CHAPTER IV

THE THESIS OF SOFYA, THE CAUCHY–KOVALEVSKAYA THEOREM

When in October 1872 Sofya explained to Weierstraß that her marriage was purely formal, he understood that she was not destined to remain an amateur mathematician supported by her husband but that she would have need of employment and thus of a diploma, and he decided to have her submit a thesis. There is general agreement, following Mittag-Leffler [1923], that this is what he wanted to convey to Sofya when he wrote on the morning of 26 October 1872 [Bölling 1993, letter 8]:

> I have been much preoccupied with you tonight—as it could not be otherwise—my thoughts have wandered in the most varied directions, have however each time returned to a single point, that I must discuss with you today. Don't be afraid that I will touch on things we have agreed not to talk about, at least for now. What I want to say to you is more closely tied to your scientific undertakings. But I am not sure that, with the admirable modesty with which you judge what you are capable of doing, you would want to agree to my plan. It is preferable to discuss this in person. Therefore permit me, although only a few hours have passed since our last meeting, which brought us so much closer, to visit you this morning for a little while (*ein Stündchen*) so you can hear me out.

And when he decided to have her submit this thesis, he took a few actions:

– He chose Göttingen University, perhaps because Göttingen had established a precedent by awarding a degree to a woman, Dorothea Schlözer Rodde, in the 18$^{\text{th}}$ century. I am not sure that this degree was a doctorate, nor do

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I know what a doctorate was in the 18$^{\text{th}}$ century. He certainly knew [1] that this university was prepared to award an honorary doctorate to Sophie Germain before she died. Göttingen seems to have been the least reactionary university in all Prussia [2] and by 1895 it was accepting women, whereas the others would wait a few years.
– He requested that the thesis be defended *in absentia* not wanting to expose timid little Sofya and her fluid but imperfect German to oral questions from a mob of old men. It seems to me that "little Sofya" would have defended herself just as courageously as she had done several years before to "old Spencer" (see page 202 ff.).
– Above all, he sent three of her memoirs, any of which would have sufficed for a thesis.

We note that Julia Lermontova will also defend at Göttingen in 1874, but that she will not be exempted from the oral examination.

The three memoirs of the thesis

The work that was the basis on which Göttingen University awarded the doctoral degree to Sofya on 29 August 1874 was comprised of three different texts, for which I give the publication references. The first appeared rather quickly, the two others later when Sofya was already in Stockholm:

– *Zur Theorie der partiallen Differentialgleichungen* (on partial differential equations), containing what is nowadays called the Cauchy–Kovalevskaya theorem [Kowalevski 1875].

– *Über die Reduction einer bestimmten Klasse abel'scher Integrale dritten Ranges auf ellitpisches Integrale* (on the reduction of a certain class of Abelian integrals to elliptic integrals) [Kowalevski 1884b].

– *Zusätze und Bemerkungen zu Laplace's Untersuchung über die Gestalt der Saturnringe* (additions and remarks on Laplace's investigations on the form of the Saturn rings) [Kowalevski 1885b].

The publication of one of the memoirs from her thesis in *Journal für die reine und angewandte Mathematik* is a great honor for a novice like Sofya.

Those who would now be called referees—whatever their titles then—were Lazarus Fuchs and Heinrich Weber [Cooke 1984, p. 21]. Weierstraß had really done his work well:

1. He mentions this in a letter to Fuchs from 27 June 1874 (see [Wentscher 1909]), to which I will have occasion to return.

2. On this subject see the commentaries of Grace Chisholm reproduced in [Cartwright 1944] and here on page 234.

> In my opinion there is no doubt at all but that *each* of these works suffices for a thesis

he wrote to Fuchs on 27 June 1874 in the letter in which he presented the works that he proposed for this thesis (this letter is published in [Wentscher 1909] (3)); but this was not at all like other theses.

I am going to talk at length in this chapter about the Cauchy–Kovalevskaya theorem because it is the most important result, based on two nice ideas (see page 239) that Sofya had in her short scientific life. But first a few words about the two other works.

