Laplacian Flow for Closed G_2 Structures

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Abstract This is an expository article based on the author's talk in *Workshop on* G_2 *Manifolds and Related Topics* held in August 2017 at The Fields Institute. The aim is to explain the results obtained recently by the author and Jason D. Lotay on the Laplacian flow for closed G_2 structures and some related progress.

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1 G_2 Structures on 7-Manifolds

The group G_2 is one of the exceptional holonomy groups and is defined as the stabilizer of the following 3-form on the 7-dimensional Euclidean space $\mathbb{R}^7$:

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ with respect to the basis $\{e^1, e^2, \ldots, e^7\}$ of $\mathbb{R}^7$. The group G_2 is a compact, connected, simply-connected, simple Lie subgroup of SO(7) of dimension 14. The group G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the Euclidean metric and orientation on $\mathbb{R}^7$. If $*_\phi$ denotes the Hodge star determined by the metric and orientation, then G_2 also preserves the 4-form $*_\phi\phi$.

Let M be a 7-manifold. We say a 3-form φ on M is definite if for $x \in M$ there exists an homomorphism $u \in \mathrm{Hom}_{\mathbb{R}}(T_xM, \mathbb{R}^7)$ such that $u^*\phi = \varphi_x$. The space of definite 3-forms on M will be denoted by $\Omega^3_+(M)$. Since ϕ is invariant under the action of the group G_2, each definite 3-form will define a G_2 structure on M. The existence of G_2 structures is equivalent to the property that the manifold M is oriented

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and spin. Note that as G_2 is a subgroup of SO(7), a G_2 structure φ defines a unique Riemannian metric $g = g_\varphi$ on M and an orientation such that

$$g_\varphi(u, v)\mathrm{vol}_{g_\varphi} = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi, \quad \forall\, u, v \in C^\infty(TM).$$

The metric and orientation determine the Hodge star operator $*_\varphi$, and we define $\psi = *_\varphi \varphi$, which is sometimes called a positive 4-form. Notice that the relationship between g_φ and φ, and hence between ψ and φ, is nonlinear.

1.1 Type Decomposition of k-Forms

The group G_2 acts irreducibly on $\mathbb{R}^7$ (and hence on $\Lambda^1(\mathbb{R}^7)^*$ and $\Lambda^6(\mathbb{R}^7)^*$), but it acts reducibly on $\Lambda^k(\mathbb{R}^7)^*$ for $2 \le k \le 5$. Hence a G_2 structure φ induces splittings of the bundles $\Lambda^k T^*M$ $(2 \le k \le 5)$ into direct summands, which we denote by $\Lambda^k_l(T^*M, \varphi)$ so that l indicates the rank of the bundle. We let the space of sections of $\Lambda^k_l(T^*M, \varphi)$ be $\Omega^k_l(M)$. We have that

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M), \qquad \Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),$$

where

$$\Omega^2_7(M) = \{\beta \in \Omega^2(M) | \beta \wedge \varphi = 2 *_\varphi \beta\} = \{X \lrcorner \varphi | X \in C^\infty(TM)\},$$
$$\Omega^2_{14}(M) = \{\beta \in \Omega^2(M) | \beta \wedge \varphi = - *_\varphi \beta\} = \{\beta \in \Omega^2(M) | \beta \wedge \psi = 0\},$$

and

$$\Omega^3_1(M) = \{f\varphi | f \in C^\infty(M)\}, \quad \Omega^3_7(M) = \{X \lrcorner \psi | X \in C^\infty(TM)\},$$
$$\Omega^3_{27}(M) = \{\gamma \in \Omega^3(M) | \gamma \wedge \varphi = 0 = \gamma \wedge \psi\}.$$

Hodge duality gives corresponding decompositions of $\Omega^4(M)$ and $\Omega^5(M)$.

The space $\Omega^3_{27}(M)$ deserves more attention. As in [3] we define a map $\mathrm{i}_\varphi :$ $\mathrm{Sym}^2(T^*M) \to \Omega^3(M)$ from the space of symmetric 2-tensors to the space of 3-forms, given locally by

$$\mathrm{i}_\varphi(h) = \frac{1}{2} h^l_i \varphi_{ljk} dx^i \wedge dx^j \wedge dx^k \tag{1.1}$$

where $h = h_{ij} dx^i dx^j \in \mathrm{Sym}^2(T^*M)$. Then $C^\infty(M) \otimes g_\varphi$ is mapped isomorphically onto $\Omega^3_1(M)$ under the map i_φ with $\mathrm{i}_\varphi(g_\varphi) = 3\varphi$, and the space of trace-free symmetric 2-tensors $\mathrm{Sym}^2_0(T^*M)$ is mapped isomorphically onto the space $\Omega^3_{27}(M)$.

1.2 Torsion of G_2 Structures

Given a G_2 structure $\varphi \in \Omega^3_+(M)$, if ∇ denotes the Levi-Civita connection with respect to g_φ, we can interpret $\nabla\varphi$ as the torsion of the G_2 structure φ. Following [25], we see that $\nabla\varphi$ lies in the space $\Omega^1_7(M) \otimes \Omega^3_7(M)$. Thus we can define a 2-tensor T which we shall call the full torsion tensor such that

$$\nabla_i \varphi_{jkl} = T_{im} g^{mn} \psi_{njkl}. \tag{1.2}$$

Using the decomposition of the spaces of forms on M determined φ, we can also decompose $d\varphi$ and $d\psi$ into types. Bryant [3] showed that there exist unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ such that

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3, \tag{1.3}$$

$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{1.4}$$

We call $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ the intrinsic torsion forms of the G_2 structure φ. The full torsion tensor T_{ij} is related to the intrinsic torsion forms by the following (see [25]):

$$T_{ij} = \frac{\tau_0}{4} g_{ij} - (\tau_1^\# \lrcorner \varphi)_{ij} - (\bar{\tau}_3)_{ij} - \frac{1}{2}(\tau_2)_{ij}, \tag{1.5}$$

where $\bar{\tau}_3$ is the trace-free symmetric 2-tensor such that $\tau_3 = \mathrm{i}_\varphi(\bar{\tau}_3)$.

