Complex and Calibrated Geometry

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Abstract This is an expository article based on a talk given by the author at the Fields Institute in August 2017 for the *Workshop on G_2 manifolds and related topics*. The aim of the article is to review some recent results of the author [11] investigating the relationship between calibrated and complex geometry.

1 Introduction

Let M be a four-dimensional Calabi–Yau manifold with Ricci-flat Kähler form ω and holomorphic volume form Ω. Since $\mathrm{Hol}(\omega) \subseteq SU(4) \subseteq Spin(7)$, we can think of M as a *Spin*(7)-manifold. In this case, the *Spin*(7)-form or Cayley calibration is given by

$$\Phi = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}\ \Omega.$$

In particular, we can see from the above expression that (M, Φ) has two special types of Cayley submanifold: two-dimensional complex submanifolds (calibrated by $\frac{1}{2}\omega \wedge \omega$) and special Lagrangian submanifolds (calibrated by $\mathrm{Re}\ \Omega$).

Of course, M may admit Cayley submanifolds that are neither complex nor special Lagrangian. One might ask whether such submanifolds can or must be built out of complex and special Lagrangian submanifolds. In this expository article, we will consider the following problem.

Given a compact complex submanifold N of a Calabi–Yau four-fold M, can one deform N as a Cayley submanifold into a Cayley submanifold N' that is not complex?

Of course, it is straightforward to see that the answer to this question is no by the following result of Harvey and Lawson.

Proposition 1.1 ([3, II.4 Thm 4.2]) *Let X be a Riemannian manifold with calibration α and let Y be a compact α-calibrated submanifold of X. Let Y' be any other compact submanifold of X homologous to Y. Then*

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$$\mathrm{vol}(Y) \leq \mathrm{vol}(Y'),$$

with equality if, and only if, Y' is also α-calibrated.

Therefore, if N is a two-dimensional compact complex submanifold of a Calabi–Yau four-fold M, given N' a Cayley deformation of N we have that both

$$\mathrm{vol}(N) \leq \mathrm{vol}(N'),$$

applying Proposition 1.1 to N and N' with calibration $\frac{1}{2}\omega \wedge \omega$, and

$$\mathrm{vol}(N') \leq \mathrm{vol}(N),$$

applying Proposition 1.1 to N' and N with the Cayley calibration Φ. But then we must have $\mathrm{vol}(N) = \mathrm{vol}(N')$ and so Proposition 1.1 with calibration $\frac{1}{2}\omega \wedge \omega$ tells us that N and N' must both be complex submanifolds.

In this article, we explore the geometric reasons for this result, and the implications for complex submanifold theory. The material in this article is based on the author's PhD thesis [12] and paper [11].

2 Deformation Theory of Calibrated Submanifolds

Given a manifold with a calibration, one would like to be able to describe its calibrated submanifolds. One way of doing this is to study the *moduli space* of a certain type of calibrated submanifold. The first study of a moduli space of calibrated submanifolds may be attributed to Kodaira [6], who studied the moduli space of compact complex submanifolds of a complex manifold, which we describe in Sect. 2.1 below, although this result predates the definition of calibration by some twenty years! Later, motivated by *Calibrated geometries*, McLean [10] sought to prove analogues of Kodaira's result for calibrated submanifolds inside manifolds with special holonomy. We will review McLean's results on compact Cayley submanifolds in Sect. 2.2.

2.1 Kodaira's Deformation Theory of Compact Complex Submanifolds

Kodaira's approach to the deformation theory of complex submanifolds uses techniques from algebraic geometry. His approach is completely different from the later method of McLean, but it will be interesting to quote and interpret Kodaira's result here in order to compare to our work later.

Let M be a complex manifold with compact complex submanifold N. Denote by $H^k(N, \nu_M^{1,0}(N))$ the kth sheaf cohomology group of the sheaf of holomorphic sections of the holomorphic normal bundle of N in M. Define the moduli space $\mathcal{M}$ of complex deformations of N in M to be the set of complex submanifolds N' of M so that there exists a diffeomorphism $N \to N'$ isotopic to the identity.

Theorem 2.1 ([6, Main Thm]) *Let M be a complex manifold with compact complex submanifold N. If $H^1(N, \nu_M^{1,0}(N)) = 0$, then $\mathcal{M}$ is a smooth manifold of dimension* $\dim_{\mathbb{R}} H^0(N, \nu_M^{1,0}(N))$.

Remark We call $H^0(N, \nu_M^{1,0}(N))$ the *infinitesimal complex deformations* of N, and $H^1(N, \nu_M^{1,0}(N))$ the *obstruction space*. Note that the vanishing of the obstruction space is sufficient, but not necessary.

We can apply Dolbeault's theorem [2, pg 45] and the Hodge decomposition theorem [4, Thm 4.1.13, Cor 4.1.14] to rephrase Kodaira's theorem in terms of a differential operator.

