Chapter 14

TRUTH, NEGATION AND OTHER BASIC NOTIONS OF LOGIC

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14.1 What is the logic of ordinary language?

According to a story, Albert Einstein was once asked how he had come upon his strange revolutionary ideas. He replied: "By asking the questions that children are discouraged to ask." If we want to follow Einstein's strategy in the philosophy of logic, we are thus led to ask such questions as we ourselves discourage our own introductory logic students to ask. But what are such questions? One of them might very well be: What is the logic of our ordinary language? It is convenient to us logic teachers to pretend initially that it is the logic we are teaching, in other words, that the notation of the usual first-order logic is nothing but a streamlined version of ordinary English. In older textbooks this claim is sometimes made explicitly. If pressed, we might appeal to Chomsky (e.g. 1986) whose Ersatz logical forms alias LFs differ only inessentially from the logical forms of ordinary first-order formulas. Yet such appeals should evoke pangs of intellectual conscience, for our actual Sprachlogik differs in several disturbing ways from the received ("Frege-Russell") first-order logic. I have shown (in Hintikka 1997) that even one of the most general notions of formal logic, the notion of scope, is not a primitive notion but one which can be applied to natural language only indirect ways. It can also be shown that the logic of natural-language conditional sentences can only be captured by going way beyond ordinary first-order logic. But even apart from such theoretical differences, there are lots of ordinary language sentences whose logic is not captured by their *prima facie* translations into first-order logical notation. For instance, consider the sentence

J. van Benthem, G. Heinzmann, M. Rebuschi and H. Visser (eds.), The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today, 195–219.

(1) For this problem, there is a person such that if he or she can solve it, anyone can.

This seems to have the logical form (if we look away from the initial demonstrative identification of the problem)

(2) $(\exists x)(A[x] \supset (\forall y)A[y])$.

But (2) turns out to be a logical truth whereas (1) is not naturally taken to be one. Likewise, C.S. Peirce already noted that the following pair of sentences clearly have a different meaning even though our logic tells us that they do not (*Collected Papers* 4.546):

(3) Someone is such that he will commit suicide if he fails in business.

(4) Someone is such that he will commit suicide if everyone fails in business.

Or, rather, Peirce noted that the following *prima facie* translations (5)–(6) of (3)–(4) into the notation of first-order logic are logically equivalent, in spite of the difference in meaning between (3)–(4):

(5) $(\exists x)(F[x] \supset S[x])$

(6) $(\exists x)((\forall y)F[y] \supset S[x])$

Likewise, the reasoning that leads to some of the best-known paradoxes seems to be impeccable reasoning, in spite of giving rise to paradoxical conclusions. The sorites paradox is a case in point. If we abbreviate "a has n hairs" as H(a,n), then the inductive inference leading to the paradox appears to be unobjectionable. It could be taken to be of the form

(7) $(H(a,0) \supset B(a))\&(\forall x)((H(a,x) \supset B(a)) \supset (H(a,x+1) \supset B(a))$
ergo $(\forall x)((H(a,x) \supset B(a))$

where $B(a)$ says that a is bald. In a simpler form, the structure of (7) can be taken to be

(8) $S[0](\forall x)(S[x] \supset S[x+1])$, ergo $(\forall x)S[x]$.

which looks like a perfectly valid instance of mathematical induction. Each of these anomalies might look insignificant, but their cumulative impact ought to be a clue that shows that the logic of our ordinary discourse is far from being adequately understood.

14.2 What is truth?

Another Einsteinian question is surely what precisely is meant by truth in first-order logic. We teach students all about truth-functions and truth-values, but are not likely to give an adequate answer when a student inquires what the mysterious notion of truth is. Maybe we tell our students to look up a

Tarski-type truth-definition from an advanced text (or from Tarski 1935), trying to suppress the guilty awareness of how a student will not find any real insight into the notion of truth from Tarski-type conditions on valuations and on infinite sequences. A student's puzzlement over such complications is part and parcel of those philosophers' dissatisfaction who claim that there is nothing in Tarski-type truth definitions that show that they are definitions of *truth*. The striking fact here nevertheless is that in the case of first-order logic (the logic of quantification) there is an accurate general answer easily available (see Hintikka 1998 and 2001). What is more, this answer is nothing more and nothing less than an explication of our natural pretheoretical notion of truth for quantificational sentences. In order to see what this answer is, consider a sentence of the form

(9) $(\forall x)(\exists y)F[x,y]$

When is (9) true? Obviously if and only if for any given value of x it is in principle possible to find a "witness individual" y depending on x such that $F[x,y]$. And this colloquial location "it is possible to find, given x" is in the eyes of a mathematician nothing but an euphemism for the existence of a function $f(x)$ which produces a suitable witness individual as its value for any given argument x, in other words a function f such that the following is true:

(10) $(\forall x)F[x,f(x)]$

The generalization of this observation is that a first-order sentence S is true if and only if there exists a full array of its Skolem functions. But what are the Skolem functions of S? In order to recognize them, let us assume that S is in a negation normal form. What this means is that its propositional constants are &, $\vee$ and $\sim$ and that all negation signs precede immediately atomic formulas or identities. Then the Skolem form of S is obtained y_1 by replacing each existentially quantified subformula $(\exists x)F[x]$ of S by $F[f(y_1,y_2,\ldots)]$ and prefixing the entire sentence with $(\exists f)$. Here f is a new function variable, different for different existential subformulas, and $(\forall y_1)$, $(\forall y_2)$, ... are all the universal quantifiers in S on which the quantifier $(\exists x)$ depends on in S. The truth-making choices of the values of the function variable f are the Skolem functions of S. And what these functions do is to produce the witness individuals (usually dependent on other such individuals), which according to our pretheoretical conception show the truth of S. Thus what we have here is a straightforward generalization of the truth-condition for (9), identified above. For some purposes, the notion of Skolem functions can - and must - be extended to relate also to the propositional connectives of S. Assuming still that S is in a negation normal form, this means replacing each disjunction $(S_1 \vee S_2)$ that occurs as a subformula of S by

(11) $(S_1 \& g(y_1,y_2,\ldots)=0) \vee (S_2 \& g(y_1,y_2,\ldots) \neq 0)$.

At the same time, the entire sentence is prefixed by $(\exists g)$. In (11), g is a new function variable, different form all the f's and different for different disjunctions and $(\forall y_1)$, $(\forall y_2)\ldots$ are all the universal quantifiers on which the disjunction in question depends on in S. Furthermore, 0 can be any designated member of the domain. In the special case of a sentence of the form $(\exists x)\,F[x]$ its sole Skolem function reduces to a constant individual. This individual serves as the "witness individual" which according to our pretheoretical conception vouchsafes the truth of the sentences in question. The general case becomes as obvious as this paradigm case as soon as we realize that in general the requisite witness individuals that show the truth of the sentence depend on other witness individuals, in mathematicians' jargon, are functions of them. These functions are precisely the Skolem functions of S. Hence appropriate witness individuals exist for S if and only if there exists an array of all the Skolem functions of S. Then and only then is S true (see Hintikka 2001). Some philosophers have played with the notion of a truth-maker. As far as quantificational languages are concerned, there is only one kind of truth-makers, and they are Skolem functions.

