Chapter 8
Sampled-data Setting

This chapter considers distributed multi-agent coordination in a sampled-data setting. We first study a distributed sampled-data coordinated tracking algorithm where a group of followers with single-integrator dynamics interacting with their neighbors at discrete-time instants intercepts a dynamic leader who is a neighbor of only a subset of the followers. We propose a PD-like discrete-time algorithm and study the condition on the interaction graph, the sampling period, and the control gain to ensure stability under directed fixed interaction and give the quantitative bound of the tracking errors. We then study convergence of two distributed sampled-data coordination algorithms with respectively, absolute damping and relative damping for double-integrator dynamics under undirected/directed fixed interaction. We show necessary and sufficient conditions on the interaction graph, the sampling period, and the control gain such that coordination is achieved using these two algorithms by using matrix theory, bilinear transformation, and Cauchy theorem. We finally study convergence of the two distributed sampled-data coordination algorithms with respectively, absolute damping and relative damping for double-integrator dynamics under directed switching interaction. We derive sufficient conditions on the interaction graph, the sampling period, and the control gain to guarantee coordination by using the property of infinity products of row-stochastic matrices. Simulation results are presented to show the effectiveness of the theoretical results.

8.1 Sampled-data Coordinated Tracking for Single-integrator Dynamics

In multi-agent coordination, agents might only be able to interact with their neighbors *intermittently* rather than *continuously* due to low bandwidth, unreliable communication channels, limited sensing capabilities, or power and cost constraints. A multi-agent system with intermittent interaction, where agents with continuous-time dynamics are controlled based on information from their neighbors updated at discrete-time instants, can be treated as a sampled-data system consisting of multi-

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ple networked subsystems. We are hence motivated to study distributed multi-agent coordination in a sampled-data setting. We explicitly consider the effect of sampled-data control on stability of the agents. In this section, we focus on sampled-data coordinated tracking for single-integrator dynamics.

8.1.1 Algorithm Design

Suppose that in addition to n followers, labeled as agents or followers 1 to n, with single-integrator dynamics given by (3.1), there exists a dynamic leader, labeled as agent 0, whose position is $r_0(t) \in \mathbb{R}^m$. Here the leader can be physical or virtual. Let $\mathscr{G} \stackrel{\triangle}{=} (\mathscr{V}, \mathscr{E})$ be the directed graph characterizing the interaction among the n followers. Let $\overline{\mathscr{G}} \stackrel{\triangle}{=} (\overline{\mathscr{V}}, \overline{\mathscr{E}})$ be the directed graph characterizing the interaction among the leader and the followers corresponding to $\mathscr{G}$.

A proportional-derivative-like (PD-like) continuous-time coordinated tracking algorithm is proposed for (3.1) in [248, Chap. 3] as

$$\begin{aligned} u_i(t) = &\frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=1}^n a_{ij} \big\{\dot{r}_j(t) - \gamma\big[r_i(t) - r_j(t)\big]\big\} \\ &+ \frac{a_{i0}}{\sum_{j=0}^n a_{ij}} \big\{\dot{r}_0(t) - \gamma\big[r_i(t) - r_0(t)\big]\big\}, \end{aligned} \tag{8.1}$$

where a_{ij}, $i, j = 1, \dots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ associated with the directed graph $\mathscr{G}$, $a_{i0} > 0$, $i = 1, \dots, n$, if the leader is a neighbor of follower i and $a_{i0} = 0$ otherwise, and γ is a positive gain. The objective of (8.1) is to guarantee that $r_i(t) - r_0(t) \to \mathbf{0}_m, i = 1, \dots, n$, as $t \to \infty$. Note that (8.1) requires each follower to obtain instantaneous measurements of its neighbors' velocities and the leader's velocity if the leader is a neighbor of the follower. This requirement might not be realistic in real applications. We next propose a PD-like discrete-time coordinated tracking algorithm.

Consider a sampled-data setting where the agents have continuous-time dynamics while the measurements are made at discrete sampling times and the control inputs are based on zero-order hold as

$$u_i(t) = u_i[k], \quad kT \le t < (k+1)T, \tag{8.2}$$

where k denotes the discrete-time index, T denotes the sampling period, and $u_i[k]$ is the control input at $t = kT$. By using direct discretization (see Sect. 1.4), the continuous-time system (3.1) can be discretized as

$$r_i[k+1] = r_i[k] + T u_i[k], \quad i = 1, \dots, n, \tag{8.3}$$

where $r_i[k]$ is the position of follower i at $t = kT$. We propose a PD-like discrete-time coordinated tracking algorithm as

$$u_i[k] = \frac{1}{\sum_{j=0}^{n} a_{ij}} \sum_{j=1}^{n} a_{ij} \left[\frac{r_j[k] - r_j[k-1]}{T} - \gamma\big(r_i[k] - r_j[k]\big) \right] + \frac{a_{i0}}{\sum_{j=0}^{n} a_{ij}} \left[\frac{r_0[k] - r_0[k-1]}{T} - \gamma\big(r_i[k] - r_0[k]\big) \right], \tag{8.4}$$

where $r_0[k]$ denotes the leader's position at $t = kT$, and $\frac{r_j[k]-r_j[k-1]}{T}$ and $\frac{r_0[k]-r_0[k-1]}{T}$ are used to approximate, respectively, $\dot{r}_j(t)$ and $\dot{r}_0(t)$ in (8.1) by noting that $r_j[k+1]$ and $r_0[k+1]$ cannot be accessed at $t = kT$. Note that using (8.4), each follower's position is updated based on its current position and its neighbors' current and previous positions as well as the leader's current and previous positions if the leader is a neighbor of the follower. As a result, (8.4) can be easily implemented in practice. In the following, we assume that all agents are in a one-dimensional space (i.e., $m = 1$) for the simplicity of presentation. However, all results hereafter are still valid for any high-dimensional space by the introduction of the Kronecker product.

8.1.2 Convergence Analysis of the Proportional-derivative-like Discrete-time Coordinated Tracking Algorithm

In this subsection, we analyze the algorithm (8.4). Define the tracking error for follower i as $\varepsilon_i[k] \triangleq r_i[k] - r_0[k]$. It follows that the closed-loop system of (8.3) using (8.4) can be written as

$$\varepsilon_i[k+1] = \varepsilon_i[k] + \frac{T}{\sum_{j=0}^{n} a_{ij}} \sum_{j=1}^{n} a_{ij} \left[\frac{\varepsilon_j[k] - \varepsilon_j[k-1]}{T} - \gamma\big(\varepsilon_i[k] - \varepsilon_j[k]\big) \right] + \frac{T a_{i0}}{\sum_{j=0}^{n} a_{ij}} \left(\frac{r_0[k] - r_0[k-1]}{T} - \gamma \varepsilon_i[k] \right) - \big(r_0[k+1] - r_0[k]\big) + \frac{\sum_{j=1}^{n} a_{ij}}{\sum_{j=0}^{n} a_{ij}} \big(r_0[k] - r_0[k-1]\big),$$

which can then be written in a vector form as

$$\varepsilon[k+1] = \big[(1 - T\gamma) I_n + (1 + T\gamma) D^{-1} \mathscr{A}\big] \varepsilon[k] - D^{-1} \mathscr{A} \varepsilon[k-1] + X^r[k], \tag{8.5}$$

where $D \triangleq \operatorname{diag}\{\sum_{j=0}^{n} a_{1j}, \ldots, \sum_{j=0}^{n} a_{nj}\}$, $\varepsilon[k] \triangleq [\varepsilon_1[k], \ldots, \varepsilon_n[k]]^T$, $\mathscr{A}$ is the adjacency matrix associated with $\mathscr{G}$, and $X^r[k] \triangleq (2r_0[k] - r_0[k-1] - r_0[k+1]) \mathbf{1}_n$. By defining $Y[k+1] \triangleq \left[\begin{smallmatrix} \varepsilon[k+1] \\ \varepsilon[k] \end{smallmatrix}\right]$, it follows from (8.5) that

$$Y[k+1] = \widetilde{A}Y[k] + \widetilde{B}X^r[k], \tag{8.6}$$

where

$$\widetilde{A} \triangleq \begin{bmatrix} (1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathscr{A} & -D^{-1}\mathscr{A} \\ I_n & 0_{n\times n} \end{bmatrix}$$

and $\widetilde{B} \triangleq \left[\begin{smallmatrix} I_n \\ 0_{n\times n} \end{smallmatrix}\right]$. It follows that the solution of (8.6) is

$$Y[k] = \widetilde{A}^k Y[0] + \sum_{i=1}^{k} \widetilde{A}^{k-i}\widetilde{B}X^r[i-1]. \tag{8.7}$$

Note that the eigenvalues of $\widetilde{A}$ play an important role in determining the value of $Y[k]$ as $k \to \infty$. In the following, we study the eigenvalues of $\widetilde{A}$. Before moving on, we first study the eigenvalues of $D^{-1}\mathscr{A}$.

Lemma 8.1. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. Then $D^{-1}\mathscr{A}$ satisfies $\|(D^{-1}\mathscr{A})^n\|_\infty < 1$ and $D^{-1}\mathscr{A}$ has all eigenvalues within the unit circle.*[1]

Proof: For the first statement, note that $D^{-1}A$ is nonnegative and each row sum of $D^{-1}A$ is less than or equal to one. Therefore, it follows that $\|D^{-1}A\|_\infty \leq 1$. Denote $\bar{i}_1$ as the set of followers that are the children of the leader, and $\bar{i}_j$, $j = 2, \ldots, \kappa$, as the set of followers that are the children of the followers in $\bar{i}_{j-1}$ but are not in $\bar{i}_r$, $r = 1, \ldots, j-2$. Because the leader has directed paths to all followers 1 to n, there are at most n edges from the leader to all followers 1 to n, which implies that $\kappa \leq n$. Let p_i and q_i^T denote, respectively, the ith column and row of $D^{-1}\mathscr{A}$. When the leader has directed paths to all followers 1 to n, without loss of generality, assume that the kth follower is a child of the leader, i.e., $a_{k0} > 0$. It follows that $q_k^T \mathbf{1}_n = 1 - \frac{a_{k0}}{\sum_{j=0}^n a_{kj}} < 1$. The same property also applies to the other elements in the set $\bar{i}_1$. Similarly, assume that the lth follower (one follower in the set $\bar{i}_2$) is a child of the kth follower (one follower in the set $\bar{i}_1$), which implies that $a_{lk} > 0$. It follows that the sum of the lth row of $(D^{-1}\mathscr{A})^2$ can be written as $q_l^T \sum_{i=1}^n p_i \leq q_l^T \mathbf{1}_n = 1 - \frac{a_{lk}}{\sum_{j=0}^n a_{lj}} < 1$. Meanwhile, the sum of the kth row of $(D^{-1}\mathscr{A})^2$ is also less than one. A similar analysis shows that each row sum of $(D^{-1}\mathscr{A})^\kappa$ is less than one when the leader has directed paths to all followers 1 to n. That is, $\|(D^{-1}A)^\kappa\|_\infty < 1$. Because $\kappa \leq n$ and $\|D^{-1}\mathscr{A}\|_\infty \leq 1$, $\|(D^{-1}\mathscr{A})^n\|_\infty < 1$ holds.

For the second statement, note from Lemma 1.25 that $\rho[(D^{-1}A)^n] \leq \|(D^{-1}A)^n\|_\infty$. Because $\|(D^{-1}A)^n\|_\infty < 1$, it follows that $\rho[(D^{-1}A)^n] < 1$, which implies that $\rho(D^{-1}A) < 1$. ■

We next study the conditions under which all eigenvalues of $\widetilde{A}$ are within the unit circle.

[1] Note that in $\overline{\mathscr{G}}$ if the leader has directed paths to all followers, then each follower has at least one neighbor, that is, $\sum_{j=0}^n a_{ij} > 0,\ i = 1, \ldots, n$. Therefore, D^{-1} exists and (8.4) is well defined.

Lemma 8.2. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. Let λ_i be the ith eigenvalue of $D^{-1}\mathscr{A}$. Then $\tau_i > 0$ holds, where $\tau_i \triangleq \frac{2|1-\lambda_i|^2(2[1-\text{Re}(\lambda_i)]-|1-\lambda_i|^2)}{|1-\lambda_i|^4+4[\text{Im}(\lambda_i)]^2}$. If the positive scalars T and γ satisfy*

$$T\gamma < \min\Big\{1, \min_{i=1,\ldots,n} \tau_i\Big\}, \tag{8.8}$$

then $\widetilde{A}$, defined by (8.1.2), has all eigenvalues within the unit circle.

