Constructions of Compact G_2-Holonomy Manifolds

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Abstract We explain the constructions for two geometrically different classes of examples of compact Riemannian 7-manifolds with holonomy G_2. One method uses resolutions of singularities of appropriately chosen 7-dimensional orbifolds, with the help of asymptotically locally Euclidean spaces. Another method uses the gluing of two asymptotically cylindrical pieces and requires a certain matching condition for their cross-sections 'at infinity'.

1 Introduction

The Lie group G_2 occurs as an exceptional case in Berger's classification of the Riemannian holonomy groups, in dimension 7. Riemannian manifolds with holonomy G_2 are Ricci-flat and admit parallel spinor fields. The purpose of these notes is to give an introduction to two methods of producing examples of compact Riemannian 7-manifolds with holonomy group G_2.

For a detailed introduction to G_2-structures on 7-manifolds and the G_2 holonomy group we refer to [25, Chap. 11], [14, Chap. 10] and the article by Karigiannis in this volume. Here we briefly recall the foundational results that we need.

The Lie group G_2 may be defined as the stabilizer, in the action of $GL(7, \mathbb{R})$, of the 3-form [4, p. 539]

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*, \quad (1)$$

where x^k are the standard coordinates on $\mathbb{R}^7$ and $dx^{klm} = dx^k \wedge dx^l \wedge dx^m$. Every linear isomorphism of $\mathbb{R}^7$ preserving φ_0 also preserves the Euclidean metric

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$\sum_{i=1}^{7}(dx^i)^2$ and orientation of $\mathbb{R}^7$, thus G_2 is a subgroup of $SO(7)$. The $GL(7,\mathbb{R})$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$.

Let M be a 7-dimensional manifold. Then every G_2-structure on M is induced by a choice of a smooth differential 3-form φ such that for each $p \in M$ there is a linear isomorphism $\iota_p : \mathbb{R}^7 \to T_pM$ with $\iota_p^*(\varphi(p)) = \varphi_0$. We say a 3-form φ is *positive* when φ satisfies the latter condition and denote by $\Omega^3_+(M) \subset \Omega^3(M)$ the subset of all positive 3-forms on M. Note that for a compact M the subset $\Omega^3_+(M)$ is open in the uniform norm topology. We shall sometimes, slightly informally, say that a differential form $\varphi \in \Omega^3_+(M)$ *is* a G_2-structure on M.

We can see from the above that every G_2-structure $\varphi \in \Omega^3_+(M)$ determines on M a metric $g(\varphi)$ and an orientation, hence also the Hodge star $*_\varphi$.

Theorem 1 (cf. [9]) *Let M be a 7-manifold endowed with a G_2-structure $\varphi \in \Omega^3_+(M)$. Then the following are equivalent.*

(a) The holonomy of the metric $g(\varphi)$ is contained in G_2.
(b) $\nabla\varphi = 0$, where ∇ is the Levi–Civita connection of $g(\varphi)$.
(c)

$$d\varphi = 0, \qquad d{*_\varphi}\varphi = 0, \tag{2}$$

(d) The intrinsic torsion of the G_2-structure φ vanishes.

Note that the second equation in (2) is non-linear because $*_\varphi$ depends non-linearly on φ.

We say that (M, φ) is a G_2-*manifold* if φ is a positive 3-form satisfying (2). If, in addition, the holonomy of $g(\varphi)$ is all of G_2, then we shall call (M, φ) an *irreducible G_2-manifold*.

Proposition 2 ([14, Proposition 10.2.2]) *A compact G_2-manifold is irreducible if and only if $\pi_1(M)$ finite.*

A key idea in the known methods of constructing irreducible G_2-manifolds is that one first achieves on M a G_2-structure φ which is, in some sense, an 'approximate' solution of (2) with $d\varphi = 0$ and $d{*_\varphi}\varphi$ having a small norm, in a suitable Banach space. In more geometric terms, the G_2-structure φ then has small torsion. Then one uses perturbative analysis to obtain a correction term $d\eta$, for a 2-form η small in the C^1 norm, so that $\varphi + d\eta$ is a valid G_2-structure and a solution of (2).

We shall explain methods of finding the desired approximate solutions of (2) by building compact Riemannian manifolds from 'simpler pieces'. These will be non-compact or singular G_2-manifolds whose metrics are flat or have holonomy $SU(2)$ or $SU(3)$, which are subgroups of G_2. These latter metrics can be obtained by using the Calabi–Yau analysis or written explicitly. The manifolds are patched together in a 'compatible' way to achieve, on the resulting compact manifolds, G_2-structures with arbitrarily small torsion.

More precisely, one obtains 1-dimensional families of metrics depending on a certain 'gluing parameter' taking values in a semi-closed interval. The limits of these families may be interpreted as boundary points in a 'partial compactification' of the

G_2 moduli space. (It is known that the moduli space of torsion-free G_2-structures on a compact 7-manifold M is a smooth manifold of dimension the third Betti number $b^3(M)$.)

In these notes, we shall explain two ways of implementing the above strategy with different respective limits in the boundary of the G_2 moduli space.

Recently, Joyce and Karigiannis [16] developed a new method of constructing holonomy G_2 manifolds using analysis on families of Eguchi–Hanson spaces. This construction is not reviewed here. It includes an application of perturbative methods for G_2-structures with small torsion but also requires significant additional methods to achieve a suitable small torsion.

2 Construction by Resolutions of Singularities

The method explained in this section was historically the first construction of compact 7-manifolds with holonomy G_2. It is due to Joyce [13, 14].

Joyce's method produces one-parameter families of holonomy G_2 metrics g_s, $0 < s \leq \varepsilon$. The limits of these families as $s \to 0$ can be interpreted as boundary points in the G_2-moduli space and are given by flat orbifolds. In particular, the limit spaces are *compact, connected* and *singular*.

More precisely, the construction proceeds via the following steps.

