{P}(R_x) \cap \mathrm{OD}_{[x]_T, f_0, \ldots, f_n}$ and

(2) f_n is a relativized prestrategy that is winning in A_n with respect to $[x]_T$,

and II wins if and only if II can play all ω rounds. We say that *strategic determinacy relativized to* $[x]_T$ *holds* (ST-$[x]_T$-*determinacy*) if II wins SG-$[x]_T$.

We caution the reader that in the context of the relativized notions we are dealing only with *definable* versions of relativized determinacy such as OD-$[x]_T$-determinacy and SG-$[x]_T$-determinacy. In fact, *full* relativized determinacy can never hold. But as we shall see both OD-$[x]_T$-determinacy and SG-$[x]_T$-determinacy can hold.

7.6 Theorem. *Assume* ZF + DC + Δ^1_2*-determinacy. Let* T *be the theory* ZFC − Replacement + Σ_2-Replacement. *There is a real* x_0 *such that for all reals* x *and for all ordinals* λ *if* $x_0 \in L_\lambda[x]$ *and* $L_\lambda[x] \models$ T, *then* $L_\lambda[x] \models$ OD-$[x]_T$*-determinacy.*

Proof. The proof is similar to that of Theorem 6.6. Assume for contradiction that for every real x_0 there is an ordinal λ and a real x such that $x_0 \in L_\lambda[x]$ and $L_\lambda[x] \models \mathrm{T} + \neg$OD-$[x]_T$-determinacy, where T = ZFC − Replacement + Σ_2-Replacement. As before, by the Löwenheim-Skolem theorem and Σ^1_2-determinacy the ordinal

$$\lambda(x) = \begin{cases} \mu\lambda\,(L_\lambda[x] \models \mathrm{T} + \neg\text{OD-}[x]_T\text{-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

is defined for a Turing cone of x. For each x such that $\lambda(x)$ is defined, let $A^x \subseteq R_x$ be the $(\mathrm{OD}_{[x]_T})^{L_{\lambda(x)}[x]}$-least counterexample.

Consider the game

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d\rangle$, I wins iff $\lambda(p)$ is defined and $L_{\lambda(p)}[p] \models$ "$a * d \in A^p$", where a and d can be thought of as strategies. This game is Σ^1_2, hence determined.

We arrive at a contradiction by showing that neither player can win.

Case 1: I has a winning strategy σ_0.

Let $x_0 \geqslant_T \sigma_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined. We claim that $L_{\lambda(x_0)}[x_0] \models$ "I has a relativized winning strategy σ in A^{x_0}", which is a contradiction. The relativized strategy σ is derived as follows: Given $d\restriction n \in \omega^n$ have II play $x_0\restriction n, d\restriction n$ in the main game. Let $a\restriction n, b\restriction n$ be σ_0's response along the way and let $a(n)$ be σ_0's next move. Then set $\sigma(d\restriction n) = a(n)$. (Clearly, σ is continuous, and the real $a = \sigma(d)$ it defines is to be thought of as coding a strategy for Player I.) This strategy σ is clearly recursive in σ_0, hence it is a relativized strategy.

It remains to show that for every $d \in R_{x_0}$, $\sigma * d \in A^{x_0}$. The point is that for $d \in R_{x_0}$, $p \equiv_T x_0$, where $p = \langle a, b, x_0, d\rangle$ is the play obtained by letting $\langle a, b\rangle = (\sigma_0 * \langle x_0, d\rangle)_I$. It follows that $\lambda(p) = \lambda(x_0)$ and hence $L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0]$ and $A^p = A^{x_0}$. Thus, $L_{\lambda(x_0)}[x_0] \models$ "$\sigma(d) * d \in A^{x_0}$". So $L_{\lambda(x_0)}[x_0] \models$ "σ is a relativized winning strategy for I in A^{x_0}".

Case 2: II has a winning strategy τ_0.

Let $x_0 \geqslant_T \tau_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. Given $a\restriction(n+1) \in \omega^{n+1}$ have I play $a\restriction(n+1), x_0\restriction(n+1)$ in the main game. Let $c\restriction n, d\restriction n$ be τ_0's response along the way. Then set $\tau(a\restriction n) = d(n)$. This strategy is clearly recursive in τ_0, hence it is a relativized strategy, and, as above, $L_{\lambda(x_0)}[x_0] \models$ "τ is a relativized winning strategy for II in A^{x_0}". $\dashv$

7.7 Theorem. *Assume* ZF + DC + Δ^1_2*-determinacy. Then for a Turing cone of* x*,*

$$L[x] \models \text{ST-}[x]_T\text{-}\textit{determinacy.}$$

Proof. The proof is a straightforward variant of the proof of Theorem 6.9. In fact it is simpler. We note the major changes.

As before we assume that $V = L[x]$ and show that there is a real z_0 with the feature that if $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$, then $L_\lambda[z] \models \text{ST-}[x]_T$-determinacy.

Assume for contradiction that this fails. For $z \in \omega^\omega$, let

$$\lambda(z) = \begin{cases} \mu\lambda\,(L_\lambda[z] \models \mathrm{T} + \neg\text{ST-}[x]_T\text{-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The following is immediate.

Claim 1. *For a Turing cone of* z*,* $\lambda(z)$ *is defined.*

For each z in the cone of Claim 1 Player I has a canonical strategy σ^z that depends only on the Turing degree of z, the point being that if $y \equiv_T z$ then $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$.

As before our aim is to obtain a contradiction by defeating σ^z for some z in the Turing cone of Claim 1. We do this by constructing a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that I must win via $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ and, for a cone of z, the winning strategies give rise to prestrategies $f^z_0, f^z_1, \ldots, f^z_n, \ldots$ that constitute a non-losing play against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Step 0. Consider (in $L[x]$) the game G_0

$$\begin{array}{lll} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is either 1 or 2 and, letting $p = \langle a, b, c, d\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A^p_0$", where $A^p_0 = \sigma^p(\varnothing)$.

Claim 2. *I has a winning strategy* σ_0 *in* G_0*.*

Proof. Assume for contradiction that I does not have a winning strategy in G_0. Then, by Σ^1_2-determinacy, II has a winning strategy τ_0 in G_0. Let $z_0 \geqslant_T \tau_0$ be such that for all $z \geqslant_T z_0$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models$ OD-$[x]_T$-determinacy (by Theorem 7.6).

Consider $A_0^{z_0} = \sigma^{z_0}(\varnothing)$. Since $L_{\lambda(z_0)}[z_0] \models$ OD-$[x]_T$-determinacy, assume without loss of generality that $L_{\lambda(z_0)}[z_0] \models$ "σ is a relativized winning strategy for I in $A_0^{z_0}$". We use τ_0 to defeat this relativized strategy. Run G_0 according to τ_0, having Player I (falsely) predict that Player I wins the auxiliary game, while steering into $L_{\lambda(z_0)}[z_0]$ by playing $b = z_0$ and using σ to respond to τ_0 on the auxiliary play:

$$\begin{array}{lll} \mathrm{I} & 1 & (\sigma * d)_I, z_0 \\ \mathrm{II} & & c, d \end{array}$$

The point is that $p \equiv_T z_0$ (since $\sigma, \tau_0 \in R_{z_0}$) and so $\lambda(p) = \lambda(z_0)$. Thus the "steering problem" is immediately solved and we have a contradiction as before. ⊣

Since the game is Σ_2^1 for Player I, Player I has a Δ_3^1 strategy σ_0, by Theorem 6.5.

Claim 3. *For every real $z \geqslant_T \sigma_0$ there is a prestrategy f_0^z such that f_0^z is recursive in σ_0 as in Claim 1 and f_0^z is a non-losing first move for II against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.*

Proof. Fix $z \geqslant_T \sigma_0$ as in Claim 1 and consider $A_0^z = \sigma^z(\varnothing)$. Let f_0^z be the prestrategy derived from σ_0 as follows: Given $y{\upharpoonright}n$ and $d{\upharpoonright}n$ have II play $y{\upharpoonright}n, d{\upharpoonright}n$ in G_0. Let $\epsilon, a{\upharpoonright}n, b{\upharpoonright}n$ be σ_0's response along the way and let $a(n)$ be σ_0's next move. Then let $f_0^z(y{\upharpoonright}n) = a(n)$. We have that f_0^z is recursive in $\sigma_0 \leqslant_T z$ and for $y \in [z]_T$, $f_0^z(y) \in R_z$. It remains to see that for $y \in [z]_T$, $f_0^z(y)$ is a relativized winning strategy for I in A_0^z. The point is that since $y \in [z]_T$, $\lambda(y) = \lambda(z)$ and so $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and $A_0^y = A_0^z$. For $d \in R_z$, by definition $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. So, letting $p = \langle a, b, y, d\rangle$ we have $p \equiv_T y$. Thus, $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f_0^z(y) * d \in A_0^z$". ⊣

Step $n+1$. Assume that we have defined (in $L[x]$) games $G_0, \ldots, G_n$ with winning strategies $\sigma_0, \ldots, \sigma_n \in \mathrm{HOD}$ such that for all $z \geqslant_T \langle \sigma_0, \ldots, \sigma_n\rangle$ as in Claim 1 there are prestrategies $f_0^z, \ldots, f_n^z$ such that f_i^z is recursive in $\langle \sigma_0, \ldots, \sigma_i\rangle$ (for all $i \leqslant n$) and $f_0^z, \ldots f_n^z$ is a non-losing partial play for II in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Consider (in $L[x]$) the game G_{n+1}

$$\begin{array}{lll} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is 1 or 2 and, letting $p = \langle a, b, c, d, \sigma_0, \ldots, \sigma_n\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A^p_{n+1}$", where A^p_{n+1} is I's response via σ^p to II's partial play $f^p_0, \dots, f^p_n$.

This game is $\Sigma^1_2(\sigma_0, \dots, \sigma_n)$ (for Player I) and hence determined (since $\sigma_0, \dots, \sigma_n \in \mathrm{HOD}$ and we have OD-determinacy).

Claim 4. *I has a winning strategy σ_{n+1} in G_{n+1}.*

Proof. The proof is as before, only now we use the relativized version of Theorem 7.6 to enforce $\mathrm{OD}_{\sigma_0,\dots,\sigma_n}$-$[x]_T$-determinacy. ⊣

Since the game is $\Sigma^1_2(\sigma_0, \dots, \sigma_n)$ for Player I, Player I has a $\Delta^1_3(\sigma_0, \dots, \sigma_n)$ strategy σ_{n+1}, by the relativized version of Theorem 6.5.

Claim 5. *For every real $z \geqslant_T \langle \sigma_0, \dots, \sigma_n \rangle$ there is a prestrategy f^z_{n+1} such that f^z_{n+1} is recursive in $\langle \sigma_0, \dots, \sigma_{n+1} \rangle$ and $f^z_0, \dots, f^z_{n+1}$ is a non-losing first move for II against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.*

Proof. The proof is just like the proof of Claim 3. ⊣

Finally, letting z^∞ as in Claim 1 be such that $z^\infty \geqslant_T z_n$ for all n we have that $f^{z^\infty}_0, \dots, f^{z^\infty}_n, \dots$ defeats σ^{z^∞} in $(SG\text{-}[x]_T)^{L_{\lambda(z^\infty)}[z^\infty]}$, which is a contradiction. ⊣

7.8 Theorem. *Assume* ZF + DC. *Then for every $x \in \omega^\omega$ and for every $\alpha < \omega_1^{L[x]}$ there is an $\mathrm{OD}_{[x]_T}$ surjection $\rho : [x]_T \to \alpha$.*

Proof. First we need to review Vopěnka's theorem. Work in $L[x]$ and let $d = [x]_T$. Let

$$\mathbb{B}'_d = \{A \subseteq d \mid A \in \mathrm{OD}_d\},$$

ordered under $\subseteq$. There is an OD_d isomorphism π between $(\mathbb{B}'_d, \subseteq)$ and a partial ordering $(\mathbb{B}_d, \leqslant)$ in HOD_d.

Claim 1. *$(\mathbb{B}_d, \leq)$ is complete in HOD_d and every real in d is HOD_d-generic for $\mathbb{B}_d$.*

Proof. For completeness consider $S \subseteq \mathbb{B}_d$ in HOD_d. We have to show that $\bigvee S$ exists. Let $S' = \pi^{-1}[S]$. Then $\bigvee S' = \bigcup S' \in \mathbb{B}'_d$ as this set is clearly OD_d. So $\bigvee S = \pi(\bigvee S')$.

Now consider $z \in d$. Let $G'_z = \{A \in \mathbb{B}'_d \mid z \in A\}$ and let $G_z = \pi[G'_z]$. We claim that G_z is HOD_d-generic for $\mathbb{B}_d$. Let $S \subseteq \mathbb{B}_d$ be a maximal antichain. So $\bigvee S = 1$. Let $S' = \pi^{-1}[S]$. Note $\bigvee S' = d$. Thus there exists a $b \in S$ such that $z \in \pi(b)$. So G_z is HOD_d-generic for $\mathbb{B}_d$. Now the map $f : \omega \to \mathbb{B}_d$, defined by $f(n) = \pi(\{x \in d \mid n \in x\})$, is in HOD_d. Moreover, $n \in z$ iff $f(n) \in G_z$. Thus $z \in \mathrm{HOD}_d[G_z]$. ⊣

Notice that $\mathrm{HOD}_d[G] = L[x]$ for every $G = G_z$ that is HOD_d-generic (where $z \in d$) since such a generic adds a real in $[x]_T$. Thus, if $\mathrm{HOD}_d \neq L[x]$ then $\mathbb{B}_d$ is non-trivial. This is a key difference between our present setting and that of Vopěnka's—in general our partial order does not have atoms.

If $L[x] = \mathrm{HOD}_d$ then clearly for each $\alpha < \omega_1^{L[x]}$ there exists a surjection $\rho : d \to \alpha$ such that $\rho \in \mathrm{OD}_d$. So we may assume that $L[x] \neq \mathrm{HOD}_d$. Thus, for every $z \in d$

$$\omega_1^{L[x]} = \omega_1^{\mathrm{HOD}_d[G_z]}.$$

Claim 2. *Assume* ZFC*. Suppose l is an uncountable regular cardinal, $\mathbb{B}$ is a complete Boolean algebra, and $V^{\mathbb{B}} \models \lambda = \omega_1$. Then for every $\alpha < \lambda$ there is an antichain in $\mathbb{B}$ of size $|\alpha|$.*

Proof. If λ is a limit cardinal then since $\mathbb{B}$ collapses all uncountable cardinals below λ it cannot be $\bar{\lambda}$-c.c. for any uncountable cardinal $\bar{\lambda} < \lambda$.

Suppose $\lambda = \bar{\lambda}^+$. We need to show that there is an antichain of size $\bar{\lambda}$. If $\bar{\lambda} > \omega$ then this is immediate since $\mathbb{B}$ collapses $\bar{\lambda}$ and so it cannot be $\bar{\lambda}$-c.c. So assume $\bar{\lambda} = \omega$. There must be an antichain of size ω since not every condition in $\mathbb{B}$ is above an atom. ⊣

Letting $\lambda = \omega_1^{L[x]}$, we are in the situation of the claim. So, for every $\alpha < \lambda$ there is an antichain S_α in $\mathbb{B}$ of size $|\alpha|$. Letting $S'_\alpha = \pi^{-1}[S_\alpha]$ we have that S'_α is an OD_d subset of $\mathbb{B}'_d$ consisting of pairwise disjoint OD_d subsets of d. Picking an element from each set we get an OD_d-surjection $\rho : d \to \alpha$. ⊣

7.9 Theorem. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone of x,*

$$\mathrm{HOD}^{L[x]}_{[x]_T} \models \omega_1^{L[x]} \textit{ is a Woodin cardinal.}$$

Proof. For a Turing cone of x, $L[x] \models \mathrm{OD}_{[x]_T}$-determinacy (by the relativized version of Theorem 6.6) and $L[x] \models \mathrm{ST}\text{-}[x]_T$-determinacy (by Theorem 7.7). Let x be in this cone and work in $L[x]$. Let $d = [x]_T$. Since $L[x] \models \mathrm{OD}_{[x]_T}$-determinacy, $\omega_1^{L[x]}$ is strongly inaccessible in HOD_d. Let $H \subseteq \omega_1^{L[x]}$ code $\mathrm{HOD}_d \cap V_{\omega_1^{L[x]}}$. Fix $T \in \mathscr{P}(\omega_1^{L[x]}) \cap \mathrm{OD}_d$ and let $T_0 \subseteq \omega_1^{L[x]}$ code T and H. Let $A = \langle A_\alpha \mid \alpha < \omega_1^{L[x]}\rangle$ be such that A_α is an OD_d prewellordering of length greater than or equal to α (by Theorem 7.8). Let $B = d$. Consider the structure

$$M = \big(L_{\omega_1^{L[x]}}(\mathbb{R})[T_0, A, B]\big)^{L[x]}.$$

We claim that

$$\mathrm{HOD}^M \models \text{There is a } T_0\text{-strong cardinal},$$

which completes the proof as before. The reason is that we are in the situation of the Generation Theorem, except with $\omega_1^{L[x]}$ replacing $\omega_2^{L[x]}$ and $\mathrm{ST}\text{-}[x]_T$-determinacy replacing ST^B-determinacy. The proof of the Generation Theorem goes through unchanged. One just has to check that all of the operations we performed before (which involved definability in various parameters) are in fact recursive in the relevant parameters. ⊣

7.2. Second Localization

We now wish to show that the above construction goes through when we replace ZF + DC with PA_2. Notice that if we had $\utilde{\Delta}^1_2$-determinacy then this would be routine.

7.10 Theorem. *Assume* $\mathrm{PA}_2 + \utilde{\Delta}^1_2$*-determinacy. Then for all reals* x*, there is a model* N *such that* $x \in N$ *and*

$$N \models \mathrm{ZFC} + \textit{There is a Woodin cardinal.}$$

Proof. Working in PA_2 if one has $\utilde{\Delta}^1_2$-determinacy then for every $x \in \omega^\omega$, $x^\#$ exists. It follows that for all $x \in \omega^\omega$, there is an ordinal $\alpha < \omega_1$ such that $L_\alpha[x] \models \mathrm{ZFC}$. Using $\utilde{\Delta}^1_2$-determinacy one can find a real x_0 enforcing OD-determinacy. Thus we have a model $L_{\alpha_0}[x_0]$ satisfying ZFC + $V = L[x_0]$ + OD-determinacy and this puts us in the situation of Theorem 6.10. ⊣

The situation where one only has Δ^1_2-determinacy is bit more involved.

7.11 Theorem. *Assume that* $\mathrm{PA}_2 + \Delta^1_2$*-determinacy is consistent. Then* ZFC + *"*On *is Woodin" is consistent.*

Proof Sketch. First we pass to a theory that more closely resembles the theory used to prove Theorem 7.9. In PA_2 one can simulate the construction of $L_{\omega_1}[x]$. Given a model M of PA_2 and a real $x \in M$, there is a definable set of reals A coding the elements of $L_{\omega_1}[x]$. One can then show that the "inner model" $L_{\omega_1}[x]$ satisfies ZFC−Power Set+$V = L[x]$ (using, for example Comprehension to get Replacement). Thus, ZFC − Power Set + $V = L[x]$ is a conservative extension of PA_2.

Next we need to arrange a sufficient amount of definable determinacy. The most natural way to secure Δ^1_2-determinacy is to let x encode winning strategies for all Δ^1_2 games. However, this approach is unavailable to us since we have not included AC in PA_2 and, in any case, we wish to work with OD-determinacy (understood schematically). For this we simultaneously run (an elaboration of) the proof of Theorem 6.6 while defining $L_{\omega_1}[x]$. In this way, for any model M of PA_2, there is a real x and an associated definable set of reals A which codes a model $L_{\omega_1}[x]$ satisfying ZFC − Power Set + $V = L[x]$ + OD-determinacy.

Working in ZFC − Power Set + $V = L[x]$ + OD-determinacy we wish now to show that

$$\mathrm{HOD}_{[x]_T} \models \mathrm{ZFC} + \mathrm{On\ is\ Woodin}.$$

So we have to localize the construction of the previous section to the structure $\langle L_{\omega_1}[x], [x]_T\rangle$. The first step is to show that

$$\langle L_{\omega_1}[x], [x]_T\rangle \models \text{ST-}[x]_T\text{-determinacy for } n \text{ moves},$$

for each n. Here by ST-$[x]_T$-determinacy we mean what we meant in the previous section. However, there is a slight metamathematical issue that arises when we work without Power Set, namely, at each stage of the game the potential moves for Player I are a proper class from the point of view of $\langle L_{\omega_1}[x], [x]_T\rangle$. So in quantifying over these moves we have to use the first-order definition of OD in $\langle L_{\omega_1}[x], [x]_T\rangle$. The winning condition for the n-move version of the game is first-order over $\langle L_{\omega_1}[x], [x]_T\rangle$ but since the complexity of the definition increases as n increases the full game is not first-order over $\langle L_{\omega_1}[x], [x]_T\rangle$. This is why we have had to restrict to the n-move version.

The proof of this version of the theorem is just like that of Theorem 7.7, only now one has to keep track of definability and verify that there is no essential use of Power Set (for example, in the proof of Third Periodicity). The proof of Theorem 7.8 goes through as before. Finally, as in the proof of Theorem 7.9, the proof of the Generation Theorem gives a structure M such that

$$\mathrm{HOD}^M_{[x]_T} \models \mathrm{ZFC} + \mathrm{On\ is\ } T\text{-strong},$$

for an arbitrary $\mathrm{OD}^{\langle L_{\omega_1}[x],[x]_T\rangle}_{[x]_T}$ class T of ordinals, which implies the final result. ⊣

This raises the following question: Are the theories $\mathrm{PA}_2 + \Delta^1_2$-determinacy and ZFC + "On is Woodin" equiconsistent? We turn to this and other more general issues in the next section.

8. Further Results

In this section we place the above results in a broader setting by discussing some results that draw on techniques that are outside the scope of this chapter. The first topic concerns the intimate connection between axioms of definable determinacy and large cardinal axioms (as mediated through inner models). The second topic concerns the surprising convergence between two very different approaches to inner model theory—the approach based on generalizations of L and the approach based on HOD. In both cases the relevant material on inner model theory can be found in Steel's chapter in this Handbook.

8.1. Large Cardinals and Determinacy

The connection between axioms of definable determinacy and inner models of large cardinals is even more intimate than indicated by the above results. We have seen that certain axioms of definable determinacy imply the existence of inner models of large cardinal axioms. For example, assuming $\mathrm{ZFC} + \utilde{\Delta}^1_2$-determinacy, for each $x \in \omega^\omega$, there is an inner model M such that $x \in M$ and

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

And, assuming $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$, in $L(\mathbb{R})$ there is an inner model M such that

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

In many cases these implications can be reversed—axioms of definable determinacy are actually *equivalent* to axioms asserting the existence of inner models of large cardinals. We discuss what is known about this connection, starting with a low level of boldface definable determinacy and proceeding upward. We then turn to lightface determinacy, where the situation is more subtle. It should be emphasized that our concern here is not merely with consistency strength but rather with outright equivalence (over ZFC).

8.1 Theorem. *The following are equivalent:*

(1) $\undertilde{\Delta}^1_2$*-determinacy.*

(2) For all $x \in \omega^\omega$*, there is an inner model* M *such that* $x \in M$ *and* $M \models$ *There is a Woodin cardinal.*

8.2 Theorem. *The following are equivalent:*

(1) PD (*Schematic*).

(2) For every $n < \omega$*, there is a fine-structural, countably iterable inner model* M *such that* $M \models$ *There are* n *Woodin cardinals.*

8.3 Theorem. *The following are equivalent:*

(1) $\mathrm{AD}^{L(\mathbb{R})}$.

(2) In $L(\mathbb{R})$*, for every set* S *of ordinals, there is an inner model* M *and an* $\alpha < \omega_1^{L(\mathbb{R})}$ *such that* $S \in M$ *and* $M \models \alpha$ *is a Woodin cardinal.*

8.4 Theorem. *The following are equivalent:*

(1) $\mathrm{AD}^{L(\mathbb{R})}$ *and* $\mathbb{R}^\#$ *exists.*

(2) $M_\omega^\#$ *exists and is countably iterable.*

8.5 Theorem. *The following are equivalent:*

(1) For all $\mathbb{B}$*,* $V^{\mathbb{B}} \models \mathrm{AD}^{L(\mathbb{R})}$.

(2) $M_\omega^\#$ *exists and is fully iterable.*

The above examples concern *boldface* definable determinacy. The situation with *lightface* definable determinacy is more subtle. For example, assuming $\mathrm{ZFC} + \Delta^1_2$-determinacy, must there exist an $\alpha < \omega_1$ and an inner model M such that α is a Woodin cardinal in M? In light of Theorem 8.1 one would expect that this is indeed the case. However, since Theorem 8.1 also holds in the context of PA_2 one would then expect that the theories $\mathrm{PA}_2 + \Delta^1_2$-determinacy and PA_2 + "There is an $\alpha < \omega_1$ and an inner model M such that $M \models \alpha$ is a Woodin cardinal" are *equivalent*, and yet this expectation is in conflict with the expectation that the theories $\mathrm{PA}_2 + \Delta^1_2$-determinacy and ZFC + "On is Woodin" are *equiconsistent*. In fact, this seems likely, but the details have not been fully checked. We state a version for third-order Peano arithmetic, PA_3, and second-order ZFC. But first we need a definition and some preliminary results.

8.6 Definition. A partial order $\mathbb{P}$ is *δ-productive* if for all δ-c.c. partial orders $\mathbb{Q}$, the product $\mathbb{P} \times \mathbb{Q}$ is δ-c.c.

8.7 Theorem. *In the fully iterable, 1-small, 1-Woodin Mitchell-Steel model the extender algebra built using all extenders on the sequence which are strong to their length is δ-productive.*

This is a warm-up since in the case of interest we do not have iterability. It is unknown if iterability is necessary.

8.8 Theorem. *Suppose δ is a Woodin cardinal. Then there is a proper class inner model $N \subseteq V$ such that*

(1) $N \models \delta$ is a Woodin cardinal and

(2) $N \models$ There is a complete δ-c.c. Boolean algebra $\mathbb{B}$ such that

$$N^{\mathbb{B}} \models \Delta^1_2\text{-}\textit{determinacy.}$$

Let ZFC_2 be second-order ZFC.

8.9 Theorem. *The following are equiconsistent*:

(1) $\mathrm{PA}_3 + \Delta^1_2$*-determinacy.*

(2) $\mathrm{ZFC}_2 + \mathrm{On}$ *is Woodin.*

We now turn from theories to models and discuss the manner in which one can pass back and forth between models of infinitely many Woodin cardinals and models of definable determinacy at the level of $\mathrm{AD}^{L(\mathbb{R})}$ and beyond. We have already dealt in detail with one direction of this—the transfer from models of determinacy to models with Woodin cardinals—and the other direction—the transfer from models with Woodin cardinals to models of determinacy—was briefly discussed in the introduction, but the situation is much more general. To proceed at the appropriate level of generality we need to introduce a potential strengthening of AD.

A set $A \subseteq \omega^\omega$ is *$^\infty$-borel* if there is a set $S \subseteq \mathrm{On}$, an ordinal α, and a formula φ such that

$$A = \{y \in \omega^\omega \mid L_\alpha[S, y] \models \varphi[S, y]\}.$$

It is fairly straightforward to show that to say that A is $^\infty$-borel is equivalent to saying that it has a "transfinite borel code". Notice that under AC every set of reals is $^\infty$-borel.

8.10 Definition. Assume ZF + $\mathrm{DC}_{\mathbb{R}}$. The theory AD^+ consists of the axioms:

(1) Every set $A \subseteq \omega^\omega$ is $^\infty$-borel.

(2) Suppose $\lambda < \Theta$ and $\pi : \lambda^\omega \to \omega^\omega$ is a continuous surjection. Then for each $A \subseteq \omega^\omega$ the set $\pi^{-1}[A]$ is determined.

