# Chapter 9
# Optimality Aspect

This chapter considers the optimality aspect in distributed multi-agent coordination. We study optimal linear coordination algorithms for multi-agent systems with single-integrator dynamics in both continuous-time and discrete-time settings from a linear quadratic regulator perspective. We propose two global cost functions, namely, interaction-free and interaction-related cost functions. With the interaction-free cost function, we derive the optimal state feedback gain matrix in both continuous-time and discrete-time settings. It is shown that the optimal gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. In addition, we show that any symmetric Laplacian matrix is inverse optimal with respect to a properly chosen cost function. With the interaction-related cost function, we derive the optimal scaling factor for a prespecified symmetric Laplacian matrix associated with an undirected interaction graph in both continuous-time and discrete-time settings. Illustrative examples are given as a proof of concept.

## 9.1 Problem Statement

Among various studies of distributed linear coordination algorithms, it is natural to ask these questions: Is there an optimal linear coordination algorithm under a given cost function? How to find the optimal linear coordination algorithm? The purpose of this chapter is to study the optimal linear coordination algorithms for agents with single-integrator dynamics from a linear quadratic regulator (LQR) perspective. Instead of studying locally optimal algorithms, we focus on globally optimal algorithms.

The contributions of this chapter are threefold. First, we mathematically prove the conditions under which the square root of a nonsymmetric Laplacian matrix is still a nonsymmetric Laplacian matrix. Second, we explicitly derive the optimal state feedback gain matrix under a given global cost function from an LQR perspective and show that the optimal state feedback gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. Third, we derive the

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optimal scaling factor for a prespecified symmetric Laplacian matrix associated with an undirected interaction graph. Although it might be intuitively true that a global optimization problem in the context of multi-agent coordination normally requires that each agent have full knowledge of all other agents, it is nontrivial to verify this fact from a theoretical perspective. In particular, for the linear coordination algorithms, it is not clear why the optimal state feedback gain matrix derived from the standard LQR solution is a nonsymmetric Laplacian matrix and why the nonsymmetric Laplacian matrix corresponds to a complete directed graph. In other words, our focus here is not to repeat the standard LQR procedure but to provide a theoretical explanation. We first propose two global cost functions, namely, interaction-free and interaction-related cost functions, in both continuous-time and discrete-time settings. With the interaction-free cost function, we derive the optimal state feedback gain matrix in both continuous-time and discrete-time settings. It is shown that the optimal state feedback gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. In addition, we show that any symmetric Laplacian matrix is inverse optimal with respect to a properly chosen cost function. With the interaction-related cost function, we derive the optimal scaling factor for a prespecified symmetric Laplacian matrix associated with an undirected interaction graph in both continuous-time and discrete-time settings.

In the continuous-time setting, consider $n$ agents with single-integrator dynamics given by (3.1). In the discrete-time setting, consider $n$ agents with discretized dynamics of (3.1) given by (8.3). Define $\Delta_{ij} = \delta_i - \delta_j$, where $\delta_i \in \mathbb{R}^m$ is constant. Here $\Delta_{ij}$ denotes the desired relative position deviation between agents $i$ and $j$. In the continuous-time setting, coordination is achieved for the $n$ agents if for all $r_i(0)$ and $i, j = 1, \ldots, n$, $r_i(t) - r_j(t) \to \Delta_{ij}$ as $t \to \infty$. In the discrete-time setting, coordination is achieved for the $n$ agents if for all $r_i[0]$ and $i, j = 1, \ldots, n$, $r_i[k] - r_j[k] \to \Delta_{ij}$ as $k \to \infty$. In the remainder of the chapter, we assume that the agents are in a one-dimensional space for simplicity. However, all results hereafter are still valid for any high-dimensional space by use of the properties of the Kronecker product.

In the continuous-time setting, similar to the cost function used in optimal control problems for systems with linear differential equations, we propose the following two coordination cost functions for (3.1) as

$$J_{fc} = \int_0^\infty \left\{ \sum_{i=1}^n \sum_{j=1}^{i-1} c_{ij} \big[ r_i(t) - r_j(t) - \Delta_{ij} \big]^2 + \sum_{i=1}^n \vartheta_i \big[ u_i(t) \big]^2 \right\} dt, \tag{9.1}$$

where $c_{ij} \geq 0$ and $\vartheta_i > 0$, and

$$J_{rc} = \int_0^\infty \left\{ \sum_{i=1}^n \sum_{j=1}^{i-1} a_{ij} \big[ r_i(t) - r_j(t) - \Delta_{ij} \big]^2 + \sum_{i=1}^n \big[ u_i(t) \big]^2 \right\} dt, \tag{9.2}$$

where $a_{ij}$ is the $(i, j)$th entry of the adjacency matrix $\mathscr{A}$ associated with the graph $\mathscr{G} \triangleq (\mathscr{V}, \mathscr{E})$ characterizing the interaction among the $n$ agents. In (9.1), both

$c_{ij} \geq 0$ and $\vartheta_i > 0$ can be chosen freely. Therefore, $J_{fc}$ is called the *interaction-free cost function*. In contrast, (9.2) depends on the adjacency matrix $\mathscr{A}$ and hence the graph $\mathscr{G}$. Therefore, $J_{rc}$ is called the *interaction-related cost function*. The motivation behind (9.1) and (9.2) is to weigh both the coordination errors $r_i(t) - r_j(t) - \Delta_{ij}$ and the control effort $u_i$. The corresponding optimization problems can be written as

$$\min_{u_i(t)} \; J_{fc}, \text{ subject to (3.1)} \tag{9.3}$$

$$\min_{\beta} \; J_{rc}, \text{ subject to (3.1) and } u_i(t) = -\sum_{j=1}^{n} \beta a_{ij} \big[r_i(t) - r_j(t) - \Delta_{ij}\big]. \tag{9.4}$$

In the discrete-time setting, we propose the following interaction-free and interaction-related cost functions for (8.3) as

$$J_{fd} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{i-1} c_{ij} \big\{r_i[k] - r_j[k] - \Delta_{ij}\big\}^2 + \sum_{k=0}^{\infty} \sum_{i=1}^{n} \vartheta_i \big(u_i[k]\big)^2, \tag{9.5}$$

where $c_{ij} \geq 0$ and $\vartheta_i > 0$, and

$$J_{rd} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{ij} \big\{r_i[k] - r_j[k] - \Delta_{ij}\big\}^2 + \sum_{k=0}^{\infty} \sum_{i=1}^{n} \big(u_i[k]\big)^2, \tag{9.6}$$

where $a_{ij}$ is defined as in (9.4). The corresponding optimization problems can be written as

$$\min_{u_i[k]} \; J_{fd} \text{ subject to (8.3)} \tag{9.7}$$

$$\min_{\beta} \; J_{rd} \text{ subject to (8.3) and } u_i[k] = -\sum_{j=1}^{n} \beta a_{ij} \big(r_i[k] - r_j[k] - \Delta_{ij}\big). \tag{9.8}$$

## 9.2 Optimal Linear Coordination Algorithms in a Continuous-time Setting from a Linear Quadratic Regulator Perspective

In this section, we derive the optimal linear coordination algorithms in a continuous-time setting from an LQR perspective. We first derive the optimal state feedback gain matrix using the continuous-time interaction-free cost function (9.1). The optimal gain matrix is later shown to be a nonsymmetric Laplacian matrix corresponding to a complete directed graph. We then find the optimal scaling factor for a prespecified symmetric Laplacian matrix associated with an undirected interaction graph using the continuous-time interaction-related cost function (9.2). Finally, illustrative examples are provided.

### 9.2.1 Optimal State Feedback Gain Matrix Using the Interaction-free Cost Function

Note that (9.3) can be written as

$$\min_{u(t)} \underbrace{\int_0^\infty \left[\tilde{r}^T(t)Q\tilde{r}(t) + u^T(t)\Theta u(t)\right] dt}_{J_{fc}} \tag{9.9}$$

$$\text{subject to:} \quad \dot{r}(t) = u(t), \tag{9.10}$$

where $\tilde{r}(t) \triangleq [\tilde{r}_1(t), \ldots, \tilde{r}_n(t)]^T$ with $\tilde{r}_i(t) \triangleq r_i(t) - \delta_i$, $r(t) \triangleq [r_1(t), \ldots, r_n(t)]^T$, $u(t) \triangleq [u_1(t), \ldots, u_n(t)]^T$, $Q \in \mathbb{R}^{n\times n}$ is symmetric with the $(i, j)$th entry and hence the $(j, i)$th entry given by $-c_{ij}$ for $i > j$ and the $(i, i)$th entry given by $\sum_{j=1}^{i-1} c_{ij} + \sum_{j=i+1}^{n} c_{ji}$, and $\Theta \in \mathbb{R}^{n\times n}$ is the positive-definite diagonal matrix with $\vartheta_i$ being the $i$th diagonal entry. It can be noted that $Q$ is a symmetric Laplacian matrix. Therefore, $Q$ is symmetric positive semidefinite. Before moving on, we need the following notations and lemmas.

