Chapter III

Monomial Resolutions

Abstract. In this chapter we discuss free resolutions of monomial ideals; we call them monomial resolutions. The problem to describe the minimal free resolution of a monomial ideal (over a polynomial ring) was posed by Kaplansky in the early 1960's. Despite the helpful combinatorial structure of monomial ideals, the problem turned out to be hard. The structure of a minimal free monomial resolution can be quite complex. There exists a minimal free monomial resolution which cannot be encoded in the structure of any CW-complex. In fact, even the minimal free resolutions of ideals generated by quadratic monomials are so complicated that it is beyond reach to obtain a description of them; we do not even know how to express the regularity of such ideals. In this situation, the guideline is to *introduce new ideas and constructions which either have strong applications or/and are beautiful.* Most proofs about monomial resolutions are easy. The key point is not to provide complicated proofs, but to introduce new beautiful ideas.

54 Examples and Notation

We will use the notation and terminology introduced in Section 26, and the tools from Section 36. Throughout the chapter M stands for a monomial ideal in S minimally generated by monomials $m_1, \dots, m_r$. We denote by L_M the set of the least common multiples of subsets of $\{m_1, \dots, m_r\}$. By convention, $1 \in L_M$ considered as the lcm of the empty set. Note that M is homogeneous with respect to the standard grading on S and with respect to the multigrading in 26.1.

I. Peeva, *Graded Syzygies*, Algebra and Applications 14,
DOI 10.1007/978-0-85729-177-6_3,

In the Running Example we will illustrate several definitions and constructions. The running example uses the ideal

$$Y = (x^2, xy, y^3)$$

in the ring $C = k[x, y]$.

Running Example 54.1. Consider the ideal $Y = (x^2, xy, y^3)$ in the ring $C = k[x, y]$. Computer computation shows that the minimal free resolution of C/Y is

$$\mathbf{F}_Y : \quad 0 \longrightarrow C^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} C^3 \xrightarrow{(\,x^2 \quad xy \quad y^3\,)} C \longrightarrow C/Y \longrightarrow 0\,.$$

We consider the basis of the free modules in $\mathbf{F}_Y$ in which the above maps are given. Denote by h the basis element of C in homological degree 0, by f_1, f_2, f_3 the basis elements of C^3 in homological degree 1, and by g_1, g_2 the basis elements of C^2 in homological degree 2. Since h has multidegree 1 and the differential is supposed to be homogeneous, it follows that f_1, f_2, f_3 have multidegrees x^2, xy, y^3 respectively. Thus, in homological degree 1 we have the free module $C(x^2) \oplus C(xy) \oplus C(y^3)$. Furthermore, $d(g_1) = -yf_1 + xf_2$ has multidegree x^2y, hence g_1 has multidegree x^2y. Similarly, since $d(g_2) = -y^2f_2 + xf_3$ has multidegree xy^3, we conclude that g_2 has multidegree xy^3. Thus, in homological degree 2 we have the free module $C(x^2y) \oplus C(xy^3)$. So the resolution can be written

$$0 \to C(x^2y) \oplus C(xy^3) \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} C(x^2) \oplus C(xy) \oplus C(y^3) \xrightarrow{(x^2\ xy\ y^3)} C\,.$$

The non-zero multigraded Betti numbers are

$$b_{1,y^3}(C/Y) = b_{1,xy}(C/Y) = b_{1,x^2}(C/Y) = 1$$
$$b_{2,x^2y}(C/Y) = b_{2,xy^3}(C/Y) = 1\,.$$

We will also write the component $(\mathbf{F}_Y)_{x^2y^2}$ of $\mathbf{F}_Y$ in multidegree x^2y^2. It is the exact sequence of k-vector spaces

$$0 \to C(x^2y)_{x^2y^2} \oplus C(xy^3)_{x^2y^2} \to C(x^2)_{x^2y^2} \oplus C(xy)_{x^2y^2} \oplus C(y^3)_{x^2y^2} \to C_{x^2y^2} \to (C/Y)_{x^2y^2} \to 0\,.$$

Note that $C(x^2y)_{x^2y^2}$ is the 1-dimensional k-vector space with basis yg_1, so we can write it as $k\{yg_1\}$. Similarly, $C(xy^3)_{x^2y^3} = 0$, $C(x^2)_{x^2y^2} = k\{y^2f_1\}$, $C(xy)_{x^2y^2} = k\{xyf_2\}$, $C(y^3)_{x^2y^3} = 0$, $C_{x^2y^2} = k\{x^2y^2\}$. Furthermore, note that $(C/Y)_{x^2y^2} = 0$ because $x^2y^2 \in Y$. Therefore, $(\mathbf{F}_Y)_{x^2y^2}$ is the exact sequence of k-vector spaces

$$0 \to k\,\{yg_1\} \to k\,\{y^2f_1\} \oplus k\,\{xyf_2\} \to k\,\{x^2y^2\} \to 0 \to 0\,.$$

By Corollary 26.9 the entries in the matrices of the differentials in the minimal free resolution $\mathbf{F}_M$ of S/M are scalar multiples of monomials. After computing a few examples, one might get the feeling that the coefficients appearing in the differential matrices are only $0, \pm 1$. Unfortunately, this is not the case, as shown by the next example.

Example 54.2. [Reiner-Welker] Assume $\mathrm{char}(k) = 0$. Consider the monomial ideal

$$\begin{aligned} T = (\, & x_1x_4x_5x_6, x_2x_4x_5x_6, x_3x_4x_5x_6, x_2x_4x_5x_7, x_3x_4x_5x_7, \\ & x_1x_3x_5x_7, x_1x_2x_4x_7, x_1x_4x_6x_7, x_1x_5x_6x_7, \\ & x_3x_4x_6x_7, x_2x_5x_6x_7, x_2x_3x_6x_7, x_1x_2x_3x_7\,)\,. \end{aligned}$$

in $A = k[x_1, \ldots, x_7]$. Computer computation shows that the minimal free resolution of A/T is

$$0 \to A \to A^{10} \to A^{21} \to A^{13} \to A \to A/T \to 0$$

and the matrix of the last differential d_4 is $\begin{pmatrix} -2x_7 \\ 2x_1 \\ -x_4 \\ x_5 \\ x_1 \\ 2x_3 \\ -2x_2 \\ x_2 \\ x_3 \\ x_6 \end{pmatrix}$ (in some fixed basis). The last matrix contains coefficients ± 2. It is shown in [Reiner-Welker] that A/T does not have a minimal multigraded free resolution with coefficients only 0, ± 1 of the monomials in the differential matrices.

Another indication that a monomial resolution could be complicated is that it might not be independent of the characteristic of k; see Example 12.4.

55 Homogenization and dehomogenization

We will explore the idea to encode the structure of a monomial resolution in a complex of vector spaces. The encoding consists of the homogenization and dehomogenization described below. We will discuss the concept of a frame, which is a complex of vector spaces with a fixed basis. By Theorem 55.7 the minimal free resolution of any monomial ideal is encoded in any of its frames. The material in this section is from [Peeva-Velasco], which was motivated by several prior constructions on monomial resolutions.

Construction 55.1. A ***frame*** (or an r-***frame***) $\mathbf{U}$ is a complex of finite k-vector spaces with differential ∂ and a fixed basis that satisfies the following conditions:

(1) $U_i = 0$ for $i < 0$ and $i \gg 0$,

(2) $U_0 = k$,

(3) $U_1 = k^r$,

(4) $\partial(w_j) = 1$ for each basis vector w_j in $U_1 = k^r$.

An M***-complex*** $\mathbf{G}$ is a multigraded complex of finitely generated free multigraded S-modules with differential d and a fixed multihomogeneous basis with multidegrees in L_M that satisfies the following conditions:

(1) $G_i = 0$ for $i < 0$ and $i \gg 0$,
(2) $G_0 = S$,
(3) $G_1 = S(m_1) \oplus \ldots \oplus S(m_r)$,
(4) $d(w_j) = m_j$ for each basis element w_j of G_1.

We need a correspondence between complexes of vector spaces and complexes of free S-modules. Such a correspondence is given by the homogenization and dehomegenization constructions described below.

Construction 55.2. Let $\mathbf{U}$ be an r-frame. We will construct an M-complex $\mathbf{G}$ of free S-modules with differential d and call it the M***-homogenization*** of $\mathbf{U}$. The construction is by induction on homological degree. Recall that mdeg stands for multidegree.

Set

$$G_0 = S \quad \text{and} \quad G_1 = S(m_1) \oplus \ldots \oplus S(m_r)\,.$$

Let $\bar{v}_1, \ldots, \bar{v}_p$ and $\bar{u}_1, \ldots, \bar{u}_q$ be the given bases of U_i and U_{i-1} respectively. Let $u_1, \ldots, u_q$ be the basis of $G_{i-1} = S^q$ chosen on the previous step of the induction. Introduce $v_1, \ldots, v_p$ that will be a basis of $G_i = S^p$. If

$$\partial(\bar{v}_j) = \sum_{1 \le s \le q} \alpha_{sj} \bar{u}_s$$

with coefficients $\alpha_{sj} \in k$, then set

$$\mathrm{mdeg}(v_j) = \mathrm{lcm}\Big(\mathrm{mdeg}(u_s) \,\Big|\, \alpha_{sj} \neq 0\Big)\,, \text{ note that } \mathrm{lcm}(\emptyset) = 1$$

$$G_i = \oplus_{1 \le j \le p}\ S(\mathrm{mdeg}(v_j))$$

$$d(v_j) = \sum_{1 \le s \le q} \alpha_{sj} \frac{\mathrm{mdeg}(v_j)}{\mathrm{mdeg}(u_s)} u_s\,.$$

Clearly, $\mathrm{Coker}(d_1) = S/M$. Note that the differential d is multihomogeneous by construction. Lemma 55.4 shows that $\mathbf{G}$ is a complex. We say that the complex $\mathbf{G}$ is obtained from $\mathbf{U}$ by ***M-homogenization***.

Running Example 55.3. Consider the 3-frame

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} k\,.$$

The Y-homogenization of this frame is

$$\mathbf{G}: \qquad 0 \to C(x^2y^3) \xrightarrow{\begin{pmatrix} y^2 \\ x \\ 1 \end{pmatrix}} C(x^2y) \oplus C(xy^3) \oplus C(x^2y^3)$$

$$\xrightarrow{\begin{pmatrix} -y & 0 & y^3 \\ x & -y^2 & 0 \\ 0 & x & -x^2 \end{pmatrix}} C(x^2) \oplus C(xy) \oplus C(y^3) \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} C\,.$$

Lemma 55.4. $\mathbf{G}$ *in Construction* 55.2 *is a complex.*

Proof. Let $\bar{v}_1, \dots, \bar{v}_p$, and $\bar{u}_1, \dots, \bar{u}_q$, and $\bar{w}_1, \dots, \bar{w}_t$ be the given bases of U_i, U_{i-1}, and U_{i-2} respectively. Let $v_1, \dots, v_p$, and $u_1, \dots, u_q$, and $w_1, \dots, w_t$ be the corresponding bases of G_i, G_{i-1}, and G_{i-2} respectively. Fix a $1 \le j \le p$. Since $\mathbf{U}$ is a complex, we have that

$$0 = \partial^2(\bar{v}_j) = \partial\Big(\sum_{1 \le s \le q} \alpha_{sj}\bar{u}_s\Big) = \sum_{1 \le s \le q} \alpha_{sj}\Big(\sum_{1 \le l \le t} \beta_{ls}\bar{w}_l\Big)$$

$$= \sum_{1 \le l \le t}\Big(\sum_{1 \le s \le q} \alpha_{sj}\beta_{ls}\Big)\bar{w}_l$$

with $\alpha_{sj}, \beta_{ls} \in k$. Hence $\sum_{1 \le s \le q} \alpha_{sj}\beta_{ls} = 0$ for each $1 \le l \le t$.

Furthermore, in **G** we have

$$\begin{aligned} d^2(v_j) &= d\Big(\sum_{1\le s\le q} \alpha_{sj}\frac{\mathrm{mdeg}(v_j)}{\mathrm{mdeg}(u_s)}u_s\Big) \\ &= \sum_{1\le s\le q} \alpha_{sj}\frac{\mathrm{mdeg}(v_j)}{\mathrm{mdeg}(u_s)}\Big(\sum_{1\le l\le t} \beta_{ls}\frac{\mathrm{mdeg}(u_s)}{\mathrm{mdeg}(w_l)}w_l\Big) \\ &= \sum_{1\le l\le t}\Big(\sum_{1\le s\le q} \alpha_{sj}\beta_{ls}\frac{\mathrm{mdeg}(v_j)}{\mathrm{mdeg}(u_s)}\frac{\mathrm{mdeg}(u_s)}{\mathrm{mdeg}(w_l)}\Big)w_l \\ &= \sum_{1\le l\le t}\Big(\sum_{1\le s\le q} \alpha_{sj}\beta_{ls}\Big)\frac{\mathrm{mdeg}(v_j)}{\mathrm{mdeg}(w_l)}w_l \\ &= 0\,. \end{aligned}$$

□

Note that **G** in Construction 55.2 may not be exact even if the frame **U** is exact.

Construction 55.5. Let **G** be an M-complex. The complex

$$\mathbf{U} = \mathbf{G} \otimes S/(x_1 - 1, \ldots, x_n - 1)$$

is called the ***frame*** of **G** or the ***dehomogenization*** of **G**. We also say that the complex **U** is obtained from **G** by ***dehomogenization***. Note that **U** is a finite complex of finite k-vector spaces with fixed basis and its differential matrices are obtained by setting $x_1 = 1, \ldots, x_n = 1$ in the differential matrices of **G**.

Exercise 55.6. *If* **G** *is the* M*-homogenization of a frame* **U***, then* **U** *is the frame of* **G***.*

A fruitful approach for constructing minimal monomial resolutions is based on the fact that the minimal free resolution of any monomial ideal can be encoded in any of its frames; this was proved in [Peeva-Velasco, Theorem 4.14]:

Theorem 55.7. *The* M*-homogenization of any frame of the minimal multigraded free resolution* **F** *of* S/M *is* **F***.*

This raises the following problem.

Open-Ended Problem 55.8. (folklore) *Find sources of frames that yield minimal free resolutions of monomial ideals.*

We will discuss some sources of frames later. In the rest of the section we provide a helpful criterion.

Construction 55.9. Let $\mathbf{G}$ be an M-complex, and let $m \in M$ be a monomial. Denote by $\mathbf{G}(\leq m)$ the subcomplex of $\mathbf{G}$ that is generated by the multihomogeneous basis elements of multidegrees dividing m.

