Chapter 5
Convolution operators

Having focused on algebras of singular integral operators in the last chapter, in this chapter convolution operator algebras will be treated. The idea is to give a general perspective of how the material in the first part of the book has been applied in the context of convolution operator algebras in recent decades, while at the same time including some previously unpublished material.

Throughout this chapter, let $1 < p < \infty$ and let w be a power weight on $\mathbb{R}$, i.e., w is of the form (4.1) with $t_i \in \mathbb{R}$ for $i = 1, \ldots, n$. We always assume that w belongs to the class $\mathcal{A}_p(\mathbb{R})$.

5.1 Multipliers and commutators

We will start this section by developing the subject begun in Section 4.2, focusing on specific classes of multipliers on the real line.

A function $a \in L^\infty(\mathbb{R})$ is called *piecewise linear* if there is a partition $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ of the real line and complex constants c_k, d_k such that $a(t) = c_0\chi_{]-\infty,t_1[} + \sum_{k=1}^{n-2}(c_k + d_k t)\chi_{]t_k,t_{k+1}[} + d_0\chi_{]t_{n-1},+\infty[}$. As usual, the function χ_I represents the characteristic function of the set I.

Since $w \in \mathcal{A}_p(\mathbb{R})$, Stechkin's equality (4.13) implies that the multiplier algebra $\mathcal{M}_{p,w}$ contains the (non-closed) algebras C_0 of all continuous and piecewise linear functions on $\dot{\mathbb{R}}$, and PC_0 of all piecewise constant functions on $\mathbb{R}$ having only finitely many discontinuities (jumps). Let $C_{p,w}$ and $PC_{p,w}$ represent the closure of C_0 and PC_0 in $\mathcal{M}_{p,w}$, respectively. When $w \equiv 1$, abbreviate $C_{p,w}$ and $PC_{p,w}$ to C_p and PC_p, and write C and PC for C_2 and PC_2, respectively. Thus, the algebras C and PC coincide with the algebras denoted by $C(\dot{\mathbb{R}})$ and $PC(\dot{\mathbb{R}})$ in previous chapters.

It is unknown for general weights w (not necessarily of power form) whether the multiplier algebra $\mathcal{M}_{p,w}$ is continuously embedded into $L^\infty(\mathbb{R}) = \mathcal{M}_2$. So it is by no means evident that functions in $PC_{p,w}$ are piecewise continuous again.

S. Roch et al., *Non-commutative Gelfand Theories*, Universitext,
DOI 10.1007/978-0-85729-183-7_5,

Proposition 5.1.1. *Let $1 < p < \infty$ and $w \in \mathcal{A}_p(\mathbb{R})$. Then the algebra $PC_{p,w}$ is continuously embedded into $L^\infty(\mathbb{R})$ and, thus, into PC. Moreover, for $a \in PC_{p,w}$,*

$$\|a\|_{L^\infty(\mathbb{R})} \leq 3\,\|S_\mathbb{R}\|_{p,w}\,\|a\|_{\mathcal{M}_{p,w}}.$$

Proof. For $a \in PC_{p,w}$, we find a sequence of piecewise constant functions a_n such that

$$\|a - a_n\|_{\mathcal{M}_{p,w}} = \|W^0(a) - W^0(a_n)\|_{\mathcal{L}(L^p(\mathbb{R},w))} \to 0$$

as $n \to \infty$. Given indices $m, n \in \mathbb{N}$ and a point $x \in \mathbb{R}$, choose piecewise constant characteristic functions $\chi_x^\pm$ with "small" support, both having a jump at x and both of total variation 2, such that

$$\chi_x^\pm(a_n - a_m) = \left(a_n(x^\pm) - a_m(x^\pm)\right)\chi_x^\pm.$$

The Stechkin inequality (4.13) yields

$$\begin{aligned} |a_n(x^\pm) - a_m(x^\pm)|\|\chi_x^\pm\|_{\mathcal{M}_{p,w}} &= \|\left(a_n(x^\pm) - a_m(x^\pm)\right)\chi_x^\pm\|_{\mathcal{M}_{p,w}} \\ &= \|\chi_x^\pm(a_n - a_m)\|_{\mathcal{M}_{p,w}} \\ &\leq 3\|S_\mathbb{R}\|_{p,w}\|a_n - a_m\|_{\mathcal{M}_{p,w}}. \end{aligned} \tag{5.1}$$

Since $\chi_x^\pm$ is real-valued, the conjugate operator to $W^0(\chi_x^\pm) \in \mathcal{L}(L^p(\mathbb{R},w))$ is $W^0(\chi_x^\pm) \in \mathcal{L}\left(L^q(\mathbb{R},w^{-1})\right)$, with $1/p + 1/q = 1$. Thus, by the Stein-Weiss interpolation theorem 4.8.1,

$$\begin{aligned} 1 &= \|\chi_x^\pm\|_{L^\infty(\mathbb{R})} = \|W^0(\chi_x^\pm)\|_{\mathcal{L}(L^2(\mathbb{R}))} \\ &\leq \|W^0(\chi_x^\pm)\|^{1/2}_{\mathcal{L}(L^p(\mathbb{R},w))}\|W^0(\chi_x^\pm)\|^{1/2}_{\mathcal{L}(L^q(\mathbb{R},w^{-1}))} \\ &= \|W^0(\chi_x^\pm)\|_{\mathcal{L}(L^p(\mathbb{R},w))} = \|\chi_x^\pm\|_{\mathcal{M}_{p,w}}, \end{aligned}$$

whence via (5.1)

$$|a_n(x^\pm) - a_m(x^\pm)| \leq 3\,\|S_\mathbb{R}\|_{p,w}\|a_n - a_m\|_{\mathcal{M}_{p,w}}. \tag{5.2}$$

So we arrive at the inequality

$$\|a_n - a_m\|_{L^\infty(\mathbb{R})} \leq 3\,\|S_\mathbb{R}\|_{p,w}\|a_n - a_m\|_{\mathcal{M}_{p,w}}, \tag{5.3}$$

from which we conclude that $a_n \to a$ in $L^\infty(\mathbb{R})$. Hence, by (5.3),

$$\begin{aligned} \|a\|_{L^\infty(\mathbb{R})} &\leq \|a_n\|_{L^\infty(\mathbb{R})} + \|a - a_n\|_{L^\infty(\mathbb{R})} \\ &\leq 3\|S_\mathbb{R}\|_{p,w}\left(\|a_n\|_{\mathcal{M}_{p,w}} + \|a - a_n\|_{\mathcal{M}_{p,w}}\right). \end{aligned}$$

Letting n go to infinity we get the assertion. ∎

Once the continuous embedding of $PC_{p,w}$ into $L^\infty(\mathbb{R})$ is established, the following propositions can be proved as in the case $w \equiv 1$ (see [43, Section 2]).

Proposition 5.1.2.

(i) *The Banach algebra $C_{p,w}$ is continuously embedded into C.*
(ii) *The maximal ideal space of $C_{p,w}$ is homeomorphic to $\dot{\mathbb{R}}$. In particular, any multiplier $a \in C_{p,w}$ is invertible in $C_{p,w}$ if and only if $a(t) \neq 0$ for all $t \in \dot{\mathbb{R}}$.*
(iii) *The algebras $C_{p,w}$ and $PC_{p,w} \cap C(\dot{\mathbb{R}})$ coincide with the closure in $\mathcal{M}_{p,w}$ of the set of all continuous functions on $\dot{\mathbb{R}}$ with finite total variation.*

Remark 5.1.3. Note that the inclusion $C_{p,w} \subseteq \mathcal{M}_{p,w} \cap C(\dot{\mathbb{R}})$ is proper for $p \neq 2$, see [53]. □

Proposition 5.1.4.

(i) *The maximal ideal space of $PC_{p,w}$ is homeomorphic to $\dot{\mathbb{R}} \times \{0,1\}$. In particular, a multiplier $a \in PC_{p,w}$ is invertible in $PC_{p,w}$ if and only if $a(t^{\pm}) \neq 0$ for all $t \in \dot{\mathbb{R}}$.*
(ii) *The algebra $PC_{p,w}$ coincides with the closure in $\mathcal{M}_{p,w}$ of the set of all piecewise continuous functions with finite total variation.*

5.2 Wiener-Hopf and Hankel operators

Denote, as before, the characteristic functions of the positive and negative half axis by χ_+ and χ_-, respectively, and let J stand for the operator $(Ju)(t) = u(-t)$ which is bounded and has norm 1 on $L^p(\mathbb{R}, w)$ if the weight function is symmetric, i.e., if $w(t) = w(-t)$ for all $t \in \mathbb{R}$.

Let $a \in \mathcal{M}_{p,w}$. The restriction of the operator $\chi_+ W^0(a)\chi_+ I$ onto the weighted Lebesgue space $L^p(\mathbb{R}^+, \chi_+ w)$ is called a *Wiener-Hopf operator* and will be denoted by $W(a)$. If, moreover, the weight on $\mathbb{R}$ is symmetric, then the restriction of the operator $\chi_+ W^0(a)\chi_- J$ onto $L^p(\mathbb{R}^+, \chi_+ w)$ is a *Hankel operator* and will be denoted by $H(a)$.

For symmetric weights, it is easy to see that for $a \in \mathcal{M}_{p,w}$ the function $\tilde{a} := Ja$, $\tilde{a}(t) = a(-t)$, is also a multiplier on $L^p(\mathbb{R}, w)$, and that the restriction of the operator $J\chi_- W^0(a)\chi_+ I$ onto $L^p(\mathbb{R}^+, \chi_+ w)$ coincides with the Hankel operator $H(\tilde{a})$. For $a, b \in \mathcal{M}_{p,w}$ one then has the fundamental identity

$$W(ab) = W(a)W(b) + H(a)H(\tilde{b}) \tag{5.4}$$

which, in a similar way to Proposition 4.5.1, follows easily from

$$\begin{aligned} W(ab) &= \chi_+ W^0(ab)\chi_+ I = \chi_+ W^0(a)W^0(b)\chi_+ I \\ &= \chi_+ W^0(a)(\chi_+ + \chi_- JJ\chi_-)W^0(b)\chi_+ I \\ &= \chi_+ W^0(a)\chi_+ \cdot \chi_+ W^0(b)\chi_+ I + \chi_+ W^0(a)\chi_- J \cdot J\chi_- W^0(b)\chi_+ I. \end{aligned}$$

The following theorems collect some basic properties of Wiener-Hopf operators with continuous generating functions. These results are the analogs of Theorems 4.1.5 and 4.1.8 for singular integral operators. Detailed proofs can be found in [21, 9.9. and 9.10].

Theorem 5.2.1. *The smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}^+))$ which contains all Wiener-Hopf operators $W(f)$ with $f \in C_p$ contains the ideal of the compact operators on $L^p(\mathbb{R}^+)$.*

Theorem 5.2.2 (Krein, Gohberg)**.** *Let $f \in C_p$. Then the Wiener-Hopf operator $W(f)$ is Fredholm on $L^p(\mathbb{R}^+)$ if and only if $f(x) \neq 0$ for all $x \in \dot{\mathbb{R}}$. If this condition is satisfied, then the Fredholm index of $W(f)$ is the negative winding number of the curve $f(\dot{\mathbb{R}})$ with respect to the origin. If the index of $W(f)$ is zero, then $W(f)$ is invertible.*

We digress for a moment and return to the context of Chapter 3. The point is that we can now give an example of a sufficiently simple algebra which is generated by three idempotents, but which does not possess a finite-dimensional invertibility symbol.

Example 5.2.3. Let $\mathscr{A}$ denote the smallest closed unital subalgebra of $\mathscr{L}(L^2(\mathbb{R}))$ which contains the operator $P_{\mathbb{R}} = (I + S_{\mathbb{R}})/2$ and the operators $\chi_{\mathbb{R}^+}I$ and $\chi_{[0,1]}I$ of multiplication by the characteristic functions of the intervals $\mathbb{R}^+$ and $[0,1]$, respectively. Thus, $\mathscr{A}$ is generated by three idempotents (actually, three orthogonal projections). Further, let $\mathscr{B}$ refer to the smallest closed subalgebra of $\mathscr{A}$ which contains all operators

$$\chi_{[0,1]}(\chi_{\mathbb{R}^+}S_{\mathbb{R}}\chi_{\mathbb{R}^+}I)^k\chi_{[0,1]}I \quad \text{with } k \in \mathbb{N}.$$

We identify $\mathscr{B}$ with a unital subalgebra of $\mathscr{L}(L^2([0, 1]))$ in the natural way. From Proposition 4.2.17 we conclude that the algebra $\mathscr{B}$ contains all operators $\chi_{[0,1]}M^0(h)\chi_{[0,1]}I$ with $h \in C(\overline{\mathbb{R}})$ with $h(\pm\infty) = 0$. The definition (4.24) of a Mellin convolution further entails that $\mathscr{B}$ contains all operators $E_2^{-1}W(a)E_2$ with $a \in C(\dot{\mathbb{R}})$. But then $\mathscr{B}$ must contain all compact operators on $L^2([0, 1])$ by Theorem 5.2.1. Since the ideal of all compact operators contains a copy of $\mathbb{C}^{l\times l}$ for all l, it is immediate that $\mathscr{B}$ (hence, $\mathscr{A}$) cannot possess a matrix symbol of any finite order.

Thus, even if the three idempotents are projections, and even if two of them commute, a matrix symbol does not need to exist. □

5.3 Commutators of convolution operators

Now we turn our attention to commutators of convolution and related operators. The commutator $AB - BA$ of two operators A and B will be denoted by $[A, B]$.

Let $\bar{L}^\infty(\mathbb{R})$ denote the set of all functions $a \in L^\infty(\mathbb{R})$ for which the essential limits at infinity exist, i.e., for which there are complex numbers $a(-\infty)$ and $a(+\infty)$ such that

$$\lim_{t\to-\infty} \operatorname*{esssup}_{s\le t} |a(s)-a(-\infty)| = 0,$$
$$\lim_{t\to+\infty} \operatorname*{esssup}_{s\ge t} |a(s)-a(+\infty)| = 0$$

and write $\dot{L}^\infty(\mathbb{R})$ for the set of all functions $a \in \bar{L}^\infty(\mathbb{R})$ such that $a(-\infty) = a(+\infty)$.

Next we define the analogous classes $\bar{\mathcal{M}}_{p,w}$ and $\dot{\mathcal{M}}_{p,w}$ of multipliers. Let Q_t denote the characteristic function of the interval $\mathbb{R}\setminus[-t,t]$. Then we let $\bar{\mathcal{M}}_{p,w}$ refer to the set of all multipliers $a \in \mathcal{M}_{p,w}$ for which there are numbers $a(-\infty)$ and $a(+\infty)$ such that

$$\lim_{t\to\infty} \|Q_t(a - a(-\infty)\chi_- - a(+\infty)\chi_+)\|_{\mathcal{M}_{p,w}} = 0. \tag{5.5}$$

Notice that this definition makes sense since, by the Stechkin inequality (4.13), the characteristic functions Q_t, χ_+ and χ_- of $\mathbb{R}\setminus[-t,t]$, $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, belong to $\mathcal{M}_{p,w}$. Also notice that the numbers $a(-\infty)$ and $a(+\infty)$ are uniquely determined by a and that $\bar{L}^\infty(\mathbb{R}) = \bar{\mathcal{M}}_2$. Further, let $\dot{\mathcal{M}}_{p,w}$ denote the class of all multipliers $a \in \bar{\mathcal{M}}_{p,w}$ such that $a(-\infty) = a(+\infty)$. Via Proposition 5.1.1 one easily gets that

$$PC_{p,w} \subseteq \bar{\mathcal{M}}_{p,w} \text{ and } C_{p,w} \subseteq \dot{\mathcal{M}}_{p,w}.$$

Recall finally that $\mathcal{K}(X)$ stands for the ideal of all compact operators on the Banach space X.

Proposition 5.3.1.

(i) *If $a \in \bar{L}^\infty(\mathbb{R})$, $b \in \dot{\mathcal{M}}_{p,w}$, and $a(\pm\infty) = b(\pm\infty) = 0$, then $aW^0(b)$ and $W^0(b)aI$ are in $\mathcal{K}(L^p(\mathbb{R},w))$.*

(ii) *If one of the conditions*

1. *$a \in C(\dot{\mathbb{R}})$ and $b \in \bar{\mathcal{M}}_{p,w}$, or*
2. *$a \in \bar{L}^\infty(\mathbb{R})$ and $b \in C_{p,w}$, or*
3. *$a \in C(\overline{\mathbb{R}})$ and $b \in C(\overline{\mathbb{R}}) \cap PC_{p,w}$*

is fulfilled, then $[aI, W^0(b)] \in \mathcal{K}(L^p(\mathbb{R},w))$.

Proof. (i) Since, by assumption, $\|Q_t a\|_\infty \to 0$ and $\|Q_t b\|_{\mathcal{M}_{p,w}} \to 0$, we can assume without loss of generality that a and b have compact support. Choose functions $u,v \in C_0^\infty(\mathbb{R})$, the space of infinitely differentiable functions with compact support, such that $u|_{\operatorname{supp} a} = 1$ and $v|_{\operatorname{supp} b} = 1$. Then

$$aW^0(b) = (au)W^0(vb) = auW^0(v)W^0(b),$$

and the assertion follows once we have shown that $uW^0(v)$ is compact. Put $k = F^{-1}v$. Then, for $f \in L^p(\mathbb{R},w)$,

$$\left(uW^0(v)f\right)(t) = \int_{-\infty}^{+\infty} u(t)k(t-s)f(s)\,ds.$$

Since k is an infinitely differentiable function for which the function $t \mapsto t^m k(t)$ is bounded for any $m \in \mathbb{N}$, we have (with $\frac{1}{p} + \frac{1}{q} = 1$)

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |u(t)k(t-s)w^{-1}(s)|^q \, ds \right)^{p/q} dt < \infty,$$

whence the compactness of $uW^0(v)$ and, hence, of $aW^0(b)$ follows. Similarly, the compactness of $W^0(b)aI$ can be established.

(ii) 1. Write $b = b(-\infty)\chi_- + b(+\infty)\chi_+ + b'$ with $b' \in \dot{\mathcal{M}}_{p,w}$ and $b'(\pm\infty) = 0$. Then $[aI, W^0(b)] = K_1 + K_2 + K_3$ with

$$K_1 = (a - a(+\infty))W^0(b'), \qquad K_2 = -W^0(b')(a - a(+\infty))I,$$

and

$$\begin{aligned} K_3 &= [aI, W^0(b(-\infty)\chi_- + b(+\infty)\chi_+)] \\ &= \frac{b(+\infty) - b(-\infty)}{2}[aI, W^0(\mathrm{sgn})]. \end{aligned}$$

The compactness of K_1 and K_2 is a consequence of part (i), and the compactness of K_3 follows from Theorem 4.1.4, since $W^0(\mathrm{sgn}) = S_{\mathbb{R}}$.

2. Write $a = a(-\infty)\chi_- + a(+\infty)\chi_+ + a'$ with $a' \in \dot{L}^\infty(\mathbb{R})$ and $a'(\pm\infty) = 0$, and write b as $b' + b(\pm\infty)$. Then $[aI, W^0(b)] = K_1 + K_2 + K_3$ with

$$K_1 = a'W^0(b'), \qquad K_2 = -W^0(b')a'I,$$

and

$$\begin{aligned} K_3 &= [(a(-\infty)\chi_- + a(+\infty)\chi_+)I, W^0(b)] \\ &= (a(+\infty) - a(-\infty))(\chi_+ W^0(b)\chi_- I - \chi_- W^0(b)\chi_+ I). \end{aligned}$$

By (i), the operators K_1 and K_2 are compact. To get the compactness of K_3 note that, by Proposition 5.1.2 (iii), we can assume, without loss of generality, that b has finite total variation. Hence, the operator $K_4 := \chi_+ W^0(b)\chi_- - \chi_- W^0(b)\chi_+$ is bounded on each space $L^p(\mathbb{R}, w)$ with $1 < p < \infty$ and $w \in \mathcal{A}_p(\mathbb{R})$. By Krasnoselskii's interpolation theorem 4.8.2, it is sufficient to verify the compactness of K_4 in $L^2(\mathbb{R})$. Since the Fourier transform is a unitary operator on $L^2(\mathbb{R})$, the operator K_4 is compact if and only if the operator

$$\begin{aligned} FK_4F^{-1} &= (F\chi_+F^{-1})b(F\chi_-F^{-1}) - (F\chi_-F^{-1})b(F\chi_+F^{-1}) \\ &= \frac{(F\mathrm{sgn}F^{-1})bI - b(F\mathrm{sgn}F^{-1})}{2} \\ &= \frac{S_{\mathbb{R}}bI - bS_{\mathbb{R}}}{2} \end{aligned}$$

is compact. The compactness of this operator has been established in Theorem 4.1.4.

3. As in 2. above we can assume without loss of generality that b is of finite total variation. By Krasnoselskii's interpolation theorem, we just have to verify the compactness of $[aI, W^0(b)]$ in $L^2(\mathbb{R})$. Put $b_\pm := (b(+\infty) \pm b(-\infty))/2$. Then

$$b(z) = b_+ + b_- \coth\big((z+\mathbf{i}/2)\pi\big) + b_0(z)$$

with $b_0 \in C(\mathbb{R})$ and $b_0(\pm\infty) = 0$. By (ii) 2., the commutator $[aI, W^0(b_0)]$ is compact, and it remains to show the compactness of $[aI, W^0(\coth(\cdot + \mathbf{i}/2)\pi)]$. Let E_2 be the operator defined in (4.21) for the weight $w \equiv 1$. Since $M^0(c) = E_2^{-1} W^0(c) E_2$ for all $c \in \mathcal{M}_p$, one has

$$E_2^{-1}\big[aI, W^0\big(\coth((\cdot + \mathbf{i}/2)\pi)\big)\big]E_2 = \big[a'I, M^0\big(\coth((\cdot + \mathbf{i}/2)\pi)\big)\big] = [a'I, S_{\mathbb{R}^+}]$$

by Proposition 4.2.11, where we wrote $a'(t) := a(\frac{1}{2\pi}\ln t)$ for $t \in \mathbb{R}^+$. The compactness of $[a'I, S_{\mathbb{R}^+}]$ on $L^2(\mathbb{R}^+)$ follows from Theorem 4.1.4 by applying the operator $\chi_+ I$ to both sides of the commutator $[cI, S_{\mathbb{R}}]$ where $c \in C(\dot{\mathbb{R}})$ is such that $c\chi_+ = a'$. ■

Let $\bar{L}^\infty(\mathbb{R}^+)$ refer to the Banach space of all measurable functions which possess essential limits at 0 and at ∞, write $\dot{L}^\infty(\mathbb{R}^+)$ for the set of these functions a from $\bar{L}^\infty(\mathbb{R}^+)$ with $a(0) = a(\infty)$, and put $\bar{C}(\mathbb{R}^+) := C(\mathbb{R}^+) \cap \bar{L}^\infty(\mathbb{R}^+)$ and $\dot{C}(\mathbb{R}^+) := C(\mathbb{R}^+) \cap \dot{L}^\infty(\mathbb{R}^+)$.

Proposition 5.3.2.
(i) *If $a \in \bar{L}^\infty(\mathbb{R}^+)$, $b \in \mathcal{M}_p$ and $a(0) = a(\infty) = b(\pm\infty) = 0$, then $aM^0(b)$ and $M^0(b)aI$ are in $\mathcal{K}(L^p(\mathbb{R}^+, w))$.*
(ii) *If one of the conditions*

1. *$a \in \dot{C}(\mathbb{R}^+)$ and $b \in \bar{\mathcal{M}}_p$,*
2. *$a \in \bar{L}^\infty(\mathbb{R}^+)$ and $b \in C_p$, or*
3. *$a \in C(\overline{\mathbb{R}})$ and $b \in C(\overline{\mathbb{R}}) \cap PC_p$*

is fulfilled, then $[aI, M^0(b)] \in \mathcal{K}(L^p(\mathbb{R}^+, w))$.

Proof. An operator A on $L^p(\mathbb{R}^+, w)$ is compact if and only if $wAw^{-1} \in \mathcal{L}(L^p(\mathbb{R}^+))$ is compact. Since $waw^{-1} = a$ for all $a \in L^\infty(\mathbb{R}^+)$ and since $wM^0(b)w^{-1}I = M^0(b)$ is bounded on $L^p(\mathbb{R}^+)$, it suffices to prove the compactness of $aM^0(b)$, $M^0(b)aI$ and $[aI, M^0(b)]$ on $L^p(\mathbb{R}^+)$ without weight. For this case, the assertions are immediate consequences of Proposition 5.3.1 via the isomorphism E_p. ■

Proposition 5.3.3. *Let $a, b \in PC_{p,w}$. Then:*

(i) *if a and b have no common discontinuities,*

$$W(ab) - W(a)W(b) = H(a)H(\tilde{b}) \in \mathcal{K}(L^p(\mathbb{R}^+, w));$$

(ii) $[W(a), W(b)] \in \mathcal{K}(L^p(\mathbb{R}^+, w))$.

Proof. By definition, the functions a and b are $\mathcal{M}_{p,w}$-limits of piecewise constant functions. So we can assume without loss of generality that a and b are piecewise constant. Since, moreover, every piecewise constant function is a finite sum of functions with one discontinuity, we can assume that a and b have at most one discontinuity. Finally, it is a clear consequence of Krasnoselskii's interpolation theorem, that it is sufficient to prove the result for $p = 2$ and $w \equiv 1$.

(i) Working on $L^2(\mathbb{R}^+)$, it is sufficient for us to consider, instead of $K_1 := W(ab) - W(a)W(b)$, the unitarily equivalent operator

$$\begin{aligned} K_2 := FK_1F^{-1} &= F\chi_+F^{-1}abF\chi_+F^{-1} - F\chi_+F^{-1}aF\chi_+F^{-1}bF\chi_+F^{-1} \\ &= Q_{\mathbb{R}}abQ_{\mathbb{R}} - Q_{\mathbb{R}}aQ_{\mathbb{R}}bQ_{\mathbb{R}} = Q_{\mathbb{R}}aP_{\mathbb{R}}bQ_{\mathbb{R}} \end{aligned}$$

with $P_{\mathbb{R}} = (I + S_{\mathbb{R}})/2$ and $Q_{\mathbb{R}} = I - P_{\mathbb{R}}$. Note that if one of the functions a and b is in $C(\dot{\mathbb{R}})$, then the result follows immediately from Theorem 4.1.4. So let both a and b be discontinuous. Let $t_a, t_b \in \dot{\mathbb{R}}$ denote the (only) points of discontinuity of a and b, respectively, and assume for definiteness that $t_a < t_b$. If χ stands for the characteristic function of the interval $[t_a, t_b]$, then a and b can be written as $a = a_1\chi + a_2$, $b = b_1\chi + b_2$, respectively, with continuous functions a_1, a_2, b_1 and b_2 such that

$$a_1(t_b) = b_1(t_a) = 0. \tag{5.6}$$

Then

$$\begin{aligned} K_2 &= Q_{\mathbb{R}}aP_{\mathbb{R}}bQ_{\mathbb{R}} \\ &= Q_{\mathbb{R}}a_1\chi P_{\mathbb{R}}b_1\chi Q_{\mathbb{R}} + Q_{\mathbb{R}}a_1\chi P_{\mathbb{R}}b_2Q_{\mathbb{R}} + Q_{\mathbb{R}}a_2P_{\mathbb{R}}b_1\chi Q_{\mathbb{R}} + Q_{\mathbb{R}}a_2P_{\mathbb{R}}b_2Q_{\mathbb{R}}, \end{aligned}$$

and it remains to verify that the operator $Q_{\mathbb{R}}a_1\chi P_{\mathbb{R}}b_1\chi Q_{\mathbb{R}}$ is compact. But

$$\begin{aligned} Q_{\mathbb{R}}a_1\chi P_{\mathbb{R}}b_1\chi Q_{\mathbb{R}} &= Q_{\mathbb{R}}\chi a_1P_{\mathbb{R}}b_1\chi Q_{\mathbb{R}} \\ &= Q_{\mathbb{R}}\chi P_{\mathbb{R}}a_1b_1\chi Q_{\mathbb{R}} + K_3 \\ &= Q_{\mathbb{R}}\chi a_1b_1\chi P_{\mathbb{R}}Q_{\mathbb{R}} + K_4 = K_4 \end{aligned}$$

with certain compact operators K_3 and K_4, because $a_1b_1\chi$ is continuous due to (5.6).