Corollary 2.2 *Let M be a complex manifold with compact complex submanifold N. Then the space of infinitesimal complex deformations of N is isomorphic to the kernel of*

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Moreover, the obstruction space is isomorphic to the kernel of

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,2}N \otimes \nu_M^{1,0}(N) \oplus \nu_M^{1,0}(N)).$$

2.2 McLean's Deformation Theory of Compact Cayley Submanifolds

McLean's goal in his 1998 paper [10] was to prove analogous results to Kodaira's Theorem 2.1 for compact calibrated submanifolds of manifolds with special holonomy. In particular, McLean proved the following result on the moduli space of Cayley deformations of a compact Cayley submanifold that admits a spin structure.

Theorem 2.3 ([10, Thm 6-3]) *Let Y be a compact Cayley submanifold of a Spin(7)-manifold X, and suppose that Y admits a spin structure. Then there exists a rank two complex vector bundle A over Y so that the Zariski tangent space to the moduli space of Cayley deformations of Y in X is given by the kernel of the twisted Dirac operator*

$$\not{D} : C^\infty(\mathbb{S}_+ \otimes A) \to C^\infty(\mathbb{S}_- \otimes A),$$

where $\mathbb{S}_+$ and $\mathbb{S}_-$ are respectively the bundles of positive and negative spinors on Y.

Here, elements of the kernel of $\not{D}$ are called *infinitesimal deformations*, while the cokernel of $\not{D}$ is called the *obstruction space*. This is because if the obstruction space is trivial, then the moduli space of Cayley deformations is a smooth manifold.

Sketch proof By the work of Harvey and Lawson [3, IV.1.C Cor 1.29] there exists a bundle-valued differential form $\tau \in \Omega^4(\Lambda^2_7)$ on X, where Λ^2_7 is the seven dimensional representation of *Spin*(7) acting on two-forms on X, satisfying for any oriented four-dimensional submanifold W of X,

$$\tau|_W \equiv 0,$$

if, and only if, W is a Cayley submanifold (up to a choice of orientation on W). This bundle-valued four-form can be described succinctly by the following expression. For orthogonal tangent vectors x, y, z, w define

$$\tau(x, y, z, w) = \pi_7(\Phi(\,\cdot\,, y, z, w) \wedge x^\flat), \tag{2.1}$$

where $\pi_7 : \Lambda^2 X \to \Lambda^2_7$ is the projection map given by

$$\pi_7(u^\flat \wedge v^\flat) = \frac{1}{2}(u^\flat \wedge v^\flat + \Phi(u, v, \,\cdot\,, \,\cdot\,))$$

and $\flat : TX \to T^*X$ denotes the musical isomorphism. Recall that if Y is a Cayley submanifold, then we can view [10, pg 741] $\Lambda^2_+ Y$ as a subbundle of $\Lambda^2_7|_Y$ via the map $\alpha \mapsto \pi_7(\alpha)$. We will denote by E the orthogonal complement to $\Lambda^2_+ Y$ in $\Lambda^2_7|_Y$, so that

$$\Lambda^2_7|_Y \cong \Lambda^2_+ Y \oplus E.$$

The tubular neighbourhood theorem [8, IV Thm 5.1] allows us to identify small normal vector fields on Y with small deformations of Y. If v is a normal vector field on Y, write $\exp_v := \exp \circ v : Y \to X$, with $Y_v := \exp_v(Y)$, the deformation corresponding to v. Then we can identify the moduli space of Cayley deformations of Y in X with the zero set of the following partial differential operator:

$$\begin{aligned} F : C^\infty(\nu_X(Y)) &\to C^\infty(E), \\ v &\mapsto \pi_E(* \exp_v^*(\tau|_{Y_v})), \end{aligned} \tag{2.2}$$

where $*$ denotes the Hodge star of Y and $\pi_E : \Lambda^2_7|_Y \to E$ denotes the projection map. The linear part of the operator at zero is

$$dF|_0(v) = \left.\frac{d}{dt}\right|_{t=0} F(tv) = \pi_E(* \mathcal{L}_v \tau|_Y).$$

This can be computed explicitly as the following operator (see for example [13, Prop 2.3]). Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal frame for TX, with dual coframe $\{e^1, e^2, e^3, e^4\}$. Define

$$\begin{aligned} D : C^\infty(\nu_X(Y)) &\to C^\infty(E), \\ v &\mapsto \sum_{i=1}^{4} \pi_7(e^i \wedge \nabla^\perp_{e_i} v), \end{aligned} \tag{2.3}$$

where $\nabla^\perp$ is the connection on the normal bundle of Y in X induced by the Levi-Civita connection of X. To deduce McLean's result, first observe that [10, pg 741] there exists a rank two complex vector bundle A so that

$$\begin{aligned} \nu_X(Y) \otimes \mathbb{C} &\cong \mathbb{S}_+ \otimes A \\ E \otimes \mathbb{C} &\cong \mathbb{S}_- \otimes A. \end{aligned}$$

