On Geometric Theorem Proving with Null Geometric Algebra

10

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Abstract

The bottleneck in symbolic geometric computation is middle expression swell. Another embarrassing problem is geometric explanation of algebraic results, which is often impossible because the results are not invariant under coordinate transformations. In classical invariant-theoretical methods, the two difficulties are more or less alleviated but stay, while new difficulties arise.

In this chapter, we introduce a new framework for symbolic geometric computing based on conformal geometric algebra: the algebra for describing geometric configuration is null Grassmann–Cayley algebra, the algebra for advanced invariant manipulation is null bracket algebra, and the algebra underlying both algebras is null geometric algebra. When used in geometric computing, the new approach not only brings about amazing simplifications in algebraic manipulation, but can be used to extend and generalize existing theorems by removing some geometric constraints from the hypotheses.

10.1 Introduction

In algebraic approaches to geometric computing, the general procedure is as follows [19]: first, the geometric configuration, including both the hypotheses and the conclusion, is translated into an algebraic formulation in a prerequisite algebraic language; second, algebraic computations are carried out to the conclusion by utilizing the computational rules of the algebra and the given hypotheses; third, the result of the computations is translated back to geometry or, in other words, is given

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a geometric interpretation. In geometric reasoning and theorem proving, the input of a geometric problem is formulated by a set of symbols and their algebraic relations, and the algebraic computing, if geometrically meaningful, is called "symbolic geometric computation" [18].

The most commonly used algebraic formulation is Cartesian coordinates and its variations. In this setting, geometric relations are represented by polynomial equalities of coordinates. When coordinates are used in geometric computation, two typical difficulties occur:

1. Middle expression swell [1]: It is quite often that both the input and output are small but the polynomials in middle steps are huge. Some computations are thus possible only theoretically, at least for the current publicly available PCs and computer algebra systems.
2. Geometric inexplicability [19]: The result of algebraic computation is usually difficult to explain geometrically. In fact, most results produced do not have any geometric meaning—they are not invariant under coordinate transformations and thus are geometrically meaningless.

In the second half of the 19th century, several algebras of geometric covariants and invariants were proposed. When used in geometric computing, such algebras may help alleviating the difficulties, because they keep more geometry within their algebraic structures [16].

Classical invariant theory deals with invariance under the transformation group $GL_n(\mathscr{K})$. The corresponding geometry is projective geometry. The corresponding algebra of covariants for describing projective incidence relations is called "Grassmann–Cayley algebra." This is an algebra equipped with two products that are dual to each other: the outer product as in exterior algebra represents the extension of geometric entities, and the meet product represents the intersection of the entities.

In classical invariant theory, the algebra of invariants is the algebra of determinants of homogeneous coordinates, called "bracket algebra" [18]. For example, let $\mathbf{1}, \mathbf{2}, \mathbf{3}$ be three points in the 2D projective plane, let their homogeneous coordinates be (x_i, y_i, z_i) for $i = 1, 2, 3$, respectively. Then

$$[\mathbf{123}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \tag{10.1}$$

Obviously, the bracket is multilinear and antisymmetric with respect to its components $\mathbf{1}, \mathbf{2}, \mathbf{3}$. The two properties, however, do not suffice to define bracket algebra completely.

In bracket algebra, people do not resort to Laplace expansions of the brackets; instead they use the brackets as basic indeterminates and take the algebraic relations among the brackets as "syzygies" [16]. Again take as example the 2D projective geometry. Let there be five points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ in the projective plane. The 3D bracket algebra generated by the five points is the bracket polynomials with $C_5^3 = 10$ indeterminates

$$[\mathbf{123}], [\mathbf{124}], [\mathbf{125}], \ldots, [\mathbf{345}].$$

The ten brackets are not algebraically independent. They satisfy five algebraic relations which generate all other relations:

$$\begin{aligned}
&[\mathbf{123}][\mathbf{145}] - [\mathbf{124}][\mathbf{135}] + [\mathbf{125}][\mathbf{134}] = 0,\\
&[\mathbf{123}][\mathbf{245}] - [\mathbf{124}][\mathbf{235}] + [\mathbf{125}][\mathbf{234}] = 0,\\
&[\mathbf{123}][\mathbf{345}] - [\mathbf{134}][\mathbf{235}] + [\mathbf{135}][\mathbf{234}] = 0,\\
&[\mathbf{124}][\mathbf{345}] - [\mathbf{134}][\mathbf{245}] + [\mathbf{145}][\mathbf{234}] = 0,\\
&[\mathbf{125}][\mathbf{345}] - [\mathbf{135}][\mathbf{245}] + [\mathbf{145}][\mathbf{235}] = 0.
\end{aligned} \tag{10.2}$$

These generating relations are the syzygies defining the bracket algebra, called the "Grassmann–Plücker syzygies" of the five coplanar points [16]. These syzygies are still not algebraically independent from each other. For example, among the five syzygies in (10.2), only three are algebraically independent, e.g., the first three. They form a "bracket basis" of the syzygies.

Brackets have obvious representational advantage over coordinates. For five points **1** to **5** in the projective plane, the corresponding bracket algebra contains monomials like $[\mathbf{123}][\mathbf{145}]$ and binomials like $[\mathbf{123}][\mathbf{145}] + [\mathbf{124}][\mathbf{135}]$. In coordinates, however, their expanded forms are much longer:

$$\begin{aligned}
&[\mathbf{123}][\mathbf{145}]\\
&\quad = x_2z_1y_3x_1y_4z_5 - x_2z_1y_3x_1z_4y_5 - x_2z_1y_3x_4y_1z_5 + x_2z_1^2y_3x_4y_5\\
&\qquad - x_1y_2z_3x_4y_1z_5 + x_1y_2z_3x_4z_1y_5 + x_1y_2z_3x_5y_1z_4 - x_1y_2z_3x_5z_1y_4\\
&\qquad - x_3z_1y_2x_1y_4z_5 + x_3z_1y_2x_1z_4y_5 + x_3z_1y_2x_4y_1z_5 - x_3z_1^2y_2x_4y_5\\
&\qquad + x_3z_1^2y_2x_5y_4 + x_1^2y_2z_3y_4z_5 - x_1^2y_2z_3z_4y_5 - x_1z_2y_3x_5y_1z_4\\
&\qquad + x_1z_2y_3x_5z_1y_4 - x_2y_1z_3x_1y_4z_5 + x_2y_1z_3x_1z_4y_5 + x_2y_1^2z_3x_4z_5\\
&\qquad - x_2y_1z_3x_4z_1y_5 - x_2y_1^2z_3x_5z_4 + x_2y_1z_3x_5z_1y_4 + x_2z_1y_3x_5y_1z_4\\
&\qquad - x_2z_1^2y_3x_5y_4 + x_3y_1z_2x_1y_4z_5 - x_3y_1z_2x_1z_4y_5 - x_3y_1^2z_2x_4z_5\\
&\qquad + x_3y_1z_2x_4z_1y_5 + x_3y_1^2z_2x_5z_4 - x_3y_1z_2x_5z_1y_4 - x_1^2z_2y_3y_4z_5\\
&\qquad + x_1^2z_2y_3z_4y_5 + x_1z_2y_3x_4y_1z_5 - x_1z_2y_3x_4z_1y_5 - x_3z_1y_2x_5y_1z_4,\\
&[\mathbf{123}][\mathbf{145}] + [\mathbf{124}][\mathbf{135}]\\
&\quad = -x_1y_2z_4x_3y_1z_5 - x_1y_2z_4x_5z_1y_3 + x_1^2z_2y_4z_3y_5\\
&\qquad - x_1z_2y_4x_5y_1z_3 - x_2y_1z_4x_1y_3z_5 + x_2y_1^2z_4x_3z_5 - x_2y_1z_4x_3z_1y_5\\
&\qquad + 2x_2z_1y_3x_1y_4z_5 - x_2z_1y_3x_1z_4y_5 - x_2z_1y_3x_4y_1z_5 + x_2z_1^2y_3x_4y_5
\end{aligned}$$