If $\nabla\varphi = 0$, we say the G_2 structure φ is torsion-free on M. The torsion-free condition clearly implies that $d\varphi = 0 = d^*_\varphi\varphi$ on M. Fernández and Gray [12] showed that $d\varphi = 0 = d^*_\varphi\varphi$ also implies $\nabla\varphi = 0$ on M, which also follows from the Eq. (1.5). The key property of a torsion-free G_2 structure φ is that the holonomy group $\mathrm{Hol}(g_\varphi) \subseteq G_2$, and thus the manifold (M, g_φ) is Ricci-flat. Moreover, one can characterise the compact G_2 manifolds (i.e., compact manifolds with torsion-free G_2 structures) with $\mathrm{Hol}(g_\varphi) = G_2$ as those with finite fundamental group. Thus understanding torsion-free G_2 structures is crucial for constructing Riemannian manifolds with holonomy G_2.

While there are some explicit examples of manifolds which admit torsion-free G_2 structures for which the holonomy of the induced metric is properly contained in G_2, for example the product of circle $\mathcal{S}^1$ with a Calabi-Yau 3-fold and the product of 3-torus $\mathbb{T}^3$ with a Calabi-Yau 2-fold, the construction of manifolds which admit torsion-free G_2 structures with holonomy equal to G_2 is a hard and important problem. The first local existence result of metrics with holonomy G_2 was obtained by Bryant [2] using the theory of exterior differential systems. Then Bryant–Salamon [4] constructed the first complete non-compact manifolds with holonomy G_2, which are the spinor bundle of $\mathcal{S}^3$ and the bundles of anti-self-dual 2-forms on $\mathcal{S}^4$ and $\mathbb{CP}^2$. In [22], Joyce constructed the first examples of compact 7-manifolds with holonomy G_2 and many further compact examples have now been constructed [7, 24, 29].

1.3 Closed G_2 Structures

If φ is closed, i.e. $d\varphi = 0$, then (1.3) implies that τ_0, τ_1 and τ_3 are all zero, so the only non-zero torsion form is $\tau_2 \in \Omega^2_{14}(M)$. In this case, we write $\tau = \tau_2$ for simplicity. Then from (1.5) we have that the full torsion tensor satisfies $T_{ij} = -\frac{1}{2}\tau_{ij}$ and is a skew-symmetric 2-tensor. By (1.4) and $\tau \in \Omega^2_{14}(M)$, we have $d\psi = \tau \wedge \varphi = -*_\varphi \tau$, which implies that

$$d^*\tau = *_\varphi d *_\varphi \tau = - *_\varphi d^2\psi = 0 \tag{1.6}$$

and the Hodge Laplacian of φ is equal to $\Delta_\varphi \varphi = -d *_\varphi d\psi = d\tau$. We computed in [34] (see also [3]) that

$$\Delta_\varphi \varphi \;=\; \mathrm{i}_\varphi(h) \;\in\; \Omega^3_1(M) \oplus \Omega^3_{27}(M) \tag{1.7}$$

where h is the symmetric 2-tensor given as follows:

$$h_{ij} = -\nabla_m T_{ni} \varphi_j^{\ mn} - \frac{1}{3}|T|^2 g_{ij} - T_{ik} g^{kl} T_{lj}. \tag{1.8}$$

Since φ determines a unique metric $g = g_\varphi$ on M, we then have the Riemann curvature tensor $\mathrm{Rm} = \{R_{ijkl}\}$, the Ricci tensor $R_{ij} = g^{kl} R_{ijkl}$ and the scalar curvature $\mathrm{R} = g^{ij} R_{ij}$ of (M, g_φ). For closed G_2 structure φ, we computed in [34] that the Ricci curvature is equal to

$$R_{ij} = \nabla_m T_{ni} \varphi_j^{\ mn} - T_{ik} g^{kl} T_{lj}, \tag{1.9}$$

and then the scalar curvature $R = -|T|^2$. With (1.9) we can write the symmetric tensor h in (1.8) as

$$h_{ij} = -R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_{ik} g^{kl} T_{lj}. \tag{1.10}$$

2 Laplacian Flow for Closed G_2 Structures

Since Hamilton [16] introduced the Ricci flow in 1982, geometric flows have been an important tool in studying geometric structures on manifolds. For example, Ricci flow was instrumental in proving the Poincaré conjecture and the $\frac{1}{4}$-pinched differentiable sphere theorem, and Kähler–Ricci flow has proved to be a useful tool in Kähler geometry, particularly in low dimensions. In 1992, Bryant (see [3]) proposed the Laplacian flow for closed G_2 structures

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t) &= \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) &= 0, \\ \varphi(0) &= \varphi_0, \end{cases} \tag{2.1}$$

where $\Delta_\varphi = dd_\varphi^* + d_\varphi^* d$ is the Hodge Laplacian with respect to g_φ and φ_0 is an initial closed G_2 structure. The stationary points of the flow are harmonic φ, which on a compact manifold are precisely the torsion-free G_2 structures, so the Laplacian flow provides a tool for studying the existence of torsion-free G_2 structures on a manifold admitting closed G_2 structures. The goal is to understand the long-time behavior of the Laplacian flow on compact manifolds M; specifically, to understand conditions under which the flow will converge to a torsion-free G_2 structure. We remark that there are other proposed flows which also have torsion-free G_2 structures as stationary points (e.g. [15, 26, 42]).

