Chapter 7
Continuum of 6-Colorings of the Plane

In 1993, another 6-coloring was found by Ilya Hoffman and me ([HS1], [HS2]). Its type was $\left(1, 1, 1, 1, 1, \sqrt{2} - 1\right)$. The story of this discovery is noteworthy. In the summer of 1993, I was visiting my cousin in Moscow, a well-known New Vienna School composer, Leonid Hoffman. His 15-year-old son Ilya was studying violin at the Gnessin Music High School. Ilya set out to learn what I was doing in mathematics and did not accept any general answers. He wanted particulars. I showed him my 6-coloring of the plane (Problem 6.4), and the teenage musician got busy. The very next day he showed me . . . the Stechkin coloring (Fig. 6.2) that he discovered on his own! "Great," I replied, "but you are 23 years late." A few days later, he came up with a new idea of using a two-square tiling. Ilya had an intuition of a virtuoso fiddler and no mathematical culture – and so I calculated the sizes the squares had to have for the 6-coloring to do the job we needed. I wanted Ilya to be the sole author, but he insisted on our joint credit. And the joint work of the unusual mathematician–musician team was born. Ilya went on to graduate from the graduate school of Moscow Conservatory in the class of the celebrated violist and conductor Yuri Bashmet and is now one of Russia's hottest violinists and violists and the winner of several international competitions.

Problem 7.1 (I. Hoffman and A. Soifer [HS1], [HS2]). There is a 6-coloring of the plane of type $\left(1, 1, 1, 1, 1, \sqrt{2} - 1\right)$.

Solution Tile the plane with squares of diagonals 1 and $\sqrt{2} - 1$ (Fig. 7.1). We use colors 1, ..., 5 for larger squares and color 6 for all smaller squares. With each square, we include half of its boundary, its left and lower sides, without the end points of this half (Fig. 7.2).

To easily verify that this coloring does the job, observe the unit of the construction that is bounded by the bold line in Fig. 7.1. Its translates tile the plane and thus define its coloring. ■

A. Soifer, *The New Mathematical Coloring Book*, https://doi.org/10.1007/978-1-0716-3597-1_7

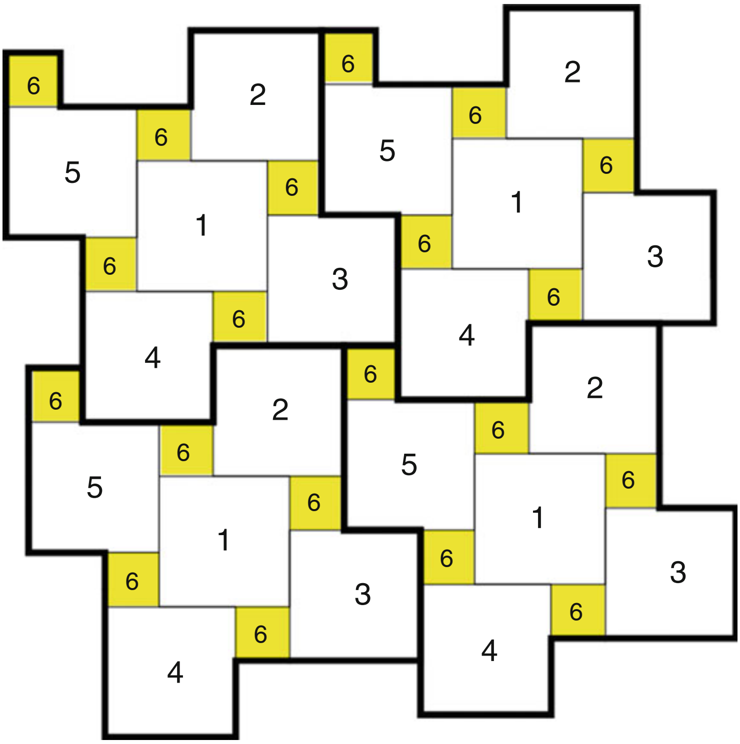


Fig. 7.1 The Hoffman–Soifer 6-coloring of the plane

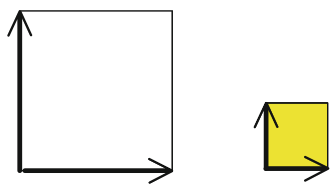


Fig. 7.2 Coloring of the boundaries

The two examples, found in the solutions of problems 6.4 and 7.1, prompted me in 1993 to introduce a new terminology for this problem and to translate the results and problems into this new language.

