Chapter 5
Elasticity Solutions for Sandwich Structures

This chapter presents the theory of elasticity solutions for sandwich plates or shells. Elasticity solutions are significant because they provide a benchmark for assessing the performance of the various plate or shell theories or various numerical methods such as the finite element method. Most of these solutions are an extension of the corresponding solutions for monolithic anisotropic bodies which have been developed primarily by Lekhnitskii (1963). This chapter does not cover all problems of the theory of elasticity for sandwich bodies, but presents only some of the most studied ones in an attempt to collect the accumulated recent progress in this field. Section 5.1 on sandwich rectangular plates is adapted from Pagano (1970a), which was extended to the case of positive discriminant materials by Kardomateas (2008a) and Section 5.2 on sandwich shells from Kardomateas (2001).

5.1 A Rectangular Sandwich Plate with Orthotropic Face Sheets and Core

We consider a sandwich plate consisting of orthotropic face sheets of thickness $h_1 = f_1$ and $h_2 = f_2$ and an orthotropic core of thickness $h_c = 2c$, such that the various axes of elastic symmetry are parallel to the plate axes x, y, and z (Figure 5.1). The plate is simply supported. A normal traction $\sigma_z = q_0(x, y)$ is applied on the upper surface but the lower surface is traction-free.

Let us denote each layer by i where $i = f_1$ for the upper face-sheet, $i = c$ for the core and $i = f_2$ for the lower face-sheet. Then, for each layer, the orthotropic strain-stress relations are

L.A. Carlsson and G.A. Kardomateas, *Structural and Failure Mechanics of Sandwich Composites*, Solid Mechanics and its Applications 121, DOI 10.1007/978-1-4020-3225-7_5,

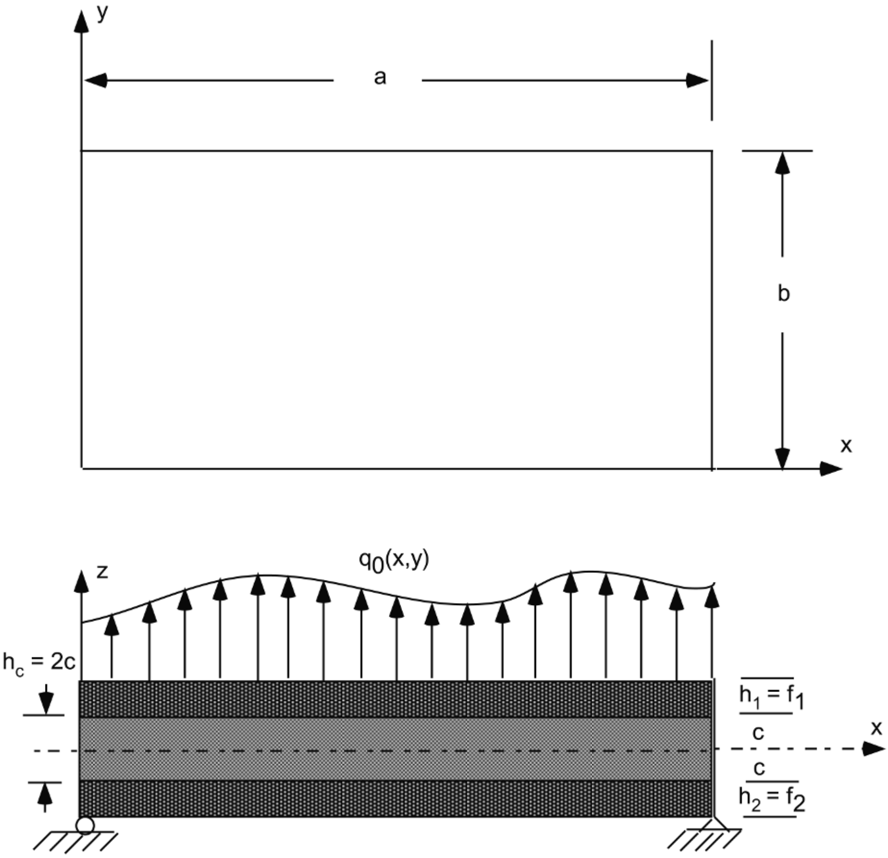

Figure 5.1 Definition of geometry and loading for the sandwich plate.

$$\begin{bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{yz}^{(i)} \\ \tau_{xz}^{(i)} \\ \tau_{xy}^{(i)} \end{bmatrix} = \begin{bmatrix} c_{11}^i & c_{12}^i & c_{13}^i & 0 & 0 & 0 \\ c_{12}^i & c_{22}^i & c_{23}^i & 0 & 0 & 0 \\ c_{13}^i & c_{23}^i & c_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^i \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(i)} \\ \varepsilon_{yy}^{(i)} \\ \varepsilon_{zz}^{(i)} \\ \gamma_{yz}^{(i)} \\ \gamma_{xz}^{(i)} \\ \gamma_{xy}^{(i)} \end{bmatrix}, \quad (i = f_1, c, f_2), \tag{5.1}$$

where c_{ij}^i are the stiffness constants (we have used the notation $1 \equiv x, 2 \equiv y, 3 \equiv z$).

Using the strain-displacement relations

$$\varepsilon_{xx} = u_{,x}; \quad \varepsilon_{yy} = v_{,y}; \quad \varepsilon_{zz} = w_{,z}, \tag{5.2a}$$

$$\gamma_{yz} = w_{,y} + v_{,z}; \quad \gamma_{xz} = u_{,z} + w_{,x}; \quad \gamma_{xy} = u_{,y} + v_{,x}, \tag{5.2b}$$

and the equilibrium relations

$$\sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} = 0, \tag{5.3a}$$

$$\tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} = 0, \tag{5.3b}$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{zz,z} = 0, \tag{5.3c}$$

leads to the following governing field equations in terms of the displacements for each layer:

$$c_{11}^i u_{,xx} + c_{66}^i u_{,yy} + c_{55}^i u_{,zz} + (c_{12}^i + c_{66}^i) v_{,xy} + (c_{13}^i + c_{55}^i) w_{,xz} = 0, \tag{5.4a}$$

$$(c_{12}^i + c_{66}^i) u_{,xy} + c_{66}^i v_{,xx} + c_{22}^i v_{,yy} + c_{44}^i v_{,zz} + (c_{23}^i + c_{44}^i) w_{,yz} = 0, \tag{5.4b}$$

$$(c_{13}^i + c_{55}^i) u_{,xz} + (c_{23}^i + c_{44}^i) v_{,yz} + c_{55}^i w_{,xx} + c_{44}^i w_{,yy} + c_{33}^i w_{,zz} = 0. \tag{5.4c}$$

In the following, we shall drop the superscript i that refers to the layers (core or face sheets) on the understanding that the derived relations will hold for each layer.

For a simply supported plate, an appropriate solution for the displacements would be in the form

$$u = U(z) \cos px \sin qy, \tag{5.5a}$$

$$v = V(z) \sin px \cos qy, \tag{5.5b}$$

$$w = W(z) \sin px \sin qy, \tag{5.5c}$$

where

$$p = n\pi/a; \quad q = m\pi/b \quad (n, m = 1, 2, 3, \ldots). \tag{5.5d}$$

These displacements, in conjunction with the corresponding strains and stresses from (5.2) and (5.1), would satisfy the simple support edge conditions:

$$\text{at} \quad x = 0, a: \quad w = v = \sigma_{xx} = 0. \tag{5.5e}$$

$$\text{at} \quad y = 0, b: \quad w = u = \sigma_{yy} = 0. \tag{5.5f}$$

Assuming that

$$[U(z), V(z), W(z)] = [U_0, V_0, W_0] e^{sz}, \tag{5.5g}$$

where U_0, V_0 and W_0 are constants, and substituting (5.5) into (5.4) results in the following system of algebraic equations:

$$(c_{11}p^2+c_{66}q^2-c_{55}s^2)U_0+(c_{12}+c_{66})pqV_0-(c_{13}+c_{55})psW_0=0, \quad (5.6a)$$

$$(c_{12}+c_{66})pqU_0+(c_{22}q^2+c_{66}p^2-c_{44}s^2)V_0-(c_{23}+c_{44})qsW_0=0, \quad (5.6b)$$

$$(c_{13}+c_{55})psU_0+(c_{23}+c_{44})qsV_0+(c_{55}p^2+c_{44}q^2-c_{33}s^2)W_0=0. \quad (5.6c)$$

Non-trivial solutions of this system exist only if the determinant of the coefficients vanishes, which leads to

$$A_0s^6+A_1s^4+A_2s^2+A_3=0, \quad (5.7)$$

where

$$A_0=-c_{33}c_{44}c_{55}, \quad (5.8a)$$

$$A_1=p^2\left[c_{44}(c_{11}c_{33}-c_{13}^2)+c_{55}(c_{33}c_{66}-2c_{13}c_{44})\right]+ \\ +q^2\left[c_{55}(c_{22}c_{33}-c_{23}^2)+c_{44}(c_{33}c_{66}-2c_{23}c_{55})\right], \quad (5.8b)$$

$$A_2=-p^4\left[c_{66}(c_{11}c_{33}-c_{13}^2)+c_{55}(c_{11}c_{44}-2c_{13}c_{66})\right] \\ +p^2q^2\big[-c_{11}(c_{22}c_{33}-c_{23}^2)-2(c_{12}+c_{66})(c_{13}+c_{55})(c_{23}+c_{44}) \\ -2c_{44}c_{55}c_{66}+2c_{11}c_{23}c_{44}+c_{12}c_{33}(c_{12}+2c_{66})+c_{13}c_{22}(c_{13}+2c_{55})\big] \\ -q^4\left[c_{66}(c_{22}c_{33}-c_{23}^2)+c_{44}(c_{22}c_{55}-2c_{23}c_{66})\right], \quad (5.8c)$$

$$A_3=p^6c_{11}c_{55}c_{66}+p^4q^2\left[c_{55}(c_{11}c_{22}-c_{12}^2)+c_{66}(c_{11}c_{44}-2c_{12}c_{55})\right] \\ +p^2q^4\left[c_{44}(c_{11}c_{22}-c_{12}^2)+c_{66}(c_{22}c_{55}-2c_{12}c_{44})\right]+q^6c_{22}c_{44}c_{66}. \quad (5.8d)$$

With the substitution

$$\beta=s^2, \quad (5.9)$$

Equation (5.7), which defines the parameter s, can be written in the form of a cubic equation as

$$\beta^3+a_1\beta^2+a_2\beta+a_3=0, \quad a_i=A_i/A_0 \quad (i=1,2,3). \quad (5.10)$$

Let

$$Q=\frac{3a_2-a_1^2}{9}; \quad R=\frac{9a_1a_2-27a_3-2a_1^3}{54}; \quad D=Q^3+R^2. \quad (5.11a)$$

The last quantity, D, is the discriminant, which determines the nature of the solution.

5.1.1 Negative Discriminant D

If $D < 0$, then all roots are real and different as follows:

$$\beta_1 = 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) - \frac{a_1}{3}\,, \tag{5.11b}$$

$$\beta_2 = 2\sqrt{-Q}\cos\left(\frac{\theta+2\pi}{3}\right) - \frac{a_1}{3}\,, \tag{5.11c}$$

$$\beta_3 = 2\sqrt{-Q}\cos\left(\frac{\theta+4\pi}{3}\right) - \frac{a_1}{3}\,, \tag{5.11d}$$

where

$$\cos\theta = R/\sqrt{-Q^3}\,. \tag{5.11e}$$

Corresponding to the three roots are the displacements functions defined in Equations (5.5a–c)

$$U(z) = \sum_{j=1}^{3} U_j(z); \quad V(z) = \sum_{j=1}^{3} V_j(z); \quad W(z) = \sum_{j=1}^{3} W_j(z). \tag{5.12}$$

If $\beta_j < 0$ then $s_j = \pm i\sqrt{|\beta_j|}$ and if we set

$$m_j = \sqrt{|\beta_j|}\,, \tag{5.13a}$$

then $s_j = \pm i m_j$. From (5.5g) for each pair of roots s_j, we can write

$$U_j(z) = U_{cj}\cos m_j z + U_{sj}\sin m_j z, \tag{5.13b}$$

$$V_j(z) = V_{cj}\cos m_j z + V_{sj}\sin m_j z, \tag{5.13c}$$

$$W_j(z) = W_{cj}\sin m_j z + W_{Sj}\cos m_j z, \tag{5.13d}$$

Substituting directly into (5.5a–c) and then into the equilibrium equations (5.4b, c), leads to the following two equations for V_{cj} and W_{sj}:

- from (5.4b) collecting terms of $\cos m_j z$:

$$V_{cj}\left(c_{66}p^2 + c_{22}q^2 + c_{44}m_j^2\right) - W_{sj}(c_{23}+c_{44})qm_j = -(c_{12}+c_{66})pqU_{cj}, \tag{5.13e}$$

- from (5.4c) collecting terms of $\sin m_j z$:

$$V_{cj}(c_{23}+c_{44})qm_j + W_{sj}\left(c_{55}p^2 + c_{44}q^2 + c_{33}m_j^2\right) = (c_{13}+c_{55})pm_jU_{cj}, \tag{5.13f}$$

