Chapter 3
Chromatic Number of the Plane: A Historical Essay

[I] cannot trace the origin of this problem.
– Paul Erdős, 1961

[This is] a long-standing open problem of Erdős.
– Hallard T. Croft, 1967

It is often easier to be precise about Ancient Egyptian history than about what happened among our contemporaries.
– Nicolaas Govert de Bruijn
Eindhoven, July 5, 1995
e-mail to A. Soifer

It happened a long time ago and is not true.
– An old Russian joke

It is natural for one to inquire into the authorship of one's favorite problem. So, in 1991, I turned to countless articles and books. Some of the information that I found appears here in Table 3.1 and Diagram 3.1 – take a look. Are you confused? I was too!

As you can see, Douglas R. Woodall credits Martin Gardner, who, in turn, refers to Leo Moser. Hallard T. Croft calls it "a long-standing open problem of Erdős", Gustavus J. Simmons credits "Erdős, [Frank] Harary, and [William Thomas] Tutte," while Paul Erdős himself "cannot trace the origin of this problem"! Later Erdős credits "Hadwiger and Nelson," while Victor Klee and Stan Wagon state that the problem was "posed in 1960–61 by M. Gardner and Hadwiger." Croft comes again, this time with Kenneth J. Falconer and Richard K. Guy, to cautiously suggest that the problem is "apparently due to E. Nelson" [CFG]. Yet, Richard Guy did not know who "E. Nelson" was and why Guy and his coauthors "apparently" attributed the problem to him (my conversation with Richard Guy on the back seat of a taxi in Keszthely, Hungary, when we both attended Paul Erdős' 80th birthday conference in August of 1993).

Thus, at least *seven* (!) mathematicians were credited with creating this problem: Paul Erdős, Martin Gardner, Hugo Hadwiger, Frank Harary, Leo Moser, Edward Nelson, and William T. Tutte – a great group of mathematicians to be sure. But it was hard for me to believe that they all created the problem, be it independently or all seven together.

A. Soifer, *The New Mathematical Coloring Book*, https://doi.org/10.1007/978-1-0716-3597-1_3

Table 3.1 Who created the chromatic number of the plane problem?

Publication	Year	Author(s)	Problem creator(s) or source named
[Gar2]	1960	Gardner	"**Leo Moser** ...writes..."
[Had4]	1961	Hadwiger (after Klee)	**Nelson**
[E61.22]	1961	Erdős	"I cannot trace the origin of this problem"
[Cro]	1967	Croft	"A long[18]-standing open problem of **Erdős**"
[Woo1]	1973	Woodall	**Gardner**
[Sim]	1976	Simmons	**Erdős**, **Harary**, and **Tutte**
[E80.38] [E81.23] [E81.26]	1980–1981	Erdős	**Hadwiger** and **Nelson**
[CFG]	1991	Croft, Falconer, and Guy	"Apparently due to **E. Nelson**"
[KW]	1991	Klee and Wagon	"Posed in 1960–61 by **M. Gardner** and **Hadwiger**"

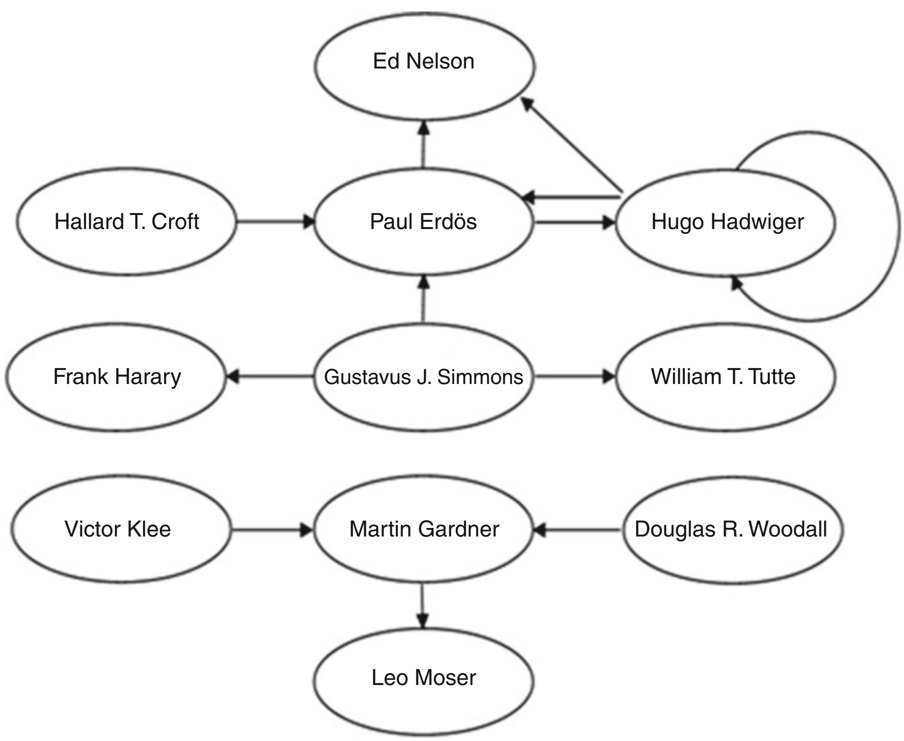


Diagram 3.1 Who created the chromatic number of the plane problem?