Proof: For the first statement, when the leader has directed paths to all followers 1 to n, it follows from the second statement in Lemma 8.1 that $|\lambda_i| < 1$. It then follows that $|1-\lambda_i|^2 > 0$ and $|1-\lambda_i|^2 = 1 - 2\text{Re}(\lambda_i) + [\text{Re}(\lambda_i)]^2 + [\text{Im}(\lambda_i)]^2 < 2[1-\text{Re}(\lambda_i)]$, which implies that $\tau_i > 0$.

For the second statement, note that the characteristic polynomial of $\widetilde{A}$ is given by

$$\begin{aligned}
&\det(zI_{2n} - \widetilde{A})\\
&\quad = \det\left(\begin{bmatrix} zI_n - [(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathscr{A}] & D^{-1}\mathscr{A} \\ -I_n & zI_n \end{bmatrix}\right)\\
&\quad = \det\big(\big[zI_n - (1-T\gamma)I_n - (1+T\gamma)D^{-1}\mathscr{A}\big]zI_n + D^{-1}\mathscr{A}\big)\\
&\quad = \det\big(\big[z^2 + (T\gamma-1)z\big]I_n + \big[1-(1+T\gamma)z\big]D^{-1}\mathscr{A}\big),
\end{aligned}$$

where we have used Lemma 1.22 to obtain the second equality because $zI_n - [(1-T\gamma)I_n + (1+T\gamma)D^{-1}\mathscr{A}]$, $D^{-1}\mathscr{A}$, $-I_n$ and zI_n commute pairwise. Noting that λ_i is the ith eigenvalue of $D^{-1}\mathscr{A}$, we can get that $\det(zI_n + D^{-1}\mathscr{A}) = \prod_{i=1}^n (z+\lambda_i)$. It thus follows that $\det(zI_{2n} - \widetilde{A}) = \prod_{i=1}^n \{z^2 + (T\gamma-1)z + [1-(1+T\gamma)z]\lambda_i\}$. Therefore, the roots of $\det(zI_{2n} - \widetilde{A}) = 0$ satisfy that

$$z^2 + \big[T\gamma - 1 - (1+T\gamma)\lambda_i\big]z + \lambda_i = 0. \tag{8.9}$$

It can be noted that each eigenvalue of $D^{-1}\mathscr{A}$, λ_i, corresponds to two eigenvalues of $\widetilde{A}$. Instead of computing the roots of (8.9) directly, we apply the bilinear transformation $z = \frac{s+1}{s-1}$ to (8.9) to get

$$T\gamma(1-\lambda_i)s^2 + 2(1-\lambda_i)s + (2+T\gamma)\lambda_i + 2 - T\gamma = 0. \tag{8.10}$$

Because the bilinear transformation is an exact one-to-one mapping from the interior of the unit circle in the complex z-plane to the open left half of the complex s-plane, it follows that (8.9) has all roots within the unit circle if and only if (8.10) has all roots in the open left half plane.

In the following, we study the condition on T and γ under which (8.10) has all roots in the open left half plane. Letting s_1 and s_2 denote the roots of (8.10), it follows from (8.10) that

$$s_1 + s_2 = -\frac{2}{T\gamma}, \tag{8.11}$$

$$s_1 s_2 = \frac{(2+T\gamma)\lambda_i + 2 - T\gamma}{T\gamma(1-\lambda_i)}. \tag{8.12}$$

Noting that (8.11) implies that $\text{Im}(s_1) + \text{Im}(s_2) = 0$, we define $s_1 = a_1 + \iota b$ and $s_2 = a_2 - \iota b$. It can be noted that s_1 and s_2 have negative real parts if and only if $a_1 a_2 > 0$ and $a_1 + a_2 < 0$. Note that (8.11) implies $a_1 + a_2 = -\frac{2}{T\gamma} < 0$ because $T\gamma > 0$. We next show a sufficient condition on T and γ such that $a_1 a_2 > 0$ holds. By substituting the definitions of s_1 and s_2 into (8.12), we have $a_1 a_2 + b^2 + \iota(a_2 - a_1)b = \frac{(2+T\gamma)\lambda_i + 2 - T\gamma}{T\gamma(1-\lambda_i)}$, which implies

$$a_1 a_2 + b^2 = -\frac{2+T\gamma}{T\gamma} + \frac{4[1-\text{Re}(\lambda_i)]}{T\gamma|1-\lambda_i|^2}, \tag{8.13}$$

$$(a_2 - a_1)b = \frac{4\text{Im}(\lambda_i)}{T\gamma|1-\lambda_i|^2}. \tag{8.14}$$

It follows from (8.14) that $b = \frac{4\text{Im}(\lambda_i)}{T\gamma(a_2-a_1)|1-\lambda_i|^2}$. Note that $(a_2 - a_1)^2 = (a_1 + a_2)^2 - 4a_1 a_2 = \frac{4}{T^2\gamma^2} - 4a_1 a_2$. After some manipulation, (8.13) can be written as

$$K_1(a_1 a_2)^2 + K_2 a_1 a_2 + K_3 = 0, \tag{8.15}$$

where $K_1 \triangleq T^2\gamma^2|1-\lambda_i|^4$, $K_2 \triangleq -|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 4[1-\text{Re}(\lambda_i)]T\gamma|1-\lambda_i|^2$, and $K_3 \triangleq \frac{1}{T\gamma}\{4[1-\text{Re}(\lambda_i)]|1-\lambda_i|^2 - (2+T\gamma)|1-\lambda_i|^4\} - 4[\text{Im}(\lambda_i)]^2$. It can be computed that $K_2^2 - 4K_1K_3 = \{|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 4[1-\text{Re}(\lambda_i)]T\gamma|1-\lambda_i|^2\}^2 + 16T^2\gamma^2|1-\lambda_i|^4[\text{Im}(\lambda_i)]^2 \geq 0$, which implies that (8.15) has two real roots. Because $|\lambda_i| < 1$, it is straightforward to show that $K_1 > 0$. Therefore, a sufficient condition for $a_1 a_2 > 0$ is that $K_2 < 0$ and $K_3 > 0$. When $0 < T\gamma < 1$, because $|1-\lambda_i|^2 < 2[1-\text{Re}(\lambda_i)]$ as shown in the proof of the first statement, it follows that $K_2 < -|1-\lambda_i|^4 + (2+T\gamma)T\gamma|1-\lambda_i|^4 - 2T\gamma|1-\lambda_i|^4 = |1-\lambda_i|^4[-1+(T\gamma)^2] \leq 0$. Similarly, when $0 < T\gamma < \tau_i$, it follows that $K_3 > 0$. Therefore, if the positive scalars γ and T satisfy (8.8), all eigenvalues of $\widetilde{A}$ are within the unit circle. ■

In the following, we apply Lemma 8.2 to derive our main result.

Theorem 8.1. *Suppose that the leader's position $r_0[k]$ satisfies that $|\frac{r_0[k]-r_0[k-1]}{T}| \leq \bar{r}$ (i.e., the changing rate of $r_0[k]$ is bounded), and in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. When the positive scalars γ and T satisfy (8.8), using (8.4) for (8.3), the maximum tracking error of the n followers is ultimately bounded by $2T\bar{r}\|(I_{2n} - \widetilde{A})^{-1}\|_\infty$.*

Proof: It follows from (8.7) that

$$\begin{aligned}\left\|Y[k]\right\|_\infty &\leq \left\|\widetilde{A}^k Y[0]\right\|_\infty + \left\|\sum_{i=1}^{k} \widetilde{A}^{k-i}\widetilde{B}X^r[i-1]\right\|_\infty \\ &\leq \|\widetilde{A}^k\|_\infty \left\|Y[0]\right\|_\infty + 2T\bar{r}\left\|\sum_{i=0}^{k-1}\widetilde{A}^i\right\|_\infty \|\widetilde{B}\|_\infty,\end{aligned}$$

where we have used the fact that

$$\left\|X^r[i]\right\|_\infty = \left\|\big(2r_0[i] - r_0[i-1] - r_0[i+1]\big)\mathbf{1}_n\right\|_\infty \leq 2T\bar{r}$$

for all i because $|\frac{r_0[k]-r_0[k-1]}{T}| \leq \bar{r}$. When the leader has directed paths to all followers 1 to n, it follows from Lemma 8.2 that $\widetilde{A}$ has all eigenvalues within the unit circle if the positive scalars T and γ satisfy (8.8). Therefore, $\lim_{k\to\infty}\widetilde{A}^k = 0_{2n\times 2n}$. Also, it follows from Lemma 1.26 that there exists a matrix norm $|||\cdot|||$ such that $|||\widetilde{A}||| < 1$. It then follows from Lemma 1.28 that $(I_{2n} - \widetilde{A})$ is invertible and $(I_{2n} - \widetilde{A})^{-1} = \sum_{i=0}^{\infty}\widetilde{A}^i$, which implies that $\lim_{k\to\infty}\|\sum_{i=0}^{k-1}\widetilde{A}^i\|_\infty = \|(I_{2n} - \widetilde{A})^{-1}\|_\infty$. Also note that $\|\widetilde{B}\|_\infty = 1$. Therefore, we have that $\|Y[k]\|_\infty$ is ultimately bounded by $2T\bar{r}\|(I_{2n} - \widetilde{A})^{-1}\|_\infty$. The theorem then follows directly by noting that $\|Y[k]\|_\infty$ denotes the maximum tracking error of the n followers. ∎

Remark 8.2 From Theorem 8.1, it can be noted that the ultimate bound of the tracking errors using the PD-like discrete-time coordinated tracking algorithm (8.4) is proportional to the sampling period T. As T approaches zero, the tracking errors will go to zero ultimately when the changing rate of the leader's position is bounded and the leader has directed paths to all followers 1 to n.

8.1.3 Comparison Between the Proportional-like and Proportional-derivative-like Discrete-time Coordinated Tracking Algorithms

A proportional-like (P-like) continuous-time coordinated tracking algorithm for (3.1) is given as[2]

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}\big[r_i(t) - r_j(t)\big] - a_{i0}\big[r_i(t) - r_0(t)\big], \tag{8.16}$$

where $a_{ij},\ i = 1,\ldots,n,\ j = 0,\ldots,n$, are defined as in (8.1). Similar to that in Sect. 8.1.1, the P-like discrete-time coordinated tracking algorithm for (8.3) is given as

[2] The algorithm is a natural extension of the consensus algorithm (2.2).

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}\big(r_i[k] - r_j[k]\big) - a_{i0}\big(r_i[k] - r_0[k]\big). \tag{8.17}$$

Letting ε_i and ε be defined as in Sect. 8.1.2, we rewrite the closed-loop system of (8.3) using (8.17) as

$$\varepsilon_i[k+1] = \varepsilon_i[k] - T\sum_{j=1}^{n} a_{ij}\big(\varepsilon_i[k] - \varepsilon_j[k]\big) - Ta_{i0}\varepsilon_i[k] - \big(r_0[k+1] - r_0[k]\big),$$

which can then be written in a vector form as

$$\varepsilon[k+1] = Q\varepsilon[k] - \big(r_0[k+1] - r_0[k]\big)\mathbf{1}_n, \tag{8.18}$$

where $Q \triangleq I_n - T\mathscr{L} - T\text{diag}(a_{10}, \ldots, a_{n0})$ with $\mathscr{L}$ being the nonsymmetric Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}$. Note that Q is nonnegative when $0 < T < \min_{i=1,\ldots,n} \frac{1}{\sum_{j=0}^{n} a_{ij}}$.