Abelian functions. The second article required a good understanding of elliptic functions for studying what in the today's geometric language would probably be called a hyperelliptic curve whose Jacobian is isogenic to a product of elliptic curves (it is an anachronism!). A deep understanding, but perhaps no more original ideas than in a typical thesis (past or present) written under the influence of a research director. We note nonetheless that a deep understanding in the new domain was not just nothing. In any case, when she later presented this work, in 1880, to a scientific meeting in Saint Petersburg, Sofya greatly impressed her listeners by her great ease in this new subject, full of unexpected subtleties for non-specialists. In his analysis of his own work on Abelian functions, Poincaré [1921] speaks of his interest in Abelian integrals that are "capable of being reduced to elliptic integrals" and notes:

In this regard see the correspondence between Poincaré and Mittag-Leffler [Nabonnand 1999].

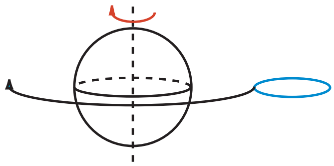
> My attention was attracted anew to this problem by a memoir of Mme Kovalevski, where two theorems of M. Weierstraß are mentioned [...]

He gives proofs of the two Weierstraß theorems and extends Sofya's results in [Poincaré 1886].

This mastery of Abelian functions will later be useful to Sofya in her work on the solid. But let's not get ahead of ourselves.

The rings of Saturn. The third memoir was more a study in applied mathematics. Laplace had shown, under the assumption that the rings of Saturn are liquid, that their cross section has, in the first approximation, the form of an ellipse. In other

3. In the same volume this letter is followed by three others to Fuchs, all from summer 1874, all regarding this thesis and published in Schlesinger [1909].



Laplace's ellipse (in blue)

words, the rings form a torus, a sort of flattened air chamber, but flattened symmetrically. See the figure.

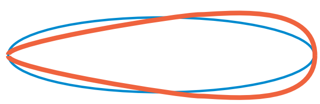
It has to do with determining the generating curve so that the liquid will be in equilibrium with the surface of the ring, under the influence of

– the attraction of the ring,
– that of Saturn,
– the centrifugal force.

I draw here from Tisserand's presentation in his book [1891], where chapters IX and XII are dedicated to the rings of Saturn (and the tenth to Sofya's memoir):

Tisserand's book appeared in 1891, written when Sofya was already famous. Recall that, if the results of the article in question were obtained in 1874 or before, the article only appeared in 1885.

> M$^{\text{me}}$ Sophie Kowalevski, already known by her good mathematical work, has brought a happy complement to Laplace's œuvre.



Laplace's ellipse, a little deformed by Sofya, has become an ovoid (in red)

Sofya had calculated the next terms of the approximation and had shown that the section of the torus was rather an ovoid (in plain English, the rings of Saturn, viewed in cross section, have the form of an egg) and moreover that the exact curve cannot be symmetric with respect to an axis parallel to the axis of revolution (vertical, in my figures). It is a rather difficult problem in hydrodynamics that is relevant to a rotating fluid rather than to the actual rings of the actual planet Saturn because, as Maxwell had conjectured in 1857 (and as Sofya realizes since she mentions it in the article), as Poincaré would prove in 1885 and as Voyager's probes confirmed a century later, they are formed of solid particles.

It is not entirely finished. We see Poincaré attack the problem. The problem interests Tisserand enough that he had an additional term calculated in a thesis ... that of Dorothea Klumpke who would be, in 1893, the first woman to defend a thesis in the mathematical sciences in France. Here is what she writes in the introduction to this thesis [1895, p. C.3]:

> [...] Later M$^{\text{me}}$ Sophie Kowalewski, taken from science prematurely, resumed this problem and added a happy complement. She shows that the ellipse found by Laplace as the equilibrium shape of the flow is transformed into an oval when terms of higher order are taken into account.
>
> Upon the invitation of M. F. Tisserand, professor in the *Faculté des Sciences* where I have taken courses, we have, following his suggestion, resumed the work of M$^{\text{me}}$ Kowalewski and we have evaluated the corresponding terms to the third approximation.

> The expression for the speed, to which M^me Kowalewski's method leads, agree with that obtained by M. Poincaré for a fluid mass subject to a rotation.

Although born in San Francisco, Dorothea Klumpke was French. The presence of Tisserand on her committee is no coincidence (see [Cooke 1984, p. 174])—it is he who has posed the problem—also not the fact that Darboux chaired the committee. It seems to me rather normal that the Dean of the Faculté des Sciences would preside over the first thesis in this institution to be defended by a woman, who moreover took his courses (see the report by Darboux on the defense on page 223).