Running Example 55.10. We continue the running example. Let $m = x^2y^2$. We have that

$$\mathbf{G}(\leq x^2y^2): \quad 0 \to C(x^2y) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} C(x^2) \oplus C(xy) \xrightarrow{(\, x^2 \quad xy\,)} C\,.$$

Proposition 55.11. *Let $m \in M$ be a monomial. Set*

$$m' = \operatorname{lcm}(m_i \,|\, m_i \ \textit{divides } m)\,.$$

Then $\mathbf{G}(\leq m) = \mathbf{G}(\leq m')$.

Proof. By Construction 55.1, all the basis elements of $\mathbf{G}$ have multidegrees in L_M . □

Theorem 55.12. *Let $\mathbf{G}$ be an M-complex and $m \in M$ be a monomial. The component of $\mathbf{G}$ of multidegree m is isomorphic to the frame of the complex $\mathbf{G}(\leq m)$.*

Proof. Note that $\mathbf{G}_m$ has basis of the form

$$\left\{ \frac{m}{\operatorname{mdeg}(g)}\, g \,\middle|\, g \text{ is in the fixed basis of } \mathbf{G}\text{, and } \operatorname{mdeg}(g) \text{ divides } m \right\}.$$

Therefore the component of $\mathbf{G}$ of multidegree m is isomorphic to the frame of the complex $\mathbf{G}(\leq m)$. □

The following criterion for exactness is very useful.

Theorem 55.13. *An M-complex $\mathbf{G}$ is a free multigraded resolution* *of S/M if and only if for all monomials $1 \neq m \in L_M$ the frame of the complex $\mathbf{G}(\leq m)$ is exact.*

Proof. Note that $G_0/d(G_1) = S/M$.

Since the complex $\mathbf{G}$ is multigraded, it suffices to check exactness in each multidegree, similarly to 3.7. As $(G_i)_m = 0$ for $i > 0$ and $m \notin M$, it suffices to check exactness in each multidegree $m \in M$. By Theorem 55.12, it suffices to check exactness of the frames $\mathbf{G}(\leq m)$ for all monomials $m \in M$.

Fix a monomial $m \in M$. Set $m' = \mathrm{lcm}(m_i \,|\, m_i \text{ divides } m)$ and apply Proposition 55.11. Hence, $\mathbf{G}(\leq m) = \mathbf{G}(\leq m')$. Therefore, it suffices to consider only the multidegrees in L_M. □

56 Subresolutions

We will present a first application of the approach in the previous section: we will show that the minimal free resolution of S/M contains as subcomplexes the minimal free resolutions of certain smaller monomial ideals.

Proposition 56.1. (Gasharov-Hibi-Peeva, Miller) *Let $u \in M$ be a monomial, and consider the monomial ideal $(M_{\leq u})$ generated by the monomials $\{m_i \,|\, m_i \text{ divides } u\}$. Fix a multihomogeneous basis of a multigraded free resolution $\mathbf{F}_M$ of S/M.*

(1) *The subcomplex $\mathbf{F}_M(\leq u)$ is a multigraded free resolution of $S/(M_{\leq u})$.*
(2) *If $\mathbf{F}_M$ is a minimal multigraded free resolution of S/M, then $\mathbf{F}_M(\leq u)$ is independent of the choice of basis.*
(3) *If $\mathbf{F}_M$ is a minimal multigraded free resolution of S/M, then the resolution $\mathbf{F}_M(\leq u)$ is minimal as well.*

Proof. Set $v = \mathrm{lcm}(m_i \,|\, m_i \text{ divides } u)$ and apply Proposition 55.11. Hence, $\mathbf{F}_M(\leq u) = \mathbf{F}_M(\leq v)$. Clearly, $(M_{\leq u}) = (M_{\leq v})$. Therefore, we can replace u by v.

By Theorem 55.13, we see that we have to show that for every monomial $1 \neq m \in L_{(M_{\leq v})}$ the frame of the complex $\big(\mathbf{F}_M(\leq v)\big)(\leq m)$ is exact. The frame of $\big(\mathbf{F}_M(\leq v)\big)(\leq m)$ is equal to the frame of

$\mathbf{F}_M(\leq w)$, where w is the maximal monomial that divides both v and m, and is in the set L_M. Since $\mathbf{F}_M$ is exact, by Theorem 55.13 it follows that the frame of $\mathbf{F}_M(\leq w)$ is exact. We proved (1).

(2) Note that the multidegrees of the basis elements in $\mathbf{F}_M$ are determined by the multigraded Betti numbers. Therefore, they are independent of the choice of basis.

(3) holds by construction. □

Theorem 56.2 is a useful particular case of the above result.

Theorem 56.2. *Let $\mathbf{F}_M$ be the minimal free multigraded resolution of S/M. Denote by N the ideal generated by the squarefree minimal monomial generators of M. The minimal free multigraded resolution of S/N is $\mathbf{F}_M(\leq x_1 \dots x_n) = \mathbf{F}_M(\leq u)$, where u is the product of the variables that appear in the minimal monomial generators of the ideal N.*

Example 56.3. We illustrate Theorem 56.2. Let $A = k[x, y, z]$, $T = (x^2, xy, xz, y^3)$, and $u = xyz$. Then $(T_{\leq xyz}) = (xy, xz)$. The minimal multigraded free resolution of A/T is

$$\mathbf{F}_T: \quad 0 \to A \xrightarrow{\begin{pmatrix} z \\ x \\ -y \\ 0 \end{pmatrix}} A^4 \xrightarrow{\begin{pmatrix} y & 0 & z & 0 \\ -x & z & 0 & y^2 \\ 0 & -y & -x & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}} A^4 \xrightarrow{(x^2 \quad xy \quad xz \quad y^3)} A\,.$$

The minimal multigraded free resolution of $A/(xy, xz)$ is the subcomplex

$$(\mathbf{F}_T)(\leq xyz): \quad 0 \to A \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} A^2 \xrightarrow{(xy \quad xz)} A\,.$$

As an application of Theorem 56.2, we will consider resolutions of squarefree Borel ideals. In the rest of this section, we will use the notation from Section 28.

Studying ideals in an exterior algebra has led to the study of squarefree Borel ideals. Their properties are similar to those of Borel ideals.

A squarefree monomial ideal N is ***squarefree Borel*** if it satisfies the ***squarefree Borel property***: whenever the conditions

- ∘ $i < j$
- ∘ g is a monomial such that $gx_j \in N$
- ∘ gx_i is squarefree,

are satisfied, we have $gx_i \in N$ as well.

Exercise 56.4. *A monomial ideal N is squarefree Borel if and only if whenever the conditions*

- ∘ $i < j$
- ∘ *g is such that gx_j is a minimal monomial generator of N*
- ∘ *gx_i is squarefree,*

are satisfied, we have $gx_i \in N$ as well.

Example 56.5. The ideal (wxy, wxz, wyz) is squarefree Borel in $k[w, x, y, z]$.

The interest in studying such special monomial ideals comes from the following result in [Aramova-Herzog-Hibi 2, Theorem 1.7].

Theorem 56.6. *The generic initial ideal of a graded ideal in an exterior algebra is squarefree Borel.*

Conjecture 56.7. (Aramova-Herzog-Hibi) *Let T' be a squarefree ideal in an exterior algebra on variables $x_1, \ldots, x_n$. Let B' be its generic initial ideal. Consider the monomial ideals T and B in S generated by the squarefree monomial generators of T' and B' respectively. For all $i \geq 0$, we have*

$$b_i^S(S/T) \leq b_i^S(S/B)\,.$$

Note that by Theorem 51.4, the ideals B and T have the same Hilbert function.

Construction 56.8. Let N be squarefree Borel, and M be the smallest Borel ideal containing N. Consider the basis elements of the Eliahou-Kervaire resolution $\mathbf{E}_M$ of S/M that have squarefree multidegrees. In the notation of 28.6, these basis elements are

$$\{1\} \cup \Big\{ (m_i; j_1, \ldots, j_p) \Big| 1 \le j_1 < \ldots < j_p < \max(m_i)\ 1 \le i \le r, \\ m_i x_{j_1} \ldots x_{j_p} \text{ is squarefree} \Big\}.$$

Let $\widetilde{\mathbf{E}}_N$ be the complex that is the essential subcomplex of $\mathbf{E}_M$ with basis the above elements (recall the definition of an essential subcomplex in Definition 3.5). We call $\widetilde{\mathbf{E}}_N$ the ***squarefree Eliahou-Kervaire resolution*** of S/N because of the next theorem, which follows immediately from Theorem 56.2 applied to the Eliahou-Kervaire resolution.

Theorem 56.9. (Aramova-Herzog) *Let N be squarefree Borel. Then $\widetilde{\mathbf{E}}_N$ is the minimal free resolution of S/N.*

Example 56.10. We will describe the squarefree Eliahou-Kervaire minimal free resolution of the squarefree Borel ideal

$$(x_1x_2,\ x_1x_3,\ x_2x_3)$$

in $A = k[x_1, x_2, x_3]$. The smallest Borel ideal, that contains it, is

$$(x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^2,\ x_2x_3)\,.$$

The basis of the squarefree Eliahou-Kervaire resolution is

$$\begin{aligned} 1 &\text{ in homological degree } 0 \\ (x_1x_2;\emptyset),\ (x_1x_3;\emptyset),\ (x_2x_3;\emptyset) &\text{ in homological degree } 1 \\ (x_1x_3;2),\ (x_2x_3;1) &\text{ in homological degree } 2\,. \end{aligned}$$

The resolution is

$$0 \to A(x_1x_2x_3)^2 \xrightarrow{\begin{pmatrix} x_3 & x_3 \\ -x_2 & 0 \\ 0 & -x_1 \end{pmatrix}} A(x_1x_2)\oplus A(x_1x_3) \oplus A(x_2x_3) \xrightarrow{(x_1x_2\ x_1x_3\ x_2x_3)} A\,.$$

Corollary 56.11. *Let N be a squarefree Borel ideal in S minimally generated by monomials $m_1, \dots, m_r$. Use the notation in Section 28. Then*

$$\begin{aligned}
\operatorname{codim}(N) &= \max\{\min(m_i) \,|\, 1 \le i \le r\,\} \\
\operatorname{reg}(N) &= \text{highest degree of a minimal generator of } N \\
\operatorname{pd}(N) &= \max\{\,\max(m_i) - \deg(m_i) \,|\, 1 \le i \le r\,\} \\
b^S_{p,p+q}(N) &= \sum_{\deg(m_i)=q} \binom{\max(m_i) - \deg(m_i)}{p} \\
b^S_p(N) &= \sum_{i=1}^{r} \binom{\max(m_i) - \deg(m_i)}{p}.
\end{aligned}$$

Proof. First, we will prove the formula for the codimension by induction on the number of variables. The proof is from [Herzog-Srinivasan, Proposition 4.1]. Set

$$q = \max\{\min(m_i) \,|\, 1 \le i \le r\,\}\,.$$

It follows that $(x_1, \dots, x_q) \supseteq N$. Hence, $\operatorname{codim}(N) \le q$.

We will show that $\operatorname{codim}(N) \ge q$. Write $N = x_1N'' \oplus N'$, where N' and N'' are squarefree Borel ideals in the ring $T = k[x_2, \dots, x_n]$. If $N' = 0$, then $\operatorname{codim}(N) = 1$. In the rest of the proof we assume that $N' \ne 0$.

By induction hypothesis,

$$\operatorname{codim}(N') = \max\{\min(m'_i) \,|\, 1 \le i \le r'\,\} - 1\,,$$

where $m'_1, \ldots, m'_{r'}$ are the minimal monomial generators of N'. The monomials $m'_1, \ldots, m'_{r'}$ are the minimal monomial generators of N not divisible by x_1. Hence,

$$q = \max\{\min(m_i) \mid 1 \leq i \leq r\} = \max\{\min(m'_i) \mid 1 \leq i \leq r'\}.$$

Thus, $\operatorname{codim}(N') = q - 1$.

If $v \in N'$ is a squarefree monomial, then $w = \dfrac{x_1 v}{x_{\min(v)}} \in x_1 N''$ and $\dfrac{v}{x_{\min(v)}} \in N''$. Clearly, $\min\Big(\dfrac{v}{x_{\min(v)}}\Big) > \min(v)$. By induction hypothesis, it follows that

$$\operatorname{codim}(N'') > \operatorname{codim}(N') = q - 1.$$

Hence, the squarefree Borel ideal $N' + N''$ has $\operatorname{codim}(N' + N'') \geq q$.

Let $P = (x_{i_1}, \ldots, x_{i_p})$ be a minimal prime containing N; we assume that $i_1 < \ldots < i_p$. We will show that $p \geq q$. If $i_1 = 1$, then $(x_{i_2}, \ldots, x_{i_p})$ contains N', so $p - 1 \geq \operatorname{codim}(N') = q - 1$. If $i_1 \neq 1$, then P is a prime ideal containing the ideal $N' + N''$ of codimension $\geq q$, so $p \geq q$.

Denote by $\operatorname{nsupp}(u) = \{j \mid x_j \text{ divides } u\}$ the numerical support of a monomial u. In order to prove the remaining formulas, note that the minimal free resolution of N has basis

$$\Big\{ (m_i; j_1, \ldots, j_p) \,\Big|\, 1 \leq j_1 < \ldots < j_p < \max(m_i), \quad 1 \leq i \leq r,\ m_i x_{j_1} \ldots x_{j_p} \text{ is squarefree} \Big\}$$

in homological degree p. For a fixed m_i, note that each j_s can take values in $\{1, \ldots, \max(m_i)\} \setminus \operatorname{nsupp}(m_i)$. Note that for the squarefree monomial m_i we have $|\operatorname{nsupp}(m_i)| = \deg(m_i)$. Therefore, for a fixed m_i, there are $\binom{\max(m_i) - \deg(m_i)}{p}$ choices for the sequence $j_1, \ldots, j_p$. □

57 Simplicial and cellular resolutions

We will explore the following idea: we will obtain frames from a standard construction in topology – homology of (simplicial) chain complexes.

Throughout this section Δ is a simplicial complex on vertices $\{m_1, \ldots, m_r\}$. Recall 36.1. Denote by $\widetilde{C}(\Delta, k)$ the augmented oriented simplicial chain complex of Δ over k; it is used in topology to compute the simplicial homology of Δ. This complex is

$$\widetilde{C}(\Delta; k) = \oplus_{\tau \in \Delta}\, k e_\tau \,,$$

where e_τ denotes the basis element corresponding to the face τ in homological degree $|\tau| - 1$, and the differential ∂ acts as

$$\partial(e_\tau) = \sum_{\tau' \text{ is a facet of } \tau} [\tau, \tau']\, e_{\tau'} \,,$$

where $[\tau, \tau']$ is the incidence (orientation) function: $[\tau, \tau'] = (-1)^i$ if $\tau \setminus \tau'$ is the $(i+1)$'st element in the sequence of the vertices of τ written in increasing order.

