Chapter 3: Masonry

Chapter 3.1

CONTINUUM MODELLING OF MASONRY STRUCTURES UNDER STATIC AND DYNAMIC LOADING

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Abstract: Ancient masonry structures can be seen as a set of rigid blocks where the deformation is taking place at the interface between the blocks. Rocking, twisting and sliding between them are possible mechanisms that actually take place under static or dynamic loading. A continuum model for regular block structures is derived by replacing the quotients of the discrete equations by corresponding differential quotients. The homogenisation procedure leads to an anisotropic Cosserat Continuum. For elastic block interactions the dispersion relations of the discrete and the continuous models are derived and compared.

Key words: masonry; block structure; 3D Cosserat; out of plane; in plane; dynamics; elasticity; homogenisation.

1. INTRODUCTION

The numerical analysis of discontinuous blocky structures can be dealt using discrete and finite element codes. In the latter case, special interface elements are needed in order to account for the unilateral kinematics of the rock joints. The interest of developing continuous models for discrete structures is that discrete type analyses are very computer time intensive and, at least for periodic structures, one might argue that a homogenised continuum model would allow a much more elegant and efficient solution. One could list the practical relevance of the development of continuum models: (a) it is extremely flexible when used with numerical methods, since no interface elements are needed and since the topology of the finite element is independent of block size and geometry (one mesh can be used to study several different structures); (b) quite a number of analytical solutions can be provided that

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can be used as benchmarks for discrete codes; (c) unconditionally stable integration through implicit algorithms can be used, unlike discrete models where conditionally stable explicit integration schemes are used.

However, the computational efficiency comes at a price. Continuum models are usually based on micro-mechanical mechanisms, which govern the material behaviour in the medium to large wavelength range. When the characteristic length of the macroscopic deformation pattern is smaller than a certain multiple of the characteristic fabric length of the material, then the applicability of the continuum model reaches its limit. Another important limitation of the homogenisation of layered or blocky structures with classical continuum theories is that they cannot account for elementary bending due to inter-layer or inter-block slip and may thus considerably overestimate the deformation. In order to overcome these limitations and to expand the domain of validity of the continuum approach one has to consider the salient features of the discontinuum within the frame of continuum theories with microstructure. The 2D-Cosserat theory has been used with some success in the recent years for analysing blocky and laminated systems[1-3]. The enriched kinematics of the Cosserat continuum allows to model microelement systems undergoing rotations, which are different from the local rotations of the continuum. Various failure modes such as inter-block slip and block tilting can then be described. In these previous studies developed in the frame of two-dimensional Cosserat theory, only one rotational mode of the block has been considered. This work is extended in this paper by considering rock twisting in addition to rock sliding. This rotational mode is frequently observed on masonry structures under dynamic loading.

The problem of a three-dimensional regular block structure is addressed here. Starting with elastic behaviour of the joints, the dynamic differential equations for the discrete 3D structure are derived. A continuum model is obtained by replacing the difference quotients of the discrete equations by corresponding differential quotients. The homogenisation procedure leads to an anisotropic 3D-Cosserat continuum. For elastic block interactions the dispersion relations of the discrete and the continuous models are derived and compared. The domain of validity of the continuous approach is discussed by comparing the dispersion function of the discrete and the continuous system.

2. THE DISCRETE MODEL FOR 3D STRUCTURE AND INFINITESIMAL DEFORMATION

An idealized model of a masonry wall is considered here (Figure 1). Six others surround each block. The main concern is the accuracy with which the continuum model reflects the domain of rigidity set by the size of the blocks. The elasticity of the blocks and the joints' elasticity are lumped at the block faces for simplicity. Fully elastic joint behaviour is assumed. Also, it is assumed that the interaction between the block is concentrated in six points of the faces as shown in Figure 1.

Moments and normal and shear forces are written as:

$$\begin{aligned} Q_{kl} &= c_Q \Delta x_{kl}^i \\ N_{kl} &= c_N \Delta x_{kl}^i \\ M_{kl} &= c_M \Delta \phi_{kl}^i \quad or \quad M_{kl} = c_T \Delta \phi_{kl}^i \end{aligned} \tag{1}$$

where c_Q, c_N, c_M and c_T are the elastic shear, normal, bending and torsional stiffness respectively and Δx and $\Delta \phi$ are the relative translations and rotations of block k with block l at various contact points.

Using d'Alembert's principle, the motion equations of block (i,j) of the three-dimensional assembly of blocks are obtained as following:

$$\begin{aligned} c_N\left(u_{i+2,j} - 2u_{i,j} + u_{i-2,j}\right) + c_Q\left(u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} - 4u_{i,j}\right) + \\ + \frac{b}{2} c_Q\left(\phi_{i+1,j+1} - \phi_{i+1,j-1} + \phi_{i-1,j+1} - \phi_{i-1,j-1}\right) = m\ddot{u}_{i,j} \end{aligned} \tag{2}$$