$$
\begin{aligned}
&- x_1y_2z_3x_4y_1z_5 + 2x_1y_2z_3x_4z_1y_5 + 2x_1y_2z_3x_5y_1z_4 - x_1y_2z_3x_5z_1y_4\\
&- x_3z_1y_2x_1y_4z_5 + 2x_3z_1y_2x_1z_4y_5 + 2x_3z_1y_2x_4y_1z_5 - 2x_3z_1^2y_2x_4y_5\\
&+ x_3z_1^2y_2x_5y_4 + x_1^2y_2z_3y_4z_5 - 2x_1^2y_2z_3z_4y_5 - x_1z_2y_3x_5y_1z_4\\
&+ 2x_1z_2y_3x_5z_1y_4 - x_2y_1z_3x_1y_4z_5 + 2x_2y_1z_3x_1z_4y_5 + x_2y_1^2z_3x_4z_5\\
&- x_2y_1z_3x_4z_1y_5 - 2x_2y_1^2z_3x_5z_4 + 2x_2y_1z_3x_5z_1y_4 + 2x_2z_1y_3x_5y_1z_4\\
&- 2x_2z_1^2y_3x_5y_4 + 2x_3y_1z_2x_1y_4z_5 - x_3y_1z_2x_1z_4y_5 - 2x_3y_1^2z_2x_4z_5\\
&+ 2x_3y_1z_2x_4z_1y_5 + x_3y_1^2z_2x_5z_4 - x_3y_1z_2x_5z_1y_4 - x_4z_1y_2x_1y_3z_5\\
&- x_4z_1y_2x_5y_1z_3 + x_4z_1^2y_2x_5y_3 - 2x_1^2z_2y_3y_4z_5 + x_1^2z_2y_3z_4y_5\\
&+ 2x_1z_2y_3x_4y_1z_5 - x_1z_2y_3x_4z_1y_5 - x_3z_1y_2x_5y_1z_4 + x_1^2y_2z_4y_3z_5\\
&- x_4y_1z_2x_1z_3y_5 + x_4y_1^2z_2x_5z_3 - x_4y_1z_2x_5z_1y_3 - x_2z_1y_4x_1z_3y_5\\
&- x_1z_2y_4x_3z_1y_5 - x_2z_1y_4x_3y_1z_5 + x_2z_1^2y_4x_3y_5. \qquad (10.3)
\end{aligned}
$$

The representational advantage of brackets does not necessarily lead to any manipulational advantage. Since the brackets are not algebraically independent, one may consider using only a minimum set of algebraically independent brackets and representing all other brackets by elements in the minimum set. If brackets are used in this way, then they are equivalent to coordinates. For example, in (10.2) the first three syzygies form a bracket basis. If only such syzygies are used, then the following algebraically independent brackets can represent the other brackets via the syzygies and are such a minimum set:

$$[\mathbf{123}], [\mathbf{124}], [\mathbf{125}], [\mathbf{234}], [\mathbf{235}], [\mathbf{134}], [\mathbf{135}]. \qquad (10.4)$$

Then essentially points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are taken as a basis of the 3D vector space realizing the 2D projective plane, and (10.4) is composed of the volume $[\mathbf{123}]$ of the basis and the homogeneous coordinates of points $\mathbf{i} = \mathbf{4}, \mathbf{5}$ with respect to the basis

$$x_i = \frac{[\mathbf{12i}]}{[\mathbf{123}]}, \qquad y_i = \frac{[\mathbf{13i}]}{[\mathbf{123}]}, \qquad z_i = \frac{[\mathbf{23i}]}{[\mathbf{123}]}.$$

Let us see how invariants are manipulated in classical invariant theory. Classical invariant-theoretical method employs a Gröbner basis of the ideal generated by the Grassmann–Plücker syzygies in bracket algebra, called "straightening syzygies" [16]. All bracket polynomials form a $\mathscr{Z}$-module with elements in the straightening syzygies as a basis. Any bracket polynomial can be written in a unique manner as a linear combination of the basis elements with integer coefficients. The latter is the "normal form" of the bracket polynomial. In the procedure of normalization, a non-straightened bracket monomial is "exploded" into many terms many times. This procedure does not have any control to the middle expression swell.

Geometric interpretation is also a problem for bracket algebra. Although each bracket, as a determinant of homogeneous coordinates of the constituent points, can be interpreted in affine geometry as the signed volumes of the simplex spanned by the points as vertices, a bracket polynomial is by no means easily interpretable with geometric terms. If the polynomial can be written as a rational monomial in a suitable covariant algebra in which the basic elements and their products are geometrically meaningful, then the polynomial finds its geometric interpretation. According to a theorem by Sturmfels [17], theoretically this procedure is always successful, called "Cayley factorization." However, there is no algorithm to produce this factorization, except for the simplest case where every point in the bracket polynomial occurs only once [18].

So in the setting of classical invariant theory, the two major problems faced by the coordinate approach are still alive, although in some special cases the algebraic manipulations can be simplified because of the simplicity in algebraic representation. Due to the algebraic dependencies among brackets, new difficulties arise, which are by no means easy to handle. In invariant-theoretical methods, people do not get rid of algebraic dependencies; otherwise it becomes a traditional coordinate method. The following are some newly invoked problems [4, 11]:

- *Representation*: A geometric entity or relation often has many representations in invariant algebra. How to choose a suitable one in computation? Can the computing be made robust against the choice?

  This problem has never been studied before. A typical example is a conic formed by five points in the projective plane. It has fifteen equal but different forms when represented as a degree-four binomial of brackets. Different choice of the representation can lead to drastic difference in complexity in subsequent algebraic manipulations.
- *Contraction*: Reduce the number of terms of a bracket polynomial.

  This problem does not exist in polynomials of coordinates. In bracket algebra this problem is wide open: people do not know how to judge and how to find a minimum-sized form for a bracket polynomial.
- *Expansion*: The reverse procedure of Cayley factorization is called "Cayley expansion" [11]. It is to translate a scalar-valued expression of the algebra of covariants into a polynomial in the algebra of invariants.

  This problem turns out to be rather complicated. A simple example is the bracket $[\mathbf{a}\mathbf{a}'\mathbf{a}'']$ formed by three intersections of pairs of lines in the projective plane:

  $$\mathbf{a} = \mathbf{12} \cap \mathbf{34}, \qquad \mathbf{a}' = \mathbf{1}'\mathbf{2}' \cap \mathbf{3}'\mathbf{4}', \qquad \mathbf{a}'' = \mathbf{1}''\mathbf{2}'' \cap \mathbf{3}''\mathbf{4}''.$$

  To compute the bracket, by substituting the expressions of the **a**'s in Grassmann–Cayley algebra into it, we get

  $$\left[\{(\mathbf{1}\wedge\mathbf{2})\vee(\mathbf{3}\wedge\mathbf{4})\}\left\{\left(\mathbf{1}'\wedge\mathbf{2}'\right)\vee\left(\mathbf{3}'\wedge\mathbf{4}'\right)\right\}\left\{\left(\mathbf{1}'\wedge\mathbf{2}'\right)\vee\left(\mathbf{3}'\wedge\mathbf{4}'\right)\right\}\right].$$

  It has 16847 different expansion results.

It is an appalling fact that classical invariant-theoretical method is far from being well developed for symbolic computation. Basic computing tasks like choosing optimal representations in the procedure of computing, different expansions of covariant expressions, contraction of invariant expressions, and factorization in both invariant and covariant algebras, are either open or overlooked. The bottleneck in symbolic computing, i.e., middle expression swell, is not taken care of. Because of this, although invariant algebra can provide simplification in algebraic description, its cost is significantly high complexity in algebraic manipulation.

When it comes to Euclidean geometry, the algebra of basic invariants is "inner-product bracket algebra" [3]. This algebra contains, besides brackets, the inner products of vectors as basic elements. In its defining syzygies there is a polynomial of the form

$$[\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_n][\mathbf{j}_1\mathbf{j}_2\cdots\mathbf{j}_n] - \det(\mathbf{i}_k\cdot\mathbf{j}_l)_{k,l=1,\ldots,n},$$

which equates the product of two determinants to the determinant of the inner products of the constituent column vectors of the two determinants. This syzygy contains as many as $n!+1$ terms. So the task to control the limit of middle expression swell is much heavier. Further to people's dismay, the invariants and covariants in geometric computing are often complicated rational polynomials of basic ones. This suggests that basic invariants are too low-level. As a result, symbolic computation in Euclidean geometry with inner-product bracket algebra is much more difficult than in projective geometry.

This is the background of our research in recent years on invariant symbolic computation in classical geometry. In the course of eight years, we have proposed a new invariant framework, called "null geometric algebra," and a new guideline for computation, called "BREEFS" [6–10, 12, 13, 15]. The former is a system of monomial representations of Euclidean incidence geometric constructions and an associated hierarchy of infinitely graded advanced invariants grown out of Clifford multiplication, and the latter is a stepwise size control strategy based on syzygies of the invariant algebra. The new framework and guideline can help achieving significant simplification in invariant algebraic manipulation and thus lead to much better computation both in efficiency (size control) and in quality (geometric interpretation), as follows:

- In projective geometry and Euclidean geometry, generally the size of an expression being computed is controlled to within two terms.
- Some geometric computing tasks, which have proved to be very hard if using only coordinates or basic invariants, can be finished with our new system.

