Chapter 8
Wrinkling and Local Instabilities

Compression loaded faces of sandwich members are sometimes subject to local instability phenomena, the most prominent being the *wrinkling or rippling* and the *intracell buckling or dimpling*. This chapter presents the mechanics associated with these phenomena and the classical formulas that predict the conditions for inducing these forms of local instability.

8.1 Wrinkling

Wrinkling refers to a particular form of local instability of the compression faces of a sandwich panel in which the wavelength of the buckled form is of the same order as the thickness of the core. This short-wavelength instability of the faces can occur at lower load levels than the ordinary global, "Euler", buckling of the structure, which is characterized by a half-wavelength of the order of the compressed length of the sandwich panel.

As shown in Figure 8.1, in the case of the global, Euler, buckling, the core may exhibit a substantial shearing deformation; in the case of local wrinkling it acts like an elastic foundation and the buckling deformation is mainly confined to the layers adjacent to the face sheets. Wrinkling of a symmetric configuration can occur in a symmetric mode or an antisymmetric one (Figure 8.1).

Wrinkling will be considered for wide sandwich panels or sandwich columns. Thus, referring to Figure 8.2, the panel is so wide that lines along the y axis can be taken as uncurved. Therefore, a unit width can be treated as a Euler column.

The classical wrinkling formulas of Hoff and Mautner (1945), Plantema (1966), and Allen (1969) will be presented in the following.

L.A. Carlsson and G.A. Kardomateas, *Structural and Failure Mechanics of Sandwich Composites*, Solid Mechanics and its Applications 121, DOI 10.1007/978-1-4020-3225-7_8,

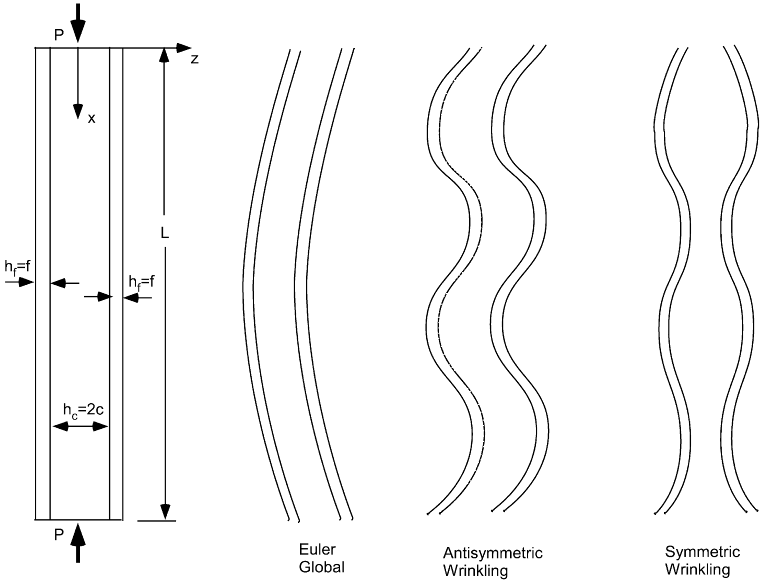

Figure 8.1 Buckling modes.

8.1.1 Hoff and Mautner's Formulas

I. Symmetric Wrinkling
The wrinkled state involves the bending of the faces of thickness $h_f = f$ and the elongation and compression of the core of thickness. $h_c = 2c$. A short wave on the surface can hardly have any effect upon the material in the middle of the core when c is large. It is assumed that the displacements occur only in marginal zones of depth d (Figure 8.3).

Assuming that the face undergoes a sinusoidal displacement and that the wave damps out linearly (linear decay) through the thickness, the transverse displacement, w of the top marginal zone is given with respect to the local coordinate system (x, z) in Figure 8.3, as follows:

$$w = \frac{Bz}{d} \sin \frac{\pi x}{a}. \tag{8.1}$$

where the origin of the z coordinate is at the boundary between the affected and unaffected core regions.

The critical load is now calculated from the requirement that the work done by the compressive force be equal to the strain energy of bending stored in the face material plus the strain energy of extension and shear stored in the core. Because of the symmetry, it is sufficient to calculate the work and the

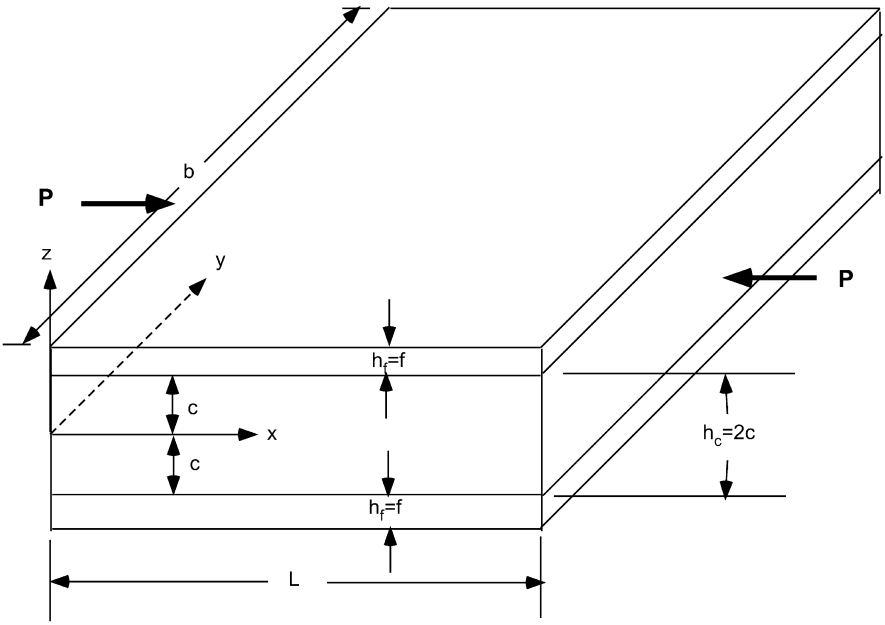

Figure 8.2 Definition of the geometry of a sandwich wide panel/beam under axial compression.

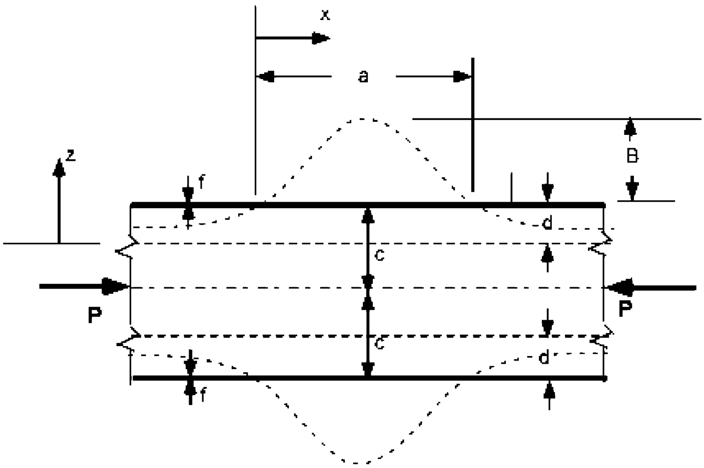

Figure 8.3 Hoff and Mautner's model for symmetric wrinkling.

strain energy for one-half of the cross-section. The width of the sandwich perpendicular to the plane of the drawing is taken as unity.

The normal (z directional) strain in the core is

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = \frac{B}{d} \sin \frac{\pi x}{a}. \tag{8.2}$$

Thus, the extensional strain energy of the core within one-half wave length becomes

$$U_e = \frac{E_c}{2} \int_0^a \int_0^d \varepsilon_{zz}^2 dx dz = \frac{E_c B^2 a}{4d}, \tag{8.3}$$

where E_c is Young's modulus of the core.