According to Lemma 1.14, if the characteristic polynomial of an M-matrix $B \in \mathbb{R}^{n\times n}$ has at most a simple zero root, then $B$ has exactly one M-matrix as its square root. In this case, we use $\sqrt{B}$ hereafter to represent the unique M-matrix that is the square root of $B$.

**Lemma 9.1.** *Let $Q$ and $\Theta$ be defined in (9.9). Suppose that $Q$ has a simple zero eigenvalue. Then $\Theta^{-1}Q$ is a nonsymmetric Laplacian matrix (and hence an M-matrix) with a simple zero eigenvalue and $\sqrt{\Theta^{-1}Q}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue.*

*Proof:* We first note that $\Theta^{-1}Q$ is a nonsymmetric Laplacian matrix because $Q$ is a symmetric Laplacian matrix, $\Theta$ is a positive-definite diagonal matrix, and $\Theta^{-1}Q\mathbf{1}_n = \Theta^{-1}\mathbf{0}_n = \mathbf{0}_n$. It thus follows from Lemma 1.15 that $\Theta^{-1}Q$ is also an M-matrix. Because $Q$ is a symmetric Laplacian matrix with a simple zero eigenvalue, it follows from Lemma 1.1 that the undirected graph associated with $Q$ is connected, which implies that the directed graph associated with $\Theta^{-1}Q$ is strongly connected. It thus follows from Lemma 1.1 that the nonsymmetric Laplacian matrix $\Theta^{-1}Q$ also has a simple zero eigenvalue. Therefore, $\sqrt{\Theta^{-1}Q}$ is the unique M-matrix that is the square root of $\Theta^{-1}Q$. We next show that $\sqrt{\Theta^{-1}Q}$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$. Let the $i$th eigenvalue of $\sqrt{\Theta^{-1}Q}$ be $\lambda_i$ with an associated eigenvector $\nu_i$. It follows that the $i$th eigenvalue of $\Theta^{-1}Q$ is $\lambda_i^2$ with an associated eigenvector $\nu_i$. Because $\Theta^{-1}Q$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$, it follows that $\sqrt{\Theta^{-1}Q}$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$. Therefore, it follows from Lemma 1.15 that $\sqrt{\Theta^{-1}Q}$ is a nonsymmetric Laplacian matrix. Combining the above arguments completes the proof. ■

We next show that the nonsymmetric Laplacian matrix $\sqrt{\Theta^{-1}Q}$ corresponds to a complete directed graph.

**Lemma 9.2.** *Let $Q$ and $\Theta$ be defined in (9.9). Suppose that $Q$ has a simple zero eigenvalue. Then the nonsymmetric Laplacian matrix $\sqrt{\Theta^{-1}Q}$ corresponds to a complete directed graph.*

*Proof:* Note from Lemma 9.1 that $\sqrt{\Theta^{-1}Q}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue. We show that each entry of $\sqrt{\Theta^{-1}Q}$ is nonzero, which implies that $\sqrt{\Theta^{-1}Q}$ corresponds to a complete directed graph. Before moving on, we let $q_{ij}$ denote the $(i,j)$th entry of $Q$. We also define $W \stackrel{\triangle}{=} \sqrt{\Theta^{-1}Q}$ and denote $w_{ij}$, $w_{i,:}$, and $w_{:,i}$ as, respectively, the $(i,j)$th entry, the $i$th row, and the $i$th column of $W$.

First, we will show that $w_{ij} \neq 0$ if $q_{ij} \neq 0$. We show this statement by contradiction. Assume that $w_{ij} = 0$. Because $\Theta^{-1}Q = W^2$, it follows that $\frac{q_{ij}}{\vartheta_i} = w_{i,:}w_{:,j}$. When $i = j$, it follows from $w_{ii} = 0$ that $w_{i,:} = \mathbf{0}_n^T$ because $W$ is a nonsymmetric Laplacian matrix, which then implies that $\frac{q_{ij}}{\vartheta_i} = w_{i,:}w_{:,j} = 0$. This contradicts the assumption that $q_{ij} \neq 0$. Because $W$ is a nonsymmetric Laplacian matrix, it follows that $w_{ik} \leq 0,\ \forall i \neq k$. When $i \neq j$, because it is assumed that $w_{ij} = 0$, it follows that $\frac{q_{ij}}{\vartheta_i} = w_{i,:}w_{:,j} = \sum_{k=1,k\neq i,k\neq j}^{n} w_{ik}w_{kj} \geq 0$. Because $Q$ is a symmetric Laplacian matrix, it follows that $q_{ij} \leq 0, \forall i \neq j$. Therefore, $\frac{q_{ij}}{\vartheta_i} \geq 0, \forall i \neq j$, implies that $q_{ij} = 0$, which also contradicts the assumption that $q_{ij} \neq 0$.

Second, we will show that $w_{ij} \neq 0$ even if $q_{ij} = 0$. We also show this statement by contradiction. Assume that $w_{ij} = 0$. To ensure that $q_{ij} = 0$, it follows from $\frac{q_{ij}}{\vartheta_i} = w_{i,:}w_{:,j} = \sum_{k=1,k\neq i,k\neq j}^{n} w_{ik}w_{kj}$ that $w_{ik}w_{kj} = 0, \forall k \neq i, \forall k \neq j, k = 1,\ldots,n$. Denote $\hat{k}_1$ as the node set such that $w_{im} \neq 0$ for each $m \in \hat{k}_1$. Then we have $w_{mj} = 0$ for each $m \in \hat{k}_1$ because $w_{ik}w_{kj} = 0$. Similarly, denote $\bar{k}_1$ as the node set such that $w_{mj} \neq 0$ for each $m \in \bar{k}_1$. Then we have $w_{im} = 0$ for each $m \in \bar{k}_1$ because $w_{ik}w_{kj} = 0$. From the discussion in the previous paragraph, when $w_{mj} = 0$, we have $q_{mj} = 0$, which implies that $w_{mp}w_{pj} = 0, \forall p \neq m, \forall p \neq j, p = 1,\ldots,n$. By following a similar analysis, we can consequently define $\hat{k}_i$ and $\bar{k}_i, i = 2,\ldots,\kappa$, where $\hat{k}_i \cap \hat{k}_j = \varnothing, \bar{k}_i \cap \bar{k}_j = \varnothing, \forall j < i$. Because both $Q$ and $W$ have a simple zero eigenvalue, it follows from Lemma 1.1 that the undirected graph associated with $Q$ is connected and the directed graph associated with $W$ has a directed spanning tree. It thus follows that $\kappa \leq n$. Therefore, each entry of $w_{:,j}$ is equal to zero by following the previous analysis for at most $n$ times. This implies that $q_{ij} = 0, \forall i \neq j$, because $\frac{q_{ij}}{\vartheta_i} = w_{i,:}w_{:,j}$. Considering the fact that $Q$ is a symmetric Laplacian matrix, it follows that $q_{ii} = 0$, which also contradicts the fact that the undirected graph associated with $Q$ is connected. ■

The main result for the optimal control problem (9.9) is given in the following theorem.

