CRITICAL LINE OF THE DECONFINEMENT PHASE TRANSITIONS

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Abstract. Phase diagram of strongly interacting matter is discussed within the exactly solvable statistical model of the quark-gluon bags. The model predicts two phases of matter: the hadron gas at a low temperature T and baryonic chemical potential μ_B, and the quark-gluon gas at a high T and/or μ_B. The nature of the phase transition depends on a form of the bag mass-volume spectrum (its pre-exponential factor), which is expected to change with the μ_B/T ratio. It is therefore likely that the line of the 1^{st} order transition at a high μ_B/T ratio is followed by the line of the 2^{nd} order phase transition at an intermediate μ_B/T, and then by the lines of "higher order transitions" at a low μ_B/T. This talk is based on a recent paper (Gorenstein, Gaździcki, and Greiner, 2005).

1. Introduction

Investigation of the properties of strongly interacting matter at a high energy density is one of the most important subjects of the contemporary physics. In particular, the hypothesis that at high energy densities the matter is in the form of quark-gluon plasma (QGP) (Collins and Perry, 1975) rather than a gas of hadrons (HG) motivated a first stage of the broad experimental program of study of ultra-relativistic nucleus–nucleus collisions. Over the last 20 years rich data were collected by experiments located at Alternating Gradient Synchrotron (AGS) and Relativistic Heavy Ion Collider (RHIC) in Brookhaven National Laboratory, USA and at Super Proton Synchrotron (SPS) in CERN, Switzerland. The results indicate that the properties of the created matter change rapidly in the region of the low SPS energies (Afanasiev et al., 2002; Gaździcki et al., 2004), $E_{lab} = 30 - 80$ GeV/nucl. The observed effects are suggestive of a transition, or perhaps rapid cross-over, from a hadron gas to a new form of strongly interacting matter (Gaździcki and Gorenstein, 1999; Gorenstein et al., 2003).

What are the properties of the transition between the two phases of strongly interacting matter? This question motivates the second stage of the investigation of nucleus-nucleus collisions. Based on the numerous examples of the well-known substances a conjecture was formulated, see review paper

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(Stephanov, 2004) and references there in, that the transition from HG to QGP is a 1^{st} order phase transition at low values of a temperature T and a high baryo-chemical potential μ_B and it is a rapid cross–over at a high T and a low μ_B. The end point of the 1^{st} order phase transition line is expected to be the 2^{nd} order critical point. This hypothesis seems to be supported by QCD–based qualitative considerations (Stephanov, 2004) and first semi–quantitative lattice QCD calculations (Fodor and Katz, 2002; Karsch et al., 2004). The properties of the deconfinement phase transition are, however, far from being well established. This stimulates our study of the transition domain within the statistical model of quark-gluon bags.

The bag model (Chodos et al., 1974) was invented in order to describe the hadron spectrum, the hadron masses and their proper volumes. This model is also successfully used for a description of the deconfined phase, see e.g. (Shuryak, 1980; Cleymans et al., 1986). Thus the model suggests a possibility for a unified treatment of both phases. Any configuration of the system and, therefore, each term in the system partition function, can be regarded as a many-bag state both below and above the transition domain.

The important properties of the statistical bootstrap model with the van der Waals repulsion are summarized in Sect. 2. In Sect. 3 the statistical model of the quark–gluon bags is presented. Within this model properties of the transition region for $\mu_B = 0$ are studied in Sect. 4 and the analysis is extended to the complete $T - \mu_B$ plane in Sect. 5. The paper is closed by the summary and outlook given in Sect. 6.

2. Statistical bootstrap model and van der Waals repulsion

The grand canonical partition function for an ideal Boltzmann gas of particles of mass m and a number of internal degrees of freedom (a degeneracy factor) g, in a volume V and at a temperature T is given by:

$$
\begin{aligned}
Z(V,T) &= \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{j=1}^{N} \int \frac{gV\, d^3k_j}{(2\pi)^3} \exp\left[- \frac{(k_j^2 + m^2)^{1/2}}{T} \right] \\
&= \sum_{N=0}^{\infty} \frac{[V\, g\phi(T,m)]^N}{N!} = \exp[Vg\phi(T,m)]\,, \qquad (1)
\end{aligned}
$$

where

$$
\phi(T,m) \equiv \frac{1}{2\pi^2} \int_0^{\infty} k^2 dk\, \exp\left[- \frac{(k^2 + m^2)^{1/2}}{T} \right] = \frac{m^2 T}{2\pi^2} K_2\left(\frac{m}{T}\right) \qquad (2)
$$

and K_2 is the modified Bessel function. The function $\phi(T,m)$ is equal to the particle number density:

$$n(T) \equiv \frac{\overline{N}(V,T)}{V} = g\phi(T,m) . \tag{3}$$

The ideal gas pressure and energy density can be derived from Eq. (1) as:

$$p(T) \equiv T\,\frac{\ln Z(V,T)}{V} = T\; g\phi(T,m) , \tag{4}$$

$$\varepsilon(T) \equiv T\,\frac{dp}{dT} - p(T) = T^2\, g\frac{d\phi(T,m)}{dT} . \tag{5}$$

One can easily generalize the ideal gas formulation (1) to the mixture of particles with masses $m_1,...,m_n$ and degeneracy factors $g_1,...,g_n$:

$$\begin{aligned} Z(V,T) &= \sum_{N_1=0}^{\infty}\;...\;\sum_{N_n=0}^{\infty}\frac{[Vg_1\phi(T,m_1)]^{N_1}}{N_1!}\;...\;\frac{[Vg_n\phi(T,m_n)]^{N_n}}{N_n!} \\ &= \exp\left[V\sum_{j=1}^{n}g_j\phi(T,m_j)\right] . \end{aligned} \tag{6}$$

The sum over different particle species j can be extended to infinity. It is convenient to introduce the mass spectrum density $\rho(m)$, so that $\rho(m)dm$ gives the number of different particle mass states in the interval $[m, m+dm]$, i.e. $\sum_{j=1}^{\infty} g_j... = \int_0^{\infty} dm...\rho(m)$. In this case the pressure and the energy density are given by:

$$p(T) = T\int_0^{\infty} dm\,\rho(m)\,\phi(T,m) , \quad \varepsilon(T) = T^2\int_0^{\infty} dm\,\rho(m)\,\frac{d\phi(T,m)}{dT} . \tag{7}$$