We now cut to the chase. I state that I am not a specialist in the Cauchy–Kovalevskaya theorem, which is the basic theorem on partial differential equations over the complex numbers. I thus refer, for the history of this theorem, to a text by Roger Cooke [2002b] (more recent and detailed than the evaluation that appeared in his book [1984]), on which I have relied in writing this chapter and, for a shorter presentation of this theorem, to [Détraz 1993, p. 251].

We briefly mention here Sofya's last article [1891] (published posthumously), where there is concensus that it goes back to the period when she was working on the theorem of her thesis (see [Cooke 1984, p. 34 and Chapter 8]).

A problem of Cauchy

The Cauchy–Kovalevskaya theorem is the principal existence and uniqueness result for solutions of partial differential equations.

For an ordinary differential equation. We first explain what such an equation is. We try to determine functions u on a variable t which satisfy a relation

$$\frac{d^m u}{dt^m} = \Phi(t, u, \dots)$$

where the dots represent the derivatives of u of order greater than 1 but less than m. We begin with a simple example (in which $m = 1$):

$$\frac{du}{dt} = u + e^t.$$

Suppose that v is a solution, which is to say a function that satisfies this relation. Then a simple calculation shows that for each real C, the function

$$u(t) = v(t) + Ce^t$$

is also a solution. We thus have lots of solutions (or none at all). If we want a unique solution, we need a supplementary constraint. Generally the problem arises in physics and we have information about the value of u at time $t = 0$, known as an "initial condition". The so-called "Cauchy problem" is the differential equation plus an initial condition. In the example,

$$\begin{cases} \dfrac{du}{dt} = u + e^t \\ u(0) = u_0, \end{cases}$$

and in the general case

$$\begin{cases} \dfrac{d^m u}{dt^m} = \Phi(t, u, \dots) \\ \dfrac{d^k u}{dt^k}(0) = u_0^k \qquad \text{for } 0 \le k \le m-1. \end{cases}$$

And the so-called Cauchy theorem asserts, under suitable conditions, the existence and uniqueness of a solution in the neighborhood of time $t = 0$.

Cauchy–who? In the tradition of the French higher education, a "Cauchy–who theorem" is an assertion of existence (and when possible of uniqueness) of a solution having this or that regularity under a hypothesis of regularity of the function Φ depending on the mathematician "who" involved. For example, Cauchy–Lipschitz if it is Lipschitzian, Cauchy–Peano if it is continuous, etc.

Another important problem, which I cannot address in this margin for lack of space, deals with the question of "maximality" of solutions: are they defined whenever the coefficients of the equation are?

What Cauchy actually proved around 1835 is the "analytic" version of the theorem. The method he used consists of finding a formal solution in the form

$$u(t) = \sum a_n t^n$$

and showing that the series obtained is convergent (has a nonzero radius of convergence) by using the "majorant series principle", so that the solution is what is called an analytic function. Here is how this works in our example. The formal series satisfies the equation if and only if

$$\sum n a_n t^{n-1} = \sum a_n t^n + \sum \frac{t^n}{n!},$$

the equality of the constant terms yielding $a_1 = a_0 + 1$, that of the terms of degree 1 gives $2a_2 = a_1 + 1$ and so on, and we obtain generally

$$(n+1)a_{n+1} = a_n + \frac{1}{n!},$$

which allows us to determine the coefficients one after the other starting with a_0,

$$a_n = \frac{a_0 + n}{n!}$$

and there is a unique solution for each $a_0 = u(0)$. For example here, with $a_0 = 0$,

$$a_n = \frac{1}{(n-1)!} \text{ and thus } u(t) = t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \cdots$$

We subsequently verify that the series obtained has a positive radius of convergence by comparing it with a geometric series. In the example considered the radius of convergence is infinite—and the solution is $u(t) = te^t$.

Cauchy (1789–1857)

The case of a partial differential equation. We wish to solve a partial differential equation of the form:

$$\frac{\partial^m u}{\partial t^m} = \Phi(t, x, u, \dots)$$

where now u is a function of several variables, a mapping from an open subset of $\mathbf{C} \times \mathbf{C}^n$ into an open subset of $\mathbf{C}^p$ and Φ is an analytic function of the time t, of the spatial variable x, of the unknown function u and of its partial derivatives of order with respect to t less than m. It is what is called a "Cauchy problem" when "initial" conditions are given, i.e. specification of the function u and its partial derivatives with respect to the "time" t at the instant $t = 0$:

$$\frac{\partial^k u}{\partial t^k}(0, x) = g_k(x) \text{ for } 0 \leq k \leq m - 1.$$

Kovalevskaya's counterexample. When Sofya attacked the problem for a partial differential equation, she began by finding a "counterexample" that much astonished Weierstraß. I will present this example and then explain why it is a counterexample. The equation is a very classical partial differential equation, the one that controls the propagation of heat and is simply called the "heat equation". It was already well known in Sofya's time because it was studied by Fourier (not a very elegant way of crediting Fourier's contribution).