8.11 Conjecture. AD implies AD^+.

It is known that the failure of this implication has strong consistency strength. For example, $\mathrm{AD} + \neg\mathrm{AD}^+$ proves $\mathrm{Con}(\mathrm{AD}_{\mathbb{R}})$.

The following theorem—the *Derived Model Theorem*—is a generalization of Theorem 1.5, mentioned in the introduction.

8.12 Theorem. *Suppose that δ is a limit of Woodin cardinals. Suppose that $G \subseteq \mathrm{Col}(\omega, <\delta)$ is V-generic and let $\mathbb{R}_G = \cup\{\mathbb{R}^{V[G\restriction\alpha]} \mid \alpha < \delta\}$. Let Γ_G be the set of $A \subseteq \mathbb{R}_G$ such that*

(1) $A \in V(\mathbb{R}_G)$,

(2) $L(A, \mathbb{R}_G) \models \mathrm{AD}^+$.

Then $L(\Gamma_G, \mathbb{R}_G) \models \mathrm{AD}^+$.

There is a "converse" to the Derived Model Theorem, the proof of which is a generalization of the proof of Theorem 6.20.

8.13 Theorem. *Assume AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$. There is a partial order $\mathbb{P}$ such that if H is $\mathbb{P}$-generic over V then there is an inner model $N \subseteq V[H]$ such that*

(1) $N \models \mathrm{ZFC}$,

(2) ω_1^V is a limit of Woodin cardinals in N,

(3) there is a g which is $\mathrm{Col}(\omega, <\omega_1^V)$-generic over N and such that

(a) $\mathbb{R}^V = \mathbb{R}_g$,

(b) $\Gamma_g = \mathscr{P}(\mathbb{R})^V$,

where $\mathbb{R}_g$ and Γ_g are as in the previous theorem with N in the role of V.

Thus, there is an intimate connection between models with infinitely many Woodin cardinals and models of definable determinacy at the level of $\mathrm{AD}^{L(\mathbb{R})}$ and beyond. Moreover, the link is even tighter in the case of fine-structural inner models with Woodin cardinals. For example, if one first applies the Derived Model Theorem to M_ω (the Mitchell-Steel model for ω-many Woodin cardinals) and then applies the "converse" theorem to the resulting derived model $L(\mathbb{R}^*)$ then one recovers the original model M_ω.

8.2. HOD-Analysis

There is also an intimate connection between the two approaches to inner model theory mentioned in the introduction—the approach based on generalizations of L and the approach based on HOD.

As mentioned in the introduction, the two approaches have opposing advantages and disadvantages. The disadvantage of the first approach is that the problem of actually defining the models that can accommodate large cardinals—*the inner model problem*—is quite a difficult problem. However, the advantage is that once the inner model problem is solved at a given level of large cardinals the inner structure of the models is quite transparent and so these models are suitable for extracting the large cardinal content inherent in a given statement. The advantage of the approach based on HOD is that this model is trivial to define and it can accommodate virtually every large cardinal. The disadvantage—*the tractability problem*—is that in general the inner structure of HOD is about as tractable as that of V and so it is not generally suitable for extracting the large cardinal content from a given statement.

Nevertheless, we have taken the approach based on HOD and we have found that $\mathrm{AD}^{L(\mathbb{R})}$ and Δ^1_2-determinacy are able to overcome (to some extent) the tractability problem for their natural models, $L(\mathbb{R})$ and $L[x]$ for a Turing (or constructibility) cone of x. For example, we have seen that under $\mathrm{AD}^{L(\mathbb{R})}$,

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \text{ is a Woodin cardinal,}$$

and that under Δ^1_2-determinacy, for a Turing cone of reals x,

$$\mathrm{HOD}^{L[x]} \models \omega_2^{L[x]} \text{ is a Woodin cardinal.}$$

Despite this progress, much of the structure of HOD in these contexts is far from clear. For example, it is unclear whether under Δ^1_2-determinacy, for a Turing cone of reals x, $\mathrm{HOD}^{L[x]}$ satisfies GCH, something that would be immediate in the case of "L-like" inner models.

Since the above results were first proved, Mitchell and Steel developed the fine-structural version of the "L-like" inner models at the level of Woodin cardinals. These models have the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of (partial) extenders and (as noted above) their inner structure is very well understood—for example, they satisfy GCH and many of the other combinatorial properties that hold in L. A natural question, then, is whether there is any connection between these radically different approaches, that is, whether HOD as computed in $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$ or in $L[x]$, for a Turing cone of x, under Δ^1_2-determinacy, bears any resemblance to the $L[\vec{E}]$ models. The remainder of this section is devoted to this question. We begin with $\mathrm{HOD}^{L(\mathbb{R})}$ and its generalizations (where a good deal is known) and then turn to $\mathrm{HOD}^{L[x]}$ (where the central question is open). Again, the situation with lightface determinacy is more subtle.

The theorems concerning $\mathrm{HOD}^{L(\mathbb{R})}$ only require $\mathrm{AD}^{L(\mathbb{R})}$ but they are simpler to state under the stronger assumption that $\mathrm{AD}^{L(\mathbb{R})}$ holds in all generic extensions of V. By Theorem 8.5, this assumption is equivalent to the statement that $M_\omega^\#$ exists and is fully iterable.

The first hint that $\mathrm{HOD}^{L(\mathbb{R})}$ is a fine-structural model is the remarkable fact that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}.$$

The agreement between $\mathrm{HOD}^{L(\mathbb{R})}$ and M_ω fails higher up but $\mathrm{HOD}^{L(\mathbb{R})}$ agrees with an iterate of M_ω at slightly higher levels. More precisely, letting N be the result of iterating M_ω by taking the ultrapower ω_1^V-many times using the (unique) normal ultrafilter on the least measurable cardinal, we have that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathscr{P}(\omega_1^V) = N \cap \mathscr{P}(\omega_1^V).$$

Steel improved this dramatically by showing that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})}}$$

is the direct limit of a directed system of iterable fine-structural inner models.

8.14 Theorem (Steel). $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ *is a Mitchell-Steel model, where* $\delta = (\underset{\sim}{\delta}{}^2_1)^{L(\mathbb{R})}$.

For a proof of this result see Steel's chapter in this Handbook. As a corollary one has that $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies GCH along with the combinatorial principles (such as $\Diamond$ and $\Box$) that are characteristic of fine-structural models.

The above results suggest that all of $\mathrm{HOD}^{L(\mathbb{R})}$ might be a Mitchell-Steel inner model of the form $L[\vec{E}]$. This is not the case.

8.15 Theorem. $\mathrm{HOD}^{L(\mathbb{R})}$ *is not a Mitchell-Steel inner model.*

Nevertheless, $\mathrm{HOD}^{L(\mathbb{R})}$ is a fine-structural inner model, one that belongs to a new, quite different, hierarchy of models. Let

$$\begin{aligned} D = \big\{L[\vec{E}] \;\big|\; & L[\vec{E}] \text{ is an iterate of } M_\omega \text{ by a countable tree} \\ & \text{which is based on the first Woodin cardinal} \\ & \text{and has a non-dropping cofinal branch}\big\}. \end{aligned}$$

Any two structures in D can be compared and the iteration halts in countably many steps (since we have full iterability) with iterates lying in D. So D is a directed system under the elementary embeddings given by iteration maps. By the Dodd-Jensen lemma the embeddings commute and hence there is a direct limit. Let $L[\vec{E}^\infty]$ be the direct limit of D. Let $\langle \delta_i^\infty \mid i < \omega \rangle$ be the Woodin cardinals of $L[\vec{E}^\infty]$.

8.16 Theorem. *Let* $L[\vec{E}^\infty]$ *be as above. Then*

(1) $L[\vec{E}^\infty] \subseteq \mathrm{HOD}^{L(\mathbb{R})}$,

(2) $L[\vec{E}^\infty] \cap V_\delta = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$, *where* $\delta = \delta_0^\infty$,

(3) $\Theta^{L(\mathbb{R})} = \delta_0^\infty$, *and*

(4) $(\underset{\sim}{\delta}{}_1^2)^{L(\mathbb{R})}$ *is the least cardinal in* $L[\vec{E}^\infty]$ *which is* λ*-strong for all* $\lambda < \delta_0^\infty$.

To reach $\mathrm{HOD}^{L(\mathbb{R})}$ we need to supplement $L[\vec{E}^\infty]$ with additional inner-model-theoretic information. A natural candidate is the iteration strategy. It turns out that by folding in the right fragment of the iteration strategy one can capture $\mathrm{HOD}^{L(\mathbb{R})}$. Let

$$\begin{aligned} T^\infty = \big\{T \mid T &\text{ is a maximal iteration tree on } L[\vec{E}^\infty] \text{ based on } \delta_0^\infty, \\ &T \in L[\vec{E}^\infty], \text{ and } \operatorname{length}(T) < \sup\{\delta_n^\infty \mid n < \omega\}\big\} \end{aligned}$$

and

$$P = \big\{\langle b, T\rangle \mid T \in T^\infty \text{ and } b \text{ is the true branch through } T\big\}.$$

8.17 Theorem. *Let* $L[\vec{E}^\infty]$ *and* P *be as above. Then*

$$\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{E}^\infty, P].$$

In fact, there is a single iteration tree $T \in T^\infty$ *such that if* b *is the branch through* T *chosen by* P *then*

$$\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{E}^\infty, b].$$

This analysis has an interesting consequence. Notice that the model $L[\vec{E}^\infty]$ is of the form $L[A]$ for $A \subseteq \delta_0^\infty$. Thus, although the addition of P does not add any new bounded subsets of $\Theta^{L(\mathbb{R})}$ it does a lot of damage to the model above $\Theta^{L(\mathbb{R})}$, for example, it collapses ω-many Woodin cardinals. One might think that this is an artifact of $L[\vec{E}^\infty]$ but in fact the situation is much more general: Suppose $L[\vec{E}]$ is ω-small, fully iterable, and has ω-many Woodin cardinals. Let P be defined as above except using the Woodin cardinals of $L[\vec{E}]$. Then $L[\vec{E}, P] \cap V_\delta = L[\vec{E}] \cap V_\delta$, where δ is the first Woodin cardinal of $L[\vec{E}]$, and $L[\vec{E}] \subsetneq L[\vec{E}, P] \subsetneq L[\vec{E}^\#]$. For example, applying this result to $L[\vec{E}] = M_\omega$, one obtains a canonical inner-model-theoretic object between M_ω and $M_\omega^\#$. In this way, what appeared to be a coarse approach to inner model theory has actually resulted in a hierarchy that supplements and refines the standard fine-structural hierarchy.

The above results generalize. We need a definition.

8.18 Definition (Mouse Capturing). MC is the statement: For all $x, y \in \omega^\omega$, $x \in \mathrm{OD}_y$ iff there is an iterable Mitchell-Steel model M of the form $L[\vec{E}, y]$ such that $x \in M$.

The *Mouse Set Conjecture*, MSC, is the conjecture that it is a theorem of AD^+ that MC holds if there is no iterable model with a superstrong cardinal. There should be a more general version of MC, one that holds for extensions of the Mitchell-Steel models that can accommodate long extenders. And this version of MC should follow from AD^+. However, the details are still being worked out. See [12].

8.19 Theorem. *Assume* $\mathrm{AD}^+ + V = L(\mathscr{P}(\mathbb{R})) + \Theta_0 = \Theta + \mathrm{MSC}$. *Then the inner model* $\mathrm{HOD}^{L(\mathscr{P}(\mathbb{R}))}$ *is of the form* $L[\vec{E}^\infty, P]$, *with the key difference being that* $L[\vec{E}^\infty]$ *need not be* ω*-small.*

8.20 Theorem. *Assume* $\mathrm{AD}^+ + V = L(\mathscr{P}(\mathbb{R})) + \Theta_0 < \Theta + \mathrm{MSC}$. *Then*

(1) Θ_0 *is the least Woodin cardinal in* HOD*,*

(2) $\mathrm{HOD} \cap V_{\Theta_0}$ *is a Mitchell-Steel model,*

(3) $\mathrm{HOD} \cap V_{\Theta_0+1}$ *is not a Mitchell-Steel model, and*

(4) $\mathrm{HOD} \cap V_{\Theta_1}$ *is a model of the form* $L[\vec{E}^\infty, P]$ (*assuming the appropriate form of the Mouse Set Theorem*).

One can move on to stronger hypotheses. For example, assuming AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$, $\mathrm{AD}_\mathbb{R}$ is equivalent to the statement that Ω (defined at the beginning of Sect. 5) is a non-zero limit ordinal. There is a minimal inner model N of $\mathrm{ZF} + \mathrm{AD}_\mathbb{R}$ that contains all of the reals. The model HOD^N has ω-many Woodin cardinals and these are exactly the members of the Θ-sequence. This model belongs to the above hierarchy and has been used to calibrate the consistency strength of $\mathrm{AD}_\mathbb{R}$ in terms of the large cardinal hierarchy. This hierarchy extends and a good deal is known about it.

We now turn to the case of lightface determinacy and the setting $L[x]$ for a Turing cone of x. Here the situation is less clear. In fact, the basic question is open.

5 Open Question. Assume Δ^1_2-determinacy. For a Turing cone of x, what is $\mathrm{HOD}^{L[x]}$ from a fine-structural point of view?

We close with partial results in this direction and with a conjecture. To simplify the discussion we state these results under a stronger assumption than is necessary: Assume Δ^1_2-determinacy and that for all $x \in \omega^\omega$, $x^\#$ exists.

It follows that M_1 and $M_1^\#$ exist. Let $x_0 \in \omega^\omega$ be such that $M_1^\# \in L[x_0]$. Let κ_{x_0} be the least inaccessible of $L[x_0]$ and let $G \subseteq \mathrm{Col}(\omega, < \kappa_{x_0})$ be $L[x_0]$-generic. The Kechris-Solovay result carries over to show that

$$L[x_0][G] \models \text{OD-determinacy}.$$

Furthermore,

$$\mathrm{HOD}^{L[x_0][G]} = \mathrm{HOD}^{L(\mathbb{R})^{L[x_0][G]}} \quad \text{and} \quad \omega_2^{L[x_0][G]} = \Theta^{L(\mathbb{R})^{L[x_0][G]}}.$$

Thus, the model $L(\mathbb{R})^{L[x_0][G]}$ is a "lightface" analogue of $L(\mathbb{R})$. In fact the conditions of the Generation Theorem hold in $L[x_0][G]$ and as a consequence one has that

$$\mathrm{HOD}^{L[x_0][G]} \models \omega_2^{L[x_0][G]} \text{ is a Woodin cardinal.}$$

For a model $L[\vec{E}]$ containing at least one Woodin cardinal let $\delta_0^{\vec{E}}$ be the least Woodin cardinal. Let

$$D = \big\{L[\vec{E}] \subseteq L[x_0][G] \bigm| L[\vec{E}] \text{ is an iterate of } M_1 \text{ and } \delta_0^{\vec{E}} < \omega_1^{L[x_0][G]}\big\}.$$

Let $L[\vec{E}^\infty]$ be the direct limit of D. Let δ^∞ be the least Woodin of $L[\vec{E}^\infty]$ and let κ^∞ be the least inaccessible above δ^∞. Let

$$T^\infty = \big\{T \bigm| T \text{ is a maximal iteration tree on } L[\vec{E}^\infty], \\ T \in L[\vec{E}^\infty], \text{ and } \mathrm{length}(T) < \kappa^\infty\big\}$$

and

$$P = \big\{\langle b, T\rangle \bigm| T \in T^\infty \text{ and } b \text{ is the true branch through } T\big\}.$$

8.21 Theorem. *Let $L[\vec{E}^\infty, P]$ be as above. Then*

(1) $\mathrm{HOD}^{L[x_0][G]} \cap V_{\delta^\infty} = L[\vec{E}^\infty] \cap V_{\delta^\infty}$,

(2) $\mathrm{HOD}^{L[x_0][G]} = L[\vec{E}^\infty, P]$, *and*

(3) $\omega_2^{L[x_0][G]} = \delta^\infty$.

A similar analysis can be carried out for other hypotheses that place one in an "$L(\mathbb{R})$-like" setting. For example, suppose again that x_0 is such that $M_1^\# \in L[x_0]$. One can "generically force" MA as follows: In $L[x_0]$ let $\mathbb{P}$ be the partial order where the conditions $\langle \mathbb{B}_\alpha \mid \alpha < \gamma\rangle$ are such that (i) for each $\alpha < \gamma$, $\mathbb{B}_\alpha$ is c.c.c., (ii) $|\mathbb{B}_\alpha| = \omega_1$, (iii) if $\alpha \leqslant \beta < \gamma$ then $\mathbb{B}_\alpha$ is a complete subalgebra of $\mathbb{B}_\beta$, and (iv) $\gamma < \omega_2$, and the ordering is by extension. The forcing is $<\!\omega_2$-closed. Let $G \subseteq \mathbb{P}$ be $L[x_0]$-generic and let $\mathbb{B}_G$ be the union of the algebras $\mathbb{B}_\alpha$ appearing in the conditions in G. It follows that $\mathbb{B}_G$ is c.c.c in $L[x_0][G]$. Now, letting $H \subseteq \mathbb{B}_G$ be $L[x_0][G]$-generic, we have that $L[x_0][G][H]$ satisfies MA. The result is that

$$\mathrm{HOD}^{L[x_0][G][H]} = L[\vec{E}^\infty, P]$$

for the appropriate $\vec{E}^\infty$ and P. However, in this context

$$\mathrm{HOD}^{L[x_0][G][H]} \models \omega_3^{L[x_0][G][H]} \text{ is a Woodin cardinal.}$$

In the case of $L(\mathbb{R})$ the non-fine-structural analysis showed that $(\underset{\sim}{\delta}{}_1^2)^{L(\mathbb{R})}$ is λ-strong in $\mathrm{HOD}^{L(\mathbb{R})}$ for all $\lambda < \Theta^{L(\mathbb{R})}$ and the HOD-analysis showed that in fact $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is the least ordinal with this feature. In the case of $L[x_0][G]$ the non-fine-structural analysis shows that some ordinal δ is λ-strong in $\mathrm{HOD}^{L[x_0][G]} = \mathrm{HOD}^{L(\mathbb{R})^{L[x_0][G]}}$ for all $\lambda < \omega_2^{L[x_0][G]} = \Theta^{L(\mathbb{R})^{L[x_0][G]}}$. Numerology would suggest that δ is $\underset{\sim}{\delta}^1_2$ as computed in $L[x_0][G]$. It turns out this analogy fails: the least cardinal δ that is λ-strong in $\mathrm{HOD}^{L[x_0][G]}$ for all $\lambda < \omega_2^{L[x_0][G]}$ is in fact strictly less δ^1_2 as computed in $L[x_0][G]$.

But there is another analogy that does hold. First we need some definitions. A set $A \subseteq \omega^\omega$ is *γ-Suslin* if there is an ordinal γ and a tree T on $\omega \times \gamma$ such that $A = p[T] = \{x \in \omega^\omega \mid \exists y \in \gamma^\omega \, \forall n \, (x\restriction n, y\restriction n) \in T\}$. A cardinal κ is a *Suslin cardinal* if there exists a set $A \subseteq \omega^\omega$ such that A is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. A set $A \subseteq \omega^\omega$ is *effectively γ-Suslin* if there is an ordinal γ and an OD tree $T \subseteq \omega \times \gamma$ such that $A = p[T]$. A cardinal κ is an *effective Suslin cardinal* if there exists a set $A \subseteq \omega^\omega$ such that A is effectively κ-Suslin but not effectively γ-Suslin for any $\gamma < \kappa$.

In $L(\mathbb{R})$, $\underset{\sim}{\delta}^2_1$ is the largest Suslin cardinal. Since $L[x_0][G]$ is a lightface analogue of $L(\mathbb{R})$ one might expect that in $L[x_0][G]$, δ^1_2 is the largest *effective* Suslin cardinal in $L[x_0][G]$. This is indeed the case.

There is one more advance on the HOD-analysis for $L[x]$ that is worth mentioning.

8.22 Theorem. *Assume $\underset{\sim}{\Delta}^1_2$-determinacy. For a Turing cone of x there is a predicate A such that*

(1) $\mathrm{HOD}_A^{L[x]}$ has the form $L[\vec{E}, P]$ where P is a fragment of the iteration strategy,

(2) $\mathrm{HOD}_A^{L[x]} \models \omega_2^{L[x]}$ is a Woodin cardinal,

(3) $\mathrm{HOD}_A^{L[x]}$ is of the form $L[\vec{E}]$ below $\omega_2^{L[x]}$,

(4) $L[x] \models \mathrm{ST}$-determinacy, and

(5) $\mathrm{HOD}^{L[x]} \cap V_\delta = \mathrm{HOD}_A^{L[x]} \cap V_\delta$ where δ is the least cardinal of $\mathrm{HOD}_A^{L[x]}$ that is λ-strong for all $\lambda < \omega_2^{L[x]}$.

Moreover, there exists a definable collection of such A and the collection has size $\omega_1^{L[x]}$.

This provides some evidence that $\mathrm{HOD}^{L[x]}$ is of the form $L[\vec{E}]$ below $\omega_2^{L[x]}$ and that $\mathrm{HOD}^{L[x]}$ is not equal to a model of the form $L[\vec{E}]$.

8.23 Conjecture. $\mathrm{HOD}^{L[x]}$ is of the form $L[\vec{E}, P]$ where P selects branches through all trees in $L[\vec{E}]$ based on the Woodin cardinal and with length less than the successor of the Woodin cardinal.

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24. Forcing over Models of Determinacy

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Contents

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The *Axiom of Determinacy* (AD) is the statement that all integer games of perfect information of length ω are determined. This statement contradicts the Axiom of Choice, and presents a radically different view of the universe of sets. Nonetheless, determinacy was a subject of intense study by the late 1960s, with an eye towards the possibility that some inner model of set theory satisfies AD (see, for example, the introductory remarks in [32]). Since strategies for these games can be coded by real numbers, the natural inner model to consider is $L(\mathbb{R})$, the smallest model of Zermelo-Fraenkel set theory containing the reals and the ordinals. This approach was validated by the following theorem of Woodin (see [14, 19]), building on work of Martin and Steel [25] and Foreman, Magidor and Shelah [7].

0.1 Theorem. *If there exists a measurable cardinal which is greater than infinitely many Woodin cardinals, then the Axiom of Determinacy holds in* $L(\mathbb{R})$.

This theorem is established in Neeman's chapter in this Handbook.

A companion to Theorem 0.1, also due to Woodin (see [19]) and building on the work of Foreman, Magidor and Shelah [7], shows that the existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ cannot be changed by set forcing. By Theorem 0.1, the Axiom of Determinacy is part of this fixed theory for $L(\mathbb{R})$.

0.2 Theorem. *If δ is a limit of Woodin cardinals and there exists a measurable cardinal greater than δ, then no forcing construction in V_δ can change the theory of* $L(\mathbb{R})$.

Theorem 0.2 has the following corollary. If P is a definable forcing construction in $L(\mathbb{R})$ which is homogeneous (i.e., the theory of the extension can be computed in the ground model), then the theory of the P-extension of $L(\mathbb{R})$ also cannot be changed by forcing (i.e., the P-extensions of $L(\mathbb{R})$ in all forcing extensions of V satisfy the same theory). This suggests that the absoluteness properties of $L(\mathbb{R})$ can be lifted to models of the Axiom of Choice, as Choice can be forced over $L(\mathbb{R})$.

In [35], Steel and Van Wesep made a major step in this direction, forcing over a model of a stronger form of determinacy than AD to produce a model of ZFC satisfying two consequences of AD, that $\underset{\sim}{\delta}^1_2$ (the supremum of the lengths of the $\underset{\sim}{\Delta}^1_2$-definable prewellorderings of the reals) is ω_2 and the nonstationary ideal NS_{ω_1} on ω_1 is saturated. Woodin [38] later improved the hypothesis to $\mathrm{AD}^{L(\mathbb{R})}$.

This theorem is proved in [5, Sect. 5.11].

In the early 1990's, Woodin proved the following theorem, showing for the first time that large cardinals imply the existence of a partial order forcing the existence of a projective set of reals giving a counterexample to the Continuum Hypothesis. The question of whether ZFC is consistent with a projective witness to $\mathfrak{c} \geq \omega_3$ remains open.

0.3 Theorem. *If NS_{ω_1} is saturated and there exists a measurable cardinal, then $\underset{\sim}{\delta}^1_2 = \omega_2$.*

One important point in this proof is the fact that if NS_{ω_1} is saturated then every member of $H(\omega_2)$ (recall that $H(\kappa)$ is the collection of sets of hereditary cardinality less than κ) appears in an iterate (in the sense of the next section) of a countable model of a suitable fragment of ZFC. Since these countable models are elements of $L(\mathbb{R})$, their iterations induce a natural partial order in $L(\mathbb{R})$. With certain technical refinements, this partial order, called $\mathbb{P}_{\max}$, produces an extension of $L(\mathbb{R})$ whose $H(\omega_2)$ is the direct limit of the structures $H(\omega_2)$ of models satisfying every forceable theory (and more). In particular, the structure $H(\omega_2)$ in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ (assuming $\mathrm{AD}^{L(\mathbb{R})}$) satisfies every Π_2 sentence ϕ (in the language with predicates for NS_{ω_1} and each set of reals in $L(\mathbb{R})$) for $H(\omega_2)$ such that for some integer n the theory ZFC + "There exist n Woodin cardinals" implies that ϕ is forceable. Furthermore, the partial order $\mathbb{P}_{\max}$ can be easily varied to produce other consistency results and canonical models.

The partial order $\mathbb{P}_{\max}$ and some of its variations (and many other related issues) are presented in [39]. The aim of this chapter is to prepare the reader for that book. First, we attempt to give a complete account of the basic analysis of the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, relative to published results. Then we briefly survey some of the issues surrounding $\mathbb{P}_{\max}$, in particular $\mathbb{P}_{\max}$ variations and forcing over larger models of determinacy. We also briefly introduce Woodin's Ω-logic, in order to properly state the maximality properties of the $\mathbb{P}_{\max}$ extension. For the most part, though, our focus is primarily on the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, and secondarily on $\mathbb{P}_{\max}$-style forcing constructions as a means of producing consistency results. For other topics, such as the Ω-conjecture and the relationship between Ω-logic and the Continuum Hypothesis, we refer the reader to [42, 40, 41, 43, 3].

The material in this chapter is due to Woodin, except where noted otherwise. The author would like to thank Howard Becker and John Steel for advising him on parts of the material in Sects. 4 and 9 respectively. He would also like to thank Andrés Caicedo, Neus Castells and Teruyuki Yorioka for making numerous helpful suggestions. The author was supported in part by the Fields Institute, FAPESP (Grant # 02/11551-3, University of São Paulo) and the Centre de Recerca Matemàtica of the Institut d'Estudis Catalans.

1. Iterations

The fundamental construction in the $\mathbb{P}_{\max}$ analysis is the iterated generic elementary embedding. These embeddings can have many forms, but we will concentrate on the following case. Suppose that I is a normal, uniform, proper ideal on ω_1 (so I is a proper subset of $\mathcal{P}(\omega_1)$ containing all the countable subsets, and such that whenever A is an I-positive set (i.e., in $\mathcal{P}(\omega_1) \setminus I$) and $f : A \to \omega_1$ is a regressive function, f is constant on an I-positive set; notationally, we are going to act as though "proper" and "uniform" are contained in the definition of *normal ideal*, and similarly for "measure" and "ultrafilter"). Then forcing with the Boolean algebra $\mathcal{P}(\omega_1)/I$ gives rise to a V-normal ultrafilter U on ω_1^V. By convention, we identify the wellfounded part of the ultrapower $\mathrm{Ult}(V, U)$ with its transitive collapse, and we note that this wellfounded part always contains ω_2^V. The corresponding elementary embedding $j : V \to \mathrm{Ult}(V, U)$ has critical point ω_1^V, and since I is normal, for each $A \in \mathcal{P}(\omega_1)^V$, $A \in U$ if and only if $\omega_1^V \in j(A)$. Under certain circumstances, the corresponding ultrapower of V is wellfounded; if every condition in $\mathcal{P}(\omega_1)/I$ forces this, then I is *precipitous*. See [5] for a general analysis of generic elementary embeddings.