Eqs. (7) were introduced by Hagedorn (Hagedorn, 1965) for the mass spectrum increasing exponentially for $m \to \infty$:

$$\rho(m) \simeq C\,m^{-a}\,\exp(bm) , \tag{8}$$

where a, b and C are model parameters. This form of the spectrum (8) was further derived from the statistical bootstrap model (Frautschi, 1971). It can be shown, that within this model the temperature $T_H \equiv 1/b$ (the “Hagedorn temperature”) is the maximum temperature of the matter. The behavior of thermodynamical functions (7) with the mass spectrum (8) depends crucially on the parameter a. In particular, in the limit $T \to T_H$ the pressure and the energy density approach:

$$p\,,\,\varepsilon \to \infty , \qquad\qquad \text{for } a \le \frac{5}{2} ; \tag{9}$$

$$p \;\rightarrow const\,,\; \varepsilon \;\rightarrow\; \infty\,, \qquad \text{for}\;\; \frac{5}{2} \;\leq\; a \;\leq\; \frac{7}{2}\,; \tag{10}$$

$$p\,,\; \varepsilon \;\rightarrow\; const\,, \qquad \text{for}\;\; a \;>\; \frac{7}{2}\,. \tag{11}$$

Up to here all particles including those with $m \rightarrow \infty$ were treated as point-like objects. Clearly this is an unrealistic feature of the statistical bootstrap model. It can be overcome by introduction of hadron proper volumes which simultaneously mimic the repulsive interactions between hadrons. The van der Waals excluded volume procedure can be applied for this purpose (Gorenstein et al., 1981). Other approaches were also discussed (Hagedorn and Rafelski, 1980; Kapusta, 1981). The volume V of the ideal gas (1) is substituted by the "available volume" $V - v_o N$, where v_o is a parameter which corresponds to a particle proper volume. The partition function then reads:

$$Z(V,T) \;=\; \sum_{N=0}^{\infty} \frac{[(V \;-\; v_o N)\; g\phi(T,m)]^N}{N!}\; \theta(V - v_o N)\,. \tag{12}$$

The pressure of the van der Waals gas can be calculated from the partition function (12) by use of its Laplace transform (Gorenstein et al., 1981; Rischke et al., 1991):

$$\hat{Z}(s,T) \;\equiv\; \int_0^{\infty} dV \exp(-sV)\; Z(V,T) \;=\; \frac{1}{s \;-\; \exp(-v_o s)\; g\phi(T,m)}\,. \tag{13}$$

In the thermodynamic limit, $V \rightarrow \infty$, the partition function behaves as $Z(V,T) \simeq \exp\left[pV/T\right]$. An exponentially increasing $Z(V,T)$ generates the farthest-right singularity $s^* \;=\; p/T$ of the function $\hat{Z}(s,T)$ in variable s. This is because the integral over V in Eq. (13) diverges at its upper limit for $s \;<\; p/T$. Consequently, the pressure can be expressed as

$$p(T) \;=\; T \lim_{V\rightarrow\infty} \frac{\ln Z(V,T)}{V} \;=\; T\; s^*(T)\,, \tag{14}$$

and the farthest-right singularity s^* of $\hat{Z}(s,T)$ (13) can be calculated from the transcendental equation (Gorenstein et al., 1981; Rischke et al., 1991):

$$s^*(T) \;=\; \exp\left[-\; v_o s^*(T)\right]\; g\phi(T,m)\,. \tag{15}$$

Note that the singularity s^* is not related to phase transitions in the system. Such a singularity exists for any statistical system. For example, for the ideal gas ($v_o \;=\; 0$ in Eq. (15)) $s^* \;=\; g\phi(T,m)$ and thus from Eq. (14) one gets $p = Tg\phi(T,m)$ which corresponds to the ideal gas equation of state (4).

3. Gas of quark-gluon bags

The van der Waals gas consisting of n hadronic species, which are called bags in what follows. is considered in this section. Its partition function reads:

$$Z(V,T) \;=\; \sum_{N_1=0}^{\infty} \ldots \sum_{N_n=0}^{\infty} \frac{[(V - v_1N_1 - \ldots - v_nN_n)\; g_1\phi(T,m_1)]^{N_1}}{N_1!} \times \ldots \tag{16}$$
$$\ldots \times \frac{[(V - v_1N_1 - \ldots - v_nN_n)\; g_n\phi(T,m_n)]^{N_n}}{N_n!}\, \theta(V - v_1N_1 - \ldots - v_nN_n)\;,$$

where $(m_1, v_1), \ldots, (m_n, v_n)$ are the masses and volumes of the bags. The Laplace transformation of Eq. (16) gives

$$\hat{Z}(s,T) \;=\; \left[s \;-\; \sum_{j=1}^{n} \exp\left(-v_j s\right)\; g_j\phi(T,m_j) \right]^{-1} . \tag{17}$$

As long as the number of types of bags, n, is finite, the only possible singularity of $\hat{Z}(s,T)$ (17) is its pole. However, in the case of an infinite number of types of bags the second singularity of $\hat{Z}(s,T)$ may appear. This case is discussed in what follows.

Introducing the bag mass-volume spectrum, $\rho(m,v)$, so that $\rho(m,v)dmdv$ gives the number of bag states in the mass-volume interval $[m, v; m + dm, v + dv]$, the sum over different bag states in definition of $Z(V,T)$ can be replaced by the integral, $\sum_{j=1}^{\infty} g_j \ldots = \int_0^{\infty} dmdv \ldots \rho(m,v)$. Then, the Laplace transform of $Z(V,T)$ reads (Gorenstein et al., 1981):

$$\hat{Z}(s,T) \;\;\equiv \int_0^{\infty} dV \exp(-sV)\, Z(V,T) \;=\; [\; s \;-\; f(T,s)]^{-1} \;, \tag{18}$$

where

$$f(T,s) \;\;=\; \int_0^{\infty} dmdv\, \rho(m,v)\; \exp(-vs)\; \phi(T,m)\;. \tag{19}$$