Then McLean's result may be deduced by showing that the following diagram commutes

$$\begin{array}{ccc} C^\infty(\mathbb{S}_+ \otimes A) & \xrightarrow{\not{D}} & C^\infty(\mathbb{S}_- \otimes A) \\ \downarrow & & \downarrow \\ C^\infty(\nu_M(N) \otimes \mathbb{C}) & \xrightarrow{D} & C^\infty(E \otimes \mathbb{C}) \end{array}$$

□

To study the kernel of the operator defined in (2.2), we can extend the map F to some Banach spaces and try to apply the Banach space implicit function theorem [7, Ch 6 Thm 2.1]. To do this, we first need the linear part of F, which is the map D defined in (2.3), to be Fredholm, which it is since D is elliptic and Y is compact. Moreover we need D to surject, which unfortunately is not true in general. The cokernel of D describes the subspace of $C^\infty(E)$ that D does not reach, and hence obstructions to elements of the kernel of D, infinitesimal Cayley deformations of Y, to be extended to true Cayley deformations of Y. However, if the obstruction space vanishes, then we may apply the implicit function theorem and deduce that every infinitesimal Cayley deformation of Y extends to a true Cayley deformation of Y.

3 Cayley Deformations of Compact Complex Submanifolds

We now focus on Cayley deformations of a compact complex surface inside a Calabi–Yau four-fold. We saw in the introduction that it is very easy to see from the work of Harvey and Lawson that there are no Cayley deformations of a compact complex surface in a Calabi–Yau four-fold that are not complex deformations. This method is highly efficient, clean and compact, but doesn't leave us with any geometric intuition for why one cannot deform a compact complex submanifold into a Cayley submanifold that isn't complex. In particular, it is known that without the assumption that the submanifold is complex, we may deform such a submanifold into not only a Cayley submanifold that is not complex, but a special Lagrangian submanifold, as we will see in the following example.

Example ([9, Ex 5.8]) Consider $\mathbb{R}^8$ with the standard *Spin*(7)-structure (Φ_0, g_0). Writing any nonzero point of $\mathbb{R}^8$ as (r, p), where $r \in (0, \infty)$ and $p \in S^7$ we define a G_2-structure (φ, h) on S^7 by

$$\Phi_0|_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 *_h \varphi|_p,$$

with h the usual round metric. Notice that since Φ_0 is closed, $d\varphi = 4 * \varphi$, so this G_2-structure is not torsion-free. Then it is easy to check that a cone $C = (0, \infty) \times L$ is a Cayley submanifold of $(\mathbb{R}^8, \Phi_0, g_0)$ if, and only if, L is an associative submanifold of (S^7, φ, h). Homogeneous associative submanifolds of S^7 were classified by Lotay [9], including the following family, diffeomorphic to $SU(2)/\mathbb{Z}_3$. The deformation theory of homogeneous associative submanifolds of S^7 was studied by Kawai [5], while a comparative study of deformations of Cayley cones can be found in a paper of the author [13, Sect. 5].

Consider the following action of $SU(2)$ on $\mathbb{C}^4$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} a^3 z_1 + \sqrt{3}a^2 b z_2 + \sqrt{3}ab^2 z_3 + b^3 z_4 \\ -\sqrt{3}a^2\bar{b}z_1 + a(|a|^2 - 2|b|^2)z_2 + b(2|a|^2 - |b|^2)z_3 + \sqrt{3}\bar{a}b^2 z_4 \\ \sqrt{3}a\bar{b}^2 z_1 - \bar{b}(2|a|^2 - |b|^2)z_2 + \bar{a}(|a|^2 - 2|b|^2)z_3 + \sqrt{3}\bar{a}^2 b z_4 \\ -\bar{b}^3 z_1 + \sqrt{3}\bar{a}\bar{b}^2 z_2 - \sqrt{3}\bar{a}^2\bar{b}z_3 + \bar{a}^3 z_4 \end{pmatrix},$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. We define $L(\theta)$ to be the orbit of the point $(\cos\theta, 0, 0, \sin\theta)^T$ under the above action, that is,

$$L(\theta) := \begin{pmatrix} a^3\cos\theta + b^3\sin\theta \\ -\sqrt{3}a^2\bar{b}\cos\theta + \sqrt{3}\bar{a}b^2\sin\theta \\ \sqrt{3}a\bar{b}^2\cos\theta + \sqrt{3}\bar{a}^2 b\sin\theta \\ -\bar{b}^3\cos\theta + \bar{a}^3\sin\theta \end{pmatrix},$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. Then for

$$\mathbb{Z}_3 := \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} \in SU(2) \mid \zeta^3 = 1 \right\},$$

$L(\theta)$ is invariant under the action of $\mathbb{Z}_3$ for all θ, therefore $L(\theta) \cong SU(2)/\mathbb{Z}_3$.

