Chapter 2
Discrete Time and Sampled Data Systems

2.1 Discrete Time Systems

In this book, we investigate model predictive control for discrete time nonlinear control systems of the form

$$x^+ = f(x, u). \tag{2.1}$$

Here, the *transition map* $f : X \times U \to X$ assigns the state $x^+ \in X$ at the next time instant to each pair of state $x \in X$ and control value $u \in U$. The *state space* X and the *control value space* U are arbitrary metric spaces, i.e., sets in which we can measure distances between two elements $x, y \in X$ or $u, v \in U$ by metrics $d_X(x, y)$ or $d_U(u, v)$, respectively. Readers less familiar with metric spaces may think of $X = \mathbb{R}^d$ and $U = \mathbb{R}^m$ for $d, m \in \mathbb{N}$ with the Euclidean metrics $d_X(x, y) = \|x - y\|$ and $d_U(u, v) = \|u - v\|$ induced by the usual Euclidean norm $\|\cdot\|$, although some of our examples use different spaces. While most of the systems we consider possess continuous transition maps f, we do not require continuity in general.

The *set of finite control sequences* $u(0), \dots, u(N-1)$ for $N \in \mathbb{N}$ will be denoted by U^N and the *set of infinite control sequences* $u(0), u(1), u(2), \dots$ by U^∞. Note that we may interpret the control sequences as functions $u : \{0, \dots, N-1\} \to U$ or $u : \mathbb{N}_0 \to U$, respectively. For either type of control sequences we will briefly write $u(\cdot)$ or simply u if there is no ambiguity. With $\mathbb{N}_\infty$ we denote the natural numbers including ∞ and with $\mathbb{N}_0$ the natural numbers including 0.

A *trajectory* of (2.1) is obtained as follows: given an initial value $x_0 \in X$ and a control sequence $u(\cdot) \in U^K$ for $K \in \mathbb{N}_\infty$, we define the trajectory $x_u(k)$ iteratively via

$$x_u(0) = x_0, \qquad x_u(k+1) = f\big(x_u(k), u(k)\big), \tag{2.2}$$

for all $k \in \mathbb{N}_0$ if $K = \infty$ and for $k = 0, 1, \dots, K-1$ otherwise. Whenever we want to emphasize the dependence on the initial value we write $x_u(k, x_0)$.

An important basic property of the trajectories is the *cocycle property*: given an initial value $x_0 \in X$, a control $u \in U^N$ and time instants $k_1, k_2 \in \{0, \dots, N-1\}$ with $k_1 \le k_2$ the solution trajectory satisfies

$$x_u(k_2, x_0) = x_{u(\cdot + k_1)}\big(k_2 - k_1, x_u(k_1, x_0)\big). \tag{2.3}$$

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Here, the *shifted* control sequence $u(\cdot + k_1) \in U^{N-k_1}$ is given by

$$u(\cdot + k_1)(k) := u(k + k_1), \quad k \in \{0, \dots, N - k_1 - 1\}, \tag{2.4}$$

i.e., if the sequence u consists of the N elements $u(0), u(1), \dots, u(N-1)$, then the sequence $\tilde{u} = u(\cdot + k_1)$ consists of the $N - k_1$ elements $\tilde{u}(0) = u(k_1), \tilde{u}(1) = u(k_1 + 1), \dots, \tilde{u}(N - k_1 - 1) = u(N - 1)$. With this definition, the identity (2.3) is easily proved by induction using (2.2).

We illustrate our class of models by three simple examples—the first two being in fact linear.

Example 2.1 One of the simplest examples of a control system of type (2.1) is given by $X = U = \mathbb{R}$ and

$$x^+ = x + u =: f(x, u).$$

This system can be interpreted as a very simple model of a vehicle on an infinite straight road in which $u \in \mathbb{R}$ is the traveled distance in the period until the next time instant. For $u > 0$ the vehicle moves right and for $u < 0$ it moves left.

Example 2.2 A slightly more involved version of Example 2.1 is obtained if we consider the state $x = (x_1, x_2)^\top \in X = \mathbb{R}^2$, where x_1 represents the position and x_2 the velocity of the vehicle. With the dynamics

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix} =: f(x, u)$$

on an appropriate time scale the control $u \in U = \mathbb{R}$ can be interpreted as the (constant) acceleration in the period until the next time instant. For a formal derivation of this model from a continuous time system, see Example 2.6, below.

Example 2.3 Another variant of Example 2.1 is obtained if we consider the vehicle on a road which forms an ellipse, cf. Fig. 2.1, in which half of the ellipse is shown.

Here, the set of possible states is given by

$$X = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\| \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} \right\| = 1 \right\}.$$

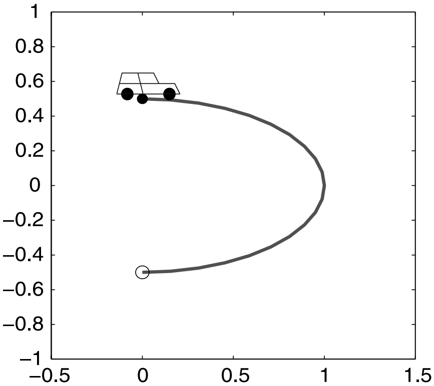

Fig. 2.1 Illustration of Example 2.3

Since X is a compact subset of $\mathbb{R}^2$ (more precisely a submanifold, but we will not need this particular geometric structure) we can use the metric induced by the Euclidean norm on $\mathbb{R}^2$, i.e., $d_X(x,y) = \|x - y\|$. Defining the dynamics

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \sin(\vartheta(x) + u) \\ \cos(\vartheta(x) + u)/2 \end{pmatrix} =: f(x,u)$$

with $u \in U = \mathbb{R}$ and

$$\vartheta(x) = \begin{cases} \arccos 2x_2, & x_1 \geq 0, \\ 2\pi - \arccos 2x_2, & x_1 < 0 \end{cases}$$

the vehicle moves on the ellipse with traveled distance $u \in U = \mathbb{R}$ in the next time step, where the traveled distance is now expressed in terms of the angle ϑ. For $u > 0$ the vehicle moves clockwise and for $u < 0$ it moves counterclockwise.

The main purpose of these very simple examples is to provide test cases which we will use in order to illustrate various effects in model predictive control. Due to their simplicity we can intuitively guess what a reasonable controller should do and often even analytically compute different optimal controllers. This enables us to compare the behavior of the NMPC controller with our intuition and other controllers. More sophisticated models will be introduced in the next section.

As outlined in the introduction, the model (2.1) will serve for generating the predictions $x_u(k, x(n))$ which we need in the optimization algorithm of our NMPC scheme, i.e., (2.1) will play the role of the model (1.1) used in the introduction. Clearly, in general we cannot expect that this mathematical model produces exact predictions for the trajectories of the real process to be controlled. Nevertheless, during Chaps. 3–7 and in Sects. 8.1–8.4 of this book we will suppose this idealized assumption. In other words, given the NMPC-feedback law $\mu : X \to U$, we assume that the resulting closed-loop system satisfies

$$x^+ = f\big(x, \mu(x)\big) \tag{2.5}$$

with f from (2.1). We will refer to (2.5) as the *nominal closed-loop system.*

There are several good reasons for using this idealized assumption: First, satisfactory behavior of the nominal NMPC closed loop is a natural necessary condition for the correctness of our controller—if we cannot ensure proper functioning in the absence of modeling errors we can hardly expect the method to work under real life conditions. Second, the assumption that the prediction is based on an exact model of the process considerably simplifies the analysis and thus allows us to derive sufficient conditions under which NMPC works in a simplified setting. Last, based on these conditions for the nominal model (2.5), we can investigate additional robustness conditions which ensure satisfactory performance also for the realistic case in which (2.5) is only an approximate model for the real closed-loop behavior. This issue will be treated in Sects. 8.5–8.9.

2.2 Sampled Data Systems

Most models of real life processes in technical and other applications are given as continuous time models, usually in form of differential equations. In order to convert these models into the discrete time form (2.1) we introduce the concept of sampling.

Let us assume that the control system under consideration is given by a finite-dimensional ordinary differential equation

$$\dot{x}(t) = f_c\big(x(t), v(t)\big) \tag{2.6}$$

with *vector field* $f_c : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, *control function* $v : \mathbb{R} \to \mathbb{R}^m$, and unknown function $x : \mathbb{R} \to \mathbb{R}^d$, where $\dot{x}$ is the usual short notation for the derivative dx/dt and $d, m \in \mathbb{N}$ are the dimensions of the state and the control vector. Here, we use the slightly unusual symbol v for the control function in order to emphasize the difference between the continuous time control function $v(\cdot)$ in (2.6) and the discrete time control sequence $u(\cdot)$ in (2.1).

Caratheodory's Theorem (see, e.g., [15, Theorem 54]) states conditions on f_c and v under which (2.6) has a unique solution. For its application we need the following assumption.

Assumption 2.4 *The vector field* $f_c : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ *is continuous and Lipschitz in its first argument in the following sense*: *for each* $r > 0$ *there exists a constant* $L > 0$ *such that the inequality*

$$\big\| f_c(x, v) - f_c(y, v)\big\| \le L\|x - y\|$$

holds for all $x, y \in \mathbb{R}^d$ *and all* $v \in \mathbb{R}^m$ *with* $\|x\| \le r$, $\|y\| \le r$ *and* $\|v\| \le r$.

Under Assumption 2.4, Caratheodory's Theorem yields that for each initial value $x_0 \in \mathbb{R}^d$, each initial time $t_0 \in \mathbb{R}$ and each locally Lebesgue integrable control function $v : \mathbb{R} \to \mathbb{R}^m$ equation (2.6) has a unique solution $x(t)$ with $x(t_0) = x_0$ defined for all times t contained in some open interval $I \subseteq \mathbb{R}$ with $t_0 \in I$. We denote this solution by $\varphi(t, t_0, x_0, v)$.

We further denote the space of locally Lebesgue integrable control functions mapping $\mathbb{R}$ into $\mathbb{R}^m$ by $L^\infty(\mathbb{R}, \mathbb{R}^m)$. For a precise definition of this space see, e.g., [15, Sect. C.1]. Readers not familiar with Lebesgue measure theory may always think of v being piecewise continuous, which is the approach taken in [7, Chap. 3]. Since the space of piecewise continuous functions is a subset of $L^\infty(\mathbb{R}, \mathbb{R}^m)$, existence and uniqueness holds for these control functions as well. Note that if we consider (2.6) only for times t from an interval $[t_0, t_1]$ then it is sufficient to specify the control function v for these times $t \in [t_0, t_1]$, i.e., it is sufficient to consider $v \in L^\infty([t_0, t_1], \mathbb{R}^m)$. Furthermore, note that two Caratheodory solutions $\varphi(t, t_0, x_0, v_1)$ and $\varphi(t, t_0, x_0, v_2)$ for $v_1, v_2 \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ coincide if v_1 and v_2 coincide for almost all $\tau \in [t_0, t]$, where *almost all* means that $v_1(\tau) \ne v_2(\tau)$ may hold for $\tau \in \mathcal{T} \subset [t_0, t]$ where $\mathcal{T}$ is a set with zero Lebesgue measure. Since, in particular, sets $\mathcal{T}$ with only finitely many values have zero Lebesgue measure, this implies that for any $v \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ the solution $\varphi(t, t_0, x_0, v)$ does not change if we change the value of $v(\tau)$ for finitely many times $\tau \in [t_0, t]$.[1]

The idea of sampling consists of defining a discrete time system (2.1) such that the trajectories of this discrete time system and the continuous time system coincide at the sampling times $t_0 < t_1 < t_2 < \cdots < t_N$, i.e.,

$$\varphi(t_n, t_0, x_0, v) = x_u(n, x_0), \quad n = 0, 1, 2, \dots, N, \tag{2.7}$$

provided the continuous time control function $v : \mathbb{R} \to \mathbb{R}^m$ and the discrete time control sequence $u(\cdot) \in U^N$ are chosen appropriately. Before we investigate how this appropriate choice can be done, cf. Theorem 2.7, below, we need to specify the discrete time system (2.1) which allows for such a choice.