The axial displacement u is assumed to be negligibly small, thus the shear strain in the core is given by

$$\gamma_{xz} = \frac{\partial w}{\partial x} = \frac{\pi B z}{ad} \cos \frac{\pi x}{a}. \tag{8.4}$$

Hence, the shear strain energy stored within a half wave length in one half the core is

$$U_s = \frac{G_c}{2} \int_0^a \int_0^d \gamma_{xz}^2 dx dz = \frac{G_c \pi^2 B^2 d}{12a}. \tag{8.5}$$

The strain energy of bending stored in one face sheet is

$$U_f = \frac{E_f I_f}{2} \int_0^a \left(\frac{\partial^2 w_f}{\partial x^2} \right)^2 dx = \frac{\pi^4 E_f B^2 f^3}{48 a^3}, \tag{8.6}$$

where w_f is the transverse displacement from (8.1) at $z = d$, E_f is Young's modulus of the face material, and $I_f = f^3/12$ is the moment of inertia of the face cross-section.

Because of the smallness of f, the displacements at $z = d + (f/2)$ are taken to be equal to those at $z = d$. With this same assumption, the shortening ΔL of the face sheet can be calculated as

$$\Delta L = \frac{1}{2} \int_0^a \left(\frac{\partial w_f}{\partial x} \right)^2 dx = \frac{\pi^2 B^2}{4a}. \tag{8.7}$$

The compressive load carried by the core is neglected since Young's modulus of the face is typically hundreds to thousands of times that of the core. Thus, the work done is

$$W = (\sigma_{\mathrm{cr}}^f f) \Delta L = \sigma_{\mathrm{cr}}^f \frac{\pi^2 B^2 f}{4a}. \tag{8.8}$$

The equation

$$W = U_e + U_s + U_f \tag{8.9}$$

can be solved for σ_{cr}^f after substitution of the expressions thus developed:

$$\sigma_{\mathrm{cr}}^{f} = \frac{E_c a^2}{\pi^2 f d} + \frac{G_c d}{3f} + \frac{\pi^2 E_f f^2}{12a^2}. \tag{8.10}$$

The critical stress in this equation depends upon the parameters d and a. The actual values of these parameters are those which make the critical stress a minimum. Consequently, $\partial\sigma_{\mathrm{cr}}^{f}/\partial d$ and $\partial\sigma_{\mathrm{cr}}^{f}/\partial a$ must vanish:

$$\partial\sigma_{\mathrm{cr}}^{f}/\partial d = -\frac{E_c a^2}{\pi^2 f d^2} + \frac{G_c}{3f} = 0, \tag{8.11a}$$

$$\partial\sigma_{\mathrm{cr}}^{f}/\partial a = \frac{2E_c a}{\pi^2 f d} - \frac{\pi^2 E_f f^2}{6a^3} = 0, \tag{8.11b}$$

Simultaneous solution of the two equations gives

$$d/f = 0.91\sqrt[3]{\frac{E_f E_c}{G_c^2}}, \tag{8.12a}$$

$$a/f = 1.65\sqrt[6]{\frac{E_f^2}{E_c G_c}}, \tag{8.12b}$$

Substitution in the equation for the critical stress yields

$$\sigma_{\mathrm{cr}}^{f} = 0.91\sqrt[3]{E_f E_c G_c}\,. \tag{8.13}$$

This formula is correct only if d is smaller than or equal to c, as may be seen from Figure 8.3. Thus, from (8.12a), the condition for validity is

$$0.91 f\sqrt[3]{\frac{E_f E_c}{G_c^2}} \leq c, \tag{8.14a}$$

that is the core to face thickness ratio,

$$\frac{2c}{f} \geq 1.82\sqrt[3]{\frac{E_f E_c}{G_c^2}}. \tag{8.14b}$$

When this inequality does not hold, then the marginal zone depth is equal to half the core thickness, so equation (8.1) is replaced by

$$w = \frac{Bz}{c}\sin\frac{\pi x}{a}. \tag{8.15}$$

The expression for the critical stress then becomes

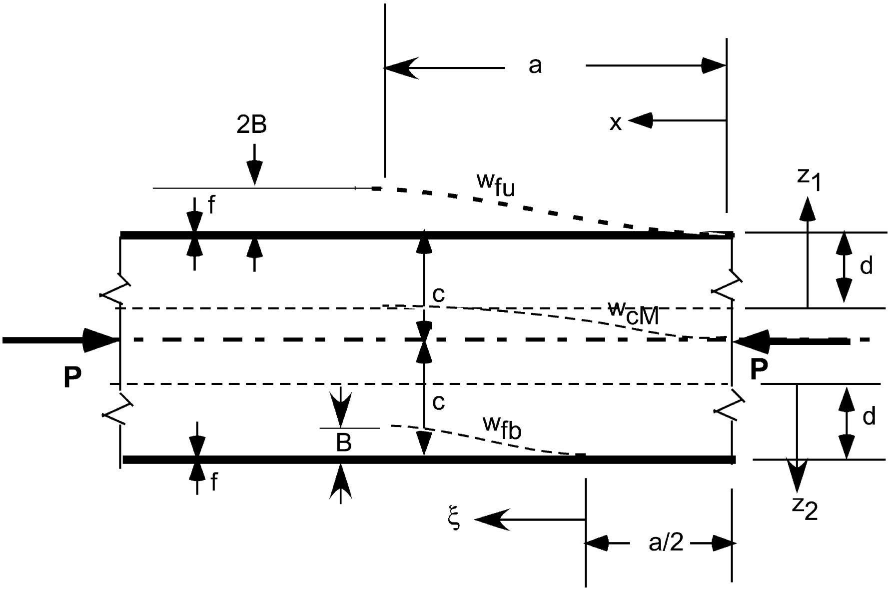

Figure 8.4 Hoff and Mautner's assumed shape for anti-symmetric wrinkling.

$$\sigma_{\text{cr}}^{f} = \frac{E_c a^2}{\pi^2 f c} + \frac{G_c c}{3f} + \frac{\pi^2 E_f f^2}{12a^2}. \tag{8.16}$$

The derivative of σ_{cr}^{f} with respect to a must again vanish. This condition yields

$$a/f = 1.42\sqrt[4]{\frac{E_f}{E_c}}\sqrt[4]{\frac{2c}{f}}. \tag{8.17}$$

Substitution in Equation (8.16) gives

$$\sigma_{\text{cr}}^{f} = 0.577\sqrt{E_f E_c}\sqrt{\frac{f}{c}} + 0.333G_c\left(\frac{c}{f}\right). \tag{8.18}$$

II. Anti-Symmetric Wrinkling
In this case the deflected shape is assumed by Hoff and Mautner as indicated by the dashed line in Figure 8.4. Again marginal zones of depth d are assumed to which all the extensions are restricted, although shear deformation occurs throughout the entire core.