I felt an urge, akin to that of a private investigator, a Sherlock Holmes, to untangle the web of conflicting accounts. It took 6 months to solve this historical puzzle. A good number of mathematicians, through conversations and e-mails, contributed their insights: Branko Grünbaum, Peter D. Johnson, Tony Hilton, Ron Graham, and Klaus Fischer first come to mind. I am especially grateful to Paul Erdős, Victor Klee, Martin Gardner, Edward Nelson, and John Isbell for contributing critically important pieces of the puzzle. Only their accounts, recollections, and congeniality made these findings possible.

What follows is my 1991 investigation into the history of the problem. Today, I still stand by this research. It gives me great sadness to see that the players and informants of this *Story of Creation* are no longer with us. As my homage to them, I would like to list them here, alphabetically:

Paul Erdős, 26 March 1913–20 September 1996
Martin Gardner, 21 October 1914–22 May 2010
Ronald L. Graham, 31 October 1935–6 July 2020
Branko Grünbaum, 2 October 1929–14 September 2018
Hugo Hadwiger, 23 December 1908–29 October 1981
John R. Isbell, 27 October 1930–6 August 2005
Victor L. Klee, Jr., 18 September 1925–17 August 2007
Leo Moser, 11 April 1921–9 February 1970
William "Willy" O.J. Moser, 5 September 1927–28 January 2009
Edward Nelson, 4 May 1932–10 September 2014

I commenced my investigation on July 6, 1991, by mailing a page-long handwritten letter to Paul Erdős [Soi91/7/6ltr]. I open it by sharing my plans and then ask several questions. The first question is of our interest here:

> Dear Paul,
>
> I am writing a book, "*Mathematical Coloring Book*."
>
> It will have two parts: one about properties of colored objects (n-colored plane and chromatic number of the plane, colored numbers and Schur Theorem, colored polygons, etc.); the other part about coloring (coloring as a mean of solving tiling problems, coloring a map, etc.)
>
> I welcome your advice, problems for inclusion, including open problems (with their history if possible).
>
> In particular, I have a few questions for you.
>
> I am trying to reconstruct the history of the problem asking for the chromatic number of the plane $\chi(E^2)$ (minimal number of colors that color the plane without monochromatic segments of length 1.
>
> Folklore has it that this is your problem. Gustavus Simmons says that in an article. Martin Gardner mentions it in 1960 and says that he heard it from Leo Moser. Victor Klee and Stan Wagon, in their almost-published book, say that this problem was born in 1960. Woodall starts the story of the problem from Gardner's October 1960 mention of Leo Moser.
>
> I think the problem is older than 1960, and it is your problem. Please, let me know the true history, as I wish to give credit where credit is due, especially since this is my

favorite problem in all of mathematics. Did you originally conjecture that $\chi(E^2) = 4$? What do you think now: 4, 5, 6, or 7?

This seems to be my first mention of an aspiration to write a book and the first time I gave this book her title, *Mathematical Coloring Book*. I never thought it would become an 18-year-long obsession that resulted, as you know, in the first edition of this book.

In the July 12, 1991, letter, Paul writes [E91/7/12ltr]:

I first heard of the chromatic number of the plane problem from Leo Moser in 1958–1962.

On August 10, 1991, Paul shares his appreciation of the problem, for which he could not claim the authorship [E91/8/10ltr]:

The problem about the chromatic number of the plane is unfortunately not mine.

In a series of letters of July 12, 1991; July 16, 1991; August 10, 1991; and August 14, 1991, Paul formulates for me a good number of problems related to the chromatic number of the plane that he did create. We will look at some of Erdős' problems in the following chapters.