We have that $L(\theta)$ is associative for $\theta \in [0, \frac{\pi}{4}]$. It is easy to check that $L(0) = L$ is the real link of a complex cone, whereas $L(\frac{\pi}{4})$ is the link of a special Lagrangian cone. Therefore $C(\theta) := (0, \infty) \times L(\theta)$ defines a family of Cayley cones in $\mathbb{C}^4$. In particular, this example shows that we can deform a complex cone into a special Lagrangian cone through Cayley cones. Notice that Harvey and Lawson's result, Proposition 1.1, doesn't apply in this situation because the cone is not compact.

3.1 The Cayley Operator on a Complex Submanifold

Let (M, J, ω, Ω) be a four-dimensional Calabi–Yau manifold and let N be a two-dimensional compact complex submanifold of M. We want to compare complex and Cayley deformations of N, but we already have some clues to help us. By Kodaira's Theorem 2.1 in combination with Dolbeault's theorem we know that infinitesimal complex deformations of N in M are given by the kernel of

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)). \tag{3.1}$$

Meanwhile, the work of McLean tells us that infinitesimal Cayley deformations of (a spin manifold) N in M are given by the kernel of the twisted Dirac operator

$$\not{D} : C^\infty(\mathbb{S}_+ \otimes A) \to C^\infty(\mathbb{S}_- \otimes A). \tag{3.2}$$

At first glance, comparing the kernels of these two operators seems like a fruitless task. However, since N is Kähler, its spin structure and Dirac operator take a special form [1, pg 82]. Given a two-dimensional Kähler manifold with a fixed spin structure, we can identify

$$\mathbb{S}_+ \cong \left(\Lambda^{0,0}N \oplus \Lambda^{0,2}N\right) \otimes S_k,$$
$$\mathbb{S}_- \cong \Lambda^{0,1}N \otimes S_k,$$

where S_k is a holomorphic line bundle satisfying $S_k \otimes S_k = \Lambda^{2,0}N$. Under these identifications, the Dirac operator is

$$\sqrt{2}\left(\bar{\partial} + \bar{\partial}^*\right) : C^\infty(S_k \oplus \Lambda^{0,2}N \otimes S_k) \to C^\infty(\Lambda^{0,1}N \otimes S_k). \tag{3.3}$$

We have already seen that McLean proved that $\nu_M(N) \otimes \mathbb{C} \cong \mathbb{S}_+ \otimes A$ and $E \otimes \mathbb{C} \cong \mathbb{S}_- \otimes A$, for some rank two complex vector bundle A. Comparing the three operators (3.1), (3.2) and (3.3) it is not unreasonable to hope that we can identify

$$\nu_M(N) \otimes \mathbb{C} \cong \nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N),$$
$$E \otimes \mathbb{C} \cong \Lambda^{0,1}N \otimes \nu_M^{1,0}(N),$$

and for us to be able to show that under these identifications infinitesimal Cayley deformations of N are given by the kernel of the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

This turns out to be true—and despite the heuristic comparison above, in fact N is not required to be spin for the following result, taken from [11, Prop 3.5 and Thm 3.9], to hold.

Theorem 3.1 *Let N be a compact complex surface inside a Calabi–Yau four-fold M. Then infinitesimal Cayley deformations of N in M are given by the kernel of the operator*

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)).$$

Moreover, the expected dimension of the moduli space of Cayley deformations of N in M is given by the index of this operator

$$\operatorname{ind}\left(\bar{\partial} + \bar{\partial}^*\right) = \frac{1}{2}\sigma(N) + \frac{1}{2}\chi(N) - [N] \cdot [N],$$

where $\sigma(N)$ is the signature of N, $\chi(N)$ is the Euler characteristic of N and $[N] \cdot [N]$ is the self-intersection number of N.

Sketch proof We have that the complex structure on M induces a natural splitting of the complexified normal bundle of N in M into holomorphic and anti-holomorphic parts

$$\nu_M(N) \otimes \mathbb{C} \cong \nu_M^{1,0}(N) \oplus \nu_M^{0,1}(N).$$

We would like to show that

$$\nu_M^{0,1}(N) \cong \Lambda^{0,2}N \otimes \nu_M^{1,0}(N).$$

To understand why this might be true, we consider the holomorphic volume form Ω of M, which is a nowhere-vanishing, parallel section of the canonical bundle of M, denoted by $K_M := \Lambda^{4,0}M$. Recall that the adjunction formula [4, Prop 2.2.17] says that