Joseph Fourier (1768–1830)

The function $u(t, x)$ represents the temperature at time t at the point with abscissa x along a rod and satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Sofya verified that the series

$$\sum_{k=0}^{+\infty} \frac{d^{2k} u_0}{dx^{2k}} \frac{(t-a)^k}{k!}$$

is a formal solution with, when $t = 0$, $u(0, x) = u_0(x)$. But she also observed that the initial data given by

$$u_0(x) = \frac{1}{1 - x},$$

a function analytic for$|x| < 1$, yields

$$u(t, x) = \sum_{k=0}^{+\infty} \frac{(2k)!}{k!} \frac{t^k}{(1 - x)^{2k+1}},$$

a series that converges only for $t = 0$. She furthermore showed that the formal solution diverges whenever the initial data u_0 has a singularity (which is to say a pole) somewhere, just like our $1/(1 - x)$ has a pole at 1.

It is the example explained here, remarkable for its simplicity, that has passed into history. The first counterexample that Sofya found was a divergent formal solution to an equation that is much more complicated, as Weierstraß will confirm for us.

This result was unexpected and did not very well fit the notions of the time. In a letter to Fuchs from 27 June 1874 that I have already mentioned [Wentscher 1909], Weierstraß says that it is “an unexpected remark and which awakens suspicion”. A young mathematician who surprises her research director (as was not said at the time) by finding something completely contrary to the intuition that the “old” director and his (“old”) colleagues may have developed, does not present an unusual circumstance. This is even what should be expected. The young look at the problem with fresh eyes and the intuitions that I just mentioned can, from their point of view, be considered prejudices. When I mention the attribution problems for the theorem, it will be seen that Sofya certainly is not the only one to have proved it. But she is the only young mathematician in the race and also the only one to have begun her work by producing a surprising counterexample. And thus the only one to have insisted on the hypotheses of the theorem. Weierstraß writes to Du Bois-Reymond on 25 September 1874 (see [Kochina 1985]):

> Except for correcting her numerous grammatical mistakes, I did not do anything other than formulate the problem for the author of the dissertation in question. And in this connection I also have to remark that as a matter of fact, I did not expect any result different to what is known from the theory of ordinary differential equations. To stay with the simplest case, I had an opinion that a power series in many variables that formally satisfies a partial differential equation must always be convergent within a certain domain and must, therefore, represent a function that really satisfies the equation. This is not true, as you can see from the example of the equation $\partial\varphi/\partial t = \partial^2\varphi/\partial x^2$

> considered in the dissertation. This was discovered, to my great surprise, by my student completely independently, first for much more involved differential equations than the one cited, so that even she doubted that it would be possible to obtain a general result; the seemingly simple means she found to overcome the obstacle I value highly as proof of her mathematical flair.

The Cauchy–Kovalevskaya theorem. The theorem uses the hypothesis that all given functions are analytic. It affirms the existence of an analytic solution to the "Cauchy problem"

$$\frac{\partial^m u}{\partial t^m} = \Phi(t, x, u, \dots)$$

with

$$\frac{\partial^k u}{\partial t^k}(0, x) = g_k(x) \text{ for } 0 \leq k \leq m-1,$$

a solution that is unique in the neighborhood of each point $(t_0, 0)$.

We remark that the theorem does not apply in the case of the heat equation with the initial condition being the distribution of heat at time 0 since this equation is of order 2 but that the derivative with respect to time that appears in it is effectively of order 1. It applies if we interchange x and t, giving as initial condition the temperature (as a function of time) at the point $x = 0$ (which does not seem very interesting, not being very realistic).