14.3 Compositionality and the meaning of quantifiers

This definition of truth is so perspicuous, and so obviously but an explication of our very own notion of truth, that one could legitimately expect that is has been acknowledged and exhaustively discussed by philosophers of our time. It boggles one's mind that this has not happened. Philosophers and logicians have discussed Tarski-type truth-definitions *ad nauseam*, notwithstanding the fact that a much simpler and much more natural truth-definition is readily available for them. Why this neglect? The reason is not that the truth definitions for first-order sentences which turn on the existence of Skolem functions cannot be formulated in the same language, for nor can a Tarski-type truth definition. The real reasons are different. One of them is Tarski's tacit insistence that truth definition must be compositional. Tarski did not spell out this requirement, but a closer examination reveals his commitment to it. (Such an examination is found in Hintikka and Sandu 1999.) In contrast, game-theoretical truth definitions of the kind explained violate prima facie the requirement of compositionality. This is shown by the definition of the Skolem form of a first-order sentence given above. In it, the selection of the arguments $y_1, y_2, \ldots$ of the new function constant do not depend only on the subformula $F[x]$, but also on which the outside universal quantifiers are on which $(\exists x)$ depends in the sentence in question. (As should be obvious, the principle of compositionality amounts essentially to the assumption of semantical context-independence.) As a consequence, the eminently natural game-theoretical definition of truth cannot be implemented without violating compositionality. In view of the popularity of compositionality among linguists and logicians, it seems likely that in different direct and indirect ways a commitment to compositionality is one of the main factors that have conspired to suppress the Skolem-function definition of truth from philosophers' attention.

Commitment to compositionality is connected with another oversight of the majority of contemporary philosophers. It is the belief that the semantics of quantifiers is exhausted by the idea that quantifiers "range over" a certain class of values. If so, the truth of a universally quantified sentence $(\forall x)\, F[x]$ reduces to the truth of all its substitution-instances F[b], where b is a member of the domain; and likewise for existentially quantified sentences. From this idea of quantificational truth it is only a short trip to Tarski-type truth definitions, which are conditioned by Tarski's requirement as to what an acceptable truth-definition must be like. What is wrong with the exclusiveness of the "ranging over" idea is that it does not address at all the other component of the semantics of quantifiers. This other component is the representation of dependencies and independencies between actual real life variables by means of the dependencies and independencies of the quantifiers to which the variables in question are bound. Such dependencies are expressed in so many words by Skolem functions. Their role in the game-theoretical truth predicate shows how the dependence relations between different variables are taken care of in the game-theoretical characterization of truth. Only when these dependence relations are looked away from will a truth-definition in terms of substitution-instances or, for that matter, a Tarski-type truth-definition, appear natural.

Another important reason for neglect of Skolem-type truth definition that logicians and philosophers have been suspicious of second-order logic and tried to stick to the first-order level. Such a goal seems to be guiding already Tarski. Now remaining on a first-order level might be a commendable aim, but it has not been implemented in the right way. Instead of second-order logic, philosophers have preferred to it set theory practiced on the first-order level. We are all familiar with Quine's misplaced quip about higher-order logic being set theory in sheep's clothing. It is turning out, Quine notwithstanding, that it is axiomatic set theory, not higher-order logic, that is the big bad wolf here. Outside philosophical fairy tales, the cold sober fact is that first-order axiomatizations of set theory cannot do an adequate job in their foundational role of capturing set-theoretical truths. Their failure is discussed in (Hintikka 2004(a)).

This failure of the only viable-looking rival means that there are no valid objections to defining truth in terms of the existence of Skolem functions. Such definitions may not be the last word on our concept of truth but they are an eminently useful first word. Their plausibility can be further enhanced by dramatizing the production of witness individuals by Skolem functions as steps in certain explicitly definable search games. The definition of these games, known as semantical games, makes the role of Skolem functions eminently intuitive. Full arrays of Skolem functions for a sentence S are precisely the winning strategies for the verifier in the semantical game $G(S)$ correlated with S and starting with S. Thus the truth of S can be defined as the existence of a winning strategy for the verifier in the game $G(S)$.

In a critical philosophical perspective, however, such a use of game-theoretical concepts is nevertheless merely a dramatization of the basic insight into the role of Skolem functions as implementing our natural notion of truth. I shall nevertheless call the definition of truth for quantificational sentences in terms of the existence of Skolem functions the game-theoretical truth definition. The game-theoretical framework is in any case useful in several respects. For one thing, it shows that the game-theoretical truth definition in not subject to criticism from an intuitionistic or from a constructivistic viewpoint. If we want for some reason or other to restrict ourselves to a constructivistic notion of truth, it can be done simply by restricting the values of Skolem function quantifiers to constructive functions (whatever they are or may be). Likewise, an intuitionistic notion of truth can be captured by restricting Skolem functions to known ones. I think that we can leave the question as to which functions are known for Brouwer and his followers to decide. One major advantage of such game-theoretical truth-definition was already noted. They allow variation in a way that captures different nonclassical conceptions of truth in a natural way. This makes it possible to compare competing logics with each other in an informed way.

14.4 IF logic as the natural basic logic

Even more importantly, when we start thinking in game-theoretical terms, we can at once see that there are lots of perfectly natural semantical games that do not correspond to any sentences of the received first-order logic. In other words, certain second-order sentences behave just like truth-conditions for nonexistent first-order sentences in terms of perfectly well-defined semantical games. For instance, the second-order sentence

(12) $(\exists f)(\exists g)(\forall x)(\forall y)F[x, f(x), y, g(x, y)]$

is the truth-condition of the sentence

(13) $(\forall x)(\exists z)(\forall y)(\exists u)F[x, z, y, u]$.

In the correlated game, the verifier is searching for a truth-making value of z on the basis of his or her (or its, if the player is a computer) knowledge of a given value of x, and searching for a value of u on the basis of his, her or its knowledge of the values of x and y. Likewise, the second-order sentence

(14) $(\exists f)(\exists g)(\forall x)(\forall y)F[x, f(x), y, g(y)]$

asserts the existence of a winning strategy in a similar game whose only novelty is that the search for the second "witness individual" is carried out with the verifier's knowledge limited to the value of y. Such games are perfectly well defined, and the second-order (14) is related to them precisely the same way as (12) is related to the semantical game played with (13). Once we see this, we can see that we can formulate first-order sentences related to (14) in the same way as (12) is related to (13), as soon as we relax our notation so as to allow a quantifier (Q_2y) to be independent of another quantifier, say (Q_1x), even though it occurs its syntactical scope. This can be done by writing it (Q_2y/Q_1x). Then the first-order counterpart to (14) is expressible as

(15) $(\forall x)(\exists z)(\forall y)(\exists u/\forall x)F[x, z, y, u]$

which is equivalent to

(16) $(\forall x)(\forall y)(\exists z/\forall y)(\exists u/\forall x)F[x, z, y, u]$.