For the most part, we will be concerned only with models of ZFC, but since occasionally we will want to deal with structures whose existence can be proved in ZFC, we define the fragment ZFC$^\circ$ to be the theory ZFC $-$ Power Set $-$ Replacement $+$ "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1). By the Axiom of Choice here we mean that every set is the bijective image of an ordinal. We will use ZFC$^\circ$ in place of the theory ZFC* from [39], which asserts closure under the Gödel operations (see [10, p. 178]) plus a scheme similar to the one above. One advantage of using ZFC* is that it holds in $H(\omega_2)$. On the other hand, it raises some technical points that we would rather avoid here. Some of these points appear in Woodin's proof of Theorem 0.3. Our concentration is on $\mathbb{P}_{\max}$, but we hope nonetheless that the reader will have no difficulty in reading the proofs of that theorem in [5, 39] after reading the material in this section.

With either theory, the point is that one needs to be able to prove the version of Łoś's theorem asserting that ultrafilters on ω_1 generate elementary embeddings, which amounts to showing the following fact. The fact follows immediately from the scheme above.

1.1 Fact (ZFC$^\circ$). Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \implies \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

If M is a model of ZFC and κ is a cardinal of M of cofinality greater than ω_1^M (in M), then $H(\kappa)^M$ satisfies ZFC$^\circ$ if it has $|\mathcal{P}(\mathcal{P}(\omega_1))|^M$ as a member.

Suppose that M is a model of ZFC$^\circ$, $I \in M$ is a normal ideal on ω_1^M and $\mathcal{P}(\mathcal{P}(\omega_1))^M$ is countable. Then there exist M-generic filters for the partial order $(\mathcal{P}(\omega_1)/I)^M$. Furthermore, if $j : M \to N$ is an ultrapower embedding of this form, then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable, and there exist N-generic filters for $(\mathcal{P}(\omega_1)/j(I))^N$. We can continue choosing generics in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition.

1.2 Definition. Let M be a model of ZFC$^\circ$ and let I be an ideal on ω_1^M which is normal in M. Let γ be an ordinal less than or equal to ω_1. An *iteration* of (M, I) of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} : M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

- $M_0 = M$,
- each G_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system $\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$, and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

If $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ is an iteration of a pair (M, I) and each $\omega_1^{M_\alpha}$ is wellfounded, then $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$ is a club subset of ω_1. Note also that if $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma \rangle$ is an iteration of a pair (M, I), then $j``\mathrm{On}^{M_0}$ is cofinal in On^{M_γ}.

The models M_α in Definition 1.2 are called *iterates* of (M, I). If M is a model of ZFC$^\circ$ then an iteration of $(M, \mathrm{NS}_{\omega_1}^M)$ is called simply an *iteration of M* and an iterate of $(M, \mathrm{NS}_{\omega_1}^M)$ is called simply an *iterate* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j : (M, I) \to (M^*, I^*)$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma \rangle$$

of (M, I), and that M^* is the final model of this iteration and $I^* = j(I)$.

If M and I are as in Definition 1.2, then the pair (M, I) is *iterable* if every iterate of (M, I) is wellfounded. In this chapter, we are in general interested only in iterable pairs (M, I). When checking for iterability it suffices to consider the countable length iterations, as any iteration of length ω_1 whose final model is illfounded contains an illfounded model at some earlier stage. The following two lemmas show that if

- M is a transitive model of ZFC$^\circ$+ Power Set containing ω_1^V,
- $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$ is countable,
- κ is a cardinal of M greater than $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$ with cofinality greater than ω_1^M in M,
- $I \in M$ is a normal precipitous ideal on ω_1^M,

then the pair $(H(\kappa)^M, I)$ is iterable. In particular, Lemma 1.6 below shows that every such pair (M, I) is iterable, and then Lemma 1.5 shows that $(H(\kappa)^M, I)$ is iterable, as every iterate of $(H(\kappa)^M, I)$ embeds into an iterate of (M, I). This argument is our primary means of finding iterable models.

1.3 Remark. Often in these notes will we use recursive codings of elements of $H(\omega_1)$ by subsets of ω. The following coding is sufficient in all cases: fixing a recursive bijection $\pi : \omega \times \omega \to \omega$, we say that $x \subseteq \omega$ codes $a \in H(\omega_1)$ if

$$\langle \omega, \{(n, m) \mid \pi(n, m) \in x\}\rangle \cong \langle \{a\} \cup \mathrm{tc}(a), \in\rangle,$$

where $\mathrm{tc}(a)$ denotes the transitive closure of a. Under this coding, the relations "$\in$" and "$=$" are both Σ^1_1, since permutations of ω can give rise to different codes for the same set.

1.4 Remark. The statement that a given pair (M, I) is iterable is Π^1_2 in any real x recursively coding the pair. One way to express this (not necessarily the most direct), is: for every countable model N of ZFC$^\circ$ with x as a member and every object $J \in N$ such that $N \models$ "J is an iteration of (M, I)" and every function f from ω to the "ordinals" of the last model of J, either N is illfounded (i.e., there exists an infinite descending sequence of "ordinals" of N) or $f(n+1)$"$\notin$"$f(n)$ for some integer n, where "$\notin$" is the negation of the $\in$-relation of the last model of J. Therefore, whether or not (M, I) is iterable is absolute between models of ZFC$^\circ$ containing the countable ordinals. Furthermore, assuming that $x^\#$ exists and letting γ denote $\omega_1^{L[x^\#]}$, any transitive model N of ZFC$^\circ$ containing $L_\gamma[x^\#]$ is correct about the iterability of (M, I), as $L[x^\#]$ is correct about it, and N thinks that $L_{\omega_1^N}[x^\#]$ is correct about it. Similarly, if γ and δ are countable ordinals coded by reals y and z, then the existence of an iteration of (M, I) of length γ which is illfounded is a Σ^1_1 fact about x and y, and the existence of an iteration of (M, I) of length γ such that the ordinals of the last model of the iteration have height at least (or, exactly) δ is a Σ^1_1 fact about x, y and z.

The first lemma is easily proved by induction. The last part of the lemma uses the assumption that N is closed under ω_1^M-sequences in M (this is the main way in which the lemma differs from the corresponding lemma in [39] (Lemma 3.8)). In our applications, N will often be $H(\kappa)^M$ for some cardinal κ of M such that $M \models \mathrm{cf}(\kappa) > \omega_1^M$, in which case $H(\kappa)^M$ is indeed closed under ω_1^M-sequences in M.

1.5 Lemma. *Suppose that M is a model of* ZFC$^\circ$ *and $I \in M$ is a normal ideal on ω_1^M. Let N be a transitive model of* ZFC$^\circ$ *in M containing $\mathcal{P}(\mathcal{P}(\omega_1)/I)^M$ and closed under ω_1^M-sequences in M. Let $\gamma \leq \omega_1$ be an ordinal and let*

$$\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

be an iteration of (N, I). Then there exists a unique iteration

$$\langle M_\alpha, G^*_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

of (M, I) such that for all $\beta < \alpha \leq \gamma$, $G_\beta = G^*_\beta$ and

$$\mathcal{P}(\mathcal{P}(\omega_1)/I)^{N_\alpha} = \mathcal{P}(\mathcal{P}(\omega_1)/I)^{M_\alpha}.$$

Furthermore, $N_\alpha = j^*_{0\alpha}(N)$ for all $\alpha \leq \gamma$.

Given ordinals α, β, the partial order $\mathrm{Col}(\alpha, \beta)$ is the set of partial functions from α to β whose domain has cardinality less than that of α, ordered by inclusion. In particular, $\mathrm{Col}(\omega, \beta)$ makes β countable. Given ordinals α and β, $\mathrm{Col}(\alpha, {<}\beta)$ is the partial order consisting of all partial functions $p : \beta \times \alpha \to \beta$ of cardinality less than α, such that for all $(\delta, \gamma) \in \mathrm{dom}(p)$, $p(\delta, \gamma) \in \delta$, ordered by inclusion (we will not use this definition until the next section).

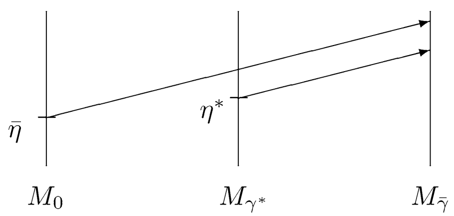
The proof of the following lemma is a modification of standard arguments tracing back to Gaifman and Solovay ([5] contains a variation of the lemma).

1.6 Lemma. *Suppose that M is a transitive model of* ZFC°+ *Power Set + Choice + Σ_1-Replacement and that $I \in M$ is a normal precipitous ideal on ω_1^M. Suppose that $j : (M, I) \to (M^*, I^*)$ is an iteration of (M, I) whose length is in $(\omega_1^V + 1) \cap M$. Then M^* is wellfounded.*

Proof. If j and M^* are as in the statement of the lemma, then M^* is the union of all sets of the form $j(H(\kappa)^M)$, where κ is a regular cardinal in M, and for each such $\kappa > |\mathcal{P}(\mathcal{P}(\omega_1))|^M$, $j{\restriction}H(\kappa)^M$ is an iteration of $(H(\kappa)^M, I)$. If the lemma fails, we may let $(\bar\gamma, \bar\kappa, \bar\eta)$ be the lexicographically least triple (γ, κ, η) such that

- κ is a regular cardinal in M greater than $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$,
- $\eta < \kappa$,
- there is an iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$ of $(H(\kappa)^M, I)$ such that $j_{0\gamma}(\eta)$ is not wellfounded.

Since I is precipitous in M, $\bar\gamma$ is a limit ordinal, and clearly $\bar\eta$ is a limit ordinal as well. Fix an iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar\gamma\rangle$ of $(H(\bar\kappa)^M, I)$ such that $j_{0\bar\gamma}(\bar\eta)$ is not wellfounded, and let $\langle M_\alpha, G_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar\gamma\rangle$ be the corresponding iteration of M as in Lemma 1.5. By the minimality of $\bar\gamma$ we have that M_α is wellfounded for all $\alpha < \bar\gamma$. Since $N_{\bar\gamma}$ is the direct limit of the iteration leading up to it, we may fix $\gamma^* < \bar\gamma$ and $\eta^* < j_{0\gamma^*}(\bar\eta)$ such that $j_{\gamma^*\bar\gamma}(\eta^*)$ is not wellfounded. By Lemma 1.5, $j'_{\gamma^*,\bar\gamma}(\eta^*) = j_{\gamma^*,\bar\gamma}(\eta^*)$ and $j'_{\gamma^*,\bar\gamma}(\bar\eta) = j'_{\gamma^*,\bar\gamma}(\bar\eta)$.



The key point is that if N is a model of ZFC°, J is a normal ideal on ω_1^N in N, γ is an ordinal and η is an ordinal in N, then the statement positing an iteration of (N, J) of length γ whose last model is illfounded below the image of η is a Σ^1_1 sentence in a real parameter recursively coding N, η and γ, and so this statement is absolute between wellfounded models of ZFC° containing such a real. In particular, if

- N is a transitive model of ZFC° + Power Set,
- J is a normal ideal on ω_1^N in N,
- κ is a regular cardinal in N greater than $|\mathcal{P}(\mathcal{P}(\omega_1))|^N$,
- $\eta < \kappa$ and γ are ordinals in N and $\beta \in N$ is an ordinal greater than or equal to $\max\{(2^\kappa)^N, \gamma\}$,

then if G is N-generic for $\mathrm{Col}(\omega, \beta)$, then $N[G]$ satisfies the correct answer for the assertion that there exists an iteration of $(H(\kappa)^N, J)$ of length γ whose last model is illfounded below the image of η. Let $\phi(\gamma, \kappa, \eta, J)$ be the formula asserting that

- J is a normal ideal on ω_1,
- κ is a regular cardinal greater than $|\mathcal{P}(\mathcal{P}(\omega_1))|$,
- $\eta < \kappa$,
- letting $\beta = \max\{2^\kappa, \gamma\}$, every condition (equivalently, some condition) in $\mathrm{Col}(\omega, \beta)$ forces that there exists an iteration of $(H(\kappa), J)$ of length γ whose last model is illfounded below the image of η.

Then, in M, $(\bar{\gamma}, \bar{\kappa}, \bar{\eta})$ is the lexicographically least triple (γ, κ, η) such that $\phi(\gamma, \kappa, \eta, I)$ holds. Furthermore, since $j'_{0\gamma^*}$ is elementary, in M_{γ^*},

$$(j'_{0\gamma^*}(\bar{\gamma}), j'_{0\gamma^*}(\bar{\kappa}), j'_{0\gamma^*}(\bar{\eta}))$$

is the least triple (γ, κ, η) such that $\phi(\gamma, \kappa, \eta, j'_{0\gamma^*}(I))$ holds. However, the tail of the iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ starting with N_{γ^*} is an iteration of

$$(H(j_{0\gamma^*}(\bar{\kappa}))^{M_{\gamma^*}}, j_{0\gamma^*}(I))$$

(note that $j'_{0\gamma^*}(H(\bar{\kappa})^M) = j_{0,\gamma^*}(H(\bar{\kappa})^M)$) of length less than or equal to $\bar{\gamma}$ which in turn is less than or equal to $j'_{0\gamma^*}(\bar{\gamma})$. Furthermore, $\eta^* < j'_{0\gamma}(\bar{\eta}) = j_{0\gamma}(\bar{\eta})$, and $j_{\gamma^*\bar{\gamma}}(\eta^*)$ is not wellfounded, which, by the correctness property mentioned above (using the fact that M_{γ^*} is wellfounded) contradicts the minimality of $j'_{0\gamma^*}(\bar{\eta})$. ⊣

1.7 Example. Let M be any countable transitive model of ZFC in which there exists a measurable cardinal κ and a normal measure $\mu \in M$ on κ such that all countable iterates of M by μ are wellfounded. Iterating M by μ ω_1

times, we obtain a model N of ZFC containing ω_1 such that $(V_\kappa)^M = (V_\kappa)^N$. Now suppose that I is a normal precipitous ideal on ω_1^M in M. By Lemmas 1.5 and 1.6, $((V_\kappa)^M, I)$ iterable.

Before moving on, we prove an important fact about iterations of iterable models which will show up later (in Lemmas 3.3, 6.2 and 7.4). This fact is a key step in Woodin's proof of Theorem 0.3.

1.8 Lemma. *Suppose that M is a countable transitive model of* ZFC° *and $I \in M$ is a normal ideal on ω_1^M such that the pair (M, I) is iterable. Let x be a real coding the pair (M, I) under some recursive coding. Let*

$$\mathcal{I} = \langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$$

be an iteration of (M, I). Then every countable ordinal γ such that $L_\gamma[x]$ satisfies ZFC *is on the critical sequence of $\mathcal{I}$.*

Proof. Fix a countable ordinal γ such that $L_\gamma[x] \models$ ZFC. We want to see that for every $\eta < \gamma$ there is a $\delta < \gamma$ such that the ordinals of the final model of every iteration of (M, I) of length η are contained in δ. To see this, fix η and let $g \subseteq \mathrm{Col}(\omega, \eta)$ be $L_\gamma[x]$-generic. Then $L_\gamma[x][g] \models$ ZFC, and in $L_\gamma[x][g]$, the set of ordertypes of the ordinals of iterates of (M, I) by iterations of length η is a Σ^1_1 set in a real coding (M, I) and g. By the boundedness lemma for $\utilde{\Sigma}^1_1$ sets of wellorderings (see [26]), then, there is a countable (in $L_\gamma[x][g]$) ordinal δ such that all of these ordertypes are less than δ. Furthermore, the nonexistence of an iteration of (M, I) of length η such that δ can be embedded in an order-preserving way into the ordinals of the final model is absolute between $L_\gamma[x][g]$ and V, by $\utilde{\Sigma}^1_1$-absoluteness. ⊣

Lemma 1.8 has the following useful corollary. The case where γ is countable follows immediately from Lemma 1.8. The case where $\gamma = \omega_1$ follows by applying the countable case to a forcing extension where ω_1 is collapsed.

1.9 Corollary. *Suppose that M is a countable transitive model of* ZFC°*, I is normal ideal on ω_1^M in M, (M, I) is iterable and x is a real coding (M, I). Suppose that γ is an x-indiscernible less than or equal to ω_1, and let $j : (M, I) \to (M^*, I^*)$ be an iteration of (M, I) of length γ. Then the ordinals of M^* have height less than the least x-indiscernible above γ.*

We note one more useful fact about sharps. The fact can be proved directly using the remarkable properties of sharps, or by noting that the two functions implicit in the fact necessarily represent the same ordinal in any generic ultrapower.

1.10 Fact. Let x be a real and let γ be the least x-indiscernible above ω_1. Let $\pi : \omega_1 \to \gamma$ be a bijection. Then the set of $\alpha < \omega_1$ such that the ordertype of $\{\pi(\beta) : \beta < \alpha\}$ is the least x-indiscernible above α contains a club.

1.11 Remark. If there exists a precipitous ideal on ω_1, then $A^{\#}$ exists for every $A \subseteq \omega_1$. To see this, note first of all that the existence of a precipitous ideal implies that for each real x there is an nontrivial elementary embedding from $L[x]$ to $L[x]$ in a forcing extension, which means that $x^{\#}$ exists already in the ground model. Furthermore, if I is a precipitous ideal on ω_1 and $j : V \to M$ is the generic embedding derived from a V-generic filter $G \subseteq \mathcal{P}(\omega_1)/I$, then $\mathcal{P}(\omega_1)^V \subseteq H(\omega_1)^M$. Therefore, for every $A \in \mathcal{P}(\omega_1)^V$, $A \in M$ and $M \models$ "$A^{\#}$ exists". Since M and $V[G]$ have the same ordinals, $V[G]$ and V then must also satisfy "$A^{\#}$ exists".

Similarly, if (M, I) is an iterable pair, then M is correct about the sharps of the reals of M, since M is elementarily embedded into a transitive model containing ω_1, and thus M is correct about the sharps of the members of $\mathcal{P}(\omega_1)^M$. In particular, if (M, I) is an iterable pair and A is in $\mathcal{P}(\omega_1)^M$, then $\mathcal{P}(\omega_1)^{L[A]} \subseteq M$, so M correctly computes $\omega_1^{L[A]}$.

2. $\mathbb{P}_{\max}$

We are now ready to define the partial order $\mathbb{P}_{\max}$. We will make one modification of the definition given in [39] and require the conditions to satisfy ZFC° instead of the theory ZFC* defined in [39]. Our $\mathbb{P}_{\max}$ is a dense suborder of the original; furthermore, the basic analysis of the two partial orders is the same, though the proofs of Lemma 7.6 and Theorem 7.7 are less elegant than they might otherwise be.

Recall that $\mathrm{MA}_{\aleph_1}$ is the version of Martin's Axiom for $\aleph_1$-many dense sets, i.e., the statement that whenever P is a c.c.c. partial order and D_α ($\alpha < \omega_1$) are dense subsets of P there is a filter $G \subseteq P$ intersecting each D_α.

2.1 Definition. The partial order $\mathbb{P}_{\max}$ consists of all pairs $\langle (M, I), a\rangle$ such that

1. M is a countable transitive model of ZFC°+ $\mathrm{MA}_{\aleph_1}$,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. $a \in \mathcal{P}(\omega_1)^M$,
5. there exists an $x \in \mathcal{P}(\omega)^M$ such that $\omega_1^M = \omega_1^{L[a,x]}$.

The order on $\mathbb{P}_{\max}$ is as follows: $\langle (M, I), a\rangle < \langle (N, J), b\rangle$ if $N \in H(\omega_1)^M$ and there exists an iteration $j : (N, J) \to (N^*, J^*)$ such that

- $j(b) = a$,
- $j, N^* \in M$,
- $I \cap N^* = J^*$.

We say that a pair (M, I) is a ($\mathbb{P}_{\max}$) *pre-condition* if there exists an a such that $\langle (M, I), a\rangle$ is in $\mathbb{P}_{\max}$.

2.2 Remark. If $\langle (M, I), a\rangle$ is a $\mathbb{P}_{\max}$ condition, then M is closed under sharps for reals (see Remark 1.11), and so a cannot be in $L[x]$ for any real x in M. Therefore, a is unbounded in ω_1^M, and this in turn implies that the iteration witnessing that a given $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ is stronger than another condition $\langle (N, J), b\rangle$ must have length ω_1^M.

2.3 Remark. To see that the order on $\mathbb{P}_{\max}$ is transitive, let j_0 be an iteration witnessing that $\langle (M_1, I_1), a_1\rangle < \langle (M_0, I_0), a_0\rangle$ and let j_1 be an iteration witnessing that $\langle (M_2, I_2), a_2\rangle < \langle (M_1, I_1), a_1\rangle$. Then j_0 is an element of M_1, and it is not hard to check that $j_1(j_0)$ witnesses that $\langle (M_2, I_2), a_2\rangle < \langle (M_0, I_0), a_0\rangle$.

2.4 Remark. The requirement that the models in $\mathbb{P}_{\max}$ conditions satisfy $\mathrm{MA}_{\aleph_1}$ is used for a particular consequence of $\mathrm{MA}_{\aleph_1}$ known as *almost disjoint coding* [12]. That is, it follows from $\mathrm{MA}_{\aleph_1}$ that if $Z = \{z_\alpha : \alpha < \omega_1\}$ is a collection of infinite subsets of ω whose pairwise intersections are finite (i.e., Z is an *almost disjoint family*), then for each $B \subseteq \omega_1$ there exists a $y \subseteq \omega$ such that for all $\alpha < \omega_1$, $\alpha \in B$ if and only if $y \cap z_\alpha$ is infinite. This will be used to show that if $\langle (M, I), a\rangle$ is a $\mathbb{P}_{\max}$ condition, then any iteration of (M, I) is uniquely determined by the image of a (see Lemma 2.7), so that there is a unique iteration witnessing the order on each pair of comparable conditions. One can vary $\mathbb{P}_{\max}$ by removing condition (5) and the requirement that $\mathrm{MA}_{\aleph_1}$ holds, and replace a with a set of iterations of smaller models into M, as in the definition of the order, satisfying this uniqueness condition. Alternately, one can require that the models satisfy the statement ψ_{AC} (see Definition 6.1 and Remark 6.4), which implies that the image of any stationary, co-stationary subset of ω_1 under an iteration determines the entire iteration.

2.5 Remark. Instead of using ideals on ω_1, we could use the stationary tower $\mathbb{Q}_{<\delta}$ (see [19]) to produce the iterations giving the order on conditions. This gives us another degree of freedom in choosing our models, since in this case a small forcing extension of a condition is also a condition, roughly speaking. The resulting extension is essentially identical.

2.6 Remark. Given a real x, $x^\dagger$ ("x dagger") is a real such that in $L[x^\dagger]$ there exists a transitive model M of ZFC containing $\omega_1^V \cup \{x\}$ in which some ordinal countable in $L[x^\dagger]$ is a measurable cardinal (see [14]; this fact about $x^\dagger$ does not characterize it, but it is its only property that we require in this chapter). By [11], if there exists a measurable cardinal, then there is a partial order forcing that NS_{ω_1} is precipitous. By [22, 13], c.c.c. forcings preserve precipitousness of NS_{ω_1}. Essentially the same arguments show that if κ is a measurable cardinal and P is a c.c.c. forcing in the $\mathrm{Col}(\omega, {<}\kappa)$-extension, then there is a normal precipitous ideal on ω_1 (which is κ) in the $\mathrm{Col}(\omega, {<}\kappa) * P$-extension. By Lemmas 1.5 and 1.6, then, the statement that

$x^\dagger$ exists for each real x implies that every real exists in the model M of some $\mathbb{P}_{\max}$ condition, and, by Lemma 2.8 below, densely many. However, the full strength of $\mathbb{P}_{\max}$ will require the consistency strength of significantly larger cardinals.

We will now prove two facts about iterations which are central to the $\mathbb{P}_{\max}$ analysis.

2.7 Lemma. *Let $\langle(M,I),a\rangle$ be a $\mathbb{P}_{\max}$ condition and let A be a subset of ω_1. Then there is at most one iteration of (M,I) for which A is the image of a. Furthermore, this iteration is in $L[\langle(M,I),a\rangle,A]$, if it exists.*

Proof. Fix a real x in M such that $\omega_1^M=\omega_1^{L[a,x]}$, and let $Z=\langle z_\alpha:\alpha<\omega_1^M\rangle$ be the almost disjoint family defined recursively from the constructibility order in $L[a,x]$ on $\mathcal{P}(\omega)^{L[a,x]}$ by letting z_α be the constructibly (in $L[a,x]$) least infinite $z\subseteq\omega$ almost disjoint from each z_β ($\beta<\alpha$) such that for no finite $a\subseteq\alpha$ does $\bigcup\{z_\beta:\beta\in a\}$ contain the complement of z (modulo finite). Suppose that

$$\mathcal{I}=\langle M_\alpha,G_\beta,j_{\alpha\delta}:\beta<\alpha\le\delta\le\gamma\rangle$$

and

$$\mathcal{I}'=\langle M'_\alpha,G'_\beta,j'_{\alpha\delta}:\beta<\alpha\le\delta\le\gamma'\rangle$$

are two iterations of (M,I) such that $j_{0\gamma}(a)=A=j'_{0\gamma'}(a)$. Then $j_{0\gamma}(Z)=j'_{0\gamma'}(Z)$ (this uses Remark 1.11 to see that the constructibility order on reals in $L[A,x]$ is computed correctly in M_γ and $M'_{\gamma'}$). Let $\langle z_\alpha:\alpha<j_{0\gamma}(\omega_1^M)\rangle$ enumerate $j_{0\gamma}(Z)$.

Without loss of generality, $\gamma\le\gamma'$. We show by induction on $\alpha<\gamma$ that, for each such α, $G_\alpha=G'_\alpha$. This will suffice. Fix α and suppose that

$$\{G_\beta:\beta<\alpha\}=\{G'_\beta:\beta<\alpha\}.$$

Then $M_\alpha=M'_\alpha$. For each $B\in\mathcal{P}(\omega_1)^{M_\alpha}$, $B\in G_\alpha$ if and only if $\omega_1^{M_\alpha}\in j_{\alpha(\alpha+1)}(B)$, and $B\in G'_\alpha$ if and only if $\omega_1^{M_\alpha}\in j'_{\alpha(\alpha+1)}(B)$. Applying almost disjoint coding, fix $x\in\mathcal{P}(\omega)^{M_\alpha}$ such that for all $\eta<\omega_1^{M_\alpha}$, $\eta\in B$ if and only if $x\cap z_\eta$ is infinite. Then $B\in G_\alpha$ if and only if $x\cap z_{\omega_1^M}$ is infinite if and only if $B\in G'_\alpha$.

For the last part of the lemma, note that the argument just given gives a definition for each G_α in terms of A, x and the iteration up to α. ⊣

One consequence of Lemma 2.7 is that, if $G\subseteq\mathbb{P}_{\max}$ is an $L(\mathbb{R})$-generic filter, and $A=\bigcup\{a\mid\langle(M,I),a\rangle\in G\}$, then $L(\mathbb{R})[G]=L(\mathbb{R})[A]$. Therefore, the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ satisfies the sentence "$V=L(\mathcal{P}(\omega_1))$" (see the discussion at the beginning of Sect. 5).