1. (a) Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ be the 7-torus with a flat G_2-structure $\varphi_0 \in \Omega^3_+(T^7)$ induced from the standard G_2-structure (1) on the Euclidean $\mathbb{R}^7$. Choose a finite group Γ of affine transformations of $\mathbb{R}^7$ which preserve φ_0 and descend to diffeomorphisms of T^7. The quotient space (T^7/Γ) is an orbifold with a torsion-free G_2-structure, still denoted by φ_0, and a flat orbifold metric g_0 induced by φ_0.
   (b) For suitable choices of Γ, all the singularities of T^7/Γ are locally modeled on $\mathbb{R}^3 \times (\mathbb{C}^2/G)$ or $\mathbb{R} \times (\mathbb{C}^3/G)$, for G a finite subgroup of respectively $SU(2)$ or $SU(3)$, and can be resolved using methods of complex algebraic geometry. Perform the resolutions to obtain a smooth compact 7-manifold M together with a resolution map $\pi : M \to T^7/\Gamma$.
2. (a) On M, one can 'explicitly' define a 1-parameter family of closed positive 3-forms $\varphi_s \in \Omega^3_+(M)$, with $d\varphi_s = 0$ for $0 < s \leq \varepsilon$, such that the G_2-structures φ_s have small torsion. The forms φ_s converge as $s \to 0$ to $\pi^*\varphi_0$ (respectively, the induced metrics $g(\varphi_s)$ converge to $\pi^* g_0$). One may also say that the Riemannian manifolds $(M, g(\varphi_s))$ converge in the Gromov–Hausdorff sense to the flat orbifold $(T^7/\Gamma, g_0)$ as $s \to 0$.
   (b) Apply perturbative analysis (more precisely, construct a convergent sequence of iterations) to show that for every small $s > 0$, the G_2-structure φ_s can be deformed to a nearby torsion-free G_2-structure $\tilde{\varphi}_s$. If $\pi_1(M)$ is finite, then the holonomy of the induced metric $\tilde{g}_s = g(\tilde{\varphi}_s)$ is precisely the group G_2, i.e. $(M, \tilde{\varphi}_s)$ is an irreducible G_2-manifold.

We illustrate this method with an example taken from [14, Sect. 12.2] (cf. also [13]) where some technical details are relatively simple. Consider the group Γ generated by

$$\begin{aligned}
\alpha : (x_1, \dots, x_7) &\mapsto (\ \ x_1, \ \ x_2, \ \ x_3, -x_4, \ -x_5, \ -x_6, \ -x_7), \\
\beta : (x_1, \dots, x_7) &\mapsto (\ \ x_1, -x_2, -x_3, \ \ x_4, \ \ x_5, \tfrac{1}{2}-x_6, \ -x_7), \\
\gamma : (x_1, \dots, x_7) &\mapsto (-x_1, \ \ x_2, -x_3, \ \ x_4, \tfrac{1}{2}-x_5, \ \ x_6, \tfrac{1}{2}-x_7).
\end{aligned}$$

The maps α, β, γ commute and each has order 2, thus Γ is isomorphic to $\mathbb{Z}_2^3$. The elements of Γ descend to T^7 and preserve φ_0, making the quotient T^7/Γ into a G_2-orbifold.

One can further check that the only elements of Γ having fixed points are α, β, γ, each fixes 16 copies of T^3 and these are all disjoint. The subgroup generated by β, γ acts freely on the 16 tori fixed by α, so these correspond to 4 copies of T^3 in the singular locus of T^7/Γ. Similar properties hold for the tori fixed by β and by γ. Thus the singular locus S of T^7/Γ is 12 disjoint copies of T^3. A neighbourhood of each 3-torus component of S is diffeomorphic to $T^3 \times (\mathbb{C}^2/\{\pm 1\})$.

The blow-up $\sigma : Y \to \mathbb{C}^2/\{\pm 1\}$ at the origin resolves the singularity with a complex surface Y biholomorphic to $T^*\mathbb{C}P^1$, with the exceptional divisor $E = \sigma^{-1}(0) \cong \mathbb{C}P^1$ corresponding to the zero section of $T^*\mathbb{C}P^1$. The canonical bundle of Y is trivial and there is a family of Ricci-flat Kähler metrics h_s on Y with holonomy equal to $SU(2)$ depending on a real parameter $s > 0$. The Kähler form of the metric h_s may be written as $\omega_s = \sigma^*(i\partial\bar{\partial} f_s)$, where

$$f_s = \sqrt{r^4 + s^4} + 2s^2 \log r - s^2 \log(\sqrt{r^4 + s^4} + s^2),$$

$r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$ and $(z_1, z_2) \in \mathbb{C}^2$. The radius function r makes sense as a smooth function on $Y \setminus E$ and the values of this function near E can be interpreted as the distance to E in the metric h_s. The forms ω_s extend smoothly over the exceptional divisor $E \subset Y$, thus the metrics h_s are well-defined on Y. These are the well-known Eguchi–Hanson metrics [8].

Comparing, for each $s > 0$, the Kähler potential f_s of h_s with the Kähler potential r^2 of the Euclidean metric h_0 on $\mathbb{C}^2$ we see that

$$\nabla^k (h_s - h_0) = O(r^{-4-k}) \quad \text{as } r \to \infty, \quad \text{for all } k = 0, 1, 2, \dots , \tag{3}$$

which means that h_s is an asymptotically locally Euclidean (ALE) metric on Y.

For each $\lambda > 0$, the dilation map $Y \to Y$ induced by $(z_1, z_2) \mapsto \lambda(z_1, z_2)$ pulls back ω_s to $\lambda^2 \omega_{\lambda s}$. It follows that s is proportional to the diameter of the exceptional divisor. One can further check that the injectivity radius of the Eguchi–Hanson metric h_s is proportional to s and that the uniform norm of the Riemannian curvature is proportional to s^{-2}.

Every Ricci-flat Kähler metric h on a complex surface is in fact hyper-Kähler: in addition to the original complex structure I there are (integrable) complex structures

J and K satisfying quaternionic relations $IJ = -JI = K$. For each $p \in Y$, there is an $\mathbb{R}$-linear isomorphism $\mathbb{R}^4 \to T_pY$ such that the linear maps $I(p)$, $J(p)$, $K(p)$ correspond to multiplication by the unit quaternions i, j, k via the standard identification $\mathbb{R}^4 \cong \mathbb{H} = \langle 1, i, j, k\rangle$ with the algebra of quaternions. Also, the metric h is Kähler with respect to each I, J, K. We shall denote by $\kappa_I, \kappa_J, \kappa_K$ the respective Kähler forms.

For a 3-torus T^3 with coordinates x_1, x_2, x_3, with a flat metric $dx_1^2 + dx_2^2 + dx_3^2$ and a hyper-Kähler 4-manifold Y as above, the Riemannian product $T^3 \times Y$ has holonomy in $SU(2)$. The product metric is induced by a torsion-free G_2 structure on $T^3 \times Y$, which is

$$\varphi_{SU(2)} = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \kappa_I + dx_2 \wedge \kappa_J - dx_3 \wedge \kappa_K. \tag{4}$$

We now define, for every small $\varepsilon > 0$, a smooth compact 7-manifold $M = M_\varepsilon$ by replacing a neighbourhood $T^3 \times \{r < 2\varepsilon\}$ of each 3-torus component in the singular locus of T^7/Γ by $T^3 \times U$, where $U = \sigma^{-1}(r < 2\varepsilon) \subset Y$ is a neighbourhood of the exceptional divisor on Y. (Note that the manifolds M_ε are diffeomorphic to each other.)