The pressure is again given by the farthest-right singularity: $p(T) = Ts^*(T)$. One evident singular point of $\hat{Z}(s,T)$ (18) is the pole singularity, $s_H(T)$:

$$s_H(T) \;=\; f\,(T, s_H(T))\;. \tag{20}$$

As mentioned above this is the only singularity of $\hat{Z}(s,T)$ if one restricts the mass-volume bag spectrum to a finite number of states. For an infinite number of mass-volume states the second singular point of $\hat{Z}(s,T)$ (18), $s_Q(T)$, can emerge, which is due to a possible singularity of the function $f(T,s)$ (19) itself. The system pressure takes then the form:

$$p(T) \;=\; Ts^*(T) \;=\; T \cdot max\{s_H(T), s_Q(T)\}\,, \tag{21}$$

and thus the farthest-right singularity $s^*(T)$ of $\hat{Z}(s,T)$ (18) can be either the pole singularity $s_H(T)$ (20) or the $s_Q(T)$ singularity of the function $f(T,s)$ (19) itself. The mathematical mechanism for possible phase transition (PT) in our model is the "collision" of the two singularities, i.e. $s_H(T) = s_Q(T)$ at PT temperature $T = T_C$ (see Fig. 1). In physical terms this can be interpreted as the existence of two phases of matter, namely, the hadron gas with the pressure, $p_H = T s_H(T)$, and the quark gluon plasma with the pressure $p_Q = T s_Q(T)$. At a given temperature T the system prefers to stay in a phase with the higher pressure. The pressures of both phases are equal at the PT temperature T_C.

An important feature of this modeling of the phase transition should be stressed here. The transition, and thus the occurrence of the two phases of matter, appears as a direct consequence of the postulated general partition function (a single equation of state). Further on, the properties of the transition, e.g. its location and order, follow from the partition function and are not assumed. This can be confronted with the well-known phenomenological construction of the phase transition, in which the existence of the two different phases of matter and the nature of the transition between them are postulated.

The crucial ingredient of the model presented here which defines the presence, location and the order of the PT is the form of the mass-volume spectrum of bags $\rho(m,v)$. In the region where both m and v are large it can be described within the bag model (Chodos et al., 1974). In the simplest case of a bag filled with the non-interacting massless quarks and gluons one finds (Gorenstein et al., 1982):

$$\rho(m,v) \simeq C\, v^{\gamma}(m-Bv)^{\delta}\, \exp\left[\frac{4}{3}\, \sigma_Q^{1/4}\, v^{1/4}\, (m-Bv)^{3/4}\right], \qquad (22)$$

where C, γ, δ and B, the so-called bag constant, $B \approx 400$ MeV/fm^3 (Shuryak, 1980), are the model parameters and

$$\sigma_Q = \frac{\pi^2}{30}\left(g_g + \frac{7}{8} g_{q\bar{q}}\right) = \frac{\pi^2}{30}\left(2\cdot 8 + \frac{7}{8}\cdot 2\cdot 2\cdot 3\cdot 3\right) = \frac{\pi^2}{30}\,\frac{95}{2} \qquad (23)$$

is the Stefan-Boltzmann constant counting gluons (spin, color) and (anti-) quarks (spin, color and u, d, s-flavor) degrees of freedom. This is the asymptotic expression assumed to be valid for a sufficiently large volume and mass of a bag, $v > V_0$ and $m > Bv + M_0$. The validity limits can be estimated to be $V_0 \approx 1$ fm^3 and $M_0 \approx 2$ GeV (Gorenstein and Lipskih, 1983; Gorenstein, 1984). The mass-volume spectrum function:

$$\rho_H(m,v) = \sum_{j=1}^{n} g_j\, \delta(m-m_j)\, \delta(v-v_j) \qquad (24)$$

should be added to $\rho(m, v)$ in order to reproduce the known low-lying hadron states located at $v < V_0$ and $m < BV_0+M_0$. The mass spectra of the resonances are described by the Breit-Wigner functions. Consequently, a general form of $f(T, s)$ (19) reads:

$$f(T,s) \equiv f_H(T,s) + f_Q(T,s) \tag{25}$$
$$= \sum_{j=1}^{n} g_j \exp(-v_j s)\, \phi(T,m_j) + \int_{V_o}^{\infty} dv \int_{M_o+Bv}^{\infty} dm\, \rho(m,v) \exp(-sv)\phi(T,m)\,,$$

where $\rho(m, v)$ is given by Eq. (22).

The behavior of $f_Q(T, s)$ is discussed in the following. The integral over mass in Eq. (25) can be calculated by the steepest decent estimate. Using the asymptotic expansion of the K_2-function one finds $\phi(T,m) \simeq (mT/2\pi)^{3/2}$ $\exp(-m/T)$ for $m \gg T$. A factor exponential in m in the last term of Eq. (25) is given by:

$$\exp\left[-\frac{m}{T} + \frac{4}{3}\sigma_Q^{1/4} v^{1/4} (m - Bv)^{3/4}\right] \equiv \exp[U(m)]\,. \tag{26}$$

The function $U(m)$ has a maximum at:

$$m_0 = \sigma_Q T^4 v + Bv\,. \tag{27}$$

Presenting $U(m)$ as

$$U(m) \simeq U(m_0) + \frac{1}{2}\left(\frac{d^2U}{dm^2}\right)_{m=m_0} (m - m_0)^2 = v s_Q(T) - \frac{(m-m_0)^2}{8\sigma_Q\, vT^5}, \tag{28}$$

with

$$s_Q(T) \equiv \frac{1}{3}\,\sigma_Q\, T^3 - \frac{B}{T}\,, \tag{29}$$

one finds

$$f_Q(T,s) \simeq u(T) \int_{V_0}^{\infty} dv\, v^{2+\gamma+\delta} \exp\left[-v\left(s - s_Q(T)\right)\right]\,, \tag{30}$$

where $u(T) = C\pi^{-1}\sigma_Q^{\delta+1/2}\, T^{4+4\delta} \left(\sigma_Q T^4 + B\right)^{3/2}$. The function $f_Q(T, s)$ (30) has the singular point $s = s_Q$ because for $s < s_Q$ the integral over dv diverges at its upper limit.

The first term of Eq. (25), f_H, represents the contribution of a finite number of low-lying hadron states. This function has no s-singularities at any temperature T. The integration over the region $m, v \to \infty$ generates the singularity $s_Q(T)$ (29) of the $f_Q(T, s)$ function (30). As follows from the discussion below this singularity should be associated with the QGP phase.

By construction the function $f(T, s)$ (19) is a positive one, so that $s_H(T)$ (20) is also positive. On the other hand, it can be seen from Eq. (29) that at a low T $s_Q(T) < 0$. Therefore, $s_H > s_Q$ at small T, and according to Eq. (21) it follows:

$$p(T) \; = \; T\, s_H \,, \quad \varepsilon(T) \; = \; T^2\, \frac{ds_H}{dT} \,. \tag{31}$$

The system of the quark-gluon bags is in a hadron phase.