2.8 Lemma (ZFC°). *If (M,I) is a pre-condition in $\mathbb{P}_{\max}$ and J is a normal ideal on ω_1 then there exists an iteration $j:(M,I)\to(M^*,I^*)$ such that $j(\omega_1^M)=\omega_1$ and $I^*=J\cap M^*$.*

Proof. First let us note that if $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ is any iteration of (M, I), then $j_{0\omega_1}(I) \subseteq J \cap M_{\omega_1}$. To see this, note that if $E \in j_{0\omega_1}(I)$, then $E \in M_{\omega_1}$ and $E = j_{\alpha\omega_1}(E')$ for some $\alpha < \omega_1$ and $E' \in j_{0\alpha}(I)$. Then for all $\beta \in [\alpha, \omega_1)$, $j_{\alpha\beta}(E') \notin G_\beta$, so $\omega_1^{M_\beta} \notin E$. Therefore, E is nonstationary, so $E \in J$ by the normality of J.

Now, noting that J is a normal ideal, let $\{A_{i\alpha} : i < \omega, \alpha < \omega_1\}$ be a collection of pairwise disjoint members of $\mathcal{P}(\omega_1) \setminus J$. We build an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ by recursively choosing the G_β's. As we do this, for each $\alpha < \omega_1$ we let the set $\{B_i^\alpha : i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{0\alpha}(I)$. Given

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma \rangle,$$

then, for some $\gamma \leq \omega_1$, if $\omega_1^{M_\gamma} \in A_{i\alpha}$ for some $i < \omega$ and $\alpha \leq \gamma$, then (noting that there can be at most one such pair (i, α)) we let G_γ be any M_γ-generic filter for $(\mathcal{P}(\omega_1)/j_{0\gamma}(I))^{M_\gamma}$ with $j_{\alpha\gamma}(B_i^\alpha)$ as a member. If $\omega_1^{M_\gamma}$ is not in $A_{i\alpha}$ for any $i < \omega$ and $\alpha \leq \gamma$, then we let G_γ be any M_γ-generic filter.

To see that this construction works, fix $E \in \mathcal{P}(\omega_1)^{M_{\omega_1}} \setminus j_{0\omega_1}(I)$. We need to see that E is not in J. We may fix $i < \omega$ and $\alpha < \omega_1$ such that $E = j_{\alpha\omega_1}(B_i^\alpha)$. Then $F = (A_{i\alpha} \cap \{\omega_1^{M_\beta} : \beta \in [\alpha, \omega_1)\}) \subseteq E$. Since F is the intersection of a club and set not in J, F is not in J, so E is not in J. ⊣

The construction in the proof of Lemma 2.8 appears repeatedly in the analysis of $\mathbb{P}_{\max}$. In order to make our presentation of $\mathbb{P}_{\max}$ more modular (i.e., to avoid having to write out the proof of Lemma 2.8 repeatedly), we give the following strengthening of the lemma in terms of games. We note that the games defined here (and before Lemmas 3.5 and 5.2 and at the end of Sect. 10.2) are not part of Woodin's original presentation of $\mathbb{P}_{\max}$.

Suppose that (M, I) is a pre-condition in $\mathbb{P}_{\max}$, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Let $\mathcal{G}((M, I), J, B)$ be the following game of length ω_1 where Players I and II collaborate to build an iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$$

of (M, I) of length ω_1. In each round α, if $\alpha \in B$, then Player I chooses a set A_α in $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{0\alpha}(I)$ and then Player II chooses an M_α-generic filter G_α contained in $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$ with $A_\alpha \in G_\alpha$. If $\alpha \notin B$, then Player II chooses any M_α-generic filter $G_\alpha \subseteq (\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$. After all ω_1 many rounds have been played, Player I wins if $j_{0\omega_1}(I) = J \cap M_{\omega_1}$.

The proof of Lemma 2.9 is almost identical to the proof of Lemma 2.8.

2.9 Lemma (ZFC°). *Suppose that (M, I) is a pre-condition in $\mathbb{P}_{\max}$, let J be a normal ideal on ω_1, and let B be a subset of ω_1. Then Player* I *has a winning strategy in $\mathcal{G}((M, I), J, B)$ if and only if $B \notin J$.*

Using Remark 2.6 and Lemmas 2.7 and 2.8, we can show that $\mathbb{P}_{\max}$ is homogeneous and countably closed. By *homogeneity* we mean the following property: for each pair of conditions p_0, p_1 in $\mathbb{P}_{\max}$ there exist conditions q_0, q_1

such that $q_0 \leq p_0$, $q_1 \leq p_1$ and the suborders of $\mathbb{P}_{\max}$ below the conditions q_0 and q_1 are isomorphic. The importance of this property is that it implies that the theory of the $\mathbb{P}_{\max}$ extension can be computed in the ground model.

2.10 Lemma. *If $x^\dagger$ exists for each real x, then $\mathbb{P}_{\max}$ is homogeneous.*

Proof. Let $p_0 = \langle (M_0, I_0), a_0\rangle$ and $p_1 = \langle (M_1, I_1), a_1\rangle$ be conditions in $\mathbb{P}_{\max}$. By Remark 2.6, we can fix a pre-condition (N, J) with $p_0, p_1 \in H(\omega_1)^N$. Applying Lemma 2.8 in N, we may fix iterations $j_0 : (M_0, I_0) \to (M_0^*, I_0^*)$ and $j_1 : (M_1, I_1) \to (M_1^*, I_1^*)$ in N such that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$. Letting $a_0^* = j_0(a_0)$ and $a_1^* = j_1(a_1)$, then,

$$q_0 = \langle (N, J), a_0^*\rangle$$

and

$$q_1 = \langle (N, J), a_1^*\rangle$$

are conditions in $\mathbb{P}_{\max}$ and j_0 and j_1 witness that $q_0 \leq p_0$ and $q_1 \leq p_1$ respectively.

Now, if $q_0' = \langle (N', J'), a'\rangle$ is a condition below q_0, then there is an iteration $j' : (N, J) \to (N^*, J^*)$ witnessing this. Then $a' = j'(a_0^*)$, and $q_1' = \langle (N', J'), j'(a_1^*)\rangle$ is a condition below q_1. Let π be the map with domain the suborder of $\mathbb{P}_{\max}$ below q_0 which sends each $\langle (N', J'), a'\rangle$ to the corresponding $\langle (N', J'), j'(a_1^*)\rangle$ as above. By Lemma 2.7, this map is an isomorphism between the suborders below q_0 and q_1 respectively. ⊣

In order to show that $\mathbb{P}_{\max}$ is countably closed, we must define a new class of iterations.

3. Sequences of Models and Countable Closure

For each $i < \omega$, let $p_i = \langle (M_i, I_i), a_i\rangle$ be a $\mathbb{P}_{\max}$ condition, and for each such i let $j_{i(i+1)} : (M_i, I_i) \to (M_i^*, I_i^*)$ be an iteration witnessing that $p_{i+1} < p_i$. Let $\{j_{ik} : i \leq k < \omega\}$ be the commuting family of embeddings generated by the $j_{i(i+1)}$'s. Let $a = \bigcup\{a_i : i < \omega\}$. By Lemma 2.7, for each $i < \omega$ there is a unique iteration $j_{i\omega} : (M_i, I_i) \to (N_i, J_i)$ sending a_i to a. Since each (M_i, I_i) is iterable, each N_i is wellfounded, and the structure $(\langle (N_i, J_i) : i < \omega\rangle, a)$ satisfies the following definition.

3.1 Definition. A *limit sequence* is a pair $(\langle (N_i, J_i) : i < \omega\rangle, a)$ such that the following hold for all $i < \omega$:

1. N_i is a countable transitive model of ZFC°,
2. $J_i \in N_i$ and in N_i, J_i is a normal ideal on ω_1,
3. $\omega_1^{N_i} = \omega_1^{N_0}$,
4. for all $k < i$, $N_k \in H(\omega_2)^{N_i}$,

5. for all $k < i$, $J_k = J_i \cap N_k$,

6. $a \in \mathcal{P}(\omega_1)^{N_0}$,

7. there exists an $x \in \mathcal{P}(\omega)^{N_0}$ such that $\omega_1^{N_0} = \omega_1^{L[a,x]}$.

A structure $\langle (N_i, J_i) : i < \omega \rangle$ is a *pre-limit sequence* if there exists a set a such that $(\langle (N_i, J_i) : i < \omega \rangle, a)$ is a limit sequence.

If we write a sequence as $\langle N_k : k < \omega \rangle$ (as we will in Sect. 10.1), the ideals are presumed to be the nonstationary ideal on ω_1.

If p_i $(i < \omega)$ is a descending sequence of $\mathbb{P}_{\max}$ conditions, then the *limit sequence corresponding to* p_i $(i < \omega)$ is the structure

$$(\langle (N_i, J_i) : i < \omega \rangle, a)$$

defined above. In this case each $\langle (N_i, J_i), a \rangle$ is a condition in $\mathbb{P}_{\max}$.

If $\langle (N_i, J_i) : i < \omega \rangle$ is a pre-limit sequence, then a filter

$$G \subseteq \bigcup \{ \mathcal{P}(\omega_1)^{N_i} \setminus J_i : i < \omega \}$$

is a $\bigcup \{ N_i : i < \omega \}$-*normal ultrafilter* for the sequence if for every regressive function f on $\omega_1^{N_0}$ in any N_i, f is constant on some member of G. Given such G and $\langle (N_i, J_i) : i < \omega \rangle$, we form the ultrapower of the sequence by letting N_i^* be the ultrapower of N_i formed from G and *all* functions $f : \omega_1^{N_0} \to N_i$ existing in any N_k (this ensures that the image of each N_i in the ultrapower of each N_k $(k > i)$ is the same as the ultrapower of N_i). As usual, we identify the transitive parts of each N_i^* with their transitive collapses. If (for each $i < \omega$) we let j_i^* be the induced embedding of N_i into N_i^* then for each $i < k < \omega$, $j_i^* = j_k^* \restriction N_i$, so we can let $j^* = \bigcup \{ j_i^* : i < \omega \}$ be the embedding corresponding to the ultrapower of the sequence.

3.2 Definition. Let $\langle (N_i, J_i) : i < \omega \rangle$ be a pre-limit sequence, and let γ be an ordinal less than or equal to ω_1. An *iteration* of $\langle (N_i, J_i) : i < \omega \rangle$ of length γ consists of pre-limit sequences $\langle (N_i^\alpha, J_i^\alpha) : i < \omega \rangle$ $(\alpha \leq \gamma)$, normal ultrafilters G_α $(\alpha < \gamma)$ and a commuting family of embeddings $j_{\alpha\beta}$ $(\alpha \leq \beta \leq \gamma)$ such that

- $\langle (N_i^0, J_i^0) : i < \omega \rangle = \langle (N_i, J_i) : i < \omega \rangle$,
- for all $\alpha < \gamma$, $G_\alpha \subseteq \bigcup \{ \mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha : i < \omega \}$ is a normal ultrafilter for the sequence $\langle (N_i^\alpha, J_i^\alpha) : i < \omega \rangle$, and $j_{\alpha(\alpha+1)}$ is the corresponding embedding,
- for each limit ordinal $\beta \leq \gamma$, $\langle (N_i^\beta, J_i^\beta) : i < \omega \rangle$ is the direct limit of the system $\{ \langle (N_i^\alpha, J_i^\alpha) : i < \omega \rangle, j_{\alpha\delta} : \alpha \leq \delta < \beta \}$ and for each $\alpha < \beta$ $j_{\alpha\beta}$ is the induced embedding.

As with iterations of single models, we sometimes describe an iteration of a pre-limit sequence by fixing only the initial sequence, the final sequence and the embedding between them. An *iterate* of a pre-limit sequence $\bar{p}$ is a pre-limit sequence appearing in an iteration of $\bar{p}$. If every iterate of a pre-limit sequence is wellfounded, then the sequence is *iterable*.

By Lemma 1.8 and Corollary 1.9, pre-limit sequences derived from descending chains $\{\langle (M_i, I_i), a_i\rangle : i < \omega\}$ in $\mathbb{P}_{\max}$ satisfy the hypotheses of the following lemma, letting x_i be any real in M_{i+1} coding M_i. Yet another way to vary $\mathbb{P}_{\max}$ is to replace the model M in the definition of $\mathbb{P}_{\max}$ conditions with sequences satisfying this hypothesis. This approach is used for the order $\mathbb{Q}^*_{\max}$ defined in Sect. 10.1.

3.3 Lemma. *Suppose that $\bar{p} = \langle (N_i, J_i) : i < \omega\rangle$ is a pre-limit sequence, and suppose that for each $i < \omega$ there is a real $x_i \in N_{i+1}$ such that $x_i^{\#} \in N_{i+1}$ and*

- *the least x_i-indiscernible above $\omega_1^{N_0}$ is greater than the ordinal height of N_i,*
- *every club subset of $\omega_1^{N_0}$ in N_i contains a tail of the x_i-indiscernibles below $\omega_1^{N_0}$.*

Then $\bar{p}$ is iterable.

Proof. First we will show that any iterate of $\bar{p}$ is wellfounded if its version of ω_1 is wellfounded. Then we will show that the ω_1 of each iterate of $\bar{p}$ is wellfounded.

For the first part, if $\langle (N_i^*, J_i^*) : i < \omega\rangle$ is an iterate of $\bar{p}$, then by elementarity the ordinals of each N_i^* embed into the least x_i-indiscernible above $\omega_1^{N_0^*}$. So, if $\omega_1^{N_0^*}$ is actually an ordinal (i.e., is wellfounded), then N_{i+1}^* constructs this next x_i-indiscernible correctly, and so N_i^* is wellfounded.

We prove the second part by induction on the length of the iteration, noting that the limit case follows immediately, and the successor case follows from the case of an iteration of length 1. What we want to see is that if G is a normal ultrafilter for $\bar{p}$ and j is the induced embedding, then $j(\omega_1^{N_0}) = \bigcup\{N_i \cap \mathrm{On} : i < \omega\}$. Notice that for each x_i, if $f_i : \omega_1^{N_0} \to \omega_1^{N_0}$ is defined by letting $f_i(\alpha)$ be the least x_i-indiscernible above α, then $j(f_i)(\omega_1^{N_0})$ is the least indiscernible of x_i above $\omega_1^{N_0}$. Thus

$$j(\omega_1^{N_0}) \geq \sup\{j(f_i)(\omega_1^{N_0}) : i < \omega\} = \bigcup\{N_i \cap \mathrm{On} : i < \omega\}\,.$$

For the other direction, let $h : \omega_1^{N_0} \to \omega_1^{N_0}$ be a function in some N_i. Then the closure points of h contain a tail of the x_i-indiscernibles, which means that $f_i > h$ on a tail of the ordinals below $\omega_1^{N_0}$, so $[f_i]_G > [h]_G$. Thus $j(\omega_1^{N_0}) = \bigcup\{N_i \cap \mathrm{On} : i < \omega\}$. ⊣

The following lemma has essentially the same proof as Lemma 2.8, and shows (given that $x^\dagger$ exists for each real x) that $\mathbb{P}_{\max}$ is countably closed. The point is that if $\langle p_i : i < \omega\rangle$ is a descending sequence of $\mathbb{P}_{\max}$ conditions, letting $\bar{p} = (\langle (N_i, J_i) : i < \omega\rangle, a)$ be the limit sequence corresponding to $\langle p_i : i < \omega\rangle$, if (M, I) is a $\mathbb{P}_{\max}$ pre-condition with $\{p_i : i < \omega\}$, $\bar{p} \in H(\omega_1)^M$, then by letting j^* be an iteration of $\bar{p}$ resulting from applying Lemma 3.4 inside of M, the embedding $j^*(j_{i\omega})$ (where $j_{i\omega}$ is as defined in the first paragraph of this section) witnesses that $\langle (M, I), j^*(a)\rangle$ is below p_i in $\mathbb{P}_{\max}$, for each $i < \omega$.

3.4 Lemma (ZFC°). *Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, and let I be a normal ideal on ω_1. Then there is an iteration*

$$j^* : \langle (N_i, J_i) : i < \omega\rangle \to \langle (N_i^*, J_i^*) : i < \omega\rangle$$

such that $j^*(\omega_1^{N_0}) = \omega_1$ and $J_i^* = I \cap N_i^*$ for each $i < \omega$.

Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, let I be a normal ideal on ω_1, and let B be a subset of ω_1. Let

$$\mathcal{G}_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, B)$$

be the following game of length ω_1 where Players I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega\rangle$ consisting of pre-limit sequences $\langle (N_i^\alpha, J_i^\alpha) : i < \omega\rangle$ $(\alpha \leq \omega_1)$, normal ultrafilters G_α $(\alpha < \omega_1)$ and a family of embeddings $j_{\alpha\beta}$ $(\alpha \leq \beta \leq \omega_1)$, as follows. In each round α, let

$$X_\alpha = \bigcup\{\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha : i < \omega\}\,.$$

If $\alpha \in B$, then Player I chooses a set $A \in X_\alpha$, and then Player II chooses a $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in X_α with $A \in G_\alpha$. If α is not in B, then Player II chooses any $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in X_α. After all ω_1 many rounds have been played, Player I wins if $J_i^{\omega_1} = I \cap N_i^{\omega_1}$ for each $i < \omega$.

Lemma 3.4 can be rephrased in terms of games as follows.

3.5 Lemma (ZFC°). *Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, let I be a normal ideal on ω_1 and let B be a subset of ω_1. Then Player* I *has a winning strategy in $\mathcal{G}_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, B)$ if and only if $B \notin I$.*

At this point, we have gone as far with the $\mathbb{P}_{\max}$ analysis as daggers can take us.

4. Generalized Iterability

The following definition gives a generalized iterability property with respect to a given set of reals. In the $\mathbb{P}_{\max}$ analysis, these sets of reals often code $\mathbb{P}_{\max}$-names for sets of reals.

4.1 Definition. Let A be a set of reals. If M is a transitive model of ZFC° and I is an ideal on ω_1^M which is normal and precipitous in M, then the pair (M, I) is A-*iterable* if

- (M, I) is iterable,
- $A \cap M \in M$,
- $j(A \cap M) = A \cap M^*$ whenever $j : (M, I) \to (M^*, I^*)$ is an iteration of (M, I).

4.2 Remark. The definition of A-iterability in [39] is more general than this one, in ways which we will not require.

In order to achieve the full effects of forcing with $\mathbb{P}_{\max}$ over a given model (for now we will deal with $L(\mathbb{R})$) we need to see (and in fact it is enough to see) that for each $A \subseteq \mathbb{R}$ in the model there exists a $\mathbb{P}_{\max}$ pre-condition (M, I) such that

- (M, I) is A-iterable,
- $\langle H(\omega_1)^M, A \cap M\rangle \prec \langle H(\omega_1), A\rangle$.

As it turns out, the existence of such a condition for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$ is equivalent to $\mathrm{AD}^{L(\mathbb{R})}$(see [39, pp. 285–290]).

There are two basic approaches to studying the $\mathbb{P}_{\max}$ extension. One can think of V as being a model of some form of determinacy, and use determinacy to analyze the $\mathbb{P}_{\max}$ forcing construction and its corresponding extension. Alternately, one can assume that Choice holds and certain large cardinals exist and use these large cardinals to analyze the $\mathbb{P}_{\max}$ extension of some inner model of ZF satisfying determinacy. Accordingly, the existence of A-iterable conditions (for a given set A) can be derived from determinacy or from large cardinals. We give here an example of each method, quoting some standard facts which we will briefly discuss.

The proof from large cardinals uses weakly homogeneous trees (see [26, 14] for more on the concepts reviewed briefly below). Recall that a *tree on* $\omega^k \times Z$ (for some integer k and some set Z) is a subset of the set of sequences of length $k+1$ whose first k elements are members of ω^n (for some integer n) and whose last element is in Z^n (as usual we require that if $\langle a_1, \ldots, a_k, u\rangle \in T$ and m is less than the length of a_1, then $\langle a_1 \restriction m, \ldots, a_k \restriction m, u \restriction m\rangle \in T$). Given such a tree T, the *projection of* T is the set of $\langle x_1, \ldots, x_k\rangle \in (\omega^\omega)^k$ for which there exists a $z \in Z^\omega$ such that for all integers n,

$$\langle x_1 \restriction n, \ldots, x_k \restriction n, z \restriction n\rangle$$

is in T.

Very briefly, a *countably complete tower* is a sequence of measures $\langle \sigma_i : i < \omega\rangle$ such that each σ_i is a measure on Z^i for some fixed underlying set

Z and for every sequence $\{A_i : i < \omega\}$ of sets such that each $A_i \in \sigma_i$ there exists a function $g \in Z^\omega$ such that $g{\restriction}i \in A_i$ for all $i < \omega$. Given a set Z, an integer k and a cardinal δ, a tree T on $\omega^k \times Z$ is *δ-weakly homogeneous* if there exists a countable family Σ of δ-complete measures on $Z^{<\omega}$ such that for each $\langle x_1, \ldots, x_k\rangle \in (\omega^\omega)^k$, $\langle x_1, \ldots, x_k\rangle \in p[T]$ if and only if there exists a sequence of measures $\{\sigma_i : i < \omega\} \subseteq \Sigma$ such that

- for all $i < \omega$, $\{z \in Z^i \mid \langle x_1{\restriction}i, \ldots, x_k{\restriction}i, z{\restriction}i\rangle \in T\} \in \sigma_i$,
- $\langle \sigma_i : i < \omega\rangle$ forms a countably complete tower.

A set of k-tuples of reals A is *δ-weakly homogeneously Suslin* if there exists a δ-weakly homogeneous tree T whose projection is A, and *weakly homogeneously Suslin* if it is δ-weakly homogeneously Suslin for some uncountable ordinal δ. For each integer k one can naturally code k-tuples of reals by single reals by interleaving coordinates. This induces an association of sets of k-tuples of reals to sets of reals which respects the property of being δ-weakly homogeneously Suslin for a given cardinal δ. This allows us to simplify notion in what follows and talk about weakly homogeneously Suslin sets of reals, knowing that these facts imply the same results for sets of k-sequences of reals.

The following fact is standard.

4.3 Theorem. *Let θ be a regular cardinal, suppose that $T \in H(\theta)$ is a weakly homogeneous tree on $\omega \times Z$ for some set Z. Let $\delta \geq 2^\omega$ be an ordinal such that there exists a countable collection Σ of δ^+-complete measures witnessing the weak homogeneity of T. Then for every elementary submodel Y of $H(\theta)$ of cardinality less than δ with $T, \Sigma \in Y$ there is an elementary submodel X of $H(\theta)$ containing Y such that $X \cap \delta = Y \cap \delta$, and such that, letting S be the image of T under the transitive collapse of X, $p[S] = p[T]$.*

Proof. Fixing θ, T, Σ and δ as in the statement of the theorem, the theorem follows from the following fact. Suppose that Y is an elementary submodel of $H(\theta)$ with $T, \Sigma \in Y$ and $|Y| < \delta$, and fix an $x \in p[T]$. Fix a countably complete tower $\{\sigma_i : i < \omega\} \subseteq \Sigma$ such that for all $i < \omega$, $\{a \in Z^i : (x{\restriction}i, a) \in T\} \in \sigma_i$, and for each $i < \omega$, let $A_i = \bigcap(\sigma_i \cap Y)$. Then since $\{\sigma_i : i < \omega\}$ is countably complete, there exists a $z \in Z^\omega$ such that for all $i < \omega$, $z{\restriction}i \in A_i$. Then the pair (x, z) forms a path through T, and, letting

$$Y[z] = \{f(z{\restriction}i) \mid i < \omega \wedge f : Z^i \to H(\theta) \wedge f \in Y\},$$

$Y[z]$ is an elementary submodel of $H(\theta)$ containing Y and $\{z{\restriction}i : i < \omega\}$, and, since each σ_i is δ^+-complete, $Y \cap \delta = Y[z] \cap \delta$. Repeated application of this fact for each real in the projection of T proves the theorem. ⊣

Proofs of the following facts about weakly homogeneous trees and weakly homogeneously Suslin sets of reals appear in [19]. Some of these facts follow directly from the definitions, and none are due to the author. Theorem 4.6 derives ultimately from [23].

4.4 Fact. For every cardinal δ, the collection of δ-weakly homogeneously Suslin sets of reals is closed under countable unions and continuous images.

4.5 Theorem (Woodin). *If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ then every set of reals in $L(\mathbb{R})$ is $<\delta$-weakly homogeneously Suslin (i.e., γ-weakly homogeneously Suslin for all $\gamma < \delta$).*

4.6 Theorem. *If δ is a cardinal and T is a δ-weakly homogeneous tree, then there is a tree S such that $p[T] = \omega^\omega \setminus p[S]$ in all forcing extensions by partial orders of cardinality less than δ (including the trivial one).*

4.7 Theorem (Woodin). *If δ is a Woodin cardinal and A is a δ^+-weakly homogeneously Suslin set of reals, then the complement of A is $<\delta$-weakly homogeneously Suslin.*

If S and T are trees whose projections are disjoint, then they remain disjoint in all forcing extensions, as there is a ranking function on the tree of attempts to build a real in both projections (likewise, the projection of a tree being nonempty is absolute to inner models containing the tree). This fact plus Theorem 4.6 gives the following corollary.

4.8 Corollary. *If δ is a cardinal and T_0 and T_1 are δ-weakly homogeneous trees with the same projection, then T_0 and T_1 still have the same projection in all forcing extensions by forcings of cardinality less than δ.*

Given a set of reals A, a set of reals B is *projective in A* if it can be defined by a projective formula (i.e., all unbounded quantifiers ranging over reals) with A as a parameter. Fact 4.4 and Theorem 4.7 together imply that if δ is a limit of Woodin cardinals then the set of $<\delta$-weakly homogeneously Suslin sets of reals is projectively closed. We separate out the following part of the proof of Theorem 4.10.

4.9 Lemma. *Let A be a set of reals, and suppose that M is a transitive model of* ZFC *such that for each set of reals B projective in A there exists a tree $S \in M$ such that $p[S] = B$. Then*

$$\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle.$$

Proof. Let C denote the set of reals which code elements of $H(\omega_1)$ under our fixed coding (this set is Π_1 and hence absolute). Given a real x in C, let $c(x)$ be the element it codes. For each integer n and each formula ϕ in the language with one additional unary predicate with n free variables, the following are equivalent.

1. For all $a_1, \ldots, a_n \in H(\omega_1)^M$, $\langle H(\omega_1), \in, A\rangle \models \phi(a_1, \ldots, a_n)$ if and only if $\langle H(\omega_1)^M, \in, A \cap M\rangle \models \phi(a_1, \ldots, a_n)$.
2. For all $x_1, \ldots, x_n \in (C \cap M)$, $\langle H(\omega_1), \in, A\rangle \models \phi(c(x_1), \ldots, c(x_n))$ if and only if $\langle H(\omega_1)^M, \in, A \cap M\rangle \models \phi(c(x_1), \ldots, c(x_n))$.

3. For all trees $S \in M$ projecting in V to the set

$$\{\langle x_1, \ldots, x_n\rangle \in C^n \mid \langle H(\omega_1), \in, A\rangle \models \phi(c(x_1), \ldots, c(x_n))\},$$

$$p[S] \cap M = \{\langle x_1, \ldots, x_n\rangle \in (C \cap M)^n \mid \\ \langle H(\omega_1)^M, \in, A \cap M\rangle \models \phi(c(x_1), \ldots, c(x_n))\}.$$

We show by induction on the complexity of formulas that these sentences hold for every formula in our language. Let $\dot{A}$ be the additional unary predicate in this language. Then the sentences above clearly hold for the basic formulas $a \in b$, $a = b$ and $\dot{A}(a)$. The induction steps for $\wedge$, $\vee$ and $\neg$ are likewise immediate. For the quantifier $\exists$, suppose that ϕ has the form $\exists \dot{z}_1 \psi(\dot{z}_1, \ldots, \dot{z}_n)$, and let $x_2, \ldots, x_n$ be elements of $C \cap M$ such that $\langle H(\omega_1), \in, A\rangle \models \phi(c(x_2), \ldots, c(x_n))$. Let S be a tree in M projecting in V to the set of $\langle y_1, \ldots, y_n\rangle \in (C \cap M)^n$ such that $\langle H(\omega_1), \in, A\rangle \models \psi(c(y_1), \ldots, c(y_n))$. The existence of a real x_1 such that $\langle x_1, \ldots, x_n\rangle \in p[S]$ is absolute between M and V, which means that there is a real $x_1 \in C \cap M$ witnessing that $\langle H(\omega_1)^M, \in, A \cap M\rangle \models \phi(c(x_2), \ldots, c(x_n))$. ⊣

The following theorem is a generalized existence result which is useful in analyzing variations of $\mathbb{P}_{\max}$.