$$\begin{aligned} c_Q\left(v_{i+2,j} - 2v_{i,j} + v_{i-2,j}\right) + c_N\left(v_{i+1,j+1} + v_{i-1,j+1} + v_{i+1,j-1} + v_{i-1,j-1} - 4v_{i,j}\right) - \\ - c_Q a\left(\phi_{i+2,j} - \phi_{i-2,j}\right) + c_N \frac{a}{2}\left(\phi_{i-1,j+1} - \phi_{i+1,j-1} + \phi_{i-1,j-1} - \phi_{i+1,j+1}\right) = m\ddot{v}_{i,j} \end{aligned} \tag{3}$$

$$\begin{aligned} a c_Q\left(v_{i+2,j} - v_{i-2,j}\right) + \frac{a}{2} c_N\left(v_{i+1,j+1} - v_{i-1,j+1} + v_{i+1,j-1} - v_{i-1,j-1}\right) - \\ - \frac{b}{2} c_Q\left(u_{i+1,j+1} - u_{i-1,j-1} + u_{i-1,j+1} - u_{i+1,j-1}\right) - a^2 c_Q\left(\phi_{i+2,j} + \phi_{i-2,j} + 2\phi_{i,j}\right) - \\ - \frac{1}{4}\left(b^2 c_Q + a^2 c_N\right)\left(\phi_{i+1,j+1} + \phi_{i-1,j+1} + \phi_{i-1,j-1} + \phi_{i+1,j-1} + 4\phi_{i,j}\right) + \\ + c_M\left(\phi_{i+2,j} + \phi_{i+1,j+1} + \phi_{i-1,j+1} + \phi_{i-2,j} + \phi_{i-1,j-1} + \phi_{i+1,j-1} - 6\phi_{i,j}\right) = J_3 \ddot{\phi}_{i,j} \end{aligned} \tag{4}$$

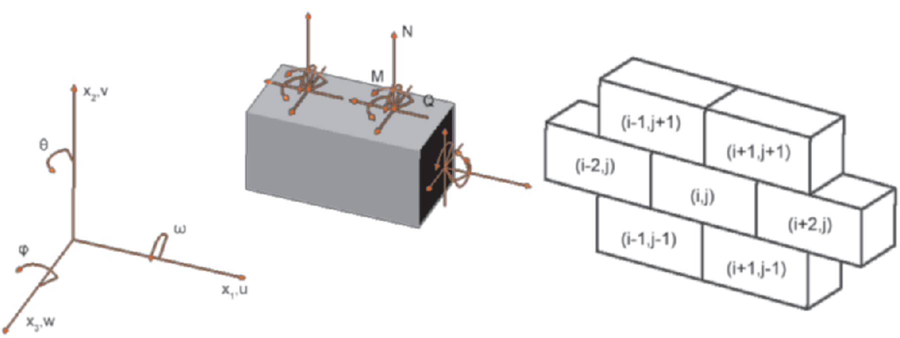

Figure 1. Statical and geometrical configuration. Each block is 2a long, b wide and b high.

$$\left(w_{i+2,j}+w_{i+1,j+1}+w_{i-1,j+1}+w_{i-2,j}+w_{i+1,j-1}+w_{i-1,j-1}-6w_{i,j}\right)+a\left(\theta_{i+2,j}-\theta_{i-2,j}\right)+ \\ +\frac{a}{2}\left(\theta_{i+1,j+1}-\theta_{i-1,j+1}+\theta_{i+1,j-1}-\theta_{i-1,j-1}\right)+\frac{b}{2}\left(\omega_{i+1,j-1}-\omega_{i-1,j+1}+\omega_{i-1,j-1}-\omega_{i+1,j+1}\right)=\frac{m}{c_Q}\ddot{w}_{i,j} \tag{5}$$

$$\frac{a}{2}c_Q\left(w_{i-1,j+1}+w_{i-1,j-1}-w_{i+1,j+1}-w_{i+1,j-1}\right)- \\ -\frac{1}{4}a^2c_Q\left(\theta_{i-1,j+1}+\theta_{i-1,j-1}+\theta_{i+1,j-1}+\theta_{i+1,j+1}+4\theta_{i,j}\right)+ac_Q\left(w_{i-2,j}-w_{i+2,j}\right)+ \\ \frac{1}{4}bac_Q\left(\omega_{i-1,j-1}-\omega_{i-1,j+1}+\omega_{i+1,j+1}-\omega_{i+1,j-1}\right)-a^2c_Q\left(\theta_{i-2,j}+\theta_{i+2,j}+2\theta_{i,j}\right)+ \\ +c_M\left(\theta_{i+2,j}+\theta_{i-2,j}-2\theta_{i,j}\right)+c_T\left(\theta_{i+1,j+1}+\theta_{i-1,j+1}+\theta_{i-1,j-1}+\theta_{i+1,j-1}-4\theta_{i,j}\right)=J_2\ddot{\theta}_{i,j} \tag{6}$$

$$\frac{1}{2}bc_Q\left(w_{i+1,j+1}+w_{i-1,j+1}-w_{i-1,j-1}-w_{i+1,j-1}\right)+ \\ +\frac{1}{4}bac_Q\left(\theta_{i+1,j+1}-\theta_{i-1,j+1}+\theta_{i-1,j-1}-\theta_{i+1,j-1}\right)+c_T\left(\omega_{i+2,j}+\omega_{i-2,j}-2\omega_{i,j}\right)- \\ -\frac{1}{4}b^2c_Q\left(\omega_{i+1,j+1}+\omega_{i-1,j+1}+\omega_{i-1,j-1}+\omega_{i+1,j-1}+4\omega_{i,j}\right)+ \\ +c_M\left(\omega_{i+1,j+1}+\omega_{i-1,j+1}+\omega_{i-1,j-1}+\omega_{i+1,j-1}-4\omega_{i,j}\right)=J_1\ddot{\omega}_{i,j} \tag{7}$$

where m and J_i the mass and the moments of inertia of each block.