We recall too that partial differential equations reach the desks of mathematicians because they model authentic physical problems and that they do not all come pre-packaged in the form

$$\frac{\partial^m u}{\partial t^m} = \Phi(t, x, u, \dots)$$

(what Sofya called a "normal form") but rather in an "implicit" form

$$\Psi(t, x, u, \dots) = 0.$$

After having considered her counterexamples, Sofya showed that *if* it is possible to put the equation into normal form, which is to say "to solve for $\partial^m u/\partial t^m$", then there will be a formal solution that converges.

Who proved the Cauchy–Kovalevskaya theorem?

Cauchy, Kovalevskaya, Darboux and the others. Cauchy in 1842, Sofya in 1873 (we have seen that the theorem served for her thesis in 1874 and that it appeared in 1875 in Crelle's journal [Kowalevski 1875]), but also Darboux who "announced" [4] a somewhat less general result in the *Comptes rendus* in 1875 and Méray, who in the same *Comptes rendus* and in the same year, announced "considerably more complete" results than those of Darboux ... Which proves that the problem was undoubtedly more in the air in 1874 than it was some thirty years before and above all that mathematicians had need of this result.

All this occurred amicably between these quality people. Weierstraß, who received his *Comptes rendus* a little late (he was late in renewing his subscription!), after considering whether it would be necessary to file a claim with the *Académie des sciences* (letter addressed to Sofya on 21 April 1875 [Bölling 1993, letter 78]), sent Sofya's article to Hermite and asked the editor of Crelle's journal, who was no longer Crelle but Borchardt, to write to Hermite to inform him that he had received Sofya's article in August 1874. Which parenthetically shows that Hermite did not know about Cauchy's article in 1875 any more than he and Weierstraß knew in 1874. The authors of another reference work on partial differential equations that was used at the time, Briot and Bouquet, do not mention it either.

The name "Crelle's journal" is still used to designate the *Journal für die reine und angewandte Mathematik* (Journal of pure and applied mathematics), a journal founded by Crelle, to distinguish it from "Liouville's journal", whose official name is *Journal de mathématiques pures et appliquées*.

Carl Borchardt was also a very close friend of Weierstraß. Moreover, just as he did with Sofya—but not with most of his other colleagues or students—Weierstraß addressed him with the familiar *Du*.
Borchardt edited Crelle's journal, also at the time called "Borchardt's journal", from 1856 until his death in 1880.

Genocchi knew about Cauchy's article in 1875. It was Genocchi who, having seen Darboux's notes appear, wrote to the *Comptes rendus* to point out Cauchy's 1842 paper. Cauchy, who lived for sixty-eight years, wrote and published eight hundred articles (we can ask ourselves how he had the time to reflect upon and conjure up and prove the next article after he had finished one of them ...), which explains why some of his results have passed unnoticed, for example Hadamard's formula expressing the radius of convergence of a power series as a limit superior figured—it too—in a course of Cauchy. In the other direction, Weierstraß in 1841 proved a theorem that bears the name of Laurent ... and which the latter announced in 1843.

Hadamard's formula, which is written nowadays

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|},$$

known to Cauchy in 1821 ... before Du Bois-Reymond invented the limit superior (lim sup), was proved in 1892 by Hadamard, who at the time knew about neither the work of Cauchy nor that of Du Bois-Reymond.

4. In the *Comptes rendus*, results are "announced", one says: I know how to prove that ... and in the best cases one adds: I use such and such a method. It serves to carve out territory. The property in question is not really established until the proof is published in a journal.

In any case, in 1875 and regarding what was not yet called the Cauchy–Kovalevskaya theorem, all the protagonists were aware of the parentage of the theorem in question.

Weierstraß pointed out to Hermite that neither Cauchy nor Darboux had made explicit the necessary condition for a partial differential equation to actually have solutions, so that neither Cauchy nor Darboux suspected that cases could exist where a formal solution did not converge.

We shall see that Sofya maintained amicable relations with the French mathematicians and notably with Darboux, who would be, thirteen years later, the referee (*rapporteur*) for the committee that awarded her the Bordin prize.

Gaston Darboux (1842–1917)

On rigor (continuation from page 4). Let us return to the story of Runge, which was told in chapter III. Here is a recent and somewhat odd version [Kozlov 2000]. After having explained the nature of the Cauchy–Kovalevskaya theorem, with the counterexample that we displayed above, the notes from Darboux in 1875 (the date is given), Kozlov writes in 2000 (translation from Russian):

> In fact, it is Carl Runge who pointed out the old results of Cauchy on the analytic solutions of differential equations; he was at the time *Dozent* at Berlin University. According to Mittag-Leffler, "Weierstraß was much astonished". For Kovalevskaya, it was especially disturbing that the information came from Runge, a young man with whom she was apparently on good terms.