This chapter intends to provide an introduction to the new system. This is done in Sect. 10.2. Section 10.3 is a practical application of the new system to the problem of "geometric factorization, decomposition, and theorem completion," that is, to explore the quantitative and geometric relationship between the hypotheses and the conclusion of a geometric theorem, and to explore and discover geometrically meaningful new conclusions by reducing the number of hypotheses in the theorem.

10.2 Null Grassmann–Cayley Algebra and Null Bracket Algebra

10.2.1 Grassmann–Cayley Algebra, Bracket Algebra and Inner-Product Bracket Algebra

First recall the definition of *Grassmann–Cayley algebra* [18]. Let $\mathscr{V}^n$ be an nD linear space over a field of characteristic not 2. Let $\Lambda(\mathscr{V}^n)$ be the Grassmann space over the base space $\mathscr{V}^n$. Define in $\Lambda(\mathscr{V}^n)$ the following meet product, which is dual to the outer product: for any $\mathbf{A}, \mathbf{B} \in \Lambda(\mathscr{V}^n)$, their meet product $\mathbf{A} \vee \mathbf{B}$ is defined by [9]

$$(\mathbf{A} \vee \mathbf{B})^* := \mathbf{B}^* \wedge \mathbf{A}^*, \tag{10.5}$$

where "*" is the dual operator in Grassmann algebra, and "$\wedge$" is the outer product.

Grassmann–Cayley algebra is a language for describing projective incidence constructions. Any vector of $\mathscr{V}^n$ represents a point of $(n-1)$D projective space, and the representation is *homogeneous* in that it is unique up to scale: any two vectors represent the same projective point if and only if they differ by scale only. The line extended by two points is represented by their outer product, and the plane extended by three points is represented by the outer product of any three vectors representing the three points. In projective space, the intersection of a line and a plane is represented by their meet product.

Let there be m projective points. An nD *bracket algebra* generated by a sequence $\mathscr{S}$ of m symbols $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$ representing the points, where $m > n+1$, is the polynomial ring generated by all subsequences of $\mathscr{S}$ of length n, denoted by square brackets, modulo the ideal generated by the left side of the following identity, called "Grassmann–Plücker syzygies":

$$\sum_{k=1}^{n+1}(-1)^{k+1}[\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{n-1}\mathbf{j}_k][\mathbf{j}_1\mathbf{j}_2\cdots\check{\mathbf{j}}_k\cdots\mathbf{j}_{n+1}] = 0, \tag{10.6}$$

where the $\mathbf{i}$'s and $\mathbf{j}$'s are symbols in $\mathscr{S}$, and $\check{\mathbf{j}}_k$ denotes that $\mathbf{j}_k$ does not occur in the subsequence. By requiring antisymmetry among them, the elements in each bracket do not need to follow their original order in the sequence $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$.

The proof of (10.6) is trivial: expand

$$(\mathbf{i}_1 \wedge \mathbf{i}_2 \wedge \cdots \wedge \mathbf{i}_{n-1}) \vee (\mathbf{j}_1 \wedge \mathbf{j}_2 \wedge \cdots \wedge \mathbf{j}_{n+1}) \tag{10.7}$$

using the "shuffle formula" in Grassmann–Cayley algebra [16] to distribute the $\mathbf{j}$'s to the sequence $\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{n-1}$, once for each $\mathbf{j}$ but with alternating signs. That (10.7) equals zero follows from the fact that any $n+1$ vectors in an nD vector space are linearly dependent, so their outer product equals zero. Thus,

$$\mathbf{j}_1 \wedge \mathbf{j}_2 \wedge \cdots \wedge \mathbf{j}_{n+1} = 0.$$

The meet product of zero with any element is zero. This proves (10.6).

Bracket algebra is established for projective geometry and, after some revision, for affine geometry. For Euclidean geometry, a new structure called inner product is needed. A bracket algebra supplemented by an inner product is an *inner-product bracket algebra* [3]. Formally, an nD inner-product bracket algebra generated by a sequence $\mathscr{S}$ of m symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, where $m \geq n$, is the quotient of the polynomial ring generated by two classes of subsequences of $\mathscr{S}$ of length 2 and n, denoted by dot and square bracket, respectively, modulo the ideal generated by the following syzygies:

- GP1:

$$\sum_{k=1}^{n+1} (-1)^{k+1} \mathbf{i} \cdot \mathbf{j}_k [\mathbf{j}_1 \mathbf{j}_2 \cdots \check{\mathbf{j}}_k \cdots \mathbf{j}_{n+1}]. \tag{10.8}$$

- GP2:

$$[\mathbf{i}_1 \mathbf{i}_2 \cdots \mathbf{i}_n][\mathbf{j}_1 \mathbf{j}_2 \cdots \mathbf{j}_n] - \det(\mathbf{i}_k \cdot \mathbf{j}_l)_{k,l=1,\ldots,n}. \tag{10.9}$$

The order of elements in the subsequences can be violated by requiring that the dot structure is symmetric while the bracket structure is antisymmetric.

The proof of the syzygies is also trivial. (10.8) is the expansion of the inner product

$$\mathbf{i} \cdot (\mathbf{j}_1 \wedge \mathbf{j}_2 \wedge \cdots \wedge \mathbf{j}_{n+1}), \tag{10.10}$$

which equals zero because the outer product of the $n+1$ $\mathbf{j}$'s is zero. (10.9) follows from

$$\begin{aligned}
&[\mathbf{i}_1 \mathbf{i}_2 \cdots \mathbf{i}_n][\mathbf{j}_1 \mathbf{j}_2 \cdots \mathbf{j}_n] \\
&\quad = (\mathbf{i}_1 \wedge \mathbf{i}_2 \wedge \cdots \wedge \mathbf{i}_n)^* (\mathbf{j}_1 \wedge \mathbf{j}_2 \wedge \cdots \wedge \mathbf{j}_n)^* \\
&\quad = \big((\mathbf{i}_1 \wedge \mathbf{i}_2 \wedge \cdots \wedge \mathbf{i}_n) \cdot (\mathbf{j}_n \wedge \mathbf{j}_{n-1} \wedge \cdots \wedge \mathbf{j}_1)\big) \\
&\quad = \det(\mathbf{i}_k \cdot \mathbf{j}_l)_{k,l=1,\ldots,n},
\end{aligned}$$

where the $\mathbf{e}$'s are a basis of the nD vector space defining the dual operator, and the dot denotes the inner product in geometric algebra. The Grassmann–Plücker syzygies (10.6) can be obtained from (10.9) directly.

To obtain geometrically explicitly meaningful results, one needs to resort to advanced algebraic invariants. An *algebraic invariant* is a polynomial function of finitely many geometric entities, each represented by coordinates, such that under all kinds of coordinate transformations specified by the defining group of the geometry, the function remains either invariant or rescaled by a power of the determinant of the transformation. It is a classical theorem that all projective algebraic invariants are generated by brackets and that all Euclidean algebraic invariants are generated by brackets and inner products of vector pairs. Conversely, it is clear that

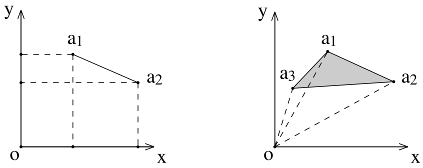

Fig. 10.1 Basic invariants in 2D Euclidean geometry

all brackets are projective invariants and all inner products are Euclidean invariants.

Brackets and inner products of vector pairs are *basic Euclidean invariants*. For example, in 2D Euclidean geometry the two basic Euclidean invariants are shown in Fig. 10.1: the distance between two points $\mathbf{a}_1 = (x_1, y_1)$, $\mathbf{a}_2 = (x_2, y_2)$:

$$\begin{aligned}|\mathbf{a}_1 - \mathbf{a}_2|^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2,\end{aligned} \tag{10.11}$$

and twice the signed volume of a simplex spanned by vertexes $\mathbf{a}_j = (x_j, y_j)$ for $1 \le j \le 3$:

$$\begin{aligned}\big[(\mathbf{a}_1 - \mathbf{a}_2)(\mathbf{a}_1 - \mathbf{a}_3)\big] &= \begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix} \\ &= x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2.\end{aligned} \tag{10.12}$$

All other algebraic invariants are polynomial functions of the basic ones.

