16
Sweep Surfaces

The surfaces described in this chapter are obtained by transforming a curve. They are not generated as interpolations or approximations of points or vectors and are consequently different from the surfaces described in previous chapters. A reader who wishes a full understanding of this chapter should first become familiar with the important three-dimensional transformations (rotation, translation, scaling, reflection, and shearing) and how they are described mathematically by a 4×4 transformation matrix. This material is discussed in Section 4.4, but the next paragraph provides a short summary, for those who only need a refresher.

A three-dimensional point $\mathbf{P} = (x, y, z)$ is transformed to a point $\mathbf{P}^* = (x^*, y^*, z^*)$ by appending a fourth coordinate of 1 to it and then multiplying it by the 4×4 transformation matrix

$$\mathbf{T} = \begin{pmatrix} a & b & c & p \\ d & e & f & q \\ h & i & j & r \\ l & m & n & s \end{pmatrix}. \tag{4.23}$$

The product $(x, y, z, 1)\mathbf{T}$ is a 4-tuple (X, Y, Z, H), where $H = xp + yq + zr + s$. The three coordinates (x^*, y^*, z^*) of $\mathbf{P}^*$ are obtained by dividing (X, Y, Z) by H. Hence, $(x^*, y^*, z^*) = (X/H, Y/H, Z/H)$. The top left 3×3 submatrix of $\mathbf{T}$ is responsible for scaling and reflection (parameters a, e, and j), shearing (b, c, f, and d, h, i), and rotation (all nine). The three quantities l, m, and n are responsible for translation, and s is a global scale factor. The three parameters p, q, and r are used for perspective projection.

D. Salomon, *The Computer Graphics Manual*, Texts in Computer Science,
DOI 10.1007/978-0-85729-886-7_16, © Springer-Verlag London Limited 2011

16.1 Sweep Surfaces

A sweep surface is obtained when a space curve $\mathbf{C}(u)$, termed the *profile*, is transformed by a transformation rule $\mathbf{T}(w)$. The transformation must include translation and/or rotation and may also include scaling and shearing (see also Section 10.7.1). We say that the surface is *swept* by the profile curve when that curve is transformed. The expression of the surface is simply the product $\mathbf{P}(u,w) = \mathbf{C}(u) \cdot \mathbf{T}(w)$. The transformation $\mathbf{T}$ is a 4×4 matrix, so vector $\mathbf{C}$ should be written in homogeneous coordinates, as the 4-tuple $\mathbf{C}(u) = (x(u), y(u), z(u), 1)$.

The simplest example is the translation of a straight line. The straight segment from the origin to $(1,0,0)$ is given by $\mathbf{C}(u) = (u,0,0,1)$ where $0 \le u \le 1$. This segment is translated along the y axis by the transformation matrix

$$\mathbf{T}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & w & 0 & 1 \end{pmatrix},$$

where $0 \le w \le 1$. The surface $\mathbf{P}(u,w) = \mathbf{C}(u)\cdot\mathbf{T}(w) = (u,w,0,1)$ swept by this segment is (after dividing by the fourth element) $\mathbf{P}(u,w) = (u,w,0)$. This surface is simply the square, on the xy plane, whose opposite corners are the origin and point $(1,1,0)$.

A more interesting example is the same segment $\mathbf{C}(u) = (u,0,0,1)$, where $0 \le u \le 1$, translated a distance α along the z axis while being rotated 360° about that axis. The transformation matrix is

$$\mathbf{T}(w) = \begin{pmatrix} \cos(2\pi w) & \sin(2\pi w) & 0 & 0 \\ -\sin(2\pi w) & \cos(2\pi w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha w & 1 \end{pmatrix}, \quad \text{for} \quad 0 \le w \le 1.$$

The expression of the surface is $\mathbf{P}(u,w) = (u\cos(2\pi w), u\sin(2\pi w), \alpha w)$ and it is displayed in Figure 16.1a. For $w = 0.5$, it reduces to the segment $(0, u, 0.5\alpha)$ (in the y direction), and for $w = 1$, it becomes the segment $(u, 0, \alpha)$ (a segment in the original x direction, but at a height α on the z axis).

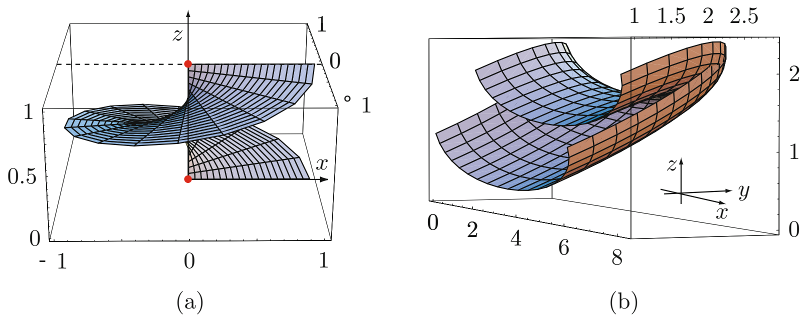
A more general example is a rectangular surface patch constructed as a sweep surface by translating an arbitrary profile along another curve, the *trajectory*. Given the two cubic Bézier curves

$$\begin{aligned} \mathbf{C}(t) &= (1-t)^3(0,1,1) + 3t(1-t)^2(1,1,0) + 3t^2(1-t)(4,2,0) + t^3(6,1,1) \\ &= (-3t^3+6t^2+3t, -3t^3+3t^2+1, 3t^2-3t+1) \\ &\text{and} \\ \mathbf{Q}(t) &= (1-t)^3(0,0,0) + 3t(1-t)^2(1,2,1) + 3t^2(1-t)(3,2,2) + t^3(2,0,1) \\ &= (-4t^3+3t^2+3t, -6t^2+6t, -2t^3+3t) \end{aligned}$$

we can create a sweep surface $\mathbf{P}(u,w)$ by translating $\mathbf{C}(u)$ along $\mathbf{Q}(w)$. The expression of the surface is the product

$$\mathbf{P}(u,w) = (-3u^3+6u^2+3u, -3u^3+3u^2+1, 3u^2-3u+1, 1)$$
$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4w^3+3w^2+3w & -6w^2+6w & -2w^3+3w & 1 \end{pmatrix}$$
$$=(3u+6u^2-3u^3+3w+3w^2-4w^3, 1+3u^2-3u^3+6w-6w^2, 1-3u+3u^2+3w-2w^3, 1).$$