These two equations can be solved for V_{cj} and W_{sj} in terms of U_{cj}. Similar equations can be derived for V_{sj} and W_{sj} by collecting terms of $\sin m_j z$ in (5.4b) and of $\cos m_j z$ in (5.4c). In the end, we obtain the following expressions:

$$U_j(z) = U_{cj} \cos m_j z + U_{sj} \sin m_j z, \tag{5.13g}$$

$$V_j(z) = \frac{B_j}{\Delta_j} U_{cj} \cos m_j z + \frac{B_j}{\Delta_j} U_{sj} \sin m_j z, \tag{5.13h}$$

$$W_j(z) = -\frac{C_j}{\Delta_j} U_{sj} \cos m_j z + \frac{C_j}{\Delta_j} U_{cj} \sin m_j z, \tag{5.13i}$$

where

$$\Delta_j = \left(c_{66}p^2 + c_{22}q^2 + c_{44}m_j^2\right)\left(c_{55}p^2 + c_{44}q^2 + c_{33}m_j^2\right) + (c_{23} + c_{44})^2 q^2 m_j^2, \tag{5.13j}$$

$$B_j = pq\big[-(c_{12} + c_{66})\left(c_{55}p^2 + c_{44}q^2 + c_{33}m_j^2\right) + (c_{13} + c_{55})(c_{23} + c_{44})m_j^2\big], \tag{5.13k}$$

$$C_j = pm_j\big[\left(c_{66}p^2 + c_{22}q^2 + c_{44}m_j^2\right)(c_{13} + c_{55}) + (c_{12} + c_{66})(c_{23} + c_{44})q^2\big]. \tag{5.13l}$$

If $\beta_j > 0$, we set

$$m_j = \sqrt{\beta_j}\,. \tag{5.14a}$$

By following an analogous procedure, we can write

$$U_j(z) = U_{cj} \cosh m_j z + U_{sj} \sinh m_j z, \tag{5.14b}$$

$$V_j(z) = \frac{B_j}{\Delta_j} U_{cj} \cosh m_j z + \frac{B_j}{\Delta_j} U_{sj} \sinh m_j z, \tag{5.14c}$$

$$W_j(z) = \frac{C_j}{\Delta_j} U_{cj} \sinh m_j z + \frac{C_j}{\Delta_j} U_{sj} \cosh m_j z, \tag{5.14d}$$

where

$$\Delta_j = \left(c_{66}p^2 + c_{22}q^2 - c_{44}m_j^2\right)\left(c_{55}p^2 + c_{44}q^2 - c_{33}m_j^2\right) + (c_{23} + c_{44})^2 q^2 m_j^2, \tag{5.14e}$$

$$B_j = -pq\big[(c_{12} + c_{66})\left(c_{55}p^2 + c_{44}q^2 - c_{33}m_j^2\right) + (c_{13} + c_{55})(c_{23} + c_{44})m_j^2\big], \tag{5.14f}$$

$$C_j = pm_j\big[-\left(c_{66}p^2 + c_{22}q^2 - c_{44}m_j^2\right)(c_{13} + c_{55}) + (c_{12} + c_{66})(c_{23} + c_{44})q^2\big]. \tag{5.14g}$$

Hence, the independent parameters are the six constants U_{c1}, U_{c2}, U_{c3}, U_{s1}, U_{s2}, U_{s3} which, for convenience, we rename g_1, g_2, g_3, g_4, g_5 and g_6, respectively. Then the displacements are as follows:

$$U(z) = d_{u1}g_1 + d_{u2}g_2 + d_{u3}g_3 + d_{u4}g_4 + d_{u5}g_5 + d_{u6}g_6, \tag{5.15a}$$

with the z-dependent coefficients defined for $j = 1, 2, 3$,

$$d_{uj} = \begin{cases} \cos m_j z, & \text{if} \quad \beta_j < 0, \\ \cosh m_j z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.15b}$$

$$d_{u(j+3)} = \begin{cases} \sin m_j z, & \text{if} \quad \beta_j < 0, \\ \sinh m_j z, & \text{if} \quad \beta_j > 0. \end{cases} \tag{5.15c}$$

In the following expressions (5.16–5.20), Δ_j, B_j and C_j refer to (5.13j–l) if $\beta_j < 0$, and to (5.14e–g) if $\beta_j > 0$. With this remark we can set $V(z)$ in the form

$$V(z) = d_{v1}g_1 + d_{v2}g_2 + d_{v3}g_3 + d_{v4}g_4 + d_{v5}g_5 + d_{v6}g_6, \tag{5.16a}$$

where, again, for $j = 1, 2, 3$,

$$d_{vj} = \begin{cases} \dfrac{B_j}{\Delta_j} \cos m_j z, & \text{if} \quad \beta_j < 0 \\ \dfrac{B_j}{\Delta_j} \cosh m_j z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.16b}$$

$$d_{v(j+3)} = \begin{cases} \dfrac{B_j}{\Delta_j} \sin m_j z, & \text{if} \quad \beta_j < 0 \\ \dfrac{B_j}{\Delta_j} \sinh m_j z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.16c}$$

and

$$W(z) = d_{w1}g_1 + d_{w2}g_2 + d_{w3}g_3 + d_{w4}g_4 + d_{w5}g_5 + d_{w6}g_6, \tag{5.17a}$$

where the z-dependent coefficients again are defined for $j = 1, 2, 3$,

$$d_{wj} = \begin{cases} \dfrac{C_j}{\Delta_j} \sin m_j z, & \text{if} \quad \beta_j < 0, \\ \dfrac{C_j}{\Delta_j} \sinh m_j z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.17b}$$

$$d_{w(j+3)} = \begin{cases} -\dfrac{C_j}{\Delta_j} \cos m_j z, & \text{if} \quad \beta_j < 0, \\ \dfrac{C_j}{\Delta_j} \cosh m_j z, & \text{if} \quad \beta_j > 0. \end{cases} \tag{5.17c}$$

The corresponding stresses are derived by substituting the above displacement expressions into (5.5), (5.1) and (5.2). We present the explicit expressions for the stresses σ_{zz}, τ_{yz} and τ_{xz} because these enter into the interface conditions. The stress σ_{zz} can be written in the form

$$\sigma_{zz} = (b_{zz1}g_1 + b_{zz2}g_2 + b_{zz3}g_3 + b_{zz4}g_4 + b_{zz5}g_5 + b_{zz6}g_6) \sin px \sin qy, \tag{5.18a}$$

The z-dependent coefficients are defined for $j = 1, 2, 3$ as

$$b_{zzj} = \begin{cases} -\left(c_{13}p + c_{23}q\dfrac{B_j}{\Delta_j} - c_{33}\dfrac{C_j}{\Delta_j}m_j \right) \cos m_j z, & \text{if} \quad \beta_j < 0 \\ -\left(c_{13}p + c_{23}q\dfrac{B_j}{\Delta_j} - c_{33}\dfrac{C_j}{\Delta_j}m_j \right) \cosh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.18b}$$

$$b_{zz(j+3)} = \begin{cases} -\left(c_{13}p + c_{23}q\dfrac{B_j}{\Delta_j} - c_{33}\dfrac{C_j}{\Delta_j}m_j \right) \sin m_j z, & \text{if} \quad \beta_j < 0 \\ -\left(c_{13}p + c_{23}q\dfrac{B_j}{\Delta_j} - c_{33}\dfrac{C_j}{\Delta_j}m_j \right) \sinh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.18c}$$

Next,

$$\tau_{yz} = \left(b_{yz1}g_1 + b_{yz2}g_2 + b_{yz3}g_3 + b_{yz4}g_4 + b_{yz5}g_5 + b_{yz6}g_6\right) \sin px \cos qy, \tag{5.19a}$$

with the z-dependent coefficients defined for $j = 1, 2, 3$ as

$$b_{yzj} = \begin{cases} c_{44}\left(q\dfrac{C_j}{\Delta_j} - m_j\dfrac{B_j}{\Delta_j} \right) \sin m_j z, & \text{if} \quad \beta_j < 0 \\ c_{44}\left(q\dfrac{C_j}{\Delta_j} + m_j\dfrac{B_j}{\Delta_j} \right) \sinh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.19b}$$

$$b_{yz(j+3)} = \begin{cases} -c_{44}\left(q\dfrac{C_j}{\Delta_j} - m_j\dfrac{B_j}{\Delta_j}\right)\cos m_j z, & \text{if} \quad \beta_j < 0 \\ c_{44}\left(q\dfrac{C_j}{\Delta_j} + m_j\dfrac{B_j}{\Delta_j}\right)\cosh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.19c}$$

Finally,

$$\tau_{xz} = (b_{xz1}g_1 + b_{xz2}g_2 + b_{xz3}g_3 + b_{xz4}g_4 + b_{xz5}g_5 + b_{xz6}g_6)\cos px \sin qy, \tag{5.20a}$$

with the z-dependent coefficients defined for $j = 1, 2, 3$ as

$$b_{xzj} = \begin{cases} c_{55}\left(p\dfrac{C_j}{\Delta_j} - m_j\right)\sin m_j z, & \text{if} \quad \beta_j < 0 \\ c_{55}\left(p\dfrac{C_j}{\Delta_j} + m_j\right)\sinh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.20b}$$

$$b_{xz(j+3)} = \begin{cases} -c_{55}\left(p\dfrac{C_j}{\Delta_j} - m_j\right)\cos m_j z, & \text{if} \quad \beta_j < 0 \\ c_{55}\left(p\dfrac{C_j}{\Delta_j} + m_j\right)\cosh m_j z, & \text{if} \quad \beta_j > 0 \end{cases} \tag{5.20c}$$

5.1.2 Positive Discriminant D

If $D > 0$, where the discriminant D is defined in (5.11a), then the cubic equation (5.10) has one real root and two complex conjugates.

With R and D defined in (5.11a), we further define

$$S = \sqrt[3]{R + \sqrt{D}}\,; \quad T = \sqrt[3]{R - \sqrt{D}}. \tag{5.21a}$$

Then with

$$\mu_R = -\frac{1}{2}(S+T) - \frac{a_1}{3}\,; \quad \mu_I = \frac{1}{2}\sqrt{3}(S-T), \tag{5.21b}$$

the two complex conjugate roots are

$$\beta_1 = \mu_R + i\mu_I; \quad \beta_2 = \mu_R - i\mu_I. \tag{5.21c}$$

The real root is

$$\beta_3 = S + T - \frac{a_1}{3}\,. \tag{5.21d}$$

The real root is dealt with in the same manner as for the case of a negative discriminant (Section 5.1.1).

Next we shall consider how to deal with the complex conjugate roots. In terms of the modulus r and amplitude θ of these complex numbers,

$$r = \sqrt{\mu_R^2 + \mu_I^2}\ ; \quad \theta = \tan^{-1}\left(\frac{\mu_I}{\mu_R}\right) \tag{5.21e}$$

these roots can be set in the form

$$\beta_1 = r(\cos\theta + i\sin\theta); \quad \beta_2 = r(\cos\theta - i\sin\theta). \tag{5.21f}$$

From (5.9), we now seek the square roots of β_1 and β_2. Thus, in terms of

$$\gamma_1 = \sqrt{r}\cos\frac{\theta}{2}\ ; \quad \gamma_2 = \sqrt{r}\sin\frac{\theta}{2}, \tag{5.21g}$$

the corresponding roots of the sixth-order equation (5.7), s_i, are

$$s_{1,2} = \pm(\gamma_1 + i\gamma_2); \quad s_{3,4} = \pm(\gamma_1 - i\gamma_2). \tag{5.21h}$$

Corresponding to these four roots, the displacement functions take the form

$$\begin{aligned} U_\eta(z) &= a_{1\eta}e^{\gamma_1 z}\cos\gamma_2 z + a_{2\eta}e^{\gamma_1 z}\sin\gamma_2 z \\ &\quad + a_{3\eta}e^{-\gamma_1 z}\cos\gamma_2 z + a_{4\eta}e^{-\gamma_1 z}\sin\gamma_2 z, \end{aligned} \tag{5.22}$$

where $\eta = u, v, w$ corresponds to the U, V, W displacements and the $a_{i\eta}$ are constants. Of the 12 constants appearing in (5.22) only four are independent. The eight relations that exist among these constants are found by substituting the displacements along with (5.5) into the equilibrium equations (5.4).

For convenience, let us set

$$r_1 = c_{44}(\gamma_1^2 + \gamma_2^2) + c_{66}p^2 + c_{22}q^2, \tag{5.23a}$$

$$r_2 = c_{44}(\gamma_1^2 + \gamma_2^2) - c_{66}p^2 - c_{22}q^2, \tag{5.23b}$$

$$r_3 = c_{55}(\gamma_1^2 + \gamma_2^2) + c_{11}p^2 + c_{66}q^2, \tag{5.23c}$$

$$r_4 = c_{55}(\gamma_1^2 + \gamma_2^2) - c_{11}p^2 - c_{66}q^2, \tag{5.23d}$$

and

$$e_1 = r_1(c_{13} + c_{55}) - q^2(c_{12} + c_{66})(c_{23} + c_{44}), \tag{5.23e}$$

$$e_2 = r_2(c_{13} + c_{55}) + q^2(c_{12} + c_{66})(c_{23} + c_{44}), \tag{5.23f}$$

$$e_3 = r_3(c_{23} + c_{44}) - p^2(c_{12} + c_{66})(c_{13} + c_{55}), \tag{5.23g}$$

$$e_4 = r_4(c_{23} + c_{44}) + p^2(c_{12} + c_{66})(c_{13} + c_{55}), \tag{5.23h}$$

In this way, we obtain the following relations for the coefficients in the displacement expression for $V(z)$, Equation (5.22), in terms of the coefficients in the expression for $U(z)$:

$$a_{1v} = \xi_{11}a_{1u} + \xi_{12}a_{2u}, \quad a_{2v} = \xi_{21}a_{1u} + \xi_{22}a_{2u}, \tag{5.24a}$$

$$a_{3v} = \xi_{33}a_{3u} + \xi_{34}a_{4u}, \quad a_{4v} = \xi_{43}a_{3u} + \xi_{44}a_{4u}, \tag{5.24b}$$

where

$$\xi_{11} = \xi_{22} = \xi_{33} = \xi_{44} = \frac{q(e_1e_3\gamma_2^2 + e_2e_4\gamma_1^2)}{p(\gamma_2^2e_1^2 + \gamma_1^2e_2^2)}. \tag{5.24c}$$

$$\xi_{12} = -\xi_{21} = -\xi_{34} = \xi_{43} = \frac{q\gamma_1\gamma_2(e_2e_3 - e_1e_4)}{p(\gamma_2^2e_1^2 + \gamma_1^2e_2^2)}. \tag{5.24d}$$

