Geometric Flows of G_2 Structures

Jason D. Lotay

Abstract Geometric flows have proved to be a powerful geometric analysis tool, perhaps most notably in the study of 3-manifold topology, the differentiable sphere theorem, Hermitian–Yang–Mills connections and canonical Kähler metrics. In the context of G_2 geometry, there are several geometric flows which arise. Each flow provides a potential means to study the geometry and topology associated with a given class of G_2 structures. We will introduce these flows, and describe some of the key known results and open problems in the field.

1 Introduction

Our understanding of G_2 structures, and particularly the question of when a G_2 structure can be deformed to become torsion-free, is very limited. It is therefore useful to look to new tools to tackle open problems in the area. An obvious avenue of attack is to use geometric flows, given their success in other geometric contexts: for example in analysing Hermitian connections (via Yang–Mills flow), convex hypersurfaces (via mean curvature flow) and perhaps most notably 3-manifolds and $\frac{1}{4}$-pinched Riemannian manifolds (via Ricci flow).

The goal of these notes is to explain some of the basics behind the geometric flow approach to studying G_2 structures and give a brief overview of what is known. It is important to note that several different flows of G_2 structures have been studied, based on various well-founded motivations. We shall attempt to give a brief description of each of these flows, the reasons behind them and some of the pros and cons in their study.

As well as giving this brief survey of the landscape in geometric flows of G_2 structures, we will provide some indication of some key open questions that we believe are worthy of further exploration.

Note These notes are based primarily on a lecture given at a Minischool on "G_2 Manifolds and Related Topics" at the Fields Institute, Toronto in August 2017.

J. D. Lotay (✉)
University of Oxford, Oxford, UK
e-mail: lotay@maths.ox.ac.uk

2 Geometric Flows

What is a geometric flow? Informally, it is a mechanism for "simplifying" or "decomposing" a given geometric structure into one or several "canonical" or "special" pieces. Thus, the primary goals of geometric flows are to show the existence of special geometric objects and to determine which geometric objects can be deformed to special ones. By answering these questions, one can then hope to understand large classes of geometric structures by understanding the much smaller class of canonical ones.

2.1 Heat Flow

Functions. The motivation for geometric flows comes from the heat flow for functions f on a Riemannian manifold:

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = 0. \tag{2.1}$$

(Here, and throughout, we will use the geometer's convention that the Laplacian Δ is a non-negative operator, so $\Delta = d^*d$ on functions.) The heat flow is parabolic, which means that if we consider (2.1) on a compact manifold M, then a short time solution to (2.1) is guaranteed to exist and the equation is "regularizing" (a notion we shall clarify in a moment).

   We now make an elementary but fundamental observation.

Proposition 2.1 *The heat flow is the negative gradient flow for the Dirichlet energy:*

$$\frac{1}{2}\int_M |df|^2 \, \mathrm{vol}_M \geq 0. \tag{2.2}$$

Proof We see that for any t-dependent family of functions we have

$$\frac{\partial}{\partial t}\frac{1}{2}\int_M |df|^2 \, \mathrm{vol}_M = \langle \frac{\partial}{\partial t}df, df\rangle_{L^2} = \langle d\frac{\partial}{\partial t}f, df\rangle_{L^2} = \langle \frac{\partial f}{\partial t}, \Delta f\rangle_{L^2}. \tag{2.3}$$

$\square$

Thus (2.2) will decrease fastest along the heat flow (2.1), and the critical points for the Dirichlet energy (which are exactly the stationary points for (2.1)) are given by the constant functions (i.e. $df = 0$) which are the absolute minimizers for the energy.

   From the gradient flow point of view we should expect that given any smooth function, by following the heat flow we should be able to deform it into a critical point for (2.2), i.e. a constant function. We now show that this is the case.

Theorem 2.2 *Suppose that* $f = f(x, t)$ *solves the heat equation* (2.1) *on a compact manifold M and* $f(x, 0)$ *is smooth. Then* $f(x, t)$ *exists for all* $t > 0$ *and* $f(x, t) \to c \in \mathbb{R}$ *smoothly as* $t \to \infty$, *where*

$$c = \frac{1}{\text{Vol}(M)} \int_M f(x, 0) \, \text{vol}_M .$$

Proof If we suppose that f is smooth at $t = 0$, then it is smooth for all $t > 0$ as well and the eigenfunctions of Δ span $L^2(M)$, so at each time t we can write

$$f(x, t) = \sum_\lambda c_\lambda(t) f_\lambda(x) \tag{2.4}$$

for functions c_λ of time t and functions f_λ on M, where $\Delta f_\lambda = \lambda f_\lambda$ for $\lambda \geq 0$ and the f_λ form a complete orthonormal system for $L^2(M)$. It quickly follows from inserting (2.4) in (2.1) that

$$c_\lambda(t) = c_\lambda(0)e^{-\lambda t}, \tag{2.5}$$

and so the solution of (2.1) actually exists for all time $t > 0$. Moreover, the solution converges as $t \to \infty$ to

$$c_0 f_0 = \frac{1}{\text{Vol}(M)} \int_M f(x, 0) \, \text{vol}_M, \tag{2.6}$$

the "average value" of f at time 0.   □

Thus, the heat flow "regularizes" the function f in that it simplifies it as much as possible (it turns it into a constant) and we see that the higher the frequency (i.e. eigenvalue) of the eigenfunction of Δ in the expansion (2.4), the faster that component of f decays under the flow by (2.5). The f_λ for high λ correspond to higher "oscillations" of f, and so these "wiggles" in f get smoothed out by (2.1), eventually giving a constant. In terms of the Dirichlet energy functional (2.2), it shows that every function can be deformed to a minimizer (so the space of smooth functions retracts onto the constant functions, which are the critical points of the functional), and the minimizer we find is determined by the average value of f.

Notice that our analysis in the proof of Theorem 2.2 implies the following.

Lemma 2.3 *The integral of* f *is constant along* (2.1).

We are therefore free to modify (2.1) and consider

$$\left(\frac{\partial}{\partial t} + \Delta - \lambda_1 \right) f = 0, \tag{2.7}$$

for functions f with $\int_M f = 0$, where λ_1 is the first positive eigenvalue of Δ on M. It is easy to see that (2.7) is still parabolic and that if $\int_M f = 0$ initially then it stays zero for all t under (2.7). However, under (2.7), we see that the flow no longer

converges to a constant, but instead to the projection of f to the λ_1-eigenspace of Δ (which may now have several components if λ_1 is not a simple eigenvalue). Again, this flow "regularizes" f, throwing away all of the higher eigenmodes of Δ in the limit.

Forms. We can also consider the heat flow on differential k-forms α on a compact manifold M:

$$\left(\frac{\partial}{\partial t} + \Delta\right)\alpha = 0, \tag{2.8}$$

where Δ is the Hodge Laplacian

$$\Delta = dd^* + d^*d. \tag{2.9}$$

The flow (2.8) is now the gradient flow for the Dirichlet energy

$$\frac{1}{2}\int_M |d\alpha|^2 + |d^*\alpha| \, \text{vol}_M \geq 0 \tag{2.10}$$

by a similar argument as before. Again decomposing $\alpha(t)$ at each time t using eigenforms for Δ, we have the following.

Theorem 2.4 *The heat equation* (2.8) *for* $\alpha(t)$ *on a compact manifold starting at a smooth form* $\alpha(0)$ *exists for all time and converges smoothly to the projection of* $\alpha(0)$ *to the* 0*-eigenforms for* Δ*, i.e. the harmonic* k*-forms*

$$d\alpha = d^*\alpha = 0. \tag{2.11}$$

The harmonic forms are precisely the critical points of (2.10) and are clearly absolute minimizers as they are zeros for the energy functional.

Now, the harmonic forms are only a finite-dimensional space in the space of k-forms, so given any initial k-form it could well be that the heat flow will just send it to zero, which is clearly a legitimate critical point for the flow (though not an interesting one!). For example if $\alpha(0)$ is exact or coexact (or the sum of forms of this type), the heat flow will just go to 0.

To ensure that we find a non-trivial critical point, we could restrict attention to closed k-forms:

$$d\alpha = 0. \tag{2.12}$$

Notice that this is preserved by (2.8) since in this case we have

$$\frac{\partial}{\partial t}\alpha = -\Delta\alpha = -(dd^* + d^*d)\alpha = -dd^*\alpha, \tag{2.13}$$

so in fact we have that $\alpha(t)$ lies in the fixed cohomology class $[\alpha(0)]$ for all time as the right-hand side of (2.13) is exact. Therefore, if we have that $[\alpha(0)] \neq 0$ is a non-

trivial cohomology class, we know that (2.13) will exist for all time and converge to the non-zero harmonic representative of that class (which we know exists and is unique by Hodge theory).

We could also equally well have restricted to coclosed k-forms

$$\mathrm{d}^*\alpha = 0, \tag{2.14}$$

since this is also preserved by a similar argument. This time $*\alpha(t)$ will lie in the fixed cohomology class $[*\alpha(0)]$ for all time and the flow will converge to the Hodge dual of the harmonic representative of $[*\alpha(0)]$.

We summarize these findings.

Proposition 2.5 *Suppose that $\alpha(t)$ is a family of k-forms on a compact n-manifold M solving* (2.8).