2.1 Gradient Flow of Volume Functional

Another motivation for studying the Laplacian flow comes from work of Hitchin [19] (see also [5]), which demonstrates its relationship to a natural volume functional. Let $\bar{\varphi}$ be a closed G_2 structure on a compact 7-manifold M and let $[\bar{\varphi}]_+$ be the open subset of the cohomology class $[\bar{\varphi}]$ consisting of G_2 structures. Define the volume functional on M by

$$\mathrm{Vol}(M,\varphi) \;=\; \frac{3}{7}\int_M \varphi \wedge *_\varphi \varphi, \quad \varphi \in [\bar{\varphi}]_+. \tag{2.2}$$

In the arXiv version of [19], Hitchin showed that $\varphi \in [\bar{\varphi}]_+$ is a critical point of $\mathrm{Vol}(M,\varphi)$ if and only if $d *_\varphi \varphi = 0$, i.e. φ is torsion-free.

Moreover, the Laplacian flow (2.1) can be viewed as the gradient flow of the volume functional (2.2). Since $\varphi(t)$ evolves in the same cohomology class with the initial data φ_0, we can write $\varphi(t) = \varphi_0 + d\eta(t)$ for some time dependent 2-form $\eta(t)$. To calculate the variation of the volume functional, we need to compute the variation of $*_{\varphi(t)}\varphi(t)$. This has already been computed in [3, 23]:

$$\frac{\partial}{\partial t}(*_{\varphi(t)}\varphi(t)) \;=\; \frac{4}{3} *_{\varphi(t)} \pi_1\left(\frac{\partial\varphi(t)}{\partial t}\right) + *_{\varphi(t)}\pi_7\left(\frac{\partial\varphi(t)}{\partial t}\right) - *_{\varphi(t)}\pi_{27}\left(\frac{\partial\varphi(t)}{\partial t}\right), \tag{2.3}$$

where π_k's are the respective projections to the invariant subspaces of $\Omega^3(M)$ and are determined by $\varphi(t)$. Then

$$\begin{aligned}\frac{d}{dt}\mathrm{Vol}(M,\varphi(t)) &= \frac{3}{7}\int_M \left(\frac{\partial\varphi(t)}{\partial t} \wedge *_{\varphi(t)}\varphi(t) + \varphi(t) \wedge \frac{\partial}{\partial t}(*_{\varphi(t)}\varphi(t))\right) \\ &= \int_M \frac{\partial\varphi(t)}{\partial t} \wedge *_{\varphi(t)}\varphi(t) \\ &= \int_M \langle \frac{\partial\eta(t)}{\partial t}, d^*_{\varphi(t)}\varphi(t)\rangle *_{\varphi(t)} 1.\end{aligned}$$

Thus gradient flow of the volume functional within the same cohomology class is given by

$$\frac{\partial\varphi(t)}{\partial t} = d\frac{\partial\eta(t)}{\partial t} = dd^*_{\varphi(t)}\varphi(t) = \Delta_{\varphi(t)}\varphi(t),$$

which is exactly the Laplacian flow. Then along the Laplacian flow, the volume will increase unless $\varphi(t)$ is torsion-free. By examining the second variation of the volume functional, Bryant [3] showed that if $\bar{\varphi}$ is torsion-free, then $\mathrm{Diff}^0(M)\cdot\bar{\varphi}$ is a local maximum of the volume functional on the moduli space $\mathrm{Diff}^0(M)\setminus[\bar{\varphi}]_+$. This gives rise to the following natural question:

Question 2.1 ([3]) *Starting from a initial data $\varphi_0 \in [\bar{\varphi}]_+$ which is sufficiently close to $\bar{\varphi}$ in an appropriate norm, does the Laplacian flow converge to a point on* $\mathrm{Diff}^0(M)\cdot\bar{\varphi}$*?*

In the statement of Question 2.1, we assumed the existence of a torsion-free G_2 structure $\bar{\varphi}$ on M. In 1996, Joyce [22] proved a criterion for the existence of torsion-free G_2 structures, which says that if one can find a G_2 structure φ with $d\varphi = 0$ on a compact 7-manifold M, whose torsion is sufficiently small in a certain sense, then there exists a torsion-free G_2-structure $\bar{\varphi}\in[\varphi]$ on M which is close to φ. This result has been used to construct compact examples of manifolds with G_2 holonomy. It would be interesting to give a new proof of Joyce's result [22] using the Laplacian flow.

Generally, one cannot expect that the Laplacian flow will converge to a torsion-free G_2 structure, even if it has long-time existence. There are compact 7-manifolds with closed G_2 structures that cannot admit holonomy G_2 metrics for topological reasons (c.f. [9, 10]), and Bryant [3] showed that the Laplacian flow starting with a particular one of these examples will exist for all time but it does not converge; for instance, the volume of the associated metrics will increase without bound. Some explicit examples of the solution to the Laplacian flow which exist for all time and converge can be found in [11, 13, 21].

2.2 Short Time Existence

Recall that the Hodge Laplacian Δ_φ is related to the analyst's Laplacian $\Delta = g^{ij}\nabla_i\nabla_j$ by the Weitzenbock formula:

$$\Delta_\varphi\omega \;=\; -\Delta\omega + \mathcal{R}(\omega) \tag{2.4}$$

for any $(0,k)$-tensor ω, where $\mathcal{R}$ is the Weitzenbock curvature operator. Since the Laplacian flow (2.1) is defined by the Hodge Laplacian, it appears at first sight to have the wrong sign for the parabolicity. However, if $d\varphi = 0$, using definition (1.2) of the torsion tensor and the divergence-free property (1.6) of τ, we see that $\Delta\varphi$ involves only up to first order derivatives of φ and thus the second order part of the