Lemma 8.3. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. When $0 < T < \min_{i=1,\ldots,n} \frac{1}{\sum_{j=0}^{n} a_{ij}}$, Q has all eigenvalues within the unit circle.*

Proof: The proof is a direct application of Lemmas 1.18 and 1.6 and is omitted here. ∎

Theorem 8.3. *Suppose that the leader's position $r_0[k]$ satisfies $|\frac{r_0[k]-r_0[k-1]}{T}| \leq \bar{r}$, and in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. When $T < \min_{i=1,\ldots,n} \frac{1}{\sum_{j=0}^{n} a_{ij}}$, using (8.17) for (8.3), the maximum tracking error of the n followers is ultimately bounded by $\bar{r}\|[\mathscr{L} + \text{diag}\{a_{10}, \ldots, a_{n0}\}]^{-1}\|_\infty$.*

Proof: The solution of (8.18) is

$$\varepsilon[k] = Q^k\varepsilon[0] - \sum_{i=1}^{k} Q^{k-i}\big(r_0[k] - r_0[k-1]\big)\mathbf{1}_n.$$

The proof then follows a similar line to that of Theorem 8.1 by noting that $\|\varepsilon[k]\|_\infty$ denotes the maximum tracking error of the n followers. ∎

Remark 8.4 In contrast to the results in Theorem 8.1, the ultimate bound of the tracking errors using the P-like discrete-time coordinated tracking algorithm (8.17) with a dynamic leader is not proportional to the sampling period T. In fact, as shown in [248, Chap. 3], even when T approaches zero, the tracking errors using (8.17) are not guaranteed to go to zero ultimately. The comparison between Theorems 8.1 and 8.3 shows the benefit of the PD-like discrete-time coordinated tracking algorithm over the P-like discrete-time consensus algorithm when there exists a dynamic leader who is a neighbor of only a subset of the followers. As a special case, when the leader's position is constant (i.e., $\bar{r} = 0$), it follows from Theorems 8.1 and 8.3

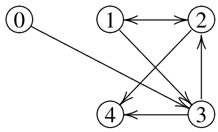


Fig. 8.1 Directed graph $\overline{\mathscr{G}}$ associated with four followers and one leader. An *arrow* from j to i denotes that agent j is a neighbor of agent i

that the tracking errors will go to zero ultimately using both the P-like and PD-like discrete-time coordinated tracking algorithms.[3]

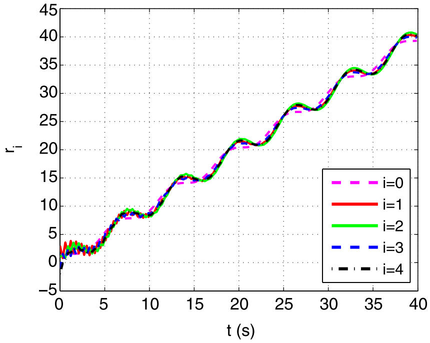
8.1.4 Simulation

In this subsection, a simulation example is presented to illustrate the PD-like discrete-time coordinated tracking algorithm (8.4). To show the benefit of the PD-like discrete-time coordinated tracking algorithm, the related simulation result obtained by applying the P-like discrete-time coordinated tracking algorithm (8.17) is also presented.

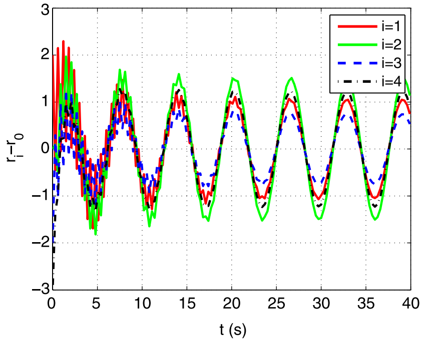
We consider a team consisting of four followers and a leader with the directed graph $\overline{\mathscr{G}}$ given by Fig. 8.1. It can be noted that the leader has directed paths to all four followers. We let $a_{ij} = 1$ if agent j is a neighbor of agent i and $a_{ij} = 0$ otherwise. For both (8.4) and (8.17), we let $r_1[0] = 3$, $r_2[0] = 1$, $r_3[0] = -1$, and $r_4[0] = -2$. For (8.4), we also let $r_i[-1] = 0$, $i = 1, \ldots, 4$. The dynamic leader's position is chosen as $r_0[k] = \sin(kT) + kT$.

Figures 8.2(a) and 8.2(b) show, respectively, the positions r_i and the tracking errors $r_i - r_0$ by using (8.4) when $T = 0.3$ s and $\gamma = 1$. From Fig. 8.2(b), it can be seen that the tracking errors are relatively large. Figures 8.2(c) and 8.2(d) show, respectively, r_i and $r_i - r_0$ by using (8.4) when $T = 0.1$ s and $\gamma = 3$. From Fig. 8.2(d), it can be seen that the tracking errors are very small ultimately. We can see that the tracking errors will become smaller if the sampling period becomes smaller. Figures 8.2(e) and 8.2(f) show, respectively, r_i and $r_i - r_0$ by using (8.4) when $T = 0.25$ s and $\gamma = 3$. Note that the product $T\gamma$ is larger than the positive upper bound derived in Theorem 8.1. It can be noted that the tracking errors become unbounded in this case. Figures 8.3(a) and 8.3(b) show, respectively, r_i and $r_i - r_0$ by using (8.17) when $T = 0.1$ s and $\gamma = 3$. By comparing Figs. 8.3(b) and 8.2(d), it can be seen that the tracking errors using (8.17) are much larger than those using (8.4) under the same condition. This shows the benefit of the PD-like discrete-time coordinated tracking algorithm over the P-like discrete-time coordinated tracking algorithm when there exists a dynamic leader who is a neighbor of only a subset of the followers.

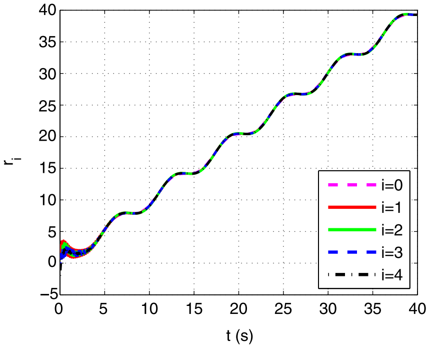
[3] In this case, the coordinated tracking problem boils down to a coordinated regulation problem because the leader's position is constant.



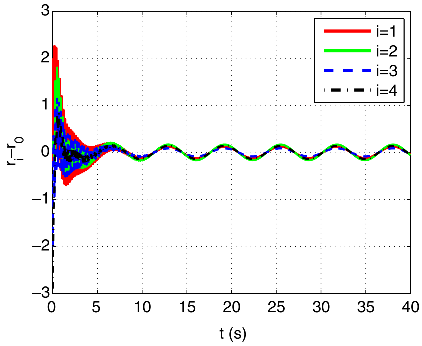
(a) Positions ($T = 0.3$ s and $\gamma = 1$)



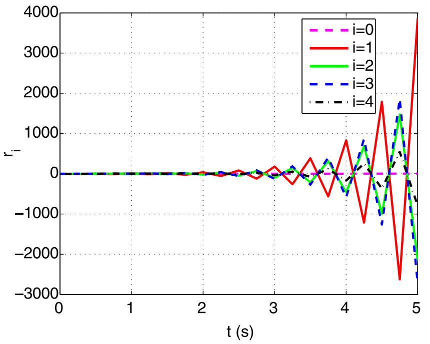
(b) Tracking errors ($T = 0.3$ s and $\gamma = 1$)



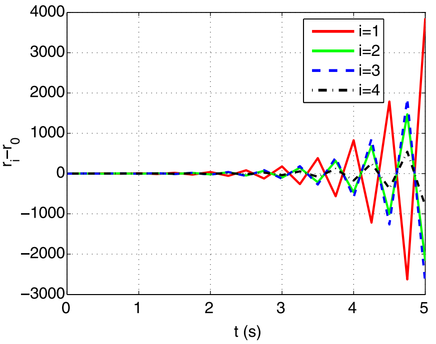
(c) Positions ($T = 0.1$ s and $\gamma = 3$)



(d) Tracking errors ($T = 0.1$ s and $\gamma = 3$)

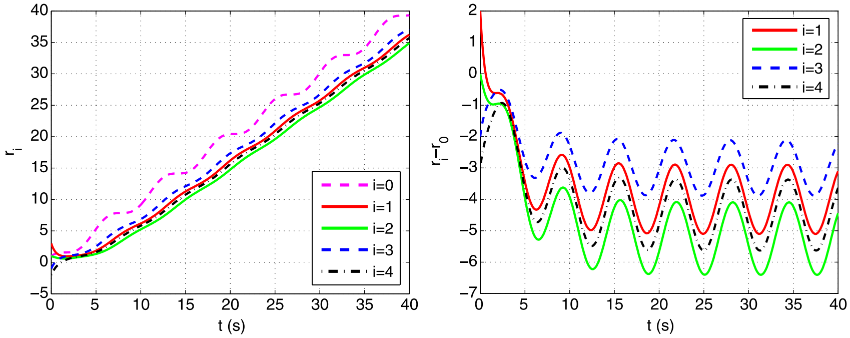


(e) Positions ($T = 0.25$ s and $\gamma = 3$)



(f) Tracking errors ($T = 0.25$ s and $\gamma = 3$)

Fig. 8.2 Distributed discrete-time coordinated tracking using the PD-like discrete-time coordinated tracking algorithm (8.4) with different T and γ



(a) Positions ($T = 0.1$ s and $\gamma = 3$)   (b) Tracking errors ($T = 0.1$ s and $\gamma = 3$)

Fig. 8.3 Distributed discrete-time coordinated tracking using the P-like discrete-time coordinated tracking algorithm (8.17)

8.2 Sampled-data Coordination for Double-integrator Dynamics Under Fixed Interaction

In this section, we study sampled-data coordination algorithms for double-integrator dynamics under fixed interaction with, respectively, absolute and relative damping.

8.2.1 Coordination Algorithms with Absolute and Relative Damping

Given n agents with dynamics given by (3.5), consider a sampled-data setting with zero-order hold as (8.2). By using direct discretization (see Sect. 1.4), the continuous-time system (3.5) can be discretized as

$$\begin{aligned} r_i[k+1] &= r_i[k] + Tv_i[k] + \frac{T^2}{2}u_i[k], \\ v_i[k+1] &= v_i[k] + Tu_i[k], \quad i = 1, \ldots, n, \end{aligned} \tag{8.19}$$

where $r_i[k] \in \mathbb{R}^m$ and $v_i[k] \in \mathbb{R}^m$ denote, respectively, the position and velocity of the ith agent at $t = kT$. Note that (8.19) is the exact discrete-time dynamics for (3.5) based on zero-order hold in a sampled-data setting.

Define $\Delta_{ij} \triangleq \delta_i - \delta_j$, where $\delta_i \in \mathbb{R}^m$ is constant. Here Δ_{ij} denotes the desired relative position deviation between agent i and agent j. We study the following two coordination algorithms

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}[k]\big[\big(r_i[k] - r_j[k]\big) - \Delta_{ij}\big] - \alpha v_i[k], \quad i = 1, \ldots, n, \tag{8.20}$$

and

$$u_i[k] = -\sum_{j=1}^{n} a_{ij}[k]\big[\big(r_i[k] - r_j[k] - \Delta_{ij}\big) + \alpha\big(v_i[k] - v_j[k]\big)\big], \quad i = 1, \ldots, n, \tag{8.21}$$

where $a_{ij}[k]$ is the (i, j)th entry of the adjacency matrix $\mathscr{A}[k]$ associated with the graph $\mathscr{G}[k] \triangleq (\mathscr{V}[k], \mathscr{E}[k])$ characterizing the interaction among the n agents at $t = kT$, and α is a position gain. Coordination is achieved for (8.20) if for all $r_i[0]$ and $v_i[0]$ and all $i, j = 1, \ldots, n$, $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to \mathbf{0}_m$ as $k \to \infty$. Coordination is achieved for (8.21) if for all $r_i[0]$ and $v_i[0]$ and all $i, j = 1, \ldots, n$, $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] - v_j[k] \to \mathbf{0}_m$ as $k \to \infty$.

In the remainder of the chapter, we assume that all agents are in a one-dimensional space (i.e., $m = 1$) for simplicity. However, all results hereafter still valid for any high-dimensional space by use of the properties of the Kronecker product.

8.2.2 Convergence Analysis of the Sampled-data Coordination Algorithm with Absolute Damping

In this subsection, we analyze the algorithm (8.20) under, respectively, an undirected fixed interaction graph and a directed fixed interaction graph. We assume that $\mathscr{A}$ is constant. In this case, using (8.20), (8.19) can be written in a vector form as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & (T - \frac{\alpha T^2}{2})I_n \\ -T\mathscr{L} & (1-\alpha T)I_n \end{bmatrix}}_{F} \begin{bmatrix} \tilde{r}[k] \\ v[k] \end{bmatrix}, \tag{8.22}$$

where $\tilde{r} \triangleq [\tilde{r}_1, \ldots, \tilde{r}_n]^T$ with $\tilde{r}_i \triangleq r_i - \delta_i$, $v \triangleq [v_1, \ldots, v_n]^T$, and $\mathscr{L}$ is the nonsymmetric Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}$. To analyze (8.22), we first study the property of F, defined in (8.22). Note that the characteristic polynomial of F is given by

$$\begin{aligned}
&\det(zI_{2n} - F) \\
&\quad = \det\left(\begin{bmatrix} zI_n - (I_n - \frac{T^2}{2}\mathscr{L}) & -(T - \frac{\alpha T^2}{2})I_n \\ T\mathscr{L} & zI_n - (1-\alpha T)I_n \end{bmatrix}\right) \\
&\quad = \det\left(\left[zI_n - \left(I_n - \frac{T^2}{2}\mathscr{L}\right)\right]\left[zI_n - (1-\alpha T)I_n\right]\right. \\
&\qquad \left. - \left\{T\mathscr{L}\left[-\left(T - \frac{\alpha T^2}{2}\right)I_n\right]\right\}\right) \\
&\quad = \det\left[\big(z^2 - 2z + \alpha T z + 1 - \alpha T\big)I_n + \frac{T^2}{2}(1+z)\mathscr{L}\right],
\end{aligned}$$

where we have used Lemma 1.22 to obtain the second equality.

