Chapter 6
Polychromatic Number of the Plane and Results Near the Upper Bound

6.1 Stechkin's 6-Coloring

In Chapter 4, we discussed the polychromatic number χ_p of the plane and looked at the 1970 paper [Rai] by Dmitry E. Raiskii, in which he was the first to prove that 4 is the lower bound of χ_p. The paper also contained the upper bound:

$$\chi_p \leq 6.$$

The example proving this upper bound was found by Sergei B. Stechkin and published with his permission by D.E. Raiskii in [Rai]. Stechkin has never gotten credit in the West for his example. Numerous articles and books credited Raiskii (except for Raiskii himself!). How did this happen? As everyone else, I read the English translation of Raiskii's paper [Rai]. It says (the words in italics are mine):

> *S.B. Stechkin noted* that the plane can be decomposed into six sets such that all distances are not realized in any one of them. A corresponding example is presented here with the *author's solution*.

The author of what? – I was wondering. The author of the paper (as everyone decided)? But there is very little need for a "solution" once the example is found. I put Sherlock Holmes's cloak on and ordered a copy of the original Russian text. I read it in disbelief:

> A corresponding example is presented here with the *author's permission*.

Stechkin *permitted* Raiskii to publish Stechkin's example! The translator mixed up two somewhat similar-looking Russian words and "innocently" created a myth (see Table 6.1):

A. Soifer, *The New Mathematical Coloring Book*, https://doi.org/10.1007/978-1-0716-3597-1_6

Table 6.1 Translator's folly

Russian word	English translation
Решение	Solution
Разрешение	Permission

This is a great example in support of the expression "lost in translation." In reality, Sergei B. Stechkin was the editor of *Matematicheskie Zametki* (*Mathematical Notes*); he received Raiskii's manuscript, came up with the example, and inserted it in the manuscript with, I am sure, the agreement of Raiskii. Let us roll back to the mathematics of this example.

Problem 6.1 (S.B. Stechkin, [Rai]). $\chi_p \leq 6$, i.e., there is a 6-coloring of the plane such that no color realizes all distances.

Solution by S.B. Stechkin [Rai]. The "unit of the construction" is a parallelogram that consists of four regular hexagons and eight equilateral triangles, all of side lengths 1 (Fig. 6.1). We color the hexagons in colors 1, 2, 3, and 4. We partition the triangles of the titling into two types: We assign color 5 to the triangles with a vertex below their horizontal base and color 6 to the triangles with a vertex above their horizontal base. While coloring, we include with every hexagon its entire boundary, except its one rightmost and two lowest vertices; and every triangle does not include any of its boundary points.

We can now tile the entire plane with translates of the unit of the construction. ■

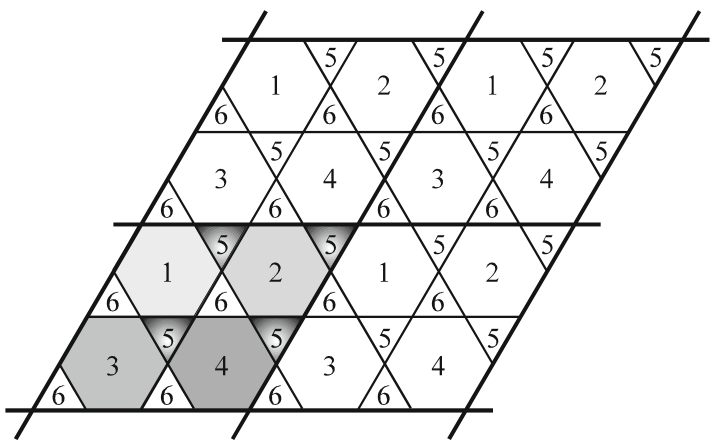

Fig. 6.1 The Stechkin 6-coloring of the plane

An easy construction solved problem 6.1 – easy to see after someone showed it to you. The trick was to find it, and Sergej Borisovich Stechkin found it first. Christopher Columbus too "just ran into" America! I got hooked.

6.2 The Best 6-Coloring of the Plane

I felt that if our ultimate goal was to find the chromatic number χ of the plane or to at least improve its known bounds ($4 \le \chi \le 7$), it may be worthwhile to somehow measure how close a given coloring of the plane is to achieving this goal. In 1992, I introduced such a measurement and named it *coloring type*.

Definition 6.2 (A. Soifer [Soi5], [Soi6], 1992). Given an n-coloring of the plane such that the color i does not realize the distance d_i ($1 \le i \le n$). Then we would say that this coloring is of *type* (d_1, d_2, ..., d_n).

This new notion of type was so natural and helpful that it received the ultimate compliment of becoming a part of the mathematical folklore: it appeared everywhere without a credit to its inventor (look, for example, p. 14 of the fundamental 991-page-long monograph [GO]).

