VORTICES IN THE FIREBALLS FORMED IN RELATIVISTIC NUCLEAR COLLISIONS

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Abstract. On the base of the system of hydrodynamic equations we consider a model of formation and development of the hydrodynamic vortices in the nuclear matter during relativistic heavy-ion collisions, in astrophysical objects, and in powerful atmospheric phenomena such as typhoons and tornados. A new class of the analytic solutions of non-relativistic hydrodynamic equations for the incompressible liquid in the presence of a bulk sink are analyzed. The main feature of these solutions is that they describe non-stationary hydrodynamic vortices with the azimuth component of velocity exponentially or explosively growing with time. A necessary attribute of a system with such a behavior is a presence of a bulk sink, which provides the existence of the non-stationary vortex regime. These solutions are obtained by nullifying the terms in the Navier-Stokes equations, which describe viscous effects, exist and represent vortex structure with "rigid-body" rotation of the core and converging radial flows. With the help of our model we explain some typical features of the above physical systems from the unique point of view.

1. Introduction

Hydrodynamic vortices and vortex flows is rather common in nature. This is constantly confirmed by formation of the powerful atmospheric vortices, typhoons, tornados, and so on. An example of the gigantic vortex structure is the big "Red Spot" on the Jupiter surface that is observed by astronomers about a several centuries.

Why such hydrodynamic vortex structures exist for an appreciable length of time in liquids and gases in spite of a finite viscosity of these media? A possible answer to this question could be associated with realization of the profiles of hydrodynamical velocities that strongly reduce the affect of the

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viscous terms in the Navier-Stokes equations. One example of a such durable motion is the axially-symmetrical Rankin vortex in an incompressible viscous liquid [1]. The velocity profile of this vortex has only one nonzero component – the azimuth component and describes the "rigid body" rotation in some cylindric domain of an arbitrary radius and differential rotation (velocity is inversely proportional to a distance from the vortex center). This profile nullifies the terms with the first viscosity in the Navier-Stokes equation and satisfies the hydrodynamic equations of an incompressible liquid at a constant velocity of the vortex rotation. However, this motion turns out to be unstable because the first derivative of the azimuth velocity has a step at the border of the rigid body rotation. In the vicinity of this point, the viscous effects result in dissipation of the vortex kinetic energy and it diffuses in the space decaying with time. It is evident that the stationary or growing with time vortices may exist only at the expense of the external energy sources. Among all possible hydrodynamic flows and vortex motions in the incompressible liquid survive only those ones which have a comparatively small dissipation of energy.

We study a new class of solutions of non-relativistic hydrodynamic equations of a incompressible liquid with a bulk sink in a finite domain of the liquid with cylindric or spherical symmetry. These solutions may be obtained by choosing the profiles of the hydrodynamic velocities that nullify the viscous terms in the Navier-Stokes equations [2]. We call them the quasi-dissipativeless solutions. They model the dynamical processes (nuclear, chemical reactions, phase transitions etc.), which move away out of the system, which performs a hydrodynamic motion, one or several species. The solutions under consideration describe the non-stationary hydrodynamic vortices with the exponentially growing velocity of rotation in some domain (core of the vortex) of the liquid with the bulk sink. The rotation acceleration of these vortices is a result of combined action of the convective and the Coriolis forces that appear due to the radial convergent flows into the bulk domain from the external region that support a constant density of the liquid. The acceleration of vortex motion may correspond to a regime of the nonlinear "explosive" instability in some special cases.

To our mind, the mechanism of formation and developing of this non-stationary hydrodynamic vortices could explain from the unique point of view a wide circle of the physical phenomena.

In this paper, we apply this hydrodynamic approach for description of:
i) the rotational motion of nuclear matter, which appears in the relativistic collisions of heavy nuclei with an initial angular momentum;
ii) the origination of non-dissipative hydrodynamic vortices in the liquid cores of planets and acceleration of the central cores of stars caused by thermonuclear reactions;

iii) the creation of powerful atmospheric vortices (whirlwinds, tornados, and typhoons).

2. The state of nuclear matter created in the relativistic nucleus-nucleus collisions

We consider the system, which is formed in the process of heavy nuclei collisions. Being accelerated to high energies ($E \gg 1$ GeV), heavy atomic nuclei with a charge $Z \gg 1$ gain the shape of ellipsoids of rotation, which are strongly contracted in the direction of motion due to the relativistic reduction of longitudinal sizes by the factor $\gamma = \sqrt{1 - v_0^2/c^2} \ll 1$ (where v_0 is the velocity of the colliding nuclei, and c is the velocity of light) (Fig. 1). A cluster of the dense hot nuclear matter (NM) that created in the nuclear collisions in the laboratory reference system takes the form of a thin disk with the initial radius $R_0 \sim 6$–7 fm $= 6$–$7 \cdot 10^{-13}$ cm (for the nuclei of Au and Pb), the thickness $h_0 = \gamma R_0 \ll R_0$, and the total mass $\mu_0 = 2Am_N/\gamma$ (where A is atomic number, and m_N is nucleon mass). A high density of particles and their small mean free paths in the disk, its evolution on the initial stage $t < t_{\rm f}$ (until the distances between particles becomes greater than the action radius of nuclear forces $r \sim \hbar/m_\pi c$, where $m_\pi \approx 140$ MeV/c^2 — mass of a π-meson) can be described in the hydrodynamic approximation. According to [3], the cluster of NM begins one-dimensional expansion along the collision axis (Fig. 1) up to the freeze-out time of the order of $t_{\rm f} \sim 10$ fm/c.

We consider the NM motion in the cylindrical coordinates with z axis along the collision line. The one-dimensionality of the expansion (along the z axis) allows us to assume that transverse hydrodynamic velocities (radial v_r and azimuthal v_φ components of the $\mathbf{v}$) are non-relativistic ones during the time, $t < t_{\rm f}$. In addition, according to the Bjorken scenario of the one-dimensional expansion [4], majority of the particles is contained in the inner layers of the fireball and move with longitudinal velocities

$$v_z = \frac{z}{t}, \quad t > t_{\rm i} \quad (t_{\rm i} \ll t_{\rm f}), \tag{1}$$

where $t_{\rm i} = 0.1$–0.3 fm/c is a time of the formation of a fireball [5]. Only in a narrow external layer, the substance moves with velocities of the order of the velocity of light $v \to c$ (see Fig. 1).

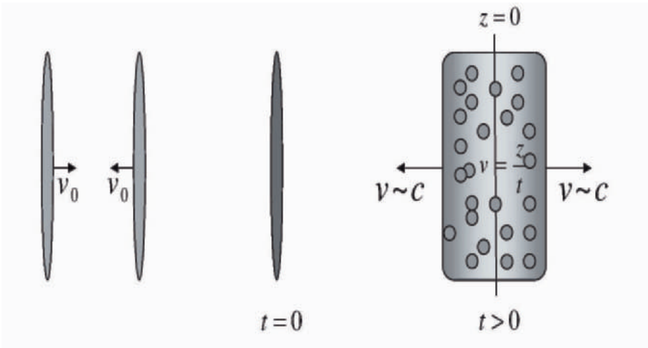

Figure 1. Sketch of the collision of relativistically contracted heavy nuclei. *Left figure*: nuclei before collision; *Figure in the center*: the nuclei just after overlapping, beginning of the formation of the fireball; *Right figure*: the fireball (cluster of hot and dense nuclear matter) which expands.