An invariant has three different forms of appearance: the *coordinate form* such as the right side of (10.12); the *expanded form* such as the right side of

$$\big[(\mathbf{a}_1 - \mathbf{a}_2)(\mathbf{a}_1 - \mathbf{a}_3)\big] = [\mathbf{a}_1\mathbf{a}_2] - [\mathbf{a}_1\mathbf{a}_3] + [\mathbf{a}_2\mathbf{a}_3]; \tag{10.13}$$

and the *compact form* such as $[(\mathbf{a}_1 - \mathbf{a}_2)(\mathbf{a}_1 - \mathbf{a}_3)]$. Obviously, the compact form is the most convenient in reading out geometric interpretation. On the other hand, without expanding the parentheses in a compact form, algebraic manipulations are much more complicated.

An *advanced invariant* is an algebraic invariant having a compact form that is a monomial in a geometric algebra. The geometric interpretation of an advanced invariant is immediate from the compact form. Advanced invariant theory studies the geometric and algebraic properties of advanced invariants. CGA (*conformal geometric algebra*) provides a natural tool to construct advanced Euclidean invariants.

10.2.2 From Conformal Geometric Algebra to Null Grassmann–Cayley Algebra and Null Bracket Algebra

To start with, consider the expansion of (10.11), i.e., the squared length of line segment $\mathbf{a}_1\mathbf{a}_2$:

$$\begin{aligned} d_{\mathbf{ab}}^2 &= |\mathbf{a}_1 - \mathbf{a}_2|^2 = (\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{a}_1 - \mathbf{a}_2) \\ &= \mathbf{a}_1 \cdot \mathbf{a}_1 + \mathbf{a}_2 \cdot \mathbf{a}_2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2. \end{aligned} \tag{10.14}$$

A geometric point can be represented by the vector drawn from the origin of the coordinate frame to the point; so $\mathbf{a}_1 \cdot \mathbf{a}_1$ represents the squared distance between the origin and the point. When the coordinate frame changes, so does the squared distance. Hence, $\mathbf{a}_1 \cdot \mathbf{a}_1$ is geometrically meaningless, and so is each term on the right side of (10.14). We are confronted with a bunch of geometrically meaningless terms when expanding a squared distance.

To avoid such expansions, a natural idea is to introduce a new inner product such that if vectors $\mathbf{a}_1, \mathbf{a}_2$ represent two geometric points, then $\mathbf{a}_1 \cdot \mathbf{a}_2$ is a geometric quantity relying on the two points only. Any such a candidate must be a function of the distance between the two points.

Then $\mathbf{a}_1 \cdot \mathbf{a}_1$ has to be independent of $\mathbf{a}_1$, because it has to be a function of the distance zero between a point and itself. The simplest choice is to set $\mathbf{a}_1 \cdot \mathbf{a}_1 = 0$, i.e., vector $\mathbf{a}_1$ is *null*. Then from (10.14) we get

$$\mathbf{a} \cdot \mathbf{b} = -\frac{d_{\mathbf{ab}}^2}{2}. \tag{10.15}$$

To realize nD Euclidean geometry in an inner-product space having property (10.15), the smallest dimension is $n+2$, and the space is Minkowski. Such a realization, called the *conformal model*, has its root in the work of Wachter (1830s) and later occurred in S. Lie's dissertation (1870s). For $n = 3$, the null-vector representation of a point $(x, y, z) \in \mathbb{R}^3$ in Minkowski space $\mathbb{R}^{4,1}$ is the following: let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis of $\mathbb{R}^3$, and let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{n}_\infty, \mathbf{n}_0)$ be a basis of $\mathbb{R}^{4,1}$ with inner-product matrix

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix}; \tag{10.16}$$

then

$$(x, y, z) \in \mathbb{R}^3 \mapsto \left(x, y, z, 1, -\frac{x^2+y^2+z^2}{2}\right) \in \mathbb{R}^{4,1} \tag{10.17}$$

is an isometry. When $x^2 + y^2 + z^2 \to \infty$, the right side of (10.17) tends to the direction of vector $\mathbf{n}_\infty$. So $\mathbf{n}_\infty$ represents a unique point at infinity compactifying a Euclidean space.

The classical *conformal model* of nD Euclidean space is the following set:

$$\mathcal{N}_{\mathbf{n}_\infty} = \left\{\mathbf{x} \in \mathcal{R}^{n+1,1} \,|\, \mathbf{x} \cdot \mathbf{x} = 0,\ \mathbf{x} \cdot \mathbf{n}_\infty = -1\right\}. \tag{10.18}$$

Here $\mathbf{n}_\infty$ is a null vector in the $(n+2)$D Minkowski space $\mathscr{R}^{n+1,1}$. Elements in $\mathscr{N}_{\mathbf{n}_\infty}$ are in one-to-one correspondence with points in nD Euclidean space. Let null vector $\mathbf{n}_0 \in \mathscr{N}_\mathbf{e}$ be the origin. In the conformal model, a point $\mathbf{x}$ in $\mathscr{R}^n$ is represented by the null vector

$$\vec{\mathbf{x}} = \mathbf{n}_0 + \mathbf{x} + \frac{\mathbf{x}^2}{2}\mathbf{n}_\infty. \tag{10.19}$$

Because they never occur simultaneously, later on, both a Euclidean vector and its null vector representation are denoted by the same letter without arrow top.

From the definition, it is clear that the conformal model depends on the choice of the origin $\mathbf{n}_0$. The *homogeneous model* [3] is a more general formulation of the classical conformal model. The model is composed of the set of null vectors

$$\mathscr{N} = \left\{\mathbf{x} \in \mathscr{R}^{n+1,1} \,|\, \mathbf{x}\cdot\mathbf{x} = 0\right\} \tag{10.20}$$

and a null vector $\mathbf{n}_\infty \in \mathscr{N}$. An element $\mathbf{x} \in \mathscr{N}$ represents a finite point if and only if $\mathbf{x}\cdot\mathbf{n}_\infty \neq 0$. Two elements in $\mathscr{N}$ represents the same point if and only if they differ by scale. This representation is homogeneous, and the model is conformal instead of isometric. Because of this, it can represent classical geometries of different metrics, where the "point at infinity" $\mathbf{n}_\infty$ remains a nonzero vector, but not necessarily a null vector. To unclutter the formulas, we will denote it by $\mathbf{e}$ in the remainder of this chapter.

The geometric algebra established upon the homogeneous model is called *conformal geometric algebra* (CGA).[1] It is the covariant algebra for Euclidean incidence relations, including collinearity, cocircularity, parallelism, perpendicularity, and tangency. CGA provides the following algebraic representations for incidence geometric constructions in Euclidean geometry:

(1) The line passing through two points $\mathbf{a}, \mathbf{b}$ is represented by $\mathbf{e}\wedge\mathbf{a}\wedge\mathbf{b}$, where the vectors representing points are null. A circle passing through three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is represented by $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$.

(2) The above constructions are "extension constructions" based on points. Duality provides intersection constructions. The intersection of two circles/lines $\mathbf{A}_1 = \mathbf{a}_1\wedge\mathbf{b}_1\wedge\mathbf{c}_1$ and $\mathbf{A}_2 = \mathbf{a}_2\wedge\mathbf{b}_2\wedge\mathbf{c}_2$ is a 0D circle represented by the meet product $\mathbf{A}_1\vee\mathbf{A}_2$.

Besides geometric constructions based on points, CGA also provides representations of geometric constructions based on symmetry generators. There are three kinds of nonzero vectors in a Minkowski space: *positive*, *null*, and *negative*. They are vectors whose inner product with itself is positive, zero, and negative, respectively. We have seen that a null vector represents in Euclidean geometry a point or the point at infinity. Below we introduce the other two kinds of vectors as symmetry generators.

[1] *Editorial note*: This characterization of CGA is more restrictive than elsewhere in this book, where (10.19) is employed.

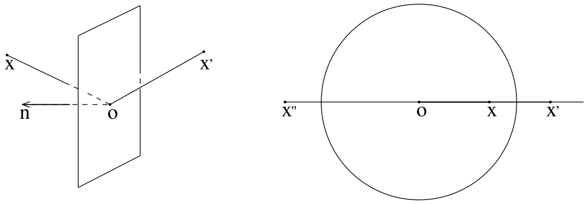

Fig. 10.2 Reflection, inversion, and antipodal inversion. *Left*: $x \mapsto x'$, reflection. *Right*: $x \mapsto x'$, inversion; $x \mapsto x''$, antipodal inversion

Any positive vector in Minkowski space $\mathbb{R}^{n+1,1}$ can be written up to scale as either $\mathbf{n}+\delta\mathbf{e}$ or $\mathbf{c}-\rho^2\mathbf{e}/2$, where $\mathbf{n} \in \mathbb{R}^n$ is a unit vector, $\mathbf{c}$ is a null vector representing a point, and $\delta, \rho \in \mathbb{R}$.