Also, the following relations for the coefficients in the expression for $W(z)$, Equation (5.22), in terms of the coefficients in the expression for $U(z)$:

$$a_{1w} = f_{11}a_{1u} + f_{12}a_{2u}, \quad a_{2w} = f_{21}a_{1u} + f_{22}a_{2u}, \tag{5.25a}$$

$$a_{3w} = f_{33}a_{3u} + f_{34}a_{4u}, \quad a_{4w} = f_{43}a_{3u} + f_{44}a_{4u}, \tag{5.25b}$$

where

$$\begin{aligned} f_{11} &= f_{22} = -f_{33} = -f_{44} \\ &= \frac{(c_{12} + c_{66})pq\gamma_1 - r_2\gamma_1\xi_{11} - r_1\gamma_2\xi_{21}}{q(c_{23} + c_{44})(\gamma_1^2 + \gamma_2^2)}, \end{aligned} \tag{5.25c}$$

$$\begin{aligned} f_{12} &= -f_{21} = f_{34} = -f_{43} \\ &= -\frac{(c_{12} + c_{66})pq\gamma_2 + r_2\gamma_1\xi_{12} + r_1\gamma_2\xi_{22}}{q(c_{23} + c_{44})(\gamma_1^2 + \gamma_2^2)}. \end{aligned} \tag{5.25d}$$

Now, coming to the real root β_3, we set

$$m_3 = \sqrt{|\beta_3|}, \tag{5.26}$$

then if $\beta_3 < 0$ and following (5.13g–l) we can write

$$U_3(z) = a_{5u}\cos m_3z + a_{6u}\sin m_3z, \tag{5.27a}$$

$$V_3(z) = \frac{B_3}{\Delta_3} a_{5u} \cos m_3 z + \frac{B_3}{\Delta_3} a_{6u} \sin m_3 z, \tag{5.27b}$$

$$W_3(z) = -\frac{C_3}{\Delta_3} a_{6u} \cos m_3 z + \frac{C_3}{\Delta_3} a_{5u} \sin m_3 z, \tag{5.27c}$$

where

$$\Delta_3 = \left(c_{66} p^2 + c_{22} q^2 + c_{44} m_3^2\right)\left(c_{55} p^2 + c_{44} q^2 + c_{33} m_3^2\right) + (c_{23} + c_{44})^2 q^2 m_3^2, \tag{5.27d}$$

$$B_3 = pq\big[-(c_{12} + c_{66})\left(c_{55} p^2 + c_{44} q^2 + c_{33} m_3^2\right) + (c_{13} + c_{55})(c_{23} + c_{44}) m_3^2 \big], \tag{5.27e}$$

$$C_3 = pm_3\big[\left(c_{66} p^2 + c_{22} q^2 + c_{44} m_3^2\right)(c_{13} + c_{55}) + (c_{12} + c_{66})(c_{23} + c_{44}) q^2 \big]. \tag{5.27f}$$

If $\beta_3 > 0$ then, following (5.14b–g)

$$U_3(z) = a_{5u} \cosh m_3 z + a_{6u} \sinh m_3 z, \tag{5.28a}$$

$$V_3(z) = \frac{B_3}{\Delta_3} a_{5u} \cosh m_3 z + \frac{B_3}{\Delta_3} a_{6u} \sinh m_3 z, \tag{5.28b}$$

$$W_3(z) = \frac{C_3}{\Delta_3} a_{5u} \sinh m_3 z + \frac{C_3}{\Delta_3} a_{6u} \cosh m_3 z, \tag{5.28c}$$

where

$$\Delta_3 = \left(c_{66} p^2 + c_{22} q^2 - c_{44} m_3^2\right)\left(c_{55} p^2 + c_{44} q^2 - c_{33} m_3^2\right) + (c_{23} + c_{44})^2 q^2 m_3^2, \tag{5.28d}$$

$$B_3 = -pq\big[(c_{12} + c_{66})\left(c_{55} p^2 + c_{44} q^2 - c_{33} m_3^2\right) + (c_{13} + c_{55})(c_{23} + c_{44}) m_3^2 \big], \tag{5.28e}$$

$$C_3 = pm_3\big[-\left(c_{66} p^2 + c_{22} q^2 - c_{44} m_3^2\right)(c_{13} + c_{55}) + (c_{12} + c_{66})(c_{23} + c_{44}) q^2 \big]. \tag{5.28f}$$

Hence, if we consider the constants a_{1u}, a_{2u}, a_{3u}, a_{4u}, a_{5u}, a_{6u} as independent, which for convenience we rename again as g_1, g_2, g_3, g_4, g_5, g_6, respectively, the displacement $U(z)$ is of the form (5.15a) with the z-dependent coefficients defined as

$$d_{u1} = e^{\gamma_1 z}\cos\gamma_2 z; \quad d_{u2} = e^{\gamma_1 z}\sin\gamma_2 z, \tag{5.29a}$$

$$d_{u3} = e^{-\gamma_1 z}\cos\gamma_2 z; \quad d_{u4} = e^{-\gamma_1 z}\sin\gamma_2 z, \tag{5.29b}$$

$$d_{u5} = \begin{cases} \cos m_3 z, & \text{if} \quad \beta_3 < 0, \\ \cosh m_3 z, & \text{if} \quad \beta_3 > 0, \end{cases} \tag{5.29c}$$

$$d_{u6} = \begin{cases} \sin m_3 z, & \text{if} \quad \beta_3 < 0, \\ \sinh m_3 z, & \text{if} \quad \beta_3 > 0. \end{cases} \tag{5.29d}$$

In the following expressions (5.30–5.34), Δ_3, B_3 and C_3 are from (5.27d–f) if $\beta_3 < 0$, and from (5.28d–f) if $\beta_3 > 0$. With this observation, the displacement $V(z)$ is of the form (5.16a), where

$$d_{v1} = (\xi_{11}\cos\gamma_2 z + \xi_{21}\sin\gamma_2 z)e^{\gamma_1 z}\,; \quad d_{v2} = (\xi_{12}\cos\gamma_2 z + \xi_{22}\sin\gamma_2 z)e^{\gamma_1 z}, \tag{5.30a}$$

$$d_{v3} = (\xi_{33}\cos\gamma_2 z + \xi_{43}\sin\gamma_2 z)e^{-\gamma_1 z}\,; \quad d_{v4} = (\xi_{34}\cos\gamma_2 z + \xi_{44}\sin\gamma_2 z)e^{-\gamma_1 z}, \tag{5.30b}$$

$$d_{v5} = \begin{cases} \dfrac{B_3}{\Delta_3}\cos m_3 z, & \text{if} \quad \beta_3 < 0 \\ \dfrac{B_3}{\Delta_3}\cosh m_3 z, & \text{if} \quad \beta_3 > 0 \end{cases} \tag{5.30c}$$

$$d_{v6} = \begin{cases} \dfrac{B_3}{\Delta_3}\sin m_3 z, & \text{if} \quad \beta_3 < 0 \\ \dfrac{B_3}{\Delta_3}\sinh m_3 z, & \text{if} \quad \beta_3 > 0. \end{cases} \tag{5.30d}$$

Similarly, the displacement $W(z)$ is of the form (5.17a) with the z-dependent coefficients:

$$d_{w1} = (f_{11}\cos\gamma_2 z + f_{21}\sin\gamma_2 z)e^{\gamma_1 z}; \quad d_{w2} = (f_{12}\cos\gamma_2 z + f_{22}\sin\gamma_2 z)e^{\gamma_1 z}, \tag{5.31a}$$

$$d_{w3} = (f_{33}\cos\gamma_2 z + f_{43}\sin\gamma_2 z)e^{-\gamma_1 z}; \quad d_{w4} = (f_{34}\cos\gamma_2 z + f_{44}\sin\gamma_2 z)e^{-\gamma_1 z}, \tag{5.31b}$$

$$d_{w5} = \begin{cases} \dfrac{C_3}{\Delta_3}\sin m_3 z, & \text{if} \quad \beta_3 < 0, \\ \dfrac{C_3}{\Delta_3}\sinh m_3 z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.31c}$$

$$d_{w6} = \begin{cases} -\dfrac{C_3}{\Delta_3}\cos m_3 z, & \text{if} \quad \beta_3 < 0, \\ \dfrac{C_3}{\Delta_3}\cosh m_3 z, & \text{if} \quad \beta_3 > 0. \end{cases} \tag{5.31d}$$

The corresponding stresses are derived by substituting the above displacement expressions into (5.5), (5.1), and (5.2). We present the explicit expressions for σ_{zz}, τ_{yz} and τ_{xz}, which enter into the interface conditions. σ_{zz} is of the form (5.18a) with the z-dependent coefficients defined as

$$b_{zz1} = [c_{33}(f_{11}\gamma_1 + f_{21}\gamma_2) - c_{13}p - c_{23}q\xi_{11}]e^{\gamma_1 z}\cos\gamma_2 z + [c_{33}(f_{21}\gamma_1 - f_{11}\gamma_2) - c_{23}q\xi_{21}]e^{\gamma_1 z}\sin\gamma_2 z, \tag{5.32a}$$

$$b_{zz2} = [c_{33}(f_{12}\gamma_1 + f_{22}\gamma_2) - c_{23}q\xi_{12}]e^{\gamma_1 z}\cos\gamma_2 z + [c_{33}(f_{22}\gamma_1 - f_{12}\gamma_2) - c_{13}p - c_{23}q\xi_{22}]e^{\gamma_1 z}\sin\gamma_2 z, \tag{5.32b}$$

$$b_{zz3} = -[c_{33}(f_{33}\gamma_1 - f_{43}\gamma_2) + c_{13}p + c_{23}q\xi_{33}]e^{-\gamma_1 z}\cos\gamma_2 z - [c_{33}(f_{43}\gamma_1 + f_{33}\gamma_2) + c_{23}q\xi_{43}]e^{-\gamma_1 z}\sin\gamma_2 z, \tag{5.32c}$$

$$b_{zz4} = -[c_{33}(f_{34}\gamma_1 - f_{44}\gamma_2) + c_{23}q\xi_{34}]e^{-\gamma_1 z}\cos\gamma_2 z - [c_{33}(f_{44}\gamma_1 + f_{34}\gamma_2) + c_{13}p + c_{23}q\xi_{44}]e^{-\gamma_1 z}\sin\gamma_2 z. \tag{5.32d}$$

$$b_{zz5} = \begin{cases} -\left(c_{13}p + c_{23}q\dfrac{B_3}{\Delta_3} - c_{33}\dfrac{C_3}{\Delta_3}m_3\right)\cos m_3 z, & \text{if } \beta_3 < 0 \\ -\left(c_{13}p + c_{23}q\dfrac{B_3}{\Delta_3} - c_{33}\dfrac{C_3}{\Delta_3}m_3\right)\cosh m_3 z, & \text{if } \beta_3 > 0 \end{cases} \tag{5.32e}$$

$$b_{zz6} = \begin{cases} -\left(c_{13}p + c_{23}q\dfrac{B_3}{\Delta_3} - c_{33}\dfrac{C_3}{\Delta_3}m_3\right)\sin m_3 z, & \text{if } \beta_3 < 0 \\ -\left(c_{13}p + c_{23}q\dfrac{B_j}{\Delta_3} - c_{33}\dfrac{C_3}{\Delta_3}m_3\right)\sinh m_3 z, & \text{if } \beta_3 > 0 \end{cases} \tag{5.32f}$$

τ_{yz} is of the form (5.19a) with the z-dependent coefficients defined as

$$b_{yz1} = c_{44}e^{\gamma_1 z}[(\xi_{11}\gamma_1 + \xi_{21}\gamma_2 + qf_{11})\cos\gamma_2 z + (\xi_{21}\gamma_1 - \xi_{11}\gamma_2 + qf_{21})\sin\gamma_2 z], \tag{5.33a}$$

$$b_{yz2} = c_{44}e^{\gamma_1 z}[(\xi_{12}\gamma_1 + \xi_{22}\gamma_2 + qf_{12})\cos\gamma_2 z + (\xi_{22}\gamma_1 - \xi_{12}\gamma_2 + qf_{22})\sin\gamma_2 z], \tag{5.33b}$$

$$b_{yz3} = c_{44}e^{-\gamma_1 z}[(qf_{33} + \xi_{43}\gamma_2 - \xi_{33}\gamma_1)\cos\gamma_2 z + (qf_{43} - \xi_{33}\gamma_2 - \xi_{43}\gamma_1)\sin\gamma_2 z], \tag{5.33c}$$

$$b_{yz4} = c_{44}e^{-\gamma_1 z}[(qf_{34} + \xi_{44}\gamma_2 - \xi_{34}\gamma_1)\cos\gamma_2 z + (qf_{44} - \xi_{34}\gamma_2 - \xi_{44}\gamma_1)\sin\gamma_2 z], \tag{5.33d}$$

$$b_{yz5} = \begin{cases} c_{44}\left(q\dfrac{C_3}{\Delta_3} - m_3\dfrac{B_3}{\Delta_3}\right)\sin m_3 z, & \text{if} \quad \beta_3 < 0, \\ c_{44}\left(q\dfrac{C_3}{\Delta_3} + m_3\dfrac{B_3}{\Delta_3}\right)\sinh m_3 z, & \text{if} \quad \beta_3 > 0, \end{cases} \tag{5.33e}$$

$$b_{yz6} = \begin{cases} -c_{44}\left(q\dfrac{C_3}{\Delta_3} - m_3\dfrac{B_3}{\Delta_3}\right)\cos m_3 z, & \text{if} \quad \beta_3 < 0, \\ c_{44}\left(q\dfrac{C_3}{\Delta_3} + m_3\dfrac{B_3}{\Delta_3}\right)\cosh m_3 z, & \text{if} \quad \beta_3 > 0. \end{cases} \tag{5.33f}$$

