Chapter 2
Chromatic Number of the Plane: The Problem

A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought and help find the path to follow.
– Henry Poincaré [Poi]

[I] can't offer money for nice problems of other people because then I will really go broke...
It is a very nice problem. If it were mine, I would offer $250 for it.
– Paul Erdős
Boca Raton, February 6, 1992

The most widely known problem in Euclidean Ramsey Theory is probably that of determining the chromatic number of the plane, $\chi(E^2)$.
– Ronald L. Graham and Eric Tressler [GT]

The unit distance graph in the plane ... is simple enough to describe to a nonmathematician, and so enigmatic that finding its chromatic number is a new four-color map problem for graph theorists.
– Ronald L. Graham and Eric Tressler (Ibid.)

If Problem 8 [the chromatic number of the plane] takes that long to settle [as the Four-Color problem], we should know the answer by the year 2084.
– Victor Klee and Stan Wagon [KW]

Our good ole Euclidean plane, don't we know all about it? What else can there be after Pythagoras and Steiner, Euclid, and Hilbert? In this chapter, we will look at an open problem that exemplifies what is best in mathematics: Anyone can understand this problem; yet, no one has been able to conquer it in 73 years.

In August 1987, I attended an inspiring talk by Paul Halmos at Chapman College in Orange, California. It was entitled "Some problems you can solve, and some you cannot." This problem is an example of a problem that "you cannot solve."

A. Soifer, *The New Mathematical Coloring Book*, https://doi.org/10.1007/978-1-0716-3597-1_2

"A fascinating problem... that combines ideas from set theory, combinatorics, measure theory, and distance geometry," write Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy in their book *Unsolved Problems in Geometry* [CFG].

"If Problem 8 takes that long to settle [as the celebrated Four-Color Conjecture], we should know the answer by the year 2084," write Victor Klee and Stan Wagon in their book *New and Old Unsolved Problems in Plane Geometry* [KW].

Are you ready? Here it is:

What is the smallest number of colors sufficient for coloring the plane in such a way that no two points of the same color are at a unit distance apart?

This number is called *the chromatic number of the plane* and is denoted by $\chi(E^2)$ or simply χ.

We will use R to denote the set of real numbers and the real line. The line equipped with the usual Euclidean distance, we will denote by E^1. Generalizing the line E^1, we get the Euclidean plane E^2 and the Euclidean space E^3, and we define the *n-dimensional* space R^n for any positive integer n as the set of all n-tuples $(x_1, x_2,..., x_n)$, where $x_1, x_2,..., x_n$ are real numbers. When the distance between two points $(x_1, x_2,..., x_n)$ and $(y_1, y_2,..., y_n)$ of R^n is defined by the equality

$$d=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2} \qquad (*)$$

we get the *Euclidean n-dimensional* space E^n. In other words, E^n is just the set R^n together with the distance d defined by (*).

To color the plane means to assign one color to every point of the plane. Please note that, here, we color without any restrictions and are not limited to "nice" tiling-like or map-like colorings. Given a positive integer n, we say that the plane is *n-colored*, if every point of the plane is assigned one of the given n colors.

Here, a *segment* will stand for just a two-point set (which are end points in a conventional treatment of a segment). Similarly, a *polygon* will stand for a finite set of points. A *monochromatic* set is a set, whose all elements are assigned the same color. In this terminology, we can formulate the chromatic number of the plane (CNP) problem as follows: What is the smallest number of colors sufficient for coloring the plane in a way that forbids monochromatic unit segments?

I do not know who first noticed the following result. Perhaps, Adam? Or Eve? To be a bit more serious, I do not think that ancient Greek geometers, for example, knew this nice fact, for they simply did not ask these kinds of questions!