Open Problem 7.2 (A. Soifer [Soi7], [Soi8]). Find the *6-realizable set* X_6 of all positive numbers α such that there exists a 6-coloring of the plane of type $(1, 1, 1, 1, 1, \alpha)$.

In this new language, the results of problems 6.4 and 7.1 can be written as follows:

$$\frac{1}{\sqrt{5}}, \sqrt{2} - 1 \in X_6.$$

We now have two examples of "working" 6-colorings. But what do they have in common? It is not obvious, is it? One uses octagons, while the other does not. After a while, I realized that they were two extreme examples of a general case and, in fact, a much better result was possible, describing a whole continuum of working 6-colorings!

Theorem 7.3 (A. Soifer [Soi7], [Soi8]).

$$\left[\sqrt{2}-1, \frac{1}{\sqrt{5}}\right] \subseteq X_6,$$

i.e., for every $\alpha \in \left[\sqrt{2}-1, \frac{1}{\sqrt{5}}\right]$, there is a 6-coloring of type $(1, 1, 1, 1, 1, \alpha)$.[1]

Proof Let a unit square be partly covered by a smaller square, which cuts off the unit square into vertical and horizontal segments of lengths x and y, respectively, and forms with it an angle ω (Fig. 7.3). These squares induce the tiling of the plane that consists of nonregular octagons and "small" squares that are congruent to each other (Fig. 7.4).

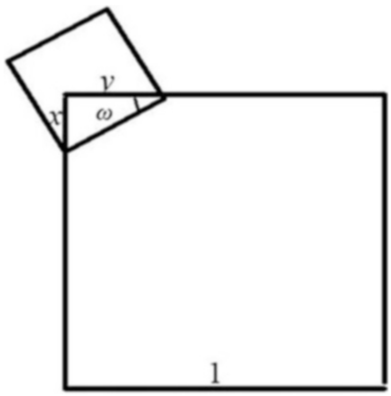


Fig. 7.3 The foundation squares

[1] Symbol $[a,b]$, $a < b$, as usual, stands for the line segment, including its end points a and b.