On each $T^3 \times U$ in M, we smoothly interpolate, for $\varepsilon < r < 2\varepsilon$, between the flat G_2-structure φ_0 induced from T^7/Γ on the complement of the regions $T^3 \times U$ and the product G_2-structure arising as in (4) from the appropriately rescaled Eguchi–Hanson hyper-Kähler h_s on $\sigma^{-1}(r < \varepsilon) \subset U$. The ALE property of the Eguchi–Hanson metric allows to take the product G_2-structure on $T^3 \times Y$ to be asymptotic to the flat G_2-structure on $T^3 \times (\mathbb{C}^2/\{\pm 1\})$. We can obtain, for each sufficiently small $s > 0$, a well-defined positive 3-form φ_s on M noting also that $\Omega^3_+(M)$ is an open subset of 3-forms in the uniform norm. Furthermore, we can choose these G_2 3-forms on M to be *closed*, $d\varphi_s = 0$. Thus the G_2-structure φ_s is torsion-free away from the interpolation region $\{\varepsilon < r < 2\varepsilon\}$ but φ_s is not co-closed in that region.

The positive 3-forms φ_s are intended as 'approximate solutions' of the torsion-free equations (2), as $s \to 0$. The parameter s may be interpreted geometrically as the maximal diameter of the pre-image of a singular point in T^7/Γ under the resolution map $M \to T^7/\Gamma$. We would like to perturb φ_s to actual solutions on M. To this end, the following two conditions satisfied by φ_s are important, cf. [14, Theorem 11.5.7].

Condition (i) One can construct a smooth 3-form ψ_s on M such that $d^*\varphi_s = d^*\psi_s$ and

$$\|\psi_s\|_{L^2} < A_1 s^4,\ \|\psi_s\|_{C^0} < A_1 s^3 \text{ and } \|d^*\psi_s\|_{L^{14}} < A_1 s^{16/7}. \tag{5}$$

Condition (ii) The injectivity radius $\delta(g_s)$ and the Riemann curvature $R(g_s)$ of the metric $g_s = g(\varphi_s)$ on M satisfy the estimates

$$\delta(g_s) > A_2\, s, \qquad \|R(g_s)\|_{C^0} < A_3 s^{-2}. \tag{6}$$

The construction of ψ_s exploits the asymptotic and scaling properties of the G_2-structure (4) on $T^3 \times U$ 'approximating' the flat G_2-structure on $T^3 \times \mathbb{C}^2/\{\pm 1\}$.

The estimates (6) follow from the properties of the metric h_s around the exceptional divisor on Y, which give the dominant contributions for small s. In the conditions (i) and (ii) the norms and the formal adjoint d^* are taken with respect to the metric $g_s = g(\varphi_s)$. The constants $A_1,\ A_2,\ A_3$ are independent of s.

We can now state the existence result for torsion-free G_2-structures.

Theorem 3 (cf. [14, Theorem 11.6.1]) *Let M be a compact 7-dimensional manifold and $\varphi_s \in \Omega^3_+(M)$, $0 < s \le s_0$, a family of G_2-structures such that $d\varphi_s = 0$ and the conditions (i) and (ii) above hold for all s.*

Then there is an $\varepsilon_0 > 0$ so that for each s with $0 < s \le \varepsilon_0$ the manifold M admits a torsion-free G_2 structure $\tilde{\varphi}_s \in \Omega^3_+(M)$ in the same cohomology class as φ_s and satisfying $\|\tilde{\varphi}_s - \varphi_s\|_{C^0} < K s^{1/2}$ with some constant K independent of s.

We next outline the Proof of Theorem 3 following [15, pp. 236–237], dropping the subscripts s to ease the notation. The desired torsion-free G_2-structure $\tilde{\varphi} = \tilde{\varphi}_s$ will be obtained in the form $\tilde{\varphi} = \varphi + d\eta$, where $d\eta$ has a small uniform norm, so $\tilde{\varphi}$ is a closed positive 3-form. We then need to satisfy the co-closed condition $d^*_{\tilde{\varphi}}\tilde{\varphi} = 0$ and this amounts to solving for a 2-form η a non-linear elliptic PDE which may be written as

$$d^* d\eta = -d^*\psi + d^* F(d\eta) \tag{7}$$

where F satisfies a quadratic estimate. A solution of (7) is achieved by using iterations to construct a sequence $\{\eta_j\}_{j=0}^{\infty}$ with $\eta_0 = 0$ and

$$d^* d\eta_{j+1} = -d^*\psi + d^* F(d\eta_j), \qquad d^*\eta_{j+1} = 0.$$

One first argues that the sequence η_j converges.

The proof of convergence is based on the following inductive estimates (all the constants C_i below are independent of s)

$$\|d\eta_{j+1}\|_{L^2} \le \|\psi\|_{L^2} + C_1 \|d\eta_j\|_{L^2} \|d\eta_j\|_{C^0}, \tag{8a}$$

$$\|\nabla d\eta_{j+1}\|_{L^{14}} \le C_2(\|d^*\psi\|_{L^{14}} + \|\nabla d\eta_j\|_{L^{14}} \|d\eta_j\|_{C^0} + s^{-4} \|d\eta_{j+1}\|_{L^2}), \tag{8b}$$

$$\|d\eta_j\|_{C^0} \le C_3(s^{1/2} \|\nabla d\eta_j\|_{L^{14}} + s^{-7/2} \|d\eta_j\|_{L^2}). \tag{8c}$$

The estimate (8a) is proved by taking the L^2 product of both sides with η_{j+1} and integrating by parts, noting also the condition (i) above. The proof of (8b) uses an elliptic regularity estimate for the operator $d + d^*$ considered for 3-forms on small balls on M with radius of order s. The condition (ii) is also required here and in (8c) which uses the Sobolev embedding of L^{14}_1 in C^0 in dimension 7 and is again achieved by working on small balls with radius of order s.