The above equations describe the dynamic behaviour of the block assembly depicted at Figure 1. Looking carefully at these equations we distinguish two uncoupled sets of equations, i.e. Eqs.(2-4) and Eqs.(5-7), which correspond to two different modes of motion-deformation of the blocky structure. These modes describe the in and the out of plane deformation of the structure. Therefore the initial problem is finally separated in two independent problems, which is a common approach in the theory of plates. The in plane mode of deformation is examined at the paper of Sulem and Mühlhaus[2].

3. THE CONTINUOUS MODEL

In the above system of equations (Eqs.(2–7)) the discrete coordinates (i,j) are replaced by the continuous ones (x_1,x_2) and instead of $u_{i\pm1,j\pm1}$, $v_{i\pm1,j\pm1}$, $w_{i\pm1,j\pm1}$, $\varphi_{i\pm1,j\pm1}$, $\theta_{i\pm1,j\pm1}$ and $\omega_{i\pm1,j\pm1}$ one writes $u(x_1\pm\alpha, x_2\pm b)$, $v(x_1\pm\alpha, x_2\pm b)$, $w(x_1\pm\alpha, x_2\pm b)$, $\phi(x_1\pm\alpha, x_2\pm b)$, $\theta(x_1\pm\alpha, x_2\pm b)$ and $\omega(x_1\pm\alpha, x_2\pm b)$. Then the u, v, w, ϕ, θ and ω degrees of freedom are developed into a Taylor series up to the second order:

$$f\left(x_1\pm a,x_2\pm b\right)=f\left(x_1,x_2\right)\pm a\frac{\partial f}{\partial x_1}\pm b\frac{\partial f}{\partial x_2}+a^2\frac{\partial^2 f}{\partial x_1^2}+b^2\frac{\partial^2 f}{\partial x_2^2}\pm ab\frac{\partial^2 f}{\partial x_1 x_2}+\mathrm{O}^3 \tag{8}$$

Applying the aforementioned procedure to Eqs.(2-7) one obtains the following system of second order differential equations:

$$2a^2\left(2c_N+c_Q\right)\frac{\partial^2 u}{\partial x_1^2}+c_Q 2b^2\frac{\partial^2 u}{\partial x_2^2}+c_Q 2b^2\frac{\partial \phi}{\partial x_2}=m\frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$2a^2\left(c_N+2c_Q\right)\frac{\partial^2 v}{\partial x_1^2}+2c_N b^2\frac{\partial^2 v}{\partial x_2^2}-2a^2\left(c_N+2c_Q\right)\frac{\partial \phi}{\partial x_1}=m\frac{\partial^2 v}{\partial t^2} \tag{10}$$

$$\begin{aligned}2a^2\left(c_N+2c_Q\right)\frac{\partial v}{\partial x_1}-2b^2c_Q\frac{\partial u}{\partial x_2}-2\left(2a^2c_Q+\frac{1}{4}b^2c_Q+\frac{1}{4}a^2c_N-3c_M\right)a^2\frac{\partial^2\phi}{\partial x_1^2}-\\ -2\left(\frac{1}{4}b^2c_Q+\frac{1}{4}a^2c_N-c_M\right)b^2\frac{\partial^2\phi}{\partial x_2^2}-2\left(b^2c_Q+2a^2c_Q+a^2c_N\right)\phi=J_3\frac{\partial^2\phi}{\partial t^2}\end{aligned} \tag{11}$$

$$6a^2\frac{\partial^2 w}{\partial x_1^2}+2b^2\frac{\partial^2 w}{\partial x_2^2}+6a^2\frac{\partial \theta}{\partial x_1}-2b^2\frac{\partial \omega}{\partial x_2}=\frac{m}{c_Q}\frac{\partial^2 w}{\partial t^2} \tag{12}$$

$$\begin{aligned}-6a^2c_Q\frac{\partial w}{\partial x_1}+a^2b^2c_Q\frac{\partial^2\omega}{\partial x_1\partial x_2}+2\left(2c_M-\frac{9}{4}a^2c_Q+c_T\right)a^2\frac{\partial^2\theta}{\partial x_1^2}+\\ +2\left(c_T-\frac{1}{4}a^2c_Q\right)b^2\frac{\partial^2\theta}{\partial x_2^2}-6a^2c_Q\theta=J_2\frac{\partial^2\theta}{\partial t^2}\end{aligned} \tag{13}$$

$$\begin{aligned}2c_Qb^2\frac{\partial w}{\partial x_2}+c_Qa^2b^2\frac{\partial^2\theta}{\partial x_1\partial x_2}+\left(c_M+2c_T-\frac{1}{4}c_Qb^2\right)2a^2\frac{\partial^2\omega}{\partial x_1^2}+\\ +\left(c_M-\frac{1}{4}c_Qb^2\right)2b^2\frac{\partial^2\omega}{\partial x_2^2}-2c_Qb^2\omega=J_1\frac{\partial^2\omega}{\partial t^2}\end{aligned} \tag{14}$$

Examining again Eqs.(9)-(14) one distinguishes two uncoupled sets of equations, i.e. Eqs.(9)-(11) and Eqs.(12)-(14), which correspond respectively to the in and the out of plane deformation of the structure.