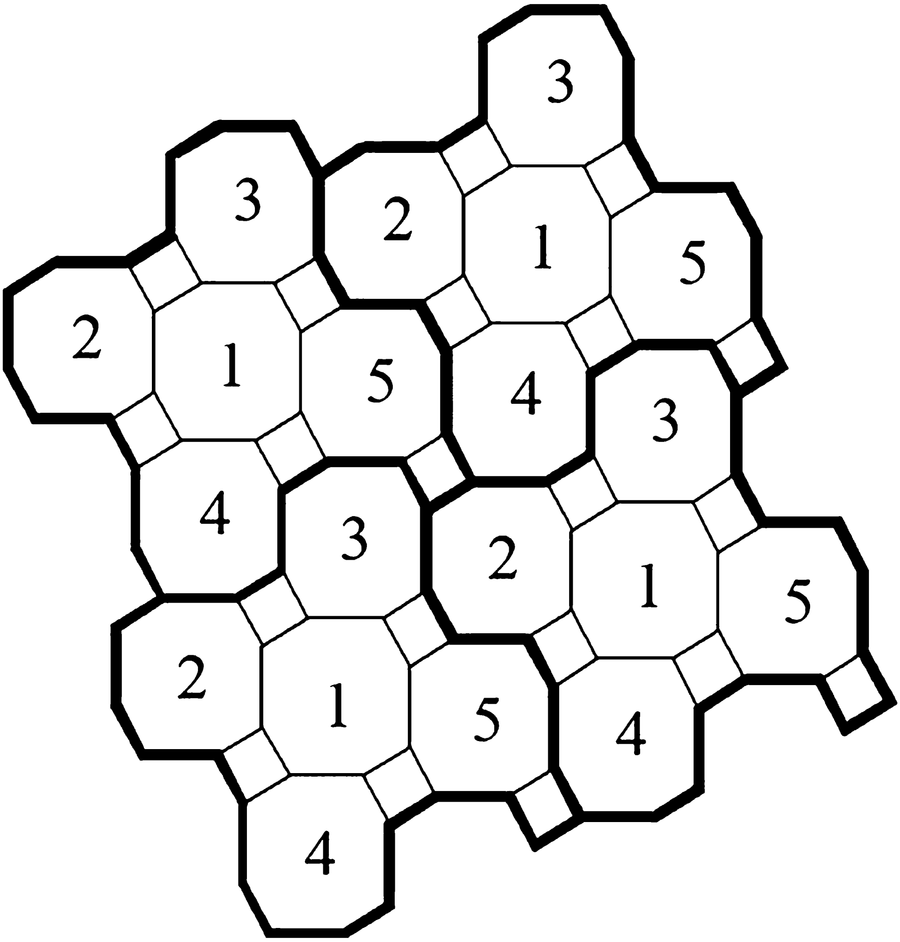


Fig. 7.4 The Soifer continuum of 6-colorings of the plane

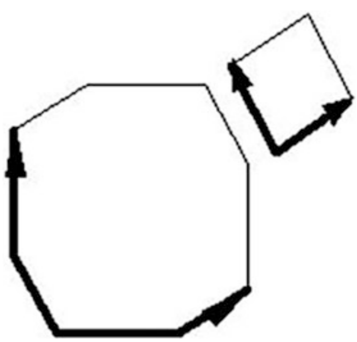


Fig. 7.5 Coloring boundaries

Now we are ready to color this tiling in 6 colors. Denote by *F* the unit of our construction, bounded by a bold line (Fig. 7.4) and consisting of five octagons and four small squares. Use colors 1 through 5 for the octagons inside *F* and color 6 for all small squares. Include in the colors of octagons and small squares the parts of their boundaries that are shown in bold in Fig. 7.5. Translates of F tile the plane and thus define the 6-coloring of the plane. We now wish to select the parameters to guarantee that each color forbids a distance.

At first, the complexity of computations appeared unassailable to me. However, a true Math Olympiad approach (i.e., good choices of variables, clever substitutions, and nice optimal properties of the chosen tiling) allowed for a successful sailing.

Let $x \leq y$ (Fig. 7.3). It is easy to see (Figs. 7.6 and 7.7) that we can split each small square into four congruent right triangles with sides x and y and a square of side $y - x$.