4.10 Theorem. *Let γ be a strongly inaccessible cardinal, let A be a set of reals, and suppose that θ is a strong limit cardinal of cofinality greater than ω_1 such that every set of reals projective in A is γ^+-weakly homogeneously Suslin as witnessed by a tree and a set of measures in $H(\theta)$. Let X be a countable elementary submodel of $H(\theta)$ with $\gamma, A \in X$, and let M be the transitive collapse of $X \cap H(\gamma)$. Let N be any forcing extension of M in which there exists a normal precipitous ideal I on ω_1^N. Let $j : (N, I) \to (N^*, I^*)$ be any iteration of (N, I). Then*

- *N^* is wellfounded,*
- *$N \cap A \in N$,*
- *$j(N \cap A) = N^* \cap A$,*
- *$\langle H(\omega_1)^{N^*}, A \cap N^*, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.*

Proof. Let $\{A_i : i < \omega\}$ be a listing in X of the sets of reals projective in A (with $A_0 = A$), and let $\{T_i : i < \omega\}$ and $\{\Sigma_i : i < \omega\}$ be sets in X such that each T_i is a γ^+-weakly homogeneous tree (as witnessed by Σ_i) projecting to A_i. By the proof of Theorem 4.3, there is an elementary submodel Y of $H(\theta)$ containing X such that $X \cap H(\gamma) = Y \cap H(\gamma)$ and such that, letting M^+ be the transitive collapse of Y, and letting, for each $i < \omega$, S_i be the image of T_i under this collapse, $p[S_i] = A_i$. Since there are sets projective in A which are not the projections of countable trees, $\omega_1 \subseteq M^+$. Now, let N be a forcing extension of M with I a normal precipitous ideal on ω_1^N in N, and let N^+ be the corresponding extension of M^+. Let $j : (N, I) \to (N^*, I^*)$ be an iteration of (N, I). By Lemmas 1.5 and 1.6, j extends to an iteration of (N^+, I) (which we will also call j), and N^* is wellfounded. Furthermore, for each $i < \omega$ there is a $j < \omega$ such that S_i and S_j project to complements. Then

- $p[S_i] \subseteq p[j(S_i)]$,
- $p[S_j] \subseteq p[j(S_j)]$,
- $p[j(S_i)] \cap p[j(S_j)] = \emptyset$,

which means that $p[S_i] = p[j(S_i)]$, so $j(N \cap A_i) = N^* \cap A_i$.

The last part of the theorem follows from Lemma 4.9. ⊣

Alternately, we can derive the existence of A-iterable conditions from determinacy. The proof from determinacy requires the following fact: if AD holds and Z is a set of ordinals, then there is an inner model of ZFC containing the ordinals with Z as a member in which some countable ordinal is a measurable cardinal. A tree of finite sequences of ordinals can easily be coded as a set of ordinals (see [27], for instance).

The following theorem of Woodin (see [15, Theorem 5.4]) is more than sufficient, but in the spirit of completeness we will not use it, since its proof is well beyond the scope of this chapter. Given a model M and a set X, HOD^M_X is the class of hereditarily ordinal definable sets (using X as a parameter), as computed in M. It is a standard fact that this model satisfies ZFC.

4.11 Theorem. *Assume* AD. *Suppose that Z is a set of ordinals. Then there exists a real x such that for all reals z with $x \in L[Z, z]$, $\omega_2^{L[Z,z]}$ is a Woodin cardinal in* $\mathrm{HOD}^{L[Z,z]}_{\{Z\}}$.

The following theorem is sufficient for our purposes.

4.12 Theorem. *Assume* AD. *For every subset Z of L, there is an inner model N of* ZFC *containing $\{Z\}$ and the ordinals such that some countable ordinal is measurable in N.*

Proof. For each increasing function $f : \omega \to \omega_1$, let $s(f)$ be the supremum of the range of f, and let $F(f)$ be the filter on $s(f)$ consisting of all subsets of $s(f)$ which contain all but finitely many members of the range of f. For each such f, let $N(f)$ be the inner model $L[Z, F(f)]$. We will find an f such that the restriction of $F(f)$ to $N(f)$ is a countably complete ultrafilter in $N(f)$, i.e., such that

(+) Every function from $s(f)$ to ω in $N(f)$ is constant on a set in $F(f)$.

Note the following facts.

1. If f_0 and f_1 are functions from ω to ω_1 whose ranges are the same modulo a finite set, then $F(f_0) = F(f_1)$ and so not only are the models $N(f_0)$ and $N(f_1)$ the same, but their canonical wellorderings are the same also.

2. Using the canonical wellordering of each $N(f)$, there is a function G choosing for each increasing $f : \omega \to \omega_1$ a function $G(f) : s(f) \to \omega$ failing condition (+) above, if one exists.

The key consequence of AD is the partition property $\omega_1 \to (\omega_1)^{\omega}_{\omega^\omega}$ (see [9] or [14, pp. 391–396]), which says that for every function from the set of increasing ω-sequences from ω_1 to ω^ω (the set of functions from ω to ω) there is an uncountable $E \subseteq \omega_1$ such that the function is constant on the set of increasing ω-sequences from E.

Now, for each increasing $f : \omega \to \omega_1$, let $P(f)$ be the constant function 0 if $F(f)$ satisfies condition (+) in $N(f)$. If f fails condition (+) in $N(f)$, then let $P(f)(0)$ be 1 and let $P(f)(n+1) = G(f)(f(n))$ for all $n \in \omega$. Let E be an uncountable subset of ω_1 such that $P(f)$ is the same for all increasing $f : \omega \to E$. We show that the constant value is the constant function 0. If the constant value corresponds to a failure of (+), then there is an $i : \omega \to \omega$ such that for all increasing $f : \omega \to E$, for all $n \in \omega$, $G(f)(f(n)) = i(n)$. Then i must be constant, since if $n \in \omega$ is such that $i(n) \neq i(0)$, then if f is an increasing function from ω to E and $g : \omega \to E$ is defined by letting $g(m) = f(m+n)$, then $G(f) \neq G(g)$, contradicting the fact that $F(f) = F(g)$. But if i is constant, then for every increasing $f : \omega \to E$, $G(f)$ is constant on a set in $F(f)$, contradicting the failure of (+). $\dashv$

4.13 Theorem. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$*, and let* A *be a set of reals in* $L(\mathbb{R})$*. Then there exists a condition* $\langle (M, I), a\rangle$ *in* $\mathbb{P}_{\max}$ *such that*

- $A \cap M \in M$,
- $\langle H(\omega_1)^M, A \cap M\rangle \prec \langle H(\omega_1), A\rangle$,
- (M, I) *is* A*-iterable.*
- *if* M^+ *is any forcing extension of* M *and* J *is a normal precipitous ideal on* $\omega_1^{M^+}$ *in* M^+ *then* $A \cap M^+ \in M^+$ *and* (M^+, J) *is* A*-iterable, and if* $j : (M^+, J) \to (M^*, J^*)$ *is any iteration of* (M^+, J)*, then*

$$\langle H(\omega_1)^{M^*}, A \cap M^*\rangle \prec \langle H(\omega_1), A\rangle.$$

Proof. Work in $L(\mathbb{R})$. If there is an $A \subseteq \mathbb{R}$ which is a counterexample to the statement of the theorem, then we may assume that there exists such an A which is $\underset{\sim}{\Delta}^2_1$. This follows from the Solovay Basis Theorem (see [9]), which says (in ZF) that every nonempty $\underset{\sim}{\Sigma}^2_1$ collection of sets of reals has a member which is $\underset{\sim}{\Delta}^2_1$. We give a quick sketch of the proof. Note first of all that for any ordinal α the transitive collapse any elementary submodel of $L_\alpha(\mathbb{R})$ containing $\mathbb{R}$ is a set of the form $L_\beta(\mathbb{R})$ for some ordinal $\beta \leq \alpha$. Now, if α is any ordinal, there exist (in $L(\mathbb{R})$) an elementary submodel X of $L_\alpha(\mathbb{R})$ containing $\mathbb{R}$ and a surjection $\pi : \mathbb{R} \to X$, so if α is the least ordinal such that a member of a given $\underset{\sim}{\Sigma}^2_1$ set exists in $L_{\alpha+1}(\mathbb{R})$ then there is a surjection (in $L(\mathbb{R})$) from $\mathbb{R}$ onto $L_{\alpha+1}(\mathbb{R})$, and a formula ϕ and a real x such that some member of the set is defined over $L_\alpha(\mathbb{R})$ by ϕ from x. By the minimality of α, that member has $\underset{\sim}{\Sigma}^2_1$ and $\underset{\sim}{\Pi}^2_1$ definitions using x and incorporating ϕ.

Towards a contradiction, fix a $\underset{\sim}{\Delta}^2_1$ counterexample A. By [24], the pointclass Σ^2_1 has the scale property in $L(\mathbb{R})$, which means that every subset of $\mathbb{R} \times \mathbb{R}$ which is $\underset{\sim}{\Delta}^2_1$ in $L(\mathbb{R})$ is the projection of a tree in $L(\mathbb{R})$ on the product of ω and some ordinal, and can be uniformized by a function which is $\underset{\sim}{\Delta}^2_1$ in $L(\mathbb{R})$. (We refer the reader to [9, 14, 26] for a discussion of scales and their corresponding trees. Briefly, if $B \subseteq \mathbb{R} \times \mathbb{R}$ is the projection of a tree T on $\omega \times \omega \times \gamma$ (for some ordinal γ) then for each real x such that there exists a y with (x, y) in B, we can recursively define functions $f(x) : \omega \to \omega$ and $g(x) : \omega \to \gamma$ as follows: if (m, α) is the lexicographically least pair in $\omega \times \gamma$ such that there exist a real y and a function $a : \omega \to \gamma$ such that

- y extends $f(x){\restriction}n$ and $y(n) = m$,
- a extends $g(x){\restriction}n$ and $a(n) = \alpha$,
- (x, y, a) is a path through T,

then $f(x)(n) = m$ and $g(x)(n) = \alpha$. Then f uniformizes B, and if T is the tree corresponding to a $\underset{\sim}{\Sigma}^2_1$ scale on B, then f is $\underset{\sim}{\Delta}^2_1$.) Now, $\underset{\sim}{\Delta}^2_1$ is closed under complements, projections and countable unions, so there exists a $\underset{\sim}{\Delta}^2_1$ set $B \subseteq \mathbb{R} \times \mathbb{R}$ such that whenever $F : \mathbb{R} \to \mathbb{R}$ is a function uniformizing B and N is a transitive model N of ZF closed under F,

$$\langle H(\omega_1)^N, A \cap N, \in \rangle \prec \langle H(\omega_1), A, \in \rangle.$$

Fix such B and F, both $\underset{\sim}{\Delta}^2_1$. Let S, S^*, T, T^* be trees (on $\omega \times \gamma$, for some ordinal γ) in $L(\mathbb{R})$ projecting to A, the complement of A, F and the complement of F respectively. Note that any transitive model of ZF with T as a member is closed under F.

Now by Theorem 4.12, we may fix a transitive model N of ZFC and a countable ordinal γ such that N contains the ordinals, S, S^*, T and T^* are elements of N and γ is a measurable cardinal in N. Since $N \subseteq L(\mathbb{R})$ and $L(\mathbb{R})$ satisfies AD, ω_1^V is a limit of strongly inaccessible cardinals in N. Let δ be any strongly inaccessible cardinal in N between γ and ω_1^V. Recall (Remark 2.6) that if we choose an N-generic filter G for the forcing consisting of $\mathrm{Col}(\omega, {<}\gamma)$ followed by the standard c.c.c. iteration to make Martin's Axiom hold, as defined in N, then if we let I be the normal ideal generated by an ideal in N dual to a fixed normal measure on γ in N, I is a precipitous ideal in $N[G]$ and

$(N_\delta[G], I)$ is iterable, by Lemmas 1.5 and 1.6. It suffices now to fix a forcing extension M^+ of $N_\delta[G]$ in which there exists a normal precipitous ideal J on $\omega_1^{M^+}$ and to show that the second part of the conclusion of the theorem holds for M^+ and J. Let N^+ be the corresponding forcing extension of $N[G]$. Since $S \in N^+$, $A \cap M^+ \in M^+$. Fix an iteration $j : (M^+, J) \to (M^*, J^*)$. By Lemma 1.6 there is an iteration $j^* : (N^+, J) \to (N^*, J^*)$ such that $j^* \restriction M^+ = j$. Now, $p[S] \subseteq p[j^*(S)]$ and $p[S^*] \subseteq p[j^*(S^*)]$, and further, by absoluteness $p[j^*(S)] \cap p[j^*(S^*)] = \emptyset$, so $p[S] = p[j^*(S)]$. Similarly, $p[T] = p[j^*(T)]$. Then N^* is closed under F, so we have that

$$\langle H(\omega_1)^{M^*}, A \cap M^*, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Furthermore, $j(A \cap M^+) = p[j^*(S)] \cap M^*$, so $j(A \cap M^+) = A \cap M^*$. This shows that A is not in fact a counterexample to the statement of the theorem. $\dashv$

Suppose that A is a set of reals and x is a real coding a condition p in $\mathbb{P}_{\max}$ by some recursive coding, and let B be the set of reals coding members of $A \times \{x\}$. Then if (M, I) is a B-iterable pair such that

$$\langle H(\omega_1)^M, B \cap M, \in\rangle \prec \langle H(\omega_1), B, \in\rangle,$$

then (M, I) is A-iterable and $p \in H(\omega_1)^M$. Therefore, the existence, for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$, of an A-iterable pair (M, I) such that

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$$

implies that for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$ the set of $\langle (M, I), a\rangle$ in $\mathbb{P}_{\max}$ such that (M, I) is A-iterable and $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$ is dense.

5. The Basic Analysis

With the existence of A-iterable conditions (for all sets of reals A in $L(\mathbb{R})$) in hand, we can now prove the most important fact about the $\mathbb{P}_{\max}$ extension, that every subset of ω_1 in the extension is the image of a member of the generic filter under the iteration of that member induced by the generic filter.

Formally, if $G \subseteq \mathbb{P}_{\max}$ is a set of pairwise compatible conditions, then since the elementary embedding witnessing the order on a pair of $\mathbb{P}_{\max}$ conditions has critical point the ω_1 of the smaller model, for each $\langle (M, I), a\rangle, \langle (N, J), b\rangle$ in G, $a \cap \gamma = b \cap \gamma$, where $\gamma = \min\{\omega_1^M, \omega_1^N\}$. For any such G, we let

$$A_G = \bigcup\{a \mid \exists (M, I)\ \langle (M, I), a\rangle \in G\}.$$

By Lemma 2.7, for any such G, for any member $\langle (M, I), a\rangle$ of G there is a unique iteration of (M, I) sending a to A_G. Using this fact, we define $\mathcal{P}(\omega_1)_G$ to be the collection of all E such that there exists a condition $\langle (M, I), a\rangle \in G$ and a set $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = E$, where j is the unique iteration of

(M, I) sending a to A_G. Likewise, we define I_G to be the collection of all E such that there exists a condition $\langle (M, I), a\rangle \in G$ and a set $e \in I$ such that $j(e) = E$, where j is the unique iteration of (M, I) sending a to A_G.

We state the following theorem from the point of view of the ground model (so in particular, the universe V in the statement of the theorem does not satisfy AC). We have seen that large cardinals and determinacy each imply that the hypothesis of the theorem is satisfied in $L(\mathbb{R})$, but as we shall see, it can hold in other models as well.

5.1 Theorem (ZF). *Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ such that (M, I) is A-iterable and*

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Suppose that $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter. Then in $V[G]$ the following hold.

(a) $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$.

(b) $\mathrm{NS}_{\omega_1} = I_G$.

(c) $\underset{\sim}{\delta}^1_2 = \omega_2$.

(d) NS_{ω_1} *is saturated.*

Before proving Theorem 5.1, we prove another iteration lemma in terms of games in order to separate out some commonly needed details.

Suppose that p is a $\mathbb{P}_{\max}$ condition, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Let $\mathcal{G}_{\omega_1}(p, J, B)$ be the game where Players I and II collaborate to build a descending ω_1-sequence of $\mathbb{P}_{\max}$ conditions $p_\alpha = \langle (M_\alpha, I_\alpha), a_\alpha\rangle$ below p, where in round $\alpha < \omega_1$, I chooses p_α if $\alpha \notin B$, and II chooses p_α if $\alpha \in B$. At the end of the game, II wins if, letting $A = \bigcup\{a_\alpha : \alpha < \omega_1\}$ and letting $j_\alpha : (M_\alpha, I_\alpha) \to (M^*_\alpha, I^*_\alpha)$ (for each $\alpha < \omega_1$) be the iteration of (M_α, I_α) sending a_α to A, $j_\alpha(I_\alpha) = J \cap M^*_\alpha$ holds for each $\alpha < \omega_1$.

5.2 Lemma (ZFC°). *Suppose that $x^\dagger$ exists for every real x. Let p be a condition in $\mathbb{P}_{\max}$, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Then Player* II *has a winning strategy in $\mathcal{G}_{\omega_1}(p, J, B)$ if and only if $B \notin J$.*

Proof. The interesting direction is showing that II has a winning strategy if $B \notin J$, and for this direction it suffices to consider the case where B consists entirely of limit ordinals (we have no use for the other direction and leave its proof to the reader). The strategy for II uses the usual trick. Partition B into J-positive sets $\{B^\alpha_i : \alpha < \omega_1, i < \omega\}$, and as the p_α are chosen, let $\{E^\alpha_i : i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$ for each α.

Fix a *ladder system* $\{h_\alpha : \alpha \in B\}$ (so each h_α is an increasing function from ω to α with cofinal range). Having constructed our sequence of p_α's up to some limit stage β in B, let

$$(\langle (N^\beta_i, J^\beta_i) : i < \omega\rangle, a^*_\beta)$$

be the limit sequence corresponding to the descending sequence

$$\langle p_{h_\beta(i)} : i < \omega \rangle,$$

and for each $i < \omega$ let $j'_{i\beta}$ be the unique iteration of $(M_{h_\beta(i)}, I_{h_\beta(i)})$ sending $a_{h_\beta(i)}$ to a^*_β. Since $x^\dagger$ exists for each real x, we may fix a $\mathbb{P}_{\max}$ pre-condition (M_β, I_β) with

$$(\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle, a^*_\beta) \in H(\omega_1)^{M_\beta}.$$

As in Lemma 3.4 (more precisely, using Lemma 3.5), we let j'_β be an iteration of $\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle$ in M_β such that

$$j'_\beta(J^\beta_i) = I_\beta \cap j'_\beta(N^\beta_i)$$

for each $i < \omega$, with the extra stipulation that if

$$\omega_1^{N^\beta_0} \in B^\gamma_k$$

for some $\gamma < \beta$ and $k < \omega$, then, letting i' be the least element i of ω such that $h_\beta(i) \geq \gamma$,

$$j'_{i'\beta}(j_{\gamma h_\beta(i')}(E^\gamma_k))$$

is in the normal filter corresponding to the first step of this iteration of $\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle$ (note that $j'_{i'\beta}(j_{\gamma h_\beta(i')}(E^\gamma_k))$ is $J^\beta_{i'}$-positive by the agreement of ideals imposed by the $\mathbb{P}_{\max}$ order). Then, letting $a_\beta = j'_\beta(a^*_\beta)$, we have that

$$\omega_1^{N^\beta_0} \in j_{\gamma\beta}(E^\gamma_k).$$

Since for each $i < \omega$ and $\alpha < \omega_1$ the set of $\beta \in B^\alpha_i$ such that $\omega_1^{N^\beta_0} = \beta$ is J-positive, by playing in this manner Player II ensures that the image of each E^α_i is J-positive. ⊣

We separate out the following standard argument as well.

5.3 Lemma. *Suppose that $x^\dagger$ exists for every real x, and let $G \subseteq \mathbb{P}_{\max}$ be an $L(\mathbb{R})$-generic filter. Let $p_0 = \langle (M, I), a \rangle$ be a $\mathbb{P}_{\max}$ condition in G, and suppose that $P \in M$ is a set of $\mathbb{P}_{\max}$ conditions such that $p \geq p_0$ for every $p \in P$. Let j be the unique iteration of (M, I) sending a to A_G. Then every member of $j(P)$ is in G.*

Proof. Let $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ be the iteration corresponding to j, and fix $q = \langle (N_0, J_0), b_0 \rangle$ in $j(P)$. Fix $\alpha_0 < \omega_1$ such that $q \in j^*_{0\alpha_0}(P)$, and let j_q (in M_{α_0}) be the iteration of (N_0, J_0) sending b_0 to $j^*_{0\alpha_0}(a)$. By the genericity of G there is a condition $p_1 = \langle (N_1, J_1), b_1 \rangle$ in G such that $p_1 \leq p_0$ and $\alpha_0 < \omega_1^{N_1}$. Then $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1^{N_1} \rangle$ is in $M_{\omega_1^{N_1}}$ and is the unique iteration of (M, I) sending a to b_1. Since

$$j_q(J_0) = j^*_{0\alpha_0}(I) \cap j_q(N_0)$$

and

$$j^*_{0\omega_1^{N_1}}(I) = J_1 \cap M_{\omega_1^{N_1}},$$

$j^*_{\alpha_0\omega_1^{N_1}}(j_q)$ witnesses that $q \geq p_1$. $\dashv$

Proof of Theorem 5.1. (a) Let τ be a $\mathbb{P}_{\max}$-name in $L(\mathbb{R})$ for a subset of ω_1, and let A be the set of reals coding (under some fixed recursive coding) the set of triples (p, α, i) such that $p \in \mathbb{P}_{\max}$, $\alpha < \omega_1$, $i \in 2$ and, if $i = 1$ then $p \Vdash \check{\alpha} \in \tau$, otherwise $p \Vdash \check{\alpha} \notin \tau$. Let $p = \langle (N, J), d\rangle$ be any condition in $\mathbb{P}_{\max}$ and let (M, I) be an A-iterable pre-condition such that

- $p \in H(\omega_1)^M$,
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.

Applying Lemma 5.2 in M, there exists a descending sequence of $\mathbb{P}_{\max}$ conditions $p_\alpha = \langle (N_\alpha, J_\alpha), d_\alpha\rangle$ ($\alpha < \omega_1^M$) such that

(1) $p_0 = p$,

(2) each $p_{\alpha+1}$ decides the sentence "$\check{\alpha} \in \tau$",

(3) letting $D = \bigcup\{d_\alpha : \alpha < \omega_1^M\}$, for each $\alpha < \omega_1$, $j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)$, where j_α is the unique iteration of (N_α, J_α) sending d_α to D.

Conditions (1) and (2) are easily satisfied, using the fact that $\langle H(\omega_1)^M$, $A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$, and we may apply Lemma 5.2 (letting B be the set of countable limit ordinals) inside M to meet Condition (3) since in M, $x^\dagger$ exists for each real x.

Now, letting D be as in Condition (3) above, $\langle (M, I), D\rangle$ is a $\mathbb{P}_{\max}$ condition below each p_α. Let e be the subset of ω_1^M in M such that for each $\alpha < \omega_1$, $\alpha \in e$ if and only if $p_{\alpha+1} \Vdash \check{\alpha} \in \tau$.

Suppose that $p' = \langle (M, I), D\rangle \in G$, and let

$$\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

be the unique iteration of (M, I) sending D to A_G. We want to see that $j^*_{0\omega_1}(e) = \tau_G$. Let $\langle q_\alpha : \alpha < \omega_1\rangle = j^*_{0\omega_1}(\langle p_\alpha : \alpha < \omega_1^M\rangle)$. By the elementarity of $j^*_{0\alpha^*}$ and the A-iterability of (M, I), for each $\gamma < \omega_1$, $q_{\gamma+1} \Vdash \check{\gamma} \in \tau$ if $\gamma \in j_{0\alpha^*}(e)$ and $q_{\gamma+1} \Vdash \check{\gamma} \notin \tau$ if $\gamma \notin j_{0\alpha^*}(e)$. By Lemma 5.3, each q_γ is in G, so $j^*(e) = \tau_G$.

(b) The fact that $I_G = \mathrm{NS}_{\omega_1}$ follows almost immediately. If $E \in I_G$, then there is a condition $\langle (M, I), a\rangle$ in G, an $e \in I$ and an iteration j of (M, I) sending e to E. Then E is disjoint from the critical sequence of this iteration and therefore nonstationary. On the other hand, if E is a nonstationary subset of ω_1 in $V[G]$, then there is a club C disjoint from E and a condition $\langle (M, I), a\rangle$ in G, sets $e, c \in \mathcal{P}(\omega_1)^M$ and an iteration j of (M, I) sending e and c to E and C respectively. Then c must be a club subset of ω_1^M in M, so $e \in I$, which means that E is in I_G.

(c) That $\underset{\sim}{\delta}^1_2 = \omega_2$ also follows almost immediately, using Corollary 1.9 and the standard fact that if $x^\#$ exists for every real x, then $\underset{\sim}{\delta}^1_2$ is equivalent to u_2, the second uniform indiscernible (the least ordinal above ω_1 which is an indiscernible of every real) (see [36, 39]). So, showing that $\underset{\sim}{\delta}^1_2 = \omega_2$ then amounts to showing that for every $\gamma < \omega_2$ there is a real x such that the least x-indiscernible above ω_1 is greater than γ. Working in $V[G]$, fix $\gamma \in [\omega_1, \omega_2)$ and a wellordering π of ω_1 of length γ. By the first part of this theorem, we may fix a condition $\langle (M, I), a\rangle \in G$ and an $e \in \mathcal{P}(\omega_1 \times \omega_1)^M$ such that $j(e) = \pi$, where j is the iteration of (M, I) sending a to A_G. Then γ is in $j(M)$, and so is less than the least indiscernible above ω_1 of any real coding (M, I), by Corollary 1.9.

(d) To show that NS_{ω_1} is saturated in $V[G]$, we show that for any set $D \subseteq \mathcal{P}(\omega_1) \setminus \mathrm{NS}_{\omega_1}$ which is dense under the subset order, there is a subset D' of D of cardinality $\aleph_1$ whose diagonal union contains a club. So, following the proof of the first part of this theorem, let τ be a name for such a set D. Let A be the set of reals coding (by a fixed recursive coding) the set of pairs (p, e) such that $p = \langle (M, I), a\rangle$ is a condition in $\mathbb{P}_{\max}$, $e \in \mathcal{P}(\omega_1)^M \setminus I$ and p forces that $j(\check{e}) \in \tau$, where j is the unique iteration of (M, I) sending a to A_G.

Let $p = \langle (N, J), b\rangle$ be any condition in $\mathbb{P}_{\max}$ and let (M, I) be an A-iterable pre-condition such that

- $p \in H(\omega_1)^M$,
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.

Fix a partition $\{B^\alpha_i : \alpha < \omega_1, i < \omega\}$ in M of pairwise disjoint I-positive sets whose diagonal union is I-large. Fix also a function $g : \omega_1^M \times \omega \to \omega_1^M$ in M such that $g(\alpha, i) \geq \alpha$ for all $(\alpha, i) \in \mathrm{dom}(g)$.