(a) *If $\mathrm{d}\alpha(0) = 0$, $\alpha(t)$ exists for all $t > 0$ satisfying $\mathrm{d}\alpha(t) = 0$ for all t and converges smoothly to the unique harmonic k-form in $[\alpha(0)] \in H^k(M)$.*

(b) *If $\mathrm{d}^*\alpha(0) = 0$, $\alpha(t)$ exists for all $t > 0$ satisfying $\mathrm{d}^*\alpha(t) = 0$ for all t and converges smoothly to the Hodge dual of the unique harmonic $(n - k)$-form in $[*\alpha(0)] \in H^{n-k}(M)$.*

We might hope, at least naively, that we could have similar good behaviour in geometric flows as for the heat flow, and thus obtain ways to canonically represent classes of geometric structures, just as harmonic forms uniquely represent all cohomology classes.

2.2 Ricci Flow and Mean Curvature Flow

Geometric flows aim to act on the same principle as the heat flow, two canonical examples being Ricci flow on metrics g and mean curvature flow on immersions F into a Riemannian manifold:

$$\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}(g) \quad \text{and} \quad \frac{\partial}{\partial t}F = H, \tag{2.15}$$

where $\mathrm{Ric}(g)$ denotes the Ricci curvature tensor of g and H denotes the mean curvature vector of the immersion F. (Two other key examples of geometric flows of significant interest where many results have been obtained are the harmonic map heat flow and Yang–Mills flow, but we do not discuss them here.) Under suitable choices of coordinates, (2.15) can be seen as "heat flows", however this time the Laplacian depends on the metric or immersion respectively, and so the flows are nonlinear.

Parabolicity. The flows (2.15) are not parabolic due to geometric invariance in the problem: in Ricci flow this is diffeomorphism invariance, and in mean curvature flow

this is invariance under reparametrisation. However, once one kills this geometric invariance, one obtains a parabolic equation.

For example, in Ricci flow, one can apply the so-called DeTurck's trick:

$$\frac{\partial}{\partial t}h = -2\operatorname{Ric}(h) + \mathcal{L}_{X(h)}h, \qquad (2.16)$$

where $X(h)$ is a suitably chosen vector field (depending on the metric h) which ensures that (2.16) is parabolic and so has a short time solution which is regularizing. We can get a solution to Ricci flow from a solution to (2.16) by considering $g = \Phi^*h$, where Φ are diffeomorphisms defined by

$$\frac{\partial}{\partial t}\Phi = -X(h) \quad \text{and} \quad \Phi(0) = \operatorname{id}. \qquad (2.17)$$

Proposition 2.6 *Suppose that h are metrics satisfying* (2.16) *and Φ are diffeomorphisms satisfying* (2.17)*. Then $g = \Phi^*h$ satisfies the Ricci flow in* (2.15)*.*

Proof By (2.16) and (2.17),

$$\frac{\partial}{\partial t}g = \frac{\partial}{\partial t}\Phi^*h = \Phi^*\frac{\partial}{\partial t}h - \Phi^*\mathcal{L}_{X(h)}h = -2\Phi^*\operatorname{Ric}(h) = -2\operatorname{Ric}(g). \qquad (2.18)$$

□

This result is great but, it is natural to ask: what is a good choice of $X(h)$? The idea is, given h, for any symmetric 2-tensor k to consider the "gravitational tensor"

$$G(k) = k - \frac{1}{2}(\operatorname{tr}k)h, \qquad (2.19)$$

whose divergence is given by the 1-form:

$$\operatorname{div}G(k) = \operatorname{div}(k) + \frac{1}{2}\operatorname{d}(\operatorname{tr}k). \qquad (2.20)$$

(Here, by the divergence we mean the formal adjoint of the map $X^\flat \mapsto \frac{1}{2}\mathcal{L}_X h$, so that $\operatorname{div}(k)$ is the negative of the trace on the first two indices of ∇k; i.e. $\operatorname{div}(k)_j = -\nabla_i k_{ij}$. The musical isomorphisms, ∇ and trace are all defined by h.) If k is a *fixed* Riemannian metric then, using h, we can view k as an invertible map on 1-forms and so

$$X(h) = (k^{-1}\operatorname{div}G(k))^\sharp, \qquad (2.21)$$

where the musical isomorphism is again given by h, is a well-defined vector field.

Theorem 2.7 *If we choose the vector field $X(h)$ as in* (2.21)*, the Ricci–DeTurck flow* (2.16) *is parabolic.*

Proof It is straightforward to compute that, with the choice of $X(h)$ in (2.21), the linearisation of (2.16) is simply

$$\frac{\partial}{\partial t} h = -\Delta h, \tag{2.22}$$

the heat equation on h. □

Moreover, (2.17) is harmonic map flow, which is parabolic, so we can solve (2.16) and (2.17) uniquely for short time by parabolic theory, and hence obtain a unique short time solution to the Ricci flow by Proposition 2.6.

Now, by analogy with the heat flow, in Ricci flow the "eigenmodes" we need to consider are solutions to

$$\mathrm{Ric}(g) = \lambda g; \tag{2.23}$$

in other words, Einstein metrics. Here the "eigenvalues" λ have no distinguished sign and so, by analogy with the heat flow analysis above, we cannot say what will happen along the flow in general. However, if the flow exists for all time and converges, then it must tend to a Ricci-flat metric—the "zero mode" in the expansion of g in Einstein metrics, if you will. Similarly, for mean curvature flow, if the flow exists for all time and converges, we would obtain a minimal immersion (and the "expansion" of the immersion should be into constant mean curvature immersions). We see in both cases that we are breaking up our geometric object into pieces of significant interest. Moreover, the long-time existence and convergence of the flow allows us both to find the special object (a Ricci-flat metric or minimal immersion) and, at the same time, show that our initial geometric object can be deformed smoothly into the special one, which again is an important and challenging problem to solve.

Compact surfaces. To see the power of geometric flows it is instructive to look at Ricci flow in dimension 2. Here, Ricci curvature is just the Gauss curvature of the surface (up to a multiple) and there are three possibilities on a compact orientable surface.

- The flow exists for all time and converges. This means that the surface has a flat metric, and so must have genus 1 (by Gauss–Bonnet).
- The flow exists for all time but does not converge. In this case, just as when we perturbed the heat flow in (2.7), we can modify the Ricci flow and show that this modified flow exists for all time and converges to a hyperbolic metric. Thus the surface must have genus at least 2.
- The flow exists for only a finite time. This does not have a direct heat flow analogue, but one can again modify the Ricci flow as in the previous case (now by adding a term with the opposite sign), and show that this converges now to a constant positive curvature metric, which means that the surface must be a sphere. This is the difficult case in the analysis of the Ricci flow and this is typical of geometric flows: the case corresponding to "negative eigenvalues" (which do not happen for the Laplacian in the heat flow) is the most challenging to understand.

Thus, the Ricci flow in dimension 2 gives an alternative means to prove the uniformization theorem. In particular, that constant curvature metrics exist on any compact orientable surface, and the topology of the surface uniquely determines the sign of the constant curvature.

Gradient flow. Finally, one can also interpret (2.15) as gradient flows: in the case of mean curvature flow this is nothing but the negative gradient flow of the volume functional on immersions, but for Ricci flow the gradient flow interpretation is more subtle and involved so we shall not describe it here.

Needless to say, the fact that they are gradient flows is very helpful, since then one has some expectation of what one might hope to happen along the flow, as one has a monotone quantity (the analogue of the Dirichlet energy) along the flow which is trying to reach a critical value for the functional.

However, even with the gradient flow point-of-view, the nonlinearity of the problem and the potential complexity of the topology of the space of geometric objects we are considering means that we cannot always hope for the analysis of our flow to be straightforward and to go as expected. For example, the Ricci flow and mean curvature flow have special features (both good and bad) due to nonlinearity which simply cannot possibly occur in the standard heat flow.

2.3 Singularities

A singularity in a geometric flow is a point where the flow cannot be continued, because some quantity blows up to infinity. We already saw this in the Ricci flow in dimension 2, where there is always a singularity in finite time if we work on a sphere. Singularities may sound bad, and they definitely can be, but they can also be very helpful because they may tell you that you need to break up your geometric object into several pieces to get canonical objects. This happens for example in 3-dimensional Ricci flow, where singularities can be used to decide how to break up the 3-manifold according to Thurston's Geometrization Conjecture (now a theorem by Perelman's work).

The question is: what happens at a singularity geometrically? In good situations the singularity will be modelled on a special solution to the flow called a soliton. By "modelled on" we mean that by appropriately rescaling the flow around the singular point, both in space and time, in the limit we should see a soliton.

Definition 2.8 A soliton is a solution to the flow which is "self-similar", meaning that it moves very simply under the flow, just under rigid motions and diffeomorphisms or reparametrisations (or whatever notion of invariance is present in the problem).

Solitons which just move under diffeomorphisms are called steady, those which rescale getting smaller are called shrinking, and those which rescale getting larger are called expanding.

That is usually all of them (like in Ricci flow), but in mean curvature flow a soliton can also just translate, which is, rather unimaginatively, called a translating soliton.

Simple examples of solitons in Ricci flow are given by constant curvature metrics: flat metrics are steady (in fact critical points), constant positive curvature metrics are shrinking (like standard round spheres) and constant negative curvature metrics are expanding (like hyperbolic space).