We consider such a region of a fireball or such a time of the hydrodynamic expansion, when the non-relativistic hydrodynamic description is valid for the longitudinal velocity v_z (in the Bjorken model, the duration of the formation of particles with great longitudinal momenta exceeds that for slow particles). In this case, we simplify our problem, which does not affect the qualitative behavior of the described hydrodynamic instability.

2.1. THE HYDRODYNAMIC INSTABILITY OF THE VORTEX MOTION IN NUCLEAR MATTER CREATED IN HEAVY-NUCLEI COLLISIONS

We describe the motion of NM with the help of the Navier–Stokes equation. In the cylindrical coordinate system it reads

$$\frac{\partial v_r}{\partial t} + (\mathbf{v}\nabla)\, v_r - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right], \tag{2}$$

$$\frac{\partial v_\varphi}{\partial t} + (\mathbf{v}\nabla)\, v_\varphi + \frac{v_r v_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left[\Delta v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right], \tag{3}$$

$$\frac{\partial v_z}{\partial t} + (\mathbf{v}\nabla)\, v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta v_z, \tag{4}$$

where p is the pressure, ν is the coefficient of kinematic viscosity, and

$$(\mathbf{v}\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\varphi}{r}\frac{\partial}{\partial \varphi} + v_z\frac{\partial}{\partial z}, \qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\,. \tag{5}$$

Different nuclear reactions are running in the fireball and accompanied by the escape of created products (secondary particles, γ-quanta, jets) and by a decrease of the total mass of the fireball $\mu(t)$ with time. These processes may be described by the the continuity equation with a bulk sink term, $q(\mathbf{r}, t)$,

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = -q\,(r, t)\,, \tag{6}$$

where $\rho(\mathbf{r}, t)$ is the density of NM. For the nuclear matter we may use the approximation of quasi-incompressible liquid. This means that in the process of hydrodynamic motion the following equation holds true

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v}\nabla)\,\rho = 0\,, \tag{7}$$

in comparison with the ordinary incompressible liquid with constant density now, $\rho(\mathbf{r}, t) \neq \mathrm{const}$. We make one more assumption that the intensity of the bulk sink in (6) is proportional to the local fireball density, $q \propto \rho$. In this case, the continuity equation (6) for the incompressible NM can be represented as

$$\mathrm{div}\,\mathbf{v} \equiv \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = -\frac{1}{\tau\,(t)}, \tag{8}$$

where the typical time $\tau = (q/\rho)^{-1}$ of the outflow of NM from a fireball at the expense of nuclear reactions (transformations of particles, annihilation, tunneling, etc.) depends on time only. The parameter τ can be estimated by the formula

$$\tau^{-1} = \sum_{\text{all processes}} \sigma_{\text{eff}}\, n v\,, \tag{9}$$

where σ_{eff} is the effective cross-section of inelastic processes or reactions, n is the bulk concentration of particles, and v is the mean velocity of the particles, which participate in the reactions. Expanding v_z in the vicinity, $z = 0$, in the middle cross-section of the fireball, where $v_z = 0$, and assuming that the fireball is rather thin, ($\gamma \ll 1$), we obtain

$$v_z\,(z, t) = \alpha\,(t)\,z\,. \tag{10}$$

Hence, according to the Bjorken model (1)

$$\alpha\,(t) = \frac{1}{t}, \quad t > t_{\mathrm{i}}\,. \tag{11}$$

Doing in the same manner as while obtaining v_z (10), we can get the profiles of the transverse velocities proportional to a distance r from the z axis (where $v_r = 0$ and $v_\phi = 0$):

$$v_r\,(r, t) = -\beta(t)\,r\,, \quad v_\phi\,(r, t) = \Omega(t)\,r\,. \tag{12}$$

The negative sign of v_r corresponds to the convergent radial flow of NM in a fireball, which ensures the validity of the continuity equation (8) in the presence of a sink and the longitudinal expansion. We note that these profiles nullify the viscous terms in Navier–Stokes equations (3)-(4).

For the axisymmetric profiles of the velocity which are independent of the azimuthal angle φ, the equation of continuity (8) with regard for (10) and (12) leads to the relation

$$\alpha(t) - 2\beta(t) = -\frac{1}{\tau(t)}\,. \tag{13}$$

On the other hand, the component of azimuthal hydrodynamic velocity $v_\varphi(r,t) = \Omega(t)r$ describing the "rigid-body" rotation with unknown $\Omega(t)$ must satisfy the Navier–Stokes equation (3). Substitution (12) $v_r(r,t)$ and $v_\varphi(r,t)$ at $r \leqslant R(t)$, ($R(t)$ is a radius of the fireball at a moment time t), in (3) yields

$$\frac{d\Omega(t)}{dt} - 2\beta(t)\,\Omega(t) = 0\,. \tag{14}$$

The solution of this equation with account of (13) takes the form

$$\Omega(t) = \Omega_0 \exp\left\{\int_{t_i}^{t}\left[\frac{1}{\tau(t')} + \alpha(t')\right]dt'\right\}, \tag{15}$$

where Ω_0 is an initial angular velocity of the rotation of nuclear matter. For the simplest case τ =const, and in the Bjorken model where $\alpha(t) = 1/t$

$$\Omega(t) = \Omega_0\frac{t}{t_\mathrm{i}}\exp\left\{\frac{t-t_\mathrm{i}}{\tau}\right\}, \quad t > t_\mathrm{i}\,. \tag{16}$$

This dependence describes the "exponential" hydrodynamic instability of the rotational motion of NM in the fireball provided that initial angular velocity, $\Omega_0 \neq 0$ (neglecting by the slow pre-exponential factor $\propto t$).

We connect the origin of nonzero Ω_0 with collisions of nuclei in highly excited rotational states with large orbital quantum numbers. Therefore, within the frame of the quasi-classical approximation there are some initial rotation of the nuclear matter with the hydrodynamic azimuthal velocity

$$v_{\varphi 0} = \Omega_0 r \quad (r \leqslant R_0)\,, \tag{17}$$

where $R_0 = R(0)$ (initial radius of the fireball), Ω_0 is the initial angular velocity of rotation connected with the initial angular momentum of the fireball M_{z0} by the relation

$$M_{z0} = I_{z0}\Omega_0. \tag{18}$$

Here, I_{z0} is the initial moment of inertia. For the fireball in the form of a disk of a thickness $h_0 = \gamma R_0 \ll R_0$, it is equal to

$$I_{z0} = \frac{1}{2}\mu_0 R_0^2, \quad \mu_0 = \pi R_0^3 \gamma \rho_0, \tag{19}$$

where $\rho_0 = \mu_0/\pi R_0^2 h_0$ is the initial density of NM.

The total mass of the fireball is given by

$$\mu(t) = \pi R^2(t) \int\limits_{-h(t)/2}^{h(t)/2} \rho(z,t)\, dz\,, \tag{20}$$

where $h(t)$ is the fireball disk thickness and $\rho(z,t)$ is its density. The fireball transverse radius $R(t)$ and its mass $\mu(t)$ decrease with time.