For every sufficiently small s, we deduce from (8) that if $d\eta_j$ satisfies

$$\|d\eta_j\|_{L^2} \le C_4 s^4, \quad \|\nabla d\eta_j\|_{L^{14}} \le C_5, \quad \|d\eta_j\|_{C^0} \le K s^{1/2}, \tag{9}$$

then these latter estimates hold for $d\eta_{j+1}$ and, by induction, for all j. Thus $d\eta_j$ is a bounded sequence in the L^{14}_1 norm on $\Lambda^3 T^* M$ and one can further show that $d\eta_j$ is a Cauchy sequence. Further, we are free to assume that the forms η_j are in the L^2-orthogonal complement $\mathcal{H}^\perp$ of harmonic forms. As the elliptic operator $d + d^*$ is bounded below on $\mathcal{H}^\perp$ it follows that the sequence η_j converges in the L^{14}_2 norm. In particular, the last inequality of (9) holds for the limit η.

Finally, a careful elliptic regularity argument shows that η is in fact a smooth solution of (7), thus completing the Proof of Theorem 3.

The metrics on M induced by φ_s have holonomy in G_2 and it remains to verify that the holonomy does not reduce further to a subgroup of G_2. In the present case, the orbifold T^7/Γ is simply-connected, therefore M is so, by the properties of the blow-up. Thus (M, φ_s) is an irreducible G_2-manifold by Proposition 2.

The discussed example may be considered as a generalization of the Kummer construction of hyper-Kähler metrics of holonomy $SU(2)$ on K3 surfaces.

It is convenient to obtain the Betti numbers of M; these are determined by $b^2(M)$ and $b^3(M)$. By considering the Γ-invariant classes in $H^*_{\mathrm{dR}}(T^7/\Gamma)$ we obtain $b^2(T^7/\Gamma) = 0$ and $b^3(T^7/\Gamma) = 7$. When resolving the singularities, we replaced a deformation retract of T^3 with $T^3 \times Y$ which is homotopy equivalent to $T^3 \times \mathbb{C}P^1$. Let S denote the singular locus of T^7/Γ. Comparing the cohomology long exact sequence for the pairs $(T^7/\Gamma, S)$ and $(M, \sqcup_{i=1}^{12}(T^3 \times U))$, we find that each of the 12 instances of a resolution adds $b^i(T^3 \times Y) - b^i(T^3)$ to the ith Betti number of M. Thus $b^2(M) = 12 \cdot 1$ and $b^3(M) = 7 + 12 \cdot 3 = 43$.

Further examples of irreducible G_2-manifolds arise by using the above method with different choices of finite groups Γ and different choices of resolutions of singularities of T^7/Γ. If every component of the singular locus of T^7/Γ has a neighbourhood diffeomorphic to $T^3 \times (\mathbb{C}^2/G)$ for a finite subgroup G of $SU(2)$ or to $S^1 \times (\mathbb{C}^3/G)$ for a finite subgroup G of $SU(3)$ acting freely on $\mathbb{C}^3 \setminus \{0\}$, then it is known from complex algebraic geometry that one can find crepant resolutions, $\sigma_2 : Y_2 \to \mathbb{C}^2/G$ or $\sigma_3 : Y_3 \to \mathbb{C}^3/G$ respectively, with the canonical bundle of Y_i holomorphically trivial.

The Ricci-flat Kähler (thus hyper-Kähler) metrics on the complex surfaces Y_2 asymptotic to $\mathbb{C}^2/G$ in the sense of (3), for each G, were constructed by Kronheimer [21] using hyper-Kähler quotients.

In complex dimension 3, the existence of ALE Ricci-flat holonomy $SU(3)$ metrics on Y_3 asymptotic to $\mathbb{C}^3/G$ follows from the solution of ALE version of the Calabi conjecture, see [14, Chap. 8] and references therein. The asymptotic rate for the metrics h is given by

$$\nabla^k(h - h_0) = O(r^{-6-k}) \quad \text{as } r \to \infty, \quad \text{for all } k = 0, 1, 2, \dots\ ,$$

where h_0 is the pull-back of the Euclidean metric on $\mathbb{C}^3/G$. The Kähler forms of h and h_0 satisfy

$$\omega - \omega_0 = i\partial\bar{\partial}u, \quad \nabla^k u = O(r^{-4-k}) \quad \text{as } r \to \infty$$

(cf. [14, Theorem 8.2.3]). The holonomy being $SU(3)$ means there is a choice of nowhere vanishing (3, 0)-form Ω on Y_3 (sometimes called a holomorphic volume form), such that $\omega^3/3! = (i/2)^3\Omega \wedge \bar{\Omega}$. A torsion-free G_2-structure on $S^1 \times Y_3$ defined by

$$\varphi_{SU(3)} = dx \wedge \omega + \mathrm{Re}\Omega, \tag{10}$$

induces a product metric corresponding to $(dx)^2$ and ω, where x is the usual coordinate on $S^1 = \mathbb{R}/\mathbb{Z}$.

The singularities of T^7/Γ can be resolved with copies of $T^3 \times U_2$ or $S^1 \times U_3$ (where U_i is a neighbourhood of $\sigma_i^{-1}(0)$ in Y_i) in a manner similar to the example above. One obtains compact smooth 7-manifolds M and closed positive 3-forms φ_s on M satisfying the hypotheses of Theorem 3. More generally, the method extends to situations when the singularities of T^7/Γ are only *locally* modeled on $\mathbb{R}^3 \times (\mathbb{C}^2/G)$ or $\mathbb{R} \times (\mathbb{C}^3/G)$. In the latter case, G need not act freely on $\mathbb{C}^3 \setminus \{0\}$ resulting in a more complicated singular locus of T^7/Γ.

Joyce found a large number of orbifolds T^7/Γ with suitable resolutions of singularities. In particular, 252 examples of topologically distinct compact 7-manifolds admitting holonomy G_2 metrics are worked out in [14, Chap. 12], including some manifolds with non-trivial fundamental group. The Betti numbers of these examples are in the range $0 \le b^2 \le 28$ and $4 \le b^3 \le 215$. There is a evidence that many more further topological types can be constructed by the same method.

3 Construction by Generalized Connected Sums

The method of constructing compact holonomy G_2 manifolds discussed in this section is sometimes called a ‘twisted connected sum’. The construction was originally developed by the author in [17] and included an important idea due to Donaldson. Generalizations and many new examples appeared in [5, 6, 18, 24].