Throughout this book we use equidistant sampling times $t_n = nT$, $n \in \mathbb{N}_0$, with *sampling period* $T > 0$. For this choice, we claim that

$$x^+ = f(x, u) := \varphi(T, 0, x, u) \tag{2.8}$$

for $x \in \mathbb{R}^d$ and $u \in L^\infty([0, T], \mathbb{R}^m)$ is the desired discrete time system (2.1) for which (2.7) can be satisfied. Clearly, $f(x, u)$ is only well defined if the solution $\varphi(t, 0, x, u)$ exists for the time $t = T$. Unless explicitly stated otherwise, we will tacitly assume that this is the case whenever using $f(x, u)$ from (2.8).

Before we explain the precise relation between u in (2.8) and $u(\cdot)$ and $v(\cdot)$ in (2.7), cf. Theorem 2.7, below, we first look at possible choices of u in (2.8). In general, u in (2.8) may be any function in $L^\infty([0, T], \mathbb{R}^m)$, i.e., any measurable continuous time control function defined on one sampling interval. This suggests that we should use $U = L^\infty([0, T], \mathbb{R}^m)$ in (2.1) when f is defined by (2.8). However, other—much simpler—choices of U as appropriate subsets of $L^\infty([0, T], \mathbb{R}^m)$ are often possible and reasonable. This is illustrated by the following examples and discussed after Theorem 2.7 in more detail.

Example 2.5 Consider the continuous time control system

$$\dot{x}(t) = v(t)$$

with $n = m = 1$. It is easily verified that the solutions of this system are given by

$$\varphi(t, 0, x_0, v) = x_0 + \int_0^t v(\tau)\, d\tau.$$

Hence, for $U = L^\infty([0, T], \mathbb{R})$ we obtain (2.8) as

$$x^+ = f(x, u) = x + \int_0^T u(\tau)\, d\tau.$$

[1]Strictly speaking, L^∞ functions are not even defined pointwise but rather via equivalence classes which identify all functions $v \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ which coincide for almost all $t \in \mathbb{R}$. However, in order not to overload the presentation with technicalities we prefer the slightly heuristic explanation given here.

If we restrict ourselves to constant control functions $u(t) \equiv u \in \mathbb{R}$ (for ease of notation we use the same symbol u for the function and for its constant value), which corresponds to choosing $U = \mathbb{R}$, then f simplifies to

$$f(x,u) = x + Tu.$$

If we further specify $T = 1$, then this is exactly Example 2.1.

Example 2.6 Consider the continuous time control system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ v(t) \end{pmatrix}$$

with $n = 2$ and $m = 1$. In this model, if we interpret $x_1(t)$ as the position of a vehicle at time t, then $x_2(t) = \dot{x}_1(t)$ is its velocity and $v(t) = \dot{x}_2(t)$ its acceleration.

Again, one easily computes the solutions of this system with initial value $x_0 = (x_{01}, x_{02})^\top$ as

$$\varphi(t,0,x_0,v) = \begin{pmatrix} x_{01} + \int_0^t x_2(\tau)\,d\tau \\ x_{02} + \int_0^t v(\tau)\,d\tau \end{pmatrix} = \begin{pmatrix} x_{01} + \int_0^t (x_{02} + \int_0^\tau v(s)\,ds)\,d\tau \\ x_{02} + \int_0^t v(\tau)\,d\tau \end{pmatrix}.$$

Hence, for $U = L^\infty([0,T],\mathbb{R})$ and $x = (x_1, x_2)^\top$ we obtain (2.8) as

$$x^+ = f(x,u) = \begin{pmatrix} x_1 + Tx_2 + \int_0^T \int_0^t u(s)\,ds\,dt \\ x_2 + \int_0^T u(t)\,dt \end{pmatrix}.$$

If we restrict ourselves to constant control functions $u(t) \equiv u \in \mathbb{R}$ (again using the same symbol u for the function and for its constant value), i.e., $U = \mathbb{R}$, then f simplifies to

$$f(x,u) = \begin{pmatrix} x_1 + Tx_2 + T^2u/2 \\ x_2 + Tu \end{pmatrix}.$$

If we further specify $T = 1$, then this is exactly Example 2.2.

In order to see how the control inputs $v(\cdot)$ in (2.6) and $u(\cdot)$ in (2.8) need to be related such that (2.8) ensures (2.7), we use that the continuous time trajectories satisfy the identity

$$\varphi(t,t_0,x_0,v) = \varphi\big(t-s, t_0-s, x_0, v(\cdot+s)\big) \tag{2.9}$$

for all $t, s \in \mathbb{R}$, provided, of course, the solutions exist for the respective times. Here $v(\cdot+s) : \mathbb{R} \to \mathbb{R}^m$ denotes the shifted control function, i.e., $v(\cdot+s)(t) = v(t+s)$, see also (2.4). This identity is illustrated in Fig. 2.2: changing $\varphi(t, t_0-s, x_0, v(\cdot+s))$ to $\varphi(t-s, t_0-s, x_0, v(\cdot+s))$ implies a shift of the upper graph by s to the right after which the two graphs coincide.

Identity (2.9) follows from the fact that $x(t) = \varphi(t-s, t_0-s, x_0, v(\cdot+s))$ satisfies

$$\begin{aligned} \dot{x}(t) &= \frac{d}{dt}\varphi\big(t-s, t_0-s, x_0, v(\cdot+s)\big) \\ &= f\big(\varphi\big(t-s, t_0-s, x_0, v(\cdot+s)\big), v(\cdot+s)(t-s)\big) = f\big(x(t), v(t)\big) \end{aligned}$$

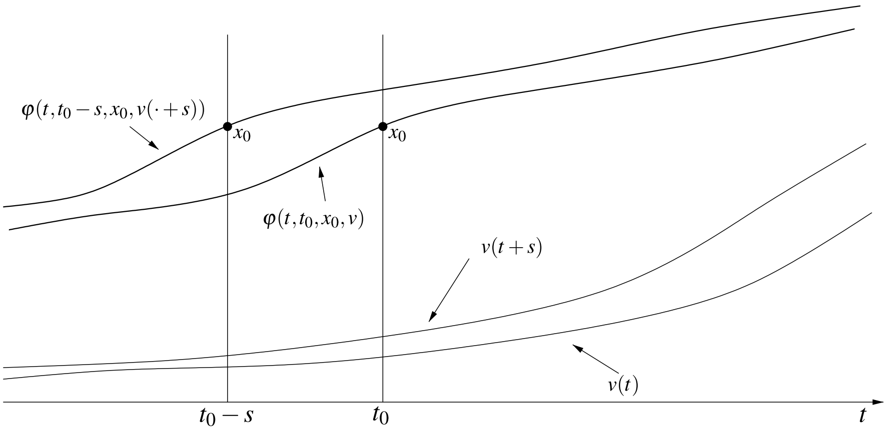

Fig. 2.2 Illustration of equality (2.9)

and

$$x(t_0) = \varphi\big(t_0 - s, t_0 - s, x_0, v(\cdot + s)\big) = x_0.$$

Hence, both functions in (2.9) satisfy (2.6) with the same control function and fulfill the same initial condition. Consequently, they coincide by uniqueness of the solution.

Using a similar uniqueness argument one sees that the solutions φ satisfy the *cocycle property*

$$\varphi(t, t_0, x_0, v) = \varphi\big(t, s, \varphi(s, t_0, x_0, v), v\big) \tag{2.10}$$

for all $t, s \in \mathbb{R}$, again provided all solutions in this equation exist for the respective times. This is the continuous time version of the discrete time cocycle property (2.3). Note that in (2.3) we have combined the discrete time counterparts of (2.9) and (2.10) into one equation since by (2.2) the discrete time trajectories always start at time 0.

With the help of (2.9) and (2.10) we can now prove the following theorem.

Theorem 2.7 *Assume that* (2.6) *satisfies Assumption* 2.4 *and let* $x_0 \in \mathbb{R}^d$ *and* $v \in L^\infty([t_0, t_N], \mathbb{R}^m)$ *be given such that* $\varphi(t_n, t_0, x_0, v)$ *exists for all sampling times* $t_n = nT$, $n = 0, \dots, N$ *with* $T > 0$. *Define the control sequence* $u(\cdot) \in U^N$ *with* $U = L^\infty([0, T], \mathbb{R}^m)$ *by*

$$u(n) = v|_{[t_n, t_{n+1}]}(\cdot + t_n), \quad n = 0, \dots, N-1, \tag{2.11}$$

where $v|_{[t_n, t_{n+1}]}$ *denotes the restriction of* v *onto the interval* $[t_n, t_{n+1}]$. *Then*

$$\varphi(t_n, t_0, x_0, v) = x_u(n, x_0) \tag{2.12}$$

holds for $n = 0, \dots, N$ *and the trajectory of the discrete time system* (2.1) *defined by* (2.8).

Conversely, given $u(\cdot) \in U^N$ *with* $U = L^\infty([0,T],\mathbb{R}^m)$, *then* (2.12) *holds for* $n = 0,\dots,N$ *for any* $v \in L^\infty([t_0,t_N],\mathbb{R}^m)$ *satisfying*

$$v(t) = u(n)(t - t_n) \quad \text{for almost all } t \in [t_n, t_{n+1}] \text{ and all } n = 0,\dots,N-1, \tag{2.13}$$

provided $\varphi(t_n, t_0, x_0, v)$ *exists for all sampling times* $t_n = nT$, $n = 0,\dots,N$.

Proof We prove the assertion by induction over n. For $n = 0$ we can use the initial conditions to get

$$x_u(t_0, u) = x_0 = \varphi(t_0, t_0, x_0, v).$$

For the induction step $n \to n+1$ assume (2.12) for t_n as induction assumption. Then by definition of x_u we get

$$\begin{aligned} x_u(n+1, x_0) &= f\big(x_u(n, x_0), u(n)\big) = \varphi\big(T, 0, x_u(n, x_0), u(n)\big) \\ &= \varphi\big(T, 0, \varphi(t_n, t_0, x_0, v), v(\cdot + t_n)\big) \\ &= \varphi\big(t_{n+1}, t_n, \varphi(t_n, t_0, x_0, v), v\big) \\ &= \varphi(t_{n+1}, t_0, x_0, v), \end{aligned}$$

where we used the induction assumption in the third equality, (2.9) in the fourth equality and (2.10) in the last equality.