If two singularities "collide", $T = T_C$ and $s_H(T_C) = s_Q(T_C)$, the singularity $s_Q(T)$ can become the farthest-right singularity at $T > T_C$. In Fig. 1 the dependence of the function $f(T, s)$ on s and its singularities are sketched for $T_1 < T_2 = T_C < T_3$. The thermodynamical functions defined by the singularity $s_Q(T)$ are:

$$p(T) \; = \; T\, s_Q \; = \; \frac{\sigma_Q}{3}\, T^4 \; - \; B \,, \quad \varepsilon(T) \; = \; T^2\, \frac{ds_Q}{dT} \; = \; \sigma_Q\, T^4 \; + \; B \tag{32}$$

and thus they describe the QGP phase.

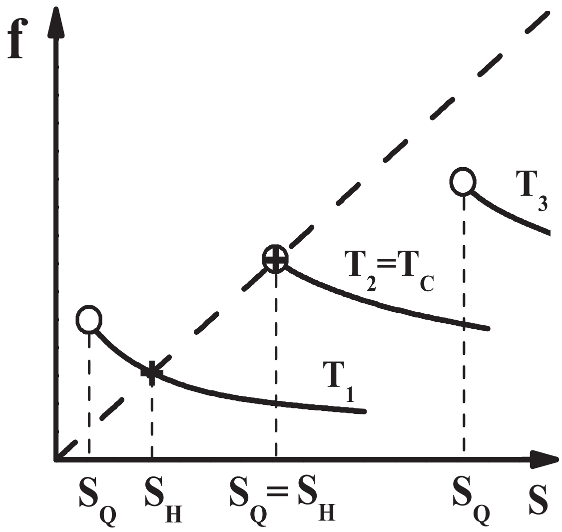

Figure 1. The dependence of $f(T, s)$ on s for three different temperatures: $T_1 < T_2 = T_C < T_3$ (solid lines). The pole singularity s_H and the singularity s_Q are denoted by circles and crosses, respectively. A PT corresponds to the "collision" of two singularities $s_H = s_Q$ at the temperature T_C.

The existence and the order of the phase transition depend on the values of the parameters of the model. From Eq. (30) follows that a PT exists, i.e. $s^* = s_Q$ at high T, provided $\gamma + \delta < -3$ (otherwise $f(T, s_Q) = \infty$, and $s_H > s_Q$ for all T, for illustration see Fig. 2, left).

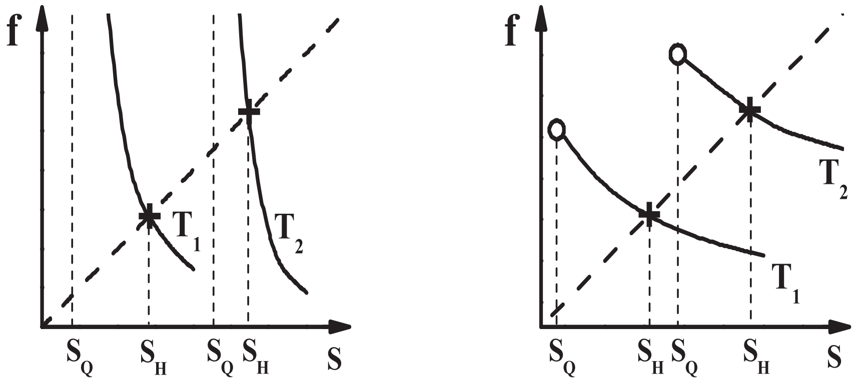

Figure 2. The dependence of $f(T, s)$ on s for two different temperatures, $T_1 < T_2$ (solid lines). The figures demonstrate the absence of the PT in the model if $f(T, s_Q) = \infty$ (left), or if $f(T, s_Q) > s_Q$ at all T (right).

In addition it is required that $f(T, s_Q) < s_Q(T)$ at $T \to \infty$, (otherwise $s^* = s_H > s_Q$ for all T, for illustration see Fig. 2, right). For $\gamma + \delta < -3$ one finds $f_Q(T, s_Q) \propto T^{10+4\delta}$ at $T \to \infty$. On the other hand, $s_Q(T) \propto T^3$ at a high T and therefore $\delta < -7/4$. Consequently the general conditions for the existence of any phase transition in the model are:

$$\gamma < -\frac{5}{4}, \qquad \delta < -\frac{7}{4}. \tag{33}$$

4. First, second, and higher order phase transitions

In this section the order of the PT in the system of quark-gluon bags is discussed.

The 1^{st} order PT takes place at $T = T_C$ provided $s_Q(T_C) = s_H(T_C)$ and:

$$\left(\frac{ds_Q}{dT}\right)_{T=T_C} > \left(\frac{ds_H}{dT}\right)_{T=T_C}. \tag{34}$$

Thus the energy density $\varepsilon = T^2 ds/dT$ has discontinuity (latent heat) at the 1^{st} order PT. Its dependence on T is shown in Fig. 3, left. In calculating ds_H/dT it is important to note that the function $s_H(T)$ (20) is only defined for $T \leq T_C$, i.e. for $s_H(T) \geq s_Q(T)$.

The 2^{nd} order PT takes place at $T = T_C$ provided $s_Q(T_C) = s_H(T_C)$ and:

$$\left(\frac{ds_Q}{dT}\right)_{T=T_C} = \left(\frac{ds_H}{dT}\right)_{T=T_C}, \quad \left(\frac{d^2 s_H}{d^2T}\right)_{T=T_C} \neq \left(\frac{d^2 s_Q}{d^2T}\right)_{T=T_C}. \tag{35}$$

Hence the energy density is a continuous function of T, but its first derivate has a discontinuity, for illustration see Fig. 3, right.

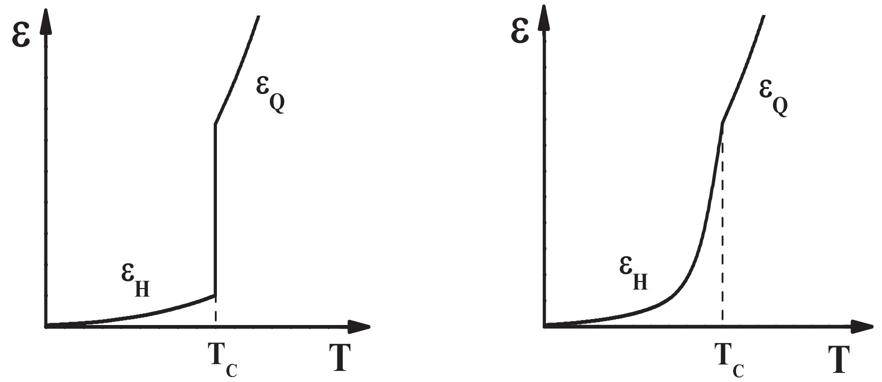

Figure 3. The dependence of the energy density ε on temperature for the 1^{st} (left) and 2^{nd} (right) order PT. The energy density $\varepsilon(T)$ has a discontinuity at $T = T_C$ for the 1^{st} order PT whereas for the 2^{nd} order PT $\varepsilon(T)$ is a continuous function of T, but $d\varepsilon/dT$ has a discontinuity at $T = T_C$.

