Synergetics: Basic Concepts

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Article Outline

Glossary

Attractor Region in the state vector space ("q-space") to which all neighboring states are attracted in the course of time.

Control parameter One or a set of (mostly externally) fixed parameters in the evolution equations.

Dynamical System System whose state vector changes in the course of time deterministically.

Evolution equations Determine the temporal evolution of the state vector. May be deterministic, stochastic or both.

Fixed point, stable Point in q space to which all neighboring trajectories converge in course of time.

Fluctuating forces Stochastic (random) forces appearing in evolution equations.

Fokker Planck equation Evolution equation for probability density function, based on drift and diffusion.

Generalized Langevin equation General evolution equations that contain both a deterministic and a stochastic part ("fluctuating forces").

Group Set of elements with specific multiplication rules (axioms).

Hamilton operator Classical Hamilton function, in which variables, e. g. position x and momentum p, are replaced by quantum mechanical operators.

Heisenberg picture in quantum mechanics The state vector is time-independent, while the operators are time-dependent and determined by Heisenberg equations of motion.

Instability Loss of stability.

Langevin equation Originally: evolution equation for velocity of a Brownian particle subject to damping and fluctuating force.

Limit cycle, stable A closed trajectory to which all neighboring trajectories converge.

Normal form Especially simple polynomial expression that still captures the essential features, e. g. of the right hand side of deterministic evolution equations.

Order parameters Collective variables that determine the macroscopic behavior of systems.

Pattern A pattern is essentially an arrangement. It is characterized by the order of the elements of which it is made rather than by the intrinsic nature of these elements (Norbert Wiener).

Probability distribution function Function that determines the probability of a random variable r to have fixed value $r = r_0$.

Quantum classical correspondence Establishes relation between quantum mechanical density

A. Hutt, H. Haken (eds.), *Synergetics*,
https://doi.org/10.1007/978-1-0716-0421-2_533

Originally published in
R. A. Meyers (ed.), *Encyclopedia of Complexity and Systems Science*, © Springer-Verlag 2009
https://doi.org/10.1007/978-3-642-27737-5_533

matrix and classical quasi-probability distribution.

Schrödinger picture of quantum mechanics In it operators are time-independent, while the wave-function ("state vector") is time-dependent and determined by the Schrödinger equation.

Self-organization Formation of spatio-temporal patterns (structures) and/or performance of functions without an "ordering hand".

Slaving principle A general theorem that allows the reduction of the variables of a system to order parameters (close to instability).

Spatial coordinate (vector x) In one, two or three dimensions.

ΔLaplace operator(in 1, 2 or 3 dimensions).

∇Vector $\left(\frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}\right)$ in 1, 2 or 3 dimensions.

Spectrum Set of eigenvalues belonging to linear stability equations with boundary conditions.

Stability of a system System returns after a (small) perturbation of its state vector into original state.

State vector Set of time- or time-independent variables that characterize the state of a system.

Symmetry Invariance of a system against specific transformations (e. g. mirror symmetry).

Synergetics Science of cooperation.

Trajectory Smooth curve q(t) of solution of evolution equation in q-space.

The Role of Synergetics in Science

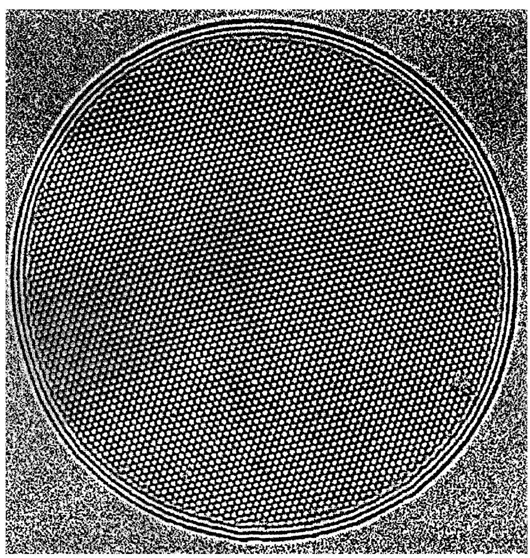
In science, we may essentially distinguish between two trends:

1. The accumulation of knowledge
2. Information reduction in the sense of finding general principles, common features.

In physics, such unifying approaches are well known: the unification of magnetism, electricity and, later on, weak and other interactions leading eventually to a unified field theory. General relativity unifies concepts of space, time and gravitation. While these unifications take place at a fundamental level, one may ask whether it is worthwhile to look also for unifications at say more macroscopic or phenomenological levels. One example is thermodynamics, another the theory of phase transitions of systems in thermal equilibrium by means of the renormalization group approach, or the concept of fractals, etc.

The main goal of Synergetics is the search for unifying principles for systems that are composed of many individual parts or components, and that may show the phenomenon of self-organization, i.e. the spontaneous formation of spatial, temporal, spatial-temporal or functional structures. The systems under discussion are, in the widest sense of the word, open physical systems whose states are maintained by an in- and outflux of energy, matter and /or information. A typical and well known example is that of a fluid in a pan that is uniformly heated from below. When the temperature difference between the lower and upper surface exceeds a critical value, the formerly homogeneous fluid develops roll or hexagonal patterns in which the fluid moves in a specific manner (Fig. 1).

As it turned out, the general principles originally elaborated in physics, can also be applied to many other systems, such as in biology, economy, ecology, sociology, management theory, psychology etc. In spite of the great variety of the



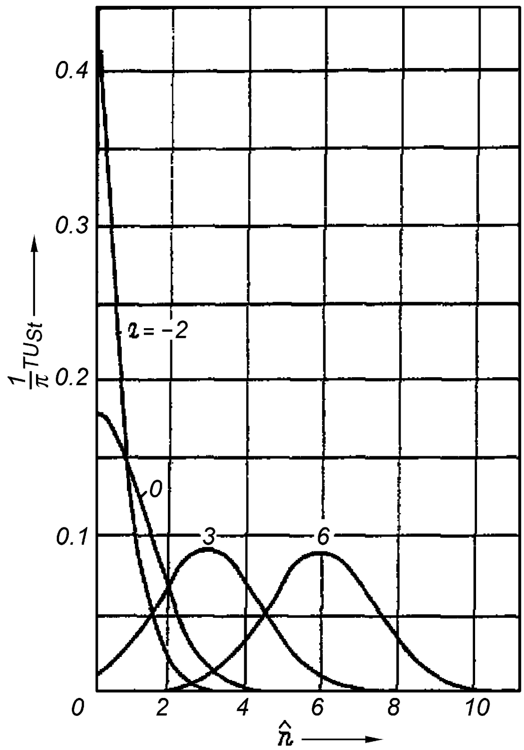
Synergetics: Basic Concepts, Fig. 1 Hexagonal pattern of a fluid (liquid helium) uniformly heated from below (Bodenschatz et al. 2000) individual systems with their components quite different in nature, such principles apply to large classes of phenomena. This is achieved by restricting the study to situations where the systems undergo qualitative changes at macroscopic scales. Here macroscopic means “with time and length scales large compared to those of the individual components”.

This leads to the definition of Synergetics as given in the preamble of the Springer Series in Synergetics: “An ever increasing number of scientific disciplines deal with complex systems. These are systems that are composed of many parts which interact with one another in a more or less complicated manner. One of the most striking features of many such systems is their ability to spontaneously form spatial or temporal structures. A great variety of these structures are found, in both the inanimate and the living world. In the inanimate world of physics and chemistry, examples include the growth of crystals, coherent oscillations of laser light, and the spiral structures formed in fluids and chemical reactions. In biology we encounter the growth of plants and animals (morphogenesis) and the evolution of species. In medicine we observe, for instance, the electromagnetic activity of the brain with its pronounced spatio-temporal structures. Psychology deals with characteristic features of human behavior ranging from simple pattern recognition tasks to complex patterns of social behavior. Examples from sociology include the formation of public opinion and cooperation or competition between social groups.”

In recent decades, it has become increasingly evident that all these seemingly quite different kinds of structure formation have a number of important features in common. The task of studying analogies as well as differences between structure formation in these different fields has proved to be an ambitious but highly rewarding endeavor. The Springer Series in Synergetics provides a forum for interdisciplinary research and discussions on this fascinating new scientific challenge. It deals with both experimental and theoretical aspects. The scientific community and the interested layman are becoming ever more conscious of concepts such as self-organization, instabilities, deterministic chaos, nonlinearity, dynamical systems, stochastic processes, and complexity. All of these concepts are facets of a field that tackles complex systems, namely Synergetics.

The Laser Paradigm

This example elucidates central concepts used in Synergetics in a qualitative fashion. An example for the laser device (an acronym for light amplification by stimulated emission of radiation, originally called optical maser (Schawlow and Townes 1958)) is the gas laser in which gas atoms are enclosed in a tube at the end-faces of which mirrors are mounted. The mirrors serve the purpose of reflecting light running in axial direction sufficiently often so that the corresponding light wave stays for an extended period in this device and can interact intensely with the atoms. The atoms are excited from the outside, e. g. by a pump light source. After having been excited, each atom can spontaneously emit a light wave track. In the usual case of a *lamp*, these wave tracks are emitted independently of each other and the amplitudes are Gaussian distributed. When the pump intensity is increased beyond a critical value, the present state gives way to a single wave with *stable amplitude* on which small *amplitude fluctuations* and *phase diffusion* are superimposed (Haken 1964). The pump intensity serves as *control parameter.* At its critical value, the old state becomes *unstable.* The emerging coherent wave acts as *order parameter* that via stimulated emission forces the electrons of the gas molecules to emit light waves in a coherent fashion. This action of the order parameter on the individual parts of the system is called *slaving principle.* If the pump power is increased further, more instabilities can appear, and a variety of temporal but also spatio-temporal patterns of light waves may appear, such as laser light chaos (Haken 1975a) or ultrashort laser pulses. The first laser threshold shows the typical features of a phase transition of a system in thermal equilibrium, namely critical slowing down, critical fluctuations and symmetry breaking (DeGiorgio and



Synergetics: Basic Concepts, Fig. 2 The stationary distribution function of the laser light intensity as a function of the normalized intensity $\hat{n}$. The individual *curves* refer to different normalized pump power values *a*, where $a < 0$ below threshold, $a = 0$ at threshold, $a > 0$ above threshold. (After Risken 1965)

Scully 1970; Graham and Haken 1968; Haken 1964, 1985; Sargent et al. 1974), as well as the emergence of a c-number amplitude of the quantized light field (Fig. 2).