The converse statement follows by observing that applying (2.11) for any v satisfying (2.13) yields a sequence of control functions $u(0),\dots,u(N-1)$ whose elements coincide with the original ones for almost all $t \in [0,T]$. □

Remark 2.8 At first glance it may seem that the condition on v in (2.13) is not well defined at the sampling times t_n: from (2.13) for $n-1$ and $t = t_n$ we obtain $v(t_n) = u(n-1)(t_n - t_{n-1})$ while (2.13) for n and $t = t_n$ yields $v(t_n) = u(n)(0)$ and, of course, the values $u(n-1)(t_n - t_{n-1})$ and $u(n)(0)$ need not coincide. However, this does not pose a problem because the set of sampling times t_n in (2.13) is finite and thus the solutions $\varphi(t, t_0, x_0, v)$ do not depend on the values $v(t_n)$, $n = 0,\dots,N-1$, cf. the discussion after Assumption 2.4. Formally, this is reflected in the words *almost all* in (2.13), which in particular imply that (2.13) is satisfied regardless of how $v(t_n)$, $n = 0,\dots,N-1$ is chosen.

Theorem 2.7 shows that we can reproduce every continuous time solution at the sampling times if we choose $U = L^\infty([0,T],\mathbb{R}^m)$. Although this is a nice property for our subsequent theoretical investigations, usually this is not a good choice for practical purposes in an NMPC context: recall from the introduction that in NMPC we want to optimize over the sequence $u(0),\dots,u(N-1) \in U^N$ in order to determine the feedback value $\mu(x(n)) = u(0) \in U$. Using $U = L^\infty([0,T],\mathbb{R}^m)$, each element of this sequence and hence also $\mu(x(n))$ is an element from a very large infinite-dimensional function space. In practice, such a general feedback concept is impossible to implement. Furthermore, although theoretically it is well possible to optimize over sequences from this space, for practical algorithms we will have

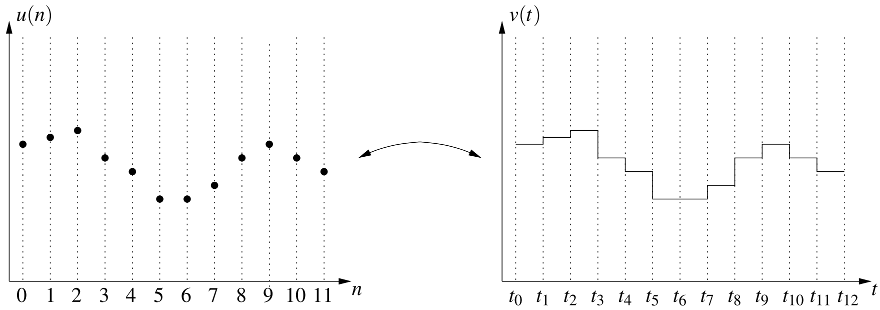

Fig. 2.3 Illustration of zero order hold: the sequence $u(n) \in \mathbb{R}^m$ on the *left* corresponds to the piecewise constant control functions with $\nu(t) = u(n)$ for almost all $t \in [t_n, t_{n+1}]$ on the *right*

to restrict ourselves to finite-dimensional sets, i.e., to subsets $U \subset L^\infty([0, T], \mathbb{R}^m)$ whose elements can be represented by finitely many parameters.

A popular way to achieve this—which is also straightforward to implement in technical applications—is via *zero order hold*, where we choose U to be the space of constant functions, which we can identify with $\mathbb{R}^m$, cf. also the Examples 2.5 and 2.6. For $u(n) \in U$, the continuous time control functions v generated by (2.13) are then piecewise constant on the sampling intervals, i.e., $v(t) = u(n)$ for almost all $t \in [t_n, t_{n+1}]$, as illustrated in Fig. 2.3. Recall from Remark 2.8 that the fact that the sampling intervals overlap at the sampling instants t_n does not pose a problem.

Consequently, the feedback $\mu(x(n))$ is a single control value from $\mathbb{R}^m$ to be used as a constant control signal on the sampling interval $[t_n, t_{n+1}]$. This is also the choice we will use in Chap. 9 on numerical methods for solving (2.6) and which is implemented in our NMPC software, cf. the Appendix. In our theoretical investigations, we will nevertheless allow for arbitrary $U \subseteq L^\infty([0, T], \mathbb{R}^m)$.

Other possible choices of U can be obtained, e.g., by polynomials $u : [0, T] \to \mathbb{R}^m$ resulting in piecewise polynomial control functions v. Yet another choice can be obtained by multirate sampling, in which we introduce a smaller sampling period $\tau = T/K$ for some $K \in \mathbb{N}$, $K \geq 2$ and choose U to be the space of functions which are constant on the intervals $[j\tau, (j+1)\tau)$, $j = 0, \dots, K-1$. In all cases the time n in the discrete time system (2.1) corresponds to the time $t_n = nT$ in the continuous time system.

Remark 2.9 The particular choice of U affects various properties of the resulting discrete time system. For instance, in Chap. 5 we will need the sets $\mathbb{X}_N$ which contain all initial values x_0 for which we can find a control sequence $u(\cdot)$ with $x_u(N, x_0) \in \mathbb{X}_0$ for some given set $\mathbb{X}_0$. Obviously, for sampling with zero order hold, i.e., for $U = \mathbb{R}^m$, this set $\mathbb{X}_N$ will be smaller than for multirate sampling or for sampling with $U = L^\infty([0, T], \mathbb{R}^m)$. For this reason, we will formulate all assumptions needed in the subsequent chapters directly in terms of the discrete time system (2.1) rather than for the continuous time system (2.6), cf. also Remark 6.7.

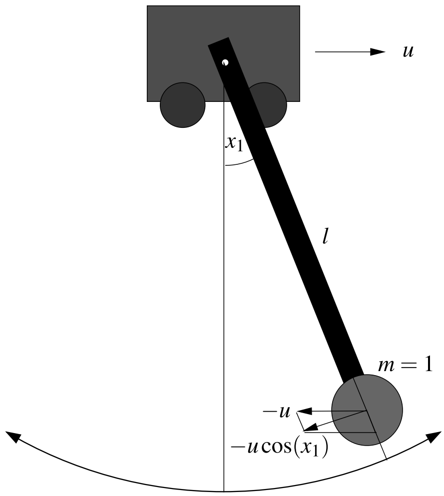

Fig. 2.4 Schematical sketch of the inverted pendulum on a cart problem: The pendulum (with unit mass $m = 1$) is attached to a cart which can be controlled using the acceleration force u. Via the joint, this force will have an effect on the dynamics of the pendulum

When using sampled data models, the map f from (2.8) is usually not available in exact analytical form but only as a numerical approximation. We will discuss this issue in detail in Chap. 9.

We end this section by three further examples we will use for illustration purposes later in this book.

Example 2.10 A standard example in control theory is the *inverted pendulum on a cart* problem shown in Fig. 2.4.

This problem has two types of equilibria, the stable downright position and the unstable upright position. A typical task is to stabilize one of the unstable upright equilibria. Normalizing the mass of the pendulum to 1, the dynamics of this system can be expressed via the system of ordinary differential equations

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\frac{g}{l}\sin\bigl(x_1(t)\bigr) - u(t)\cos\bigl(x_1(t)\bigr) - \frac{k_L}{l}x_2(t)\bigl|x_2(t)\bigr| - k_R\,\mathrm{sgn}\bigl(x_2(t)\bigr), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= u(t)
\end{aligned}$$

with gravitational force g, length of the pendulum l, air friction constant k_L and rotational friction constant k_R. Here, x_1 denotes the angle of the pendulum, x_2 the angular velocity of the pendulum, x_3 the position and x_4 the velocity of the cart. For this system the upright unstable equilibria are of the form $((2k+1)\pi, 0, 0, 0)^\top$ for $k \in \mathbb{Z}$.

Our model thus presented deviates from other variants often found in the literature, see, e.g., [2, 9], in terms of the types of friction we included. Instead of the linear friction model often considered, here we use a nonlinear air friction term $\frac{k_L}{l}x_2(t)|x_2(t)|$ and a rotational discontinuous Coulomb friction term $k_R\,\mathrm{sgn}(x_2(t))$.

The air friction term captures the fact that the force induced by the air friction grows quadratically with the speed of the pendulum mass. The Coulomb friction term is derived from first principles using Coulomb's law, see, e.g., [17] for an introduction and a description of the mathematical and numerical difficulties related to discontinuous friction terms. We consider this type of modeling as more appropriate in an NMPC context, since it describes the evolution of the dynamics more accurately, especially around the upright equilibria which we want to stabilize. For short time intervals, these nonlinear effect may be neglected, but within the NMPC design we have to predict the future development of the system for rather long periods, which may render the linear friction model inappropriate.

Unfortunately, these friction terms pose problems both theoretically and numerically:

$$\dot{x}_2(t) = -\frac{g}{l}\sin\big(x_1(t)\big) - u(t)\cos\big(x_1(t)\big) - \underbrace{\frac{k_L}{l}x_2(t)\big|x_2(t)\big|}_{\text{not } C^2} - \underbrace{k_R\,\mathrm{sgn}\big(x_2(t)\big)}_{\text{discontinuous}}.$$

The rotational Coulomb friction term is discontinuous in $x_2(t)$, hence Assumption 2.4, which is needed for Caratheodory's existence and uniqueness theorem, is not satisfied. In addition, the air friction term is only once continuously differentiable in $x_2(t)$, which poses problems when using higher order numerical methods for solving the ODE for computing the NMPC predictions, cf. the discussion before Theorem 9.5 in Chap. 9.

Hence, for the friction terms we use smooth approximations, which allow us to approximate the behavior of the original equation:

$$\dot{x}_1(t) = x_2(t), \tag{2.14}$$

$$\begin{aligned}\dot{x}_2(t) = &-\frac{g}{l}\sin\big(x_1(t)\big) - \frac{k_L}{l}\arctan\big(1000x_2(t)\big)x_2^2(t) - u(t)\cos\big(x_1(t)\big) \\ &- k_R\left(\frac{4ax_2(t)}{1+4(ax_2(t))^2} + \frac{2\arctan(bx_2(t))}{\pi}\right),\end{aligned} \tag{2.15}$$

$$\dot{x}_3(t) = x_4(t), \tag{2.16}$$

$$\dot{x}_4(t) = u(t). \tag{2.17}$$

In some examples in this book we will also use the linear variant of this system. To obtain it, a transformation of coordinates is applied which shifts one unstable equilibrium to the origin and then the system is linearized. Using a simplified set of parameters including only the gravitational constant g and a linear friction constant k, this leads to the linear control system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} u(t). \tag{2.18}$$

Example 2.11 In contrast to the inverted pendulum example where our task was to stabilize one of the upright equilibria, the control task for the arm/rotor/platform

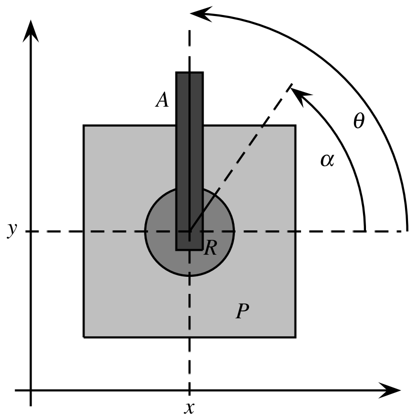

Fig. 2.5 Graphical illustration of the arm/rotor/platform (ARP) problem, see also [1, Sect. 7.3]: The arm (A) is driven by a motor (R) via a flexible joint. This motor is mounted on a platform (P) which is again flexibly connected to a fixed base (B). Moreover, we assume that there is no vertical force and that the rotational motion of the platform is not present

(ARP) model illustrated in Fig. 2.5 (the meaning of the different elements A, R, P and B in the model is indicated in the description of this figure) is a digital redesign problem, see [4, 12].