One more example of the absence of rigor ... The discussion with Darboux and Hermite, the intervention of Genocchi pointing out Cauchy's article, the fact that the whole history was known and set since 1875, all have disappeared. And for what might be credible, there is no date; the fact that Runge, who was but nineteen in 1875, did not intervene until ten years later has also disappeared. To make way for an insinuation that has nothing to do with Cauchy's article.

Jacques Hadamard (1865–1963)

It was Hadamard who named the theorem. We have understood that, with its hypotheses, its proof, its conclusion and the examples showing the importance of the given hypotheses, this particular theorem is due to Sofya. The name of Cauchy–Kovalevskaya is well suited. First because Cauchy was the first person to study the problem, announce a result and give a proof. Next because we can understand it as a variant of "Cauchy–who" (see page 76).

The theorem in question is—further evidence that mathematicians needed it—quickly taught. For example to examinees for teaching positions in Toulouse in 1889, who are however—as students are of course everywhere—completely ignorant:

> And this year I explained to some of my students who, having obtained the teaching certificate, now aspire to the *agrégation* [competitive examination for a teaching position], the theorem of Cauchy and M^{me} Kovalewski on partial differential equations! Of course it was first necessary to give them supplementary lessons on the theory of functions.

wrote Stieltjes to Hermite on 22 March 1889 ([Baillaud & Bourguet 1905a, p. 376]). The hyphen is not there yet but the theorem has already been attributed.

According to Roger Cooke, in whom I have enduring confidence, the unnatural coupling (my term, Roger Cooke has not expressed it thus) of the name of the old reactionary Cauchy with that of a young (female) revolutionary (thanks to the hyphen) must be the work of that great progressive Hadamard in lectures he gave at the beginning of the 20th century in New York (later published in [Hadamard 1923]).

The terminology "Cauchy problem" was already well established. In paging through the third volume of the Œuvres of Hadamard [1968], I have noted that the terminology "Cauchy–Kovalevskaya" (up to the spelling of Sofya's name) was also well established since the 1920s. Here are some of the ways in which the theorem appears, all the quotes come from [Hadamard 1968], the year designating the year of the appearance of the article from which the phrase was taken:

> The work of Cauchy and, in clearer and more easily accessible form, the famous proof of Sophie Kowalewski have established a fundamental existence theorem for partial differential equations (1926, p. 1457).

> The Cauchy–Kowalewsky theorem led to determination of a solution of this equation (1933, p. 1574).

> For classical analysis the problem was supposed to have a preliminary answer, simple and general, given by the theorem of Cauchy, for which we have Sophie KOWALEWSKI's celebrated and beautiful proof (1935, p. 1594).

> On the contrary, the conclusion of Cauchy–Kowalewski remains exact, without hypothesis of analyticity, for equations of hyperbolic type (1937, p. 1661).

It is known that this problem, always possible and determined (by virtue of Cauchy–Kowalewski) (1945, p. 1669).

On rigor (sequel)

Here is the entirety of the text that Gårding [1998, p. 93] dedicates to the Cauchy–Kovalevskaya theorem (as translated from Swedish into English by its author):

> The first fruit of Sonya Kovalevski's studies with Weierstraß in Berlin was the Cauchy–Kovalevski theorem, which is the basic proof of the existence of analytic solutions for analytic differential equations.
>
> Let $f(x)$ be a function of n variables $x = (x_1, \ldots, x_n)$. The Cauchy initial data of order m of f on a surface $S : s(x) = 0$ are defined as the restriction to the surface of the function and its normal derivatives of orders $< m$. These Cauchy initial data are generically mutually independent and determine the derivatives of order $< m$ of the function restricted to the surface. For a general differential equation
>
> $$F(x, u, \partial u, \ldots, \partial^m u) = 0$$
>
> of order m in several variables $x = (x_1, \ldots, x_n)$, Cauchy formulated a boundary value problem that is called Cauchy's problem: to find a solution of the equation with Cauchy initial data of order $< m$ on a given surface. The problem makes sense only when the equation gives the normal derivative as a function of the others. If we introduce coordinates such that S is the plane $x_1 = 0$, which is to say that the equation can be written locally as
>
> $$\partial^m u/\partial x_1^m = G(x, u, \partial u, \ldots, \partial^m u)$$
>
> where the term on the left does not appear among the derivatives $\partial^m u$ of the term on the right. In an equation of this form we can calculate—by differentiation—all the derivatives of a solution u restricted to S when the Cauchy data is known. Kovalevski shows that the formal solution, calculated in this way, is analytic at a point x_0 on $x_1 = 0$ if the Cauchy data is analytic and, in addition, the function G is analytic [(incomprehensible) in all the variables] for the values of the derivatives $u, \partial u, \ldots, \partial^m u$ corresponding to the Cauchy data at the point x_0. The