We assume that the value of the integral

$$\int\limits_{-h(t)/2}^{h(t)/2} \rho(z,t)\, dz \equiv \frac{\mu(t)}{\pi R^2(t)} = \frac{\mu_0}{\pi R_0^2} = \text{const}$$

is not changed during the expansion of the fireball, which would be fulfiled exactly upon the one-dimensional expansion in the absence of nuclear reactions, where $h(t)\rho(t) = h_0\rho_0 = \text{const}$. In this case, the decrease in $R(t)$ will occur only due to the presence of a bulk sink in the fireball (rather than at the expense of its expansion). This yields that the inertia moment drops proportionally to the fourth degree of the radius $R(t)$:

$$I_z(t) = \frac{1}{2}\mu(t) R^2(t) = \frac{\pi}{2}\rho_0 h_0 R^4(t)\,. \tag{21}$$

By virtue of the law of conservation of the angular momentum $M_z = M_{z0} = \text{const}$ and according to (18), there exists the interrelation between the angular velocity of rotation of the cluster $\Omega(t)$ and its actual radius $R(t)$:

$$\Omega(t) = \Omega_0 \left[\frac{R_0}{R(t)}\right]^4. \tag{22}$$

According to (22), this corresponds to the exponential decrease (by neglecting the slower power behavior) of the cluster radius,

$$R(t) = R_0 \left(\frac{t_\mathrm{i}}{t}\right)^{1/4} \exp\left\{-\frac{t - t_\mathrm{i}}{4\tau}\right\}, \quad t > t_\mathrm{i}, \tag{23}$$

and to the exponential growth in time of the maximal azimuthal velocity of NM on the boundary of the cluster ($t > t_\mathrm{i}$):

$$v_\varphi^{\max}(t) = \Omega(t) R(t) = \Omega_0 R_0 \left(\frac{t}{t_\mathrm{i}}\right)^{3/4} \exp\left\{\frac{3}{4}\frac{t - t_\mathrm{i}}{\tau}\right\}. \tag{24}$$

The latter must lead to a change in the distributions of the momenta of outgoing secondary particles and to the appearance of the angular momentum in the distributions of those products of nuclear reactions which will take away a part of the rotational moment of the system with themselves.

For the velocity profiles (10)—(17), Eq. (2) defines a change in time of the radial pressure of NM in the cluster $r \leqslant R(t)$:

$$p(r,t) = p_0 + \frac{\rho r^2}{2}\left[\Omega^2(t) - \beta^2(t) + \dot{\beta}(t)\right], \tag{25}$$

where p_0 — pressure at $r = 0$, and the dot above a letter stands for the derivative with respect to time.

It is worth to note that, in the Bjorken model, the pressure turns out to be homogeneous along the axis of the expansion of NM by virtue of the equality $\dot{\alpha} = -1/t^2 = -\alpha^2$. With regard for (11) and (13), we get (at $\tau = \text{const}$):

$$p(r,z,t) = p_0 + \frac{\rho r^2}{2}\left[\Omega^2(t) - \frac{1}{4}\left(\frac{1}{t} + \frac{1}{\tau}\right)^2 - \frac{1}{2t^2}\right]. \tag{26}$$

Upon the noncentral collision of heavy nuclei (nonzero impact parameter), there appears the region of the overlapping of nuclei which is characterized by the asymmetry in the plane transverse to the collision axis. This leads to the appearance of the asymmetry of the spatial gradients of pressure and density, which causes the appearance of an asymmetry in the momentum distribution of particles [5]. In this case, the azimuthal angular distributions of secondary particles are usually represented by the formula [6]

$$\frac{dN}{d\varphi} = \frac{N_0}{2\pi}\left(1 + 2v_1\cos\varphi + 2v_2\cos 2\varphi + ...\right). \tag{27}$$

The case with $v_2 \neq 0$ corresponds to the elliptic flow. The parameter v_2 depends monotonically on the transverse momentum of outgoing particles $p_\perp = \sqrt{p_r^2 + p_\varphi^2}$ and varies as follows: $v_2 = 0 \div 0.2$ [6]. We note that a similar azimuthal asymmetry indicates once more the presence of the effect of the collective character of the nuclear matter which is created in relativistic heavy-ion collisions and can be described in the hydrodynamic approximation.

The form of an axially nonsymmetric cluster of NM can be described as $R(t,\varphi) = R(t)(1 + \xi\cos 2\varphi)$, and its velocity components are

$$v_r(r,\varphi,t) = -\beta r(1 + \varepsilon\cos 2\varphi), \tag{28}$$

$$v_\varphi(r,\varphi,t) = \Omega r(1 + \delta\sin 2\varphi), \tag{29}$$

$$v_z(z,t) = \alpha z\,. \tag{30}$$

Here, $\xi(t), \varepsilon(t), \delta(t)$ — some functions of time. For such angular and radial dependences, the terms describing the viscosity in the Navier–Stokes equations are identically equal to zero under the condition $\beta\varepsilon = \Omega\delta$:

$$\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right] \equiv 0\,, \tag{31}$$

$$\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right] \equiv 0\,. \tag{32}$$

In this case, the equation of continuity (10) leads again to relation (13), and Eq. (3) yields (14).

Thus, in the hydrodynamic model under consideration, the noncentral character of the collisions of heavy nuclei does not practically affect the mechanism of development of a hydrodynamic rotational instability. The last is defined by the presence of the term describing a sink in the equation of continuity and by the initial nonzero vorticity of the cluster of NM and corresponds to the hydrodynamic velocity profiles which nullify the terms describing the viscosity in the Navier–Stokes equation.

3. Spherically symmetrical vortex structures in astrophysics (planets, stars)

In this section, we consider the possibility of existence of the dissipativeless vortex structures and the flow with zeroth viscosity in the spherically symmetric objects, particularly, in the planets and the Earth, where the melted liquid nucleus exists and in the central cores of stars and the Sun where the thermonuclear fusion reactions take place.

It is known, that in depths of the Earth and other planets of the Earth group the liquid cores exist. They consist of melted liquid minerals and metals. This core is enveloped by the rigid crust (lithosphere and asthenosphere)of a radius R_0, and limited from inside by the surface of rigid core of a radius r_0. On action of the gravity forces from other celestial bodies (in particular, for the Earth from the Moon and the Sun), the tidal waves may appear in the liquid cores that could cause breaks of the rigid crust forming powerful earthquakes.

We show that a spherical layer of the viscous liquid inside the liquid planet core in the domain $r_0 \leqslant r \leqslant R_0$ the close global hydrodynamical flows (three dimensional vortices) may appear that in the incompressible liquid correspond to the zeroth bulk viscosity and may be excited by the external gravitational fields. In stationary conditions, due to a finite viscosity and friction with the rigid walls, the liquid core of the planets must take part in the daily rotation as a whole with a definite angular velocity Ω_0 (for the Earth $\Omega_0 \approx 7.3 \cdot 10^{-5}$ c^{-1}). The hydrodynamic azimuth velocity $v_{\varphi 0}$ of this "rigid" rotation of the liquid in the layer $r_0 \leqslant r \leqslant R_0$ with account of the spherically symmetric gravitational forces of the planet satisfies the following equations:

$$\frac{v_{\varphi 0}^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r} + g(r), \quad v_{\varphi 0}^2 \cot\theta = \frac{1}{\rho}\frac{\partial P}{\partial \theta}, \tag{33}$$

where P and ρ are the equilibrium pressure and density of the liquid , $g(r) = \frac{4\pi}{3}G\rho r + g_0$is the gravity acceleration inside the liquid core, $g_0 = GM_0/r_0^2$ is the gravity acceleration formed by the inner rigid core of a mass $M_0 = \frac{4\pi}{3}\rho_0 r_0^3$ and a density ρ_0, G is the gravity constant, and θ is a polar angle of the spherical coordinates.