**Theorem 9.1.** *In the optimal control problem* (9.9)*, suppose that* $Q$ *has a simple zero eigenvalue. The optimal coordination algorithm is*

$$u(t) = -\sqrt{\Theta^{-1}Q}\tilde{r}(t). \tag{9.11}$$

*The matrix* $\sqrt{\Theta^{-1}Q}$ *is a nonsymmetric Laplacian matrix with a simple zero eigenvalue corresponding to a complete directed graph. Using* (9.11) *for* (9.10)*, coordination is achieved.*

*Proof:* Note that $\dot{r} = u(t)$ is equivalent to $\dot{\tilde{r}} = u(t)$. Consider the following standard LQR problem

$$\min_{u(t)} J_{fc} \quad \text{subject to:} \quad \dot{\tilde{r}}(t) = A\tilde{r}(t) + Bu(t), \tag{9.12}$$

where $J_{fc}$ is given in (9.9), $A = 0_{n\times n}$, and $B = I_n$. It can be noted that $(A, B)$ is controllable, which implies that there exists a unique positive-semidefinite matrix $P \in \mathbb{R}^{n\times n}$ satisfying the continuous-time algebraic Riccati equation

$$A^T P + PA - PB\Theta^{-1}B^T P + Q = 0_{n\times n}. \tag{9.13}$$

It follows from (9.13) that $P\Theta^{-1}P = Q$ by noting that $A = 0_{n\times n}$ and $B = I_n$, which implies that $\Theta^{-1}P\Theta^{-1}P = \Theta^{-1}Q$. It then follows from Lemma 9.1 that $\Theta^{-1}P = \sqrt{\Theta^{-1}Q}$ is also a nonsymmetric Laplacian matrix with a simple zero eigenvalue when $Q$ has a simple zero eigenvalue. Therefore, the optimal control is $u(t) = -\Theta^{-1}B^T P\tilde{r}(t) = -\sqrt{\Theta^{-1}Q}\tilde{r}(t)$. It also follows from Lemma 9.2 that $\sqrt{\Theta^{-1}Q}$ corresponds to a complete directed graph. Note that using (9.11) for (9.10), the closed-loop system becomes $\dot{\tilde{r}}(t) = \dot{r}(t) = -\sqrt{\Theta^{-1}Q}\tilde{r}(t)$. Because $\sqrt{\Theta^{-1}Q}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue, it follows from Lemma 1.3 that $\tilde{r}_i(t) - \tilde{r}_j(t) \to 0$ as $t \to \infty$, which implies that $r_i(t) - r_j(t) \to \Delta_{ij}$ as $t \to \infty$. ■

**Remark 9.2** Note that $Q$ is a symmetric Laplacian matrix. It follows from Lemma 1.1 that the assumption that $Q$ has a simple zero eigenvalue is equivalent to the assumption that the undirected graph corresponding to $Q$ is connected.

**Remark 9.3** In fact, the assumption that the symmetric Laplacian matrix $Q$ has a simple zero eigenvalue is also necessary to ensure coordination. If otherwise, the undirected graph corresponding to $Q$ is not connected. It thus follows that $\sum_{i=1}^{n}\sum_{j=1}^{i-1} c_{ij}[r_i(t) - r_j(t) - \Delta_{ij}]^2$ in (9.1) can be written as a sum of at least two independent terms, where each term involves an independent subset of the agents. As a result, the optimal control problem (9.3) and hence (9.9) can be decoupled into at least two independent optimal control problems. By solving the independent optimal control problems, coordination is only guaranteed for each independent subset of the agents but not for all agents.

**Remark 9.4** From Theorem 9.1, it can be noted that the graph corresponding to $\sqrt{\Theta^{-1}Q}$ is in general different from that corresponding to $Q$. Note that $\sqrt{\Theta^{-1}Q}$

is not necessarily symmetric in general. When $\Theta$ is a diagonal matrix with identical diagonal entries (i.e., $\Theta = cI_n$, where $c > 0$), $\sqrt{\Theta^{-1}Q}$ is symmetric.

We next show that any symmetric Laplacian matrix $\mathscr{L}$ with a simple zero eigenvalue is inverse optimal with respect to some given cost function.

**Theorem 9.5.** *Any symmetric Laplacian matrix $\mathscr{L} \in \mathbb{R}^{n\times n}$ with a simple zero eigenvalue is the optimal state feedback gain matrix under the cost function $J = \int_0^\infty [\tilde{r}^T(t)\mathscr{L}^2\tilde{r}(t) + u^T(t)u(t)]\,dt$.*

*Proof:* By letting $Q = \mathscr{L}^2$ and $\Theta = I_n$, it follows directly from the proof of Theorem 9.1 that $\mathscr{L}$ is the optimal state feedback gain matrix. ■

## 9.2.2 Optimal Scaling Factor Using the Interaction-related Cost Function

Suppose that the graph $\mathscr{G}$ is undirected. With the interaction-related cost function (9.2), the optimal control problem (9.4) can be written as

$$\begin{aligned} &\min_\beta \underbrace{\int_0^\infty \left[\tilde{r}^T(t)\mathscr{L}\tilde{r}(t) + u^T(t)u(t)\right] dt}_{J_{rc}} \\ &\text{subject to:} \quad \dot{r}(t) = u(t),\ u(t) = -\beta\mathscr{L}\tilde{r}(t), \end{aligned} \tag{9.14}$$

where $\mathscr{L}$ is the prespecified symmetric Laplacian matrix associated with the adjacency matrix $\mathscr{A}$ and hence the undirected graph $\mathscr{G}$, and $\beta$ is the scaling factor.

**Theorem 9.6.** *In the optimal control problem (9.14), suppose that $\mathscr{L}$ has a simple zero eigenvalue. Then the optimal $\beta$, denoted by $\beta_{\text{opt}}$, is $\sqrt{\frac{\tilde{r}^T(0)\tilde{r}(0) - \frac{1}{n}[\mathbf{1}_n^T\tilde{r}(0)]^T[\mathbf{1}_n^T\tilde{r}(0)]}{\tilde{r}^T(0)\mathscr{L}\tilde{r}(0)}}$.*[1]

*Proof:* Note that $u(t) = -\beta\mathscr{L}\tilde{r}(t)$. It follows that $\dot{\tilde{r}}(t) = \dot{r}(t) = -\beta\mathscr{L}\tilde{r}(t)$. It thus follows that $\tilde{r}(t) = e^{-\beta\mathscr{L}t}\tilde{r}(0)$ and hence $u(t) = -\beta\mathscr{L}e^{-\beta\mathscr{L}t}\tilde{r}(0)$. The cost function $J_{rc}$ can then be written as

$$J_{rc} = \int_0^\infty \tilde{r}^T(0)\left[e^{-\beta\mathscr{L}t}\mathscr{L}e^{-\beta\mathscr{L}t} + \beta^2 e^{-\beta\mathscr{L}t}\mathscr{L}^2 e^{-\beta\mathscr{L}t}\right]\tilde{r}(0)\,dt.$$

Taking the derivative of $J_{rc}$ with respect to $\beta$ gives

$$\begin{aligned} \frac{dJ_{rc}}{d\beta} = \int_0^\infty \tilde{r}^T(0)\big[ &- 2\mathscr{L}te^{-\beta\mathscr{L}t}\mathscr{L}e^{-\beta\mathscr{L}t} + 2\beta e^{-\beta\mathscr{L}t}\mathscr{L}^2 e^{-\beta\mathscr{L}t} \\ &- 2\beta^2\mathscr{L}te^{-\beta\mathscr{L}t}\mathscr{L}^2 e^{-\beta\mathscr{L}t}\big]\tilde{r}(0)\,dt. \end{aligned}$$

[1] Note that coordination is obviously achieved when $\beta = \beta_{\text{opt}}$.

Setting $\frac{dJ_{rc}}{d\beta} = 0$ gives

$$\begin{aligned}
&\beta^2 \tilde{r}^T(0)\left[\int_0^\infty \mathscr{L} t e^{-\beta \mathscr{L} t} \mathscr{L}^2 e^{-\beta \mathscr{L} t}\, dt\right]\tilde{r}(0) \\
&- \beta \tilde{r}^T(0)\left[\int_0^\infty e^{-\beta \mathscr{L} t} \mathscr{L}^2 e^{-\beta \mathscr{L} t}\, dt\right]\tilde{r}(0) \\
&+ \tilde{r}^T(0)\left[\int_0^\infty \mathscr{L} t e^{-\beta \mathscr{L} t} \mathscr{L} e^{-\beta \mathscr{L} t}\, dt\right]\tilde{r}(0) = 0,
\end{aligned} \tag{9.15}$$

where we have used the fact that $\mathscr{L}$ and $e^{-\beta \mathscr{L} t}$ commute. Because $\mathscr{L}$ is symmetric, $\mathscr{L}$ can be diagonalized as

$$\mathscr{L} = M \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\Lambda} M^T, \tag{9.16}$$

where $M$ is an orthogonal matrix, and $\lambda_i$ is the $i$th eigenvalue of $\mathscr{L}$. Note that $\mathscr{L}$ has a simple zero eigenvalue. Without loss of generality, we let $\lambda_1 = 0$ and hence $\lambda_i > 0$, $i = 2, \ldots, n$ (see Lemma 1.1). Note that the columns of $M$ can be chosen as the normalized right eigenvectors of $\mathscr{L}$. Also note that $\mathbf{1}_n$ is a right eigenvector of $\mathscr{L}$ associated with the zero eigenvalue. Therefore, we let the first column of $M$ be $\frac{\mathbf{1}_n}{\sqrt{n}}$. Note that

$$\begin{aligned}
&\int_0^\infty \mathscr{L} t e^{-\beta \mathscr{L} t} \mathscr{L}^2 e^{-\beta \mathscr{L} t}\, dt \\
&= \int_0^\infty M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{-2\beta\lambda_2 t}\lambda_2^3 t & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_n t}\lambda_n^3 t \end{bmatrix} M^T\, dt \\
&= \frac{1}{4\beta^2} M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} M^T \qquad (9.17) \\
&= \frac{1}{4\beta^2} \mathscr{L}. \qquad (9.18)
\end{aligned}$$