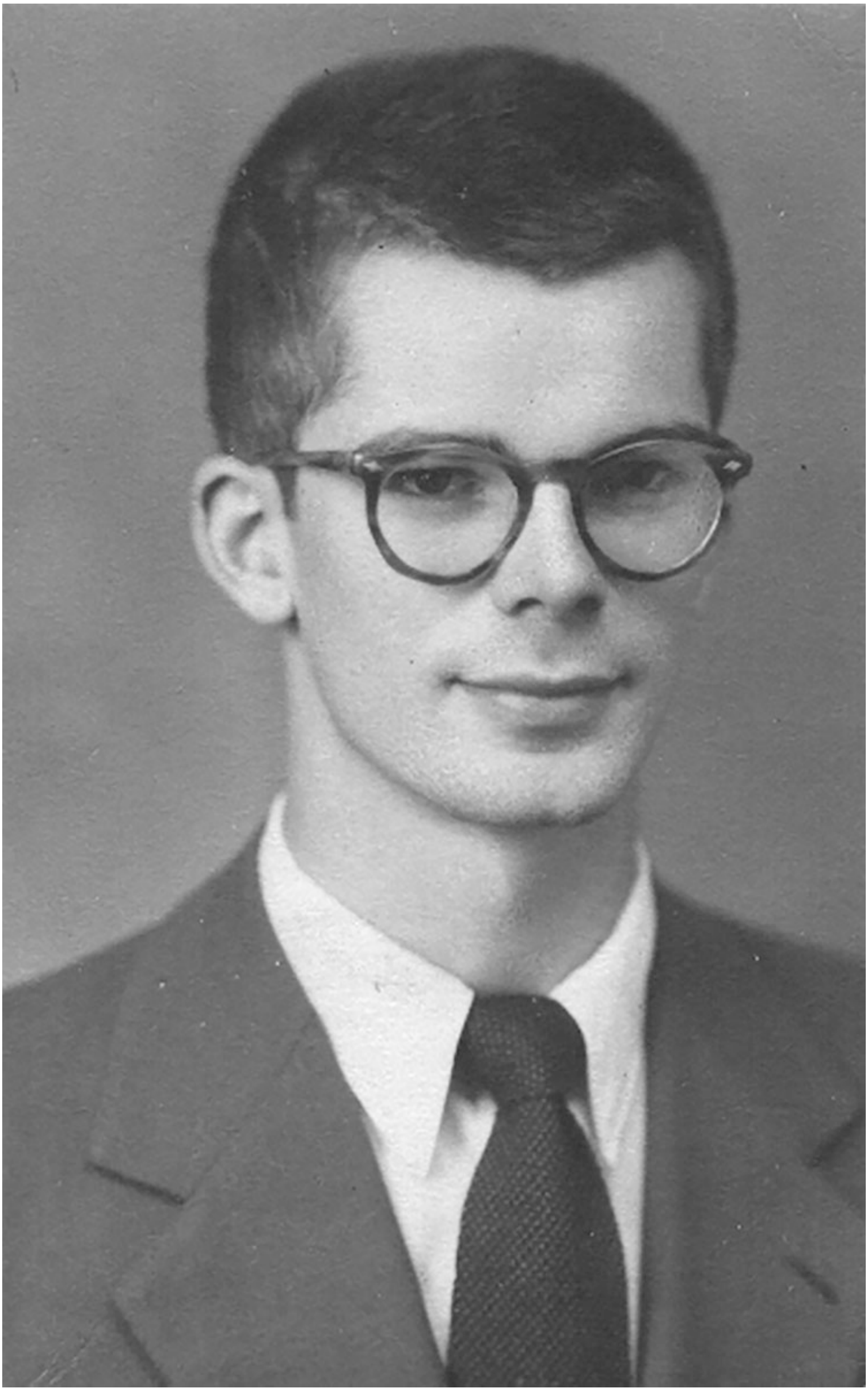
Having established that the author was not Paul Erdős, I moved down the list of the "candidates," and, on August 8, 1991, and again on August 30, 1991, I wrote to Victor Klee, Edward Nelson, and John Isbell. I shared with them my Table 3.1 and asked them to tell me what they knew about the birth of the problem. I also interviewed Princeton Professor Nelson over the phone on September 18, 1991.

Edward Nelson created what he named "a second four-color problem" (the first being the famous four-color problem of map coloring), which we will discuss in Part 4). In his October 5, 1991, letter [Nel2], he conveys the *Story of Creation*:

Dear Professor Soifer:

In the autumn of 1950, I was a student at the University of Chicago and among other things was interested in the Four-Color problem, the problem of coloring graphs topologically embedded in the plane. These graphs are visualizable as nodes connected by wires. I asked myself whether a sufficiently rich class of such graphs might possibly be subgraphs of one big graph whose coloring could be established once and for all, for example, the graph of all points in the plane with the relation of being unit distance apart (so that the wires become rigid, straight, of the same length, but may cross). The idea did not hold up, but the other problem was interesting in its own right and I mentioned it to several people.

I asked Professor Nelson for his photograph from the time when he created this problem. He referred me to his photograph published in *Time Magazine* from December 1949. I found the article about Eddie Nelson's successes with his photograph and called *Time Magazine*. They informed me that *if* they find that photograph in their archive, which was not given, they will charge me $300 for a one-time right to reproduce it in my book. Nelson was clearly unhappy with the *Time*'s reply. He found a photograph of himself ca. 1950 and generously sent it to me as a gift, which I share with you here.