From this point of view we see that shrinking solitons are the ones we should be most concerned with, since they will become singular in finite time, shrinking away. However, steady and expanding solitons also play an important role.

Steady solitons are potentially where the flow can get "stuck" going round and round under diffeomorphisms and never converging. Non-stationary compact examples of steady solitons are ruled out if you have a standard gradient flow interpretation of the flow, since the corresponding functional would be constant for steady solitons, which is a contradiction unless they are stationary. This shows one of the benefits of knowing that your geometric flow is the gradient flow of some functional.

On the other hand, expanding solitons give a potential mechanism to escape from a singularity, since they expand away from a singular geometric object.

It is therefore clear that understanding singularities and solitons is an important part of the study of any geometric flow.

3   G₂ Structures

Given this discussion of geometric flows, we are now motivated to ask the question: are there (useful) geometric flows of G₂ structures on a (compact) 7-manifold M and what do they want to achieve? We should perhaps not expect there to be just one useful flow to consider: for immersions, both mean curvature flow and inverse mean curvature flow have important geometric uses, for example. We therefore need to think about what are the important classes of G₂ structures that we want to analyse and what we expect to be "canonical" representatives for these classes. (For details about G₂ structures, which are equivalent to positive 3-forms, see for example [21].)

3.1   *Torsion-Free and Torsion Forms*

Clearly the most important class of G₂ structures are the torsion-free ones, given by positive 3-forms φ on M satisfying

$$\nabla_\varphi \varphi = 0 \quad \Leftrightarrow \quad \mathrm{d}\varphi = \mathrm{d}_\varphi^* \varphi = 0 \quad \Leftrightarrow \quad \mathrm{Hol}(g_\varphi) \subseteq \mathrm{G}_2 . \tag{3.1}$$

(We are being slightly sloppy in the last equivalence, since given a metric there are infinitely many G₂ structures inducing that metric, so we mean that the holonomy $\mathrm{Hol}(g_\varphi)$ of g_φ is contained in G₂ if and only if there is some G₂ structure φ inducing g_φ which is closed and coclosed.) We also know that we may equivalently define G₂ structures on oriented, spin, Riemannian 7-manifolds using unit spinors σ, and the

condition for the G_2 structure to be torsion-free is that σ is parallel with respect to the spin connection :

$$\nabla \sigma = 0. \tag{3.2}$$

(We can recall the relationship between unit spinors and positive 3-forms:

$$4\sigma \otimes \sigma = 1 + \varphi + *_\varphi \varphi + \mathrm{vol}_\varphi, \tag{3.3}$$

up to appropriate normalizations and sign conventions.)

Closed and coclosed G_2 structures. However, there are other obvious (potentially) important classes of G_2 structures, for example closed G_2 structures

$$\mathrm{d}\varphi = 0, \tag{3.4}$$

or coclosed G_2 structures

$$\mathrm{d}_\varphi^* \varphi = 0. \tag{3.5}$$

It is worth noting that, on the face of it, (3.4) is much stronger than (3.5): the first is a condition on a 4-form in 7-dimensions (so 35 equations at each point), whereas the second is a condition on a 5-form in 7-dimensions (so 21 equations at each point).

Both conditions (3.4) and (3.5) can be satisfied independently on any open 7-manifold admitting a G_2 structure by a straightforward h-principle argument; thus one can say (in some sense) that these conditions are only truly meaningful on compact 7-manifolds. In fact, (3.5) can always be satisfied on a compact 7-manifold admitting a G_2 structure also by an h-principle [7], but it is currently unknown whether the same is true for condition (3.4) or not: this again reflects the fact that (3.4) is a stronger condition than (3.5).

Theorem 3.1 *Let φ be a G_2 structure on M.*

(a) If M is open, then there exists a G_2 structure $\tilde{\varphi}$ C^0-close to φ satisfying (3.4).
(b) If M is either open or compact, then there exists a G_2 structure $\tilde{\varphi}$ C^0-close to φ satisfying (3.5).

One can interpret the h-principle result for coclosed G_2 structures as both positive and negative. On the one hand, it is good because we can always assume the condition (3.5) holds for φ if we want, though we have very little control on the φ produced by the h-principle: we can assume it is C^0-close to our original G_2 structure but the method only produces φ which will be very far away in the C^1-topology. On the other hand, it says that the condition (3.5) is, in a sense, meaningless and that talking about coclosed G_2 structures is the same as talking about all G_2 structures, which becomes a topological rather than a geometric question.

Torsion forms. One could also conceivably look at other special torsion classes by setting various combinations of the intrinsic torsion forms to vanish, recalling that these are given by

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3 \quad \text{and} \quad d *_\varphi \varphi = 4\tau_1 \wedge *_\varphi \varphi + \tau_2 \wedge \varphi, \qquad (3.6)$$

where $\tau_0 \in C^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$, $\tau_3 \in \Omega^3_{27}(M)$, with the standard notation referring to the type decomposition of forms determined by φ (see, for example, [5]). Recall that $\beta \in \Omega^2_{14}(M)$ if and only if

$$\beta \wedge \varphi = - *_\varphi \beta \quad \Leftrightarrow \quad \beta \wedge *_\varphi \varphi = 0, \qquad (3.7)$$

and that $\gamma \in \Omega^3_{27}(M)$ if and only if

$$\gamma \wedge \varphi = \gamma \wedge *_\varphi \varphi = 0. \qquad (3.8)$$

Moreover, recall that we have an isomorphism $i_\varphi : S^2 T^* M = \text{Span}\{g_\varphi\} \oplus S^2_0 T^* M \to \Lambda^3_1 \oplus \Lambda^3_{27}$ given on decomposable elements $\alpha \circ \beta$ by

$$i_\varphi(\alpha \circ \beta) = \alpha \wedge *_\varphi(\beta \wedge *_\varphi \varphi) + \beta \wedge *_\varphi(\alpha \wedge *_\varphi \varphi).$$

We also have an explicit way to invert i_φ using $j_\varphi : \Lambda^3 T^* M \to S^2 T^* M$ given by

$$j_\varphi(\gamma)(u, v) = *_\varphi(u \lrcorner \varphi \wedge v \lrcorner \varphi \wedge \gamma).$$

Notice that $i_\varphi(g_\varphi) = 6\varphi$, $j_\varphi(\varphi) = 6g_\varphi$ and $\text{Ker } j_\varphi = \Lambda^3_7$.

Other classes. A particular class of G$_2$ structures one could consider are the nearly parallel G$_2$ structures

$$d\varphi = \tau_0 *_\varphi \varphi \qquad (3.9)$$

for a constant τ_0. These structures define Einstein metrics with non-negative scalar curvature, and so there is a potential relation between these structures and our discussion of the Ricci flow above.

One could also view matters in terms of spinors, and study geometric flows of unit spinors. One can then try studying parallel spinors or, more generally, Killing spinors, as well as other special types of spinors (e.g. twistor spinors).

3.2   General Flows

Based on this discussion, it is clear that there are many possible geometric flows one could write down, and each one could potentially tackle different open problems in G$_2$ geometry.

That said, one can describe how various key quantities vary under a general flow of G$_2$ structures (see [5, 22]). By the type decomposition of 3-forms, any geometric flow of G$_2$ structures can be written

$$\frac{\partial}{\partial t}\varphi = 3f_0\varphi + *_\varphi(f_1 \wedge \varphi) + f_3 = i_\varphi(h) + X \lrcorner *_\varphi \varphi \qquad (3.10)$$

where $f_0 \in C^\infty(M)$, $f_1 \in \Omega^1(M)$ and $f_3 \in \Omega^3_{27}(M)$, $h \in C^\infty(S^2T^*M)$ and $X \in C^\infty(TM)$ at each time t. From this one can see that the metric and 4-form evolve as follows.

Proposition 3.2 *Along* (3.10) *we have*

$$\frac{\partial}{\partial t}g_\varphi = 2f_0 g + \frac{1}{2}j_\varphi(f_3) = 2h \qquad (3.11)$$

and

$$\frac{\partial}{\partial t} *_\varphi \varphi = 4f_0 *\varphi + f_1 \wedge \varphi - *_\varphi f_3. \qquad (3.12)$$

In particular, along any flow (3.10), the evolution of the metric is independent of the vector field X, and the volume form evolves as follows.

Proposition 3.3 *Along* (3.10) *we have*

$$\frac{\partial}{\partial t} \mathrm{vol}_\varphi = 7f_0 \mathrm{vol}_\varphi = \frac{1}{3}\frac{\partial}{\partial t}\varphi \wedge *_\varphi\varphi. \qquad (3.13)$$

This formula is useful to study Hitchin's volume functional [17] on a compact manifold M. (Note that the published version [18] of [17] omits the material on G_2 structures.)