Figure 16.1b shows the resulting surface patch.




```
(* 2 sweep surface examples *)
alf=1;
ParametricPlot3D[{u Cos[2Pi w],u Sin[2Pi w],alf w},{u,0,1},{w,0,1},
 ViewPoint->{3.369,-2.693,0.479},PlotPoints->20]
m={-3u^3+6u^2+3u,-3u^3+3u^2+1,3u^2-3u+1,1}.
 {{1,0,0,0},{0,1,0,0},{0,0,1,0},{-4w^3+3w^2+3w,-6w^2+6w,-2w^3+3w,1}};
ParametricPlot3D[Drop[m,-1],{u,0,1},{w,0,1},
 ViewPoint->{4.068,-1.506,0.133},PlotPoints->20]
```

Figure 16.1: Two Sweep Surfaces.

◇ **Exercise 16.1:** Calculate the sweep surface obtained when line $\mathbf{C}(u) = (3u, 0, 0, 1)$ is translated along the z axis and at the same time translated in the y direction along a sine curve.

◇ **Exercise 16.2:** Calculate the half-sphere produced when the quarter circle

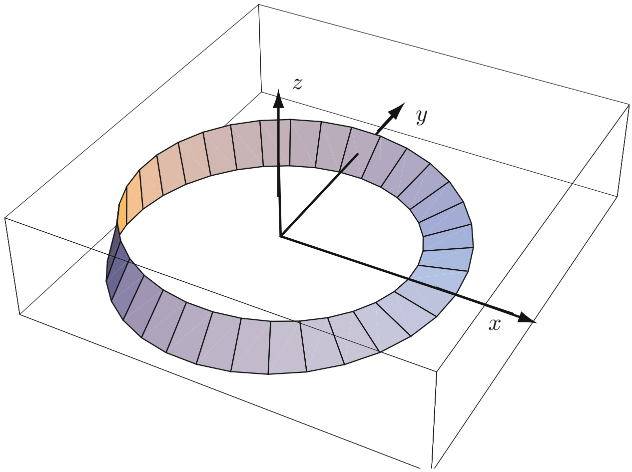
$$\mathbf{C}(u) = \left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, 0\right), \quad \text{where} \quad 0 \le u \le 1,$$

is rotated 360° about the y axis.

◇ **Exercise 16.3:** Calculate the expression of a cone as a sweep surface. Assume that the cone is created by constructing the line from the origin to point $(R, 0, H)$, and rotating it 360° about the z axis.

> ...treat Nature by the sphere, the cylinder and the cone ...
>
> —Paul Cézanne.

Example: A Möbius strip can be constructed as a sweep surface by rotating a short straight segment in a big circle (i.e., through an angle of 2π radians) while also rotating it about itself at half speed (i.e., through π radians). We start with the segment $\texttt{segm}(t) = (t, 0, 0)$. When t is varied from, say, -3 to 3, this becomes a short segment along the x axis from $(-3, 0, 0)$ to $(3, 0, 0)$. Note that it is centered on the origin. The segment is rotated in steps about the z-axis by varying a variable ϕ from 0 to 2π. At each step of this rotation, the segment starts at its original position, it is rotated about the y-axis through an angle of $\phi/2$, it is then translated 20 units in the positive x direction, and is finally rotated by ϕ about the z-axis. Figure 16.2 shows the resulting surface swept by this segment and the code that does the computations.

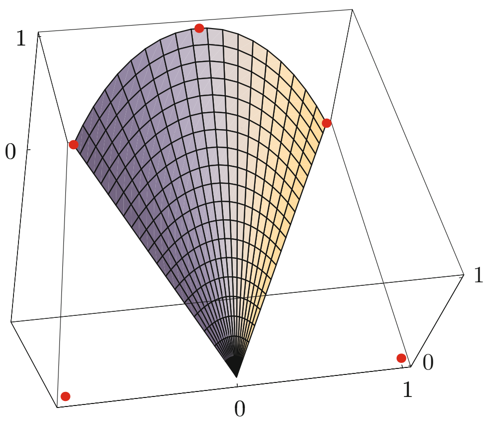



```
(* Mobius strip as a sweep surface *)
Clear[r,roty,rotz,segm];
segm[t_]:={t,0,0}; (* a short line segment *)
roty[phi_]:={{Cos[phi],0,-Sin[phi]},{0,1,0},{Sin[phi],0,Cos[phi]}};
rotz[phi_]:={{Cos[phi],-Sin[phi],0},{Sin[phi],Cos[phi],0},{0,0,1}};
ParametricPlot3D[Evaluate[rotz[phi].(roty[phi/2].segm[t]+{20,0,0})],
 {phi,0,2Pi}, {t,-3,3}, Boxed->True, PlotPoints->{35,2}, Axes->False]
Show[{%,Graphics3D[{AbsoluteThickness[1], (* show the 3 axes *)
 Line[{{0,0,30},{0,0,0},{30,0,0},{0,0,0},{0,30,0}}]}]},
  PlotRange -> All]
```

Figure 16.2: A Möbius Strip.

The basic sweep surface $\mathbf{C}(u)\mathbf{T}(w)$ can be extended to the product $\mathbf{C}(u,w)\mathbf{T}(w)$ of a surface and a transformation. This product is still a sweep surface, since $\mathbf{C}(u,w)$ reduces to a curve for any value of w. We can think of $\mathbf{C}(u,w)$ as a curve that's a function of the parameter u but whose shape depends on w. As w is varied, $\mathbf{C}(u,w)$ yields different curves and each is transformed differently.