Definition 57.1. [Bayer-Peeva-Sturmfels] We use the notation above. After shifting $\widetilde{C}(\Delta; k)$ in homological degree, we get that $\widetilde{C}(\Delta; k)[-1]$ is a frame. Denote by $\mathbf{F}_\triangle$ the M-homogenization of $\widetilde{C}(\Delta; k)[-1]$ (see Construction 55.2). We say that $\mathbf{F}_\triangle$ is ***supported*** on Δ, or that Δ ***supports*** $\mathbf{F}_\triangle$. The complex $\mathbf{F}_\triangle$ is a ***simplicial resolution*** if it is exact. Simplicial resolutions are interesting because they are usually nicely combinatorially structured. They were introduced in [Bayer-Peeva-Sturmfels].

For each vertex m_i of Δ, we set that m_i has multidegree $\mathrm{mdeg}(m_i) = m_i$. We define that a face τ has multidegree

$$\mathrm{mdeg}(\tau) = \mathrm{lcm}(m_i \,|\, m_i \in \tau) \,.$$

By convention, $\mathrm{mdeg}(\emptyset) = 1$.

We think of Δ as a simplicial complex with labeled faces: each face is labeled by its multidegree.

Theorem 57.2. [Bayer-Peeva-Sturmfels] *For each face τ of dimension i the complex $\mathbf{F}_\triangle$ has the generator e_τ in homological degree $i+1$.*

(1) *We have* $\mathrm{mdeg}(e_\tau) = \mathrm{mdeg}(\tau)$.
(2) *The differential in* $\mathbf{F}_\Delta$ *is*

$$\begin{aligned}\partial(e_\tau) &= \sum_{\tau' \text{ is a facet of } \tau} [\tau, \tau'] \, \frac{\mathrm{mdeg}(\tau)}{\mathrm{mdeg}(\tau')} \, e_{\tau'} \\ &= \sum_{\tau' \text{ is a facet of } \tau} [\tau, \tau'] \, \frac{\mathrm{lcm}(m_i | m_i \in \tau)}{\mathrm{lcm}(m_i | m_i \in \tau')} \, e_{\tau'} \,.\end{aligned}$$

Proof. (2) follows from (1) and the fact that the differential is multi-homogeneous. We will prove (1) by induction on homological degree. Clearly, $\mathrm{mdeg}(e_{m_i}) = m_i$ holds for each vertex m_i of Δ. Since

$$\partial(e_\tau) = \sum_{\tau' \text{ is a facet of } \tau} [\tau, \tau'] \, e'_\tau \,,$$

it follows by Construction 55.2 that

$$\begin{aligned}\mathrm{mdeg}(e_\tau) &= \mathrm{lcm}(\mathrm{mdeg}(e_{\tau'}) \,|\, \tau' \text{ is a facet of } \tau\,) \\ &= \mathrm{lcm}(\mathrm{mdeg}(\tau') \,|\, \tau' \text{ is a facet of } \tau\,) \\ &= \mathrm{lcm}\Big(\mathrm{lcm}(m_i | m_i \in \tau') \,\Big|\, \tau' \text{ is a facet of } \tau\,\Big) \\ &= \mathrm{lcm}(m_i | m_i \in \tau) = \mathrm{mdeg}(\tau)\,.\end{aligned}$$

□

Running Example 57.3. Consider the triangle Δ with vertices x^2, xy, y^3. We label each edge by the least common multiple of its vertices, so we get labels x^2y, xy^3, x^2y^3 on the edges. We label the triangle by the least common multiple x^2y^3 of its vertices. See Figure 8 below.

The following is an augmented oriented chain complex of the triangle:

$$0 \longrightarrow k \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}} k^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} k\,.$$

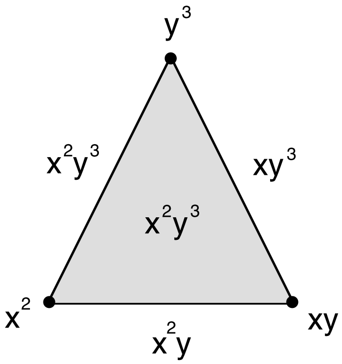

Figure 8.

The corresponding Y-homogenized complex $\mathbf{T}_Y$ is

$$\mathbf{T}_Y :\ 0 \longrightarrow A \xrightarrow{\begin{pmatrix} y^2 \\ x \\ 1 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} -y & 0 & y^3 \\ x & -y^2 & 0 \\ 0 & x & -x^2 \end{pmatrix}} A^3 \xrightarrow{(\,x^2 \quad xy \quad y^3\,)} A\,.$$

We see that $\mathbf{T}_Y = \mathbf{F}_Y \oplus \big(0 \to A \to A \to 0\big)$, so it is exact. Thus, $\mathbf{T}_Y$ is a simplicial resolution, which is non-minimal.

The minimal free resolution $\mathbf{F}_Y$ is also simplicial and corresponds to the simplicial complex with vertices labeled by x^2, xy, y^3 and with two edges $\{x^2, xy\}$ and $\{xy, y^3\}$.

We have

$$\mathbf{F}_Y :\quad 0 \longrightarrow A^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & -y^2 \\ 0 & x \end{pmatrix}} A^3 \xrightarrow{(\,x^2 \quad xy \quad y^3\,)} A\,.$$

For each multidegree m, define the following two subcomplexes of Δ:

$$\Delta_{\le m} = \{\tau \in \Delta \,|\, \mathrm{mdeg}(\tau) \text{ divides } m\}$$
$$\Delta_{< m} = \{\tau \in \Delta \,|\, \mathrm{mdeg}(\tau) \text{ strictly divides } m\}.$$

Running Example 57.4. See Figure 8 above. The subcomplex

$$\Delta_{\le x^2y} = \{\tau \in \Delta \,|\, \mathrm{mdeg}(\tau) \text{ divides } x^2y\}$$

is the edge $\{x^2, xy\}$. The subcomplex

$$\Delta_{<x^2y^3} = \{\tau \in \Delta \,|\, \mathrm{mdeg}(\tau) \text{ strictly divides } x^2y^3\}$$

consists of the two edges $\{x^2, xy\}$ and $\{xy, y^3\}$.

Proposition 57.5. [Bayer-Peeva-Sturmfels] *The complex F_Δ is a free resolution of S/M if and only if for all multidegrees $1 \ne m \in L_M$ the complex $\Delta_{\le m}$ is acyclic over k. Note that for $m \notin M$, the complex $\Delta_{\le m}$ is empty.*

Proof. This follows from Theorem 55.13 because if $m \in L_M$ then the frame of $F_\Delta(\le m)$ is $\widetilde{C}(\Delta_{\le m}; k)[-1]$ by 57.1. □

Theorem 57.6. [Bayer-Sturmfels] *Let F_Δ be a resolution of S/M. For $i \ge 1$ and multidegree $m \ne 1$ we have*

$$b^S_{i,m}(S/M) = \begin{cases} \dim \widetilde{\mathrm{H}}_{i-2}(\Delta_{<m}; k) & \textit{if } \Delta_{\le m} \ne \emptyset \\ 0 & \textit{if } \Delta_{\le m} = \emptyset. \end{cases}$$

Proof. We will compute $\mathrm{Tor}_i(S/M, k)_m$ using F_Δ. Note that $(F_\Delta)_m$ has basis

$$\left\{ \frac{m}{\mathrm{mdeg}(e_\tau)}\, e_\tau \,\middle|\, \tau \in \Delta_{\le m} \right\}.$$

Hence, $(F_\Delta)_m \otimes k$ has basis

$$\{\, e_\tau \,|\, \mathrm{mdeg}(\tau) = m \,\}.$$

The complex $(F_\Delta)_m \otimes k$ of k-vector spaces is isomorphic to the chain complex $\widetilde{C}(\Delta_{\le m}, \Delta_{<m}; k)$, which computes the reduced relative simplicial homology with coefficients in k of the pair $(\Delta_{\le m}, \Delta_{<m})$. We get

$$\mathrm{Tor}_i(S/M, k)_m = \widetilde{\mathrm{H}}_{i-1}(\Delta_{\le m}, \Delta_{<m}; k).$$

If $\Delta_{\le m} = \emptyset$, then $\Delta_{<m} = \emptyset$ and $\widetilde{\mathrm{H}}_{i-1}(\Delta_{\le m}, \Delta_{<m}; k) = 0$.

If $\Delta_{\leq m} \neq \emptyset$, then $\Delta_{\leq m}$ is acyclic by Proposition 57.5. Therefore, the long exact sequence

$$\ldots \to \widetilde{\mathrm{H}}_{i-1}(\Delta_{\leq m}; k) \to \widetilde{\mathrm{H}}_{i-1}(\Delta_{\leq m}, \Delta_{<m}; k) \to \widetilde{\mathrm{H}}_{i-2}(\Delta_{<m}; k) \to \widetilde{\mathrm{H}}_{i-2}(\Delta_{\leq m}; k) \to \ldots$$

implies the isomorphism $\widetilde{\mathrm{H}}_{i-1}(\Delta_{\leq m}, \Delta_{<m}; k) \simeq \widetilde{\mathrm{H}}_{i-2}(\Delta_{<m}; k)$. □

Example 57.7. Consider $T = (ab, ac, ae, bc)$ in $A = k[a, b, c, e]$. The simplicial complex $\Theta_{<abce}$ is shown in Figure 9. We have $b^A_{3,abce}(A/T) = 1$.

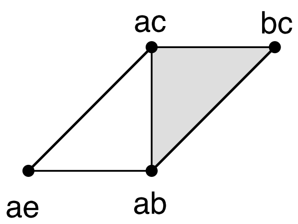

Figure 9.

Proposition 57.8. *Let F_Δ be a resolution of S/M, and let $u \in M$ be a monomial. Consider the monomial ideal $(M_{\leq u})$ generated by the monomials $\{m_i \mid 1 \leq i \leq r,\ m_i \text{ divides } u\}$. The complex $F_{\Delta_{\leq u}}$ is a resolution of $S/(M_{\leq u})$.*

Proof. This follows from Theorem 56.1 since $F_{\Delta_{\leq u}} = F_\Delta(\leq u)$. □

Theorem 57.9. [Bayer-Peeva-Sturmfels] *Denote by Θ the simplex with r vertices $m_1, \ldots, m_r$.*

(1) *Taylor's resolution* 26.5 *is supported on* Θ.
(2) *For $i \geq 1$, the Betti numbers of S/M are*

$$b^S_{i,m}(S/M) = \begin{cases} \dim \widetilde{\mathrm{H}}_{i-2}(\Theta_{<m}; k) & \text{if } m \text{ divides } \mathrm{lcm}(m_1, \ldots, m_r) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply Theorem 57.6 to Taylor's resolution. □

Example 57.10. Example 54.2 provides an example of a monomial ideal with a non-simplicial minimal free resolution since the coefficients of the monomials in the differential matrices cannot be chosen in $\{0, \pm 1\}$.

Cellular resolutions are a natural generalization of simplicial resolutions. They were introduced and studied in [Bayer-Sturmfels]. Let X be a finite regular cell complex on vertices $m_1, \ldots, m_r$. It is equipped with a (non-unique) incidence function, cf. [Bruns-Herzog, Lemma 6.2.1]. The augmented oriented chain complex $\widetilde{C}(X, ; k)$ is used in topology to compute the reduced homology of X. This complex is $\widetilde{C}(X; k) = \oplus_{\tau \in X} k\, e_\tau$, where the basis element e_τ is placed in homological degree $\dim(\tau)$, and the differential ∂ acts as

$$\partial(e_\tau) = \sum_{\tau' \text{ is a facet of } \tau} [\tau, \tau']\, e_{\tau'}\,,$$

where $[\tau, \tau']$ is the incidence function. Clearly, $\widetilde{C}(X; k)[-1]$ is a frame. Denote by $\mathbf{F}_X$ the M-homogenization of $\widetilde{C}(X; k)[-1]$. We say that $\mathbf{F}_X$ is ***supported*** on X, or that X ***supports*** $\mathbf{F}_X$. The complex $\mathbf{F}_X$ is a ***cellular resolution*** if it is exact.

Example 54.2 provides an example of a monomial ideal with a non-cellular minimal free resolution.

A finite regular cell complex is a ***polyhedral cell complex*** if each closed cell is homeomorphic to a convex polytope on the vertices contained in the cell. If X is a polyhedral cell complex, then $\mathbf{F}_X$ is a ***polyhedral resolution***.

It is natural to consider CW-***cellular resolutions*** that are supported by CW-complexes. This is a significant generalization; the structure of CW-cellular resolutions is more complex than that of cellular resolutions. For example, it is no longer true that the coefficients appearing in the differential matrices are only 0, ± 1. CW-cellular resolutions are introduced and studied in [Batzies-Welker] and [Jöllenbeck-Welker]. But even this generalization is not sufficient to cover all minimal monomial resolutions: in [Velasco] it is shown that there exists a monomial ideal whose minimal free resolution does not admit any CW-cellular structure.

58 The lcm-lattice

We will discuss the lcm-lattice, which was introduced in [Gasharov-Peeva-Welker 2] and plays a key role in the study of monomial resolutions. The idea to use the lcm-lattice was inspired by the role of the intersection lattice in computing cohomology of subspace arrangements.

A ***lattice*** is a poset P for which every pair of elements has a join (least upper bound) and a meet (greatest lower bound). If P has a bottom element $\hat{0}$, then the elements in P covering $\hat{0}$ are called ***atoms***.

Lemma 58.1. *Let P be a finite poset with bottom element $\hat{0}$. If every pair of elements has a join, then P is a lattice.*

Proof. Let x and y be two elements in P. The set

$$T = \{\, z \in P \,|\, z \leq x,\ z \leq y \,\}$$

is finite and non-empty. The meet of x and y is the join of all elements in T. □

Construction 58.2. [Gasharov-Peeva-Welker 2] We denote by L_M the lattice with elements the least common multiples of subsets of $m_1, \ldots, m_r$ ordered by divisibility. The atoms in L_M are $m_1, \ldots, m_r$. The top element is $m_M = \mathrm{lcm}(m_1, \ldots, m_r)$. The bottom element is 1 regarded as the lcm of the empty set. The least common multiple of elements in L_M is their join. By Lemma 58.1, L_M is a lattice. We call L_M the ***lcm-lattice*** of M. For $m \in L_M$ we denote by $(1, m)_{L_M}$ the open interval in L_M below m; it consists of all non-unit monomials in L_M that strictly divide m.