Proposition 3.4 *Along* (3.10)*, the volume functional*

$$\mathrm{Vol}(\varphi) = \frac{1}{7}\int_M \varphi \wedge *\varphi = \int_M \mathrm{vol}_\varphi = \mathrm{Vol}(M, g_\varphi). \qquad (3.14)$$

satisfies (by (3.13)*)*

$$\frac{\partial}{\partial t} \mathrm{Vol}(\varphi) = 7\int_M f_0 \mathrm{vol}_\varphi = \frac{1}{3}\langle \frac{\partial}{\partial t}\varphi, \varphi \rangle_{L^2}. \qquad (3.15)$$

This shows in particular that $\mathrm{Vol}(\varphi)$ will be monotone along any flow for which f_0 has a sign, and the critical points of the functional will be characterised by the G_2 structures for which $f_0 = 0$. It also shows that the obvious gradient flow for $\mathrm{Vol}(\varphi)$ is

$$\frac{\partial}{\partial t}\varphi = \lambda\varphi \qquad (3.16)$$

for some $\lambda > 0$ (or $\lambda < 0$ for the negative gradient flow). This is clearly useless since all it does is rescale φ! Therefore, if one wants to think about making use of the volume functional for a gradient flow, we should consider restricting the class of G_2 structures we work with.

4 Laplacian Flow

The geometric flow of G_2 structures that has received the most attention is the Laplacian flow due to Bryant [5].

Definition 4.1 The Laplacian flow is given by

$$\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi, \tag{4.1}$$

where

$$\Delta_\varphi = dd_\varphi^* + d_\varphi^* d \tag{4.2}$$

is the Hodge Laplacian.

On a compact manifold, we see that

$$\Delta_\varphi \varphi = 0 \quad \Leftrightarrow \quad d\varphi = d_\varphi^* \varphi = 0 \tag{4.3}$$

by integration by parts, and so torsion-free G_2 structures will be the critical points of (4.1).

4.1 Closed G_2 Structures

Bryant's suggestion is to restrict (4.1) to closed G_2 structures φ as in (3.4). A key motivation is the usual one in G_2 geometry: namely that the torsion-free condition naturally splits into a linear condition (3.4) and a nonlinear condition (3.5). Thus, it is useful to assume the linear condition is satisfied then try to solve the nonlinear one. This strategy is the only one that has proved to be successful, by the work of Joyce [21, Chap. 11]. Hence, it clearly makes sense to follow the same approach in a geometric flow.

It will turn out that when (4.1) exists the closed condition is preserved. In that case

$$\frac{\partial}{\partial t}\varphi = dd_\varphi^* \varphi, \tag{4.4}$$

so in fact (4.1) stays within a fixed cohomology class $[\varphi(0)]$.

Proposition 4.2 *If $\varphi(t)$ satisfies (4.1) on M and $d\varphi(t) = 0$ then $\varphi(t) \in [\varphi(0)] \in H^3(M)$ for all t for which the flow exists.*

This is perhaps reminiscent of the Kähler–Ricci flow on a manifold with $c_1 = 0$, which starts with a Kähler form and stays within the Kähler class, but it is not clear at all whether such an analogy is pertinent or a red herring.

When we restrict to closed G_2 structures we can decompose

$$\Delta_\varphi \varphi = \frac{1}{7}|\tau_2|^2 \varphi + f_3. \tag{4.5}$$

We can see this because

$$\mathrm{d} *_\varphi \varphi = \tau_2 \wedge \varphi = - *_\varphi \tau_2 \tag{4.6}$$

by (3.6) and (3.7), and therefore

$$\Delta_\varphi \varphi = \mathrm{dd}_\varphi^* \varphi = \mathrm{d}\tau_2 \tag{4.7}$$

and differentiating (4.6) gives

$$0 = \mathrm{d}\tau_2 \wedge \varphi = \Delta_\varphi \varphi \wedge \varphi = 0. \tag{4.8}$$

This means that $f_1 = 0$ in (3.10). Moreover, (4.6) and (3.7) imply that

$$\mathrm{d}\tau_2 \wedge *_\varphi \varphi = \mathrm{d}(\tau_2 \wedge *_\varphi \varphi) - \tau_2 \wedge \mathrm{d} *_\varphi \varphi = -\tau_2 \wedge \tau_2 \wedge \varphi = |\tau_2|^2 \, \mathrm{vol}_\varphi. \tag{4.9}$$

Hence, for closed G_2 structures, (4.5) implies the following.

Lemma 4.3 *For a closed* G_2 *structure* φ, *we have*

$$\Delta_\varphi \varphi = 0 \quad \Leftrightarrow \quad \mathrm{d}_\varphi^* \varphi = 0. \tag{4.10}$$

Therefore, the critical points of (4.1) on closed G_2 structures are precisely the torsion-free G_2 structures, without assuming compactness. This might seem like a minor point (since we will mainly only care about the Laplacian flow on compact manifolds) but it appears to hint at the special character of the Laplacian flow because it is restricted to closed G_2 structures.

Moreover, we can show that

$$\Delta_\varphi \varphi = i_\varphi \Big(-\mathrm{Ric}(g_\varphi) + \frac{4}{21}|\tau_2|^2 g_\varphi + \frac{1}{8} j_\varphi \big(*_\varphi (\tau_2 \wedge \tau_2) \big) \Big) \tag{4.11}$$

so that by (3.11) we have the following.

Proposition 4.4 *Along* (4.1) *we have*

$$\frac{\partial}{\partial t} g_\varphi = -2\,\mathrm{Ric}(g_\varphi) + \frac{8}{21}|\tau_2|^2 g_\varphi + \frac{1}{4} j_\varphi \big(*_\varphi (\tau_2 \wedge \tau_2) \big). \tag{4.12}$$

4.2 Volume Functional

Lemma 4.5 *Along* (4.1) *for closed* G_2 *structures, the volume functional* $\mathrm{Vol}(\varphi)$ *is monotonically increasing.*

Proof We can already see from (4.5) that, in terms of (3.10),

$$f_0 = \frac{1}{7}|\tau_2|^2 \geq 0. \tag{4.13}$$

Hence, by (3.15), the volume functional $\text{Vol}(\varphi)$ is monotonically increasing along the Laplacian flow. □

This allows us to give a very quick alternative proof of the main result of [27].

Proposition 4.6 *Let $\varphi(t)$ be a steady or shrinking soliton to the Laplacian flow* (4.1) *for closed* G_2 *structures on a compact manifold. Then $\varphi(t)$ is stationary, i.e. $\varphi(t) = \varphi(0)$ is torsion-free for all t.*

Proof If $\varphi(t)$ is steady or shrinking then

$$\frac{\partial}{\partial t}\text{Vol}(\varphi(t)) \leq 0. \tag{4.14}$$

Therefore, by Lemma 4.5, $\text{Vol}(\varphi(t))$ is constant. Hence, by (3.15) and (4.13), we must have that

$$f_0 = \frac{1}{7}|\tau_2|^2 = 0, \tag{4.15}$$

which means $\tau_2 = 0$, as required. □

It is important to note that this result fails in the non-compact setting: Lauret [26] has constructed examples of non-compact shrinking and steady solitons which are not stationary. The proof of Proposition 4.6 is not valid here since the volume is not well-defined for these non-compact examples.

We now show that the Laplacian flow for closed G_2 structures is actually the gradient flow for $\text{Vol}(\varphi)$ where, from now on, we only consider the volume functional restricted to a given cohomology class.

Proposition 4.7 *The Laplacian flow* (4.1) *for closed* G_2 *structures φ is the gradient flow for the volume functional $\text{Vol}(\varphi)$ in* (3.14) *restricted to* $[\varphi]$.

Proof We know that the flow stays within a given cohomology class, so we can write

$$\varphi(t) = \varphi(0) + d\eta(t) \tag{4.16}$$

for some 2-forms η and the Laplacian flow is really, in some sense, a flow on 2-forms. (Again, this is reminiscent of Kähler–Ricci flow with $c_1 = 0$ as the flow becomes a flow on Kähler potentials, but this analogy is made with the usual caveats.) Then (3.15) gives us that

$$\frac{\partial}{\partial t}\text{Vol}(\varphi) = \frac{1}{3}\langle\frac{\partial}{\partial t}\varphi, \varphi\rangle_{L^2} = \frac{1}{3}\langle d\frac{\partial}{\partial t}\eta, \varphi\rangle_{L^2} = \frac{1}{3}\langle\frac{\partial}{\partial t}\eta, d_\varphi^*\varphi\rangle_{L^2}. \tag{4.17}$$

Hence, the gradient flow for $\mathrm{Vol}(\varphi)$ is

$$\frac{\partial}{\partial t}\eta = \mathrm{d}_\varphi^* \varphi \quad \Rightarrow \quad \frac{\partial}{\partial t}\varphi = \mathrm{d}\frac{\partial}{\partial t}\eta = \mathrm{d}\mathrm{d}_\varphi^*\varphi = \Delta_\varphi\varphi, \tag{4.18}$$

the Laplacian flow. (We have ignored the factor $\frac{1}{3}$ which amounts to rescaling time t.) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.7 immediately yields the following.