The Hierarchical Structure of Synergetics

Before I discuss the mathematical approach in detail and to provide the ground for farther reaching applications, I hint at the three levels of Synergetics:

1. *The microscopic theory*, based either on microscopic equations, such as in the laser example, those of quantum mechanics and quantum field theory, or in biology on mathematical models on the behavior of individual parts of a system. At this level, concepts, such as order parameters and enslavement (cf. section "The Laser Paradigm"), can be mathematically derived.
2. *Phenomenological Synergetics* directly starts from concepts, such as order parameters and enslavement, which then may be cast into mathematical relations.
3. *Semantic Synergetics* deals with cases where a mathematical formulation is (at present or in principle) not possible, but still formulations using concepts and relationships unearthed in Synergetics are applicable.

A general goal of Synergetics consists in elaborating relationships between levels 1, 2, 3.

In the present article I will mainly focus my attention on the mathematical formulation dealing with 1 and 2.

Basic Equations

The basic equations are classical or quantum mechanical evolution equations, in which the temporal evolution of the microscopic quantities under consideration is described by ordinary or partial differential equations. Since the systems are open, the inputs and outputs of energy, matter and/or information must be taken care of, which, quite often, appears in the form of coupling to heat baths in the sense of thermodynamics. In open systems, these heat baths must be kept at different temperatures, in order to maintain the non-equilibrium state of the system. The heat bath variables can be eliminated which gives rise to differential equations which contain "pumping" and "damping" terms as well as fluctuating (stochastic) forces. In the case of quantum mechanical equations the stochastic forces are operators. With the inclusion of stochastic forces, the classical or quantum mechanical equations acquire the character of stochastic differential equations which may be called "generalized Langevin equations".

Depending on the definition of the random forces, we may distinguish between the *I* to, the Statonovich and the Klimontovich approach (Haken 2004b; Îto 1969; Stratonovich 1963). As is well known in statistical physics, Langevin equations can be converted into equations for distribution functions, such as e. g. the Fokker–Planck equation.

A further approach, mainly used in quantum mechanics, but also in models on sociodynamics, is the master equation.

In order not to overload this article, I will focus my attention on the treatment of evolution equations.

This approach seems to be particularly suited for the treatment of phase transition- like phenomena, i.e. the transitions between qualitatively different states of a system. If noise is neglected and transients are not treated, these transitions are called bifurcations (Arnold et al. 1999; Chow and Hale 1982; Guckenheimer and Holmes 1983; Iooss and Joseph 1980; Kielhöfer 2004; Kuznetsov 1995; Ma and Wang 2005).

At the microscopic level the systems are described by a state vector q with components $q_1, \ldots, q_n$ which may also be space dependent, $q_j = q_j\ (x, t)$, where x is a one, two or three dimensional vector. The time dependence is described by evolution equations of the form of a vector equation.

$$\dot{q} = N(q, \nabla, \alpha) + F(q, \nabla, \alpha). \tag{1}$$

The dot $\dot{}$ means time-derivative. N is a vector valued function that depends on q in a nonlinear fashion. ∇ indicates spatial derivatives (of any order) or non-local integrations e. g. of the form

$$\int K(x, x')q(x')\mathrm{d}x' \tag{2}$$

where K is a matrix.

α represents a set of fixed control parameters. If not otherwise stated, we explicitly treat only one control parameter. Equation (1) must be supplemented by appropriate boundary and initial conditions. F is a vector valued stochastic function of time with vanishing mean.

Method of Solution

We assume that for a certain control parameter value α_0 the state vector as solution of Eq. (1) is known, $q = q_0$. The following cases have been considered, see e. g. (Haken 2004b):

(a) q_0 is a stable fixed point (section "Instability of a Fixed Point")
(b) q_0 is a stable limit cycle (section "Instability of a Limit Cycle, $q_0(t)$ (Haken 2004b)")
(c) q_0 is a stable n-dimensional torus. (section "Instability of Tori (Haken 2004b)")

Now the control parameter value is changed and the stability of the system is checked by means of linear stability analysis (Hahn 1967).

Instability of a Fixed Point

We first elucidate our general procedure by means of the instability of an originally stable fixed point. This procedure differs from the classical approach of bifurcation theory (Lyapunov 1906; Schmidt 1908) in two important aspects:

1. The role of the fluctuating forces is fully taken into account in order to be able to make contact with the theory of phase transitions in the Landau sense (Landau and Lifshitz 1959).

2. The approach covers the surrounding of the fixed point in order to deal with relaxation processes towards the newly evolving stable states.

The hypothesis

$$q(t) = q_0 + W(t) \tag{3}$$

is inserted into (1) and the Eq. (1) with $F \equiv 0$ linearized with respect to $W(t)$,

$$\dot{W} = LW \tag{4}$$

where L may be a linear differential (or integral) linear operator.

The solutions are of the form

$$W(x, t) = e^{\lambda_k t} \sum_{d=0}^{D} t^d v_{k,d}(x) \tag{5}$$

where $D > 0$ may happen if the corresponding eigenvalue λ_k is degenerate. In the following we consider $D = 0$ and $v_{k,d} = v_k$. The unstable modes $v_k \equiv v_u$ are connected with

$$\mathrm{Re}\,\lambda_k \geq 0, \tag{6}$$

the stable modes $v_k \equiv v_s$ with

$$\mathrm{Re}\,\lambda_k < 0. \tag{7}$$

It is assumed that Re $\lambda_k < A < 0$, A fixed, if the eigenvalues are discrete.

We decompose the wanted solution to the original non-linear and stochastic equations into a super position of modes determined by the instability analysis whereby we distinguish between the unstable and stable modes. The amplitudes of the unstable modes are the order parameters. Inserting

$$q(t) = q_0 + \sum_u \xi_u(t) v_u(x) + \sum_s \xi_s(t) v_s(x) \tag{8}$$

into the Eqs. (1) and projecting both sides of the resulting equation on the stable and unstable modes, we obtain equations of the form

$$\dot{\xi}_u = \lambda_u \xi_u + \widehat{N}_u(\{\xi_u\},\{\xi_s\}) + \widehat{F}_u(\{\xi_u\},\{\xi_s\}) \tag{9}$$

$$\dot{\xi}_u = \lambda_s \xi_s + \widehat{N}_s(\{\xi_u\},\{\xi_s\}) + \widehat{F}_s(\{\xi_u\},\{\xi_s\}). \tag{10}$$

λ_u, λ_s are the eigenvalues (6), (7), which are assumed to be discrete. By a suitable, in general nonlinear, transformation to new variables, $\tilde{N}$ $(\{\xi_u\})$ can be cast into a particularly simple form ("normal form" theory (Murdock 2002; Nayfeh 1993), initiated by Poincaré (1960)).

If the eigenvalues λ_u, $0 > \mathrm{Re}\,\lambda_s > -|B|$ are a continuous function of an index, e. g. a wave number k, wave packets of $\xi_u(t)$ are used as new order parameter variables Ξ and $\lambda_u(k)$ is replaced by an operator $\Lambda_u\left(-i\frac{\mathrm{d}}{\mathrm{d}x}\right)$ in one space-dimension or, more generally, $\Lambda_u = (-i\nabla)$ (Haken 2004b). For a related approach in fluid dynamics cf. (Newell and Whitehead 1969).

The central idea of further procedure consists in eliminating the amplitudes of the stable modes. This is achieved by the *slaving principle* (Haken 1975b, 2004b; Haken and Wunderlin 1982; Wunderlin and Haken 1981) which allows us to express the amplitudes of the stable modes in terms of the unstable modes

$$\xi_s(t) = f_s(\{\xi_u(t)\},t), \tag{11}$$

where ξ_s, ξ_u, are taken at the same time t. The explicit time-dependence of f_s stems exclusively from that of the fluctuating forces. f_s can be explicitly calculated in terms of a series expansion in powers of the order parameters. For practical purposes, in general only a few terms are needed. For a general discussion of the convergence of this series see (Haken 2004b). When noise is neglected, contact can be made with center manifold theory (Kelley 1967; Pliss 1964), which originally was a mere existence theory and was not constructive. For more recent developments, see books on bifurcation theory. A related approach is based on time-scale separation: The slowly damped or undamped modes serve as order parameters, which enslave the rapidly damped modes. A special case is adiabatic elimination.