Running Example 58.3. The lcm-lattice of (x^2, xy, y^3) is given in Figure 10.

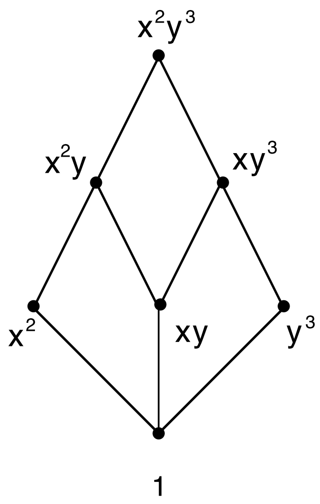

Figure 10.

Exercise 58.4. *If M_{pol} is the polarization of M, then $L_M \cong L_{M_{pol}}$.*

One might wonder what lattices appear as lcm-lattices. The answer to this question was given by Phan in his Ph.D. thesis.

Construction 58.5. [Phan] Let L be a finite atomic lattice (atomic means that each non-bottom element is a joint of atoms). An element in L is meet-irreducible if it is not the meet of two elements in L, and it is not the top or bottom element in L. Suppose that L has n meet-irreducible elements. Label them by $x_1, \ldots, x_n$. Now, label an element $c \in L$ by the monomial

$$\operatorname{mon}(c) = \frac{x_1 \ldots x_n}{\prod_{x_i \geq c} x_i}.$$

Let N_L be the monomial ideal generated by the labels of the atoms in L. This monomial ideal is called the ***L*-ideal.**

Theorem 58.6. [Phan] *Let L be a finite atomic lattice. There exists a monomial ideal whose lcm-lattice is L.*

Proof. We use the notation in Construction 58.5. We will show that the L-ideal N_L has lcm-lattice L.

We will show that the monomial mon(c) is the least common multiple of the labels of the atoms below c. Let $p_1, \dots, p_q$ be the atoms below c. If x_i does not divide mon(c), then $x_i \geq c$; so $x_i \geq p_j$ for each $1 \leq j \leq q$; hence mon(p_j) divides mon(c) for each j.

On the other hand, if x_i divides mon(c), then $x_i \not\geq c$; so there exists an atom p_j such that $x_i \not\geq p_j$; hence, x_i divides some mon(p_j).

Denote by L' the lcm-lattice of N_L. Let $\psi : L \to L'$ be the map that maps an element $c \in L$ to mon(c) $\in L'$. The map is order-preserving and surjective. We will show that it is injective. Let $c, c' \in L$ be such that mon(c) = mon(c'). Therefore, the set $\mathcal{M}_c$ of meet-irreducible elements over c coincides with the the set $\mathcal{M}_{c'}$ of meet-irreducible elements over c'. Note that c is the meet of the elements in $\mathcal{M}_c$, and c' is the meet of the elements in $\mathcal{M}_{c'}$. Hence, $c = c'$. □

Example 58.7. Consider the lattice in Figure 11. The meet-irreducible elements are labeled by variables.

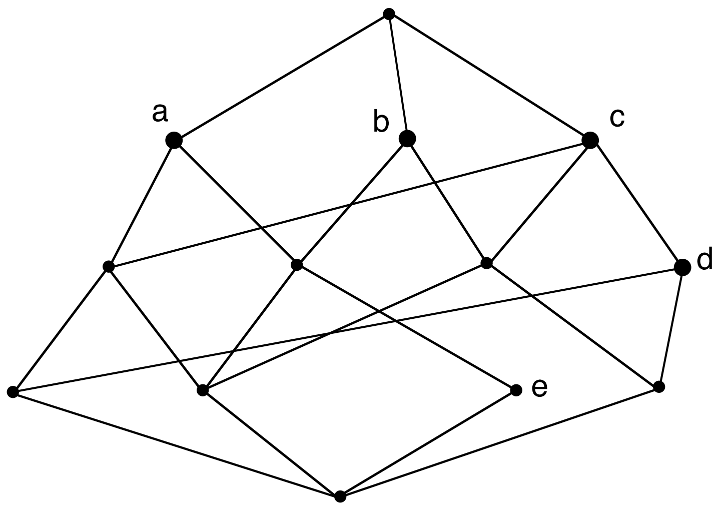

Figure 11.

In Figure 12 we show the same lattice but labeled as the lcm-lattice of the ideal constructed in the proof of the theorem above.

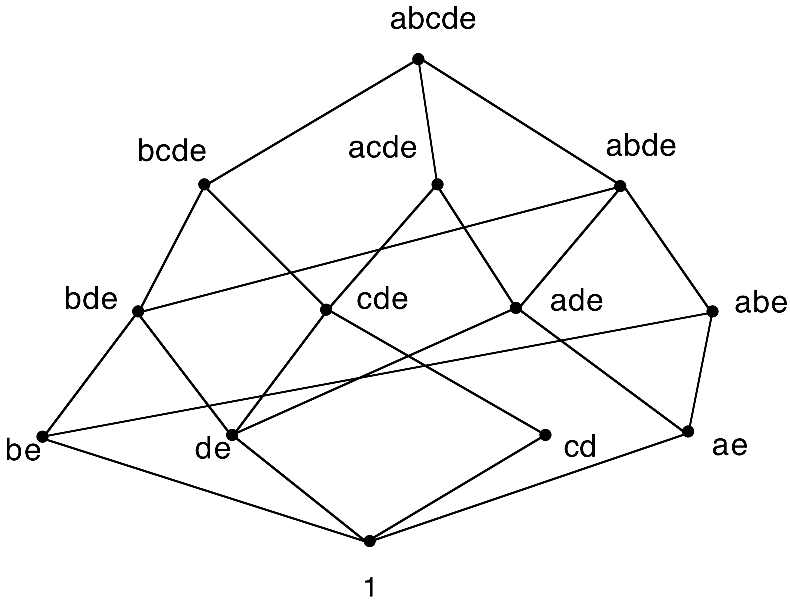

Figure 12.

Theorem 58.8 is the main result in this section.

Theorem 58.8. [Gasharov-Peeva-Welker 2] *For* $i \geq 1$ *we have*

$$b^S_{i,m}(S/M) = \begin{cases} \dim \widetilde{\mathrm{H}}_{i-2}\big((1,m)_{L_M}; k\big) & \text{if } 1 \neq m \in L_M \\ 0 & \text{if } m \notin L_M \ . \end{cases}$$

Note that $\widetilde{\mathrm{H}}_{i-2}\big((1,m)_{L_M}; k\big)$ means $\widetilde{\mathrm{H}}_{i-2}\big(O\big((1,m)_{L_M}\big); k\big)$; recall that $O\big((1,m)_{L_M}\big)$ is the order complex of $(1,m)_{L_M}$ (see Section 36).

Proof. By 57.9 we have that $b^S_{i,m}(S/M) = 0$ if $m \notin L_M$.

Let Θ be the simplex with r vertices labeled by $m_1, \ldots, m_r$. Fix a monomial $m \in L_M$, $m \neq 1$. The formula for the Betti numbers in Theorem 57.9 is $b^S_{i,m}(S/M) = \dim \widetilde{\mathrm{H}}_{i-2}\big(\Theta_{<m}; k\big)$ for $i \geq 1$.

Recall 36.14. The set C of the minimal monomial generators of M that divide m forms a crosscut of the poset $(1,m)_{L_M}$. Its crosscut complex has faces the subsets of C whose lcm is in $(1,m)_{L_M}$, that is the lcm strictly divides m. So the crosscut complex coincides with the complex $\Theta_{<m}$. By Theorem 36.16 the crosscut complex $\Theta_{<m}$ is homotopic to the order complex of $(1, m)_{L_M}$. □

Running Example 58.9. See Figure 10. Consider the open interval $(1, x^2y^3)$ in the lcm-lattice of Y. Its order complex $O(1, x^2y^3)$ has 5 vertices x^2, xy, y^3, x^2y, xy^3 and 4 edges. It is contractible, so we get that $b^C_{i,x^2y^3}(A/Y)$ vanish for all i. Now, consider the open interval $(1, x^2y)$. Its order complex $O(1, x^2y)$ has 2 vertices and no edge. Hence $b^C_{2,x^2y}(C/Y) = 1$.

Forgetting about the multigrading in the Theorem 58.8 we obtain the following result.

Corollary 58.10. *For $i \geq 1$ we have*

$$b^S_i(S/M) = \sum_{\substack{m \in L_M \\ m \neq 1}} \dim \widetilde{\mathrm{H}}_{i-2}\big((1, m)_{L_M}; k\big)\,.$$

This formula is an analogue of the Goresky-MacPherson Formula, which expresses the dimensions of the cohomology groups of the complement of a subspace arrangement in terms of the dimensions of the homology groups of the lower intervals in the intersection lattice.

Next, we will show that in order to compute the Betti numbers one can use the lcm-lattice built on any set of monomial generators of the ideal M.

Proposition 58.11. [Gasharov-Peeva-Welker 2] *Let L' be the lattice of the least common multiples of subsets of a set of monomials generating M. For $i \geq 1$ we have*

$$b^S_{i,m'}(S/M) = \begin{cases} \dim \widetilde{\mathrm{H}}_{i-2}\big((1, m')_{L'}; k\big) & \text{if } m' \in L' \\ 0 & \text{if } m' \notin L'. \end{cases}$$

Proof. If $m' \in L' \setminus L_M$, then the order complex of $(1, m')_{L'}$ is a cone over $\mathrm{lcm}\{m_i | m_i \text{ divides } m'\}$. Hence,

$$\dim \widetilde{\mathrm{H}}_{i-2}\big((1, m')_{L'}; k\big) = 0 = b^S_{i,m'}(S/M)\,.$$

Consider the map

$$f: \; L' \to L_M \subseteq L'$$
$$m' \mapsto \operatorname{lcm}\{m_i | m_i \text{ divides } m'\}\,.$$

This map is order-preserving and it is a closure operator (see 36.6). By Theorem 36.6 it follows that $(1, m')_{L'}$ and $(1, f(m'))_{L_M}$ are homotopic. If $m' \in L_M$, then $f(m') = m'$ and

$$\dim \widetilde{\mathrm{H}}_{i-2}\big((1, m')_{L'}; k\big) = \dim \widetilde{\mathrm{H}}_{i-2}\big((1, m')_{L_M}; k\big) = b^S_{i,m'}(S/M)\,.$$

□

Running Example 58.12. The lcm-lattice of Y is given in Figure 10. We can also use the lcm-lattice in Figure 13 consisting of the lcm's of the monomials $x^2, xy, y^3, x^2y^2, xy^5$.

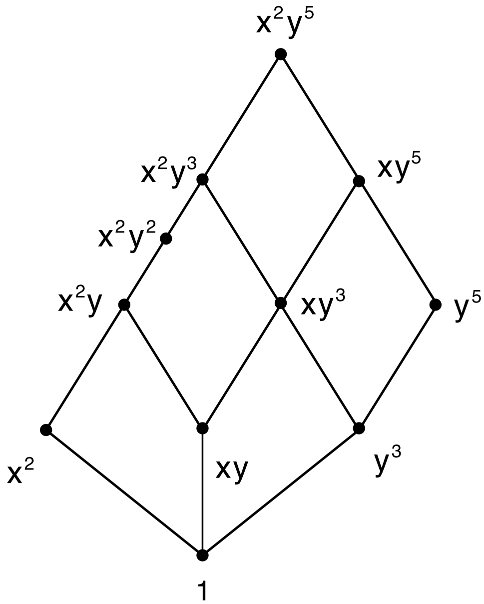

Figure 13.

59 The Scarf complex

We will show that generically the minimal free resolutions of monomial ideals are simplicial. For this purpose we will use the Scarf complex, introduced in [Bayer-Peeva-Sturmfels]. This complex is always contained in the minimal free resolution of a monomial ideal. We will discuss the ideals for which the Scarf complex provides the minimal free resolution; such ideals are called Scarf ideals. Unless otherwise stated, the material in this section is from [Bayer-Peeva-Sturmfels].

Construction 59.1. Recall that m_τ stands for $\operatorname{lcm}(m_i | \, m_i \in \tau)$. The ***Scarf complex*** of M is the simplicial complex

$$\Omega_M = \Big\{ \tau \subseteq \{m_1, \dots, m_r\} \;|\; m_\tau \neq m_\sigma \text{ for all } \sigma \subseteq \{m_1, \dots, m_r\} \text{ other than } \tau \Big\}.$$

In [Bayer-Peeva-Sturmfels] it is shown that Ω_M equals a simplicial complex introduced by Scarf in the context of mathematical economics. Denote by $\mathbf{F}_{\Omega_M}$ the M-homogenization of the augmented oriented simplicial chain complex of Ω_M (see 57.1).

The multidegree of a vertex m_i in Ω_M is the monomial m_i. The multidegree of a face $\tau \in \Omega_M$ is $\operatorname{mdeg}(\tau) = \operatorname{lcm}(\, m_i \,|\, m_i$ is a vertex of $\tau\,)$. By Theorem 57.2, the multidegree of the basis element e_τ in $\mathbf{F}_{\Omega_M}$ is $\operatorname{mdeg}(\tau)$. The multidegrees of the faces of Ω_M are called ***Scarf multidegrees***.

Theorem 59.2. *If* $\operatorname{mdeg}(\tau)$ *is a Scarf multidegree, then*

$$b^S_{i,\operatorname{mdeg}(\tau)}(S/M) = \begin{cases} 1 & \textit{if } i = \dim(\tau) + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\operatorname{mdeg}(\tau) \neq 1$. The closed interval $[1, \operatorname{mdeg}(\tau)]$ in L_M is the face lattice of the simplex τ. Hence, the open interval $(1, \deg(\tau))$ is homotopic to the boundary of the simplex τ. Therefore,

$$b^S_{i,\operatorname{mdeg}(\tau)}(S/M) = \widetilde{\mathrm{H}}_{i-2}((1, \operatorname{mdeg}(\tau)); k) = \begin{cases} 1 & \text{if } i-2 = \dim(\tau) - 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Running Example 59.3. The Scarf complex Ω_Y of Y has the three vertices x^2, xy, y^3 and the two edges $\{x^2, xy\}$, $\{xy, y^3\}$.

Proposition 59.4. *The complex* $\mathbf{F}_{\Omega_M}$ *is an essential subcomplex of* *the minimal free resolution* $\mathbf{F}_M$ *of* S/M*, that is, there exists a basis of* $\mathbf{F}_{\Omega_M}$ *that is part of a basis of* $\mathbf{F}_M$*.*

Proof. Consider Taylor's resolution $\mathbf{T}_M$.

By Theorem 59.2, it follows that for each multidegree m, that divides a Scarf multidegree, we have that $\mathbf{F}_M$, $\mathbf{T}_M$, and $\mathbf{F}_{\Omega_M}$ have the same number (0 or 1) of basis elements in multidegree m.