Example: The lofted surface of Figure Ans.32 is multiplied by a transformation matrix that scales in the x dimension. The result is shown in Figure 16.3. (This is one way to obtain a triangular surface patch, but it looks bad as a wireframe because one family of curves converges to a point.)




```
(* Sweep surface example. Lofted surface with scaling transform *)
pnts={{-1,-1,0},{1,-1,0},{-1,1,0},{0,1,1},{1,1,0}};
{2u-1,2w-1,4u w (1-u)}.{{w,0,0},{0,1,0},{0,0,1}};
g1=ParametricPlot3D[%,{u,0,1},{w,0,1},AspectRatio->Automatic,
 Ticks->{{0,1},{0,1},{0,1}}];
g2=Graphics3D[{Red,AbsolutePointSize[6],Table[Point[pnts[[i]]],{i,1,5}]}];
Show[g1,g2,ViewPoint->{-0.139,-1.179,1.475},PlotRange->All]
```

Figure 16.3: A Lofted Swept Surface.

Example: A sweep surface that's a product of the surface $\mathbf{C}(u,w)=(u,1,u+2)w+(-u,1,u-2)(1-w)$ and a rotation about the z axis. Note that $\mathbf{C}(u,w)$ varies from the curve $\mathbf{C}(u,0)=(-u,1,u-2)$ to the straight line $\mathbf{C}(u,1)=(u,1,u+2)$ while being rotated. This is shown in Figure 16.4.

An even more general (and interesting) sweep surface is generated when a profile curve $\mathbf{C}(u)$ is swept along a trajectory curve $\mathbf{Q}(w)=(Q_x(w),Q_y(w),Q_z(w))$ and is also rotated about a certain axis by a rotation matrix $\mathbf{R}(w)$. Such a surface is called a *swung*

```
(* A Sweep Surface. Curve Cu[u,w] times matrix Trn[w] *)
Clear[Cu,Trn];
Cu[u_,w_]:={u,1,u+2}w+{-u,1,u-2}(1-w);
Trn[w_]:={{Cos[2Pi w],Sin[2Pi w],0},{-Sin[2Pi w],
 Cos[2Pi w],0},{0,0,1}};
ParametricPlot3D[{Cu[u,w].Trn[w][[1]],Cu[u,w].
 Trn[w][[2]],Cu[u,w].Trn[w][[3]]},{u,0,1},{w,0,1},
 Ticks->None,PlotRange->All,
 AspectRatio->Automatic,ViewPoint->{-0.510,-1.365,1.210}]
```

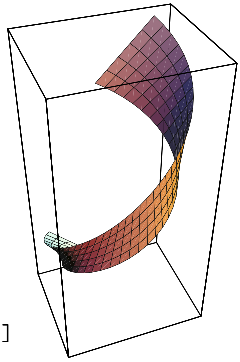


Figure 16.4: Sweeping while Rotating.

surface and its expression is

$$\mathbf{P}(u,w)=\mathbf{C}(u)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\Q_x(w)&Q_y(w)&Q_z(w)&1\end{pmatrix}\mathbf{R}(w),$$

where parameter w is related to the rotation angle θ in a simple way, such as $\theta=2\pi w$ (for a 360° rotation when w varies from 0 to 1) or $\theta=\pi w$ (for a 180° rotation).

In order to construct a useful, meaningful surface, the profile, trajectory, and axis of rotation have to be selected with care. A simple example is a profile curve in the yz plane, a trajectory curve in the xy plane, and a rotation about the z-axis. Figure 16.5 is an example.

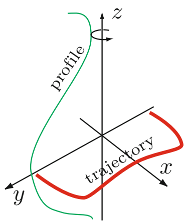


Figure 16.5: A Swung Surface.

16.2 Surfaces of Revolution

A surface of revolution is a special case of a swept surface. It is obtained when a space curve (termed the *profile* of the surface) is rotated about an axis $\mathbf{r} = (r_x, r_y, r_z)$ in space. The rotation angle can be 360° or less. A general rotation in three dimensions is fully specified by the axis of rotation (a vector) and the rotation angle (a number). If the rotation angle is θ and the rotation axis (as a unit vector) is $\mathbf{r}$, then the rotation matrix $\mathbf{T}(\theta)$ about $\mathbf{u}$ is given by

$$\begin{pmatrix} r_x^2+\cos\theta(1-r_x^2) & r_xr_y(1-\cos\theta)-r_z\sin\theta & r_xr_z(1-\cos\theta)+r_y\sin\theta \\ r_xr_y(1-\cos\theta)+r_z\sin\theta & r_y^2+\cos\theta(1-r_y^2) & r_yr_z(1-\cos\theta)-r_x\sin\theta \\ r_xr_z(1-\cos\theta)-r_y\sin\theta & r_yr_z(1-\cos\theta)+r_x\sin\theta & r_z^2+\cos\theta(1-r_z^2) \end{pmatrix}.$$

If the space curve is expressed by $\mathbf{P}(u)$, where $0 \le u \le 1$, then the surface of revolution has the form $\mathbf{P}(u,\theta) = \mathbf{P}(u)\mathbf{T}(\theta)$, where $0 \le u \le 1$ and $0 \le \theta \le 2\pi$. Varying u moves us along the curve and varying θ moves us in a circle (or a circular arc) about the rotation axis.

Example: Given the parametric curve $\mathbf{P}(u) = (f(u), 0, g(u))$ in the xz plane, we can revolve it around the z axis using the rotation matrix

$$\mathbf{T}_z(w) = \begin{pmatrix} \cos w & \sin w & 0 \\ -\sin w & \cos w & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16.1}$$

to get the surface

$$\mathbf{P}(u)\mathbf{T}_z(w) = (f(u)\cos w, f(u)\sin w, g(u)), \quad \text{where } 0 \le u \le 1 \text{ and } 0 \le w \le 2\pi.$$

Example: Given the five points $\mathbf{P}_1 = (0,1,0)$, $\mathbf{P}_2 = (1,1,0)$, $\mathbf{P}_3 = (2,2,0)$, $\mathbf{P}_4 = (1.5,3,0)$, and $\mathbf{P}_5 = (1.5,5,0)$, we construct $\mathbf{P}(u)$ as their Bézier curve

$$\mathbf{P}(u) = \bigl(4t-6t^3+2t^4+(3/2)t^4, -4(t-1)^3t+12(t-1)^2t^2-12(t-1)t^3+5t^4+(t-1)^4, 0\bigr).$$