Hodge Laplacian $\Delta_\varphi \varphi$ lies in the part $\mathcal{R}(\varphi)$ of (2.4). Using DeTurck's trick in the Ricci flow, Bryant–Xu [5] modified the Laplacian flow by an operator of the form $\mathcal{L}_{V(\varphi)}\varphi = d(V \lrcorner \varphi) + V \lrcorner d\varphi = d(V \lrcorner \varphi)$ for some vector field $V(\varphi)$ and showed that the Laplacian–DeTurck flow

$$\frac{\partial \varphi(t)}{\partial t} = \Delta_{\varphi(t)}\varphi(t) + \mathcal{L}_{V(\varphi)}\varphi(t) \tag{2.5}$$

is strictly parabolic in the direction of closed forms by choosing a special vector field $V(\varphi)$. In fact, if $d\theta = 0$, they calculated that the linearization of RHS of (2.5) is

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \left(\Delta_{\varphi+\epsilon\theta}(\varphi+\epsilon\theta) + \mathcal{L}_{V(\varphi+\epsilon\theta)}(\varphi+\epsilon\theta)\right) \;=\; -\Delta_\varphi \theta + d\Phi(\theta) \tag{2.6}$$

where $d\Phi(\theta)$ is algebraic linear in θ and $d\Phi(\theta) = 0$ if φ is torsion-free. However, no existing theory of parabolic equations can be used directly since the parabolicity of (2.5) is only true in the direction of closed forms. Fortunately, by using the Nash Moser inverse function theorem [17] for tame Féchet spaces, Bryant and Xu proved the following short time existence theorem.

Theorem 2.2 (Bryant–Xu [5]) *Assume that M is compact and φ_0 is a closed* G_2 *structure on M. Then the Laplacian flow has a unique solution for a short time $t \in [0, \varepsilon)$ with ε depending on φ_0.*

As in the Ricci flow, we can also write the Laplacian–DeTurck flow (2.5) explicitly in local coordinates. Let $\tilde{g}$ be a fixed Riemannian metric on M and $\tilde{\nabla}$, $\tilde{\Gamma}^k_{ij}$ be the corresponding Levi-Civita connection and Christoffel symbols. We know that the difference $\Gamma^j_{kl} - \tilde{\Gamma}^j_{kl}$ of the Levi-Civita connections of the metrics g and $\tilde{g}$ is a well-defined tensor on M. This gives us a vector field V on M with

$$V_i = g_{ij} g^{kl}(\Gamma^j_{kl} - \tilde{\Gamma}^j_{kl}), \tag{2.7}$$

which is just the vector field chosen in Ricci-DeTurck flow [41]. By a direct but lengthy computation, we can show that if $d\varphi = 0$, the Laplacian–DeTurck flow Eq. (2.5) with V given by (2.7) has the following expression in local coordinates:

$$\frac{\partial}{\partial t}\varphi_{ijk} = g^{pq}\tilde{\nabla}_p \tilde{\nabla}_q \varphi_{ijk} + l.o.t \tag{2.8}$$

and the associated metric g_{ij} evolves by

$$\frac{\partial}{\partial t} g_{ij} = g^{pq}\tilde{\nabla}_p \tilde{\nabla}_q g_{ij} + l.o.t \tag{2.9}$$

where the lower order terms only involve the φ, g, $\tilde{\nabla} g$ and $\tilde{\nabla}\varphi$ and can be written down explicitly. The readers may find that the vector field V is different at first sight with the one chosen by Bryant–Xu [5]. However, we can see that they are essentially the same by considering the linearization of V in the direction of closed forms (see also [15, pp. 400–401]).

2.3 Evolution Equations

Since each G_2 structure induces a unique Riemannian metric on the manifold, the Laplacian flow (2.1) induces a flow for the associated Riemannian metric $g(t) = g_{\varphi(t)}$. Recall that under a general flow for G_2 structures

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{i}_{\varphi(t)}(h(t)) + X\lrcorner\psi(t), \tag{2.10}$$

where $h(t) \in \mathrm{Sym}^2(T^*M)$ and $X(t) \in C^\infty(TM)$, it is well known that (see [3, 23] and explicitly [25]) the associated metric tensor $g(t)$ evolves by

$$\frac{\partial}{\partial t}g(t) = 2h(t). \tag{2.11}$$

By (1.7) and (1.10), we deduce that the associated metric $g(t)$ of the solution $\varphi(t)$ of the Laplacian flow evolves by

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} - \frac{2}{3}|T|^2 g_{ij} - 4T_{ik}g^{kl}T_{lj}, \tag{2.12}$$

which corresponds to the Ricci flow plus some lower order terms involving the torsion tensor, as already observed in [3]. Then it's easy to see that the volume form $\mathrm{vol}_{g(t)}$ evolves by

$$\frac{\partial}{\partial t}\mathrm{vol}_{g(t)} = \frac{1}{2}\mathrm{tr}_g(\frac{\partial}{\partial t}g(t))\mathrm{vol}_{g(t)} = \frac{2}{3}|T|^2\mathrm{vol}_{g(t)}, \tag{2.13}$$

where we used the fact that the scalar curvature $R = -|T|^2$. Hence, along the Laplacian flow, the volume of M with respect to the associated metric $g(t)$ will nondecrease (as already noted in Sect. 2.1). Since the torsion tensor T is defined by the first covariant derivative of φ and the Riemannian curvature tensor Rm involves up to second order derivatives of the metric, we calculated in [34] that the evolution equations of the torsion tensor and Riemannian curvature tensor along the Laplacian flow are of the form

$$\frac{\partial}{\partial t}T = \Delta T + Rm * T + Rm * T * \psi + \nabla T * T * \varphi + T * T * T, \tag{2.14}$$

$$\frac{\partial}{\partial t}Rm = \Delta Rm + Rm * Rm + Rm * T * T + \nabla^2 T * T + \nabla T * \nabla T, \qquad (2.15)$$

where we use $*$ to mean some contraction using the metric $g(t)$ associated with $\varphi(t)$.