Taylor's resolution is possibly non-minimal, so $\mathbf{T}_M \cong \mathbf{F}_M \oplus \mathbf{P}$, where $\mathbf{P}$ is a sum of trivial complexes of the form $0 \to S(u) \to S(u) \to 0$. It follows that such a multidegree u cannot divide a Scarf multidegree. Therefore, $\big(\mathbf{T}_M\big)_{\leq m} \cong \big(\mathbf{F}_M\big)_{\leq m}$ for each multidegree m, that divides a Scarf multidegree.

By the construction of Taylor's resolution, we have that $\mathbf{F}_{\Omega_M}$ is an essential subcomplex of $\mathbf{T}_M$. Hence, $\mathbf{F}_{\Omega_M}$ is an essential subcomplex of the minimal free resolution $\mathbf{F}_M$. □

We call M a ***Scarf ideal*** if $\mathbf{F}_{\Omega_M}$ is the minimal free resolution of S/M, and we say that $\mathbf{F}_{\Omega_M}$ is its ***Scarf resolution***. The definition of Ω_M immediately implies the following properties of the Scarf resolution.

Corollary 59.5. *Let* M *be a Scarf ideal.*

(1) *The number of* j*-faces of the Scarf complex* Ω_M *equals the Betti number* $b^S_{j+1}(S/M)$*.*

(2) $\mathbf{F}_{\Omega_M}$ *is multigraded and in each multidegree the Betti number is either* 0 *or* 1*.*

Theorem 59.6. *An ideal* M *is Scarf if and only if all non-zero Betti numbers of* S/M *are in Scarf multidegrees.*

Proof. Suppose that all non-zero Betti numbers of S/M are in Scarf multidegrees. If $\mathbf{F}_M$ is strictly larger than $\mathbf{F}_{\Omega_M}$, then there exists a face $\tau \in \Omega_M$ such that $\mathbf{F}_M$ has at least two basis elements in multidegree $\mathrm{mdeg}(\tau)$. This contradicts Theorem 59.2.

Theorem 59.7. [Peeva-Velasco]

(1) *A simplicial complex with* r *vertices is the Scarf complex of a*

monomial ideal if and only if it is not the boundary of the simplex with r *vertices.*

(2) *A finite simplicial complex* Ω *supports a Scarf resolution if and only if* Ω *is acyclic.*

In the proof we use the following construction.

Construction 59.8. (Mermin) Let $\Omega \neq \emptyset$ be a finite simplicial complex. For each face τ of Ω we introduce a variable x_τ, and then we set $B = k[x_\tau \,|\, \tau \in \Omega,\, \tau \neq \emptyset\,]$. We will construct a monomial ideal in this polynomial ring. Set z to be the product of all the variables.

Set the multidegree of a vertex v of Ω to be

$$\mathrm{mdeg}(v) = \prod_{v \notin \tau \in \Omega} x_\tau \,.$$

Denote by Θ the simplex on the vertices of Ω. It follows that a face $\sigma \in \Theta$ has multidegree

$$\mathrm{mdeg}(\sigma) = \mathrm{lcm}(\mathrm{mdeg}(v) \,|\, v \in \sigma) = \prod_{\sigma \not\subseteq \tau \in \Omega} x_\tau \,.$$

If $\sigma \notin \Omega$ then $\mathrm{mdeg}(\sigma) = z$. Every two faces in Ω have distinct multidegrees.

Denote by J_Ω the ideal generated by the multidegrees of the vertices. We say that J_Ω is the ***nearly Scarf ideal*** of Ω. It is easy to see that the lcm-lattice L_{J_Ω} consists of the top element z and the face poset of Ω.

Proof. Let Ω be a finite simplicial complex. Both (1) and (2) hold if Ω is either a point or $\emptyset$. We will assume that Ω has at least two vertices.

(1) The complex Ω is the Scarf complex of the monomial ideal J_Ω, constructed in Construction 59.8, if and only if Ω is a simplex or Ω has at least two non-faces. This happens if and only if Ω is not the boundary of the simplex Θ.

(2) If Ω supports a Scarf resolution, then it is acyclic by Theorem 55.13 applied to the multidegree m that is the lcm of all the minimal monomial generators of the ideal.

Suppose that the simplicial complex Ω is acyclic. We will show that the ideal J_Ω, constructed in Construction 59.8, is a Scarf ideal with Scarf complex Ω. The lcm of its minimal monomial generators is z.

We want to apply Theorem 59.6. Thus, we have to show that $b^S_{i,z}(S/J_\Omega) = 0$ for every i. Compute these Betti numbers using Theorem 58.8. The lcm-lattice of J_Ω consists of the Scarf multidegrees (including the bottom element 1) and the top element z. The interval $[1, z)$ is the face poset of Ω. The order complex of $(1, z)$ is the barycentric subdivision of Ω by 36.8, and is homotopic to Ω by 36.9, so it is acyclic. Therefore, the simplicial complex Ω supports the Scarf resolution of J_Ω. □

Theorem 59.9 provides a wide class of ideals which are Scarf ideals and which have the advantage of being defined by a simple combinatorial property; note that the ideals are defined by a generic condition on the exponents of the minimal monomial generators.

Theorem 59.9. *Suppose that no variable x_i appears at the same non zero exponent in two distinct minimal monomial generators of M. Then M is a Scarf ideal.*

Proof. For $\tau \subseteq \{m_1, \ldots, m_r\}$, set $m_\tau = \mathrm{mdeg}(\tau) = \mathrm{lcm}(m_i \,|\, i \in \tau)$. Consider a multidegree m_τ with $\tau \notin \Omega_M$. By Theorem 59.6 we have to show that all Betti numbers in multidegree m_τ vanish. We compute the Betti numbers of S/M using the Koszul complex $\mathbf{K}$ that is the minimal free resolution of k over S. We use the notation in Construction 26.4. The component of $\mathbf{K}$ in multidegree m_τ has basis

$$\left\{ \frac{m_\tau}{x_{j_1} \ldots x_{j_i}} e_{j_1} \wedge \cdots \wedge e_{j_i} \;\middle|\; x_{j_p} \text{ divides } m_\tau \text{ for } 1 \le p \le i, \quad 1 \le j_1 < \ldots < j_i \le n \right\}.$$

Fix an element $f = \dfrac{m_\tau}{x_{j_1} \ldots x_{j_i}} e_{j_1} \wedge \cdots \wedge e_{j_i}$ in this basis. Choosing τ minimal with respect to inclusion, we may assume $m_\tau = m_{\tau \cup m_s}$ for some $m_s \in \{m_1, \ldots, m_r\} \backslash \tau$. We have that m_s divides m_τ. On the other hand, by assumption the monomials m_s and m_τ have different non-zero exponents in each variable. Hence, the monomial m_s divides $\dfrac{m_\tau}{\prod_{\{i \,|\, x_i \text{ divides } m_\tau\}} x_i}$. Therefore, the image of f in $(S/M \otimes \mathbf{K}_i)_{m_\tau}$ vanishes. We conclude that $(S/M \otimes \mathbf{K}_i)_{m_\tau} = 0$. Thus, the Scarf complex is exact and $\mathbf{F}_{\Omega_M}$ is a fee resolution of S/M.

If $\sigma \in \Omega_M$ and $m_i \in \sigma$, then by the definition of the Scarf complex it follows that $\mathrm{mdeg}(\sigma \setminus m_i)$ strictly divides $\mathrm{mdeg}(\sigma)$. Therefore, $d(\mathbf{F}_{\Omega_M}) \subseteq (x_1, \ldots, x_n)\mathbf{F}_{\Omega_M}$. Thus, the resolution is minimal. □

60 Rootings and Lyubeznik's resolution

We will construct a simplicial resolution which is smaller than Taylor's resolution. The material in this section is from [Novik].

A ***rooting map*** on the lcm-lattice L_M is a map

$$h: \ L_M \setminus \{1\} \to \{m_1, \ldots, m_r\}$$

such that the following two conditions are satisfied:

(1) for every monomial m, we have that $h(m)$ divides m.

(2) if $m, m' \in L_M \setminus \{1\}$ are such that $h(m)$ divides m' and m' divides m, then $h(m) = h(m')$.

For every nonempty set $\mathcal{B} \subseteq \{m_1, \ldots, m_r\}$ we define

$$h(\mathcal{B}) = h(\mathrm{lcm}(m_i | m_i \in \mathcal{B}))\,.$$

We say that $\mathcal{B}$ is ***unbroken*** if $h(\mathcal{B}) \in \mathcal{B}$. A set $\mathcal{B}$ is ***rooted*** if all nonempty subsets of $\mathcal{B}$ are unbroken. The collection of all rooted sets is the ***rooted complex*** of h, and is denoted RC_h. It is a subcomplex of the simplex Θ on vertices $m_1, \ldots, m_r$.

Exercise 60.1. *If h is a rooting map on L_M, then RC_h is a simplicial complex.*

Theorem 60.2. *If h is a rooting map on L_M, then the rooted complex RC_h supports a simplicial free resolution of S/M.*

Proof. We will apply Proposition 57.5.

Let $m \in L_M \setminus \{1\}$. We will prove that the simplicial complex $(\mathrm{RC}_h)_{\leq m}$ is a cone with apex $h(m)$.

Let $\tau \in (\mathrm{RC}_h)_{\leq m}$ be a face. We will show that either it contains the vertex $h(m)$ or $\tau \cup h(m) \in (\mathrm{RC}_h)_{\leq m}$. Suppose that $h(m) \notin \tau$. We will prove that every subset σ of $\tau \cup h(m)$ is unbroken. Note that σ either is a subset of τ and so is unbroken, or it contains $h(m)$. Suppose the latter case holds. Set $m' = \mathrm{lcm}(m_i | m_i \in \sigma)$. Then $h(m)$ divides m' since $h(m) \in \sigma$. As $\tau \in (\mathrm{RC}_h)_{\leq m}$ and $h(m)$ divides m, it follows that $m' = \mathrm{lcm}(m_i | m_i \in \sigma)$ divides m. By the definition of a rooting map it follows that $h(\sigma) = h(m') = h(m)$. Since $h(m) \in \sigma$, we conclude that σ is unbroken. Therefore, $\tau \cup h(m)$ is rooted. Thus, $\tau \cup h(m) \in (\mathrm{RC}_h)_{\leq m}$.

Since $(\mathrm{RC}_h)_{\leq m}$ is a cone, it is acyclic. The theorem follows by Proposition 57.5. □

For a set $\tau \subseteq \{m_1, \ldots, m_r\}$, set $\min(\tau) = \min\{i \,|\, m_i \in \tau)$ and recall that $\mathrm{mdeg}(\tau) = \mathrm{lcm}(m_i | m_i \in \tau)$.

Exercise 60.3. *Define a map*

$$\begin{aligned} h:\ L_M \setminus \{1\} &\longrightarrow \{m_1, \ldots, m_r\} \\ m &\mapsto \min\{i \,|\, m_i \textit{ divides } m)\,. \end{aligned}$$

(1) *The map h is a rooting map.*
(2) *A set $\tau \subseteq \{m_1, \ldots, m_r\}$ is unbroken if and only if m_q does not divide* $\mathrm{mdeg}(\tau)$ *for every $q < \min(\tau)$.*

The following result is an immediate corollary of Theorem 60.2 and Exercise 60.3.

Theorem 60.4. *Let h be the rooting map constructed in Exercise* 60.3. *The rooted complex RC_h supports a simplicial free resolution of S/M denoted $\mathbf{L}_M$. It is the essential subcomplex of Taylor's resolution such that the free module in homological degree j in $\mathbf{L}_M$ has basis*

$$\left\{ e_\tau \,\middle|\, m_q \textit{ does not divide } \mathrm{mdeg}(\sigma) \textit{ for all } \sigma \subseteq \tau \textit{ and } q < \min(\sigma) \right\}.$$

This simplicial free resolution of S/M is called the ***Lyubeznik***

resolution. It was introduced in [Lyubeznik] and proved in a different way.

Running Example 60.5. Order the minimal monomial generators of Y by $m_1 = xy, m_2 = y^3, m_3 = x^2$. Consider the rooting map defined in Exercise 60.3. The edge $\{x^2, y^3\}$ is not unbroken, so this edge and the triangle $\{x^2, y^3, xy\}$ are not rooted. Therefore, the rooted complex has facets the edges $\{x^2, xy\}$ and $\{xy, y^3\}$. Thus, the rooted complex supports the minimal free resolution of C/Y. It is strictly smaller than Taylor's resolution.

61 Betti numbers via simplicial complexes

The Betti numbers of S/M can be computed using various simplicial complexes. It is helpful to have formulas based on different simplicial complexes since different complexes are useful in different situations.

Construction 61.1. Let $\Gamma(m)$ be the simplicial complex on vertices $x_1, \dots, x_n$ and with faces

$$\left\{\tau \subseteq \{x_1, \dots, x_n\} \;\middle|\; \frac{m}{\prod_{x_i \in \tau} x_i} \in M\right\}.$$

Sometimes, it is more convenient to denote the vertices of $\Gamma(m)$ by $\{1, \dots, n\}$ and then the faces are

$$\left\{\tau \subseteq \{1, \dots, n\} \;\middle|\; \frac{m}{\prod_{i \in \tau} x_i} \in M\right\}.$$

Let Θ be the simplex with r vertices $m_1, \dots, m_r$. Also, let L_M be the lcm-lattice of M, and $O(1, m)$ be the order complex of the open interval $(1, m)$ in the lcm-lattice L_M.

Running Example 61.2. Let $m = x^2y^3$. The complex $\Theta_{<x^2y^3}$ has three vertices x^2, xy, y^3 and the two edges $\{x^2, xy\}$, $\{xy, y^3\}$. The complex $\Gamma(m)$ has two vertices x, y and the edge $\{x, y\}$. The complex $O(1, m)$ has 5 vertices x^2, xy, y^3, x^2y, xy^3 and 4 edges $\{x^2, x^2y\}$, $\{xy, x^2y\}$, $\{xy, xy^3\}$, $\{y^3, xy^3\}$. See Figure 14.