It would have been a great improvement in our search for the chromatic number of the plane if we were to find a 6-coloring of type (1, 1, 1, 1, 1, 1) or to show that one does not exist. With the appropriate choice of a unit, we can make the 1970 Stechkin coloring to have type $(1, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$. Three years later, in 1973, Douglas R. Woodall [Woo1] found the second 6-coloring of the plane with all distances not realized in any color. Woodall's coloring had a special property that the author desired for his purposes: each of the six monochromatic sets was closed. His example, however, had three distinct "missing distances": It had type $(1, 1, 1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}})$. Woodall unsuccessfully tried to reduce the number of distinct distances, for he wrote "I have not managed to make two of the three 'missing distances' equal in this way" ([Woo1], p. 193).

In 1991, in search of a "good" coloring, I looked at the tiling with regular octagons and squares that tiled floors in many Russian public places (Fig. 6.2).

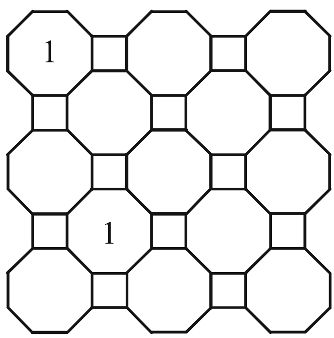

Fig. 6.2 Tiling used in many public places

But the "Russian public tiling" did not work! See it for yourself:

Problem 6.3 Prove that the set of all squares in the tiling of Fig. 6.2 (even without their boundaries) realizes all distances.

I then decided to shrink the squares until their diagonal became equal to the distance between the two closest squares. Simultaneously (!), the diagonal of the now nonregular octagon became equal to the distance between the two octagons marked with 1 in Fig. 6.2. I was in business!

Problem 6.4 (A. Soifer [Soi3], 1991). There is a 6-coloring of the plane of type (1, 1, 1, 1, 1, $\frac{1}{\sqrt{5}}$).

Solution We start with two squares, one of side 2 and the other of diagonal 1 (Fig. 6.3). We can use them to create the tiling of the plane with squares and (nonregular) octagons (Fig. 6.5). Colors 1, ..., 5 will consist of octagons; we will color all squares in color 6. With each octagon and each square, we include half of its boundary (bold lines in Fig. 6.4) without the end points of that half. It is easy to verify (please do) that the distance $\sqrt{5}$ is not realized in any of the colors 1, ..., 5 and the distance 1 is not realized in the color 6. By shrinking all linear sizes by a factor of $\sqrt{5}$, we get the 6-coloring of type (1,1,1,1,1,$\frac{1}{\sqrt{5}}$).

To simplify a verification, observe that the unit of my construction is bounded by the bold line in Fig. 6.5; its translates tile the plane. ■

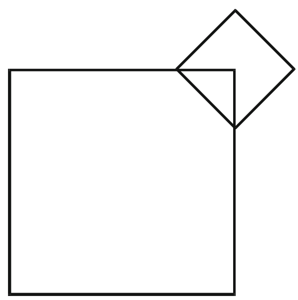

Fig. 6.3 Foundation squares

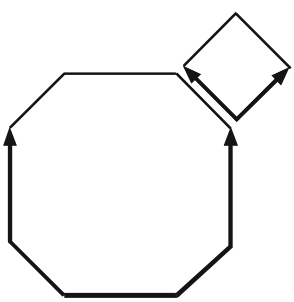

Fig. 6.4 Coloring of the boundaries

I had mixed feelings when I obtained the result of problem 6.4 in early August 1991. On the one hand, I knew the result was "close but no cigar": after all, a 6-coloring of type (1, 1, 1, 1, 1, 1) was not found. On the other hand, I thought that the latter 6-coloring may not exist, and, if so, my 6-coloring would be the best possible. There was another consideration as well. While in a PhD program in Moscow, I hoped to produce the longest paper that would still be refereed in by a major journal (and I got one published in 1973 that in manuscript was 56 pages long). This time, I was interested in a "dual record": how short can a paper be and still contain enough "stuff" to be refereed in and published? The paper [Soi6] solving problem 6.4 was two pages long, *including* three pictures. It was received on August 8, 1991, and

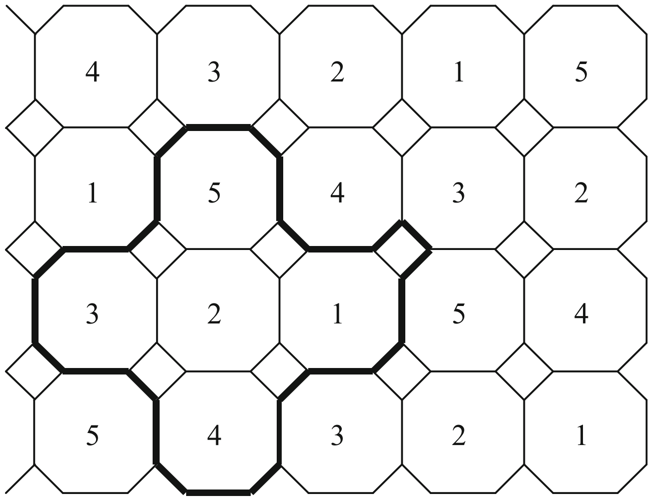

Fig. 6.5 The Soifer 6-coloring of the plane

accepted the next day by the *Journal of Combinatorial Theory, Series A* by the managing editor Bruce Rothschild of UCLA (University of California, Los Angeles). As nearly all journal editors have nearly always, Professor Rothschild insisted on objectivity. Referring to the chromatic number of the plane problem (CNP), I wrote that it was "my favorite open problem." Rothschild changed it in pencil to "an old problem." I accepted the edit as it was a condition for publication. In my book, I can finally declare that CNP is still my favorite open problem in all of mathematics.