What is the order of the PT in the model? To answer this question Eq. (20) should be rewritten as:

$$s_H = f_H + u \int_{V_o}^{\infty} dv v^{-\alpha} \exp\left[-v\left(s_H - s_Q\right)\right] , \qquad (36)$$

where $\alpha \equiv -(\gamma + \delta + 2) > 1$. Differentiating both sides with respect to T (the prime denotes the d/dT derivative) one gets:

$$\begin{aligned} s'_H &= f'_H + u' \int_{V_o}^{\infty} dv v^{-\alpha} \exp\left[-v\left(s_H - s_Q\right)\right] \\ &+ u \int_{V_o}^{\infty} dv v^{-\alpha+1} \left(s'_Q - s'_H\right) \exp\left[-v\left(s_H - s_Q\right)\right] , \end{aligned} \qquad (37)$$

from which follows:

$$s'_H = \frac{G + F \cdot s'_Q}{1 + F} , \qquad (38)$$

where

$$\begin{aligned} G &\equiv f'_H + u' \int_{V_o}^{\infty} dv v^{-\alpha} \exp\left[-v\left(s_H - s_Q\right)\right] , \\ F &\equiv u \int_{V_o}^{\infty} dv v^{-\alpha+1} \exp\left[-v\left(s_H - s_Q\right)\right] . \end{aligned} \qquad (39)$$

It is easy to see that the transition is of the 1^{st} order, i.e. $s'_Q(T_C) > s'_H(T_C)$, provided $\alpha > 2$. The 2^{nd} or higher order phase transition takes place provided

$s'_Q(T_C) = s'_H(T_C)$ at $T = T_C$. This condition is satisfied when $F(T)$ diverges to infinity at $T \to (T_C - 0)$, i.e. for T approaching T_C from below.

One notes that the exponential factor in $\rho(m, v)$ (22) generates the singularity $s_Q(T)$ (29) of the $\hat{Z}(s, T)$ function (18). Due to this there is a possibility of a phase transition in the model. Whether it does exist and what the order is depends on the values of the parameters γ and δ in the pre-exponential power-like factor, $v^\gamma(m - Bv)^\delta$, of the mass-volume bag spectrum $\rho(m, v)$ (22). This resembles the case of the statistical bootstrap model discussed in Sec. II. The limiting temperature $T_H = 1/b$ appears because of the exponentially increasing factor $\exp(bm)$, in the mass spectrum (8), but the thermodynamical behavior (9-11) at $T = T_H$ depends crucially on the value of the parameter a in the pre-exponential factor, m^{-a} (8).

What is the physical difference between $\alpha > 2$ and $1 < \alpha \leq 2$ in the model? From Eq. (36) it follows that the volume distribution function of quark-gluon bags at $T < T_C$ has the form

$$W(v) \propto v^{-\alpha} \exp\left[-v\left(s_H - s_Q\right)\right] . \tag{40}$$

Consequently the average bag volume:

$$\bar{v} = \int_{V_o}^{\infty} dv\, v\, W(v) , \tag{41}$$

at $T \to (T_Q - 0)$ approaches:

$$\bar{v} = const , \qquad \text{for} \quad \alpha > 2 ; \tag{42}$$

$$\bar{v} \to \infty , \qquad \text{for} \quad 1 < \alpha \leq 2 . \tag{43}$$

Thus in the vicinity of the 1^{st} order PT the finite volume bags (hadrons) dominate at $T < T_C$. There is a single infinite volume bag (QGP) at $T > T_C$ and a mixed phase at $T = T_C$ (Gorenstein et al., 1998). For the 2^{nd} order and/or “higher order transitions” the dominant bag configurations include the large volume bags already in a hadron phase $T < T_C$, and the average bag volume increases to infinity at $T \to (T_C - 0)$.

The condition for the 2^{nd} order PT can be derived as following. The integral for the function F reads

$$\int_{V_o}^{\infty} dv v^{-\alpha+1} \exp\left[-v\left(s_H - s_Q\right)\right] = \left(s_H - s_Q\right)^{-2+\alpha}\, \Gamma\left[2 - \alpha, (s_H - s_Q)V_o\right] ,$$

where $\Gamma(k, x)$ is the incomplete Gamma-function. Thus using Eq. (38) one finds at $T \to (T_C - 0)$:

$$s'_Q - s'_H \propto \left(s_H - s_Q\right)^{2-\alpha} , \qquad \text{for} \quad \alpha < 2 , \tag{44}$$

$$s'_Q - s'_H \propto -\ln^{-1}\left(s_H - s_Q\right) , \qquad \text{for} \quad \alpha = 2 , \tag{45}$$

and consequently:

$$s''_H - s''_Q \propto (s_H - s_Q)^{3-2\alpha} \qquad \text{for} \qquad \alpha < 2\,; \tag{46}$$

$$s''_H - s''_Q \propto -\frac{\ln^{-3}(s_H - s_Q)}{s_H - s_Q} \qquad \text{for} \qquad \alpha = 2\,. \tag{47}$$

Therefore for $3/2 < \alpha \leq 2$ the 2^{nd} order PT with $s''_H(T_C) = \infty$ takes place whereas for $\alpha = 3/2$ the 2^{nd} order PT with the finite value of $s''_H(T_C)$ is observed. The dependence of the specific heat $C \equiv d\varepsilon/dT$ on T is shown in Fig. 4, left.