The connected sum construction produces one-parameter families of holonomy G_2 metrics g_T, $T_0 \le T < \infty$, on compact manifolds with ‘long necks’. The parameter T here is asymptotic, as $T \to \infty$, to the diameter of the metric g_T. We may think of the respective families of torsion-free G_2-structures as paths in the G_2 moduli space, going to the boundary as one ‘stretches the neck’, the limit boundary point corresponding to the disjoint union of the initial two asymptotically cylindrical pieces. So, in this construction, the limit spaces are *disconnected, non-compact* and *smooth*.

A twisted connected sum is an instance of generalized connected sum of a pair of asymptotically cylindrical Riemannian manifolds which, in the present case, are G_2-manifolds. The asymptotically cylindrical G_2-manifolds we require are Riemannian products $W \times S^1$, where W is a Ricci-flat Kähler manifold with cylindrical end asymptotic to a Riemannian product $D \times S^1 \times [0, \infty)$ with D a K3 surface with a hyper-Kähler metric. For certain pairs of the K3 surfaces D_1, D_2 there is a way to ‘join’ the two latter asymptotically cylindrical manifolds at their ends. We obtain a compact simply-connected manifold M and a G_2-structure with small torsion on M to which a perturbative analysis can be applied.

We now describe the key steps in the construction in more detail, starting with the asymptotically cylindrical Calabi–Yau threefolds W.

Theorem 4 ([11, 17, 27]) *Let $\overline{W}$ be a compact Kähler threefold with Kähler form $\overline{\omega}$ and suppose that a K3 surface $D \in |-K_{\overline{W}}|$ is an anticanonical divisor on $\overline{W}$ with holomorphically trivial normal bundle $N_{D/\overline{W}}$. Denote by z a complex coordinate around D vanishing to order one precisely on D. Suppose that $\overline{W}$ is simply-connected and the fundamental group of $W = \overline{W} \setminus D$ is finite.*

Then W admits a complete Ricci-flat Kähler metric, with holonomy $SU(3)$, with Kähler form ω and a non-vanishing holomorphic $(3,0)$-form Ω. These are asymptotic to the product cylindrical Ricci-flat Kähler structure on $D \times S^1 \times \mathbb{R}_{>0}$

$$\omega = \kappa_I + dt \wedge d\theta + d\psi,$$
$$\Omega = (\kappa_J + i\,\kappa_K) \wedge (dt + i d\theta) + d\Psi,$$

where $\exp(-t - i\theta) = z$, for $(\theta, t) \in S^1 \times \mathbb{R}_{>0}$ and the forms ψ, Ψ exponentially decay as $t \to \infty$. Also κ_I is the Ricci-flat Kähler metric on D in the class $[\overline{\omega}|_D]$ and $\kappa_J + i\kappa_K$ is a non-vanishing holomorphic $(2,0)$-form on D.

Remark Any threefold $\overline{W}$ satisfying the hypotheses of Theorem 4 is necessarily projective and algebraic [18, Proposition 2.2]. The holomorphic coordinate z extends to a meromorphic function $\overline{W} \to \mathbb{C}P^1$ vanishing precisely on D.

Theorem 4 extends to higher dimensions $m \geq 3$ with D replaced by a compact simply-connected Calabi–Yau $(m-1)$-fold. The result may be regarded as a solution of an 'asymptotically cylindrical version' of the Calabi conjecture.

It will be convenient to extend the parameter t along the cylindrical end in Theorem 4 to a smooth function t defined on all of W with $t < 0$ away from a tubular neighbourhood of D. We shall also assume that the holomorphic 2-form on a Kähler K3 surface D is normalized so that $\kappa_I^2 = \kappa_J^2 = \kappa_K^2$, with the implied normalization of a holomorphic 3-form Ω on W. The Ricci-flat Kähler (hyper-Kähler) structure on D is in fact determined by the triple $\kappa_I, \kappa_J, \kappa_K$ (cf. [10, p. 91].

The following relation between K3 surfaces is crucial for the connected sum construction of G_2-manifolds.

Definition 1 We say that two Ricci-flat Kähler K3 surfaces $(D_1, \kappa'_I, \kappa'_J + i\kappa'_K)$, $(D_2, \kappa''_I, \kappa''_J + i\kappa''_K)$ satisfy the *Donaldson matching condition* if there exists an isometry of lattices $h : H^2(D_2, \mathbb{Z}) \to H^2(D_1, \mathbb{Z})$, so that the $\mathbb{R}$-linear extension of h satisfies

$$h : \; [\kappa''_I] \mapsto [\kappa'_J], \qquad [\kappa''_J] \mapsto [\kappa'_I], \qquad [\kappa''_K] \mapsto [-\kappa'_K]. \tag{11}$$

It follows, by application of the Torelli theorem for K3 surfaces, that there is a smooth map

$$f : D_1 \to D_2, \;\text{ such that } h = f^*.$$

Note that f is *not* a holomorphic map between D_1 and D_2 (with complex structures I), though f is an isometry of the underlying Riemannian 4-manifolds. In particular, the pull back f^* rotates the 2-forms of the hyper-Kähler triple (not just their cohomology classes), $\kappa''_I \mapsto \kappa'_J$, $\kappa''_J \mapsto \kappa'_I$, $\kappa''_K \mapsto -\kappa'_K$.

Now if (W, ω, Ω) is an asymptotically cylindrical Calabi–Yau manifold given by Theorem 4, then $W \times S^1$ has a torsion-free G_2-structure given by (10)

$$\varphi_W = d\tilde{\theta} \wedge \omega + \mathrm{Re}\Omega,$$

where $\tilde{\theta}$ is the standard coordinate on the S^1 factor. The form φ_W is asymptotic to a cylindrical product torsion-free G_2-structure φ_∞ on the cylindrical end $D \times [0, \infty) \times S^1 \times S^1 \subset W \times S^1$,

$$\varphi_\infty = dt \wedge d\theta \wedge d\tilde{\theta} + d\tilde{\theta} \wedge \kappa_I + dt \wedge \kappa_J - d\theta \wedge \kappa_K.$$

corresponding to the hyper-Kähler structure $(\kappa_I, \kappa_J, \kappa_K)$ on D (cf. (4)).

For $i = 1, 2$ and $T > 0$, let $W_{i,T}$ be a compact manifold with boundary obtained by truncating W_i at $t_i = T + 1$ (where t_i is the parameter along the cylindrical end as in Theorem 4). We can smoothly cut off each φ_{W_i} to obtain on $W_{i,T}$ a closed G_2-structure $\varphi_{W,T}$ so that $\varphi_{W_i,T}$ equals its cylindrical asymptotic model φ_∞ on a collar neighbourhood $D_i \times S^1 \times S^1 \times [T, T+1]$ of the boundary.