3 Foundational Results of Laplacian Flow

In this section, we discuss several foundational results on the Laplacian flow, which are important for further studies.

3.1 Shi-type Estimates

The first result is the derivative estimates of the solution to the Laplacian flow. For a solution $\varphi(t)$ of the Laplacian flow (2.1), we define the quantity

$$\Lambda(x, t) = \left(|\nabla T(x, t)|^2_{g(t)} + |Rm(x, t)|^2_{g(t)}\right)^{\frac{1}{2}}. \qquad (3.1)$$

Notice that the torsion tensor T is determined by the first order derivative of φ and the curvature tensor Rm is second order in the metric g_φ, so both Rm and ∇T are second order in φ. We show that a bound on $\Lambda(x, t)$ will induce a priori bounds on all derivatives of Rm and ∇T for positive time. More precisely, we have the following.

Theorem 3.1 ([34]) *Suppose that $K > 0$ and $\varphi(t)$ is a solution of the Laplacian flow (2.1) for closed G_2 structures on a compact manifold M^7 for $t \in [0, \frac{1}{K}]$. For all $k \in \mathbb{N}$, there exists a constant C_k such that if $\Lambda(x, t) \le K$ on $M^7 \times [0, \frac{1}{K}]$, then*

$$|\nabla^k Rm(x, t)|_{g(t)} + |\nabla^{k+1} T(x, t)|_{g(t)} \le C_k t^{-\frac{k}{2}} K, \quad t \in (0, \frac{1}{K}]. \qquad (3.2)$$

We call the estimates (3.2) Shi-type estimates for the Laplacian flow, because they are analogues of the well-known Shi derivative estimates in the Ricci flow. In Ricci flow, a Riemann curvature bound will imply bounds on all the derivatives of the Riemann curvature: this was proved by Bando [1] and comprehensively by Shi [41] independently. The techniques used in [1, 41] were introduced by Bernstein (in the early twentieth century) for proving gradient estimates via the maximum principle, and was also the key in [34] to prove Theorem 3.1. A key motivation for defining $\Lambda(x, t)$ as in (3.1) is that the evolution equations of $|\nabla T(x, t)|^2$ and $|Rm(x, t)|^2$ both have some bad terms, but the chosen combination kills these terms and yields an effective evolution equation for $\Lambda(x, t)$ which looks like

$$\frac{\partial}{\partial t}\Lambda(x,t)^2 \leq \Delta\Lambda(x,t)^2 + C\Lambda(x,t)^3$$

for some positive constant C. This shows that the quantity Λ has similar properties to Riemann curvature under Ricci flow. Moreover, it implies that the assumption $\Lambda(x,t) \leq K$ in Theorem 3.1 is reasonable as $\Lambda(x,t)$ cannot blow up quickly. We remark that the constant C_k depends on the order of differentiation. In a joint work with Lotay [36], we showed that C_k are of sufficiently slow growth in the order k and then we deduced that the G_2 structure $\varphi(t)$ and associated metric $g_{\varphi(t)}$ are real analytic at each fixed time $t > 0$.

The Shi-type estimates could be used to study finite-time singularities of the Laplacian flow. Given an initial closed G_2 structure φ_0 on a compact 7-manifold, Theorem 2.2 tells us there exists a solution $\varphi(t)$ of the Laplacian flow on a maximal time interval $[0, T_0)$. If T_0 is finite, we call T_0 the singular time. Using our global derivative estimates (3.2), we have the following long time existence result on the Laplacian flow.

Theorem 3.2 ([34]) *If $\varphi(t)$ is a solution of the Laplacian flow (2.1) on a compact manifold M^7 in a maximal time interval $[0, T_0)$ with $T_0 < \infty$, then*

$$\lim_{t\nearrow T_0} \sup_{x\in M} \Lambda(x,t) = \infty.$$

Moreover, there exists a positive constant C such that the blow-up rate satisfies

$$\sup_{x\in M} \Lambda(x,t) \geq \frac{C}{T_0 - t}.$$

In other words, Theorem 3.2 shows that the solution $\varphi(t)$ of the Laplacian flow for closed G_2 structures will exist as long as the quantity $\Lambda(x,t)$ in (3.1) remains bounded.

3.2 Uniqueness

Given a closed G_2 structure φ_0 on a compact 7-manifold, Theorem 2.2 says that there exists a unique solution to the Laplacian flow for a short time interval $t \in [0, \varepsilon)$. The proof in [5] relies on the Nash–Moser inverse function theorem [16] and the DeTurck's trick. In [34], we gave a new proof the forward uniqueness by adapting an energy approach used previously by Kotschwar [28] for Ricci flow. The idea is to define an energy quantity $\mathcal{E}(t)$ in terms of the differences of the G_2 structures, metrics, connections, torsion tensors and Riemann curvatures of two Laplacian flows, which vanishes if and only if the flows coincide. By deriving a differential inequality for $\mathcal{E}(t)$, it can be shown that $\mathcal{E}(t) = 0$ if $\mathcal{E}(0) = 0$, which gives the forward uniqueness. We also proved in [34] a backward uniqueness result for the solution of Laplacian flow by applying a general backward uniqueness theorem in [27] for time-dependent sections of vector bundles satisfying certain differential inequalities.

Theorem 3.3 ([34]) *Suppose $\varphi(t)$, $\tilde{\varphi}(t)$ are two solutions to the Laplacian flow (2.1) on a compact manifold M^7 for $t \in [0, \epsilon]$, $\epsilon > 0$. If $\varphi(s) = \tilde{\varphi}(s)$ for some $s \in [0, \epsilon]$, then $\varphi(t) = \tilde{\varphi}(t)$ for all $t \in [0, \epsilon]$.*

An application of Theorem 3.3 is that on a compact manifold M^7, the subgroup $I_{\varphi(t)}$ of diffeomorphisms of M isotopic to the identity and fixing $\varphi(t)$ is unchanged along the Laplacian flow. Since I_φ is strongly constrained for a torsion-free G_2 structure φ on M, this gives a test for when the Laplacian flow with a given initial condition could converge.