Since all the z coordinates are zero, the curve lies in the xy plane. We arbitrarily decide to rotate it about the y axis, so the rotation matrix is

$$\mathbf{T}_y(w) = \begin{pmatrix} \cos w & 0 & \sin w \\ 0 & 1 & 0 \\ -\sin w & 0 & \cos w \end{pmatrix}. \tag{16.2}$$

The surface expression is

$$\begin{aligned}\mathbf{P}(u)\mathbf{T}_y(w) = &\bigl((4t-6t^3+7t^4/2)\cos w, \\ &(t-1)^4-4(-1+t)^3t+12(t-1)^2t^2-12(t-1)t^3+5t^4, \\ &(4t-6t^3+7t^4/2)\sin w\bigr).\end{aligned}$$

Such a surface is easy to display. To display it as a wire frame, just perform a double loop in which u is varied from 0 to 1, and w is varied from 0 to 2π, in any desired steps (Section 8.11.2). To display it as a solid surface, a similar double loop should cover every pixel (i.e., should iterate in very small steps) and should calculate the normal to the surface at the pixel and, from it, the intensity of light reflected from the pixel.

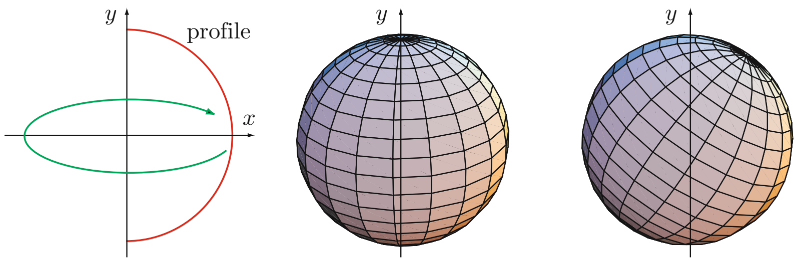
Following are other examples of surfaces of revolution (see also Exercise 8.30):

Example: A sphere of radius R is generated by rotating a half-circle 360° about the axis that passes through the half-circle's endpoints. Figure 16.6a shows the half-circle $\mathbf{P}(u) = (R\cos u, R\sin u, 0)$ in the xy plane. A sphere $\mathbf{P}(u,w)$ is obtained when this half-circle is rotated about the y axis:

$$\mathbf{P}(u,w) = \mathbf{P}(u)\mathbf{T}_y(w) = (R\cos u\cos w, R\sin u, R\cos u\sin w), \qquad (16.3)$$

where $-\pi/2 \le u \le \pi/2$ and $0 \le w \le 2\pi$. It is obvious, from Figure 16.6b, that curves of constant w are meridians of longitude. As u varies from $-\pi/2$ to $\pi/2$, we travel on a semicircle (the profile of the surface) on the sphere. Similarly, varying w for a constant u takes us along a latitude. The north pole is obtained for $u = \pi/2$ (and any w). The equator is the curve obtained when varying w for $u = 0$.

◇ **Exercise 16.4:** Derive the expression for the same sphere centered at (x_0, y_0, z_0).



(a) (b) (c)

Figure 16.6: A Sphere as a Surface of Revolution.

◇ **Exercise 16.5:** Tilt the sphere of Equation (16.3) θ degrees about the z axis (Figure 16.6c).

◇ **Exercise 16.6:** Derive the expression of the sphere that's obtained when the half-circle in the xz plane is rotated 360° about the z axis.

Example: An ellipsoid with radii a and b is obtained by rotating, for example, the ellipse $\mathbf{P}(u) = (a\cos u, b\sin u, 0)$ about the y axis. After translating by (x_0, y_0, z_0), the result is

$$(x_0 + a\cos u\cos w, y_0 + b\sin u, z_0 + a\cos u\sin w),$$

where $-\pi/2 \le u \le \pi/2$ and $0 \le w \le 2\pi$.

◇ **Exercise 16.7:** Derive the equation of a torus as a surface of revolution. Assume that the torus is centered at the origin, and its two radii are R and r (Figure 16.7). The surface is created by drawing the circle of radius r centered at $(R, 0, 0)$, and rotating it 360° about the z axis.

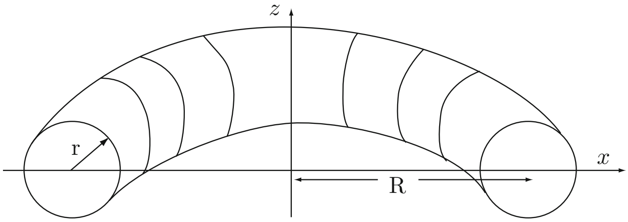
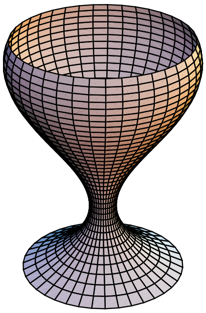


Figure 16.7: Torus as a Surface of Revolution.

Example: Figure 16.8a,b shows a chalice as a surface of revolution and its profile.



```
(* A Chalice *)
(*the profile*)
ParametricPlot[{.5u^3-.3u^2-.5u-.2,u+1},{u,-1,1},
 AspectRatio->Automatic]
(*the surface*)
RevolutionPlot3D[{.5u^3-.3u^2-.5u-.2,u+1},{u,-1,1},
 PlotPoints->40]
```

Figure 16.8: A Chalice as a Surface of Revolution.

16.3 An Alternative Approach

Generating surfaces of revolution with a rotation matrix is simple but slow, since it requires the use of trigonometric functions. An alternative method is described here.

Two given curves $\mathbf{P}(u) = (P_x(u), P_y(u), P_z(u))$ and $\mathbf{C}(w) = (C_x(w), C_y(w), C_z(w))$ can be combined as follows:

$$\mathbf{S}(u,w) = \bigl(P_x(u)C_x(w), P_y(u)C_y(w), P_z(u)C_z(w)\bigr), \tag{16.4}$$

and it is easy to show that $\mathbf{S}(u,w)$ is a surface. When u is fixed at a value u_0, expression (16.4) becomes

$$\begin{aligned}\mathbf{S}(u_0,w) &= \bigl(P_x(u_0)C_x(w), P_y(u_0)C_y(w), P_z(u_0)C_z(w)\bigr)\\ &= \bigl(\alpha C_x(w), \beta C_y(w), \gamma C_z(w)\bigr),\end{aligned}$$

which is a curve in the w direction. For each u_0 we therefore have a curve in the w direction. Similarly, for each value w_0 we have a curve going in the u direction. The only condition is that none of the components of the curves be identical to zero. If, for example, $C_x(w) \equiv 0$, then the x component of $\mathbf{S}(u_0,w)$ is always zero, so it degenerates from a surface to a curve in the yz plane.