Suppose that D_1 and D_2 satisfy the Donaldson matching condition. Then we can define a compact 7-manifold

$$M = M_T = (W_{1,T+1} \times S^1) \cup_F (W_{2,T+1} \times S^1) \tag{12}$$

by identifying the collar neighbourhoods of the boundaries using a map

$$\begin{aligned} F : D_1 \times S^1 \times S^1 \times [T, T+1] &\to D_2 \times S^1 \times S^1 \times [T, T+1], \\ (y, \theta, \tilde{\theta}, T+t) &\mapsto (f(y), \tilde{\theta}, \theta, T+1-t). \end{aligned} \tag{13}$$

The form $\varphi_\infty|_{[T,T+1]}$ is preserved by F, so the G_2-structures $\varphi_{i,T}$ agree on the overlap and patch together to a well-defined closed 3-form φ_T on M. It is easy to see that φ_T is a well-defined G_2-structure on M for every large T.

Another important property of the map F is that F identifies the S^1 factor in $W_{1,T+1} \times S^1$ with a circle around the divisor on the other threefold W_2 and vice versa. This eliminates the possibility of an infinite fundamental group of M, in particular, M will be simply-connected when the threefolds W_1 and W_2 are so.

The G_2-structure form on M satisfies $d\varphi_T = 0$, one of the two equations in (2), but the co-derivative $d *_T \varphi_T$ in general will not vanish. The cut-off functions introduce 'error terms' which depend on the difference between the $SU(3)$-structures on the end of W_i and on its cylindrical asymptotic model, and can be estimated as

$$\|d *_T \varphi_T\|_{L^p_k} < C_{p,k} e^{-\lambda T},$$

with $\lambda > 0$. Here $*_T$ denotes the Hodge star of the metric $g(\varphi_T)$.

The next result shows that for a sufficiently long neck the G_2-structure φ_T on M can be made torsion-free by adding a small correction term.

Theorem 5 *Suppose that each of $\overline{W}_1, D_1$ and $\overline{W}_2, D_2$ satisfies the hypotheses of Theorem 4 and the $K3$ surfaces $D_j \in |-K_{\overline{W}_j}|$ satisfy the Donaldson matching condition. Let M be the compact 7-manifold M defined in (12) with a closed G_2-structure φ_T induced from φ_{W_1}, φ_{W_2}.*

Then M has finite fundamental group. Furthermore, there exists $T_0 \in \mathbb{R}$ and for every $T \geq T_0$ a unique smooth 2-form η_T on M, orthogonal to the closed forms, so that the following holds.
(a) $\|\eta_T\|_{C^1} < A \cdot e^{-\mu T}$, for some constants $A, \mu > 0$ independent of T, where the norm is defined using the metric $g(\varphi_T)$. In particular, $\varphi_T + d\eta_T$ is a valid G_2-structure on M.
(b) The closed 3-form $\varphi_T + d\eta_T$ satisfies

$$d *_{\varphi_T + d\eta_T} (\varphi_T + d\eta_T) = 0. \tag{14}$$

and so $\varphi_T + d\eta_T$ defines a metric with holonomy G_2 on M.

As discussed in the previous section, the perturbative problem (14) can be rewritten as a non-linear elliptic PDE for the 2-form η. When η has a small norm this PDE takes the form $a(\eta) = a_0 + A\eta + Q(\eta) = 0$, where $a_0 = d *_T \varphi_T$, the linear elliptic operator $A = A_T$ is a compact perturbation of the Hodge Laplacian of the form $dd^* + d^*d + O(e^{-\varepsilon T})$, $\varepsilon > 0$ and $Q(\eta)$ satisfies a quadratic estimate in $d\eta$.

One can use elliptic theory for manifolds with cylindrical ends and the gluing analysis for the problem at hand is then simplified, compared to the general situation of Theorem 3. The central idea in the proof of Theorem 5 may be informally described as follows. For small η, the map $a(\eta)$ is approximated by its linearization and so there would be a unique small solution η to the equation $a(\eta) = 0$, for every small a_0 in the range of A. This perturbative approach requires the invertibility of A and a suitable upper bound on the operator norm $\|A_T^{-1}\|$, as $T \to \infty$. This bound determines what is meant by 'small a_0' above.

As we actually need the value of $d\eta$ rather than η we may consider the equation for η in the orthogonal complement of harmonic 2-forms on M where the Laplacian is invertible. We use the technique similar to [20, Sect. 4.1] based on Fredholm theory for the asymptotically cylindrical manifolds and weighted Sobolev spaces to find an upper bound $\|A_T^{-1}\| < Ge^{\delta T}$. Here the constant G is independent of T and $\delta > 0$ can be taken arbitrarily small. So, for large T, the growth of $\|A_T^{-1}\|$ is negligible compared to the decay of $\|d *_T \varphi_T\|$ and the 'inverse function theorem' strategy applies to give the required small solution η_T in a (appropriately chosen) Sobolev space. Standard elliptic methods show that this η_T is in fact smooth.

3.1 Some Examples and Further Results

In order to make irreducible G_2-manifolds using the connected sum construction, we require pairs $\overline{W}_1$, $\overline{W}_2$ of complex algebraic threefolds with matching anticanonical K3 divisors $D_i \subset \overline{W}_i$. We begin with an example based on some classical algebraic geometry.

Example 1 The intersection of three generically chosen quadric hypersurfaces in $\mathbb{C}P^6$ defines a smooth Kähler threefold X_8. It is simply-connected and the characteristic class $c_1(X_8)$ of its anticanonical bundle is the pull-back to X_8 of the positive generator of the cohomology ring $H^*(\mathbb{C}P^6)$. This tells us that the anticanonical bundle $K_{X_8}^{-1}$ is the restriction to X_8 of the tautological line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^6$. It follows that any anticanonical divisor D on X_8 is obtained by taking an intersection $D = X_8 \cap H$ with a hyperplane H in $\mathbb{C}P^6$. A generic such hyperplane section D is a complex surface, isomorphic to a smooth complete intersection of three quadrics in $\mathbb{C}P^5$. This is a well-known example of a K3 surface.

Conversely, starting from a smooth intersection D of three quadrics in $\mathbb{C}P^5$ we can write down a smooth threefold $X_8 \subset \mathbb{C}P^6$ as above containing the K3 surface D as an anticanonical divisor.