3.3 Compactness and κ-Non-collapsing

In the study of Ricci flow, Hamilton's compactness theorem [18] and Perelman's κ-non-collapsing estimate [38] are two essential tools to study the behavior of the flow near a singularity. We also have the analogous results for the Laplacian flow, which were proved by the author and Lotay [34] and Chen [6] respectively.

Theorem 3.4 ([34]) *Let M_i be a sequence of compact 7-manifolds and let $p_i \in M_i$ for each i. Suppose that, for each i, $\varphi_i(t)$ is a solution to the Laplacian flow (2.1) on M_i for $t \in (a, b)$, where $-\infty \le a < 0 < b \le \infty$. Suppose that*

$$\sup_i \sup_{x \in M_i, t \in (a,b)} \Lambda_{\varphi_i}(x, t) < \infty \tag{3.3}$$

and

$$\inf_i inj(M_i, g_i(0), p_i) > 0. \tag{3.4}$$

Then there exists a 7-manifold M, a point $p \in M$ and a solution $\varphi(t)$ of the Laplacian flow on M for $t \in (a, b)$ such that, after passing to a subsequence, $(M_i, \varphi_i(t), p_i)$ converge to $(M, \varphi(t), p)$ as $i \to \infty$.

To prove Theorem 3.4, we first proved in [34] a Cheeger–Gromov-type compactness theorem for the space of G_2 structures, which states that the space of G_2 structures with bounded $|\nabla^{k+1}T| + |\nabla^k Rm|$, $k \ge 0$, and bounded injectivity radius is compact. Given this, Theorem 3.4 follows from a similar argument for the analogous compactness theorem in Ricci flow as in [18], with the help of the Shi-type estimate in Theorem 3.1.

The κ-non-collapsing estimate is an estimate on the volume ratio which only involves the Riemannian metric. A Riemannian metric g on a manifold M is κ-non-collapsed relative to an upper bound on the scalar curvature of the metric on the scale ρ if for any geodesic ball $B_g(p, r)$ with $r < \rho$ such that $\sup_{B_g(p,r)} R_g \le r^{-2}$, there holds $\mathrm{Vol}(B_g(p,r)) \geq \kappa r^n$. By using the same $\mathcal{W}$ functional, Chen [6] generalized Perelman's κ-non-collapsing theorem [38] for Ricci flow to any flow

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) + E(t) \tag{3.5}$$

for the Riemannian metric $g(t)$, where $E(t)$ is a symmetric 2-tensor.

Theorem 3.5 ([6]) *If $|E(t)|_{g(t)}$ is bounded along the flow (3.5) for $t \in [0,s)$ with $s < \infty$, then there exists $\kappa > 0$ such that for all $t \in [0,s)$, $g(t)$ is κ-non-collapsed relative to the upper bound on the scalar curvature on the scale $\rho = \sqrt{s}$.*

Theorem 3.5 applies effectively to our Laplacian flow since the induced metric flow is just a perturbation of the Ricci flow, see (2.12). The κ-non-collapsing estimate is useful to estimate the lower bound on the injectivity radius, which together with the Shi-type estimate in Theorem 3.1 guarantees the condition of the compactness theorem for the purpose of the blow up analysis.

3.4 Solitons

Given a 7-manifold M, a Laplacian soliton on M is a triple (φ, X, λ) satisfying

$$\Delta_\varphi \varphi = \lambda\varphi + \mathcal{L}_X \varphi, \tag{3.6}$$

where $d\varphi = 0$, $\lambda \in \mathbb{R}$, X is a vector field on M and $\mathcal{L}_X\varphi$ is the Lie derivative of φ in the direction of X. Laplacian solitons give self-similar solutions to the Laplacian flow. Specifically, suppose (φ_0, X, λ) satisfies (3.6). Define $\rho(t) = (1 + \frac{2}{3}\lambda t)^{\frac{3}{2}}$, $X(t) = \rho(t)^{-\frac{2}{3}} X$, and let ϕ_t be the family of diffeomorphisms generated by the vector fields $X(t)$ such that ϕ_0 is the identity. Then $\varphi(t)$ defined by $\varphi(t) = \rho(t)\phi_t^*\varphi_0$ is a solution of the Laplacian flow (2.1), which only differs by a scaling factor $\rho(t)$ and pull-back by a diffeomorphism ϕ_t for different times t. We say a Laplacian soliton (φ, X, λ) is expanding if $\lambda > 0$; steady if $\lambda = 0$; and shrinking if $\lambda < 0$.

The soliton solutions of the Laplacian flow are expected to play a role in understanding the behavior of the flow near singularities. Thus the classification is an important problem. In this direction, Lin [30] proved that there are no compact shrinking solitons, and the only compact steady solitons are given by torsion-free G_2 structures. In [34], we show that any Laplacian soliton that is an eigenform (i.e., $X = 0$ in (3.6)) must be an expander or torsion-free. Hence, stationary points of the Laplacian flow on 7-manifold (not necessarily compact) are given by torsion-free G_2 structures. Moreover, we show that there are no compact Laplacian solitons that are eigenforms unless φ is torsion-free. Combining this with Lin's result, any nontrivial Laplacian soliton on a compact manifold M (if it exists) must satisfy (3.6) for $\lambda > 0$ and $X \neq 0$. This phenomenon is somewhat surprising, since it is very different from

Ricci solitons $\mathrm{Ric} + \mathcal{L}_X g = \lambda g$: when $X = 0$, the Ricci soliton equation is just the Einstein equation and there are many examples of compact Einstein metrics.