Finally, τ_{xz} is of the form (5.20a) with the z-dependent coefficients defined as

$$b_{xz1} = c_{55}e^{\gamma_1 z}[(\gamma_1 + pf_{11})\cos\gamma_2 z + (pf_{21} - \gamma_2)\sin\gamma_2 z], \tag{5.34a}$$

$$b_{xz2} = c_{55}e^{\gamma_1 z}[(\gamma_2 + pf_{12})\cos\gamma_2 z + (pf_{22} + \gamma_1)\sin\gamma_2 z], \tag{5.34b}$$

$$b_{xz3} = c_{55}e^{-\gamma_1 z}[(pf_{33} - \gamma_1)\cos\gamma_2 z + (pf_{43} - \gamma_2)\sin\gamma_2 z], \tag{5.34c}$$

$$b_{xz4} = c_{55}e^{-\gamma_1 z}[(pf_{34} + \gamma_2)\cos\gamma_2 z + (pf_{44} - \gamma_1)\sin\gamma_2 z], \tag{5.34d}$$

$$b_{xz5} = \begin{cases} c_{55}\left(p\dfrac{C_3}{\Delta_3} - m_3\right)\sin m_3 z, & \text{if} \quad \beta_3 < 0\,, \\ c_{55}\left(p\dfrac{C_3}{\Delta_3} + m_3\right)\sinh m_3 z, & \text{if} \quad \beta_3 > 0, \end{cases} \tag{5.34e}$$

$$b_{xz6} = \begin{cases} -c_{55}\left(p\dfrac{C_3}{\Delta_3} - m_3\right)\cos m_3 z, & \text{if} \quad \beta_3 < 0, \\ c_{55}\left(p\dfrac{C_3}{\Delta_3} + m_3\right)\cosh m_3 z, & \text{if} \quad \beta_3 > 0. \end{cases} \tag{5.34f}$$

5.1.3 Isotropic Layers

In the event that one of the layers in the sandwich panel is isotropic (this is more common for the core) with extensional modulus E and Poisson's ratio ν, then the following relationships for the material constants hold:

$$c_{11} = c_{22} = c_{33} = E\frac{1-\nu}{(1-2\nu)(1+\nu)}, \tag{5.35a}$$

$$c_{12} = c_{13} = c_{23} = c_{11}\frac{\nu}{1-\nu}\,; \quad c_{66} = c_{55} = c_{44} = c_{11}\frac{1-2\nu}{2(1-\nu)}. \tag{5.35b}$$

In this case we find that D vanishes and the solution to Equation (5.10) consists of three equal roots, $\beta_i = p^2 + q^2$. Therefore, the solutions to (5.7) occur in the form of three repeated pairs of roots, $s_i = \pm\lambda$, where

$$\lambda = (p^2 + q^2)^{1/2}. \tag{5.36}$$

In this case, the displacement functions take the form

$$U_\eta(z) = (a_{1\eta} + a_{3\eta} z + a_{5\eta} z^2)e^{\lambda z} + (a_{2\eta} + a_{4\eta} z + a_{6\eta} z^2)e^{-\lambda z}, \tag{5.37}$$

where $\eta = u, v, w$ corresponds to the U, V, W displacements and the $a_{i\eta}$ are constants. Of the 18 constants appearing in (5.37), only six are independent. The various relations that exist among these constants are found by substituting (5.37) and (5.5) into (5.4), in which the relations (5.35) for the isotropic material constants are used. In this way we deduce the following 12 relations:

$$a_{5\eta} = a_{6\eta} = 0; \quad \eta = u, v, w, \tag{5.38a}$$

$$q a_{3u} = p a_{3v}; \quad \lambda a_{3u} = p a_{3w}, \tag{5.38b}$$

$$q a_{4u} = p a_{4v}; \quad \lambda a_{4u} = -p a_{4w}, \tag{5.38c}$$

$$p a_{1u} + q a_{1v} - \lambda a_{1w} = -\frac{\lambda}{p}(4\nu - 3) a_{3u}, \tag{5.38d}$$

$$p a_{2u} + q a_{2v} + \lambda a_{2w} = \frac{\lambda}{p}(4\nu - 3) a_{4u}. \tag{5.38e}$$

Hence, if we consider the constants a_{1u}, a_{2u}, a_{3u}, a_{4u}, a_{1v}, and a_{2v} as independent, which for convenience we rename g_1, g_2, g_3, g_4, g_5, g_6, respectively, the displacement $U(z)$ is of the form (5.15a) with the z-dependent coefficients defined as

$$d_{u1} = e^{\lambda z}; \quad d_{u2} = e^{-\lambda z}; \quad d_{u3} = z e^{\lambda z}; \quad d_{u4} = z e^{-\lambda z}; \quad d_{u5} = d_{u6} = 0. \tag{5.39}$$

The displacement $V(z)$ is of the form (5.16a) where

$$d_{v1} = d_{v2} = 0; \;\; d_{v3} = \frac{q}{p} z e^{\lambda z}; \;\; d_{v4} = \frac{q}{p} z e^{-\lambda z}; \;\; d_{v5} = e^{\lambda z}; \;\; d_{v6} = e^{-\lambda z}, \tag{5.40}$$

and the displacement $W(z)$ is of the form (5.17a) where,

$$d_{w1} = \frac{p}{\lambda} e^{\lambda z}; \quad d_{w2} = -\frac{p}{\lambda} e^{-\lambda z}; \quad d_{w3} = \left(\frac{4\nu - 3}{p} + \frac{\lambda}{p} z\right) e^{\lambda z}, \tag{5.41a}$$

$$d_{w4} = \left(\frac{4\nu - 3}{p} - \frac{\lambda}{p}z\right) e^{-\lambda z}\,; \quad d_{w5} = \frac{q}{\lambda} e^{\lambda z}\,; \quad d_{w6} = -\frac{q}{\lambda} e^{-\lambda z}. \tag{5.41b}$$

The corresponding stresses are derived by substituting the above displacement expressions into (5.5), (5.1), and (5.2). We present again the explicit expressions for σ_{zz}, τ_{yz}, and τ_{xz}, which come into the interface conditions. σ_{zz} is of the form (5.18a) with the z-dependent coefficients defined as

$$b_{zz1} = c_{11} p \frac{1-2\nu}{1-\nu} e^{\lambda z}\,; \quad b_{zz2} = c_{11} p \frac{1-2\nu}{1-\nu} e^{-\lambda z}, \tag{5.42a}$$

$$b_{zz3} = c_{11} \frac{\lambda(1-2\nu)}{p(1-\nu)} e^{\lambda z} \left[\lambda z - 2(1-\nu)\right], \tag{5.42b}$$

$$b_{zz4} = c_{11} \frac{\lambda(1-2\nu)}{p(1-\nu)} e^{-\lambda z} \left[\lambda z + 2(1-\nu)\right], \tag{5.42c}$$

$$b_{zz5} = c_{11} q \frac{1-2\nu}{1-\nu} e^{\lambda z}\,; \quad b_{zz6} = c_{11} q \frac{1-2\nu}{1-\nu} e^{-\lambda z}. \tag{5.42d}$$

τ_{yz} is of the form (5.19a) with the z-dependent coefficients defined as

$$b_{yz1} = \frac{c_{11}(1-2\nu)}{2(1-\nu)} \frac{pq}{\lambda} e^{\lambda z}\,; \quad b_{yz2} = -\frac{c_{11}(1-2\nu)}{2(1-\nu)} \frac{pq}{\lambda} e^{-\lambda z}, \tag{5.43a}$$

$$b_{yz3} = \frac{c_{11}(1-2\nu)}{(1-\nu)} (2\nu - 1 + \lambda z) \frac{q}{p} e^{\lambda z}, \tag{5.43b}$$

$$b_{yz4} = \frac{c_{11}(1-2\nu)}{(1-\nu)} (2\nu - 1 - \lambda z) \frac{q}{p} e^{-\lambda z}, \tag{5.43c}$$

$$b_{yz5} = \frac{c_{11}(1-2\nu)}{2(1-\nu)} \left(\frac{q^2}{\lambda} + \lambda\right) e^{\lambda z}; \;\; b_{yz6} = -\frac{c_{11}(1-2\nu)}{2(1-\nu)} \left(\frac{q^2}{\lambda} + \lambda\right) e^{-\lambda z}. \tag{5.43d}$$

τ_{xz} is of the form (5.20a) with the z-dependent coefficients defined as

$$b_{xz1} = \frac{c_{11}(1-2\nu)}{2(1-\nu)} \left(\frac{p^2}{\lambda} + \lambda\right) e^{\lambda z}; \;\; b_{xz2} = -\frac{c_{11}(1-2\nu)}{2(1-\nu)} \left(\frac{p^2}{\lambda} + \lambda\right) e^{-\lambda z}, \tag{5.44a}$$

$$b_{xz3} = \frac{c_{11}(1-2\nu)}{(1-\nu)} (2\nu - 1 + \lambda z) e^{\lambda z}, \tag{5.44b}$$

$$b_{xz4} = \frac{c_{11}(1-2\nu)}{(1-\nu)} (2\nu - 1 - \lambda z) e^{-\lambda z}, \tag{5.44c}$$

$$b_{xz5} = \frac{c_{11}(1-2\nu)}{2(1-\nu)} \frac{pq}{\lambda} e^{\lambda z}; \quad b_{xz6} = -\frac{c_{11}(1-2\nu)}{2(1-\nu)} \frac{pq}{\lambda} e^{-\lambda z}. \tag{5.44d}$$

From this analysis, we can see that the stresses in each layer (i), where $i = f_1, c, f_2$, are described by six constants: $g_j^{(i)}$, $g_{j+3}^{(i)}$, $j = 1, 2, 3$. Therefore, for the sandwich panel, a total of 18 constants are to be determined.

There are three traction conditions at each of the two core/face-sheet interfaces, giving a total of six conditions. In a similar fashion, there are three displacement continuity conditions at each of the two core/face-sheet interfaces, giving another six conditions. Finally, there are three traction boundary conditions on each of the two plate outer surfaces, giving an another six conditions, i.e. a total of 18 equations.

Finally, for completeness, we also give the detailed expressions for the in-plane stresses σ_{xx}, σ_{yy} and τ_{xy}. σ_{xx} can be written in the form

$$\sigma_{xx} = (b_{xx1}g_1 + b_{xx2}g_2 + b_{xx3}g_3 + b_{xx4}g_4 + b_{xx5}g_5 + b_{xx6}g_6) \sin px \sin qy, \tag{5.45}$$

where the z-dependent coefficients b_{xxj} are found from the b_{zzj} expressions (5.18b–c) and (5.32a–f) by replacing c_{33} with c_{13}, c_{13} with c_{11} and c_{23} with c_{12}. In the same manner, σ_{yy} is given by

$$\sigma_{yy} = \left(b_{yy1}g_1 + b_{yy2}g_2 + b_{yy3}g_3 + b_{yy4}g_4 + b_{yy5}g_5 + b_{yy6}g_6\right) \sin px \sin qy, \tag{5.46}$$

where the z-dependent coefficients b_{yyj} are again found from the b_{zzj} expressions (5.18b–c) and (5.32a–f) by now replacing c_{33} with c_{23}, c_{13} with c_{12} and c_{23} with c_{22}. Finally, the shear stress, τ_{xy}, is

$$\tau_{xy} = \left(b_{xy1}g_1 + b_{xy2}g_2 + b_{xy3}g_3 + b_{xy4}g_4 + b_{xy5}g_5 + b_{xy6}g_6\right) \cos px \sin qy. \tag{5.47}$$

For orthotropic layers with $D < 0$, the z-dependent coefficients are defined for $j = 1, 2, 3$ as

$$b_{xyj} = \begin{cases} c_{66}\left(q + p\dfrac{B_j}{\Delta_j}\right)\cos m_j z, & \text{if} \quad \beta_j < 0, \\ c_{66}\left(q + p\dfrac{B_j}{\Delta_j}\right)\cosh m_j z, & \text{if} \quad \beta_j > 0, \end{cases} \tag{5.48a}$$

$$b_{xy(j+3)} = \begin{cases} c_{66}\left(q + p\dfrac{B_j}{\Delta_j}\right)\sin m_j z, & \text{if} \quad \beta_j < 0, \\ c_{66}\left(q + p\dfrac{B_j}{\Delta_j}\right)\sinh m_j z, & \text{if} \quad \beta_j > 0. \end{cases} \tag{5.48b}$$

In the expressions (5.48) and (5.49), Δ_j and B_3 refer to (5.13j–k) if $\beta_j < 0$, and to (5.14e–f) if $\beta_j > 0$. Further, Δ_3 and B_3 refer to (5.27d–e) if $\beta_3 < 0$ and to (5.28d–e) if $\beta_3 > 0$. With this note, for orthotropic layers with $D > 0$, the z-dependent coefficients are

$$b_{xy1} = c_{66}e^{\gamma_1 z}[(q + p\xi_{11})\cos\gamma_2 z + p\xi_{21}\sin\gamma_2 z], \tag{5.49a}$$

$$b_{xy2} = c_{66}e^{\gamma_1 z}[p\xi_{12}\cos\gamma_2 z + (q + p\xi_{22})\sin\gamma_2 z], \tag{5.49b}$$

$$b_{xy3} = c_{66}e^{-\gamma_1 z}[(q + p\xi_{33})\cos\gamma_2 z + p\xi_{43}\sin\gamma_2 z], \tag{5.49c}$$

$$b_{xy4} = c_{66}e^{-\gamma_1 z}[p\xi_{34}\cos\gamma_2 z + (q + p\xi_{44})\sin\gamma_2 z], \tag{5.49d}$$

$$b_{xy5} = \begin{cases} c_{66}\left(q + p\dfrac{B_3}{\Delta_3}\right)\cos m_3 z, & \text{if} \quad \beta_3 < 0 \\ c_{66}\left(q + p\dfrac{B_3}{\Delta_3}\right)\cosh m_3 z, & \text{if} \quad \beta_3 > 0 \end{cases} \tag{5.49e}$$

$$b_{xy6} = \begin{cases} c_{66}\left(q + p\dfrac{B_3}{\Delta_3}\right)\sin m_3 z, & \text{if} \quad \beta_3 < 0 \\ c_{66}\left(q + p\dfrac{B_3}{\Delta_3}\right)\sinh m_3 z, & \text{if} \quad \beta_3 > 0 \end{cases} \tag{5.49f}$$