Working in M, we are going to build a descending sequence of $\mathbb{P}_{\max}$ conditions $p_\alpha = \langle (N_\alpha, J_\alpha), b_\alpha\rangle$ (with the order on conditions witnessed by a commuting family of embeddings $j_{\alpha\beta}$), enumerations $\{e^\alpha_i : i < \omega\}$ in M of each set $\mathcal{P}(\omega_1)^{N_\alpha} \setminus J_\alpha$ and sets d_α ($\alpha \leq \beta \leq \omega_1^M$) such that

(4) $p_0 = p$,

(5) each $d_\alpha \in \mathcal{P}(\omega_1)^{N_{\alpha+1}} \setminus J_{\alpha+1}$ and, if $\alpha = g(\beta, i)$ for some $\beta \leq \alpha$ and $i < \omega$, then $d_\alpha \subseteq j_{\beta(\alpha+1)}(e^\beta_i)$ and $(p_{\alpha+1}, d_\alpha)$ is coded by a real in A,

(6) for each $(\beta, i) \in \mathrm{dom}(g)$, $B^\alpha_i \setminus j_{(g(\beta,i)+1)\omega_1^M}(d_{g(\beta,i)})$ is nonstationary.

Conditions (5) and (6) together imply that our sequence will satisfy Condition (3) from part (a) of this proof. Furthermore, Conditions (4) and (5) here are easily achieved, by the assumptions on τ. In particular, for each $\alpha < \omega_1^M$, by the assumptions on τ there exists a pair (p^*, d^*) such that $p^* \leq p_\alpha$ and Condition (5) holds with p^* in the role of $p_{\alpha+1}$ and d^* in the role of d_α, and we let $(p_{\alpha+1}, d_\alpha)$ be any such pair. Condition (6) implies that the diagonal union of the sets $j_{(g(\beta,i)+1)\omega_1^M}(d_{g(\beta,i)})$ will be I-large.

Condition (6) is achieved in almost exactly the same way as Condition (3) in the first part of the proof (but not exactly the same way; unfortunately we cannot quote Lemma 5.2). Fix a ladder system $\{h_\alpha : \text{limit } \alpha < \omega_1\}$ in M. Having constructed our sequence of p_α's up to some limit stage β, let $(\langle(N_i^\beta, J_i^\beta) : i < \omega\rangle, b^*_\beta)$ be the limit sequence corresponding to the descending sequence $\langle p_{h_\beta(i)} : i < \omega\rangle$, and again for each $i < \omega$ let $j'_{i\beta}$ be the unique iteration of $(N_{h_\beta(i)}, J_{h_\beta(i)})$ sending $b_{h_\beta(i)}$ to b^*_β. Fix a pre-condition (N_β, J_β) in M with

$$(\langle N_i^\beta : i < \omega\rangle, b^*_\beta) \in H(\omega_1)^{N_\beta}.$$

As in Lemma 3.4, we let j'_β be an iteration of $\langle(N_i^\beta, J_i^\beta) : i < \omega\rangle$ in N_β such that

$$j'_\beta(J_i^\beta) = J_\beta \cap j'_\beta(N_i^\beta)$$

for each $i < \omega$, with the extra stipulation that if

$$\omega_1^{N_0^\beta} \in B_k^\gamma$$

for some $\gamma < \beta$ and $k < \omega$ with $g(\gamma, k) < \beta$, then, letting i' be the least $i \in \omega$ such that $h_\beta(i) \geq g(\gamma, k)$,

$$j'_{i'\beta}(j_{(g(\gamma,k)+1)h_\beta(i')}(d_{g(\gamma,k)}))$$

is in the filter corresponding to the first step of this iteration of the sequence $\langle(N_i^\beta, J_i^\beta) : i < \omega\rangle$, ensuring (once we let $b_\beta = j'_\beta(b^*_\beta)$) that

$$\omega_1^{N_0^\beta} \in j_{(g(\gamma,k)+1)\beta}(d_{g(\gamma,k)}).$$

Then since $\{\omega_1^{N_0^\beta} : \text{limit } \beta < \omega_1\}$ is a club subset of ω_1^M, Condition (6) is satisfied.

Now, letting $B = \bigcup\{b_\alpha : \alpha < \omega_1^M\}$, $\langle(M, I), B\rangle$ is a $\mathbb{P}_{\max}$ condition below each p_α. For each $\alpha < \omega_1$ and $i < \omega$, let $d'_{\alpha i} = j_{(g(\alpha,i)+1)\omega_1^M}(d_{g(\alpha,i)})$. Then the diagonal union of

$$\mathcal{A} = \{d'_{\alpha i} : \alpha < \omega_1^M, i < \omega\}$$

contains an I-large subset of ω_1^M in M.

Suppose that $\langle(M, I), B\rangle \in G$, and let $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ be the unique iteration of (M, I) sending B to A_G. Then the diagonal union of $j^*_{0\omega_1}(\mathcal{A})$ contains the critical sequence of $j^*_{0\omega_1}$, which is a club. We want to see that $j^*_{0\omega_1}(\mathcal{A}) \subseteq \tau_G$.

Let $\langle q_\alpha : \alpha < \omega_1\rangle = j^*_{0\omega_1}(\langle p_\alpha : \alpha < \omega_1^M\rangle)$. By Lemma 5.3, each q_α is in G. Since (M, I) is A-iterable, each member of $j^*(\mathcal{A})$ is forced to be in τ_G by some q_α, so $j^*(\mathcal{A}) \subseteq \tau_G$. ⊣

5.4 Remark. It is shown in [18] that, under the hypothesis of Theorem 5.1, Todorcevic's Open Coloring Axiom [37] holds in the $\mathbb{P}_{\max}$ extension. The proof in that paper can be greatly simplified by using Lemmas 5.2 and 5.3 to separate out the standard details.

6. ψ_{AC} and the Axiom of Choice

We have not yet shown that the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ satisfies the Axiom of Choice. We shall do this by showing (assuming $\mathrm{AD}^{L(\mathbb{R})}$) that the following axiom holds there.

We let $\mathrm{ot}(X)$ denote the ordertype of a linear order X.

6.1 Definition. ψ_{AC} is the statement that for every pair A, B of stationary, co-stationary subsets of ω_1, there exists a bijection π between ω_1 and some ordinal γ such that the set $\{\alpha < \omega_1 \mid \alpha \in A \iff \mathrm{ot}(\pi[\alpha]) \in B\}$ contains a club.

Using a partition $\{A_\alpha : \alpha < \omega_1\}$ of ω_1 into stationary sets, ψ_{AC} allows us to define an injection from 2^{ω_1} into ω_2. Since the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ satisfies the sentence "$V = L(\mathcal{P}(\omega_1))$", this is enough to see that AC holds there. Let B^* be any stationary, co-stationary subset of ω_1. For each $X \subseteq \omega_1$, let $A_X = \bigcup\{A_\alpha : \alpha \in X\}$, and let γ_X be the ordinal given by ψ_{AC}, where A_X is in the role of A, and B^* is in the role of B. Let X_0 and X_1 be distinct subsets of ω_1, and let E be the (stationary) symmetric difference of A_{X_0} and A_{X_1}. Supposing towards a contradiction that $\gamma_{X_0} = \gamma_{X_1}$, let π_0 and π_1 be bijections and C_0 and C_1 club subsets of ω_1 witnessing ψ_{AC} for the pairs A_{X_0}, B^* and A_{X_1}, B^* respectively. Then there is a club subset D of ω_1 such that $\mathrm{ot}(\pi_0[\alpha]) = \mathrm{ot}(\pi_1[\alpha])$ for all $\alpha \in D$. Then $E \cap C_0 \cap C_1 \cap D$ is nonempty, which gives a contradiction, since $\alpha \in A_{X_0} \iff \mathrm{ot}(\pi_0[\alpha]) \in B \iff \mathrm{ot}(\pi_1[\alpha]) \in B \iff \alpha \in A_{X_1}$ for all $\alpha \in C_0 \cap C_1 \cap D$. Therefore, ψ_{AC} implies that $2^{\omega_1} = \omega_2$. In fact, it also implies that $2^\omega = 2^{\omega_1}$, but we will not take the time to show this (it follows from a result of Shelah proved in [39, Sect. 3.2]); we already know from Theorem 5.1 that the Continuum Hypothesis fails in the $\mathbb{P}_{\max}$ extension (assuming $\mathrm{AD}^{L(\mathbb{R})}$).

That ψ_{AC} holds in the $\mathbb{P}_{\max}$ extension follows from part (a) of Theorem 5.1 and the following lemma.

6.2 Lemma (ZFC°). *Suppose that (M, I) is a pre-condition in $\mathbb{P}_{\max}$, and let $A, B \in M$ be I-positive subsets of ω_1^M whose complements in ω_1^M are also I-positive. Let J be a normal ideal on ω_1. Then there exist an iteration $j : (M, I) \to (M^*, I^*)$ of (M, I) of length ω_1, an ordinal $\gamma < \omega_2$, and a bijection $\pi : \omega_1 \to \gamma$ such that $I^* = J \cap M^*$ and*

$$\{\alpha < \omega_1 \mid \alpha \in j(A) \iff \mathrm{ot}(\pi[\alpha]) \in j(B)\}$$

contains a club.

Proof. Let x be a real coding (M, I). Using Fact 1.10, it suffices to construct an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ such that for every α which is a limit of countable x-indiscernibles, $j_{0\alpha}(A) \in G_\alpha$ if and only if $j_{0\alpha^*}(B) \in G_{\alpha^*}$, where α^* is the least x-indiscernible above α. By the proof of Lemma 1.8, $\omega_1^{M_\gamma} = \gamma$ for each x-indiscernible γ, so in particular, each such γ is on the critical sequence.

We construct our iteration using the winning strategy for Player I in $\mathcal{G}(\omega_1 \setminus E)$ from Lemma 2.9, where E is the set of countable ordinals of the form α^* as above, where α is a limit of x-indiscernibles. This ensures that $j_{0\omega_1}(I) = J \cap M_{\omega_1}$. To complete the construction, we recursively choose each G_{α^*} ($\alpha^* \in E$) in such a way that $j_{0\alpha^*}(B) \in G_{\alpha^*}$ if and only if $j_{0\alpha}(A) \in G_\alpha$. Fact 1.10 implies that any iteration satisfying these conditions satisfies the conclusion of the lemma. $\dashv$

Stated in the fashion of Theorem 5.1, we have shown the following.

6.3 Theorem (ZF). *Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ such that (M, I) is A-iterable and*

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Suppose that $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter. Then ψ_{AC} holds in $V[G]$.

6.4 Remark. Suppose that A, B are stationary, co-stationary subsets of ω_1, $\pi : \omega_1 \to \gamma$ is a bijection (for some $\gamma < \omega_2$) and the set

$$\{\alpha < \omega \mid \alpha \in A \iff \mathrm{ot}(\pi[\alpha]) \in B\}$$

contains a club subset of ω_1. Then for any normal ideal I on ω_1, A is the Boolean value in the partial order $\mathcal{P}(\omega_1)/I$ that $\gamma \in j(B)$, where j is the induced embedding. It follows that if (M, I) is any iterable pair with M a countable transitive model of ZFC$^\circ$+ ψ_{AC} and B is any stationary, costationary subset of ω_1^M in M, then the image of B under any iteration of (M, I) determines the entire iteration. This in turn implies that one can replace $\mathrm{MA}_{\aleph_1}$ with ψ_{AC} in the definition of $\mathbb{P}_{\max}$ without significantly changing the corresponding analysis; in some cases the analysis is easier with ψ_{AC}.

7. Maximality and Minimality

In this section we will show that if certain large cardinals exist in V then the $\mathbb{P}_{\max}$ extension of the inner model $L(\mathbb{R})$ is maximal, in that all forceable Π_2 sentences for $H(\omega_2)$ hold there, and that it is minimal, in that every subset of ω_1 added by the generic filter for $\mathbb{P}_{\max}$ generates the entire extension. We will also show that a certain form of this maximality characterizes the $\mathbb{P}_{\max}$ extension.

To show that the $\mathbb{P}_{\max}$ extension is Π_2-maximal, we will use the following theorem of Woodin (see [8, 6, 30, 39]; Foreman, Magidor and Shelah originally proved the theorem from the existence of supercompact cardinal [7]). An ideal I on ω_1 is *presaturated* if for any $A \in \mathcal{P}(\omega_1) \setminus I$ and any sequence $\langle \mathcal{A}_i : i < \omega\rangle$ of maximal antichains in $\mathcal{P}(\omega_1) \setminus I$ there exists a $B \in \mathcal{P}(A) \setminus I$ such that there are at most $\aleph_1$ many $X \in \bigcup\{\mathcal{A}_i : i < \omega\}$ such that $X \cap B \notin I$. It is straightforward to check that normal presaturated ideals on ω_1 are precipitous.

7.1 Theorem. *If δ is a Woodin cardinal, then every condition in the partial order* $\mathrm{Col}(\omega_1, {<}\delta)$ *forces that* NS_{ω_1} *is presaturated.*

7.2 Definition. Given a cardinal κ, a set of reals A is *κ-universally Baire* if there exist trees S and T (contained in $\omega \times Z$ for some set Z) such that $p[S] = A$ and S and T project to complements in all extensions by forcing constructions of cardinality less than or equal to κ. The set A is *${<}\kappa$-universally Baire* if it is γ-universally Baire for all $\gamma < \kappa$.

Theorem 4.6 shows that for any cardinal κ, κ^+-weakly homogeneously Suslin sets of reals are κ-universally Baire.

If κ is a cardinal, A is a κ-universally Baire set of reals and $V[G]$ is an extension of V by a forcing construction of cardinality less than or equal to κ, then we let $A(G)$ be the union of all sets of the form $(p[S])^{V[G]}$, where S is a tree in V whose projection in V is contained in A. (The notation A_G is often used here, but we are already using that for something else.) For any pair of trees S and T in V witnessing that A is κ-universally Baire (i.e., such that $p[S] = A$ and S and T project to complements in all extensions by forcing constructions of cardinality less than κ), $(p[S])^{V[G]} = A(G)$.

The following theorem is an immediate consequence of part (a) of Theorem 5.1, and it implies in particular that $\mathrm{MA}_{\aleph_1}$ holds in the $\mathbb{P}_{\max}$ extension.

7.3 Theorem. *Suppose that δ is a limit of Woodin cardinals, and $\kappa > \delta$ is measurable. Let A be a set of reals in $L(\mathbb{R})$. Suppose that ϕ is a Π_2 sentence in the expanded language with two additional unary predicates, and that P is a partial order in V_δ forcing that ϕ holds in the structure $\langle H(\omega_2), \in, A(G)\rangle$. Then ϕ holds in the structure $\langle H(\omega_2), \in A\rangle$ in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.*

Proof. Suppose that ϕ has the form $\forall X \exists Y \psi(X, Y)$. By part (a) of Theorem 5.1, it suffices to show that for every $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ and every $x \in H(\omega_2)^M$ there exists a $\mathbb{P}_{\max}$ condition $q = \langle (N, J), b\rangle$ such that

- $q \leq p$,
- (N, J) is A-iterable,
- if $j : (M, I) \to (M^*, I^*)$ is the unique iteration of (M, I) sending a to b, then
$$\langle H(\omega_2)^N, J, A \cap N, \in\rangle \models \exists y\ \psi(j(x), y).$$

Let Z be a countable elementary submodel with $\langle (M, I), a\rangle$, A, P and δ as a members, and let N be the transitive collapse of Z. By Theorem 4.10, any forcing extension of M in which the nonstationary ideal is precipitous will be A-iterable with respect to the nonstationary ideal. Let $N[g_0]$ be a forcing extension of N by P, and let $j : (M, I) \to (M^*, I^*)$ be an iteration of (M, I) in $N[g_0]$ such that $I^* = \mathrm{NS}_{\omega_1}^{N[g_0]} \cap M^*$. Since ϕ holds in the structure $\langle H(\omega_2)^{N[g_0]}, \in, A \cap N[g_0]\rangle$, there is a $y \in H(\omega_2)^{N[g_0]}$ such that $\psi(j(x), y)$ holds there (and in the structure $\langle H(\omega_2)^{N^*}, \in, A \cap N^*\rangle$ of every outer model N^* of

$N[g_0]$ which agrees with $N[g_0]$ about stationarity for subsets of ω_1. In $N[g_0]$ there exists a Woodin cardinal γ, so let $N[g_1]$ be a $\mathrm{Col}(\omega_1, {<}\gamma)$-extension of $N[g_0]$. Finally, let $N[g_2]$ be a c.c.c. forcing extension of $N[g_1]$ in which $\mathrm{MA}_{\aleph_1}$ holds. Then $\langle(N[g_2], \mathrm{NS}_{\omega_1}^{N[g_2]}), j(a)\rangle$ is the desired condition q. ⊣

Theorem 7.8 below is a sort of converse to Theorem 7.3. First we will show that every new subset of ω_1 in the $\mathbb{P}_{\max}$ extension generates the entire generic filter (Theorem 7.7).

The conclusion of the following lemma corresponds to Condition 5 in the definition of $\mathbb{P}_{\max}$.

7.4 Lemma. *Suppose that $x^\dagger$ exists for each real x. Let $\langle(M', I'), a'\rangle$ be a $\mathbb{P}_{\max}$ condition and let e be an element of $\mathcal{P}(\omega_1)^{M'}$. Then there exist a $\mathbb{P}_{\max}$ pre-condition (N, J) with $(M', I') \in H(\omega_1)^N$ and an iteration $j : (M', I') \to (M^*, I^*)$ in N such that*

- $j(\omega_1^{M'}) = \omega_1^N$,
- $I^* = J^* \cap M^*$,

and either

1. *for some $x \in \mathcal{P}(\omega)^N$, $j(e) \in L[x]$, or*
2. *for some $x \in \mathcal{P}(\omega)^N$, $\omega_1^N = \omega_1^{L[j(e),x]}$.*

Proof. Fix a limit sequence $(\langle(M_i, I_i) : i < \omega\rangle, a)$ corresponding to any descending ω-sequence in $\mathbb{P}_{\max}$ starting with $\langle(M', I'), a'\rangle$, and let (N, J) be a $\mathbb{P}_{\max}$ pre-condition with $\{(M', I'), (\langle(M_i, I_i) : i < \omega\rangle, a)\} \in H(\omega_1)^N$. Let j' be the iteration of (M', I') sending a' to a. Now one of two things must hold. Either there exist $i < \omega$, $\gamma < \omega_2^{M_i}$ and a bijection $f : \omega_1^{M_0} \to \gamma$ in M_i such that $\{\alpha < \omega_1^{M_0} : \mathrm{ot}(f``\alpha) \in j'(e)\}$ and $\{\alpha < \omega_1^{M_0} : \mathrm{ot}(f``\alpha) \notin j'(e)\}$ are both I_i-positive subsets of $\omega_1^{M_0}$ in M_i, or there are no such i, γ, f.

If there is no such triple, then the image of $j'(e)$ is the same under every iteration of $\langle(M_i, I_i) : i < \omega\rangle$ of length ω_1^N. Let x be a real in N coding $\langle(M_i, I_i) : i < \omega\rangle$. There exist iterations of $\langle(M_i, I_i) : i < \omega\rangle$ of length ω_1^N in forcing extensions of $L[x]$ by the partial order $\mathrm{Col}(\omega, {<}\omega_1^N)$, and since this partial order is homogeneous, this fixed image of $j'(e)$ exists in $L[x]$. Letting j be any suitable (for example, using a strategy for Player I in $\mathcal{G}(\omega_1)$ as in Theorem 3.5) such iteration in N of length ω_1^N, then, $j(j')$ is an iteration of (M', I') satisfying the first conclusion of the lemma.

If there is such a triple, note that there is a real y in M_{i+1} such that γ is definable in M_{i+1} (absolutely, in fact) from $\omega_1^{M_0}$ and y (for instance, we could let y be the sharp of any real whose least indiscernible above $\omega_1^{M_0}$ is greater than γ). In particular, we may fix a ternary formula ϕ such that γ is the unique ordinal such that $\phi(\gamma, y, \omega_1^{M_0})$ holds in $L[y]$. Let

$$A = \{\alpha < \omega_1^{M_0} : \mathrm{ot}(f``\alpha) \in j'(e)\}.$$

Then A is the Boolean value of the statement that γ is in the image of $j'(e)$. Let x be a real in N coding $(\langle (M_i, I_i) : i < \omega \rangle, a)$. Then just as in the proof of Lemma 1.8, the indiscernibles of x are on the critical sequence of any iteration of $\langle (M_i, I_i) : i < \omega \rangle$. Fix a set $B \subseteq \omega_1^N$ in N such that $\omega_1^N = \omega_1^{L[B]}$. Working in N, build an iteration j (with partial iterations $j_{\alpha\beta}$ ($\alpha \leq \beta \leq \omega_1^N$) and normal filters G_α ($\alpha < \omega_1^N$)) of $\langle (M_i, I_i) : i < \omega \rangle$, using a winning strategy for Player I in the game $\mathcal{G}(\omega_1^N \setminus E)$ from Theorem 3.5, where E is the set of countable x-indiscernibles which are not limits of x-indiscernibles (note that $x^\# \in N$, as (N, J) is iterable, so N contains the sharps for all its reals). When $j_{0\alpha}(\omega_1^{M_0})$ is in E, we put $j_{0\alpha}(A)$ in the normal filter G_α if and only if $\eta \in B$, where $j_{0\alpha}(\omega_1^{M_0})$ is the ηth successor x-indiscernible. Having completed the construction of our iteration, we have that B is constructible from $j(j'(e)), y$ and $x^\#$: B is the set of $\eta < \omega_1^N$ such that, letting ι_η be the η-th successor x-indiscernible, the unique ordinal γ^* satisfying $\phi(\gamma^*, y, \iota_\eta)$ in $L[y]$ is in $j(j'(e))$. Then $j(j')$ is an iteration of (M', I') satisfying the second conclusion of the lemma. ⊣

For the rest of this section we fix the following notation: if B is a subset of ω_1, we let F_B be the set of conditions $\langle (M, I), b \rangle$ in $\mathbb{P}_{\max}$ such that there exists an iteration $j : (M, I) \to (M^*, I^*)$ such that $j(b) = B$ and $I^* = \mathrm{NS}_{\omega_1} \cap M^*$.

Woodin defines the following axiom.

7.5 Definition. Axiom $(*)$ is the statement that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.

The proofs of Theorems 7.7 and 7.8 use the following lemma.

7.6 Lemma. *Assume that axiom $(*)$ holds, and let B be a subset of ω_1 such that there exists a real z such that $\omega_1 = \omega_1^{L[z,B]}$. Then the set F_B is a filter.*

Proof. Fix an $L(\mathbb{R})$-generic filter $G \subseteq \mathbb{P}_{\max}$ such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$. Fix a real x such that $\omega_1 = \omega_1^{L[x,A_G]}$. As in the proof of Lemma 2.7, let $\{a_\alpha : \alpha < \omega_1\}$ be the almost disjoint family of subsets of ω constructed in $L[x, A_G]$ by recursively taking a_α to be the first real in the $L[x, A_G]$ constructibility order almost disjoint from each a_β ($\beta < \alpha$). Now let $y \subseteq \omega$ be such that for all $\alpha < \omega$, $a_\alpha \cap y$ is infinite if and only if $\alpha \in B$. Let

$$p_0 = \langle (M_0, I_0), b_0 \rangle \quad \text{and} \quad p_1 = \langle (M_1, I_1), b_1 \rangle$$

be members of F_B, as witnessed by iterations j_0 and j_1 respectively, and let C_0 and C_1 be the respective critical sequences of j_0 and j_1. Let $\langle (N, J), a \rangle$ be a member of G with $x, y, z, p_0, p_1 \in H(\omega_1)^N$ and sets c_0 and c_1 in $\mathcal{P}(\omega_1)^N$ such that, for j the unique iteration of (N, J) sending a to A_G, $j(c_0) = C_0$ and $j(c_1) = C_1$. Then $c_0 = C_0 \cap \omega_1^N$, $c_1 = C_1 \cap \omega_1^N$, and c_0 and c_1 are both club subsets of ω_1^N. Since $\omega_1^{L[z,B]} = \omega_1$, $\omega_1^{L[z,b_0]} = \omega_1^{M_0}$ and $\omega_1^{L[z,b_1]} = \omega_1^{M_1}$. Since $x \in N$, $\{a_\alpha : \alpha < \omega_1^N\}$ is in N (it satisfies the same definition in N

relative to x and a that $\{a_\alpha : \alpha < \omega_1\}$ satisfies relative to x and A_G). Since $y \in N$, $B \cap \omega_1^N \in N$, and the unique iterations

$$j_0^* : (M_0, I_0) \to (M_0^*, I_0^*)$$

and

$$j_1^* : (M_1, I_1) \to (M_1^*, I_1^*)$$

sending b_0 to $B \cap \omega_1^N$ and b_1 to $B \cap \omega_1^N$ respectively are in N, and furthermore $j(B \cap \omega_1^N) = B$. Once we see that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$ we will be done. The two proofs are the same. For (M_0, I_0), if $E \in J \cap M_0^*$, then since $E \in J$, $j(E) \in \mathrm{NS}_{\omega_1}^{L(\mathbb{R})[G]}$. Now, $j(j^*)$ is an iteration of (M_0, I_0) sending b_0 to B, and so it is equal to j_0. Then $j(E)$ is the image of E under the tail of the iteration j_0 starting with (M_0^*, I_0^*). So $j(E) \in j_0(M_0)$, and since $j_0(I_0) = \mathrm{NS}_{\omega_1}^{L(\mathbb{R})[G]} \cap j_0(M_0)$, $j(E) \in j_0(I_0)$, and so $E \in I_0^*$. ⊣

If $G \subseteq \mathbb{P}_{\max}$ is an $L(\mathbb{R})$-generic filter, then $F_{(A_G)}$ is a filter containing G, and so by the genericity of G, $F_{(A_G)} = G$.

Now we show that any new subset of ω_1 added by forcing with $\mathbb{P}_{\max}$ generates the entire extension.

7.7 Theorem. *Assume that axiom* $(*)$ *holds. Then for every* $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$*,* F_B *is an* $L(\mathbb{R})$*-generic filter for* $\mathbb{P}_{\max}$*, and* $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[F_B]$*.*

Proof. By Lemma 7.4, there is a real z such that $L[z, B]$ correctly computes ω_1. By Lemma 7.6, F_B is a filter. Now, let $\langle (M, I), a\rangle$ be a condition in G such that $z \in M$ and for some $b \in \mathcal{P}(\omega_1)^M$, $j(b) = B$, for j the unique iteration of (M, I) sending a to A_G. As in Lemma 2.10, the mapping π sending each condition $\langle (N, J), c\rangle$ below $\langle (M, I), a\rangle$ to the condition $\langle (N, J), b^*\rangle$, where b^* is the image of b by the iteration of (M, I) sending a to c, is an isomorphism. The image of G under π, F_B, is then an $L(\mathbb{R})$-generic filter in $\mathbb{P}_{\max}$. Furthermore, π is in $L(\mathbb{R})$, so G is in $L(\mathbb{R})[F_B]$. ⊣

7.8 Theorem. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and that for every* Π_2 *sentence* ϕ *in the language with two additional unary predicates, if* $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ *and*

$$\langle H(\omega_2), A, \mathrm{NS}_{\omega_1}, \in\rangle^{L(\mathbb{R})^{\mathbb{P}_{\max}}} \models \phi$$

then

$$\langle H(\omega_2), A, \mathrm{NS}_{\omega_1}, \in\rangle \models \phi\,.$$

Then for every $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$*,* F_B *is an* $L(\mathbb{R})$*-generic filter and*

$$H(\omega_2) = H(\omega_2)^{L(\mathbb{R})[F_B]}.$$

Proof. Let $\mathcal{P}$ denote $\mathcal{P}(\omega_1) \setminus \bigcup\{L[x] : x \in \mathbb{R}\}$ (under $\mathrm{AD}^{L(\mathbb{R})}$ this is the same as $\mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, but we want to make the relevant syntax more explicit). The sentence asserting that F_B is a filter for every $B \in \mathcal{P}$ is Π_2 in $H(\omega_2)$ with parameters for NS_{ω_1} and the set of $\mathbb{P}_{\max}$ conditions, and by Lemma 7.6, this sentence holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$. If $X \in L(\mathbb{R})$ is a dense subset of $\mathbb{P}_{\max}$, then the statement that $F_B \cap X$ is nonempty for every $B \in \mathcal{P}$ is Π_2 in $H(\omega_2)$ with parameters for $\S_{\omega_1}$, X and the set of $\mathbb{P}_{\max}$ conditions, and this sentence holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ by Theorem 7.7. Thus, for every $B \in \mathcal{P}$, F_B is $L(\mathbb{R})$-generic. Finally, the following statement is Π_2 in $H(\omega_2)$ with parameters for NS_{ω_1} and the set of $\mathbb{P}_{\max}$ conditions, and holds in the $\mathbb{P}_{\max}$ extension: for every $E \subseteq \omega_1$ and for every $B \in \mathcal{P}$ there is a $\mathbb{P}_{\max}$ condition $\langle (M, I), b\rangle$ and an iteration $j : (M, I) \to (M^*, I^*)$ such that $E \in \mathcal{P}(\omega_1)^{M^*}$, $j(b) = B$ and $I^* = \mathrm{NS}_{\omega_1} \cap M^*$. Fixing a set $B \in \mathcal{P}$, then, since $\{x, B\} \in L(\mathbb{R})[F_B]$, $H(\omega_2) \subseteq H(\omega_2)^{L(\mathbb{R})[F_B]}$. ⊣

Theorem 7.7 gives us another way to characterize the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, this time without mention of $\mathbb{P}_{\max}$. For the definition below, we fix the following notation. If g is a filter contained in $\mathrm{Col}(\omega, {<}\omega_1)$, then for each $\alpha < \omega_1$ we let

$$S^g_\alpha = \{\beta \mid \exists p \in g\ p(0, \beta) = \alpha\}$$

and, for each $\tau \subseteq \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$, we let

$$I_g(\tau) = \{\alpha \mid \exists p \in g\ (\alpha, p) \in \tau\}.$$

7.9 Definition. Axiom $\binom{*}{*}$ is the statement that $x^\#$ exists for every real number x and if X is a nonempty subset of $\mathcal{P}(\omega_1)$ which is definable from real and ordinal parameters then there exists a real x and a set

$$\tau \subseteq \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$$

such that $\tau \in L[x]$ and such that for all filters $g \subseteq \mathrm{Col}(\omega, {<}\omega_1)$, if g is $L[x]$-generic and if for each $\alpha < \omega_1$, S^g_α is stationary, then $I_g(\tau) \in X$.