Then the semantical game correlated with (15) is such that (14) expresses the existence of a winning strategy for the verifier in it.

This can obviously be generalized. The result is what has been called independence-friendly (IF) logic. For its theory, the reader is referred to Hintikka (2002(b)). IF logic is our natural basic logic. It is richer in its expressive capacities than the received first-order logic, which can be thought of as the slash-free fragment of IF first-order logic. (But cf. below.) In it, several crucial notions can be expressed that were not expressible in the received old first-order logic. For instance, the equicardinality of two sets, say α and β, can be expressed on the first-order level as follows:

(17) $(\forall x)(\forall z)(\exists y/\forall z)(\exists u/\forall x)$
$((x \in \alpha \supset y \in \beta)\&(z \in \beta \supset u \in \alpha)\&((y = z) \leftrightarrow (x = u)))$

This may be compared with the second-order sentence serving the same purpose:

(18) $(\exists f)(\exists g)(\forall x)(\forall z)$
$((x \in \alpha \supset f(x) \in \beta)\&(z \in \supset g(z) \in \alpha)\&((f(x) = z) \leftrightarrow (x = g(z))))$

The equivalence of (17) and (18) illustrates a most remarkable thing about IF first-order logic: It is tantamount to the $\sum_1^1$ (sigma one-one) fragment of second-order logic. It is easily shown that each sentence of this fragment has an equivalent IF first-order sentence. Conversely, each IF first-order sentence is equivalent to its on game-theoretical truth condition, which is a $\sum_1^1$-sentence. It can also be seen that the truth conditions of different IF first-order sentences can be integrated into a $\sum_1^1$-truth predicate. (It is assumed here that we are dealing with an IF first-order language strong enough to express its own syntax.) Since that predicate has an equivalent in the corresponding IF first-order language, which can admit of a truth predicate definable in the same language (see Hintikka 1998 and 2001). Thus the notion of truth and its definability are put to a radically new light by the simple step of allowing quantifiers to be independent of each other. In every other respects, we can preserve all the classical semantical rules, as they must be formulated in game-theoretical terms.

14.5 Two negations

But the notion of truth is not the only one which now has to be re-examined. Another one is the notion of negation. With respect to it, we are in for an intriguing surprise. The concept of negation that results from perfectly "classical" semantical rules where independence is allowed does not obey the law of excluded middle. Is IF logic therefore "nonclassical"? The truth is that there is no obvious definition of "classical" that we could appeal here to decide the issue, unless we resort to the quaint old sense of the word as referring to what is taught in classrooms. Since the negation $\sim$ used in IF logic obeys the most classical semantical rules imaginable and yet violates *tertium non datur*, the right conclusion to be drawn here that the law of excluded middle is not part and parcel of "classical" logic.

This strong negation $\sim$ has to be distinguished from the familiar contradictory negation $\neg$. The same distinctions must be extended to conditionals and equivalences. A conditional "If A, then B" may have the logical force of either $(\sim A \vee B)$ or $(\neg A \vee B)$, and an equivalence can mean either $(A\&B) \vee (\sim A \& \sim B)$ or $(A\&B) \vee (\neg A\&\neg B)$.

Here we are witnessing yet another apparently trivial question which nevertheless leads to surprising new perspectives. What has been found out is that there is a strong (dual) negation implicit in all our use of the basic logical notions. It is the negation that naturally goes together with the game-theoretical concept of truth which was seen to be but an implementation of our pretheoretical notion of truth. Such a strong negation must thus be tacitly present also in the logic of ordinary language. This strong negation is in a game-theoretical perspective even more fundamental than contradictory negation. But if so, how is the contradictory negation to be handled in our explicit logic? How come that the negation that is present in natural language in the sense of having syntactical markers for it is the contradictory one? How is the contradictory negation to be interpreted semantically in GTS? And what is there to be said in the light of the distinction about the conditionals of ordinary language?

Three possibilities can be investigated here separately. The first is suggested by the naturalness of the game-theoretical truth condition also when applied to natural language. It suggests that appearances notwithstanding it is the game-theoretical semantics that governs also the semantics of natural language, including the behavior of negation and conditionals in them. But how can this make much difference? When no slashes are present, the difference between $\sim$ and $\neg$ should not make any difference. Indeed, the received first-order logic can apparently be identified with the slash-free fragment of IF first-order logic. So how can the distinction between $\sim$ and $\neg$ make any difference for slash-free sentences like (1)–(6) or for inferences like (8)? An answer here is that the distinction between $\sim$ and $\neg$ makes no difference for slash-free formulas only if it is assumed that atomic sentences obey the law of excluded middle. If they do not, there is a difference after all. Among other things, the same sentences are no longer logically true. And it is in fact easy to ascertain that then (2) is no longer logically true, (5) and (6) no longer logically equivalent and (8) no longer a valid inference. (The "then" here means of course taking $(A \supset B)$ to mean $(\sim A \vee B)$.)

Thus independence-friendly logic offers an interesting general perspective on the different mini-paradoxes of first-order logic. They can be dissolved if we assume that the conditionals of natural language are of the form $(\sim A \vee B)$ rather than $(\neg A \vee B)$, that is, that they are IF conditionals (as we will call them) rather than traditional ones. This dissolution strongly suggests that the logic of ordinary language is primarily independence-friendly logic rather than Frege-Russell one. This result is especially interesting philosophically in the case of (7)-(8), that is, in the case of sororities paradox. There exists a large and inconclusive literature on this paradox and on its variants. It is often surmised that the paradox should not arise in connection with predicates like "bald" which are unsharp, that is, whose attribution to a particular case need not always be either true or false. (This is sometimes expressed by speaking of "truth-value gaps".) However, no simple way of implementing this idea is found in the literature. Now we can see that the failure of *tertium non datur* for the predicate in question is after all that is needed to disarm the paradox, assuming that the basic logic of natural language is IF first-order logic. This assumption is in fact strengthened by its success in disarming the prima facie paradoxes of first-order logic. This observation can be generalized. The IF first-order logic that has been examined promises to be a far more natural logic of unsharp concepts than the so-called fuzzy logic of Lofti Zadeh. (see e.g. Zadeh and Yager 1991.) Of perhaps what can be said here is that the logic of natural language we are in effect already using can serve as a "fuzzy logic" better than its trade name variant without any additional assumptions or constructions.