For isotropic materials, the z-dependent coefficients are:

$$b_{xy1} = \frac{c_{11}(1-2\nu)}{2(1-\nu)}qe^{\lambda z}\,; \quad b_{xy2} = \frac{c_{11}(1-2\nu)}{2(1-\nu)}qe^{-\lambda z}, \tag{5.50a}$$

$$b_{xy3} = \frac{c_{11}(1-2\nu)}{1-\nu}qze^{\lambda z}, \tag{5.50b}$$

$$b_{xy4} = \frac{c_{11}(1-2\nu)}{1-\nu}qze^{-\lambda z}, \tag{5.50c}$$

$$b_{xy5} = \frac{c_{11}(1-2\nu)}{2(1-\nu)}pe^{\lambda z}\,; \quad b_{xy6} = \frac{c_{11}(1-2\nu)}{2(1-\nu)}pe^{-\lambda z}. \tag{5.50d}$$

5.1.4 Examples

As an illustration of the above, let us consider a symmetric sandwich plate with unidirectional carbon/epoxy faces and a hexagonal glass/phenolic honeycomb core. This material combination is very common in the aerospace/rotorcraft industry (although the faces would be multidirectional for most applications). The orthotropic face moduli are (in GPa):

$E_1^f = 181$, $E_2^f = E_3^f = 10.3$, $G_{23}^f = 5.96$, $G_{12}^f = G_{31}^f = 7.17$; and the face Poisson's ratios: $\nu_{12}^f = \nu_{13}^f = 0.277$, $\nu_{32}^f = 0.400$. The orthotropic honeycomb core moduli are (in GPa): $E_1^c = E_2^c = 0.032$, $E_3^c = 0.300$, $G_{23}^c = G_{31}^c = 0.048$, $G_{12}^c = 0.013$; and the core Poisson's ratios: $\nu_{12}^c = \nu_{32}^c = \nu_{31}^c = 0.25$. The thickness of each face sheet is $f_1 = f_2 = 2$ mm and the core thickness $2c = 16$ mm. The plate is square with $a = b = 10h$, where h is the total thickness of the sandwich plate.

We further assume that a transverse load is applied at the top face sheet of the form

$$q_0(x, y) = Q_0 \sin px \sin qy, \tag{5.51}$$

and in the definition of p and q in (5.5d), we further assume $m = n = 1$. Note that a general load can be expanded in a series of terms of the type (5.51).

For each layer, the compliance constants are given by

$$a_{11} = \frac{1}{E_1}; \quad a_{12} = \frac{-\nu_{21}}{E_2}; \quad a_{13} = \frac{-\nu_{31}}{E_3}, \tag{5.52a}$$

$$a_{22} = \frac{1}{E_2}; \quad a_{23} = \frac{-\nu_{32}}{E_3}; \quad a_{33} = \frac{1}{E_3}, \tag{5.52b}$$

$$a_{44} = \frac{1}{G_{23}}; \quad a_{55} = \frac{1}{G_{13}}; \quad a_{66} = \frac{1}{G_{12}}. \tag{5.52c}$$

The stiffness matrix is the inverse of the compliance matrix. The inversion leads to the following formulas for the c_{ij}:

$$c_{11} = E_1 \frac{(1 - \nu_{23}\nu_{32})}{C_0}; \quad c_{12} = E_2 \frac{(\nu_{12} + \nu_{13}\nu_{32})}{C_0}; \quad c_{13} = E_3 \frac{(\nu_{13} + \nu_{12}\nu_{23})}{C_0}, \tag{5.52d}$$

$$c_{22} = E_2 \frac{(1 - \nu_{13}\nu_{31})}{C_0}; \quad c_{23} = E_3 \frac{(\nu_{23} + \nu_{21}\nu_{13})}{C_0}; \quad c_{33} = E_3 \frac{(1 - \nu_{12}\nu_{21})}{C_0}, \tag{5.52e}$$

$$c_{44} = G_{23}; \quad c_{55} = G_{13}; \quad c_{66} = G_{12}, \tag{5.52f}$$

where

$$C_0 = 1 - (\nu_{12}\nu_{21} + \nu_{23}\nu_{32} + \nu_{13}\nu_{31}) - (\nu_{12}\nu_{23}\nu_{31} + \nu_{21}\nu_{13}\nu_{32}). \tag{5.52g}$$

Substituting the corresponding constants leads to the following β's:

- *Face sheets*, $D > 0$, therefore two complex conjugate roots and one real root:

$$\beta_1^f = 342.5 + i316.3; \quad \beta_2^f = 342.5 - i316.3; \quad \beta_3^f = 6150.2.$$

- *Core*, $D > 0$, therefore again two complex conjugate roots and one real root:

$$\beta_1^c = 158.9 + i49.2; \quad \beta_2^c = 158.9 - i49.2\,; \quad \beta_3^c = 131.6.$$

Since we have a positive discriminant for both the face sheet and the core, the corresponding positive discriminant formulas for the coefficients in the expressions of the displacements and stresses are applicable. The solution is obtained by imposing the following.

There are three traction conditions at the lower face-sheet/core interface, $z = -c$:

(a) $\sigma_{zz}^{(c)} = \sigma_{zz}^{(f_2)}|_{z=-c}$, which gives

$$\sum_{j=1}^{6} b_{zzj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} b_{zzj}^{(f_2)}|_{z=-c} g_j^{(f_2)}, \tag{5.53a}$$

(b) $\tau_{yz}^{(c)} = \tau_{yz}^{(f_2)}|_{z=-c}$, which gives

$$\sum_{j=1}^{6} b_{yzj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} b_{yzj}^{(f_2)}|_{z=-c} g_j^{(f_2)}, \tag{5.53b}$$

and

(c) $\tau_{xz}^{(c)} = \tau_{xz}^{(f_2)}|_{z=-c}$, which gives

$$\sum_{j=1}^{6} b_{xzj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} b_{xzj}^{(f_2)}|_{z=-c} g_j^{(f_2)}. \tag{5.53c}$$

There are also three displacement continuity conditions at the lower core/face-sheet interfaces:

(a) $U^{(c)} = U^{(f_2)}$ at $z = -c$, which results in

$$\sum_{j=1}^{6} d_{uj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} d_{uj}^{(f_2)}|_{z=-c} g_j^{(f_2)}, \tag{5.53d}$$

(b) $V^{(c)} = V^{(f_2)}$ at $z = -c$, which gives

$$\sum_{j=1}^{6} d_{vj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} d_{vj}^{(f_2)}|_{z=-c} g_j^{(f_2)}, \tag{5.53e}$$

and

(c) $W^{(c)} = W^{(f_2)}$ at $z = -c$, which gives

$$\sum_{j=1}^{6} d_{wj}^{(c)}|_{z=-c} g_j^{(c)} = \sum_{j=1}^{6} d_{wj}^{(f_2)}|_{z=-c} g_j^{(f_2)}. \tag{5.53f}$$

Next, there are three traction conditions at the upper face-sheet/core interface, $z = c$:

(a) $\sigma_{zz}^{(f_1)} = \sigma_{zz}^{(c)}|_{z=c}$, which gives

$$\sum_{j=1}^{6} b_{zzj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} b_{zzj}^{(f_1)}|_{z=c} g_j^{(f_1)}, \tag{5.54a}$$

(b) $\tau_{yz}^{(f_1)} = \tau_{yz}^{(c)}|_{z=c}$, which gives

$$\sum_{j=1}^{6} b_{yzj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} b_{yzj}^{(f_1)}|_{z=c} g_j^{(f_1)}, \tag{5.54b}$$

and

(c) $\tau_{xz}^{(f_1)} = \tau_{xz}^{(c)}|_{z=c}$, which gives

$$\sum_{j=1}^{6} b_{xzj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} b_{xzj}^{(f_1)}|_{z=c} g_j^{(f_1)}. \tag{5.54c}$$

The corresponding displacement continuity conditions at the upper face-sheet/core interface, $z = c$ are

(a) $U^{(f_1)} = U^{(c)}$ at $z = c$, which gives

$$\sum_{j=1}^{6} d_{uj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} d_{uj}^{(f_1)}|_{z=c} g_j^{(f_1)}, \tag{5.54d}$$

(b) $V^{(f_1)} = V^{(c)}$ at $z = c$, which gives

$$\sum_{j=1}^{6} d_{vj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} d_{vj}^{(f_1)}|_{z=c} g_j^{(f_1)}, \tag{5.54e}$$

and

(c) $W^{(f_1)} = W^{(c)}$ at $z = c$, which gives

$$\sum_{j=1}^{6} d_{wj}^{(c)}|_{z=c} g_j^{(c)} = \sum_{j=1}^{6} d_{wj}^{(f_1)}|_{z=c} g_j^{(f_1)}. \tag{5.54f}$$

Finally, three traction conditions exist on each of the two outer surfaces. The traction free conditions at the lower outer surface, $z = -(c + f_2)$, can be written as follows:

(a) $\sigma_{zz}|_{z=-(c+f_2)} = 0$, which gives

$$\sum_{j=1}^{6} b_{zzj}^{(f_2)}|_{z=-(c+f_2)} g_j^{(f_2)} = 0, \tag{5.55a}$$

(b) $\tau_{yz}|_{z=-(c+f_2)} = 0$, which gives

$$\sum_{j=1}^{6} b_{yzj}^{(f_2)}|_{z=-(c+f_2)} g_j^{(f_2)} = 0, \tag{5.55b}$$

and

(c) $\tau_{xz}|_{z=-(c+f_2)}$, which gives

$$\sum_{j=1}^{6} b_{xzj}^{(f_2)}|_{z=-(c+f_2)} g_j^{(f_2)} = 0. \tag{5.55c}$$

For the upper surface, where the transverse pressure q_0 is applied:

(a) $\sigma_{zz}|_{z=(c+f_1)} = q_0$, which gives

$$\sum_{j=1}^{6} b_{zzj}^{(f_1)}|_{z=(c+f_1)} g_j^{(f_1)} = Q_0, \tag{5.55d}$$

(b) $\tau_{yz}|_{z=(c+f_1)} = 0$, which gives

$$\sum_{j=1}^{6} b_{yzj}^{(f_1)}|_{z=(c+f_1)} g_j^{(f_1)} = 0, \tag{5.55e}$$

and

(c) $\tau_{xz}|_{z=(c+f_1)} = 0$, which gives

$$\sum_{j=1}^{6} b_{xzj}^{(f_1)}|_{z=(c+f_1)} g_j^{(f_1)} = 0. \tag{5.55f}$$

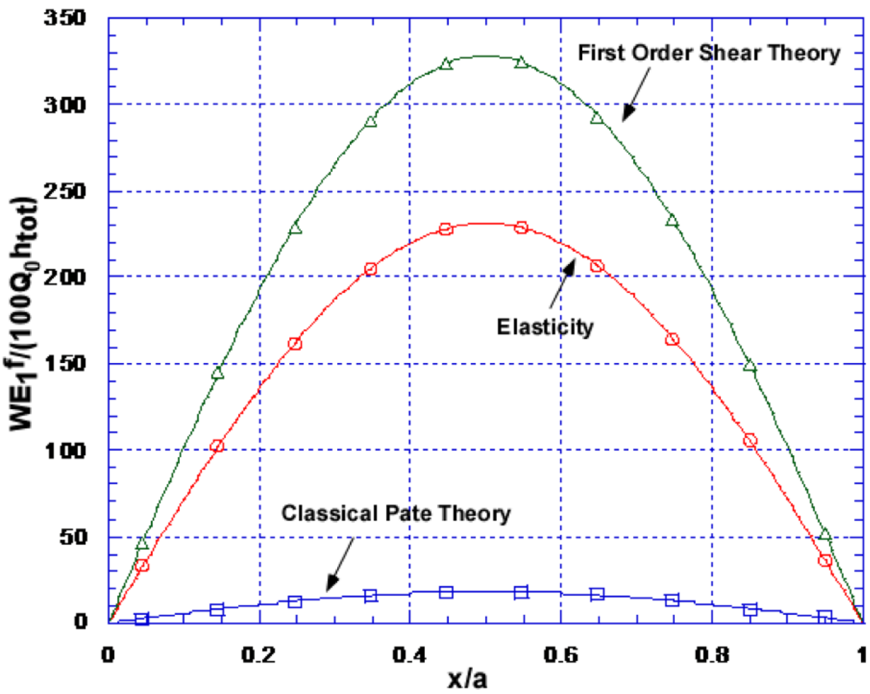

Figure 5.2 Transverse displacement, W, at the top face sheet and at $y = b/2$, as a function of x for $a = b = 10h_{\text{tot}}$ (carbon/epoxy faces and glass/phenolic honeycomb core).

Therefore, we have a system of 18 linear algebraic equations with 18 unknowns, $g_j^{(f_2)}$, $g_j^{(c)}$ and $g_j^{(f_1)}$, $j = 1, 6$.

A square sandwich panel with size $a = b = 10h_{\text{tot}}$, where h_{tot} $(= h)$ is the thickness of the panel, was considered first. The resulting transverse displacement profile w at the top surface, $z = c + f_1$, and at $y = b/2$, is shown in Figure 5.2. The displacement is normalized with $100hQ_0/E_1^f$. In this figure, we also show the predictions from the classical plate theory which does not include transverse shear. Furthermore, the displacement profile obtained from the first-order core shear theory is also shown. The classical and first-order shear theories are outlined in detail in Chapter 3. It can be seen that the classical plate theory is unconservative and quite inaccurate. Furthermore, the first-order shear is too conservative and also quite inaccurate (although considerably better than the classical plate theory).