Eddie Nelson, c. 1950. (Courtesy of Edward Nelson)

One of the people Eddie Nelson mentioned the problem to was John Isbell. Almost half a century later, Isbell still remembers the story very vividly when on August 26, 1991, he shares it with me [Isb1]:

> Dear Professor Soifer,
>
> I should certainly like to receive any future *Geombinatorici*, and I might contribute. There is an annoying problem I talked with Erdős and Pach about a few (5–6) years back; I sort of promised Pach a preprint, but didn't get enough results to publish. Maybe I could write it up as a problem or two and put in the bits of result I did get as a background.
>
> Authorship of plane chromatic no. problem: Of course I can't comment on Croft's attribution to Erdős "long ago" except that it is a vague reference (but you cite Croft et al. 1991 against early Croft, maybe because he learned [from Klee?] about the following.
>
> Ed Nelson told me the problem and $\chi \geq 4$ in November 1950, unless it was October – we met in October. I said what upper bound have you, he said none, and I worked out 7. I was a senior at the time (B.S., 1951). I think Ed had just entered U. Chicago as a nominal sophomore and taken placement exams which placed him a bit ahead of me, say a beginning graduate student with a gap or two in his background. I certainly mentioned the problem to other people between 1950 and 1957; Hugh Spencer Everett III, the author of the many-worlds interpretation of quantum mechanics, would certainly be one, and Elmer Julian Brody who did a doctorate under Fox and has long been at the Chinese University of Hong Kong and is said to be into classical Chinese literature would be another. I mentioned it to Vic Klee in 1958 ± 1 ...
>
> I don't see that Woodall's attribution to Gardner, who attributes it elsewhere, is worth a plugged nickel. If you said Erdős, Moser, and Nelson independently, you would probably be accurate in the eyes of the Recording Angel.

In September 1991, I had a most enjoyable phone conversation with the distinguished geometer Victor Klee. He too remembered hearing the problem from John Isbell in 1957–1958. In fact, it took place before September 1958 when Professor Klee left for Europe. There, *Klee passed this problem to Hugo Hadwiger*, who was collecting problems for the book *Open Problems in Intuitive Geometry,* to be written jointly by Erdős, Fejes Toth, Hadwiger, and Klee. To my great regret, this great book-to-be has never materialized.

Gustavus J. Simmons [Sim], in giving credit for the problem to "Erdős, Harary, and Tutte," no doubt had in mind their joint 1965 paper in which the three famous authors defined the dimension of a graph (see Chapter 13 on this). The year of 1965 was way too late for our problem's creation, and, besides, the three authors have never made any claims to such a discovery.

What were the roles of Paul Erdős, Martin Gardner, and Leo Moser in the *Story of Creation*? I am prepared to answer these questions, all except one: I am leaving to others to research Leo Moser's archive (it used to be maintained by his brother Willy Moser at McGill University in Montreal) and find out when and from whom Leo Moser came by the problem. What is important to me is that he did not create it independently from Edward Nelson, as Paul Erdős informed me in his July 16, 1991, letter [E91/7/16]:

I do not remember whether Moser in 1958 [possibly on June 16, 1958, the date from which we are lucky to have a photo record] told me how he heard the problem on the chromatic number of the plane, I only remember that it was not his problem.

Paul Erdős (left) and Leo Moser, June 16, 1958. (Courtesy of Paul Erdős)

Yet, Leo Moser made a valuable contribution to the survival of the problem. He gave it to both Paul Erdős and the wonderful mathematics expositor Martin Gardner. Gardner, due to his fine taste, recognized the value of this problem and included it in his October 1960 *Mathematical Games* column in *Scientific American* ([Gar2]), with the acknowledgment that he received it from Leo Moser of the University of Alberta. Thus, the credit for the first *publication* of the problem goes to Martin Gardner. It is beyond me why so many authors of articles and books, as late as 1973 ([Woo1], for example), gave credit for the *creation* of the problem to Martin Gardner, something he himself had never claimed. In my 1991 telephone conversation with him, Martin told me for a fact that the problem was not his, and he promptly listed Leo Moser as his source, both in print and in his archive, which he checked as I was waiting on the line.

Moreover, some authors ([KW], for example) who knew Edward Nelson's authorship, still credited Martin Gardner and Hugo Hadwiger as late as in 1991 because, it seems, only written, preferably published, word was acceptable to them. Following this logic, the creation of the celebrated four-color map coloring problem (4CP) must be attributed to Augustus De Morgan, who first *wrote* about it in his October 23, 1852, letter to William Rowan Hamilton, or better yet to Arthur Cayley, whose 1878 abstract included the *first non-anonymous publication* of the problem.[1] Yet, we all seem to agree that the 20-year-old Francis Guthrie created 4CP, even though he did not publish or even write a word about it! (See Part IV for more on this.)