> method, which is borrowed from Cauchy, consists of majorizing the coefficients of the Taylor series of u ["method of majorants"].
>
> The theorem extends to systems of differential equations for a certain number of unknown functions $u_1, \ldots, u_N$. The condition is that the system can be solved for the highest order derivatives of all the functions and that no derivative of the corresponding terms on the right has order greater than these normal derivatives. If this condition is not satisfied, for example in the case of the heat equation
>
> $$u_t = u_{xx}$$
>
> then the theorem does not apply. The solutions may be analytic in x without being analytic in t.

This scarcely comprehensible text seem to me more interesting for what it does not say than for what it does say. The sole contribution it attributes to Sofya is to have "borrowed" the method of majorant series from Cauchy to show that a formal solution is convergent. It is hard to understand that one's name would be given to a theorem for so little, especially so tardily as in 1874. It does not tell us the significance of the problem making sense, nor too that it was Sofya who brought forth the normal form condition (described here in a rather complicated way) and still less that it was she who showed that the theorem does not apply to the heat equation and who indeed had shown the necessity of the normal form hypothesis, and that it was this that was new and original in her work.

⋆

And now,
after so much mathematics
and before a chapter with still more mathematics
a literary pause.

⋆

Italo Calvino (1923–1985)

Pause: The rings of Saturn

LAPLACE *showed that the cross sections of the rings of Saturn are elliptical. It was thought at the time that the rings were liquid. Today science allows that they satisfy the equilibrium conditions of a fluid, which conforms to the idea that Cassini had come up with in the seventeenth century about the rings of Saturn, that they are neither gaseous nor liquid, but that they are made up of solid particles of matter, discontinuous, separated by great distances, a multitude of little satellites linked only by their mutual attraction, very weak in comparison with that of the planet.*

Not at all, yelled old Qfwfq! I remember very well that it was a liquid, a very thick liquid, like a piece of mozzarella cheese, a thick soup, like milk, yes, milk, that's what it was. If you see pieces now, it's because the milk curdled. That's not very surprising after so much time ... Anyway, at that time, it was like that, the rings were liquid. As for the cross section, it was elliptical, it's very true! You can trust me. I know them well, these rings, by dint of having them around my head.

At first we use them only to create shade, like parasols. You can't imagine what they were like, these days on Saturn, always in bright sunlight without ever a cloud. Good that they didn't last too long and that nightfall came rather soon. But nonetheless, during the day, we placed ourselves so that the rings would shelter us a bit. We didn't yet know what they were made of. But of course they were liquid. Even the first imbecile who came along could tell that.

We all were there, my aunt M_i, who made us huge plates of tagliatelle, my deaf cousin, old captain XarlraX and his two sisters, with little S_0Ph(i), an agreeable company if it hadn't been for that plague of a 0-b^{e}Ll, always turning around the plate and mostly about S_0Ph(i).

How nice it was then to see little S_0Ph(i) amusing herself with nothing, looking at these little pieces of wood turning, absorbed in her thoughts, counting and recounting the moons, dreaming, her eyes on the rings, covering sheets of paper with mathematical symbols, of which this plague of a 0-b^{e}Ll never missed saying that it was not right for a cute girl to spend her time like that and that she'd do better by tramping around with him.

Except for watching little S_0Ph(i), for filling one's heart with her joy, there was not anything to do except to admire the round arms of my aunt M_i going back and forth over the big chunks of egg dough, her white arms smeared with oil right up to the elbows. Because, for making tagliatelle, for that, there was space on Saturn. Not only space for spreading out the dough, but space for gardens or ripening tomatoes, for fields for growing wheat, for mountains for the water to come down to irrigate them, and sun for ripening the wheat, for there was no lack of sun.