Such problems consist of two separate steps: First, a continuous time control signal $v(t)$ derived from a continuous time feedback law is designed which—in the case considered here—solves a tracking problem. Since continuous time control laws may perform poorly under sampling, in a second step, the trajectory corresponding to $v(t)$ is used as a reference function to compute a digital control using NMPC such that the resulting sampled data closed-loop mimics the behavior of the continuous time reference trajectory. Compared to a direct formulation of a tracking problem, this approach is advantageous since the resulting NMPC problem is easier to solve. Here, we describe the model and explain the derivation of continuous time control function $v(t)$. Numerical results for the corresponding NMPC controller are given in Example 7.21 in Chap. 7.

Using the Lagrange formalism and a change of coordinates detailed in [1, Sect. 7.3], the ARP model can be described by the differential equation system

$$\dot{x}_1(t) = x_2(t) + x_6(t)x_3(t), \tag{2.19}$$

$$\dot{x}_2(t) = -\frac{k_1}{M}x_1(t) - \frac{b_1}{M}x_2(t) + x_6(t)x_4(t) - \frac{mr}{M^2}b_1x_6(t), \tag{2.20}$$

$$\dot{x}_3(t) = -x_6(t)x_1(t) + x_4(t), \tag{2.21}$$

$$\dot{x}_4(t) = -x_6(t)x_2(t) - \frac{k_1}{M}x_3(t) - \frac{b_1}{M}x_4(t) + \frac{mr}{M^2}k_1, \tag{2.22}$$

$$\dot{x}_5(t) = x_6(t), \tag{2.23}$$

$$\dot{x}_6(t) = -a_1x_5(t) - a_2x_6(t) + a_1x_7(t) + a_3x_8(t) - p_1x_1(t) - p_2x_2(t), \tag{2.24}$$

$$\dot{x}_7(t) = x_8(t), \tag{2.25}$$

$$\dot{x}_8(t) = a_4x_5(t) + a_5x_6(t) - a_4x_7(t) - (a_5 + a_6)x_8(t) + \frac{1}{J}v(t) \tag{2.26}$$

where

$$a_1 = \frac{k_3 M}{MI - (mr)^2}, \qquad a_4 = \frac{k_3}{J}, \qquad p_1 = \frac{mr}{MI - (mr)^2} k_1,$$
$$a_2 = \frac{b_3 M^2 - b_1 (mr)^2}{M[MI - (mr)^2]}, \qquad a_5 = \frac{b_3}{J}, \qquad p_2 = \frac{mr}{MI - (mr)^2} b_1.$$
$$a_3 = \frac{b_3 M}{MI - (mr)^2}, \qquad a_6 = \frac{b_4}{J},$$

Here, M represents the total mass of arm, rotor and platform and m is the mass of arm, r denotes the distance from the A/R joint to the arm center of mass and I, J and D are the moment of inertia of the arm about the A/R joint, of the rotor and of the platform, respectively. Moreover, k_1, k_2 and k_3 denote the translational spring constant of the P/B connection as well as the rotational spring constants of the P/B connection and the A/R joint. Last, b_1, b_2, b_3 and b_4 describe the translational friction coefficient of P/B connection as well as the rotational friction coefficients of the P/B, A/R and R/P connection, respectively. The coordinates x_1 and x_2 correspond to the (transformed) x position of P and its velocity of the platform in direction x whereas x_3 and x_4 represent the (transformed) y position of P and the respective velocity. The remaining coordinates x_5 and x_7 denote the angles θ and α and the coordinates x_6 and x_8 the corresponding angular velocities.

Our design goal is to regulate the system such that the position of the arm relative to the platform, i.e. the angle x_5, tracks a given reference signal. Note that this task is not simple since both connections of the rotor are flexible. Here, we assume that the reference signal and its derivatives are known and available to the controller. Moreover, we assume that the relative positions and velocities x_5, x_6, x_7 and x_8 are supplied to the controller.

In order to derive the continuous time feedback, we follow the backstepping approach from [1] using the output

$$\zeta(t) = x_5(t) - \frac{a_3}{a_1 - a_2 a_3}\big[x_6(t) - a_3 x_7(t)\big]. \tag{2.27}$$

The output has relative degree 4, that is, the control $v(t)$ appears explicitly within the fourth derivative of $\zeta(t)$. Expressing $\zeta^{(4)}(t)$ by the known data, we obtain the continuous time input signal[2]

$$v(t) = \frac{J}{a_1^2 + a_3[p] \cdot [[\frac{\partial F(x_6(t))}{\partial x_6(t)}] \cdot [\eta(t)] + [\frac{\partial G(x_6(t))}{\partial x_6(t)}]]}$$
$$\Big(-\big(-a_1 x_5(t) - a_2 x_6(t) + a_1 x_7(t) + a_3 x_8(t) - [p] \cdot \big[\eta(t)\big]\big)$$
$$\Big(-a_1^2 + a_1 a_2 (a_2 - a_3) + \big(a_3 [p] \cdot \big[F\big(x_6(t)\big) - (a_1 + a_2 a_3)[p]\big]\big)$$
$$\bigg[\bigg[\frac{\partial F(x_6(t))}{\partial x_6(t)}\bigg] \cdot \big[\eta(t)\big] + \bigg[\frac{\partial G(x_6(t))}{\partial x_6(t)}\bigg]\bigg]$$

[2]For details of the derivation see [13, Sect. 7.3].

$$
\begin{aligned}
&+ 2a_3[p]\left[\frac{\partial F(x_6(t))}{\partial x_6(t)}\right]\cdot\big[\big[F\big(x_6(t)\big)\big]\cdot\big[\eta(t)\big]+\big[G\big(x_6(t)\big)\big]\big]\Big)\\
&-\big(a_4x_5(t)+a_5x_6(t)-a_4x_7(t)-(a_5+a_6)x_8(t)\big)\\
&\bigg(a_1^2+a_3\bigg(a_3[p]\cdot\bigg[\bigg[\frac{\partial F(x_6(t))}{\partial x_6(t)}\bigg]\cdot\big[\eta(t)\big]\\
&+\bigg[\frac{\partial G(x_6(t))}{\partial x_6(t)}\bigg]\bigg]-a_1(a_2-a_3)\bigg)\bigg)\\
&-\big(a_3[p]\cdot\big[F\big(x_6(t)\big)\big]-a_1[p]\big)\cdot\big[F\big(x_6(t)\big)\big]\cdot\big[\big[F\big(x_6(t)\big)\big]\cdot\big[\eta(t)\big]\\
&+\big[G\big(x_6(t)\big)\big]\big]\\
&-\big(-a_1\big(x_6(t)-x_8(t)\big)-[p]\cdot\big[\big[F\big(x_6(t)\big)\big]\cdot\big[\eta(t)\big]+\big[G\big(x_6(t)\big)\big]\big]\big)\\
&\bigg(-a_1(a_2-a_3)+a_3[p]\cdot\bigg[\bigg[\frac{\partial F(x_6(t))}{\partial x_6(t)}\bigg]\cdot\big[\eta(t)\big]+\bigg[\frac{\partial G(x_6(t))}{\partial x_6(t)}\bigg]\bigg]\bigg)\\
&+(a_1-a_2a_3)\hat{v}(t)\bigg)
\end{aligned}
\tag{2.28}
$$

where we used the abbreviations

$$
\begin{aligned}
\big[\eta(t)\big] &:= \big(x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\big)^T,\\
\big[\chi(t)\big] &:= \big(x_5(t)\ x_6(t)\ x_7(t)\ x_8(t)\big)^T,\\
\big[F\big(x_6(t)\big)\big] &:= \begin{pmatrix} 0 & 1 & x_6(t) & 0\\ -\frac{k_1}{M} & -\frac{b_1}{M} & 0 & x_6(t)\\ -x_6(t) & 0 & 0 & 1\\ 0 & -x_6(t) & -\frac{k_1}{M} & -\frac{b_1}{M}\end{pmatrix},\\
\big[G\big(x_6(t)\big)\big] &:= \begin{pmatrix} 0\\ -\frac{mrb_1}{M^2}x_6(t)\\ 0\\ \frac{mrk_1}{M^2}\end{pmatrix},\\
[A] &:= \begin{pmatrix} 0 & 1 & 0 & 0\\ -a_1 & -a_2 & a_1 & a_3\\ 0 & 0 & 0 & 1\\ a_4 & a_5 & -a_4 & -(a_5+a_6)\end{pmatrix},\\
[E] &:= \begin{pmatrix} 0 & 0\\ -p_1 & -p_2\\ 0 & 0\\ 0 & 0\end{pmatrix},\qquad [B] := \begin{pmatrix} 0\\ 0\\ 0\\ \frac{1}{J}\end{pmatrix}
\end{aligned}
$$

as well as the row vector $[p] := (p_1\ p_2\ 0\ 0)$. In (2.28), we added the function $\hat{v}(t)$, which we will now use as the new input. Given a desired reference $\zeta_{\text{ref}}(\cdot)$ for the output (2.27), we can track this reference by setting $\hat{v}$ in (2.28) as

$$
\begin{aligned}
\hat{v}(t) := \zeta_{\text{ref}}^{(4)}(t) &- c_3\big(\zeta^{(3)}(t)-\zeta_{\text{ref}}^{(3)}(t)\big)-c_2\big(\ddot{\zeta}(t)-\ddot{\zeta}_{\text{ref}}(t)\big)\\
&- c_1\big(\dot{\zeta}(t)-\dot{\zeta}_{\text{ref}}(t)\big)-c_0\big(\zeta(t)-\zeta_{\text{ref}}(t)\big)
\end{aligned}
$$

with design parameters $c_i \in \mathbb{R}$, $c_i \geq 0$. These parameters are degrees of freedom within the design of the continuous time feedback which can be used as tuning parameters, e.g., to reduce the transient time or the overshoot.

Example 2.12 Another class of systems fitting our framework, which actually goes beyond the setting we used for introducing sampled data systems, are infinite-dimensional systems induced by partial differential equations (PDEs). In this example, we slightly change our notation in order to be consistent with the usual PDE notation.

In the following controlled parabolic PDE (2.29) the solution $y(t,x)$ with $y : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ depends on time t as well as on a one-dimensional state variable $x \in \Omega = (0, L)$ for a parameter $L > 0$. Thus, the state of the system at each time t is now a continuous function $y(t,\cdot) : \overline{\Omega} \to \mathbb{R}$ and x becomes an independent variable. The control v in this example is a so-called distributed control, i.e., a measurable function $v : \mathbb{R} \times \Omega \to \mathbb{R}$. The evolution of the state is defined by the equation

$$y_t(t,x) = \theta y_{xx}(t,x) - y_x(t,x) + \rho\big(y(t,x) - y(t,x)^3\big) + v(t,x) \tag{2.29}$$

for $x \in \Omega$ and $t \geq 0$ together with the initial condition $y(0,x) = y_0(x)$ and the boundary conditions $y(t,0) = y(0,L) = 0$.

Here y_t and y_x denote the partial derivatives with respect to t and x, respectively and y_{xx} denotes the second partial derivative with respect to x. The parameters θ and ρ are positive constants. Of course, in order to ensure that (2.29) is well defined, we need to interpret this equation in an appropriate weak sense and make sure that for the chosen class of control functions a solution to (2.29) exists in appropriate function spaces. For details on these issues we refer to, e.g., [10] or [18]. As we will see later in Example 6.27, for suitable values of the parameters θ and ρ the uncontrolled equation, i.e., (2.29) with $v \equiv 0$, has an unstable equilibrium $y_* \equiv 0$ which can be stabilized by NMPC.