(ii) If a and b have no common points of discontinuity, then the assertion is an immediate consequence of the preceding one. So let a and b have common discontinuities. As above, we may assume that both a and b have exactly one point $s \in \mathbb{R}$ of discontinuity. Then there exist a constant β and a continuous function f such that $a = \beta b + f$. Thus,

$$[W(a), W(b)] = [W(\beta b + f), W(b)] = [W(f), W(b)],$$

and the assertion follows from part (i) of this proposition. ■

Proposition 5.3.4. *Let the weight function w_α be given by $w_\alpha(t) = |t|^\alpha$. Then:*

(i) *if $a \in C_{p,w_\alpha}$, $b \in \dot{\mathcal{M}}_p$ and $a(0) = a(\pm\infty) = b(\pm\infty) = 0$, we have $W(a)M^0(b) \in \mathcal{K}(L^p(\mathbb{R}^+, w_\alpha))$ and $M^0(b)W(a) \in \mathcal{K}(L^p(\mathbb{R}^+, w_\alpha))$;*

(ii) *each of the following conditions is sufficient for the compactness of the commutator $[W(a), M^0(b)]$ on $L^p(\mathbb{R}^+, w_\alpha)$:*

1. *$a \in PC_{p,w_\alpha}$ and $b \in C(\overline{\mathbb{R}}) \cap PC_p$;*
2. *$a \in C_{p,w_\alpha}$ with $a(\pm\infty) = a(0)$ and $b \in \bar{\mathcal{M}}_p$.*

Proof. (i) Since $\|Q_t b\|_{\mathcal{M}_{p,w_\alpha}} \to 0$ as $t \to \infty$, we can assume that b has compact support. Let f_b be a continuous function with total variation 2 and such that $f_b(t) = 1$ for $t \in \operatorname{supp} b$. Then

$$M^0(b) = M^0(f_b b) = M^0(f_b)M^0(b),$$

so that it remains to show the compactness of $W(a)M^0(f_b)$. Due to Proposition 4.2.10, we can approximate f_b by functions of the form

$$f(t) = \sum_{k=0}^{n} \beta_k \coth^k \big((t + \mathbf{i}(1/p + \alpha))\pi\big), \tag{5.7}$$

for which

$$M^0(f) = \sum_{k=0}^{n} \beta_k S_{\mathbb{R}^+}^k = \sum_{k=0}^{n} \beta_k (W(\mathrm{sgn}))^k.$$

Since $a \cdot \mathrm{sgn}$ is a continuous function, we deduce from Proposition 5.3.3 (i) that

$$\begin{aligned} W(a)M^0(f) &= \sum_{k=0}^{n} \beta_k W(a)(W(\mathrm{sgn}))^k \\ &= \sum_{k=0}^{n} \beta_k W(a(\mathrm{sgn})^k) + K_1 \\ &= W\left(a \sum_{k=0}^{n} \beta_k (\mathrm{sgn})^k\right) + K_2 \end{aligned}$$

with compact operators K_1 and K_2. The assumption $b(\pm\infty) = 0$ and (5.7) imply that $\sum_{k=0}^n \beta_k (\pm 1)^k = 0$. Thus,

$$\sum_{k=0}^{n} \beta_k (\mathrm{sgn})^k \equiv 0,$$

which gives our claim. The inclusion $M^0(b)W(a) \in \mathcal{K}(L^p(\mathbb{R}^+, w_\alpha))$ can be proved similarly.

(ii) By Proposition 4.2.10, every Mellin convolution $M^0(b)$ with $b \in C(\overline{\mathbb{R}}) \cap PC_p$ belongs to the algebra $\mathcal{E}_{p,\alpha}$ generated by the operators I and $S_{\mathbb{R}^+}$. Thus, $M^0(b)$ is contained in the algebra generated by all Wiener-Hopf operators $W(a)$ with $a \in PC_p$. The latter algebra is commutative modulo the compact operators by Proposition 5.3.3 (ii), which implies the first assertion of (ii). For a proof of the second assertion, write $b_\pm = (b(+\infty) \pm b(-\infty))/2$ again. Then

$$b(z) = b_+ + b_- \coth\big((z + \mathbf{i}(1/p+\alpha))\,\pi\big) + b_0(z)$$

with $b_0 \in \dot{\mathcal{M}}_p$ and $b_0(\pm\infty) = 0$. By part (i) of this assertion, the operators

$$W(a) \quad \text{and} \quad M^0\Big(b_+ + b_- \coth\big((z+\mathbf{i}(1/p+\alpha))\pi\big)\Big)$$

commute modulo a compact operator, and we have only to deal with the commutator $[W(a), M^0(b_0)]$. Put $a' = a - a(\infty)$. Then $a' \in \dot{C}_{p,w}$ with $a'(\pm\infty) = a'(0) = 0$ and $[W(a), M^0(b_0)] = [W(a'), M^0(b_0)]$. Part (i) now gives that $[W(a'), M^0(b_0)]$ is compact, and the proof is finished. ■

Proposition 5.3.5. *Let $w_\alpha(t) = t^\alpha$, $a \in \bar{L}^\infty(\mathbb{R}^+)$, $b \in \dot{C}_{p,w}$ and $c \in \bar{\mathcal{M}}_p$. Then each of the conditions:*

(i) $a(0) = b(0) = c(\pm\infty) = 0$;
(ii) $a(0) = c(-\infty) = 0$ *and* $b(t) = 0$ *for all* $t > 0$;
(iii) $a(0) = c(+\infty) = 0$ *and* $b(t) = 0$ *for all* $t < 0$

is sufficient for the compactness of $aW(b)M^0(c)$ on $L^p(\mathbb{R}^+, w_\alpha)$.

Proof. First we show that it is sufficient to prove the assertion in the case when $a \in \bar{C}(\mathbb{R}^+)$ and $c \in C(\overline{\mathbb{R}}) \cap PC_p$. Let the function $a' \in \bar{C}(\mathbb{R}^+)$ have the same limits at 0 and ∞ as the function a, and let $c' \in C(\overline{\mathbb{R}}) \cap PC_p$ have the same limits at $\pm\infty$ as the function c. Then

$$\begin{aligned} aW(b)M^0(c) &= a'W(b)M^0(c) + (a-a')W(b)M^0(c) \\ &= a'W(b)M^0(c') + a'W(b)M^0(c-c') \\ &\quad + (a-a')W(b)M^0(c') + (a-a')W(b)M^0(c-c'). \end{aligned}$$

The functions $a - a'$ and $c - c'$ can be approximated (in the supremum and the multiplier norm, respectively) by functions $a_0 \in \bar{L}^\infty(\mathbb{R}^+)$ and $c_0 \in \bar{\mathcal{M}}_p$ with compact support in $[0, +\infty[$ and $]-\infty, +\infty[$, respectively. Thus, $aW(b)M^0(c)$ can be approximated (in the operator norm) as closely as desired by operators of the form

$$a'W(b)M^0(c') + a'W(b)M^0(c_0) + a_0W(b)M^0(c') + a_0W(b)M^0(c_0). \tag{5.8}$$

Choose continuous functions f_0 and g_0 with total variation 2 such that $f_0 \equiv 1$ on $\operatorname{supp} a_0$ and $g_0 \equiv 1$ on $\operatorname{supp} c_0$. Then the operator (5.8) can be written as

$$\begin{aligned} &a'W(b)M^0(c') + a'W(b)M^0(g_0)M^0(c_0) \\ &\quad + a_0f_0W(b)M^0(c') + a_0f_0W(b)M^0(g_0)M^0(c_0). \end{aligned}$$

It is easy to check that if the triple (a,b,c) satisfies the conditions of the proposition, then so does each of the triples

$$(a',b,g_0),\ (a',b,c'),\ (f_0,b,g_0) \text{ and } (f_0,b,c').$$

Since each of the functions in these triples is continuous, we are indeed left with the proof of the assertion in the continuous setting.

We start the proof in the continuous setting by proving the assertion for a special choice of the functions a, b, c. Let

$$a_1(t) := e^{-1/t^2}, \qquad b_1(t) := \frac{t^4}{(1+t^2)^2}, \qquad c_1(t) := e^{-t^2/4}.$$

Then $c_1 = Mk$ with

$$k(x) = \frac{2}{\pi} x^{-1/p-\alpha} e^{-\ln^2 x}.$$

Indeed, since the function $z \mapsto e^{-z^2}$ is analytic in any strip $-m < \Im(z) < m$ and vanishes as $z \to \infty$ in that strip, one has, by the Cauchy integral theorem,

$$\begin{aligned}(Mk)(t) &= \frac{2}{\pi}\int_0^{+\infty} s^{-\mathbf{i}t} e^{-\ln^2 s} s^{-1}\, ds = \frac{2}{\pi}\int_{-\infty}^{+\infty} e^{-\mathbf{i}yt-y^2}\, dy \\ &= \frac{2}{\pi} e^{-t^2/4}\int_{-\infty}^{+\infty} e^{-(y+\frac{\mathbf{i}}{2}t)^2}\, dy = \frac{2}{\pi} e^{-t^2/4}\int_{\mathbb{R}+\frac{t}{2}\mathbf{i}} e^{-z^2}\, dz \\ &= \frac{2}{\pi} e^{-t^2/4}\int_{-\infty}^{+\infty} e^{-z^2}\, dz = e^{-t^2/4} = c_1(t).\end{aligned}$$

Let

$$g(t) := \left(F^{-1}(1-b_1)\right)(t) = \int_{-\infty}^{+\infty} e^{2\pi\mathbf{i}\lambda t}\frac{1+2\lambda^2}{(1+\lambda^2)^2}\, d\lambda.$$

For $u \in L^p(\mathbb{R}^+, w_\alpha)$, we then have

$$(W(b_1)u)(t) = u(t) - \int_0^\infty g(t-s)u(s)\, ds$$

and

$$\left(M^0(c_1)u\right)(t) = \int_0^\infty k\left(\frac{t}{s}\right) u(s) s^{-1}\, ds,$$

and the kernel $\mathfrak{K}$ of the integral operator $T := a_1 W(b_1) M^0(c_1) a_1 I$ is given by

$$\begin{aligned}\mathfrak{K}(x,y) &= a_1(x)a_1(y)\left(y^{-1}k\left(\frac{x}{y}\right) - \int_0^{+\infty} k\left(\frac{t}{y}\right) g(x-t) y^{-1}\, dt\right) \\ &= a_1(x)a_1(y)\left(y^{-1}h\left(\frac{x}{y}\right) - \int_{-\infty}^{+\infty} h\left(\frac{t}{y}\right) g(x-t) y^{-1}\, dt\right)\end{aligned}$$

with

$$h(z) = \begin{cases} k(z) & \text{if } z > 0, \\ 0 & \text{if } z < 0. \end{cases}$$

It is easy to see that h belongs to the Schwartz space $\mathscr{S}(\mathbb{R})$ of the rapidly decreasing infinitely differentiable functions on $\mathbb{R}$. Set $d := Fh \in \mathscr{S}(\mathbb{R})$. A simple calculation shows that then

$$\frac{1}{y} h\left(\frac{x}{y}\right) = (F^{-1} d_y)(x)$$

where $d_y(z) := d(yz)$ for $y > 0$ and $z \in \mathbb{R}$. Taking into account that $g = F^{-1}(1 - b_1)$, we can write $\mathfrak{K}(x,y)$ as

$$a_1(x)a_1(y)\left(\int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} zx} d(yz)\, dz - \int_{-\infty}^{+\infty} (F^{-1}d_y)(t) \cdot (F^{-1}(1-b_1))(x-t)\, dt\right)$$

or, equivalently,

$$a_1(x)a_1(y)\left((F^{-1}d_y)(x) - (F^{-1}d_y) * (F^{-1}(1-b_1))(x)\right).$$

By the convolution theorem, we thus get

$$\begin{aligned} \mathfrak{K}(x,y) &= a_1(x)a_1(y)\left((F^{-1}d_y)(x) - F^{-1}(d_y(1-b_1))(x)\right) \\ &= a_1(x)a_1(y)F^{-1}(d_y b_1)(x) \\ &= a_1(x)a_1(y)\int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} zx} d(yz) b_1(z)\, dz. \end{aligned}$$

Integrating twice by parts we find

$$\mathfrak{K}(x,y) = -\frac{a_1(x)a_1(y)}{(2\pi \mathbf{i} x)^2} \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} zx} \frac{\partial^2}{\partial z^2}\left(d(yz)\frac{z^4}{(1+z^2)^2}\right) dz.$$

Thus, there are constants C_1, C_2 and functions $d_1, d_2, d_3 \in \mathscr{S}(\mathbb{R})$ such that

$$\begin{aligned} |\mathfrak{K}(x,y)| &\le C_1 \frac{|a_1(x)||a_1(y)|}{x^2} \sum_{m=0}^{2} y^m \int_{-\infty}^{+\infty} |d_m(yz)| z^{m+2}\, dz \\ &= C_1 \frac{|a_1(x)|}{x^2} \frac{|a_1(y)|}{y^2} \sum_{m=0}^{2} y^m \int_{-\infty}^{+\infty} |d_m(yz)|(yz)^{m+2}\, dz \\ &\le C_2 \frac{|a_1(x)|}{x^2} \frac{|a_1(y)|}{y^2}. \end{aligned}$$

Now insert $a_1(x) = e^{-1/x^2}$ and $q = p/(p-1)$ to obtain

$$\int_0^{+\infty} \left(\int_0^{+\infty} |\mathfrak{K}(x,y)x^{-\alpha}|^q dx\right)^{p/q} dy < \infty.$$

This estimate implies that the operator $a_1 W(b_1) M^0(c_1)$ is compact on $L^p(\mathbb{R}^+, w_\alpha)$, which settles the assertion for the specific functions a_1, b_1 and c_1.

Now we return to the general case when $a \in \bar{C}(\mathbb{R}^+)$, $b \in \dot{C}_{p,w}$ and $c \in C(\overline{\mathbb{R}}) \cap PC_p$. (i) Let $a(0) = b(0) = c(\pm\infty) = 0$. Standard approximation arguments show that it is sufficient to consider functions with $a(t) = 0$ for $0 < t < \delta$ with some $\delta > 0$, $b(t) = 0$ for $|t| < \varepsilon$ with some $\varepsilon > 0$, and $c(t) = 0$ for $|t| > N$ with some $N > 0$. Due to Propositions 5.3.1–5.3.4,

$$aW(b)M^0(c) = \frac{a}{a_1^2} W(b/b_1) M^0(c/c_1) \left(a_1 W(b_1) M^0(c_1) a_1 I\right) + K_1$$

with a compact operator K_1. Since the operator $a_1 W(b_1) M^0(c_1)$ is compact, the assertion follows.

(ii) Write

$$c(t) = c(+\infty) \frac{1 + \coth\left((t + \mathbf{i}(1/p + \alpha))\pi\right)}{2} + c'(t).$$

Then

$$\begin{aligned} aW(b)M^0(c) &= aW(b)M^0(c') + c(+\infty) aW(b) M^0 \left(\frac{1 + \coth\left((t + \mathbf{i}(1/p + \alpha))\pi\right)}{2} \right) \\ &= aW(b)M^0(c') + c(+\infty) aW(b\chi_+) + K_2 \end{aligned}$$

with a compact operator K_2 (here we took into account Proposition 4.2.11). Since $b\chi_+ \equiv 0$ by assumption, and since $c'(\pm\infty) = 0$, this reduces our claim to the case previously considered in part (i). The proof of part (iii) proceeds analogously. ■

The next proposition concerns commutators with Hankel operators. Since the flip operator is involved, we assume the weight to be symmetric.

Proposition 5.3.6. *Let w be a symmetric weight on $\mathbb{R}$, i.e., $w(x) = w(-x)$. Then:*

- (i) *if $b \in C_{p,w}$ then $H(b) \in \mathcal{K}(L^p(\mathbb{R}^+, w))$;*
- (ii) *if $a \in \bar{L}^\infty(\mathbb{R}^+)$ and $b \in \dot{\mathcal{M}}_{p,w}$ with $a(+\infty) = 0$ and $b(\pm\infty) = 0$, then $aH(b)$ and $H(b)aI$ are in $\mathcal{K}(L^p(\mathbb{R}^+, w))$;*
- (iii) *if $a \in \bar{C}(\mathbb{R}^+)$ and $b \in \bar{\mathcal{M}}_{p,w}$, then $[aI, H(b)] \in \mathcal{K}(L^p(\mathbb{R}^+, w))$;*
- (iv) *if $a \in C_{p,w}$ and $b \in \mathcal{M}_{p,w}$ with a even, then $[W(a), H(b)] \in \mathcal{K}(L^p(\mathbb{R}^+, w))$;*
- (v) *if $a \in C_{p,w}$ and $b \in \mathcal{M}_{p,w}$, then $[W(a), W(b)] \in \mathcal{K}(L^p(\mathbb{R}^+, w))$.*

Proof. (i) As in the proof of Proposition 5.3.3, we can restrict ourselves to the case when $p = 2$ and $w \equiv 1$. Then $H(b)$ is unitarily equivalent to the operator

$$FH(b)F^{-1} = F\chi_+ F^{-1} bFJ\chi_+ F^{-1} = F\chi_+ F^{-1} bFJ\chi_- F^{-1} J = Q_{\mathbb{R}} b P_{\mathbb{R}} J,$$

which is compact.

(ii) Extend a symmetrically onto the whole axis. Then $a \in \dot{L}^\infty(\mathbb{R})$ with $a(\pm\infty) = 0$, and from Proposition 5.3.1 (i) we conclude

$$a\chi_+ W^0(b)\chi_- J = \chi_+ aW^0(b)\chi_- J \in \mathscr{K}(L^p(\mathbb{R},w))$$

and

$$\chi_+ W^0(b)\chi_- JaI = \chi_+ W^0(b)a\chi_- J \in \mathscr{K}(L^p(\mathbb{R},w)).$$

(iii) This follows from Proposition 5.3.1 (ii) via the same arguments as in (ii).
(iv) The identity

$$H(ab) = W(a)H(b) + H(a)W(\tilde{b}) \tag{5.9}$$

can be shown as (5.4). Consequently, if $\tilde{a} = a$, then

$$\begin{aligned} W(a)H(b) &= H(ab) - H(a)W(\tilde{b}) = H(ba) - H(a)W(\tilde{b}) \\ &= W(b)H(a) + H(b)W(a) - H(a)W(\tilde{b}) \end{aligned}$$

which implies the assertion since $H(a)$ is compact by (i).
(v) Finally, by (5.4),

$$W(a)W(b) - W(b)W(a) = H(b)H(\tilde{a}) - H(a)H(\tilde{b}).$$

Since a and $\tilde{a}$ are continuous, the assertion follows from (i). ■

5.4 Homogenization of convolution operators

Here we continue the technical preparation for the local study of algebras of convolution operators. In particular we show that the homogenization technique from Section 4.2.5 applies to large classes of convolutions. Recall the definitions of the operators U_t, V_s and Z_τ in (4.19), (4.20) and (4.37), respectively.

For $s \in \mathbb{R}$ and $-1/p < \alpha < 1 - 1/p$, let $w_{\alpha,s}$ be the weight on $\mathbb{R}$ defined by $w_{\alpha,s}(t) := |t-s|^\alpha$ and write w_α for $w_{\alpha,0}$. Let $A \in \mathscr{L}(L^p(\mathbb{R}, w_{\alpha,s}))$. If the strong limit

$$\operatorname*{s-lim}_{\tau\to+\infty} Z_\tau V_{-s} A V_s Z_\tau^{-1} \tag{5.10}$$

exists, we denote it by $\mathsf{H}_{s,\infty}(A)$. Analogously, if $A \in \mathscr{L}(L^p(\mathbb{R}, w_\alpha))$ and if the strong limit

$$\operatorname*{s-lim}_{\tau\to+\infty} Z_\tau^{-1} U_t A U_{-t} Z_\tau \tag{5.11}$$

exists for some $t \in \mathbb{R}$, we denote it by $\mathsf{H}_{\infty,t}(A)$. It is easy to see that the set of all operators for which the strong limits $\mathsf{H}_{s,\infty}(A)$ (resp. $\mathsf{H}_{\infty,t}(A)$) exist forms a Banach algebra, that

$$\|\mathsf{H}_{s,\infty}(A)\|_{\mathscr{L}(L^p(\mathbb{R},w_{\alpha,0}))} \le \|A\|_{\mathscr{L}(L^p(\mathbb{R},w_{\alpha,s}))} \tag{5.12}$$

respectively

$$\|\mathsf{H}_{\infty,t}(A)\|_{\mathscr{L}(L^p(\mathbb{R},w_{\alpha,0}))} \le \|A\|_{\mathscr{L}(L^p(\mathbb{R},w_{\alpha,0}))} \tag{5.13}$$

for all operators in these algebras and that, hence, the operators $\mathsf{H}_{s,\infty}$ and $\mathsf{H}_{\infty,t}$ are bounded homomorphisms.

For $x \in \mathbb{R}$, let $b(x^{\pm})$ denote the right/left one-sided limit of the piecewise continuous function b at x.

Proposition 5.4.1. *Let $a \in \bar{L}^{\infty}(\mathbb{R})$, $b \in PC_{p,w_\alpha}$ and $c \in \bar{\mathcal{M}}_p$. Then, for $t \in \mathbb{R}$:*

(i) $\mathsf{H}_{\infty,t}(aI) = a(-\infty)\chi_- I + a(+\infty)\chi_+ I$;
(ii) $\mathsf{H}_{\infty,t}(W^0(b)) = b(t^-)Q_{\mathbb{R}} + b(t^+)P_{\mathbb{R}}$;
(iii) $$\mathsf{H}_{\infty,t}(\chi_+ M^0(c)\chi_+ I + \chi_- I) = \begin{cases} c(+\infty)\chi_+ I + \chi_- I & \text{if } t > 0,\\ \chi_+ M^0(c)\chi_+ I + \chi_- I & \text{if } t = 0,\\ c(-\infty)\chi_+ I + \chi_- I & \text{if } t < 0. \end{cases}$$

Proof. Assertion (i) is immediate from Proposition 4.2.22 (ii) since $U_t a U_{-t} = aI$.
(ii) It is sufficient to prove the assertion for $t = 0$. Write b as $b(0^-)\chi_- + b(0^+)\chi_+ + b_0$ where the function $b_0 \in PC_{p,w_\alpha}$ is continuous at 0 and $b_0(0) = 0$. Since

$$W^0\left(b(0^-)\chi_- + b(0^+)\chi_+\right) = b(0^-)Q_{\mathbb{R}} + b(0^+)P_{\mathbb{R}}$$

and the operators $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$ commute with Z_τ, it remains to show that

$$Z_\tau^{-1} W^0(b_0) Z_\tau \to 0 \quad \text{strongly as } \tau \to \infty.$$

By the definition of the class PC_{p,w_α} and by Proposition 5.1.1, we can approximate the function b_0 in the multiplier norm as closely as desired by a piecewise constant function b_{00} which is zero in an open neighborhood U of 0. It is thus sufficient to show that

$$Z_\tau^{-1} W^0(b_{00}) Z_\tau \to 0 \quad \text{strongly as } \tau \to \infty.$$

Since the operators on the left-hand side are uniformly bounded with respect to τ, it is further sufficient to show that

$$Z_\tau^{-1} W^0(b_{00}) Z_\tau u \to 0$$

for all functions u in a certain dense subset of $L^p(\mathbb{R}, w_\alpha)$. For, consider the set of all functions in the Schwartz space $\mathscr{S}(\mathbb{R})$, the Fourier transform of which has compact support. This space is indeed dense in $L^p(\mathbb{R}, w_\alpha)$ since the space $\mathscr{D}(\mathbb{R})$ of the compactly supported infinitely differentiable functions is dense in $\mathscr{S}(\mathbb{R})$ ([171, Theorem 7.10]), since the Fourier transform F is a continuous bijection on $\mathscr{S}(\mathbb{R})$, and since $\mathscr{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R}, w_\alpha)$. In this special setting, the latter fact can easily be proved by hand. Note in this connection that already $\mathscr{D}(\mathbb{R})$ is dense in $L^p(\mathbb{R}, w)$ for *every* Muckenhoupt weight w; see [80, Exercise 9.4.1]. If u is a function with these properties, then

$$Z_\tau^{-1} W^0(b_{00}) Z_\tau u = F^{-1} Z_\tau b_{00} Z_\tau^{-1} F u. \tag{5.14}$$

If τ is sufficiently large, then the support of Fu is contained in U; hence, the function on the right-hand side of (5.14) is the zero function.