$$K_M|_N \cong \Lambda^{2,0}N \otimes \Lambda^2 \nu_M^{*1,0}(N). \tag{3.4}$$

In particular, $\overline{\Omega}|_N$ is a well-defined nowhere-vanishing section of $\Lambda^{0,2}N \otimes \Lambda^2 \nu_M^{*0,1}(N)$. So given any section of $\nu_M^{0,1}(N)$ is is easy to check that

$$\overline{\Omega}(v, \cdot, \cdot, \cdot)|_N, \tag{3.5}$$

is a well-defined section of $\Lambda^{0,2}N \otimes \nu_M^{*0,1}(N)$. Finally, the Riemannian metric on M defines a musical isomorphism $\sharp : \nu_M^{*0,1}(N) \to \nu_M^{1,0}(N)$. It is easy to verify that these objects provide the desired isomorphism.

It is simple to check using local coordinates that $E \cong \Lambda^{0,1}N \otimes \nu_M^{1,0}(N)$, again with the help of the musical isomorphism, and moreover that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) & \xrightarrow{\bar{\partial}+\bar{\partial}^*} & C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)) \\ \downarrow & & \downarrow \\ C^\infty(\nu_M(N) \otimes \mathbb{C}) & \xrightarrow{\quad D \quad} & C^\infty(E \otimes \mathbb{C}) \end{array}$$

where D was defined in (2.3).

The index formula follows from the Hirzebruch–Riemann–Roch Theorem [4, Cor 5.1.4]. □

Example Let

$$M := \{[z_0 : \cdots : z_5] \in \mathbb{C}P^5 \mid z_0^6 + \cdots + z_5^6 = 0\},$$

and take

$$N = \{z \in M \mid f_1(z) = f_2(z) = 0\},$$

where f_i are irreducible homogeneous polynomials of degree d_i such that the Jacobian of $g = (f_1, f_2)$ has rank two at each point of N. Then we can compute that

$$\begin{aligned}[N] \cdot [N] &= 6d_1^2 d_2^2,\\ \chi(N) &= 90d_1 d_2 + 6d_1^3 d_2 + 6d_2^2 d_1 + 6d_1^2 d_2^2,\\ \sigma(N) &= -60d_1 d_2 - 2d_1^3 d_2 - 2d_2^3 d_1,\end{aligned}$$

so that

$$\text{ind}(\bar{\partial} + \bar{\partial}^*) = d_1 d_2 (15 + 2d_1^2 + 2d_2^2 - 3d_1 d_2).$$

Examining this expression, we see that the expected dimension of the moduli space of Cayley deformations of N in M will be strictly positive and even for any $d_1, d_2 \in \mathbb{N}$.

4 Complex Deformations of a Compact Complex Surface

We would like to compare complex and Cayley deformations of a compact complex surface N in a Calabi–Yau four-fold (M, J, ω, Ω). So far we have seen, by Kodaira's Theorem 2.1, that infinitesimal complex deformations of N are given by holomorphic normal vector fields in the kernel of

$$\bar{\partial} : C^\infty(\nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)), \tag{4.1}$$

with the dimension of the space of infinitesimal complex deformations of N given by the real dimension of the kernel of (4.1).

We saw in Sect. 3 that the infinitesimal Cayley deformations of N are given by forms in the kernel of the operator

$$\bar{\partial} + \bar{\partial}^* : C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)), \tag{4.2}$$

with the dimension of the space of infinitesimal Cayley deformations of N given by the *complex* dimension of the kernel of (4.2) (since we complexified the normal bundle of N in M to find this operator).

At first glance, comparing the above operators this seems to be a mistake, but it turns out that there is an isomorphism between the kernel of (4.1) and the kernel of

$$\bar{\partial}^* : C^\infty(\Lambda^{0,2}N \otimes \nu_M^{1,0}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N)). \tag{4.3}$$

The following result is taken from [11, Lem 4.6].

Lemma 4.1 *Let N be a complex surface in a Calabi–Yau four-fold (M, J, ω, Ω). Then the kernels of (4.1) and (4.3) are isomorphic.*

Proof (*Sketch*) Similar to the proof of Theorem 3.1 where we constructed an isomorphism $\nu_M^{0,1}(N) \to \Lambda^{0,2}N \otimes \nu_M^{1,0}(N)$, we take the map

$$\begin{aligned} \nu_M^{1,0}(N) &\to \Lambda^{0,2}N \otimes \nu_M^{1,0}(N), \\ v &\mapsto \left(\bar{v} \lrcorner \overline{\Omega}\right)^\sharp, \end{aligned}$$

where $\sharp : \nu_M^{*0,1}(N) \to \nu_M^{1,0}(N)$ is the standard musical isomorphism. That this map sends Ker $\bar{\partial}$ to Ker $\bar{\partial}^*$ is essentially a consequence of Ω being parallel. □

What we have seen so far suggests therefore that an infinitesimal Cayley deformation of N that is not an infinitesimal complex deformation of N looks like $v \oplus w \in C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2}N \otimes \nu_M^{1,0}(N))$ with

$$\bar{\partial}v = -\bar{\partial}^* w \neq 0.$$

We know by Hodge theory that this cannot happen when N is compact—so this would explain why we cannot deform a compact complex surface into a Cayley submanifold that is not complex.