Similarly, it follows that

$$\int_0^\infty e^{-\beta\mathscr{L}t}\mathscr{L}^2 e^{-\beta\mathscr{L}t}\,dt \tag{9.19}$$

$$= \int_0^\infty M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{-2\beta\lambda_2 t}\lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_n t}\lambda_n^2 \end{bmatrix} M^T\,dt \tag{9.20}$$

$$= -\frac{1}{2\beta} M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 e^{-2\beta\lambda_2 t}|_0^\infty & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n e^{-2\beta\lambda_n t}|_0^\infty \end{bmatrix} M^T$$

$$= \frac{1}{2\beta} M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} M^T = \frac{1}{2\beta}\mathscr{L} \tag{9.21}$$

and

$$\int_0^\infty \mathscr{L}te^{-\beta\mathscr{L}t}\mathscr{L}e^{-\beta\mathscr{L}t}\,dt$$

$$= \int_0^\infty M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{-2\beta\lambda_2 t}\lambda_2^2 t & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_n t}\lambda_n^2 t \end{bmatrix} M^T dt$$

$$= -\frac{1}{2\beta} M$$

$$\times \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 te^{-2\beta\lambda_2 t}|_0^\infty - \int_0^\infty e^{-2\beta\lambda_2 t}\lambda_2 dt & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda_n te^{-2\beta\lambda_n t}|_0^\infty - \int_0^\infty e^{-2\beta\lambda_n t}\lambda_n dt \end{bmatrix} M^T$$

$$= -\frac{1}{4\beta^2} M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{-2\beta\lambda_2 t}|_0^\infty & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & e^{-2\beta\lambda_n t}|_0^\infty \end{bmatrix} M^T$$

$$= -\frac{1}{4\beta^2} M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} M^T = \frac{I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T}{4\beta^2}, \tag{9.22}$$

where we have used the fact that the first column of $M$ is $\frac{\mathbf{1}_n}{\sqrt{n}}$. By substituting (9.18), (9.21), and (9.22) into (9.15), it follows that the optimal $\beta$ is $\beta_{\text{opt}}$. ■

**Remark 9.7** In Theorem 9.6, we consider a simple case when all agents have the same coupling gain and find the optimal coupling gain explicitly. It is also possible to consider the case when the coupling gains for each agent are different. However, it is, in general, hard to find the optimal coupling gains explicitly. Instead, numerical solutions can be obtained accordingly.

### *9.2.3 Illustrative Examples*

In this subsection, we provide two illustrative examples about the optimal state feedback gain matrix and the optimal scaling factor derived in, respectively, Sect. 9.2.1 and Sect. 9.2.2.

In (9.9), we simply choose

$$Q = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

It then follows from Theorem 9.1 that the optimal state feedback gain matrix is given by

$$\sqrt{\Theta^{-1}Q} = \begin{bmatrix} 1.3134 & -0.5459 & -0.5964 & -0.1711 \\ -0.2730 & 0.8491 & -0.4206 & -0.1556 \\ -0.1988 & -0.2804 & 0.8218 & -0.3426 \\ -0.0428 & -0.0778 & -0.2570 & 0.3775 \end{bmatrix}.$$

Note that the optimal gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. Also note that the graph associated with $Q$ is different from that associated with $\sqrt{\Theta^{-1}Q}$.

In (9.14), we simply choose

$$\mathscr{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and the initial state $\tilde{r}(0) = [1, 2, 3, 4]^T$. Figure 9.1 shows how the cost function $J_{rc}$ evolves as the scaling factor $\beta$ increases. From Theorem 9.6, it can be computed that the optimal scaling factor is $\beta_{\text{opt}} = 0.845$, which is consistent with the result shown in Fig. 9.1.

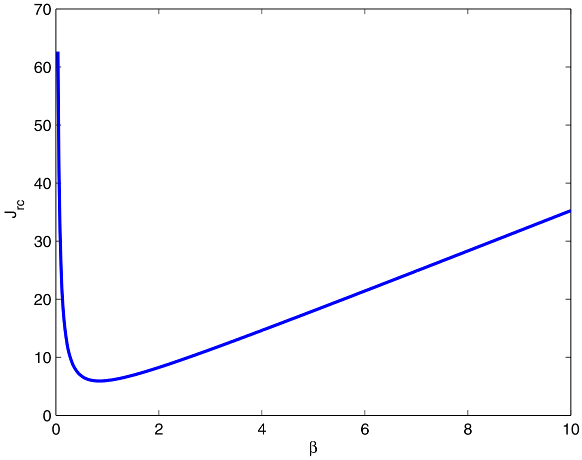


**Fig. 9.1** Evolution of the cost function $J_{rc}$ as a function of $\beta$

## 9.3 Optimal Linear Coordination Algorithms in a Discrete-time Setting from a Linear Quadratic Regulator Perspective

In this section, we study the optimal linear coordination algorithms in a discrete-time setting from an LQR perspective. As shown later, the analysis in the discrete-time case is more challenging than that in the continuous-time case. We will first derive the optimal state feedback gain matrix using the discrete-time interaction-free cost function (9.5). The optimal gain matrix is later shown to be a nonsymmetric Laplacian matrix corresponding to a completed directed graph. We then find the optimal scaling factor for a prespecified symmetric Laplacian matrix associated with an undirected interaction graph using the discrete-time interaction-related cost function (9.6). Finally, illustrative examples are provided.

### *9.3.1 Optimal State Feedback Gain Matrix Using the Interaction-free Cost Function*

Note that (9.7) can be written as

$$\min_{u[k]} \underbrace{\sum_{k=0}^{\infty}\left(\tilde{r}^T[k]Q\tilde{r}[k] + u^T[k]\Theta u[k]\right)}_{J_{fd}} \tag{9.23}$$

$$\text{subject to:} \quad r[k+1] = r[k] + Tu[k], \tag{9.24}$$

where $\tilde{r}[k] \triangleq [\tilde{r}_1[k], \ldots, \tilde{r}_n[k]]^T$ with $\tilde{r}_i[k] \triangleq r_i[k] - \delta_i$, $r[k] \triangleq [r_1[k], \ldots, r_n[k]]^T$, $u[k] \triangleq [u_1[k], \ldots, u_n[k]]^T$, and $Q$ and $\Theta$ are defined as in (9.9). Before moving on, we need the following lemmas.

**Lemma 9.3.** *Let $P_1 \in \mathbb{R}^{n\times n}$ be a row-stochastic matrix with positive diagonal entries satisfying that $P_1$ has a simple eigenvalue equal to one and all other eigenvalues are within the unit circle. Let $P_2 \in \mathbb{R}^{n\times n}$ be a nonnegative matrix satisfying that $\rho(P_2) < 1$. Denote*

$$X_{i+1,j} = \frac{1}{2}\big[P_j + (X_{i,j})^2\big], \qquad X_{0,j} = 0_{n\times n}, \tag{9.25}$$

*for $j = 1, 2$. Then $\lim_{i\to\infty} X_{i,j},\ j = 1, 2$, exists. Denote $X_j^\star \triangleq \lim_{i\to\infty} X_{i,j}, j = 1, 2$. If $P_1$ and $P_2$ commute, the following statements hold:*

1. *$X_j^\star$ and $P_k$ commute for $j, k = 1, 2$;*
2. *$X_1^\star$ and $X_2^\star$ commute.*