To illustrate the effect of plate size, Figure 5.3 shows the displacement profiles for a plate five times larger, i.e., with $a = b = 50h$. For this case,

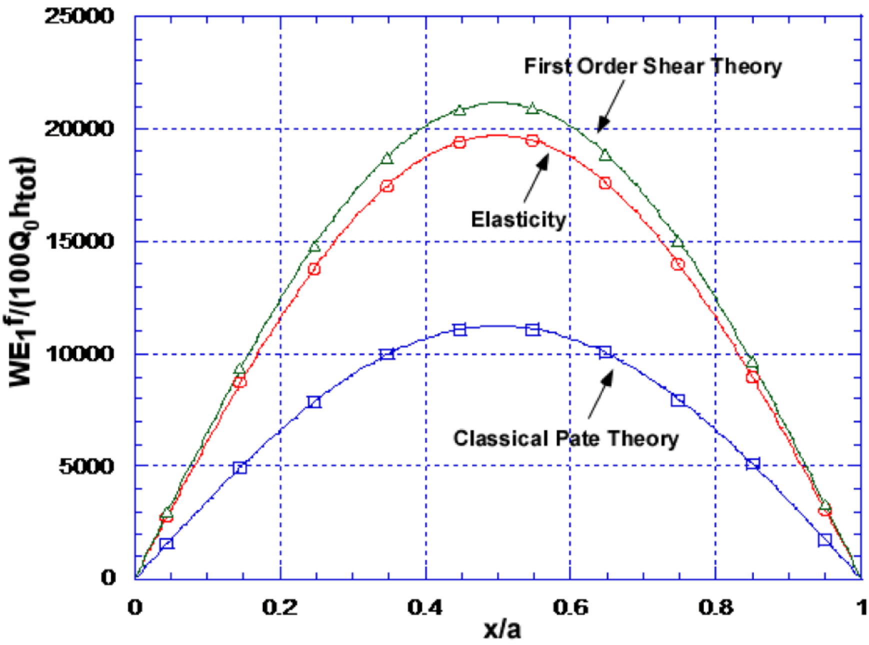

Figure 5.3 Transverse displacement, W, at the top face sheet and at $y = b/2$, as a function of x for $a = b = 50h_{\text{tot}}$.

the first-order shear theory is close to the elasticity, as expected. The classical plate theory is still quite inaccurate. These figures demonstrate clearly the large effect of transverse shear, which is an important feature of sandwich structures.

5.2 A Cylindrical Sandwich Shell with Orthotropic Layers

We consider next the elastic equilibrium of a body in the form of a hollow round cylinder (a tube) of sandwich construction which consists of two face-sheets and a core (Figure 5.4). All three layers are made from a material with cylindrical orthotropy. The body is under the influence of stresses distributed along the lateral surface and on the ends. Let us assume that (1) the axis of orthotropy coincides with the geometric axis of the body; (2) there are planes of elastic symmetry normal to the axis of the cylinder; (3) the stresses acting on the outer and inner surfaces are normal and distributed uniformly, and (4) the stresses which act on the end surfaces reduce to forces

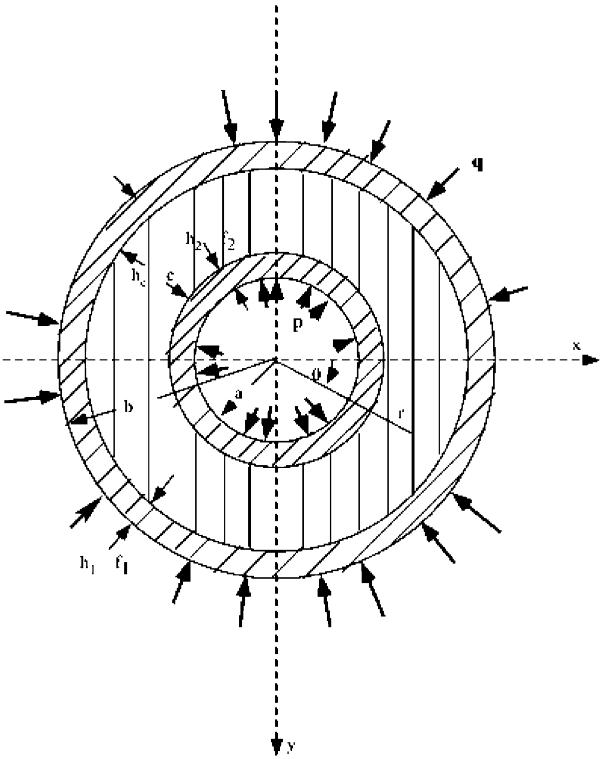

Figure 5.4 Cross-section of a cylindrical sandwich shell under internal pressure (p), external pressure (q), and axial loading (P, normal to the plane of the figure).

which are directed along the axis and to twisting moments. We denote the thickness of the outer face-sheet by $h_1 = f_1$, that of the inner face-sheet by $h_2 = f_2$, and that of the core by $h_c = c$. The inner radius is a and the outer b, where, of course, $b = a + f_2 + c + f_1$. The shell thickness is $h = b - a$.

Let us denote each layer by i where $i = f_1$ for the outer face-sheet, $i = c$ for the core and $i = f_2$ for the inner face-sheet. Then, for each layer, the orthotropic strain-stress relations are

$$\begin{bmatrix} \varepsilon_{rr}^{(i)} \\ \varepsilon_{\theta\theta}^{(i)} \\ \varepsilon_{zz}^{(i)} \\ \gamma_{\theta z}^{(i)} \\ \gamma_{rz}^{(i)} \\ \gamma_{r\theta}^{(i)} \end{bmatrix} = \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & 0 & 0 & 0 \\ a_{12}^i & a_{22}^i & a_{23}^i & 0 & 0 & 0 \\ a_{13}^i & a_{23}^i & a_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66}^i \end{bmatrix} \begin{bmatrix} \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{\theta z}^{(i)} \\ \tau_{rz}^{(i)} \\ \tau_{r\theta}^{(i)} \end{bmatrix}, \quad (i = f_1, c, f_2) \tag{5.56}$$

where a_{ij}^i are the compliance constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

We have taken the axis of the body as the z axis of the cylindrical coordinate system, and the polar x axis is arbitrary in the plane of one of the end sections. The following notations are introduced: p and q are the internal and external pressures, respectively; P is the axial force; M is the twisting moment. Let us introduce the following notation for certain constants which enter into the stress formulas and depend on the elastic constants:

$$\beta_{11}^i = a_{11}^i - \frac{a_{13}^{i\,2}}{a_{33}^i}; \quad \beta_{22}^i = a_{22}^i - \frac{a_{23}^{i\,2}}{a_{33}^i}, \tag{5.57a}$$

$$\beta_{12}^i = a_{12}^i - \frac{a_{13}^i a_{23}^i}{a_{33}^i}, \tag{5.57b}$$

and

$$k_i = \sqrt{\frac{\beta_{11}^i}{\beta_{22}^i}}; \quad \mu_i = \frac{1}{a_{44}^i}; \quad \xi_i = \frac{a_{13}^i - a_{23}^i}{\beta_{22}^i - \beta_{11}^i}, \tag{5.58}$$

where $i = f_1, c, f_2$.

Remark. In the case of isotropy ($a_{13}^i = a_{23}^i$ and $\beta_{22}^i = \beta_{11}^i$), ξ equals zero and all the formulas in this section will still be valid.

Now, if we assume that the applied external stresses are the same at all the cross-sections (do not vary with z) and, in addition, that the stresses depend only on the distance r from the axis, then the stresses in each of the orthotropic layers can be written in terms of two stress functions, $F^{(i)}(r)$ and $\Psi^{(i)}(r)$, $(i = f_1, c, f_2)$ so that

$$\sigma_{rr}^{(i)}(r) = \frac{F^{(i)\prime}(r)}{r}; \quad \sigma_{\theta\theta}^{(i)}(r) = F^{(i)\prime\prime}(r), \tag{5.59a}$$

$$\tau_{r\theta}^{(i)} = 0; \quad \tau_{rz}^{(i)} = 0; \quad \tau_{\theta z}^{(i)} = -\Psi^{(i)\prime}(r), \tag{5.59b}$$

$$\sigma_{zz}^{(i)} = C^{(i)} - \frac{1}{a_{33}} \left[a_{13}^{i} \sigma_{rr}^{(i)} + a_{23}^{i} \sigma_{\theta\theta}^{(i)} \right], \tag{5.59c}$$

where $i = f_1, c, f_2$.

Under the aforementioned assumptions, the equations of equilibrium and the condition that the displacements are single-valued functions of the coordinates, will be satisfied if

$$F^{(i)}(r) = \frac{C^{(i)}}{2} \xi_i r^2 + \frac{C_2^{(i)}}{1+k_i} r^{1+k_i} + \frac{C_3^{(i)}}{1-k_i} r^{1-k_i}, \tag{5.60a}$$

$$\Psi^{(i)}(r) = -\frac{\bar{\theta}^{(i)} \mu_i}{2} r^2. \tag{5.60b}$$

where $i = f_1, c, f_2$.

The constants $C^{(i)}$, $C_2^{(i)}$, $C_3^{(i)}$, $\bar{\theta}^{(i)}$ are found from the conditions on the cylindrical lateral surfaces (e.g. applied uniform internal and/or external pressure) and the conditions on the ends (e.g. applied axial load or axial strain or twisting moment).

Therefore, from Equations (5.59), the stresses are

$$\sigma_{rr}^{(i)}(r) = C^{(i)} \xi_i + C_2^{(i)} r^{k_i - 1} + C_3^{(i)} r^{-k_i - 1}, \tag{5.61a}$$

$$\sigma_{\theta\theta}^{(i)}(r) = C^{(i)} \xi_i + C_2^{(i)} k_i r^{k_i - 1} - C_3^{(i)} k_i r^{-k_i - 1}, \tag{5.61b}$$

$$\tau_{\theta z}^{(i)}(r) = \bar{\theta}^{(i)} \mu_i r, \tag{5.61c}$$

$$\begin{aligned} \sigma_{zz}^{(i)}(r) = C^{(i)} \left[1 - \frac{(a_{13}^{i} + a_{23}^{i})}{a_{33}^{i}} \xi_i \right] - C_2^{(i)} \frac{(a_{13}^{i} + a_{23}^{i} k_i)}{a_{33}^{i}} r^{k_i - 1} \\ - C_3^{(i)} \frac{(a_{13}^{i} - a_{23}^{i} k_i)}{a_{33}^{i}} r^{-k_i - 1}. \end{aligned} \tag{5.61d}$$

where $i = f_1, c, f_2$.

Denoting by $u_r^{(i)}$, $u_\theta^{(i)}$ and $w^{(i)}$ the displacements in the radial, circumferential and axial direction, respectively, the displacement field for this case, excluding rigid body translation and rotation, is given as

$$\begin{aligned} & u_r^{(i)}(r, z) = U^{(i)}(r) \; ; \quad u_\theta^{(i)}(r, z) = \bar{\theta}^{(i)} z r + V^{(i)}(r); \\ & w^{(i)}(r, z) = C^{(i)} a_{33}^{i} z + W^{(i)}(r), \end{aligned} \tag{5.62}$$

where $U^{(i)}$, $V^{(i)}$ and $W^{(i)}$ are found from the strain-displacement relations and the stress field (5.59) from the following:

$$\frac{\partial U^{(i)}}{\partial r} = \beta_{11}^i \sigma_{rr}^{(i)} + \beta_{12}^i \sigma_{\theta\theta}^{(i)} + a_{13}^i C^{(i)}, \tag{5.63a}$$

$$\frac{1}{r}\frac{\partial V^{(i)}}{\partial \theta} + \frac{U^{(i)}}{r} = \beta_{12}^i \sigma_{rr}^{(i)} + \beta_{22}^i \sigma_{\theta\theta}^{(i)} + a_{23}^i C^{(i)}, \tag{5.63b}$$

$$\frac{1}{r}\frac{\partial U^{(i)}}{\partial \theta} + \frac{\partial V^{(i)}}{\partial r} - \frac{V^{(i)}}{r} = 0; \quad \frac{\partial W^{(i)}}{\partial r} = 0; \quad \frac{1}{r}\frac{\partial W^{(i)}}{\partial \theta} = 0. \tag{5.63c}$$

Therefore, with the definitions (5.57) for k_i and ξ_i, the displacement field which satisfies these equations and would result in strains, is found by integrating (5.63), as

$$U^{(i)}(r) = C^{(i)}\left[a_{13}^i + \xi_i(\beta_{11}^i + \beta_{12}^i)\right] r + C_2^{(i)} \frac{(\beta_{11}^i + k_i \beta_{12}^i)}{k_i} r^{k_i} - C_3^{(i)} \frac{(\beta_{11}^i - k_i \beta_{12}^i)}{k_i} r^{-k_i}, \tag{5.64a}$$

$$V^{(i)}(r) = 0\,; \qquad W^{(i)}(r) = 0. \tag{5.64b}$$

5.2.1 Generalized Plane Deformation of an Orthotropic Sandwich Tube Subjected to Internal and/or External Pressures

Let us assume that the sandwich cylinder considered in the previous section is subject to pressures p and q distributed uniformly on the inner and outer surfaces, respectively, and has infinite length (generalized plane deformation assumption). Then, not only the stresses, but also the displacements do not depend on z. Alternatively, this is the assumption we would make if the cylinder were securely fixed at the ends ($\varepsilon_z = 0$). Consequently, we can assume

$$C^{(i)} = \bar{\theta}^{(i)} = 0. \tag{5.65}$$

The traction conditions at the core/face-sheet interfaces give

$$\sigma_{rr}^{(f_2)}|_{r=a+f_2} = \sigma_{rr}^{(c)}|_{r=a+f_2}; \quad \sigma_{rr}^{(c)}|_{r=b-f_1} = \sigma_{rr}^{(f_1)}|_{r=b-f_1}. \tag{5.66}$$