For $0 \le x \le a/2$ (i.e. the right segment of the figure), the displacement of the upper face is assumed to be

$$w_{fu} = B\left(1 - \cos\frac{\pi x}{a}\right), \tag{8.19}$$

whereas the displacement of the bottom face is

$$w_{fb} = 0. \tag{8.20}$$

Consequently, the displacement w_{cM} of the median line of the core is assumed to be

$$w_{cM} = \frac{B}{2}\left(1 - \cos\frac{\pi x}{a}\right). \tag{8.21}$$

There are two marginal zones, one for each face sheet. The displacement of any point in the upper marginal zone is assumed to be

$$w_{cu} = \frac{B}{2}\left(1 - \cos\frac{\pi x}{a}\right) + \frac{B}{2}\frac{z_1}{d}\left(1 - \cos\frac{\pi x}{a}\right), \quad \text{for } 0 \le z_1 \le d. \tag{8.22a}$$

In the bottom marginal zone

$$w_{cb} = -\frac{B}{2}\left(1 - \cos\frac{\pi x}{a}\right) + \frac{B}{2}\frac{z_2}{d}\left(1 - \cos\frac{\pi x}{a}\right), \quad \text{for } 0 \le z_2 \le d. \tag{8.22b}$$

In the middle portion of the core, the displacement depends on z and is assumed to be as in (8.21), i.e.

$$w_{cm} = w_{cM} = \frac{B}{2}\left(1 - \cos\frac{\pi x}{a}\right), \tag{8.22c}$$

The extensional strain in the upper marginal zone is

$$\varepsilon_{cu} = \frac{\partial w_{cu}}{\partial z_1} = \frac{B}{2d}\left(1 - \cos\frac{\pi x}{a}\right). \tag{8.23a}$$

In the lower marginal zone,

$$\varepsilon_{cb} = \frac{\partial w_{cb}}{\partial z_2} = \frac{B}{2d}\left(1 - \cos\frac{\pi x}{a}\right) = \varepsilon_{cu}. \tag{8.23b}$$

In the middle portion, the extensional strain is zero.

Consequently, the extensional strain energy is

$$U_\varepsilon = 2\frac{E_c}{2}\int_0^{a/2}\int_0^d \varepsilon_{cu}^2 \, dx\, dz_1 = E_c\frac{B^2 a(3\pi - 8)}{16\pi d}. \tag{8.24}$$

The shear strain in the upper marginal zone is (axial displacement again assumed to be negligibly small):

$$\gamma_{cu} = \frac{\partial w_{cu}}{\partial x} = \frac{B}{2}\left(1 + \frac{z_1}{d}\right)\frac{\pi}{a}\sin\frac{\pi x}{a}. \tag{8.25a}$$

In the bottom marginal zone

$$\gamma_{cb} = \frac{\partial w_{cb}}{\partial x} = -\frac{B}{2}\left(1 - \frac{z_2}{d}\right)\frac{\pi}{a}\sin\frac{\pi x}{a}. \tag{8.25b}$$

In the middle portion of the core

$$\gamma_{cm} = \frac{\partial w_{cm}}{\partial x} = \frac{B}{2}\frac{\pi}{a}\sin\frac{\pi x}{a}. \tag{8.25c}$$

The shear strain energy can be now calculated from

$$U_\gamma = \frac{G_c}{2}\int_0^{a/2}\left[\int_0^d \gamma_{cu}^2 dz_1 + \int_0^d \gamma_{cb}^2 dz_2 + \gamma_{cm}^2(2c - 2d)\right]dx. \tag{8.26a}$$

Integration gives

$$U_\gamma = \frac{\pi^2 B^2 G_c}{48a}(3c + d). \tag{8.26b}$$

In the middle segment of the figure, i.e. for

$$\frac{a}{2} \le x \le a, \quad 0 \le \xi \le \frac{a}{2},$$

the displacements of the faces and the median line of the core, respectively, are assumed to be

$$w_{fu} = B\left(1 + \sin\frac{\pi\xi}{a}\right), \tag{8.27a}$$

$$w_{fb} = B\left(1 - \cos\frac{\pi\xi}{a}\right). \tag{8.27b}$$

$$w_{cM} = B\left(1 - \frac{1}{2}\cos\frac{\pi\xi}{a} + \frac{1}{2}\sin\frac{\pi\xi}{a}\right). \tag{8.27c}$$

The displacements of points in the upper marginal zone, the middle portion and the bottom marginal zone, respectively, are

$$w_{cu} = B\left(1 - \frac{1}{2}\cos\frac{\pi\xi}{a} + \frac{1}{2}\sin\frac{\pi\xi}{a}\right) + \frac{Bz_1}{d}\left(\frac{1}{2}\sin\frac{\pi\xi}{a} + \frac{1}{2}\cos\frac{\pi\xi}{a}\right), \tag{8.28a}$$

$$w_{cm} = w_{cM} = B\left(1 - \frac{1}{2}\cos\frac{\pi\xi}{a} + \frac{1}{2}\sin\frac{\pi\xi}{a}\right), \tag{8.28b}$$

$$w_{cb} = B\left(-1 + \frac{1}{2}\cos\frac{\pi\xi}{a} - \frac{1}{2}\sin\frac{\pi\xi}{a}\right) + \frac{Bz_2}{d}\left(\frac{1}{2}\sin\frac{\pi\xi}{a} + \frac{1}{2}\cos\frac{\pi\xi}{a}\right). \tag{8.28c}$$

The corresponding extensional strains are

$$\varepsilon_{cu} = \frac{\partial w_{cu}}{\partial z_1} = \frac{B}{2d}\left(\sin\frac{\pi\xi}{a} + \cos\frac{\pi\xi}{a}\right) = \varepsilon_{cb} = \frac{\partial w_{cb}}{\partial z_2}; \quad \varepsilon_{cm} = 0. \quad (8.29)$$

The shear strains are

$$\gamma_{cu} = \frac{\partial w_{cu}}{\partial \xi} = \frac{B\pi}{2a}\left(\sin\frac{\pi\xi}{a} + \cos\frac{\pi\xi}{a}\right) + \frac{B\pi}{2a}\frac{z_1}{d}\left(\cos\frac{\pi\xi}{a} - \sin\frac{\pi\xi}{a}\right). \quad (8.30a)$$

$$\gamma_{cm} = \frac{\partial w_{cm}}{\partial \xi} = \frac{B\pi}{2a}\left(\sin\frac{\pi\xi}{a} + \cos\frac{\pi\xi}{a}\right). \quad (8.30b)$$

$$\gamma_{cb} = \frac{\partial w_{cb}}{\partial \xi} = -\frac{B\pi}{2a}\left(\sin\frac{\pi\xi}{a} + \cos\frac{\pi\xi}{a}\right) + \frac{B\pi}{2a}\frac{z_2}{d}\left(\cos\frac{\pi\xi}{a} - \sin\frac{\pi\xi}{a}\right). \quad (8.30c)$$

The strain energy can be calculated as before (Equations (8.24) and (8.26a) with ξ in place of x). Integration yields

$$U_\varepsilon = E_c\frac{B^2 a(\pi + 2)}{8d\pi}, \quad (8.31a)$$

$$U_\gamma = G_c\frac{B^2\pi}{8a}\left[(\pi + 2)c + (\pi - 2)\frac{d}{3}\right]. \quad (8.31b)$$

Because of the point symmetry of the distorted shape, the strain energy in the left half shown in Figure 8.4 is the same as in the right half. The total strain energy U_c stored in the core is, therefore, the sum of the strain energies given in Equations (8.31) plus twice those given in Equations (8.24) and (8.26b), i.e.,

$$U_c = E_c\frac{B^2 a(2\pi - 3)}{4\pi d} + G_c\frac{B^2\pi}{12a}\left[3c(\pi + 1) + (\pi - 1)d\right]. \quad (8.32)$$