Let μ_i be the ith eigenvalue of $-\mathscr{L}$, we get that $\det(zI_n + \mathscr{L}) = \prod_{i=1}^n (z - \mu_i)$. It thus follows that $\det(zI_{2n} - F) = \prod_{i=1}^n (z^2 - 2z + \alpha T z + 1 - \alpha T - \frac{T^2}{2}(1 + z)\mu_i)$. Therefore, the roots of $\det(zI_{2n} - F) = 0$ (i.e., the eigenvalues of F) satisfy that

$$z^2 + \left(\alpha T - 2 - \frac{T^2}{2}\mu_i\right) z + 1 - \alpha T - \frac{T^2}{2}\mu_i = 0. \tag{8.23}$$

Note that each eigenvalue of $-\mathscr{L}$, μ_i, corresponds to two eigenvalues of F, denoted by λ_{2i-1} and λ_{2i}. Note that $\mathscr{L}$ has at least one zero eigenvalue, without loss of generality, let $\mu_1 = 0$. It follows from (8.23) that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. Therefore, F has at least one eigenvalue equal to one. Let $[p^T, q^T]^T$, where $p, q \in \mathbb{R}^n$, be a right eigenvector of F associated with the eigenvalue $\lambda_1 = 1$. It follows that

$$\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & (T - \frac{\alpha T^2}{2})I_n \\ -T\mathscr{L} & (1 - \alpha T)I_n \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

After some manipulation, it follows from Lemma 1.1 that we can choose $p = \mathbf{1}_n$ and $q = \mathbf{0}_n$. Similarly, it can be shown that $[\mathbf{p}^T, (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T]^T$, where $\mathbf{p} \in \mathbb{R}^n$ is defined in Lemma 1.1, is a left eigenvector of F associated with the eigenvalue $\lambda_1 = 1$.

Lemma 8.4. *Using* (8.20) *for* (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}$ *and* $v_i[k] \to 0$ *if and only if one is the unique eigenvalue of* F*, where* F *is defined in* (8.22)*, with the maximum modulus. In particular,* $r_i[k] \to \delta_i + \mathbf{p}^T \tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0]$ *and* $v_i[k] \to 0$ *as* $k \to \infty$*, where* $\mathbf{p} \in \mathbb{R}^n$ *is defined in Lemma* 1.1.

Proof: (Sufficiency) Note that $p = [\mathbf{1}_n^T, \mathbf{0}_n^T]^T$ and $q = [\mathbf{p}^T, (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T]^T$ are, respectively, a right and left eigenvector of F associated with the eigenvalue one. Also note that $p^T q = 1$. If one is the unique eigenvalue with the maximum modulus, then it follows from Lemma 1.7 that $\lim_{k\to\infty} F^k = \left[\begin{smallmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{smallmatrix}\right][\mathbf{p}^T, (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T]$. Therefore, it follows that $\lim_{k\to\infty} \left[\begin{smallmatrix} \tilde{r}[k] \\ v[k] \end{smallmatrix}\right] = \lim_{k\to\infty} F^k \left[\begin{smallmatrix} \tilde{r}[0] \\ v[0] \end{smallmatrix}\right] = \left[\begin{smallmatrix} \tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0] \\ \mathbf{0}_n \end{smallmatrix}\right]$.

(Necessity) Note that F can be written in the Jordan canonical form as $F = PJP^{-1}$, where J is the Jordan block matrix. If $\tilde{r}_i[k] \to \mathbf{p}^T \tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0]$ and $v_i[k] \to 0$ as $k \to \infty$, it follows that $\lim_{k\to\infty} F^k = \left[\begin{smallmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{smallmatrix}\right][\mathbf{p}^T, (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T]$, which has rank one. It thus follows that $\lim_{k\to\infty} J^k$ has rank one, which implies that all but one eigenvalue of F are within the unit circle. Noting that F has at least one eigenvalue equal to one, it follows that one is the unique eigenvalue of F with the maximum modulus. ■

We first show necessary and sufficient conditions on α and T such that coordination is achieved using (8.20) under an undirected interaction graph. Note that all eigenvalues of $\mathscr{L}$ are real for undirected graphs because $\mathscr{L}$ is symmetric in this case. Before moving on, we need the following lemma.

Lemma 8.5. *The polynomial*

$$z^2 + az + b = 0, \tag{8.24}$$

where $a, b \in \mathbb{C}$, has all roots within the unit circle if and only if all roots of

$$(1 + a + b)s^2 + 2(1 - b)s + b - a + 1 = 0 \tag{8.25}$$

are in the open left half plane.

Proof: By applying bilinear transformation $z = \frac{s+1}{s-1}$, (8.24) can be rewritten as

$$(s + 1)^2 + a(s + 1)(s - 1) + b(s - 1)^2 = 0,$$

which implies (8.25). Note that the bilinear transformation maps the open left half plane one-to-one onto the interior of the unit circle. The lemma follows directly. ■

Lemma 8.6. *Suppose that the undirected graph $\mathscr{G}$ is connected. All eigenvalues of F, where F is defined in (8.22), are within the unit circle except one eigenvalue equal to one if and only if the positive α and T are chosen from the set*

$$S_r \triangleq \left\{ (\alpha, T) \middle| -\frac{T^2}{2} \min_i \mu_i < \alpha T < 2 \right\}.^{4} \tag{8.26}$$

Proof: When the undirected graph $\mathscr{G}$ is connected, it follows from Lemma 1.1 that $\mu_i < 0, i = 2, \ldots, n$, by noting that $\mu_1 = 0$. Also note that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. To ensure $|\lambda_2| < 1$, it is required that $0 < \alpha T < 2$. Let $a \triangleq \alpha T - 2 - \frac{T^2}{2}\mu_i$ and $b \triangleq 1 - \alpha T - \frac{T^2}{2}\mu_i$. It follows from Lemma 8.5 that for $\mu_i < 0, i = 2, \ldots, n$, the roots of (8.23) are within the unit circle if and only if all roots of

$$-T^2\mu_i s^2 + \left(T^2\mu_i + 2\alpha T\right)s + 4 - 2\alpha T = 0 \tag{8.27}$$

are in the open left half plane. Because $-T^2\mu_i > 0$, $i = 2, \ldots, n$, the roots of (8.27) are always in the open left half plane if and only if $T^2\mu_i + 2\alpha T > 0$ and $4 - 2\alpha T > 0$, which implies that $-\frac{T^2}{2}\mu_i < \alpha T < 2$. Combining the above arguments proves the lemma. ■

Theorem 8.5. *Suppose that the undirected graph $\mathscr{G}$ is connected. Let $\mathbf{p} \in \mathbb{R}^n$ be defined in Lemma 1.1. Using (8.20) for (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to 0$ if and only if α and T are chosen from S_r, where S_r is defined by (8.26). In particular, $r_i[k] \to \delta_i + \mathbf{p}^T\tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0]$ and $v_i[k] \to 0$ as $k \to \infty$.*

Proof: The statement follows directly from Lemmas 8.4 and 8.6. ■

Remark 8.6 From Lemma 8.6, we can get that $T < \frac{2}{\sqrt{-\mu_i}}$. From Lemma 1.18, it follows that $|\mu_i| \le 2\max_i \ell_{ii}$, where ℓ_{ii} is the ith diagonal entry of $\mathscr{L}$. Therefore, if $T < \sqrt{\frac{2}{\max_i \ell_{ii}}}$, then we have that $T < \frac{2}{\sqrt{-\mu_i}}$. Note that $\max_i \ell_{ii} = \max_i \sum_{j=1, j\neq i}^{n} a_{ij}$ represents the maximal in-degree of the nodes in the graph $\mathscr{G}$ under the assumption that $a_{ii} = 0$. Therefore, the sufficient bound of the sampling period is related to the maximal in-degree of the nodes in $\mathscr{G}$.

[4] Note that S_r is nonempty.

We next show necessary and sufficient conditions on α and T such that coordination is achieved using (8.20) under a directed interaction graph. Because it is not easy to find the explicit bounds for α and T such that the necessary and sufficient conditions are satisfied, we also present sufficient conditions that can be used to compute the explicit bounds for α and T. Note that the eigenvalues of $\mathscr{L}$ might be complex for directed graphs, which makes the analysis more challenging.

Lemma 8.7. *Suppose that the directed graph $\mathscr{G}$ has a directed spanning tree. There exist positive α and T such that the following three conditions are satisfied:*

1. $0 < \alpha T < 2$;
2. *When* $\mu_i < 0$, $(\alpha, T) \in S_r$, *where* S_r *is defined by* (8.26);
3. *When* $\mathrm{Re}(\mu_i) < 0$ *and* $\mathrm{Im}(\mu_i) \neq 0$, α *and* T *satisfy that*

   (i) *If* $\alpha > \frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, *then* $0 < T < \frac{-2\alpha\mathrm{Re}(\mu_i)}{|\mu_i|^2}$.

   (ii) *If* $\frac{|\mathrm{Im}(\mu_i)|}{\sqrt{-\mathrm{Re}(\mu_i)}} \le \alpha \le \frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, *then* $0 < T < \min\{\overline{T}_{i1}, \frac{-2\alpha\mathrm{Re}(\mu_i)}{|\mu_i|^2}\}$, *where*

$$\overline{T}_{i1} \triangleq \frac{-2\alpha[\mathrm{Re}(\mu_i)]^2 - 2\,|\mathrm{Im}(\mu_i)|\,\sqrt{[-\mathrm{Re}(\mu_i)][\alpha^2\mathrm{Re}(\mu_i) + |\mu_i|^2]}}{\mathrm{Re}(\mu_i)|\mu_i|^2}. \tag{8.28}$$

   (iii) *If* $0 < \alpha < \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, *then* $0 < T < \min\{\overline{T}_{i2}, \frac{-2\alpha\mathrm{Re}(\mu_i)}{|\mu_i|^2}\}$, *where*

$$\overline{T}_{i2} \triangleq \frac{-2\alpha[\mathrm{Re}(\mu_i)]^2 + 2\,|\mathrm{Im}(\mu_i)|\,\sqrt{[-\mathrm{Re}(\mu_i)][\alpha^2\mathrm{Re}(\mu_i) + |\mu_i|^2]}}{\mathrm{Re}(\mu_i)|\mu_i|^2}. \tag{8.29}$$

In addition, all eigenvalues of F, where F is defined in (8.22)*, are within the unit circle except for one eigenvalue equal to one if and only if the previous three conditions are satisfied.*

Proof: For the first statement, when T is sufficiently small, there always exists a positive α such that Conditions 1, 2, and 3 are satisfied.