Let $\mathbf{x}$ be a null vector representing a point. Let $\mathbf{s}$ be a positive vector. Then $\mathbf{x}\cdot\mathbf{s} = 0$ iff point $\mathbf{x}$ is on the hyperplane or sphere represented by $\mathbf{s}^*$. When $\mathbf{s} = \mathbf{n}+\delta\mathbf{e}$, where $\delta = -(\mathbf{a}\cdot\mathbf{n})/(\mathbf{a}\cdot\mathbf{e})$ for a vector $\mathbf{a} \in \mathscr{N}$ representing a point, then $\mathbf{s}^*$ is the hyperplane normal to $\mathbf{n}$ and passing through point $\mathbf{a}$; when $\mathbf{s} = \mathbf{c} - \rho^2\mathbf{e}/2$, then $\mathbf{s}^*$ is the sphere centering at $\mathbf{c}$ with radius $|\rho|$.

When $\mathbf{x}\cdot\mathbf{s} \neq 0$, then $\mathbf{x}\wedge\mathbf{s}$ represents a pair of points: $\mathbf{x}$ and the reflection/inversion of $\mathbf{x}$ with respect to hyperplane/sphere $\mathbf{s}^*$, as shown in Fig. 10.2.

Any negative vector $\mathbf{t} \in \mathbb{R}^{n+1,1}$ can be written up to scale as $\mathbf{c}+\rho^2\mathbf{e}/2$, where $\mathbf{c}$ is a null vector representing a point, and $\rho \in \mathbb{R}^+$. Let $\mathbf{x}$ be a null vector representing a point. Let $\mathbf{t}$ be a negative vector. Then $\mathbf{x}\wedge\mathbf{t}$ represents a pair of points, $\mathbf{x}$ and the *antipodal inversion* of $\mathbf{x}$ with respect to sphere $(\mathbf{c}, \rho)$. As shown in Fig. 10.2 (right): $\mathbf{x}$ is mapped to $\mathbf{x}''$ such that $\overrightarrow{\mathbf{ox}}'' = -\rho\overrightarrow{\mathbf{ox}}^{-1}$.

Hence in CGA, the extension product is generalized to include not only generating objects (e.g., points), but also symmetries (e.g., reflection, inversion, and antipodal inversion) of the resulting object. Dually, the intersection of two geometric objects can be their common symmetry. CGA extends Grassmann's original extension of linear (flat) objects to include not only round objects, but also symmetries.

Now consider the simplest 2D case and the representations of points of intersection. Let $\mathbf{ab}_1\mathbf{c}_1$ and $\mathbf{ab}_2\mathbf{c}_2$ be two circles/lines. Their intersection is a pair of points, one of which may be the point at infinity:

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b}_1\wedge\mathbf{c}_1)\vee(\mathbf{a}\wedge\mathbf{b}_2\wedge\mathbf{c}_2) &= \mathbf{a}\wedge\big([\mathbf{ab}_1\mathbf{c}_1\mathbf{c}_2]\mathbf{b}_2-[\mathbf{ab}_1\mathbf{c}_1\mathbf{b}_2]\mathbf{c}_2\big)\\ &= \mathbf{a}\wedge\big([\mathbf{ab}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{c}_1-[\mathbf{ac}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{b}_1\big).\end{aligned} \quad (10.21)$$

The point other than $\mathbf{a}$ at the intersection is called the *second point of intersection*.

Suppose that we are already given one point of intersection $\mathbf{a}$, and we want to have an expression for the second point of intersection. First of all, the expression is neither vector $[\mathbf{ab}_1\mathbf{c}_1\mathbf{c}_2]\mathbf{b}_2-[\mathbf{ab}_1\mathbf{c}_1\mathbf{b}_2]\mathbf{c}_2$ nor vector $[\mathbf{ab}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{c}_1-[\mathbf{ac}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{b}_1$, because both vectors are not null. Second, the second point of intersection must be of the form $[\mathbf{ab}_1\mathbf{c}_1\mathbf{c}_2]\mathbf{b}_2-[\mathbf{ab}_1\mathbf{c}_1\mathbf{b}_2]\mathbf{c}_2+\lambda\mathbf{a}$ or $[\mathbf{ab}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{c}_1-[\mathbf{ac}_1\mathbf{b}_2\mathbf{c}_2]\mathbf{b}_1+\mu\mathbf{a}$, where λ, μ are scalars to make the whole expression into a null vector. In order to obtain a monomial and symmetric representation with respect to $\mathbf{ab}_1\mathbf{c}_1$ and $\mathbf{ab}_2\mathbf{c}_2$, we need to introduce a new meet product to unify the two different forms, and another product to convert a non-null vector into a null one.

Hence, to represent the second intersection point by a null vector multiplicatively in a monomial manner, we need to introduce two more products into CGA, the *reduced meet product* and the *nullification product*. The reduced meet product of $\mathbf{b}_1 \wedge \mathbf{c}_1$ and $\mathbf{b}_2 \wedge \mathbf{c}_2$ with base $\mathbf{a}$ is denoted by $(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)$ and defined by

$$\mathbf{a} \wedge \left\{(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)\right\} = (\mathbf{a} \wedge \mathbf{b}_1 \wedge \mathbf{c}_1) \vee (\mathbf{a} \wedge \mathbf{b}_2 \wedge \mathbf{c}_2). \tag{10.22}$$

The above identity indicates that the reduced meet product defined in the CGA over $\mathbb{R}^{3,1}$ with base $\mathbf{a}$ is unique only *modulo* $\mathbf{a}$, i.e., if both $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{3,1}$ satisfy

$$\mathbf{a} \wedge \mathbf{u} = \mathbf{a} \wedge \mathbf{v} = (\mathbf{a} \wedge \mathbf{b}_1 \wedge \mathbf{c}_1) \vee (\mathbf{a} \wedge \mathbf{b}_2 \wedge \mathbf{c}_2), \tag{10.23}$$

then $\mathbf{u} = \mathbf{v} + \lambda\mathbf{a}$ for some scale λ. Despite the uncertainty, $\mathbf{a}\{(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)\}$ and $\{(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)\}\mathbf{a}$ are both fixed.

An important property of the Minkowski plane is that it has two and only two null directions, and the two directions can be interchanged by any reflection in the plane. Since in Geometric Algebra a reflection is generated by the graded adjoint action of an invertible vector [2], the second intersection of circles/lines $\mathbf{a}\mathbf{b}_1\mathbf{c}_1$ and $\mathbf{a}\mathbf{b}_2\mathbf{c}_2$ can be represented by reflecting vector $\mathbf{a}$ with respect to vector $(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)$:

$$\frac{1}{2}\left\{(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)\right\}\mathbf{a}\left\{(\mathbf{b}_1 \wedge \mathbf{c}_1) \vee_{\mathbf{a}} (\mathbf{b}_2 \wedge \mathbf{c}_2)\right\}. \tag{10.24}$$

In CGA, the *nullification product* of $\mathbf{a}$ by $\mathbf{b}$ is defined by

$$N_{\mathbf{b}}(\mathbf{a}) := \frac{1}{2}\mathbf{a}\mathbf{b}\mathbf{a}. \tag{10.25}$$

An nD *null Grassmann–Cayley algebra* refers to a Grassmann–Cayley algebra whose generating vectors are null and whose algebraic operators include not only the outer product, meet product, dual and bracket operators, but also i-grading operators where i takes values in $\{0, 1, n-1, n\}$, the reduced meet product, and the nullification product.

Null Grassmann–Cayley algebra is a language for describing Euclidean incidence constructions. For example, in the plane there are three points $\mathbf{1}, \mathbf{2}, \mathbf{3}$. The line passing through point $\mathbf{1}$ and parallel to line $\mathbf{23}$ has the following monomial representation:

$$\mathbf{e} \wedge \mathbf{1} \wedge \langle \mathbf{e23} \rangle_1, \tag{10.26}$$

where $\langle \mathbf{e23} \rangle_1$ is a vector describing the direction of line $\mathbf{23}$. As a second example, the line passing through point $\mathbf{1}$ and perpendicular to line $\mathbf{23}$ has the following monomial representation:

$$\mathbf{e} \wedge \mathbf{1} \wedge \langle \mathbf{e23} \rangle_3^*, \tag{10.27}$$

where $\langle \mathbf{e23} \rangle_3^*$ is a vector describing the normal direction of line $\mathbf{23}$.