To make these ideas more formal, we will argue in the style of McLean to characterise complex deformations of a compact complex surface. We will first look for a differential form that vanishes exactly when restricted to a complex surface.

Firstly, let us take a Cayley submanifold N' of a Calabi–Yau four-fold (M, J, ω, Ω). We have that

$$\tau|_{N'} \equiv 0,$$

and

$$\Phi|_{N'} = \text{Re } \Omega|_{N'} + \frac{1}{2}\omega \wedge \omega|_{N'} = \text{vol}_{N'}.$$

So to further ensure that N' is complex, we must ask that

$$\text{Re } \Omega|_{N'} \equiv 0.$$

So we see that $v \in C^\infty(\nu_M(N) \otimes \mathbb{C})$ defines a complex deformation of N if, and only if

$$G(v) = (\exp_v^* \tau|_{N_v}, \exp_v^* \operatorname{Re} \Omega|_{N_v}) = (0, 0).$$

We ask how the linearisation of G at zero differs from the linearisation of F defined in (2.2) at zero. Finding the linear part of $\exp_v^* \operatorname{Re} \Omega|_{N_v}$, we see that

$$\left.\frac{d}{dt}\right|_{t=0} \exp_{tv} \operatorname{Re} \Omega|_{N_{tv}} = \mathcal{L}_v \operatorname{Re} \Omega = \frac{1}{2} d(v \lrcorner \Omega + v \lrcorner \overline{\Omega}).$$

Writing $v = v_1 \oplus v_2$, where $v_1 \in C^\infty(\nu_M^{1,0}(N))$ and $v_2 \in C^\infty(\nu_M^{0,1}(N))$, we see that

$$\frac{1}{2} d(v \lrcorner \Omega + v \lrcorner \overline{\Omega}) = \frac{1}{2} d(v_1 \lrcorner \Omega + v_2 \lrcorner \overline{\Omega}) = \frac{1}{2} \bar{\partial}(v_1 \lrcorner \Omega) + \frac{1}{2} \partial(v_2 \lrcorner \overline{\Omega}),$$

since $v_1 \lrcorner \Omega \in C^\infty(\Lambda^{2,0} N \otimes \nu_M^{*1,0}(N))$ and $v_2 \lrcorner \overline{\Omega} \in C^\infty(\Lambda^{0,2} N \otimes \nu_M^{*0,1}(N))$. Therefore a normal vector field $v = v_1 \oplus v_2$ is an infinitesimal complex deformation of N if, and only if, the linearisation of the first component of G vanishes, which by Theorem 3.1 is

$$\bar{\partial} v_1 + \bar{\partial}^*(v_2 \lrcorner \overline{\Omega}) = 0,$$

where we recall the isomorphism of $\nu_M^{0,1}(N)$ and $\Lambda^{0,2} N \otimes \nu_M^{1,0}(N)$ given in Eq. (3.5), and the linearisation of the second component of G vanishes, which as we've just seen is

$$\bar{\partial}(v_1 \lrcorner \Omega) = 0 = \partial(v_2 \lrcorner \overline{\Omega}),$$

since $\bar{\partial}(v_1 \lrcorner \Omega)$ and $\partial(v_2 \lrcorner \overline{\Omega})$ take values in different vector bundles. Similarly to Lemma 4.1, we can show that this is equivalent to

$$\bar{\partial} v_1 = 0 = \bar{\partial}^*(v_2 \lrcorner \overline{\Omega}),$$

which moreover by definition of the isomorphism $\nu_M^{0,1}(N) \to \Lambda^{0,2} N \otimes \nu_M^{1,0}(N)$ in the proof of Theorem 3.1 is equivalent to the set of $v \oplus w \in C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2} N \otimes \nu_M^{1,0}(N))$ such that

$$\bar{\partial} v = 0 = \bar{\partial}^* w.$$

This informal argument shows that our assertion that infinitesimal Cayley deformations of N expressed in the form $v \oplus w \in C^\infty(\nu_M^{1,0}(N) \oplus \Lambda^{0,2} N \otimes \nu_M^{1,0}(N))$ that are not infinitesimal complex deformations must satisfy

$$\bar{\partial} v = -\bar{\partial}^* w,$$

is correct.

It turns out that we can study the complex deformations of N in M without thinking about Cayley deformations at all.