Consider another anticanonical divisor $D' = X_8 \cap H'$ and let $\tilde{X}_8 \to X_8$ be the blow-up of the second hyperplane section $C = D \cap D' = X_8 \cap H \cap H'$. (It is convenient, though not strictly necessary, to choose D' so that C is a non-singular connected complex curve.) The pencil defined by D and D' lifts, via the proper transform, to a pencil consisting of the fibres of a holomorphic map $\tilde{X}_8 \to \mathbb{C}P^1$. In particular, the K3 divisor D lifts to an isomorphic K3 surface $\tilde{D}$ which is an anticanonical divisor on $\tilde{X}_8$ and has trivial normal bundle. Moreover, a Kähler metric on $\tilde{X}_8$ may be chosen so that $\tilde{D}$ and D are isometric Kähler K3 surfaces.

It is not difficult to check that $\tilde{X}_8 \setminus \tilde{D}$ is simply-connected, noting that $\tilde{D}$ and X_8 are so and considering an exceptional curve in the blow up $\tilde{X}_8$. The pair $\tilde{X}_8$, $\tilde{D}$ thus satisfies all the hypotheses of Theorem 4, and so the quasiprojective threefold $W = \tilde{X}_8 \setminus \tilde{D}$ admits an asymptotically cylindrical Ricci-flat Kähler metric with holonomy $SU(3)$. Note that the cylindrical asymptotic model for this metric is determined by the Ricci-flat Kähler structure in the Kähler class of D in X_8.

We would like to choose two octic threefolds $X_8^{(i)}$, $i = 1, 2$ and a K3 surface D_i in each, so as to satisfy the Donaldson matching condition. We do this by applying some general theory of K3 surfaces and their moduli (see [2, Chap. VIII]). The key point is that one can determine a Ricci-flat Kähler K3 surface D, up to isomorphism, by a data of the integral second cohomology $H^2(D, \mathbb{Z})$.

Recall that all K3 surfaces are diffeomorphic and the intersection form makes $H^2(D, \mathbb{Z})$ into a lattice. There is an isomorphism, called a marking, $p : H^2(D, \mathbb{Z}) \to L$ to a fixed non-degenerate even unimodular lattice L with signature (3, 19). We shall refer to L as the *K3 lattice*; its bilinear form is given by the orthogonal direct sum $L = 3H \oplus 2(-E_8)$ of 3 copies of the hyperbolic plane lattice $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and 2 copies of the negative definite root lattice $-E_8$ of rank 8. Now if $D \subset \mathbb{C}P^5$ is an octic K3 surface, then the image $p(\kappa_I)$ of the Kähler class of D is primitive (non-divisible in L by an integer > 1) and $p(\kappa_I) \cdot p(\kappa_I) = 8$, computed in the bilinear form of L. The images $p(\kappa_J)$, $p(\kappa_K)$ span a positive 2-plane P orthogonal to $p(\kappa_I)$ in the real vector space $L \otimes \mathbb{R}$. Conversely, the positive 2-planes P arising in this way form a dense open set in the Grassmannian of positive 2-planes orthogonal to $p(\kappa_I)$ in $L \otimes \mathbb{R}$.

It is known that the group of lattice isometries of L acts transitively on the set of all primitive vectors with a fixed value of $v \cdot v$ in L. We can therefore choose two octic K3 surfaces with hyper-Kähler structures (in the respective Kähler classes) $(D_1; \kappa'_I, \kappa'_J, \kappa'_K)$, $(D_2; \kappa''_I, \kappa''_J, \kappa''_K)$ and the markings p_1, p_2 with $p_1(\kappa'_I) = p_2(\kappa''_J)$, $p_1(\kappa'_J) = p_2(\kappa''_I)$ in L, and $p_1(\kappa'_K) = -p_2(\kappa''_K)$ in $L \otimes \mathbb{R}$ thereby achieving a matching.

Choosing the ambient octic threefolds X'_8, X''_8 for the latter D_1, D_2, blowing up these threefolds to obtain asymptotically cylindrical Ricci-flat threefolds by Theorem 4, and applying Theorem 5 to the respective connected sum, we obtain a simply-connected compact 7-manifold M with a metric of holonomy G_2.

We may consider in a very similar way, in place of one of both X_8's above, a smooth intersection X_6 of a quadric and a cubic in $\mathbb{C}P^5$. The respective K3 divisor then is an intersection of a quadric and a cubic in $\mathbb{C}P^4$ and the image of the Kähler class of this divisor has square 6 in the bilinear form L.

More generally, it was shown in [17, Sect. 6,7] that in place of X_8, X_6 in the above example we can consider any non-singular *Fano threefold* V, i.e. a projective complex 3-dimensional manifold such that the image of the first Chern class $c_1(V)$ in the de Rham cohomology can be represented by some Kähler form on V. Equivalently, the anticanonical bundle K_V^{-1} is ample. Smooth Fano threefolds are completely classified; up to deformations, there are 105 algebraic types [12, 22].

Every Fano threefold V is simply-connected and a generic anticanonical divisor D on V is a (smooth) K3 surface [26]. A threefold $\overline{W}$ is obtained by blowing up a connected complex curve representing the self-intersection cycle $D \cdot D$ (in the sense of the Chow ring). Then $\overline{W}$ and the proper transform of D satisfy the hypotheses of Theorem 4. Alternatively, if $D \cdot D$ is represented by a finite sequence of curves, then $\overline{W}$ may be defined by successively blowing up these curves. We shall refer to any such threefold $\overline{W}$ to be of *Fano type*.

A Kähler K3 surface D and its proper transform in $\overline{W}$ can be assumed isomorphic by choosing an appropriate Kähler metric on $\overline{W}$. Then the cylindrical asymptotic model for W is determined by the K3 surface D with the Kähler metric restricted from V.

For a general Fano V, the class of anticanonical K3 surfaces D arising in the deformations of V will correspond to an open dense subset of *lattice-polarized* K3 surfaces. This latter class is defined by the condition that the Picard lattice $H^{1,1}(D, \mathbb{R}) \cap H^2(D, \mathbb{Z})$ contains a sublattice isomorphic to a fixed lattice N and this sublattice contains a class of some Kähler form. In the case of algebraic K3 surfaces of a fixed degree, as in the example above, N is generated by the Kähler form κ_I induced from the embedding of D in the projective space. In general, N

arises as $\iota^* H^2(V, \mathbb{Z})$ from the embedding $\iota : D \to V$. The rank of N is the Betti number $b^2(V)$ as ι^* in injective by the Lefschetz hyperplane theorem.