Of course, a lone self-serving statement would be too weak a foundation for a historical claim. On the other hand, independent disinterested testimonies corroborating each other comprise as solid a foundation for the attribution of the credit as any publication. This is precisely what my inquiry has produced. Here is just one example of Nelson and Isbell's selflessness. Edward Nelson writes to me on August 23, 1991 [Nel1]:

> I proved nothing at all about the problem.

John Isbell corrects Nelson in his September 3, 1991, letter [Isb2]:

> Ed Nelson's statement which you quote, "I proved nothing at all about the problem," can come only from a failure of memory. He proved to me that the number we are talking about is ≥ 4, by precisely the argument in Hadwiger 1961. Hadwiger's attribution (on Klee's authority) of that inequality to me can only be Hadwiger's or Klee's mistake.

This brings us to the issue of the authorship of the bounds for χ:

$$4 \leq \chi \leq 7.$$

Once again, the entire literature is off the mark by giving credit for the first proofs to Hadwiger and the Mosers. Yes, in 1961, the famous Swiss geometer Hugo Hadwiger published [Had4] the chromatic number of the plane problem together with proofs of both bounds. But he writes there (*and nobody reads!!*):

> We thank Mr. V. L. Klee (Seattle, USA) for the following information. The problem is due to E. Nelson; the inequalities are due to J. Isbell.

Hadwiger does go on to say:

> Some years ago the author [i.e., Hadwiger] discussed with P. Erdős questions of this kind.

Did Hadwiger insinuate that he created the problem independently from Nelson? We will never know for sure, but I have my doubts about Hadwiger's (co)authorship. Hadwiger jointly with Hans Debrunner published an excellent, long problem paper in 1955 [HD1] that was extended to their wonderful, famous book in 1959 [HD2]; see also its 1964 English translation [HDK] with Victor Klee and the 1965 Russian translation [HD3] edited by the famous Russian geometer and expositor Isaac Moiseevich Yaglom. All these books (and Hadwiger's other papers) included a number of "questions of this kind," but did not once

[1] First publication could be attributed to De Morgan, who mentioned the problem in his 1860 book review in *Athenaeum* [DeM4], albeit anonymously – see more on this in Section 18.

include the chromatic number of the plane problem. Moreover, it seems to me that the problem in question is somewhat out of Hadwiger's "character": in all problems "of this kind," he preferred to consider closed sets rather than arbitrary sets, in order to take advantage of topological tools.

I shared with Paul Erdős these twofold doubts about Hadwiger independently creating the problem. It was especially important because Hadwiger in the above-quoted text mentioned Erdős as his witness or coauthor of sorts. Paul replied to me in his July 16, 1991, letter [E91/7/16] as follows:

> I met Hadwiger only after 1950, thus I think Nelson has priority (Hadwiger died a few years ago, thus I cannot ask him, but I think the evidence is convincing).

During his talk at the 25th Southeastern International Conference on Combinatorics, Graph Theory, and Computing at Florida Atlantic University, Boca Raton, Florida, at 9:30–10:30 a.m. on Thursday, March 10, 1994, Paul Erdős summarized the results of my historical research in the characteristically Erdősian style ([E94.60])[2]:

> There is a mathematician called Nelson who in 1950 when he was an epsilon, that is he was 18, discovered the following question. Suppose you join two points in the plane whose distance is 1. It is an infinite graph. What is [the] chromatic number of this graph?
>
> Now, de Bruijn and I showed that if an infinite graph, which is chromatic number *k*, it always has a finite subgraph, which is chromatic number *k*. So, this problem is really [a] finite problem and not an infinite problem. And it was not difficult to prove that the chromatic number of the plane is between 4 and 7. I would bet it is bigger than 4, but I am not sure. And the problem is still open.
>
> If it would be my problem, I would certainly offer money for it. You know, I can't offer money for every nice problem because I would go broke immediately. I was asked once what would happen if all your problems would be solved, could you pay? Perhaps not, but it doesn't matter. What would happen to the strongest bank if all the people who have money there would ask for money back? Or what would happen to the strongest country if they suddenly ask for money? Even Japan or Switzerland would go broke. You see, Hungary would collapse instantly. Even the United States would go broke immediately . . .
>
> Actually, it was often attributed to me, this problem. It is certain that I had nothing to do with the problem. I first learned the problem, the chromatic number of the plane, in 1958, in the winter, when I was visiting [Leo] Moser. He did not tell me from where this nor the other problems came from. It was also attributed to Hadwiger, but Soifer's careful research showed that the problem is really due to Nelson.

The leader of Ramsey Theory, Ronald L. Graham, also endorses the results of my historical investigation in his important 2004 problem paper [Gra6] in *Geombinatorics*:

> It is certainly not necessary to point out to readers of this journal any facts concerning the history and current status of this problem (which [is] due to Nelson in 1950) since

[2]Thanks to Professor Fred Hoffman, the tireless organizer of this annual conference, I have a videotape of this memorable Paul Erdős' talk and thus have transcribed Paul's words exactly.

the Editor Alexander Soifer has written a scholarly treatment of this subject in this journal [Soi18], [Soi19], [SS2].