Let us think about the holomorphic volume form. The adjunction formula tells us that, for N, a complex surface inside a four-dimensional Calabi–Yau manifold M with holomorphic volume form Ω, we have

$$\Omega|_N \in K_M|_N \cong K_N \otimes \Lambda^2 \nu_M^{*1,0}(N).$$

So this tells us that given any three tangent vector fields v_1, v_2 and v_3 on N we must have that

$$\Omega(v_1, v_2, v_3, \cdot) = 0.$$

It is natural to wonder whether conversely, if given any three tangent vector fields v_1, v_2 and v_3 to a real oriented four manifold W in a Calabi–Yau four-fold M we have

$$\Omega(v_1, v_2, v_3, \cdot) = 0,$$

then W must a complex submanifold of M. This is not quite right, but a similar result turns out to be true, as we show in [11, Prop 4.2].

Proposition 4.2 *An oriented four-dimensional real submanifold X of a four-dimensional Calabi–Yau manifold (M, J, ω, Ω) is a complex submanifold if, and only if,*

$$\sigma|_X \equiv 0,$$

where $\sigma \in C^\infty(\Lambda^3 M \otimes T^*M|_X)$ is defined by

$$\sigma(v_1, v_2, v_3) = \text{Re}\, \Omega(v_1, v_2, v_3, \cdot),$$

for any $v_1, v_2, v_3 \in C^\infty(TX)$.

This result is purely an exercise in linear algebra. It suffices to check that the proposition holds for a linear subspace of $\mathbb{C}^4$.

Example Let ω and Ω be the standard Kähler form and holomorphic volume form on $\mathbb{C}^3$. Then we can define a G_2-structure on $\mathbb{R}^7$ by

$$\varphi = dx \wedge \omega + \text{Re}\, \Omega,$$
$$*\varphi = \frac{1}{2}\omega \wedge \omega - dx \wedge \text{Im}\, \Omega,$$

where x is the coordinate on $\mathbb{R}$ in $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$. Introducing another factor of $\mathbb{R}$ with coordinate t, we can take the following Calabi–Yau structure on $\mathbb{C}^4$

$$\tilde{\omega} = dt \wedge dx + \omega,$$
$$\tilde{\Omega} = (dt + i dx) \wedge \Omega.$$

Let N be a four-dimensional real submanifold of $\mathbb{C}^3 \subseteq \mathbb{C}^4$. By Proposition 4.2, N is complex in $\mathbb{C}^4$ if, and only if,

$$\text{Re } \tilde{\Omega} = dt \wedge \text{Re } \Omega - dx \wedge \text{Im } \Omega = 0,$$

as a three-form on N. So in particular, we must have that

$$\text{Re } \Omega|_N = 0 = \text{Im } \Omega|_N.$$

This in combination with the fact that $N \subseteq \mathbb{C}^3$ shows that

$$\varphi|_N = 0,$$

and so N is a coassociative submanifold of $\mathbb{R}^7$, and moreover

$$\text{vol}_N = *\varphi|_N = \frac{1}{2}\omega \wedge \omega|_N - dx \wedge \text{Im } \Omega|_N = \frac{1}{2}\omega \wedge \omega|_N,$$

so if N is a complex submanifold of $\mathbb{C}^4$ and is contained in $\mathbb{C}^3$ then it is also a complex submanifold of $\mathbb{C}^3$.

It turns out that we can generalise this idea to study any complex submanifold of a Calabi–Yau manifold.

Proposition 4.3 *Let (M, J, ω, Ω) be an m-dimensional Calabi–Yau manifold and let $p \in \mathbb{N}$ be such that $p < m - 1$. Then an oriented $2p$-dimensional real submanifold X of M is a complex submanifold of M if, and only if,*

$$\sigma|_X = 0,$$

where $\sigma \in C^\infty(\Lambda^{p+1}M \otimes \Lambda^{m-p-1}M)$ is given by

$$\sigma(v_1, \dots v_{p+1}) = \text{Re } \Omega(v_1, \dots, v_{p+1}, \cdot, \dots, \cdot),$$

for any $v_1, \dots v_{p+1} \in C^\infty(TX)$. If $p + 1 = m$, then we must have that

$$\text{Re } \Omega|_X = 0 = \text{Im } \Omega|_X.$$

Example Applying Proposition 4.3 to the previous example, a complex surface N in $\mathbb{C}^3$ must satisfy

$$\text{Re } \Omega|_N = 0 = \text{Im } \Omega|_N,$$

so considering N as a submanifold of $\mathbb{C}^4$ this implies that as a three-form

$$\text{Re } \tilde{\Omega}|_N = (dx \wedge \text{Re } \Omega)|_N - (dx \wedge \text{Im } \Omega)|_N = 0,$$

so N is also a complex submanifold of $\mathbb{C}^4$ as one would expect.