4. IDENTIFICATION OF A COSSERAT ANISOTROPIC CONTINUUM

In a three-dimensional Cosserat continuum each material point has three translational degrees of freedom (*u*; *v*; *w*) and three rotational degrees of freedom ω^c_i. The index c is used to distinguish the Cosserat rotation from the rotation:

$$\omega_{ij}=\frac{1}{2}\left(u_{i,j}-u_{j,i}\right);\ (.)_{,i}=\frac{\partial(.)}{\partial x_i}\ i=1,2,3 \tag{15}$$

For the formulation of the constitutive relationships one needs deformation measures, which are invariant to rigid body motion, i.e. the conventional strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{16}$$

The relative rotation of each material point is given by:

$$\omega_i^{rel} = \omega_{ji} n_j - \omega_i^c \tag{17}$$

and the gradient of the Cosserat rotation, which is called the curvature of the deformation, is given by:

$$\kappa_{ij} = \frac{\partial \omega_j^c}{\partial x_i} \tag{18}$$

Eqs.(16,17) are combined to give the following deformation quantities for the 3D-Cosserat continuum:

$$\begin{array}{l|l|l|l} \gamma_{11} = \dfrac{\partial u_1}{\partial x_1} & \gamma_{12} = \dfrac{\partial u_1}{\partial x_2} + \omega_3^c & \gamma_{13} = \dfrac{\partial u_1}{\partial x_3} - \omega_2^c & \gamma_{23} = \dfrac{\partial u_2}{\partial x_3} + \omega_1^c \\ \gamma_{22} = \dfrac{\partial u_2}{\partial x_2} & \gamma_{21} = \dfrac{\partial u_2}{\partial x_1} - \omega_3^c & \gamma_{31} = \dfrac{\partial u_3}{\partial x_1} + \omega_2^c & \gamma_{32} = \dfrac{\partial u_3}{\partial x_2} - \omega_1^c \\ \gamma_{33} = \dfrac{\partial u_3}{\partial x_3} & & & \end{array} \tag{19}$$

The twelve deformation quantities (Eqs.(18, 19)) are conjugate in energy with twelve stress quantities. First, one has the nine components of the non-symmetric stress tensor σ_{ij}, which are conjugate to the non-symmetric deformation tensor γ_{ij}, and second one has nine moment stresses (moment per unit area), which are conjugate with the nine components of the deformation curvature tensor κ_{ij}. Force and moment equilibrium at the element (dx_1, dx_2, dx_3) lead to (Figure 2):

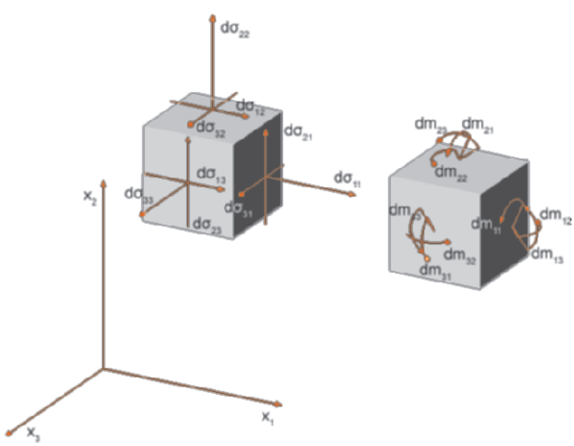

Figure 2. Stresses on element (dx_1, dx_2, dx_3).

$$
\begin{aligned}
&\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} - \rho \ddot{u}_1 = 0 \\
&\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} - \rho \ddot{u}_2 = 0 \\
&\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} - \rho \ddot{u}_3 = 0 \\
&m_{11,1} + m_{21,2} + m_{31,3} + \sigma_{32} - \sigma_{23} - J_1 \ddot{\omega}_1^c = 0 \\
&m_{12,1} + m_{22,2} + m_{32,3} + \sigma_{13} - \sigma_{31} - J_2 \ddot{\omega}_2^c = 0 \\
&m_{13,1} + m_{23,2} + m_{33,3} + \sigma_{21} - \sigma_{12} - J_3 \ddot{\omega}_3^c = 0
\end{aligned}
\tag{20}
$$

Following the hereunder constitutive relations an equivalent Cosserat continuum is identified that describes the studied blocky structure.