The stationary solutions that satisfy equations (33) take the following form

$$v_{\varphi 0}(r,\theta) = r\Omega_0 \sin\theta, \tag{34}$$

$$P(r,\theta) = P_0 + \frac{\rho r^2}{2}\left[\Omega_0^2 \sin^2\theta - \frac{4\pi}{3}G\rho\right] - \rho g_0 r. \tag{35}$$

However, this ground state of the "solid " rotation may be perturbed by the external gravitational fields. To describe this perturbed hydrodynamic motion of the liquid in the domain $r_0 \leqslant r \leqslant R_0$ we apply equations of the Navier-Stokes and continuity equation for the incompressible viscous equation in the spherical coordinate system [2]. Keeping in mind that the radial velocity of the liquid v_r on the rigid walls at $r = r_0$ and $r = R_0$ must be zero, we set $v_r = 0$ in the whole bulk of the liquid core. In this case the hydrodynamic equations take the form:

$$-\frac{v_\varphi^2 + v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - g(r) + \widetilde{g}_r - \frac{2\nu}{r^2}\left[\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + v_\theta \mathrm{ctg}\,\theta\right], \tag{36}$$

$$\frac{\partial v_\varphi}{\partial t} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi v_\theta \mathrm{ctg}\,\theta}{r} = -\frac{1}{r\rho\sin\theta}\frac{\partial P}{\partial \varphi} + \widetilde{g}_\varphi + \tag{37}$$

$$+\nu\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r v_\varphi) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta^2}(\sin\theta v_\varphi) + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{r^2\sin^2\theta}\right], \tag{38}$$

$$\frac{\partial v_\theta}{\partial t} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} - \frac{v_\varphi^2 \mathrm{ctg}\,\theta}{r} = -\frac{1}{r\rho}\frac{\partial P}{\partial\theta} + \widetilde{g}_\theta + \nu\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r v_\theta) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\varphi^2} + \right. \tag{39}$$

$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta^2}(\sin\theta v_\theta) - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{r^2\sin^2\theta}\right]. \tag{40}$$

Here v_θ is the meridian velocity of the liquid, $\widetilde{g}_r$, $\widetilde{g}_\varphi$ and $\widetilde{g}_\theta$ are the radial, azimuth, and meridian components of the perturbing acceleration that formed by the external gravity fields, respectively.

The continuity equation at $v_r = 0$ reduces to

$$\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} + v_\theta \cot\theta = 0. \tag{41}$$

According to equation (41), the expression in the square brackets with the coefficient of kinematic viscosity ν on the r.h.s. of equation (36) identically equals to zero. The continuity equation (41) is satisfied by the following angular dependencies of the angular velocities

$$v_\varphi(r, \theta, \varphi) = [r\Omega_0 + F(r)]\sin\theta + f(r)\cos\theta\cos\varphi, \tag{42}$$

$$v_\theta(r, \varphi) = f(r)\sin\varphi, \tag{43}$$

where $F(r)$ and $f(r)$ are arbitrary functions of r. Taking these functions in the form

$$F(r) = \Omega(r + \alpha/r^2), \quad f(r) = \omega(r + \beta/r^2), \tag{44}$$

where Ω and ω are some corrections to the constant angular velocity Ω_0 of the "rigid " rotation of the liquid and α i β are arbitrary parameters that are dictated by the boundary conditions the both expressions in the square brackets with ν in the right hand parts of equations (38) and (40). This means that the hydrodynamic flows with velocities (42) and (43) at the radial distribution (44) in the incompressible viscous liquid actually correspond to the zeroth bulk viscosity.

If we select parameters α and β equal to $\alpha = \beta = -R_0^3$, when the perturbed azimuth and meridional velocities equal to zero at $r = R_0$,, then the friction between the liquid and the inner wall of rigid crust disappears. Moreover, if we assume that the rigid central planet core can rotate with the angular velocity different from the Ω_0, then the friction between the liquid and rigid cores will be minimal at velocities that coincide at the point $r = r_0$. For example, on the ecuador at $\theta = \pi/2$ and $\varphi = 0$:

$$v_\varphi(r_0, \pi/2, 0) = [r_0\Omega_0 + \Omega(r_0 - R_0^3/r_0^2)]. \tag{45}$$

Now we show that these three dimensional global vortexes in the liquid cores of planets may be born under action of the external gravitational perturbations. Substituting expressions (42)—(44) at $\alpha = \beta = -R_0^3$ in equation (36) and taking account of the relation for the equilibrium state (34) and (35), for the perturbation of pressure $\widetilde{P}$ we have the following equation:

$$r\left(1-\frac{R_0^3}{r^3}\right)^2\left[\Omega\left(2\Omega'+\Omega\right)\sin^2\theta+2\omega\left(\Omega'+\Omega\right)\sin\theta\cos\theta\cos\varphi+\omega^2\right.$$
$$\left.\left(\cos^2\theta\cos^2\varphi+\sin^2\varphi\right)\right] \quad (46)$$

$$=-\frac{1}{r}\frac{\partial\widetilde{P}}{\partial r}+\widetilde{g}_r\,, \qquad (47)$$

Let us consider the case of a maximum gravitational force acting on the Earth from the Moon and the Sun at the complete solar eclipse when the resultant gravitational potential is equal to

$$\Phi(r,\widetilde{\theta})=-G\left[\frac{M_1}{\sqrt{R_1^2+r^2-2R_1r\cos\widetilde{\theta}}}+\frac{M_2}{\sqrt{R_2^2+r^2-2R_2r\cos\widetilde{\theta}}}\right], \qquad (48)$$

where M_1 i M_2 are the masses od the Sun and the Moon that are located a distances R_1 i R_2 from the Earth center respectively , **r** is a radius-vector of an arbitrary point in the Earth bulk (in particular, in the liquid core $r < R_0$), and $\widetilde{\theta}$ is the angle between vectors **r** and $\mathbf{R}_{1,2}$. With account of the strong inequalities $R_0 \ll R_2 \ll R_1$ (at $M_1 \gg M_2$) keeping the terms of the second order of smallness, according to (48) we get

$$\Phi(r,\widetilde{\theta})\cong-\left[\Phi_0+\Phi_1 r\cos\widetilde{\theta}+\Phi_2\frac{r^2}{2}(3\cos^2\widetilde{\theta}-1)\right], \qquad (49)$$

where

$$\Phi_0=G\left(\frac{M_1}{R_1}+\frac{M_2}{R_2}\right),\quad \Phi_1=G\left(\frac{M_1}{R_1^2}+\frac{M_2}{R_2^2}\right),\quad \Phi_2=G\left(\frac{M_1}{R_1^3}+\frac{M_2}{R_2^3}\right). \qquad (50)$$

In the spherical coordinates with the polar axis coinciding with the Earth rotation axis, the angle $\widetilde{\theta}$ can be expressed through the polar (θ i θ') and azimuth (φ i φ') angles of the vectors **r** and $\mathbf{R}_{1,2}$ as follows

$$\cos\widetilde{\theta}=\cos\theta\cos\theta'+\sin\theta\sin\theta'(\cos\varphi\cos\varphi'+\sin\varphi\sin\varphi')\,. \qquad (51)$$