The converse of the following theorem also holds (see [39, Sects. 5.7, 5.8]), though its proof is beyond the scope of this chapter.

7.10 Theorem. *Axiom $(*)$ implies that axiom $\binom{*}{*}$ holds in $L(\mathcal{P}(\omega_1))$.*

Proof. First note that AD implies that the sharp of every real exists. Now let G be an $L(\mathbb{R})$-generic filter such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$, and fix a set X as in the statement of axiom $\binom{*}{*}$. Let $p = \langle (M, I), a\rangle$ be a condition in G such that for some $b \in \mathcal{P}(\omega_1)^M$, p forces that $j(b) \in X$, for j the unique iteration of (M, I) sending a to A_G. Let x be a real such that $\langle (M, I), a\rangle \in H(\omega_1)^{L[x]}$. Now, if $g \subseteq \mathrm{Col}(\omega, {<}\omega_1)$ is $L[x]$-generic, then as in the proof of Lemma 2.8, in $L[x][g]$ there is an iteration

$$\{M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\}$$

of (M, I) such that for each $\beta < \omega_1$ and each set $e \in \mathcal{P}(\omega_1)^{M_\beta} \setminus j_{0\beta}(I)$ there is an $\alpha < \omega_1$ such that $S^g_\alpha \setminus j_{\beta\omega_1}(e)$ is nonstationary. Let σ be an $\mathrm{Col}(\omega, {<}\omega_1)$-name in $L[x]$ for the embedding $j_{0\omega_1}$ corresponding to such an iteration, and let τ be the set of pairs $(\alpha, p) \in \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$ such that $p \Vdash \check{\alpha} \in \sigma(\check{b})$. Now suppose that $g \subseteq \mathrm{Col}(\omega, {<}\omega_1)$ is $L[x]$-generic and that each S^g_α is stationary. Then σ_g is an iteration of (M, I), and since there exists a real z such that

$$\omega_1^{L[z,\sigma_g(a)]} = \omega_1,$$

$\sigma_g(a)$ is not in $L(\mathbb{R})$. Then Theorem 7.7 implies that $G_{\sigma_g(a)}$ (as in the statement of that theorem) is an $L(\mathbb{R})$-generic filter for $\mathbb{P}_{\max}$. Since each S^g_α is stationary, σ_g witnesses that $\langle (M, I), a\rangle$ is in $G_{\sigma_g(a)}$, which means that $\sigma_g(b)$ is in X. Since $\sigma_g(b) = I_g(\tau)$, we are done. ⊣

Theorem 7.10 has the following immediate corollary. By a perfect subtree of $2^{<\omega_1}$ we mean a tree of height ω_1 such that every node is extended by a pair of incompatible nodes, and such that every countable increasing sequence has a node extending it.

7.11 Corollary. *Assume $\mathrm{AD}^{L(\mathbb{R})}$, and let $G \subseteq \mathbb{P}_{\max}$ be an $L(\mathbb{R})$-generic filter. Let ϕ be a unary formula with parameters for elements of $L(\mathbb{R})$ and suppose that there exists a subset of ω_1 in $L(\mathbb{R})[G] \setminus L(\mathbb{R})$ satisfying ϕ. Then there is a perfect subtree T of $2^{<\omega_1}$ such that every subset of ω_1 corresponding to a path through T satisfies ϕ.*

Given an ordinal β, Martin's Maximum$^{+\beta}$ ($\mathrm{MM}^{+\beta}$, derived from [7]) is the statement that whenever P is a partial order such that forcing with P preserves stationary subsets of ω_1, $\langle D_\alpha : \alpha < \omega_1\rangle$ is a sequence of dense subsets of P and $\langle \tau_\alpha : \alpha < \beta\rangle$ is a sequence of P-names for stationary subsets of ω_1, there is a filter $G \subseteq P$ such that $G \cap D_\alpha$ is nonempty for each $\alpha < \omega_1$ and $\{\gamma < \omega_1 \mid \exists p \in G\ p \Vdash \check{\gamma} \in \tau_\alpha\}$ is stationary for each $\alpha < \beta$.

It is shown in [17] that $\mathrm{MM}^{+\omega}$ does not imply axiom $(*)$, if the existence of a supercompact limit of supercompact cardinals is consistent with ZFC. The question of whether $\mathrm{MM}^{+\omega_1}$ implies axiom $(*)$ remains open. We mention the following two test cases, consequences of axiom $(*)$ which have not been shown from large cardinals to be provably forceable by a semi-proper partial order. We omit the proofs, as they appear in full in [39] (Theorem 7.12 appears in [39] as Theorem 5.74(5) and Theorem 7.15 appears as Theorem 6.124).

7.12 Theorem. *Suppose that axiom $\binom{*}{*}$ holds. Then for every $A \subseteq \omega_1$ which is not constructible from a real, there exist a real x and an $L[x]$-generic filter $g \subseteq \mathrm{Col}(\omega, {<}\omega_1)$ such that $L[x][g] = L[x, A]$.*

The statement of Theorem 7.15 requires the following definitions.

7.13 Definition. A tree $T \subseteq \{0,1\}^{<\omega_1}$ is *weakly special* if for all countable $X \prec \langle H(\omega_2), T, \in\rangle$, if $b : \omega_1 \cap X \to \{0,1\}$ is a cofinal branch of T_X not in M_X, then there is a bijection $\pi : \omega \to \omega_1^{M_X}$ definable in the structure $\langle M_X, T_X, b, \in\rangle$, where $\langle M_X, T_X, \in\rangle$ is the transitive collapse of X.

7.14 Definition. $\Phi^+_\diamondsuit$ is the statement that for each $A \subseteq \omega_1$ there exists a $B \subseteq \omega_1$ such that, letting $T_B = \{0,1\}^{<\omega_1} \cap L[B]$,

- $A \in L[B]$,
- T_B is weakly special,
- every branch of T_B is in $L[B]$.

7.15 Theorem. *Axiom* $(*)$ *implies* $\Phi^+_\diamondsuit$.

One consequence of Theorem 7.15 is that there are no weak Kurepa trees (subtrees of $\{0,1\}^{<\omega_1}$ of cardinality $\aleph_1$ with $\aleph_2$ many cofinal branches) in any $\mathbb{P}_{\max}$ extension.

8. Larger Models

The forcing construction $\mathbb{P}_{\max}$ can be applied to larger models than $L(\mathbb{R})$, if they satisfy (ostensibly) stronger forms of determinacy.

8.1 Definition. A set of reals A is *$^\infty$-borel* if there exists a set of ordinals S, an ordinal α and a binary formula ϕ such that

$$A = \{y \in \mathbb{R} \mid L_\alpha[S, y] \models \phi(S, y)\}.$$

The ordinal Θ is defined to be the least ordinal which is not a surjective image of $\mathbb{R}$. The notion of continuity in the definition below refers to the discrete topology on λ, not the interval topology. *Dependent Choice* (DC) is a weak form of the Axiom of Choice saying that every tree of height ω with no terminal nodes has a cofinal branch; *Dependent Choice for Sets of Reals* ($\mathrm{DC}_\mathbb{R}$) is the restriction of DC to trees on the reals.

8.2 Definition (ZF + $\mathrm{DC}_\mathbb{R}$). AD^+ is the conjunction of the following two statements.

- Every set of reals is *$^\infty$-borel*.
- If $\lambda < \Theta$ and $\pi : \lambda^\omega \to \omega^\omega$ is a continuous function, then $\pi^{-1}(A)$ is determined for every $A \subseteq \omega^\omega$.

It is an open question whether AD implies AD^+, though it is known that AD^+ holds in all models of AD of the form $L(A, \mathbb{R})$, where A is a set of reals (some of the details of the argument showing this appear in [9]).

The following consequences of AD^+ are enough to prove that $\mathbb{P}_{\max}$ conditions exist in suitable generality.

8.3 Theorem (ZF + $\mathrm{DC}_\mathbb{R}$). *If* AD^+ *holds and* $V = L(\mathcal{P}(\mathbb{R}))$ *then*

- *the pointclass* Σ^2_1 *has the scale property,*
- *every* Σ^2_1 *set of reals is the projection of a tree in* HOD*,*
- *every true* Σ_1*-sentence is witnessed by a* $\underset{\sim}{\Delta}^2_1$ *set of reals.*

Adapting the proof of Theorem 4.13, then, we have the following.

8.4 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages and that $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}^+$. Then for every set of reals A in $L(\Gamma, \mathbb{R})$ there is a $\mathbb{P}_{\max}$ precondition (M, I) such that*

- $A \cap M \in M$,
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$,
- (M, I) *is A-iterable.*

The corresponding parts of the proof of Theorem 5.1 then go through to give the following.

8.5 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}^+$. Suppose that $G \subseteq \mathbb{P}_{\max}$ is $L(\Gamma, \mathbb{R})$-generic. Then the following hold in $L(\Gamma, \mathbb{R})[G]$:*

- $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$,
- I_G *is the nonstationary ideal,*
- $\utilde{\delta}^1_2 = \omega_2$,
- I_G *is saturated.*

If there is no surjection in $L(\Gamma, \mathbb{R})$ from $\mathbb{R} \times \mathrm{On}$ onto Γ, then Γ is not wellordered in the $\mathbb{P}_{\max}$ extension of $L(\Gamma, \mathbb{R})$. Producing a model of Choice then requires the following step, which appears with proof in [39] as Theorem 9.36. The statement ω_2-DC says that $<\!\omega_2$-closed trees of height ω_2 with no terminal nodes have cofinal branches.

8.6 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}^+ +$ "Θ is regular". Suppose that $G \subseteq \mathbb{P}_{\max}$ is $L(\Gamma, \mathbb{R})$-generic. Then $L(\Gamma, \mathbb{R})[G] \models \omega_2$-DC.*

The axiom $\mathrm{AD}_{\mathbb{R}}$ is the statement that all two player games of perfect information of length ω where the players play real numbers are determined. This statement easily implies $\mathrm{DC}_{\mathbb{R}}$ and in the context of DC is properly stronger than AD^+. Theorem 8.7 below lists some properties of the $\mathbb{P}_{\max}$ extension of a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular". Many of the corresponding proofs proceed by finding a $\mathbb{P}_{\max}$ condition satisfying axiom $(*)$ and satisfying the conclusion of Theorem 8.4 for a suitable set A. We emphasize that the first conclusion of Theorem 8.7 says that in the $\mathbb{P}_{\max}$ extension of $L(\Gamma, \mathbb{R})$, $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, not (merely) $L(\Gamma, \mathbb{R})$.

Martin's Maximum$^{++}(\mathfrak{c})$ is the restriction of Martin's Maximum$^{+\omega_1}$ to partial orders of cardinality the continuum, which it implies is $\aleph_2$. (The notation comes from Woodin [39, p. 36], where $\mathrm{MM}^{+\omega_1}$ is called Martin's Maximum^{++}.) The statement $\diamondsuit(S^{\omega_2}_\omega)$ says that there is a sequence

$$\{A_\gamma : \gamma < \omega_2 \wedge \mathrm{cf}(\gamma) = \omega\}$$

such that each A_γ is a subset of γ and such that for every $B \subseteq \omega_2$, the set of $\alpha < \omega_2$ of countable cofinality such that $B \cap \alpha = A_\alpha$ is stationary. Woodin shows in [39, Sect. 5.2] that $\diamondsuit(S^{\omega_2}_\omega)$ follows from Martin's Maximum. Part (4) of the conclusion of Theorem 8.7 is due to Daniel Seabold [28]. *Chang's Conjecture* is the statement that for each function $F : [\omega_2]^{<\omega} \to \omega_2$ there exists an $X \subseteq \omega_2$ of ordertype ω_1 such that $F``[X]^{<\omega} \subseteq X$ (i.e., that the set of subsets of ω_2 of ordertype ω_1 is stationary, in the sense of [19]). It is an open question whether Chang's Conjecture holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ whenever $L(\mathbb{R})$ satisfies AD. This question has been resolved (negatively) for $\mathbb{Q}_{\max}$ (see Remark 10.7).

Parts (5), (6) and (7) of Theorem 8.7 show that $\mathbb{P}_{\max}$ can be used to produce consistency results at ω_2 as well as at ω_1. We let $\mathrm{NS}^\omega_{\omega_2}$ denote the nonstationary ideal on ω_2 concentrating on the ordinals of cofinality ω. The ideal $\mathrm{NS}^\omega_{\omega_2}$ is *weakly presaturated* if for every $S \in \mathcal{P}(\omega_2) \setminus \mathrm{NS}^\omega_{\omega_2}$ and every function $f : S \to \omega_2$ there exist a ordinal $\gamma < \omega_3$ and a bijection $\pi : \omega_2 \to \gamma$ such that

$$\{\alpha \in S \mid f(\alpha) < \mathrm{ot}(\pi[\alpha])\} \notin \mathrm{NS}^\omega_{\omega_2}.$$

A normal ideal I on ω_2 is *semi-saturated* if whenever U is a set generic V-normal ultrafilter on ω_2 contained in $\mathcal{P}(\omega_2) \setminus I$, $\mathrm{Ult}(V, U)$ is wellfounded.

8.7 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_\mathbb{R} +$ "Θ is regular". Suppose that $G \subseteq \mathbb{P}_{\max}$ is $L(\Gamma, \mathbb{R})$-generic, and let*

$$H \subseteq \mathrm{Col}(\omega_3, H(\omega_3))^{L(\Gamma,\mathbb{R})[G]}$$

be an $L(\Gamma, \mathbb{R})[G]$-generic filter. Then the following hold in $L(\Gamma, \mathbb{R})[G][H]$:

1. *Axiom $(*)$,*
2. *Martin's Maximum$^{++}(\mathfrak{c})$,*
3. *$\diamondsuit(S^{\omega_2}_\omega)$,*
4. *Chang's Conjecture,*
5. *$\mathrm{NS}^\omega_{\omega_2}$ is precipitous,*
6. *$\mathrm{NS}^\omega_{\omega_2}$ is weakly presaturated,*
7. *there is a normal semi-saturated ideal on ω_2 containing $\mathrm{NS}^\omega_{\omega_2}$.*

9. Ω-Logic

In this section we will briefly describe the relationship between $\mathbb{P}_{\max}$ and Woodin's Ω-logic as presented in [39] (our presentation of Ω-logic, however, will follow the one in [44]; many basic facts about Ω-logic are proved in [1]). Let T be a set of sentences and let ϕ be a sentence, both in the language of set theory. Then $T \models_\Omega \phi$ (ϕ is Ω_T-valid) if for every forcing construction P and every ordinal α, if $V^P_\alpha \models T$ then $V^P_\alpha \models \phi$. We will define below the conjectured proof-theoretic complement to this model-theoretic notion.

A set of reals A is *universally Baire* if it is κ-universally Baire for all cardinals κ (see Definition 7.2). Woodin has shown that if δ is a limit of Woodin cardinals, then a set of reals is $<\delta$-universally Baire if and only if it is $<\delta$-weakly homogeneously Suslin (a proof is given in [19]). Given a universally Baire set of reals A, a transitive model N of ZFC is said to be A-*closed* if, whenever P is a partial order in N and $G \subseteq P$ is V-generic (not just N-generic), then $N[G] \cap A(G)$ is in $N[G]$. Lemmas 9.2 and 9.3 give useful reformulations of A-closure, and are relatively easy to prove (see [1]). The proof of Lemma 9.2 uses the following fact, which will show up again in the proof of Theorem 9.4 and in Sect. 10.1. For a proof of Theorem 9.1, see [10, p. 516] or [19, Appendix].

9.1 Theorem (McAloon). *If $\mathbb{P}$ is a partial order and forcing with $\mathbb{P}$ makes $\mathbb{P}$ countable, then $\mathbb{P}$ is forcing-equivalent to* $\mathrm{Col}(\omega, |\mathbb{P}|)$.

Theorem 9.1 implies that every partial order $\mathbb{P}$ regularly embeds into $\mathrm{Col}(\omega, |\mathbb{P}|)$, which is forcing-equivalent to $\mathbb{P} \times \mathrm{Col}(\omega, |\mathbb{P}|)$.

9.2 Lemma. *Given a universally Baire set of reals A, a model M of* ZFC *is A-closed if and only if for all ordinals $\gamma \in M$, the set of pairs $(\tau, p) \in H(|\gamma|^+)^M$ such that τ is a* $\mathrm{Col}(\omega,\gamma)$*-name in M for a real, p is a condition in* $\mathrm{Col}(\omega,\gamma)$ *and p forces in V that the realization of τ is in $A(G)$ is in M.*

Lemma 9.3 shows that for countable models, it is not necessary to consider V-generic filters. The point is that, even without assuming the existence of large cardinals, if

- A is a universally Baire set of reals,
- M is a countable transitive model of ZFC,
- P is a partial order in M,
- p is a condition in P, and
- τ is a P-name in M for a real number

then p forces in V that τ_G is in $A(G)$ if and only if there exists (in V) a collection $\{D_i : i < \omega\}$ of dense subsets of P such that $\tau_g \in A$ for every M-generic filter $g \subseteq P$ containing p and intersecting each D_i.

9.3 Lemma. *Let A be a universally Baire set of reals and let M be a countable transitive model of* ZFC*. Then M is A-closed if and only if for each partial order P in M there exists (in V) a collection $\{D_i : i < \omega\}$ of dense subsets of P such that $M[G] \cap A \in M[G]$ for every M-generic filter $g \subseteq P$ intersecting each D_i.*

By Lemma 9.3 (and the fact that the set of wellfounded ordinals of an illfounded model of ZFC is not an element of the model), if A is the set of reals coding wellorderings of ω (under some fixed recursive coding), then (expanding to the class of ω-models of ZFC) A-closure is equivalent to wellfoundedness.

Let T be a theory containing ZFC and let ϕ be a sentence, both in the language of set theory. Then $T \vdash_\Omega \phi$ (T implies ϕ in Ω-logic) if there exists a set of reals A such that

- $L(A, \mathbb{R}) \models \mathrm{DC}_\mathbb{R} + \mathrm{AD}^+$,
- every set of reals in $L(A, \mathbb{R})$ is universally Baire,
- for every countable A-closed model M and every ordinal $\alpha \in M$, if V_α^M satisfies T then V_α^M satisfies ϕ.

A sentence ϕ is Ω_{ZFC}-*consistent* if ZFC $\nvdash_\Omega \neg\phi$. The first two conditions above ensure that the set of reals A is sufficiently canonical, and hold of all universally Baire sets of reals in the presence of a proper class of Woodin cardinals. The third condition says that A serves as a sort of proof of ϕ, in the sense that ϕ holds in all models which are closed under a certain function corresponding to A.

The following theorem shows that statements which can be forced to hold (along with ZFC) in suitable initial segments of the universe are Ω_{ZFC}-consistent. The proof shows the stronger fact that for every universally Baire set of reals A, all forceable statements hold in models N which are A-closed in the stronger sense that $N[G] \cap A \in N[G]$ for all N-generic filters G.

9.4 Theorem. *Suppose that A is a universally Baire set of reals and that κ is a strongly inaccessible cardinal. Then any forcing extension (in V) of any transitive collapse of any elementary submodel of V_κ containing A is A-closed.*

Proof. Since A is universally Baire and κ is strongly inaccessible, A is universally Baire in V_κ. To see this, fix a partial order P in V_κ and trees S and T witnessing the universal Baireness of A for P. Let ρ be a P-name in V_κ for all the reals of the P-extension, let θ be a regular cardinal greater than $|S^{<\kappa}|$, $|T^{<\kappa}|$ and κ and let X be an elementary submodel of $H(\theta)$ of cardinality less than κ containing $\{S, T\}$ and the transitive closure of ρ. Then the images of S and T under the transitive collapse of X are in V_κ and witness the universal Baireness of A for P.

Now let X be an elementary submodel of V_κ with A as an element, and let M be the transitive collapse of X. Let P be a partial order in X, let $\bar{P}$ be the image of P under the transitive collapse of X, and let $g \subseteq \bar{P}$ be an M-generic filter. Let τ be a P-name in X for a partial order, and let $\bar{\tau}$ be the image of τ under the transitive collapse of X. We want to see that whenever $h \subseteq \bar{\tau}_g$ is a V-generic filter, then $M[g][h] \cap A(h)$ is in $M[g][h]$. Let $\gamma \in X \cap \kappa$ be a cardinal greater than $|P * \tau|$ and let S and T be trees in X witnessing the universal Baireness of A for $\mathrm{Col}(\omega, \gamma)$. Then S and T project to complements in any forcing extension of V by either $P * \tau$ or $\bar{\tau}_g$.

Let σ be a $\bar{\tau}_g$-name in $M[g]$ for a real. Let $\bar{S}$ and $\bar{T}$ be the images of S and T under the transitive collapse of X. Let $h \subseteq \bar{\tau}_g$ be V-generic. Then σ_h is in exactly one of $(p[S])^{V[h]}$ and $(p[T])^{V[h]}$, and by the elementarity of the collapsing map, σ_h is in exactly one of $(p[\bar{S}])^{M[g][h]}$ and $(p[\bar{T}])^{M[g][h]}$. Since $(p[\bar{S}])^{M[g][h]} \subseteq (p[S])^{V[h]}$ and $(p[\bar{T}])^{M[g][h]} \subseteq (p[T])^{V[h]}$, and since $A(h) = (p[S])^{V[h]}$, σ_h is in $A(h)$ if and only it is in $(p[\bar{S}])^{M[g][h]}$. Putting all of this together, we have that

$$M[g][h] \cap A(h) = (p[\bar{S}])^{M[g][h]},$$

which shows that $M[g]$ is A-closed. $\dashv$

Woodin has shown that the axiom $(*)$ is Ω_{ZFC}-consistent.

9.5 Theorem. *Suppose that there exists a proper class of Woodin cardinals and that there is an inaccessible cardinal which is a limit of Woodin cardinals. Then the theory*

$$\mathrm{ZFC} + (*)$$

is Ω_{ZFC}-consistent.

The proof of Theorem 9.5 requires one to force axiom $(*)$ over larger models than $L(\mathbb{R})$, in particular, models of the form $L(S, \mathbb{R})$, where for some strongly inaccessible limit of Woodin cardinals κ, S is a $<\kappa$-weakly homogeneous tree. A proof that such models can satisfy AD^+ appears in [19]. It is not known whether there are large cardinals whose existence implies that one can force over V to make axiom $(*)$ hold. Woodin has conjectured that (ordertype) ω^2 many Woodin cardinals are sufficient. Of course, if $\mathrm{MM}^{+\omega_1}$ implies axiom $(*)$ (we discussed this question in Sect. 7) then one supercompact cardinal is sufficient.

9.6 Definition. Woodin's Ω *Conjecture* asserts that if there exist proper class many Woodin cardinals then for every sentence ϕ, $\emptyset \models_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$.

Recall that for a set x, $x^\dagger$ is a set of the same cardinality as x coding a theory extending ZFC + “There exists a measurable cardinal” with constants for each member of x and for two classes of indiscernibles (above and below the measurable cardinal). If there exist proper class many Woodin cardinals, then the set D of reals coding (under some fixed recursive coding) pairs (x, i), where x is a real number, i is an integer and $i \in x^\dagger$ is universally Baire. Any D-closed model M then has the property that for any set x, $x^\dagger$ exists in any forcing extension of M where x is countable, which since $x^\dagger$ is unique means that $x^\dagger$ exists in M already (an easy way to say this uses the fact that $\mathrm{Col}(\omega, |x|)$ is homogeneous, though this is not necessarily the most direct way). Thus for every ordinal $\alpha \in M$, there exist an inner model N of M containing V_α^M (definable in M), an ordinal $\kappa > \alpha$ which is a measurable cardinal in N and a set μ which is a κ-complete nonprincipal measure on κ in N such that all iterates of N by μ are wellfounded. As in Example 1.7, then, if M is a D-closed model and I is a normal precipitous ideal on ω_1^M in M, then every rank initial segment of M satisfying ZFC° is a rank initial segment of a model N such that (N, I) is iterable, and so (M, I) is also iterable. Using this we have that every Π_2 sentence for $\langle H(\omega_2), \mathrm{NS}_{\omega_1}, \in\rangle$ which is Ω_{ZFC}-consistent with the existence of a precipitous ideal on ω_1 holds in the $\mathbb{P}_{\max}$ extension. Using the canonical inner models for Woodin cardinals one can do more, however.

9.7 Theorem. *If there is a proper class of Woodin cardinals, then for every set of reals A in $L(\mathbb{R})$, every Ω_{ZFC}-consistent Π_2 sentence for $\langle H(\omega_2), \mathrm{NS}_{\omega_1}, A, \in\rangle$ holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.*

In the next few paragraphs we will sketch a proof of Theorem 9.7 (an actual proof is beyond the scope of this chapter). This will require introducing some concepts from inner model theory (see [34, 27], for instance).

Given a set of reals A such that $A = L(A) \cap \mathbb{R}$, $A^\#$ is a set of reals coding the theory of $L(A)$ in the language with constants for each real and ω many ordinal indiscernibles (see [33]; again, this is not a characterization of $A^\#$, which is unique if it exists). If $\mathbb{R}^\#$ exists then each set of reals in $L(\mathbb{R})$ is definable in $L(\mathbb{R})$ from a real and a finite set of these indiscernibles.

The following theorem (due to Woodin) is proved in [19].

