Chapter 5
De Bruijn–Erdős Reduction to Finite Sets and Results Near the Lower Bound

We can expand the notion of the chromatic number to any subset S of the plane. *The chromatic number* $\chi(S)$ of S is the smallest number of colors sufficient for coloring the points of S in such a way that forbids monochromatic unit segments.

In 1951, Nicolaas Govert de Bruijn and Paul Erdős published a highly powerful tool [BE2] that will help us with this and other problems. We will formulate and prove it in Part V. In our setting here, it implies the following.

Compactness Theorem 5.1[1] (N.G. de Bruijn, P. Erdős). The chromatic number of the plane is equal to the maximum chromatic number of its *finite* subsets.

Thus, as Paul Erdős used to say, the problem of finding the chromatic number of the plane is a problem about finite sets in the plane.[2]

There are easy questions about finite sets in the plane. Solve the following two problems on your own.

Problem 5.2 Find the smallest number δ_3 of points in a plane set whose chromatic number is equal to 3.

Problem 5.3 (L. Moser and W. Moser, [MM]). Find the smallest number δ_4 of points in a plane set whose chromatic number is 4. (Answer: $\delta_4 = 7$).

Victor Klee and Stan Wagon posed the following open problem in [KW]:

Open Problem 5.4 When k is 5, 6, or 7, what is the smallest number δ_k of points in a plane set whose chromatic number is equal to k?

Of course, problem 5.4 makes sense only if $\chi > 4$. In the latter case, this problem suggests a way to attack the chromatic number of the plane problem by constructing new "spindles."

When you worked on problems 5.2 and 5.3, you probably remembered our problems 2.1 and 2.2. Indeed, those problems provide optimal configurations (Figs. 2.1 and 2.2) for

[1] The axiom of choice is assumed in this result.

[2] Or so we all thought. Because of that, I choose to leave this chapter as it was written in the early 1990s. See Part XII of this book for axiomatic developments.

A. Soifer, *The New Mathematical Coloring Book*, https://doi.org/10.1007/978-1-0716-3597-1_5

problems 5.2 and 5.3. Both optimal configurations were built of equilateral triangles of side 1. Can we manage without them?

Problem 5.5 Find the smallest number σ_3 of points in a plane set without equilateral triangles of side 1 whose chromatic number is equal to 3.

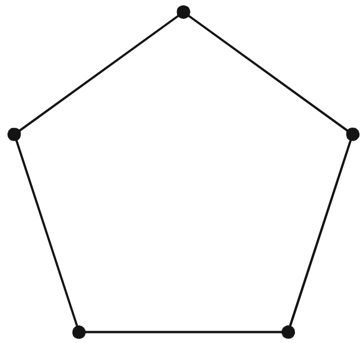

Fig. 5.1 An equilateral pentagon of side 1

Solution $\sigma_3 = 5$. The regular pentagon of side 1 (Fig. 5.1) delivers a minimal configuration of chromatic number 3.

It is easy to 2-color any four-point set A, B, C, D without equilateral triangles of side 1. Just color A red. All points at a distance 1 from A, color blue; these are second-generation points. All uncolored points at a distance 1 from any point of the second generation, we color red, and these are third-generation points. All uncolored points at a distance 1 from the points of the third generation, we color blue. If we did not color all four points, then we start this process all over again by coloring any uncolored point red. If this algorithm were not to define the color of any point uniquely, we would have an odd-sided n-gon with all sides 1, i.e., an equilateral triangle (since $n \leq 4$), which cannot be present, and thus would provide the desired contradiction. ■

For four colors, this question for a while was an open problem first posed by Paul Erdős in July 1975 (and published in 1976), who, as was usual for him, offered to "buy" the first solution – for $25.

Paul Erdős' $25 Problem 5.6 [E76.49]. Let S be a subset of the plane, which contains no equilateral triangles of size 1. Join two points of S if their distance is 1. Does this graph have chromatic number 3?

If the answer is no, assume that the graph defined by S contains no C_l (cycles of length l) for $3 \leq l \leq t$ and ask the same question.