Resulting Langevin Equations The enslaved mode amplitudes can be expressed by the order parameters and inserted in (9), so that closed equations for the order parameters alone result.

$$\dot{\xi}_u = \lambda_u \xi_u + \tilde{N}_u(\{\xi_u\}) + \tilde{F}_u(\{\xi_u\},t) \tag{12}$$

where $\tilde{N}$is a polynominal of $\xi(x, t)$ starting with at least second order. $\tilde{F}$ is a stochastic force. A simple, yet prototypical example is (with a single order parameter $\xi = \xi_u$)

$$\dot{\xi} = \lambda\xi + a\xi^2 - b\xi^2 + F(t), \quad b > 0 \tag{13}$$

$$\dot{\xi} = -\frac{\partial V(\xi)}{\partial \xi} + F(t), \tag{14}$$

with the potential

$$V = -\frac{\lambda}{2}\xi^2 - \frac{a}{3}\xi^3 + \frac{b}{4}\xi^4. \tag{15}$$

If λ_u, λ_s (6, 7) represent a continuous spectrum, (generalized) Ginzburg–Landau equations result (Haken 2004b). For example, the complex Ginzburg–Landau equation with fluctuating force reads (Aronson and Kramer 2002).

$$\begin{aligned} &(\xi(x,t) \equiv \xi_u, \text{complex order parameter}) \\ &\dot{\xi} = \lambda\xi + a\Delta\xi - c|\xi|^2\xi + F(t). \end{aligned} \tag{16}$$

A further example is given by the Swift–Hohenberg equation (Swift and Hohenberg

1977), see also (Cross and Hohenberg 1993) (which was derived differently, however)

$$\dot{\xi}(x,t) = (a - b\Delta)^2 \xi(x,t) + c\xi(x,t) - d\xi(x,t)^3. \quad (17)$$

The Eqs. (12, 13, 16, 17) allow for a great variety of solutions. In the case of real λ and a single order parameter, a nonequilibrium phase transition occurs (see below). In case of λ complex, and (at least) one complex order parameter, Landau-Hopf bifurcation (Hopf 1942, 1948), i.e. formation of a limit cycle may happen. In case of (at least) three order parameters and no noise, deterministic chaos may occur (Lorenz 1963; Ruelle and Takens 1971; Sparrow 1982) (in the presence of noise, mixed effects may occur).

Fokker–Planck Equation Below and above the instability point in control parameters space, in a first step the fluctuations can be neglected and then, in the next step, taken care of by means of lowest order perturbation theory. In order to cover the transition region, under well defined conditions a Fokker–Planck equation for the probability density function $f(\{\xi_u\})$ of the order parameters can be derived. For details see (Haken 2004b; Haken and Graham 1971) and the article by T. Frank, this volume.

The Fokker–Planck equation is of the general form

$$\dot{f}(\{\xi_u\}) = -\sum_u \frac{\partial}{\partial \xi_u}(\tilde{N}_u f) + \frac{1}{2}\sum_{uv} \frac{\partial^2}{\partial \xi_u \partial \xi_v}(Q_{uv} f). \quad (18)$$

It is assumed that $\tilde{F}_u$ in (12) is δ correlated in time,

$$\langle \tilde{F}_u(t)\tilde{F}_v(t')\rangle = Q_{uv}\,\delta(t-t'). \quad (19)$$

If $\tilde{F}_u$ depends on ξ_u, the $\hat{I}$ to, Stratonovich or Klimontovich procedure must be applied.

In the case of a single order parameter, where the Langevin equation (Langevin 1908), originally with $\tilde{N} = -\alpha\xi$) is given by

$$\dot{\xi} = \tilde{N}(\xi) + F(t),\ \langle F(t)F(t')\rangle = Q\,\delta(t-t'). \quad (20)$$

The steady state distribution function of (18) is given by (Haken 2004b; Risken 1965)

$$f(\xi) = N \exp\left(-2\int^{\xi} (\tilde{N}(\xi')/Q\mathrm{d}\xi') \equiv N \exp\left(-2V(\xi)/Q\right)\right) \quad (21)$$

provided the boundary conditions are

$$f(\xi) \to 0 \text{ for } |\xi| \to \infty. \quad (22)$$

In the second Eq. (21), $Q =$ const. is assumed. N is a normalization constant. A generalization of (18) to continuous variables, $\xi_u(x,\ t)$, gives rise to a functional Fokker–Planck equation. An explicit solution of the Fokker–Planck equation in the case of several discrete or continuous order parameters can be found if the drift and diffusion coefficients obey the rules of detailed balance (Graham 1981; Graham and Haken 1971).

Nonequilibrium Phase Transition. Connection with Landau Theory The explicit form of the solution of the Fokker–Planck Eq. (21) allows us to make contact with the theory of phase transitions in the sense of the Landau theory (Landau and Lifshitz 1959) where

$$f(\xi) = N \exp\left(-F(\xi,T)/(kT)\right),$$
$$F(\xi,T) = F(0,T) + a(T-T_c)\xi^2 + \frac{\beta}{4}\xi^4. \quad (23)$$

In (21), V corresponds to the free energy F and the noise strength Q corresponds to *absolute* temperature T. T_c is the critical temperature, and (23) refers to a second order phase transition. In case of a first order phase transition, an additional term $\gamma\xi^3$ appears in (23).

An important difference between phase transitions at thermal equilibrium and in the present case of non-equilibrium should be mentioned, however. The decisive constants in the case of non-equilibrium (Haken 2004b) phase transitions are rate constants in contrast to thermodynamic quantities in (23). While non-equilibrium phase transitions described by (21) were experimentally very well verified for instance in the case of lasers (Risken 1965) (Fig. 1), in the case of thermal equilibrium the Landau theory can not be considered as a good approximation and had been replaced by the concept of critical exponents etc. as dealt with by renormalization group theory (Kadanoff et al. 1967; Wilson and Kogut 1974). For a treatment of the time dependent Fokker–Planck equations see Risken (1989).

In a number of cases the drift- and diffusion coefficients of the Fokker–Planck equation are by themselves expectation values, defined on the probability density function so that the Fokker–Planck equation becomes non-linear. For more details see the article by T.D. Frank in this volume.

Instability of a Limit Cycle, $q_0(t)$ (Haken 2004b)

The instability is checked by linear stability analysis by means of the hypothesis

$$q(t) = q_0(t) + W(t), \tag{24}$$

where $q_0(t)$ is a time-periodic solution to (1) with $\alpha = \alpha_0$, $W(t)$ a small deviation.

Inserting (24) into (1) with $F \equiv 0$ and linearization leads to an equation of the form (4), where L because of $q_0(t)$ has become also a time-periodic function with the same period as $q_0(t)$. According to Floquet theory (Floquet 1883), the solutions to (4) with periodic $L(t)$ are given by

$$W(t) = e^{\lambda_j t} v_j(t) \tag{25}$$

(in the case of nondegeneracy), where $v_j(t)$ has the same period as q_0, i.e. L.

Depending on Re $\lambda_j \geq 0$ or < 0 we distinguish between unstable and stable modes (6, 7), respectively. One eigenvalue is $= 0$ and corresponds to an indeterminate phase shift, which in nonlinear analysis is taken care of by a phase $\phi(t)$ that acts as additional order parameter. In order to solve the fully nonlinear and stochastic equations, the hypothesis

$$\begin{aligned} q(t) &= q_0(t + \phi(t)) \\ &\quad + \sum_u \xi_u(t) v_u(t + \phi(t)) \\ &\quad + \sum_s \xi_s(t) v_s(t + \phi(t)) \end{aligned} \tag{26}$$

is inserted in the Eqs. (1). The subsequent procedure follows the lines outlined above and leads to order parameter equations of the form

$$\dot{\xi}_u = \lambda_u \xi_u + \widehat{N}_u(\{\xi_u\}, \phi) + \widehat{F}_u(\{\xi_u\}, \phi) \tag{27}$$

$$\dot{\phi} = M(\{\xi_u\}, \phi) + G(\{\xi_u\}, \phi) \tag{28}$$

where $\widehat{N}$, $\widehat{F}$, M, G are polynominals in $\{\xi_u\}$ and periodic functions of ϕ.

The novelty as compared to the case of an unstable fixed point consists in the introduction of a phase as order parameter.

When noise is neglected, the newly evolving, i.e. bifurcating solutions are either two (or several) limit cycles or tori. Also basically, depending on the system, also a "back bifurcation" to a stable focus can happen.

Instability of Tori (Haken 2004b)

The corresponding theory is rather complex so that a few words must suffice here. The basic idea (Haken 2004b) is based on an extension of (24, 26) where q_0 is chosen as a quasi periodic function

$$q_0 = q_0(\omega_1 t, \omega_2 t, \ldots, \omega_M t) \tag{29}$$

where the $\omega's$ must be sufficiently irrational in the sense of the KAM (Kolmogorov (1954), Arnold (1963), Moser (1967)) theorem. Besides amplitudes as order parameters, also phases $\phi_1(t), \ldots, \phi_M(\tau)$ are introduced. For details cf. (Haken 2004b), and for alternative approaches (Chenciner and Iooss 1979; Sell 1979).

A Remark on the Method of Solution of Evolution Eq. (1)

In this article the central role of order parameters is stressed because this allows us to establish profound analogies between quite different systems. In practical applications it may be preferable, however, to apply other methods of solution, analytical, numerical or mixed, in order to derive the spatial, temporal or spatio-temporal patterns. In this way, the Springer Series in Synergetics have developed a "tool box" of models (Mikhailov 1993).

Quantum Theoretical Formulation

In a quantum theoretical treatment one deals with quantum mechanical Langevin equations which are Heisenberg equations of motion for operators to which pumping and damping terms as well as random noise sources are added. Here, according to quantum theory, the system's observables are represented by time-dependent quantum mechanical operators, Ω_j. For instance, by the position operator $\widehat{x}$ and the momentum operator $\widehat{p}$ of a particle, or, in quantum field theory, by creation and annihilation operators $\widehat{b}^+$, $\widehat{b}$, respectively. The quantum mechanical Langevin equations read (see, for instance (Haken 1970, 1985)):

$$\dot{\Omega}_j = \frac{i}{\hbar}\left[H, \Omega_j\right] + \text{damping} + F_j(t), \tag{30}$$

where H is the Hamilton operator, and $F_j(t)$ are stochastic operators which usually are assumed to be δ-correlated in time. The quantum mechanical properties can be determined by the postulate of quantum mechanical consistency of Ω_j, (cf. (Haken 1970), appendix).