At the points of the Earth orbits (ecliptic) that corresponds to winter and summer solstice when the vectors $\mathbf{R}_{1,2}$ and the inclined axis of the Earth lies in the same plane so that $\varphi'=0$, and $\theta'=\pi/2-\theta_1$ (where θ_1 is the angle of Earth incline to the ecliptic plane), with the help (51) we get

$$\cos\widetilde{\theta}(\theta,\varphi)=\cos\theta\sin\theta_1+\sin\theta\cos\varphi\cos\theta_1\,. \qquad (52)$$

At the points of spring and fall solstice when the vectors $\mathbf{R}_{1,2}$ under complete solar eclipse are directed so that $\varphi' = \theta' = \pi/2$, we have:

$$\cos\widetilde{\theta}(\theta,\varphi) = \sin\theta\sin\varphi. \tag{53}$$

The perturbations of the gravity force with account of (49) and (52) or (53) may be calculated by using the formulas:

$$\widetilde{g}_r = -\frac{\partial\Phi}{\partial r}, \quad \widetilde{g}_\varphi = -\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}, \quad \widetilde{g}_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta}. \tag{54}$$

In particular, for $\widetilde{g}_r$ according to (49) we get

$$\widetilde{g}_r = \Phi_1\cos\widetilde{\theta}(\theta,\varphi) + \Phi_2 r\left[3\cos^2\widetilde{\theta}(\theta,\varphi) - 1\right]. \tag{55}$$

With account of relations (52) and (53) we can see that the second term in (55) contains the same space—angular dependencies as the left hand part of the equation (47). In fact, according to (52) and (53), we obtain

$$\cos^2\widetilde{\theta} = \begin{cases} \cos^2\theta\sin^2\theta_1 + \sin\theta\cos\theta\cos\varphi\sin 2\theta_1 + \\ +(1-\cos^2\theta)\cos^2\varphi\cos^2\theta_1, \\ 1-(\cos^2\theta+\cos^2\varphi)+\cos^2\theta\cos^2\varphi. \end{cases} \tag{56}$$

The correspondent terms in (56) with the angular dependencies of the type $\sin\theta\cos\theta\cos\varphi$ (at $\sin 2\theta_1$) and $\cos^2\theta\cos^2\varphi$ coincide in their structure with some terms in the left hand side of equation (47). In other words, a peculiar space—time resonance appears between the gravitation perturbations (55) and the dissipativeless hydrodynamic flows. It may excite these vortex flows in the liquid core of the Earth at the moments when the complete solar eclipse coincides with the above specified location of the Earth on the ecliptic.

The correspondent inhomogeneous perturbation of the pressure $\widetilde{P}$ must act on the rigid at $r = R_0$ and along with regular tidal waves can cause detraction (break) of the lithosphere forming conditions for appearance of the power earthquakes on the land or for birth of tsunami in an ocean, and amplification of the volcanic processes as well.

3.1. INFLUENCE OF THE THERMONUCLEAR REACTIONS ON THE HYDRODYNAMICS OF ROTATION OF THE CENTRAL CORE OF A STAR

It is known, that in the star cental cores, particularly, in the sun, where the temperature could reach ten million degrees, the thermonuclear reaction resulting in transformation of hydrogen into helium take place. Formation of one helium nucleus ^{4}He (or ^{3}He) from four (or three) nuclei of hydrogen or protons H, leads to decreasing of the specific volume that occupies the fully ionized and compressed by the gravity forces hot hydrogen-helium plasma (analogously to the condensation process of water droplets). In other words, due to the thermonuclear fusion inside the star central core of a radius r_0 the bulk sink of substance appears. Under conditions of the dynamic and chemical equilibrium between the star core and its external envelop of a radius $R_0 > r_0$ convergent radial flows of hydrogen appear. They equalize the density and the chemical constitution of the plasma inside the star. If a typical time of the thermonuclear reactions is τ_0, then at $\rho \approx$ const we can write the following effective continuity equation for the incompressible liquid (gas) with acco8unt of the spherical symmetry of the problem

$$\frac{\partial v_r}{\partial r} + \frac{2}{r} v_r = \begin{cases} -1/\tau_0, & r \leqslant r_0, \\ 0, & r_0 < r \leqslant R_0. \end{cases} \tag{57}$$

A solution of the equation (57) for the radial hydrodynamic velocity takes the form:

$$v_r = \begin{cases} -r/3\tau_0, & r \leqslant r_0, \\ -r_0^3/3\tau_0 r^2, & r_0 < r \leqslant R_0. \end{cases} \tag{58}$$

Below we neglect a high conductivity of the hot plasma and different electromagnetic effects and interactions with magnetic fields. In this situation, from the macroscopic point of view the hydrodynamic rotational motion of the dense electro-neutral plasma inside a spherically symmetric star can be approximately described by the conventual Navier-Stokes equations with account of the gravity forces. We can see that the radial dependencies (58) provide nullification of the term describing the bulk viscosity, i.e.

$$v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - g(r). \tag{59}$$

On the other hand, if we chose the azimuth velocity v_φ in the form

$$v_\varphi(r, \theta) = \omega \sin\theta \begin{cases} r, & r \leqslant r_0, \\ r_0^2/r, & r_0 < r \leqslant R_0, \end{cases} \tag{60}$$

the term with the first viscosity will be nullified also. Substituting (58) and (60) into (59) and performing integration, we obtain the pressure distribution with account of gravitation inside the star:

$$P = \begin{cases} P_0 - \frac{\rho r^2}{2} \left[\frac{1}{9\tau_0^2} + \frac{4\pi}{3} G\rho - \omega^2 \sin^2\theta \right], & r \leqslant r_0, \\ P_0 - \frac{\rho r^2}{2} \left[\frac{1}{9\tau_0^2} \frac{r_0^6}{r^6} + \frac{4\pi}{3} G\rho - \omega^2 \sin^2\theta \right] - \\ -\rho g_0 (r - r_0), \quad r_0 < r \leqslant R_0. \end{cases} \tag{61}$$

Here, for the sake of simplicity, we assume that the in the star bulk is homogeneous.

Substituting (60) in the l.h.s. of the second Navier-Stokes equation in spherical coordinates, we get:

$$\frac{d\omega}{dt} = \begin{cases} 2\omega/3\tau_0, & r \leqslant r_0, \\ 0, & r_0 < r \leqslant R_0. \end{cases} \tag{62}$$

From this at $\tau_0 = \text{const}$ we again obtain the exponential law of increasing of the angular velocity of "rigid" rotation of the substance in the bulk of the central core and a constant value of ω in the external domain:

$$\omega(t) = \begin{cases} \omega(0)\, e^{2t/3\tau_0}, & r \leqslant r_0, \\ \omega(0) = \text{const}, & r_0 < r \leqslant R_0. \end{cases} \tag{63}$$

This means that at the core border $r = r_0$ a step of the azimuth velocity increases in time:

$$\triangle v_\varphi(t) = [\omega(t) - \omega(0)] r_0 \sin\theta. \tag{64}$$

According to the calculations it causes the unstable surface perturbations and to development of the strong turbulence. This results in the stationary turbulent regime in some domain near the core border where at the expense of the anomalous turbulent viscosity $\nu^* \gg \nu$ when there is a constant difference between the velocities of rotation of the star central core and its envelop. The developed turbulent pulsations on the border between the core and its envelop may stimulate the active turbulence that is observed in the Sun atmosphere. However, an analysis of these phenomena with account of the Sun magnetic field perturbations requires the equations of magnetic hydrodynamics.