Equation (16.4) can be used to construct a surface of revolution if $\mathbf{C}(w)$ is a circle or an arc. To explain our approach, let's first restrict the discussion to curves that are cubic polynomial segments. Such a curve has the form $\mathbf{P}(u) = (u^3, u^2, u, 1)\mathbf{MP}$, where $\mathbf{M}$ is a 4×4 basis matrix and $\mathbf{P}$ is a geometry vector, a 4-tuple of points and/or vectors. We can write such a curve in the form

$$\begin{aligned}\mathbf{P}(u) &= \bigl(F_0(u), F_1(u), F_2(u), F_3(u)\bigr)\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\end{pmatrix}\\ &= F_0(u)\mathbf{P}_0 + F_1(u)\mathbf{P}_1 + F_2(u)\mathbf{P}_2 + F_3(u)\mathbf{P}_3\\ &= \sum_{i=0}^{3} F_i(u)\mathbf{P}_i.\end{aligned}$$

(See, for example, Equations (11.5), (13.7), and (14.11).) Similarly, curve $\mathbf{C}(w)$ can be expressed as

$$\begin{aligned}\mathbf{C}(w) &= (w^3, w^2, w, 1)\mathbf{NC}\\ &= \bigl(G_0(w), G_1(w), G_2(w), G_3(w)\bigr)\begin{pmatrix}\mathbf{C}_0\\ \mathbf{C}_1\\ \mathbf{C}_2\\ \mathbf{C}_3\end{pmatrix}\\ &= G_0(w)\mathbf{C}_0 + G_1(w)\mathbf{C}_1 + G_2(w)\mathbf{C}_2 + G_3(w)\mathbf{C}_3\\ &= \sum_{i=0}^{3} G_i(w)\mathbf{C}_i.\end{aligned}$$

Now, consider the x component of the surface resulting from the product of two such curves:

$$\begin{aligned}
\mathbf{S}_x(u,w) &= \Big[\sum_{i=0}^{3} F_i(u)P_{xi}\Big]\Big[\sum_{j=0}^{3} G_j(w)C_{xj}\Big]\\
&= \sum_{i,j=0}^{3} F_i(u)P_{xi}C_{xj}G_j(w)\\
&= \sum_{i,j=0}^{3} F_i(u)Q_{xij}G_j(w)\\
&= \big(F_0(u),F_1(u),F_2(u),F_3(u)\big)\mathbf{Q}_x\begin{pmatrix}G_0(w)\\G_1(w)\\G_2(w)\\G_3(w)\end{pmatrix},
\end{aligned}$$

where $Q_{xij}=P_{xi}C_{xj}$ and similarly for the y and z components. The elements $\mathbf{Q}_{ij}$ of matrix $\mathbf{Q}$ are therefore triplets of the form

$$\mathbf{Q}_{ij}=(Q_{xij},Q_{yij},Q_{zij})=\big(P_{xi}C_{xj},P_{yi}C_{yj},P_{zi}C_{zj}\big) \tag{16.5}$$

and the entire surface can be expressed as a typical bicubic patch

$$\begin{aligned}
\mathbf{S}(u,w) &= \big(F_0(u),F_1(u),F_2(u),F_3(u)\big)\mathbf{Q}\begin{pmatrix}G_0(w)\\G_1(w)\\G_2(w)\\G_3(w)\end{pmatrix}\\
&= (u^3,u^2,u,1)\mathbf{M}\mathbf{Q}\mathbf{N}^T\begin{pmatrix}w^3\\w^2\\w\\1\end{pmatrix}.
\end{aligned} \tag{16.6}$$

Equation (16.6) can be generalized to cases where the constructing curves $\mathbf{C}(w)$ and $\mathbf{P}(u)$ are not cubic polynomials.

Once the designer has an idea of the shape of the surface, it may not be too difficult to select two curves that will produce this shape. The problem is to position the surface at the right location in space. The location of the surface depends both on the types and the locations of the curves used. Imagine, for example, that two cubic Bézier curves are used to construct such a surface. One curve starts and ends at control points $\mathbf{P}_0$ and $\mathbf{P}_3$, and the other goes from $\mathbf{C}_0$ to $\mathbf{C}_3$. The resulting surface will be a bicubic Bézier patch anchored at the four corner points:

$$\begin{aligned}
\mathbf{Q}_{00}&=(P_{x0}C_{x0},P_{y0}C_{y0},P_{z0}C_{z0}), &\quad \mathbf{Q}_{01}&=(P_{x0}C_{x1},P_{y0}C_{y1},P_{z0}C_{z1}),\\
\mathbf{Q}_{10}&=(P_{x1}C_{x0},P_{y1}C_{y0},P_{z1}C_{z0}), &\quad \mathbf{Q}_{11}&=(P_{x1}C_{x1},P_{y1}C_{y1},P_{z1}C_{z1}).
\end{aligned}$$

There is no reason why these points will happen to be in the right locations and it may take some effort to vary the coordinates of all the control points to move the curves to other locations without changing their shape, in order to move points $\mathbf{Q}_{ij}$ to the right locations. The use of this surface method may therefore be limited, but it is useful for surfaces of revolution. Imagine the problem of designing a machine part with circular symmetry. If the part is to be manufactured under computer control, the location of the part in three-dimensional space may be irrelevant because the machine making it is only interested in its shape.