If the non-commutativity of operators is taken care of, the procedure to derive order parameter equations is formally the same as in the case of classical Langevin equations as indicated above. The Fokker–Planck equation, however, must be replaced by a density matrix equation, originally introduced as master equation (Pauli 1928). For nonequilibrium systems, such as the laser, see (Scully and Lamb 1967; Weidlich and Haake 1965), also (Haken 1970; Sargent et al. 1974). Using methods of quantum classical correspondence, this density matrix equation can be converted into a Fokker–Planck equation under specific conditions. The basic idea is this:

Quantum-Classical Correspondence

There are several ways to define quantum classical correspondence. In the case of position operator $\widehat{x}$ and momentum operator $\widehat{p}$ with the commutator $[\widehat{p}, \widehat{x}] = \frac{\hbar}{i}$ and the density matrix ρ, the Wigner distribution function $W(x, p)$ (Wigner 1932) is defined by

$$W(x,p) = \frac{1}{(2\pi)^2} \cdot \int \int_{-\infty}^{\infty} e^{-ikx-ilp} \cdot \operatorname{tr}\left(e^{ik\widehat{x}+il\widehat{p}}\rho\right) \mathrm{d}k\,\mathrm{d}l \tag{31}$$

where "tr" means trace.

Thus a relation is established between the quantum mechanical density matrix and a classical quasi-density $W(x, p)$. Based on (31) or equation (34, 35, 36), a density matrix equation can be converted into a generalized Fokker–Planck equation (Haken 1964).

By the transformation of $\widehat{x}$, $\widehat{p}$ to creation and annihilation operators b^+, b by means of

$$\widehat{b}^+ = \frac{1}{\sqrt{2\hbar}}(\widehat{x} + i\widehat{p}) \tag{32}$$

$$\widehat{b} = \frac{1}{\sqrt{2\hbar}}(\widehat{x} - i\widehat{p}) \tag{33}$$

an alternative form to (32) is given by

$$P(\beta, \beta*) = \frac{1}{\pi^2} \int \int_{-\infty}^{\infty} e^{-i\beta k - i\beta * l} \cdot \operatorname{tr}\left(e^{ik\widehat{b}^+ + il\widehat{b}}\rho\right) \mathrm{d}\,k\,\mathrm{d}\,l. \tag{34}$$

Because $\widehat{b}^+$, $\widehat{b}$ are noncommuting operators, $[\widehat{b}^+, \widehat{b}] = 1$, different "quasiprobability" distributions P result, if

$$e^{ik\widehat{b}^+ + il\widehat{b}}$$

is replaced by

$$e^{ik\widehat{b}^+} e^{il\widehat{b}} \tag{35}$$

or

$$e^{ik\widehat{b}} e^{il\widehat{b}^+}. \tag{36}$$

(35) gives rise to the Glauber–Sudarshan representation. For details and references see (Haken 1970).

Regular Spatial and Spatio-Temporal Patterns

One of the most striking features of non-equilibrium systems in physics, chemistry and biology is their capability of forming (more or less) regular spatial pattern (for explicit examples see below). (There is a rich literature on pattern formation in physics, especially fluids (Chandrasekhar 1961; Cross and Hohenberg 1993; Manneville 1990; Swinney and Gollub 1981), but also semiconductors (Schöll 2001) and nonlinear optics (Staliunas et al. 2003), chemistry (Epstein and Pojman 1998; Fife 1979; Kuramoto 1984) and biology (Babloyantz 1986; Meinhardt 1982, 1990; Murray 1989) and general (Horsthemke and Lefever 1983; Hoyle 2006; Mikhailov 1993; Nekorkin and Velarde 2002; Pismen 1999, 2006; Rabinovich et al. 2000; Vavilin et al. 1967). Furthermore, the patterns exhibit striking similarities in spite of the fact that the individual parts are quite different. The methodology of Synergetics (e. g. (Haken 2004b)) provides us with a basic insight into the causes of such analogies.

Pattern formation is determined by at least three causes:

1. internal mechanism, such as e. g. the interplay between reactions and diffusion in large scale chemical processes,
2. the influence of boundaries,
3. initial conditions.

Concerning (1) and (2) between two (limiting) cases can be distinguished.

1. dimensions of the internally evolving patterns are of the same or larger order as those of the boundaries. Here a strong influence of the boundaries must be expected.
2. dimensions of evolving patterns are small compared to those of the boundaries (boundaries $\to \infty$).

To bring out the essential features we consider that originally for a control parameter value α_0 the system is homogeneous and quiescent. The approach can, however, be extended to a space dependent reference state (which, e. g. resulted from a first bifurcation leading to $q_0 = q_0(x)$) and the cases of a limit cycle or torus. The space may be 1, 2 or three dimensional Euclidian or, e. g., a 2 or 3 sphere.

Infinite Boundaries

We start with 2 infinite boundaries, the medium is homogeneous and isotropic. We assume a continuous transition from the homogeneous to the "bifurcating" state. The evolving patterns are determined by the leading terms in (8) that we call the "mode skeleton"

$$q(x,t) = q_0 + \sum \xi_u(t) v_u(x) \qquad (37)$$

and the order parameter Eq. (12). The functions $v_u(x)$ are *the space-dependent part of the* solutions to (4) where L is a differential (or integral) operator which is invariant against translation and rotation. Thus, e. g., L commutes with the displacement operator

$$\Omega_a : x \to x + a,\ a \text{ constant vector}.$$

Thus $v_u(x)$ can be chosen as eigenfunction to Ω_a,

$$\Omega_a v_u(x) = \Lambda v_u(x) \qquad (38)$$

with

$$v_u = e^{ikx} \qquad (39)$$

$$\Lambda = e^{ika} \qquad (40)$$

i.e. plane waves. Which waves must be considered in (37) is determined by λ_u in (6) as well as by the order parameter Eq. (12).

The condition Re $\lambda_u(k) = 0$ defines $k = k_{\text{crit}}$. As was shown by means of many examples $k \approx k_{\text{crit}} \neq 0$. If the boundaries are finite, such a discrete k must be chosen which comes closest to k_{crit}. If the boundaries tend to infinity, a continuous set k is taken care of by (generalized)

Ginzburg–Landau equation (see above). If the boundaries are "narrow" in 1 or 2 dimensions, but large in the remaining dimensions, the wave vector k must be split into k_{II} and $k_\perp$ where k_{II} is practically continuous and $k_\perp$ discrete. Quite often only one $k_\perp$ (the most critical) needs to be considered. This leads to practically 2 (or 1) dimensional patterns connected with k_{II}. In the 2-dimensional case, the modes with $|k_{\text{II}}| = k_{\text{crit}}$. are degenerate. This degeneracy can be lifted by a weak influence of boundaries (leading to roll patterns), by specific initial condition which (by chance) prefers a specific roll pattern, or by terms in the order parameter-equations that lead to specific combinations, e. g.

$$k_1 + k_2 + k_3 = 0, \tag{41}$$

where k_j, $j = 1, 2, 3$ belong to k_{II}.

This gives rise to the formation of hexagons. This is the case if the leading term of $\tilde{N}$ contains

$$\int v_{k_1} v_{k_2} v_{k_3}\, \mathrm{d}^2 x \neq 0. \tag{42}$$

In three dimensions this mechanism may lead to plane wave fronts stabilizing each other which gives rise to icosaeders, as observed in diatomea.

An important class of spatio-temporal patterns (in 2 dimensions) results when the system utilizes *rotation symmetry.* This can best be explained by the following example:

In many cases of practical interest, N in (1) and thus L in (4) contain the Laplace operator Δ When written in planar polar coordinates r, ϑ, solutions to (4) are of the general form

$$v \propto \mathrm{e}^{i(m\vartheta - kr - \omega t)} \tag{43}$$

(times a rotation symmetric function *g(r))* which represents *spirals.* $m = 0$ represents concentric rings, while an integer $m > 0$ represents the number of spiral arms. $\omega = 0$ represents standing spirals, $\omega \neq 0$ rotating spirals.

The mode skeleton (37) is composed of functions of the form (43). Which of the functions (43) appear in (37) depends on the competition Eqs. (12) for order parameters, which may also allow for a super position of counter rotating spirals (such as in the sunflower head). As group theory shows (see below), solutions (43) with different m's belong to different irreducible representations, and do not coexist in (37). This does not exclude the coexistence of differently rotating spirals in *different* regions of space, however.

The above results can be cast into the isomorphy principle:

While the "true" q is represented by (we omit the homogeneous q_0)

$$q = \sum_u \xi_u v_u(x) \\ + \text{enslaved modes, with same symmetry.} \tag{44}$$

and v_u "true modes", its symmetry features can be replaced by a "representative" q':

$$q' = \sum_k \xi_k R_k(x), \tag{45}$$

where R_k represent the "elementary" functions showing the symmetry under consideration. While the material significance and explicit form of q according to (44) may be quite different for different material substrates, q' (45) shows the *same* patterns for different systems.

These results can be deepened by invoking group theory, in which also the effect of the boundaries is taken into account.

Theory, Representation Theory, Finite Boundaries

Consider a set of transformations G_j of space variables $x \to x'$ so that

$$G_j q \to q' \tag{46}$$

Example 1 G_j induces the translation

$$x \to x + a \quad \text{so that} \quad G_j q(x) = q(x + a). \tag{47}$$

The transformations must be so that they are compatible with the internal properties of the system (1) and the boundary conditions. Example: when dealing with a problem on a 2-dimensional sphere, the transformed coordinates x must not leave the sphere.

Because of the symmetry of the problem, the transformations G_j form a group defined by

1. existence of unity E such

$$G_j E = G_j \text{ for all } j \tag{48}$$

2. the product of two group elements is again an element of the group,

$$G_j G_k = G_l \text{ for all } j, k \tag{49}$$

3. existence of an inverse G_j^{-1} for all j so that

$$G_j^{-1} G_j = E, \tag{50}$$

4. associative law

$$(G_k G_l) G_j = G_k (G_l G_j) \tag{51}$$

for all group elements.