Applying (5.61) and (5.65), this gives

$$C_2^{(f_2)}(a+f_2)^{k_{f_2}-1} + C_3^{(f_2)}(a+f_2)^{-k_{f_2}-1} = C_2^{(c)}(a+f_2)^{k_c-1} + C_3^{(c)}(a+f_2)^{-k_c-1}, \tag{5.67a}$$

$$C_2^{(c)}(b-f_1)^{k_c-1} + C_3^{(c)}(b-f_1)^{-k_c-1} = C_2^{(f_1)}(b-f_1)^{k_{f_1}-1} + C_3^{(f_1)}(b-f_1)^{-k_{f_1}-1}. \tag{5.67b}$$

The displacement continuity at the core/face-sheet interfaces is, in turn,

$$U^{(f_2)}|_{r=a+f_2} = U^{(c)}|_{r=a+f_2}; \quad U^{(c)}|_{r=b-f_1} = U^{(f_1)}|_{r=b-f_1}. \tag{5.68}$$

which, by use of (5.64a) and (5.65), gives

$$C_2^{(f_2)} \frac{(\beta_{11}^{f_2} + k_{f_2}\beta_{12}^{f_2})}{k_{f_2}}(a+f_2)^{k_{f_2}} - C_3^{(f_2)} \frac{(\beta_{11}^{f_2} - k_{f_2}\beta_{12}^{f_2})}{k_{f_2}}(a+f_2)^{-k_{f_2}}$$
$$= C_2^{(c)} \frac{(\beta_{11}^{c} + k_c\beta_{12}^{c})}{k_c}(a+f_2)^{k_c} - C_3^{(c)} \frac{(\beta_{11}^{c} - k_c\beta_{12}^{c})}{k_c}(a+f_2)^{-k_c}, \tag{5.69a}$$

$$C_2^{(c)} \frac{(\beta_{11}^{c} + k_c\beta_{12}^{c})}{k_c}(b-f_1)^{k_c} - C_3^{(c)} \frac{(\beta_{11}^{c} - k_c\beta_{12}^{c})}{k_c}(b-f_1)^{-k_c}$$
$$= C_2^{(f_1)} \frac{(\beta_{11}^{f_1} + k_{f_1}\beta_{12}^{f_1})}{k_{f_1}}(b-f_1)^{k_{f_1}} - C_3^{(f_1)} \frac{(\beta_{11}^{f_1} - k_{f_1}\beta_{12}^{f_1})}{k_{f_1}}(b-f_1)^{-k_{f_1}}. \tag{5.69b}$$

The conditions of applied internal and external pressures on the inner and outer surfaces ($r = a, b$) are

$$\sigma_{rr}^{(f_2)}|_{r=a} = -p\ ; \quad \sigma_{rr}^{(f_1)}|_{r=b} = -q, \tag{5.70}$$

which gives

$$C_2^{(f_2)} a^{k_{f_2}-1} + C_3^{(f_2)} a^{-k_{f_2}-1} = -p; \quad C_2^{(f_1)} b^{k_{f_1}-1} + C_3^{(f_1)} b^{-k_{f_1}-1} = -q, \tag{5.71}$$

The six unknowns $C_2^{(i)}, C_3^{(i)}$ ($i = f_1, c, f_2$) are solved in terms of p and q using a system of six linear equations formed by Equations (5.67a, b), (5.69a, b) and (5.71). Then, the stresses are found by Equations (5.61).

Since there is no stress $\tau_{\theta z}$, there is no resultant twisting moment. The stresses σ_{zz} on the ends and in any cross-section reduce to an axial force P which can be found from

$$\frac{P}{2\pi} = \int_a^b \sigma_{zz} r dr = \int_a^{a+f_2} \sigma_{zz}^{(f_2)} r dr + \int_{a+f_2}^{b-f_1} \sigma_{zz}^{(c)} r dr + \int_{b-f_1}^{b} \sigma_{zz}^{(f_1)} r dr. \tag{5.72}$$

Using (5.61d), this becomes

$$\frac{P}{2\pi} = -(D_2 + D_3). \tag{5.73a}$$

where

$$D_2 = C_2^{(f_1)} \frac{(a_{13}^{f_1} + a_{23}^{f_1} k_{f_1})}{a_{33}^{f_1}(k_{f_1}+1)} \left[b^{(k_{f_1}+1)} - (b-f_1)^{(k_{f_1}+1)} \right]$$
$$+ C_2^{(c)} \frac{(a_{13}^{c} + a_{23}^{c} k_{c})}{a_{33}^{c}(k_{c}+1)} \left[(b-f_1)^{(k_c+1)} - (a+f_2)^{(k_c+1)} \right]$$
$$+ C_2^{(f_2)} \frac{(a_{13}^{f_2} + a_{23}^{f_2} k_{f_2})}{a_{33}^{f_2}(k_{f_2}+1)} \left[(a+f_2)^{(k_{f_2}+1)} - a^{(k_{f_2}+1)} \right], \quad (5.73b)$$

$$D_3 = C_3^{(f_1)} \frac{(a_{13}^{f_1} - a_{23}^{f_1} k_{f_1})}{a_{33}^{f_1}(-k_{f_1}+1)} \left[b^{(-k_{f_1}+1)} - (b-f_1)^{(-k_{f_1}+1)} \right]$$
$$+ C_3^{(c)} \frac{(a_{13}^{c} - a_{23}^{c} k_{c})}{a_{33}^{c}(-k_{c}+1)} \left[(b-f_1)^{(-k_c+1)} - (a+f_2)^{(-k_c+1)} \right]$$
$$+ C_3^{(f_2)} \frac{(a_{13}^{f_2} - a_{23}^{f_2} k_{f_2})}{a_{33}^{f_2}(-k_{f_2}+1)} \left[(a+f_2)^{(-k_{f_2}+1)} - a^{(-k_{f_2}+1)} \right]. \quad (5.73c)$$

5.2.2 An Orthotropic Hollow Sandwich Cylinder Loaded by an Axial Force

We now assume that the shell is loaded by stresses distributed on the ends and which reduce to a tensile force P. The stresses at the ends are applied so that a uniformly distributed constant axial strain, ε_0, exists throughout the section. Note also that no resultant twisting moment is assumed to exist and $\bar{\theta}^i = 0$.

From (5.62) the axial strain is $C^{(i)} a_{33}^{i}$, and the first condition is

$$C^{(f_2)} a_{33}^{f_2} = C^{(c)} a_{33}^{c} = C^{(f_1)} a_{33}^{f_1} = \varepsilon_0. \quad (5.74)$$

i.e., the constants $C^{(i)}$ are non-zero.

Next, the traction conditions (5.66) at the face-sheet/core interfaces give by use of (5.61a) and (5.74):

$$\varepsilon_0 \frac{\xi_{f_2}}{a_{33}^{f_2}} + C_2^{(f_2)} (a+f_2)^{k_{f_2}-1} + C_3^{(f_2)} (a+f_2)^{-k_{f_2}-1}$$
$$= \varepsilon_0 \frac{\xi_c}{a_{33}^{c}} + C_2^{(c)} (a+f_2)^{k_c-1} + C_3^{(c)} (a+f_2)^{-k_c-1}, \quad (5.75a)$$

$$\varepsilon_0 \frac{\xi_c}{a_{33}^c} + C_2^{(c)}(b - f_1)^{k_c - 1} + C_3^{(c)}(b - f_1)^{-k_c - 1}$$
$$= \varepsilon_0 \frac{\xi_{f_1}}{a_{33}^{f_1}} + C_2^{(f_1)}(b - f_1)^{k_{f_1} - 1} + C_3^{(f_1)}(b - f_1)^{-k_{f_1} - 1}. \quad (5.75b)$$

The displacement continuity at the face-sheet/core interfaces, (5.68), by use of (5.64a) and (5.74) becomes

$$\varepsilon_0 \frac{(a_{13}^{f_2} + \xi_{f_2}(\beta_{11}^{f_2} + \beta_{12}^{f_2}))}{a_{33}^{f_2}}(a + f_2) + C_2^{(f_2)} \frac{(\beta_{11}^{f_2} + k_{f_2}\beta_{12}^{f_2})}{k_{f_2}}(a + f_2)^{k_{f_2}}$$
$$- C_3^{(f_2)} \frac{(\beta_{11}^{f_2} - k_{f_2}\beta_{12}^{f_2})}{k_{f_2}}(a + f_2)^{-k_{f_2}} = \varepsilon_0 \frac{(a_{13}^c + \xi_c(\beta_{11}^c + \beta_{12}^c))}{a_{33}^c}(a + f_2)$$
$$+ C_2^{(c)} \frac{(\beta_{11}^c + k_c\beta_{12}^c)}{k_c}(a + f_2)^{k_c} - C_3^{(c)} \frac{(\beta_{11}^c - k_c\beta_{12}^c)}{k_c}(a + f_2)^{-k_c}, \quad (5.76a)$$

$$\varepsilon_0 \frac{(a_{13}^c + \xi_c(\beta_{11}^c + \beta_{12}^c))}{a_{33}^c}(b - f_1) + C_2^{(c)} \frac{(\beta_{11}^c + k_c\beta_{12}^c)}{k_c}(b - f_1)^{k_c}$$
$$- C_3^{(c)} \frac{(\beta_{11}^c - k_c\beta_{12}^c)}{k_c}(b - f_1)^{-k_c} = \varepsilon_0 \frac{(a_{13}^{f_1} + \xi_{f_1}(\beta_{11}^{f_1} + \beta_{12}^{f_1}))}{a_{33}^{f_1}}(b - f_1)$$
$$+ C_2^{(f_1)} \frac{(\beta_{11}^{f_1} + k_{f_1}\beta_{12}^{f_1})}{k_{f_1}}(b - f_1)^{k_{f_1}} - C_3^{(f_1)} \frac{(\beta_{11}^{f_1} - k_{f_1}\beta_{12}^{f_1})}{k_{f_1}}(b - f_1)^{-k_{f_1}}. \quad (5.76b)$$

Next, the condition of traction-free lateral surfaces is expressed by

$$\sigma_{rr}^{(f_2)}|_{r=a} = 0; \quad \sigma_{rr}^{(f_1)}|_{r=b} = 0, \quad (5.77)$$

which gives

$$\varepsilon_0 \frac{\xi_{f_2}}{a_{33}^{f_2}} + C_2^{(f_2)} a^{k_{f_2} - 1} + C_3^{(f_2)} a^{-k_{f_2} - 1} = 0, \quad (5.78a)$$

$$\varepsilon_0 \frac{\xi_{f_1}}{a_{33}^{f_1}} + C_2^{(f_1)} b^{k_{f_1} - 1} + C_3^{(f_1)} b^{-k_{f_1} - 1} = 0. \quad (5.78b)$$

Again, the solution is found by solving for the six constants $C_2^{(i)}$, $C_3^{(i)}$, $(i = f_1, c, f_2)$ in terms of ε_0, from the six linear equations (5.75a, b), (5.76a, b) and (5.78a, b).

An expression for the resultant applied force P in terms of ε_0 can be found by integrating σ_{zz} as in (5.72), and this now gives, by using (5.61d),

$$\frac{P}{2\pi} = -(D_1 + D_2 + D_3), \tag{5.79a}$$

where D_2 and D_3 are given by (5.73b, c) and

$$\begin{aligned} D_1/\varepsilon_0 = {} & \left[1 - \frac{(a_{13}^{f_1} + a_{23}^{f_1})}{a_{33}^{f_1}}\xi_{f_1}\right] \frac{\left[b^2 - (b - f_1)^2\right]}{2a_{33}^{f_1}} \\ & + \left[1 - \frac{(a_{13}^{c} + a_{23}^{c})}{a_{33}^{c}}\xi_{c}\right] \frac{\left[(b - f_1)^2 - (a + f_2)^2\right]}{2a_{33}^{c}} \\ & + \left[1 - \frac{(a_{13}^{f_2} + a_{23}^{f_2})}{a_{33}^{f_2}}\xi_{f_2}\right] \frac{\left[(a + f_2)^2 - a^2\right]}{2a_{33}^{f_2}}. \end{aligned} \tag{5.79b}$$

Of course, the axial stress σ_{zz} is non-uniformly distributed over the cross-section as opposed to the axial strain, ε_0, assumed to be uniform.

5.2.3 Sandwich Shell Theory Expressions

We refer to a cylindrical coordinate system z, θ and r, in which z and θ are in the axial and circumferential directions and r is in the (radial) direction. The corresponding displacements at any point are denoted by w, v and u.

In addition to Equation (5.56) which is in terms of the compliance constants, we shall use the stress-strain relations in terms of the stiffness constants, as follows:

$$\begin{bmatrix} \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{\theta z}^{(i)} \\ \tau_{rz}^{(i)} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11}^i & c_{12}^i & c_{13}^i & 0 & 0 & 0 \\ c_{12}^i & c_{22}^i & c_{23}^i & 0 & 0 & 0 \\ c_{13}^i & c_{23}^i & c_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^i \end{bmatrix} \begin{bmatrix} \varepsilon_{rr}^{(i)} \\ \varepsilon_{\theta\theta}^{(i)} \\ \varepsilon_{zz}^{(i)} \\ \gamma_{\theta z}^{(i)} \\ \gamma_{rz}^{(i)} \\ \gamma_{r\theta}^{(i)} \end{bmatrix}, \quad (i = f_1, c, f_2), \tag{5.80}$$

where we have again used the notation $1 \equiv r, 2 \equiv \theta, 3 \equiv z$.

The sandwich shell theory employed is a version of Love's (1927) shell theory extended to shear deformable structures (but note the absence of shear in this case of orthotropy). The core is assumed to carry only shear stresses and the face sheets carry the normal stresses, therefore the extensional and bending stiffnesses of the shell are based exclusively on the face-sheet stiffnesses. On the contrary, transverse shear stress resultants (should they exist) are based exclusively on the shear stiffnesses of the core.

Taking into account the displacement distribution through the thickness assumed in the shell theory, we can easily see that in the generalized plane deformation problems under consideration, the displacement field throughout the shell is

$$u(r,\theta,z) = u_0\,; \quad v(r,\theta,z) = 0 \quad w(r,\theta,z) = \varepsilon_0 z, \tag{5.81a}$$

where u_0 is a constant and ε_0 is the uniform axial strain.