In order to apply Equation (16.6) to create a surface of revolution we need one curve $\mathbf{P}(u)$ to serve as a "profile" and another curve $\mathbf{C}(w)$ that's a circle, an ellipse, or an arc. As an example, consider the approximate circles obtained by cubic uniform B-splines of Section 14.15. We place four points $\mathbf{C}_i$ in the way explained in that section to make curve $\mathbf{C}(w)$ an approximate circle or circular arc. If curve $\mathbf{P}(u)$ is also expressed as a cubic B-spline, then Equation (16.6) becomes the bicubic B-spline patch:

$$\mathbf{S}(u,w) = \left[\frac{1}{6}\right]^2 [u^3, u^2, u, 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix} \qquad (16.7)$$

(compare with Equation (14.51)). The surface is created in two steps. In step 1, the surface control points $\mathbf{Q}_{ij}$ are selected. If $\mathbf{P}(u)$ is based on the $n+1$ points $\mathbf{P}_0$ through $\mathbf{P}_n$ and $\mathbf{C}(w)$ is based on the $m+1$ control points $\mathbf{C}_0$ through $\mathbf{C}_m$, then matrix $\mathbf{Q}$ is of order $(n+1)\times(m+1)$. In step 2, Equation (16.7) is applied $(n-1)\times(m-1)$ times to calculate all the surface patches. If the surface should make a complete revolution, then curve $\mathbf{C}(w)$ should be closed. The number of control points in this case is the same, but the number of patches is $(n-1)\times(m+1)$. If curve $\mathbf{P}(u)$ is also closed (as in a torus), then $(n+1)\times(m+1)$ surface patches are needed.

If $\mathbf{C}(w)$ should be a full circle, at least four control points $\mathbf{C}_i$ are needed and the (closed) curve consists of four segments. If curve $\mathbf{P}(u)$ (the "profile" of the surface) is open and is defined by $n+1$ points, it consists of $n-1$ segments. In such a case, the total number of surface control points $\mathbf{Q}_{ij}$ is $4\times(n+1)$ and the entire surface of revolution consists of $4\times(n-1)$ patches.

Example: We select the third quarter-circle segment $\mathbf{P}_4(t)$ of Equation (Ans.41) and denote it by $\mathbf{C}(w)$:

$$\mathbf{C}(w) = \frac{1}{4}(2t^3 - 6t^2 + 4, -2t^3 + 6t, 1).$$

It is defined by the four control points $\mathbf{C}_0 = (0,-3/2,1)$, $\mathbf{C}_1 = (3/2,0,1)$, $\mathbf{C}_2 = (0,3/2,1)$, and $\mathbf{C}_3 = (-3/2,0,1)$ and it goes from $(1,0,1)$ to $(0,1,1)$. Notice that we have located $\mathbf{C}(w)$ on the $z=1$ plane, so none of its components are identical to zero. For the curve profile $\mathbf{P}(u)$ we select the cubic B-spline segment defined by the four control points $\mathbf{P}_0 = (0,0,0)$, $\mathbf{P}_1 = (-1,1,0)$, $\mathbf{P}_2 = (-1,1,3)$, and $\mathbf{P}_3 = (0,0,3)$. These points are located on the $x=-y$ plane and go from $z=0$ to $z=3$, so none of the three components of $\mathbf{P}(u)$ is zero. Matrix $\mathbf{Q}$ is shown in Table 16.9. Figure 16.10 shows the surface itself and the code that generated it.

The location of this surface in space may sometimes be a problem and should therefore be discussed. Since our quarter circle goes from $(1,0,1)$ to $(0,1,1)$, we intuitively

	$(0,0,0)$	$(-1,1,0)$	$(-1,1,3)$	$(0,0,3)$
$(0,-3/2,1)$	$(0,0,0)$	$(0,-3/2,0)$	$(0,-3/2,3)$	$(0,0,3)$
$(3/2,0,1)$	$(0,0,0)$	$(-3/2,0,0)$	$(-3/2,0,3)$	$(0,0,3)$
$(0,3/2,1)$	$(0,0,0)$	$(0,3/2,0)$	$(0,3/2,3)$	$(0,0,3)$
$(3/2,0,1)$	$(0,0,0)$	$(3/2,0,0)$	$(3/2,0,3)$	$(0,0,3)$

Table 16.9: Matrix $\mathbf{Q}$ for Surface of Revolution Example.

expect the profile $\mathbf{P}(u)$ to be rotated from direction $(1,0)$ (the positive x axis) to direction $(0,1)$ (the positive y axis). A direct check, however, shows that the four corners of this patch are $\mathbf{S}(0,0) = (-0.833, 0, 0.5)$, $\mathbf{S}(0,1) = (-0.833, 0, 2.5)$, $\mathbf{S}(1,0) = (0, 0.833, 0.5)$, and $\mathbf{S}(1,1) = (0, 0.833, 2.5)$. Thus, the profile has been rotated from direction $(-0.833, 0)$ to direction $(0, 0.833)$ because of its particular original location (as defined, the profile is located on the $x = -y$ plane).

Because of the high symmetry of surfaces of revolution, especially those that go through a complete revolution, their precise location in space may not be important, so our method may be useful for this type of surface.

The method developed here can be used with any type of parametric curves, not just B-splines and not just PCs. Equation (16.8) shows how a standard quadratic Lagrange polynomial (Equation (10.13)) can be combined with a degree-4 Bézier curve to form a surface patch based on 3×5 points

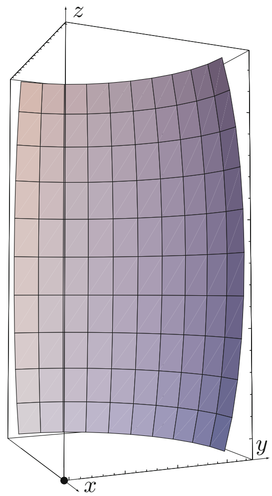
$$\mathbf{Q}_{ij} = (Q_{xij}, Q_{yij}, Q_{zij}) = \bigl(P_{xi}C_{xj}, P_{yi}C_{yj}, P_{zi}C_{zj}\bigr).$$