In the following we first ignore random forces, i.e. we consider (1) with $F \equiv 0$.

$$\dot{q}(x,t) = N(q, \Delta, \alpha). \tag{52}$$

Jointly with the boundary conditions, (52) defines a function space S in which all functions to be considered must lie (i.e. can be represented by linear combinations of a complete set of (vector valued) basic functions of S; example: S is a Hilbert space)

Definition 1 The system is invariant against G_j if for all f ∈ S

$$G_j N\left(G_j^{-1} f\right) = N(f). \tag{53}$$

Example 2

$$G_j : x \to x + a, \tag{54}$$

$$N(f) = \Delta f + V(x) f + f^2. \tag{55}$$

Then

$$\begin{aligned} G_j \cdot N\left(G_j^{-1} f\right) = \Delta G_j^{-1} f(x+a) \\ + V(x+a) G_j^{-1} f(x+a) \\ + \left(G_j^{-1} f(x+a)\right)^2 \end{aligned} \tag{56}$$

$$= \Delta f(x) + V(x+a) f(x) + f(x)^2 \tag{57}$$

$$\neq N(f) = \Delta f(x) + V(x) f(x) + f(x)^2 \tag{58}$$

unless $V(x+a) = V(x)$. If a in (54) is arbitrary, N is not invariant against (54).

Application of G_j to q in (52) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(G_j q\right) = N\left(G_j q\right) \tag{59}$$

or because of (53), (with $f = G_j q$), to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(G_j q\right) = G_j N(q). \tag{60}$$

In the spirit of representation theory of groups the action of G_j on f can be understood as an abstract operation, but also as a matrix acting on the vector f in S- space.

By appropriate transformation of basis of q, and using the representation theory of symmetry groups, all matrices U_j belonging to all group elements j can simultaneously be decomposed into “irreducible” representations so that (in the example of 3 irreducible representations)

$$U_j = \begin{pmatrix} \square & \bigcirc & \bigcirc \\ \bigcirc & \square & \bigcirc \\ \bigcirc & \bigcirc & \square \end{pmatrix} \tag{61}$$

Each box □ is a matrix $U_j^{(k)}$ with dimension Dk, so that

$$D_1 + D_2 + \cdots + Dk = \text{dimension } U_j.$$

Example 3 Rotation group applied to 2-sphere (e. g. earth surface). Basis functions are spherical harmonics Y_m^l with “quantum numbers” l, m. Subspace l fixed, $m = 0, \ldots, l-1$. As a consequence, the mode skeleton reduces to (q_0 dropped)

$$q^l = \sum_m \xi_m(t) Y_m^l. \tag{62}$$

There is no coupling between different *ls,* which implies a low dimensional dynamics of ξ_m.

Generally, the original function space S is decomposed into subspaces forming the basis of each irreducible representation. This implies a symmetry reduction beyond bifurcation point, compared to the situation below bifurcation point, where

$$G_j\, q = q \text{ for all } j, \tag{63}$$

i.e. q fully symmetric under G.

In our example beyond the bifurcation point q is given by q^l where Y_m^l transforms according to the subgroup G^l, which leaves the space spanned by Y_m^l invariant. If, however, group elements not belonging to G^l are applied to q^l, this space is left. In other words, q^l is connected with a lower symmetry than q (63). By bifurcations, the symmetry of q is lowered and one speaks of "symmetry breaking instability". If fluctuating forces in (1), i.e. in (52) are taken into account, the full symmetry can be restored (under specific conditions on the fluctuating forces).

While group theory has found important and widespread applications to quantum theory, it is less frequently used in problems of Synergetics, though there it may lead to deep insights as pointed out above. (For an in-depth approach see (Golubitsky and Schaeffer 1988; Golubitsky et al. 1988; Sattinger 1980).)

On top of, or jointly with, regular patterns, a variety of defects as well as boundaries between different patterns may occur (cf. contribution by Pismen, this volume and (Pismen 1999, 2006)).

A Further Mathematical Tool: Shannon Information and the Maximum (Information) Entropy Principle

While evolution equations are the backbone of Synergetics, also other tools are invoked to deal with complex systems. Such a tool is Shannon information (Shannon and Weaver 1949) which is defined by

$$i = -\sum_j p_j \log_2 p_j \tag{64}$$

where p_j is the relative frequency of the event j or, in a different interpretation, the probability of finding the realization j in an experiment. The maximum (information) entropy principle as formulated by Jaynes (1957, 1967), for an earlier proposal see (Elsasser 1937)), allows one to make unbiased guesses on systems on which only incomplete data are known by maximizing the informations, i.e. (64) = max! or = extremum! under given constraints.

A simple example is provided by a gas composed of N particles, where the total kinetic energy $\mathrm{E}_{\mathrm{kin}}^{\mathrm{tot}}$ is fixed. Denoting the kinetic energy of a particle with mass m and velocity v_i by $f_i = (m/2)v_i^2$, the mean kinetic energy per particle is

$$\sum_i p_i f_i = \mathrm{E}_{\mathrm{kin}}^{\mathrm{tot}}/\mathrm{N} \tag{65}$$

To fix p_i, (64) must be maximized under the normalization condition

$$\sum_i p_i = 1 \tag{66}$$

and the constraint (65).

Using Lagrange multipliers, λ, λ_1, the result reads

$$p_i = \exp\left(-\lambda - \lambda_1 m v_i^2/2\right) \tag{67}$$

i.e. the Maxwell–Boltzmann distribution function. Also relations between the Lagrange multipliers λ, λ_1 can be established which, evidently, have fundamental thermodynamic significance.

This approach has been extended to the treatment of nonequilibrium phase transitions, i.e. determination of order parameters, enslaved modes and emerging patterns (Haken 2000). The crucial idea consists in the proper choice of constraints, as which the moments of the variables q_i are chosen:

$< \ldots >$ means average over the joint distribution function $f(q_1, q_2, \ldots, q_n)$ which replaces p_j and the vector $(q_1, \ldots, q_N)$ replaces j. The variables q_j may be discrete or continuous.

$$f_i = < q_i > , i = 1, 2, \ldots, N. \tag{68}$$

$$f_{ij} = < q_i q_j >. \tag{69}$$

$$f_{ijkl} = < q_i q_j q_k q_l >, i, j, k, l = 1, 2, \ldots, N. \tag{70}$$

The resulting distribution function is given by

$$q = \exp V(\lambda, q) \tag{71}$$

with

$$V(\lambda, q) = \lambda + \sum_i \lambda_i q_i + \cdots + \sum_{ijkl} \lambda_{ijkl} q_i q_j q_k q_l. \tag{72}$$

(71) is a starting point to make contact with the Landau or Ginzburg–Landau theory of phase transitions (Landau and Lifshitz 1959), and to guessing Fokker–Planck equations. The approach allows one to calculate the efficiency of self-organizing systems close to their instability points.

The method has been extended to the "unbiased modeling" of stochastic processes: how to guess path integrals, Fokker–Planck equations and Langevin-Îto equations (Haken 1996). The central quantity to be searched for is the probability density P_n of paths.

Let $q(t)$ be the state vector $q = (q_1, \ldots, q_n)$ at time t, then

$$\begin{aligned} &P_n(t_n, t_{n-1}, \ldots, t_0) \\ &= P_n(q(t_n), t_n; q(t_{n-1}), t_{n-1}; \ldots; q(t_0), t_0), \\ &\quad t_n > t_{n-1} > \ldots > t_0. \end{aligned} \tag{73}$$

This task is simplified if the Markov hypothesis on the process holds, i.e.

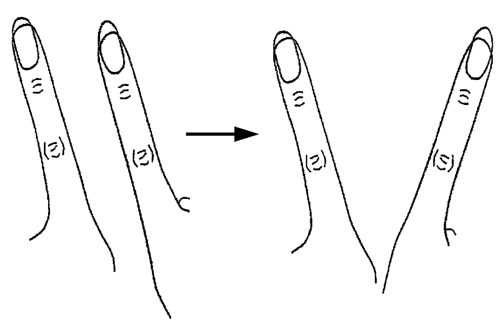
$$P_n(t_n, t_{n-1}, \ldots, t_0) = \widehat{P}(q(t_n), t_n | q(t_{n-1}) t_{n-1}) \cdot P_{n-1} \tag{74}$$

where $\widehat{P}$ is the transition probability so that only transition probabilities between subsequent states (with $\Delta t \to 0$) must be guessed in addition to P_0.

In the frame of the present approach, this task is fulfilled by use of the maximum information principle. The constraints to be used are essentially conditional first order moments and two-time correlation functions of the state vectors $q(t)$, $q(t')$.

Phenomenological Synergetics

In many fields of science, including medicine, the microscopic variables and their dynamics are not well-known or not known at all. Nevertheless, in quite a number of cases, namely where dramatic macroscopic changes of the system's behavior take place, general insights, gained by Synergetics, can be invoked. A paradigm for this procedure is the modeling of Kelso's finger experiments (Kelso 1981, 1995) (Fig. 3). He instructed subjects to move their index fingers in parallel which was accordingly performed. However, when the speed of the fingers was increased, the parallel movement was replaced by a symmetric movement quite involuntarily and spontaneously. In other words, a transition from a parallel to an anti-parallel phase takes place. In terms of Synergetics, the interpretation is simple: the control parameter consists in the prescribed frequency ω of the finger movement, whereas the macroscopic quantity, i.e. the order parameter that changes dramatically is provided by the relative phase of the two index fingers. According to the experience made in Synergetics, the order



Synergetics: Basic Concepts, Fig. 3 Transition between finger movements from parallel to symmetric in Kelso's experiment (Haken et al. 1985) parameter, here called ϕ obeys a typical order parameter equations of the form (Haken et al. 1985)

$$\dot{\phi} = -\frac{\partial V}{\partial \phi} + F(t), \tag{75}$$

where $V(\phi, \omega)$)is a potential function and F a fluctuating force. When the control parameter ω is changed, the potential runs through a series of forms as depicted in Fig. 4. As was shown in detail, at a critical value of ω, the transition from one potential minimum to another one occurs, as related to the change of the kind of finger movement. The mathematical analysis shows hysteresis, critical slowing down and critical fluctuations (Haken 1996) which reject the idea that the brain acts like a computer via a motor program but rather via self-organization.