Since a G_2 structure φ determines a unique metric g, it is natural to ask what condition the Laplacian soliton Eq. (3.6) on φ will impose on g. By writing $\mathcal{L}_X\varphi$ with respect to the type decomposition of 3-forms, we derived from the Laplacian soliton Eq. (3.6) that the induced metric g_φ satisfies, in local coordinates,

$$-R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_{ik}g^{kl}T_{lj} = \frac{1}{3}\lambda g_{ij} + \frac{1}{2}(\mathcal{L}_X g)_{ij} \tag{3.7}$$

and the vector field X satisfies $d^*(X\lrcorner\varphi) = 0$. In particular, we deduce that any Laplacian soliton (φ, X, λ) must satisfy $7\lambda + 3\mathrm{div}(X) = 2|T|^2 \geq 0$, which leads to a new short proof of Lin's result [30] for the closed case.

Remark 3.6 We remark that there are many new results concerning the soliton solutions of the Laplacian flow. We refer the readers to [11, 31–33, 37] for details.

4 Extension Theorem

As we said in Sect. 3, the compactness theorem and the non-collapsing estimate could be used to study the singularities of the Laplacian flow. Theorem 3.2 already characterized the finite time singularities as the points where the quantity $\Lambda(x, t)$ (defined in (3.1)) blow up. This means that the solution of the Laplacian flow exists as long as $\Lambda(x, t)$ remains bounded. The quantity $\Lambda(x, t)$ consists of the full information of the G_2 structure $\varphi(t)$ up to second derivatives. It's interesting to see whether some weaker quantity can control the behavior of the flow. Using the compactness theorem, we improved Theorem 3.2 to the following desirable result, which states that the Laplacian flow will exist as long as the velocity of the flow remains bounded.

Theorem 4.1 ([34]) *Let M be a compact 7-manifold and $\varphi(t)$, $t \in [0, T_0)$, where $T_0 < \infty$, be a solution to the Laplacian flow (2.1) with associated metric $g(t)$ for each t. If the velocity of the flow satisfies $\sup_{M\times[0,T_0)} |\Delta_\varphi\varphi(x, t)|_{g(t)} < \infty$, then the solution $\varphi(t)$ can be extended past time T_0.*

Note that for closed G_2 structures, the velocity $\Delta_\varphi\varphi = d\tau$ is just some components of the first derivative of the torsion tensor. Theorem 4.1 is the G_2 analogue of Sesum's [39] theorem that the Ricci flow exists as long as the Ricci tensor remains bounded. It is an open question whether the scalar curvature (the trace of the Ricci tensor) is enough to control the behavior of the Ricci flow, though it is known for Type-I Ricci flow [8] and Kähler–Ricci flow [44]. For a closed G_2 structure φ, the velocity $\Delta_\varphi\varphi = \mathrm{i}_\varphi(h)$ is equivalent to a symmetric 2-tensor h with trace equal to $\frac{2}{3}|T|^2$. Since the scalar curvature of the metric induced by φ is $-|T|^2$, comparing with Ricci flow one may ask whether the Laplacian flow for closed G_2 structures will exist as long as the torsion tensor remains bounded. This is also the natural question to ask from the point of view of G_2 geometry. However, even though $-|T|^2$ is the scalar curvature, it is only *first order* in φ, rather than second order like $\Delta_\varphi\varphi$, so it would be a major step forward to control the Laplacian flow using just a bound on the torsion tensor.

The Proof of Theorem 4.1 involves a standard blow up analysis using the compactness theorem in Sect. 3. However, the non-collapsing estimate is not required for the proof. In fact, for a closed G_2 structure φ, $\Delta_\varphi\varphi = \mathrm{i}_\varphi(h)$ and $|\Delta_\varphi\varphi|_g^2 = (\mathrm{tr}_g(h))^2 + 2|h|^2$ with h given by (1.10). Then the condition $|\Delta_{\varphi(t)}\varphi(t)|_{g(t)} < \infty$ is equivalent to $\sup_{M\times[0,T_0)} |h(t)| < \infty$, which implies the uniform continuity of the metric $g(t)$. A desired injectivity radius estimate then follows and the blow up analysis works.

Remark 4.2 By applying the compactness theorem and the non-collapsing estimate and using the method in [43], Chen [6] improved the result in Theorem 4.1. See [6] for the details. Moreover, Fine and Yao studied in [14] the hypersymplectic flow on a compact 4-manifold X related to the Laplacian flow on the 7-manifold $X \times \mathbb{T}^3$ and proved that the flow extends as long as the scalar curvature of the corresponding G_2 structure remains bounded.

5 Stability of Torsion-Free G_2 Structures

As we stated in Question 2.1, Bryant asked the question whether the Laplacian flow with initial G_2 structure φ_0 which is sufficiently close to a torsion-free G_2 structure $\bar{\varphi}$ will converge to a point in the diffeomorphism orbit of $\bar{\varphi}$. Jointly with Lotay, we gave a positive answer in [35].