The surface expression is

$$\mathbf{S}(u,w) = (u^2, u, 1)\begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{Q} \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} w^4 \\ w^3 \\ w^2 \\ w \\ 1 \end{pmatrix}, \tag{16.8}$$

where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{Q}_{02} & \mathbf{Q}_{03} & \mathbf{Q}_{04} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{Q}_{20} & \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} & \mathbf{Q}_{24} \end{pmatrix}.$$




```
(*Surface of revolution*)
Clear[basis,Cubi];
(*as a combination of 2 cubic B-splines*)
(*matrix ‘basis’ has dimensions 4x4x3*)
basis={{{0,0,0},{0,-3/2,0},{0,-3/2,3},{0,0,3}},
{{0,0,0},{-3/2,0,0},{-3/2,0,3},{0,0,3}},
{{0,0,0},{0,3/2,0},{0,3/2,3},{0,0,3}},
{{0,0,0},{3/2,0,0},{3/2,0,3},{0,0,3}}};
Cubi={{-1,3,-3,1},{3,-6,3,0},{-3,0,3,0},{1,4,1,0}};
prt[i_]:=basis[[Range[1,4],Range[1,4],i]];
(*‘prt’ extracts component i from the 3rd dimen of ‘basis‘*)
coord[i_]:={u^3,u^2,u,1}.Cubi.prt[i].Transpose[Cubi].{w^3,w^2,w,1};
ParametricPlot3D[{coord[1],coord[2],coord[3]}/36,{u,0,1},{w,0,1},
 Prolog->AbsoluteThickness[.5],ViewPoint->{1.736,-0.751,-0.089}]
```

Figure 16.10: A Quarter-Circle Surface of Revolution made of B-Splines.

16.4 Skinned Surfaces

In many practical applications, the surface designer starts with only a rough idea of the shape of the required surface. The designer wants to compute and display the result of this idea, and then manipulate and improve it interactively. A common example is a pipe that winds its way inside an engine compartment, avoiding hot areas. The pipe has to have a complex shape in order to go around the various parts of the engine and may even have to change its cross section as it travels through narrow passages. One approach to such a design problem is a *skinned surface*. The designer starts by specifying several curves $\mathbf{C}_i(u)$ that become profiles (or cross sections) of the surface, and the resulting surface $\mathbf{P}(u, w)$ is an interpolation of these cross sections.

It is intuitively clear that the precise shape of the surface depends on the method used to interpolate the cross sections. Thus, general-purpose software for skinned surfaces should give the user a choice of several interpolation and approximation methods. Three examples are discussed here.

Section 10.6.2 shows how to select four points on each of four given curves and employ bicubic interpolation (Section 10.6) to compute a bicubic surface that passes through the four curves. Such a surface is also a skinned surface that interpolates the four curves.

Given a set of $n+1$ Bézier curves, each defined by a set of $m+1$ control points, we can use the Bézier approximation method of Section 13.17 to compute a rectangular surface patch that's an interpolation of the curves. This surface passes only through the four corner points, but it passes through all $n+1$ given curves. Figure 16.11 shows how a set of nine similar (but not identical) Bézier curves can be used as cross sections to construct the surface of a boat as a skinned surface. Each curve must be defined by the same number of control points (five in our example) and the fact that they are similar suggests that we can start by constructing one curve, and then duplicate it as many times as necessary and scale, move, and shear the copies as needed. In this type of work it makes sense to start with the most complex curve and use it as the basis of all the other cross sections. (Each of the curves in this example is actually two mirror image Bézier curves joined at one point.)

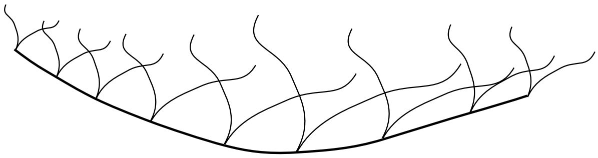


Figure 16.11: Nine Cross Sections of a Boat.

Similarly, given a set of $n+1$ B-spline or NURBS curves, each defined by a set of $m+1$ control points, we can use the approximation methods of Chapter 14 to compute a skinned surface with the curves as its cross sections. The computations in this case are more intensive, but the advantage is that such a surface may have sharp corners and edges.

> . . . and as the astronomer has his grand telescope with which to sweep the skies, and, as it were, bring the stars nearer for his inspection, so I had a smaller one, of pocket size, for the use of my observatory, with which I could sweep the regions below . . .
>
> —Washington Irving, *The Alhambra* (1832)

Plate J.1. Water Font With Colors and Backgrounds (Modo).

Plate J.2. Sink and Faucet Spraying (Modo).

Plate J.3. Image Wrapped Around Various Objects (Modo).

Plate K.1. Morphing Blond to Red (Morph Age).

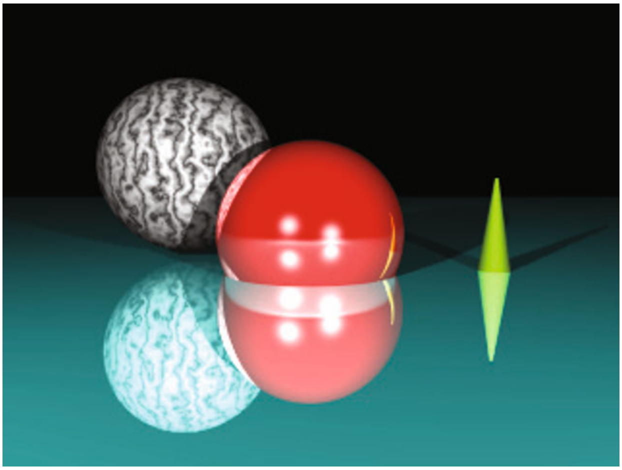


Plate K.2. Ray Tracing (MegaPOV).

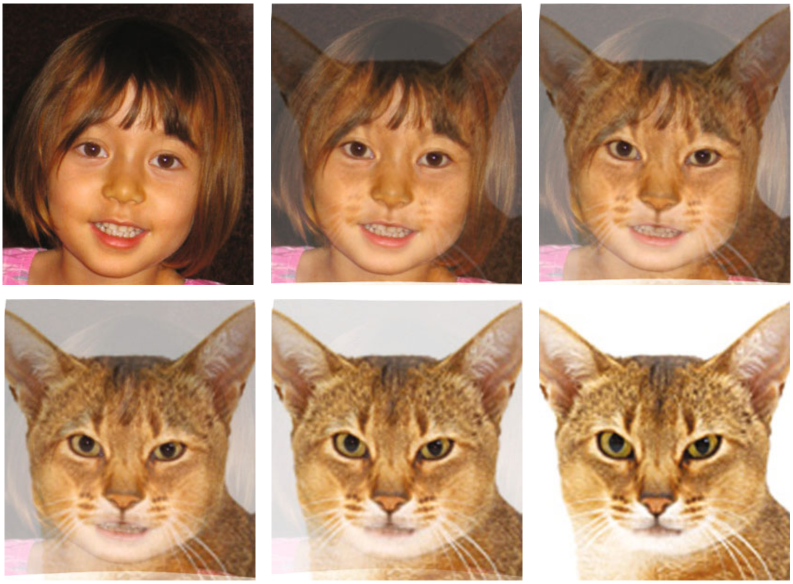


Plate K.3. Morphing Girl to Cat (Morph Age).