Using the letter z for the state of the discrete time system associated to the sampled data solution of (2.29), we can abstractly write this system as

$$z^+ = f(z,u)$$

with z and z^+ being continuous functions from $\overline{\Omega}$ to $\mathbb{R}$. The function f maps $y_0 = z$ to the solution $y(T,x)$ of (2.29) at the sampling time T using the measurable control function $u = v : [0,T] \times \Omega \to \mathbb{R}$. Thus, it maps continuous functions to continuous functions; again we omit the exact details of the respective functions spaces.

As in the ordinary differential equation case, we can restrict ourselves to the zero order hold situation, i.e., to control functions $u(t,x)$ which are constant in $t \in [0,T]$. The corresponding control functions v generated via (2.11) are again constant in t on each sampling interval $[t_n, t_{n+1})$. Note, however, that in our distributed control context both u and v are still arbitrary measurable—i.e., in particular non-constant—functions in x.

For sampled data systems, the nominal closed-loop system (2.5) corresponds to the closed-loop sampled data system

$$\dot{x}(t) = f_c\big(x(t), \mu\big(x(t_n)\big)(t - t_n)\big), \quad t \in [t_n, t_{n+1}),\ n = 0, 1, 2, \ldots \tag{2.30}$$

whose solution with initial value $x_0 \in X$ we denote by $\varphi(t, t_0, x_0, \mu)$. Note that the argument "$(t - t_n)$" of $\mu(x(t_n))$ can be dropped in case of sampling with zero order hold when—as usual—we interpret the control value $\mu(x(t_n)) \in U = \mathbb{R}^m$ as a constant control function.

2.3 Stability of Discrete Time Systems

In the introduction, we already specified the main goal of model predictive control, namely to control the state $x(n)$ of the system toward a reference trajectory $x^{\mathrm{ref}}(n)$ and then keep it close to this reference. In this section we formalize what we mean by "toward" and "close to" using concepts from stability theory of nonlinear systems.

We first consider the case where x^{ref} is constant, i.e., where $x^{\mathrm{ref}} \equiv x_*$ holds for some $x_* \in X$. We assume that the states $x(n)$ are generated by a difference equation of the form

$$x^+ = g(x) \tag{2.31}$$

for a not necessarily continuous map $g : X \to X$ via the usual iteration $x(n+1) = g(x(n))$. As before, we write $x(n, x_0)$ for the trajectory satisfying the initial condition $x(0, x_0) = x_0 \in X$. Allowing g to be discontinuous is important for our NMPC application, because g will later represent the nominal closed-loop system (2.5) controlled by the NMPC-feedback law μ, i.e., $g(x) = f(x, \mu(x))$. Since μ is obtained as an outcome of an optimization algorithm, in general we cannot expect μ to be continuous and thus g will in general be discontinuous, too.

Nonlinear stability properties can be expressed conveniently via so-called comparison functions, which were first introduced by Hahn in 1967 [5] and popularized in nonlinear control theory during the 1990s by Sontag, particularly in the context of input-to-state stability [14]. Although we mainly deal with discrete time systems, we stick to the usual continuous time definition of these functions using the notation $\mathbb{R}_0^+ = [0, \infty)$.

Definition 2.13 We define the following classes of comparison functions:

$$\begin{aligned}
\mathcal{K} &:= \left\{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \alpha \text{ is continuous \& strictly increasing with } \alpha(0) = 0\right\},\\
\mathcal{K}_\infty &:= \left\{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \alpha \in \mathcal{K},\ \alpha \text{ is unbounded}\right\},\\
\mathcal{L} &:= \left\{\delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \delta \text{ is continuous \& strictly decreasing with } \lim_{t\to\infty} \delta(t) = 0\right\},\\
\mathcal{KL} &:= \left\{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L}\right\}.
\end{aligned}$$

The graph of a typical function $\beta \in \mathcal{KL}$ is shown in Fig. 2.6.

Using this function, we can now introduce the concept of asymptotic stability. Here, for arbitrary $x_1, x_2 \in X$ we denote the distance from x_1 to x_2 by

$$|x_1|_{x_2} := d_X(x_1, x_2).$$

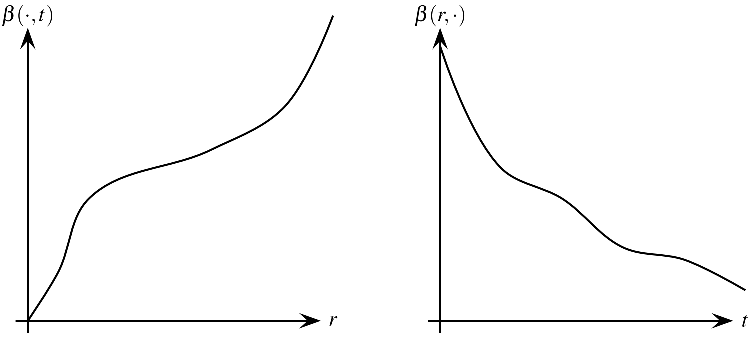

Fig. 2.6 Illustration of a typical class $\mathcal{KL}$ function

Furthermore, we use the ball

$$\mathcal{B}_\eta(x_*) := \left\{x \in X \,\middle|\, |x|_{x_*} < \eta\right\}$$

and we say that a set $Y \subseteq X$ is *forward invariant* for (2.31) if $g(x) \in Y$ holds for all $x \in Y$.

Definition 2.14 Let $x_* \in X$ be an equilibrium for (2.31), i.e., $g(x_*) = x_*$. Then we say that x_* is *locally asymptotically stable* if there exist $\eta > 0$ and a function $\beta \in \mathcal{KL}$ such that the inequality

$$\left|x(n, x_0)\right|_{x_*} \le \beta\big(|x_0|_{x_*}, n\big) \tag{2.32}$$

holds for all $x_0 \in \mathcal{B}_\eta(x_*)$ and all $n \in \mathbb{N}_0$.

We say that x_* is *asymptotically stable on a forward invariant set Y* with $x_* \in Y$ if there exists $\beta \in \mathcal{KL}$ such that (2.32) holds for all $x_0 \in Y$ and all $n \in \mathbb{N}_0$ and we say that x_* is *globally asymptotically stable* if x_* is asymptotically stable on $Y = X$.

If one of these properties holds then β is called *attraction rate*.

Note that asymptotic stability on a forward invariant set Y implies local asymptotic stability if Y contains a ball $\mathcal{B}_\eta(x_*)$. However, we do not necessarily require this property.

Asymptotic stability thus defined consists of two main ingredients.

(i) The smaller the initial distance from x_0 to x_* is, the smaller the distance from $x(n)$ to x_* becomes for all future n, or formally: for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(n, x_0)|_{x_*} \le \varepsilon$ holds for all $n \in \mathbb{N}_0$ and all $x_0 \in Y$ (or $x_0 \in \mathcal{B}_\eta(x_*)$) with $|x_0|_{x_*} \le \delta$.

This fact is easily seen by choosing δ so small that $\beta(\delta, 0) \le \varepsilon$ holds, which is possible since $\beta(\cdot, 0) \in \mathcal{K}$. Since β is decreasing in its second argument, for $|x_0|_{x_*} \le \delta$ from (2.32) we obtain

$$\left|x(n, x_0)\right|_{x_*} \le \beta\big(|x_0|_{x_*}, n\big) \le \beta\big(|x_0|_{x_*}, 0\big) \le \beta(\delta, 0) \le \varepsilon.$$

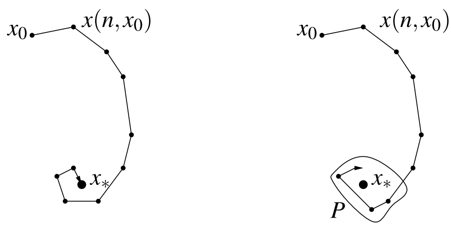

Fig. 2.7 Sketch of asymptotic stability (*left*) as opposed to practical asymptotic stability (*right*)

(ii) As the system evolves, the distance from $x(n, x_0)$ to x_* becomes arbitrarily small, or formally: for each $\varepsilon > 0$ and each $R > 0$ there exists $N > 0$ such that $|x(n, x_0)|_{x_*} \le \varepsilon$ holds for all $n \ge N$ and all $x_0 \in Y$ (or $x_0 \in \mathcal{B}_\eta(x_*)$) with $|x_0|_{x_*} \le R$. This property easily follows from (2.32) by choosing $N > 0$ with $\beta(R, N) \le \varepsilon$ and exploiting the monotonicity properties of β.

These two properties are known as (i) stability (in the sense of Lyapunov) and (ii) attraction. In the literature, asymptotic stability is often defined via these two properties. In fact, for continuous time (and continuous) systems (i) and (ii) are known to be equivalent to the continuous time counterpart of Definition 2.14, cf. [8, Sect. 3]. We conjecture that the arguments in this reference can be modified in order to prove that equivalence also holds for our discontinuous discrete time setting.

Asymptotic stability includes the desired properties of the NMPC closed loop described earlier: whenever we are already close to the reference equilibrium we want to stay close; otherwise we want to move toward the equilibrium.

Asymptotic stability also includes that eventually the distance of the closed-loop solution to the equilibrium x_* becomes arbitrarily small. Occasionally, this may be too demanding. In the following chapters, this is for instance the case if the system is subject to perturbations or modeling errors, cf. Sects. 8.5–8.9 or if in NMPC without stabilizing terminal constraints the system cannot be controlled to x_* sufficiently fast, cf. Sect. 6.7. In this case, one can relax the asymptotic stability definition to practical asymptotic stability as follows. Here we only consider the case of asymptotic stability on a forward invariant set Y.

Definition 2.15 Let Y be a forward invariant set and let $P \subset Y$ be a subset of Y. Then we say that a point $x_* \in P$ is *P-practically asymptotically stable on Y* if there exists $\beta \in \mathcal{KL}$ such that (2.32) holds for all $x_0 \in Y$ and all $n \in \mathbb{N}_0$ with $x(n, x_0) \notin P$.

Figure 2.7 illustrates practical asymptotic stability (on the right) as opposed to "usual" asymptotic stability (on the left).

This definition is typically used with P contained in a small ball around the equilibrium, i.e., $P \subseteq \mathcal{B}_\delta(x_*)$ for some small $\delta > 0$. In this case one obtains the estimate

$$\left|x(n, x_0)\right|_{x_*} \le \max\left\{\beta\left(|x_0|_{x_*}, n\right), \delta\right\} \tag{2.33}$$

for all $x_0 \in Y$ and all $n \in \mathbb{N}_0$, i.e., the system behaves like an asymptotically stable system until it reaches the ball $\mathcal{B}_\delta(x_*)$. Note that x_* does not need to be an equilibrium in Definition 2.15.

For general non-constant reference functions $x^{\text{ref}} : \mathbb{N}_0 \to X$ we can easily extend Definition 2.14 if we take into account that the objects under consideration become time varying in two ways: (i) the distance under consideration varies with n and (ii) the system (2.31) under consideration varies with n. While (i) is immediate, (ii) follows from the fact that with time varying reference also the feedback law μ is time varying, i.e., we obtain a feedback law of the type $\mu(n, x(n))$. Consequently, we now need to consider systems

$$x^+ = g(n, x) \tag{2.34}$$

with g of the form $g(n, x) = f(x, \mu(n, x))$. Furthermore, we now have to take the initial time n_0 into account: while the solutions of (2.31) look the same for all initial times n_0 (which is why we only considered $n_0 = 0$) now we need to keep track of this value. To this end, by $x(n, n_0, x_0)$ we denote the solution of (2.34) with initial condition $x(n_0, n_0, x_0) = x_0$ at time n_0. The appropriate modification of Definition 2.14 then looks as follows. Here we say that a time-dependent family of sets $Y(n) \subseteq X$, $n \in \mathbb{N}_0$ is *forward invariant* if $g(n, x) \in Y(n+1)$ holds for all $n \in \mathbb{N}_0$ and all $x \in Y(n)$.