Problem 2.1 (Adam and Eve). No matter how the plane is two-colored, it contains a monochromatic segment of length 1, i.e.,

$$\chi \geq 3.$$

Proof Toss on the two-colored plane an equilateral triangle T of side 1 (Fig. 2.1). We have only two colors, while T has three vertices (I trust you have not forgotten the Pigeonhole principle). Two of the vertices must be of the same color. They *are* at a distance 1 apart. ■

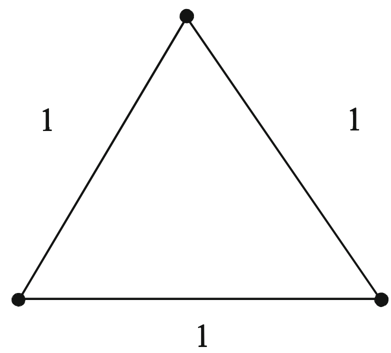

Fig. 2.1 At least 3 colors are necessary

We can do better than Adam and Eve:

Problem 2.2 No matter how the plane is three-colored, it contains a monochromatic segment of length 1, i.e.,

$$\chi \geq 4.$$

Proof by the Canadian Geometers, Brothers Leo and William Moser (1961, [MM]). Toss on the three-colored plane what we now call *the Mosers Spindle* (Fig. 2.2). Every edge in the spindle has the length 1.

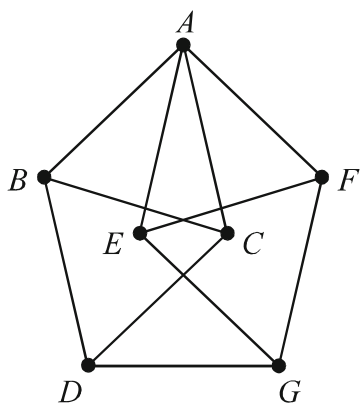

Fig. 2.2 The Mosers Spindle

Assume that the seven vertices of the spindle do not contain a monochromatic unit segment. Call the colors used in coloring the plane red, white, and blue. The solution now will faithfully follow the children's alphabet song "A B C D E F G .... ".

Let the point A be red, then B and C must be one white and one blue, respectively, and therefore, D must be red. Similarly, E and F must be one white and one blue, respectively, and therefore, G must be red. We have a monochromatic unit segment DG in contradiction to our assumption. ■

Observe The Mosers Spindle has worked for us in solving problem 2.2 precisely because *any* three vertices of the spindle contain two vertices that are at a distance 1 apart. This implies that *in* *a Mosers Spindle that forbids a monochromatic unit segment, at most two points can be of the same color.* Let us record this observation as a tool, which we will need later in Chapters 4 and 40.

Mosers' Tool 2.3 *Any* three vertices of the Mosers Spindle contain a unit segment. Consequently, in a Mosers Spindle that forbids a monochromatic unit segment, at most two vertices can be of the same color.

When I presented the Mosers' solution to high school mathematicians, everyone agreed that it was beautiful and simple. "But how do you come up with a thing like the spindle?", I was asked. As a reply, I presented a less elegant but a more naturally found solution. In fact, I would call it a second version of the same solution. Here, we touch on a curious aspect of mathematics. In mathematical texts, we often see the terms "second solution" and "third solution." However, which two solutions ought to be called distinct? We do not know. It is not defined and is thus a judgment call. Distinct solutions for one person could be viewed as versions of the same for another. It is interesting to notice that both versions were published in the same year, 1961, one in Canada and the other in Switzerland.

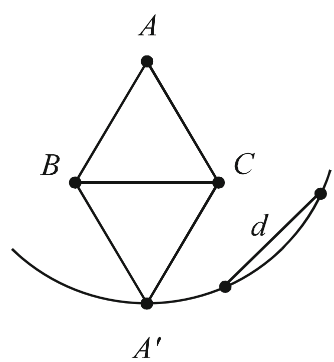

Fig. 2.3 At least 4 colors are necessary

Second Version of the Proof (Hugo Hadwiger, 1961, [Had4]). Assume that a three-colored red–white–blue plane does not contain a monochromatic unit segment. Then an equilateral triangle ABC of side 1 will have one vertex of each color (Fig. 2.3). Let A be red, then B and C must be one white and one blue, respectively. The vertex A' symmetric to A with respect to the side BC must be red as well. As we rotate our rhombus $ABA'C$ through *any* angle about A, the vertex A' will have to remain red due to the above argument. Thus, we get a whole red circle of radius AA'. Surely, it contains a cord d of length 1, both end points of which are red, in contradiction to our assumption. ■

Does an upper bound exist for χ? It is not immediately obvious. Can you find one? Think of tiling the plane with square tiles.