*Proof:* It follows from Lemma 1.7 and Definition 1.1 that $P_1$ is semiconvergent. Also it follows from Lemma 1.27 and Definition 1.1 that $P_2$ is also semiconvergent. It then follows from Property (c) in Lemma 1.13 that (9.25) is convergent. That is, $\lim_{i\to\infty} X_{i,j}, j = 1, 2$, exists. We next show that Statements 1 and 2 hold by induction. It can be computed from (9.25) that $X_{1,1} = \frac{1}{2}P_1$ and $X_{1,2} = \frac{1}{2}P_2$. Therefore, it is easy to verify that $P_k$ and $X_{1,j}$ commute for $j, k = 1, 2$. Similarly, $X_{1,1}$ and $X_{1,2}$ also commute. Assume that $P_k$ and $X_{\ell,j}$ commute for $j, k = 1, 2$ and $X_{\ell,1}$ and $X_{\ell,2}$ commute. It can be computed from (9.25) that $X_{\ell+1,j} = \frac{1}{2}[P_j + (X_{\ell,j})^2]$ for $j = 1, 2$. It can also be easily verified that $X_{\ell+1,j}$ and $P_k$ commute for $j, k = 1, 2$. In addition, we also have that

$$\begin{aligned}
X_{\ell+1,1}X_{\ell+1,2} &= \frac{1}{4}\big[P_1 + (X_{\ell,1})^2\big]\big[P_2 + (X_{\ell,2})^2\big] \\
&= \frac{1}{4}\big[P_1P_2 + (X_{\ell,1})^2P_2 + P_1(X_{\ell,2})^2 + (X_{\ell,1})^2(X_{\ell,2})^2\big] \\
&= \frac{1}{4}\big[P_2P_1 + P_2(X_{\ell,1})^2 + (X_{\ell,2})^2P_1 + (X_{\ell,2})^2(X_{\ell,1})^2\big] \\
&= X_{\ell+1,2}X_{\ell+1,1},
\end{aligned}$$

where we have used the assumption that $P_k$ and $X_{\ell,j}$ commute for $j, k = 1, 2$ and $X_{\ell,1}$ and $X_{\ell,2}$ commute to derive the final result. Therefore, $X_{\ell+1,1}$ and $X_{\ell+1,2}$ also commute. By induction, $P_k$ and $\lim_{i\to\infty} X_{i,j}$ commute for $j, k = 1, 2$ and $\lim_{i\to\infty} X_{i,1}$ and $\lim_{i\to\infty} X_{i,2}$ commute. Because $X_j^\star = \lim_{i\to\infty} X_{i,j},\ j = 1, 2$, the lemma holds clearly. ■

**Lemma 9.4 ([152]).** *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. If $AB = BA$, then $\rho(A+B) \leq \rho(A) + \rho(B)$.*

**Lemma 9.5.** *Let $G \in \mathbb{R}^{n\times n}$ be a nonsymmetric Laplacian matrix with a simple zero eigenvalue. Suppose that $\gamma \geq 0$. Then $\sqrt{G+\gamma I_n}\sqrt{G}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue.*[2]

*Proof:* When $\gamma = 0$, the proof is trivial. When $\gamma > 0$, the proof follows tow steps:

Step 1. *The off-diagonal entries of $\sqrt{G+\gamma I_n}\sqrt{G}$ are nonpositive.* Because $G = [g_{ij}]$ is a nonsymmetric Laplacian matrix, $G$ can be written as $G = s(I_n - P)$, where $s > 2\max_i g_{ii}$, and $P$ is a row-stochastic matrix with positive diagonal entries. Because $G$ has a simple zero eigenvalue, it follows that $P$ has a simple eigenvalue equal to one and all other eigenvalues are within the unit circle. Therefore, it follows from Lemma 1.7 and Definition 1.1 that $P$ is semiconvergent. It follows from Property (c) in Lemma 1.13 that the iteration (1.3) is convergent. According to part (a) in Lemma 1.13, $\sqrt{G} = \sqrt{s}(I_n - X^\star)$, where $X^\star = \lim_{i\to\infty} X_i$ with $\alpha = 1$ in (1.3). Similarly, $G+\gamma I_n$ can be written as $G+\gamma I_n = (s+\gamma)(I_n - \frac{s}{s+\gamma}P)$, where $s > 2\max_i g_{ii}$. By following a similar analysis to that of $G$, it follows that $\sqrt{G+\gamma I_n} = \sqrt{s+\gamma}(I_n - \widehat{X}^\star)$, where $\widehat{X}^\star = \lim_{i\to\infty} X_i$ with $P$ replaced with $\frac{s}{s+\gamma}P$ and $\alpha = 1$ in (1.3). With $P$ and $\frac{s}{s+\gamma}P$ playing the role of, respectively, $P_1$ and $P_2$ in Lemma 9.3, it follows from parts (a) and (c) in Lemma 1.13 and Lemma 9.3 that $X^\star$ and $\widehat{X}^\star$ commute because $P$ and $\frac{s}{s+\gamma}P$ commute. Then we have

$$\begin{aligned}
&\frac{1}{\sqrt{s(s+\gamma)}}\sqrt{G+\gamma I_n}\sqrt{G}\\
&= (I_n - X^\star - \widehat{X}^\star + X^\star\widehat{X}^\star)\\
&= I_n - \frac{1}{2}[P + (X^\star)^2] - \frac{1}{2}\left[\frac{s}{s+\gamma}P + (\widehat{X}^\star)^2\right] + X^\star\widehat{X}^\star \qquad (9.26)\\
&= I_n - \frac{1}{2}\left[P + \frac{s}{s+\gamma}P + (X^\star - \widehat{X}^\star)^2\right], \qquad (9.27)
\end{aligned}$$

where we have used the fact that $X^\star = \frac{1}{2}[P + (X^\star)^2]$ and $\widehat{X}^\star = \frac{1}{2}[\frac{s}{s+\gamma}P + (\widehat{X}^\star)^2]$ as shown in part (c) of Lemma 1.13 to derive (9.26) and the fact that $X^\star$ and $\widehat{X}^\star$ commute to derive (9.27).

From (9.27), a sufficient condition to show that the off-diagonal entries of $\sqrt{G+\gamma I_n}\sqrt{G}$ are nonpositive is to show that $X^\star - \widehat{X}^\star$ is nonnegative because $P$ is a row-stochastic matrix. We next show that this condition can be satisfied. It follows from part (a) of Lemma 1.13 that $I - P = (I_n - X^\star)^2$ and $I - \frac{s}{s+\gamma}P = (I_n - \widehat{X}^\star)^2$ when $\alpha = 1$. Therefore, we have

[2] Note that $G+\gamma I_n$, $\gamma \geq 0$, is an M-matrix with at most one zero eigenvalue. Note also that $G$ is an M-matrix with a simple zero eigenvalue. Therefore, $\sqrt{G+\gamma I_n}$ and $\sqrt{G}$ are well defined.

$$\begin{aligned}\frac{\gamma}{s+\gamma}P &= (I_n-\widehat{X}^\star)^2-(I_n-X^\star)^2\\ &= 2(X^\star-\widehat{X}^\star)-(X^\star-\widehat{X}^\star)(X^\star+\widehat{X}^\star)\\ &= (X^\star-\widehat{X}^\star)(2I_n-X^\star-\widehat{X}^\star).\end{aligned} \tag{9.28}$$

We next show that $2I_n - X^\star - \widehat{X}^\star$ is a nonsingular M-matrix and then use Lemma 1.17 to show that $X^\star - \widehat{X}^\star$ is nonnegative. Because $G+\gamma I_n$ is a nonsingular M-matrix from Definition 1.2, it follows from Lemma 1.16 that $\sqrt{G+\gamma I_n}$ is also a nonsingular M-matrix. Because $\sqrt{G+\gamma I_n} = \sqrt{s+\gamma}(I_n - \widehat{X}^\star)$, it follows that $\rho(\widehat{X}^\star) < 1$ according to Definition 1.2. Similarly, it follows from Lemma 1.14 that $\sqrt{G}$ is an M-matrix. Because $\sqrt{G} = \sqrt{s}(I_n - X^\star)$, it follows that $\rho(X^\star) \le 1$ according to Definition 1.2. Because $\widehat{X}^\star$ and $X^\star$ commute, it then follows from Lemma 9.4 that $\rho(\widehat{X}^\star + X^\star) \le \rho(\widehat{X}^\star) + \rho(X^\star) < 2$. Therefore, $2I_n - X^\star - \widehat{X}^\star$ is a nonsingular M-matrix according to Definition 1.2. Because $2I_n - X^\star - \widehat{X}^\star$ is a nonsingular M-matrix, it follows from Lemma 1.17 that $(2I_n - X^\star - \widehat{X}^\star)^{-1}$ is nonnegative, which implies that $X^\star - \widehat{X}^\star$ is nonnegative because $X^\star - \widehat{X}^\star = \frac{\gamma}{s+\gamma}P(2I_n - X^\star - \widehat{X}^\star)^{-1}$ and $P$ is a row-stochastic matrix. Therefore, it follows from (9.27) that the off-diagonal entries of $\sqrt{G+\gamma I_n}\sqrt{G}$ are nonpositive.