(iii) For $t = 0$, the assertion follows immediately from Lemma 4.2.13 and the fact that $Z_\tau \chi_- I = \chi_- Z_\tau$. Let $t \neq 0$. Then, clearly, $U_{-t} Z_\tau = Z_\tau U_{-t\tau}$ and $Z_\tau^{-1} U_t = U_{t\tau} Z_\tau^{-1}$, whence

$$\begin{aligned} Z_\tau^{-1} U_t (\chi_+ M^0(c) \chi_+ I + \chi_- I) U_{-t} Z_\tau &= U_{t\tau} Z_\tau^{-1} (\chi_+ M^0(c) \chi_+ I + \chi_- I) Z_\tau U_{-t\tau} \\ &= U_{t\tau} (\chi_+ M^0(c) \chi_+ I + \chi_- I) U_{-t\tau} \end{aligned}$$

due to Lemma 4.2.13. Thus,

$$\begin{aligned} &\mathsf{H}_{\infty,t}(\chi_+ M^0(c) \chi_+ I + \chi_- I) \\ &= \begin{cases} \text{s-lim}_{\tau \to +\infty} U_\tau (\chi_+ M^0(c) \chi_+ I + \chi_- I) U_{-\tau} & \text{if } t > 0, \\ \text{s-lim}_{\tau \to -\infty} U_\tau (\chi_+ M^0(c) \chi_+ I + \chi_- I) U_{-\tau} & \text{if } t < 0. \end{cases} \end{aligned} \tag{5.15}$$

To deal with the strong limits (5.15), suppose first that c is a polynomial in the function $\coth\big((\cdot + \mathbf{i}(1/p + \alpha))\pi\big)$, i.e.

$$c(t) = \sum_{k=0}^{n} c_k \coth^k \big((t + \mathbf{i}(1/p + \alpha))\pi\big),$$

with certain constants c_k. Then

$$\chi_+ M^0(c) \chi_+ I + \chi_- I = \chi_+ \left(\sum_{k=0}^{n} c_k (W(\text{sgn}))^k \right) \chi_+ I + \chi_- I. \tag{5.16}$$

So it remains to consider the strong limits

$$\operatorname*{s-lim}_{\tau \to \pm\infty} U_\tau (\chi_+ W^0(\text{sgn}) \chi_+ I + \chi_- I) U_{-\tau}.$$

Since $U_\tau W^0(\text{sgn}) U_{-\tau} = W^0(V_{-\tau} \text{sgn} V_\tau)$ by Lemma 4.2.4, we have just to check the strong convergence of $V_{-\tau} \text{sgn} V_\tau$ as $\tau \to \pm\infty$. One has

$$V_{-\tau} a V_\tau \to \begin{cases} a(+\infty) I & \text{as } \tau \to +\infty, \\ a(-\infty) I & \text{as } \tau \to -\infty \end{cases} \tag{5.17}$$

for every function $a \in \bar{\mathcal{M}}_{p,w_\alpha}$. Thus,

$$U_\tau W^0(\text{sgn}) U_{-\tau} \to \pm I \quad \text{as } \tau \to \pm\infty$$

whence, via (5.16),

$$\operatorname*{s-lim}_{\tau \to \pm\infty} U_\tau (\chi_+ M^0(c) \chi_+ I + \chi_- I) U_{-\tau} = \chi_+ \sum_{k=0}^{n} c_k (\pm 1)^k \chi_+ I + \chi_- I = c(\pm\infty) \chi_+ I + \chi_- I$$

for each polynomial c. Since the set of all polynomials is dense in $\bar{C}_p$, we obtain assertion (iii) for functions $c \in \bar{C}_p$. To treat the general case, let $c \in \bar{\mathcal{M}}_p$ and write c as

$$c(t) = c_+ + c_- \coth\big((t + \mathbf{i}(1/p+\alpha))\pi\big) + c'(t)$$

with

$$c_\pm := (c(+\infty) \pm c(-\infty))/2$$

and with a function $c' \in \dot{\mathcal{M}}_p$ with $c'(\pm\infty) = 0$. After what has just been proved, we are left to verify that

$$\mathsf{H}_{\infty,t}(\chi_+ M^0(c')\chi_+ I + \chi_- I) = \chi_- I \quad \text{for } t \neq 0.$$

Without loss of generality, we can assume that the support of c' is compact. Choose a function $u \in C_0^\infty(\mathbb{R})$ with total variation 2 which is identically 1 on the support of c'. Then

$$\begin{aligned} U_\tau \chi_+ M^0(c')\chi_+ U_{-\tau} &= U_\tau \chi_+ M^0(c'u)\chi_+ U_{-\tau} \\ &= \left(U_\tau \chi_+ M^0(c')\chi_+ U_{-\tau}\right)\left(U_\tau \chi_+ M^0(u)\chi_+ U_{-\tau}\right). \end{aligned}$$

The operators in the first parentheses are uniformly bounded with respect to τ, whereas the operators in the second ones tend strongly to zero as $\tau \to \pm\infty$ by what we have already shown. ■

The assertions of the following lemma are either taken directly from the preceding proof, or they follow by repeating some arguments of that proof.

Lemma 5.4.2.
(i) *If $a \in \bar{L}^\infty(\mathbb{R})$, then $V_{-\tau} a V_\tau \to a(\pm\infty)I$ as $\tau \to \pm\infty$.*
(ii) *If $b \in \bar{\mathcal{M}}_{p,w_\alpha}$, then $U_\tau W^0(b) U_{-\tau} \to b(\pm\infty)I$ as $\tau \to \pm\infty$.*
(iii) *If $c \in \bar{\mathcal{M}}_p$, then $U_\tau M^0(c) U_{-\tau} \to c(\pm\infty)I$ as $\tau \to \pm\infty$.*

The following is the analog of Proposition 5.4.1 for the second family of strong limits.

Proposition 5.4.3. *Let $s \in \mathbb{R}$, $a \in PC$, $b \in \bar{\mathcal{M}}_{p,w_{\alpha,s}}$ and $c \in \bar{\mathcal{M}}_p$. Then:*

(i) $\mathsf{H}_{s,\infty}(aI) = a(s^-)\chi_- I + a(s^+)\chi_+ I$;
(ii) $\mathsf{H}_{s,\infty}(W^0(b)) = b(-\infty)Q_\mathbb{R} + b(+\infty)P_\mathbb{R}$;
(iii) $$\mathsf{H}_{s,\infty}(\chi_+ M^0(c)\chi_+ I + \chi_- I) = \begin{cases} c(-\infty)Q_\mathbb{R} + c(+\infty)P_\mathbb{R} & \text{if } s > 0, \\ \chi_+ M^0(c)\chi_+ I + \chi_- I & \text{if } s = 0, \\ I & \text{if } s < 0. \end{cases}$$

Proof. Assertion (i) is immediate from Proposition 4.2.22 (ii) since $(V_{-s} a V_s)(t) = a(t+s)$. For assertion (ii), one uses Lemma 4.2.4 and Proposition 4.2.22 (ii) as in the proof of Proposition 5.4.1(ii). Note that $V_{-s} W^0(b) V_s = W^0(b)$. For $s = 0$, assertion (iii) is a consequence of Lemma 4.2.13 and the commutativity of $\chi_- I$ and Z_τ, and for $s < 0$ it follows from the already proved part (i). For $s > 0$, write

$$c(t) = c_+ + c_- \coth\big((t + \mathbf{i}(1/p+\alpha))\pi\big) + c'(t)$$

with $c_\pm := (c(+\infty) \pm c(-\infty))/2$. First note that

$$\mathsf{H}_{s,\infty}\Big(\chi_+ M^0\Big(c_+ + c_- \coth\big((\cdot + \mathbf{i}(1/p+\alpha))\pi\big) + c'(t)\Big)\chi_+ I + \chi_- I\Big)$$
$$= c_+ I + c_- S_{\mathbb{R}} \;=\; c(-\infty)\frac{I - S_{\mathbb{R}}}{2} + c(+\infty)\frac{I + S_{\mathbb{R}}}{2}.$$

Indeed, since

$$M^0\Big(c_+ + c_- \coth\big((\cdot + \mathbf{i}(1/p+\alpha))\pi\big)\Big) = c_+\chi_+ I + c_- S_{\mathbb{R}^+} = c_+\chi_+ I + c_- W(\mathrm{sgn}),$$

this follows easily from what we proved in parts (i) and (ii). Next, similarly to the proof of part (iii) of Proposition 5.4.1, one verifies the assertion for c being a polynomial in $\coth\big((\cdot + \mathbf{i}(1/p+\alpha))\pi\big)$. Finally, one shows that $\mathsf{H}_{s,\infty}(\chi_+ M^0(c')\chi_+ I + \chi_- I) = 0$, again by employing the approximation arguments from the proof of Proposition 5.4.1. ∎

Proposition 5.4.4.
(i) *If K is a compact operator on $L^p(\mathbb{R}, w_\alpha)$, then $\mathsf{H}_{\infty,s}(K) = 0$ for all $s \in \mathbb{R}$.*
(ii) *If K is compact on $L^p(\mathbb{R}, w_{\alpha,s})$, then $\mathsf{H}_{s,\infty}(K) = 0$ for all $s \in \mathbb{R}$.*

The proof runs as that of Proposition 4.2.22 (iii).

5.5 Algebras of multiplication, Wiener-Hopf and Mellin operators

Given subsets $X \subseteq L^\infty(\mathbb{R}^+)$, $Y \subseteq \mathcal{M}_{p,w_\alpha}$ and $Z \subseteq \mathcal{M}_p$, we let $\mathscr{A}(X,Y,Z)$ denote the smallest closed subalgebra of the algebra of all bounded linear operators on $L^p(\mathbb{R}^+, w_\alpha)$ which contains all multiplication operators aI with $a \in X$, all Wiener-Hopf operators $W(b)$ with $b \in Y$, and all Mellin convolutions $M^0(c)$ with $c \in Z$. By $\mathscr{A}^{\mathscr{K}}(X,Y,Z)$ we denote the image of $\mathscr{A}(X,Y,Z)$ in the Calkin algebra over $L^p(\mathbb{R}^+, w_\alpha)$, and we write Φ for the corresponding canonical homomorphism.

The invertibility of elements of the algebra $\mathscr{A}^{\mathscr{K}}(X,Y,Z)$ will again be studied by using Allan's local principle. Thus we must single out central subalgebras of this algebra which are suitable for localization. For special choices of X, Y and Z, this is done in the next proposition, which follows immediately from Propositions 5.3.1–5.3.6.

Proposition 5.5.1. *In each of the cases below, $\mathscr{B}$ is a central subalgebra of $\mathscr{A}$:*

(i) $\mathscr{A} = \mathscr{A}^{\mathscr{K}}\big(L^\infty(\mathbb{R}^+), \bar{\mathcal{M}}_{p,w_\alpha}, \bar{\mathcal{M}}_p\big)$ *and* $\mathscr{B} = \mathscr{A}^{\mathscr{K}}\big(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, C(\overline{\mathbb{R}}) \cap PC_p\big)$;
(ii) $\mathscr{A} = \mathscr{A}^{\mathscr{K}}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ *and* $\mathscr{B} = \mathscr{A}^{\mathscr{K}}\big(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, C_p\big)$;

(iii) $\mathscr{A} = \mathscr{A}^{\mathscr{K}}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\right)$ *and* $\mathscr{B} = \mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$*;*
(iv) $\mathscr{A} = \mathscr{A}^{\mathscr{K}}\left(\bar{C}(\mathbb{R}^+), PC_{p,w_\alpha}, C(\overline{\mathbb{R}}) \cap PC_p\right)$ *and* $\mathscr{B} = \mathscr{A}$*;*
(v) $\mathscr{A} = \mathscr{A}^{\mathscr{K}}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, \emptyset\right)$ *and* $\mathscr{B} = \mathscr{A}^{\mathscr{K}}\left(\bar{C}(\mathbb{R}^+), C_{p,w_\alpha}, \emptyset\right)$.

It is evident that the setting of case (i) is too general for a successful analysis. It will be our goal in this section to examine cases (ii) and (iii). One peculiarity of the present context is that, in general, $(\mathscr{A}, \mathscr{B})$ is not a faithful localizing pair unless $p = 2$ (since one has to localize over algebras of multipliers). Thus, one cannot expect the same elegant and complete results as for the algebra $\mathscr{A}^{\mathscr{K}}(\Gamma, w)$ considered in the previous chapter. The objectives of this section are quite modest when compared with Chapter 4: We will only derive necessary and sufficient conditions for the invertibility of cosets in $\mathscr{A}^{\mathscr{K}}$, and we will show that this algebra is inverse-closed in the Calkin algebra, i.e., that invertibility in $\mathscr{A}^{\mathscr{K}}$ is equivalent to the Fredholm property. On the other hand, we will at least be able to establish *isometrically isomorphic* representations of the *local algebras* that arise. If $p = 2$, the localizing pairs become faithful, and one gets an isometrically isomorphic representation of the (global) algebra $\mathscr{A}^{\mathscr{K}}$.

We derive the maximal ideal spaces for some algebras $\mathscr{B}$ which appear in the above proposition. Let us start with two very simple situations. Corollary 1.4.9 and Proposition 1.4.11 (which we need here for weighted L^p-spaces) imply that the maximal ideal space of the commutative Banach algebra $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), \emptyset, \emptyset\right)$ is homeomorphic to the one-point compactification $\dot{\mathbb{R}}^+$ of $\mathbb{R}^+$ by the point $\infty = 0$ (and, thus, homeomorphic to a circle). The maximal ideal which corresponds to $s \in \dot{\mathbb{R}}^+$ is equal to $\{\Phi(aI) : a \in \dot{C}(\mathbb{R}^+), a(s) = 0\}$.

Taking into account Proposition 5.1.2 (i), it is also not hard to see that the maximal ideal space of the commutative Banach algebra $\mathscr{A}^{\mathscr{K}}\left(\emptyset, C^0_{p,w_\alpha}, \emptyset\right)$ is homeomorphic to the compactification of $\mathbb{R}$ which arises by identifying the three points $-\infty$, 0, and $+\infty$. We denote this compactification by $\dot{\mathbb{R}}_0$. One can think of $\dot{\mathbb{R}}_0$ as the

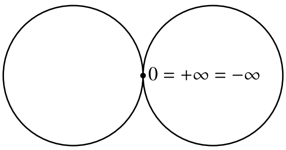

Fig. 5.1 The maximal ideal space of the algebra $\mathscr{A}^{\mathscr{K}}\left(\emptyset, C^0_{p,w_\alpha}, \emptyset\right)$.

the union of two circles which have exactly one point, ∞ say, in common (see Figure 5.1). The maximal ideal of $\mathscr{A}^{\mathscr{K}}\left(\emptyset, C^0_{p,w_\alpha}, \emptyset\right)$ which corresponds to the point $s \in \dot{\mathbb{R}}_0$ is then

$$\begin{aligned} &\left\{\Phi(W(a)) : a \in C^0_{p,w_\alpha},\, a(s) = 0\right\} && \text{if } s \neq \infty, \\ &\left\{\Phi(W(a)) : a \in C^0_{p,w_\alpha},\, a(0) = a(\pm\infty) = 0\right\} && \text{if } s = \infty. \end{aligned}$$

For the next result, we have to combine the maximal ideal spaces $\dot{\mathbb{R}}^+$ and $\dot{\mathbb{R}}_0$ of these algebras.

Proposition 5.5.2. *The maximal ideal space of the commutative Banach algebra* $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$ *is homeomorphic to that subset of the "double torus"* $\dot{\mathbb{R}}^+ \times \dot{\mathbb{R}}_0$ *which consists of the circle* $\dot{\mathbb{R}}^+ \times \{\infty\}$ *and the "double circle"* $\{\infty\} \times \dot{\mathbb{R}}_0$. *In particular, the value of the Gelfand transform of the coset* $\Phi(aW(b))$ *with* $a \in \dot{C}(\mathbb{R}^+)$ *and* $b \in C^0_{p,w_\alpha}$ *at the point* $(s,t) \in (\dot{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}}_0)$ *is* $a(s)b(t)$.

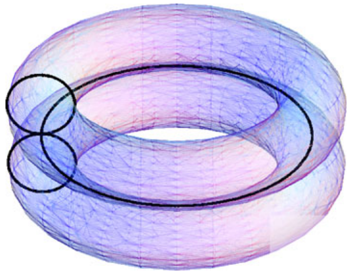

Fig. 5.2 The maximal ideal space of the algebra $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$. The intersection point is $\infty \times \infty$ (or 0×0); the points on the single circle are of the form $s \times \infty$, and the ones on the double circle of the form $\infty \times t$.

Proof. Let $\mathscr{J}$ be a maximal ideal of $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$. By Proposition 2.2.1, $\mathscr{J} \cap \mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), \emptyset, \emptyset\right)$ and $\mathscr{J} \cap \mathscr{A}^{\mathscr{K}}\left(\emptyset, C^0_{p,w_\alpha}, \emptyset\right)$ are maximal ideals of $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), \emptyset, \emptyset\right)$ and $\mathscr{A}^{\mathscr{K}}\left(\emptyset, C^0_{p,w_\alpha}, \emptyset\right)$, respectively. Thus, there are points $s \in \dot{\mathbb{R}}^+$ and $t \in \dot{\mathbb{R}}_0$ such that the value of the Gelfand transform of the coset $\Phi(aW(b))$ at the ideal $\mathscr{J}$ equals $a(s)b(t)$ for each choice of $a \in \dot{C}(\mathbb{R}^+)$ and $b \in C^0_{p,w_\alpha}$. Hence, the maximal ideal space of the algebra $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$ can be identified with a subset of the double torus $\dot{\mathbb{R}}^+ \times \dot{\mathbb{R}}_0$.

Now let $s \in \dot{\mathbb{R}}^+ \setminus \{\infty\}$ and $t \in \dot{\mathbb{R}}_0 \setminus \{\infty\}$. Given functions $a \in \dot{C}(\mathbb{R}^+)$ and $b \in C^0_{p,w_\alpha}$, choose functions $a' \in C_0^\infty(\mathbb{R}^+)$ and $b' \in C_0^\infty(\mathbb{R})$ of finite total variation such that $a(s) = a'(s)$, $b(t) = b'(t)$ and $0 \notin \operatorname{supp} b'$. Then,

$$aW(b) = (a-a')W(b-b') + (a-a')W(b') + a'W(b-b') + a'W(b').$$

The first three items of the sum on the right-hand side belong to the ideal $\mathscr{J} = (s,t)$, whereas while the fourth item is compact by Proposition 5.3.1. Thus, the smallest closed ideal of $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$ which corresponds to (s,t) with $s \in \dot{\mathbb{R}}^+ \setminus \{\infty\}$ and $t \in \dot{\mathbb{R}}_0 \setminus \{\infty\}$ coincides with the whole algebra. So, the maximal ideals of the algebra under consideration can only correspond to points (s,t) from $(\dot{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}}_0)$. On the other hand, each of these points gives a maximal ideal of $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset\right)$, which is a consequence of Theorem 2.1.9 (ii). Since the

Shilov boundaries of the algebras $\mathscr{A}^{\mathscr{K}}\left(\dot{C}(\mathbb{R}^+),\emptyset,\emptyset\right)$ and $\mathscr{A}^{\mathscr{K}}\left(\emptyset,C^0_{p,w_\alpha},\emptyset\right)$ coincide with $\dot{\mathbb{R}}^+$ and $\dot{\mathbb{R}}_0$, respectively (recall Exercise 2.1.7), the assertion follows. ■

For $s \in \dot{\mathbb{R}}^+ \times \{\infty\}$ and $t \in \{\infty\} \times \dot{\mathbb{R}}_0$, let $\mathscr{I}_{s,t}$ denote the smallest closed ideal of the Banach algebra $\mathscr{A}^{\mathscr{K}}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$ which contains the ideal (s,t), and let $\Phi^{\mathscr{K}}_{s,t}$ refer to the canonical homomorphism from $\mathscr{A}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$ onto the local quotient algebra

$$\mathscr{A}^{\mathscr{K}}_{s,t}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right) := \mathscr{A}^{\mathscr{K}}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)/\mathscr{I}_{s,t}.$$

As in Section 4.2.3, $\mathscr{E}_{p,\alpha}$ denotes the smallest closed subalgebra of $\mathscr{L}\left(L^p(\mathbb{R}^+,t^\alpha)\right)$ which contains the identity operator $I=\chi_+$ and the operator $S_{\mathbb{R}^+}$. We provide a description of the local algebras of $\mathscr{A}^{\mathscr{K}}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$ in a couple of separate statements.

Theorem 5.5.3. *Let $s=\infty$ and $t \in \dot{\mathbb{R}}_0 \setminus \{\infty\}$. Then the local algebra*

$$\mathscr{A}^{\mathscr{K}}_{s,t}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$$

is isometrically isomorphic to $\mathscr{E}_{p,\alpha}$. The isomorphism is given by

$$\Phi^{\mathscr{K}}_{s,t}(A) \mapsto \mathsf{H}_{s,t}(A) \tag{5.18}$$

for each operator $A \in \mathscr{A}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$. In particular, for $a \in PC(\mathbb{R}^+)$, $b \in PC_{p,w_\alpha}$ and $c \in PC_p$,

$$\begin{aligned}
\mathsf{H}_{s,t}(aI) &= a(+\infty)I,\\
\mathsf{H}_{s,t}(W(b)) &= b(t^-)\frac{I-S_{\mathbb{R}^+}}{2} + b(t^+)\frac{I+S_{\mathbb{R}^+}}{2},\\
\mathsf{H}_{s,t}(M^0(c)) &= \begin{cases} c(+\infty)I & \text{if } t>0,\\ c(-\infty)I & \text{if } t<0.\end{cases}
\end{aligned}$$

Proof. Let $A \in \mathscr{A}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$. First we show that the mapping (5.18) is correctly defined in the sense that the operator $\mathsf{H}_{s,t}(A)$ depends only on the local coset $\Phi^{\mathscr{K}}_{s,t}(A)$ of A. Indeed, by Proposition 5.4.4, the ideal $\mathscr{K}\left(L^p(\mathbb{R}^+,w_\alpha)\right)$ is contained in the kernel of the operator $\mathsf{H}_{s,t}$. Hence, $\mathsf{H}_{s,t}(A)$ depends only on the coset $\Phi(A)$ of A. Moreover, if $b \in C_{p,w_\alpha}$ and $b(t)=0$ then, by Proposition 5.4.1 (ii), $\mathsf{H}_{s,t}\left(\Phi(W(b))\right)=0$. Consequently, the operator $\mathsf{H}_{s,t}(A)$ depends only on the coset $\Phi^{\mathscr{K}}_{s,t}(A)$ of A in the local algebra.

It follows from the definition of $\mathsf{H}_{s,t}$ that (5.18) is a bounded algebra homomorphism with a norm not greater than 1. The images of the operators aI, $W(b)$ and $M^0(c)$ under this homomorphism were studied in Proposition 5.4.1. From the concrete form of these images, one concludes that $\mathsf{H}_{s,t}$ is in fact a mapping onto $\mathscr{E}_{p,\alpha}$.

It remains to show that the homomorphism (5.18) is an isometry and, hence, an isomorphism. To that end we prove that, for each $A \in \mathscr{A}\left(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\right)$,

$$\Phi_{s,t}^{\mathcal{K}}(A) = \Phi_{s,t}^{\mathcal{K}}(U_{-t}\mathsf{H}_{s,t}(A)U_t). \tag{5.19}$$

Once this equality is verified, the assertion will follow from

$$\|\Phi_{s,t}^{\mathcal{K}}(A)\| = \|\Phi_{s,t}^{\mathcal{K}}(U_{-t}\mathsf{H}_{s,t}(A)U_t)\| \le \|U_{-t}\mathsf{H}_{s,t}(A)U_t\| \le \|\mathsf{H}_{s,t}(A)\| \le \|\Phi_{s,t}^{\mathcal{K}}(A)\|$$

by (5.12). So we are left to verify the identity (5.19). Since $\Phi_{s,t}^{\mathcal{K}}$ and $\mathsf{H}_{s,t}$ are continuous homomorphisms, it suffices to check (5.19) with A replaced by the operators aI, $W(b)$ and $M^0(c)$. Let $A = aI$ with $a \in PC(\mathbb{R}^+)$. Then (5.19) reduces to

$$\Phi_{s,t}^{\mathcal{K}}(aI) = \Phi_{s,t}^{\mathcal{K}}(a(\infty)I). \tag{5.20}$$

Choose $f \in \dot{C}(\mathbb{R}^+)$ with $f(\infty) = f(0) = 1$ and such that the support of f is contained in $[0,1] \cup [N,\infty]$ with N large enough, and write f as $f_0 + f_\infty$ with

$$f_0(t) = \begin{cases} f(t) & \text{if } t \in [0,1], \\ 0 & \text{if } t \in]1,\infty] \end{cases} \quad \text{and} \quad f_\infty(t) = \begin{cases} f(t) & \text{if } t \in [N,\infty], \\ 0 & \text{if } t \in [0,N[\,. \end{cases}$$

Further choose $g \in C_{p,w_\alpha}^0$ with $g(t) = 1$ and $g(\infty) = g(0) = 0$. Then, obviously, $\Phi_{s,t}^{\mathcal{K}}(fW(g)) = \Phi_{s,t}^{\mathcal{K}}(I)$. From this equality and from

$$fW(g) = f_0W(g) + f_\infty W(g) = f_\infty W(g) + \text{compact},$$

by Proposition 5.3.1 (i) we obtain

$$\begin{aligned} \|\Phi_{s,t}^{\mathcal{K}}\big((a-a(\infty))I\big)\| &= \|\Phi_{s,t}^{\mathcal{K}}\big((a-a(\infty))fW(g)\big)\| \\ &= \|\Phi_{s,t}^{\mathcal{K}}\big((a-a(\infty))f_\infty W(g)\big)\| \\ &\le \|(a-a(\infty))f_\infty\|_\infty \|W(g)\|. \end{aligned}$$

The right-hand side of this estimate can be made as small as desired if N is chosen large enough. Now let $A = W(b)$ with $b \in PC_{p,w_\alpha}$. Let χ_t refer to the characteristic function of the interval $[t,+\infty]$, and choose the function g as above, but with the additional property that g has total variation 2. Using Proposition 5.4.1 (ii), we then conclude that

$$\begin{aligned} &\left\|\Phi_{s,t}^{\mathcal{K}}\Big(W(b) - U_{-t}\mathsf{H}_{s,t}\big(W(b)\big)U_t\Big)\right\| \\ &= \left\|\Phi_{s,t}^{\mathcal{K}}\Big(W(b) - U_{-t}\big(W(b(t^-)\chi_- + b(t^+)\chi_+)\big)U_tW(g)\Big)\right\| \\ &= \left\|\Phi_{s,t}^{\mathcal{K}}\Big(W\big(b - (b(t^-)(1-\chi_t) + b(t^+)\chi_t)\big)W(g)\Big)\right\| \\ &\le \left\|\big((b - (b(t^-)(1-\chi_t) + b(t^+)\chi_t))g\big)\right\|_{\mathcal{M}_{p,w_\alpha}}. \end{aligned}$$

The right-hand side of this estimate becomes as small as desired if the support of g is chosen small enough. Finally, let $A = M^0(c)$ with $c \in PC_p$. For definiteness, let $t > 0$. Then (5.19) reduces to

$$\Phi_{s,t}^{\mathscr{K}}\left(M^0(c)\right) = \Phi_{s,t}^{\mathscr{K}}\left(c(+\infty)I\right).$$

To verify this equality, choose f, f_0, f_∞ and g as above and suppose that supp $g \subseteq \mathbb{R}^+$. Then

$$\begin{aligned}\Phi_{s,t}^{\mathscr{K}}\left(M^0(c-c(+\infty))\right) &= \Phi_{s,t}^{\mathscr{K}}\left(fW(g)M^0(c-c(+\infty))\right)\\ &= \Phi_{s,t}^{\mathscr{K}}\left(f_\infty W(g)M^0(c-c(+\infty))\right) = 0\end{aligned}$$

since $f_\infty W(g)M^0(c-c(+\infty))$ is a compact operator by Proposition 5.3.5 (iii). ■

Let $\operatorname{alg}\{I, \chi_+ I, S_\mathbb{R}\}$ denote the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}))$ which contains the operators $I, \chi_+ I$ and $S_\mathbb{R}$. The following theorem identifies a second family of the local algebras.

Theorem 5.5.4. *Let $s \in \dot{\mathbb{R}}^+ \setminus \{\infty\}$ and $t = \infty$. Then the local algebra*

$$\mathscr{A}_{s,t}^{\mathscr{K}}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\right)$$

is isometrically isomorphic to the subalgebra $\operatorname{alg}\{I, \chi_+ I, S_\mathbb{R}\}$ *of* $\mathscr{L}(L^p(\mathbb{R}))$. *The isomorphism is given by*

$$\Phi_{s,t}^{\mathscr{K}}(A) \mapsto \mathsf{H}_{s,t}(A) \tag{5.21}$$

for each operator $A \in \mathscr{A}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\right)$. In particular, for $a \in PC(\mathbb{R}^+)$, $b \in PC_{p,w_\alpha}$ and $c \in PC_p$,

$$\begin{aligned}\mathsf{H}_{s,t}(aI) &= a(s^-)\chi_- I + a(s^+)\chi_+ I,\\ \mathsf{H}_{s,t}(W(b)) &= b(-\infty)\frac{I-S_\mathbb{R}}{2} + b(+\infty)\frac{I+S_\mathbb{R}}{2},\\ \mathsf{H}_{s,t}(M^0(c)) &= c(-\infty)\frac{I-S_\mathbb{R}}{2} + c(+\infty)\frac{I+S_\mathbb{R}}{2}.\end{aligned}$$

Proof. Taking into account that the weight function w_α is locally non-trivial only at the points 0 and ∞, one can show by repeating the arguments of the proof of Proposition 4.3.2 that the local algebras $\mathscr{A}_{s,t}^{\mathscr{K}}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\right)$ generated by operators acting on $L_p(\mathbb{R}^+, w_\alpha)$ and $\mathscr{A}_{s,t}^{\mathscr{K}}\left(PC(\mathbb{R}^+), PC_p, PC_p\right)$ generated by operators on $L_p(\mathbb{R}^+)$ are isometrically isomorphic. So we shall only deal with the latter algebra.