The strain energy of bending stored in the two faces, U_f, is found by substituting w_f into $(E_f I_f/2)\int w_f''^2 dx$ where w_f is obtained from (8.27a) and (8.27b) plus twice the contributions from (8.19) and (8.20), i.e.,

$$U_f = E_f\frac{B^2\pi^4 f^3}{24a^3}. \quad (8.33)$$

Similarly, the work W done by the external forces is found from $(\sigma_{\mathrm{cr}}^f A_f/2)\int w_f'^2 dx$ where A_f is the cross-sectional area of each face sheet, and w_f is given by Equations (8.27a) and (8.27b) plus twice the contributions from (8.19) and (8.20), i.e.,

$$W = \sigma_{\mathrm{cr}}^f\frac{B^2\pi^2 f}{2a}. \quad (8.34)$$

Substitution in the equation

$$W = U_f + U_c$$

and solution for σ_{cr}^f yields

$$\sigma_{\mathrm{cr}}^f = \beta_1 \left(\frac{f}{a}\right)^2 + \beta_2 \left(\frac{a^2}{df}\right) + \beta_3 \left(\frac{c}{f}\right) + \beta_4 \left(\frac{d}{f}\right), \tag{8.35a}$$

where

$$\beta_1 = \frac{\pi^2 E_f}{12}; \quad \beta_2 = \frac{(2\pi - 3)E_c}{2\pi^3}; \quad \beta_3 = \frac{(\pi + 1)G_c}{2\pi}; \quad \beta_4 = \frac{(\pi - 1)G_c}{6\pi}. \tag{8.35b}$$

Minimization with respect to a and d, yields

$$\frac{\partial \sigma_{\mathrm{cr}}^f}{\partial a} = -\frac{\beta_1 f^2}{a^4} + \frac{\beta_2}{df} = 0; \quad \frac{\partial \sigma_{\mathrm{cr}}^f}{\partial d} = -\frac{\beta_2 a^2}{d^2 f} + \frac{\beta_4}{f} = 0. \tag{8.36}$$

Solution of the two equations yields

$$a/f = \left(\frac{\beta_1^2}{\beta_2 \beta_4}\right)^{1/6} = 2.19 \sqrt[6]{\frac{E_f^2}{E_c G_c}}, \tag{8.37a}$$

$$d/f = \left(\frac{\beta_1 \beta_2}{\beta_4^2}\right)^{1/3} = 1.50 \sqrt[3]{\frac{E_f E_c}{G_c^2}}. \tag{8.37b}$$

Substitution into (8.35a) results in the expression

$$\sigma_{\mathrm{cr}}^f = 3\,(\beta_1 \beta_2 \beta_4)^{1/3} + \beta_3 (c/f) = 0.51 \sqrt[3]{E_f E_c G_c} + 0.66 G_c (c/f). \tag{8.38}$$

These formulas are valid only if d proves to be smaller than or equal to c, hence if

$$c/f \leq 1.50 \sqrt[3]{\frac{E_f E_c}{G_c^2}}. \tag{8.39}$$

When this inequality is not satisfied, d must be replaced by c in the expressions assumed for the deflected shapes. If the calculations are carried out on the basis of this assumption, the following expression is obtained for the critical stress (by replacing d with c in (8.35a)):

$$\sigma_{\mathrm{cr}}^f = \frac{\beta_1 f^2}{a^2} + \frac{\beta_2}{cf} a^2 + (\beta_3 + \beta_4) \frac{c}{f}. \tag{8.40a}$$

The critical stress is a minimum when

$$\partial \sigma_{\mathrm{cr}}^{f}/\partial a = -\frac{\beta_1 f^2}{a^4} + \frac{\beta_2}{cf} = 0. \tag{8.40b}$$

Consequently,

$$a^4 = \frac{\beta_1 f^3 c}{\beta_2}$$

and

$$a/f = 1.98\sqrt[4]{E_f/E_c}\sqrt[4]{c/f}. \tag{8.41}$$

Substitution into Equation (8.40a) gives

$$\begin{aligned}\sigma_{\mathrm{cr}} &= 2\sqrt{\beta_1\beta_2 f/c} + (\beta_3 + \beta_4)c/f \\ &= 0.417\sqrt{E_f E_c}\sqrt{f/c} + 0.773 G_c(c/f).\end{aligned} \tag{8.42}$$

Summary: Hoff and Mautner's wrinkling formulas are (i) for symmetric wrinkling, the critical stress is from Equation (8.13) provided the inequality (8.14b) is fulfilled; if not then the critical wrinkling stress is from Equation (8.18); (ii) for anti-symmetric wrinkling, the critical stress is from Equation (8.38) provided the inequality (8.39) is fulfilled; if not then the critical wrinkling stress is obtained from Equation (8.42). Calculations made by Hoff and Mautner (1945) for a sandwich consisting of papreg faces (a paper/plastic laminate) and a cellular cellulose acetate core showed that the anti-symmetric wrinkling would dominate (i.e. the critical stress corresponding to anti-symmetry would be less than that for symmetry) if the core thickness of face thickness ratio, $2c/f$, is smaller than 20.6. This is, of course, only valid for this particular sandwich material system and is not a general conclusion, but it gives some idea of the level of the parameters influencing the occurrence of symmetric vs. anti-symmetric wrinkling. Nevertheless, Equation (8.13) for symmetric wrinkling is the most popular wrinkling formula and the one mostly used, albeit with a factor of 0.5 instead of 0.91 for safety (e.g. Zenkert, 1997).

8.1.2 Plantema's Formula

The Plantema (1966) analysis assumes that the points of the core undergo vertical displacements with an exponential decay. With z defined from the upper face sheet (Figure 8.5) and n denoting the number of half

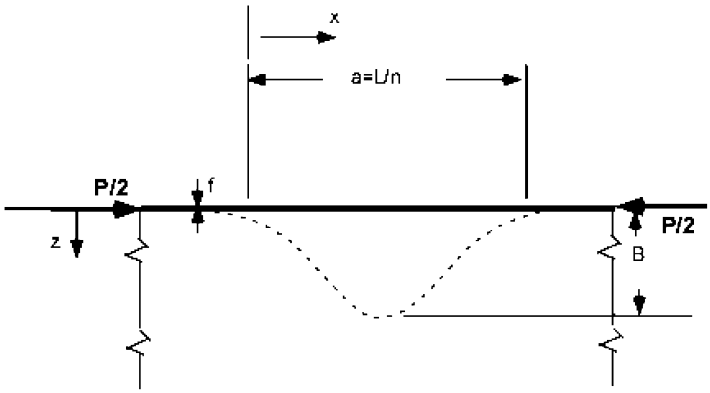

Figure 8.5 Plantema's wrinkling model.

waves over the length L of the panel, the transverse displacement expression (8.1) becomes

$$w = Be^{-kz} \sin \frac{n\pi x}{L}. \tag{8.43}$$

Strictly speaking, this equation implies that the core is infinitely thick. It is also assumed that vertical lines remain vertical during wrinkling ($\partial u/\partial z = 0$). Notice that no marginal depth is assumed in this model.

The transverse normal stress σ_{zz} and the shear stress τ_{xz} in the core are

$$\sigma_{zz} = E_c \frac{\partial w}{\partial z} \quad \text{and} \quad \tau_{xz} = G_c \frac{\partial w}{\partial x}. \tag{8.44}$$

The total strain energy per unit width consists of the bending strain energy in the face and the strain energy in the core associated with the stresses σ_{zz} and τ_{xz} computed from

$$\begin{aligned} U &= \frac{E_f I_f}{2} \int_0^L \left(\frac{\partial^2 w_f}{\partial x^2} \right)^2 dx + \frac{1}{2E_c} \int_0^L \int_0^\infty \sigma_{zz}^2 dx dz \\ &\quad + \frac{1}{2G_c} \int_0^L \int_0^\infty \tau_{xz}^2 dx dz \\ &= B^2 \left(\frac{\pi^4 n^4}{4L^3} E_f I_f + \frac{kL}{8} E_c + \frac{\pi^2 n^2}{8kL} G_c \right), \end{aligned} \tag{8.45}$$

where $I_f = f^3/12$ is the moment of inertia of the face cross-section. For a plate, the first term in the above equation, $E_f I_f$ should be replaced by $E_f I_f/(1 - \nu_f^2)$ where ν_f is the Poisson ratio of the face.