Given Proposition 4.3, we can study complex deformations of complex submanifolds of Calabi–Yau manifolds in a similar style to McLean's Theorem 2.3. We focus on the special case of compact complex surfaces inside Calabi–Yau four-folds here, but the result below holds for any compact complex submanifold of a Calabi–Yau manifold, see [11, Prop 4.4, Prop 4.5]. Notice that this recovers a special case of Kodaira's theorem [6, Main Thm] using a completely different method.

Theorem 4.4 *Let N be a two-dimensional compact complex submanifold of a four-dimensional Calabi–Yau manifold M. Then the moduli space of complex deformations of N in M is locally homeomorphic to the zero set of a partial differential operator*

$$G : C^\infty(\nu_M^{1,0}(N) \oplus \nu_M^{0,1}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N) \oplus \Lambda^{1,0}N \otimes \nu_M^{0,1}(N)),$$

with linearisation at zero given by

$$dG|_0(v_1 \oplus v_2) = \bar{\partial} v_1 \oplus \partial v_2. \tag{4.4}$$

Remark Notice that the kernels of $\bar{\partial}$ and ∂ acting on holomorphic and anti-holomorphic vector fields are naturally isomorphic by complex conjugation.

Sketch proof It is clear by Proposition 4.2 that given a complexified normal vector field v, the corresponding deformation N_v is a complex submanifold of M if, and only if,

$$\exp_v^*(\sigma|_{N_v}) = 0.$$

This is a three-form on N that takes values in $T^*M|_N \otimes \mathbb{C}$. A local argument, [11, Prop 4.4] shows that is suffices to check only the parts of this form that take values in the space $\Lambda^{2,1}N \otimes \nu_M^{*^{1,0}}(N) \oplus \Lambda^{1,2}N \otimes \nu_M^{*^{0,1}}(N)$. Denote the projection onto this vector bundle by π. Then N_v is a complex submanifold if, and only if,

$$\pi(\exp_v^*(\sigma|_{N_v})) = 0.$$

Finally, we have that the maps

$$\Lambda^{0,1}N \otimes \nu_M^{1,0}(N) \to \Lambda^{2,1}N \otimes \nu_M^{*1,0}(N),$$
$$\Lambda^{1,0}N \otimes \nu_M^{0,1}(N) \to \Lambda^{1,2}N \otimes \nu_M^{*0,1}(N),$$

given respectively by

$$\alpha \otimes v \mapsto \alpha \wedge (v \lrcorner \Omega)|_N,$$
$$\tilde{\alpha} \otimes \tilde{v} \mapsto \tilde{\alpha} \wedge (\tilde{v} \lrcorner \overline{\Omega})|_N,$$

define vector bundle isomorphisms [11, Lem 4.3]. Denoting these isomorphisms by Ψ, we finally define the partial differential operator whose kernel can be identified with the moduli space of complex deformations of N in M to be

$$G : C^\infty(\nu_M^{1,0}(N) \oplus \nu_M^{0,1}(N)) \to C^\infty(\Lambda^{0,1}N \otimes \nu_M^{1,0}(N) \oplus \Lambda^{1,0}N \otimes \nu_M^{0,1}(N)),$$
$$v \mapsto \Psi^{-1} \circ \pi(\exp_v^*(\sigma|_{N_v})).$$

A short computation [11, Prop 4.5] shows that the linearisation of G at zero is the operator (4.4) as claimed. □

5 Further and Future Work

5.1 Can We Describe the Moduli Space of Complex Submanifolds in Any Ambient Complex Manifold Using These Techniques?

As long as the ambient manifold is Kähler, its complex manifolds are calibrated submanifolds. In the work described here the existence of a parallel, nowhere vanishing $(m, 0)$-form on the ambient (complex m-dimensional) manifold is essential. If the ambient manifold were Kähler with a nowhere vanishing $(m, 0)$-form that was not parallel, one could repeat the above argument, with the linearised operator having additional zero-order terms.

5.2 Can These Results Be Extended to Noncompact Complex Submanifolds?

As was mentioned in Sect. 3, we expect a noncompact complex submanifold of a Calabi–Yau four-fold will admit Cayley deformations that are not complex, as evidenced by the given example. The author has studied *conically singular* Cayley and complex submanifolds [13], and has shown that infinitesimal Cayley and complex deformations of a conically singular complex surface in a Calabi–Yau four-fold are still of the same type. Note that Harvey and Lawson's result stated in Proposition 1.1 continues to hold for *currents* with compact support and therefore will apply in the setting of conically singular calibrated submanifolds.

The analytic techniques available for studying deformations of a conically singular submanifold only allow one to consider somewhat rigid classes of deformations. For example, it is not possible to deform a conically singular calibrated submanifold into a non-singular calibrated submanifold using current techniques. An interesting problem would be to produce new techniques to study wider classes of deformations of singular calibrated submanifolds, perhaps inspired by techniques from algebraic geometry.

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