Another source of examples for Theorem 4 was given by Lee and the author in [18]. The construction uses K3 surfaces S with *non-symplectic involution*, a holomorphic map $\rho : S \to S$, such that ρ^* restricts to -1 on $H^{2,0}(S)$. The K3 surfaces of this type were completely classified up to deformation by Nikulin [1], who determined the complete system of invariants and fixed point set of ρ for each deformation family. We require the fixed point set of ρ to be non-empty; this occurs in all but one of the 75 deformation families.

Let $\psi : \mathbb{C}P^1 \to \mathbb{C}P^1$ denote the holomorphic involution $\psi(z_0 : z_1) = z_1 : z_0$ fixing exactly two points. The quotient $Z = (S \times \mathbb{C}P^1)/(\rho, \psi)$ is then an orbifold whose singular locus is a disjoint union of smooth curves. The desired 3-fold $\overline{W}$ is defined by the resolution of singularities diagram for Z,

$$\begin{array}{ccc} \widetilde{W} & \longrightarrow & \overline{W} \\ \downarrow & & \downarrow \\ S \times \mathbb{C}P^1 & \longrightarrow & Z, \end{array}$$

where the vertical arrows correspond to blowing up the fixed locus of (ρ, ψ) in $S \times \mathbb{C}P^1$ and the singular locus of Z and the horizontal arrows are the quotient maps.

The anticanonical divisor D on $\overline{W}$ arises as the (pre-)image of $S \times \{p\}$, via the above diagram, where $\psi(p) \neq p$. Such D is clearly isomorphic to the Kähler K3 surface S and has trivial normal bundle in $\overline{W}$. It can be checked that $\overline{W}$ and $W = \overline{W} \setminus D$ are simply-connected (the condition that ρ have fixed points is needed here). Thus W has an asymptotically cylindrical Ricci-flat Kähler metric by Theorem 4.

The pull-back $\iota^* : H^2(\overline{W}, \mathbb{Z}) \to H^3(D, \mathbb{Z})$ defined by the embedding of D makes D into a lattice polarized K3 surface with N corresponding to the sublattice of all classes fixed by ρ^* in $H^2(D, \mathbb{Z})$. On the other hand, ι^* has a kernel of dimension at least 2. A threefold $\overline{W}$ obtained from K3 surface with non-symplectic involution is therefore *never* deformation equivalent to any threefold of Fano type (assuming $D \cdot D$ in the latter threefold was represented by a single curve).

The matching problem in all the examples becomes entirely a consideration on the K3 lattice L, as illustrated by the example in the beginning of this subsection. In general, the argument is more technical and requires results on the lattice embeddings [23].

One simple sufficient (though not necessary) condition for the existence of the Donaldson matching for representatives in the two classes of lattice polarized K3 surfaces is that the rank of each polarizing lattice N_i is ≤ 5.

All the irreducible G_2-manifolds M constructed from threefolds in the above examples are simply-connected. The cohomology of compact irreducible G_2-manifolds M coming from the connected sum construction may be determined by application of the Mayer–Vietoris exact sequence and generally depends on the choice of matching. However, the sum of the Betti numbers

$$b^2(M) + b^3(M) = b^3(\overline{W}_1) + b^3(\overline{W}_2) + 2d_1 + 2d_2 + 23, \tag{15}$$

for any matching, depends only on the threefolds $\overline{W}_i$ and the dimensions d_i of the kernel of $\iota^* : H^2(\overline{W}_i, \mathbb{R}) \to H^3(D_i, \mathbb{R})$. The quantities in (15) can be determined by standard methods (adjunction formula, Lefschetz–Bott hyperplane theorem) from known algebraic invariants of Fano threefolds or, respectively, of non-symplectic involutions.

In particular, the Fano threefold X_8 discussed in Example 1 above has $b^2(X_8) = 1$, $b^3(X_8) = 28$ and its blow-up has $b^2(\overline{W}) = 2$, $b^3(\overline{W}) = 38$. An irreducible compact G_2-manifold M constructed from a pair of X_8's has $b^2(M) = 0$ and then $b^3(M) = 99$ as d_i vanish in this case. This irreducible G_2-manifold is topologically distinct from the examples given by Joyce via resolution of singularities; the only irreducible G_2-manifold in [14] with $b^2 = 0$ has $b_3 = 215$. The property $b^2(M) = 0$ holds in many other examples coming from pairs of threefolds of Fano type and these latter G_2-manifolds typically have smaller b^2 and larger b^3 than the examples given by Joyce. (Note also that every compact irreducible G_2-manifold M must have $b^1(M) = 0$ by Proposition 2 but $b^3(M)$ cannot vanish as the G_2 3-form φ on M is harmonic.)

Corti, Haskins, Nordström and Pacini [5, 6] generalized the class of threefolds of Fano type by considering *weak Fano* threefolds V whose anticanonical bundle K_V^{-1} is only required to be big and nef. (Every such V may be obtained as a resolution of an appropriate singular Fano threefold.) They identified a large subclass called semi-Fano threefolds and generalized for this class the properties required in the construction of G_2-manifolds from threefolds W of Fano type. This generalization dramatically increased the number of examples of connected sum G_2-manifolds. Some of the examples were shown to be 2-connected which allows to determine their diffeomorphism type by computing certain standard invariants.

More recently, Braun [3] gave a toric geometry construction, from certain lattice polytops, of examples of pairs $\overline{W}$, D defining asymptotically cylindrical Calabi–Yau threefolds by Theorem 4. Useful invariants of $\overline{W}$ e.g. the Hodge numbers can be computed by combinatorial formulae.

Nordström [24] gave an interesting generalization of the connected sum construction, by replacing (11) with a different 'hyper-Kähler rotation' and taking finite quotients of asymptotically cylindrical Calabi–Yau threefolds W. Applications of the construction include topologically new examples of compact irreducible G_2-manifolds some of which have a non-trivial finite fundamental group.

In conclusion, we mention two works which contain results concerning relations between the two types of construction of G_2-manifolds discussed in these notes.

Nordström and the author identified in [19] an example of a compact irreducible G_2-manifold given by Joyce [14] where the underlying 7-manifold is diffeomorphic to one obtainable from the construction in [18]. Further, the two respective families of G_2-metrics on this manifold are connected in the G_2-moduli space.

On the other hand, some of the G_2-manifolds given by Joyce cannot possibly be obtained by the connected sum construction. The result is due to Crowley and Nordstrom [7] who constructed an invariant of G_2-structures which is equal to 24 for each connected sum (12) but is odd for some examples in [14].

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