Proposition 4.8 *A closed* G_2 *structure on a compact manifold is a critical point of* $\mathrm{Vol}(\varphi)$ *within a fixed cohomology class if and only if* φ *is torsion-free.*

We can even say more about the critical points of the volume functional. If we look at the second derivative at a critical point $\varphi(0)$, then by (3.15) along any variation $\varphi(s) = \varphi(0) + \mathrm{d}\eta(s)$ in the cohomology class

$$\frac{\partial^2}{\partial s^2}\mathrm{Vol}(\varphi)|_{s=0} = \frac{1}{3}\int_M \frac{\partial}{\partial s}\varphi \wedge \frac{\partial}{\partial s} *_\varphi \varphi|_{s=0}, \tag{4.19}$$

since

$$\frac{1}{3}\int_M \frac{\partial^2}{\partial s^2}\varphi \wedge *_\varphi\varphi|_{s=0} = \frac{1}{3}\langle \mathrm{d}\frac{\partial^2}{\partial s^2}\eta, *_\varphi\varphi\rangle_{L^2}|_{s=0} = \frac{1}{3}\langle\frac{\partial^2}{\partial s^2}\eta, \mathrm{d}_\varphi^*\varphi\rangle_{L^2}|_{s=0} = 0 \tag{4.20}$$

as $\mathrm{d}_\varphi^*\varphi = 0$ at $s = 0$ by assumption. Now if we write the variation of $\varphi(s)$ at $s = 0$ as in formula (3.10) then $*_\varphi\varphi(s)$ varies by (3.12) and so we see that

$$\int_M \frac{\partial}{\partial s}\varphi \wedge \frac{\partial}{\partial s} *_\varphi\varphi|_{s=0} = c_0\|f_0\|_{L^2}^2 + c_1\|f_1\|_{L^2}^2 - \|f_3\|_{L^2}^2 \tag{4.21}$$

for some positive constants c_0, c_1. If the variation $\varphi(s)$ is orthogonal to the action by diffeomorphisms (meaning that $\frac{\partial\varphi}{\partial s}|_{s=0}$ and the tangent directions to the diffeomorphism orbit through $\varphi(0)$ are orthogonal), then a slice theorem argument forces $f_0 = f_1 = 0$ (see [17]). In other words, the slice condition

$$\mathrm{d}_\varphi^*\mathrm{d}\eta \in \Omega_{14}^2(M) \quad \Rightarrow \quad \mathrm{d}\eta \in \Omega_{27}^3(M), \tag{4.22}$$

so $f_3 = \mathrm{d}\eta(0)$. Putting this observation together with (4.19) and (4.21), we see that, orthogonal to the action by diffeomorphisms, we have

$$\frac{\partial^2}{\partial s^2}\mathrm{Vol}(\varphi)|_{s=0} = -\|\mathrm{d}\eta(0)\|_{L^2}^2 \leq 0. \tag{4.23}$$

Thus, we have the following.

Theorem 4.9 *Critical points of* $\mathrm{Vol}(\varphi)$ *on* $[\varphi]$ *are strict local maxima (modulo the action of diffeomorphisms).*

This suggests that the gradient flow of the volume functional (i.e. (4.1)) could be well-behaved since its only critical points are maxima.

4.3 Short-Time Existence

A key issue we have avoided in our discussion thus far is the question of whether the Laplacian flow exists or not. Certainly, if we look at (4.1) and compare it to (2.1) we would seem to have the wrong sign! In general, (4.1) does not seem to be parabolic in any sense, which is very bad news analytically.

However, again the fact that we are restricting to closed G$_2$ structures comes to our rescue. In this case, we have already seen in (4.12) that the metric evolves by Ricci flow plus lower order terms and so its flow is parabolic modulo diffeomorphisms.

If we do DeTurck's trick for the Laplacian flow, using d$\varphi = 0$:

$$\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi + \mathcal{L}_{X(\varphi)}\varphi = \Delta_\varphi \varphi + \mathrm{d}(X(\varphi)\lrcorner\varphi), \qquad (4.24)$$

(as in the Ricci flow case, and with the same vector field X given in (2.21), in fact) then we might hope that we end up with a genuine parabolic equation in (4.24). However, this is not the case!

In fact, (4.24) is only parabolic in the direction of closed forms, so one has to consider the restricted flow in order to prove short-time existence. This is a little bit tricky but was done by Bryant–Xu [6]. Their paper, which definitely gives a correct result that is fundamental to the subject, has never been published, so we give an account of the proof here, which is essentially the same as in [6].

Theorem 4.10 *Let φ_0 be a smooth closed* G$_2$ *structure on a compact manifold M. There exists $\epsilon > 0$ so that a unique solution $\varphi(t)$ to the Laplacian flow* (4.1) *with $\varphi(0) = \varphi_0$ and d$\varphi(t) = 0$ exists for all $t \in [0, \epsilon]$, where ϵ depends on φ_0.*

Proof We know that if (4.1) exists then it will stay in the cohomology class $[\varphi_0]$ by Proposition 4.2. Therefore, we could write

$$\varphi(t) = \varphi_0 + \mathrm{d}\eta(t) \qquad (4.25)$$

for a family of exact 3-forms d$\eta(t)$, and (4.1) with the initial condition $\varphi(t) = \varphi_0$ is equivalent to

$$\frac{\partial}{\partial t}\mathrm{d}\eta = \Delta_{\varphi_0+\mathrm{d}\eta}\mathrm{d}\eta \quad \text{and} \quad \mathrm{d}\eta(0) = 0. \qquad (4.26)$$

Let $X(\varphi)$ be the vector field given by $X(g_\varphi)$ in (2.21), where we can choose the fixed background metric $k = g_{\varphi_0}$ for example. Suppose we can solve

$$\frac{\partial}{\partial t}\mathrm{d}\eta(t) = \Delta_{\varphi_0+\mathrm{d}\eta}\mathrm{d}\eta + \mathrm{d}(X(\varphi_0 + \mathrm{d}\eta)\lrcorner\mathrm{d}\eta) \quad \text{and} \quad \mathrm{d}\eta(0) = 0 \qquad (4.27)$$

uniquely for short time. Then we can find a unique family of diffeomorphisms Φ solving (2.17) (since this is the harmonic map heat flow). Therefore, just as in Proposition 2.6, it follows that $\varphi = \Phi^*(\varphi_0 + \mathrm{d}\eta)$ satisfies (4.1) with $\varphi(0) = \varphi_0$, $\mathrm{d}\varphi(t) = 0$ for short time, and the solution is unique. (We have used here that the condition for a 3-form to be positive is open, so for t sufficiently small, $\varphi_0 + \mathrm{d}\eta(t)$ will be a positive 3-form.) We are therefore left to show that (4.27) has a unique short-time solution.

We know from (4.11) and [28, Lemma 9.3] that

$$
\Delta_\varphi \varphi + \mathcal{L}_{X(\varphi)}\varphi = \frac{1}{2} i_\varphi \Big(- 2\operatorname{Ric}(g_\varphi) + \mathcal{L}_{X(g_\varphi)}g_\varphi + \frac{2}{21}|\tau_2|^2 g_\varphi + \frac{1}{4} j_\varphi (*_\varphi(\tau_2 \wedge \tau_2)) \Big)
$$
$$
+ \frac{1}{2} \big(\mathrm{d}^*(X(\varphi)\lrcorner\varphi) \big)^\sharp \lrcorner *_\varphi \varphi. \tag{4.28}
$$

Given that the terms with τ_2 in them in (4.28) are lower order, and the Ricci–DeTurck flow (2.16) is parabolic, it would seem likely that the linearisation of (4.28) is parabolic when restricted to closed forms, and hence that (4.27) is parabolic. In fact, one may explicitly compute as in [6] that the linearisation of (4.27) in the direction of exact forms is

$$
\frac{\partial}{\partial t}\mathrm{d}\zeta = -\Delta_{\varphi_0 + \mathrm{d}\eta}\mathrm{d}\zeta + \mathrm{d}\big(Q_{\varphi_0 + \mathrm{d}\eta}(\mathrm{d}\zeta) \big), \tag{4.29}
$$

where $Q_{\varphi_0 + \mathrm{d}\eta}(\mathrm{d}\zeta)$ is order zero in $\mathrm{d}\zeta$ (meaning it depends on just $\mathrm{d}\zeta$ and not its derivatives). Somewhat surprisingly, the sign has switched and (4.29) is manifestly parabolic.

However, we have only shown that (4.27) is parabolic in the direction of exact forms, so we cannot apply standard parabolic theory. Instead we will invoke the Nash–Moser Inverse Function Theorem (see [16] for a detailed discussion of this theorem).

We start by setting up the notation. We let

$$
\mathcal{X} = \mathrm{d}\big(C^\infty([0, \epsilon] \times M, \Lambda^2 T^*M) \big) \quad \text{and} \quad \mathcal{Y} = \mathrm{d}\Omega^2(M), \tag{4.30}
$$

and

$$
\mathcal{U} = \{ \mathrm{d}\eta \in \mathcal{X} \ : \ \varphi_0 + \mathrm{d}\eta(t) \text{ is a } G_2 \text{ structure for all } t \}, \tag{4.31}
$$

which is an open set in $\mathcal{X}$ containing 0. We define $\mathcal{F} : \mathcal{U} \to \mathcal{X} \times \mathcal{Y}$ by

$$
\mathcal{F}(\mathrm{d}\eta) = \left(\frac{\partial}{\partial t}\mathrm{d}\eta(t) - \Delta_{\varphi_0 + \mathrm{d}\eta}\mathrm{d}\eta - \mathrm{d}(X(\varphi_0 + \mathrm{d}\eta)\lrcorner\mathrm{d}\eta), \mathrm{d}\eta(0) \right), \tag{4.32}
$$

so that $\mathcal{F}(\mathrm{d}\eta) = (0, 0)$ if and only if $\mathrm{d}\eta$ solves (4.27). If we can show that $\mathcal{F}$ is locally invertible near 0, we have that (4.27) has a unique short-time solution and so the proof is complete.