$$
\begin{aligned}
&\sigma_{11} = \frac{a}{b^2}\left(c_Q + 2c_N\right)\cdot \gamma_{11} && m_{11} = \frac{a}{b^2}\left(c_M + 2c_T - \frac{1}{4}b^2 c_Q\right)\cdot \kappa_{11} \\
&\sigma_{22} = \frac{1}{a}c_N \cdot \gamma_{22} && m_{22} = \frac{1}{a}\left(c_T - \frac{1}{4}a^2 c_Q\right)\cdot \kappa_{22} \\
&\sigma_{33} = 0 && m_{33} = 0 \\
&\sigma_{12} = \frac{1}{a}c_Q \cdot \gamma_{12} && m_{12} = \frac{a}{b^2}\left(2c_M + c_T - \frac{9}{4}a^2 c_Q\right)\cdot \kappa_{12} + \frac{1}{2}ac_Q \cdot \kappa_{21} \\
&\sigma_{21} = \frac{a}{b^2}\left(c_N + 2c_Q\right)\cdot \gamma_{21} && m_{21} = \frac{1}{2}ac_Q \cdot \kappa_{12} + \frac{1}{a}\left(c_M - \frac{1}{4}b^2 c_Q\right)\cdot \kappa_{21} \\
&\sigma_{23} = 0 && m_{23} = -\frac{1}{a}\left(\frac{1}{4}a^2 c_N + \frac{1}{4}b^2 c_Q - c_M\right)\cdot \kappa_{23} \\
&\sigma_{32} = \frac{1}{a}c_Q \cdot \gamma_{32} && m_{32} = 0 \\
&\sigma_{13} = 0 && m_{13} = -\frac{a}{b^2}\left(2a^2 c_Q + \frac{1}{4}b^2 c_Q + \frac{1}{4}a^2 c_N - 3c_M\right)\cdot \kappa_{13} \\
&\sigma_{31} = \frac{3a}{b^2}c_Q \cdot \gamma_{31} && m_{31} = 0
\end{aligned}
\tag{21}
$$

The geometric configuration of the considered structure is anisotropic even for square blocks as each block has four neighbours in the x_2 direction and only two neighbours in the x_1 direction (Figure 1).

5. THE DISPERSION FUNCTION

The domain of validity of the above representation of a blocky structure by a Cosserat continuum is evaluated by comparing the dynamic response of the discrete and the homogenized structures. The dynamic response of a structure is characterized by its dispersion function, which relates the wave propagation velocity to the wavelength of the input signal. For elastic behaviour it is possible to derive analytical solutions for the dispersion function of the discrete and the continuous systems by using discrete and continuous Fourier transforms as presented below. Here, one is interested in the out of plane dynamic response of the structure. For the in plane comparison of the blocky structure the reader is referred to the previous work of Sulem and Mühlhaus[2], where a similar analysis is followed showing that the Cosserat model gives a good approximation for wavelengths bigger than five times the size of the block. First we will derive the expression of the dispersion function for the continuous system.

The Fourier transform of a function is:

$$\mathrm{G}(\theta_1,\theta_2,\nu)=\mathcal{F}\{\mathrm{g}(x_1,x_2,\mathrm{t})\}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{g}(x_1,x_2,\mathrm{t})\mathrm{e}^{2\pi\mathrm{i}(\theta_1 ax+\theta_2 by+\nu t)}dx_1dx_2dt \tag{22}$$

where $\mathrm{i}=\sqrt{-1}$, θ_1 and θ_2 the wave numbers at x_1 and x_2 direction respectively and v the frequency.

The inverse transform is:

$$\mathrm{g}(\mathrm{x}_1,\mathrm{x}_2,\mathrm{t})=\mathcal{F}^{-1}\{\mathrm{G}(\mathrm{p},\mathrm{q},\nu)\}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{G}(\mathrm{p},\mathrm{q},\nu)\mathrm{e}^{-2\pi\mathrm{i}(\frac{\theta_1}{a}\mathrm{x}_1+\frac{\theta_2}{b}x_2+\nu t)}d\theta_1 d\theta_2 d\nu \tag{23}$$

The Fourier transform will be denoted as:

$$g(x_1,x_2,t)\rightleftharpoons G(\theta_1,\theta_2,\nu) \tag{24}$$

In general the wave speed V depends on the wave numbers θ_1 and θ_2. The dispersion function $V(\theta_1, \theta_2)$ is determined by insertion of the transformed unknown functions w, ω, and θ and their transformed derivatives into Eqs. (12)-(14). This leads to the following homogeneous system of equations:

$$\begin{pmatrix} a_1+Y & \mathrm{i}a_2 & \mathrm{i}a_3 \\ -\mathrm{i}a_2 & b_1+b_3Y & b_2 \\ -\mathrm{i}a_3 & b_2 & c_1+c_2Y \end{pmatrix}\begin{pmatrix} W(\theta_1,\theta_2,\nu) \\ \Theta(\theta_1,\theta_2,\nu) \\ \Omega(\theta_1,\theta_2,\nu) \end{pmatrix}=\mathbf{0} \tag{25}$$