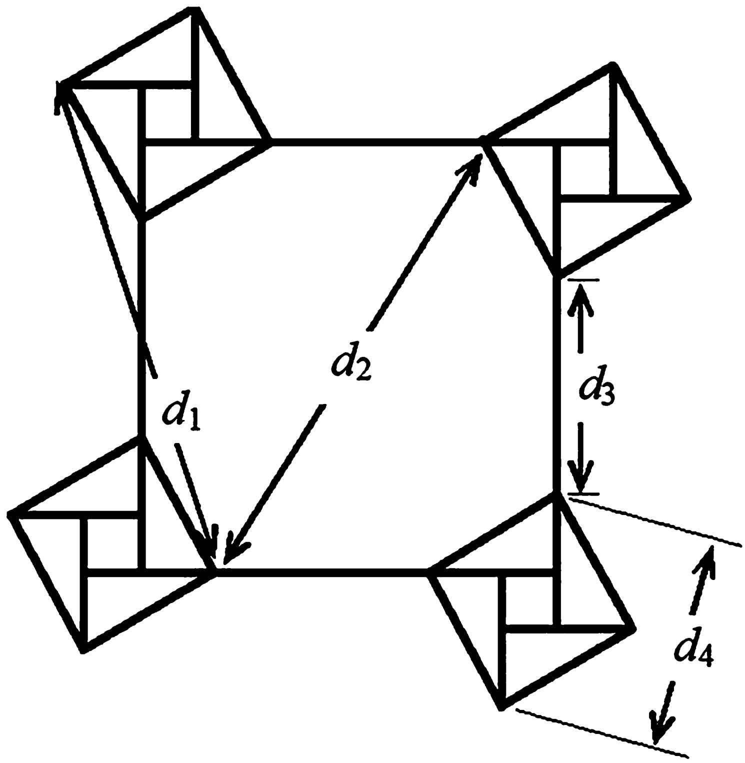


Fig. 7.6 A closer look at the tiling's foundation

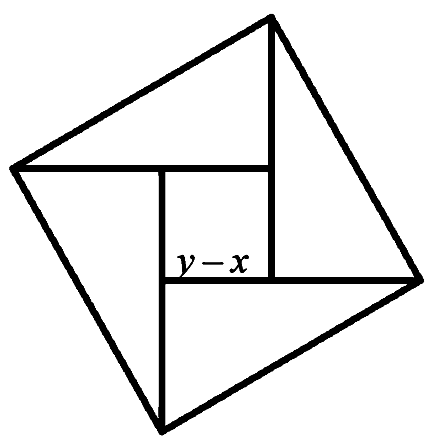


Fig. 7.7 A foundation close-up

The requirement for each color to forbid a distance produces the following system of two inequalities (see Fig. 7.6):

$$\begin{cases} d_1 \geq d_2 \\ d_3 \geq d_4 \end{cases}$$

Figures 7.6 and 7.7 allow for an easy representation of all d_i $(i = 1, 2, 3, 4)$ in terms of x and y. As a result, we get the following system of inequalities:

$$\left.\begin{array}{r} \sqrt{(1+y-x)^2+(2x)^2} \geq \sqrt{1+(1-2x)^2} \\ 1-x-y \geq \sqrt{2(x^2+y^2)} \end{array}\right\} \tag{7.2}$$

Solving for x in each of the two inequalities in (7.2) separately, we unexpectedly get the following system:

$$x^2+2(1-y)x+\left(y^2+2y-1\right) \geq 0$$
$$x^2+2(1-y)x+\left(y^2+2y-1\right) \leq 0.$$

Therefore, we get the equation (!) in x and y:

$$x^2+2(1-y)x+\left(y^2+2y-1\right) = 0.$$

Treating this as the equation in variable x, we obtain a *unique* (!) solution for x as a function of y that satisfies the system (7.2) of inequalities:

$$x = \sqrt{2-4y}+y-1, \text{where } 0 \leq y \leq 0.5. \tag{7.3}$$

Since $0 \leq x \leq y$, we get even narrower bounds for y: $0.25 \leq y \leq \sqrt{2}-1$. For any value of y within these bounds, x is uniquely determined by (7.3) and is accompanied by the *equalities* (!) $d_1 = d_2$ and $d_3 = d_4$.