In the same manner as for Hoff and Mautner, the stress $\sigma_{xx,c}$ in the core is neglected in comparison with the stress $\sigma_{xx,f}$ in the faces, and the work done by the external load, $P/2$ per unit width, becomes

$$W = \frac{P}{4}\int_0^L \left(\frac{\partial w_f}{\partial x}\right)^2 dx = \frac{\pi^2 n^2 B^2}{8L} P. \tag{8.46}$$

The total potential is $U - W$ and it contains the parameters B, n and k. Minimizing this total potential with respect to B, n, and k, and assuming n to be a continuous variable, we obtain

$$a = \frac{L}{n} = 1.26\pi \left[\frac{(E_f I_f)^2}{E_c G_c}\right]^{1/6}; \quad k = \left[\frac{G_c^2}{2(E_f I_f)E_c}\right]^{1/3};$$
$$P = 3[2(E_f I_f)E_c G_c]^{1/3}. \tag{8.47}$$

Again, for a plate, $E_f I_f$ should be replaced by $E_f I_f/(1-\nu_f^2)$.

Strictly speaking, these equations are valid only for a plate length $L = \infty$, but in practice, n will be so large that they may be used for finite values of L. The ratio $a = L/n$ is the half wavelength of the wrinkles.

For a sandwich plate of unit width, substituting $I_f = f^3/(1-\nu_f^2)$, with $\nu_f = 0.3$, yields

$$a = 1.78f\left(\frac{E_f^2}{E_c G_c}\right)^{1/6}; \quad k = \frac{1.76}{f}\left(\frac{G_c^2}{E_f E_c}\right)^{1/3};$$
$$P = 1.70f(E_f E_c G_c)^{1/3}. \tag{8.48}$$

It should be noted that the wrinkling stress, $P/(2f)$, is independent of the face thickness, f.

8.1.3 Allen's Formula

Allen's (1969) formula is based on simplifying the face wrinkling problem to that of an infinitely long strut (face sheet) attached to an elastic medium (core), which extends to infinity on one side of the strut. To this extent, the geometry is the same as that in Plantema's model (Figure 8.5). The strut (representing the face sheet of a sandwich with a core of infinite thickness) is assumed to be of rectangular section, of thickness f and width b in the y direction.

The differential equation of the face is

$$D_f \frac{d^4 w_f}{dx^4} + (P/2)\frac{d^2 w_f}{dx^2} = b\sigma_{zz}^0, \tag{8.49}$$

where D_f is the flexural rigidity of the face, P is the axial force, w_f is the displacement of the face in the z direction, and $b\sigma_{zz}^0$ is the transverse load on the face (σ_{zz}^0 is the normal stress between face and core).

Suppose that the face buckles into sinusoidal waves with half-wavelength a, as in Section 8.2, i.e.

$$w_f = B \sin \frac{\pi x}{a}. \tag{8.50}$$

An Airy stress function approach is used to derive the normal stress in the core, σ_{zz}. In particular, the stress function

$$\phi(x, z) = A(1 - dz)e^{-\pi z/a} \sin \frac{\pi x}{a}, \tag{8.51}$$

satisfies the bi-harmonic

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \phi}{\partial z^4} = 0$$

and results in stresses

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial x^2} = -A(1 - dz)\frac{\pi^2}{a^2} e^{-\pi z/a} \sin \frac{\pi x}{a}, \tag{8.52a}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial z^2} = \left[(1 - dz)\frac{\pi}{a} + 2d\right] A \frac{\pi}{a} e^{-\pi z/a} \sin \frac{\pi x}{a}. \tag{8.52b}$$

At the face/core interface, $z = 0$

$$\sigma_{zz}^0 = \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{z=0} = -A\frac{\pi^2}{a^2} \sin \frac{\pi x}{a}, \tag{8.53a}$$

$$\sigma_{xx}^0 = \left.\frac{\partial^2 \phi}{\partial z^2}\right|_{z=0} = A\frac{\pi}{a}\left(2d + \frac{\pi}{a}\right) \sin \frac{\pi x}{a}. \tag{8.53b}$$

It is assumed that the face is attached to the surface of the core and is permitted to deform in the z direction only; there are no x displacements. Consequently, the axial strain at the face/core interface, e_{xx}^0, is zero, which gives

$$\begin{aligned} e_{xx}^0 &= \left.\frac{\partial u}{\partial x}\right|_{z=0} = \frac{1}{E_c}\left(\sigma_{xx} - \nu_c \sigma_{zz}\right)|_{z=0} \\ &= \frac{A}{E_c}\frac{\pi}{a}\left[2d + \frac{\pi}{a}(1 + \nu_c)\right] \sin \frac{\pi x}{a} = 0, \end{aligned} \tag{8.54}$$

where E_c and ν_c are the modulus of elasticity and the Poisson ratio of the core.

Therefore, the constant d must take the value

$$d = -\frac{\pi}{2a}(1 + \nu_c). \tag{8.55}$$

The displacement w is found by integrating the normal strain equation

$$\frac{\partial w}{\partial z} = e_{zz} = \frac{1}{E_c}\left(\sigma_{zz} - \nu_c \sigma_{xx}\right), \tag{8.56a}$$

resulting in

$$w = \left\{\left[(1 + \nu_c)\frac{\pi}{a} + (\nu_c - 1)d\right]\frac{a}{\pi} - d(1 + \nu_c)z\right\}\frac{A}{E_c}\frac{\pi}{a}e^{-\pi z/a}\sin\frac{\pi x}{a}. \tag{8.56b}$$

Substituting the expression for d, (8.55), and evaluating w at the face/core interface, $z = 0$, gives

$$w_f = w|_{z=0} = \frac{\pi}{2a}(1 + \nu_c)(3 - \nu_c)\frac{A}{E_c}\sin\frac{\pi x}{a}. \tag{8.57a}$$

Combining with (8.50) gives A in terms of B:

$$A = \frac{BE_c}{(1 + \nu_c)(3 - \nu_c)}\frac{2a}{\pi}, \tag{8.57b}$$

which can be substituted in the expression for σ_{zz}^0 in (8.53a) which gives

$$\sigma_{zz}^0 = -\frac{B}{a}\frac{2\pi E_c}{(3 - \nu_c)(1 + \nu_c)}\sin\frac{\pi x}{a}. \tag{8.58}$$

Substituting for w_f and σ_{zz}^0 from Equations (8.50) and (8.58) into (8.49) yields an expression in which $(B \sin \pi x/a)$ cancels, leaving the result

$$D_f\frac{\pi^4}{a^4} - (P/2)\frac{\pi^2}{a^2} = -\frac{2\pi E_c b}{(3 - \nu_c)(1 + \nu_c)a}. \tag{8.59}$$

This equation defines the critical value of P for a given a.

It is convenient to write

$$D_f = \frac{E_f bf^3}{12}; \quad P/2 = \sigma_f bf, \tag{8.60}$$

where E_f is the modulus of elasticity of the face and σ_f is the compressive stress in the face. Then from (8.59) (and after cancelling b), we obtain

$$\sigma_f = \frac{\pi^2 E_f}{12}\left(\frac{f}{a}\right)^2 + \frac{2E_c}{\pi(3-\nu_c)(1+\nu_c)}\left(\frac{a}{f}\right), \tag{8.61}$$

i.e., a function of a/f.