Definition 2.16 Let $x^{\text{ref}} : \mathbb{N}_0 \to X$ be a trajectory for (2.31), i.e., $x^{\text{ref}}(n+1) = g(x^{\text{ref}}(n))$ for all $n \in \mathbb{N}_0$. Then we say that x^{ref} is *locally uniformly asymptotically stable* if there exists $\eta > 0$ and a function $\beta \in \mathcal{KL}$ such that the inequality

$$\bigl|x(n, n_0, x_0)\bigr|_{x^{\text{ref}}(n)} \le \beta\bigl(|x_0|_{x^{\text{ref}}(n_0)}, n - n_0\bigr) \tag{2.35}$$

holds for all $x_0 \in \mathcal{B}_\eta(x^{\text{ref}}(n_0))$ and all $n_0, n \in \mathbb{N}_0$ with $n \ge n_0$.

We say that x_* is *uniformly asymptotically stable on a forward invariant family of sets* $Y(n)$ with $x^{\text{ref}}(n) \in Y(n)$ if there exists $\beta \in \mathcal{KL}$ such that (2.35) holds for all $n_0, n \in \mathbb{N}_0$ with $n \ge n_0$ and all $x_0 \in Y(n_0)$ and we say that x_* is *globally uniformly asymptotically stable* if x_* is asymptotically stable on $Y(n) = X$ for all $n_0 \in \mathbb{N}_0$.

If one of these properties hold then β is called (*uniform*) *attraction rate*.

The term "uniform" describes the fact that the bound $\beta(|x_0|_{x^{\text{ref}}(n_0)}, n - n_0)$ only depends on the elapsed time $n - n_0$ but not on the initial time n_0. If this were the case, i.e., if we needed different β for different initial times n_0, then we would call the asymptotic stability "nonuniform". For a comprehensive discussion of nonuniform stability notions and their representation via time-dependent $\mathcal{KL}$ functions we refer to [3].

As in the time-invariant case, asymptotic stability on a forward invariant family of sets $Y(n)$ implies local asymptotic stability if each $Y(n)$ contains a ball $\mathcal{B}_\eta(x^{\text{ref}}(n))$. Again, we do not necessarily require this property.

The time varying counterpart of P-practical asymptotic stability is defined as follows.

Definition 2.17 Let $Y(n)$ be a forward invariant family of sets and let $P(n) \subset Y(n)$ be subsets of $Y(n)$. Then we say that a reference trajectory x^{ref} with $x^{\text{ref}}(n) \in P(n)$ is P-practically uniformly asymptotically stable on $Y(n)$ if there exists $\beta \in \mathcal{KL}$ such that (2.35) holds for all $x_0 \in Y(n_0)$ and all $n_0, n \in \mathbb{N}_0$ with $n \geq n_0$ and $x(n, n_0, x_0) \notin P(n)$.

Analogous to the time-invariant case, this definition is typically used with $P(n) \subseteq \mathcal{B}_\delta(x^{\mathrm{ref}}(n))$ for some small value $\delta > 0$, which then yields

$$\left|x(n, n_0, x_0)\right|_{x^{\mathrm{ref}}(n)} \leq \max\left\{\beta\left(|x_0|_{x^{\mathrm{ref}}(n_0)}, n - n_0\right), \delta\right\}. \tag{2.36}$$

In order to verify that our NMPC controller achieves asymptotic stability we will utilize the concept of Lyapunov functions. For constant reference $x^{\mathrm{ref}} \equiv x_* \in X$ these functions are defined as follows.

Definition 2.18 Consider a system (2.31), a point $x_* \in X$ and let $S \subseteq X$ be a subset of the state space. A function $V : S \to \mathbb{R}_0^+$ is called a *Lyapunov function* on S if the following conditions are satisfied:

(i) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1\left(|x|_{x_*}\right) \leq V(x) \leq \alpha_2\left(|x|_{x_*}\right) \tag{2.37}$$

holds for all $x \in S$.

(ii) There exists a function $\alpha_V \in \mathcal{K}$ such that

$$V\left(g(x)\right) \leq V(x) - \alpha_V\left(|x|_{x_*}\right) \tag{2.38}$$

holds for all $x \in S$ with $g(x) \in S$.

The following theorem shows that the existence of a Lyapunov function ensures asymptotic stability.

Theorem 2.19 *Let x_* be an equilibrium of* (2.31) *and assume there exists a Lyapunov function V on S. If S contains a ball $\mathcal{B}_\nu(x_*)$ with $g(x) \in S$ for all $x \in \mathcal{B}_\nu(x_*)$ then x_* is locally asymptotically stable with $\eta = \alpha_2^{-1} \circ \alpha_1(\nu)$. If $S = Y$ holds for some forward invariant set $Y \subseteq X$ containing x_* then x_* is asymptotically stable on Y. If $S = X$ holds then x_* is globally asymptotically stable.*

Proof The idea of the proof lies in showing that by (2.38) the function $V(x(n, x_0))$ is strictly decreasing in n and converges to 0. Then by (2.37) we can conclude that $x(n, x_0)$ converges to x_*. The function β from Definition 2.14 will be constructed from α_1, α_2 and α_V. In order to simplify the notation, throughout the proof we write $|x|$ instead of $|x|_{x_*}$.

First, if S is not forward invariant, define the value $\gamma := \alpha_1(\nu)$ and the set $\widetilde{S} := \{x \in X \mid V(x) < \gamma\}$. Then from (2.37) we get

$$x \in \widetilde{S} \quad \Rightarrow \quad \alpha_1\left(|x|\right) \leq V(x) < \gamma \quad \Rightarrow \quad |x| < \alpha_1^{-1}(\gamma) = \nu \quad \Rightarrow \quad x \in \mathcal{B}_\nu(x_*),$$

observing that each $\alpha \in \mathcal{K}_\infty$ is invertible with $\alpha^{-1} \in \mathcal{K}_\infty$.

Hence, for each $x \in \widetilde{S}$ Inequality (2.38) applies and consequently $V(g(x)) \leq V(x) < \gamma$ implying $g(x) \in \widetilde{S}$. If $S = Y$ for some forward invariant set $Y \subseteq X$ we define $\widetilde{S} := S$. With these definitions, in both cases the set $\widetilde{S}$ becomes forward invariant.

Now we define $\alpha'_V := \alpha_V \circ \alpha_2^{-1}$. Note that concatenations of $\mathcal{K}$-functions are again in $\mathcal{K}$, hence $\alpha'_V \in \mathcal{K}$. Since $|x| \geq \alpha_2^{-1}(V(x))$, using monotonicity of α_V this definition implies

$$\alpha_V\big(|x|\big) \geq \alpha_V \circ \alpha_2^{-1}\big(V(x)\big) = \alpha'_V\big(V(x)\big).$$

Hence, along a trajectory $x(n, x_0)$ with $x_0 \in \widetilde{S}$, from (2.38) we get the inequality

$$\begin{aligned} V\big(x(n+1, x_0)\big) &\leq V\big(x(n, x_0)\big) - \alpha_V\big(\big|x(n, x_0)\big|\big) \\ &\leq V\big(x(n, x_0)\big) - \alpha'_V\big(V\big(x(n, x_0)\big)\big). \end{aligned} \tag{2.39}$$

For the construction of β we need the last expression in (2.39) to be strictly increasing in $V(x(n, x_0))$. To this end we define

$$\tilde{\alpha}_V(r) := \min_{s\in[0,r]} \big\{\alpha'_V(s) + (r-s)/2\big\}.$$

Straightforward computations show that this function satisfies $r_2 - \tilde{\alpha}_V(r_2) > r_1 - \tilde{\alpha}_V(r_1) \geq 0$ for all $r_2 > r_1 \geq 0$ and $\min\{\alpha'_V(r/2), r/4\} \leq \tilde{\alpha}_V(r) \leq \alpha'_V(r)$ for all $r \geq 0$. In particular, (2.39) remains valid and we get the desired monotonicity when α'_V is replaced by $\tilde{\alpha}_V$.

We inductively define a function $\beta_1 : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}_0^+$ via

$$\beta_1(r, 0) := r, \qquad \beta_1(r, n+1) = \beta_1(r, n) - \tilde{\alpha}_V\big(\beta_1(r, n)\big). \tag{2.40}$$

By induction over n using the properties of $\tilde{\alpha}_V(r)$ and Inequality (2.39) one easily verifies the following inequalities:

$$\beta_1(r_2, n) > \beta_1(r_1, n) \geq 0 \quad \text{for all } r_2 > r_1 \geq 0 \text{ and all } n \in \mathbb{N}_0, \tag{2.41}$$

$$\beta_1(r, n_1) > \beta_1(r, n_2) > 0 \quad \text{for all } n_2 > n_1 \geq 0 \text{ and all } r > 0, \tag{2.42}$$

$$V\big(x(n, x_0)\big) \leq \beta_1\big(V(x_0), n\big) \quad \text{for all } n \in \mathbb{N}_0 \text{ and all } x_0 \in \widetilde{S}. \tag{2.43}$$

From (2.42) it follows that $\beta_1(r, n)$ is monotone decreasing in n and by (2.41) it is bounded from below by 0. Hence, for each $r \geq 0$ the limit $\beta_1^\infty(r) = \lim_{n\to\infty} \beta_1(r, n)$ exists. We claim that $\beta_1^\infty(r) = 0$ holds for all r. Indeed, convergence implies $\beta_1(r, n) - \beta_1(r, n+1) \to 0$ as $n \to \infty$, which together with (2.40) yields $\tilde{\alpha}_V(\beta_1(r, n)) \to 0$. On the other hand, since $\tilde{\alpha}_V$ is continuous, we get $\tilde{\alpha}_V(\beta_1(r, n)) \to \tilde{\alpha}_V(\beta_1^\infty(r))$. This implies

$$\tilde{\alpha}_V\big(\beta_1^\infty(r)\big) = 0,$$

which, because of $\tilde{\alpha}_V(r) \geq \min\{\alpha'_V(r/2), r/4\}$ and $\alpha'_V \in \mathcal{K}$, is only possible if $\beta_1^\infty(r) = 0$.