Another well-known paradox is likewise disarmed by IF first-order logic. It is the liar paradox. When we use IF logic in a theory of elementary arithmetic, we can of course formulate a truth predicate $W[x]$ for it in the same arithmetic. Hence by means of the diagonal lemma we can formulate the Gödel-type sentence

(19) $\sim W[\mathbf{g}]$

whose Gödel number is g and which says that the sentence with the Gödel number g is false. (In (19) $\mathbf{g}$ is the numeral that represents g.) The sentence (19) is true if false, and false if true. Hence it must be neither true nor false, which is perfectly possible in IF logic. No contradiction is hence forthcoming.

But now it might at first seem that the extended IF first-order logic must run afoul of the so-called strong liar paradox. In elementary arithmetic using IF logic we can formulate a truth predicate, that is a predicate $W[x]$ that applies to the Gödel number $x = g(S)$ of an arithmetical sentence S if and only if S is true. Why cannot we apply the diagonal argument to the contradictory negation of $W[x]$ so as to obtain a sentence that so to speak says "I am not (i.e. contradictorily not) true"? The answer is that one cannot prefix $\neg$ to an open formula like $W[x]$, only to closed sentences. Hence the crucial liar sentence (Gödel-type self-referential sentence) is in this case ill formed. Again, no contradiction is in the offing.

Thus we have found an excellent first approximation to the logic of natural language. It is not the "ordinary" (i.e. received) first-order logic but the slash-free part of IF first-order logic, with the tacit provision that the predicate constants may be unsharp, that is, may fail to obey the law of excluded middle. The only negation used in this logic is the dual negation $\sim$. This nevertheless makes a difference only when the given predicate constants fail to conform to the *tertium non datur*.

Thus we have found exceedingly simple solutions to some of the oldest and most intriguing puzzles of the entire canon of logic. These solutions might at first seem too good meaning too simple to be true. Now I firmly believe that these solutions are definitive ones, but I also believe that further discussion is needed to back them up and to put them into perspective. But in order not to trivialize the issues that discussion must not pertain to the details of the paradoxes or to the purported lines of reasoning that lead into them. The solutions I have explained depend essentially on only one assumption. This assumption is that the natural, preferred logic of ordinary language is IF logic. Hence the further discussion that is needed here should pertain to the status of IF logic as compared with alternatives to it, especially when it comes to the treatment of negation. The rest of this paper is accordingly devoted to certain extensions of IF logic that might seem to have a claim to be our genuine *Sprachlogik*.

Indeed, it is unmistakable that the contradictory negation is needed in the semantics of natural languages. Hence we have to develop an explicit formal logic that will involve $\neg$ and not only $\sim$. A minimal step in that direction is to introduce $\neg$ by a fiat. The result is what has been called the extended IF logic. Studying it is the second one of the three lines of thought mentioned above. However, since there cannot be any game rules for $\neg$, the semantics of extended IF logic will have to be introduced by the bland metalinguistic stipulation that $\neg S$ is true if and only if S is not true. And here the italized not must itself be a contradictory negation. Hence the semantics of $\neg$ can along these lines be specified only by relying on the same notion in a metalanguage.

By the same token, the contradictory negation can in the extended IF logic occur only sentence-initially. For if it occurred otherwise, its semantics would presuppose a game rule for it. With proper care, it is possible to relax this requirement somewhat, however, as long as does not occur within the syntactical scope of any quantifier. If we now assume that the logic of natural language is like the extended IF logic, a number of phenomena in natural language become explainable. Some of them are mentioned in Hintikka (2002(a)), especially the fact that contradictory negation is in natural languages a barrier to anaphora.

The extended IF first-order logic is an interesting logic in its own right. It is obviously equivalent to the $(\sum_1^1 \cup \prod_1^1)$ fragment of second order logic. It might at first sight seem rather similar to the unextended IF first-order logic. On a closer examination, however, the differences are seen to be profound. Most of the "nice" metatheorems that hold for IF first-order logic are no longer valid in the extended IF logic, such as compactness, upwards Skolem-Löwenheim theorem, and the separation theorem. We will return to this matter, but it can already now be seen that the extension in question is important.

This is connected with the expressive richness of the extended IF first-order logic. In order to see this richness, consider an attempt to reconstruct the entire simple theory of types on the first-order level, construing it as a many-sorted first-order logic with different sorts. The structure of types is easy to specify on the first-order level. The only thing that cannot be expected by ordinary first-order logic is the requirement that for each arbitrary class of n-tuples of entities of a certain type there exists the embodiment of that class on the next higher type (order) level. This requirement can obviously be implemented by means of sentences of the extended IF first-order logic. This logic is therefore in a sense as rich as the entire theory of all finite types, and hence capable of codifying most traditional mathematics.

14.6 Extending IF logic with the help of tertium non datur

However, it seems clear that the extended IF logic cannot be the last word here. On the one hand, it is unsatisfactory simply to introduce $\neg$ by a fiat, without giving any account of the actual rules by means of which its semantics is determined. On the other hand, it can easily be seen that the logic of natural language is richer than even the extended IF first-order logic, in that what is unmistakably a contradictory negation can occur within the scope of quantifiers. The most obvious case in point is offered by negative quantifiers like no. If someone says

(20) nobody has the winning lottery ticket

it does not mean that everybody has something else. It simply means that it is not the case that someone has the winning ticket. Such a sentence therefore has the logical form

(21) $(\forall x)\neg W[x]$

Since W[x] is allowed here to be an IF formula which is not necessarily true or false for different substitution values for x, (21) is not necessarily equivalent to

(22) $\neg(\exists x)W[x]$

Now the semantics of (21) is not determined by the game-theoretical rules of IF first-order logic. It was just seen that it is not determined by the semantics of the extended IF first-order logic, either. How, then, can a natural semantics be defined for sentences like (21)? An eminently natural answer is available here. It can be approached from two different directions. The general question concerns the interpretation of sentences S_0 where $\neg$ is allowed to occur within the scope of quantifiers. One way of doing so is by considering a hierarchy of semantical games. The first begins with S_0 and comes to an end either with an atomic sentence or with a sentence (closed sentence) of the form $\neg S_1$. (This sentence usually is a substitution-instance of a subformula of S_0.) The truth-value of $\neg S_1$ is either true or false, and it is determined by the facts of the subordinate game $G(S_1)$ which can then be handled in the same way. In other words, $\neg S_1$ is deemed true for the purposes of $G(S_0)$ if and only if there exists no winning strategy for the verifier in $G(S_1)$, otherwise false.