There is one last step in order to determine the critical wrinkling stress. We have to minimize σ_f from (8.61) with respect to a/f:

$$\frac{d\sigma_f}{d(a/f)} = 0. \tag{8.62}$$

This gives the following values for the critical stress and the half-wavelength at which it occurs:

$$\sigma_f^{\mathrm{cr}} = B_1 E_f^{1/3} E_c^{2/3}; \quad \text{where} \quad B_1 = 3\left[12(3-\nu_c)^2(1+\nu_c)^2\right]^{-1/3}. \tag{8.63a}$$

$$\left(\frac{a}{f}\right)_{\mathrm{cr}} = C\left(\frac{E_f}{E_c}\right)^{1/3}; \quad \text{where} \quad C = \pi\left[(3-\nu_c)(1+\nu_c)/12\right]^{1/3}. \tag{8.63b}$$

8.1.4 The Winkler Elastic Foundation Approach

This approach assumes that the core supports the faces as an elastic foundation, i.e. an array of continuously distributed linear springs, as shown in Figure 8.6. In the anti-symmetrical case, it is seen that the springs remain unloaded even after wrinkling and, furthermore, the mode of deformation in the core is shear rather than tension/compression, which the set of springs is unable to model and, hence, no solution can be derived in this case.

In the symmetrical case, on the other hand, the model becomes more realistic since the mode of deformation in the core is both tension/compression and shear. We can refer again to Figure 8.5 and we suppose that the elastic foundation modulus is λ, which is the force needed to displace a unit area of the face through a unit distance in the z direction. Then the corresponding normal stress at the face/core interface (tension positive) is

$$\sigma_{zz}^0 = -\lambda w_f. \tag{8.64}$$

Substituting Equation (8.64) into the differential equation of the face, Equation (8.49), leads to

$$\frac{d^4 w_f}{dx^4} + \frac{(P/2)}{D_f}\frac{d^2 w_f}{dx^2} + \frac{b\lambda}{D_f} w_f = 0. \tag{8.65}$$

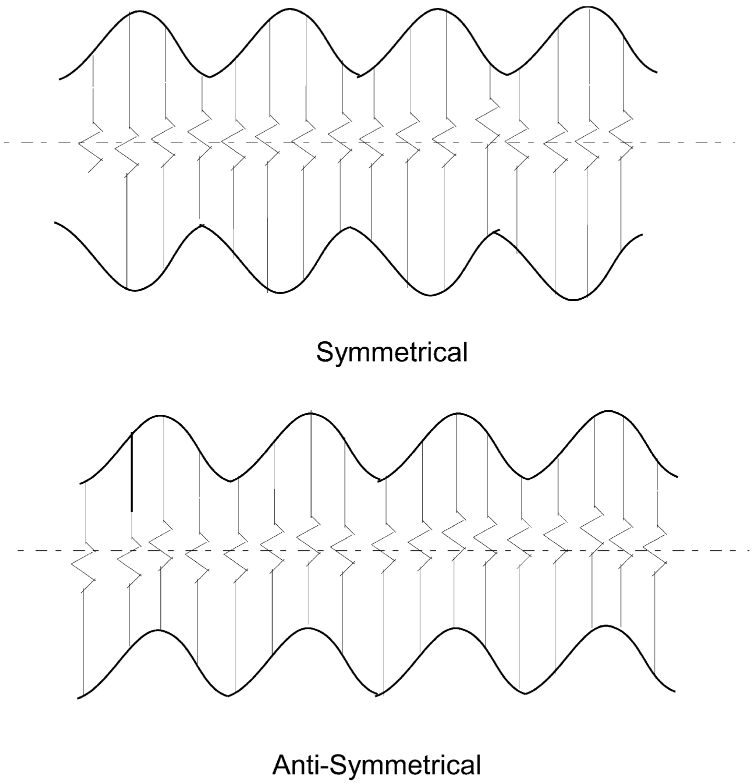

Figure 8.6 Winkler elastic foundation model.

The face will assume the same displacement as in Equation (8.50), which, when substituted into Equation (8.65), yields an expression for P:

$$P/2 = D_f\left(\frac{\pi}{a}\right)^2 + b\lambda\left(\frac{a}{\pi}\right)^2. \tag{8.66}$$

To find the critical load we must further minimize P with respect to the unknown wavelength, a, by setting $dP/da = 0$, which gives

$$a^4 = \frac{\pi^4 D_f}{b\lambda} \tag{8.67}$$

and the corresponding critical load from Equation (8.66) is

$$P_{\mathrm{cr}}/2 = 2\sqrt{D_f b\lambda}. \tag{8.68}$$

An expression for λ in terms of the properties of the core can be found as follows: since the springs are assumed to be linear, the core stress must be independent of z, therefore the core strain is also independent of z and equal to

$$\varepsilon_{zz}^{c}(x,z) = -\frac{\sigma_{zz}^{0}}{E_c} = -\frac{\lambda w_f(x)}{E_c}. \tag{8.69}$$

Integrating with respect to z gives the core displacement as

$$w_c(x,z) = -\frac{\lambda w_f(x)}{E_c} z + e(x). \tag{8.70}$$

Next we use the fact that w_c must be zero for $z = c$ (Figure 8.6), which gives:

$$e(x) = \frac{\lambda w_f(x)}{E_c} c. \tag{8.71}$$

Therefore,

$$w_c(x,z) = -\frac{\lambda w_f(x)}{E_c}(z - c). \tag{8.72}$$

At $z = 0$,

$$w_c(x,z)|_{z=0} = w_f, \tag{8.73}$$

which gives

$$\lambda = \frac{E_c}{c}. \tag{8.74}$$

This provides the critical load from Equation (8.68), after substituting the expression for D_f from (8.60), as

$$P_{\mathrm{cr}}/2 = 2bf\sqrt{\frac{E_f E_c f}{12c}}, \tag{8.75}$$

or the critical stress in the face

$$\sigma_{\mathrm{cr}}^{f} = \frac{P_{\mathrm{cr}}/2}{bf} = 2\sqrt{\frac{E_f E_c f}{12c}}. \tag{8.76}$$

8.1.5 Example and Comparison of the Wrinkling Formulas

Let us now perform a comparison of the predictions from these different formulas. We consider a sandwich with isotropic face and core materials with $E_f/E_c = 1{,}000$ and $\nu_c = 0$ (this case has been historically emphasized in the early sandwich literature). Also, the plate length to total thickness ratio, $L/h = 5$. We will examine a range of face thicknesses defined by the

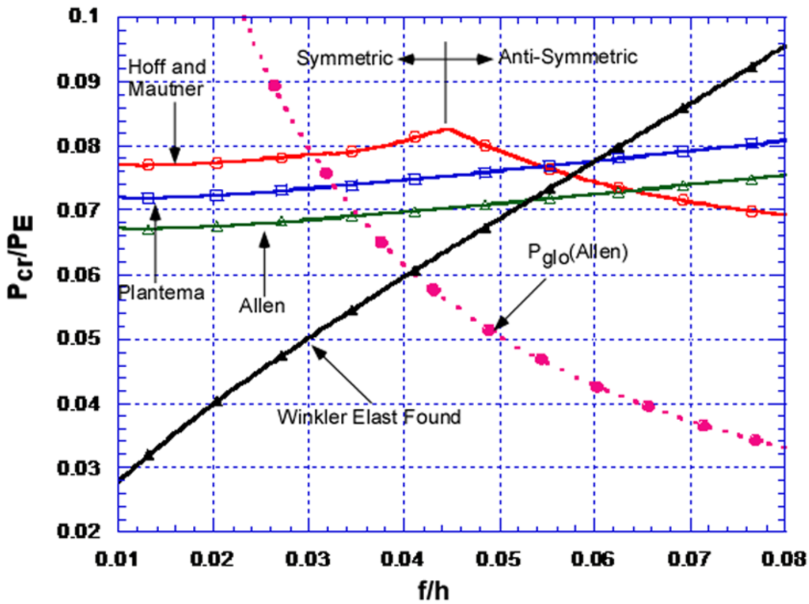

Figure 8.7 Wrinkling load calculated from different formulas for a range of thickness ratios, f/h.

ratio of face sheet thickness over total thickness, f/h, between 0.01 and 0.08. Figure 8.7 shows the critical wrinkling load, normalized with the Euler load for a simply supported configuration (without transverse shear), $P_E = \pi^2(EI)_{\text{eq}}/L^2$. The figure also shows the critical global buckling load, based on Allen's formula, which includes transverse shear (see Chapter 7).