Ron confirmed the validity of my historical research in his January 26, 2007, email:

Hi Sasha,
. . . I think it is clear by your historical research that Nelson gets credit for the chromatic number of the plane problem.
Best regards,
Ron

The results of my historical research are summarized in Diagram 3.2, where arrows show passing of the problem from one mathematician to another. In the end, Paul Erdős shares the problem with the world in numerous talks and articles.

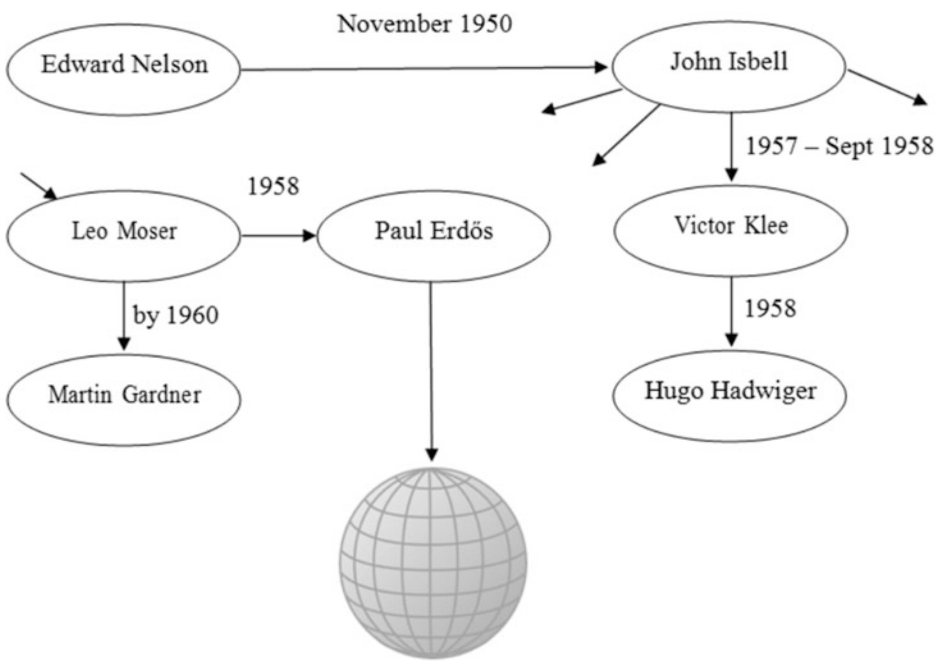


Diagram 3.2 Passing the baton of the chromatic number of the plane problem

Paul Erdős' and Ron Graham's acceptance of my research on the history of this fascinating problem has had a significant effect: most researchers and expositors now give credit to Edward Nelson for the chromatic number of the plane problem. There are, however, exceptions. László Lovász and K. Vesztergombi, for example, state in 2002 [LV] that

in 1944 Hadwiger and Nelson raised the question of finding the chromatic number of the plane.

Of course, the problem did not exist in 1944, in Hadwiger's cited paper or anywhere else. Moreover, Eddie Nelson was just an 11–12-year-old boy in 1944! In the same book, dedicated to the memory of Paul Erdős, one of the leading researchers of the problem László

Székely (who had in 1992 already attended my talk on the history of the problem in Boca Raton, where I presented the proof of Nelson's authorship), goes even further than Lovász and Vesztergombi in creating a myth [Sze3]:

> E. Nelson and J. R. Isbell, and independently Erdős and H. Hadwiger, posed the following problem . . .

The fine Russian researcher of this and related problems Andrei M. Raigorodskii copies Székely in his 2003 book [Raig6, p. 3], despite citing (and thus presumably knowing) my historical investigation in his survey [Raig3]:

> There were several authors. First of all, already in the early 1940s the problem was posed by remarkable mathematicians Hugo Hadwiger and Paul Erdős; secondly, E. Nelson and J. P. Isbell worked on the problem independently from Erdős and Hadwiger.[3]

Raigorodskii then "discovers" a nonexistent connection between World War II (!) and the popularity of the chromatic number of the plane problem[4]:

> In the 1940s there was W.W.II, and this circumstance is responsible for the fact that at first chromatic numbers [sic] did not raise too thunderous an interest.

In 2019, [BRa] L.I. Bogolyubsky and A.M. Raigorodskii drop John Isbell and call it "the Nelson–Erdős–Hadwiger problem."