Problem 2.4 There is a 9-coloring of the plane that contains no monochromatic segments of length 1, i.e.,

$$\chi \leq 9.$$

Proof Tile the plane with unit squares. Now, we color one square in color 1 and its eight neighbors in colors 2, 3, ..., 9 (Fig. 2.4). The union of these 9 unit squares is a 3×3 square S, shown in bold. Translates of S (i.e., images of S under translations) tile the plane and determine how we color it in nine colors.

You can easily verify (do) that no distance d in the range $\sqrt{2} < d < 2$ is realized monochromatically in the plane. Thus, by shrinking all linear sizes by the factor of, say, 1.5, we get a 9-coloring that contains no monochromatic segments of length 1. (Observe: due to the above inequality, we have enough cushion so that it does not matter in which of the two adjacent colors we color the boundaries of the unit squares.) ■

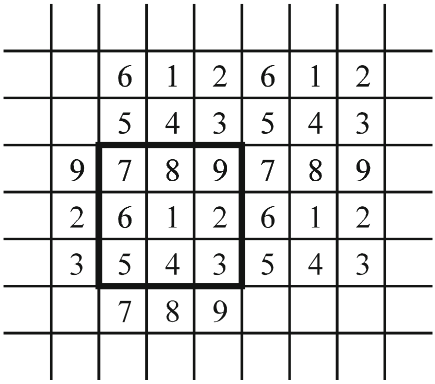

Fig. 2.4 9 colors suffice

Now that a tiling has helped us solve the above problem, it is natural to ask whether another tiling can help us improve the upper bound. One can indeed.

Problem 2.5 There is a 7-coloring of the plane that contains no monochromatic unit segments, i.e.,

$$\chi \leq 7.$$

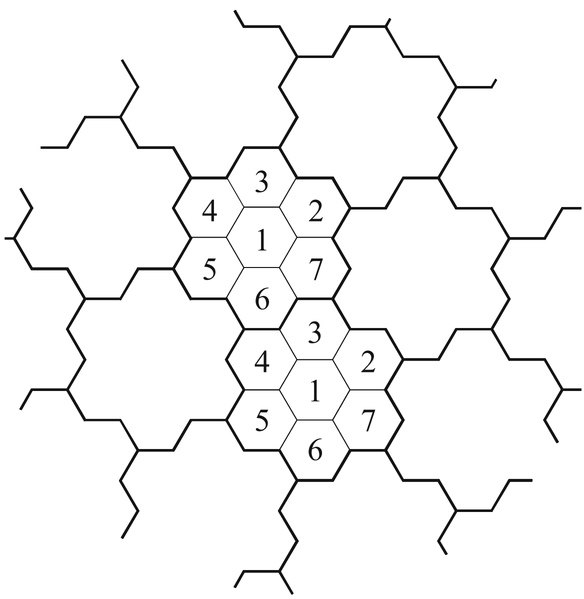

Fig. 2.5 A 7-coloring using a hexagonal tiling

Proof [Had3]. We can tile the plane by regular hexagons of side 1. Now, we color one hexagon in color 1 and its six neighbors in colors 2, 3, ..., 7 (Fig. 2.5). The union of these seven hexagons forms a "flower" P, a highly symmetric polygon P of 18 sides. Translates of P tile the plane and determine how we color the plane in seven colors. It is easy to compute (please do) that each color does not have monochromatic segments of any length d, where $2 < d < \sqrt{7}$. Thus, if we shrink all linear sizes by a factor of, say, 2.1, we will get a 7-coloring of the plane that has no monochromatic segments of length 1. (Observe: due to the above inequality, we have enough cushion so that it does not matter in which of the two adjacent colors we color the boundaries of the hexagons.) ■

This is the way the upper bound is proved in every book I know ([CFG] and [KW], for example). Yet, in 1982, the Hungarian mathematician László A. Székely found a clever way to prove the upper bound 7 without using hexagonal tiling.

Problem 2.6 (L. A. Székely, [Sze1]). Prove the upper bound $\chi \leq 7$ by tiling the plane with squares again.