This short paper also gave birth to a new definition and an open problem.

Definition 6.5 [HS1]. *An almost chromatic number* χ_a of the plane is the minimum number of colors that are required for coloring the plane so that almost all (i.e., all but one) colors forbid a unit distance and the remaining color forbids a distance.

We have the following inequalities for χ_a:

$$4 \leq \chi_a \leq 6.$$

The lower bound follows from Dmitry Raiskii's [Rai]. I proved the upper bound in 6.4 above [Soi6]. This naturally gave birth to a new problem, which is still open:

Open Problem 6.6 [HS1]. Find χ_a.

6.3 The Age of Tiling

Hadwiger's, Stechkin's, and my ornaments (Figs. 2.4, 6.2, and 6.6, respectively) delivered new mathematical results. They were also aesthetically pleasing. Have we contributed something, however little, to the arts? Not really. Nothing is new in the world of arts. We can find Henry Moore's aesthetics in pre-Columbian art and Picasso's cubistic geometrization

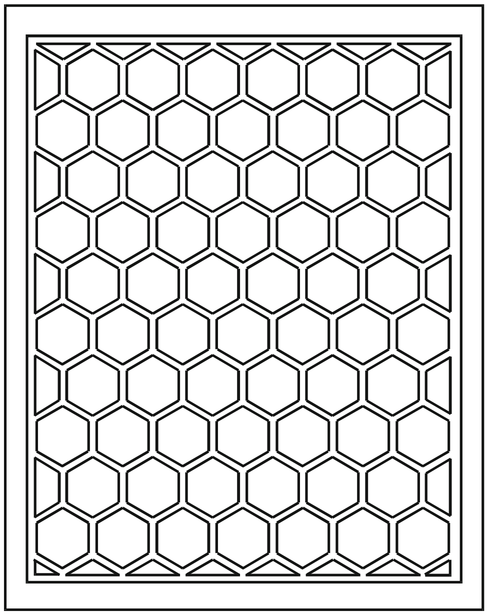

Fig. 6.6 Chinese lattice 1

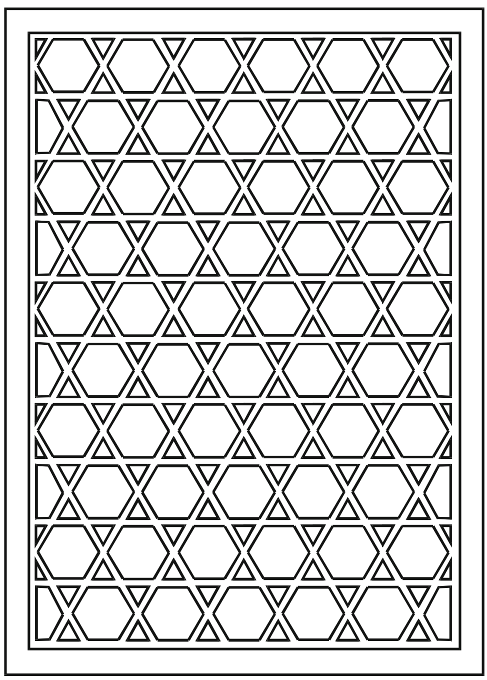

Fig. 6.7 Chinese lattice 2

of form in the art of sub-Saharan Africa. Our ornaments too were known for over 1000 years to the artists of China, India, Persia, Turkey, and Europe. Figures 6.6, 6.7, and 6.8, reproduced with kind permission from the Harvard-Yenching Institute from the wonderful 1937 book *A*

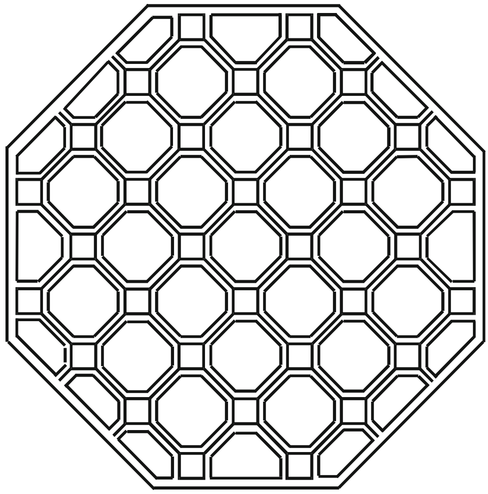

Fig. 6.8 Chinese lattice 3

Grammar of Chinese Lattice by Daniel Sheets Dye [Dye], show how those ornaments were implemented in old Chinese lattices.

If it is any consolation, I can point out that the Chinese ancestors did not invent the beauty and strength of honeycombs either: Bees were here first!