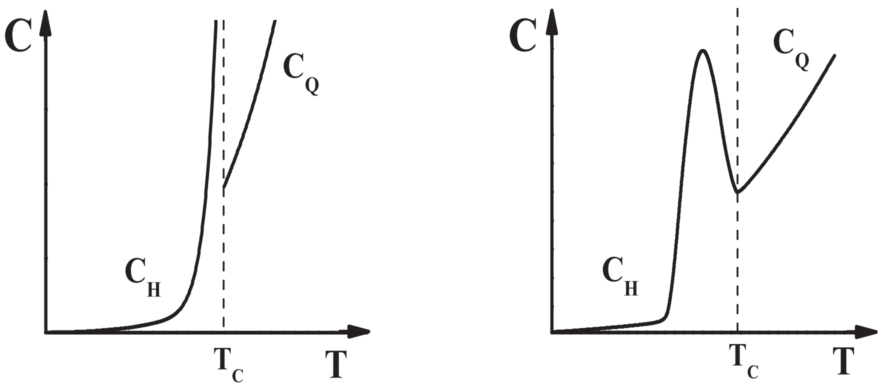

Figure 4. The dependence of the specific heat on temperature for the 2^{nd} order PT (left) and for the 3^{rd} order PT (right). Large values of C in the hadron phase reflect the fact that large derivative $d\varepsilon_H/dT$ at T close to T_C is needed for the 2^{nd} and 3^{rd} order PT to reach the value of $\varepsilon_Q(T_C)$ without energy density discontinuity.

An infinite value of a specific heat C at $T \to (T_C - 0)$ obtained within the model for $3/2 < \alpha \leq 2$ is typical for the 2^{nd} order PTs.

From Eq. (46) it follows that $s''_H(T_C) = s''_Q(T_C)$ for $\alpha < 3/2$. Using Eqs. (44, 46) one finds at $T \to (T_C - 0)$:

$$s'''_H - s'''_Q \propto (s_H - s_Q)^{4-3\alpha}\,. \tag{48}$$

Thus, for $4/3 \leq \alpha < 3/2$ there is a 3^{rd} order transition with $s_Q(T_C) = s_H(T_C)$ and:

$$s'_H(T_C) = s'_Q(T_C)\,, \quad s''_H(T_C) = s''_Q(T_C)\,, \quad s'''_H(T_C) \neq s'''_Q(T_C)\,, \tag{49}$$

with $s'''_H(T_C) = \infty$ for $4/3 < \alpha < 3/2$ and with a finite value of $s'''_H(T_C)$ for $\alpha = 4/3$. The dependence of the specific heat $C \equiv d\varepsilon/dT$ on temperature for the 3^{rd} order transition is shown in Fig. 4, right.

By calculating higher order derivatives of s_H and s_Q with respect to T it can be shown that for $(n+1)/n \leq \alpha < n/(n-1)$ $(n = 4, 5, ...)$ there is a n^{th} order transition with $s_Q(T_C) \ = \ s_H(T_C)$ and:

$$s'_H(T_C) \ = \ s'_Q(T_C) \ , \ \ ... \ \ s_H^{(n-1)}(T_C) \ = \ s_Q^{(n-1)}(T_C) \ , \ s_H^{(n)}(T_C) \ \neq \ s_Q^{(n)}(T_C), \quad (50)$$

with $s_H^{(n)}(T_C) = \infty$ for $(n+1)/n < \alpha < n/(n-1)$ and with a finite value of $s_H^{(n)}(T_C)$ for $\alpha = (n+1)/n$.

The 3^{rd} and higher order PTs correspond to a continuous specific heat function C with its maximum at T near T_C (see Fig. 4, right). This maximum appears due to the fact that large values of derivative $d\varepsilon_H/dT$ at T close to T_C are needed to reach the value of $\varepsilon_Q(T_C)$ without discontinuities of energy density and specific heat. The so called crossover point is usually defined as a position of this maximum. Note that in the present model the maximum of a specific heat is always inside the hadron phase.

5. Non-zero baryonic number

At a non-zero baryonic density the grand canonical partition function for the system of quark-gluon bags can be presented in the form

$$Z(V,T,\mu_B) \ = \ \equiv \sum_{b=-\infty}^{\infty} \ \exp\left(\frac{b\,\mu_B}{T}\right) \ Z(V,T,B) \ , \quad (51)$$

where μ_B is the baryonic chemical potential (for simplicity strangeness is neglected in the initial discussion). The Laplace transform of $Z(V,T,\mu_B)$ (51) reads (Gorenstein et al., 1998)

$$\hat{Z}(s,T,\mu_B) \ \equiv \ \int_0^\infty dV \exp(-\ sV) Z(V,T,\mu_B) \ = \ \frac{1}{s \ - \ f(s,T,\mu_B)} \ , \quad (52)$$

where

$$f(T,\mu_B,s) \ = \ f_H(T,\mu_B,s) + \int_{V_o}^\infty dv \int_{M_o+Bv}^\infty dm \, \rho(m,v;\mu_B/T) \ \exp(-sv)\phi(T,m) \ , \quad (53)$$

with

$$f_H(T,\mu_B,s) \ = \ \sum_{j=1}^{n} g_j \ \exp\left(\frac{b_j\mu_B}{T}\right) \ \exp(-v_j s) \ \phi(T,m_j) \ , \quad (54)$$

$$\rho(m,v;\mu_B/T) \ \equiv \ \sum_{b=-\infty}^{\infty} \exp\left(\frac{b\mu_B}{T}\right) \rho(m,v;b) \ . \quad (55)$$

Similar to the case of $\mu_B = 0$ discussed in the previous sections one finds that the pressure is defined by the farthest-right singularity, s^*:

$$p(T,\mu_B) \;=\; \lim_{V\to\infty} \frac{T}{V} \ln Z(V,T,\mu_B) \;=\; T\ s^*(T,\mu_B) \;=\; T \cdot max\{s_H,\ s_Q\}\ ,(56)$$

and it can be either given by the pole singularity, s_H:

$$s_H(T,\mu_B) \;=\; f(s_H,T,\mu_B)\ , \tag{57}$$

or the singularity s_Q of the function $f(s,T,\mu_B)$ (53) itself:

$$\begin{aligned} s_Q(T,\mu_B) \quad &= \frac{\pi^2}{90} T^3 \left[\frac{95}{2} + \frac{10}{\pi^2} \left(\frac{\mu_B}{T}\right)^2 + \frac{5}{9\pi^4} \left(\frac{\mu_B}{T}\right)^4 \right] - \frac{B}{T} \\ &\equiv \frac{1}{3}\, \overline{\sigma}_Q(\mu_B)\, T^3 - \frac{B}{T}\ . \end{aligned} \tag{58}$$