For the second statement, because $\mu_1 = 0$, it follows that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. Therefore, λ_2 is within the unit circle if and only if Condition 1 is satisfied. When $\mu_i < 0$, $i \neq 1$, it follows from a similar line to that in Lemma 8.6 that all roots of F corresponding to μ_i are within the unit circle if and only if Condition 2 is satisfied. We next consider the case when $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, $i \neq 1$. Letting s_1 and s_2 be the two roots of (8.27), it follows that $\mathrm{Re}(s_1) + \mathrm{Re}(s_2) = 1 + 2\frac{\alpha}{T}\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2}$. Therefore, a necessary condition to guarantee that both s_1 and s_2 are in the open left half plane is that $1 + 2\frac{\alpha}{T}\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2} < 0$, i.e., $\frac{\alpha}{T} > -\frac{|\mu_i|^2}{2\mathrm{Re}(\mu_i)}$. To find the exact bound on T, we assume that one root of (8.27) is on the imaginary axis. Without loss of generality, let $s_1 = \chi\iota$, where $\chi \in \mathbb{R}$. Substituting $s_1 = \chi\iota$ into (8.27) and separating the corresponding real and imaginary parts give that

$$T^2\mathrm{Re}(\mu_i)\chi^2 - T^2\mathrm{Im}(\mu_i)\chi + 4 - 2\alpha T = 0, \tag{8.30}$$
$$T^2\mathrm{Im}(\mu_i)\chi^2 + \left[T^2\mathrm{Re}(\mu_i) + 2\alpha T\right]\chi = 0. \tag{8.31}$$

It follows from (8.31) that

$$\chi = -\frac{T\mathrm{Re}(\mu_i) + 2\alpha}{T\mathrm{Im}(\mu_i)}. \tag{8.32}$$

Substituting (8.32) into (8.30) gives

$$\frac{\mathrm{Re}(\mu_i)[T\mathrm{Re}(\mu_i) + 2\alpha]^2}{[\mathrm{Im}(\mu_i)]^2} + T\left[T\mathrm{Re}(\mu_i) + 2\alpha\right] + 4 - 2\alpha T = 0.$$

After some simplification, we get

$$\mathrm{Re}(\mu_i)|\mu_i|^2T^2 + 4\alpha\left[\mathrm{Re}(\mu_i)\right]^2T + 4\alpha^2\mathrm{Re}(\mu_i) + 4\left[\mathrm{Im}(\mu_i)\right]^2 = 0. \tag{8.33}$$

When $\alpha > \frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, it can be computed that

$$\begin{aligned}
&\left\{4\alpha\left[\mathrm{Re}(\mu_i)\right]^2\right\}^2 - 4\mathrm{Re}(\mu_i)|\mu_i|^2\left(4\alpha^2\mathrm{Re}(\mu_i) + 4\left[\mathrm{Im}(\mu_i)\right]^2\right) \\
&\quad = -16\left\{\alpha^2\left[\mathrm{Re}(\mu_i)\right]^2\left[\mathrm{Im}(\mu_i)\right]^2 + \mathrm{Re}(\mu_i)|\mu_i|^2\left[\mathrm{Im}(\mu_i)\right]^2\right\} \\
&\quad = -16\mathrm{Re}(\mu_i)\left[\mathrm{Im}(\mu_i)\right]^2\left[\alpha^2\mathrm{Re}(\mu_i) + |\mu_i|^2\right] < 0.
\end{aligned}$$

Therefore, there does not exist a positive T such that one root of (8.27) is on the imaginary axis, which implies that s_1 (respectively, s_2) is always on the open left or right half plane. Because $\mathrm{Re}(s_1) + \mathrm{Re}(s_2) = 1 + 2\frac{\alpha}{T}\frac{\mathrm{Re}(\mu_i)}{|\mu_i|^2}$, when α is sufficiently large, it follows that $\mathrm{Re}(s_1) + \mathrm{Re}(s_2) < 0$. This implies that s_1 (respectively, s_2) is always on the open left half plane when $\alpha > \frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$. When $\frac{|\mathrm{Im}(\mu_i)|}{\sqrt{-\mathrm{Re}(\mu_i)}} \leq \alpha \leq \frac{|\mu_i|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, it follows that $4\alpha^2\mathrm{Re}(\mu_i) + 4[\mathrm{Im}(\mu_i)]^2 \geq 0$. Noting that $\mathrm{Re}(\mu_i)|\mu_i|^2 < 0$, it follows that there exists a unique positive $\overline{T}_{i1}$ such that (8.33) holds when $T = \overline{T}_{i1}$, where $\overline{T}_{i1}$ is given by (8.28). Similarly, when $0 < \alpha < \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{-\mathrm{Re}(\mu_i)}}$, it follows that $4\alpha^2\mathrm{Re}(\mu_i) + 4[\mathrm{Im}(\mu_i)]^2 < 0$. Noting also that $\mathrm{Re}(\mu_i)|\mu_i|^2 < 0$, it follows that there are two positive solutions with the smaller one given by $\overline{T}_{i2}$ defined by (8.29).

Combining the previous arguments completes the proof. ∎

Theorem 8.7. *Suppose that the directed graph $\mathscr{G}$ has a directed spanning tree. Let $\mathbf{p} \in \mathbb{R}^n$ be defined in Lemma* 1.1. *Using* (8.20) *for* (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}$ *and* $v_i[k] \to 0$ *if and only if α and T are chosen satisfying the conditions in Lemma* 8.7. *In particular,* $r_i[k] \to \delta_i + \mathbf{p}^T\tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^Tv[0]$ *and* $v_i[k] \to 0$ *as* $k \to \infty$.

Proof: The statement follows directly from Lemmas 8.4 and 8.7. ∎

From Lemma 8.7, it is not easy to find α and T explicitly such that the conditions in Lemma 8.7 are satisfied. We next present a sufficient condition that can be used to determine the bounds for α and T explicitly. Before moving on, we need the following lemmas and corollary.

Lemma 8.8 ([50, 252]). *All zeros of the complex polynomial*

$$P(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n$$

satisfy $|z| \leq z_0$*, where* z_0 *is the unique nonnegative solution of the equation*

$$z^n - |\alpha_1| z^{n-1} - \cdots - |\alpha_{n-1}| z - |\alpha_n| = 0.$$

The bound z_0 *is attained if* $\alpha_i = -|\alpha_i|$.

Corollary 8.1. *The roots of* (8.24) *are within the unit circle if* $|a| + |b| < 1$. *Moreover, if* $|a+b| + |a-b| < 1$*, the roots of* (8.24) *are still within the unit circle.*

Proof: According to Lemma 8.8, the roots of (8.24) are within the unit circle if the unique nonnegative solution z_0 of $z^2 - |a|z - |b| = 0$ satisfies $z_0 < 1$. It is straightforward to show that $z_0 = \frac{|a|+\sqrt{|a|^2+4|b|}}{2}$. Therefore, the roots of (8.24) are within the unit circle if

$$|a| + \sqrt{|a|^2 + 4|b|} < 2. \tag{8.34}$$

We next discuss the condition under which (8.34) holds. If $b = 0$, then the statements of the corollary hold trivially. If $|b| \neq 0$, we have that

$$\frac{(|a| + \sqrt{|a|^2 + 4|b|})(-|a| + \sqrt{|a|^2 + 4|b|})}{-|a| + \sqrt{|a|^2 + 4|b|}} < 2.$$

After some computation, it follows that (8.34) is equivalent to $|a| + |b| < 1$. Therefore, the first statement of the corollary holds. For the second statement, because $|a| + |b| \leq |a+b| + |a-b|$, if $|a+b| + |a-b| < 1$, then $|a| + |b| < 1$, which implies that the second statement of the corollary also holds. ■

The following lemma presents a sufficient condition that can be used to find α and T explicitly.

Lemma 8.9. *Suppose that the directed graph* $\mathscr{G}$ *has a directed spanning tree. There exist positive* α *and* T *such that* $S_c \cap S_r$ *is nonempty, where*

$$S_c \triangleq \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0} \left\{ (\alpha, T) \middle| \left|1 + T^2 \mu_i\right| + |3 - 2\alpha T| < 1 \right\}, \tag{8.35}$$

and

$$S_r \triangleq \bigcap_{\forall \mu_i \leq 0} \left\{ (\alpha, T) | -\frac{T^2}{2}\mu_i < \alpha T < 2 \right\}. \tag{8.36}$$

If α *and* T *are chosen from* $S_c \cap S_r$*, then all eigenvalues of* F *are within the unit circle except one eigenvalue equal to one.*

Proof: For the first statement, we let $\alpha T = \frac{3}{2}$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, $|1+T^2\mu_i|+|3-2\alpha T| < 1$ implies that $|1+T^2\mu_i| < 1$ because $\alpha T = \frac{3}{2}$. It thus follows that $0 < T < \frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}$ for all $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$. When $\mu_i \leq 0$, $-\frac{T^2}{2}\mu_i < \alpha T < 2$ can be simplified as $-T^2\mu_i < \frac{3}{2}$ because $\alpha T = \frac{3}{2}$. It thus follows that $0 < T < \sqrt{\frac{3}{-\mu_i}}$ for all $\mu_i \leq 0$. Let $T_c \triangleq \bigcap_{\forall \mathrm{Re}(\mu_i)<0 \text{ and } \mathrm{Im}(\mu_i)\neq 0}\{T|0 < T < \frac{\sqrt{-2\mathrm{Re}(\mu_i)}}{|\mu_i|}\}$ and $T_r \triangleq \bigcap_{\forall \mu_i \leq 0}\{T|0 < T < \sqrt{\frac{3}{-\mu_i}}\}$.[5] It is straightforward to see that $T_c \cap T_r$ is nonempty. Recalling that $\alpha T = \frac{3}{2}$, it follows that $S_c \cap S_r$ is nonempty as well.

For the second statement, note that if the directed graph $\mathscr{G}$ has a directed spanning tree, then it follows from Lemma 1.1 that $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$, by noting that $\mu_1 = 0$. Also note that $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha T$. To ensure that $|\lambda_2| < 1$, it is required that $0 < \alpha T < 2$. When $\mathrm{Re}(\mu_i) < 0$ and $\mathrm{Im}(\mu_i) \neq 0$, it follows from Corollary 8.1 that the roots of (8.23) are within the unit circle if $|1 + T^2\mu_i| + |3 - 2\alpha T| < 1$, where we have used the second statement of Corollary 8.1 by letting $a = \alpha T - 2 - \frac{T^2}{2}\mu_i$ and $b = 1 - \frac{T^2}{2}\mu_i - \alpha T$. When $\mu_i < 0$, it follows from the proof of Lemma 8.6 that the roots of (8.23) are within the unit circle if $-\frac{T^2}{2}\mu_i < \alpha T < 2$. Combining the above arguments proves the second statement. ∎

Remark 8.8 According to Lemmas 8.4 and 8.9, if α and T are chosen from $S_c \cap S_r$, where S_c is defined by (8.35) and S_r is defined by (8.36), and the directed graph $\mathscr{G}$ has a directed spanning tree, coordination is achieved ultimately. An easy way to choose α and T is to let $\alpha T = \frac{3}{2}$. It then follows that T can be chosen from $T_c \cap T_r$, where T_c and T_r are defined in the proof of Lemma 8.9.

8.2.3 *Convergence Analysis of the Sampled-data Coordination Algorithm with Relative Damping*

In this subsection, we analyze the algorithm (8.21) under, respectively, an undirected fixed interaction graph and a directed fixed interaction graph. We assume that $\mathscr{A}$ is constant. In this case, using (8.21), (8.19) can be written in a vector form as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & TI_n - \frac{T^2}{2}\mathscr{L} \\ -T\mathscr{L} & I_n - \alpha T\mathscr{L} \end{bmatrix}}_{G} \begin{bmatrix} \tilde{r}[k] \\ v[k] \end{bmatrix}, \tag{8.37}$$

where $\tilde{r}$, v, and $\mathscr{L}$ are defined as in (8.22). A similar analysis to that for (8.22) shows that the roots of $\det(zI_{2n} - G) = 0$ (i.e., the eigenvalues of G) satisfy

[5] When $\mu_i = 0$, $T > 0$ can be chosen arbitrarily.

$$z^2 - \left(2 + \alpha T \mu_i + \frac{1}{2}T^2\mu_i\right)z + 1 + \alpha T \mu_i - \frac{1}{2}T^2\mu_i = 0. \tag{8.38}$$

Similarly, each eigenvalue of $-\mathscr{L}$, μ_i, corresponds to two eigenvalues of G, denoted by ρ_{2i-1} and ρ_{2i}. Note that $\mathscr{L}$ has at least one zero eigenvalue. Without loss of generality, let $\mu_1 = 0$, which implies that $\rho_1 = \rho_2 = 1$. Therefore, G has at least two eigenvalues equal to one.

Lemma 8.10. *Using* (8.21) *for* (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}$ *and* $v_i[k] - v_j[k] \to 0$ *as* $k \to \infty$ *if and only if* G, *where* G *is defined in* (8.37), *has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one. In particular,* $r_i[k] - \delta_i - (\mathbf{p}^T\tilde{r}[0] + kT\mathbf{p}^T v[0]) \to 0$ *and* $v_i[k] \to \mathbf{p}^T v[0]$ *as* $k \to \infty$, *where* $\mathbf{p} \in \mathbb{R}^n$ *is defined in Lemma* 1.1.