This system possesses non-trivial solutions when its determinant vanishes, that is:

$$\det\begin{pmatrix} a_1+Y & \mathrm{i}a_2 & \mathrm{i}a_3 \\ -\mathrm{i}a_2 & b_1+b_3Y & b_2 \\ -\mathrm{i}a_3 & b_2 & c_1+c_2Y \end{pmatrix}=0 \tag{26}$$

where

$$\begin{aligned} a_1 &= -6c_Q\theta_1^2-2c_Q\theta_2^2 \\ a_2 &= -6c_Q a\theta_1 \\ a_3 &= 2c_Q b\theta_2 \\ b_1 &= -2\left(2c_M-\frac{9}{4}a^2c_Q+c_T\right)\theta_1^2-2\left(c_T-\frac{1}{4}a^2c_Q\right)\theta_2^2-6a^2c_Q \end{aligned} \tag{27}$$

$$
\begin{aligned}
b_2 &= c_Q ab\theta_1\theta_2 \\
b_3 &= {J_y}/{m} \\
c_1 &= -2\left(c_M + 2c_T - \frac{1}{4}c_Q b^2\right)\theta_1^{\,2} - 2\left(c_M - \frac{1}{4}c_Q b^2\right)\theta_2^{\,2} - 2b^2 c_Q \\
c_2 &= {J_x}/{m}
\end{aligned}
\tag{27 cont'd}
$$

and

$$
Y = m\left(\frac{\theta_1 V_1}{a} + \frac{\theta_2 V_2}{b}\right)^2 \tag{28}
$$

Y is the solution of the following characteristic polynomial equation:

$$
AY^3 + BY^2 + CY + D = 0 \tag{29}
$$

where the coefficients are:

$$
\begin{aligned}
A &= b_3 c_2 \\
B &= a_1 b_3 c_2 + b_3 c_1 + b_1 c_2 \\
C &= a_1 b_1 c_2 + a_1 b_3 c_1 - a_2^{\,2} c_2 + b_1 c_1 - a_3^{\,2} b_3 - b_2^{\,2} \\
D &= a_1 b_1 c_1 - a_3^{\,2} b_1 + 2a_2 b_2 a_3 - a_2^{\,2} c_1 - a_1 b_2^{\,2}
\end{aligned}
\tag{30}
$$

Concerning the discrete system a function $g_{\mathrm{i,j}}(t)$ can be written as:

$$
g_{i,j}(t) = g(x_{1i}, x_{2j}, t) = Ш_{\mathrm{a,b}} f(x_1, x_2, t) \tag{31}
$$

where:

$$
\mathrm{x}_{1\mathrm{j}} = j \cdot a \,, \quad x_{2\mathrm{j}} = j \cdot b \,,
$$

$$
Ш_{\mathrm{a,b}} = \sum_{n,k=-\infty}^{n,k=+\infty} \delta(x_1 - n\,a)\,\delta(x_2 - k\,b)
$$

(Dirac comb) and δ is the Dirac function. The discrete Fourier transform of function $g_{i,j}(t)$ is denoted as:

$$
g_{i,j}(t) \rightleftharpoons G(\theta_1, \theta_2, \nu) \tag{32}
$$

and consequently:

$$
g_{i\pm1,j\pm1}(t) \rightleftharpoons e^{\mp \mathrm{i}\theta_1 \mp \mathrm{i}\theta_2} G(\theta_1, \theta_2, \nu) \tag{33}
$$

Similarly to the continuous case, the dispersion function is determined by applying the Fourier transform to the discrete system of Eqs. (5)-(7). This leads to the following homogeneous system of equations:

$$\begin{pmatrix} a_1+Y & \mathrm{i}a_2 & \mathrm{i}a_3 \\ -\mathrm{i}a_2 & b_1+b_3Y & b_2 \\ -\mathrm{i}a_3 & b_2 & c_1+c_2Y \end{pmatrix} \begin{pmatrix} W(\theta_1,\theta_2,\nu) \\ \Theta(\theta_1,\theta_2,\nu) \\ \Omega(\theta_1,\theta_2,\nu) \end{pmatrix} = \mathbf{0} \tag{34}$$

This system has the same form as the one derived from the continuum model and it similarly possesses non-trivial solutions when its determinant vanishes:

$$\det \begin{pmatrix} a_1+Y & \mathrm{i}a_2 & \mathrm{i}a_3 \\ -\mathrm{i}a_2 & b_1+b_3Y & b_2 \\ -\mathrm{i}a_3 & b_2 & c_1+c_2Y \end{pmatrix} = 0 \tag{35}$$

where

$$\begin{aligned}
a_1 &= 4c_Q\cos\theta_1\left(\cos\theta_1+\cos\theta_2\right)-8c_Q \\
a_2 &= -2c_Q a\sin\theta_1\left(2\cos\theta_1+\cos\theta_2\right) \\
a_3 &= 2c_Q b\cos\theta_1\sin\theta_2 \\
b_1 &= \left(4c_T-a^2c_Q\right)\cos\theta_1\cos\theta_2-4a^2c_Q\cos^2\theta_1-4c_M\sin^2\theta_1-a^2c_Q-4c_T \\
b_2 &= bac_Q\sin\theta_1\sin\theta_2 \\
b_3 &= J_y/m \\
c_1 &= \left(4c_M-b^2c_Q\right)\cos\theta_1\cos\theta_2-4c_T\sin^2\theta_1-b^2c_Q-4c_M \\
c_2 &= J_x/m
\end{aligned} \tag{36}$$

and

$$Y = m\left(\frac{\theta_1 V_1}{a}+\frac{\theta_2 V_2}{b}\right)^2 \tag{37}$$

Y is the solution of the following characteristic polynomial equation:

$$AY^3+BY^2+CY+D=0 \tag{38}$$

where the coefficients are:

$$\begin{aligned}
A &= b_3c_2 \\
B &= a_1b_3c_2+b_3c_1+b_1c_2 \\
C &= a_1b_1c_2+a_1b_3c_1-a_2{}^2c_2+b_1c_1-a_3{}^2b_3-b_2{}^2 \\
D &= a_1b_1c_1-a_3{}^2b_1+2a_2b_2a_3-a_2{}^2c_1-a_1b_2{}^2
\end{aligned} \tag{39}$$