Note that for $\mu_B = 0$ Eq. (58) is transformed back to $s_Q(T)$ (29), as $\overline{\sigma}_Q(\mu_B = 0) = \sigma_Q$. The energy density and baryonic number density are equal to

$$\varepsilon(T,\mu_B) \quad = T^2\, \frac{\partial s^*(T,\mu_B)}{\partial T} + T\mu_B\, \frac{\partial s^*(T,\mu_B)}{\partial \mu_B}\ , \tag{59}$$

$$n_B(T,\mu_B) \quad = T\, \frac{\partial s^*(T,\mu_B)}{\partial \mu_B}\ . \tag{60}$$

The second term in Eq. (53), the function $f_Q(T,\mu_B,s)$, can be approximated as

$$f_Q(T,s,\mu_B) \;\simeq\; u(T,\mu_B/T) \int_{V_o}^{\infty} dv\ v^{-\alpha}\ \exp\left[-v\left(s - s_Q(T,\mu_B)\right)\right]\ , \tag{61}$$

i.e., it has the same form as $f_Q(T,s)$ (30). At $\mu_B/T = const$ and $T \to \infty$ one finds $s_Q(T,\mu_B) \propto T^3$ and $u(T,\mu_B/T) \propto T^{10+4\delta}$. This is the same behavior as in the case of $\mu_B = 0$, (30). At a small T and μ_B one finds $s_H > s_Q$, so that the farthest-right singularity s^* equals to s_H. This pole-like singularity, $s_H(T,\mu_B)$, of the function $\hat{Z}(s,T,\mu_B)$ (52) should be compared with the singularity $s_Q(T,\mu_B)$ of the function $f_Q(T,s,\mu_B)$ (53) itself. The dependence of s_Q on the variables T and μ_B is known in an explicit form (58). If conditions (33) are satisfied it can be shown that with an increasing T along the lines of $\mu_B/T = const$ one reaches the point $T_C(\mu_B)$ at which the two singularities collide, $s_Q = s_H$, and $s_Q(T,\mu_B)$ becomes the farthest-right singularity of the function $\hat{Z}(s,T,\mu_B)$ (52) at $T \to \infty$ and fixed μ_B/T. Therefore, the line of phase transitions $s_H(T,\mu_B) = s_Q(T,\mu_B)$ appears in the $T - \mu_B$ plane. Below the phase transition line one observes $s_H(T,\mu_B) > s_Q(T,\mu_B)$ and the system is in a hadron phase. Above this line the singularity $s_Q(T,\mu_B)$ becomes the farthest-right singularity of the function $\hat{Z}(s, T, \mu_B)$ (52) and the system is in a QGP phase. The corresponding thermodynamical functions are given by:

$$p(T,\mu_B) \;=\; T\; s_Q(T,\mu_B) \;=\; \frac{\pi^2}{90}\cdot\frac{95}{2}\,T^4 \;+\; \frac{1}{9}\,\mu_B^2 T^2 \;+\; \frac{1}{162\pi^2}\,\mu_B^4 - B, \quad (62)$$

$$\varepsilon(T,y_B) \;=\; T^2\,\frac{\partial s_Q(T,\mu_B)}{\partial T} \;+\; T\mu_B\,\frac{\partial s_Q(T,\mu_B)}{\partial \mu_B} \;=\; 3\,p(T,\mu_B) \;+\; 4B\,, \quad (63)$$

$$n_B(T,\mu_B) \;=\; T\,\frac{\partial s_Q(T,\mu_B)}{\partial \mu_B} \;=\; \frac{2}{9}\,\mu_B\,T^2 \;+\; \frac{2}{81\pi^2}\mu_B^3\,. \quad (64)$$

The analysis similar to that in the previous section leads to the conclusion that one has the 1^{st} order PT for $\alpha > 2$ in Eq. (61), for $3/2 \le \alpha \le 2$ there is the 2^{nd} order PT, and, in general, for $(n+1)/n \le \alpha < n/(n-1)$ $(n = 3, 4, ...)$ there is a n^{th} order transition. Note that $s_H(T, \mu_B)$ found from by Eq. (57) is only weakly dependent on α. This means that for $\alpha > 1$ the hadron gas pressure $p_H = T s_H$ and thus the position of the phase transition line,

$$s_H(T,\mu_B) \;=\; s_Q(T,\mu_B)\,, \quad (65)$$

in the $T - \mu_B$ plane is not affected by the contribution from the large volume bags. The main contribution to s_H (57) comes from small mass (volume) bags, i.e. from known hadrons included in f_H (54). This is valid for all $\alpha > 1$, so that the line (65) calculated within the model is similar for transitions of different orders. On the other hand, the behavior of the derivatives of s_H (57) with respect to T and/or μ_B near the critical line (65), and thus the order of the phase transition, may crucially depend on the contributions from the quark-gluon bags with $v \to \infty$. For $\alpha > 2$ one observes the 1^{st} order PT, but for $1 < \alpha \le 2$ the 2^{nd} and higher order PTs are found. In this latter case the energy density, baryonic number density and entropy density have significant contribution from the large volume bags in the hadron phase near the PT line (65).

The actual structure of the "critical" line on the $T - \mu_B$ plane is defined by a dependence of the parameter α on the μ_B/T ratio. This dependence can not be reliably evaluated within the model and thus an external information is needed in order to locate the predicted "critical" line in the phase diagram. The lattice QCD calculations indicate that at zero μ_B there is rapid but smooth cross-over. Thus this suggests a choice $1 < \alpha < 3/2$ at $\mu_B = 0$, i.e. the transition is of the 3^{rd} or a higher order. Numerous models predict the strong 1^{st} order PT at a high μ_B/T ratio, thus $\alpha > 2$ should be selected in this domain. As a simple example in which the above conditions are satisfied one may consider a linear dependence, $\alpha = \alpha_0 + \alpha_1\mu_B/T$, where $\alpha_0 = 1 + \epsilon$ $(0 < \epsilon << 1)$ and $\alpha_1 \approx 0.5$. Then the line of the 1^{st} order PT at a high μ_B/T

ends at the point $\mu_B/T \approx 2$, where the line of the 2^{nd} order PT starts. Further on at $\mu_B/T \approx 1$ the lines of the 3^{rd} and higher order transitions follow on the "critical" line. This hypothetical "critical" line of the deconfinement phase transition in the $T - \mu_B$ plane is shown in Fig. 5.