Consequently, $\beta_1(r, n)$ has all properties of a $\mathcal{KL}$ function except that it is only defined for $n \in \mathbb{N}_0$. Defining the linear interpolation

$$\beta_2(r, t) := (n + 1 - t)\beta_1(r, n) + (t - n)\beta_1(r, n + 1)$$

for $t \in [n, n+1)$ and $n \in \mathbb{N}_0$, we obtain a function $\beta_2 \in \mathcal{KL}$ which coincides with β_1 for $t = n \in \mathbb{N}_0$. Finally, setting

$$\beta(r,t) := \alpha_1^{-1} \circ \beta_2\big(\alpha_2(r), t\big)$$

we can use (2.43) in order to obtain

$$\begin{aligned}\big|x(n,x_0)\big| &\le \alpha_1^{-1}\big(V\big(x(n,x_0)\big)\big) \le \alpha_1^{-1} \circ \beta_1\big(V(x_0), n\big) \\ &= \alpha_1^{-1} \circ \beta_2\big(V(x_0), n\big) \le \alpha_1^{-1} \circ \beta_2\big(\alpha_2\big(|x_0|\big), n\big) = \beta\big(|x_0|, n\big),\end{aligned}$$

for all $x_0 \in \widetilde{S}$ and all $n \in \mathbb{N}_0$. This is the desired Inequality (2.32). If $\widetilde{S} = S = Y$ this shows the claimed asymptotic stability on Y and global asymptotic stability if $Y = X$. If $\widetilde{S} \neq S$, then in order to satisfy the local version of Definition 2.14 it remains to show that $x \in \mathcal{B}_\eta(x_*)$ implies $x \in \widetilde{S}$. Since by definition of η and γ we have $\eta = \alpha_2^{-1}(\gamma)$, we get

$$x \in \mathcal{B}_\eta(x_*) \quad \Rightarrow \quad |x| < \eta = \alpha_2^{-1}(\gamma) \quad \Rightarrow \quad V(x) \le \alpha_2\big(|x|\big) < \gamma \quad \Rightarrow \quad x \in \widetilde{S}.$$

This finishes the proof. □

Likewise, P-practical asymptotic stability can be ensured by a suitable Lyapunov function condition provided the set P is forward invariant.

Theorem 2.20 *Consider forward invariant sets Y and $P \subset Y$ and a point $x_* \in P$. If there exists a Lyapunov function V on $S = Y \setminus P$ then x_* is P-practically asymptotically stable on Y.*

Proof The same construction of β as in the proof of Theorem 2.19 yields

$$\big|x(n,x_0)\big|_{x_*} \le \beta\big(|x|_{x_*}, n\big) \tag{2.32}$$

for all $n = 0, \ldots, n^* - 1$, where $n^* \in \mathbb{N}_0$ is minimal with $x(n^*, x_0) \in P$. This follows with the same arguments as in the proof of Theorem 2.19 by restricting the times considered in (2.39) and (2.43) to $n = 0, \ldots, n^* - 2$ and $n = 0, \ldots, n^* - 1$, respectively.

Since forward invariance of P ensures $x(n, x_0) \in P$ for all $n \ge n^*$, the times n for which $x(n, x_0) \notin P$ holds are exactly $n = 0, \ldots, n^* - 1$. Since these are exactly the times at which (2.32) is required, this yields the desired P-practical asymptotic stability. □

In case of a time varying reference x^{ref} we need to use the time varying asymptotic stability from Definition 2.16. The corresponding Lyapunov function concept is as follows.

Definition 2.21 Consider a system (2.34), reference points $x^{\mathrm{ref}}(n)$, subsets of the state space $S(n) \subseteq X$ and define $\mathcal{S} := \{(n,x) \,|\, n \in \mathbb{N}_0,\ x \in S(n)\}$. A function $V : \mathcal{S} \to \mathbb{R}_0^+$ is called a *uniform time varying Lyapunov function* on $S(n)$ if the following conditions are satisfied:

(i) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1\big(|x|_{x^{\text{ref}}(n)}\big) \le V(n,x) \le \alpha_2\big(|x|_{x^{\text{ref}}(n)}\big) \tag{2.44}$$

holds for all $n \in \mathbb{N}_0$ and all $x \in S(n)$.

(ii) There exists a function $\alpha_V \in \mathcal{K}$ such that

$$V\big(n+1, g(n,x)\big) \le V(n,x) - \alpha_V\big(|x|_{x^{\text{ref}}(n)}\big) \tag{2.45}$$

holds for all $n \in \mathbb{N}_0$ and all $x \in S(n)$ with $g(n,x) \in S(n+1)$.

Theorem 2.22 *Let x^{ref} be a trajectory of (2.34) and assume there exists a uniform time varying Lyapunov function V on $S(n)$. If each $S(n)$ contains a ball $\mathcal{B}_\nu(x^{\text{ref}}(n))$ with $g(n,x) \in S(n+1)$ for all $x \in \mathcal{B}_\nu(x^{\text{ref}}(n))$ then x^{ref} is locally asymptotically stable with $\eta = \alpha_2^{-1} \circ \alpha_1(\nu)$. If the family of sets $S(n)$ is forward invariant in the sense stated before Definition 2.16, then x^{ref} is asymptotically stable on $S(n)$. If $S(n) = X$ holds for all $n \in \mathbb{N}_0$ then x^{ref} is globally asymptotically stable.*

Proof The proof is analogous to the proof of Theorem 2.19 with the obvious modifications to take $n \in \mathbb{N}_0$ into account. □

Indeed, the necessary modification in the proof are straightforward because the time varying Lyapunov function is uniform, i.e., α_1, α_2 and α_V do not depend on n. For the more involved nonuniform case we again refer to [3].

The P-practical version of this statement is provided by the following theorem in which we assume forward invariance of the sets $P(n)$. Observe that here x^{ref} does not need to be a trajectory of the system (2.34).

Theorem 2.23 *Consider forward invariant families of sets $Y(n)$ and $P(n) \subset Y(n)$, $n \in \mathbb{N}_0$, and reference points $x^{\text{ref}}(n) \in P(n)$. If there exists a uniform time varying Lyapunov function V on $S(n) = Y(n) \setminus P(n)$ then x^{ref} is P-practically asymptotically stable on $Y(n)$.*

Proof The proof is analogous to the proof of Theorem 2.20 with the obvious modifications. □

2.4 Stability of Sampled Data Systems

We now investigate the special case in which (2.31) represents the nominal closed-loop system (2.5) with f obtained from a sampled data system via (2.8). In this case, the solutions $x(n, x_0)$ of (2.31) and the solutions $\varphi(t_n, t_0, x_0, \mu)$ of the sampled data closed-loop system (2.30) satisfy the identity

$$x(n, x_0) = \varphi(t_n, t_0, x_0, \mu) \tag{2.46}$$

for all $n \in \mathbb{N}_0$. This implies that the stability criterion from Definition 2.14 (and analogous for the other stability definitions) only yields inequalities for the continuous state of the system at the sampling times t_n, i.e.,

$$\big|\varphi(t_n, t_0, x_0, \mu)\big|_{x_*} \le \beta\big(|x_0|_{x_*}, n\big) \quad \text{for all } n = 0, 1, 2, \dots \tag{2.47}$$

for a suitable $\beta \in \mathcal{KL}$. However, for a continuous time system it is in general desirable to ensure the existence of $\overline{\beta} \in \mathcal{KL}$ such that the continuous time asymptotic stability property

$$\big|\varphi(t, t_0, x_0, \mu)\big|_{x_*} \le \overline{\beta}\big(|x_0|_{x_*}, t\big) \quad \text{for all } t \ge 0 \tag{2.48}$$

holds.

In the remainder of this chapter we will show that under a reasonable additional assumption (2.47) implies the existence of $\overline{\beta} \in \mathcal{KL}$ such that (2.48) holds. For simplicity, we restrict ourselves to local asymptotic stability and to the case of time-invariant reference $x^{\mathrm{ref}} \equiv x_*$. The arguments can be modified to cover the other cases, as well.

The necessary additional condition is the following boundedness assumption on the solutions in between two sampling instants.

Definition 2.24 Consider a sampled data closed-loop system (2.30) with sampling period $T > 0$. If there exists a function $\gamma \in \mathcal{K}$ and a constant $\eta > 0$ such that for all $x \in X$ with $|x|_{x_*} \le \eta$, the solutions of (2.30) exist on $[0, T]$ and satisfy

$$\big|\varphi(t, 0, x, \mu)\big|_{x_*} \le \gamma\big(|x|_{x_*}\big)$$

for all $t \in [0, T]$ then the solutions of (2.30) are called *uniformly bounded over T*.

Effectively, this condition demands that in between two sampling times t_n and t_{n+1} the continuous time solution does not deviate too much from the solution at the sampling time t_n. Sufficient conditions for this property formulated directly in terms of the vector field f_c in (2.30) can be found in [11, Lemma 3]. A sufficient condition in our NMPC setting is discussed in Remark 4.13.

For the subsequent analysis we introduce the following class of $\mathcal{KL}$ functions, which will allow us to deal with the inter sampling behavior of the continuous time solution.

Definition 2.25 A function $\beta \in \mathcal{KL}$ is called *uniformly incrementally bounded* if there exists $P > 0$ such that $\beta(r, k) \le P\beta(r, k + 1)$ holds for all $r \ge 0$ and all $k \in \mathbb{N}$.

Uniformly incrementally bounded $\mathcal{KL}$ functions exhibit a nice bounding property compared to standard $\mathcal{KL}$ functions which we will use the proof of Theorem 2.27. Before, we show that any $\mathcal{KL}$ function β—like the one in (2.47)—can be bounded from above by a uniformly incrementally bounded $\mathcal{KL}$ function.

Lemma 2.26 *For any $\beta \in \mathcal{KL}$ the function*

$$\tilde\beta(r,t) := \max_{\tau\in[0,t]} 2^{-\tau}\beta(r,t-\tau)$$

is a uniformly incrementally bounded $\mathcal{KL}$ function with $\beta(r,t) \le \tilde\beta(r,t)$ for all $r \ge 0$ and all $t \ge 0$ and $P = 2$.

Proof The inequality $\beta \le \tilde\beta$ follows immediately from the definition. Uniform incremental boundedness with $P = 2$ follows from the inequality

$$\begin{aligned}\tilde\beta(r,t) &= \max_{\tau\in[0,t]} 2^{-\tau}\beta(r,t-\tau) = \max_{\tau\in[1,t+1]} 2^{1-\tau}\beta(r,t-\tau+1)\\ &= 2\max_{\tau\in[1,t+1]} 2^{-\tau}\beta(r,t-\tau+1) \le 2\max_{\tau\in[0,t+1]} 2^{-\tau}\beta(r,t-\tau+1)\\ &= 2\tilde\beta(r,t+1).\end{aligned}$$

It remains to show that $\tilde\beta \in \mathcal{KL}$.

Since $\beta \in \mathcal{KL}$ it follows that $\tilde\beta$ is continuous and $\tilde\beta(0,t) = 0$ for any $t \ge 0$. For any $r_2 > r_1 \ge 0$, $\beta \in \mathcal{KL}$ implies $2^{-\tau}\beta(r_2,t-\tau) > 2^{-\tau}\beta(r_1,t-\tau)$. This shows that $\tilde\beta(r_2,t) > \tilde\beta(r_1,t)$ and hence $\tilde\beta(\cdot,t) \in \mathcal{K}$.