By (4.29), the linearisation of $\mathcal{F}$ at $\mathrm{d}\eta \in \mathcal{U}$ is given by

$$\mathrm{d}\mathcal{F}|_{\mathrm{d}\eta}(\mathrm{d}\zeta) = \left(\frac{\partial}{\partial t}\mathrm{d}\zeta + \Delta_{\varphi_0 + \mathrm{d}\eta}\mathrm{d}\zeta - \mathrm{d}\big(Q_{\varphi_0 + \mathrm{d}\eta}(\mathrm{d}\zeta)\big), \mathrm{d}\zeta(0) \right). \tag{4.33}$$

Since (4.29) is parabolic, we see that if $\mathrm{d}\mathcal{F}|_{\mathrm{d}\eta}(\mathrm{d}\zeta) = (0,0)$ then $\mathrm{d}\zeta = 0$ as this is the unique solution to the linear parabolic equation (4.29) with zero initial condition. Further, given $(\mathrm{d}\xi, \mathrm{d}\xi_0) \in \mathcal{X} \times \mathcal{Y}$, we see that $\mathrm{d}\mathcal{F}|_{\mathrm{d}\eta}(\mathrm{d}\zeta) = (\mathrm{d}\xi, \mathrm{d}\xi_0)$ if and only if

$$\frac{\partial}{\partial t}\mathrm{d}\zeta = -\Delta_{\varphi_0 + \mathrm{d}\eta}\mathrm{d}\zeta + \mathrm{d}\big(Q_{\varphi_0 + \mathrm{d}\eta}(\mathrm{d}\zeta)\big) + \mathrm{d}\xi \quad \text{and} \quad \mathrm{d}\zeta(0) = \mathrm{d}\xi_0, \tag{4.34}$$

which can be solved by linear parabolic theory.

We therefore have that $\mathrm{d}\mathcal{F}|_{\mathrm{d}\eta} : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ is invertible for all $\mathrm{d}\eta \in \mathcal{U}$. However, this is not yet enough to show that $\mathcal{F}$ is locally invertible. Notice that the family of inverses provides a map $\mathcal{G} : \mathcal{U} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ given by

$$\mathcal{G}(\mathrm{d}\eta, \mathrm{d}\xi, \mathrm{d}\xi_0) = \mathrm{d}\mathcal{F}|_{\mathrm{d}\eta}^{-1}(\mathrm{d}\xi, \mathrm{d}\xi_0). \tag{4.35}$$

To deduce that $\mathcal{F}$ is locally invertible, as we said, we want to invoke the Nash–Moser Inverse Function Theorem which means that we need the following:

- $\mathcal{X}$ and $\mathcal{Y}$ are tame Fréchet spaces and $\mathcal{F}$ is a smooth tame map;
- $\mathrm{d}\mathcal{F}|_{\mathrm{d}\eta}$ is invertible for all $\mathrm{d}\eta \in \mathcal{U}$ and $\mathcal{G}$ is a smooth tame map.

The fact that $C^\infty([0,\epsilon] \times M, \Lambda^2 T^*M)$ and $\Omega^2(M)$ are naturally tame Fréchet spaces is standard, and it therefore quickly follows (from Hodge theory) that $\mathcal{X}$ and $\mathcal{Y}$ are also tame Fréchet spaces. Smooth partial differential operators are smooth tame maps, so $\mathcal{F}$ is a smooth tame map.

The last thing we need to show is that $\mathcal{G}$ is a smooth tame map, but this follows immediately from a general result about the family of inverses given by solutions of a smooth family of parabolic partial differential equations, due to Hamilton [15].

Thus, the Nash–Moser Inverse Function Theorem applies to $\mathcal{F}$ and so it is locally invertible as desired. □

Actually, before DeTurck's trick the same method of Bryant–Xu was used by Hamilton [15] to prove short-time existence of the Ricci flow. The reason is that the Ricci flow is not a flow amongst all symmetric 2-tensors really, since the Ricci tensor always satisfies the contracted Bianchi identity. Therefore, one could consider the flow restricted to those which satisfy this identity. To prove rigorously that one can do this, one must employ the Nash–Moser Inverse Function Theorem, as Hamilton did. This is not needed as we have said for the Ricci flow, since DeTurck's trick already removes the issue caused by the Bianchi identity there, but it is needed currently for the Laplacian flow.

4.4 Results and Questions

Results. There are several important areas in the Laplacian flow for closed G_2 structures where progress has been made.

- Long-time existence criteria based on curvature and torsion estimates along the flow, uniqueness and compactness theory, and real analyticity of the flow [28, 30].
- Stability of the critical points [29].
- Non-collapsing under assumption of bounded torsion [8].
- Explicit study of the flow in homogeneous situations and other symmetric cases such as nilmanifolds and warped products with a circle [9, 11, 25, 26, 31, 33].
- Examples and non-existence results for solitons [25–28].
- Eternal solutions for the flow arising from extremally Ricci pinched G_2 structures [12].
- Reduction of the flow to 4 dimensions, with improved long-time existence criteria [10] and analysis of the 4-torus case [19].
- Reduction of the flow to 3 dimensions, with striking long-time existence and convergence results [24].

It is worth remarking that the scalar curvature here is given by

$$R(g_\varphi) = -\frac{1}{2}|\tau_2|^2, \tag{4.36}$$

so having a bound on torsion is equivalent to a bound on scalar curvature.

Questions. There are many open problems in the area.

- Does the flow exist as long as the torsion is bounded?
- Can a volume bound be used to control the flow?
- Are there any compact examples which develop a singularity in finite time?
- Is there a relationship between the flow and calibrated submanifolds, specifically coassociative submanifolds?

For the last two points, there is an example due to Bryant [5] which shows that singularities can happen at infinite time (i.e. the flow exists for all time but does not converge), and that the singularity is related to coassociative geometry.

We can also ask whether the Laplacian flow is potentially useful to study other classes of G_2 structures. For example, naively if we assume φ is coclosed and the flow exists then it should stay coclosed since then

$$\Delta_\varphi \varphi = d_\varphi^* d\varphi. \tag{4.37}$$

So, an obvious question is: does this flow exist? Gavin Ball has informed the author that, in general, the Laplacian flow will *not* preserve the coclosed condition, so the answer would appear to be negative.

5 Laplacian Coflow

Another approach to studying G$_2$ structures was introduced in [23].

Definition 5.1 The Laplacian coflow for G$_2$ structures is given by:

$$\frac{\partial}{\partial t} *_\varphi \varphi = \Delta_{*_\varphi \varphi} *_\varphi \varphi, \qquad (5.1)$$

where $\Delta_{*_\varphi \varphi}$ is the Hodge Laplacian of the metric determined by $*_\varphi \varphi$. (Actually, in [23], they introduced (5.1) with a minus sign on the right-hand side by analogy with the heat equation (2.1), but as we shall see below the "correct" sign in the equation is that given, just as in (4.1).)

Here one has to be a little careful since the 4-form $*_\varphi \varphi$ is not quite equivalent to the 3-form φ. In particular, the 4-form does not determine the orientation, but we can assume we have an initial orientation which stays fixed along the flow.

Again by integration by parts it is easy to see that on a compact manifold

$$\Delta_{*_\varphi \varphi} *_\varphi \varphi = 0 \quad \Leftrightarrow \quad \mathrm{d}\varphi = \mathrm{d}_\varphi^* \varphi = 0, \qquad (5.2)$$

so the critical points are again the torsion-free G$_2$ structures.

5.1 Coclosed G$_2$ Structures

The proposal in [23] is to restrict (5.1) to closed 4-forms (so coclosed G$_2$ structures). If the flow exists, meaning it preserves closed forms as in the Laplacian flow setting, we would have that:

$$\frac{\partial}{\partial t} *_\varphi \varphi = \mathrm{d}\mathrm{d}_\varphi^* *_\varphi \varphi. \qquad (5.3)$$

Thus, again, the flow will stay in the given cohomology class $[*_\varphi \varphi(0)]$ as long as it exists. Therefore, the Laplacian coflow can be seen as a possible means to deform $*_\varphi \varphi$ in its cohomology class so that it becomes torsion-free.

Proposition 5.2 *If $*_\varphi \varphi(t)$ satisfies (5.1) on M and $\mathrm{d}*_\varphi \varphi(t) = 0$ then $*_\varphi \varphi \in [*_\varphi \varphi(0)]$ $\in H^4(M)$ for all t for which the flow exists.*

We can easily modify our discussion of the volume functional Vol(φ), given in (3.14), and the Laplacian flow in Proposition 4.7 and Theorem 4.9 to show the following.

Theorem 5.3 *The flow (5.1) for coclosed G$_2$ structures $*_\varphi \varphi$ is the gradient flow of the volume functional in (3.14) restricted to $[*_\varphi \varphi]$ and the critical points are strict local maxima for the volume functional (modulo diffeomorphisms).*

At this point, things are looking quite good in the study of the Laplacian coflow.