The relationships for the strains throughout the shell, corresponding to Love's (1927) shell theory are

$$\varepsilon_{rr} = 0\,; \quad \varepsilon_{\theta\theta} = \frac{u_0}{R}\,; \quad \varepsilon_{zz} = \varepsilon_0, \tag{5.81b}$$

where R is the mid-surface radius. The shear strains are all zero. Notice that in these simplified, axisymmetric, generalized plane deformation problems, there is no difference between first-order shear deformation and classical solutions.

The stress resultants of interest are

$$N_\theta = C_{22}\varepsilon^0_{\theta\theta} + C_{23}\varepsilon^0_{zz}; \quad N_z = C_{23}\varepsilon^0_{\theta\theta} + C_{33}\varepsilon^0_{zz}; \quad N_{z\theta} = 0, \tag{5.81c}$$

where ε^0_{ij} are the mid-surface strains, identical to the ones in (5.81b). Moreover, the C_{ij} are the shell stiffness constants, determined by the face-sheets (in the context of sandwich shell formulation) by

$$C_{ij} = f_1 c^{f_1}_{ij} + f_2 c^{f_2}_{ij}, \quad (i,j = 2,3). \tag{5.81d}$$

For *external pressure*, the equilibrium equations in terms of the stress resultants are satisfied if

$$N_\theta = -qR. \tag{5.82a}$$

Furthermore, based on the assumptions of the problem for the external pressure case, $\varepsilon_0 = 0$. Then (5.81c) and (5.81b) give

$$u_0 = -qR^2/C_{22}\,; \quad \varepsilon_{\theta\theta} = -qR/C_{22}. \tag{5.82b}$$

Subsequently by using (5.80) the stresses are:

$$\sigma_{rr} = -q\frac{c_{12}^i R}{C_{22}}; \quad \sigma_{\theta\theta} = -q\frac{c_{22}^i R}{C_{22}}; \quad \sigma_{zz} = -q\frac{c_{23}^i R}{C_{22}} \quad (i = f_2, c, f_1), \tag{5.82c}$$

For *axial loading* with a uniform axial strain ε_0, the equilibrium equations are satisfied if $N_\theta = 0$, which, by using (5.81c) and (5.81b), gives

$$u_0 = -\varepsilon_0 R C_{23}/C_{22}\,; \quad \varepsilon_{\theta\theta} = -\varepsilon_0 C_{23}/C_{22}. \tag{5.83a}$$

Subsequently, N_z can be obtained from (5.81c) as

$$N_z = \varepsilon_0\left(C_{33} - \frac{C_{23}^2}{C_{22}}\right). \tag{5.83b}$$

Then the stresses are found by using (5.80):

$$\sigma_{rr} = \varepsilon_0\left(c_{13}^i - c_{12}^i\frac{C_{23}}{C_{22}}\right); \quad \sigma_{\theta\theta} = \varepsilon_0\left(c_{23}^i - c_{22}^i\frac{C_{23}}{C_{22}}\right);$$

$$\sigma_{zz} = \varepsilon_0\left(c_{33}^i - c_{23}^i\varepsilon_0\frac{C_{23}}{C_{22}}\right), \tag{5.83c}$$

where $i = f_1, c, f_2$.

As an illustrative example, the stress and displacement distribution was determined for a sandwich composite circular cylindrical shell of outer radius $b = 1$ m, a ratio of outside over inside radii, $b/a = 1.20$, ratios of face-sheet thicknesses over shell thickness, $f_2/h = f_1/h = 0.10$.

The face sheets are made from unidirectional E-glass/polyester with the fiber direction along the circumference, with moduli in GPa: $E_2^{(f_1,f_2)} = 40$, $E_1^{(f_1,f_2)} = E_3^{(f_1,f_2)} = 10$, $G_{13}^{(f_1,f_2)} = 3.5$, $G_{12}^{(f_1,f_2)} = G_{23}^{(f_1,f_2)} = 4.5$, and Poisson's ratios $\nu_{31}^{(f_1,f_2)} = 0.40$, $\nu_{21}^{(f_1,f_2)} = \nu_{23}^{(f_1,f_2)} = 0.26$. Note that 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction. The core modulus and Poisson's ratio are assumed to be $E^c = 75$ MPa and $\nu^c = 0.30$. Notice that the compliance constants for the orthotropic face sheets are given by (5.32a–c).

For the case of pure *external pressure*, q, Figure 5.5 shows the radial displacement $U(r)$, normalized with qR^2/C_{22} (C_{22} is defined in (5.81d)) plotted vs. r/R. The elasticity solution (Section 5.2.1) predicts a non-uniform displacement as opposed to the shell theory.

For the case of pure *axial loading* by a uniform applied axial strain ε_0, Figure 5.6 shows the displacement, $U(r)$, normalized with $\varepsilon_0 R C_{23}/C_{22}$. Again, the elasticity solution (Section 5.2.2) predicts a non-uniform displacement distribution as opposed to the shell theory.

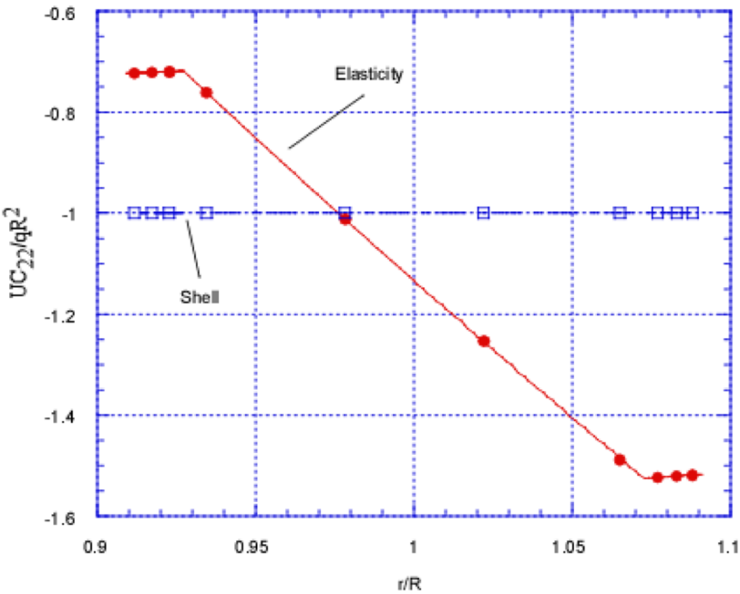

Figure 5.5 Radial displacement, $U(r)$ vs. normalized radius (r/R), for a cylindrical sandwich shell of mean radius, R, under uniform external pressure, q.

Also note that because of the orthotropy and the axisymmetric geometry, there are no shear stresses generated from internal/external pressure and axial loading. Therefore even a first-order shear deformation theory would not result in improved shell theory predictions.

Finally, it should be pointed out that the concept of sandwich construction may not be ideal for the loading and structure analyzed. This is because in the case considered there is no shear in the core and to really take advantage of the sandwich concept, the core should carry the shear and the face sheet should support the normal stresses. If the cylinder is loaded in compression, however, and buckling occurs, then the core would support the shear, and the solution presented can be used as the exact pre-buckling state of stress and displacement in the formulation of the buckling problem.

5.2.4 Torsion of a Sandwich Shaft

Let us consider the more general case of off-axis orientation of the material, but with one plane of elastic symmetry normal to the cylinder axis. Hence, a^i_{45}, a^i_{16}, a^i_{26} and a^i_{36} are non-zero, and the strain-stress relations becomes

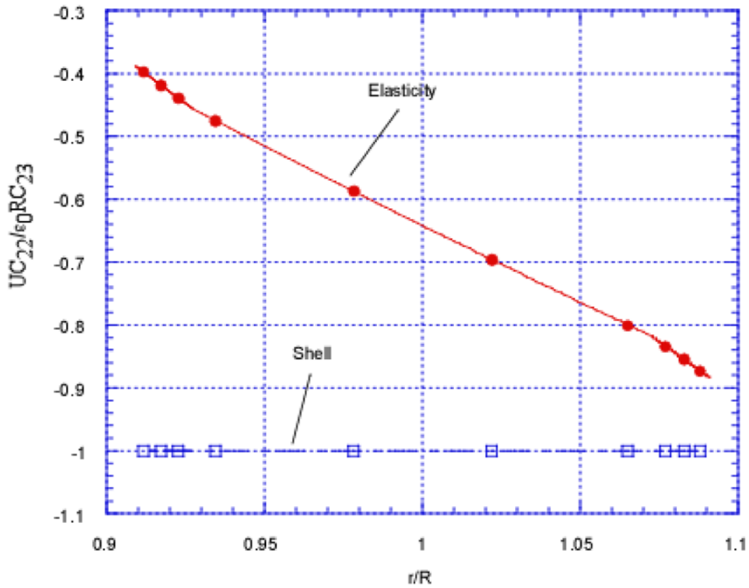

Figure 5.6 Radial displacement, $U(r)$ vs. normalized radius (r/R), for a cylindrical sandwich shell of mean radius, R, under uniformly applied axial strain, ε_0.

$$\begin{bmatrix} \varepsilon_{rr}^{(i)} \\ \varepsilon_{\theta\theta}^{(i)} \\ \varepsilon_{zz}^{(i)} \\ \gamma_{\theta z}^{(i)} \\ \gamma_{rz}^{(i)} \\ \gamma_{r\theta}^{(i)} \end{bmatrix} = \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & 0 & 0 & a_{16}^i \\ a_{12}^i & a_{22}^i & a_{23}^i & 0 & 0 & a_{26}^i \\ a_{13}^i & a_{23}^i & a_{33}^i & 0 & 0 & a_{36}^i \\ 0 & 0 & 0 & a_{44}^i & a_{45}^i & 0 \\ 0 & 0 & 0 & a_{45}^i & a_{55}^i & 0 \\ a_{16}^i & a_{26}^i & a_{36}^i & 0 & 0 & a_{66}^i \end{bmatrix} \begin{bmatrix} \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{\theta z}^{(i)} \\ \tau_{rz}^{(i)} \\ \tau_{r\theta}^{(i)} \end{bmatrix}, \quad (i = f_1, c, f_2). \tag{5.84a}$$

The equations of equilibrium are satisfied for

$$\sigma_{rr}^{(i)} = \sigma_{\theta\theta}^{(i)} = \sigma_{zz}^{(i)} = \tau_{r\theta}^{(i)} = \tau_{rz}^{(i)} = 0\,; \quad \tau_{\theta z}^{(i)} = \frac{\bar{\theta}^{(i)}}{a_{44}^i} r, \tag{5.84b}$$

and the displacement field (excluding rigid body rotation and translation) that results from these stresses can be found from the strain-displacement relations and the strain-stress relations, which in this case become

$$\frac{\partial U^{(i)}}{\partial r} = 0\,; \quad \frac{1}{r}\frac{\partial V^{(i)}}{\partial \theta} + \frac{U^{(i)}}{r} = 0\,; \quad \frac{1}{r}\frac{\partial U^{(i)}}{\partial \theta} + \frac{\partial V^{(i)}}{\partial r} - \frac{V^{(i)}}{r} = 0, \tag{5.85a}$$

where $i = f_1, c, f_2$.

$$\frac{\partial W^{(i)}}{\partial r} = a_{45}^{i}\tau_{\theta z}^{(i)} = a_{45}^{i}\frac{\bar{\theta}^{(i)}}{a_{44}^{i}}r\;;\qquad \frac{1}{r}\frac{\partial W^{(i)}}{\partial \theta} = a_{44}^{i}\tau_{\theta z}^{(i)} - \bar{\theta}^{(i)}r = 0. \quad (5.85b)$$

The resulting displacement field obtained by integrating the above relations is

$$u_r^{(i)} = 0\;;\quad u_\theta = \bar{\theta}^{(i)}rz\;;\quad w^{(i)} = \bar{\theta}^{(i)}\frac{a_{45}^{(i)}}{a_{45}^{(i)}}\frac{r^2}{2} + d_i. \quad (5.86)$$

where d_i are constants to be determined from face/core interface displacement continuity requirements.

The continuity of displacement, u_θ, at the face-sheet/core interfaces results in a constant relative angle of twist, $\bar{\theta}^{(i)}$:

$$\bar{\theta}^{(i)} = \bar{\theta} \quad (i = f_1, c, f_2). \quad (5.87)$$

The continuity of the displacement, w, at the face-sheet/core interfaces in turn results in equations for the constants d_i in terms of the axial displacement, w, expressions (5.86).

The resultant twisting moment, M, is then found from

$$\frac{M}{2\pi} = \int_a^b \tau_{\theta z}r^2dr = \int_a^{a+f_2}\tau_{\theta z}^{(f_2)}r^2dr + \int_{a+f_2}^{b-f_1}\tau_{\theta z}^{(c)}r^2dr + \int_{b-f_1}^{b}\tau_{\theta z}^{(f_1)}r^2dr. \quad (5.88)$$

Using (5.84b) and (5.87) results in the following expression:

$$\frac{M}{2\pi} = \frac{\bar{\theta}}{4}\left\{\frac{[(a+f_2)^4 - a^4]}{a_{44}^{(f_2)}} + \frac{[(b-f_1)^4 - (a+f_2)^4]}{a_{44}^{(c)}} + \frac{[b^4 - (b-f_1)^4]}{a_{44}^{(f_1)}}\right\}. \quad (5.89)$$

If $a_{45}^{i} = 0$ for all three layers, then $w^{(i)} = 0$, and the cross-sections will remain planar and not warp.

We have presented in this chapter some fundamental cases regarding three-dimensional elasticity of sandwich structures. Elasticity solutions for other cases, e.g. a hollow orthotropic sandwich cylinder loaded by bending moments applied at the ends, or an orthotropic sandwich curved bar, loaded by couples or terminal forces, can be found by extending these solutions.