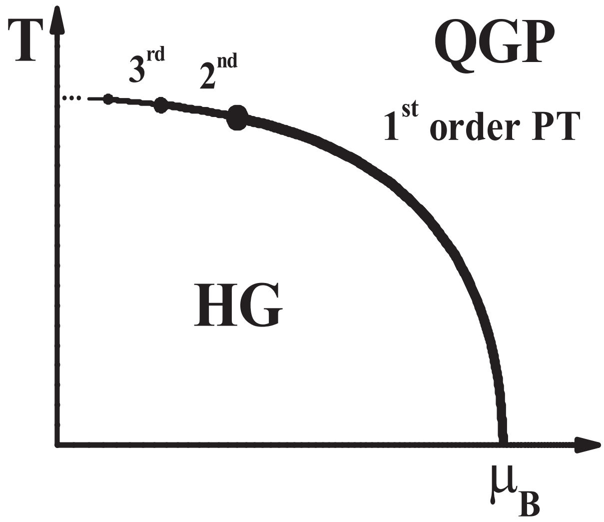

Figure 5. The hypothetical phase diagram of strongly interacting matter in the $T - \mu_B$ plane within the quark-gluon bag model. The influence of strangeness is neglected. The line of the 1^{st} order phase transition at a high μ_B/T ratio is followed by the line of the 2^{nd} order PT at an intermediate μ_B/T values and by the lines of higher order PTs at a low μ_B/T.

In the case of the non-zero strange chemical potential μ_S the pole singularity, s_H, and the singularity s_Q become dependent on μ_S. The system created in nucleus-nucleus collisions has zero net strangeness and consequently,

$$n_S(T,\mu_B,\mu_S) = T\,\frac{\partial s^*(T,\mu_B,\mu_S)}{\partial \mu_S} = 0\,. \tag{66}$$

At a small T and μ_B, when $s_H > s_Q$, Eq. (66) with $s^* = s_H$ defines the strange chemical potential $\mu_S = \mu_S^H(T,\mu_B)$ which guarantees a zero value of the net strangeness density in a hadron phase. When the singularity s_Q becomes the farthest-right singularity the requirement of zero net strangeness (66) with $s^* = s_Q$ leads to $\mu_S = \mu_S^Q = \mu_B/3$. The functions $\mu_S^H(T,\mu_B)$ and $\mu_S^Q(T,\mu_B) = \mu_B/3$ are different. Consequently, the line (65) of the 1^{st} order PT in the $T - \mu_B$ plane is transformed into a "strip" (Lee and Heinz, 1993). The phase diagram in which influence of the strangeness is taken into account is shown schematically in Fig. 5.

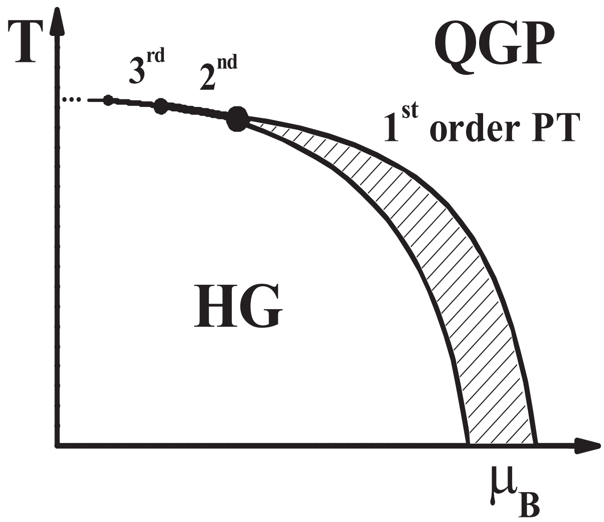

Figure 6. The hypothetical phase diagram of strongly interacting matter in the $T - \mu_B$ plane with strangeness taken into account. The line of the 1^{st} order phase transition is transformed into a "strip". For further details see caption of Fig. 5.

The lower and upper limits of the "strip" are defined, respectively, by the conditions:

$$s_H(T,\mu_B,\mu_S^H) \;=\; s_Q(T,\mu_B,\mu_S^H)\,, \quad s_Q(T,\mu_B,\mu_S^Q) \;=\; s_H(T,\mu_B,\mu_S^Q)\,. \quad (67)$$

The T and μ_B values inside the "strip" correspond to the mixed HG-QGP phase for which the following conditions have to be satisfied:

$$s_H(T,\mu_B,\mu_S) \;=\; s_Q(T,\mu_B,\mu_S)\,, \qquad (68)$$

$$n_S^{mix} \equiv \delta \cdot n_S^Q \;+\; (1 \;-\; \delta) \cdot n_S^H \;=\; \delta \cdot T\,\frac{\partial s_Q}{\partial \mu_S} \;+\; (1-\delta) \cdot T\,\frac{\partial s_H}{\partial \mu_S} \;=\; 0\,, \quad (69)$$

where δ is a fraction of the mixed phase volume occupied by the QGP. Eq. (68) reflect the Gibbs equilibrium between the two phases: the thermal equilibrium (equal temperatures), the mechanical equilibrium (equal pressures), the chemical equilibrium (equal chemical potentials). The net strangeness does not vanish in each phase separately, but the total net strangeness of the mixed phase (69) is equal to zero. This means the strangeness–anti-strangeness separation inside the mixed phase (Greiner et al., 1987). The line (65) for the 2^{nd} and higher order PTs (i.e. for $1 < \alpha \leq 2$) remains unchanged as μ_S^H becomes equal to μ_S^Q along this line.

6. Summary and outlook

The phase diagram of strongly interacting matter was studied within the exactly solvable statistical model of the quark-gluon bags. The model predicts two phases of matter: the hadron gas at a low temperature T and baryonic chemical potential μ_B, and the quark-gluon gas at a high T and/or μ_B. The order of the phase transition is expected to alter with a change of the μ_B/T ratio. The line of the 1^{st} order transition at a high μ_B/T is followed by the line of the 2^{nd} order phase transition at an intermediate μ_B/T, and then by the lines of the "higher order transitions" at a low μ_B/T. The condition of the strangeness conservation transforms the 1^{st} order transition line into the "strip" in which the strange chemical potential varies between the QGP and HG values.

In the high and low temperature domains the approach presented here reduces to two well known and successful models: the hadron gas model and the bag model of QGP. Thus, one may hope the obtained results concerning properties of the phase transition region may reflect the basic features of nature.

Clearly, a further development of the model is possible and required. It is necessary to perform quantitative calculations using known hadron states. These calculations should allow to establish a relation between the model parameters and the T and μ_B values and hence locate the critical line in the $T-\mu_B$ plane. Further on one should investigate various possibilities to study experimentally a rich structure of the transition line predicted by the model.

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