We can observe the following: (a) for this configuration, wrinkling definitely dominates for ratios f/h below about 0.03. Above this level, global buckling would dominate, although there is a "gray" area between 0.03 and about 0.04, in which global buckling dominance would depend on the wrinkling formula used; (b) the Winkler elastic foundation formula seems to be outside the "cluster" of the other formulas; (c) for the range of f/h where wrinkling dominates, all formulas with the exception of the Winkler formula, are close to each other; among these, the Allen formula is most conservative and Hoff and Mautner's the least; (d) based on Hoff and Mautner's approach, there is a switch from symmetric to anti-symmetric wrinkling at a ratio f/h of about 0.044. This is very evident on the curve from the marked discontinuity in the slope of the curve at this point.

An elasticity solution of the wrinkling problem was presented by Kardomateas (2005) and the same configuration as the above was examined.

This solution can serve as a benchmark against which all these theories can be compared. In that work, the elasticity solution was found to be closest to the Hoff and Mautner prediction, which, however, was above the elasticity solution. On the contrary, Allen's formula although not the most accurate, was always below the elasticity value. The case of orthotropic layers was also examined and again Allen's formula was always conservative.

It should be emphasized that the possibility of global buckling should always be considered as transverse shear drives the critical load for global buckling to just a fraction of the Euler load (see Figure 8.7).

8.2 Intracell Buckling in Honeycomb Core Sandwich Structures

Intracell buckling is a local buckling phenomenon which can take place in honeycomb sandwich structures. Intracell buckling, also found in the literature by the terms "dimpling" or "intercell buckling", is the buckling of the face sheet within an individual honeycomb cell. Although the honeycomb structure will retain part of its load-carrying capacity after intracell buckling has occurred, the buckled face sheet may have undesirable influences, for example, it can adversely affect the aerodynamic properties of the structure; thus, it is important to know the load at which this intracell buckling occurs.

A contrast can be made with the previously treated phenomenon of wrinkling, which is local face sheet buckling over a row of several honeycomb cells (i.e., the buckling wavelength is greater than the honeycomb cell size). Wrinkling would appear as a sharp trough on the face sheet and would be accompanied either by the crushing of the honeycomb core or the separation of the core and face sheet at this sharp trough. When wrinkling occurs, for all practical purposes, the honeycomb structure has lost its load carrying capacity.

The oldest intracell buckling formula is the one suggested by Norris and Krommer (1950), as derived empirically from tests. An early theoretical investigation was also done by Weikel and Kobayashi (1959).

8.2.1 The Norris Formula

We shall consider two cases of a sandwich panel with square, honeycomb cells loaded in uniaxial compression: (a) the honeycomb cells are oriented such that this load is parallel to the diagonal of the cell (Figure 8.8) and

(b) the honeycomb cells are oriented such that this load is parallel to a cell wall (Figure 8.9). In both cases, we assume that the honeycomb cell is square in shape, a being the length of each side. We also assume that, in both cases, the face sheet is simply supported along its four edges at the cell and that the thickness of the honeycomb core is large compared to the cell size.

Within these assumptions, let us approximate the face sheet deflection by a one-term double trigonometric series:

$$w = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}. \tag{8.77}$$

In connection with this assumed displacement profile (8.77), we generally assume a deflection of the form $\sin(m\pi x/a)\sin(n\pi y/b)$, where a and b are the sides of the rectangle. However, for the simple square geometry considered here, the minimum load is obtained for $m = n = 1$.

Denoting by D_f the flexural rigidity of the face sheet and by ν_f Poisson's ratio of the face sheet (considered isotropic), the strain energy due to bending of the face sheet is

$$U_B = \frac{D_f}{2} \int_0^a \int_0^a \left[w_{,xx}^2 + w_{,yy}^2 + 2\nu_f w_{,xx} w_{,yy} + 2(1-\nu_f) w_{,xy}^2 \right] dxdy. \tag{8.78}$$

The work done by the external force is

$$W = \int_0^a \int_0^a \left(N_{xx} \frac{w_{,x}^2}{2} + N_{yy} \frac{w_{,y}^2}{2} + N_{xy} w_{,x} w_{,y} \right) dxdy. \tag{8.79}$$

For both loading configurations, the strain energy of bending is

$$U_B = \frac{D_f \pi^4}{2a^2} A^2. \tag{8.80}$$

The first loading configuration yields (Figure 8.8): $N_{xx} = N_{yy} = N/\sqrt{2}$ and $N_{xy} = -N/\sqrt{2}$, therefore the work done by the external force is

$$W = \frac{N\pi^2\sqrt{2}}{8} A^2. \tag{8.81}$$

Setting the work done by the external forces equal to the strain energy of bending, $U_B = W$, and substituting the expression for the bending rigidity of the face, $D_f = E_f f^3/[12(1-\nu_f^2)]$, gives the critical load

$$N_{\mathrm{cr}} = \frac{E_f}{1-\nu^2} f \left(\frac{f}{a}\right)^2 \frac{\pi^2}{3\sqrt{2}} \simeq 2.3 \frac{E_f}{1-\nu^2} f \left(\frac{f}{a}\right)^2. \tag{8.82}$$

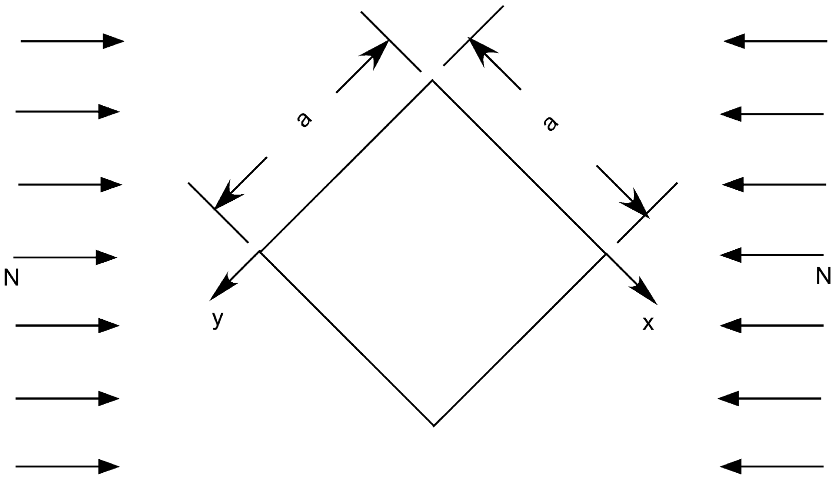

Figure 8.8 Intracell buckling model with the load parallel to the diagonal of the honeycomb cell.

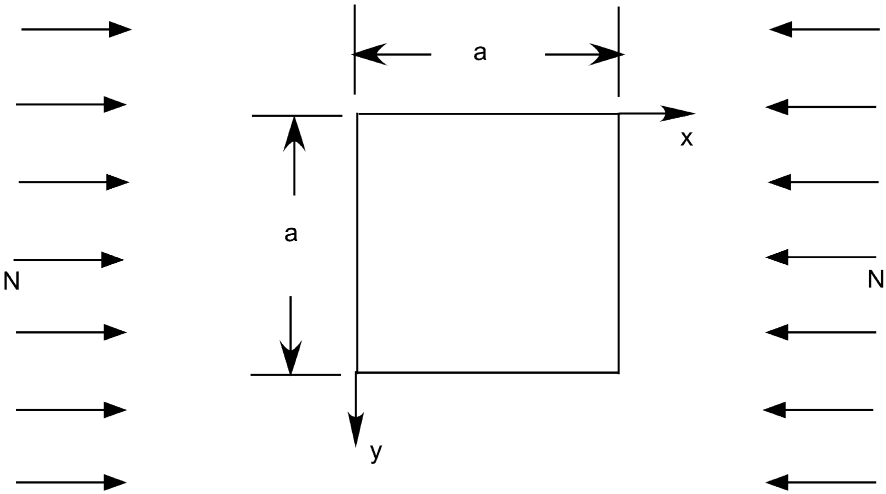

Figure 8.9 Intracell buckling model with the load parallel to a cell wall of the honeycomb core.