What was missing were flocks and, if there were, of prairies for them to frolic and graze. "And meat?", you're going to ask me. Well, no, there was not anyone to give us any meat. Oh, we had all sorts of birds, but we didn't eat them. On other, more advanced planets there were perhaps livestock, but we on Saturn didn't have any, none at all, so we savored tagliatelle with tomato sauce and were perfectly satisfied, except perhaps that this plague of a 0-b^{e}Ll was always complaining, the old grump. And it lasted forever like that until the evening, I recall that it was an evening, but of course on Saturn at that time the evenings didn't last so long, you couldn't call them long evenings, on that evening my aunt M_i exclaimed: "My children, if only I had a little milk or cream, how I would like to make you a Sicilian cassata!"

That's when little S_0Ph(i), although a modest and shy girl, had an idea. A brilliant idea I can tell you. And if this plague of a 0-b^{e}Ll tries to tell you that it was not she who had this idea, and that it was for example the old captain XarlraX, that would be plain meanness, don't believe it. Little S_0Ph(i)'s idea, it was the rings. Because, by having looked at them, she had understood, and she alone, that the rings were of milk. And because she was not lacking in practical sense, she also imagined a way of recovering some, some of this milk. "We're going to milk the rings", she said to us.

I have to tell you, the rings, they weren't so far away. They almost grazed us. So, that's how we proceeded. We climbed right to the top of the Zinc mountains, several of us went, old captain XarlraX, my deaf cousin and myself, following little S_0Ph(i), who skipped along in front of us, holding her milkpot in her hands, sometimes on her head. Evidently this plague of a 0-b^{e}Ll walked behind us. How unpleasant it was to have that one on our heels! This is how little S_0Ph(i) proposed to realize her idea, this is how we would do the milking. We would bring a ladder, she would take off her shoes, climb it, fasten her tin milkpot to the left side, perched on the last rung, her left foot above the milkpot, yelling "I'm there!", she would manage to touch the bottom of the ring by reaching with her left arm, you can imagine that the whole thing was unstable and that our role was to hold the ladder so that she wouldn't fall. You should have seen her, little S_0Ph(i), a sense of balance, a competence, a tenacity, you wouldn't have believed it, in such a pretty little girl. And pretty she was, even if she was hidden by her big hat. She had to protect herself from the sun, you can't imagine what it was like, the sun, on Saturn, at that time. When, with her left forefinger, she would reach the ring, it, by a sort of capillarity phenomenon, would begin to run gently along her arm, along her body and her left leg right until her foot, and it would fill the milkpot. When it would be completely full, she would bring it gently down and we would go back home.

And my aunt M_i not only made us a Sicilian cassata, but also some *straciatella*, some Neapolitan bars, coffee and vanilla ice cream with tiramisu, chocolate, nougat, rum and raisins, and even one day in a vein of exoticism, a tutti frutti. So much so that the rings started to shrink.

One milking day, little S_0Ph(i) launched into a new calculation, you could see that she had been thinking about something for several days. Then she put down her pencil and said to my aunt M_i: I have to tell you, aunty, the rings have become ovoids. That is to say, she added, egg-shaped with a little part and a big part, and that my aunt M_i could understand, because we have some on Saturn, with all those birds. It must be said that for explaining something, little S_0Ph(i) was the champ. Aunty, we need to stop, concluded little S_0Ph(i).

Since that time, the rings have had this form. Ovoids, as little S_0Ph(i) said! And since that time, aunt M_i hasn't made ices for us. We have dispersed. Now when I feel like eating a Sicilian cassata, I go buy one at Nico's, on the *Zattere*. I've happened to run into that plague of a 0-b^{e}Ll there, but I pretend not to recognize him. It could be that one time or another I've run across little S_0Ph(i), but I've never seen her, whether because

I had bent down to tie one of my shoelaces or that I've turned my head to watch a pigeon fly off or that I've started running because my *vaporetto* has arrived.

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This text is inspired by the *Cosmicomics* of Italo Calvino [1976], particularly by drawing of the moon (in *The Distance of the Moon*) and by the Big Bang (in *All at One Point*) which occurs, as is known, because a woman would have liked to have room for cooking a tagliatelle dish. You will undoubtedly find here too in the last phrase an echo of another Italian novel that I like a lot, *L'amante Senza Fissa Dimora* [Fruttero & Lucentini 1988].