Step 2. *$\sqrt{G+\gamma I_n}\sqrt{G}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue.* Similar to the analysis in Lemma 9.1, it follows that $\sqrt{G}$ has a simple zero eigenvalue with a corresponding eigenvector $\mathbf{1}_n$. Then $\sqrt{G+\gamma I_n}\sqrt{G}$ also has a simple zero eigenvalue with a corresponding eigenvector $\mathbf{1}_n$ because $\sqrt{G+\gamma I_n}$ is a nonsingular M-matrix as shown in Step 1. Combining with Step 1 indicates that $\sqrt{G+\gamma I_n}\sqrt{G}$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue. ■

**Lemma 9.6.** *Let $G \in \mathbb{R}^{n\times n}$ be a nonsymmetric Laplacian matrix with a simple zero eigenvalue. Suppose that $\gamma > 0$. Then $\sqrt{G+\gamma I_n}\sqrt{G} - G$ is also a nonsymmetric Laplacian matrix with a simple zero eigenvalue.*

*Proof:* It follows from Step 1 in the proof of Lemma 9.5 that $\sqrt{G} = \sqrt{s}(I_n - X^\star)$, $\sqrt{G+\gamma I_n} = \sqrt{s+\gamma}(I_n - \widehat{X}^\star)$, and $X^\star$ and $\widehat{X}^\star$ commute. It follows that $\sqrt{G}$ and $\sqrt{G+\gamma I_n}$ also commute. It can be computed that $\overline{P} \triangleq \sqrt{G+\gamma I_n}\sqrt{G} + G$ is the solution of the following matrix equation

$$P^2 - 2PG - \gamma G = 0_{n\times n}, \tag{9.29}$$

where we have used the fact that $\sqrt{G+\gamma I_n}\sqrt{G}$ and $G$ commute because $\sqrt{G+\gamma I_n}$ and $\sqrt{G}$ commute and $G = \sqrt{G}\sqrt{G}$. From Lemma 9.5, we know that $\sqrt{G+\gamma I_n}\sqrt{G}$ is a nonsymmetric Laplacian matrix, which implies that $\overline{P}$ is also a nonsymmetric Laplacian matrix. Therefore, $\gamma I_n + 2\overline{P}$ is a nonsingular M-matrix according to Definition 1.2. From (9.29), we can get that

$$G = (\gamma I_n + 2\overline{P})^{-1}\overline{P}^2 = \frac{1}{2}(\gamma I_n + 2\overline{P})^{-1}(\gamma I_n + 2\overline{P} - \gamma I_n)\overline{P}$$
$$= \frac{1}{2}\big[I_n - \gamma(\gamma I_n + 2\overline{P})^{-1}\big]\overline{P},$$

which implies that

$$\frac{1}{2}\gamma(\gamma I_n + 2\overline{P})^{-1}\overline{P} = \frac{1}{2}\overline{P} - G = \frac{1}{2}\big(\sqrt{G + \gamma I_n}\sqrt{G} - G\big). \tag{9.30}$$

Note also that

$$\gamma(\gamma I_n + 2\overline{P})^{-1}\overline{P} = \frac{1}{2}\gamma\big[I_n - \gamma(\gamma I_n + 2\overline{P})^{-1}\big]. \tag{9.31}$$

Combining (9.30) and (9.31) gives that

$$\sqrt{G + \gamma I_n}\sqrt{G} - G = \gamma\big[I_n - \gamma(\gamma I_n + 2\overline{P})^{-1}\big]. \tag{9.32}$$

Because $\gamma I_n + 2\overline{P}$ is a nonsingular M-matrix, it follows from Lemma 1.17 that $(\gamma I_n + 2\overline{P})^{-1}$ is nonnegative. It then follows from (9.32) that the off-diagonal entries of $\sqrt{G + \gamma I_n}\sqrt{G} - G$ are nonpositive.

Because the off-diagonal entries of $\sqrt{G + \gamma I_n}\sqrt{G} - G$ are nonpositive, to show that $\sqrt{G + \gamma I_n}\sqrt{G} - G$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue, it is sufficient to show that $\sqrt{G + \gamma I_n}\sqrt{G} - G$ has a simple zero eigenvalue with an associated eigenvector $\mathbf{1}_n$. Letting $\mu$ be an eigenvalue of $G$ with an associated eigenvector $\nu$, it can be computed that $(G + \gamma I_n)\nu = (\mu + \gamma)\nu$, which implies that the corresponding eigenvalue of $\sqrt{G + \gamma I_n}$ is given by $\sqrt{\mu + \gamma}$ with an associated eigenvector $\nu$. Therefore, the corresponding eigenvalue of $\sqrt{G + \gamma I_n}\sqrt{G} - G$ is given by $\sqrt{\mu + \gamma}\sqrt{\mu} - \mu$ with an associated eigenvector $\nu$. Noting that the nonsymmetric Laplacian matrix $G$ has a simple zero eigenvalue, it follows that $\sqrt{G + \gamma I_n}\sqrt{G} - G$ also has a simple zero eigenvalue because $\sqrt{\mu + \gamma}\sqrt{\mu} - \mu \neq 0$ if $\mu \neq 0$. Note also that $(\sqrt{G + \gamma I_n}\sqrt{G} - G)\mathbf{1}_n = \mathbf{0}_n$ because $\sqrt{G}\mathbf{1}_n = \mathbf{0}_n$ and $G\mathbf{1}_n = \mathbf{0}_n$. Therefore, $\sqrt{G + \gamma I_n}\sqrt{G} - G$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue. ■

**Lemma 9.7.** *Let $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. If each off-diagonal entry of $B$ is not equal to zero (and hence negative), $B^{-1}$ is positive.*

*Proof:* From Definition 1.2, $B = \alpha I_n - C$. By choosing $\alpha > \max_i b_{ii}$, it follows that $C$ is positive. Because $B$ is a nonsingular M-matrix, it follows from Definition 1.2 that $\rho(C) < \alpha$ and hence $\rho(\frac{C}{\alpha}) < 1$. Note that $B^{-1} = \alpha^{-1}(I_n - \frac{C}{\alpha})^{-1}$. It follows from Lemmas 1.28 and 1.26 that $(I_n - \frac{C}{\alpha})^{-1} = \sum_{i=0}^{\infty}(\frac{C}{\alpha})^i$. Because $C$ is positive, it follows directly that $B^{-1}$ is positive. ■

**Lemma 9.8.** *Let $Q$ and $\Theta$ be defined in (9.23). Suppose that $Q$ has a simple zero eigenvalue. Suppose that $\gamma > 0$. Then $W \triangleq \sqrt{\Theta^{-1}Q + \gamma I_n}\sqrt{\Theta^{-1}Q} - \Theta^{-1}Q$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue corresponding to a complete directed graph.*

*Proof:* Note that $Q$ has a simple zero eigenvalue. It follows from Lemmas 9.1 and 9.6 that $W$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue. We study how $W$ evolves when $\gamma$ increases. Taking the derivative of $\Theta^{-1}Q + \gamma I_n$ with respect to $\gamma$ gives

$$\frac{d(\Theta^{-1}Q + \gamma I_n)}{d\gamma} = I_n. \tag{9.33}$$

Note that $\Theta^{-1}Q + \gamma I_n$, where $\gamma > 0$, is a nonsingular M-matrix. It follows from Lemma 1.16 that $\sqrt{\Theta^{-1}Q + \gamma I_n}$ is also a nonsingular M-matrix. We also have

$$\frac{d(\sqrt{\Theta^{-1}Q + \gamma I_n})^2}{d\gamma} = 2\sqrt{\Theta^{-1}Q + \gamma I_n}\frac{d\sqrt{\Theta^{-1}Q + \gamma I_n}}{d\gamma}. \tag{9.34}$$

Therefore, it follows from (9.33) and (9.34) that $\frac{d\sqrt{\Theta^{-1}Q+\gamma I_n}}{d\gamma} = \frac{1}{2} \times (\sqrt{\Theta^{-1}Q + \gamma I_n})^{-1}$. It then follows that

$$\begin{aligned}
&\frac{d\sqrt{\Theta^{-1}Q + \gamma I_n}\sqrt{\Theta^{-1}Q}}{d\gamma} \\
&= \frac{d\sqrt{\Theta^{-1}Q + \gamma I_n}}{d\gamma}\sqrt{\Theta^{-1}Q} \\
&= \frac{1}{2}(\sqrt{\Theta^{-1}Q + \gamma I_n})^{-1}\sqrt{\Theta^{-1}Q} \\
&= \frac{1}{2}I_n - \frac{1}{2}(\sqrt{\Theta^{-1}Q + \gamma I_n})^{-1}(\sqrt{\Theta^{-1}Q + \gamma I_n} - \sqrt{\Theta^{-1}Q}) \\
&= \frac{1}{2}I_n - \frac{\gamma}{2}(\sqrt{\Theta^{-1}Q + \gamma I_n})^{-1}(\sqrt{\Theta^{-1}Q + \gamma I_n} + \sqrt{\Theta^{-1}Q})^{-1}.
\end{aligned} \tag{9.35}$$

By following a similar analysis to that of Lemma 9.2, we can show that each entry of $\sqrt{\Theta^{-1}Q + \gamma I_n}$ is not equal to zero. It follows from Lemma 9.7 that each entry of $(\sqrt{\Theta^{-1}Q + \gamma I_n})^{-1}$ is positive. Similarly, each entry of $(\sqrt{\Theta^{-1}Q + \gamma I_n} + \sqrt{\Theta^{-1}Q})^{-1}$ is also positive. It then follows from (9.35) that each off-diagonal entry of $\frac{d\sqrt{\Theta^{-1}Q+\gamma I_n}\sqrt{\Theta^{-1}Q}}{d\gamma}$ is negative, which implies that the off-diagonal entries of $W$ will decrease when $\gamma$ increases. Noting that $W = 0_{n\times n}$ when $\gamma = 0$, it follows that all off-diagonal entries of $W$ are less than zero for all $\gamma > 0$. $W$ corresponds to a complete directed graph. ■

The main result for the optimal control problem (9.23) is given in the following theorem.