For instance, a play of the game with the sentence (21), i.e. of the game $G((\forall x)\neg W[x])$, will stop after a choice of an individual (say b) by the falsifier. This endpoint sentence is of the form $\neg W[b]$. It is true if and only if there exists no winning strategy for the verifier in the game $G(W[b])$, otherwise false.

It is immediately seen that on this interpretation (21) is true if and only if all the sentences of the form $\neg W[b]$ are true, where b is a constant representing some member of the domain. This assigns a meaning to (21) game-theoretically if $W[x]$ does not contain $\neg$, for in $\neg W[b]$ the contradictory negation is then sentence-initial (i.e. prefixed to a closed formula). Otherwise we are dealing with a clause in a recursive truth-definition.

What all this amounts to is that the interpretation that extends our semantics to nested contradictory negations is a kind of substitutional interpretation quantifier. It will be called in the following the substitutional interpretation quantifier, but without thereby prejudicing its precise relation to what has in the past been called the substitutional interpretation quantifier of quantifiers. In the simplest cases it does coincide with substitutional interpretations in the received sense. In such cases, the truth of a universally quantified sentence is tantamount to the truth of all its substitution-instance, and the truth of an existentially quantified one with the truth of at least one of its substitution-instances. Restricting one's attention to such simple cases has led some philosophers to the conclusion that there is no deep difference between substitutional and objectual interpretations of quantifiers. (For this kind of view, see Kripke 1976.) This is nevertheless a mistake belied by what is found in IF logic. Even in the absence of contradictory negation, a strict inside-out (recursive) definition of truth in a substitutional sense is impossible in the presence of irreducible independence. This is especially blatant in the case of mutually dependent quantifiers. Their logic can be considered a strict counter-example to the substitutional interpretation quantifier of quantifiers.

In any case, in the presence of independence and dependence indicators there will have to become restrictions on the occurrence of $\neg$. The main such restriction is that no quantifier outside the scope of a given occurrence of $\neg$ can depend on a quantifier inside its scope. More generally, the scopes of different occurrences of $\neg$ must be nested, that is, they must form a tree structure.

14.7 Elementary versus non elementary logics

The logic definable in this way will be called the fully extended IF first-order logic. It calls for a number of explanations and comments.

First, it might be tempting to consider the fully extended first-order logic as the natural logic of ordinary language. This temptation is perhaps strengthened by the belief that something like the substitutional interpretation quantifier of quantifiers constitutes their natural semantics. There is perhaps a true element to this temptation. However, it is not the whole story. For one thing, the usual substitutional interpretation quantifier of quantifiers relies on the assumption that the semantics of quantifiers is exhausted by the "ranging over" idea. This interpretation hence cannot do justice to the role of quantifiers as expressing relations of dependence and independence between variables. It is therefore only a part of the story. Indeed, it is a secondary part, for the solutions to the mini-paradoxes of first-order logic outlined above strongly suggests that our basic logic operates like the unextended IF logic and not like its full version. The substitutional component in the truth definition for the full IF logic is thus an additional ingredient over and above the game-theoretical conception of truth codified by the existence of Skolem functions. The naturalness of this game-theoretical truth definition is the best testimony against relying too much on the "ranging over" idea alone. Indeed, we can now begin to appreciate the reasons for the complexity of the semantics of natural languages. This semantics is a mixture of different ingredients, where the basic game-theoretical conception of truth is supplemented by essentially different substitutional ideas.

The specious plausibility of the substitutional interpretation quantifier of quantifiers may perhaps be partly dispelled by asking what has to be known in order to understand a quantificational sentence or such an interpretation. The most important part of the answer is that the domain of individuals (aka the universe of discourse) has to be known. This rules out all uses of quantifiers where their range is open-ended. Such an open-endedness does not make it impossible to play semantical games and to understand statements as to what can or cannot happen in them. Accordingly, a quantificational language can be understood and used even when the language users do not know precisely what the domain is. Hence game-theoretical semantics of quantifiers is more widely applicable than a substitutional one.

If it strikes you as outlandish to apply a quantificational language without a sharply defined universe of discourse, you can contemplate Aristotle's syllogistic logic. No idea of a sharply delineated range of quantifiers was presupposed there, and existential force was not carried primarily by the particular "quantifier" but by the predicate term. Or think of higher-order quantifiers in our own logic. What precisely is their range? Believers in the standard interpretation in Henkin's sense will give you one answer, believers in nonstandard interpretation another one. And in this case the idea of a quantifier "ranging over" its values does not help us very much.

These remarks are not calculated to show that the substitutional interpretation quantifier is not possible or that it is not interesting. However, they suggest strongly that it is not the whole story of our pretheoretical understanding of quantifiers.

This point can be elaborated further. In a sense semantical games are what I shall call concrete processes. They are playable by actual humans. Indeed, Charles S. Peirce already envisaged them as games between a human "proponent" and an equally human "interpreter" (e.g. *Collected Papers* 3.479-482, 5.542, see Hilpinen 1982). (For most applications, it is nevertheless more natural to think of them as games between a human agent ("knower") and nature in the familiar game-theoretical sense of "games against nature".) Each play of these games which is connected with a finite sentence consists of a finite number of concrete moves. The players need not be initially familiar with all the members of the domain in which the game is played. One can in fact develop an instructive fallibilist epistemology starting from the assumption that the only information an inquirer receives about the world are the outcomes of the semantical games the inquirer plays against nature with different strategies. All this is possible even when the given domain is infinite. The crucial point is that the domain never plays any role in these activities as a closed totality.

It need not be assumed that every member of the domain has a name. It suffices to require that when a player of a semantical game chooses an individual from the domain, the players can give it a name and amplify their language by adjoining the newly coined name to it. Only a finite number of such extensions is needed in any play of a semantical game. Admittedly, for a truth definition we need the totality of the strategies that the inquirer has at his, her or its disposal. But if one's logical conscience is sensitive, one can restrict this strategy set to strategies that are constructive, known, computable or otherwise in conformity with one's principles of logical morality. (Or should I say, one's moral logic?) Apart from that qualification, the basic features of IF logic should be acceptable to everyone. That everyone includes Wittgenstein, for whom (as was pointed out) basic semantical games must be humanly playable. In the spirit of virtual history, I cannot help wondering how much greater progress the philosophy of logic would have experienced if Wittgenstein had realized that semantical games are the true logical home of our basic logical concepts. (Wittgenstein associated an importance to the activities of seeking and finding, but he related them to the notion of object rather than to the nature of quantifiers. see Hintikka, forthcoming (c).)

The game-theoretical truth definition and unextended IF logic should likewise be acceptable to intuitionists, at least if we allow them to restrict the verifier's strategies to known ones. No infinite operations are involved in playing the semantical games that are the logical home of unextended IF first-order logic.