However, it is now worth going back to the earlier discussion of the coclosed condition (3.5). Taking a positive interpretation of the h-principle result (Theorem 3.1), we can always assume that our G_2 structure is coclosed and therefore (5.1) potentially allows us to study the space of all G_2 structures whilst restricting to the coclosed ones. A more negative outlook is to say that studying the Laplacian coflow means that we are effectively studying all G_2 structures, which only has topological rather than geometric content, and so perhaps it is unrealistic to expect the flow to be well-behaved in such generality.

5.2   Short-Time Existence: A Modification

Given this discussion it is worth confronting the key issue of whether the flow (5.1) even exists. To tackle this, we would employ the DeTurck's trick and hope to show that (5.1) is parabolic, possibly only in the direction of closed forms, just as in the case of (4.1) as in Theorem 4.10. Unfortunately, this does not work! Therefore, we cannot currently say (regardless of which sign we choose in (5.1)) that the Laplacian coflow even exists. Nonetheless, one can find solutions to it in special cases [23], so we can continue to ask the question: does (5.1) exist restricted to coclosed G_2 structures?

An approach taken in [13] is to modify (5.1) to get a family of flows depending on an arbitrary constant c.

Definition 5.4 The modified Laplacian coflow(s) (recalling the torsion forms in (3.6)) for $c \in \mathbb{R}$ is defined by

$$\frac{\partial}{\partial t} *_\varphi \varphi = \Delta_{*_\varphi \varphi} *_\varphi \varphi + \mathrm{d}\left(\left(c - \frac{7}{2}\tau_0\right)\varphi\right), \tag{5.4}$$

again restricted to coclosed G_2 structures.

This again moves in the cohomology class and has the added benefit of defining a flow which is parabolic in the direction of closed forms (modulo diffeomorphisms), and therefore (5.4) is guaranteed to exist on a compact 7-manifold by essentially the same proof as Theorem 4.10.

Theorem 5.5 *Let φ_0 be a smooth coclosed G_2 structure on a compact manifold M. There exists $\epsilon > 0$ so that a unique solution $*_\varphi \varphi(t)$ to the modified Laplacian coflow (5.4) with $*_\varphi \varphi(0) = *_{\varphi_0} \varphi_0$ and $\mathrm{d} *_\varphi \varphi(t) = 0$ exists for all $t \in [0, \epsilon]$.*

Critical points. The short-time existence is of course important, but (5.4) is no longer obviously a gradient flow and there is also no reason why the critical points of the flow are torsion-free G_2 structures. It is clear that if φ is torsion-free then the right-hand side of (5.4) vanishes (since $\tau_0 = 0$ and $\mathrm{d}\varphi = 0$). However, suppose we

choose a nearly parallel G$_2$ structure φ as in (3.9), recalling that τ_0 is constant in that case. Then we see that

$$\Delta_{*_\varphi\varphi} *_\varphi \varphi = dd_\varphi^* *_\varphi \varphi = d*_\varphi d\varphi = d*_\varphi \tau_0 *_\varphi \varphi = \tau_0 d\varphi = \tau_0^2 *_\varphi \varphi \qquad (5.5)$$

and

$$d\left(\left(c - \frac{7}{2}\tau_0\right)\varphi\right) = \tau_0\left(c - \frac{7}{2}\tau_0\right) *_\varphi \varphi. \qquad (5.6)$$

Hence,

$$\Delta_{*_\varphi\varphi} *_\varphi \varphi + d\left(\left(c - \frac{7}{2}\tau_0\right)\varphi\right) = \tau_0\left(c - \frac{5}{2}\tau_0\right) *_\varphi \varphi, \qquad (5.7)$$

which vanishes if $c = \frac{5}{2}\tau_0$. Therefore, we will also get certain nearly parallel G$_2$ structures as critical points. Notice here that we can change τ_0 by rescaling our nearly parallel G$_2$ structure, so the flow will distinguish a certain scale for the nearly parallel G$_2$ structures. For example, the 7-sphere has a canonical nearly parallel G$_2$ structure, and only the 7-sphere of a certain size (depending on a choice of positive c) will be a critical point whereas others will not.

Altogether, this is a rather strange situation, which shows that the modified Laplacian coflow, though parabolic, has some potentially undesirable properties.

5.3 Results and Questions

The Laplacian coflow (5.1) and its modification (5.4) have so far received rather little attention, but the key results in the area include the following.

- Soliton solutions arising from warped products and symmetries [14, 23].
- Explicit study of the flow for symmetric situations [3, 4, 31].
- Long-time existence criteria based on curvature and torsion estimates along the modified Laplacian coflow, and non-collapsing for (5.4) under assumption of bounded scalar curvature [8].

  There are also many open problems.

- Does the Laplacian coflow exist (possibly under some further assumptions)?
- What are the critical points of the modified Laplacian coflow?
- What do dimensional reductions of the Laplacian coflow or its modification look like?

6 Dirichlet Energies and Spinorial Flows

The final set of flows we will discuss are explicitly constructed as gradient flows. We have already seen gradient flows of the volume functional (3.14) for G_2 structures (when restricted to closed or coclosed G_2 structures) but these are actually atypical examples of gradient flows. The reason is that $\mathrm{Vol}(\varphi)$ is of order 0 in φ. Typically, gradient flows are with respect to functionals which are first order in the geometric quantity in question. For example, mean curvature flow is the gradient flow for the volume functional which is first order in the immersion.

6.1 Dirichlet Energies

In light of (3.1), there are two clear choices for possible functionals on a compact 7-manifold to consider.

Definition 6.1 We let

$$\mathcal{C}(\varphi) = \frac{1}{2} \int_M |\nabla_\varphi \varphi|^2_{g_\varphi} \,\mathrm{vol}_\varphi \geq 0 \quad \text{and} \quad \mathcal{D}(\varphi) = \frac{1}{2} \int_M |\mathrm{d}\varphi|^2_{g_\varphi} + |\mathrm{d}^*_\varphi \varphi|^2_{g_\varphi} \,\mathrm{vol}_\varphi \geq 0. \tag{6.1}$$

Observe that $\mathcal{D}$ and $\mathcal{C}$ differ by the total scalar curvature functional (see [37]):

$$\mathcal{D}(\varphi) - \mathcal{C}(\varphi) = \int_M R(g_\varphi) \,\mathrm{vol}_\varphi. \tag{6.2}$$

The functionals $\mathcal{C}$ and $\mathcal{D}$ are both what one might call "Dirichlet energies", by analogy with (2.2). Therefore, one could potentially call the gradient flows of these quantities "heat flows". There is also the possibility to generalise slightly and consider (in the notation of (3.6)):

$$\mathcal{D}_\nu(\varphi) = \sum_i \frac{\nu_i}{2} \int_M |\tau_i|^2_{g_\varphi} \,\mathrm{vol}_\varphi \geq 0 \tag{6.3}$$

for positive constants ν_i: this encompasses $\mathcal{C}$ and $\mathcal{D}$ for appropriate choices of ν_i.

All of these functionals are considered in [36, 37] where the authors show the following.

Theorem 6.2 *The critical points of the Dirichlet energies $\mathcal{D}_\nu$ in (6.3) are the torsion-free G_2 structures, which are the absolute minimizers for the functionals (since they are precisely zero at these points).*

They have also shown short-time existence of the gradient flows of the functionals $\mathcal{D}_\nu$, since they are parabolic (modulo diffeomorphisms) and so the standard DeTurck's trick approach can be used.

The key main results are the following (again in [36, 37]).

- Stability of the critical points.
- Some simple examples of solitons.

In [37] the following observation is made.

Proposition 6.3 *The volume functional* $\mathrm{Vol}(\varphi)$ *in* (3.14) *is monotone decreasing along the gradient flow of* $\mathcal{C}$ *in* (6.1)*. In fact, it is a convex function along the flow.*

This is contrary to our earlier results for the Laplacian flow and coflow which viewed $\mathrm{Vol}(\varphi)$ as having strict maxima (modulo diffeomorphisms). In our view, this indicates a key drawback in this Dirichlet energy approach which we shall return to later.

6.2 Spinorial Flow

Up to a constant multiplicative factor, the functional $\mathcal{C}$ can also be written in terms of unit spinors σ on a 7-manifold as

$$\mathcal{E}(\sigma) = \frac{1}{2} \int_M |\nabla_g \sigma|_g^2 \, \mathrm{vol}_\sigma \geq 0. \tag{6.4}$$

This formulation of the Dirichlet energy is something which can clearly be extended beyond G$_2$ geometry.

Definition 6.4 Define the following functional on pairs of metrics and unit spinors on a compact oriented spin manifold M:

$$\mathcal{E}(g, \sigma) = \frac{1}{2} \int_M |\nabla_g \sigma|_g^2 \, \mathrm{vol}_g \geq 0. \tag{6.5}$$

The gradient flow of $\mathcal{E}$ is called the spinorial or spinor flow.

Here, unlike the G$_2$ case, one can vary the metric and spinor independently, with the caveat that the spinor must remain a g-spinor and be unit length.

In [1] the authors show that the critical points of $\mathcal{E}(g, \sigma)$, when the dimension of the manifold M is at least 3, are given by parallel unit spinors σ and so the metric g is Ricci-flat of special holonomy. They also show that the associated gradient flow (the spinorial flow) is parabolic modulo diffeomorphisms and so has short-time existence. This time the analysis is more involved because the space of unit spinors varies as the metric varies, but despite these complications the final result is as one would expect.