The correctness of the definition (5.21) as well as the fact that it defines a bounded algebra homomorphism with norm not greater than 1 can be checked as in the preceding proof. The values of this homomorphism at the operators aI, $W(b)$ and $M^0(c)$ are a consequence of Proposition 5.4.3, from which we also conclude that (5.21) maps the local algebra onto $\operatorname{alg}\{I, \chi_+ I, S_\mathbb{R}\}$. That this homomorphism is an isometry and, hence, an isomorphism, will follow once we have shown that

$$\Phi_{s,t}^{\mathscr{K}}(A) = \Phi_{s,t}^{\mathscr{K}}\left(\chi_+ V_s \mathsf{H}_{s,t}(A) V_{-s} \chi_+ I|_{L^p(\mathbb{R}^+)}\right) \tag{5.22}$$

for all operators A in $\mathscr{A}\left(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\right)$. To verify this identity, it is again sufficient to check it for the generating operators aI, $W(b)$ and $M^0(c)$ in place of A.

Let $A = aI$ with $a \in PC(\mathbb{R}^+)$. Then (5.22) states that

$$\Phi_{s,t}^{\mathscr{K}}(aI) = \Phi_{s,t}^{\mathscr{K}}\left(a(s^-)(1-\chi_s)\chi_+ I + a(s^+)\chi_s I\right). \tag{5.23}$$

To prove this equality, choose a function $f \in \dot{C}(\mathbb{R}^+)$ with $f(s) = 1$ and compact support and a function $g \in C_p^0$ with $g(\infty) = 1$ with support in $[-\infty, -N] \cup [-1, 1] \cup [N, +\infty]$ where N is chosen sufficiently large. Then

$$\begin{aligned} &\left\|\Phi_{s,t}^{\mathscr{K}}\left(aI - a(s^-)(1-\chi_s)\chi_+ I + a(s^+)\chi_s I\right)\right\| \\ &= \left\|\Phi_{s,t}^{\mathscr{K}}\left((aI - a(s^-)(1-\chi_s)\chi_+ + a(s^+)\chi_s)fW(g)\right)\right\| \\ &\leq \left\|\left((aI - a(s^-)(1-\chi_s)\chi_+ + a(s^+)\chi_s)f\right)\right\|_\infty \|W(g)\|, \end{aligned}$$

and the right-hand side of this estimate becomes as small as desired if supp f is chosen small enough.

Now let $A = W(b)$ with $b \in PC_{p,w_\alpha}$. Choose f and g as above and write g as $g_0 + g_\infty$ with a function g_0 vanishing outside the interval $[-1, 1]$. Then, according to Proposition 5.3.1 (i), $\Phi_{s,t}^{\mathscr{K}}(I) = \Phi_{s,t}^{\mathscr{K}}(fW(g)) = \Phi_{s,t}^{\mathscr{K}}(fW(g_\infty))$, whence

$$\begin{aligned} &\left\|\Phi_{s,t}^{\mathscr{K}}\left(W(b - b(-\infty)\chi_- - b(+\infty)\chi_+)\right)\right\| \\ &= \left\|\Phi_{s,t}^{\mathscr{K}}\left(W\left((b - b(-\infty)\chi_- - b(+\infty)\chi_+)g_\infty\right) fI\right)\right\| \\ &\leq \left\|(b - b(-\infty)\chi_- - b(+\infty)\chi_+)g_\infty\right\|_{\mathcal{M}_p} \|f\|_\infty. \end{aligned}$$

Again, the norm on the right-hand side becomes arbitrarily small if N is chosen large enough. Hence,

$$\Phi_{s,t}^{\mathscr{K}}(W(b)) = \Phi_{s,t}^{\mathscr{K}}\left(b(-\infty)W(\chi_-) + b(+\infty)W(\chi_+)\right) \tag{5.24}$$

which verifies (5.22) for $A = W(b)$. Finally, let $A = M^0(c)$ with $c \in PC_p$. Now one has to show that

$$\Phi_{s,t}^{\mathscr{K}}(M^0(c)) = \Phi_{s,t}^{\mathscr{K}}\left(c(-\infty)\frac{I - S_{\mathbb{R}^+}}{2} + c(+\infty)\frac{I + S_{\mathbb{R}^+}}{2}\right). \tag{5.25}$$

For, write c as

$$c(t) = c(-\infty)\frac{1 - \coth\left((t + \mathbf{i}/p)\pi\right)}{2} + c(+\infty)\frac{1 + \coth\left((t + \mathbf{i}/p)\pi\right)}{2} + c'(t).$$

Then $M^0(c) = c(-\infty)\frac{I - S_{\mathbb{R}^+}}{2} + c(+\infty)\frac{I + S_{\mathbb{R}^+}}{2} + M^0(c')$, and it remains to show that $\Phi_{s,t}^{\mathscr{K}}\left(M^0(c')\right)$ is the zero coset. For, choose f and $g = g_0 + g_\infty$ as above and take into account that $\Phi^{\mathscr{K}}_{s,t}(fW(g_\infty)) = \Phi^{\mathscr{K}}_{s,t}(I)$ and that the operator $fW(g_\infty)M^0(c')$ is compact by Proposition 5.3.5. ■

One can show by the same arguments as above that Theorems 5.5.3 and 5.5.4 remain valid if the algebra $\mathscr{A}^{\mathscr{K}}(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p)$ is replaced by the larger algebra $\mathscr{A}^{\mathscr{K}}(PC(\mathbb{R}^+), PC_{p,w_\alpha}, \bar{\mathfrak{M}}_p)$ and if the same subalgebra $\mathscr{A}^{\mathscr{K}}(\dot{C}(\mathbb{R}^+), C^0_{p,w_\alpha}, \emptyset)$ is used for localizing both algebras.

Let us now turn to the local algebra at (∞,∞), which has a more involved structure than the local algebras already studied. For this reason we start with analyzing the smaller algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p)$ before dealing with the full local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p)$.

We shall need a few more strong limit operators. For $A \in \mathscr{L}(L^p(\mathbb{R},w))$, let

$$\mathsf{H}^{+\pm}(A) := \operatorname*{s-lim}_{t\to\pm\infty} \operatorname*{s-lim}_{s\to+\infty} U_t V_{-s} A V_s U_{-t}, \tag{5.26}$$

provided that this strong limit exists.

Proposition 5.5.5. *For $A \in \mathscr{A}(PC(\dot{\mathbb{R}}), PC_{p,w}, C_p)$, the strong limits (5.26) exist, and the mappings*

$$\mathsf{H}^{+\pm} : \mathscr{A}(PC(\dot{\mathbb{R}}), PC_{p,w}, C_p) \to \mathbb{C}I$$

are algebra homomorphisms. In particular, for $a \in PC(\dot{\mathbb{R}})$, $b \in PC_{p,w}$, $c \in C_p$ and $K \in \mathscr{K}(L^p(\mathbb{R},w))$,

$$\mathsf{H}^{+\pm}(aI) = a(+\infty)I, \qquad \mathsf{H}^{+\pm}(W^0(b)) = b(\pm\infty)I,$$

$$\mathsf{H}^{+\pm}(M^0(c)) = c(\pm\infty)I \quad \textit{and} \quad \mathsf{H}^{+\pm}(K) = 0.$$

Proof. The first assertion comes from Lemma 5.4.2. The multiplicativity of $\mathsf{H}^{+\pm}$ is due to the uniform boundedness of $U_{-t}V_{-s}AV_sU_t$. The existence of the first three of the strong limits was established in Lemma 5.4.2. The last assertion follows from Lemma 1.4.6 since the V_s tend weakly to zero and the V_{-s} are uniformly bounded. ■

We have to introduce some new notation in order to give a description of the local algebras at (∞,∞). Let f be a function in $\dot{C}(\mathbb{R}^+)$ with $f(\infty) = f(0) = 1$ the support of which is contained in $[0,1] \cup [N,\infty]$ with some sufficiently large N, and write f as $f_0 + f_\infty$ with

$$f_0(t) = \begin{cases} f(t) & \text{if } t \in [0,1], \\ 0 & \text{if } t \in \,]1,\infty] \end{cases} \quad \text{and} \quad f_\infty(t) = \begin{cases} f(t) & \text{if } t \in [N,\infty], \\ 0 & \text{if } t \in [0,N[. \end{cases}$$

Further, let $g \in C^0_{p,w_\alpha}$ with $g(\infty) = g(0) = 1$ and $\operatorname{supp} g \subseteq [-\infty,-N] \cup [-1,1] \cup [N,+\infty]$, and write $g = g_0 + g_\infty$ with

$$g_0(t) = \begin{cases} g(t) & \text{if } t \in [-1,1], \\ 0 & \text{if } t \in \mathbb{R} \setminus [-1,1] \end{cases} \quad \text{and} \quad g_\infty(t) = \begin{cases} 0 & \text{if } t \in [-N,N], \\ g(t) & \text{if } t \in \mathbb{R} \setminus [-N,N]. \end{cases}$$

Set $g_\infty^\pm := \chi_\pm g_\infty$. Since the operator $f_0 W(g_0)$ is compact by Proposition 5.3.1, one gets

$$\begin{aligned} \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(I) &= \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(fW(g)) \\ &= \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(f_0 W(g_\infty)) + \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(f_\infty W(g_0)) + \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(f_\infty W(g_\infty)). \end{aligned}$$

Denote the first, second and third item in the sum of the right-hand side by $P_{0,\infty}$, $P_{\infty,0}$ and $P_{\infty,\infty}$, respectively, and define for $(x,y) \in \{(0,\infty),(\infty,0),(\infty,\infty)\}$,

$$\mathscr{A}^{x,y}_{\infty,\infty} := P_{x,y} \mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big) P_{x,y}.$$

Theorem 5.5.6. *Let $s = \infty$ and $t = \infty$. Then:*

(i) *the sets $\mathscr{A}^{0,\infty}_{\infty,\infty}$, $\mathscr{A}^{\infty,0}_{\infty,\infty}$ and $\mathscr{A}^{\infty,\infty}_{\infty,\infty}$ are Banach algebras, and*

$$\mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big) = \mathscr{A}^{0,\infty}_{\infty,\infty} \dotplus \mathscr{A}^{\infty,0}_{\infty,\infty} \dotplus \mathscr{A}^{\infty,\infty}_{\infty,\infty}$$

where the sums are direct;

(ii) *the algebra $\mathscr{A}^{0,\infty}_{\infty,\infty}$ is isometrically isomorphic to $\mathscr{E}_{p,\alpha}$, and the isomorphism is given by*

$$P_{0,\infty} \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(A) P_{0,\infty} \mapsto \mathsf{H}_{0,\infty}(A)$$

for each $A \in \mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big)$;

(iii) *the algebra $\mathscr{A}^{\infty,0}_{\infty,\infty}$ is isometrically isomorphic to $\mathscr{E}_{p,\alpha}$, and the isomorphism is given by*

$$P_{\infty,0} \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(A) P_{\infty,0} \mapsto \mathsf{H}_{\infty,0}(A)$$

for each $A \in \mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big)$;

(iv) *the algebra $\mathscr{A}^{\infty,\infty}_{\infty,\infty}$ is commutative and finitely generated. Its generators are the cosets $\boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(f_\infty W(g_\infty^\pm))$. For $A \in \mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big)$, the coset $P_{\infty,\infty} \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(A) P_{\infty,\infty}$ is invertible if and only if the operators $\mathsf{H}^{+\pm}(A)$, which are constant multiples of the identity, are invertible.*

Proof. (i) It is easy to see that $P_{\infty,0}$, $P_{0,\infty}$ and $P_{\infty,\infty}$ are idempotents which satisfy

$$P_{\infty,0} + P_{0,\infty} + P_{\infty,\infty} = \boldsymbol{\Phi}^{\mathscr{K}}_{\infty,\infty}(I),$$

and

$$P_{x_1,y_1} P_{x_2,y_2} = 0 \quad \text{if} \quad (x_1,y_1) \neq (x_2,y_2).$$

Hence, assertion (i) will follow once we have shown that $P_{\infty,0}$, $P_{0,\infty}$ and $P_{\infty,\infty}$ belong to the center of the local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, C_p\big)$. By Proposition 4.2.10, every Mellin convolution $M^0(c)$ with $c \in C_p$ can be approximated by a polynomial in $S_{\mathbb{R}^+}$. Since $S_{\mathbb{R}^+} = W(\text{sgn})$, it thus suffices to check whether $P_{\infty,0}$, $P_{0,\infty}$ and $P_{\infty,\infty}$

commute with $\Phi^{\mathscr{K}}_{\infty,\infty}(aI)$ and $\Phi^{\mathscr{K}}_{\infty,\infty}(W(b))$ for all functions $a \in PC(\mathbb{R}^+)$ and $b \in PC_{p,w_\alpha}$.

First consider the commutator $[P_{0,\infty}, \Phi^{\mathscr{K}}_{\infty,\infty}(aI)]$. A little thought shows that there is a function $a_\infty \in \bar{C}(\mathbb{R}^+)$ such that $\Phi^{\mathscr{K}}_{\infty,\infty}(aI) = \Phi^{\mathscr{K}}_{\infty,\infty}(a_\infty I)$. So the assertion follows immediately from the compactness of $[W(g_\infty), a_\infty]$, which we infer from Proposition 5.3.1 (ii). Now consider $[P_{0,\infty}, \Phi^{\mathscr{K}}_{\infty,\infty}(W(b))]$ where $b \in PC_{p,w_\alpha}$. Since g_∞ is continuous on $\dot{\mathbb{R}}$, we conclude via Proposition 5.3.3 (i) that

$$W(g_\infty)W(b) = W(g_\infty b) + K_1 = W(b)W(g_\infty) + K_2$$

with compact operators K_1 and K_2. As above, one finds a function $b_\infty \in PC_{p,w_\alpha} \cap C(\overline{\mathbb{R}})$ such that $\Phi^{\mathscr{K}}_{\infty,\infty}(W(b)) = \Phi^{\mathscr{K}}_{\infty,\infty}(W(b_\infty))$. Since the commutator $[f_0 W(b_\infty)]$ is compact by Proposition 5.3.1 (ii), it follows that $P_{0,\infty}$ commutes with all elements of the local algebra.

To get that the commutator $[P_{\infty,0}, \Phi^{\mathscr{K}}_{\infty,\infty}(aI)]$ vanishes, one can argue as above. So we are left to verify that $[P_{\infty,0}, \Phi^{\mathscr{K}}_{\infty,\infty}(W(b))] = 0$ for all $b \in PC_{p,w_\alpha}$. Using Proposition 5.3.3 again, we obtain that $[W(g_0), W(b)]$ is compact, and from Proposition 5.3.1 (ii) we infer that $[f_\infty, W(b)]$ is compact. Thus, $P_{\infty,0}$ is also in the center of the local algebra. Since $P_{\infty,\infty} = I - P_{\infty,0} - P_{0,\infty}$, the coset $P_{\infty,\infty}$ belongs to the center, too.

(ii) Propositions 5.4.1 and 5.4.3 imply that the operator $\mathsf{H}_{0,\infty}(A)$ depends only on the coset $P_{0,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}(A)P_{0,\infty}$. The specific form of $\mathsf{H}_{0,\infty}(A)$ is also a consequence of Propositions 5.4.1 and 5.4.3. The identity,

$$P_{0,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}(A)P_{0,\infty} = P_{0,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}(\mathsf{H}_{0,\infty}(A))P_{0,\infty}$$

can be checked by repeating arguments from the proofs of Theorems 5.5.3 and 5.5.4. This proves assertion (ii), and assertion (iii) of the theorem follows in a similar way.

(iv) Let $a \in PC(\mathbb{R}^+)$ and $b \in PC_{p,w_\alpha}$. Then

$$\begin{aligned} &P_{\infty,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}\big(aW(b)\big)P_{\infty,\infty} \\ &\quad= P_{\infty,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}\big(a(\infty)f_\infty W(b(-\infty)g^-_\infty + b(+\infty)g^+_\infty)\big)P_{\infty,\infty} \\ &\quad= a(\infty)b(-\infty)\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^-_\infty)\big) + a(\infty)b(+\infty)\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^+_\infty)\big). \end{aligned} \tag{5.27}$$

Taking into account that $\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^-_\infty)\big) + \Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^+_\infty)\big) = P_{\infty,\infty}$ is the identity element in $\mathscr{A}^{\infty,\infty}_{\infty,\infty}$ and that

$$\begin{aligned} \Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^-_\infty)\big)\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^+_\infty)\big) &= \Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g^-_\infty)f_\infty W(g^+_\infty)f_\infty I\big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W^0(g^-_\infty)f_\infty W^0(g^+_\infty)f_\infty I\big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W^0(g^-_\infty)W^0(g^+_\infty)f_\infty I\big) = 0 \end{aligned}$$

by Proposition 5.3.1 (ii), we conclude that every element B of $\mathscr{A}^{\infty,\infty}_{\infty,\infty}$ can be written in the form

$$B = \alpha_-(B)\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g_\infty^-)\big) + \alpha_+(B)\Phi^{\mathscr{K}}_{\infty,\infty}\big(f_\infty W(g_\infty^+)\big)$$

with uniquely determined complex numbers $\alpha_\pm(B)$. Since the existence of the strong limits follows from Proposition 5.5.5, it remains to show that

$$\mathsf{H}_{+\pm}(A) = \alpha_\pm\big(P_{\infty,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}(A)P_{\infty,\infty}\big)I. \tag{5.28}$$

The mappings $A \mapsto \mathsf{H}_{+\pm}(A)$ and $A \mapsto \alpha_\pm\big(P_{\infty,\infty}\Phi^{\mathscr{K}}_{\infty,\infty}(A)P_{\infty,\infty}\big)$ are continuous homomorphisms. It is thus sufficient to verify (5.28) with A replaced by aI and $W(b)$. For these operators, the assertion follows immediately from Proposition 5.5.5 and equality (5.27). ■

Now we turn our attention to the larger algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$. Again one can show that the idempotent $P_{\infty,\infty}$ belongs to the center of this algebra, but the idempotents $P_{0,\infty}$ and $P_{\infty,0}$ no longer possess this property. Therefore, this larger local algebra does not admit as simple a decomposition as that one observed in Theorem 5.5.6. We shall study the local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ via a second localization. To that end notice that, for $c \in C(\overline{\mathbb{R}}) \cap PC_p$, the coset $\Phi^{\mathscr{K}}_{\infty,\infty}(M^0(c))$ belongs to the center of this algebra by Propositions 5.3.2 (ii) and 5.3.4 (ii) (take into account that, for each $a \in PC(\mathbb{R}^+)$, there is an $a_\infty \in \bar{C}(\mathbb{R}^+)$ such that $\Phi^{\mathscr{K}}_{\infty,\infty}(aI - a_\infty I) = 0$). Thus, using Allan's local principle, we can localize $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ with respect to the maximal ideal space of the Banach algebra

$$\Big\{\Phi^{\mathscr{K}}_{\infty,\infty}(M^0(c)) : c \in C(\overline{\mathbb{R}}) \cap PC_p\Big\},$$

which can be identified with the two-point compactification $\overline{\mathbb{R}}$ of the real axis in an obvious way. For $x \in \overline{\mathbb{R}}$, let $\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x}$ denote the corresponding bilocal algebra, and write $\Phi^{\mathscr{K}}_{\infty,\infty,x}$ for the canonical homomorphism from $\mathscr{A}$ onto $\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x}$. Further, let the functions f_0, f_∞, g_0 and g_∞ be defined as before Theorem 5.5.6. For $x \in \{\pm\infty\}$ and $(y,z) \in \{(0,\infty), (\infty,0), (\infty,\infty)\}$, set

$$P_x^{y,z} := \Phi^{\mathscr{K}}_{\infty,\infty,x}\big(f_y W(g_z)\big)$$

and abbreviate

$$\mathscr{A}^{y,z}_{\infty,\infty,x} := P_x^{y,z}\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x}P_x^{y,z}.$$

Finally, let $\mathscr{B}_{p,\alpha}$ denote the smallest closed subalgebra of $\mathscr{L}\big(L^p(\mathbb{R}, w_\alpha)\big)$ which contains $S_{\mathbb{R}}$ and $\chi_+ I$. The following theorem identifies the local algebras $\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x}$.

Theorem 5.5.7.

(i) *Let $x \in \mathbb{R}$. For each $A \in \mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$, there is an operator $A_\infty \in \mathscr{A}\big(PC(\mathbb{R}^+), \emptyset, PC_p\big)$ such that $\Phi^{\mathscr{K}}_{\infty,\infty,x}(A - A_\infty) = 0$. The local algebra $\mathscr{A}^{y,z}_{\infty,\infty,x}$ is isometrically isomorphic to the algebra $\mathscr{B}_{p,\alpha}$, and the isomorphism is given by*

$$\mathsf{H}_{\infty,\infty,x} : \Phi^{\mathscr{K}}_{\infty,\infty,x}(A) \mapsto \mathsf{H}_{\infty,x}(E_{p,w_\alpha}A_\infty E^{-1}_{p,w_\alpha}).$$

In particular,

$$\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(aI)\big) = a(+\infty)\chi_{-}I + a(0^{+})\chi_{+}I,$$

$$\begin{aligned}\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(W(b))\big) &= \left(b(-\infty)\frac{1-d(x)}{2} + b(+\infty)\frac{1+d(x)}{2}\right)\chi_{-}I \\ &\quad + \left(b(0^{-})\frac{1-d(x)}{2} + b(0^{+})\frac{1+d(x)}{2}\right)\chi_{+}I\end{aligned}$$

with $d(x) := \coth\big((x+\mathbf{i}(1/p+\alpha))\pi\big)$*, and*

$$\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(M^{0}(c))\big) = c(x^{-})Q_{\mathbb{R}} + c(x^{+})P_{\mathbb{R}}.$$

(ii) *Let* $x \in \{\pm\infty\}$. *Then* $\mathscr{A}^{0,\infty}_{\infty,\infty,x}$, $\mathscr{A}^{\infty,0}_{\infty,\infty,x}$ *and* $\mathscr{A}^{\infty,\infty}_{\infty,\infty,x}$ *are Banach algebras, and the algebra* $\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x}$ *decomposes into the direct sum*

$$\mathscr{A}^{\mathscr{K}}_{\infty,\infty,x} = \mathscr{A}^{0,\infty}_{\infty,\infty,x} \dotplus \mathscr{A}^{\infty,0}_{\infty,\infty,x} \dotplus \mathscr{A}^{\infty,\infty}_{\infty,\infty,x}.$$

Moreover, for $(y,z) \in \{(0,\infty),(\infty,0),(\infty,\infty)\}$*, there is an isomorphism* $\mathsf{H}^{y,z}_{x}$ *from* $\mathscr{A}^{y,z}_{\infty,\infty,x}$ *onto* $\mathbb{C}$*. In particular,*

$$\mathsf{H}^{y,z}_{x}\big(P^{y,z}_{\pm\infty}\Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(aW(b)M^{0}(c))P^{y,z}_{\pm\infty}\big) = a(y)b(z^{\pm})c(x)$$

where $\infty^{\pm} := \pm\infty$.

Proof. (i) Choose $c_x \in C(\overline{\mathbb{R}}) \cap PC_p$ so that supp c_x is compact and $c_x(x) = 1$. Then the operator $f_\infty W(g_\infty)M^{0}(c_x)$ is compact by Proposition 5.3.5 (i). Hence, for every function $b \in PC_{p,w_\alpha}$, which is continuous at the point 0 and satisfies $b(0) = 0$, we obtain

$$\begin{aligned}\Phi^{\mathscr{K}}_{\infty,\infty,x}\big(W(b)\big) &= \Phi^{\mathscr{K}}_{\infty,\infty,x}\Big(W(b)\big(f_0W(g_\infty) + f_\infty W(g_0) + f_\infty W(g_\infty)\big)M^{0}(c_x)\Big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty,x}\big(W(bg_\infty)f_0M^{0}(c_x) + W(bg_0)f_\infty M^{0}(c_x)\big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty,x}\big(W(bg_\infty)f_0M^{0}(c_x)\big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty,x}\big(W(b(-\infty)\chi_{-} + b(+\infty)\chi_{+})f_0M^{0}(c_x)\big) \\ &= \Phi^{\mathscr{K}}_{\infty,\infty,x}\big((b(-\infty)W(\chi_{-}) + b(+\infty)W(\chi_{+}))f_0M^{0}(c_x)\big).\end{aligned} \tag{5.29}$$

If now b is an arbitrary function in PC_{p,w_α}, then we write

$$W(b) = b(0^{-})W(\chi_{-}) + b(0^{+})W(\chi_{+}) + W(b') \tag{5.30}$$

with b' being continuous at zero and $b'(0) = 0$. Then the first part of assertion (i) follows, since the $W(\chi_{\pm})$ are also Mellin operators.

Since the images of aI and $M^{0}(c)$ under the mapping $A \mapsto E_{p,w_\alpha}AE^{-1}_{p,w_\alpha}$ are the operators $\breve{a}I$ with $\breve{a}(t) = a(e^{2\pi t})$ and $W^{0}(c)$, respectively, the strong limits

$\mathsf{H}_{\infty,x}(E_{p,w_\alpha}A_\infty E_{p,w_\alpha}^{-1})$ exist for each operator $A_\infty \in \mathscr{A}\big(PC(\mathbb{R}^+),0,PC_p\big)$ by Proposition 5.4.1.

Let $\mathscr{J}_{\infty,\infty,x}$ stand for the closed ideal generated by all cosets $\Phi^{\mathscr{K}}_{\infty,\infty}(M^0(c))$ with $c \in C(\overline{\mathbb{R}}) \cap PC_p$ and $c(x) = 0$. In order to get the correctness of the definition of $\mathsf{H}_{\infty,\infty,x}$, we must show that if $A \in \mathscr{A}\big(PC(\mathbb{R}^+),PC_{p,w_\alpha},PC_p\big)$ and $\Phi^{\mathscr{K}}_{\infty,\infty,x}(A) = 0$, then the strong limit $\mathsf{H}(A) := \mathsf{H}_{\infty,x}(E_{p,w_\alpha}A_\infty E_{p,w_\alpha}^{-1})$ exists and is equal to 0.