4. One-component liquid with accelerated flows: the mechanism of formation of a funnel and a windspout

One of the most interesting paradoxes in hydrodynamics is the so-called "funnel effect" (see [1, 7]). It is assumed, that this effect is caused by the conservation laws of the angular momenta of the incompressible liquid (gas) inside the given contour, and accompanied by the accelerated rotation of a vortex at concentration of vorticity a flow $\omega = \text{rot}\,\mathbf{v}$ due to the narrowing of the channel. Another approach to the problem of a funnel formation lies in the assumption (see [8]) about origination of the angular momentum at a zero initial vorticity as a result of instability of cylindrically-symmetric flow in a liquid (the flooded jet) in relation to axially-asymmetric left- and right-spiral perturbations with carrying out of rotation of a certain sign to the infinity at the expense of a flow (convective instability) and accumulation of a rotation motion of another sign (absolute instability).

Let us show, that there exist one more simple mechanism of a vortex formation in the incompressible liquid (gas), which is in a gravitational field and includes vertical ascending or descending flows, whose velocities depend on the coordinate z (along the vertical axis of a vortex). We consider Navier-Stokes equation for axially-symmetric motion of the incompressible viscous liquid (gas) in cylindrical coordinates:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right), \tag{65}$$

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \nu \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} \right), \tag{66}$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + +\nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right), \tag{67}$$

where v_r, v_φ and v_z are radial, azimuthal and axial components of hydrodynamical velocity $\mathbf{v}$, P and ρ — pressure and density of a liquid (gas), $\nu = \eta/\rho$ — a coefficient of kinematic viscosity, and g — the gravity acceleration, which is directed opposite to axis z.

Equations (65)–(67) must be completed with the continuity equation

$$\operatorname{div} \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0. \tag{68}$$

Let us notice that in equations (65) and (66) the dependencies of v_r and v_φ on z are not taken into account for simplification reasons. We shall also notice, that despite the axial symmetry, the expression near the factor ν in the right hand parts of equations (65) and (66) we formally have a kind of the Laplace operator for a certain scalar complex function $f(r)\, e^{i\varphi}$, where the azimuthal angle φ plays a role of a phase, which by physical sense is similar to the well-known Berry phase.

In case of a plane vortical rotation, when $v_r = v_z = 0$, equations (65) and (66) become a form:

$$\frac{v_\varphi^2}{r} = \frac{1}{\rho}\frac{dP}{dr}, \tag{69}$$

$$\frac{\partial v_\varphi}{\partial t} = \nu \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} \right). \tag{70}$$

If a radial dependence of the azimuthal velocity $v_\varphi(r)$, according to the Rankine vortex model (see [1]), is chosen as

$$v_\varphi(r) = \begin{cases} \omega r, & r \leqslant R_0, \\ \omega R_0^2/r, & r > R_0, \end{cases} \tag{71}$$

than the right hand part of equation (70) equals identically to zero, what corresponds to the rotation of an incompressible liquid (gas) with a constant angular velocity $\omega = \text{const}$.

At this, a distribution of hydrodynamical pressure $P(r)$, according to (65), in the so-called cyclostrophic rotation regime has a form:

$$P(r) = \begin{cases} P_0 + \rho\omega^2 r^2/2, & r \leqslant R_0, \\ P_\infty - \rho\omega^2 R_0^4/2r^2, & r > R_0, \end{cases} \tag{72}$$

where $P_0 = (P_\infty - \rho\omega^2 R_0^2)$ is the pressure on the vortex axis, and P_∞ — the pressure on large distances (at $r \to \infty$).

In the Rankine model a radius of a vortex core R_0 is not determined, but if in a liquid (gas) a cylindrically-symmetric flow (the flooded jet) exists with a velocity v_z, than its radius will determine the size R_0 in (71). Let us assume, that the flow velocity depends linearly on the coordinate z and does not depend on r in the area $r \leqslant R_0$, i.e.

$$v_z(z) = \begin{cases} v_{z0} + \alpha z, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{73}$$

In this case the continuity equation (68) is satisfied for the following radial dependence of velocity v_r, continuous at $r = R_0$:

$$v_r(r) = \begin{cases} -\dfrac{1}{2}\alpha r, & r \leqslant R_0, \\ -\dfrac{1}{2}\alpha \dfrac{R_0^2}{r}, & r > R_0. \end{cases} \tag{74}$$

Let us notice, that the above mentioned structures of velocities (71), (73) and (74) nullify viscous terms in equations (65)–(67). At the same time, the diagonal components of the viscous stress tensor for these profiles are distinct from zero and result in the following expression for the change of kinetic energy of a vortex due to dissipation (by unit of length of a vortex along the axis z):

$$\left(\frac{dE_{\text{kin}}}{dt}\right)_{\text{dis}} = -4\pi\rho\nu\left[\frac{3}{2}\alpha^2 + \omega^2(0)\right]R_0^2. \tag{75}$$

Substituting expressions (71) and (74) in the equation (66), we get:

$$\frac{d\omega}{dt} = \begin{cases} \alpha\omega, & r \leqslant R_0, \\ 0, & r > R_0. \end{cases} \tag{76}$$

Equation (76) is a result of that in the inner region ($r \leqslant R_0$) the convertive and Coriolis forces at $v_r \neq 0$ are added up, but in the external region ($r > R_0$) they mutually compensate one another.

If the parameter α is constant in time ($\alpha = \text{const}$, $\dfrac{d\alpha}{dt} = 0$) and positive $\alpha > 0$, than from equation (76) it follows, that inside the area $r \leqslant R_0$ an angular velocity of a vortex grows in time due to the exponential law

$$\omega(t) = \omega(0)\, e^{\alpha t}, \tag{77}$$

provided that a nonzero initial vorticity $\omega(0) \neq 0$ exist in a liquid (gas), whereas in the external area $\omega = \omega(0) = \text{const}$. Thus, on the border of a vortex core $r = R_0$ a jump of azimuthal velocity, which exponentially grows in time, arises (see section 4).

Let us emphasize, that energy dissipation (75) at the non-stationary vortical motion does not depend on time t, and is determined only by the initial vorticity $2\omega(0)$. Thus, a dissipation remains small, despite the fast increase of an angular velocity of a "rigid-body" rotation of a vortex core. This means, that non-stationary vortices are not suppressed by the dissipation at the initial stage of their developments.