Eq.(37) is formally the same as Eq.(29).

6. VALIDATION OF THE COSSERAT CONTINUUM - HOMOGENISED vs. DISCRETE MODEL

In order to compare the dynamic response of the discrete and the continuous systems, Eq.(29) and Eq.(37) are solved. The following dimensionless quantities are defined:

dimensionless wave velocity: $$V = \sqrt{\frac{1}{c_Q}\frac{Y}{\theta_1^2+\theta_2^2}} \quad (40)$$

phase velocity: $$\omega = \sqrt{\frac{Y}{m}} \quad (41)$$

dimensionless wave lengths: $$w_1 = \frac{2\pi}{\theta_1}, \; w_2 = \frac{\pi}{\theta_2} \quad (42)$$

The wave propagation is analysed successively in the x_1 (θ_2=0) and the x_2 direction (θ_1=0). The dimensionless wave shear velocity versus the dimensionless wavelength for the Discrete and the Cosserat models are presented in Figures 3 and 4. In Figures 5 and 6 the phase velocity for the three different modes corresponding to the three real roots of the corresponding characteristic Eqs.(29) and (38) are depicted. It is observed that the Cosserat model gives a good approximation for wavelengths bigger than five times the size of the block ($\theta_2 < 0.6$).

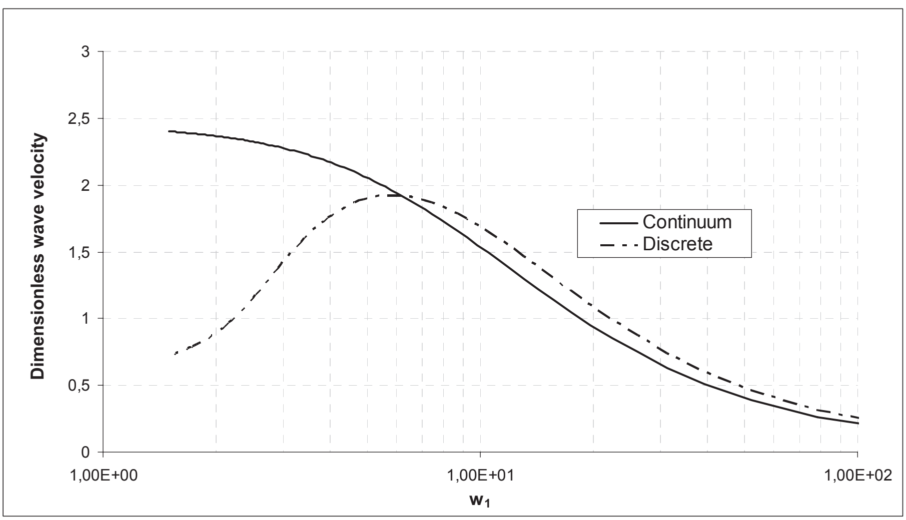

Figure 3. Dimensionless wave velocity for the Discrete and the Cosserat models (c_N=5c_Q, b=a) for wave propagation in x_1-direction.

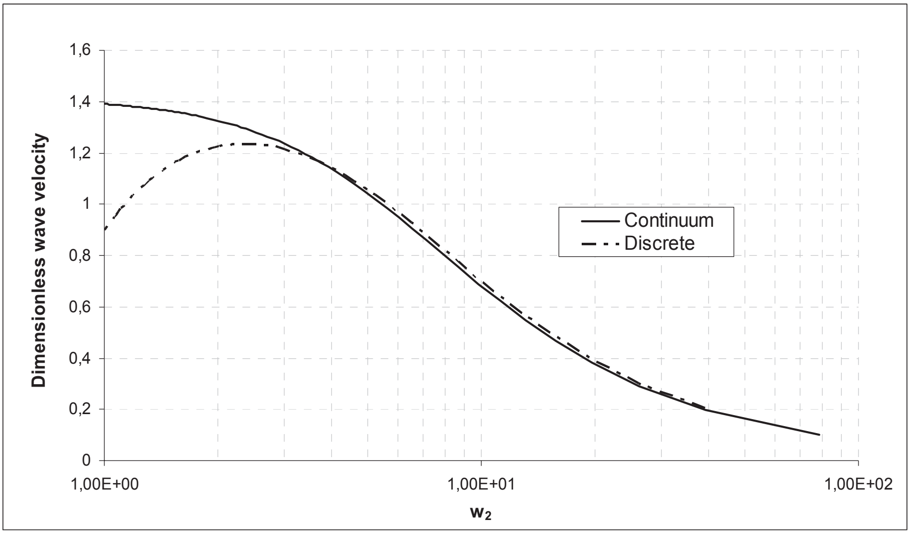

Figure 4. Dimensionless wave velocity for the Discrete and the Cosserat models ($c_N=5c_Q$, b=a) for wave propagation in x_2-direction.