**Theorem 9.8.** *In the optimal control problem* (9.23)*, suppose that* $Q$ *has a simple zero eigenvalue. The optimal coordination algorithm is*

$$u[k] = -K\tilde{r}[k], \tag{9.36}$$

*where* $K \triangleq \frac{T[\sqrt{\Theta^{-1}Q+4I_n/T^2}\sqrt{\Theta^{-1}Q}-\Theta^{-1}Q]}{2}$. *The matrix* $K$ *is a nonsymmetric Laplacian matrix with a simple zero eigenvalue corresponding to a complete directed graph. Using* (9.36) *for* (9.24)*, coordination is achieved.*

*Proof:* Note that $r[k+1] = r[k] + Tu[k]$ is equivalent to $\tilde{r}[k+1] = \tilde{r}[k] + Tu[k]$. Consider the following LQR problem

$$\min_{u[k]} J_{fd} \quad \text{subject to:} \quad \tilde{r}[k+1] = A\tilde{r}[k] + Bu[k],$$

where $J_{fd}$ is defined in (9.23), $A = I_n$, and $B = TI_n$. It can be noted that $(A, B)$ is controllable, which implies that there exists a unique positive-semidefinite matrix $P \in \mathbb{R}^{n\times n}$ satisfying the discrete-time algebraic Riccati equation

$$P = Q + A^T\big[P - PB\big(\Theta + B^T PB\big)^{-1} B^T P\big]A. \tag{9.37}$$

Noting that $A = I_n$ and $B = TI_n$, we can simplify (9.37) as

$$Q = T^2 P\big(\Theta + T^2 P\big)^{-1} P. \tag{9.38}$$

By multiplying $\Theta^{-1}$ on both sides of (9.38), after some manipulation, we can get that

$$\Theta^{-1}Q = T^2\Theta^{-1}P\big(I_n + T^2\Theta^{-1}P\big)^{-1}\Theta^{-1}P. \tag{9.39}$$

Note that

$$\big(I_n + T^2\Theta^{-1}P\big)^{-1}T^2\Theta^{-1}P = I_n - \big(I_n + T^2\Theta^{-1}P\big)^{-1}. \tag{9.40}$$

By substituting (9.40) into (9.39), after some manipulation, we can get that

$$\Theta^{-1}Q = \Theta^{-1}P\big[I_n - \big(I_n + T^2\Theta^{-1}P\big)^{-1}\big], \tag{9.41}$$

which can be simplified as

$$\big(\Theta^{-1}P\big)^2 - \Theta^{-1}Q\big(\Theta^{-1}P\big) - \frac{1}{T^2}\Theta^{-1}Q = 0_{n\times n}. \tag{9.42}$$

It can be computed that (9.42) holds when $\Theta^{-1}P = \frac{\Theta^{-1}Q+\sqrt{\Theta^{-1}Q+4I_n/T^2}\sqrt{\Theta^{-1}Q}}{2}$. The optimal control strategy is given by $u[k] = -F\tilde{r}[k]$, where

$$F = \big(I_n + T^2\Theta^{-1}P\big)^{-1}T\Theta^{-1}P = T\big(\Theta^{-1}P - \Theta^{-1}Q\big) = K,$$

where we have used (9.41) to derive the third equality. It follows from Lemma 9.8 that $K$ is a nonsymmetric Laplacian matrix that corresponds to a complete directed graph.

We next show that coordination is achieved using (9.36) for (9.24) (i.e., $r_i[k] - r_j[k] \to \Delta_{ij}$ or equivalently $\tilde{r}_i[k] - \tilde{r}_j[k] \to 0$ as $k \to \infty$). Note that $K$ is a nonsymmetric Laplacian matrix with a simple zero eigenvalue and $I_n - TK$ has nonnegative off-diagonal entries and all row sums equal to one. According to Lemmas 1.1 and 1.11, coordination is achieved if $I_n - TK$ has positive diagonal entries. With $\frac{T^2}{4}\Theta^{-1}Q$ playing the role of $G$ and $\gamma = 1$, it follows from a similar argument to that in the beginning of the proof of Lemma 9.6 that $\sqrt{\frac{T^2}{4}\Theta^{-1}Q + I_n}$ and $\sqrt{\frac{T^2}{4}\Theta^{-1}Q}$ commute. After some manipulation, we have

$$I_n - TK = \left(\sqrt{\frac{T^2}{4}\Theta^{-1}Q + I_n} - \sqrt{\frac{T^2}{4}\Theta^{-1}Q}\right)^2.$$

By following a similar proof to that of Lemma 9.8, we have that $\sqrt{\frac{T^2}{4}\Theta^{-1}Q + I_n} - \sqrt{\frac{T^2}{4}\Theta^{-1}Q}$ is an M-matrix with each entry not equal to zero. Combining with Definition 1.2 shows that all diagonal entries of $I_n - TK$ are positive. Therefore, coordination is achieved when using (9.36) for (9.24). ■

**Remark 9.9** From Theorem 9.8, it is easy to verify that when $T$ approaches zero, the optimal state feedback gain matrix is the same as that in the continuous-time case in Theorem 9.1. In addition, the matrix $K$ is not necessarily symmetric. When $\Theta$ is a diagonal matrix with identical diagonal entries (i.e., $\Theta = cI_n$, where $c > 0$), $K$ is symmetric.

**Remark 9.10** In Theorem 9.1 (correspondingly, Theorem 9.8), a standard LQR problem is solved. The solution can be solved using the standard Matlab command. However, it is not clear why the optimal state feedback gain matrix derived from the standard LQR perspective is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. The contribution of Sect. 9.2.1 (correspondingly, Sect. 9.3.1) is that we mathematically prove the conditions under which the square root of a nonsymmetric Laplacian matrix is still a nonsymmetric Laplacian matrix, explicitly derive the optimal state feedback gain matrix under a given global cost function, and show that the gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph. Although it might be intuitively true that a global optimization problem in the context of multi-agent coordination normally requires that each agent have full knowledge of all other agents, it is nontrivial to theoretically prove this fact. We have provided a theoretical explanation.

Similar to the discussion in Sect. 9.2, we next show that any symmetric Laplacian matrix $\mathscr{L}$ with a simple zero eigenvalue is inverse optimal with respect to some given cost function.

**Theorem 9.11.** *Any symmetric Laplacian matrix $\mathscr{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with a simple zero eigenvalue is the optimal state feedback gain matrix under the cost function $J = \sum_{k=0}^{\infty}(\tilde{r}[k]Q\tilde{r}[k] + u[k]u[k])$, where $Q \triangleq (I_n - T\mathscr{L})^{-1}\mathscr{L}^2$ and $0 < T < \frac{1}{2}\min_i \frac{1}{\ell_{ii}}$.*

*Proof:* When $0 < T < \frac{1}{2}\min_i \frac{1}{\ell_{ii}}$, it follows from Lemma 1.18 that $\rho(T\mathscr{L}) < 1$. It then follows from Lemmas 1.26 and 1.28 that $I_n - T\mathscr{L}$ is invertible and $(I_n -$

$T\mathscr{L})^{-1} = \sum_{i=0}^{\infty}(T\mathscr{L})^i$. Because $\mathscr{L}$ is symmetric positive semidefinite with a simple zero eigenvalue, it then follows that $Q$ is symmetric positive semidefinite with a simple zero eigenvalue by noting that $Q = (I_n - T\mathscr{L})^{-1}\mathscr{L}^2$. Also note that $(I_n - T\mathscr{L})Q = \mathscr{L}^2$, i.e., $Q = T\mathscr{L}Q + \mathscr{L}^2$, which implies that

$$(2\mathscr{L} + TQ)^2 = 4\mathscr{L}^2 + 4T\mathscr{L}Q + (TQ)^2 = 4\big(\mathscr{L}^2 + T\mathscr{L}Q\big) + (TQ)^2 \\ = 4Q + (TQ)^2. \tag{9.43}$$

By taking the square root of both sides of (9.43) and some simplification, we can get $\frac{T(\sqrt{Q+4I_n/T^2}\sqrt{Q}-Q)}{2} = \mathscr{L}$. Applying Theorem 9.8 finishes the proof. ∎

### 9.3.2 Optimal Scaling Factor Using the Interaction-related Cost Function

Suppose that the graph $\mathscr{G}$ is undirected. With the interaction-related cost function (9.6), the optimal control problem (9.8) can be written as:

$$\min_{\beta} \underbrace{\sum_{k=0}^{\infty}\big(\tilde{r}^T[k]\mathscr{L}\tilde{r}[k] + u^T[k]u[k]\big)}_{J_{rd}} \tag{9.44}$$
$$\text{subject to:} \quad r[k+1] = r[k] + Tu[k],\ u[k] = -\beta\mathscr{L}\tilde{r}[k],$$

where $\mathscr{L}$ is the prespecified symmetric Laplacian matrix associated with the adjacency matrix $\mathscr{A}$ and hence the undirected graph $\mathscr{G}$, and $\beta$ is the scaling factor.