Thus, we showed that for every $y \in \left[0.25, \sqrt{2}-1\right]$, there is a 6-coloring of type $(1, 1, 1, 1, 1, \alpha)$. But what values can α take on? Surely,

$$\alpha = \frac{d_4}{d_2}. \tag{7.4}$$

Let us introduce a new variable $Y = \sqrt{2-4y}$, where $Y \in \left[2-\sqrt{2}, 1\right]$ and figure out x and y from (7.3) as functions of Y:

$$4y = -Y^2+2$$
$$4x = -Y^2+4Y-2 \tag{7.5}$$

Now substituting from (7.1) and (7.2) the expressions for d_4 and d_2 into (7.4), and using the two equalities (7.5) to get rid of x and y everywhere, we get a "nice" expression for α^2 as a function of Y (do verify my algebraic manipulations on your own):

$$\alpha^2 = \frac{Y^4 - 4Y^3 + 8Y^2 - 8Y + 4}{Y^4 - 8Y^3 + 24Y^2 - 32Y + 20}.$$

By substituting $Z = Y - 2$, where $Z \in \left[-\sqrt{2}, -1\right]$, we get a simpler function α^2 of Z:

$$\alpha^2 = 1 + \frac{4Z\left(Z^2 + 2Z + 2\right)}{Z^4 + 4}.$$

To observe the behavior of the function α^2, we compute its derivative:

$$\left(\alpha^2\right)' = -\frac{4}{\left(Z^4 + 4\right)^2}\left(Z^6 + 4Z^5 + 6Z^4 - 12Z^2 - 16Z - 8\right).$$

Normally, there is nothing promising about finding the exact roots of an algebraic polynomial of a degree greater than 4. But we are positively lucky here, for this sixth-degree polynomial can be nicely decomposed into factors:

$$\left(\alpha^2\right)' = -\frac{4}{\left(Z^4 + 4\right)^2}\left(Z^2 - 2\right)\left[(Z+1)^2 + 1\right]^2.$$

Hence, the derivative has only two zeros. In fact, in the segment of our interest, $Z \in \left[-\sqrt{2}, -1\right]$, the only extremum of α^2 occurs when $Z = -\sqrt{2}$. Going back from Z to Y to y, we see that on the segment $y \in \left[0.25, \sqrt{2} - 1\right]$, the function $\alpha = \alpha(y)$ decreases from $\alpha = \frac{1}{\sqrt{5}} \approx 0.44721360$ (i.e., a 6-coloring of problem 6.4) to $\alpha = \sqrt{2} - 1 \approx 0.41421356$ (i.e., a 6-coloring of problem 7.1). Since the function $\alpha = \alpha(y)$ is continuous and increasing on $\left[0.25, \sqrt{2} - 1\right]$, it takes on *each* intermediate value from the segment $\left[\sqrt{2} - 1, \frac{1}{\sqrt{5}}\right]$ and only *once*.

We have proved the required result and much more:

For every angle ω between the small and the large squares (see Fig. 7.3)*, there are unique sizes of the two squares (and unique square intersection of parameters x and y) such that the constructed 6-coloring has type* $(1, 1, 1, 1, 1, \alpha)$ *for a uniquely determined α.*

This is a remarkable fact: the working solutions barely exist – they form something of a curve in a three-dimensional space formed by the angle ω and two linear variables x and y! We thus found a continuum of permissible values for α and a continuum of working 6-colorings of the plane. ■

Remark The problem of finding the 6-realizable set X_6 has a close relationship with the problem of finding the chromatic number χ of the plane. Its solution would shed light – if not solve – the chromatic number of the plane problem:

If $1 \notin X_6$, then $\chi = 7$;
If $1 \in X_6$, then $\chi \leq 6$.

Open Problem 7.4 (A. Soifer [Soi5]). Find X_6.