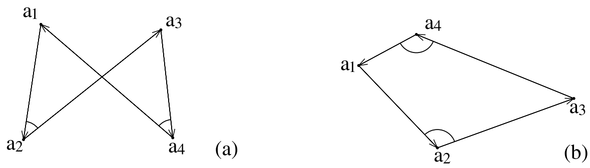

Fig. 10.3 Geometric interpretations of $\langle \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle$ and $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4]$

CGA also provides a hierarchy of advanced invariants for geometric computing. In the previous subsection, we have shown that there are two basic invariants, $\langle \mathbf{a}_1\mathbf{a}_2\rangle = \mathbf{a}_1\cdot\mathbf{a}_2 = \langle \mathbf{a}_1\mathbf{a}_2\rangle_0$, and $[\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n] = (\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_n)^* = \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n\rangle_n^*$, where $\langle\ \rangle_0$ and $\langle\ \rangle_n$ are grading operators of grade 0 and n, respectively. The geometric product prolongs the two basic invariants to the following two sequences of advanced invariants: for any $k, l \geq 0$,

$$\begin{aligned} \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{2k}\rangle &:= \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{2k}\rangle_0, \\ [\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{n+2l}] &:= \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{n+2l}\rangle_n^*. \end{aligned} \tag{10.28}$$

The two kinds of advanced invariants have nice geometric interpretations and algebraic properties. For example, if the $\mathbf{a}_i$ are null vectors representing points such that $\mathbf{a}_i\cdot\mathbf{e} = -1$, then

$$\begin{aligned} \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{2k}\rangle &= \frac{1}{2}\langle \overrightarrow{\mathbf{a}_1\mathbf{a}_2}\,\overrightarrow{\mathbf{a}_2\mathbf{a}_3}\cdots\overrightarrow{\mathbf{a}_{2k-1}\mathbf{a}_{2k}}\,\overrightarrow{\mathbf{a}_{2k}\mathbf{a}_1}\rangle, \\ [\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{n+2l}] &= (-1)^n\frac{1}{2}[\overrightarrow{\mathbf{a}_1\mathbf{a}_2}\,\overrightarrow{\mathbf{a}_2\mathbf{a}_3}\cdots\overrightarrow{\mathbf{a}_{n+2l}\mathbf{a}_1}]. \end{aligned} \tag{10.29}$$

Here $\overrightarrow{\mathbf{a}_i\mathbf{a}_j}$ denotes the displacement vector from point $\mathbf{a}_i$ to point $\mathbf{a}_j$ in Euclidean geometry. In particular, when $n = 2$, $k = 2$, and $l = 1$, then

$$\begin{aligned} \langle \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4\rangle &= -\frac{d_{\mathbf{a}_1\mathbf{a}_2}d_{\mathbf{a}_2\mathbf{a}_3}d_{\mathbf{a}_3\mathbf{a}_4}d_{\mathbf{a}_4\mathbf{a}_1}}{2}\cos(\angle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 + \angle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1), \\ [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4] &= -\frac{d_{\mathbf{a}_1\mathbf{a}_2}d_{\mathbf{a}_2\mathbf{a}_3}d_{\mathbf{a}_3\mathbf{a}_4}d_{\mathbf{a}_4\mathbf{a}_1}}{2}\sin(\angle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 + \angle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1), \end{aligned} \tag{10.30}$$

where $d_{\mathbf{ab}}$ denotes the Euclidean distance between points $\mathbf{a}$, $\mathbf{b}$.

In the plane, if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_3$ are the sequence of vertexes of a quadrilateral (Fig. 10.3(a)), then $\angle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3, \angle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1$ have opposite signs; if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are the sequence of vertexes of a quadrilateral (Fig. 10.3(b)), then $\angle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3, \angle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1$ have the same sign. By (10.30), $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4] = 0$ iff $\angle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 + \angle\mathbf{a}_3\mathbf{a}_4\mathbf{a}_1 = 0 \bmod \pi$ or, equivalently, iff points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are on the same circle or line.

The two advanced invariants have the following reversion symmetries and shift symmetries:

$$\begin{aligned} \langle \mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{2k}\rangle &= \langle \mathbf{a}_{2k}\mathbf{a}_{2k-1}\cdots\mathbf{a}_1\rangle = \langle \mathbf{a}_{2k}\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{2k-1}\rangle, \\ [\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_{n+2l}] &= (-1)^{\frac{n(n-1)}{2}}[\mathbf{a}_{n+2l}\cdots\mathbf{a}_2\mathbf{a}_1] = (-1)^{n-1}[\mathbf{a}_{n+2l}\mathbf{a}_1\cdots\mathbf{a}_{n+2l-1}]. \end{aligned} \tag{10.31}$$

Clifford bracket algebra [3] is the algebra of advanced invariants generated by the above two kinds of brackets. The elements are naturally graded by their lengths. Formally, an nD Clifford bracket algebra generated by a sequence $\mathscr{S}$ of m symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, where $m \geq n$, is the quotient of the polynomial ring generated by (1) subsequences of length n, denoted by square brackets, (2) symmetric pairs of vectors, denoted by angular brackets, (3) repeatable permutations of vectors of length $n+2k$ for $k > 0$, denoted by square brackets, (4) another group of repeatable permutations of vectors of length $2l+2$ for $l > 0$, denoted by angular brackets, modulo the ideal generated by GP1 in (10.8), GP2 in (10.9), where the dot products are replaced by angular brackets of length two, and the following SB and AB:

- SB:

$$[\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{n+2k}] - \sum_{1\leq\sigma\leq n+2k} \text{sign}(\sigma, \check{\sigma})\langle\mathbf{i}_{\sigma(1)}\mathbf{i}_{\sigma(2)}\cdots\mathbf{i}_{\sigma(2k)}\rangle \times [\mathbf{i}_{\check{\sigma}(1)}\mathbf{i}_{\check{\sigma}(2)}\cdots\mathbf{i}_{\check{\sigma}(n)}], \tag{10.32}$$

  where $\sigma, \check{\sigma}$ run over all permutations of $1, 2, \ldots, n+2k$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(2k)$ and $\check{\sigma}(1) < \check{\sigma}(2) < \cdots < \check{\sigma}(n)$.

- AB:

$$\langle\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{2l}\rangle - \sum_{k=2}^{2l}(-1)^k\langle\mathbf{i}_1\mathbf{i}_k\rangle\,\langle\mathbf{i}_2\cdots\check{\mathbf{i}}_k\cdots\mathbf{i}_{2l}\rangle. \tag{10.33}$$

In fact, $\langle\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{2l}\rangle$ is just the Pfaffian of the $\mathbf{i}$'s, and (10.33) is the recursive relation of Pfaffians. (10.32) is the nth-grade *Caianiello expansion* [5] of the geometric product of the $\mathbf{i}$'s. The term Pfaffian was introduced by A. Cayley, who used the term in 1852 to honor of the German mathematician J. Pfaff [14].

The Clifford bracket algebra generated by points and the point at infinity $\mathbf{e}$ in conformal geometric algebra is *null bracket algebra* [7]. It is an $(n+2)$D Clifford bracket algebra with the special requirement that all symbolic vectors are null, i.e., the following *null syzygies*:

$$\langle\mathbf{ii}\rangle = 0. \tag{10.34}$$

10.3 Applications: Geometric Factorization, Decomposition, and Theorem Completion

In this section, we present an example of automated discovering of new geometric theorems to show the essential role played by null geometric algebra. The scenario is as follows [7]: for a geometric theorem, if one of its hypotheses is removed, then the conclusion is no longer true. However, the conclusion should contain the removed hypothesis as a factor. If the other factors of the conclusion are all geometrically meaningful, the factorization of the conclusion is called *geometric*.

If more than one hypothesis is removed, by Hilbert's Nullstellensatz, some power of the conclusion can be written as a linear combination of the removed hypotheses. If the coefficients in the combination are geometrically meaningful, then this decomposition is called *geometric*. However, a geometric decomposition does not provide any clear geometric interpretation to the conclusion other than the quantitative contribution of each hypothesis to the conclusion in geometrical terms. If instead the conclusion can be written in some suitable covariant algebra into the form $f=0$ where f is a monomial, then it has clear geometric interpretation, and a new theorem is created (or discovered), called the geometric *completion* of the original theorem. It generalizes an existing theorem by reducing its hypotheses.