**Theorem 9.12.** *Let $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $\mathscr{L}$. In the optimal control problem (9.44), suppose that $\mathscr{L}$ has a simple zero eigenvalue. Then the optimal $\beta$, denoted by $\beta_{\text{opt}}$, satisfies* $\frac{-T+\sqrt{T^2+\frac{4}{\lambda_n}}}{2} \le \beta_{\text{opt}} \le \frac{-T+\sqrt{T^2+\frac{4}{\lambda_2}}}{2}$.[3]

*Proof:* Note that $u[k] = \beta\mathscr{L}\tilde{r}[k]$. It follows that $\tilde{r}[k+1] = \tilde{r}[k] - \beta T\mathscr{L}\tilde{r}[k]$. It follows that $\tilde{r}[k] = (I_n - \beta T\mathscr{L})^k\tilde{r}[0]$ and hence $u[k] = -\beta\mathscr{L}(I_n - \beta T\mathscr{L})^k\tilde{r}[0]$. Therefore, $J_{fd}$ can be written as $J_{fd} = \sum_{k=0}^{\infty}\tilde{r}^T[0][(I_n - \beta T\mathscr{L})^k\mathscr{L}(I_n - \beta T\mathscr{L})^k + \beta^2(I_n - \beta T\mathscr{L})^k\mathscr{L}^2(I_n - \beta T\mathscr{L})^k]\tilde{r}[0]$. By rewriting $\mathscr{L}$ in a diagonal form as shown in (9.16), $J_{rd}$ can be further written as

$$J_{rd} = \sum_{k=0}^{\infty}\tilde{r}^T[0]M\big[(I_n - \beta T\Lambda)^k\Lambda(I_n - \beta T\Lambda)^k \\ + \beta^2(I_n - \beta T\Lambda)^k\Lambda^2(I_n - \beta T\Lambda)^k\big]M^T\tilde{r}[0].$$

[3] Note that there always exists a positive $\beta$ such that coordination is achieved. In this case, $J_r$ is finite. Therefore, when $\beta = \beta_{\text{opt}}$ coordination is always guaranteed because otherwise $J_r$ will go to infinity, which will then result in a contradiction.

Because $\mathscr{L}$ has a simple zero eigenvalue, it follows that $\lambda_i > 0, i = 2, \ldots, n$. After some manipulation, we have that

$$J_{rd} = \tilde{r}^T[0]M \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\frac{1}{T}}{\frac{2\beta+T}{1+\beta^2\lambda_2}-T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\frac{1}{T}}{\frac{2\beta+T}{1+\beta^2\lambda_n}-T} \end{bmatrix} M^T\tilde{r}[0]$$
$$= \sum_{i=2}^{n} \frac{\frac{1}{T}}{\frac{2\beta+T}{1+\beta^2\lambda_i} - T} y_i^2,$$

where $y_i$ is the $i$th component of $M^T\tilde{r}[0]$. For $i = 2, \ldots, n$, taking the derivative of $\frac{2\beta+T}{1+\beta^2\lambda_i} - T$ with respect to $\beta$ and setting the derivative to zero gives

$$\frac{2(1+\beta^2\lambda_i) - 2\beta\lambda_i(2\beta+T)}{(1+\beta^2\lambda_i)^2} = 0.$$

It can be computed that $\beta = \frac{-T+\sqrt{T^2+\frac{4}{\lambda_i}}}{2}$. Note that for $\beta < \frac{-T+\sqrt{T^2+\frac{4}{\lambda_n}}}{2}$, $J_{rd}$ will decrease when $\beta$ increases because $\frac{\frac{1}{T}}{\frac{2\beta+T}{1+\beta^2\lambda_i}-T}$ increases when $\beta$ increases, $i = 2, \ldots, n$. Similarly, for $\beta > \frac{-T+\sqrt{T^2+\frac{4}{\lambda_2}}}{2}$, $J_{\text{rd}}$ will increase when $\beta$ increases because $\frac{\frac{1}{T}}{\frac{2\beta+T}{1+\beta^2\lambda_i}-T}$ decreases when $\beta$ increases, $i = 2, \ldots, n$. Combining the previous arguments shows that $\frac{-T+\sqrt{T^2+\frac{4}{\lambda_n}}}{2} \le \beta_{\text{opt}} \le \frac{-T+\sqrt{T^2+\frac{4}{\lambda_2}}}{2}$. ■

**Remark 9.13** The problem stated in Theorem 9.12 is essentially a polynomial optimization problem. Numerical optimization methods [162] can be used to solve this problem.

## *9.3.3 Illustrative Examples*

In this subsection, we provide two illustrative examples about the optimal state feedback gain matrix and the optimal scaling factor derived in, respectively, Sect. 9.3.1 and Sect. 9.3.2.

In (9.23), let $Q$ and $\Theta$ be chosen as in Sect. 9.2.3 and the sampling period $T = 0.1$ s. It then follows from Theorem 9.8 that the optimal state feedback gain matrix is

$$\begin{bmatrix} 1.2173 & -0.498 & -0.5484 & -0.1709 \\ -0.249 & 0.8007 & -0.3963 & -0.1554 \\ -0.1828 & -0.2642 & 0.7734 & -0.3264 \\ -0.0427 & -0.0777 & -0.2448 & 0.3653 \end{bmatrix}.$$

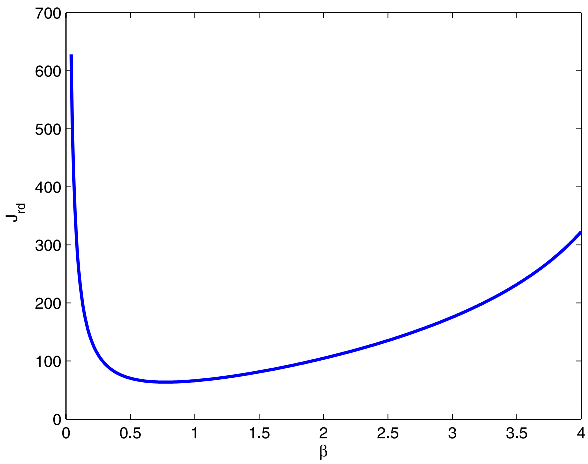


**Fig. 9.2** Evolution of the cost function $J_{rd}$ as a function of $\beta$

Note that the optimal gain matrix is a nonsymmetric Laplacian matrix corresponding to a complete directed graph.

In (9.44), let $\mathscr{L}$ and the initial state $\tilde{r}[0]$ be chosen as in Sect. 9.2.3. Figure 9.2 shows how the cost function $J_{rd}$ evolves as the scaling factor $\beta$ increases. From Theorem 9.12, it can be computed that the optimal scaling factor satisfies $0.45 \leq \beta_{\text{opt}} \leq 0.95$, which is consistent with the result shown in Fig. 9.2.

## 9.4 Notes

The results in this chapter are based mainly on [31, 35]. For further results on the optimality aspect in distributed multi-agent coordination, see [19, 72, 148, 259, 308]. In particular, in [19], a locally optimal nonlinear consensus algorithm is proposed by imposing individual objectives. In [72], an optimal interaction graph, a de Bruijn's graph, is proposed in the average consensus problem. In [259], a semi-decentralized optimal control strategy is designed by minimizing the individual cost functions. In addition, cooperative game theory is employed to ensure cooperation in the presence of a team cost function. In [148], an iterative algorithm is proposed to maximize the second smallest eigenvalue of a symmetric Laplacian matrix to optimize the control system performance. In [308], the fastest converging linear iteration is studied by using semidefinite programming.