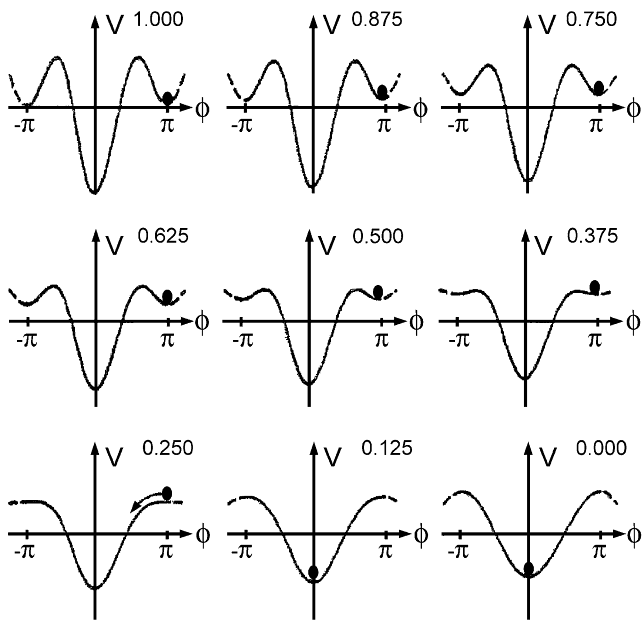
Another application is made by the Synergetic computer (Haken 2004a) (Figs. 5 and 6), where to each pattern to be recognized a specific order parameter is attached. Pattern recognition is then achieved via a competition between order parameters. The competition equations are given by

$$\dot{\xi}_k = \frac{\partial V}{\partial \xi_k} V(\xi_1, \ldots, \xi_M)$$
$$= -\frac{1}{2}\sum_k \lambda_k \xi_k^2 + \beta \sum_{k,k'} \xi_k^2 \xi_{k'}^2 - C \sum_k \xi_k^4. \tag{76}$$

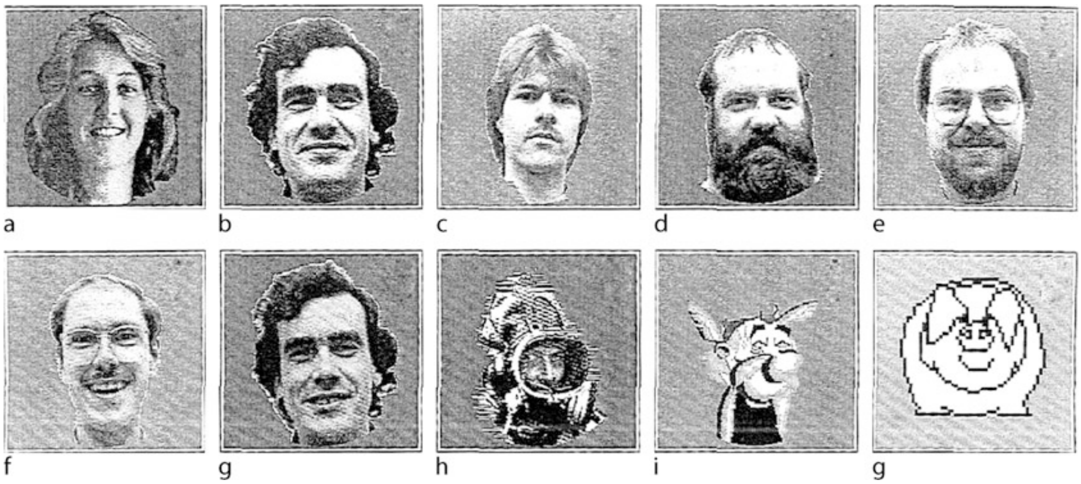
This approach may serve also for modeling of brain functions: both recognition as well as movements are governed by the establishing of order parameters which may wander from one quasi attractor to another one. Quasi attractors are defined as attractors that vanish after the task has been accomplished, e. g. after a pattern has been recognized or movement performed.

Based on the concept of order parameters, a learning procedure for Synergetic computers has been developed (Haken 2004a). Here the number of patterns to be recognized is prescribed and then a special functional must be minimized. In the case of the Synergetic computer, it is possible to make contact between the microscopic and the mesoscopic description, i.e. the microscopic variables are pixel values q_j, j pixel index, whereas the mesoscopic (or macroscopic) quantities are the order parameters ξ_k.

The relation between ξ_k, q_j is given by



Synergetics: Basic Concepts, Fig. 4 Sequence of potential curves of the Haken–Kelso–Bunz model of Kelso's experiment (Haken et al. 1985)



Synergetics: Basic Concepts, Fig. 5 Recognition of faces by the synergetic computer: stored or learned prototype patterns (Haken 2004a)

Synergetics: Basic Concepts, Fig. 6 Pattern recognition by the synergetic computer: recognition of a specific face of which initially only a subset of pixels is presented (Haken 2004a)

$$\xi_k = \sum_j v_j^{k+} q_j, \tag{77}$$

where v_j^{k+} are adjoint prototype patterns, with k pattern index, j pixel index.

The relation between prototype patterns v_j^k and their adjoints is given by

$$\sum_j v_j^{k+} v_j^{k'} = \delta_{kk'} \tag{78}$$

At the phenomenological level the order parameter concept allows us to interpret and model complex movement patterns, e. g. learning to ride on a pedalo (Haken 1996). In the experiments, LED's are fixed at the joints of the subject and their positions measured which gives rise to a series of time-dependent tracks. Then, in a first step, a principle component analysis is performed, in the next step, by means of a variational principle, the best fit is searched in terms of order parameters and their equations of motion, in order to mimic the actual tracks. While in the learning phase several order parameters are needed, at the end the whole movement is governed by a rather simple equation for a complex order parameter.

During the development of Synergetics it turned out that there are strong relations to gestalt theory (Köhler 1920) as well as to psycho physics. A typical example is provided by ambivalent figures where (Fig. 7) (Fisher 1967) shows an example. An observer may either perceive a young woman or an old woman, but not both simultaneously, rather the perception switches between these two percepts. In the mathematical modeling to each percept an order parameter is attached (Haken 2004a), which obeys the typical equations of Synergetics. The control parameter invoked here is attention. According to an early suggestion by Wolfgang Köhler (1920), when a pattern is recognized, the corresponding attention fades away. This has been modeled mathematically based on a competition dynamics between two order parameters, when the control parameter (attention) of one pattern fades away, the other pattern gets the possibility of being perceived. Then in the next step the corresponding attention parameter fades away and the first pattern may re-appear (Fig. 8) (Haken 2004a). This model describes details of the observed phenomena, such as the dependence of the duration of the perception of one face as compared to that of the other face, dependent on the bias which face is recognized first. Also, one may distinguish between slow, medium and fast observers, depending on the individual parameters.

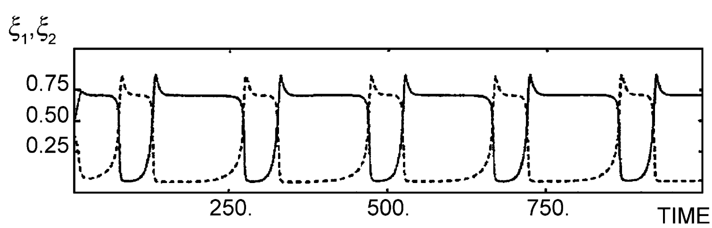
Quite generally, order parameters may have properties of gestalt in the sense that they are invariant against size, orientation and perception of objects in space.

In medicine, a syndrome has the characteristic features of an order parameter. On the one hand it is generated by the co-operation, or at least by the simultaneous presence of specific features, on the other hand once the syndrome (order parameter) is established, it acts on the individual parts of the system, where the slaving principle induces specific phenomena at the level of individual parts. Clearly, the concept of circular causality plays an important role here. It shows that the syndrome, at least in general, can not be cured by curing an individual symptom, but rather by curing a decisive majority of individual causes.