Next we show that for any fixed $r > 0$ the function $t \mapsto \tilde\beta(r,t)$ is strictly decreasing to 0. To this end, in the following we use that for all $t \ge s \ge q \ge 0$ and all $r \ge 0$ the inequality

$$\max_{\tau\in[q,s]} 2^{-\tau}\beta(r,t-\tau) \le 2^{-q}\beta(r,t-s)$$

holds. In order to show the strict decrease property for $r > 0$, let $t_2 > t_1 \ge 0$. Defining $d := t_2 - t_1$ we obtain

$$\begin{aligned}\tilde\beta(r,t_2) &= \max_{\tau\in[0,t_2]} 2^{-\tau}\beta(r,t_2-\tau)\\ &= \max\Big\{\max_{\tau\in[0,d/2]} 2^{-\tau}\beta(r,t_2-\tau),\ \max_{\tau\in[d/2,d]} 2^{-\tau}\beta(r,t_2-\tau),\\ &\qquad \max_{\tau\in[d,t_2]} 2^{-\tau}\beta(r,t_2-\tau)\Big\}\\ &\le \max\Big\{\beta(r,t_2-d/2), 2^{-d/2}\beta(r,t_2-d),\ \max_{\tau\in[0,t_1]} 2^{-\tau-d}\beta(r,t_1-\tau)\Big\}\\ &= \max\big\{\beta(r,t_1+d/2), 2^{-d/2}\beta(r,t_1), 2^{-d}\tilde\beta(r,t_1)\big\}.\end{aligned}$$

Now the strict monotonicity $\tilde\beta(r,t_2) < \tilde\beta(r,t_1)$ follows since $\beta(r,t_1+d/2) < \beta(r,t_1) \le \tilde\beta(r,t_1)$, $2^{-d/2}\beta(r,t_1) < \beta(r,t_1) \le \tilde\beta(r,t_1)$ and $2^{-d}\tilde\beta(r,t_1) < \tilde\beta(r,t_1)$.

Finally, we prove $\lim_{t\to\infty}\tilde\beta(r,t) = 0$ for any $r > 0$. Since

$$\begin{aligned}\tilde\beta(r,t) &\le \max\Big\{\max_{\tau\in[0,t/2]} 2^{-\tau}\beta(r,t-\tau),\ \max_{\tau\in[t/2,t]} 2^{-\tau}\beta(r,t-\tau)\Big\}\\ &\le \max\big\{\beta(r,t/2), 2^{-t/2}\beta(r,0)\big\} \to 0 \quad \text{as } t\to\infty\end{aligned}$$

the assertion follows. □

Now, we are ready to prove the final stability result.

Theorem 2.27 *Consider the sampled data closed-loop system (2.30) with sampling period $T > 0$ and the corresponding discrete time closed-loop system (2.5) with f from (2.8). Then (2.30) is locally asymptotically stable, i.e., there exists $\eta > 0$ and $\overline{\beta} \in \mathcal{KL}$ such that (2.48) holds for all $x \in \mathcal{B}_\eta(x_*)$, if and only if (2.5) is locally asymptotically stable and the solutions of (2.30) are uniformly bounded over T.*

Proof If (2.30) is locally asymptotically stable with some $\overline{\beta} \in \mathcal{KL}$, then by (2.46) it immediately follows that the discrete time system (2.5) is asymptotically stable with $\beta(r,k) = \overline{\beta}(r,kT)$ and that the solutions of (2.30) are uniformly bounded with $\gamma(r) = \overline{\beta}(r,0)$.

Conversely, assume that (2.5) is locally asymptotically stable and that the solutions of (2.30) are uniformly bounded over T. Denote the values $\eta > 0$ from Definition 2.14 and Definition 2.24 by η^s and η^b, respectively. These two properties imply that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x|_{x_*} \le \eta^s \quad \Longrightarrow \quad \big|\varphi(kT,0,x,\mu)\big|_{x_*} \le \beta\big(|x|_{x_*},k\big) \quad \text{for all } k \ge 0, \tag{2.49}$$

$$|x|_{x_*} \le \eta^b \quad \Longrightarrow \quad \big|\varphi(t,0,x,\mu)\big|_{x_*} \le \gamma\big(|x|_{x_*}\big) \quad \text{for all } t \in [0,T]. \tag{2.50}$$

In order to show the assertion we have to construct $\eta > 0$ and $\overline{\beta} \in \mathcal{KL}$ with

$$|x|_{x_*} \le \eta \quad \Longrightarrow \quad \big|\varphi(t,0,x,\mu)\big|_{x_*} \le \overline{\beta}\big(|x|_{x_*},t\big) \quad \text{for all } t \ge 0. \tag{2.51}$$

Define $\gamma_0(r) := \beta(r,0)$ and let $\eta = \min\{\eta^s, \gamma_0^{-1}(\eta^b)\}$. This definition implies $\beta(\eta,0) \le \eta^b$ and $\eta \le \eta^s$. In what follows we consider arbitrary $x \in X$ with $|x|_{x_*} \le \eta$. For these x, (2.49) and $\eta \le \eta^s$ yield

$$\big|\varphi(kT,0,x,\mu)\big|_{x_*} \le \beta\big(\|x\|_{x_*},k\big) \le \beta(\eta,0) \le \eta^b \quad \text{for all } k \ge 0. \tag{2.52}$$

For any $k \ge 0$ and $t \in [kT,(k+1)T]$ the definition of (2.30) implies

$$\varphi(t,0,x,\mu) = \varphi\big(t-kT,0,\varphi(kT,0,x,\mu),\mu\big).$$

Since (2.52) implies $|\varphi(kT,0,x,\mu)|_{x_*} \le \eta^b$ for all $k \ge 0$, (2.50) holds for $x = \varphi(kT,0,x,\mu)$ and from (2.50) and (2.52) we obtain

$$\big|\varphi(t,0,x,\mu)\big|_{x_*} \le \gamma\big(\big\|\varphi(kT,0,x,\mu)\big\|\big) \le \gamma\big(\beta\big(\|x\|_{x_*},k\big)\big) \tag{2.53}$$

for all $t \in [kT,(k+1)T]$ and all $k \ge 0$.

Now we define $\hat{\beta}(r,t) := \gamma(\beta(r,t))$. Clearly, $\hat{\beta} \in \mathcal{KL}$ and by Lemma 2.26 we can assume without loss of generality that $\hat{\beta}$ is uniformly incrementally bounded; otherwise we replace it by $\tilde{\beta}$ from this lemma.

Hence, for $k \in \mathbb{N}_0$ and $s \in [0,1]$ we obtain

$$\hat{\beta}(r,k) \le P\hat{\beta}(r,k+1) \le P\hat{\beta}(r,k+s). \tag{2.54}$$

Now pick an arbitrary $t \ge 0$ and let $k \in \mathbb{N}_0$ be maximal with $k \le t/T$. Then (2.53) and (2.54) with $s = t/T - k \in [0,1]$ imply

$$\big|\varphi(t,0,x,\mu)\big|_{x_*} \le \hat{\beta}\big(\|x\|_{x_*},k\big) \le P\hat{\beta}\big(|x|_{x_*},k+(t/T-k)\big) = P\hat{\beta}\big(|x|_{x_*},t/T\big).$$

This shows the assertion with $\overline{\beta}(r,t) = P\hat{\beta}(r,t/T)$. □

Concluding, if we can compute an asymptotically stabilizing feedback law for the discrete time system induced by the sampled data system, then the resulting continuous time sampled data closed loop is also asymptotically stable provided its solutions are uniformly bounded over T.

2.5 Notes and Extensions

The general setting presented in Sect. 2.1 is more or less standard in discrete time control theory, except maybe for the rather general choice of the state space X and the control value space U which allows us to cover infinite-dimensional systems as illustrated in Example 2.12 and sampled data systems without the zero order hold assumption as discussed after Theorem 2.7.

This definition of sampled data systems is not so frequently found in the literature, where often only the special case of zero order hold is discussed. While zero order hold is usually the method of choice in practical applications and is also used in the numerical examples later in this book, for theoretical investigations the more general approach given in Sect. 2.2 is appealing, too.

The discrete time stability theory presented in Sect. 2.3 has a continuous time counterpart, which is actually more frequently found in the literature. Introductory textbooks on this subject in a control theoretic setting are, e.g., the books by Khalil [7] and Sontag [15]. The proofs in this section are not directly taken from the literature, but they are based on standard arguments, which appear in many books and papers on the subject. Formulating asymptotic stability via $\mathcal{KL}$-function goes back to Hahn [5] and became popular in nonlinear control theory during the 1990s via the input-to-state stability (ISS) property introduced by Sontag in [14]. A good survey on this theory can be found in Sontag [16].

While here we only stated direct Lyapunov function theorems which state that the existence of a Lyapunov function ensures asymptotic stability, there is a rather complete converse theory, which shows that asymptotic stability implies the existence of Lyapunov functions. A collection of such results—again in a control theoretic setting—can be found in the PhD thesis of Kellett [6].

The final Sect. 2.4 on asymptotic stability of sampled data systems is based on the Paper [11] by Nešić, Teel and Sontag, in which this topic is treated in a more general setting. In particular, this paper also covers ISS results for perturbed systems.

2.6 Problems

1. Show that there exists no differential equation $\dot{x}(t) = f_c(x(t))$ (i.e., without control input) satisfying Assumption 2.4 and $f_c(0) = 0$ such that the difference equation $x^+ = f(x)$ with

$$f(x) = \begin{cases} \frac{x}{2}, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

is the corresponding sampled data system.

2. (a) Show that $x^{\text{ref}}(n) = \sum_{k=0}^{n} \frac{1}{2^{n-k}} \sin(k)$ is a solution of the difference equation

$$x(n+1) = \frac{1}{2}x(n) + \sin(n).$$

   (b) Prove that x^{ref} from (a) is uniformly asymptotically stable and derive a comparison function $\beta \in \mathcal{KL}$ such that (2.35) holds. Here it is sufficient to derive a formula for $\beta(r, n)$ for $n \in \mathbb{N}_0$.
   (c) Show that $x^{\text{ref}}(n) = \sum_{k=0}^{n} \frac{k+1}{n+1} \sin(k)$ is a solution of the difference equation

$$x(n+1) = \frac{n+1}{n+2}x(n) + \sin(n).$$

   (d) Can you also prove uniform asymptotic stability for x^{ref} from (c)?
   Hint for (b) and (d): One way to proceed is to derive a difference equation for $z(n) = x(n, n_0, x_0) - x^{\text{ref}}(n)$ and look at the equilibrium $x_* = 0$ for this new equation.
3. Consider the two-dimensional difference equation

$$x^+ = \big(1 - \|x\|\big)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

   with $x = (x_1, x_2)^\top \in \mathbb{R}^2$.
   (a) Prove that $V(x) = x_1^2 + x_2^2$ is a Lyapunov function for the equilibrium $x_* = 0$ on $S = \{x \in \mathbb{R}^2 \mid \|x\| \le 1\}$.
   (b) Is V also a Lyapunov function on $S = \mathbb{R}^2$?
   (c) Solve (a) and (b) for the difference equation

$$x^+ = \frac{1}{1 + \|x\|}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x.$$

4. Consider a globally asymptotically stable difference equation (2.31) with equilibrium $x_* \in X$ and a Lyapunov function V on $S = X$ with $\alpha_1(r) = 2r^2$, $\alpha_2(r) = 3r^2$ and $\alpha_V(r) = r^2$.
   Compute the rate of attraction $\beta \in \mathcal{KL}$ such that (2.32) holds. Here it is sufficient to derive a formula for $\beta(r, n)$ for $n \in \mathbb{N}_0$.
   Hint: Follow the construction of β from the proof of Theorem 2.19. Why can you use $\tilde{\alpha}_V = \alpha_V'$ for this problem?
5. Consider a difference equation (2.31) with equilibrium $x_* \in X$ and a function $V : X \to \mathbb{R}_0^+$ which satisfies (2.37) but only

$$V\big(g(x)\big) \le V(x)$$

   instead of (2.38).
   (a) Prove that there exists $\alpha_L \in \mathcal{K}_\infty$ such that the solutions of (2.1) satisfy the inequality

$$\big|x(n, x_0)\big|_{x_*} \le \alpha_L\big(|x_0|\big).$$

   (b) Conclude from (a) that the system is stable in the sense of Lyapunov, cf. the discussion after Definition 2.14.

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