I have won many battles in my life, for example, the change of the "Rolf Nevanlinna Prize" to "IMU Abacus Medal." However, I am giving up trying to do justice here and get the only correct name, **The Edward Nelson Problem** in this case. Even though Hugo Hadwiger admitted in print that he was not the author of the problem, the name "Hadwiger–Nelson" got stuck to the problem, just as Cardano did not author the Cardano formula and the Pythagoras theorem was known a millennium before the great Greek was born. Such is life with credits in mathematics. Most mathematicians view history as Cinderella that does not merit the respect they hold toward mathematics. History requires and deserves rigor and respect, gentlemen.

Not only Hadwiger but also the two famous Canadian problem people, the brothers Leo and William Moser, published in 1961 [MM] the proof of the lower bound $4 \le \chi$ while solving a different problem. Although, in my opinion, their proof is not distinct from those by Nelson and by Hadwiger, the Mosers' emphasis on a finite set and their invention of the seven-point configuration, now called *The Mosers Spindle* (plural, "Mosers," for we have here two brothers) proved to be highly productive.

Now, we can finally give due credit to Edward Nelson for being the first in 1950 to prove the lower bound $4 \le \chi$. Because of the bound 4, John Isbell recalls in his letter [Isb1] that Nelson "liked calling it a second four-color problem!" Nelson shared with me that he thought the chromatic number of the plane to be 4.

In phone interviews with Edward Nelson on September 18 and 30, 1991, I learned some information about the problem creator.

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[3] My translation from Russian.

[4] Ibid.

Joseph Edward Nelson was born on May 4, 1932 (an easy number to remember: 5/4/32), in Decatur, Georgia, near Atlanta. The son of the secretary of the Italian YMCA,[5] Ed Nelson had studied at a *liceo* (Italian prep school) in Rome. In 1949, Eddie returned to the United States and entered the University of Chicago. The visionary president of the university, Robert Hutchins,[6] allowed students to avoid "doing time" at the university by passing lengthy placement exams instead. Ed Nelson had done so well on so many exams that he was allowed to go right on to graduate school without working on his bachelor's degree.

Time Magazine reported young Nelson's fine achievements in 14 exams on December 26, 1949 [Time], next to the report on the completion of the last war crime trials of World War II (Field Marshal Fritz Erich von Manstein was sentenced to 18 years in prison), assurances by General Dwight D. Eisenhower that he would *not* be a candidate in the 1952 presidential election (he certainly was – and won it), and promise to announce *Time*'s "A Man of the Half-Century" in the next issue (subsequently, the *Time*'s choice was Winston Churchill).

Upon obtaining his doctorate from the University of Chicago in 1955, Edward Nelson became the National Science Foundation's postdoctoral fellow at Princeton's Institute for Advanced Study in 1956. Three years later, he became a professor of mathematics at Princeton University. His main areas of interest were analysis and logic. In 1975, Edward Nelson was elected to the American Academy of Arts and Sciences and in 1997 to the National Academy of Sciences. Ed shared with me, with excitement, that he gave an invited mathematical talk in Vatican. During my 2002–2004 and 2006–2007 work at Princeton, I had the pleasure of interacting with Professor Nelson almost daily. We became friends and shared lunches in the established company of senior Princetonians. Ed had a wonderful contagious smile. He enjoyed smoking his pipe and sipping wine. Ed attended my talk on the chromatic number of the plane problem at Princeton's Discrete Mathematics Seminar that I dedicated "To Edward Nelson, who created this celebrated problem for us all." He passed away on September 10, 2014, in Princeton.

John Rolfe Isbell (October 27, 1930–August 6, 2005) was the first in 1950 to prove the upper bound $\chi \leq 7$. He used the same hexagonal 7-coloring of the plane that Hadwiger published in 1961 [Had4]. Please note that Hadwiger first used this coloring of the plane in 1945 [Had3] but for a different problem: His goal was to show that there are seven congruent closed sets that cover the plane (he also proved there that no five congruent closed sets cover the plane). Professor Isbell, PhD Princeton University 1954 under Albert Tucker, had been for decades on the faculty of mathematics at the State University of New York at Buffalo, where he later became professor emeritus. John Isbell passed away on 6 August 2005.

Presently, in expanding the book for this new edition, I simply ought to pose a Hadwigerian open problem:

Plane Covering Problem 3.0 Is there a closed set S such that there is a distance not realizable between any pair of points of S and the plane can be covered by six sets congruent to S?

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[5]The Young Men's Christian Association (YMCA) is one of the oldest and largest not-for-profit community service organizations in the world.

[6]Robert Maynard Hutchins (1899–1977) was president (1929–1945) and chancellor (1945–1951) of the University of Chicago.

Paul Erdős' contribution to the survival and popularity of this problem is twofold. First of all, Paul kept the flaming torch of the problem brightly lit. He made the chromatic number of the plane problem well-known by posing it in his countless problem talks and many publications, for example, we see it in [E61.22], [E63.21], [E75.24], [E75.25], [E76.49], [E78.50], [E79.04], [ESi], [E80.38], [E80.41], [E81.23], [E81.26], [E85.01], [E91.60], [E92.19], [E92.60], and [E94.60].