To see this, note first that the ideal $\mathscr{K}$ of the compact operators belongs to the kernel of H. Thus, H depends only on the coset $\Phi(A)$. Further, since $\mathsf{H}(aI) = 0$ for each continuous function a with $a(0) = a(\infty) = 0$ by Proposition 5.4.1 (i), the local ideal $\mathscr{I}_{\infty,\infty}$ lies in the kernel of H. Notice that for this conclusion we do *not* need to know whether the strong limit $\mathsf{H}(\Phi(W(b))$ exists: indeed, each operator A with $\Phi(A) \in \mathscr{I}_{\infty,\infty}$ can be approximated by finite sums

$$\sum_j A_j a_j I + K$$

where $a_j(0) = a_j(\infty) = 0$ and K is compact. If A is of this form then

$$\begin{aligned}\mathsf{H}(A) &= \operatorname*{s-lim}_{\tau\to+\infty} Z_\tau^{-1}U_{-x}E_{p,w_\alpha}AE_{p,w_\alpha}^{-1}U_xZ_\tau \\ &= \operatorname*{s-lim}_{\tau\to+\infty} \sum_j (Z_\tau^{-1}U_{-x}E_{p,w_\alpha}A_jE_{p,w_\alpha}^{-1}U_xZ_\tau)(Z_\tau^{-1}U_{-x}E_{p,w_\alpha}a_jE_{p,w_\alpha}^{-1}U_xZ_\tau),\end{aligned}$$

from which the conclusion follows since $Z_\tau^{-1}U_{-x}E_{p,w_\alpha}a_jE_{p,w_\alpha}^{-1}U_xZ_\tau \to 0$ and since the norms of $Z_\tau^{-1}U_{-x}E_{p,w_\alpha}A_jE_{p,w_\alpha}^{-1}U_xZ_\tau$ are uniformly bounded with respect to τ. Hence, $\mathsf{H}(A)$ depends only on $\Phi^{\mathscr{K}}_{\infty,\infty}(A)$. The same arguments show that the local ideal $\mathscr{J}_{\infty,\infty,x}$ is contained in the kernel of the mapping $\Phi^{\mathscr{K}}_{\infty,\infty}(A) \mapsto \mathsf{H}(A)$ (take into account that $\mathsf{H}(M^0(c)) = 0$ whenever $c \in C(\overline{\mathbb{R}}) \cap PC_p$ and $c(x) = 0$). This observation establishes the correctness of the definition of $\mathsf{H}_{\infty,\infty,x}$.

We further have to show that the invertibility of

$$\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(A)\big) = \mathsf{H}_{\infty,x}(E_{p,w_\alpha}A_\infty E_{p,w_\alpha}^{-1})$$

implies the invertibility of $\Phi^{\mathscr{K}}_{\infty,\infty,x}(A)$. But this is an easy consequence of the identity

$$\Phi^{\mathscr{K}}_{\infty,\infty,x}(A) = \Phi^{\mathscr{K}}_{\infty,\infty,x}\big(E_{p,w_\alpha}^{-1}U_x\mathsf{H}_{\infty,x}(E_{p,w_\alpha}A_\infty E_{p,w_\alpha}^{-1})U_xE_{p,w_\alpha}\big)$$

which can be verified in a similar way as the corresponding identity in the proof of Theorem 5.5.3. Finally, the special values of $\mathsf{H}_{\infty,\infty,x}$ at the generators of the algebra follow from the equalities (4.21), (5.29) and (5.30) and from Proposition 5.4.1.

(ii) Since $\Phi^{\mathscr{K}}_{\infty,\infty,x}(M^0(c)) = c(x)\Phi^{\mathscr{K}}_{\infty,\infty,x}(I)$, the proof of the first part of this assertion runs as that of Theorem 5.5.6. The second part of the assertion will follow immediately from the identity

$$P^{y,z}_{\pm\infty}\Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(aW(b)M^0(c)) = a(y)b(z^\pm)c(\pm\infty)P^{y,z}_{\pm\infty}$$

which we shall verify only for the basic case when $a \equiv 1$ and $c \equiv 1$. For definiteness, let $(y,z) = (\infty, 0)$. Then

$$\begin{aligned}
P^{\infty,0}_{\pm\infty} \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}\big(W(b)\big) &= \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}\big(W(bg_0)f_\infty I\big) \\
&= \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}\Big(W\big(b(0^-)\chi_- + b(0^+)\chi_+\big)\Big) P^{\infty,0}_{\pm\infty} \\
&= \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}\Big(b(0^-)W(\chi_-) + b(0^+)W(\chi_+)\Big) P^{\infty,0}_{\pm\infty} \\
&= \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}\left(b(0^-)M^0\left(\frac{1-d}{2}\right) + b(0^+)M^0\left(\frac{1+d}{2}\right)\right) P^{\infty,0}_{\pm\infty} \\
&= \left(b(0^-)\frac{1-d(\pm\infty)}{2} + b(0^+)\frac{1+d(\pm\infty)}{2}\right) P^{\infty,0}_{\pm\infty} \\
&= b(0^\pm) P^{\infty,0}_{\pm\infty}.
\end{aligned}$$

Similarly one gets that, for every operator $A \in \mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$, there is a function $c \in C(\overline{\mathbb{R}}) \cap PC_p$ such that

$$P^{y,z}_{\pm\infty} \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(A) = \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(M^0(c)) P^{y,z}_{\pm\infty} = c(\pm\infty) P^{y,z}_{\pm\infty}.$$

This observation finishes the proof. ■

We summarize the results obtained in this section in the following theorem.

Theorem 5.5.8. *Let $A \in \mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$. The coset $A + \mathscr{K}(L^p(\mathbb{R}^+, w_\alpha))$ is invertible in $\mathscr{A}^{\mathscr{K}}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ if and only if the operators*

$$\begin{aligned}
&\mathsf{H}_{\infty,t}(A) \in \mathscr{E}_{p,\alpha} && \text{for } r \in \dot{\mathbb{R}}_0 \setminus \{\infty\}, \\
&\mathsf{H}_{s,\infty}(A) \in \mathscr{B}_p && \text{for } s \in \dot{\mathbb{R}}^+ \setminus \{\infty\}, \\
&\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(A)\big) \in \mathscr{B}_{p,\alpha} && \text{for } x \in \mathbb{R}
\end{aligned}$$

are invertible in the respective algebras and if the complex numbers

$$\mathsf{H}^{y,z}_x\big(P^{y,z}_{\pm\infty} \Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(A)\big) \qquad \text{for } (y,z) \in \{(0,\infty), (\infty,0), (\infty,\infty)\}$$

are not zero.

The following theorem establishes the relation of this result to the Fredholm property of operators in $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$.

Theorem 5.5.9. *The algebra $\mathscr{A}^{\mathscr{K}}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ is inverse-closed in the Calkin algebra $\mathscr{L}(L^p(\mathbb{R}^+, w_\alpha)) / \mathscr{K}(L^p(\mathbb{R}^+, w_\alpha))$.*

Proof. There are several ways to verify the inverse-closedness. One way is to consider the smallest (non-closed) subalgebra $\mathscr{A}_0$ of $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ which contains all operators aI, $W(b)$ and $M^0(c)$ with piecewise constant functions a, b and c. Applying Theorem 5.5.8 to an operator $A \in \mathscr{A}_0$, we find that the spectrum of the coset $A + \mathscr{K}(L^p(\mathbb{R}^+, w_\alpha))$ in $\mathscr{A}^{\mathscr{K}}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ is a thin subset of the complex plane. Since $\mathscr{A}_0$ is dense in $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$, the assertion follows from Corollary 1.2.32.

For another proof, one shows that, for every Fredholm operator A in the algebra $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$, the H-limits quoted in Theorem 5.5.8 are invertible (as operators on the respective Banach spaces). Then one employs the inverse-closedness of the algebras $\mathscr{E}_{p,\alpha}$ and $\mathscr{B}_{p,\alpha}$ in the algebra $\mathscr{L}(L^p(\mathbb{R}^+, w_\alpha))$ and applies Theorem 5.5.8. ■

Corollary 5.5.10. *Let $A \in \mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$. Then A is a Fredholm operator on $L^p(\mathbb{R}^+, w_\alpha)$ if and only if the operators*

$$
\begin{aligned}
&\mathsf{H}_{\infty,t}(A) \in \mathscr{E}_{p,\alpha} && \textit{for } r \in \dot{\mathbb{R}}_0 \setminus \{\infty\} \\
&\mathsf{H}_{s,\infty}(A) \in \mathscr{B}_p && \textit{for } s \in \dot{\mathbb{R}}^+ \setminus \{\infty\} \\
&\mathsf{H}_{\infty,\infty,x}\big(\Phi^{\mathscr{K}}_{\infty,\infty,x}(A)\big) \in \mathscr{B}_{p,\alpha} && \textit{for } x \in \mathbb{R}
\end{aligned}
$$

are invertible (as operators on the respective Banach spaces) and if the complex numbers

$$
\mathsf{H}^{y,z}_x\big(P^{y,z}_{\pm\infty}\Phi^{\mathscr{K}}_{\infty,\infty,\pm\infty}(A)\big) \qquad \textit{for } (y,z) \in \{(0,\infty), (\infty,0), (\infty,\infty)\}
$$

are not zero.

Combining this result with the results of Section 4.2 one easily gets a matrix-valued symbol for the Fredholmness of operators in $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$.

Remark 5.5.11. In this section we constructed representations of the local algebras by employing a basic property of the operators which constitute the local algebras: their local homogeneity. This property enabled us to identify the local algebras via homogenizing strong limits. It would also have been possible to identify the local algebras by means of the concepts developed in Section 2.6 and Chapter 3: PI-algebras and, in particular, algebras generated by idempotents. We will illustrate the use of those concepts in Section 5.7 to identify some of the local algebras that appear there. □

5.6 Algebras of multiplication and Wiener-Hopf operators

Let the weight function w be given by (4.8). In this section we address the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}, w))$ which contains all operators aI of multiplication by a function $a \in PC(\dot{\mathbb{R}})$ and all Fourier convolutions $W^0(b)$ where $b \in PC_{p,w}$. We denote this algebra by $\mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}\big)$, and we write $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}\big)$ for the image of this algebra in the Calkin algebra and Φ for the canonical homomorphism from $\mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ onto $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$.

If $f \in C(\dot{\mathbb{R}})$ and $g \in C_{p,w}$ then the coset $\Phi\left(fW^0(g)\right)$ belongs to the center of $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ by Proposition 5.3.1. So we can localize this algebra with respect to the maximal ideal space of $\mathscr{A}^{\mathscr{K}}\left(C(\dot{\mathbb{R}}), C_{p,w}\right)$, which is homeomorphic to the subset $(\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ of the torus $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$. The proof of the latter fact is similar to the proof of Proposition 5.5.2.

Given $(s,t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$, let $\mathscr{I}_{s,t}$ denote the smallest closed ideal of the Banach algebra $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ which contains the point (s,t), and let $\Phi^{\mathscr{K}}_{s,t}$ refer to the canonical homomorphism from $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ onto the local quotient algebra

$$\mathscr{A}^{\mathscr{K}}_{s,t} := \mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)/\mathscr{I}_{s,t}.$$

Further, for each weight w of the form (4.8) and for each $x \in \mathbb{R}$, define the local weight $w_{\alpha(x)}$ at x by $w_{\alpha(x)}(t) := |t|^{\alpha(x)}$ with

$$\alpha(x) := \begin{cases} 0 & \text{if } x \notin \{t_1, \ldots, t_n, \infty\}, \\ \alpha_j & \text{if } x = t_j \text{ for some } (j = 1, \ldots, n), \\ \sum_{j=0}^{n} \alpha_j & \text{if } x = \infty. \end{cases} \tag{5.31}$$

To describe the local algebras $\mathscr{A}^{\mathscr{K}}_{s,t}$, we have to introduce some new strong limit operators. For $A \in \mathscr{L}\left(L^p(\mathbb{R}, w)\right)$, let

$$\mathsf{H}^{\pm\pm}(A) := \operatorname*{s-lim}_{t\to\pm\infty} \operatorname*{s-lim}_{s\to\pm\infty} U_t V_{-s} A V_s U_{-t} \tag{5.32}$$

provided that the strong limits exist. Here, by convention, the first superscript in $\mathsf{H}^{\pm\pm}$ refers to the strong limit with respect to $s \to \pm\infty$ and the second one to $t \to \pm\infty$.

Proposition 5.6.1. *The strong limits (5.32) exist for $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$, and the mappings $\mathsf{H}^{\pm\pm}$ are algebra homomorphisms from $\mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ onto the algebra $\mathbb{C}I$. In particular, for $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_{p,w}$,*

$$\mathsf{H}^{+\pm}(aI) = a(+\infty)I, \qquad \mathsf{H}^{-\pm}(aI) = a(-\infty)I, \tag{5.33}$$

$$\mathsf{H}^{\pm+}(W^0(b)) = b(+\infty)I, \qquad \mathsf{H}^{\pm-}(W^0(b)) = b(-\infty)I, \tag{5.34}$$

and

$$\mathsf{H}^{\pm\pm}(K) = 0 \qquad \text{for} \quad K \in \mathscr{K}\left(L^p(\mathbb{R}, w)\right). \tag{5.35}$$

Proof. The first assertion comes from Lemma 5.4.2. The multiplicativity of $\mathsf{H}^{\pm\pm}$ is due to the uniform boundedness of $U_t V_{-s} A V_s U_{-t}$. Finally, if K is compact then, by Lemma 1.4.6, KV_s goes strongly to zero, as V_s tends weakly to zero. The result then follows from the uniform boundedness of V_{-s}. ∎

Theorem 5.6.2. *Let $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$.*

(i) *The coset $A + \mathscr{K}(L^p(\mathbb{R}, w))$ is invertible in $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ if and only if the coset $\Phi^{\mathscr{K}}_{s,t}(A)$ is invertible in $\mathscr{A}^{\mathscr{K}}_{s,t}$ for each $(s,t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$.*

(ii) *For $s \in \mathbb{R}$, the local algebra $\mathscr{A}^{\mathscr{K}}_{s,\infty}$ is isometrically isomorphic to the subalgebra $\mathrm{alg}\{I, \chi_+ I, S_{\mathbb{R}}\}$ of $\mathscr{L}\left(L^p(\mathbb{R}, w_{\alpha(s)})\right)$, and the isomorphism is given by*

$$\Phi^{\mathscr{K}}_{s,\infty}(A) \mapsto \mathsf{H}_{s,\infty}(A) \tag{5.36}$$

for each operator $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$.

(iii) *For $t \in \mathbb{R}$, the local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,t}$ is isometrically isomorphic to the subalgebra $\mathrm{alg}\{I, \chi_+ I, S_{\mathbb{R}}\}$ of $\mathscr{L}\left(L^p(\mathbb{R}, w_{\alpha(\infty)})\right)$, and the isomorphism is given by*

$$\Phi^{\mathscr{K}}_{\infty,t}(A) \mapsto \mathsf{H}_{\infty,t}(A) \tag{5.37}$$

for each operator $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$.

(iv) *The local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ is generated by the four idempotent elements*

$$\Phi^{\mathscr{K}}_{\infty,\infty}(W(\chi_\pm)\chi_\pm I),$$

and the coset $\Phi^{\mathscr{K}}_{\infty,\infty}(A)$ is invertible if and only if the four operators

$$\mathsf{H}^{\pm\pm}(A),$$

which are complex multiples of the identity operator, are invertible.

Proof. Assertion (i) is just a reformulation of Allan's local principle. For the proof of assertion (ii), one employs the same arguments as in the first and second step of the proof of Proposition 4.3.2 to obtain that the algebras $\mathscr{A}^{\mathscr{K}}_{s,\infty}$ corresponding to the spaces $L^p(\mathbb{R}, w)$ and $L^p(\mathbb{R}, w_s)$ with $w_s(x) = |x-s|^{\alpha(s)}$ are isometrically isomorphic. The remainder of the proof of assertion (ii) can be done as in Theorem 5.5.4.

The proof of part (iii) runs parallel to that of Theorem 5.5.3. One only has to take into account that the local algebras related to $L^p(\mathbb{R}, w)$ and $L^p(\mathbb{R}, w_\infty)$ with $w_\infty(x) = |x|^{\alpha(\infty)}$ are isometrically isomorphic. To prove assertion (iv), note that there are functions $f \in C(\overline{\mathbb{R}})$ and $g \in C(\overline{\mathbb{R}}) \cap PC_{p,w}$ such that

$$\Phi^{\mathscr{K}}_{\infty,\infty}(fI - \chi_+ I) = 0 \quad \text{and} \quad \Phi^{\mathscr{K}}_{\infty,\infty}(W^0(g - \chi_+)) = 0.$$

From Proposition 5.3.1(ii) we infer that the commutator $[fI, W^0(g)]$ is compact. Thus, the cosets $\Phi^{\mathscr{K}}_{\infty,\infty}(\chi_+ I)$ and $\Phi^{\mathscr{K}}_{\infty,\infty}(W^0(\chi_+))$ commute. Since

$$\Phi^{\mathscr{K}}_{\infty,\infty}(aI) = \Phi^{\mathscr{K}}_{\infty,\infty}(a(-\infty)\chi_- I + a(+\infty)\chi_+ I)$$

for every $a \in PC(\dot{\mathbb{R}})$ and

$$\Phi^{\mathscr{K}}_{\infty,\infty}(W^0(b)) = \Phi^{\mathscr{K}}_{\infty,\infty}(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)),$$

for every $b \in PC_{p,w}$, we find that, for each $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$, the coset $\Phi^{\mathscr{K}}_{\infty,\infty}(A)$ can be represented in the form

$$\Phi^{\mathscr{K}}_{\infty,\infty}\left(h_{--}W(\chi_-)\chi_- I + h_{-+}W(\chi_+)\chi_- I + h_{+-}W(\chi_-)\chi_+ I + h_{++}W(\chi_+)\chi_+ I\right)$$

with uniquely determined complex numbers $h_{\pm\pm} = h_{\pm\pm}(A)$. Thereby,

$$h_{+\pm}(aI) = a(+\infty)I, \qquad h_{-\pm}(aI) = a(-\infty)I, \tag{5.38}$$

$$h_{\pm+}(W^0(b)) = b(+\infty)I, \qquad h_{\pm-}(W^0(b)) = b(-\infty)I, \tag{5.39}$$

and the coset $\Phi^{\mathscr{K}}_{\infty,\infty}(A)$ is invertible if and only if the numbers $h_{\pm\pm}(A)$ are not zero. Since $\mathsf{H}^{\pm\pm}(A) = h_{\pm\pm}(A)I$ by Proposition 5.6.1, the result follows. ■

The following corollary can be proved by repeating the arguments from the proof of Theorem 5.5.9.

Corollary 5.6.3. *The algebra $\mathscr{A}^{\mathscr{K}}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ is inverse-closed in the Calkin algebra $\mathscr{L}(L^p(\mathbb{R},w))/\mathscr{K}(L^p(\mathbb{R},w))$, and an operator $A \in \mathscr{A}\left(PC(\dot{\mathbb{R}}), PC_{p,w}\right)$ is Fredholm if and only if the operators $\mathsf{H}_{s,\infty}(A)$, $\mathsf{H}_{\infty,t}(A)$ and $\mathsf{H}^{\pm\pm}(A)$ are invertible for all $s,t \in \mathbb{R}$.*

To illustrate the previous results, we consider a particular class of operators, the so-called paired convolution operators. These are operators of the form

$$A = a_1 W^0(b_1) + a_2 W^0(b_2) \tag{5.40}$$

with $a_1, a_2 \in PC(\dot{\mathbb{R}})$ and $b_1, b_2 \in PC_{p,w}$. The following result is an immediate consequence of Corollary 5.6.3.

Theorem 5.6.4. *The operator A in (5.40) is Fredholm on $L^p(\mathbb{R},w)$ if and only if the following three conditions are fulfilled:*

(i) *the operator $c_+ P_{\mathbb{R}} + c_- Q_{\mathbb{R}}$ with*

$$\begin{aligned} c_\pm(s) := {} & \left(a_1(s^-)b_1(\pm\infty) + a_2(s^-)b_2(\pm\infty)\right)\chi_- I \\ & + \left(a_1(s^+)b_1(\pm\infty) + a_2(s^+)b_2(\pm\infty)\right)\chi_+ I \end{aligned}$$

is invertible on $L^p(\mathbb{R}, w_{\alpha(s)})$ for each $s \in \mathbb{R}$;

(ii) *the operator $d_+ P_{\mathbb{R}} + d_- Q_{\mathbb{R}}$ with*

$$\begin{aligned} d_\pm(t) := {} & \left(a_1(-\infty)b_1(t^\pm) + a_2(-\infty)b_2(t^\pm)\right)\chi_- I \\ & + \left(a_1(+\infty)b_1(t^\pm) + a_2(+\infty)b_2(t^\pm)\right)\chi_+ I \end{aligned}$$

is invertible on $L^p(\mathbb{R}, w_{\alpha(\infty)})$ for each $t \in \mathbb{R}$;

(iii) *none of the following numbers is zero:*

$$a_1(+\infty)b_1(\pm\infty) + a_2(+\infty)b_2(\pm\infty), \quad a_1(-\infty)b_1(\pm\infty) + a_2(-\infty)b_2(\pm\infty).$$

Of particular interest are paired operators of the form

$$A = a_1 W^0(\chi_+) + a_2 W^0(\chi_-) = a_1 P_{\mathbb{R}} + a_2 Q_{\mathbb{R}} \tag{5.41}$$

with $a_1, a_2 \in PC(\dot{\mathbb{R}})$, which can also be written as the singular integral operator

$$\frac{a_1 + a_2}{2} I + \frac{a_1 - a_2}{2} S_{\mathbb{R}}.$$

For these operators, Corollary 5.6.3 implies the following.

Corollary 5.6.5. *Let $a_1, a_2 \in PC(\dot{\mathbb{R}})$. The singular integral operator $a_1 P_{\mathbb{R}} + a_2 Q_{\mathbb{R}}$ is Fredholm on $L^p(\mathbb{R}, w)$ if and only if*

(i) *the operator $(a_2(s^-)\chi_- + a_2(s^+)\chi_+)Q_{\mathbb{R}} + (a_1(s^-)\chi_- + a_1(s^+)\chi_+)P_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R}, w_{\alpha(s)})$ for each $s \in \mathbb{R}$ and*
(ii) *the operator $(a_2(-\infty)\chi_- + a_2(+\infty)\chi_+)Q_{\mathbb{R}} + (a_1(-\infty)\chi_- + a_1(+\infty)\chi_+)P_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R}, w_{\alpha(\infty)})$.*

The corresponding result for operators on the semi-axis reads as follows.

Corollary 5.6.6. *Let $a_1, a_2 \in PC(\mathbb{R}^+)$. The singular integral operator $a_1 P_{\mathbb{R}^+} + a_2 Q_{\mathbb{R}^+}$ is Fredholm on $L^p(\mathbb{R}^+, w)$ if and only if*

(i) *the operator $a_1(0^+)P_{\mathbb{R}^+} + a_2(0^+)Q_{\mathbb{R}^+}$ is invertible on $L^p(\mathbb{R}, w_{\alpha(0)})$,*
(ii) *the operator $(a_2(s^-)\chi_- + a_2(s^+)\chi_+)Q_{\mathbb{R}} + (a_1(s^-)\chi_- + a_1(s^+)\chi_+)P_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R}, w_{\alpha(s)})$ for each $s \in \mathbb{R}^+ \setminus \{0\}$, and*
(iii) *the operator $a_1(+\infty)P_{\mathbb{R}^+} + a_2(+\infty)Q_{\mathbb{R}^+}$ is invertible on $L^p(\mathbb{R}^+, w_{\alpha(\infty)})$.*

Proof. This follows by applying Corollary 5.6.3 to the operator

$$\left(a_1 W^0(\chi_+) + a_1 W^0(\chi_-)\right) \chi_+ I + \chi_- I \in \mathscr{L}(L^p(\mathbb{R}, w))$$

with a_1 and a_2 extended to the whole line by zero. This operator is equivalent to the singular integral operator $a_1 P_{\mathbb{R}^+} + a_2 Q_{\mathbb{R}^+}$ in the sense that these operators are Fredholm, or not, simultaneously. (Of course, one could also apply Corollary 5.5.10 directly.) ■

Note that Propositions 4.2.11 and 4.2.19 combined with the above results give a matrix-valued symbol for the Fredholmness of the operators considered. Note further that one can derive similar results for operators of the form $a_1 M(b_1) + a_2 M(b_2)$ with $a_1, a_2 \in PC([0,1])$ and $b_1, b_2 \in PC_{p,w_\alpha}$ considered on the space $L^p([0,1], w_\alpha)$. The easiest way to do this is to reduce them to the operators considered above via the mapping $A \mapsto E_{p,w}^{-1} A E_{p,w}$.

5.7 Algebras of multiplication, convolution and flip operators

Let $\tilde{w}$ be a weight function on $\mathbb{R}^+$ of the form (4.8), and let w denote its symmetric extension to $\mathbb{R}$, i.e.

$$w(t) := \begin{cases} \tilde{w}(t) & \text{if } t \geq 0, \\ \tilde{w}(-t) & \text{if } t < 0. \end{cases}$$

The symmetry of the weight implies that the flip operator J given by $(Ju)(t) := u(-t)$ is bounded on $L^p(\mathbb{R}, w)$. It thus makes sense to consider the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}, w))$ which contains all operators aI of multiplication by a function $a \in PC(\dot{\mathbb{R}})$, all Fourier convolutions $W^0(b)$ where $b \in PC_{p,w}$, and the flip J. We denote this algebra by $\mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$. Note that this algebra contains the Hankel operators $H(b) := \chi_+ W^0(b) J \chi_+ I$ with $b \in PC_{p,w}$. Further, we let $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ refer to the image of $\mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ in the Calkin algebra and write $\boldsymbol{\Phi}$ for the corresponding canonical homomorphism.

Let $\tilde{C}(\dot{\mathbb{R}})$ and $\tilde{C}_{p,w}$ denote the subalgebras of $C(\dot{\mathbb{R}})$ and $C_{p,w}$, respectively, which are constituted by the even functions, i.e., by the functions f with $Jf = f$.

Proposition 5.7.1. *If $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_{p,w}$, then the coset $\boldsymbol{\Phi}(fW^0(g))$ belongs to the center of $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$.*

Proof. It easy to see that $fW^0(g)J = JfW^0(g)$. From

$$\begin{aligned} fW^0(g) &= (f - f(\infty))W^0(g - g(\infty)) + (f - f(\infty))W^0(g(\infty)) \\ &\qquad + f(\infty)W^0(g - g(\infty)) + f(\infty)W^0(g(\infty)) \\ &= (f - f(\infty))W^0(g - g(\infty)) + (f - f(\infty))g(\infty)I \\ &\qquad + f(\infty)W^0(g - g(\infty)) + f(\infty)g(\infty)I \end{aligned}$$

and from Proposition 5.3.1 it becomes clear that $fW^0(g)$ also commutes with the other generators of the algebra modulo compact operators. ■

Let $\overline{\mathbb{R}}^+$ denote the compactification of $\mathbb{R}^+$ by the point $\{\infty\}$, i.e., $\overline{\mathbb{R}}^+$ is homeomorphic to $[0,1]$. The maximal ideal space of the algebra generated by all cosets $\boldsymbol{\Phi}(fW^0(g))$ with $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_{p,w}$ is homeomorphic to the subset $(\overline{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \overline{\mathbb{R}}^+)$ of the square $\overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+$, which can be checked as in the proof of Proposition 5.5.2. The maximal ideal corresponding to $(s,t) \in (\overline{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \overline{\mathbb{R}}^+)$ is just the class of all cosets $\boldsymbol{\Phi}(fW^0(g))$ where $f \in \tilde{C}(\dot{\mathbb{R}})$ with $f(s) = 0$ and $g \in \tilde{C}_{p,w}$ with $g(t) = 0$.

We proceed by localization over this maximal ideal space. Let $\mathscr{I}_{s,t}$ stand for the smallest closed ideal of the Banach algebra $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ which contains the maximal ideal $(s,t) \in (\overline{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \overline{\mathbb{R}}^+)$, and write $\mathscr{A}^{\mathscr{K}}_{s,t}$ for the quotient algebra

$$\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big) / \mathscr{I}_{s,t}$$

and $\Phi_{s,t}^{\mathscr{K}}$ for the canonical homomorphism from $\mathscr{A}\big(PC(\dot{\mathbb{R}}),PC_{p,w},J\big)$ onto $\mathscr{A}_{s,t}^{\mathscr{K}}$. For $x\in\overline{\mathbb{R}}^{+}$, let $\alpha(x)$ be the local exponent defined by (5.31).