On the other hand, substituting (71), (73) and (74) in equations (65) and (67), we get the following equations for the determination of a pressure:

$$\frac{\partial P}{\partial r} = \begin{cases} \rho r\left[\omega^2(t) - \dfrac{\alpha^2(t)}{4} + \dfrac{1}{2}\dfrac{d\alpha}{dt}\right], & r \leqslant R_0, \\ \dfrac{\rho R_0^4}{r^3}\left[\omega^2(0) + \dfrac{\alpha^2(t)}{4} + \dfrac{1}{2}\dfrac{d\alpha}{dt}\dfrac{r^2}{R_0^2}\right], & r > R_0, \end{cases} \tag{78}$$

$$\frac{\partial P}{\partial z} = \begin{cases} -\rho\left[g + \alpha v_{z0} + z\left(\dfrac{d\alpha}{dt} + \alpha^2\right)\right], & r \leqslant R_0, \\ -\rho g, & r > R_0. \end{cases} \tag{79}$$

Let us notice, that equations (78) and (79) indicate on the existence and increasing in time of step of the first derivatives of the pressure $\partial P/\partial r$ and $\partial P/\partial z$ on a surface of a vortex core $r = R_0$. From equation (79) a possibility of existence of the non-stationary solution with $\frac{d\alpha}{dt} \neq 0$ follows, namely:

$$\frac{d\alpha}{dt} + \alpha^2(t) = 0, \qquad \frac{\partial P}{\partial z} \pm \rho g = 0. \tag{80}$$

The first equation (80) at $\alpha > 0$ has a solution:

$$\alpha(t) = \frac{\alpha_0}{1 + \alpha_0 t}, \qquad \alpha_0 \equiv \alpha(0) > 0, \tag{81}$$

whereas the second equation (80) corresponds to hydrostatic pressure distribution in the whole space, and the sign (+) corresponds to an ascending flow (the axis z is directed upwards), and the sign (−) — to a descending flow (the axis z is directed downwards). In this case equation (76) in the area $r \leqslant R_0$ has a solution

$$\omega(t) = \omega(0) \exp\left\{\int_0^t \frac{\alpha_0\, dt'}{1 + \alpha_0 t'}\right\} = \omega(0)(1 + \alpha_0 t), \tag{82}$$

which corresponds to the linear growth in time of velocity of a vortex rotation.

At last, we should notice that equation (76) under condition of $\alpha(t) = \omega(t) > 0$ takes a form:

$$\frac{d\omega}{dt} - \omega^2(t) = 0. \tag{83}$$

The solution of the nonlinear equation (83) corresponds to the so-called "explosive" instability:

$$\omega(t) \equiv \alpha(t) = \frac{\omega(0)}{1 - \omega(0)t}, \tag{84}$$

when in a final time interval $t_0 = 1/\omega(0) \equiv 1/\alpha(0)$ the angular velocity of a liquid rotation $\omega(t)$ and the derivative of the axial velocity on z, $\alpha(t) \equiv \partial v_z/\partial z$, rise formally to infinity, although they are actually limited from the above due to satisfying of the incompressibility condition of a liquid (see below).

The positive sign of α corresponds to the growth of velocity of a flow along the axis z directed by velocity v_{z0}. Thus, in a descending flow of a liquid which flows out through a hole at the bottom under the action of gravitation and accelerates along the axis z by a linear law (73), we get the exponential (77), linear (82), or "explosive" (84) laws of vortex rotation velocity growths in time. At this, the amount of a liquid, which flows out, is completely compensated by the inflow of the same amount of a liquid with velocity of a converging radial flow (74) from the surrounding volume, which is considered as an enough large reservoir of the substance.

Such a simple model explains the funnel formation in a bath at the opening of a hole or an whirlpool formation on a small river in the place of a sharp deepening of the bottom. The going out of exponential rotation velocity growth of a liquid (water) on the stationary regime is caused by the friction with a solid fixed surfaces, and also by the energy dissipation on the tangential step of the azimuthal velocity v_φ at the point $r = R_0$ (see below).

This model can also explain the origination of sandy tornados in deserts. Due to a strong heating by the sunlight of some sites of a surface of a sandy ground (the darkest or located perpendicularly to the solar rays), the nearby air gets warm locally and starts to rise upwards with acceleration under the action of the Archimedian force, as more light. If this acceleration in a quasi-stationary conditions corresponds approximately to the linear by z law (73), than at $\alpha > 0$ we again obtain the exponential law of rotation velocity growth of a vortex (77). At this, the accelerated decreasing in time of the pressure in the vortex axis $r = 0$ with $\omega(t)$ and $\alpha(t)$ increasing leads to the suction of sand deep into the vortex and formation of a visible tornado, which is frequently observed in deserts.

4.1. FORMATION OF A TORNADO FUNNEL WITH ACCOUNT OF GRAVITY AND VERTICAL FLOWS OF AIR

As marked above, for flows of the air, which flow into a rain cloud of a cylindrical form at the time of its condensation, the following hydrodynamical velocities are characteristic:

$$v_r = \begin{cases} -\beta r, & r \leqslant R_0, \\ -\beta R_0^2/r, & r > R_0, \end{cases} \quad v_\varphi = \begin{cases} \omega r, & r \leqslant R_0, \\ \omega R_0^2/r, & r > R_0, \end{cases} \quad v_z = \begin{cases} v_{z0} + \alpha z, & r \leqslant R_0, \\ 0, & r > R_0, \end{cases} \tag{85}$$

where $2\beta = \alpha + |Q|/\rho$. In the case of exponential instability we have $\omega(t) = \omega(0)\, e^{2\beta t}$ (at $\alpha = \text{const}$ and $|Q| = \text{const}$). At this, bulk viscous forces in equations (65)–(67) equal identically to the zero, so that we get:

$$\frac{\partial P}{\partial r} = \begin{cases} \rho r \left[\omega^2(t) - \beta^2\right], & r \leqslant R_0, \\ \dfrac{\rho R_0^4}{r^3}\left[\omega^2(0) + \beta^2\right], & r > R_0, \end{cases} \tag{86}$$

$$\frac{\partial P}{\partial z} = \begin{cases} -\rho\tilde{g} - \rho\alpha^2 z, \quad \tilde{g} = g + \alpha v_{z0}, & r \leqslant R_0, \\ -\rho g, & r > R_0. \end{cases} \tag{87}$$

The integration of equations (86) and (87) determines a difference of the pressure between two arbitrary points

$$P_2 - P_1 = \begin{cases} -\rho\tilde{g}(z_2 - z_1) - \dfrac{\rho}{2}\,\alpha^2(z_2^2 - z_1^2) + \dfrac{\rho}{2}\left[\omega^2(t) - \beta^2\right](r_2^2 - r_1^2), & r_{1,2} \leqslant R_0, \\ -\rho\tilde{g}(z_2 - z_1) - \dfrac{\rho R_0^4}{2}\left[\omega^2(0) + \beta^2\right]\left(\dfrac{1}{r_2^2} - \dfrac{1}{r_1^2}\right), & r_{1,2} > R_0. \end{cases} \tag{88}$$

From (88) it follows, that the form of a surface of the constant pressure (an isobar), which corresponds to the point of water drops evaporation P_{evp}, in the internal area $r \leqslant R_0$ is determined by the equation

$$z^2(r,t) + \frac{2\tilde{g}}{\alpha^2} z(r,t) + \frac{R_0^2}{\alpha^2}\left[\omega^2(t) + \omega^2(0)\right] - \frac{r^2}{\alpha^2}\left[\omega^2(t) - \beta^2\right] - \frac{2\left(P_\infty - P_{\text{evp}}\right)}{\rho\alpha^2} = 0, \tag{89}$$

where the coordinate z is counted from the initial flat surface $P_0 = P_\infty = P_{\text{evp}}$, and in the external area $r > R_0$ is set by the relation

$$z(r,t) = z_0(t) - \frac{\left[\omega^2(0) + \beta^2\right] R_0^4}{2gr^2} + \frac{\left(P_\infty - P_{\text{evp}}\right)}{\rho g}, \tag{90}$$

where the function $z_0(t)$ is determined from the condition of isobar continuity in the point $r = R_0$. According to equation (89), we find value coordinates $z(r,t)$ in the point $r = R_0$:

$$z(R_0,t) = -\frac{\tilde{g}}{\alpha^2} + \sqrt{\frac{\tilde{g}^2}{\alpha^4} - \frac{[\omega^2(0)+\beta^2]\,R_0^2}{\alpha^2} + \frac{2\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho\alpha^2}} = \mathrm{const}, \tag{91}$$

so that

$$z_0 = z(R_0) + \frac{R_0^2}{2g}\left[\omega^2(0)+\beta^2\right] - \frac{\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho g} = \mathrm{const}, \tag{92}$$

i.e. in the external area $r > R_0$ a form of the isobar does not depend on time.