- All of the ingredients enable one to prove that the critical points of (6.5) are stable under its gradient flow, as shown in [34].
- The special case of the spinorial flow on Berger spheres is studied in detail in [38].
- The full analysis of the 2-dimensional case, which has special features not covered in [1], is part of work in progress at the time of writing.

6.3 Questions

Morse functionals. The motivation for studying gradient flows of functionals is that one would hope (at least formally) that the functional is Morse (or Morse–Bott) on the space of geometric objects in question. Therefore, the critical points would be encoded by the topology of the space of geometric objects. Very often making this formal picture rigorous is very challenging, but still motivational. For example, in the study of surfaces in 3-manifolds (or, more generally, hypersurfaces in n-manifolds), since the topology of the space of such surfaces is infinite and the volume functional is (in some sense) formally a Morse function on this space, one might hope to construct infinitely many minimal surfaces in any 3-manifold: this is a conjecture due to Yau from 1982, which has been proved when the Ricci curvature of the 3-manifold is positive or generic [20, 32], and recently claimed (at the time of writing) for all metrics on 3-manifolds in [35].

Here, the Dirichlet functionals are defined on the space of G_2 structures (modulo diffeomorphism). However, unlike the case of the volume functional on hypersurfaces, the only critical points of the functionals are absolute minimizers. Therefore, if the functional is a Morse function then the best we can hope for is that the space of G_2 structures on our given manifold could be contractible onto the torsion-free G_2 structures.

Scaling. More than that, just as we saw with the volume functional in (3.14), the best way to reduce the Dirichlet energy is to send the 3-form to zero by scaling, which is clearly useless. The same thing of course happens when studying hypersurfaces under mean curvature flow, but we can stop the hypersurface from being contracted to a point by simply choosing a nontrivial homology class for our initial hypersurface, which then obviously cannot contain the "zero" hypersurface. The same happens in the Laplacian flow and coflow: the cohomology class is fixed so that one kills the action of rescaling.

Unfortunately, when studying the Dirichlet energies, the class of G_2 structure is not preserved and so all one can do is look at the homotopy class of the initial G_2 structure φ: this class is always homotopic to 0 in the space of G_2 structures just by rescaling (although 0 is, of course, not a G_2 structure). Therefore, one might expect for generic initial conditions that the Dirichlet energy gradient flows just send the 3-form to 0, which is certainly an absolute minimizer of the energy, but does not appear to provide any meaningful content.

This discussion leads to the following question: is there a way to modify or restrict the gradient flows of (6.1) to ensure that the 3-form does not go to 0?

7 Conclusions

The study of geometric flows of G_2 structures has seen some important progress, but it is fair to say that at the time of writing the subject is still in its relative infancy. The flows we have described have both pros and cons, and seek to tackle different

problems, so it is potentially interesting to further investigate all of them to see what we can learn about G$_2$ structures. There are also potentially further flows of G$_2$ structures that could be useful, such as the flow of isometric G$_2$ structures introduced in [2]. The field is clearly vibrant and wide open for discovery and progress.

In particular, it we would be very exciting if by studying geometric flows we can uncover a new criteria (geometric or topological) for the existence or otherwise of torsion-free G$_2$ structures. Whilst this is an ambitious goal, by seeking to solve it we may well acquire a much better understanding of the space of G$_2$ structures.

References

1. Amman, B., Weiss, H., & Witt, F. (2016). A spinorial energy functional: Critical points and gradient flow. *Mathematische Annalen*, *365*, 1559–1602.
2. Bagaglini, L. (2019). The energy functional of G$_2$structures compatible with a background metric. *The Journal of Geometric Analysis*.
3. Bagaglini, L., Fernández, M., & Fino, A. Laplacian coflow on the 7-dimensional Heisenberg group. *Asian Journal of Mathematics*.
4. Bagaglini, L., & Fino, A. (2018). The Laplacian coflow on almost-abelian Lie groups. *Annali di Matematica Pura ed Applicata*, *197*, 1855–1873.
5. Bryant, R. L. (2005). Some remarks on G$_2$-structures. In *Proceedings of Gökova Geometry-Topology Conference* (pp. 75–109).
6. Bryant, R. L., & Xu, F. Laplacian flow for closed G$_2$-structures: Short time behavior. arXiv:1101.2004.
7. Crowley, D., & Nordström, J. (2015). New invariants of G$_2$ structures. *Geometry & Topology*, *19*, 2949–2992.
8. Chen, G. (2018). Shi-type estimates and finite time singularities of flows of G$_2$ structures. *The Quarterly Journal of Mathematics*, *69*, 779–797.
9. Fernández, M., Fino, A., & Manero, V. (2016). Laplacian flow of closed G$_2$-structures inducing nilsolitons. *The Journal of Geometric Analysis*, *26*, 1808–1837.
10. Fine, J., & Yao, C. (2018). Hypersymplectic 4-manifolds, the G$_2$-Laplacian flow and extension assuming bounded scalar curvature. *Duke Mathematical Journal, 167*(18), 3533–3589.
11. Fino, A., & Raffero, A. (2020). Closed warped G$_2$-structures evolving under the Laplacian flow. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 5*(20), 315–348.
12. Fino. A., & Raffero, A. A class of eternal solutions to the G$_2$-Laplacian flow. arXiv:1807.01128.
13. Grigorian, S. (2013). Short-time behaviour of a modified Laplacian coflow of G$_2$-structures. *Advances in Mathematics*, *248*, 378–415.
14. Grigorian, S. (2016). Modified Laplacian coflow of G$_2$-structures on manifolds with symmetry. *Differential Geometry and Its Applications*, *46*, 39–78.
15. Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, *17*, 255–306.
16. Hamilton, R. S. (1982). The inverse function theorem of Nash and Moser. *Bulletin of the American Mathematical Society*, *7*(1), 65–222.
17. Hitchin, N. The geometry of three-forms in six and seven dimensions. arXiv:math/0010054.
18. Hitchin, N. (2000). The geometry of three-forms in six dimensions. *Journal of Differential Geometry*, *55*(3), 547–576.
19. Huang, H., Wang, Y., & Yao, C. (2018). Cohomogeneity-one G$_2$-Laplacian flow on 7-torus. *Journal of the London Mathematical Society, 98*(2), 349–368.
20. Irie, K., Marques, F., & Neves, A. (2018). Density of minimal hypersurfaces for generic metrics. *Annals of Mathematics* (2) **187**(3), 963–972.

21. Joyce, D. D. (2000). *Compact manifolds with special holonomy*. Oxford: OUP.
22. Karigiannis, S. (2009). Flows of G$_2$-structures. I. *Quarterly Journal of Mathematics* **60**(4), 487–522.
23. Karigiannis, S., McKay, B., & Tsui, M.-P. (2012). Soliton solutions for the Laplacian coflow of some G$_2$-structures with symmetry. *Differential Geometry and Its Applications*, *30*, 318–333.
24. Lambert, B., & Lotay, J. D. (2019). Spacelike mean curvature flow. *The Journal of Geometric Analysis*.
25. Lauret, J. (2017). Laplacian flow of homogeneous G$_2$ structures and their solitons. *Proceedings of the London Mathematical Society, 114*(3), 527–560.
26. Lauret, J. (2017). Laplacian flow: Questions and homogeneous examples. *Differential Geometry and Its Applications*, *54*, 345–360.
27. Lin, C. (2013). Laplacian solitons and symmetry in G$_2$-geometry. *Journal of Geometry and Physics*, *64*, 111–119.
28. Lotay, J. D., & Wei, Y. (2017). Laplacian flow for closed G$_2$ structures: Shi-type estimates, uniqueness and compactness. *Geometric and Functional Analysis*, *27*, 165–233.
29. Lotay, J. D., & Wei, Y. (2019). Stability of torsion-free G$_2$ structures along the Laplacian flow. *Journal of Differential Geometry, 111*(3), 495–526.
30. Lotay, J. D., & Wei, Y. (2019). Laplacian flow for closed G$_2$ structures: Real analyticity. *Communications in Analysis and Geometry, 27*, 73–109.
31. Manero, V., Otal, A., & Villacampa, R. Solutions of the Laplacian flow and coflow of a locally conformal parallel G$_2$-structure. arXiv:1711.08644.
32. Marques, F. C., & Neves, A. (2017). Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Inventiones Mathematicae*, *209*, 577–616.
33. Nicolini, M. (2018). Laplacian solitons on nilpotent Lie groups. *Bulletin of the Belgian Mathematical Society Simon Stevin*, *25*, 183–196.
34. Schiemanowski, L. Stability of the spinor flow. arXiv:1706.09292.
35. Song, A. Existence of infinitely many minimal hypersurfaces in closed manifolds. arXiv:1806.08816.
36. Weiss, H., & Witt, F. (2012). A heat flow for special metrics. *Advances in Mathematics*, *231*, 3288–3322.
37. Weiss, H., & Witt, F. (2012). Energy functionals and soliton equations for G$_2$-forms. *Annals of Global Analysis and Geometry*, *42*, 585–610.
38. Wittmann, J. (2016). The spinorial energy functional: Solutions of the gradient flow on Berger spheres. *Annals of Global Analysis and Geometry*, *49*, 329–348.