Let $A\in\mathscr{A}\big(PC(\dot{\mathbb{R}}),PC_{p,w},J\big)$. Allan's local principle implies that the coset $A+\mathscr{K}(L^p(\mathbb{R},w))$ is invertible in $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}),PC_{p,w},J\big)$ if and only if the local cosets $\Phi_{s,t}^{\mathscr{K}}(A)$ are invertible for all $(s,t)\in(\overline{\mathbb{R}}^{+}\times\{\infty\})\cup(\{\infty\}\times\overline{\mathbb{R}}^{+})$. We are thus left to analyze the local algebras $\mathscr{A}_{s,t}^{\mathscr{K}}$. Some care is in order since, in contrast to the previous sections, the strong limits $\mathsf{H}_{s,\infty}(J)$ and $\mathsf{H}_{\infty,t}(J)$ exist only at $s=0$ and $t=0$, respectively.

We are going to start with the local algebras $\mathscr{A}_{0,\infty}^{\mathscr{K}}$ and $\mathscr{A}_{\infty,0}^{\mathscr{K}}$.

Proposition 5.7.2. *The local algebra $\mathscr{A}_{0,\infty}^{\mathscr{K}}$ is isometrically isomorphic to the closed subalgebra* $\mathrm{alg}\{I,\chi_+I,P_{\mathbb{R}},J\}$ *of $\mathscr{L}(L^p(\mathbb{R}^+,w_{\alpha(0)}))$, with the isomorphism given by $\Phi_{0,\infty}^{\mathscr{K}}(A)\mapsto\mathsf{H}_{0,\infty}(A)$. In particular, for $a\in PC(\dot{\mathbb{R}})$ and $b\in PC_{p,w}$,*

$$\begin{aligned}\Phi_{0,\infty}^{\mathscr{K}}(aI)&\mapsto a(0^-)\chi_-I+a(0^+)\chi_+I,\\ \Phi_{0,\infty}^{\mathscr{K}}(W(b))&\mapsto b(-\infty)Q_{\mathbb{R}}+b(+\infty)P_{\mathbb{R}},\\ \Phi_{0,\infty}^{\mathscr{K}}(J)&\mapsto J.\end{aligned}$$

Proof. First note that the algebras $\mathscr{A}_{0,\infty}^{\mathscr{K}}$ related to $L^p(\mathbb{R}^+,w)$ and to $L^p(\mathbb{R}^+,|t|^{\alpha(0)})$, respectively, are isometrically isomorphic, as can be seen by the same arguments as in Proposition 4.3.2, steps 1 and 2. From this fact one deduces the independence of $\mathsf{H}_{0,\infty}(A)$ of operators belonging to the local ideal $\mathscr{I}_{0,\infty}$, whence the correctness of the definition of the homomorphism follows. The concrete form of the values of the homomorphism at the generators comes from Proposition 5.4.3 and the fact that the operator J is homogeneous. Thus we conclude that $\Phi_{0,\infty}^{\mathscr{K}}(A)\mapsto\mathsf{H}_{0,\infty}(A)$ is a mapping onto $\mathrm{alg}\{I,\chi_+I,P_{\mathbb{R}},J\}$. Finally, since $\Phi_{0,\infty}^{\mathscr{K}}(A)=\Phi_{0,\infty}^{\mathscr{K}}(\mathsf{H}_{0,\infty}(A))$, this mapping is an isometry. ■

In a similar way, one gets the following description of the local algebra at $(\infty,0)$.

Proposition 5.7.3. *The local algebra $\mathscr{A}_{\infty,0}^{\mathscr{K}}$ is isometrically isomorphic to the closed subalgebra* $\mathrm{alg}\{I,\chi_+I,P_{\mathbb{R}},J\}$ *of $\mathscr{L}(L^p(\mathbb{R}^+,w_{\alpha(\infty)}))$, and the isomorphism is given by $\Phi_{\infty,0}^{\mathscr{K}}(A)\mapsto\mathsf{H}_{\infty,0}(A)$. In particular, for $a\in PC(\dot{\mathbb{R}})$ and $b\in PC_{p,w}$,*

$$\begin{aligned}\Phi_{\infty,0}^{\mathscr{K}}(aI)&\mapsto a(-\infty)\chi_-I+a(+\infty)\chi_+I,\\ \Phi_{\infty,0}^{\mathscr{K}}(W(b))&\mapsto b(0^-)Q_{\mathbb{R}}+b(0^+)P_{\mathbb{R}},\\ \Phi_{\infty,0}^{\mathscr{K}}(J)&\mapsto J.\end{aligned}$$

Now we turn to the local algebras $\mathscr{A}_{s,\infty}^{\mathscr{K}}$ and $\mathscr{A}_{\infty,t}^{\mathscr{K}}$ where $s,t>0$. As already mentioned, the mappings $\mathsf{H}_{s,\infty}$ and $\mathsf{H}_{\infty,t}$ are not well defined on $\mathscr{A}\big(PC(\dot{\mathbb{R}}),PC_{p,w},J\big)$ for $s,t\neq0$. So we will have to use a modified approach which is based on the fact that every operator A in $\mathscr{A}\big(PC(\dot{\mathbb{R}}),PC_{p,w},J\big)$ can be approximated as closely as desired by operators of the form $A_1 + JA_2$ where A_1 and A_2 belong to the algebra $\mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}\big)$ without flip. To be precise, $\mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}\big)$ stands for the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R},w))$ which contains all multiplication operators aI with $a \in PC(\dot{\mathbb{R}})$ and all convolution operators $W^0(b)$ with $b \in PC_{p,w}$. Note that the decomposition of an operator A in the form $A_1 + JA_2$ is not unique in general.

We start with verifying that the homomorphisms $\mathsf{H}_{s,\infty}$ and $\mathsf{H}_{\infty,t}$ *are* well defined on the elements of the ideals $\mathscr{I}_{s,t}$. Note that $\mathsf{H}_{s,\infty}(K) = 0$ and $\mathsf{H}_{\infty,t}(K) = 0$ for every compact operator K.

Proposition 5.7.4. *If $A + \mathscr{K} \in \mathscr{I}_{s,t}$, then $\mathsf{H}_{s,\infty}(A + \mathscr{K}) = 0$ and $\mathsf{H}_{\infty,t}(A + \mathscr{K}) = 0$.*

Proof. It is evident from the definition of the ideal $\mathscr{I}_{s,t}$ that each of its elements can be approximated as closely as desired by operators of the form $A_1 + JA_2$ with $A_1, A_2 \in \mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}\big)$ belonging to the smallest closed ideal of that algebra which contains all cosets $\Phi(fW^0(g))$ with $f \in \tilde{C}(\dot{\mathbb{R}})$ with $f(s) = 0$ and $g \in \tilde{C}_{p,w}$ with $g(t) = 0$. So we can assume, without loss of generality, that A is of this form. Then

$$\begin{aligned}
\mathsf{H}_{s,\infty}(A + \mathscr{K}) &= \operatorname*{s-lim}_{\tau \to +\infty} Z_\tau V_{-s} A V_s Z_\tau^{-1} \\
&= \operatorname*{s-lim}_{\tau \to +\infty} Z_\tau V_{-s} (A_1 + JA_2) V_s Z_\tau^{-1} \\
&= \operatorname*{s-lim}_{\tau \to +\infty} Z_\tau V_{-s} A_1 V_s Z_\tau^{-1} + \operatorname*{s-lim}_{\tau \to +\infty} (Z_\tau V_{-s} J V_s Z_\tau^{-1})\,(Z_\tau V_{-s} A_2 V_s Z_\tau^{-1}).
\end{aligned}$$

For $i = 1, 2$, one has $\text{s-lim}\, Z_\tau V_{-s} A_i V_s Z_\tau^{-1} = 0$, and the operators $Z_\tau V_{-s} J V_s Z_\tau^{-1}$ are uniformly bounded with respect to τ. Thus, $\mathsf{H}_{s,\infty}(A + \mathscr{K}) = 0$. The proof of the second assertion is similar. ■

Let f_s be a continuous function with support in $\mathbb{R}^+$ and such that $f_s(s) = 1$. Set $p := \Phi^{\mathscr{K}}_{s,\infty}(f_s I)$, $j := \Phi^{\mathscr{K}}_{s,\infty}(J)$, and $e := \Phi^{\mathscr{J}}_{s,\infty}(I)$. Then $p^2 = p$, p commutes with all generators of the algebra except with j, and $jpj = e - p$. Thus, by Corollary 1.1.20, every element of $\mathscr{A}^{\mathscr{K}}_{s,\infty}$ can be (uniquely) written as $a = a_1 + a_2 j$, where the a_i belong to the corresponding local algebra without flip, and we can employ this corollary to eliminate the flip by doubling the dimension. Let L denote the mapping defined before Proposition 1.1.19 and consider the mapping

$$\dot{\mathsf{H}}_{s,\infty} := \mathsf{H}_{s,\infty} L \;:\; \mathscr{A}^{\mathscr{K}}_{s,\infty} \to [\operatorname{alg}\{I,\, \chi_+ I,\, P_{\mathbb{R}}\}]^{2\times 2}, \tag{5.42}$$

where $\mathsf{H}_{s,\infty}$ now refers to the canonical (diagonal) extension for matrix operators of the strong limit defined in (5.10). The mapping $\dot{\mathsf{H}}_{s,\infty}$ is well defined due to Proposition 5.7.4, and it acts as an homomorphism between the algebras mentioned. In what follows, the notation $-s^{\pm}$ is understood as $(-s)^{\pm}$.

Proposition 5.7.5. *Let $s > 0$. The local algebra $\mathscr{A}^{\mathscr{K}}_{s,\infty}$ is isomorphic to the matrix algebra $[\operatorname{alg}\{I,\, \chi_+ I,\, P_{\mathbb{R}}\}]^{2\times 2}$ with entries acting on $L^p(\mathbb{R}, w_{\alpha(s)})$. The isomorphism* *is given by* $\Phi^{\mathcal{K}}_{s,\infty}(A) \mapsto \dot{\mathsf{H}}_{s,\infty}(A)$. *In particular, for* $a \in PC(\dot{\mathbb{R}})$ *and* $b \in PC_{p,w}$,

$$\begin{aligned}
\Phi^{\mathcal{K}}_{s,\infty}(aI) &\mapsto \begin{bmatrix} a(s^-)\chi_- I + a(s^+)\chi_+ I & 0 \\ 0 & a(-s^+)\chi_- I + a(-s^-)\chi_+ I \end{bmatrix}, \\
\Phi^{\mathcal{K}}_{s,\infty}\left(W^0(b)\right) &\mapsto \begin{bmatrix} b(-\infty)Q_{\mathbb{R}} + b(+\infty)P_{\mathbb{R}} & 0 \\ 0 & b(+\infty)Q_{\mathbb{R}} + b(-\infty)P_{\mathbb{R}} \end{bmatrix}, \\
\Phi^{\mathcal{K}}_{s,\infty}(J) &\mapsto \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.
\end{aligned}$$

Proof. The mapping $\dot{\mathsf{H}}_{s,\infty}$ is well defined on $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ by Proposition 5.7.4. The values of the homomorphism can be derived from Corollary 1.1.20 and Proposition 5.4.3. To see that the homomorphism $\dot{\mathsf{H}}_{s,\infty}$ is injective, define

$$\begin{aligned}
&\dot{\mathsf{H}}'_{s,\infty}\colon [\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2} \to \mathcal{A}^{\mathcal{K}}_{s,\infty}, \\
&\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \mapsto L^{-1}\left(\begin{bmatrix} p\Phi^{\mathcal{K}}_{s,\infty}(V_s A_{11} V_{-s}) & p\Phi^{\mathcal{K}}_{s,\infty}(V_s A_{12} V_{-s}) \\ p\Phi^{\mathcal{K}}_{s,\infty}(V_s A_{21} V_{-s}) & p\Phi^{\mathcal{K}}_{s,\infty}(V_s A_{22} V_{-s}) \end{bmatrix}\right).
\end{aligned} \tag{5.43}$$

The injectivity will follow once we have shown that

$$\dot{\mathsf{H}}'_{s,\infty}\left(\dot{\mathsf{H}}_{s,\infty}(\Phi^{\mathcal{K}}_{s,\infty}(A))\right) = \Phi^{\mathcal{K}}_{s,\infty}(A) \quad \text{for all} \quad \Phi^{\mathcal{K}}_{s,\infty}(A) \in \mathcal{A}^{\mathcal{K}}_{s,\infty}.$$

It is sufficient to check this equality for the generating cosets of $\mathcal{A}^{\mathcal{K}}_{s,\infty}$, i.e., for $\Phi^{\mathcal{K}}_{s,\infty}(I)$, $\Phi^{\mathcal{K}}_{s,\infty}(J)$, $\Phi^{\mathcal{K}}_{s,\infty}(P_{\mathbb{R}})$, and $\Phi^{\mathcal{K}}_{s,\infty}(\chi_s I)$, where χ_s stands for the characteristic function of $]-\infty, s]$. This check is straightforward.

Finally, to verify the surjectivity of the homomorphism $\dot{\mathsf{H}}_{s,\infty}$, we again rely on $\dot{\mathsf{H}}'_{s,\infty}$. Indeed, this mapping is well defined on all of $[\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2}$, and one has $\dot{\mathsf{H}}_{s,\infty}\left(\dot{\mathsf{H}}'_{s,\infty}(A)\right) = A$ for all $A \in [\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2}$. ■

Now let $t > 0$. For the local algebras $\mathcal{A}^{\mathcal{K}}_{\infty,t}$, we again apply Corollary 1.1.20 to eliminate the flip by doubling the dimension. Let f_t be a continuous function with support in $\mathbb{R}^+$ such that $f_t(t) = 1$, and put $p := \Phi^{\mathcal{K}}_{\infty,t}(W^0(f_t))$, $j := \Phi^{\mathcal{K}}_{\infty,t}(J)$, and $e := \Phi^{\mathcal{K}}_{\infty,t}(I)$. Then p is an idempotent which commutes with all generators of the algebra except with j, for which one has $jpj = e - p$. Every element of $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ can be (uniquely) written as $a = a_1 + a_2 j$, where the a_i belong to the corresponding local algebra without flip. Define the homomorphism

$$\dot{\mathsf{H}}_{\infty,t} := \mathsf{H}_{\infty,t} L \,:\, \mathcal{A}^{\mathcal{K}}_{\infty,t} \to [\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2} \tag{5.44}$$

where $\mathsf{H}_{\infty,t}$ now refers to the canonical (diagonal) extension for matrix operators of the strong limit defined in (5.11), and where L is again the mapping defined before Proposition 1.1.19.

The proof of the next result is the same as that of Proposition 5.7.5.

Proposition 5.7.6. *Let $t > 0$. The local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,t}$ is isomorphic to the matrix algebra $[\mathrm{alg}\{I,\, \chi_+ I,\, P_{\mathbb{R}}\}]^{2\times 2}$ with entries acting on $L^p(\mathbb{R}, w_{\alpha(\infty)})$. The isomorphism is given by $\Phi^{\mathscr{K}}_{\infty,t}(A) \mapsto \dot{\mathsf{H}}_{\infty,t}\,(A)$. In particular, for $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_{p,w}$,*

$$\Phi^{\mathscr{K}}_{\infty,t}(aI) \quad\mapsto\quad \begin{bmatrix} a(-\infty)\chi_- I + a(+\infty)\chi_+ I & 0 \\ 0 & a(+\infty)\chi_- I + a(-\infty)\chi_+ I \end{bmatrix},$$

$$\Phi^{\mathscr{K}}_{\infty,t}\left(W^0(b)\right) \quad\mapsto\quad \begin{bmatrix} b(t^-)Q_{\mathbb{R}} + b(t^+)P_{\mathbb{R}} & 0 \\ 0 & b(-t^+)Q_{\mathbb{R}} + b(-t^-)P_{\mathbb{R}} \end{bmatrix},$$

$$\Phi^{\mathscr{K}}_{\infty,t}(J) \quad\mapsto\quad \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Our final concern is the local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$. We are going to show that $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ is a unital algebra generated by two commuting projections and a flip.

Proposition 5.7.7. *The local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ is generated by the elements $e := \Phi^{\mathscr{K}}_{\infty,\infty}(I)$, $p := \Phi^{\mathscr{K}}_{\infty,\infty}(\chi_+ I)$, $r := \Phi^{\mathscr{K}}_{\infty,\infty}(W^0(\chi_+))$ and $j := \Phi^{\mathscr{K}}_{\infty,\infty}(J)$.*

Proof. For $a \in PC(\dot{\mathbb{R}})$, write $\Phi^{\mathscr{K}}_{\infty,\infty}(aI)$ as

$$\Phi^{\mathscr{K}}_{\infty,\infty}(a(-\infty)\chi_- I + a(+\infty)\chi_+ I) - \Phi^{\mathscr{K}}_{\infty,\infty}((a - a(-\infty)\chi_- - a(+\infty)\chi_+)I).$$

Since the function $a - a(-\infty)\chi_- - a(+\infty)\chi_+$ is continuous at infinity and has the value 0 there, we obtain

$$\Phi^{\mathscr{K}}_{\infty,\infty}(aI) = \Phi^{\mathscr{K}}_{\infty,\infty}((a(-\infty)\chi_- + a(+\infty)\chi_+)I) = 0.$$

For $b \in PC_{p,w}$, one gets similarly

$$\Phi^{\mathscr{K}}_{\infty,\infty}(W^0(b)) = \Phi^{\mathscr{K}}_{\infty,\infty}(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)).$$

For the other generators, the result is obvious. ■

The generators of the algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ satisfy the relations

$$rp = pr, \qquad jrj = e - r \quad \text{and} \quad jpj = e - p.$$

Only the first of these relations is not completely evident. It can be verified by repeating arguments from the proof of Theorem 5.6.2 (iv). Thus, the algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ is generated by two commuting projections and a flip.

To get a matrix-valued symbol for the invertibility in the algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$, one can apply Proposition 1.1.19 to eliminate the flip by doubling the dimension, or, one refers formally to Theorem 3.3.13,. For the latter, note that the elements b and c defined in (3.53) and (3.54) are given by $b = pr + (e-p)(e-r)$ and $c = (pr - rp)j = 0$ in the present context, and that the spectrum of b is $\{0, 1\}$ since b is a non-trivial idempotent. Thus, Theorem 3.3.13 applies with $y = 0$ and $x = \pm 1$. In each case, we arrive at the following.

Proposition 5.7.8. *The local algebra $\mathscr{A}^{\mathscr{K}}_{\infty,\infty}$ is generated by the commuting projections $p = \Phi^{\mathscr{K}}_{\infty,\infty}(\chi_+ I)$ and $r = \Phi^{\mathscr{K}}_{\infty,\infty}(W^0(\chi_+))$ and by the flip $j = \Phi^{\mathscr{K}}_{\infty,\infty}(J)$. There is a symbol mapping which assigns with e, p, j and r a matrix-valued function on $\{0,1\}$ by*

$$(\operatorname{smb} e)(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\operatorname{smb} p)(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$(\operatorname{smb} j)(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; (\operatorname{smb} r)(x) = \begin{bmatrix} x & 0 \\ 0 & 1-x \end{bmatrix}.$$

Combining the previous results with Allan's local principle, we arrive at the following.

Theorem 5.7.9. *Let $A \in \mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$. The coset $A + \mathscr{K}(L^p(\mathbb{R}, w))$ is invertible in the quotient algebra $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ if and only if:*

(i) *the operator $\mathsf{H}_{0,\infty}(A)$ is invertible in the subalgebra* $\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}, J\}$ *of* $\mathscr{L}(L^p(\mathbb{R}, w_{\alpha(0)}))$*;*
(ii) *the operator $\mathsf{H}_{\infty,0}(A)$ is invertible in the subalgebra* $\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}, J\}$ *of* $\mathscr{L}(L^p(\mathbb{R}, w_{\alpha(\infty)}))$*;*
(iii) *the operator $\dot{\mathsf{H}}_{s,\infty}(A)$ is invertible in the subalgebra* $[\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}\}]^{2\times 2}$ *of* $[\mathscr{L}(L^p(\mathbb{R}, w_{\alpha(s)}))]^{2\times 2}$ *for every $s > 0$;*
(iv) *the operator $\dot{\mathsf{H}}_{\infty,t}(A)$ is invertible in the subalgebra* $[\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}\}]^{2\times 2}$ *of* $[\mathscr{L}(L^p(\mathbb{R}, w_{\alpha(\infty)}))]^{2\times 2}$ *for every $t > 0$;*
(v) *the matrix* $\operatorname{smb} \Phi^{\mathscr{K}}_{\infty,\infty}(A)$ *is invertible in $\mathbb{C}^{2\times 2}$.*

The following corollary can be proved by repeating the arguments from the proof of Theorem 5.5.9.

Corollary 5.7.10. *The algebra $\mathscr{A}^{\mathscr{K}}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ is inverse-closed in the Calkin algebra $\mathscr{L}(L^p(\mathbb{R}, w))/\mathscr{K}(L^p(\mathbb{R}, w))$. An operator $A \in \mathscr{A}\big(PC(\dot{\mathbb{R}}), PC_{p,w}, J\big)$ is Fredholm if and only if:*
(i) *the operator $\mathsf{H}_{0,\infty}(A)$ is invertible on $L^p(\mathbb{R}, w_{\alpha(0)})$;*
(ii) *the operator $\mathsf{H}_{\infty,0}(A)$ is invertible on $L^p(\mathbb{R}, w_{\alpha(\infty)})$;*
(iii) *the operator $\dot{\mathsf{H}}_{s,\infty}(A)$ is invertible on $L^p_2(\mathbb{R}, w_{\alpha(s)})$ for every $s > 0$;*
(iv) *the operator $\dot{\mathsf{H}}_{\infty,t}(A)$ is invertible on $L^p_2(\mathbb{R}, w_{\alpha(\infty)})$ for every $t > 0$;*
(v) *the matrix* $\operatorname{smb} \Phi^{\mathscr{K}}_{\infty,\infty}(A)$ *is invertible in $\mathbb{C}^{2\times 2}$.*

In the remainder of this section we are going to apply this corollary to a class of operators of particular interest: the Wiener-Hopf plus Hankel operators. These are the operators $W(b) + H(c)$ on $L^p(\mathbb{R}^+, w)$ with $b, c \in PC_{p,w}$. Equivalently, one can think of a Wiener-Hopf plus Hankel operator $W(b) + H(c)$ as the operator

$$\chi_+ W^0(b) \chi_+ I + \chi_+ W^0(c) J \chi_+ I + \chi_- I, \tag{5.45}$$

acting on $L^p(\mathbb{R}, w)$.

Theorem 5.7.11. *Let $b, c \in PC_{p,w}$. The operator (5.45) is Fredholm on $L^p(\mathbb{R}, w)$ if and only if $b(\pm\infty) \neq 0$ and if the functions*

(i) $y \mapsto (b(+\infty) + b(-\infty)) \sinh\big((y + \mathbf{i}\upsilon)\pi\big) + (b(+\infty) - b(-\infty)) \cosh\big((y + \mathbf{i}\upsilon)\pi\big) + c(+\infty) - c(-\infty)$ *with* $\upsilon := 1/p + \alpha(0)$,

(ii) $y \mapsto (b(0^+) + b(0^-)) \sinh\big((y + \mathbf{i}\upsilon)\pi\big) + (b(0^+) - b(0^-)) \cosh\big((y + \mathbf{i}\upsilon)\pi\big) + c(0^+) - c(0^-)$ *with* $\upsilon := 1/p + \alpha(\infty)$, *and*

(iii) $y \mapsto b_t^+ b_{-t}^+ + (b_t^- b_{-t}^+ + b_t^+ b_{-t}^-) \coth\big((y + \mathbf{i}\upsilon)\pi\big) + b_t^- b_{-t}^- \Big(\coth\big((y + \mathbf{i}\upsilon)\pi\big)\Big)^2 - c_t c_{-t} \Big(\sinh\big((y + \mathbf{i}\upsilon)\pi\big)\Big)^{-2}$ *with* $\upsilon := 1/p + \alpha(\infty)$, $b_t^\pm := b(t^+) \pm b(t^-)$, $b_{-t}^\pm := b(-t^+) \pm b(-t^-)$ *and* $c_{\pm t} := c(\pm t^+) - c(\pm t^-)$ *for* $t > 0$

do not vanish on $\overline{\mathbb{R}}$.

Proof. By Corollary 5.7.10, the operator (5.45) is Fredholm if and only if a collection of related operators, labeled by the points of the set $(\overline{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \overline{\mathbb{R}}^+)$, is invertible. We are going to examine the invertibility of the related operators for each point in this set.

For the point $(0, \infty)$, the related operator is

$$\chi_+ \left(b(-\infty)Q_{\mathbb{R}} + b(+\infty)P_{\mathbb{R}} + (c(-\infty)Q_{\mathbb{R}} + c(+\infty)P_{\mathbb{R}})J\right) \chi_+ I + \chi_- I.$$

This operator is invertible on $L^p(\mathbb{R}, w_{\alpha(0)})$ if and only if the operator

$$\frac{b(+\infty) + b(-\infty)}{2} I + \frac{b(+\infty) - b(-\infty)}{2} S_{\mathbb{R}^+} + \frac{c(+\infty) - c(-\infty)}{2} H_\pi$$

is invertible on $L^p(\mathbb{R}^+, w_{\alpha(0)})$. The latter condition is equivalent to condition (i) which can easily be seen by inserting the Mellin symbols of the operators $S_{\mathbb{R}^+}$ and H_π quoted in Section 4.2.2.

For the point $(\infty, 0)$, the related operator

$$\chi_+ \left(b(0^-)Q_{\mathbb{R}} + b(0^+)P_{\mathbb{R}} + (c(0^-)Q_{\mathbb{R}} + c(0^+)P_{\mathbb{R}})J\right) \chi_+ I + \chi_- I$$

is invertible on $L^p(\mathbb{R}, w_{\alpha(\infty)})$ if and only if the operator

$$\frac{b(0^+) + b(0^-)}{2} I + \frac{b(0^+) - b(0^-)}{2} S_{\mathbb{R}^+} + \frac{c(0^+) - c(0^-)}{2} H_\pi$$

is invertible on $L^p(\mathbb{R}^+, w_{\alpha(\infty)})$. Condition (ii) states the conditions for the invertibility of the Mellin symbol of this operator.

For (s, ∞) with $s > 0$, we have

$$b(-\infty)Q_{\mathbb{R}} + b(+\infty)P_{\mathbb{R}}$$

as the related operator. This operator is invertible on $L^p(\mathbb{R}, w_{\alpha(s)})$ if and only if $b(\pm\infty) \neq 0$.