The second loading configuration yields (Figure 8.9): $N_{xx} = N$, $N_{yy} = N_{xy} = 0$, therefore

$$W = \frac{N\pi^2}{8} A^2. \tag{8.83}$$

Again, setting the work done by the external forces equal to the strain energy of bending, and substituting the expression for the bending rigidity of the face, gives the critical load

$$N_{\mathrm{cr}} = \frac{E_f}{1-\nu^2} f \left(\frac{f}{a}\right)^2 \frac{\pi^2}{3} \simeq 3.3 \frac{E_f}{1-\nu^2} f \left(\frac{f}{a}\right)^2. \tag{8.84}$$

Norris provided the formula in terms of an empirical factor K_d, as

$$N_{\mathrm{cr}} = K_d \frac{E_f}{1-\nu^2} f \left(\frac{f}{s}\right)^2, \tag{8.85}$$

where s is the diameter of the circle that can be inscribed in the honeycomb cell and K_d was specified to be about $K_d \simeq 2$. Obviously, for square cells, $s = a$.

From the two derived formulas (8.82) and (8.84), it can be seen that the orientation of the cells is important, i.e. it makes a difference whether the uniaxial compression is parallel to the diagonal of the cells (factor of 2.3 in place of K_d) or parallel to a side of the cells (factor of 3.3 in place of K_d).

Weikel and Kobayashi (1959) examined the conditions that would make intracell buckling dominate over wrinkling and they concluded that intracell buckling would dominate over wrinkling for a value of $\lambda a^4/D_f > 100$, where λ is the apparent core stiffness (when viewed as an elastic foundation, see Equation (8.74)). In other words, the predominant mode of buckling failure will be wrinkling for a relatively soft core, but with increased core stiffness, λ, intracell buckling becomes the predominant mode of local buckling failure.

8.2.2 The Fokker Dimpling Formula

The foregoing formulas can be easily revised for the case of orthotropic faces. Let us consider the case of the honeycomb cells oriented such that the applied compressive load is parallel to a cell wall (Figure 8.9). If $x \equiv 1$, $y \equiv 2$, then the strain energy of bending for an orthotropic plate is

$$U_B = \frac{1}{2}\int_0^a \int_0^a \left[D_{11} w_{,xx}^2 + D_{22} w_{,yy}^2 + 2D_{12} w_{,xx} w_{,yy} + 4D_{66} w_{,xy}^2\right] dxdy, \tag{8.86a}$$

where

$$D_{11} = \frac{E_1 f^3}{12(1-\nu_{12}\nu_{21})}; \quad D_{22} = \frac{E_2 f^3}{12(1-\nu_{12}\nu_{21})}, \tag{8.86b}$$

$$D_{12} = D_{11}\nu_{21}; \quad D_{66} = \frac{G_{12}f^3}{12}. \tag{8.86c}$$

Therefore, by using the displacement profile (8.77), the strain energy of bending is

$$U_B = (D_{11} + D_{22} + 2D_{12} + 4D_{66})\frac{\pi^4}{8a^2}A^2. \tag{8.87}$$

The work done by the external load is the same as in the isotropic face case and is given by (8.83). Equating the strain energy of bending with the work done gives the critical load as

$$N_{\text{cr}} = [D_{11} + 2(D_{12} + 2D_{66}) + D_{22}]\,\frac{\pi^2}{a^2}. \tag{8.88}$$

This is known as the "Fokker Dimpling Formula", as it was first suggested in a study conducted by the Fokker Aircraft BV for the European Space Agency (Blaas et al., 1984).

For the case of the honeycomb cells oriented such that the applied compressive load is parallel to the diagonal of the cell (Figure 8.8), we have to transform the elastic constants to the x and y system oriented at an angle $\phi = 45°$ to the principal elastic axes of the face (Lekhnitskii, 1968):

$$\begin{aligned} D'_{11} &= D_{11}\cos^4\phi + 2(D_{12} + 2D_{66})\sin^2\phi\cos^2\phi + D_{22}\sin^4\phi \\ &= \frac{1}{4}(D_{11} + D_{22} + 2D_{12} + 4D_{66}), \end{aligned} \tag{8.89a}$$

$$D'_{22} = D_{11}\sin^4\phi + 2(D_{12} + 2D_{66})\sin^2\phi\cos^2\phi + D_{22}\cos^4\phi = D'_{11}, \tag{8.89b}$$

$$\begin{aligned} D'_{12} &= D_{12} + [D_{11} + D_{22} - 2(D_{12} + 2D_{66})]\sin^2\phi\cos^2\phi \\ &= \frac{1}{2}D_{12} + \frac{1}{4}\left(D_{11} + D_{22} - 4D_{66}\right), \end{aligned} \tag{8.89c}$$

$$\begin{aligned} D'_{66} &= D_{66} + [D_{11} + D_{22} - 2(D_{12} + 2D_{66})]\sin^2\phi\cos^2\phi \\ &= \frac{1}{4}\left(D_{11} + D_{22} - 2D_{12}\right), \end{aligned} \tag{8.89d}$$

$$\begin{aligned} D'_{16} &= \frac{1}{2}[D_{22}\sin^2\phi - D_{11}\cos^2\phi + (D_{12} + 2D_{66})\cos 2\phi]\sin 2\phi \\ &= \frac{1}{4}\left(D_{22} - D_{11}\right), \end{aligned} \tag{8.89e}$$

$$D'_{26} = \frac{1}{2}[D_{22}\cos^2\phi - D_{11}\sin^2\phi - (D_{12} + 2D_{66})\cos 2\phi]\sin 2\phi = D'_{16}. \tag{8.89f}$$

In this case, the plate behaves as being anisotropic rather than orthotropic. The strain energy of bending is

$$U_B = \frac{1}{2}\int_0^a\int_0^a [D'_{11}w^2_{,xx} + D'_{22}w^2_{,yy} + 2D'_{12}w_{,xx}w_{,yy} + 4D'_{66}w^2_{,xy} + 4D'_{16}w_{,xx}w_{,xy} + 4D'_{26}w_{,yy}w_{,xy}]dxdy. \tag{8.90}$$

With the displacement profile (8.77), the strain energy of bending becomes

$$U_B = (D'_{11} + D'_{22} + 2D'_{12} + 4D'_{66})\frac{\pi^4}{8a^2}A^2. \tag{8.91}$$

Equating with the work done by the external load, Equation (8.81), gives the critical load as

$$N_{\mathrm{cr}} = (D'_{11} + D'_{22} + 2D'_{12} + 4D'_{66})\frac{\pi^2}{a^2\sqrt{2}}. \tag{8.92}$$

Substituting the expressions of the transformed elastic constants from (8.89a–d) we obtain

$$N_{\mathrm{cr}} = 2(D_{11} + D_{22})\frac{\pi^2}{a^2\sqrt{2}}. \tag{8.93}$$

Equation (8.93) is a newly derived formula, given for the first time herein, that can be used with orthotropic faces when the honeycomb cells are oriented such that the applied compressive load is parallel to the diagonal of the cell (Figure 8.8).