In the following example, first the geometric factorization is carried out by removing one hypothesis, then a geometric completion is reached by removing one more hypothesis, and finally a geometric decomposition is obtained by expanding the completion. The original theorem is very easy:

Example 10.1 In the plane two circles intersect at points $\mathbf{1}, \mathbf{1}'$, respectively. Draw two secant lines through them, which intersect the two circles at points $\mathbf{2}, \mathbf{3}$ and $\mathbf{2}', \mathbf{3}'$, respectively, then $\mathbf{22}'//\mathbf{33}'$. See Fig. 10.4 (left).

The first question is this: if one constraint is absent, say $\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{2}'$ are no longer cocircular (see Fig. 10.4 (right)), then how far are lines $\mathbf{22}'$ and $\mathbf{33}'$ away from being parallel?

A beautiful formula is obtained with null bracket algebra:

$$\frac{1}{2}\,\frac{[\mathbf{e22'e33'}]}{(\mathbf{e}\cdot\mathbf{2})(\mathbf{e}\cdot\mathbf{2}')(\mathbf{3}\cdot\mathbf{3}')} = \frac{[\mathbf{e13}]}{[\mathbf{e12}]}\,\frac{[\mathbf{e131'}]}{[\mathbf{e31'2'}]}\,\frac{[\mathbf{11'22'}]}{(\mathbf{1}\cdot\mathbf{1}')(\mathbf{1}\cdot\mathbf{3})}. \tag{10.35}$$

Its geometric interpretation follows the list below:

(1) $\mathbf{e}\cdot\mathbf{2} = \mathbf{e}\cdot\mathbf{2}' = -1$;

(2) $\mathbf{3}\cdot\mathbf{3}' = \dfrac{d_{\mathbf{33}'}}{2}, \quad \mathbf{1}\cdot\mathbf{1}' = \dfrac{d_{\mathbf{11}'}}{2}, \quad \mathbf{1}\cdot\mathbf{3} = \dfrac{d_{\mathbf{13}}}{2}$;

(3) $[\mathbf{e13'1'}] = 2\,S_{\mathbf{131}'}, \quad [\mathbf{e31'2'}] = 2\,S_{\mathbf{31'2'}}$;

(4) $[\mathbf{e22'e33'}] = -2(\mathbf{23}\times\mathbf{2'3'})$;

(5) $\dfrac{[\mathbf{e13}]}{[\mathbf{e12}]} = \dfrac{d_{\mathbf{13}}}{d_{\mathbf{12}}}\varepsilon_{\mathbf{123}}$.

Here (a) $S_{\mathbf{131}'}$ denotes the signed area of triangle $\mathbf{131}'$ with respect to the orientation of the plane; (b) $\overrightarrow{\mathbf{22}'}\times\overrightarrow{\mathbf{33}'}$ denotes the signed length of the vector product of vectors $\overrightarrow{\mathbf{22}'}, \overrightarrow{\mathbf{33}'}$ with respect to the unit normal of the plane; (c) $\varepsilon_{\mathbf{123}} = 1$ if point $\mathbf{1}$ is inside line segment $\mathbf{23}$ and -1 otherwise.

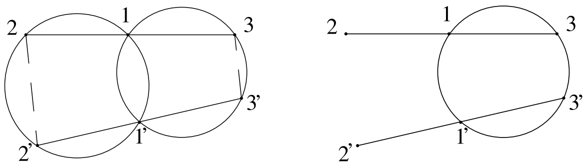

Fig. 10.4 *Left*: the original theorem; *right*: one hypothesis removed

The computing of (10.35) is executed by a general algorithm as follows:

Input (1) Geometric objects constructed sequentially, with free objects first. (2) Target $conc = [\mathbf{e22'e33'}]$, which is a Clifford algebraic expression.

The construction sequence of the configuration is as follows:
Free points $\mathbf{1}, \mathbf{2}, \mathbf{1}', \mathbf{2}'$.
Semifree point $\mathbf{3}$ on line $\mathbf{12}$.
Intersection $\mathbf{3}' = \mathbf{1}'\mathbf{e2}' \cap \mathbf{1}'\mathbf{13}$.
This means that $\mathbf{1}', \mathbf{3}'$ are the intersection of circle $\mathbf{1}'\mathbf{13}$ and line $\mathbf{1}'\mathbf{2}'$.

Output $conc/conc'$ after canceling their common factors, where $conc'$ is an expression to homogenize $conc$, and for which we choose $\mathbf{3} \cdot \mathbf{3}'$. Below we explain the term "homogenization."

In the homogeneous model, any geometric relation occurs as a homogeneous equality. Unfortunately, this is no longer true in algebraic representations of geometric entities. For example, let $\mathbf{a}$ be the intersection of lines $\mathbf{12}$ and $\mathbf{1}'\mathbf{2}'$. In Grassmann–Cayley algebra,

$$\mathbf{a} = (\mathbf{1} \wedge \mathbf{2}) \vee \left(\mathbf{1}' \wedge \mathbf{2}'\right). \tag{10.36}$$

Obviously, this is not a homogeneous relation. The five vectors can be scaled arbitrarily and independently, so the equality can only be understood as an equality up to an arbitrary scale. When we compute the quantitative relations among geometric objects, we certainly do not want a result with arbitrary scale.

There is a remedy for this. For example, in a rational expression in which the degree of point $\mathbf{a}$ in the numerator equals that in the denominator, the substitution of (10.36) into the expression does not cause any arbitrary scaling. If we compute like this:

$$\frac{[\mathbf{abc}]}{[\mathbf{ab}'\mathbf{c}']} = \frac{(\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{1}' \wedge \mathbf{2}') \vee (\mathbf{b} \wedge \mathbf{c})}{(\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{1}' \wedge \mathbf{2}') \vee (\mathbf{b}' \wedge \mathbf{c}')},$$

then we get an equality invariant under the scaling of $\mathbf{a}$.

Homogenization is to change a nonhomogeneous equality into a homogeneous one. To achieve this, we need to compute a second expression $conc'$ containing the same constrained vector variables with their degrees inclusive as those in $conc$. Then we compute $conc/conc'$.

Part 1. Elimination (1) Eliminate the last entity from *conc*. Expand and simplify the result. (2) Go to the beginning of Step 1 if *conc* contains any constrained entity.

In this example there are two constructions, the second point of intersection

$$\mathbf{3}' = \frac{1}{2}\{(\mathbf{e}\wedge\mathbf{2}')\vee_{\mathbf{1}'}(\mathbf{1}\wedge\mathbf{3})\}\mathbf{1}'\{(\mathbf{e}\wedge\mathbf{2}')\vee_{\mathbf{1}'}(\mathbf{1}\wedge\mathbf{3})\} \tag{10.37}$$

and a free point $\mathbf{3}$ on line $\mathbf{12}$; from $\mathbf{e}\wedge\mathbf{1}\wedge\mathbf{2}\wedge\mathbf{3}=0$ we get the following Cramer's rule:

$$[\mathbf{e12}]\mathbf{3} = [\mathbf{123}]\mathbf{e} - [\mathbf{e23}]\mathbf{1} + [\mathbf{e13}]\mathbf{2}, \tag{10.38}$$

where the brackets are based on the 3D space spanned by $\mathbf{e}, \mathbf{1}, \mathbf{2}, \mathbf{3}$.

The eliminations are made by substituting the expressions of the constructions into the conclusion expression and then making simplification:

$$\begin{aligned}
[\mathbf{e22'e33'}] &\overset{\text{eliminate } \mathbf{3}'}{=} \frac{1}{2}[\mathbf{e22'e3}\{(\mathbf{e}\wedge\mathbf{2}')\vee_{\mathbf{1}'}(\mathbf{1}\wedge\mathbf{3})\}\mathbf{1}'\{(\mathbf{e}\wedge\mathbf{2}')\vee_{\mathbf{1}'}(\mathbf{1}\wedge\mathbf{3})\}]\\
&\overset{\text{expand}}{=} \frac{1}{2}\underbrace{[\mathbf{e31'2'}][\mathbf{e131'}]}[\mathbf{e22'e311'2'}]\\
&\overset{\text{simplify}}{=} \underbrace{\mathbf{e}\cdot\mathbf{2}'}[\mathbf{2'2e311'}]\\
&\overset{\text{eliminate } \mathbf{3}}{=} \underbrace{\frac{[\mathbf{e13}]}{[\mathbf{e12}]}}[\mathbf{2'2e211'}]\\
&\overset{\text{simplify}}{=} 2\underbrace{(\mathbf{e}\cdot\mathbf{2})}[\mathbf{121'2'}].
\end{aligned} \tag{10.39}$$

Explanation:

(1) Step 1 substitutes the expression of $\mathbf{3}'$ into *conc*.