It appears that Paul Erdős was not sure of the outcome, which was rare for him. Moreover, from the next publication of this problem in 1979 [E79.04], it is clear that Paul expected that triangle-free unit distance graphs had chromatic number at most 3 or else chromatic number 3 can be forced by prohibiting all small cycles up to C_k for a sufficiently large k:

Paul Erdős' $25 Problem 5.6' [E79.04]. "Let our n points [in the plane] be such that they do not contain an equilateral triangle of side 1. Then their chromatic number is probably at most 3, but I do not see how to prove this. If the conjecture would unexpectedly [sic] turn out to be false, the situation can perhaps be saved by the following new conjecture:

There is a k so that if the girth of $G(x_1,\ldots,x_n)$ is greater than k, then its chromatic number is at most three – in fact, it will probably suffice to assume that $G(x_1,\ldots,x_n)$ has no odd circuit of length $\leq k$."[3]

Erdős' first surprise arrived in 1979 from down under: Nicholas Wormald, then of the University of Newcastle, Australia, disproved the first, easier, triangle-free conjecture. Erdős paid \$25 reward for the surprise and promptly reported it in his next 1978 talk (published 3 years later [E81.23]):

> Wormald in a recent paper (which is not yet published) disproved my original conjecture – he found a [set] S for which [the unit distance graph] $G_1(S)$ has girth 5 and chromatic number 4. Wormald's construction uses elaborate computations and is fairly complicated.

In his paper [Wor], Wormald proved the existence of a set S of 6448 (!) points without triangles and quadrilaterals with all sides 1, whose chromatic number was 4. He was aided by a computer. I would like to give you a taste of the initial Wormald construction or, more precisely, the Blanche Descartes construction that Wormald was able to embed in the plane, but it is a better fit in Chapter 12 – so, see it there.

The size of Wormald's example, of course, did not appear to be anywhere near optimal. Surely, it must have been possible to do the job with less than 6448 points! In my March–1992 talk at the Southeastern International Conference on Combinatorics, Graph Theory, and Computing at Florida Atlantic University, I shared Paul Erdős' old question, but I put it in a form of competition:

A graph is called *unit-distance* if its two vertices are connected by an edge if and only if they are at distance 1 apart.

Open Problem 5.7 Find the smallest (in the number of vertices) unit-distance graph in the plane without equilateral triangles, whose chromatic number is 4. Construct such a graph.

The result exceeded my wildest dreams. A number of young mathematicians, including graduate students, were inspired by this talk and entered the race I proposed. Coincidentally, during that academic year, with the participation of the celebrated geometer Branko Grünbaum, and of Paul Erdős, whose problem papers set the style, I started a new and unique quarterly *Geombinatorics*, dedicated to problem-posing essays on discrete and combinatorial geometry and related areas. *Geombinatorics* is still alive and well now, 32 years later. The aspirations of the journal were clear from my 1991 Editor's Page in Issue 3 of Volume I:

> In a regular journal, a paper appears 1 to 2 (or more) years after the research is completed. By then even the author may not be excited any more about his results. In *Geombinatorics* we can exchange open problems, conjectures, aspirations, work-in-progress that is still exciting to the author, and therefore inspiring to the reader.

A true World Series played out on the pages of *Geombinatorics* around problem 5.7. The graphs obtained by the record setters were as mathematically significant as they were beautiful. I have to show them to you – see them discussed in detail in Chapters 14 and 15.

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[3] The symbol $G(x_1,\ldots,x_n)$ denotes the graph on the listed inside parentheses n vertices, with two vertices adjacent if and only if they are a unit distance apart.

Many attempts to increase the lower bound of the chromatic number of the plane had not achieved their goal. Rutgers University's PhD student Rob Hochberg believed that the chromatic number of the plane was 4, while his roommate and fellow PhD student Paul O'Donnell was of the opposite opinion. They managed to get along despite this disagreement of the mathematical kind. On January 7, 1994, Rob sent me an e-mail to that effect:

> Alex, hello. Rob Hochberg here. (The one who's gonna prove $\chi(E^2) = 4$.) . . . It seems that Paul O'Donnell is determined to do his Ph. D. thesis by constructing a 5-chromatic unit-distance graph in the plane. He's got several interesting 4-chromatic graphs and great plans. We still get along.

Two months later, Paul O'Donnell's abstract in the *Abstracts* book of the Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton, Florida, included the following announcement:

> The chromatic number of the plane is between four and seven. A five-chromatic subgraph would raise the lower bound. If I discover such a subgraph, I will present it.

We all came to his talk of course (it was easy for me, as I spoke immediately before Paul in the same room). At the start of his talk, however, Paul simply said, "not yet," and went on to show his impressive 4-chromatic graph of girth 4. Five years later, on May 25, 1999, Paul O'Donnell defended his doctorate at Rutgers University.

Much was learned about 4-chromatic unit distance graphs. The best of these results, in my opinion, was contained in O'Donnell's dissertation. He completely solved Paul Erdős' problem 5.6 and delivered to Paul Erdős an ultimate surprise by negatively answering Erdős' general conjecture:

O'Donnell's Theorem 5.8 [Odo3, Odo4, Odo5]. There exist 4-chromatic unit distance graphs of arbitrary finite girth.

I choose to divide the proof of this result between Parts III and IX. See you there!