Second, Paul Erdős created a good number of fabulous, related problems. We will discuss one of them in the next chapter.

In February 1992 at the 23rd Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton, during his traditional Thursday morning talk, I asked Paul Erdős how much he would offer for the first solution of the chromatic number of the plane problem. Paul replied (while I was jotting down on paper his words):

I can't offer money for nice problems of other people because then I will really go broke.

I then transformed my question into the realm of mathematics and asked Paul "*Assume* this is *your* problem, Paul; how much would you initially offer for its first solution?" Paul answered:

It is a very nice problem. If it were mine, I would offer $250 for it.

Later, the price went up for the first solution of just the lower bound part of the chromatic number of the plane problem. On Saturday, May 4, 2002, which by the way was precisely Edward Nelson's 70th birthday, Ronald L. Graham gave a talk on Ramsey theory at the Massachusetts Institute of Technology to about 200 high school participants in the USA Mathematical Olympiad. During the talk, he offered $1000 for the first proof or disproof of what he called, after Nelson, "Another Four-Color Conjecture." The talk commenced at 10:30 a.m. (as a member of the USA Mathematical Olympiad Subcommittee, I was in attendance and took notes).

Another Four-Color $1000 Problem 3.1 (R.L. Graham, May 4, 2002). Is it possible to four-color the plane to forbid a monochromatic distance 1?

In August 2003, in his talk "What is Ramsey Theory?" at Berkeley [Gra4], Graham asked for much more work for $1000:

$1000 Open Problem 3.2 (R.L. Graham, August 2003). Determine the value of the chromatic number χ of the plane.

It seems that Graham believed that the chromatic number of the plane takes on an intermediate value, between of its known boundaries, for in his two surveys [Gra7], [Gra8], he offered the following open problems:

$100 Open Problem 3.3 (R.L. Graham [Gra7], [Gra8]). Show that $\chi \geq 5$.[7]

$250 Open Problem 3.4 (R.L. Graham [Gra7], [Gra8]). Show that $\chi \leq 6$.

This prompted me to look at all published Erdős' predictions of the chromatic number of the plane. Let me summarize them here for you. First Erdős believed – and communicated it

[7] Ron Graham cites the O'Donnell theorem 50.4 (see it later in this book) as "perhaps, the evidence that χ is at least 5."

in 1961 [E61.22] and in 1975 [E75.24] – that the problem creator Nelson conjectured that the chromatic number was 4; Paul enters no prediction of his own. In 1976 [E76.49], Erdős asks:

Is this graph 4-chromatic?

In 1979 [E79.04], Erdős becomes more assertive:

It seems likely that the chromatic number is greater than four. By a theorem of de Bruijn and myself this would imply that there are n points $x_1, \ldots, x_n$ in the plane so that if we join any two of them whose distance is 1, then the resulting graph $G(x_1, \ldots, x_n)$ has chromatic number > 4. I believe such an n exists but its value may be very large.

A certainty comes in 1980 [E80.38] and [E80.41]:

I am sure that [the chromatic number of the plane] $\alpha_2 > 4$ but cannot prove it.

In 1981 [E81.23] and [E81.26], we read Erdős, respectively:

It has been conjectured [by E. Nelson] that $\alpha_2 = 4$, but now it is generally believed that $\alpha_2 > 4$.

It seems likely that $\chi(E^2) > 4$.

In 1985 [E85.01], Paul Erdős writes:

I am almost sure that $h(2) > 4$.

Once – just once – Erdős expresses mid-value expectations. It happened on Thursday, March 10, 1994 at the 25th Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton. Following Erdős' plenary talk (9:30–10:30 a.m.), I was giving my talk at 10:50 a.m., when suddenly Paul Erdős said (and I jotted it down):

Excuse me for interrupting, I am almost sure that the chromatic number of the plane is greater than 4. It is not a proof, but any measurable set without distance 1 in a very large circle has measure less than ¼. I also do not think that it is 7.

It is time for me to speak on the record and predict the chromatic number of the plane. In 2002, I was leaning toward predicting 7 or else 4 – somewhat disjointly from Graham and Erdős' apparent expectations. Limiting myself to just one value, still in 2002, I conjectured:

Chromatic Number of the Plane Conjecture 3.5 (A. Soifer, 2002)[8]. $\chi = 7$.

On January 26, 2007, in a personal e-mail to me, Ron Graham clarified the terms of awarding his prizes:

I always assume that we are working in ZFC (for the chromatic number of the plane!). My monetary awards can vary depending on which audience I am talking to. I always give the maximum of whatever I have announced (and not the sum!).

[8] See more on the predictions in Chapter 62.