(2) Step 2 expands the reduced meet products in a monomial manner: since

$$\begin{aligned}
(\mathbf{e}\wedge\mathbf{2}')\vee_{\mathbf{1}'}(\mathbf{1}\wedge\mathbf{3}) &= [\mathbf{1'e2'3}]\mathbf{1} - [\mathbf{1'e2'1}]\mathbf{3}\\
&= [\mathbf{1'e13}]\mathbf{2}' - [\mathbf{1'2'13}]\mathbf{e} \mod \mathbf{1}',
\end{aligned}$$

if $\mathbf{3}'$ occurs in an expression where $\mathbf{3}'$ is neighbor to either null vector $\mathbf{1}$ or null vector $\mathbf{3}$, by selecting the first expansion one can control the conclusion expression to be 1-termed. Alternatively, if $\mathbf{3}'$ is neighbor to either $\mathbf{e}$ or $\mathbf{2}'$ in the conclusion expression, then choosing the last expansion leads to a monomial result. This is called *monomial expansion* in null Grassmann–Cayley algebra.

In the above computing, the first meet product in the expression is neighbor to $\mathbf{3}$, while the second is neighbor to $\mathbf{e}$ by shift symmetry, so up to scale, the two meet products are replaced by vectors $\mathbf{1}$ and $\mathbf{2}'$, respectively. The under braced factors in the result do not participate in further eliminations and simplifications, and are removed from succeeding computing procedure.

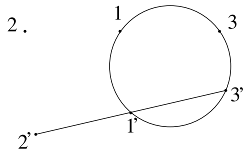

Fig. 10.5 Two hypotheses removed

(3) Step 3 is simplification after expansions and is based on the symmetries

$$\left[\mathbf{e22'e311'2'}\right] = -\left[\mathbf{2'e22'e311'}\right] = \left[\mathbf{2'e2'2e311'}\right]$$

and monomial expansion $\mathbf{2'e2'} = 2(\mathbf{e}\cdot\mathbf{2'})\mathbf{2'}$.

(4) Step 4 substitutes the expression of $\mathbf{3}$ from Cramer's rule into *conc*.

(5) The last step is based on $\mathbf{2e2} = 2(\mathbf{e}\cdot\mathbf{2})\mathbf{2}$ and antisymmetry within a bracket of length $n+2=4$.

Part 2. Homogenization Use the algorithm in Part 1 to compute $conc' = \mathbf{3}\cdot\mathbf{3'}$.

$$\begin{aligned}
\mathbf{3}\cdot\mathbf{3'} &\overset{\text{eliminate } \mathbf{3'}}{=} \frac{1}{2}\langle \mathbf{3}\{(\mathbf{e}\wedge\mathbf{2'})\vee_{\mathbf{1'}}(\mathbf{1}\wedge\mathbf{3})\}\mathbf{1'}\{(\mathbf{e}\wedge\mathbf{2'})\vee_{\mathbf{1'}}(\mathbf{1}\wedge\mathbf{3})\}\rangle \\
&\overset{\text{expand}}{=} \frac{1}{2}\underbrace{[\mathbf{e31'2'}]^2}\langle\mathbf{311'1}\rangle \\
&\overset{\text{simplify}}{=} (\mathbf{1}\cdot\mathbf{1'})(\mathbf{1}\cdot\mathbf{3}).
\end{aligned}$$

The ratio $conc/conc'$ gives the desired identity (10.35), which is an extended theorem. It provides a quantization of the dependency of the conclusion upon the cocircularity of points $\mathbf{1},\mathbf{2},\mathbf{1'},\mathbf{2'}$.

In the above computing procedure, the conclusion expression remains 1-termed. The result is naturally in factored form containing the desired factor $[\mathbf{121'2'}]$. Furthermore, (10.35) is a quantitative description of the conclusion.

Next, we remove one more hypothesis. We remove straight line $\mathbf{123}$. The new configuration, as shown in Fig. 10.5, has only two constraints: cocircularity $[\mathbf{131'3'}]=0$ and collinearity $[\mathbf{e1'2'3'}]=0$ in the homogeneous model.

The new configuration can be constructed as follows: points $\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1'},\mathbf{2'}$ are free in the plane, and points $\mathbf{1'},\mathbf{3'}$ are at the intersection of line $\mathbf{1'2'}$ and circle $\mathbf{131'}$. In (10.39), we have already obtained the result of eliminating $\mathbf{3'}$ after the third step. Again, choose $conc' = \mathbf{3}\cdot\mathbf{3'}$. We have

$$\frac{conc}{conc'} = \frac{[\mathbf{e22'e33'}]}{\mathbf{3}\cdot\mathbf{3'}} = \frac{\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{e311'2'2}]}{(\mathbf{1}\cdot\mathbf{1'})(\mathbf{1}\cdot\mathbf{3})[\mathbf{e31'2'}]}. \tag{10.40}$$

(10.40) is the *geometric completion* of the original theorem under the constraints that $\mathbf{1},\mathbf{3},\mathbf{1'},\mathbf{3'}$ are cocircular and $\mathbf{1'},\mathbf{2'},\mathbf{3'}$ are collinear. Its geometric meaning is immediate from

$$\left[\mathbf{e311'2'2}\right]=\frac{1}{2}d_{\mathbf{31}}d_{\mathbf{11'}}d_{\mathbf{1'2'}}d_{\mathbf{2'2}}\sin\left(\angle\left(\overrightarrow{\mathbf{31}},\overrightarrow{\mathbf{11'}}\right)+\angle\left(\overrightarrow{\mathbf{1'2'}},\overrightarrow{\mathbf{2'2}}\right)\right). \tag{10.41}$$

The *geometric decomposition* of $[\mathbf{e22'e33'}]$ with respect to the two removed hypotheses $[\mathbf{121'2'}]=0$ and $[\mathbf{e123}]$ is obtained by the following *rational binomial expansion* [9] of $[\mathbf{e311'2'2}]$:

$$\begin{aligned}\left[\mathbf{e311'2'2}\right]&=-\left[\mathbf{2e311'2'}\right]\\&=-\frac{1}{2(\mathbf{1}\cdot\mathbf{2})}\left[\mathbf{2e3}\underline{\mathbf{121}}\mathbf{1'2'}\right]\\&=-\frac{1}{2(\mathbf{1}\cdot\mathbf{2})}\left(\langle\mathbf{2e31}\rangle\left[\mathbf{211'2'}\right]+\left\langle\mathbf{211'2'}\right\rangle[\mathbf{2e31}]\right).\end{aligned} \tag{10.42}$$

An algebraic proof is said to be a *monomial* (or *binomial*) one if throughout the proving procedure, the expressions in process are monomials (or binomials at most). By now we have tested over 100 theorems in Euclidean geometry involving circles and angles. About two thirds are given binomial proofs, and about one third are given monomial proofs.

10.4 Conclusion

In symbolic geometric computation, the bottleneck is middle expression swell, which makes many computations possible only theoretically. Another problem is geometric explanation of algebraic results. Often this is impossible if using coordinates, especially when the results are not invariant under coordinate transforms. In classical invariant-theoretical methods the two problems remain, and new difficulties arise.

In this chapter, we introduce a new invariant framework based on Clifford algebra and the homogeneous model of classical geometry. In geometric computing, the advanced invariants introduced in this framework bring about amazing simplifications in algebraic manipulations. The proofs generated by such advanced invariants have the features that the symbolic manipulations are easy and succinct, the input and output are both geometrically meaningful, and the proofs provide *quantitative descriptions* of the relations among the conclusion and the hypotheses.

Still this is just the beginning. A variety of open problems, old and new, are waiting there for us to solve. Their solving may ultimately lead to a revolution in symbolic geometric computing, which is a revitalization of synthetic covariant approach to classical geometry.

10.5 Exercises

10.1 Using the nullification product show that $N_{\mathbf{e}}(\mathbf{a})$ for a non-null vector $\mathbf{a}$ representing a sphere is proportional to the point at the center of the sphere (remembering that $\mathbf{e}$ represents the point at infinity).

10.2 Verify the geometric interpretation of (10.30).

10.3 Verify the geometric interpretation of all factors in (10.35).

10.4 Verify the geometric interpretation of all factors in (10.40) using hint (10.42).

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