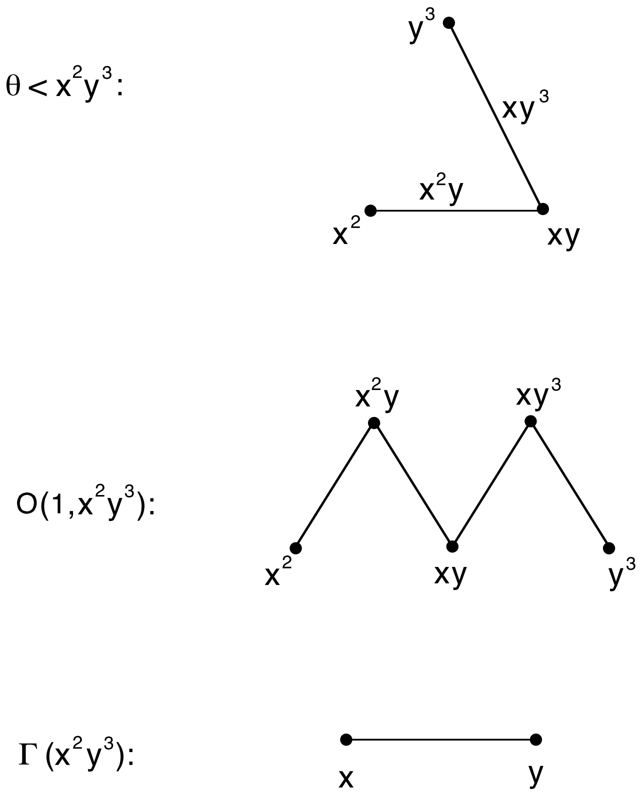

Figure 14.

Theorem 61.3. [Bayer-Sturmfels], [Bruns-Herzog 2], [Gasharov-Peeva-Welker 2] *The simplicial complexes* $\Theta_{<m}$, $\Gamma(m)$, *and* $O(1,m)$ *are homotopy equivalent. For* $i \geq 0$ *and any monomial* $m \neq 1$, *we have*

$$\begin{aligned} b^S_{i,m}(M) &= \dim \widetilde{\mathrm{H}}_{i-1}\big(\Theta_{<m};k\big) \\ &= \dim \widetilde{\mathrm{H}}_{i-1}\big(\Gamma(m);k\big) \\ &= \dim \widetilde{\mathrm{H}}_{i-1}\big(O(1,m);k\big)\,. \end{aligned}$$

Proof. The proof of Theorem 58.8 shows that $O(1,m)$ and $\Theta_{<m}$ are homotopy equivalent. We will show that $\Theta_{<m}$ and $\Gamma(m)$ are homotopy equivalent. We are going to use the Nerve Theorem 36.11. For each

$1 \leq i \leq n$, consider the simplicial complex $\Lambda_{x_i} \subseteq \Theta$ with faces

$$\Big\{ \{m_{j_1}, \dots, m_{j_q}\} \Big| \, m_{j_p} \text{ divides } \frac{m}{x_i} \text{ for } 1 \leq p \leq q \Big\} .$$

Each Λ_{x_i} is a simplex. The simplices $\Lambda_{x_1}, \dots, \Lambda_{x_n}$ cover $\Theta_{<m}$. We have

$$\cap_{x_i \in \mathcal{A}} \Lambda_{x_i} = \Big\{ \{m_{j_1}, \dots, m_{j_q}\} \Big| \, m_{j_p} \text{ divides } \frac{m}{\prod_{x_i \in \mathcal{A}} x_i} \text{ for } 1 \leq p \leq q \Big\}$$

so if an intersection is non-empty then it is a simplex, so contractible. The nerve of this cover is $\Gamma(m)$. Hence $\Gamma(m)$ and $\Theta_{<m}$ are homotopy equivalent by the Nerve Theorem 36.11.

The formula for the Betti numbers follows from Theorem 57.9.

Here is another proof following [Bruns-Herzog 2, 1.1]. Apply Construction 26.4 and use its notation. Denote by $\mathbf{T}$ the augmented oriented simplicial chain complex computing the reduced homology of the simplicial complex $\Gamma(m)$. Then

$$\begin{aligned} (M \otimes \mathbf{K}_i)_m \ &\longrightarrow \mathbf{T}_{i-1} \\ \frac{m}{x_{j_1} \dots x_{j_i}} e_{j_1} \wedge \dots \wedge e_{j_i} \ &\mapsto \text{the face with vertices } x_{j_1}, \dots, x_{j_i} \end{aligned}$$

is an isomorphism of complexes. Hence, $b^S_{i,m}(M) = \dim \widetilde{\mathrm{H}}_{i-1}\big(\Gamma(m); k\big)$ as desired. □

62 The Stanley-Reisner correspondence

The first peak in the study of monomial resolutions was in the 1970's. The Stanley-Reisner theory was introduced by Hochster [Hochster] and Reisner [Reisner], and had applications in combinatorics, cf. [Stanley]. The main idea in the Stanley-Reisner theory is to use simplicial complexes in order to compute the Betti numbers of a squarefree monomial ideal. Polarization (see Section 21) can be used to reduce to the squarefree case, that is, to reduce the study of resolutions of monomial ideals to the study of resolutions of squarefree monomial ideals.

In this section we consider squarefree monomial ideals.

The ***support*** of a squarefree monomial m is the set

$$\operatorname{supp}(m) = \{x_i \,|\, x_i \text{ divides } m\}$$

For $\tau \subseteq \{x_1, \dots, x_n\}$, set

$$\mathbf{x}_\tau = \prod_{i \in \tau} x_i \,.$$

Recall the definition of the Stanley-Reisner ideal (see Section 51). Let Δ be a simplicial complex with vertices $x_1, \dots, x_n$. The Stanley-Reisner ideal in S of Δ is

$$I_\Delta \;=\; (x_{i_1} \dots x_{i_p} \,|\, \{x_{i_1}, \dots, x_{i_p}\} \notin \Delta\,)\,.$$

The Stanley-Reisner ring of Δ is S/I_Δ.

Note that any squarefree monomial ideal is the Stanley-Reisner ideal for some simplicial complex. In [Bruns-Gubeladze] it is proved that if two Stanley-Reisner rings are isomorphic as k-algebras, then their simplicial complexes are isomorphic.

The following basic equality is proved in Theorem 51.5.

Theorem 62.1. $\dim(S/I_\Delta) = \dim(\Delta) + 1$.

Construction 62.2. The ***Alexander dual complex*** of Δ is

$$\Delta^\vee \;=\; \Big\{\, \{x_1, \dots, x_n\} \setminus \tau \,\Big|\, \tau \notin \Delta \,\Big\}\,.$$

Exercise 62.3. $\Delta^{\vee\vee} = \Delta$.

The Stanley-Reisner ideal of the Alexander dual complex is

$$I_{\Delta^\vee} \;=\; \left(\left\{ \frac{x_1 \cdots x_n}{\mathbf{x}_\tau} \,\middle|\, \tau \in \Delta \right\} \right)$$

$$= \Big(\big\{ \text{monomial } m \,|\, \gcd(m, m') \neq 1 \text{ for each monomial } m' \in I_\Delta \,\big\} \Big)\,.$$

Exercise 62.4. *The facets of* Δ *correspond bijectively to the minimal monomial generators of* $I_{\Delta^\vee}$.

The key idea in this section is that the algebraic properties of the minimal free resolution of I_Δ are closely related to the topological and combinatorial properties of $\Delta^\vee$.

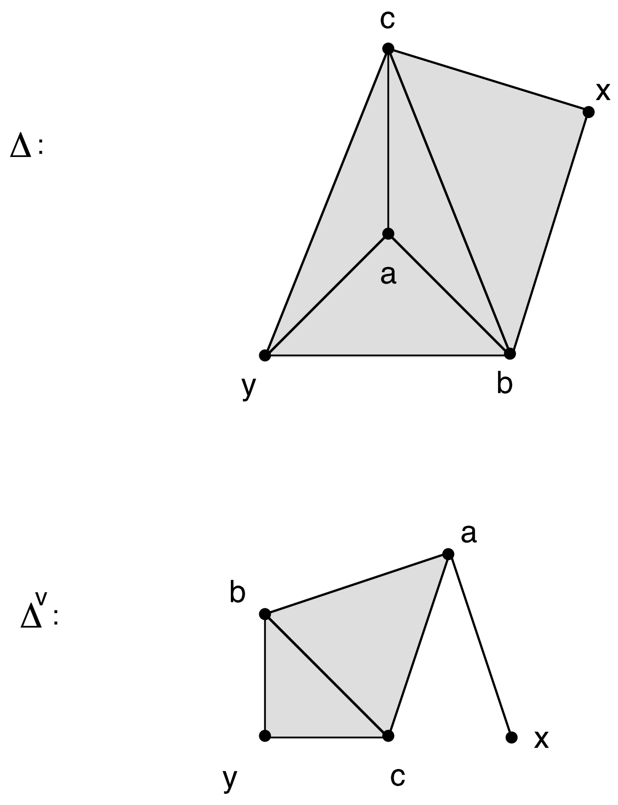

Figure 15.

Running Example 62.5. Polarizing Y we obtain the squarefree ideal (xa, xy, ybc). It is the Stanley-Reisner ideal of the simplicial complex Δ on vertices x, y, a, b, c and with facets the triangles $\{x, c, b\}$, $\{a, y, c\}$, $\{a, b, y\}$, $\{a, b, c\}$. The facets of the Alexander dual complex

$\Delta^\vee$ are the supports of the monomials $\dfrac{xyabc}{xa} = ybc$, $\dfrac{xyabc}{xy} = abc$, and $\dfrac{xyabc}{ybc} = xa$, so they are $\{y, b, c\}$, $\{a, b, c\}$, and $\{x, a\}$. See Figure 15 above.

Theorem 62.6 is presented without a proof, and is a useful tool from Algebraic Topology; cf. [Bayer-Charalambous-Popescu, 2.1].

Alexander Duality Theorem 62.6.

$$\dim \widetilde{\mathrm{H}}_{n-i-2}(\Delta; k) = \dim \widetilde{\mathrm{H}}^{i-1}(\Delta^\vee; k) = \dim \widetilde{\mathrm{H}}_{i-1}(\Delta^\vee; k)\,.$$

For a subset τ of $\{1, \ldots, n\}$, let Δ_τ be the ***restriction*** of Δ on τ, that is the maximal subcomplex of Δ on vertices τ. Note that $(\Delta_\tau)^\vee = (\Delta^\vee)_\tau$, and denote it $\Delta^\vee_\tau$.

Theorem 62.7. *Let τ be a subset of $\{1, \ldots, n\}$.*

(1) *Recall Construction* 61.1 *defining $\Gamma(\mathbf{x}_\tau)$ to be the simplicial complex with faces $\left\{\sigma \subseteq \tau \,\middle|\, \dfrac{\mathbf{x}_\tau}{\mathbf{x}_\sigma} \in I_\Delta\right\}$. We have*

$$\Gamma(\mathbf{x}_\tau) = \Delta^\vee_\tau$$

and

$$b^S_{i,\mathbf{x}_\tau}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{i-1}\big(\Delta^\vee_\tau; k\big)\,.$$

(2) *We have*

$$b^S_{i,\mathbf{x}_\tau}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{|\tau|-i-2}(\Delta_\tau; k)\,.$$

Proof. First, we prove (1). The simplicial complex $\Gamma(\mathbf{x}_\tau)$ has faces

$$\left\{\sigma \subseteq \tau \,\middle|\, \frac{\mathbf{x}_\tau}{\mathbf{x}_\sigma} \in I_\Delta\right\} = \{\sigma \subseteq \tau \,|\, \sigma \in \Delta^\vee_\tau\}\,.$$

Therefore, $\Gamma(\mathbf{x}_\tau) = \Delta^\vee_\tau$. From Theorem 61.3 it follows that

$$b^S_{i,\mathbf{x}_\tau}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{i-1}\big(\Gamma(\mathbf{x}_\tau); k\big) = \dim \widetilde{\mathrm{H}}_{i-1}\big(\Delta^\vee_\tau; k\big)\,.$$

Now, we prove (2). By (1) and the Alexander Duality Theorem 62.6, we get

$$b^S_{i,\mathbf{x}_\tau}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{i-1}\big(\Delta^\vee_\tau;k\big) = \dim \widetilde{\mathrm{H}}_{|\tau|-i-2}\big(\Delta_\tau;k\big)\,.$$

□

Theorem 62.8. [Terai], [Bayer-Charalambous-Popescu]

$$\mathrm{pd}(S/I_{\Delta^\vee}) = \mathrm{reg}(I_\Delta)\,.$$

Proof.

$$\begin{aligned}
\mathrm{reg}(I_\Delta) &= \max\{\, j \mid b^S_{i,i+j}(I_\Delta) \neq 0\,\}\\
&= \max\{\, j \mid b^S_{i,\mathbf{x}_\tau}(I_\Delta) \neq 0 \text{ and } |\tau| = i+j\,\}\\
&= \max\{\, j \mid \widetilde{\mathrm{H}}_{|\tau|-i-2}\big(\Delta_\tau;k\big) \neq 0 \text{ and } |\tau| = i+j\,\} \quad \textit{by } 62.7(2)\\
&= \max\{\, j \mid \widetilde{\mathrm{H}}_{j-2}\big(\Delta_\tau;k\big) \neq 0\,\}\\
&= \max\{\, j \mid b^S_{j-1,\mathbf{x}_\tau}(I_{\Delta^\vee}) \neq 0\,\} \quad \textit{by Theorem } 62.7(1)\\
&= \mathrm{pd}(I_{\Delta^\vee}) + 1 = \mathrm{pd}(S/I_{\Delta^\vee})\,.
\end{aligned}$$

□

Corollary 62.9. (Eagon-Reiner) *The ideal I_Δ has a linear minimal free resolution if and only if $S/I_{\Delta^\vee}$ is Cohen-Macaulay.*

Proof. By 62.1, $\dim(S/I_{\Delta^\vee}) = \dim(\Delta^\vee) + 1$. Suppose that $S/I_{\Delta^\vee}$ is Cohen-Macaulay. Then,

$$\begin{aligned}
\mathrm{reg}(I_\Delta) &= \mathrm{pd}(S/I_{\Delta^\vee}) = n - \mathrm{depth}(S/I_{\Delta^\vee}) = n - \dim(S/I_{\Delta^\vee})\\
&= n - \dim(\Delta^\vee) - 1\\
&= \text{the minimal degree of a minimal monomial generator of } I_\Delta\,.
\end{aligned}$$

Hence, I_Δ has a linear minimal free resolution.