For (∞, t) with $t > 0$, the invertibility of the related operator

$$\begin{bmatrix} \chi_+(b(t^-)Q_{\mathbb{R}}+b(t^+)P_{\mathbb{R}})\chi_+I+\chi_-I & \chi_+(c(t^-)Q_{\mathbb{R}}+c(t^+)P_{\mathbb{R}})\chi_-I \\ \chi_-(c(-t^+)Q_{\mathbb{R}}+c(-t^-)P_{\mathbb{R}})\chi_+I & \chi_-(b(-t^+)Q_{\mathbb{R}}+b(-t^-)P_{\mathbb{R}})\chi_-I+\chi_+I \end{bmatrix}$$

on $L_2^p(\mathbb{R}, w_{\alpha(\infty)})$ is equivalent to the invertibility of the operator

$$\begin{bmatrix} b(t^-)Q_{\mathbb{R}^+}+b(t^+)P_{\mathbb{R}^+} & \frac{c(t^+)-c(t^-)}{2}H_\pi \\ \frac{c(-t^+)-c(-t^-)}{2}H_\pi & b(-t^+)P_{\mathbb{R}^+}+b(-t^-)Q_{\mathbb{R}^+} \end{bmatrix}.$$

The Mellin symbol of this operator is

$$\begin{bmatrix} b_t^+ + b_t^- \coth\big((y+\mathbf{i}\upsilon)\pi\big) & \frac{c_t}{2}\Big(\sinh\big((y+\mathbf{i}\upsilon)\pi\big)\Big)^{-1} \\ \frac{c_{-t}}{2}\Big(\sinh\big((y+\mathbf{i}\upsilon)\pi\big)\Big)^{-1} & b_{-t}^+ + b_{-t}^- \coth\big((y+\mathbf{i}\upsilon)\pi\big) \end{bmatrix}, \tag{5.46}$$

and condition (iii) states exactly the conditions for the invertibility of this matrix function. Finally, the matrix related to the point (∞,∞) is invertible if and only if $b(\pm\infty) \neq 0$. ■

What the above results tell us about the essential spectrum of the Wiener-Hopf plus Hankel operator is in some sense expected. The local spectrum at each point where both b and c are continuous corresponds to the value of the function b at that point. If only b is discontinuous at some point, then the local spectrum corresponds to the left and right one-sided limits, joined by a circular arc, the shape of which depends on the space and weight (see Figure 4.4). If only c is discontinuous at some point t, but not at $-t$, there is no effect on the essential spectrum. If both b and c share a point of discontinuity at 0 or ∞, the effect on the essential spectrum is the "sum" of the circular arc with the "water drop" arc (see Figure 4.5 (a)). The more complex effects occur when both b and c share points of discontinuity on $\pm t$, $t \in \mathbb{R}^+$. In this case, the essential spectrum is given by the spectrum of the matrix function (5.46).

Example 5.7.12. Let the weight w be such that $1/p + \alpha(\infty) \in \{1/2, 1/2 + 0.01, 2/3\}$. Define $b \in PC_p$ by

$$b(t) = \begin{cases} \frac{t+10}{10} + \frac{t+10}{2}\mathbf{i} & \text{for } -20 < t < 0, \\ \frac{t-10}{10} + \frac{t-10}{2}\mathbf{i} & \text{for } 0 < t < 20, \\ -3\mathbf{i} & \text{for all other } t \end{cases}$$

and consider a function c which is continuous at all points except the integer points in $[-19, 19] \setminus \{-10, 0, 10\}$, where it satisfies $c(t^+) - c(t^-) = 1$, $c(-t^+) - c(-t^-) = 1$ if $1 \leq t \leq 9$ and $c(t^+) - c(t^-) = 1$, $c(-t^+) - c(-t^-) = -1$ in the case $11 \leq t \leq 19$. Then the essential spectrum of the operator $W(b) + H(c)$ on the space $\mathscr{L}(L^p(\mathbb{R}, w))$ is given by Figure 5.3. The three arcs joining the points of discontinuity of the function b are clear, as is the variation of the size of the "water drop" arcs derived from the distance between $b(-t)$ and $b(t)$. When the jumps of c have the same sign, they will actually interfere with one another and form other geometric figures. If we change p or the weight, such that $\upsilon = 1/p + \alpha(\infty)$ approaches $1/2$, all curves from discontinuities turn into line segments. □

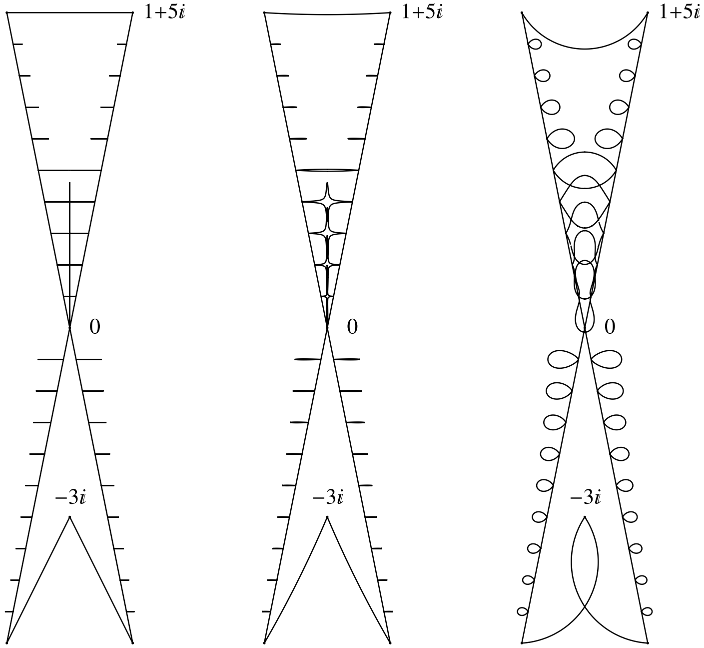

Fig. 5.3 The essential spectrum of $W(b) + H(c)$ for $\upsilon = 1/2$, $1/2 + 0.01$ and $2/3$.

Remark 5.7.13. In contrast to Section 4.5, the proof of Theorem 5.7.11 yields immediately a 2-symbol for the Wiener-Hopf plus Hankel operators with piecewise generating functions due to the finer localization used to obtain Theorem 5.7.9. □

5.8 Multidimensional convolution type operators

Now we turn our attention to the Fredholm property of multidimensional convolution type operators on $\mathbb{R}^N$. We will see that the techniques developed so far –

localization and homogenization – work well also in the multidimensional context, but that a new difficulty appears if $N > 1$: already the generators of the algebra will have massive spectra. Hence, Corollary 1.2.32 does not apply to prove the inverse-closedness of the operator algebra under consideration (and, actually, we still do not know if this algebra is inverse-closed). In this section, we will point out one way to overcome this difficulty.

Let N be a positive integer. We denote the Euclidean norm on $\mathbb{R}^N$ by $|\cdot|$ and write $\langle\cdot,\cdot\rangle$ for the related scalar product on $\mathbb{R}^N$. Thus, $|x|^2 = \langle x, x\rangle$ for $x \in \mathbb{R}^N$. The unit sphere in $\mathbb{R}^N$ will be denoted by $\mathbb{S}^{N-1}$, and the open unit ball by $\mathbb{B}^N$.

It is easy to see that the mapping

$$\xi : \mathbb{B}^N \to \mathbb{R}^N, \quad x \mapsto \frac{x}{1-|x|}, \tag{5.47}$$

is a homeomorphism with inverse

$$\xi^{-1} : \mathbb{R}^N \to \mathbb{B}^N, \quad x \mapsto \frac{x}{1+|x|}. \tag{5.48}$$

In particular, a function f on $\mathbb{R}^N$ is continuous if and only if the function $f \circ \xi$ is continuous on $\mathbb{B}^N$. We denote by $C(\overline{\mathbb{R}^N})$ the set of all continuous complex-valued functions f on $\mathbb{R}^N$ for which the (continuous) function $f \circ \xi$ on $\mathbb{B}^N$ possesses a continuous extension $f^\sim$ onto the closed ball $\overline{\mathbb{B}^N}$. Provided with pointwise operations and the supremum norm, $C(\overline{\mathbb{R}^N})$ forms a commutative C^*-algebra, and this algebra is isomorphic to $C(\overline{\mathbb{B}^N})$. Thus, the maximal ideal spaces of these algebras are homeomorphic. The maximal ideal space of $C(\overline{\mathbb{B}^N})$ is the closed unit ball $\overline{\mathbb{B}^N}$, which is a union of the open ball $\mathbb{B}^N$ and the unit sphere $\mathbb{S}^{N-1}$. Analogously, one can think of the maximal ideal space of $C(\overline{\mathbb{R}^N})$ as the union of $\mathbb{R}^N$ and of an "infinitely distant" sphere. More precisely, every multiplicative linear functional on $C(\overline{\mathbb{R}^N})$ is either of the form

$$f \mapsto f(x) \quad \text{with} \quad x \in \mathbb{R}^N$$

or of the form

$$f \mapsto (f \circ \xi)^\sim(\theta) \quad \text{with} \quad \theta \in \mathbb{S}^{N-1}.$$

We denote the latter functional by θ_∞ and write $f(\theta_\infty)$ in place of $\theta_\infty(f)$. Clearly, $f(\theta_\infty) = \lim_{t\to\infty} f(t\theta)$, and a sequence $h \in \mathbb{R}^N$ converges to θ_∞ if $\xi^{-1}(h_n)$ converges to θ. A basis of neighborhoods of θ_∞ is provided by the sets of the form

$$\begin{aligned} U_{R,\varepsilon}(\theta_\infty) := \quad & \{|x|\psi \in \mathbb{R}^N : |x| > R,\ \psi \in \mathbb{S}^{N-1} \text{and } |\psi - \theta| < \varepsilon\} \\ & \bigcup \{\psi_\infty : \psi \in \mathbb{S}^{N-1} \text{and } |\psi - \theta| < \varepsilon\}. \end{aligned} \tag{5.49}$$

We denote the maximal ideal space of $C(\overline{\mathbb{R}^N})$ by $\overline{\mathbb{R}^N}$.

Every function $a \in L^1(\mathbb{R}^N)$ defines an operator W_a^0 of convolution by a by

$$W_a^0 : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad g \mapsto \int_{\mathbb{R}^N} a(t-s)g(s)\,ds. \tag{5.50}$$

The operator of convolution by $a \in L^1(\mathbb{R}^N)$ is bounded on $L^p(\mathbb{R}^N)$, and

$$\|W_a^0\|_{\mathscr{L}(L^p(\mathbb{R}^N))} \leq \|a\|_{L^1(\mathbb{R}^N)}.$$

The goal of this section is to study the Fredholm property of operators which belong to the smallest closed subalgebra $\mathscr{A}_p$ of $\mathscr{L}(L^p(\mathbb{R}^N))$ which contains

- all convolution operators W_a^0 with $a \in L^1(\mathbb{R}^N)$,
- all operators of multiplication by a function in $C(\overline{\mathbb{R}^N})$,
- all operators $\chi_{\mathbb{H}(\theta)}I$ of multiplication by the characteristic function of a half-space
$$\mathbb{H}(\theta) := \{x \in \mathbb{R}^N : \langle x, \theta\rangle \geq 0\} \quad \text{with} \quad \theta \in \mathbb{S}^{N-1}.$$

Proposition 5.8.1. *The algebra $\mathscr{A}_p$ contains the ideal $\mathscr{K}(L^p(\mathbb{R}^N))$ of the compact operators.*

Proof. Let $\mathscr{A}_p'$ denote the smallest closed subalgebra of $\mathscr{L}(L^p(\mathbb{R}^N))$ which contains all operators W_a^0 with $a \in L^1(\mathbb{R}^N)$ and all operators of multiplication by a function in $C_0^\infty(\mathbb{R}^N)$. We will show $\mathscr{K}(L^p(\mathbb{R}^N))$ is already contained in $\mathscr{A}_p'$, which implies the assertion .

It is sufficient to show that $\mathscr{A}_p'$ contains all operators of rank one. Every operator of rank one on $L^p(\mathbb{R}^N)$ has the form

$$(Ku)(t) = a(t)\int_{\mathbb{R}^N} b(s)u(s)ds, \quad t \in \mathbb{R}^N, \tag{5.51}$$

where $a \in L^p(\mathbb{R}^N)$ and $b \in L^q(\mathbb{R}^N)$ with $1/p + 1/q = 1$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ (with respect to the corresponding norms), it is further sufficient to show that every operator (5.51) with $a, b \in C_0^\infty(\mathbb{R}^N)$ belongs to $\mathscr{A}_p'$.

Let $a, b \in C_0^\infty(\mathbb{R}^N)$, and choose a function $k \in L^1(\mathbb{R}^N)$ which is 1 on the compact set $\{t - s : t \in \operatorname{supp} f, s \in \operatorname{supp} g\}$. Then the operator (5.51) can be written as

$$(Ku)(t) = a(t)\int_{\mathbb{R}^N} k(t-s)b(s)u(s)ds, \quad t \in \mathbb{R}^N.$$

Evidently, this operator belongs to $\mathscr{A}_p'$. ■

As already mentioned, we do not know if the algebra $\mathscr{A}_p^{\mathscr{K}} := \mathscr{A}_p/\mathscr{K}(L^p(\mathbb{R}^N))$ is inverse-closed in the Calkin algebra. Therefore we have to apply the local principle in a larger algebra which we are going to introduce in a couple of steps.

Let $\Lambda_p \subset \mathscr{L}(L^p(\mathbb{R}^N))$ denote the Banach algebra of all operators of local type with respect to the algebra $C(\overline{\mathbb{R}^N})$, that is, an operator $A \in \mathscr{L}(L^p(\mathbb{R}^N))$ belongs to Λ_p if and only if

$$fA - AfI \in \mathscr{K}\left(L^p(\mathbb{R}^N)\right) \quad \text{for every} \quad f \in C(\overline{\mathbb{R}^N}).$$

In order to show that $\mathscr{A}_p \subset \Lambda_p$, we need the following lemma.

Lemma 5.8.2. *Let F_1 and F_2 be disjoint closed subsets of $\overline{\mathbb{R}^N}$. Then there exists a $\delta > 0$ such that $|x-y| > (R+1)\delta$ for all $R > 0$, all $x \in F_1 \cap \mathbb{R}^N$ with $|x| > R$ and all $y \in F_2 \cap \mathbb{R}^N$ with $|y| > R$.*

Proof. Let $\widetilde{\xi}$ stand for the homeomorphism from $\overline{\mathbb{B}^N}$ onto $\overline{\mathbb{R}^N}$ which coincides with ξ on $\mathbb{B}^N$, and set $\tilde{F}_1 := \widetilde{\xi}^{-1}(F_1)$ and $\tilde{F}_2 := \widetilde{\xi}^{-1}(F_2)$. Then $\tilde{F}_1$ and $\tilde{F}_2$ are disjoint compact subsets of $\overline{\mathbb{B}^N}$; hence, $\delta := \operatorname{dist}(\tilde{F}_1, \tilde{F}_2) > 0$. Thus, for $x \in F_1$ and $y \in F_2$ one has $|\xi^{-1}(x) - \xi^{-1}(y)| > \delta$ or, equivalently,

$$\left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| > \delta. \tag{5.52}$$

Let $y \neq 0$ and consider the function $f(t) := |x - ty|$ on $\mathbb{R}$. This function attains its minimum at the point $t^* := \langle x, y\rangle / \langle y, y\rangle$ and is therefore monotonically increasing on the interval $[t^*, \infty[$. Since $t^* \le |x|\,|y|/|y|^2 = |x|/|y|$ and

$$\frac{|x|}{|y|} \le \frac{1+|x|}{1+|y|} \le 1 \quad \text{if} \quad |x| \le |y|,$$

we conclude that $f(\frac{1+|x|}{1+|y|}) \le f(1)$. Thus, by (5.52),

$$|x-y| \ge \left| x - \frac{1+|x|}{1+|y|} y \right| = (1+|x|) \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \ge (1+|x|)\delta \tag{5.53}$$

for $|x| \le |y|$. Analogously, if $|y| \le |x|$, then $|x-y| \ge (1+|y|)\delta$. So one gets

$$|x-y| \ge \min\{1+|x|,\, 1+|y|\}\,\delta,$$

which implies the assertion. ∎

Proposition 5.8.3. *If $f \in C(\overline{\mathbb{R}^N})$ and $a \in L^1(\mathbb{R}^N)$, then $W_a^0 fI - fW_a^0$ is a compact operator. Thus, $\mathscr{A}_p \subset \Lambda_p$.*

Proof. By Krasnoselskii's interpolation theorem, it is sufficient to verify the compactness of $W_a^0 fI - fW_a^0$ on $L^2(\mathbb{R})$. By Theorem 2.5.6, this operator is compact if and only if the operator $\chi_{F_1} W_a^0 \chi_{F_2} I$ is compact for each choice of closed disjoint subsets F_1 and F_2 of $\overline{\mathbb{R}^N}$. For g in $L^2(\mathbb{R}^N)$, one has

$$(\chi_{F_1} W_a^0 \chi_{F_2} g)(s) = \int_{\mathbb{R}^N} \chi_{F_1}(s) a(s-t) \chi_{F_2}(t) g(t)\, dt.$$

Since functions in $L^1(\mathbb{R}^N)$ can be approximated by continuous functions with compact support, we can assume that a is a continuous function with support contained in the centered ball of radius M. Set $\widetilde{a}(s,t) := \chi_{F_1}(s) a(s-t) \chi_{F_2}(t)$. By the previous lemma, there exists $\delta > 0$ such that $|s-t| > R\delta$ for every $R > 0$ and for arbitrary points $s \in F_1 \cap \mathbb{R}^N$ and $t \in F_2 \cap \mathbb{R}^N$ with $|s| > R$ and $|t| > R$. Choose R such that

$R > M$ and $R\delta > M$. Then $\widetilde{a}(s,t) = 0$ if $s \in F_1$ and $|s| > 2R$, or if $t \in F_2$ and $|t| > 2R$. Indeed, if $|s| > 2R$ and $|t| \le R$, then $|s-t| \ge |s| - |t| > R > M$, and if $|s| > 2R$ and $|t| > R$, then $|s-t| > R\delta > M$ due to the choice of R. Similarly, $|t| > 2R$ implies that $|s-t| > M$. Hence, $\widetilde{a}$ is a compactly supported bounded function, whence $\widetilde{a} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$. Therefore, $\chi_{F_1} W_a^0 \chi_{F_2} I$ is a Hilbert-Schmidt operator, and thus compact. ■

Our next goal is to introduce certain strong limit operators which will be used later to identify local algebras. For $k \in \mathbb{R}^N$, we define the shift operator

$$V_k : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad (V_k u)(s) = u(s-k),$$

and for $t > 0$, the dilation operator

$$Z_t : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad (Z_t u)(s) := t^{-N/p} u(s/t).$$

Both operators act as bijective isometries, with inverses given by $V_k^{-1} = V_{-k}$ and $Z_t^{-1} = Z_{t^{-1}}$.

Proposition 5.8.4. *Let $x \in \mathbb{R}^N$. Then the strong limit*

$$\mathsf{H}_x(A) := \operatorname*{s-lim}_{t\to\infty} Z_t V_{-x} A V_x Z_t^{-1} \tag{5.54}$$

exists for every operator $A \in \mathscr{A}_p$, the mapping H_x defines a homomorphism on $\mathscr{A}_p$, and

(i) $\mathsf{H}_x(W_a^0) = 0$ *for* $a \in L^1(\mathbb{R}^N)$;
(ii) $\mathsf{H}_x(fI) = f(x)I$ *for* $f \in C(\overline{\mathbb{R}^N})$;
(iii) $\mathsf{H}_x(\chi_{\mathbb{H}(\theta)} I)$ *is* 0, I *or* $\chi_{\mathbb{H}(\theta)} I$, *depending on whether x is outside, in the interior or on the boundary of* $\mathbb{H}(\theta)$, *respectively, where* $\theta \in \mathbb{S}^{N-1}$;
(iv) $\mathsf{H}_x(K) = 0$ *for K compact.*

The proof follows as that of Proposition 5.4.3 (but is actually much simpler since all the functions are continuous). The details are left to the reader.

For $\theta \in \mathbb{S}^{N-1}$, consider the sequence $h^\theta : \mathbb{N} \to \mathbb{R}^N$ defined by $h^\theta(n) := n\theta$.

Proposition 5.8.5. *Let $\theta \in \mathbb{S}^{N-1}$. Then the strong limit*

$$\mathsf{H}_\theta^\circ(A) := \operatorname*{s-lim}_{n\to\infty} V_{-h^\theta(n)} A V_{h^\theta(n)} \tag{5.55}$$

exists for every operator $A \in \mathscr{A}_p$, the mapping H_θ° defines a homomorphism on $\mathscr{A}_p$, and

(i) $\mathsf{H}_\theta^\circ(W_a^0) = W_a^0$ *for* $a \in L^1(\mathbb{R}^N)$;
(ii) $\mathsf{H}_\theta^\circ(fI) = f(\theta_\infty)I$ *for* $f \in C(\overline{\mathbb{R}^N})$;
(iii) $\mathsf{H}_\theta^\circ(\chi_{\mathbb{H}(\psi)} I)$ *is* 0, I *or* $\chi_{\mathbb{H}(\psi)} I$, *depending on whether θ is outside, inside or on the boundary of* $\mathbb{H}(\psi)$, *respectively, where* $\psi \in \mathbb{S}^{N-1}$;
(iv) $\mathsf{H}_\theta^\circ(K) = 0$ *for K compact.*

Proof. Assertion (i) is evident since convolution operators are shift invariant. For assertion (ii), let g be a function in $L^p(\mathbb{R}^N)$ the support of which is in the ball $B_M(0)$ with radius M for some $M > 0$. By definition, we have

$$(V_{-h^\theta(n)} f V_{h^\theta(n)} g)(x) = f(x + h^\theta(n))g(x).$$

Since f is continuous at θ_∞ there is, for every $\varepsilon > 0$, a neighborhood $U_{R,\delta}(\theta_\infty)$ of θ_∞ as in (5.49) such that $|f(x) - f(\theta_\infty)| < \varepsilon$ for every $x \in U_{R,\delta}(\theta_\infty)$. The compactness of $B_M(0)$ guarantees that there is an $n_0 \in \mathbb{N}$ such that $B_M(0) + h^\theta(n)$ is contained in $U_{R,\delta}(\theta_\infty)$ whenever $n \geq n_0$. Thus, for $n \geq n_0$,

$$\begin{aligned}&\|(V_{-h^\theta(n)} f V_{h^\theta(n)} - f(\theta_\infty))g\|_{L^p} \\ &\qquad \leq \sup_{x\in B_M(0)} |f(x + h^\theta(n)) - f(\theta_\infty)| \, \|g\|_{L^p} \leq \varepsilon \|g\|_{L^p}.\end{aligned}$$

Since the functions with compact support are dense in $L^p(\mathbb{R}^N)$, we get

$$\|(V_{-h^\theta(n)} f V_{h^\theta(n)} - f(\theta_\infty))g\|_{L^p} \to 0$$

for every g in $L^p(\mathbb{R}^N)$.

Assertion (iii) is again evident, since $V_{-h^\theta(n)} \chi_{\mathbb{H}(\psi)} V_{h^\theta(n)}$ is the operator of multiplication by the characteristic function of the shifted half space $-h^\theta(n) + \mathbb{H}(\psi)$. Assertion (iv) follows from the compactness of T and from the weak convergence of the operators $V_{h(n)}$ to zero as $|h(n)| \to \infty$. ■

Let Λ_p^{hom} stand for the set of all operators $A \in \mathcal{L}(L^p(\mathbb{R}^N))$ which are subject to the following conditions:

- A is of local type, i.e., $A \in \Lambda_p$;
- the strong limits $\mathsf{H}_x(A)$ on $L^p(\mathbb{R}^N)$ and $\mathsf{H}_x(A^*)$ on $(L^p(\mathbb{R}^N))^*$ defined by (5.54) exist for every $x \in \mathbb{R}^N$;
- the strong limits $\mathsf{H}_\theta^\circ(A)$ on $L^p(\mathbb{R}^N)$ and $\mathsf{H}_\theta^\circ(A^*)$ on $(L^p(\mathbb{R}^N))^*$ defined by (5.55) exist for every $\theta \in \mathbb{S}^{N-1}$.

Proposition 5.8.6.
(i) *Λ_p^{hom} is a closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}^N))$ which contains $\mathcal{A}_p$;*
(ii) *the algebra Λ_p^{hom} is inverse-closed in $\mathcal{L}(L^p(\mathbb{R}^N))$; and*
(iii) *the quotient algebra $\Lambda_p^{hom}/\mathcal{K}(L^p(\mathbb{R}^N))$ is inverse-closed in the Calkin algebra $\mathcal{L}(L^p(\mathbb{R}^N))/\mathcal{K}(L^p(\mathbb{R}^N))$.*

Proof. The proof of the first part of assertion (i) is straightforward, and the second part is a consequence of Propositions 5.8.3, 5.8.4 and 5.8.5. Assertion (ii) follows from assertion (iii) via Lemma 1.2.33 (note that the ideal of the compact operators is included in $\mathcal{A}_p \subseteq \Lambda_p^{hom}$ by Proposition 5.8.1).

So we are left with verifying assertion (iii). Let A be an operator in Λ_p^{hom} which has the Fredholm property, i.e., the coset $A + \mathcal{K}(L^p(\mathbb{R}^N))$ is invertible in the Calkin algebra $\mathscr{L}(L^p(\mathbb{R}^N))/\mathscr{K}(L^p(\mathbb{R}^N))$. Let $R \in \mathscr{L}(L^p(\mathbb{R}^N))$ be an operator such that $RA - I =: K$ and $AR - I =: L$ are compact. We have to show that $R \in \Lambda_p^{hom}$. Let $f \in C(\overline{\mathbb{R}^N})$. Then the operator

$$fR - RfI = (RA-K)fR - Rf(AR-L) = R(AfI - fA)R - KfR + RfL$$

is compact, whence $R \in \Lambda_p$. It remains to show that all required strong limits of R exist. We will verify this for the strong limit H_0; the proof for the other limits proceeds analogously.

First we show that $\mathsf{H}_0(A)$ is an invertible operator. Since A is Fredholm, there is a positive number c and a compact operator T such that

$$\|Au\| + \|Tu\| \ge c\|u\| \quad \text{for all} \quad u \in L^p(\mathbb{R}^N)$$

(see Exercise 1.4.7). Since the operators Z_t are isometries, this estimate implies

$$\|Z_tAZ_t^{-1}u\| + \|Z_tTZ_t^{-1}u\| \ge c\|u\|$$

for all $u \in L^p(\mathbb{R}^N)$. Passing to the strong limit as $t \to \infty$ we finally obtain

$$\|\mathsf{H}_0(A)u\| \ge c\|u\| \quad \text{for all} \quad u \in L^p(\mathbb{R}^N).$$

Thus, $\mathsf{H}_0(A)$ is bounded below. Applying the same argument to the adjoint operator A^* (which is Fredholm, too) we find that $\mathsf{H}_0(A^*) = \mathsf{H}_0(A)^*$ is also bounded below. Hence, $\mathsf{H}_0(A)$ is invertible.