Synergetics: Basic Concepts, Fig. 7 Example of an ambivalent figure: young/or old woman? (Fisher 1967)

Semantic Synergetics

In soft sciences, but also in medicine and other fields, a mathematical modeling, even at the level of order parameters may not be possible. Nevertheless, Synergetics may provide us with qualitative insights into basic mechanisms. In psychology and psychiatry (Schiepek 1999), quite often specific mental states can be ascribed to a patient. For instance in bipolar patients a depressive phase or a manic phase may appear or in depressive patients a normal phase and a depressive phase. Another example is provided by patients with a compulsory action. In the spirit of Synergetics, as a theory of indirect control, one may ask, whether there are appropriate control parameters by means of which the behavior of a person can be changed. Let the two states be

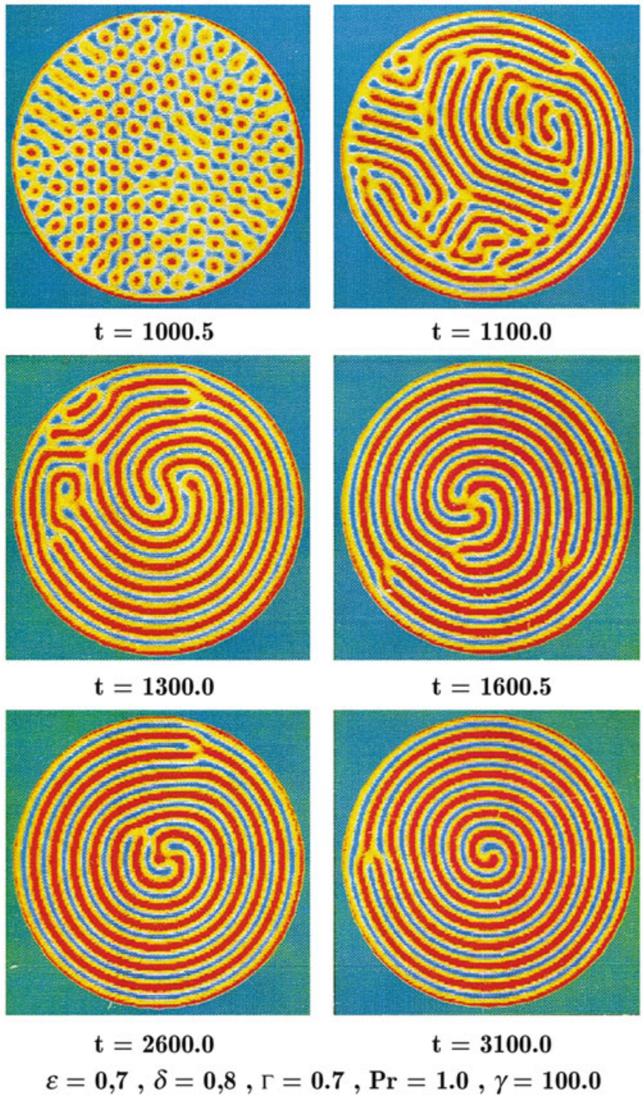


Synergetics: Basic Concepts, Fig. 8 Order parameter oscillations belonging to the recognition young woman/old woman with bias towards the young woman (Haken 1996) represented by the positions of a ball in a landscape with two valleys. In this situation, direct control means to push the ball from the unwanted position to the wanted. Indirect control means to lower the potential hill between the two valleys so that the wanted transition may occur via self-organization. This may happen through interventions used in cognitive psychology, a change of environmental conditions, or/and by specific medication. The central issue here is that the patient is not directly influenced, e. g. by saying you must do this or that, but rather by a soft changing of his/her point of view. A number of successes have been reported about this method which is, to some extent, well known in psychiatry, but finds here a scientific theoretical basis. For more details see the article by G. Schiepek and V. Perlitz, this section, and in a somewhat related form (Hansch 2002).

Some Selected Examples

The study of nonlinear, self-sustained oscillations (Abraham and Marsden 1978; Andronov et al. 1966; Bogoliubov and Mitropolsky 1961) be it in radio-engineering, mechanics or other fields, has a long tradition. In the context of bifurcation theory, their origin was unearthed by Hopf (1942, 1948).

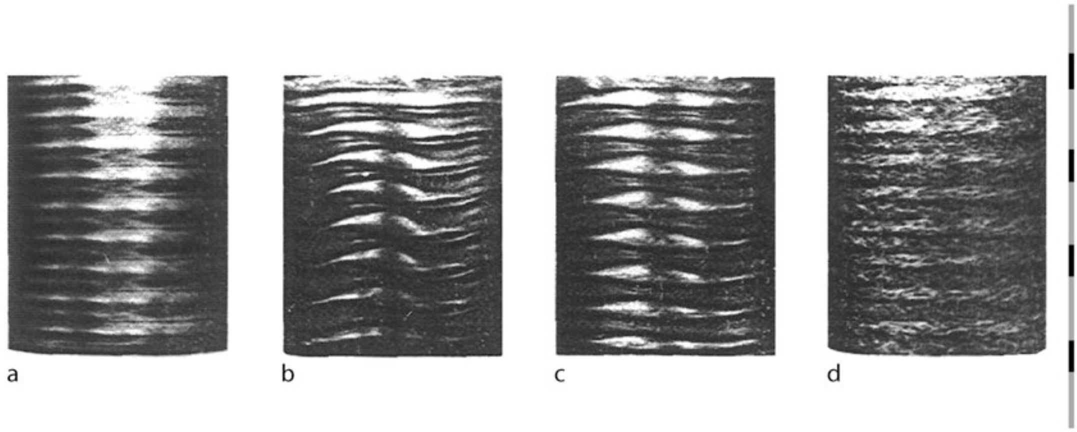
Nonlinear optics (Mills 1991) and, when quantum effects are important, quantum optics (Haken 1979; Meystre and Sargent 1990; Schleich 2001; Walls and Milburn 1994) provide us with a wealth of phenomena, in particular of the formation of coherent oscillations. A device, closely related to the laser, is the parametric oscillator (Graham 1970), in which, within a nonlinear crystal, incoming pumplight is split into a signal and an idler. Then, similar to the laser light, the signal light becomes amplified, and its generation can be described as that of a nonlinear quantum-mechanical oscillator. Fluid dynamics is rich of pattern formations (including chaos) (Bodenschatz et al. 2000; Busse 1972; Bénard 1900a, b; Fenstermacher et al. 1979; Gollup and Benson 1979; Lorenz 1963; Manneville 1990; Newell and Whitehead 1969; Rabinovich et al. 2000; Ruelle and Takens 1971; Segel 1969; Swift and Hohenberg 1977; Swinney and Gollub 1981), to mention just a few. In a fluid heated uniformly from below, with increasing temperature difference, several instabilities may occur for instance giving rise to stationary patterns, such as rolls, hexagons (Fig. 9) or squares. In the next step the rolls may start to show oscillations, and still more complex patterns may occur (Fig. 3). In the case of the Taylor instability (Taylor 1923), a liquid is placed in between two coaxial cylinders, where the outer one is rotating. With increasing rotation speed, a hierarchy of instabilities is reached, first the formation of roles, then oscillating rolls at one frequency, then oscillation of rolls at two frequencies, and finally weak turbulence, i.e. chaos occurs (Fig. 10) (Fenstermacher et al. 1979; Marx 1987). Important phenomena are the establishing of boundaries and of defects as described in the article by Pismen (1999, 2006) and other articles of this Encyclopedia. A rich variety of pattern formation may occur in semiconductors (Schöll 2001), where electrons and holes as well as currents form specific spatio-temporal patterns. In meteorology, atmospheric convection patterns and other instabilities are treated (Giaiotti et al. 2007). In chemistry, oscillations and large scale patterns arise by means of the interplay of chemical reactions and diffusions (Belousov 1959; Bray 1921; Epstein and Pojman 1998; Field et al. 1972; Fife 1979; Zaikin and Zhabotinsky 1970), e.g. concentric ring patterns, each starting from a center, which then annihilate each other when colliding. An important class is provided by spiral patterns which may have one to several arms (Fig. 11). In biology, specific models on morphogenesis were treated, such as the formation of stripe or spot patterns on animal furs or skins of fish (Fig. 12) or still more complicated patterns on sea shells (Gierer and Meinhard 1972; Haken 2004b; Meinhardt 1982; Meinhardt 1990; Murray 1989). The basic idea which can be traced back to Turing (Turing 1952) is this: originally unspecialized cells produce activator and inhibitor molecules which by reaction and diffusion form a prepattern, a morphogenetic field (Wolpert 1969). At positions of high



Synergetics: Basic Concepts, Fig. 9 Model calculation of the motion of a fluid in a circular pan uniformly heated from below. (After Fantz et al. 1993). Upper left corner: above a critical temperature difference between lower and upper surface of the fluid layer, a hexagonal pattern appears. If the boundary is also heated uniformly, a transition to the spiral pattern with one or several arms can be found (lower right corner*)*

activator concentration, genes are switched on which then leads to cell differention producing e. g. pigments. In aggregating slime mold, spiral or concentric ring patterns are observed (Bonner et al. 1972; Gerisch and Hess 1974). Mathematical models on prebiotic evolution (Eigen and Schuster 1977) study the competition between species of biomolecules and the "survival" of the fittest, where pronounced analogies with the dynamics of laser photons can be unearthed, fully in line with Synergetics (Haken 2004b). In the understanding of brain function, for instance, steering of movements, pattern recognition or decision making, the reduction of degrees of freedom of the numerous neurons to few order parameters is central (Haken 1996).

The concepts and principles of Synergetics shed new light on important relationships in economy, such as cooperation and competition between companies, the important role of indirect steering by means of control parameters, such as taxes, interest rates. It can be shown, that a fusion of companies does not necessarily lead to so called synergy effects, but rather critically



Synergetics: Basic Concepts, Fig. 10 Pattern hierarchy in the Taylor–Couette instability. A fluid in between two vertical coaxial cylinders of which the outer one rotates, shows no macroscopic movement pattern, if the movement of the outer cylinder is slow. When the rotation speed is increased, first a role pattern appears in which the fluid moves outwards at one height, and then inwards at another height. This movement pattern is periodic with respect to height (Taylor 1923). At a further critical rotation speed, the pattern shows oscillations which at a further speed transform into a motion with two frequencies until eventually chaotic motion appears. The experiments were done by (Fenstermacher et al. 1979), the modeling was done for the first transition (homogeneous to roles) and especially the second transition (roles to oscillating roles) by (Marx 1987)

Synergetics: Basic Concepts, Fig. 11 Belousov–Shabotinsky reaction: the occurrence of spirals. (Courtesy A.T. Winfree). They may show one to several arms. The centers of the spirals may occur at different positions. Spirals hitting each other, annihilate each other

Synergetics: Basic Concepts, Fig. 12 Stripe pattern on a tropical fish

depends on initial conditions and details of the cooperation between the previously separated firms. Important insights are also gained into fundamental processes of climatology, as well as in ecology such as the by now well-known and publicly discussed effects that even small concentrations of chemicals in the atmosphere can change the climate dramatically. The same is true for lakes, in which beyond a critical pollution, fish population dies out entirely.