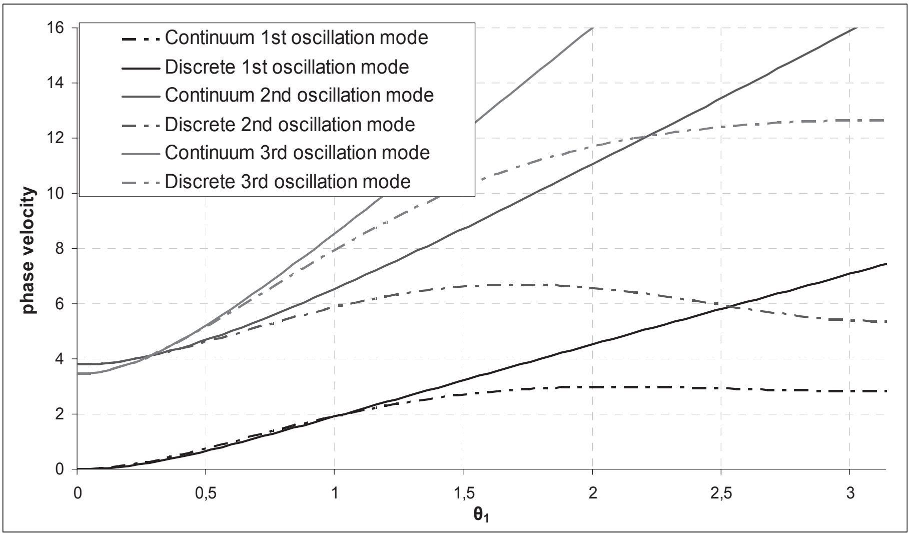

Figure 5. Phase velocity for the Discrete and the Cosserat models ($c_N=5c_Q$, b=a) for wave propagation in x_1-direction.

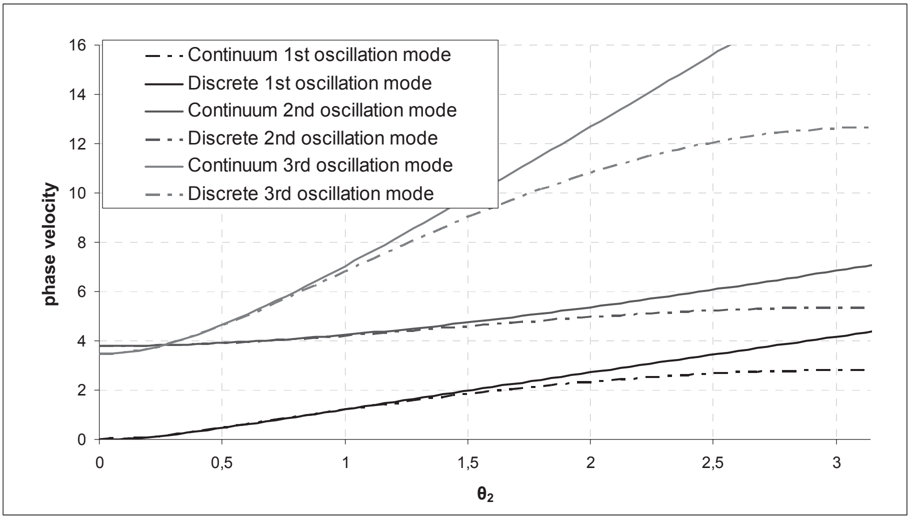

Figure 6. Phase velocity for the Discrete and the Cosserat model (c_N=5c_Q, b=a) for wave propagation in x_2-direction.

7. CONCLUSIONS

When dealing with blocky or layered structures or more generally with any structure where inhomogeneities are visible, one can address the question of modelling the behaviour of such a structure either by considering each inhomogeneity individually and solving the problem as in the distinct or discrete-element methods, or by considering the salient features of the discontinuum within the framework of generalised continuum theory.

The three-dimensional dynamic behaviour of a wall was studied here. Both the discrete and the 3D Cosserat model of the aforementioned structure are described, similarly to plate theory, by two uncoupled sets of equations, which correspond to the in- and to the out-of-plane motion of the blocky structure. The in plane motion of the wall was previously studied by Sulem and Mühlhaus[2]. Concerning the out of plane wall behaviour, the homogenisation procedure leads to similar conclusions with the in-plane analysis of the structure. The Discrete and the 3D Cosserat approaches coincide for wavelengths five times bigger the size of the block. However, the Cosserat model becomes increasingly inaccurate for smaller wavelengths. Generally one could assert that the Cosserat model appears to be the natural starting point for the development of continuum models for blocky structures. In the future, the derived continuous model is going to be extended in the plasticity domain. Multi-yield plasticity criteria will be developed and applied to express and simulate the interblock sliding, tilting and twisting failure modes.

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