Now we show that the strong limit $\mathsf{H}_0(R)$ exists and that $\mathsf{H}_0(R) = \mathsf{H}_0(A)^{-1}$. Indeed, let $RA - I =: K$ as before. Then, for each $u \in L^p(\mathbb{R}^N)$,

$$\begin{aligned}
&\|(Z_tRZ_t^{-1} - \mathsf{H}_0(A)^{-1})u\| \\
&= \|(Z_tRZ_t^{-1} - Z_t(RA-K)Z_t^{-1}\mathsf{H}_0(A)^{-1})u\| \\
&= \|(Z_tRZ_t^{-1} - (Z_tRZ_t^{-1}Z_tAZ_t^{-1} - Z_tKZ_t^{-1})\mathsf{H}_0(A)^{-1})u\| \\
&\le \|Z_tRZ_t^{-1}\|\,\|u - Z_tAZ_t^{-1}\mathsf{H}_0(A)^{-1}u\| + \|Z_tKZ_t^{-1}\mathsf{H}_0(A)^{-1}u\| \\
&\le \|R\|\,\|\mathsf{H}_0(A)v - Z_tAZ_t^{-1}v\| + \|Z_tKZ_t^{-1}v\|
\end{aligned}$$

with $v := \mathsf{H}_0(A)^{-1}u$. Since the right-hand side of this estimate tends to zero as $t \to \infty$, the assertion follows. ■

Thus, an operator $A \in \Lambda_p^{hom}$ is Fredholm if and only if its coset modulo compact operators is invertible in $\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N))$. For operators $A \in \mathscr{A}_p$, we will study the invertibility of the coset $A + \mathscr{K}(L^p(\mathbb{R}^N))$ in this quotient algebra by localizing the algebra $\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N))$ by Allan's local principle over its central subalgebra which consists of all cosets $fI + \mathscr{K}(L^p(\mathbb{R}^N))$ with $f \in C(\overline{\mathbb{R}^N})$.

Proposition 5.8.7. *The algebra $\mathscr{C} := \{fI + \mathscr{K}(L^p(\mathbb{R}^N)) : f \in C(\overline{\mathbb{R}^N})\}$ is isometrically isomorphic to the algebra $C(\overline{\mathbb{R}^N})$ in a natural way.*

Proof. One has only to prove that

$$\|f\| = \|fI + \mathscr{K}(L^p(\mathbb{R}^N))\| := \inf_{K \in \mathscr{K}(L^p(\mathbb{R}^N))} \|fI + K\|$$

for every $f \in C(\overline{\mathbb{R}^N})$. Proposition 5.8.4 (ii), (iv) ensure that $\|fI + K\| \geq |f(x)|$ for every $f \in C(\overline{\mathbb{R}^N})$, $K \in \mathscr{K}(L^p(\mathbb{R}^N))$ and $x \in \mathbb{R}^N$. Hence, $\|fI + \mathscr{K}(L^p(\mathbb{R}^N))\| \geq |f(x)|$ for all $x \in \mathbb{R}^N$. Since $\mathbb{R}^N$ is dense in $\overline{\mathbb{R}^N}$ with respect to the Gelfand topology, we get

$$\|f\| \geq \|fI + \mathscr{K}(L^p(\mathbb{R}^N))\| \geq \|f\|,$$

which is the assertion. ∎

In particular, the maximal ideal space of the algebra $\mathscr{C}$ is homeomorphic to $\overline{\mathbb{R}^N}$. The maximal ideal which corresponds to $x \in \overline{\mathbb{R}^N}$ is the set of all cosets $fI + \mathscr{K}(L^p(\mathbb{R}^N))$ with $f(x) = 0$. We denote this maximal ideal by x and let $\mathscr{J}_x$ stand for the smallest closed ideal of $\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N))$ which contains x. Further we write Φ_x for the canonical homomorphism

$$\Lambda_p^{hom} \to (\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_x, \quad A \mapsto (A + \mathscr{K}(L^p(\mathbb{R}^N))) + \mathscr{J}_x.$$

Note that the compact operators lie in the kernel of each homomorphism H_x and H_θ° with $x \in \mathbb{R}^N$ and $\theta \in \mathbb{S}^{N-1}$ by (iv) in Propositions 5.8.4 and 5.8.5. Thus, the mappings

$$A + \mathscr{K}(L^p(\mathbb{R}^N)) \mapsto \mathsf{H}_x(A) \quad \text{and} \quad A + \mathscr{K}(L^p(\mathbb{R}^N)) \mapsto \mathsf{H}_\theta^\circ(A)$$

are correctly defined for each operator $A \in \Lambda_p^{hom}$. We denote them again by H_x and H_θ°, respectively. Further, by (ii) in Propositions 5.8.4 and 5.8.5, the local ideal $\mathscr{J}_x$ lies in the kernel of $\mathsf{H}_x : \Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)) \to \mathscr{L}(L^p(\mathbb{R}^N))$ for every $x \in \mathbb{R}^N$, and the local ideal $\mathscr{J}_{\theta_\infty}$ lies in the kernel of $\mathsf{H}_\theta^\circ : \Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)) \to \mathscr{L}(L^p(\mathbb{R}^N))$ for every $\Theta \in \mathbb{S}^{N-1}$. Hence, the mappings

$$(A + \mathscr{K}(L^p(\mathbb{R}^N))) + \mathscr{J}_x \mapsto \mathsf{H}_x(A) \quad \text{and} \quad (A + \mathscr{K}(L^p(\mathbb{R}^N))) + \mathscr{J}_{\theta_\infty} \mapsto \mathsf{H}_\theta^\circ(A)$$

are correctly defined for each $A \in \Lambda_p^{hom}$, and we denote them again by H_x and H_θ°, respectively. The following propositions identify the algebras $\Phi_x(\mathscr{A}_p)$ for $x \in \overline{\mathbb{R}^N}$.

Proposition 5.8.8. *Let $x = 0 \in \mathbb{R}^N$. Then:*

(i) *the local algebra $\Phi_0(\mathscr{A}_p)$ is isometrically isomorphic to the smallest closed subalgebra $PC(\mathbb{S}^{N-1})$ of $\mathscr{L}(L^p(\mathbb{R}^N))$ which contains all operators $\chi_{\mathbb{H}(\psi)}$ with $\psi \in \mathbb{S}^{N-1}$;*
(ii) *for every operator $A \in \mathscr{A}_p$, the coset $\Phi_0(A)$ is invertible in the local algebra $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_0$ if and only if the operator $\mathsf{H}_0(A)$ is invertible (in $\mathscr{L}(L^p(\mathbb{R}^N))$);*
(iii) *the algebra $\Phi_0(\mathscr{A}_p)$ is inverse-closed in $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_0$.*

The notation $PC(\mathbb{S}^{N-1})$ has been chosen since $PC(\mathbb{S}^1)$ can be identified (by restriction) with the algebra of all piecewise continuous functions on the (one-dimensional) unit sphere.

Proof. (i) It follows from Proposition 5.8.4 that H_0 is a homomorphism from $\Phi_0(\mathscr{A}_p)$ into $PC(\mathbb{S}^{N-1})$. This homomorphism is onto since $PC(\mathbb{S}^{N-1})$ is a subalgebra of $\mathscr{A}_p$ and $\mathsf{H}_0(A) = A$ for every operator $A \in PC(\mathbb{S}^{N-1})$. Further, since the Z_t are isometries, it is clear that the mapping

$$\mathsf{H}_0 : \Phi_0(\mathscr{A}_p) \to PC(\mathbb{S}^{N-1})$$

is a contraction. In order to show that this mapping is an isometric isomorphism, we claim that

$$\Phi_0(A) = \Phi_0(\mathsf{H}_0(A)) \quad \text{for every} \quad A \in \mathscr{A}_p. \tag{5.56}$$

Since the mappings Φ_0 and H_0 are continuous homomorphisms, it is sufficient to check (5.56) for the generating operators of the algebra $\mathscr{A}_p$.

For the operators $A = fI$ with $f \in C(\overline{\mathbb{R}^N})$ one has $\mathsf{H}_0(fI) = f(0)I$ by Proposition 5.8.4; so one has to check that $\Phi_0(fI) = \Phi_0(f(0)I)$, which is immediate from the definition of the local ideals. For $A = \chi_{\mathbb{H}(\psi)}I$ with $\psi \in \mathbb{S}^{N-1}$, the claim (5.56) is evident.

So we are left with the case when $A = W_a^0$ with $a \in L^1(\mathbb{R}^N)$. Then we have to show that $\Phi_0(W_a^0) = 0$. Let f be a continuous function on $\mathbb{R}^N$ with compact support and with $f(0) = 1$. The operator $W_a^0 fI$ which acts on $L^p(\mathbb{R}^N)$ by

$$[W_a^0 f(g)](x) = \int_{\mathbb{R}^N} a(x-t)f(t)g(t)dt, \quad x \in \mathbb{R}^N,$$

is compact. Indeed, we can suppose, without loss of generality, that a is a continuous function with compact support, because the functions with these properties are dense in $L^1(\mathbb{R}^N)$. Further, by Krasnoselskii's interpolation theorem, we can also suppose that $p = 2$. Since then the kernel of the integral operator $W_a^0 fI$ is a continuous and compactly supported function, we conclude that $W_a^0 fI$ is a Hilbert-Schmidt operator, and therefore compact. Thus, $\Phi_x(W_a^0 fI) = 0$. Since $\Phi_x(fI)$ is the identity element of the local algebra, we have $\Phi_x(W_a^0) = 0$, which proves the claim.

(ii) Let A be an operator in $\mathscr{A}_p$ for which the coset $\Phi_0(A)$ is invertible in $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_0$. Since H_0 acts as a homomorphism on that algebra, we conclude that $\mathsf{H}_0(A)$ is an invertible operator.

Conversely, let the operator $\mathsf{H}_0(A)$ be invertible (in $\mathscr{L}(L^p(\mathbb{R}^N))$). From part (i) we know that $\mathsf{H}_0(A)$ belongs to the algebra Λ_p^{hom}, and from Proposition 5.8.6 (ii) we infer that the algebra Λ_p^{hom} is inverse-closed in $\mathscr{L}(L^p(\mathbb{R}^N))$. Hence, the inverse operator $\mathsf{H}_0(A)^{-1}$ belongs to Λ_p^{hom}. Applying the local mapping Φ_0 to the equality

$$\mathsf{H}_0(A)^{-1}\mathsf{H}_0(A) = \mathsf{H}_0(A)\mathsf{H}_0(A)^{-1} = I$$

and recalling (5.56) we conclude that $\Phi_0(A)$ is invertible.

(iii) This is the same proof as before if one takes into account that the algebra

$PC(\mathbb{S}^{N-1})$ is inverse-closed in $\mathscr{L}(L^p(\mathbb{R}^N))$. The latter fact can easily be proved via Corollary 1.2.32; it follows also from Theorem 2.2.8. ■

Proposition 5.8.9. *Let $x \in \mathbb{R}^N \setminus \{0\}$. Then:*

(i) *the local algebra $\Phi_x(\mathscr{A}_p)$ is isometrically isomorphic to the smallest closed subalgebra $\mathscr{B}_x$ of $\mathscr{L}(L^p(\mathbb{R}^N))$ which contains all operators $\chi_{\mathbb{H}(\psi)}I$ for which x lies on the boundary of $\mathbb{H}(\psi)$;*
(ii) *for every operator $A \in \mathscr{A}_p$, the coset $\Phi_x(A)$ is invertible in the local algebra $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_x$ if and only if the operator $\mathsf{H}_x(A)$ is invertible (in $\mathscr{L}(L^p(\mathbb{R}^N))$);*
(iii) *the algebra $\Phi_x(\mathscr{A}_p)$ is inverse-closed in $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_x$.*

Clearly, if $N = 2$, there are only two values of ψ such that x lies on the boundary of $\mathbb{H}(\psi)$. If $\psi(x)$ is one of these values, then $-\psi(x)$ is the other one, and the algebra $\mathscr{B}_x$ consists of all linear combinations of $\chi_{\mathbb{H}(\psi(x))}I$ and $\chi_{\mathbb{H}(-\psi(x))}I$. Thus, $\mathscr{B}_x \cong \mathbb{C}^2$ in this case. If $N > 2$, then x lies on the boundary of each half space $\mathbb{H}(\psi)$ with ψ being orthogonal to x. The set of these ψ can be identified with $\mathbb{S}^{N-2}$.

Proof. The proof proceeds as that of the preceding proposition. In place of (5.56), one now has to verify that

$$\Phi_x(A) = \Phi_x(\mathsf{H}_x(A)) \quad \text{for every} \quad A \in \mathscr{A}_p. \tag{5.57}$$

We only note that $\chi_{\mathbb{H}(\psi)}$ is continuous and equal to one in a neighborhood of x if x is in the interior of $\mathbb{H}(\psi)$. Thus, $\Phi_x(\chi_{\mathbb{H}(\psi)}I)$ is the local identity element in this case. Similarly, if x is in the exterior of $\mathbb{H}(\psi)$, then $\Phi_x(\chi_{\mathbb{H}(\psi)}I)$ is the local zero element. In the case x lies on the boundary of $\mathbb{H}(\psi)$ then $\Phi_x(\chi_{\mathbb{H}(\psi)}I)$ is a proper idempotent (i.e., the spectrum of this local coset is $\{0, 1\}$), by Proposition 5.8.4 (iii). ■

Proposition 5.8.10. *Let $\theta \in \mathbb{S}^{N-1}$. Then:*

(i) *the local algebra $\Phi_{\theta_\infty}(\mathscr{A}_p)$ is isometrically isomorphic to the smallest closed subalgebra $\mathscr{B}_{\theta_\infty}$ of $\mathscr{L}(L^p(\mathbb{R}^N))$ which contains all convolutions W_a^0 with $a \in L^1(\mathbb{R}^N)$ and all operators $\chi_{\mathbb{H}(\psi)}I$ for which θ lies on the boundary of $\mathbb{H}(\psi)$;*
(ii) *for every operator $A \in \mathscr{A}_p$, the coset $\Phi_{\theta_\infty}(A)$ is invertible in the local algebra $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_{\theta_\infty}$ if and only if the operator $\mathsf{H}_\theta^\circ(A)$ is invertible in $\mathscr{L}(L^p(\mathbb{R}^N))$;*
(iii) *for every operator $A \in \mathscr{A}_p$, the coset $\Phi_{\theta_\infty}(A)$ is invertible in the local algebra $\Phi_{\theta_\infty}(\mathscr{A}_p)$ if and only if the operator $\mathsf{H}_\theta^\circ(A)$ is invertible in $\mathscr{B}_{\theta_\infty}$.*

Proof. The proof follows the same lines as that of the preceding propositions, where one has now to check that

$$\Phi_{\theta_\infty}(A) = \Phi_{\theta_\infty}(\mathsf{H}_\theta^\circ(A)) \quad \text{for every} \quad A \in \mathscr{A}_p. \tag{5.58}$$

Note that in the case at hand, we do not know if the algebras $\mathscr{B}_{\theta_\infty}$ are inverse-closed in $\mathscr{L}(L^p(\mathbb{R}^N))$. That is why we give two invertibility criteria: one for invertibility in $(\Lambda_p^{hom}/\mathscr{K}(L^p(\mathbb{R}^N)))/\mathscr{J}_{\theta_\infty}$, and one for invertibility in $\Phi_{\theta_\infty}(\mathscr{A}_p)$. Assertion (iii) can be proved as assertion (ii) of Proposition 5.8.8. ■

Now one can formulate and prove the main result of this section.

Theorem 5.8.11. *An operator $A \in \mathscr{A}_p$ is Fredholm if and only if all operators $\mathsf{H}_x(A)$ with $x \in \mathbb{R}^N$ and all operators $\mathsf{H}^\circ_\theta(A)$ with $\theta \in \mathbb{S}^{N-1}$ are invertible (as operators on $L^p(\mathbb{R}^N)$).*

Proof. The proof follows immediately from Allan's local principle and from the criteria for invertibility in the corresponding local algebras which are stated in assertions (ii) of Propositions 5.8.8, 5.8.9 and 5.8.10. ■

For completeness, let us mention that the coset $A + \mathscr{K}(L^p(\mathbb{R}^N))$ of an operator $A \in \mathscr{A}_p$ is invertible in the quotient algebra $\mathscr{A}_p/\mathscr{K}(L^p(\mathbb{R}^N))$ if and only if the operator $\mathsf{H}_x(A)$ is invertible in $\mathscr{L}(L^p(\mathbb{R}^N))$ for every $x \in \mathbb{R}^N$ and if the operator $\mathsf{H}^\circ_\theta(A)$ is invertible in $\mathscr{B}_{\theta_\infty}$ for every $\theta \in \mathbb{S}^{N-1}$. This follows again from Allan's local principle, but now applied in $\mathscr{A}_p/\mathscr{K}(L^p(\mathbb{R}^N))$, and from assertions (iii) of Propositions 5.8.8, 5.8.9 and 5.8.10.

To illustrate the previous results we let $N = 2$ and consider restrictions of convolution operators to half-planes and cones. By a *cone* in $\mathbb{R}^2$ with vertex at the origin we mean a set of the form $\mathbb{K}(\psi_1, \psi_2) := \mathbb{H}(\psi_1) \cap \mathbb{H}(\psi_2)$ with $\psi_1, \psi_2 \in \mathbb{S}^1$. To avoid trivialities, we assume that neither $\psi_1 = \psi_2$ nor $\psi_1 = -\psi_2$. Thus, $\mathbb{K}(\psi_1, \psi_2)$ is neither a half-plane nor a line.

Let χ_M refer to the characteristic function of a measurable subset M of $\mathbb{R}^2$. The following is an immediate consequence of the Fredholm criterion in Theorem 5.8.11 and of Propositions 5.8.4 and 5.8.5.

Corollary 5.8.12. *Let $a \in L^1(\mathbb{R}^2)$ and $f \in C(\overline{\mathbb{R}^2})$, and let $\psi, \psi_1, \psi_2 \in \mathbb{S}^1$ be subject to the above agreement.*

(i) *The operator $\chi_{\mathbb{H}(\psi)}(W^0_a + fI)\chi_{\mathbb{H}(\psi)}I + (1 - \chi_{\mathbb{H}(\psi)})I$ is Fredholm on $L^p(\mathbb{R}^2)$ if and only if*

- *$f(x) \neq 0$ for all $x \in \mathbb{H}(\psi)$,*
- *$W^0_a + f(\theta_\infty)I$ is invertible for every $\theta \in \mathbb{S}^1$ in the interior of $\mathbb{H}(\psi)$,*
- *$\chi_{\mathbb{H}(\psi)}(W^0_a + f(\theta_\infty)I)\chi_{\mathbb{H}(\psi)}I + (1 - \chi_{\mathbb{H}(\psi)})I$ is invertible for every $\theta \in \mathbb{S}^1$ on the boundary of $\mathbb{H}(\psi)$.*

(ii) *The operator $\chi_{\mathbb{K}(\psi_1,\psi_2)}(W^0_a + fI)\chi_{\mathbb{K}(\psi_1,\psi_2)}I + (1 - \chi_{\mathbb{K}(\psi_1,\psi_2)})I$ is Fredholm on $L^p(\mathbb{R}^2)$ if and only if*

- *$f(x) \neq 0$ for all $x \in \mathbb{K}(\psi_1, \psi_2)$,*
- *$W^0_a + f(\theta_\infty)I$ is invertible for every $\theta \in \mathbb{S}^1$ in the interior of $\mathbb{K}(\psi_1, \psi_2)$,*
- *$\chi_{\mathbb{H}(\psi_i)}(W^0_a + f(\theta_\infty)I)\chi_{\mathbb{H}(\psi_i)}I + (1 - \chi_{\mathbb{H}(\psi_i)})I$ is invertible for every $\theta \in \mathbb{S}^1$ on the boundary of $\mathbb{K}(\psi_1, \psi_2)$.*

Note that the invertibility of the half-plane operators in (i) and (ii) can be effectively checked by means of a result by Goldenstein and Gohberg [77] which states that the following conditions are equivalent for $a \in L^1(\mathbb{R}^2)$, $\lambda \in \mathbb{C}$ and $\psi \in \mathbb{S}^1$:

(i) the operator $\chi_{\mathbb{H}(\psi)}(W_a^0 + \lambda I)\chi_{\mathbb{H}(\psi)} I + (1 - \chi_{\mathbb{H}(\psi)})I$ is invertible on $L^p(\mathbb{R}^2)$,
(ii) the operator $W_a^0 + \lambda I$ is invertible on $L^p(\mathbb{R}^2)$,
(iii) $\lambda \neq 0$, and the function $Fa + \lambda$, with F standing for the Fourier transform on $\mathbb{R}^2$, does not vanish on $\mathbb{R}^2$.

With this additional information, Corollary 5.8.12 can be reformulated as follows.

Corollary 5.8.13. *Let $a \in L^1(\mathbb{R}^2)$ and $f \in C(\overline{\mathbb{R}^2})$, and let $\psi, \psi_1, \psi_2 \in \mathbb{S}^1$ be subject to the above agreement.*

(i) *The operator $\chi_{\mathbb{H}(\psi)}(W_a^0 + fI)\chi_{\mathbb{H}(\psi)} I + (1 - \chi_{\mathbb{H}(\psi)})I$ is Fredholm on $L^p(\mathbb{R}^2)$ if and only if*
   - *$f(x) \neq 0$ for all $x \in \mathbb{H}(\psi)$,*
   - *$W_a^0 + f(\theta_\infty)I$ is invertible for every $\theta \in \mathbb{S}^1 \cap \mathbb{H}(\psi)$.*

(ii) *The operator $\chi_{\mathbb{K}(\psi_1,\psi_2)}(W_a^0 + fI)\chi_{\mathbb{K}(\psi_1,\psi_2)} I + (1 - \chi_{\mathbb{K}(\psi_1,\psi_2)})I$ is Fredholm on $L^p(\mathbb{R}^2)$ if and only if*
   - *$f(x) \neq 0$ for all $x \in \mathbb{K}(\psi_1, \psi_2)$,*
   - *$W_a^0 + f(\theta_\infty)I$ is invertible for every $\theta \in \mathbb{S}^1 \cap \mathbb{K}(\psi_1, \psi_2)$.*

Note in this connection also that the following assertions are equivalent for $a \in L^1(\mathbb{R}^2)$, $\lambda \in \mathbb{C}$ and $\psi_1, \psi_2 \in \mathbb{S}^1$ (see [21, Section 9.53]):

(i) $\chi_{\mathbb{K}(\psi_1,\psi_2)}(W_a^0 + \lambda I)\chi_{\mathbb{K}(\psi_1,\psi_2)} I + (1 - \chi_{\mathbb{K}(\psi_1,\psi_2)})I$ is Fredholm on $L^p(\mathbb{R}^2)$,
(ii) $\chi_{\mathbb{H}(\psi_i)}(W_a^0 + \lambda I)\chi_{\mathbb{H}(\psi_i)} I + (1 - \chi_{\mathbb{H}(\psi_i)})I$ is invertible on $L^p(\mathbb{R}^2)$ for $i \in \{1, 2\}$.

Corollary 5.8.14. *Let A belong to the smallest closed subalgebra on $\mathcal{L}(L^p(\mathbb{R}^2))$ which contains all operators W_a^0 with $a \in L^1(\mathbb{R}^2)$ and the operator $\chi_{\mathbb{H}(\theta)} I$ for a fixed $\theta \in \mathbb{S}^1$. Then A is Fredholm if and only if A is invertible.*

Proof. If A is Fredholm then by (the easy half of) Theorem 5.8.11, the limit operator $\mathsf{H}_\theta^\circ(A)$ is invertible. From Proposition 5.8.5 we infer that $\mathsf{H}_\theta^\circ(A) = A$. Thus, A is invertible. ■

Let us finally mention that the algebras $\Lambda_p/\mathcal{K}(L^p(\mathbb{R}^N))$ and $C(\overline{\mathbb{R}^N})$ constitute a faithful localization pair by Theorem 2.5.13. Thus, the machinery of norm-preserving localization and local enclosement theorems as well as Simonenko's theory of local operators work in the present setting.

5.9 Notes and comments

Wiener-Hopf integral equations of the type

$$cu(t) + \int_0^\infty k(t-s)u(s)\,ds = v(t) \;\; (t > 0),$$

with $k \in L^1(\mathbb{R})$, had been the subject of detailed studies by many people, including Wiener and Hopf [201], Paley and Wiener [133], Smithies [190], Reissner [161], Fock [57], Titchmarsh [193], Rapoport [158] and Noble [129]. The fundamental 1958 paper [105] of Krein, translated into English by the American Mathematical Society in 1962, presented a clear and complete theory of this topic at the time. The case of systems of Wiener-Hopf integral equations with kernels belonging to $L^1(\mathbb{R})$ was studied by Gohberg and Krein in [68]. Gohberg and Feldman's book [66], published originally in Russian in 1971, is devoted to a unified approach to different kinds of convolution equations with continuous generating functions. But Duduchava's book [43] marked the start of a new era in the topic, with the study of convolution operators with piecewise continuous generating functions and of algebras generated by such operators.

The results of Sections 5.1, 5.2 and 5.3 are taken from Duduchava's works [43, 44] with exception of Proposition 5.1.2 which is due to Schneider [176]. The results from Section 5.4 go back to two of the authors [168]. In Sections 5.3 and 5.4 some of the proofs are streamlined with respect to their original versions.

Duduchava [44] studied the algebra $\mathscr{A} := \mathscr{A}(X,Y,Z)$ in the particular case $X = C(\mathbb{R}^+)$, $Y = PC_{p,w_\alpha}$ and $Z = C_p$. These restrictions imply the commutativity of the related quotient algebra $\mathscr{A}^{\mathscr{K}}$. Duduchava further wrote that *...the same methods make it possible to investigate a more complicated algebra...*, namely the algebra $\mathscr{A}^{\mathscr{K}}\big(PC(\mathbb{R}^+), PC_{p,w_\alpha}, PC_p\big)$ in our notation. As far as we know, he never published these results. Moreover, even in the case of continuous generating functions, the approach presented above in this chapter gives a more precise information than Duduchava's approach: it allows us to characterize the local algebras up to isometry as algebras of Mellin convolution operators.

The algebra $\mathscr{A}\big(PC(\mathbb{R}^+), PC_{p,w}, \emptyset\big)$ studied in Section 5.5 was the subject of Duduchava's investigations in the particular case of unweighted spaces or for weights $w(t) := |t|^\alpha$. For Khvedelidze weights, Schneider proved a criterion for the Fredholm property of the Wiener-Hopf operator $W(a)$ with $a \in PC_{p,w}$.

The results of Section 5.6 are again taken from [168]. These results can be extended to algebras generated by Wiener-Hopf and Hankel operators with piecewise continuous generating functions. But in Section 5.7 we present more general results based on considering the flip as an independent generator for the algebra. That possibility was used initially in [165] by the authors to analyze algebras resulting from approximating methods, the theme of Chapter 6. Algebras generated by Wiener-Hopf and Hankel operators have an analog in algebras generated by Toeplitz and Hankel operators with piecewise continuous generating function (defined on the unit circle $\mathbb{T}$). That problem was studied by Power in [147, 149] and by one of the authors [182] by different means, but only for the C^*-case. Interestingly, the approach of [182] is based upon the two projections theorem. Moreover, Theorem 4.4.5 leads to the description of the Fredholm properties of operators belonging to the Banach algebras generated by Toeplitz and Hankel operators with piecewise continuous generating functions and acting on Hardy spaces with weight. In particular, the essential spectrum of Hankel operators with piecewise continuous generating functions acting on a variety of Banach spaces has been known since 1990 [168]. Some of these results were reproved in [94].

Section 5.8 is devoted to the reproduction of some results obtained by Simonenko [187] who derived them by using the local principle named after him. Our exposition is based on a combination of Allan's local principle and limit operator techniques and is in the spirit of the previous sections.

The methods described in Chapter 5 also apply to other classes of operators such as multidimensional singular integral operators, singular integral operators with fixed singularities, singular integro-differential operators and certain classes of boundary integral operators (e.g., single and double layer potential operators). Concerning the investigation of multidimensional operators on $L^p(\mathbb{R}^n)$ by local principles see also the nice recent book by Simonenko [188]. It contains a complete study of shift-invariant operators as well as a description of a few important subclasses of such operators. Simonenko also considered Banach algebras generated by operators which are locally equivalent to operators in one of the mentioned subclasses.