9.8 Theorem. *Suppose that δ is a limit of Woodin cardinals below a measurable cardinal. Then $\mathbb{R}^\#$ is $<\delta$-weakly homogeneous, and if M is any forcing extension of V by a forcing construction in V_δ then $(\mathbb{R}^\#)^V = (\mathbb{R}^\#)^M \cap V$.*

For each set a, let $M(a)$ denote the minimal model of ZFC + "There exist infinitely many Woodin cardinals below a measurable cardinal". (i.e., the unique fine structural, fully iterable model of this theory which comes out shorter in comparison with every other such model of this theory). This theory implies that $\mathbb{R}^\#$ is $<\lambda$-weakly homogeneously Suslin, and so there exist in $M(a)$ trees S and T on $\omega \times \lambda$ witnessing in $M(a)$ that $\mathbb{R}^\#$ and its complement are $<\lambda$-universally Baire. Furthermore, from the point of view of V, S and T project to a subset of $\mathbb{R}^\#$ and a subset of $\mathbb{R} \setminus \mathbb{R}^\#$, respectively.

The property of $M(a)$ that we need is the following: if δ is a Woodin cardinal in $M(a)$ below λ, γ is an ordinal below δ and y is a subset of ω, then there exist a partial order P (this partial order was discovered by Woodin and is usually called the *extender algebra*) of cardinality δ in $M(a)$ and an elementary embedding $j : M(a) \to M'$ with critical point greater than γ such that

- y is M'-generic for $j(P)$,
- $p[j(S)] \subseteq \mathbb{R}^{\#}$,
- $p[j(T)] \subseteq \mathbb{R} \setminus \mathbb{R}^{\#}$.

There is a universally Baire function f taking a real x coding an a in $H(\omega_1)$ to a real $f(x)$ coding $M(a)$. If x and y code the same element of $H(\omega_1)$ then $f(x)$ and $f(y)$ code the same model. If B is the set of reals coding pairs (x, i) such that $i \in f(x)$, then, every B-closed model of ZFC contains $M(a)$ for every set a in M.

Now let ϕ be a Π_2 sentence for $H(\omega_2)$ (of the form $\exists X \forall Y \psi(X, Y)$) with predicates for NS_{ω_1} and a given set of reals A in $L(\mathbb{R})$. Let z be a real number coding a given $\mathbb{P}_{\max}$ condition and a real which codes A relative to $\mathbb{R}^{\#}$. Suppose that N is a countable B-closed model of ZFC satisfying ϕ and containing z. Let a be a wellordering of $H(\omega_2)^N$ in N. Then $H(\omega_2)^{M(a)} = H(\omega_2)^N$. Let γ be the least strongly inaccessible cardinal in $M(a)$ above the least Woodin cardinal. Let S and T be trees in $M(a)$ witnessing the $<\lambda$-universal Baireness of $\mathbb{R}^{\#}$ and its complement, where λ is the least limit of Woodin cardinals in $M(a)$. We want to see that whenever we make NS_{ω_1} precipitous by any forcing in $V_\gamma^{M(a)}$ (getting a generic filter g) and then iterate $V_\gamma^{M(a)[g]}$ by NS_{ω_1}, we iterate correctly for $\mathbb{R}^{\#}$. Given this, if g is such a generic filter for a forcing preserving stationary subsets of $\omega_1^{M(a)}$ then $V_\gamma^{M(a)[g]}$ is an A-iterable model such that $\exists Y \psi(X, Y)$ holds in $H(\omega_2)^{V_\gamma^{M(a)[g]}}$ for all $X \in H(\omega_2)^{V_\gamma^{M(a)}}$, and by a density argument then, ϕ holds in the $\mathbb{P}_{\max}$ extension.

Towards a contradiction, choose a bad generic filter g and bad iteration k. Let $j : M(a) \to M'$ be an embedding (with critical point above γ) such that we can add g and k to M' by forcing with the extender algebra for the image of the least Woodin cardinal in $M(a)$ above γ. Then $M'[g, k]$ has a bad iteration of $V_\gamma^{M(a)[g]}$ in it, and by Lemma 1.5 this iteration extends to an iteration of $M'[g]$ (which we will also call k), which means that

$$k(\mathbb{R}^{\#} \cap V_\gamma^{M(a)[g]}) = p[k(j(S))] \cap k(V_\gamma^{M(a)[g]})$$

and

$$k((\mathbb{R} \setminus \mathbb{R}^{\#}) \cap V_\gamma^{M(a)[g]}) = p[k(j(T))] \cap k(V_\gamma^{M(a)[g]}).$$

But $j(S)$ and $j(T)$ are $<j(\lambda)$-universally Baire in M', so they project to complements in $M'[g, k]$. Furthermore,

$$p[j(S)] \subseteq \mathbb{R}^{\#}$$

and

$$p[j(T)] \subseteq \mathbb{R} \setminus \mathbb{R}^{\#}.$$

Since $p[j(S)] \subseteq p[k(j(S))]$ and $p[j(T)] \subseteq p[k(j(T))]$, $p[j(S)] = p[k(j(S))]$ and $p[j(T)] = p[k(j(T))]$, contradicting that k is a bad iteration. This completes our sketch of the proof of Theorem 9.7.

Another strengthening of Theorem 0.2, using the absoluteness of $\mathbb{R}^{\#}$, is the following.

9.9 Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either* ZFC $\vdash_\Omega L(\mathbb{R}) \models \phi$ *or* ZFC $\vdash_\Omega L(\mathbb{R}) \not\models \phi$.

Since $\mathbb{P}_{\max}$ is a homogeneous forcing extension of $L(\mathbb{R})$, this gives the following.

9.10 Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either*

$$\text{ZFC} + (*) \vdash_\Omega L(\mathcal{P}(\omega_1)) \models \phi$$

or

$$\text{ZFC} + (*) \vdash_\Omega L(\mathcal{P}(\omega_1) \not\models \phi.$$

Since $\mathbb{R}^{\#}$ is not in $L(\mathbb{R})$, the Continuum Hypothesis (plus the existence of $\mathbb{R}^{\#}$) implies that $L(\mathcal{P}(\omega_1))$ is not contained in a forcing extension of $L(\mathbb{R})$. Moreover, Woodin has shown (see Theorem 10.183 of [39]) that if ψ is any sentence for which Theorem 9.10 holds with ψ in the place of axiom $(*)$, then ZFC + ψ implies in Ω-logic that the Continuum Hypothesis is false.

10. Variations

The $\mathbb{P}_{\max}$ method is fairly flexible, and the partial order $\mathbb{P}_{\max}$ can be varied in a number of ways. We present here two types of variations. The first is an example of the utility of $\mathbb{P}_{\max}$ for manipulating ideals on ω_1. The second illustrates a method for producing extensions which are Π_2-maximal for $H(\omega_2)$ relative to a fixed Σ_2 sentence. Several other variations appear in [39, 45]. Still others appear in [4, 20].

10.1. Variations for NS_{ω_1}

An ideal I on ω_1 is $\aleph_1$-*dense* if the Boolean algebra $\mathcal{P}(\omega_1)/I$ has a dense subset of cardinality $\aleph_1$. In unpublished work, Woodin showed that starting from a huge cardinal one can force the existence of a normal $\aleph_1$-dense ideal on ω_1. Shelah later showed [29] that, starting from a supercompact cardinal, one can force that the nonstationary ideal restricted to a fixed stationary subset of ω_1 is $\aleph_1$-dense. The $\mathbb{P}_{\max}$ variation $\mathbb{Q}^*_{\max}$ discussed here, when applied to a model of the form $L(\mathbb{R})$ satisfying AD, produces a model of ZFC in which NS_{ω_1} is $\aleph_1$-dense; by unpublished work of Woodin, this shows that the Axiom of Determinacy and the $\aleph_1$-density of NS_{ω_1} are equiconsistent. To date, $\mathbb{P}_{\max}$ variations are the only known means for producing models in which NS_{ω_1} is $\aleph_1$-dense.

Using the result of Shelah mentioned above, the partial order $\mathbb{Q}_{\max}$ below can be used to obtain the $\aleph_1$-density of NS_{ω_1} from a supercompact cardinal below ω Woodin cardinals below a measurable. This hypothesis is not optimal, but unlike with $\mathbb{Q}^*_{\max}$, we can give all the details here (aside from one argument, we have already done so).

By Theorem 9.1, the $\aleph_1$-density of a σ-ideal on ω_1 is witnessed by a function from ω_1 to $H(\omega_1)$ of the following form.

10.1 Definition. Given a normal $\aleph_1$-dense ideal I on ω_1, $Y_{\mathrm{Col}}(I)$ is the set of functions $f : \omega_1 \to H(\omega_1)$ satisfying the following conditions (where for each $p \in \mathrm{Col}(\omega, \omega_1)$ we let $S^f_p = \{\alpha < \omega_1 \mid p \in f(\alpha)\}$):

- for each $\alpha < \omega_1$, $f(\alpha)$ is a filter in $\mathrm{Col}(\omega, 1 + \alpha)$
- for each $p \in \mathrm{Col}(\omega, \omega_1)$, $S^f_p \notin I$,
- for each $S \in \mathcal{P}(\omega_1)/I$, there exists a condition $p \in \mathrm{Col}(\omega, \omega_1)$ such that $S^f_p \setminus S \in I$.

10.2 Definition. The partial order $\mathbb{Q}_{\max}$ consists of the set of pairs of the form $\langle (M, I), f\rangle$ satisfying the following conditions:

1. M is a countable transitive model of ZFC°,
2. I is a normal $\aleph_1$-dense ideal on ω_1^M in M,
3. (M, I) is iterable,
4. $f \in (Y_{\mathrm{Col}}(I))^M$.

The order on $\mathbb{Q}_{\max}$ is as follows: $\langle (N, J), g\rangle < \langle (M, I), f\rangle$ if $M \in H(\omega_1)^N$ and there exists an iteration $j : (M, I) \to (M^*, I^*)$ such that

- $j(f) = g$,
- $j, M^* \in N$,
- $I^* = M^* \cap J$.

If $\langle (M, I), f\rangle$ is a $\mathbb{Q}_{\max}$ condition, then by the normality of I in M, the image of f under any iteration of (M, I) determines the entire iteration.

The only new argument we need to give in the $\mathbb{Q}_{\max}$ analysis is the following. The corresponding versions for iterating sequences of models and for building descending ω_1-sequences of conditions are essentially the same.

10.3 Lemma. *Suppose that J is a normal $\aleph_1$-dense ideal on ω_1, and let g be a function in $Y_{\mathrm{Col}}(J)$. Suppose that $\langle (M, I), f\rangle$ is a condition in $\mathbb{Q}_{\max}$. Then there is an iteration $j : (M, I) \to (M^*, I^*)$ of length ω_1 such that*

- $\{\alpha < \omega_1 \mid j(f)(\alpha) \neq g(\alpha)\} \in J$,
- $I^* = M^* \cap J$.

Proof. The second conclusion follows from the first. Let

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

be any iteration of (M, I) satisfying the condition that whenever $\beta < \omega_1$ is such that $j_{0\beta}(\omega_1^M) = \beta$ and $g(\beta)$ is M_β-generic for $\mathrm{Col}(\omega, \beta)$, then G_β is the corresponding filter in $\mathcal{P}(\omega_1)^{M_\beta}/j_{0\beta}(I)$, i.e., for each $p \in \mathrm{Col}(\omega, \beta)$, $S_p^{j_{0\beta}(f)} \in G_\beta$ if and only if $p \in g(\beta)$. It is immediate that G_β is M_β-generic, and that the choice of G_β makes $j_{0(\beta+1)}(f)(\beta) = g(\beta)$. It remains to see that the set of $\beta < \omega_1$ such that $g(\beta)$ is not M_β-generic for $\mathrm{Col}(\omega, \beta)$ is in J. To see this, let A be subset of ω_1 coding $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ under some fixed recursive coding. Then for club many $\eta < \omega_1$, $j_{0\eta}(\omega_1)^M = \eta$ and $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \eta\rangle \in L[A \cap \eta]$. Every condition in $\mathcal{P}(\omega_1)/J$ forces that (letting k be the induced elementary embedding) $k(g)(\omega_1^V)$ is a V-generic (and thus $L[A]$-generic) filter in $\mathrm{Col}(\omega, \omega_1^V)$, which means that the set of $\eta < \omega_1$ such that $g(\eta)$ is not $L[A \cap \eta]$-generic is in J. Since $M_\eta \in L[A \cap \eta]$ for club many η, we are done. ⊣

Theorem 4.10 plus the result of Shelah mentioned above gives the following.

10.4 Theorem. *Suppose that there exists a supercompact cardinal below infinitely many Woodin cardinals below a measurable cardinal. Then for every set of reals A in $L(\mathbb{R})$ there exists a $\mathbb{Q}_{\max}$ condition $\langle (M, I), f\rangle$ such that*

- $A \cap M \in M$,
- (M, I) *is A-iterable,*
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1, A, \in\rangle$.

The proof of the following is essentially the same as for $\mathbb{P}_{\max}$. The $\aleph_1$-density of I_G follows immediately from $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$ and the definition of $\mathbb{Q}_{\max}$ (letting I_G and $\mathcal{P}(\omega_1)_G$ have the definitions here analogous to those used for $\mathbb{P}_{\max}$).

10.5 Theorem (ZF). *Suppose that for every set of reals A there exists a $\mathbb{Q}_{\max}$ condition $\langle (M, I), f\rangle$ such that*

- $A \cap M \in M$,
- (M, I) *is A-iterable,*
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1, A, \in\rangle$.

Then $\mathbb{Q}_{\max}$ is ω-closed and homogeneous. Furthermore, if G is an V-generic filter for $\mathbb{Q}_{\max}$, then the following hold in $V[G]$:

- $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$,
- $\mathrm{NS}_{\omega_1} = I_G$,
- ψ_{AC},
- $\underset{\sim}{\delta}^1_2 = \omega_2$,
- NS_{ω_1} *is* $\aleph_1$-*dense.*

To obtain the $\aleph_1$-density of NS_{ω_1} from the optimal hypothesis, we can use the partial order $\mathbb{Q}^*_{\max}$ below. Conditions in $\mathbb{Q}^*_{\max}$ are similar to the limit sequences used in the $\mathbb{P}_{\max}$ analysis. The utility of this approach here is that the existence of $\mathbb{Q}^*_{\max}$ conditions does not require the existence of a model with an $\aleph_1$-dense ideal on ω_1. The analyses of $\mathbb{Q}^*_{\max}$ and $\mathbb{Q}_{\max}$ are the same, once we show that $\mathbb{Q}^*_{\max}$ conditions exist in suitable generality. Showing this, however, is beyond the scope of this chapter.

10.6 Definition. $\mathbb{Q}^*_{\max}$ is the set of pairs $(\langle M_k : k < \omega\rangle, f)$ such that the following hold.

1. The set f is a function from $\omega_1^{M_0}$ to M_0 in M_0 such that for all $\alpha < \omega_1^{M_0}$, $f(\alpha)$ is a filter in $\mathrm{Col}(\omega, 1+\alpha)$.
2. Each $M_k \models \mathrm{ZFC}^\circ$.
3. Each $M_k \in M_{k+1}$.
4. For all $k < \omega$, $\omega_1^{M_k} = \omega_1^{M_0}$.
5. For all $k < \omega$, $\mathrm{NS}_{\omega_1}^{M_{k+1}} \cap M_k = \mathrm{NS}_{\omega_1}^{M_{k+2}} \cap M_k$.
6. The sequence $\langle M_k : k < \omega\rangle$ is iterable.
7. For each $p \in \mathrm{Col}(\omega, \omega_1^{M_0})$, $\{\alpha < \omega_1^{M_0} \mid p \in f(\alpha)\} \notin \mathrm{NS}_{\omega_1}^{M_1}$.
8. For each $k < \omega$ and for each $a \subseteq \omega_1^{M_0}$ such that $a \in M_k \setminus \mathrm{NS}_{\omega_1}^{M_{k+1}}$, there exists a $p \in \mathrm{Col}(\omega, \omega_1^{M_0})$ such that
$$\{\alpha < \omega_1^{M_0} \mid p \in f(\alpha)\} \cap (\omega_1^{M_0} \setminus a) \in \mathrm{NS}_{\omega_1}^{M_{k+1}}.$$

The ordering on $\mathbb{Q}^*_{\max}$ is given by letting
$$(\langle N_k : k < \omega\rangle, g) < (\langle M_k : k < \omega\rangle, f)$$
if $\langle M_k : k < \omega\rangle \in H(\omega_1)^{N_0}$ and there exists an iteration
$$j : \langle M_k : k < \omega\rangle \to \langle M^*_k : k < \omega\rangle$$
in N_0 such that

- $j(f) = g$,
- $\mathrm{NS}_{\omega_1}^{M^*_{k+1}} \cap M^*_k = \mathrm{NS}_{\omega_1}^{N_1} \cap M^*_k$ for all $k < \omega$.

Condition (5) above says that the models in the sequence need not agree about stationary sets, but rather, each subset of $\omega_1^{M_0}$ in each M_k which is stationary in M_{k+1} is stationary in all further M_j's. This extra degree of freedom is essential in constructing $\mathbb{Q}^*_{\max}$ conditions without presupposing the existence of $\mathbb{Q}_{\max}$ conditions. Conditions (7) and (8) ensure that if

$$G \subseteq \bigcup\{\mathcal{P}(\omega_1^{M_0})^{M_k} \setminus \mathrm{NS}_{\omega_1}^{M_{k+1}} : k < \omega\}$$

is a $\bigcup\{M_k : k < \omega\}$-normal filter, then (letting j be the induced embedding) $j(f)(\omega_1^{M_0})$ is a filter in $\mathrm{Col}(\omega, \omega_1^{M_0})$ meeting every dense set in each M_k, and vice-versa: if g is a filter in $\mathrm{Col}(\omega, \omega_1^{M_0})$ meeting every dense set in each M_k, then there is a $\bigcup\{M_k : k < \omega\}$-normal filter G contained in $\bigcup\{\mathcal{P}(\omega_1^{M_0})^{M_k} \setminus \mathrm{NS}_{\omega_1}^{M_{k+1}} : k < \omega\}$ such that $j(f)(\omega_1^{M_0}) = g$.

10.7 Remark. If Γ is a pointclass closed under continuous images such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} +$ "Θ is regular", then the $\mathbb{Q}_{\max}$ extension of $L(\Gamma, \mathbb{R})$ satisfies Chang's Conjecture. However, for consistency strength reasons one cannot prove that Chang's Conjecture holds in the $\mathbb{Q}_{\max}$ extension of $L(\mathbb{R})$ from the assumption $\mathrm{AD}^{L(\mathbb{R})}$(see [39, p. 651]).

The utility of the $\mathbb{P}_{\max}$ approach for manipulating ideals on ω_1 is applied in other several ways in [39], notably to create a model in which the saturation of NS_{ω_1} can be destroyed without adding a subset of ω_1. In [20], a variation of $\mathbb{P}_{\max}$ is used to produce a model in which the saturation of NS_{ω_1} can be destroyed by forcing with a Suslin tree. As far as we know, these results have not been reproduced by other methods.

10.2. Conditional Variations for Σ_2 sentences

As we saw in Sect. 7, the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ (assuming that there exists a proper class of Woodin cardinals) satisfies all forceable Π_2 sentences for $H(\omega_2)$ with parameters for NS_{ω_1} and sets of reals in $L(\mathbb{R})$. In some cases, one can fix a Σ_2 sentence for this structure and produce a model satisfying all Π_2 sentences forceably consistent with it (and in some cases one cannot). If ϕ is a Σ_2 sentence of the form $\exists A \forall B \psi(A, B)$, where all quantifiers in ψ are bounded, the *optimal iteration lemma* for ϕ is the following statement: if

- M is a countable transitive model of ZFC°,
- I is normal ideal on ω_1^M in M,
- (M, I) is iterable,
- $a \in H(\omega_2)^M$ and $H(\omega_2)^M \models \forall b \psi(a, b)$,

- $H(\omega_2) \models \exists A \forall B \psi(A, B)$,
- J is a normal ideal on ω_1,

then there exists an iteration $j : (M, I) \to (M^*, I^*)$ of length ω_1 such that

- $I^* = J \cap M^*$,
- $H(\omega_2) \models \forall B \psi(j(a), B)$.

Roughly, the optimal iteration lemma for ϕ says that given a countable transitive iterable model of ϕ and a fixed witness for ϕ in this model, in order to prove that there is an iteration of this model mapping this witness to a witness for ϕ is V, we need assume only that ϕ holds in V. Since this assumption is necessary, in the cases where the lemma holds, it is optimal. In [31], the optimal iteration lemma is proved for the following sentences (the first four of which are defined in [2]; we direct the reader to [31] for the other two).

- The dominating number ($\mathfrak{d}$) is $\aleph_1$.
- The bounding number ($\mathfrak{b}$) is $\aleph_1$.
- The cofinality of the meager ideal is $\aleph_1$.
- The cofinality of the null ideal is $\aleph_1$.
- There exists a coherent Suslin tree.
- There exists a free Suslin tree.

Given a Σ_2 sentence ϕ as above, we can define the $\mathbb{P}_{\max}$ variation $\mathbb{P}^{\phi}_{\max}$ as follows. Since ϕ may contradict $\mathrm{MA}_{\aleph_1}$, we remove the requirement that the models satisfy $\mathrm{MA}_{\aleph_1}$ and ensure the uniqueness of iterations directly (alternately, we can usually replace $\mathrm{MA}_{\aleph_1}$ with ψ_{AC}). The partial order $\mathbb{P}^{\phi}_{\max}$ is defined recursively on the ω_1 of the selected model M.

10.8 Definition. The partial order $\mathbb{P}^{\phi}_{\max}$ consists of all pairs $\langle (M, I), a, X\rangle$ such that

1. M is a countable transitive model of ZFC°,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. $a \in \mathcal{P}(\omega_1)^M$ and $H(\omega_2)^M \models \forall b \psi(a, b)$,
5. $X \in M$ and X is a set (possibly empty) of pairs $(\langle (N, J), b, Y\rangle, j)$ such that
    - $\langle (N, J), b, Y\rangle \in \mathbb{P}^{\phi}_{\max} \cap H(\omega_1)^M$,

- j is an iteration of (N, J) of length ω_1^M such that $j(J) = I \cap j(N)$ and $j(b) = a$,
- $j(Y) \subseteq X$,

with the property that for each $p \in \mathbb{P}^{\phi}_{\max}$ there is at most one j such that $(p, j) \in X$.

The order on $\mathbb{P}^{\phi}_{\max}$ is implicit in the conditions on X:

$$\langle (M, I), a, X\rangle < \langle (N, J), b, Y\rangle$$

if there exists a j such that $(\langle (N, J), b, Y\rangle, j) \in X$.

For ϕ as above, we have games $\mathcal{G}^{\phi}_{\omega}$ and $\mathcal{G}^{\phi}_{\omega_1}$ which are strictly analogous to the games $\mathcal{G}_{\omega}$ and $\mathcal{G}_{\omega_1}$ for $\mathbb{P}_{\max}$.

For $\mathcal{G}^{\phi}_{\omega}$, suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence (in the sense of $\mathbb{P}_{\max}$) and that there exists an $a \in \mathcal{P}(\omega_1)^{N_0}$ such that $H(\omega_2)^{N_i} \models \forall b\psi(a, b)$ for each $i < \omega$. Then given a normal ideal I on ω_1 and a set $E \subseteq \omega_1$, we define $\mathcal{G}^{\phi}_{\omega}(\langle (N_i, J_i) : i < \omega\rangle, I, E)$ to be the following game of length ω_1 where Players I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega\rangle$ consisting of pre-limit sequences $\langle (N^{\alpha}_i, J^{\alpha}_i) : i < \omega\rangle$ $(\alpha \leq \omega_1)$, normal ultrafilters G_{α} $(\alpha < \omega_1)$ and a commuting family of embeddings $j_{\alpha\beta}$ $(\alpha \leq \beta \leq \omega_1)$, as follows. In each round α, let

$$Q_{\alpha} = \bigcup\{\mathcal{P}(\omega_1)^{N^{\alpha}_i} \setminus J^{\alpha}_i : i < \omega\}\,.$$

If $\alpha \in E$, then Player I chooses a set $A \in Q_{\alpha}$, and then Player II chooses a $\bigcup\{N^{\alpha}_i : i < \omega\}$-normal filter G_{α} contained in Q_{α} with $A \in G_{\alpha}$. If α is not in E, then Player II chooses any $\bigcup\{N^{\alpha}_i : i < \omega\}$-normal filter G_{α} contained in Q_{α}. After all ω_1 many rounds have been played, Player I wins if $H(\omega_2) \models \forall B\psi(j_{0\omega_1}(a), B)$ and if $J^{\omega_1}_i = I \cap N^{\omega_1}_i$ for each $i < \omega$.

Similarly, given a $\mathbb{P}^{\phi}_{\max}$ condition $p = \langle (M, I), a, X\rangle$, a normal ideal J on ω_1 and a subset of ω_1 E, we let $\mathcal{G}^{\phi}_{\omega_1}(p, J, E)$ be game of length ω_1 where players I and II collaborate to build a descending ω_1-chain of conditions $p_{\alpha} = \langle (M_{\alpha}, I_{\alpha}), a_{\alpha}, X_{\alpha}\rangle$ $(\alpha < \omega_1)$ of $\mathbb{P}^{\phi}_{\max}$ conditions below p as follows. In each round α, all p_{β} $(\beta < \alpha)$ have been defined. If α is a successor ordinal (or 0), Player II chooses a condition $p_{\alpha} < p_{\alpha-1}$ $(< p)$. If α is a limit ordinal, then Player I picks a condition p_{α} below each p_{β} $(\beta < \alpha)$. Then, letting $j_{\alpha\beta}$ $(\alpha < \beta \leq \omega_1)$ be the induced commuting family of embeddings (and letting j be the embedding witnessing that $p_0 < p$), Player I wins the game if $H(\omega_2) \models \forall B\psi(j_{0\omega_1}(j(a)), B)$, and if for all $\alpha < \omega_1$, $j_{\alpha\omega_1}(I_{\alpha}) = J \cap j_{\alpha\omega_1}(M_{\alpha})$.

The arguments in [31] show that $\diamondsuit_{\omega_1}$ implies that Player I has a winning strategy in each game $\mathcal{G}^{\phi}_{\omega}(\langle (N_i, J_i) : i < \omega\rangle, I, E)$ and each game $\mathcal{G}^{\phi}_{\omega_1}(p, J, E)$ for each of the sentences listed before Definition 10.8 (typically these arguments are essentially the same as the proof of the corresponding optimal iteration lemma).

The proof of the following theorem then is a straightforward generalization of the arguments we have given for $\mathbb{P}_{\max}$.

10.9 Theorem. *Assume* $\mathrm{AD}^{L(\mathbb{R})}$*, and let* ϕ *be an* Ω_{ZFC}*-consistent* Σ_2 *sentence for* $H(\omega_2)$*. Suppose that the optimal iteration lemma for* ϕ *holds, and that the following sentences are* Ω_{ZFC}*-consistent:*

- *for all iterable pre-limit sequences* $\langle (N_i, I_i) : i < \omega\rangle$ *and for all normal ideals* I *on* ω_1*, Player* I *has a winning strategy in*

$$\mathcal{G}^{\phi}_{\omega}(\langle (N_i, J_i) : i < \omega\rangle, I, \omega_1);$$

- *for all* $\mathbb{P}^{\phi}_{\max}$ *conditions* p *and for all normal ideals* J *on* ω_1*, Player* I *has a winning strategy in* $\mathcal{G}_{\omega_1}(p, J, \omega_1)$*.*

Let $G \subseteq \mathbb{P}^{\phi}_{\max}$ *be* $L(\mathbb{R})$*-generic. Then in* $L(\mathbb{R})[G]$ *the following hold:*

- ϕ*,*
- $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$*,*
- $\mathrm{NS}_{\omega_1} = I_G$*,*
- NS_{ω_1} *is saturated.*

Furthermore, for every set of reals A *in* $L(\mathbb{R})$*,* $L(\mathbb{R})[G]$ *satisfies every* Π_2*-sentence for the structure* $\langle H(\omega_2), \mathrm{NS}_{\omega_1}, A, \in\rangle$ *which is* Ω_{ZFC}*-consistent with* ϕ*.*

The variation $\mathbb{P}^{\phi}_{\max}$ where ϕ asserts the existence of a coherent Suslin tree is studied in [16, 21].

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