Suppose that I_Δ has a linear minimal free resolution. Then

$$\begin{aligned}
n - \mathrm{depth}(S/I_{\Delta^\vee}) &= \mathrm{pd}(S/I_{\Delta^\vee}) = \mathrm{reg}(I_\Delta)\\
&= \text{the minimal degree of a minimal monomial generator of } I_\Delta\\
&= n - \dim(\Delta^\vee) - 1\\
&= n - \dim(S/I_{\Delta^\vee})\,.
\end{aligned}$$

Hence, $\operatorname{depth}(S/I_{\Delta^\vee}) = \dim(S/I_{\Delta^\vee})$. Thus, the Stanley-Reisner ring $S/I_{\Delta^\vee}$ is Cohen-Macaulay. $\square$

Corollary 62.10. [Stanley] *If $S/I_{\Delta^\vee}$ is Cohen-Macaulay, then the complex $\Delta^\vee$ is pure (that is, all maximal faces of $\Delta^\vee$ have the same dimension).*

Proof. This follows from the first part of the proof of Corollary 62.9. Since $\operatorname{reg}(I_\Delta)$ is equal to the minimal degree of a minimal monomial generator of I_Δ, it follows that all minimal monomial generators of I_Δ have the same degree. $\square$

For $\sigma \subseteq \tau \in \Delta$ we define the ***closed interval***

$$[\sigma, \tau] = \{\mu \,|\, \sigma \subseteq \mu \subseteq \tau\}\,.$$

A ***partition*** of Δ, is a disjoint union

$$\Delta = \bigsqcup_{1 \le i \le s} [\sigma_i, \tau_i]\,,$$

where $\tau_1, \ldots, \tau_s$ are the facets of Δ. Thus, the closed intervals $[\sigma_i, \tau_i]$ are disjoint and cover Δ. We say that Δ is ***partitionable*** if it has a partition.

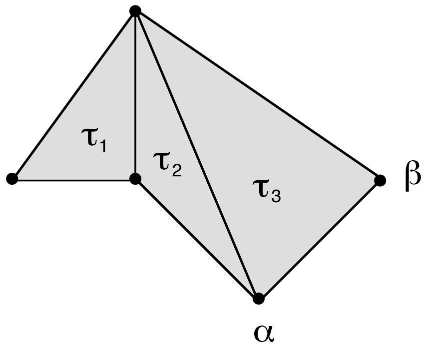

Figure 16.

Example 62.11. Consider the simplicial complex Δ in Figure 16 above. Let τ_1, τ_2, τ_3 be the three facets. Then Δ has the partition

$$\Delta = [\emptyset, \tau_1] \sqcup [\alpha, \tau_2] \sqcup [\beta, \tau_3],$$

where α and β are the vertices labeled in Figure 16.

A simplicial complex Δ is ***Cohen-Macaulay*** if its Stanley-Reisner ring is Cohen-Macaulay. An open conjecture, central in combinatorics, cf. [Stanley, Stanley 2], states that a Cohen-Macaulay simplicial complex is partitionable. By Corollary 62.9, we have the following equivalent conjecture.

Conjecture 62.12. [Stanley], [Stanley 2, Problem 6] *If $I_{\Delta^\vee}$ has a linear resolution, then Δ is partitionable.*

Exercise 62.13. *If $I_{\Delta^\vee}$ is squarefree Borel and has a linear resolution, then Δ is partitionable.*

In the next theorem we show how Alexander duality is related to the lcm-lattice. The ***proper part*** of a lattice P, with bottom element $\hat{0}$ and top element $\hat{1}$, is $P \setminus \{\hat{0}, \hat{1}\}$.

Theorem 62.14. [Gasharov-Peeva-Welker 2] *Let $L_{\Delta^\vee}$ be the lattice of all non-empty intersections of the facets of $\Delta^\vee$ ordered by reverse inclusion, and enlarged by an additional bottom element $\hat{0}$ and an additional top element $\hat{1}$. The lattices L_{I_Δ} and $L_{\Delta^\vee}$ are isomorphic. Furthermore, $\Delta^\vee$ is homotopy equivalent to the order complex of the proper part of L_{I_Δ}.*

Proof. Let $\tau_1, \dots, \tau_p$ be the facets of $\Delta^\vee$. For any $\emptyset \neq A \subseteq \{1, \dots, p\}$ we consider the bijective correspondence

$$\bigcap_{i \in A} \tau_i \quad \longleftrightarrow \quad \frac{x_1 \dots x_n}{\bigcap_{i \in A} \mathrm{supp}(\mathbf{x}_{\tau_i})} = \mathrm{lcm}\left(\frac{x_1 \dots x_n}{\mathbf{x}_{\tau_i}} \,\middle|\, i \in A \right).$$

The lattices $L_{\Delta^\vee}$ and L_{I_Δ} are isomorphic via the above correspondence.

In particular, for the minimal monomial generators of I_Δ we have

$$\tau_i \in \Delta^\vee \quad \longleftrightarrow \quad \frac{x_1 \dots x_n}{\mathbf{x}_{\tau_i}} \in I_\Delta.$$

By Corollary 36.13 and Corollary 36.9 it follows that $\Delta^\vee$ is homotopy equivalent to the order complex of the proper part of L_{I_Δ}. □

Recall by 36.17 that for $\tau \in \Delta$, the link of τ is

$$\mathrm{link}_\Delta(\tau) = \{\sigma \in \Delta \,|\, \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \emptyset\}\,.$$

Theorem 62.15. [Hochster] *Consider an open interval* $(1, m)$ *in the lcm-lattice* L_{I_Δ}. *Let* $\tau = \{x_1, \dots, x_n\} \setminus \mathrm{supp}(m)$ *be the complement of* $\mathrm{supp}(m)$. *The order complex* $O(1, m)$ *is homotopic to the link* $\mathrm{link}_{\Delta^\vee}(\tau)$. *In particular,*

$$\dim \widetilde{\mathrm{H}}_j(O(1, m); k) = \dim \widetilde{\mathrm{H}}_j(\mathrm{link}_{\Delta^\vee}(\tau); k) \quad \textit{for all } j\,.$$

For $i \geq 0$, *we have that*

$$b^S_{i,m}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{i-1}(\mathrm{link}_{\Delta^\vee}(\tau); k)\,.$$

The above formula for the Betti numbers is called the ***Hochster formula***.

Proof. As shown in the proof of Theorem 62.14, the open interval $(1, m)$ is isomorphic to the open interval $(\hat{0}, \tau)$ in the lattice $L_{\Delta^\vee}$ of all non-empty intersections of the facets of $\Delta^\vee$ ordered by reverse inclusion. By Corollary 36.13, the order complex of $(\hat{0}, \tau)$ is homotopic to the order complex of $(\tau, \hat{1})$ in the face lattice W of $\Delta^\vee$ (since W is ordered by inclusion, while $L_{\Delta^\vee}$ is ordered by reverse inclusion). Each element μ in the poset $(\tau, \hat{1})$ can be written as $\mu = \sigma_\mu \cup \tau \in \Delta$ so that $\sigma_\mu \cap \tau = \emptyset$. We get an isomorphism between $(\tau, \hat{1})$ and the face poset $F(\mathrm{link}_\Delta(\tau))$ which maps μ to σ_μ. Therefore, the order complex of $(\tau, \hat{1})$ is homotopic to the simplicial complex $\mathrm{link}_\Delta(\tau)$. The formula for the Betti numbers follows from Theorem 61.3:

$$b^S_{i,m}(I_\Delta) = \dim \widetilde{\mathrm{H}}_{i-1}(O(1, m); k) = \dim \widetilde{\mathrm{H}}_{i-1}(\mathrm{link}_{\Delta^\vee}(\tau); k)\,.$$

□

Example 62.16. Consider $N = (ac, ah, be, ce)$ in $A = k[a, b, c, e, h]$. Its Stanley-Reisner simplicial complex Δ and the Alexander dual complex $\Delta^\vee$ are shown in Figure 17. Note that the non-faces are not shaded, so $\{c, e, h\}$ is not a face in $\Delta^\vee$.

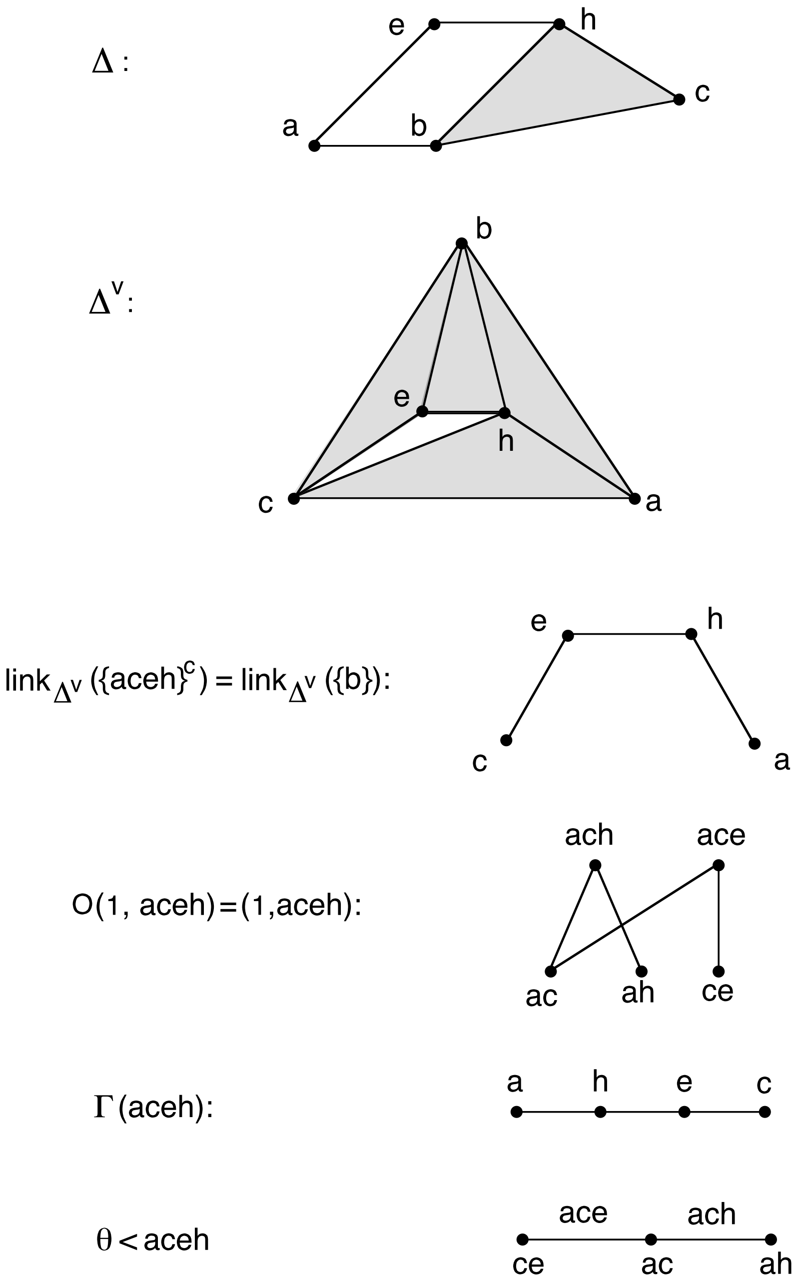
Δ :
e
h
c
a
b
Δ^v:
b
e
h
c
a
link_Δ^v ({aceh}^c) = link_Δ^v ({b}):
e
h
c
a
O(1, aceh) = (1,aceh):
ach
ace
ac
ah
ce
Γ (aceh):
a
h
e
c
θ < aceh
ace
ach
ce
ac
ah

Figure 17.

The complexes $\Theta_{<aceh}$, $\Gamma(aceh)$, $O(1, aceh)$, and $\mathrm{link}_{\Delta^\vee}(\{aceh\}^c)$ are shown in Figure 17 as well. By Theorem 61.3 and Theorem 62.15, the multigraded Betti numbers $b^A_{i,aceh}(N)$ can be computed using any of these simplicial complexes. We have that all the Betti numbers in multidegree $aceh$ vanish.

Corollary 62.17. [Reisner] *S/I_Δ is Cohen-Macaulay if and only if for each face $\tau \in \Delta$ we have $\widetilde{\mathrm{H}}_i(\mathrm{link}_\Delta(\tau); k) = 0$ for $i \neq \dim(\mathrm{link}_\Delta(\tau))$.*

Proof. By Corollary 62.9, S/I_Δ is Cohen-Macaulay if and only if the ideal $I_{\Delta^\vee}$ has a linear minimal free resolution, if and only if for every monomial m we have that $b^S_{i,m}(I_{\Delta^\vee}) = 0$ for $\deg(m) \neq p + i$, where p is the minimal degree of a minimal monomial generator of $I_{\Delta^\vee}$. By Theorem 62.15, this is equivalent to $\widetilde{\mathrm{H}}_i(\mathrm{link}_\Delta(\tau); k) = 0$ for $i \neq \dim(\mathrm{link}_\Delta(\tau))$, where $\tau = \{x_1, \ldots, x_n\} \setminus \mathrm{supp}(m)$. □

Define a Betti number $b_{i,m}$ to be ***i*-extremal** if $b_{i,m'} = 0$ for all monomials m' strictly divisible by m. The following result is proved in [Bayer-Charalambous-Popescu].

Theorem 62.18. *If $b^S_{i,m}(I_{\Delta^\vee})$ is i-extremal, then*

$$b^S_{i,m}(I_{\Delta^\vee}) \geq b^S_{\deg(m)-i-1,m}(I_\Delta).$$

Proposition 62.19. *Denote by* $\mathrm{mingens}(I_{\Delta^\vee})$ *the set of minimal monomial generators of $I_{\Delta^\vee}$. The irredundant primary decomposition of I_Δ is*

$$I_\Delta = \bigcap_{x_{j_1}\cdots x_{j_s} \in \mathrm{mingens}\,(I_{\Delta^\vee})} (x_{j_1}, \ldots, x_{j_s}).$$

Proof. The ideal I_Δ is radical, so it equals the intersection of its minimal prime divisors.The associated primes of S/I_Δ are its minimal prime divisors. An ideal P is an associated prime of S/I_Δ exactly when

$$P = \Big\{ (x_{i_1}, \ldots, x_{i_r}) \,|\, \{x_1, \ldots, x_n\} \setminus \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a facet of } \Delta \Big\}.$$

□

63 Quadratic monomial ideals

One might expect that the simplest minimal free monomial resolutions are those of the ideals generated by quadratic monomials, and that it might be nearly an exercise to describe them. However, these resolutions are so complicated that it is beyond reach to obtain a description of them; we do not even know how to express the regularity. In this section we will apply the mapping cone construction to determine which quadratic monomial ideals have 2-linear free resolutions.

It is easy to encode a set of squarefree quadratic monomials in a graph. Throughout the section, we consider a simple (that is, with no loops and no multiple edges) graph G on vertices $x_1, \dots, x_n$. The ***edge ideal*** I_G is

$$I_G = \big(x_i x_j \mid x_i x_j \text{ is an edge in } G\big)\,.$$

Exercise 63.1. *The polarization of any quadratic monomial ideal is an edge ideal.*

Thus, studying the minimal free resolutions of quadratic monomial ideals is equivalent to studying the minimal free resolutions of edge ideals. The following problems are open.