Theorem 5.1 ([35]) *Let $\bar{\varphi}$ be a torsion-free* G_2 *structure on a compact 7-manifold M. Then there is a neighborhood $\mathcal{U}$ of $\bar{\varphi}$ such that for any $\varphi_0 \in [\bar{\varphi}]_+ \cap \mathcal{U}$, the Laplacian flow* (2.1) *with initial value φ_0 exists for all $t \in [0,\infty)$ and converges to $\varphi_\infty \in \mathrm{Diff}^0(M)\cdot\bar{\varphi}$ as $t \to \infty$. In other words, torsion-free* G_2 *structures are (weakly) dynamically stable along the Laplacian flow for closed* G_2 *structures.*

The Proof of Theorem 5.1 is inspired by the proof of an analogous result in Ricci flow: Ricci-flat metrics are dynamically stable along the Ricci flow. The idea is to combine arguments for the Ricci flow case [20, 40] with the particulars of the geometry of closed G_2 structures and new higher order estimates for the Laplacian flow derived by the author with Lotay in [34]. We first look at the Laplacian–DeTurck flow (2.5). By linearizing (2.5) at the torsion-free G_2 structure $\bar{\varphi}$, we have (see (2.6)):

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\Delta_{\bar{\varphi}+\epsilon\theta}(\bar{\varphi}+\epsilon\theta) + \mathcal{L}_{V(\bar{\varphi}+\epsilon\theta)}(\bar{\varphi}+\epsilon\theta)\right) \;=\; -\,\Delta_{\bar{\varphi}}\theta, \tag{5.1}$$

where θ is an exact 3-form. Note that the operator $-\Delta_{\bar{\varphi}}$ is strictly negative on the space of exact 3-forms by Hodge decomposition theorem. Let $\tilde{\varphi}(t)$ be the solution of Laplacian–DeTurck flow and denote $\theta(t) = \tilde{\varphi}(t) - \bar{\varphi}$. By the linearization (5.1), there exists $\epsilon > 0$ such that for all t for which $\|\theta(t)\|_{C^k_{\bar{g}}} < \epsilon$, we have

$$\frac{\partial}{\partial t}\theta(t) = -\Delta_{\bar{\varphi}}\theta + dF(\bar{\varphi}, \tilde{\varphi}(t), \theta(t), \bar{\nabla}\theta(t)),$$

where F is a 2-form which is smooth in the first two arguments and linear in the last two arguments. The idea is that if $\theta(t)$ is sufficiently small, the behavior of the Laplacian–DeTurck flow is dominated by the linear term $-\Delta_{\bar{\varphi}}\theta$. If the initial φ_0 is sufficiently close to $\bar{\varphi}$, i.e., $\theta(0)$ is sufficiently small, by estimating the velocity of the Laplacian–DeTurck flow we can show that the solution exists and remains small at least for time $t \in [0, 1]$. By using the strict negativity of the operator $-\Delta_{\bar{\varphi}}$, we show that $\theta(t)$ has an exponential decay in L^2 norm as long as the solution exists and remains small. By deriving higher order integral estimates, we can in fact show that the solution of the Laplacian–DeTurck flow exists for all time and also converges to $\bar{\varphi}$ exponentially and smoothly as time goes to infinity. The final step is to transform back to Laplacian flow via time-dependent diffeomorphisms $\phi(t)$ determined by the vector field $V(\tilde{\varphi}(t))$. The Shi-type esimate and compactness result apply here to show the smooth convergence of Laplacian flow and completes the proof.

As we mentioned in Sect. 2, Joyce [22] proved an existence result for torsion-free G_2 structures, which states that if we control the C^0 and L^2-norms of γ and the L^{14}-norm of $d^*_{\varphi_0}\gamma = d^*_{\varphi_0}\varphi_0$, we can deform φ_0 in its cohomology class to a unique C^0-close torsion-free G_2 structure $\bar{\varphi}$. By choosing a neighbourhood $\mathcal{U}$ appropriately, controlling derivatives up to at least order 8, we can ensure that we can apply both the theory in [22] and Theorem 5.1, and thus deduce the following corollary.

Corollary 5.2 ([35]) *Let φ_0 be a closed G_2 structure on a compact 7-manifold M. There exists an open neighbourhood $\mathcal{U}$ of 0 in $\Omega^3(M)$ such that if $d^*_{\varphi_0}\varphi_0 = d^*_{\varphi_0}\gamma$ for some $\gamma \in \mathcal{U}$, then the Laplacian flow (2.1) with initial value φ_0 exists for all time and converges to a torsion-free G_2 structure.*

The neighbourhood $\mathcal{U}$ given by Corollary 5.2 is not optimal, and one would like to be able to prove this result directly using the Laplacian flow with optimal conditions and without recourse to [22], but nevertheless, Corollary 5.2 gives significant evidence that the Laplacian flow will play an important role in understanding the problem of existence of torsion-free G_2 structures on 7-manifolds admitting closed G_2 structures.

Our results also motivate us to study an approach to the following problem, as pointed out by Thomas Walpuski. The work of Joyce [22] shows that the natural map from the moduli space $\mathcal{M}$ of torsion-free G_2 structures to $H^3(M)$ given by $\mathrm{Diff}^0(M) \cdot \bar{\varphi} \mapsto [\bar{\varphi}]$ is locally injective, but the question of whether this map is globally injective, raised by Joyce (c.f. [23]), is still open. Suppose we have two torsion-free G_2 structures $\bar{\varphi}_0$ and $\bar{\varphi}_1$ which lie in the same cohomology class, so we can write $\bar{\varphi}_1 = \bar{\varphi}_0 + d\eta$ for some 2-form η. We would like to see whether $\bar{\varphi}_1 \in \mathrm{Diff}^0(M) \cdot \bar{\varphi}_0$. By our main theorem (Theorem 5.1) we know that the Laplacian flow starting at $\varphi_0(s) = \bar{\varphi}_0 + s d\eta$ (which is closed) will exist for all time and converge to

$\phi_s^* \bar{\varphi}_0$ for some $\phi_s \in \mathrm{Diff}^0(M)$ when s is sufficiently small. Similarly, the Laplacian flow starting at $\varphi_0(s)$ for s near 1 will also exist for all time and now converge to $\phi_s^* \bar{\varphi}_1$ for some $\phi_s \in \mathrm{Diff}^0(M)$. The aim would be to study long-time existence and convergence of the flow starting at any $\varphi_0(s)$.

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