In contrast, the substitutional interpretation quantifier (and its objectual counterparts) involve the given domain as a complete totality. For instance, the substitutional truth condition (21) requires that $W[b]$ is true, for each and every name b of a member of the domain. If different variables range over different classes of values, all the totalities of such values have to be considered as completed totalities. It has to be assumed, importantly, that all the individuals (members of the domain) have names. Thus substitutional truth conditions are infinitistic and nonconstructive in a way the game-theoretical truth predicate is not. With a side-glance at Hilbert, it may be said that the classes of values of the different first-order quantifiers are the only true "ideal objects" we need in mathematics.

What has been found has major repercussions for the traditional philosophy of mathematics. For one thing, it is seen that the main source of trouble is not the infinity of the domain of numbers. Semantical games can be played on infinite models and not only on finite ones. It makes sense to speak for instance of seeking and finding a rational number of a certain kind. Conversely, IF logic is one of the few rivals of the ordinary first-order logic that affects also the theory of finite models for first-order formulas. For such reasons, I am not calling the approach favored here finitistic. The infinity of the domain is not the main issue.

However, in contrast Brouwer seems to have been right in blaming a large part of the interpretational problems in the foundations on the unrestricted use of the tertium non datur principle in mathematics. For the substitutional interpretation quantifier with all its infinitistic burdens becomes unavoidable only when we have to interpret contradictory negations occurring within the scope of quantifiers.

This point must be pushed deeper, however. As I have argued elsewhere, intuitionistic logic should be thought of, not as the logic of mathematical truths per se, but as a logic of our knowledge of mathematical (and logical) entities. Now the game-theoretical truth definition shows that the crucial entities in the logic of mathematics are Skolem functions. Therefore we can hope to interpret sentences in a logical language intuitionistically only as long as their semantics can be formulated by reference to Skolem functions. (Of course we may have to restrict them to known functions.) Now it is precisely when we begin to use contradictory negation in arbitrary positions that we cannot any longer interpret our sentences by reference to Skolem functions.

However, Brouwer's insights have not been implemented in an instructive way in the earlier discussion. The best known way of attempting to do so has been to set up an "intuitionistic logic" to replace ordinary first-order logic. (see e.g. Heyting 1956.) But this is barking up the wrong logic. Of course ordinary first-order logic has to be generalized so as to become a fragment of IF first-order logic. But understood as such a fragment, there is nothing wrong with ordinary first-order logic.

On the other hand, even when an explicit intuitionistic logic is formulated, it does not do the job of IF first-order logic, either. It does not capture adequately the epistemic element in Brouwer's thinking, and more importantly it does not deal any better than ordinary first-order logic with the representation of dependence and independence relations between variables, either.

This failure is not automatically avoided by the introduction of the substitutional interpretation quantifier of quantifiers, either. Even though the substitutional interpretation quantifier is objectionable to an intuitionist, it does not make any difference in ordinary first-order logic. Hence Brouwer's objections to classical logic can be met by using a game-theoretical interpretation rather than a substitutional one. It is only when independence indicators are present that the two interpretations, the game-theoretical and the substitutional one, differ from each other. It is therefore only then that the tertium non datur principle begins to depend on distinctly infinitistic element into the logic of classical mathematics. It does so because the tertium non datur principle can only be backed up by the substitutional interpretation quantifier. But this is a much deeper issue than what can be handled by tinkering with the inference rules of ordinary first-order logic. Hence the real target of intuitionistic criticism ought to be the substitutional interpretation quantifier of quantifiers rather than the inference rules of first-order logic.

This point is somewhat obscured by the fact that prima facie failures of the law of excluded middle can also be caused by the epistemic element in intuitionistic logic. A failure of the law of excluded middle is hence merely a symptom of trouble. The need of a substitutional interpretation quantifier is the trouble.

The infinitistic character of substitutional first-order logic also makes it a poor candidate for the role of the true logic of ordinary language and ordinary discourse. Here I can in fact appeal to many of the standard finitistic arguments once one of the persisting mistakes in this area is eliminated. This mistake confuses the infinity of the domain with the infinity of the operations we need to carry out to apply our logic and our languages to it . This mistake is undoubtedly due to the more general mistake of assuming that the semantics of quantifiers must be explained by reference to their "ranging over" all the members of the domain. What matters is the question whether infinite operations are needed to apply our logic, not what the cardinality of the domain is to which it is being applied. As long as our logic is purely game-theoretical, its application can be thought of as being implemented by plays of semantical games. Such plays involve a finite number of moves that in principle can be carried out by human players. Hence the possible infinity of the domain does not matter.

Thus there is no obstacle to thinking of IF logic as the natural logic of our colloquial language. But the application of a substitutionally interpreted quantificational language in an infinite domain presupposes infinite operations. If so, it cannot very well be the logic of ordinary discourse for in our ordinary thinking we cannot even in principle rely on the assumption that infinite operations actually are carried out. IF logic is the natural logic of natural language, and thereby supports the solutions of the paradoxes discussed above.

14.8 Fully extended IF logic is equivalent to second-order logic

We nevertheless have to take the substitutional interpretation quantifier seriously in general logical theory. There are in fact interesting further insights to be reached here concerning first-order logics that rely on substitutional interpretation quantifier. We have defined a hierarchy of *first-order* sentences with an increasingly complex structure of nested *contradictory negations*. How is it related to the *quantificational* hierarchy ($\sum_n^1 - \prod_n^1$ hierarchy) of *second-order* sentences? The surprising answer turns out to be: the two hierarchies are equivalent.

The validity of this result is in fact fairly easy to see. It is well known that (and how) a $\sum_1^1$-sentence can be reduced to an IF first-order sentence. Furthermore, a $\prod_1^1$-sentence is equivalent to a sentence of the form $\neg S$, where S is an IF first-order sentence without contradictory negations. By the same token, if each $\sum_n^1$-sentence S is equivalent to a sentence of the full IF first-order logic with $n-1$ layers of contradictory negations, then clearly each $\prod_n^1$ sentence has an equivalent translation of the form $\neg S$.

Furthermore, consider a $\sum_n^1$ sentence. By definition, it has the form of a string of existential second-order quantifiers followed by a $\prod_{n-1}^1$ formula. Now these second-order existential quantifiers can be replaced by first-order independent quantifiers in the same way as in showing that each $\sum_1^1$ sentence is equivalent to an IF first-order sentence.