In this way, the numerous examples collected in the field of Synergetics, provide not only scientists but also the public with impressive examples of dramatic changes (instabilities) provoked by even a slight change of control parameters. Clearly, an important research subject of

Synergetics is a detailed study of which control parameters are critical and to which control parameters a system is rather insensitive. Sociology is an important field for the application of stochastic models (Bartholomew 1967). In particular, basic concepts of Synergetics have proven useful in the developing field of sociodynamics, where e. g. phase transition-like phenomena may occur (Weidlich 2000).

History and Relations to Other Fields

The term Synergetics was coined by H. Haken in 1969 in a lecture at University of Stuttgart. A first description of the goals of this field was given by H. Haken and R. Graham in 1971 (Haken and Graham 1971) where the unifying role of the concept of order parameters is outlined. A relationship exists to the general system theory due to von Bertalanffi (1950), which also aims at the exploration of analogies between different systems, but on the level of the individual elements rather than on the level of order parameters. Von Bertalanffi coined the term flux equilibrium (Fließgleichgewicht) in order to characterize homeostasis in active systems (von Bertalanffi 1953). A general mathematical frame for Synergetics is provided by dynamic systems theory (see, for instance, (Guckenheimer and Holmes 1983)) which, however, in the traditional approach ignores stochastic processes (mainly chance events) which are also of great relevance for Synergetics. Here the theory of Markov processes with their typical equations, such as Langevin equations, Fokker–Planck equations, Chapman–Kolmogorov equations, the Kramers–Moyal expansion etc. is important (see for instance (Stratonovich 1963) and Linear and Non-linear Fokker–Planck Equations by T. Frank).

A basic feature of Synergetics consists in dealing with nonlinearities in complex systems and studying, mainly quantitatively, qualitative changes at macroscopic scales. Qualitative changes of systems at macroscopic levels are studied also by catastrophe theory (Arnold et al. 1999; Thom 1975), which may be interpreted as a study of the surfaces of equilibrium points of few order parameters, where different cases are classified according to the (low) number of control and order parameters. Chaos theory studies the mostly irregular dynamics of deterministic low dimensional continuous (Lorenz 1963; Newhouse et al. 1978; Ruelle and Takens 1971; Sparrow 1982) or discrete dynamic systems (Collet and Eckmann 1980; Feigenbaum 1978; Grossmann and Thomae 1977; May 1976; Smale 1967), where the behavior is mainly characterized by so called Lyapounov exponents, various kinds of fractal dimensions and chaotic attractors. The slaving principle of Synergetics provides a basis for an application of chaos theory to multi-component systems in that Synergetics shows the possibility of reducing the degrees of freedom. Synergetics shares some of its topics with singularity theory (Arnold 1993; Golubitsky and Schaeffer 1988; Golubitsky et al. 1988), which applies to bifurcation points and their surrounding. Another point of contact is bifurcation theory (see the quotations in previous chapters), in which the branching of solutions of the dynamic system close to instability points is studied. The term dissipative structure was coined by Prigogine (Glansdorff and Prigogine 1971) to characterize evolving structures in systems away from thermal equilibrium where as in all such non-equilibrium systems dissipation occurs. A typical example is that of the convection instability. Prigogine tried to base his approach on thermodynamics, introducing concepts of entropy production and excess entropy production. As we now know, these concepts are, however, insufficient to deal with structure formation in such systems (Landauer 1975). Based on a fundamental idea of A. Turing (1952), Prigogine and Nicolis (1967), see also (Nicolis and Prigogine 1977), treated macroscopic pattern formation in a specific chemical reaction model. For more recent work see (Nicolis 1995).

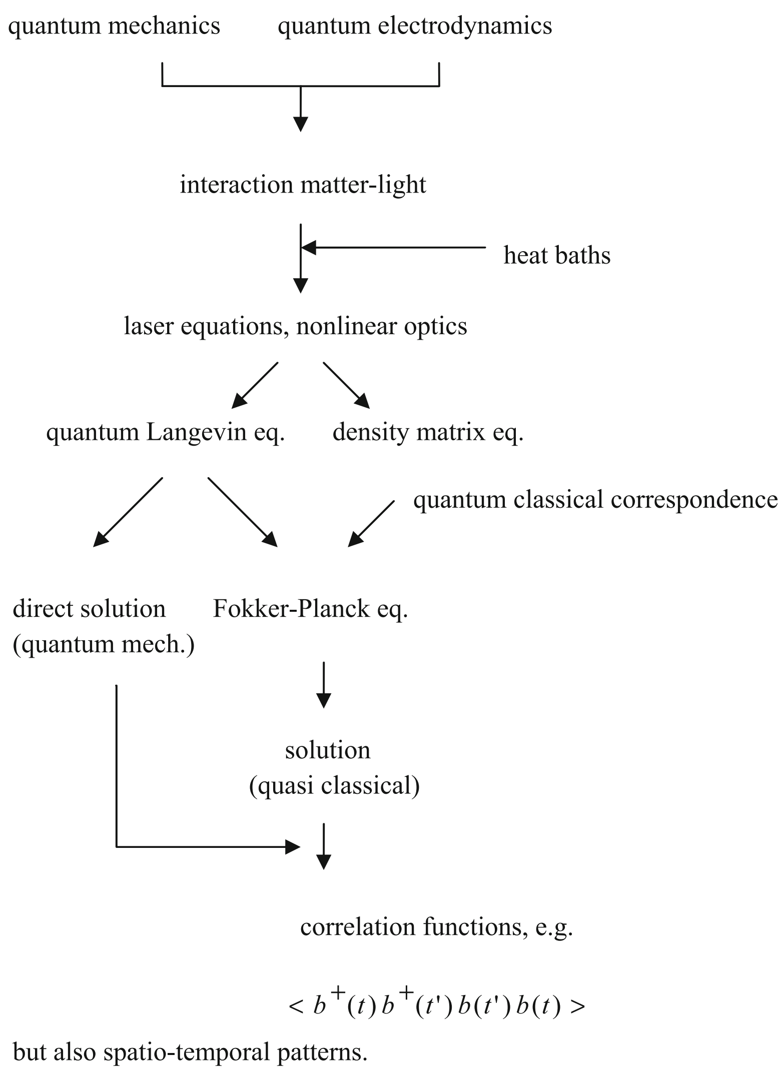
Because of the fundamental importance of thermodynamics, we elucidate its relationship to Synergetics more closely.

Thermodynamics (see for instance (Callen 1960)) deals with systems in and out of thermal equilibrium. A central concept is entropy. In a closed system, it tends to its maximum value. Thermal equilibrium is characterized by the equipartition theorem: each degree of freedom has an average energy of 1/2 kT, k = Boltzmann constant, T absolute temperature. This may refer e. g. to gas atoms as well as to collective excitations in crystals. These systems are in thermal equilibrium with their surrounding (heatbaths, reservoirs).

Irreversible thermodynamics (Haase 1969) treats systems which are not in thermal equilibrium but close to it. It mainly deals with transport and relaxation processes. A central concept is entropy production.

In the domains of physics, chemistry, biology, Synergetics deals with systems far from (thermal) equilibrium. This state is caused and maintained by an in- and outflow of matter, energy and/or information. This is achieved by a coupling of the "proper system" to heat baths (reservoirs) at different temperatures. The former concepts of thermodynamics, in particular the first and second law, are still valid for the total system ("proper" plus reservoirs), but no more sufficient to deal with the kinetics of the proper system. Now the central concept is growth and decay rates. In systems far from thermal equilibrium, collective modes are formed. One or

Synergetics: Basic Concepts, Scheme 1 Quantum optics, example laser. (After Haken 1970)



several of them compete best for the external supply of matter, energy, information and grow at the expense of all other degrees of freedom (or modes). Thus the equipartion theorem is no more valid. In general, the behavior of the system is governed by few degrees of freedom (order parameters). Incidentally, this "growth and competition" principle applies to a great variety of fields out of physics, chemistry and biology, where "modes" may not only be special physical structures, but may mean behavioral patterns, special functions etc. Quite often, a "mode" is initiated by a chance event (fluctuation). Clearly, a generalized Darwinian principle can be seen: The interplay between *mutations* (microscopic chance events) and *selection* (competition between mascropic modes) leads to macroscopic patterns (structures) in the widest sense of the word (Scheme 1).

In present days research, a new name is spreading, namely complexity or complexity theory. There seems to be no precise definition of this field available in the scientific community. Of what is known so far, we may conclude that this field has strong ties to the original field of Synergetics in that it searches also for general principles but, in addition, it allows the collection or accumulation of knowledge on all kinds of complex systems, as is witnessed in the excellent Complexity Digest, weekly edited by Gottfried Mayer. What "complexity" eventually might mean is reflected by the present encyclopedia.

Future Directions

Synergetics is surely not a closed scientific discipline but quite open to further research. On the one hand we may think of further applications of the principles of Synergetics that have been hitherto elaborated on, such as order parameters etc. Here a wide field of application is provided by robotics, construction of prostheses, automatic steering of cars etc. On the other hand, new ideas to endow systems with self-organizing properties are needed, e. g. groups of mobile agents for the execution of specific tasks. First steps have been done for instance by Kornienko (Kernbach 2008).

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Basically, the individual volumes of the Springer Series in Synergetics and selected volumes of the Springer Series on Complexity provide further information. Here, because of lack of space, only a few basic papers can be quoted.

The number of books, reviews and original papers on the topic of the present article is enormous. I have tried to achieve a fair balance between original contributions and reviews/books. Nevertheless, my selection must remain – to some extent – arbitrary and incomplete.

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