The coordinate of a point of the isobar on a vortex axis at $r = 0$, according to (89)–(92), is determined by the expression

$$z(0,t) = -\frac{\tilde{g}}{\alpha^2} + \sqrt{\frac{\tilde{g}^2}{\alpha^4} - \frac{R_0^2\,[\omega^2(t)+\omega^2(0)]}{\alpha^2} + \frac{2\left(P_\infty - P_{\mathrm{evp}}\right)}{\rho\alpha^2}}. \tag{93}$$

From here it follows, that at $\tilde{g} > 0$, with growth of the angular velocity $\omega(t)$ in result of the exponential instability there is an increasing of the absolute value of the negative coordinate $z(0,t)$, what corresponds to the deepening of a minimum of function $z(r,t)$ in the area $z < 0$.

We can conclude, that the account of gravity and vertical flows allows to describe one of the main observable phenomena — a funnel formation on the bottom edge of a cloud during origination and development of tornados.

4.2. DISCUSSION OF APPLICABILITY OF THE DERIVED RESULTS FOR THE DESCRIPTION OF TORNADOS AND TYPHOONS

With the purpose of finding-out of applicability of the obtained new non-stationary solutions of Navier-Stokes and continuity equations for incompressible viscous media with a bulk sink and free inflow of the substance for the description of atmospheric vortices (tornados and typhoons) we will carry out numerical estimations of characteristic times of development of the considered above hydrodynamical instabilities of the vortical motion in conditions of intensive condensation of moisture inside a cloud.

Let us consider a round cylindrical cloud of a radius $R_0 \approx 1 \div 10$ km, which slowly rotates in the twirled air flow, with equal by order initial values of azimuthal and radial velocities on the border of a cloud $v_{r0} \approx v_{\varphi 0} \approx (1\div 10)$ m/s. This corresponds to absolute values of initial angular velocity $\omega(0)$ and velocities of a converging radial flow β in the range $(10^{-4} \div 10^{-2})$ s^{-1}. At humidity of atmospheric air of about 100%, its density almost twice exceeds density of the dry air $\rho = 1.3 \times 10^{-3}$ g/sm^3, so that the mentioned values of the parameter β are equivalent to capacities of a bulk sink Q due to condensation of a moisture of the order of $\sim 5 \times (10^{-7} \div 10^{-5})$ g/s·sm^3. In these conditions a characteristic time of acceleration of a vortex rotation in a result of exponential instability equals $\tau = 1/2\beta \approx (1 \div 100)$ min.

The maximal velocity of the order of c_s is reached for a time interval of $t \approx (3 \div 500)$ min. Typical observable times of origination and development of a tornado lie just in such a time interval (from several minutes to several hours).

For large-scale atmospheric vortices such as cyclones, hurricanes and typhoons, which originate in cloud masses with sizes of about 100 km and more, characteristic times of development of the instability increase by two – three orders and can reach several days, that also agrees with observable times of hurricane and typhoon existence. At the same time, the decreased pressure on the axis of a vortex caused by a cyclostrophic rotation regime explains the characteristic "suck" effect of a tornado.

Moreover, as it was marked above, in a two-phase system "air — water drops" a decreasing of the pressure below the boiling point of water at the given temperature should result in the termination of condensation of moisture and in evaporation of the drops. This explains the formation of a clear tornado "core" or a typhoon "eye" in the central part of a cloud system in the paraxial area of a vortex. Inside this area the velocity of a vortical rotation of the air slows down in time, as at a condition of $P < P_{\text{evp}}$ instead of a bulk sink ($Q < 0$) there is a source of a gas phase ($Q > 0$), and the parameter β changes its sign ($\beta < 0$), what corresponds to an exponential decelerating of the air. Such a condition of a dead calm is observed in the center "eye" of a typhoon.

It is possible to estimate the time of a tornado funnel contact with the surface of the Earth. In the case of exponential instability, at $z = -H$ (where H is the height of the bottom edge of a cloud above the ground), according to equation (93), at the conditions $\omega^2(t) \gg \beta^2$ and $\tilde{g}/\alpha^2 \gg H$ we find a time interval till this contact

$$t_H = \frac{1}{4\beta} \ln\left[\frac{\tilde{g}}{\omega^2(0)R_0^2}\right]. \tag{94}$$

At $H = 1$ km for the above mentioned values for $v_{\varphi 0}$ we get an estimation: $t_H \approx (1 \div 250)$ minutes, what in good correspondence with the observed phenomena.

5. Conclusions

In this paper a new class of exact solutions of hydrodynamics equations for an incompressible liquid (gas) at presence of a bulk sink and ascending flows of a substance has been considered. It is essential that one or several species of the system are gone under some dynamical process, for instance nuclear (chemical) reactions or phase transitions, from the general collective hydrodynamic motion. It is shown that those profiles, which nullify the terms in Navier-Stokes equation which describe viscous effects, exist and represent vortex structures with "rigid-body" rotation of the core and converging radial flows. In the case of constant bulk sink and inflow of the matter from the outside, the azimuthal velocity of a "rigid-body" rotation v_φ increases exponentially with time. At simultaneous infinite increasing of the sink and inflow rates, v_φ increases by scenario of the "explosive" instability when during a finite time interval the infinite rotation velocity is reached. We apply this hydrodynamic approach for description of the rotational motion of nuclear matter, which appears in the relativistic collisions of heavy nuclei with an initial angular momentum. We show that the acceleration of the rotation elocity of non-stationary vortices in nuclear matter has an explosive character.

On the basis of the developed theory of unstable hydrodynamical vortices an offered in [2] mechanism of origination and development of powerful atmospheric vortices — tornados and typhoons — during the intensive condensation of water vapor from the cooled below a dew point humid air at formation of dense rain clouds is considered. Within the framework of this mechanism it is possible to explain the basic characteristics of tornados and typhoons. Estimations of characteristic times of instability development of vortical motion are agreed by orders with the corresponding times of origination and existence of tornados (from several minutes till several hours) and typhoons (several days). With the account of gravity the given model describes the main phenomenon of a tornado — formation of a lengthening funnel on the bottom edge of a cloud in result of changing form of the surface of a constant pressure (isobar), that limits from the below the area of intensive condensation of moisture. The velocity step on the border of a tornado core results in the development of a strong turbulence, which can be described with the help of anomalous coefficient of turbulent viscosity, which in many orders exceeds the usual viscosity of the air.

Thus, the above considered non-stationary vortical structures are characteristic for all viscous liquids. In particular, in the classical hydrodynamics a favorable condition for the origination and existence of such vortices is nullification of the terms, which describe kinematic viscosity of the incompressible liquid, in the cases of cylindrical and spherical symmetries. Such flows have the minimal energy dissipation, i.e. correspond to a peculiar "minimal entropy production principle", and therefore relatively easily realize in the corresponding natural conditions.

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