To illustrate this step, assume that the given $\sum_n^1$ sentence is

(23) $(\exists f)F[f]$

where f is a zero-argument function variable and $F[f]$ is a $\prod_{n-1}^{1}$ formula. It is assumed that $F[f]$ is in the negation normal form. Then (23) is clearly equivalent to the following sentence:

(24) $(\forall x)(\forall y)(\exists z/\forall y)(\exists u/\forall x)((x = y) \supset (z = u)) \,\&\, F^{*}[x, y, z, u])$

Here x, y, z, u are new variables not occurring in $F[f]$ and F^{*} is obtained from $F[f]$ as follows:

(i) Every occurrence of a subformula of the form $(f(w) = v)$ is replaced by $((x = w) \supset (z = v))$ and likewise for subformulas of the forms

$$(v = f(w))$$

$$(f(w) = a)\,(a = f(w)), \text{ etc}$$

(ii) Every occurrence of an atomic subformula of the form $A(f(w))$ is replaced by $((x = w) \supset (A(z)))$, and likewise for other kinds of atomic subformulas containing an argument of the form $f(w)$ or $f(a)$. Nested functions are handled in the same way as in the $\sum_{1}^{1}$ case. Predicates can be handled by means of their characteristic functions.

Thus the entire second-order logic turns out to be equivalent to the substitutional first-order logic. We shall call this result the *negation reduction* of second-order logic to the first-order level. (An essentially equivalent result is proved in Väänänen 2001.) It throws light on several questions in the foundations of logic and mathematics. One of them is the relation of first-order logic to higher-order logics. We may in fact think of this problem as one of the Einsteinian questions mentioned in the beginning of this paper, that is, questions that are so subtle that they appear trivial. The obvious-looking way of answering this question by saying that what distinguishes the two is the ontological status of the entities which our quantifiers range over in the two kinds of logic. In first-order logic, they are particulars (individuals); in the second-order logic, the values of quantified variables can also be sets or relations of particulars or functions from particulars to particulars. What can be simpler that this? Admittedly, in first-order axiomatic set theory sets are admitted as values of first-order variables. However, such a set theory is turning out to be a disaster area when considered as a foundational project. (see Hintikka, 2004(a).) Again, the peculiarities of higher-order logic come to play only when a standard interpretation in Henkin's sense is imposed on ranges of higher-order variables. If that is not done, higher-order logic can in effect be dealt with as if it were a many-sorted first-order logic. Apart from such qualifications, the distinction between first-order logic and higher-order logic seems to be exhaustively characterized by reference to the categorical (ontological) status of the values of the variables of quantification.

The result we have reached shows that the first-order vs. higher-order distinction is in reality more complicated than that. (This point has been emphasized aptly by Jouko Väänänen (2001).) The reduction to the first-order level marks a definite gain in conceptual clarity. Any philosophical nominalist will rejoice at this reduction. Among other things, the reduction shows that foundationally speaking we do not have to worry in second-order logic about the thorny question of the existence or nonexistence of different kinds of higher-order entities, such as sets. All we are trafficking in are different kinds of structures of particular objects. This satisfies one of the major desiderata of Hilbert (1996, p. 1121) who blamed the entire *Grundlagenkrise* on the use of higher-order entities by Frege, Dedekind and Cantor.

Hilbert's worry about mathematicians' reliance on higher-order quantification has not received the attention it deserves. Apparently the problem of the existence of higher-order objects has been tacitly transformed to the technical-looking question as to what existence assumptions to make in axiomatic set theory. The reappearance of serious problems in the foundations of set theory shows that this attempted transformation does not help us. The negation reduction shows that substitutional first-order logic satisfies Hilbert's wishes, even though it involves serious other problems.

In a different direction, the negation reduction vindicates the status of second-order logic as genuine logic. Since the existence of higher-order entities like sets plays no role in it, we do not need any set theory to back it up. This reinforces our reversal of Quine's quip. To deal with sets as if they were particular objects is to admit dangerous higher-order conceptualizations in sheep's clothing. (How very dangerous they are is shown in Hintikka, 2004(a).) In contrast, the prima facie dangerous second-order quantifiers turn out to be reducible in a sense to the sheepish first-order level.

14.9 Logic and mathematical reasoning

In a foundational perspective results like the negation reduction aid and abet mightily the cause of logicism. Admittedly, some of the earlier formulations of the tenets of logicism are now inapplicable, including those that claim that all the axioms we need in mathematics are theorems of a suitable axiomatization of logic. There is no such thing as a complete axiomatization of even the (unextended) IF first-order logic. In other words, our basic logic is semantically incomplete. Hence it makes no sense to speak of an axiomatic reduction of mathematics to logic. The real question is whether all the different conceptualizations and all the different modes of reasoning used in mathematics can be reconstructed by means of logic.

It is usually thought and said that all modes of reasoning needed in mathematics can be represented by means of second-order logic. If so, the negation reduction theorem shows that they can in at least one natural sense be reduced to modes of logical reasoning, which is precisely what logicists are supposed to claim.

This is perhaps not the last word on the subject, however. On the one hand, the substitutional first-order logic which is the target of the reduction is not unproblematic philosophically. On the other hand, it is not obvious that literally all assumptions that can be considered in mathematics can in fact be captured by means of second-order logic. For instance, it is not immediately clear that maximality assumptions like Hilbert's Axiom of Completeness (Hilbert 1903) can be so formulated. However, even apart from such qualifications, the results reached here, especially when they are combined with the realization of the failure of first-order axiomatic set theory to capture set-theoretical truths (see Hintikka 2004(a)), show impressively the fundamental role of logic in mathematical reasoning.

At the same time, the negation reduction raises our awareness of what separates concrete unproblematic reasoning from questionable one. What makes the difference was seen not to be the finitude of the domain. Now it is seen not to lie (Hilbert notwithstanding) in the first-order character of unproblematic reasoning, either, for all second-order reasoning and hence virtually all mathematical reasoning can in principle be conducted on first-order level. The crucial step is to allow contradictory negations into the scopes of quantifiers or, to use logicians' jargon, to "quantify into" a context governed by a contradictory negation.

The negation reduction theorem is of interest also from the vantage point of hierarchy theory. (For it see Addison 1961, and forthcoming). In this theory, different quantifier hierarchies are studied comparatively. Now by utilizing the notion of informational independence (independence of a quantifier on another one) it can be shown that certain important quantifier hierarchies are equivalent to hierarchies of contradictory negation. This seems to open a possibility of extending the scope of the entire hierarchy theory.

There has been in the literature some discussion of the question whether IF first-order logic is perhaps "really" (part of) higher-order logic. The results reached here show that the entire question is ill formulated. By the ontological criterion, IF first-order logic is unproblematically first-order, for all values of bound variables in its semantics are individuals (particular members of the domain). But if we do not go by this criterion, the notions of "first-order" and "second-order" have to be redefined. In the light of the results reached here, it might be maintained with a greater plausibility that the entire second-order logic is "in reality" but full IF first-order logic in a different notation. It seems that those philosophers who claim that IF logic is "in reality" second-order logic are tacitly requiring that the semantics of genuine first-order logic must be compositional and must rely exclusively on the "ranging over" idea. If so, the upshot of the line of though carried out here is to show how hopelessly restrictive such a conception of first-order is. Such a truncated first-order logic will not cut much ice even as the supposed logic of ordinary discourse.