Proof: (Sufficiency) Note from (8.38) that if G has exactly two eigenvalues equal to one (i.e., $\rho_1 = \rho_2 = 1$), then $-\mathscr{L}$ has exactly one eigenvalue equal to zero. Let $[p^T, q^T]^T$, where $p, q \in \mathbb{R}^n$, be a right eigenvector of G associated with the eigenvalue one. It follows that

$$\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L} & TI_n - \frac{T^2}{2}\mathscr{L} \\ -T\mathscr{L} & I_n - \alpha T\mathscr{L} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

After some computation, it follows that the eigenvalue one has geometric multiplicity equal to one even if it has algebraic multiplicity equal to two. It also follows from Lemma 1.1 that we can choose $p = \mathbf{1}_n$ and $q = \mathbf{0}_n$. In addition, a generalized right eigenvector associated with the eigenvalue one can be chosen as $[\mathbf{0}_n^T, \frac{1}{T}\mathbf{1}_n^T]^T$. Similarly, it can be shown that $[\mathbf{0}_n^T, T\mathbf{p}_n^T]^T$ and $[\mathbf{p}^T, \mathbf{0}_n^T]^T$ are, respectively, a left eigenvector and generalized left eigenvector associated with the eigenvalue one. Note that G can be written in the Jordan canonical form as $G = PJP^{-1}$, where the columns of P, denoted by p_k, $k = 1, \ldots, 2n$, can be chosen to be the right eigenvectors or generalized right eigenvectors of G, the rows of P^{-1}, denoted by q_k^T, $k = 1, \ldots, 2n$, can be chosen to be the left eigenvectors or generalized left eigenvectors of G such that $p_k^T q_k = 1$ and $p_k^T q_\ell = 0$, $k \neq \ell$, and J is the Jordan block diagonal matrix with the eigenvalues of G being the diagonal entries. Note that $\rho_1 = \rho_2 = 1$ and $|\rho_k| < 1$, $k = 3, \ldots, 2n$. Also note that we can choose $p_1 = [\mathbf{1}_n^T, \mathbf{0}_n^T]^T$, $p_2 = [\mathbf{0}_n^T, \frac{1}{T}\mathbf{1}_n^T]^T$, $q_1 = [\mathbf{p}^T, \mathbf{0}_n^T]^T$, and $q_2 = [\mathbf{0}_n^T, T\mathbf{p}_n^T]^T$. Because $\left[\begin{smallmatrix} \tilde{r}[k] \\ v[k] \end{smallmatrix}\right] = G^k \left[\begin{smallmatrix} \tilde{r}[0] \\ v[0] \end{smallmatrix}\right] = PJ^kP^{-1}\left[\begin{smallmatrix} \tilde{r}[0] \\ v[0] \end{smallmatrix}\right]$ and $\lim_{k\to\infty}\left\|PJ^kP^{-1} - \left[\begin{smallmatrix} \mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & \frac{1}{T}\mathbf{1}_n \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{p}^T & \mathbf{0}_n^T \\ \mathbf{0}_n^T & T\mathbf{p}^T \end{smallmatrix}\right]\right\| = \lim_{k\to\infty}\left\|PJ^kP^{-1} - \left[\begin{smallmatrix} \mathbf{1}_n\mathbf{p}^T & kT\mathbf{1}_n\mathbf{p}^T \\ 0_{n\times n} & \mathbf{1}_n\mathbf{p}^T \end{smallmatrix}\right]\right\| = 0$, it follows that $|\tilde{r}_i[k] - \mathbf{p}^T\tilde{r}[0] - kT\mathbf{p}^T v[0]| \to 0$ and $v_i[k] \to \mathbf{p}^T v[0]$ as $k \to \infty$.

(Necessity) Note that G has at least two eigenvalues equal to one. If $\tilde{r}_i[k] - \mathbf{p}^T\tilde{r}[0] - kT\mathbf{p}^T v[0] \to 0$ and $v_i[k] \to \mathbf{p}^T v[0]$ as $k \to \infty$, it follows that F^k has rank two as $k \to \infty$, which in turn implies that J^k has rank two as $k \to \infty$. It follows that G has exactly two eigenvalues equal to one and all other eigenvalues have modulus smaller than one. ■

We first show necessary and sufficient conditions on α and T such that coordination is achieved using (8.21) under an undirected interaction graph.

Lemma 8.11. *Suppose that the undirected graph $\mathscr{G}$ is connected. All eigenvalues of G, where G is defined in (8.37), are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from the set*

$$Q_r \triangleq \left\{(\alpha,T)\middle|\frac{T^2}{2} < \alpha T < -\frac{2}{\min_i \mu_i}\right\}.^{6} \tag{8.39}$$

Proof: Because the undirected graph $\mathscr{G}$ is connected, it follows from Lemma 1.1 that $\mu_i < 0$, $i = 2,\dots,n$, by noting that $\mu_1 = 0$. Also note that $\rho_1 = \rho_2 = 1$. Let $a = -(2+\alpha T\mu_i + \frac{1}{2}T^2\mu_i)$ and $b = 1+\alpha T\mu_i - \frac{1}{2}T^2\mu_i$. It follows from Lemma 8.5 that for $\mu_i < 0$, $i = 2,\dots,n$, the roots of (8.38) are within the unit circle if and only if all roots of

$$-T^2\mu_i s^2 + \big(T^2\mu_i - 2\alpha T\mu_i\big)s + 4 + 2\alpha T\mu_i = 0, \tag{8.40}$$

are in the open left half plane. Because $-T^2\mu_i > 0$, the roots of (8.40) are always in the open left half plane if and only if $4 + 2\alpha T\mu_i > 0$ and $T^2\mu_i - 2\alpha T\mu_i > 0$, which implies that $\frac{T^2}{2} < \alpha T < -\frac{2}{\mu_i}$, $i = 2,\dots,n$. Combining the above arguments proves the lemma. ∎

Theorem 8.9. *Suppose that the undirected graph $\mathscr{G}$ is connected. Let $\mathbf{p} \in \mathbb{R}^n$ be defined in Lemma 1.1. Using (8.21), $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] - v_j[k] \to 0$ as $k \to \infty$ if and only if α and T are chosen from Q_r, where Q_r is defined by (8.39). In particular, $r_i[k] - \delta_i - (\mathbf{p}^T\tilde{r}[0] + kT\mathbf{p}^T v[0]) \to 0$ and $v_i[k] \to \mathbf{p}^T v[0]$ as $k \to \infty$.*

Proof: The statement follows directly from Lemmas 8.10 and 8.11. ∎

We next show necessary and sufficient conditions on α and T such that coordination is achieved using (8.21) under a directed interaction graph. Note again that the eigenvalues of $\mathscr{L}$ might be complex for directed graphs, which makes the analysis more challenging.

Lemma 8.12. *Suppose that $\mathrm{Re}(\mu_i) < 0$. The roots of (8.38) are within the unit circle if and only if $\frac{\alpha}{T} > \frac{1}{2}$ and $B_i < 0$, where*

$$B_i \triangleq \left(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2T^2} + \frac{2\alpha}{T}\right)\left(1 - \frac{2\alpha}{T}\right)^2 + \frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4T^4}. \tag{8.41}$$

Proof: As in the proof of Lemma 8.11, the roots of (8.38) are within the unit circle if and only if the roots of (8.40) are in the open left half plane. Letting s_1 and s_2 denote the roots of (8.40), it follows that

$$s_1 + s_2 = 1 - 2\frac{\alpha}{T} \tag{8.42}$$

[6] Note that Q_r is nonempty.

and

$$s_1 s_2 = -\frac{4}{\mu_i T^2} - 2\frac{\alpha}{T}. \tag{8.43}$$

Noting that (8.42) implies that $\mathrm{Im}(s_1) + \mathrm{Im}(s_2) = 0$, we define $s_1 = a_1 + \iota b$ and $s_2 = a_2 - \iota b$. Note that s_1 and s_2 have negative real parts if and only if $a_1 + a_2 < 0$ and $a_1 a_2 > 0$. Note from (8.42) that $a_1 + a_2 < 0$ is equivalent to $\frac{\alpha}{T} > \frac{1}{2}$. We next show conditions on α and T such that $a_1 a_2 > 0$ holds. Substituting the definitions of s_1 and s_2 into (8.43) gives $a_1 a_2 + b^2 + \iota(a_2 - a_1)b = -\frac{4}{\mu_i T^2} - 2\frac{\alpha}{T}$, which implies

$$(a_2 - a_1)b = \frac{4\mathrm{Im}(\mu_i)}{|\mu_i|^2 T^2}, \tag{8.44}$$

$$a_1 a_2 + b^2 = \frac{-4\mathrm{Re}(\mu_i)}{|\mu_i|^2 T^2} - 2\frac{\alpha}{T}. \tag{8.45}$$

It follows from (8.44) that $b = \frac{4\mathrm{Im}(\mu_i)}{|\mu_i|^2 T^2 (a_2 - a_1)}$. Consider also the fact that $(a_2 - a_1)^2 = (a_2 + a_1)^2 - 4a_1 a_2 = (1 - 2\frac{\alpha}{T})^2 - 4a_1 a_2$. After some manipulation, (8.45) can be written as

$$4(a_1 a_2)^2 + A_i a_1 a_2 - B_i = 0, \tag{8.46}$$

where $A_i \triangleq 4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2 T^2} + 2\frac{\alpha}{T}) - (1 - 2\frac{\alpha}{T})^2$ and B_i is defined by (8.41). It follows that $A_i^2 + 16B_i = [4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2 T^2} + 2\frac{\alpha}{T}) + (1 - 2\frac{\alpha}{T})^2]^2 + \frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4 T^4} \geq 0$, which implies that (8.46) has two real roots. Therefore, the necessary and sufficient conditions for $a_1 a_2 > 0$ are $B_i < 0$ and $A_i < 0$. Because $\frac{16\mathrm{Im}(\mu_i)^2}{|\mu_i|^4 T^4} > 0$, if $B_i < 0$ then $4(\frac{4\mathrm{Re}(\mu_i)}{|\mu_i|^2 T^2} + 2\frac{\alpha}{T}) < 0$, which implies $A_i < 0$ as well. Combining the previous arguments proves the lemma. ■

Lemma 8.13. *Suppose that the directed graph $\mathscr{G}$ has a directed spanning tree. There exist positive α and T such that Q_c is nonempty, where*

$$Q_c \triangleq \bigcap_{\forall \mathrm{Re}(\mu_i) < 0} \left\{ (\alpha, T) \middle| \frac{1}{2} < \frac{\alpha}{T}, B_i < 0 \right\}, \tag{8.47}$$

where B_i is defined by (8.41). All eigenvalues of G, where G is defined in (8.37), are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from Q_c.

Proof: For the first statement, we let $\alpha > T > 0$, which implies that $\frac{\alpha}{T} > \frac{1}{2}$ holds apparently. Note that $\alpha > T$ implies that $(T - 2\alpha)^2 > \alpha^2$. Therefore, a sufficient condition for $B_i < 0$ is

$$\alpha T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4 \alpha^2} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2}. \tag{8.48}$$

To ensure that there are feasible $\alpha > 0$ and $T > 0$ satisfying (8.48), we first need to ensure that the right side of (8.48) is positive, which requires that $\alpha > \frac{2|\mathrm{Im}(\mu_i)|}{|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)}}$ for all $\mathrm{Re}(\mu_i) < 0$. It also follows from (8.48) that $T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha}$ for all $\mathrm{Re}(\mu_i) < 0$. Therefore, (8.47) is ensured to be nonempty if α and T are chosen from, respectively, $\alpha_c \triangleq \bigcap_{\forall \mathrm{Re}(\mu_i)<0}\{\alpha | \alpha > \frac{2|\mathrm{Im}(\mu_i)|}{|\mu_i|\sqrt{-\mathrm{Re}(\mu_i)}}\}$ and

$$T_c \triangleq \bigcap_{\forall \mathrm{Re}(\mu_i)<0} \left\{ T \middle| T < -\frac{8\mathrm{Im}(\mu_i)^2}{|\mu_i|^4\alpha^3} - \frac{2\mathrm{Re}(\mu_i)}{|\mu_i|^2\alpha} \text{ and } 0 < T < \alpha \right\}.$$

It is straightforward to see that both α_c and T_c are nonempty. Combining the above arguments shows that Q_c is nonempty.

For the second statement, note that if the directed graph $\mathscr{G}$ has a directed spanning tree, it follows from Lemma 1.1 that $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$, by noting that $\mu_1 = 0$. Also note that $\rho_1 = 1$ and $\rho_2 = 1$. When $\mathrm{Re}(\mu_i) < 0$, it follows from Lemma 8.12 that the roots of (8.38) are within the unit circle if and only if $\frac{\alpha}{T} > \frac{1}{2}$ and $B_i < 0$. It thus follows that all eigenvalues of G are within the unit circle except two eigenvalues equal to one if and only if α and T are chosen from Q_c. ■

Remark 8.10 From the proof of the first statement of Lemma 8.13, an easy way to choose α and T is to let $\alpha > T$. Then α is chosen from α_c and T is chosen from T_c, where α_c and T_c are defined in the proof of Lemma 8.13.