Problems 63.2. (folklore)

(1) *Express* $\operatorname{reg}(I_G)$ *in terms of properties of the graph* G.

(2) *Find upper* (*and lower*) *bounds on* $\operatorname{reg}(I_G)$ *in terms of properties of the graph* G.

(3) *Find upper* (*and lower*) *bounds on the Betti numbers of* I_G *in terms of properties of the graph* G.

For the next theorem we need a few definitions about graphs. The ***complement graph*** G^c of G is the graph on the same set of vertices, and with edges

$$\{x_i x_j \mid x_i x_j \text{ is not an edge in } G\}\,.$$

We say that a simple graph T contains a ***q-cycle*** $(x_{i_1} \dots x_{i_q})$ if $x_{i_q} x_{i_1} \in T$ and $x_{i_j} x_{i_{j+1}} \in T$ for all $1 \le j \le q-1$. A ***chord*** in the cycle is an edge between two non-consecutive vertices. A cycle is called ***minimal*** (or ***induced***) if it has no chords. A cycle with three vertices is called a triangle.

Theorem 63.3. [Fröberg] *The following properties are equivalent.*
(1) I_G *has a 2-linear minimal free resolution.*
(2) $\operatorname{reg}(S/I_G) = 1$.
(3) *Every minimal cycle in* G^c *is a triangle.*

Proof. (1) and (2) are equivalent.

We will show that (3) implies (2). Dirac's Theorem, cf. [Herzog-Hibi-Zheng] and [Horwitz], states that if every minimal cycle in G^c is a triangle then there exists an order of the vertices so that the following property holds: if $x_i x_j \in G$ and x_p is a vertex with $i, j < p$, then either $x_i x_p$, or $x_j x_p$, or both are edges in G.

For $p \geq 1$, denote by G_p the induced subgraph of G on the vertices $x_1, \ldots, x_p$. Our proof is by induction on the number of vertices p.

Let $p \geq 2$. Set

$$J = (x_p x_q \,|\, 1 \leq q < p,\ x_p x_q \in G)\,.$$

Consider the short exact sequence

$$(*)\ \ 0 \to J/(I_{G_{p-1}} \cap J) \to S/I_{G_{p-1}} \to S/(I_{G_{p-1}} + J) = S/I_{G_p} \to 0\,.$$

We will show that $I_{G_{p-1}} \cap J = x_p I_{G_{p-1}}$. Consider a monomial $x_p x_q x_i x_j$ such that $x_p x_q \in J$ and $x_i x_j \in G_{p-1}$. Since $i, j < p$, by Dirac's order of the variables, we have that either $x_i x_p$, or $x_j x_p$, or both are edges in G_p. Therefore, $x_p x_i x_j \in I_{G_{p-1}} \cap J$. It follows that the ideal $I_{G_{p-1}} \cap J$ is generated by the monomials $\{x_p x_i x_j \,|\, x_i x_j \in G_{p-1}\}$. Hence,

$$\begin{aligned} I_{G_{p-1}} \cap J &= x_p I_{G_{p-1}} \\ J &= x_p(x_q \,|\, 1 \leq q < p,\ x_p x_q \in G)\,. \end{aligned}$$

The minimal free resolution of the ideal $(x_q \,|\, 1 \leq q < p,\ x_p x_q \in G)$ is given by a Koszul complex. Therefore, we get

$$\begin{aligned} \operatorname{reg}(I_{G_{p-1}} \cap J) &= 1 + \operatorname{reg}(I_{G_{p-1}}) = 3 \\ \operatorname{reg}(J) &= 1 + \operatorname{reg}(x_q \,|\, 1 \leq q < p,\ x_p x_q \in G) = 2\,. \end{aligned}$$

Now, Corollary 18.6 applied to the short exact sequence

$$0 \to I_{G_{p-1}} \cap J \to J \to J/(I_{G_{p-1}} \cap J) \to 0$$

implies that $\operatorname{reg}\big(J/(I_{G_{p-1}} \cap J)\big) = 2$. By induction hypothesis, we have $\operatorname{reg}(S/I_{G_{p-1}}) = 1$. Therefore, Corollary 18.6 applied to the short exact sequence $(*)$ implies that $\operatorname{reg}(S/I_{G_p}) = 1$.

(1) implies (3) by the next exercise. □

Exercise 63.4. *If G^c contains a minimal cycle $(x_{i_1} \dots x_{i_q})$ with $q > 3$, then $b^S_{q-2, x_{i_1} \dots x_{i_q}}(S/I_G) \neq 0$.*

64 Infinite free monomial resolutions

Infinite free resolutions related to monomial ideals have been studied much less than finite ones. So far, the three main results in that area are Theorem 35.6 on the rate, Backelin's Theorem 64.2 on the rationality of the Poincaré series, and Berglund's Theorem 64.4 on computing the Betti numbers by simplicial complexes.

In this section, we study the multigraded minimal free resolution $\mathbf{G}$ of k over the quotient ring S/M. It is infinite (unless M is generated by variables) and starts with

$$\dots \to (S/M)^n \xrightarrow{(x_1 \quad x_2 \quad \dots \quad x_n)} S/M \to k \to 0\,.$$

Theorem 64.1. *The entries in the matrices of the differentials in* $\mathbf{G}$ *are scalar multiples of monomials.*

Proof. Since $\mathbf{G}$ is multigraded we have that each entry f in the matrices of the differentials in $\mathbf{G}$ is homogeneous. Let m be the multidegree of f. Since R_m is one dimensional, it follows that f is a scalar multiple of the unique monomial in multidegree m. □

Problems of rationality of Poincaré and Hilbert series were stated by several mathematicians: by Serre and Kaplansky for local noetherian rings, by Kostrikin and Shafarevich for nilpotent algebras, by

Govorov for associative graded algebras, by Serre and Moore for simply-connected complexes. It is of interest to find explicit formulas in some cases and to establish rational relations between various Poincaré and Hilbert series. The Serre-Kaplansky problem, "*Is the total Poincaré series of a finitely generated commutative local Noetherian ring rational?*", was one of the central questions in Commutative Algebra for many years. An example of irrational Poincaré series was first constructed in [Anick]. In contrast, the following result is proved in [Backelin].

Theorem 64.2. *The multigraded Poincaré series of k over S/M can be written as*

$$\mathrm{P}_k^{S/M}(t, x_1, \dots, x_n) = \frac{(1+tx_1)\cdots(1+tx_n)}{1+F(t, x_1, \dots, x_n)}\,,$$

where the denominator $1+F(t, x_1, \dots, x_n)$ is a polynomial. The degree of the polynomial $F(t, x_1, \dots, x_n)$ in t is bounded above by the degree of the monomial $\operatorname{lcm}(m_1, \dots, m_r)$. The monomials in $x_1, \dots, x_n$ appearing in $F(t, x_1, \dots, x_n)$ (that is, the monomial coefficients of the powers of t) divide $\operatorname{lcm}(m_1, \dots, m_r)$.

Open-Ended Problem 64.3. (folklore) *Understand the denominator $1 + F(t, \mathbf{x})$.*

Based on a substantial amount of computational evidence Charalambous and Reeves conjectured the form of the terms of the polynomial denominator $1 + F(t, \mathbf{x})$ of the Poincaré series $\mathrm{P}_k^{S/I}(t)$. Their conjecture is proved by Berglund, who provides a beautiful construction on how to compute the denominator using simplicial complexes as described in Theorem 64.4. For the formulation of that theorem, we need some terminology. Let G be the graph on vertices $m_1, \dots, m_r$ and with edges $\{\, m_i m_j \mid \gcd(m_i, m_j) \neq 1 \,\}$; we call it the ***gcd-graph*** of the ideal M. If $\mathcal{M}$ is a nonempty subset of $\{m_1, \dots, m_r\}$, then we denote by $G_{\mathcal{M}}$ the induced subgraph of G on the vertices in $\mathcal{M}$. Let $c_{\mathcal{M}}$ be the number of connected components of $G_{\mathcal{M}}$, and denote by $G_{\mathcal{M}}(1), \dots, G_{\mathcal{M}}(c_{\mathcal{M}})$ the connected components; we say that $\mathcal{M}$ is

connected if $G_{\mathcal{M}}$ is. Set $m_{\mathcal{M}} = \operatorname{lcm}(\, m_i \,|\, m_i \in \mathcal{M}\,\}$. We say that $\mathcal{M}$ is ***saturated*** if for every m_i and every connected subset $\mathcal{N} \subseteq \mathcal{M}$ we have that m_i divides $m_{\mathcal{N}}$ implies $m_i \in \mathcal{M}$.

Theorem 64.4. [Berglund] *Suppose that M is generated by monomials of degree ≥ 2. For a subset $\mathcal{M}$ of $\{m_1, \ldots, m_r\}$ define the simplicial complex $\Delta_{\mathcal{M}}$ to have vertices the elements in $\mathcal{M}$ and faces*

$$\{\, \mathcal{K} \subseteq \mathcal{M} \,|\, m_{\mathcal{K}} \neq m_{\mathcal{M}} \text{ or } G_{\mathcal{K}} \cap G_{\mathcal{M}}(i) \ \text{ is disconnected for some } i\,\}\,.$$

(1) *The multigraded Poincarè series $P_k^{S/M}(t, u_1, \ldots, u_n)$ of k over S/M is*

$$\frac{\prod_{i=1}^{n}\,(1+tu_i)}{1+\sum_{\text{ saturated } \mathcal{M}\subseteq\{m_1,\ldots,m_r\}}\Big(m_{\mathcal{M}}(-t)^{c_{\mathcal{M}}+2}\sum_i \widetilde{\mathrm{H}}_i(\Delta_{\mathcal{M}};k)\,t^i\Big)}\,.$$

The Poincarè series $P_k^{S/M}(t)$ of k over S/M is

$$\frac{(1+t)^n}{1+\sum_{\text{saturated } \mathcal{M}\subseteq\{m_1,\ldots,m_r\}}\sum_i (-1)^{c_{\mathcal{M}}}\widetilde{\mathrm{H}}_i(\Delta_{\mathcal{M}};k)\,t^{c_{\mathcal{M}}+2+i}}\,.$$

(2) *Let $\mathcal{P}$ be the poset of saturated subsets of $\{m_1, \ldots, m_r\}$ ordered by inclusion. If $\mathcal{M}$ is a saturated subset of $\{m_1, \ldots, m_r\}$, then*

$$\widetilde{\mathrm{H}}_*(\Delta_{\mathcal{M}};k) = \widetilde{\mathrm{H}}_*(\,(\emptyset,\mathcal{M})_{\mathcal{P}};k)\,,$$

where $(\emptyset,\mathcal{M})_{\mathcal{P}}$ is the open interval below $\mathcal{M}$ in $\mathcal{P}$.

Using simplicial complexes in order to compute the Betti numbers of finite monomial minimal free resolutions has a long and fruitful tradition. Very little is known about infinite resolutions. Theorem 64.4 shows that the Poincarè series of k over S/M can be computed using simplicial complexes.

Corollary 64.5. (Avramov) *Let M and M' be monomial ideals in the polynomial rings S and S' respectively. If there exists an isomorphism of the lcm-lattices of M and M' which induces an isomorphism of the gcd-graphs, then $\mathrm{P}_k^{S/M}(t)$ and $\mathrm{P}_k^{S'/M'}(t)$ have the same denominator (when written as in Theorem* 64.2).

Open-Ended Problem 64.6. (folklore) *Obtain information on the* *real roots of the polynomial denominator* $1 + F(t, (1, \ldots, 1))$ *of the Poincaré series* $\mathrm{P}_k^{S/M}(t)$.

The following construction provides the minimal free resolution of k over S/M explicitly in the case when M is a Borel ideal.

Construction 64.7. [Peeva] Let M be a Borel monomial ideal. The minimal free resolution of k over S/M can be described as follows.

Let $\mathbf{K}$ be the Koszul complex that resolves k over S. Consider $\mathbf{K}$ as the exterior algebra on basis $e_1, e_2, \ldots, e_n$ with differential $d(e_i) = x_i$. Denote by E_{p+2} the k-space with basis

$$\left\{ (\, m_i; j_1, \ldots, j_p) \mid 1 \leq j_1 < \ldots < j_p < \max(m_i),\ 1 \leq i \leq r \right\}.$$

Set $E = E_2 \oplus E_3 \oplus \ldots \oplus E_{n+1}$. Define $\mathbf{G} = S/M \otimes \mathbf{K} \otimes T(E)$, where

$$T(E) = k \,\oplus\, E \,\oplus\, (\, E \,\otimes\, E\,) \,\oplus \ldots$$

is the tensor algebra of E. A basis element in $\mathbf{G}$ has the form

$$t \otimes (z_1; i_1, \ldots, i_{p_1}) \otimes (z_2; l_1, \ldots, l_{p_2}) \otimes \,\ldots\, \otimes (z_s; j_1, \ldots, j_{p_s})\,,$$

where $t \in S/M \otimes \mathbf{K}$ and $z_1, \ldots, z_s$ are among the minimal monomial generators of M. Define a differential ∂ on the basis elements in $\mathbf{G}$ as follows:

$$\begin{aligned} &\partial\Big(t \otimes (z_1; i_1, \ldots, i_{p_1}) \otimes (z_2; l_1, \ldots, l_{p_2}) \otimes \,\ldots\, \otimes (z_s; j_1, \ldots, j_{p_s})\Big) \\ &= d(t) \otimes (z_1; i_1, \ldots, i_{p_1}) \otimes (z_2; l_1, \ldots, l_{p_2}) \otimes \,\ldots\, \otimes (z_s; j_1, \ldots, j_{p_s}) \\ &\quad + (-1)^{\deg(t)}\, t\ \frac{z_1}{x_{\max(z_1)}} e_{i_1} \wedge \ldots \wedge e_{i_{p_1}} \wedge e_{\max(z_1)} \otimes (z_2; l_1, \ldots, l_{p_2}) \otimes \\ &\qquad \ldots \otimes (z_s; j_1, \ldots, j_{p_s})\,, \end{aligned}$$

where $d(t)$ is the differential in $S/M \otimes \mathbf{K}$ if $t \notin S/M \otimes K_0$, and we set $d(t) = 0$ in case $t \in S/M \otimes K_0$. Extend the differential by linearity. It is proved in [Peeva] that $\mathbf{G}$ is the minimal free resolution of k over S/M. The Poincaré series of the resolution is

$$\mathrm{P}_k^{S/M}(t) = \frac{(1+t)^n}{1 - t^2 \sum_{1 \leq i \leq r} (1+t)^{\max(m_i)-1}}\ .$$