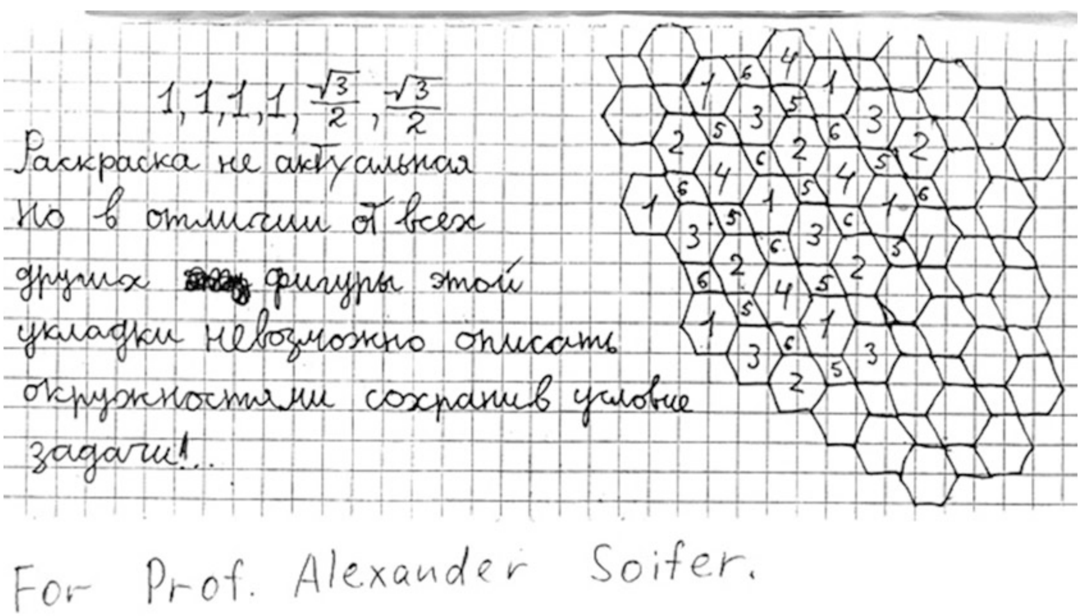
I am sure you understand that this problem, formulated in just two words, is extremely difficult.

In 1999, the Russian authorities accused my young coauthor and fine young violinist Ilya Hoffman of computer hacking (even though he did not pocket any money) and imprisoned him before the trial as a "danger to the society." I flew to Moscow, met with the presiding judge, a middle-aged pretty lady, in a black gown, of course. We were alone in her office. I asked, "What danger to the society does my nephew-violinist present?" The judge replied that she was not at liberty to do what she thought was right. I understood: she could have lost her job for that – or worse.

I met with Valery Vasilyevich Borshchev, a member of the Russian Parliament "Duma" and a human rights supporter. I also met with the vice president of the Russian Academy of Sciences and the head of the Judicial Division of the Academy, Vladimir Nikolaevich Kudryavtsev, who listened to me and generously volunteered to write a "Friend of the Court" opinion if the case were to reach the level of the City of Moscow Court or higher. Permit me to tell you a few words about the celebrated Jurist Kudryavtsev (10 April 1923–5 October 2007).

In 1951, Stalin's prosecutor general, Vyshinsky, announced a new legal doctrine: "One is guilty whom the court finds guilty." He called the presumption of innocence "bourgeois superstition." A young senior lieutenant rose to speak against the new Stalin's doctrine announced by Vyshinsky. This extraordinary hero was Vladimir N. Kudryavtsev. It was unforgettable to meet this brave man and get his full understanding and support.

When the trial finally took place, Ilya was released home from the courtroom. While in prison, he was not allowed to play his viola and violin, so Ilya wrote music and mathematics. The following page he sent to me from his prison cell:


1, 1, 1, 1, $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$
Раскраска не актуальная
Но в отличии от всех
других фигуры этой
укладки невозможно описать
окружностями сохранив условие
задачи!..
For Prof. Alexander Soifer.

Ilya discovered a new 6-coloring of the plane. Four colors consist of regular hexagons of diameter 1 and two colors occupy rhombuses. By carefully assigning colors to the boundaries, we get a 6-coloring of type $(1, 1, 1, 1, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$.

When my writings require an English translation of brilliant Russian poetry, I connect with Ilya for a joint translation work. "Always invite me to play linguistic combinatorics – I'm very pleased," wrote Ilya to me on New Year's Day, January 1, 2023.