Theorem 8.11. *Suppose that the directed graph $\mathscr{G}$ has a directed spanning tree. Using* (8.21) *for* (8.19), *$r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] - v_j[k] \to 0$ as $k \to \infty$ if and only if α and T are chosen from Q_c, where Q_c is defined by* (8.47). *In particular, $r_i[k] - \delta_i - (\mathbf{p}^T\tilde{r}[0] + kT\mathbf{p}^T v[0]) \to 0$ and $v_i[k] \to \mathbf{p}^T v[0]$ as $k \to \infty$.*

Proof: The proof follows directly from Lemmas 8.11 and 8.13. ■

8.2.4 Simulation

In this section, we present simulation results to validate the theoretical results derived in Sects. 8.2.2 and 8.2.3. We consider a team of four agents with the directed graph $\mathscr{G}$ shown by Fig. 8.4. Note that $\mathscr{G}$ has a directed spanning tree. The nonsymmetric Laplacian matrix associated with $\mathscr{G}$ is chosen as

$$\mathscr{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1.5 & -1.5 & 0 \\ -2 & 0 & 2 & 0 \\ -2.5 & 0 & 0 & 2.5 \end{bmatrix}.$$

It can be computed that for $\mathscr{L}$, $\mathbf{p} = [0.4615, 0.3077, 0.2308, 0]^T$. Here for simplicity, we have chosen $\delta_i = 0,\ i = 1, \ldots, 4$, which implies that $\Delta_{ij} = 0$.

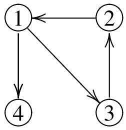
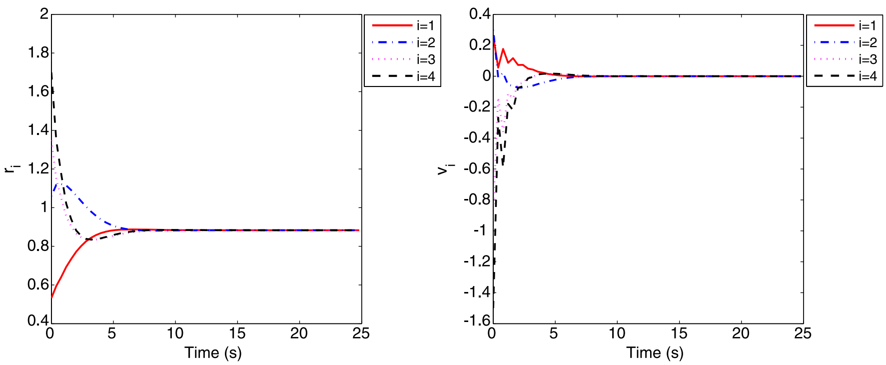
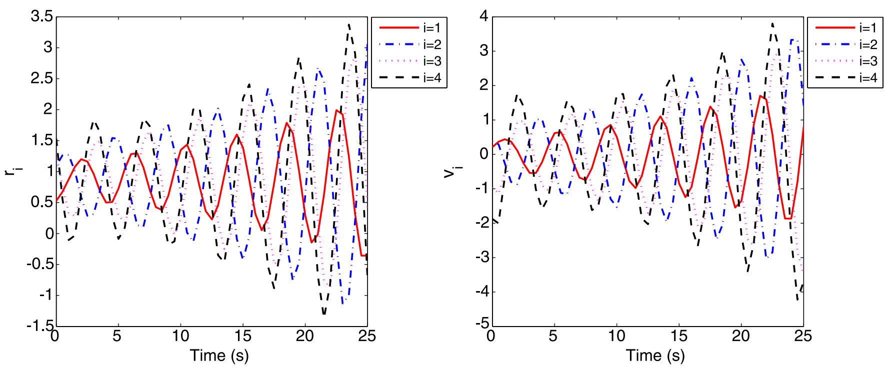


Fig. 8.4 Directed graph $\mathscr{G}$ for four agents. An *arrow* from j to i denotes that agent j is a neighbor of agent i



(a) Positions (b) Velocities

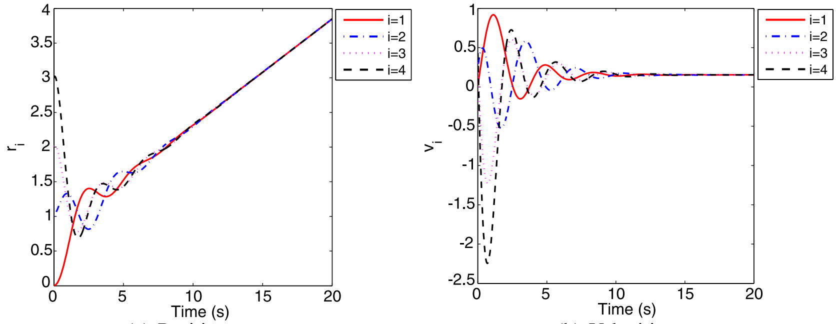
Fig. 8.5 Convergence results using (8.20) with $\alpha = 4$ and $T = 0.4$ s



(a) Positions (b) Velocities

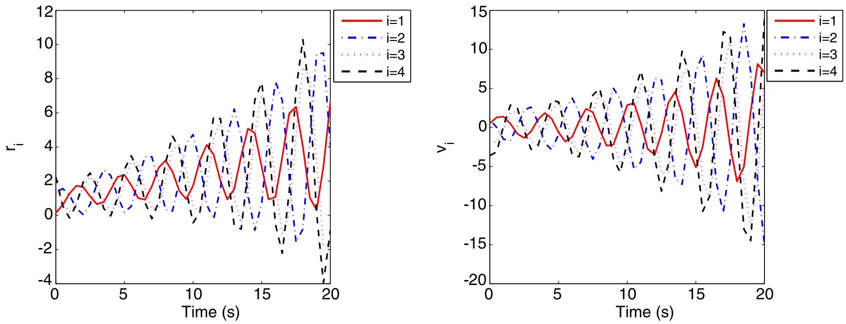
Fig. 8.6 Convergence results using (8.20) with $\alpha = 1.2$ and $T = 0.5$ s

For the coordination algorithm (8.20), let $r_i[0] = 0.5i$, $i = 1, \ldots, 4$, $v_1[0] = -0.1$, $v_2[0] = 0$, $v_3[0] = 0.1$, and $v_4[0] = 0$. Figure 8.5 shows the convergence result using (8.20) with $\alpha = 4$ and $T = 0.4$ s. Note that the conditions in Theorem 8.7 are satisfied. It can be seen that coordination is achieved with the final equilibrium for $r_i[k]$ being 0.8835, which is equal to $\delta_i + \mathbf{p}^T \tilde{r}[0] + (\frac{1}{\alpha} - \frac{T}{2})\mathbf{p}^T v[0]$ as stated in Theorem 8.7. Figure 8.6 shows the convergence result using (8.20) with $\alpha = 1.2$ and $T = 0.5$ s. Note that coordination is not achieved in this case.



(a) Positions (b) Velocities

Fig. 8.7 Convergence results using (8.21) with $\alpha = 0.6$ and $T = 0.02$ s



(a) Positions (b) Velocities

Fig. 8.8 Convergence results using (8.21) with $\alpha = 0.6$ and $T = 0.5$ s

For the coordination algorithm (8.21), let $r_i[0] = i - 1$ and $v_i[0] = 0.2(i - 1)$, $i = 1, \ldots, 4$. Figure 8.7 shows the convergence result using (8.21) with $\alpha = 0.6$ and $T = 0.02$ s. Note that the conditions in Theorem 8.13 are satisfied. It can be seen that coordination is achieved with the final equilibrium for $v_i[k]$ being 0.1538, which is equal to $\mathbf{p}^T v[0]$ as stated in Theorem 8.13. Figure 8.8 shows the convergence result using (8.21) with $\alpha = 0.6$ and $T = 0.5$ s. Note that coordination is not achieved in this case.

8.3 Sampled-data Coordination for Double-integrator Dynamics Under Switching Interaction

In this section, we study (8.20) and (8.21) under directed switching interaction. Note that there are a finite number of possible directed graphs for n agents. We assume that for each possible directed graph, there are a finite number of adjacency matrices associated with the directed graph. Therefore, all nonzero $a_{ij}[k]$ in (8.20) and (8.21) are chosen from a finite set.

8.3.1 Convergence Analysis of the Sampled-data Coordination Algorithm with Absolute Damping

In this subsection, we analyze (8.20) under a directed switching interaction graph. Here $\mathscr{A}[k]$ is switching. In this case, using (8.20), (8.19) can be written in a vector form as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L}[k] & (T - \frac{\alpha T^2}{2})I_n \\ -T\mathscr{L}[k] & (1-\alpha T)I_n \end{bmatrix}}_{F_k} \begin{bmatrix} \tilde{r}[k] \\ v[k] \end{bmatrix}, \tag{8.49}$$

where $\tilde{r}$ and v are defined as in (8.22), and $\mathscr{L}[k]$ is the nonsymmetric Laplacian matrix associated with $\mathscr{A}[k]$ and hence $\mathscr{G}[k]$. Note that the solution of (8.49) can be written as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \begin{bmatrix} B_k & C_k \\ D_k & E_k \end{bmatrix} \begin{bmatrix} \tilde{r}[0] \\ v[0] \end{bmatrix}, \tag{8.50}$$

where $\left[\begin{smallmatrix} B_k & C_k \\ D_k & E_k \end{smallmatrix}\right] \triangleq \prod_{i=0}^{k} F_i$. Therefore, B_k, C_k, D_k, and E_k satisfy

$$\begin{bmatrix} B_k \\ D_k \end{bmatrix} = F_k \begin{bmatrix} B_{k-1} \\ D_{k-1} \end{bmatrix} \tag{8.51}$$

and

$$\begin{bmatrix} C_k \\ E_k \end{bmatrix} = F_k \begin{bmatrix} C_{k-1} \\ E_{k-1} \end{bmatrix}. \tag{8.52}$$

Lemma 8.14. *Assume that $\alpha T \neq 2$. Using (8.20) for (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to 0$ as $k \to \infty$ if $\lim_{k\to\infty} B_k$ exists and all rows of $\lim_{k\to\infty} B_k$ are the same for arbitrary initial conditions.*

Proof: When $\lim_{k\to\infty} B_k$ exists and all rows of $\lim_{k\to\infty} B_k$ are the same for arbitrary initial conditions, it follows that $\lim_{k\to\infty} C_k$ exists and all rows of $\lim_{k\to\infty} C_k$ are the same for arbitrary initial conditions as well because (8.51) and (8.52) have the same structure. It then follows from (8.51) that

$$B_k = \left(I_n - \frac{T^2}{2}\mathscr{L}[k]\right)B_{k-1} + \left(T - \frac{\alpha T^2}{2}\right)D_{k-1}. \tag{8.53}$$

Because $\mathscr{L}[k]\mathbf{1}_n = \mathbf{0}_n$ and all rows of $\lim_{k\to\infty} B_{k-1}$ are the same, it follows that $\lim_{k\to\infty} \mathscr{L}[k]B_{k-1} = 0_{n\times n}$. It thus follows that

$$\lim_{k\to\infty}\left(T - \frac{\alpha T^2}{2}\right)D_{k-1} = \lim_{k\to\infty}(B_k - B_{k-1}) = 0_{n\times n}.$$

Because $\alpha T \neq 2$, i.e., $T - \frac{\alpha T^2}{2} \neq 0$, it follows that $\lim_{k\to\infty} D_k = 0_{n\times n}$ for arbitrary initial conditions. Similarly, it follows that $\lim_{k\to\infty} E_k = 0_{n\times n}$ for arbitrary initial conditions because (8.51) and (8.52) have the same structure. Combining the previous arguments with (8.50) shows that $\tilde{r}_i[k] - \tilde{r}_j[k] \to 0$ and $v_i[k] \to 0$ as $k \to \infty$, which implies that $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to 0$ as $k \to \infty$. ∎

Note from (8.53) that

$$B_{k-1} = \left(I_n - \frac{T^2}{2}\mathscr{L}[k-1]\right)B_{k-2} + \left(T - \frac{\alpha T^2}{2}\right)D_{k-2}. \tag{8.54}$$

It follows from (8.51) that

$$D_{k-1} = -T\mathscr{L}[k-1]B_{k-2} + (1-\alpha T)D_{k-2}. \tag{8.55}$$

Therefore, it follows from (8.53) and (8.54) that

$$\begin{aligned} B_k - (1-\alpha T)B_{k-1} = &\left(I_n - \frac{T^2}{2}\mathscr{L}[k]\right)B_{k-1} \\ &- (1-\alpha T)\left(I_n - \frac{T^2}{2}\mathscr{L}[k-1]\right)B_{k-2} \\ &+ \left(T - \frac{\alpha T^2}{2}\right)\big[D_{k-1} - (1-\alpha T)D_{k-2}\big]. \end{aligned} \tag{8.56}$$

By substituting (8.55) into (8.56), (8.56) can be simplified as

$$B_k = \Phi_{k1}B_{k-1} + \Phi_{k2}B_{k-2}, \tag{8.57}$$

where

$$\Phi_{k1} \triangleq (2-\alpha T)I_n - \frac{T^2}{2}\mathscr{L}[k] \tag{8.58}$$

and

$$\Phi_{k2} \triangleq (\alpha T - 1)I_n - \frac{T^2}{2}\mathscr{L}[k-1]. \tag{8.59}$$

We next study the conditions on $\mathscr{G}[k]$, T, and α such that $\lim_{k\to\infty} B_k$ exists and all rows of $\lim_{k\to\infty} B_k$ are the same for arbitrary initial conditions. Before moving on, we need the following lemma.