It has to be admitted, however, that the borderline between first-order logic and second-order logic is much less sharp than first meets an untrained eye. This interplay of the two logics is manifested in the role of Skolem functions in the theory of first-order logic. It is also natural to generalize the rule of existential instantiation so as to allow the introduction of new function constants and not only new individual constants. (The instantiating "witness individuals" may depend on other individuals.) Moreover, first-order formulas can entail second-order formulas, even existential ones. This is exemplified by the fact that each first-order sentence logically implies its own Skolem form, as (9) implies

(25) $(\exists f)(\forall x)F[x, f(x)]$

Similar crossings are not found where two parts of logic are truly separated. For instance, no positive epistemic conclusion is implied by non-epistemic premises, and similarly for other parts of logic.

14.10 A prescriptive postscript

But what do all these results have to do with the title notion of this volume, the notion of alternative logic? The answer depends on what this singularly ill-defined term is taken to mean. In its most superficial sense, an alternative logic is any logic different from the received logic which is usually taken to be the "classical" first-order logic. What has been found in this paper (and in its predecessors) shows that in this sense the term "alternative logic" is an oxymoron. This is shown once and for all by the independence-friendly (IF logic) examined in this paper. IF logic is not an alternative to the received first-order logic. Rather, IF logic replaces the received "classical" first-order logic and accommodates it as a special case. The received first-order logic turns out to be a result of logicians' failure to acknowledge the important limitations that restrict the expressive power of the "classical" first-order logic. These limitations also show that the received first-order logic does not deserve this honorific appellation, unless the term "classical" is taken in one of its earlier senses as "what is taught in class-rooms".

In a somewhat less insipid sense an alternative logic is often taken to be a logic whose system of axioms and inference rules is different from the "classical" one. But what is a valid rule of inference? It is in its usual sense a rule that preserves truth. (It is already a symptom of an invidious confusion that in the literature the preservation of truth is not always distinguished from the preservation of logical truth.) Admittedly, in some cases what is to be preserved is merely probable truth or truthlikeness. However, those variations do not make an essential difference to the line of thought pursued here. If a putative rule of inference does not satisfy such a preservation requirement, it can scarcely serve any realistic purpose in the applications of logic and hence should not be called rule of inference. But if so, if truth-preservation is a condition sine qua non of a rule of inference, the validity of rules of inference has to be studied in a semantical theory of the language in which the inferences are couched, for the notion of truth belongs to semantics (model theory). More fully expressed, the study of inferences involving certain logical notions must turn on the role these notions play in the way reality is represented in language. Now what is in this perspective the semantical task of quantifiers, those central notions of our basic logic? It is usually thought that the semantical function of quantifiers is exhausted by their variables' "ranging over" a class of values. This idea is among other places epitomized by Frege's unfortunate idea that quantifiers are higher-order predicates whose task is to express whether lower-order predicates are empty or not.

In reality, this "ranging over" is only a part of the real job description of quantifiers. The other part of what quantifiers do is to express through their formal dependence or independence of each other the real-life dependence or independence of their respective variables. Once this is realized, it is easily seen that the received Frege-Russell notation does not allow the representation of all possible patterns of dependence and independence between variables. It hence fails to do full justice to the meaning of quantifiers. What IF logic does is to eliminate this shortcoming. Unlike the "classical" first-order logic, it fulfills the whole task of quantifier logic and not only a part of it.

Since this is precisely the task that any general logic of quantifiers has to accomplish, there cannot be any genuine alternatives to IF logic, either. What look like such alternatives, principally intuitionistic logic and constructivistic logic, can be construed as resulting from restricting the modes of dependence between variables, perhaps to knowable ones or constructive ones. If this is taken to be a sufficient reason to label them "alternative logics", there is no need to object, as long as their character as variants of IF logic is acknowledged.

Thus the verdict is clear as far as independence-friendly logic is concerned. IF logic is not a logic of certain kinds of games. It is a study of Skolem functions of quantifier sentences. These functions receive their significance from the fact that they codify relations of dependence between different variables. Rightly understood, IF logic is not alternative to any other logic, nor does it have any genuine alternatives.

This leaves most of the so-called alternative logics still unexplained. It is impossible to do justice to all of them here, but it might be in order to try to indicate what is interesting about them. As an example, what is known as the theory of circumscription will do. An inference from certain premises to a conclusion by circumscription relies, over and above the information that the premises convey, on a tacit assumption to the effect that the premises provide all the relevant information. This is a contingent assumption, not a logical or even necessarily a common sense truth. As any puzzle fan knows, often the solution of a puzzle requires precisely a violation of the sufficiency presumption in that it requires the presence of a factor not foreshadowed in the given information.

Now at first such inference might not seem to require any new principles of reasoning. Indeed, reasoning from partly tacit premises is one of the oldest topics of logical theory. It has an established name, viz. enthymemic reasoning. Why should circumscriptive reasoning nevertheless require a special alternative logic? Perhaps it does not. What is peculiar and interesting about it is that there does not seem to be any way of expressing the tacit sufficiency premise in the language in which the circumscriptive inferences are carried out.

The theory of circumscriptive inference is therefore an attempt to elicit information from a premise which is not only unspoken but unspeakable in the language used, by introducing special rules of inference. This is an intriguing enterprise, independently of whether it is deemed fully successful or not, but it need not involve a logic alternative to our old ones. It is a branch of the theory of enthymemic inference, viz. the branch which studies inferences in which the tacit premise is not expressible in the language in which the reasoning is carried out.

This can be generalized to several other highly interesting "logics". Probabilistic inductive logic depends on assumptions concerning the orderliness of one's universe of discourse. In the simplest case, such an assumption is codified in the constant of Carnap's λ-continuum. The choice of a value of λ codifies such a regularity or irregularity assumption, even when it is not expressed in the form of a proposition in the explicit language of inductive reasoning.

In mathematics, we find fascinating assumptions that are not expressible, or at least not easily expressible, by means of the usual mathematical and logical concepts in the mathematical notation itself. They are assumptions of extremality (maximality and minimality). Hilbert's struggles with his "Axiom of Completeness" in geometry vividly illustrate this problem. Extremality assumptions have not given rise to a new logic except perhaps in Hintikka (1993). They can nevertheless play the same role as the tacit premises of circumscriptive logic or inductive logic. What is common to all these "alternative logics" is that they are methods of eliciting consequences of certain tacit assumptions. They are not theories of inference in general; they are chapters of a theory of enthymemic reasoning. It is not even clear that they involve in the last analysis any peculiar modes of logical inference.

In the light of these observations, perhaps what you should do next time when you are tempted to use the expression "alternative logic", is to look for an alternative locution.

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