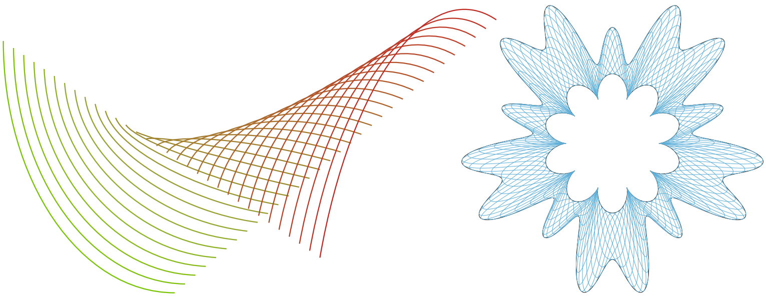


Plate K.4. Blending and Guilloche (Excentro and Adobe Illustrator).

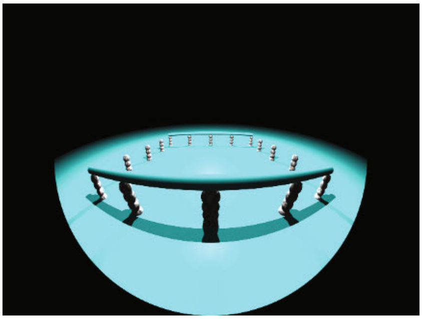
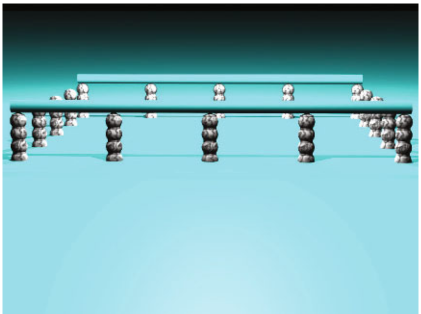
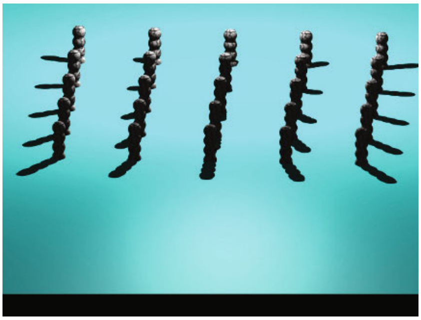





Plate L.1. Fisheye, Isometric, and Perspective Projections of Columns (MegaPOV).

Plate L.2. A Complex Knot (KnotMaker).

Plate L.3. A Fisheye Photo Taken Through a Peephole.

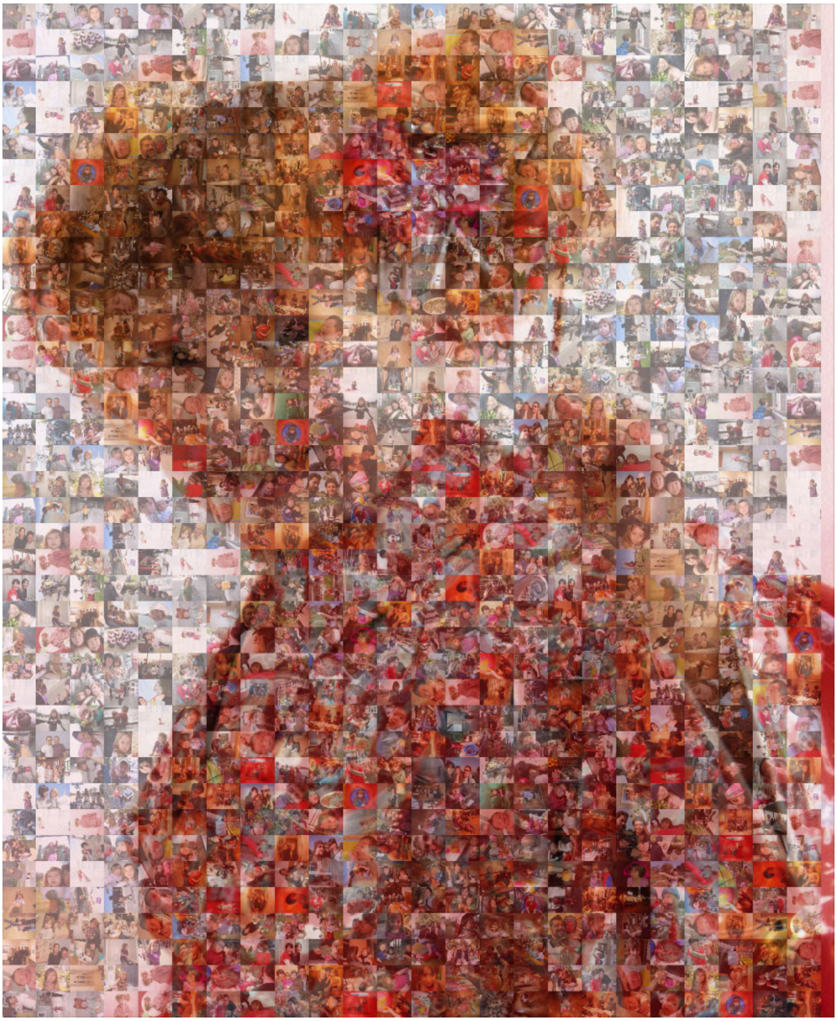


Plate L.4. A Mosaic of Images (MozoDojo).