Proof This is László Székely's proof from [Sze1]. His original picture needs a small correction in his Fig. 1, and boundary coloring needs to be addressed, which I am doing here. We start with a row of squares of diagonal 1, with cyclically alternating colors of the squares $1, 2, \ldots, 7$ (Fig. 2.6). We then obtain consecutive rows of colored squares by shifting the preceding row to the right through 2.5 square sides.

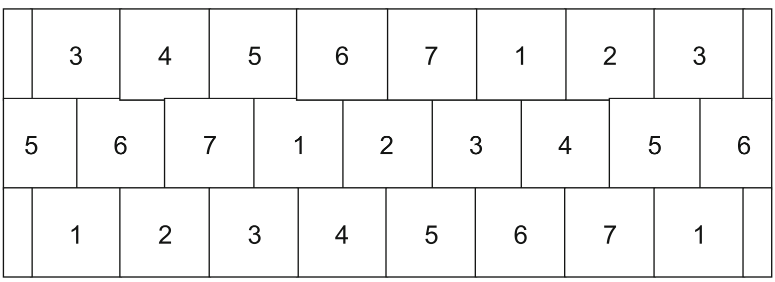

Fig. 2.6 A 7-coloring using square tiling

The upper and right boundaries are included in the color of each square, except the square's upper left and lower right corners. ■

In 1995, my former student and now a well-known puzzlist Edward Pegg, Jr. sent me two distinct 7-colorings of the plane. In the one I am sharing with you (Fig. 2.7), Ed uses 7-gons for six of the colors and tiny squares for the seventh color. In fact, the seventh color occupies only about one-third of 1% of the plane.

In Fig. 2.7, all thick black bars have a unit length. A unit of the tiling uses a heptagon and half a square.

The area of each square is 0.00412220518993071681 62...

The area of each heptagon is 0.622651271646476296 46...

Thus, the area ratio is 302.0962048019455285300783627265828...

If one-third of 1% of the plane is removed, then the remainder can be six-colored with this tiling!

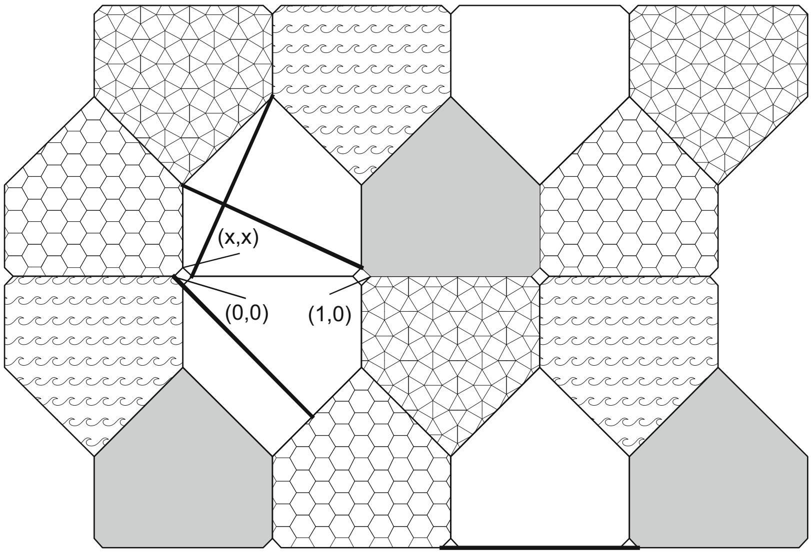

Fig. 2.7 Ed Pegg's 7-coloring with a small use of color 7

The lower bound for the chromatic number of the plane (problem 2.2) also has proofs that are fundamentally different from using the Mosers Spindle. In the early 1990s, I received from my colleague and friend Klaus Fischer of George Mason University a finite configuration of the chromatic number 4, different from the Mosers Spindle. Klaus had no idea who created it, so I commenced backtracking this construction. Klaus got it from our friend and colleague Heiko Harborth of Braunschweig Technical University, Germany, who, in turn, referred me to his source, Solomon W. Golomb of the University of Southern California, the famous inventor of polyomino. Solomon invented this graph as well and described it in the September 10, 1991, letter to me [Gol1]:

> The example you sketched of a 4-chromatic unit-distance graph with ten vertices is original with me. I originally thought of it as a 3-dimensional structure (the regular hexagon below, the equilateral triangle above it in a plane parallel to it), and all connected by unit-length toothpicks. The structure is then allowed to collapse down into the plane, to form the final Figure (Fig. 2.8). I have shown it to a number of people, including the late Leo Moser, Martin Gardner, and Paul Erdős, as well as Heiko Harborth. It is possible that Martin Gardner may have used it in one of his columns, but I don't remember. Besides my example and Mosers' original example (which I'm reasonably sure I have seen in Gardner's column), I have not seen any other "fundamental" examples. I believe what I had suggested to Dr. Harborth in Calgary was the possibility of finding a 5-chromatic unit-distance graph, having a much larger number of edges and vertices.

"The possibility of finding a 5-chromatic unit-distance graph" was on the minds of most of us, who worked on this problem. Does it exist? You will find a definitive answer later in this book.

In the consequent September 25, 1991, letter [Gol2], Sol Golomb informed me that he likely found this example, which I will naturally call *the Golomb Graph*, in the period 1960–1965.

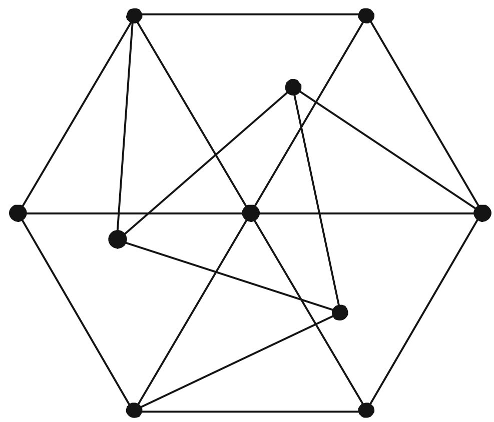

Fig. 2.8 The Golomb graph

Second Solution of Problem 2.2 Just toss the Golomb graph with all edges of unit length on a three-colored (red, white, and blue) plane (Fig. 2.8). Assume that in the graph, there are no adjacent vertices of the same color. Let the center vertex be colored red, then, since it is connected by unit edges to all vertices of the regular hexagon *H*, the vertices of *H* must be colored white and blue in an alternating manner. All vertices of the central equilateral triangle *T* are connected by unit edges to the three vertices of *H* of the same color, say, white. However, then, white cannot be used in coloring *T*, and, thus, *T* is colored red and blue. However, this implies that two of the vertices of *T* are assigned the same color. This contradiction proves that 3 colors are not enough to properly color the 10 vertices of the Golomb graph, let alone the whole plane. ■

It is amazing that the pretty easy solutions of problems 2.2 and 2.4 provided us with the best bounds known to mathematics prior to 2018 for the chromatic number of the plane χ *in the general case*. They were published more than 60 years ago (in fact, they are older than that: see the next chapter for an intriguing historical account). Still, all we knew at the time of the first edition of this book was

$$\chi = 4, \text{or } 5, \text{or } 6, \text{or } 7.$$

A very broad spread! Which do you think is the exact value of χ? The legendary Paul Erdős believed that it was $\chi \geq 5$.

The renown American geometer Victor Klee of the University of Washington shared with me in 1991 a highly intriguing story. In 1980, he lectured in Zürich, Switzerland. The celebrated 77-year-old mathematician Bartel L. van der Waerden (whom we will frequently meet later in this book) was in attendance. When Vic presented the state of this problem, Van der Waerden became very interested. Right there and then, during Vic's lecture, Bartel started working on the problem. He tried to prove that $\chi = 7$.

For many years, I believed that $\chi = 7$ (you will find my thoughts on the matter in *Predicting the Future*, later in this book). Paul Erdős used to say that

> God has a transfinite Book, which contains all theorems and their best proofs, and if He is well intentioned toward those, He shows them the Book for a moment.

If I ever deserved the honor and had a choice, I would have asked to peek at the page with the chromatic number of the plane problem. Wouldn't you?