Lemma 8.15. *Suppose that a nonnegative matrix $A \in \mathbb{R}^{n\times n}$ has the same row sum. Let $\overline{A} \triangleq \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \otimes A$. If the directed graph of A has a directed spanning tree, the directed graph of $\overline{A}$ also has a directed spanning tree.*

Proof: Note that the eigenvalues of $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ are $\lambda_1 = 0$ and $\lambda_2 = 2$. Let μ_j be the jth eigenvalue of A. Because $\overline{A} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \otimes A$, it follows from Lemma 1.21 that the eigenvalues of $\overline{A}$ are $\lambda_i\mu_j, i = 1, 2, j = 1, \dots, n$. It thus follows that $\rho(\overline{A}) = 2\rho(A)$. If the directed graph of A has a directed spanning tree, it follows from Lemma 1.10 that A has a simple eigenvalue equal to $\rho(A)$, which implies that $\overline{A}$ also has a simple eigenvalue equal to $\rho(\overline{A})$. Therefore, it follows again from Lemma 1.10 that the directed graph of $\overline{A}$ has a directed spanning tree. ∎

Lemma 8.16. *Let Φ_{k1} and Φ_{k2} be defined by, respectively,* (8.58) *and* (8.59)*. There exist positive α and T such that both Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal entries. If the positive α and T are chosen such that both Φ_{k1} and Φ_{k2} are nonnegative with positive diagonal entries, and there exists a positive integer κ such that for any nonnegative integer k_0, the union of $\mathscr{G}[k]$ across $k \in [k_0, k_0+\kappa]$ has a directed spanning tree, the iteration* (8.57) *is stable for arbitrary initial conditions* (*i.e.,* $\lim_{k\to\infty} B_k$ *exists*) *and all rows of* $\lim_{k\to\infty} B_k$ *are the same.*

Proof: For the first statement, consider $\alpha T = \frac{3}{2}$. It follows that if $T^2 < \min_i \frac{1}{\ell_{ii}[k]}$, $k = 0, 1, \ldots$, where $\ell_{ii}[k]$ is the ith diagonal entry of $\mathscr{L}[k]$, then both Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal entries.

For the second statement, rewrite (8.57) as

$$\begin{bmatrix} B_k \\ B_{k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi_{k1} & \Phi_{k2} \\ I_n & 0_{n\times n} \end{bmatrix}}_{H_k} \begin{bmatrix} B_{k-1} \\ B_{k-2} \end{bmatrix}. \tag{8.60}$$

Note that $\Phi_{k1}\mathbf{1}_n = 2 - \alpha T$, $\Phi_{k2}\mathbf{1}_n = \alpha T - 1$, and $(\Phi_{k1} + \Phi_{k2})\mathbf{1}_n = 1$. When the positive α and T are chosen such that both Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal entries, it follows that H_k is a row-stochastic matrix. It then follows that $H_{k+1}H_k = \left[\begin{smallmatrix} \Phi_{(k+1)1}\Phi_{k1} + \Phi_{(k+1)2} & \Phi_{(k+1)1}\Phi_{k2} \\ \Phi_{k1} & \Phi_{k2} \end{smallmatrix}\right]$ is also a row-stochastic matrix because the product of row-stochastic matrices is also a row-stochastic matrix. In addition, the diagonal entries of $H_{k+1}H_k$ are positive because both Φ_{k1} and Φ_{k2} are nonnegative matrices with positive diagonal entries. Similarly, for any positive integer m and nonnegative integer ℓ_0, the matrix product $\prod_{i=0}^m H_{\ell_0+i}$ is also a row-stochastic matrix with positive diagonal entries. From Lemma 1.8, we have that

$$\begin{aligned} H_{k+1}H_k &\geq \begin{bmatrix} \gamma_1(\Phi_{(k+1)1} + \Phi_{k1}) + \Phi_{(k+1)2} & \gamma_2(\Phi_{(k+1)1} + \Phi_{k2}) \\ \Phi_{k1} & \Phi_{k2} \end{bmatrix} \\ &\geq \gamma \begin{bmatrix} \Phi_{(k+1)1} + \Phi_{k1} + \Phi_{(k+1)2} & \Phi_{(k+1)1} + \Phi_{k2} \\ \Phi_{k1} & \Phi_{k2} \end{bmatrix} \end{aligned}$$

for some positive γ that is determined by γ_1, γ_2, Φ_{k1}, Φ_{k2}, $\Phi_{(k+1)1}$, and $\Phi_{(k+1)2}$, where γ_1 is determined by $\Phi_{(k+1)1}$ and Φ_{k1}, and γ_2 is determined by $\Phi_{(k+1)1}$ and Φ_{k2}. Note also that the directed graph of $\Phi_{(k-1)1}$ is the same as that of Φ_{k2}. We can thus replace Φ_{k2} with $\Phi_{(k-1)1}$ without changing the directed graph of H_k and vice versa. Therefore, it follows from the definitions of Φ_{k1} and Φ_{k2} that $H_{k+1}H_k \geq \hat{\gamma}\left[\begin{smallmatrix} \Phi_{(k+1)1} + \Phi_{k1} & \Phi_{(k+1)1} + \Phi_{(k-1)1} \\ \Phi_{k1} & \Phi_{(k-1)1} \end{smallmatrix}\right]$ for some positive $\hat{\gamma}$ that is determined by Φ_{k1}, Φ_{k2}, $\Phi_{(k+1)1}$, $\Phi_{(k+1)2}$, and γ. Similarly, $\prod_{i=0}^m H_{\ell_0+i}$ satisfies

$$\prod_{i=0}^m H_{\ell_0+i} \geq \tilde{\gamma} \begin{bmatrix} \sum_{i=\ell_0}^{\ell_0+m} \Phi_{i1} & \sum_{i=\ell_0+1}^{\ell_0+m} \Phi_{i1} + \Phi_{(\ell_0-1)1} \\ \sum_{i=\ell_0}^{\ell_0+m-1} \Phi_{i1} & \sum_{i=\ell_0+1}^{\ell_0+m-1} \Phi_{i1} + \Phi_{(\ell_0-1)1} \end{bmatrix}$$

$$\geq \tilde{\gamma} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes \sum_{i=\ell_0+1}^{\ell_0+m-1} \Phi_{i1} \tag{8.61}$$

for some positive $\tilde{\gamma}$.

Because there exists a positive integer κ such that for any nonnegative integer k_0, the union of $\mathscr{G}[k]$ across $k \in [k_0, k_0+\kappa]$ has a directed spanning tree, it follows that the directed graph of $\sum_{i=k_0}^{k_0+\kappa} \Phi_{i1}$ also has a directed spanning tree. Note from (8.61) that $\prod_{i=0}^{\kappa+2} H_{k_0-1+i} \geq \tilde{\gamma}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \otimes \sum_{i=k_0}^{k_0+\kappa} \Phi_{i1}$. It follows from Lemma 8.15 that $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \otimes \sum_{i=k_0}^{k_0+\kappa} \Phi_{i1}$ has a directed spanning tree, which implies that the directed graph of $\prod_{i=0}^{\kappa+2} H_{k_0-1+i}$ also has a directed spanning tree. Also note that $\prod_{i=0}^{\kappa+2} H_{k_0-1+i}$ is a row-stochastic matrix with positive diagonal entries. It follows from Lemma 1.9 that $\prod_{i=0}^{\kappa+2} H_{k_0-1+i}$ is SIA. It then follows from Lemma 1.12 that $\lim_{k\to\infty} \prod_{i=2}^{k} H_i = \mathbf{1}_{2n} y^T$ for some column vector $y \in \mathbb{R}^{2n}$. Therefore, it follows from (8.60) that $\lim_{k\to\infty} B_k$ exists and all rows of $\lim_{k\to\infty} B_k$ are the same. ■

Theorem 8.12. *Suppose that there exists a positive integer κ such that for any nonnegative integer k_0, the union of $\mathscr{G}[k]$ across $k \in [k_0, k_0+\kappa]$ has a directed spanning tree. Let Φ_{k1} and Φ_{k2} be defined by, respectively, (8.58) and (8.59). If the positive α and T are chosen such that both Φ_{k1} and Φ_{k2} are nonnegative with positive diagonal entries, $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to 0$ as $k \to \infty$.*

Proof: It follows from Lemma 8.16 that $\lim_{k\to\infty} B_k$ exists and all rows of $\lim_{k\to\infty} B_k$ are the same under the condition of the theorem. Because Φ_{k1} is nonnegative with positive diagonal entries, it follows that $\alpha T < 2$ (and hence $\alpha T \neq 2$). It then follows from Lemma 8.14 that $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] \to 0$ as $k \to \infty$ under the condition of the theorem. ■

8.3.2 Convergence Analysis of the Sampled-data Coordination Algorithm with Relative Damping

In this subsection, we analyze (8.21) under a directed switching interaction graph. Here $\mathscr{A}[k]$ is switching. In this case, using (8.21), (8.19) can be written in a vector form as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} I_n - \frac{T^2}{2}\mathscr{L}[k] & TI_n - \frac{T^2}{2}\mathscr{L}[k] \\ -T\mathscr{L}[k] & I_n - \alpha T \mathscr{L}[k] \end{bmatrix}}_{G_k} \begin{bmatrix} \tilde{r}[k] \\ v[k] \end{bmatrix}, \tag{8.62}$$

where $\tilde{r}$ and v are defined as in (8.22), and $\mathscr{L}[k]$ is defined as in (8.49). Note that G_k can be written as

$$G_k = \underbrace{\begin{bmatrix} (1-T)I_n - \frac{T^2}{2}\mathscr{L}[k] & TI_n - \frac{T^2}{2}\mathscr{L}[k] \\ \sqrt{T}I_n - T\mathscr{L}[k] & (1-\sqrt{T})I_n - \alpha T\mathscr{L}[k] \end{bmatrix}}_{R_k} + \underbrace{\begin{bmatrix} TI_n & 0_{n\times n} \\ -\sqrt{T}I_n & \sqrt{T}I_n \end{bmatrix}}_{S}. \tag{8.63}$$

In the following, we study the property of the matrix product $\prod_{i=0}^{k} G_i$ defined as

$$\prod_{i=0}^{k} G_i \triangleq \begin{bmatrix} \widetilde{G}_{k1} & \widetilde{G}_{k2} \\ \widetilde{G}_{k3} & \widetilde{G}_{k4} \end{bmatrix}, \tag{8.64}$$

where $\widetilde{G}_{ki} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, 4$.

Lemma 8.17. *Suppose that the directed graph $\mathscr{G}[k]$, $k = 0, 1, \ldots,$ has a directed spanning tree. There exist positive α and T such that the following two conditions are satisfied:*

1. $(1-T)I_n - \frac{T^2}{2}\mathscr{L}[k]$ *and* $(1-\sqrt{T})I_n - \alpha T\mathscr{L}[k]$, $k = 0, 1, \ldots,$ *are nonnegative matrices with positive diagonal entries, and* $TI_n - \frac{T^2}{2}\mathscr{L}[k]$ *and* $\sqrt{T}I_n - T\mathscr{L}[k]$, $k = 0, 1, \ldots,$ *are nonnegative matrices.*
2. $\|S\|_\infty < 1$, *where S is defined in (8.63).*

In addition, if the positive α and T are chosen such that Conditions 1 and 2 are satisfied, the matrix product $\prod_{i=0}^{k} G_i$ has the property that all rows of each $\widetilde{G}_{ki}$, $i = 1, \ldots, 4$, become the same as $k \to \infty$.

Proof: For the first statement, it can be noted that when T is sufficiently small, Condition 1 is satisfied. Similarly, when $0 < T < \frac{1}{4}$, it follows that $\|S\|_\infty < 1$. Therefore, there exist positive α and T such that Conditions 1 and 2 are satisfied.

For the second statement, it is assumed that α and T are chosen such that Conditions 1 and 2 are satisfied. It can be computed that R_k, $k = 0, 1, \ldots,$ are row-stochastic matrices with positive diagonal entries when Condition 1 is satisfied. Note that

$$\prod_{i=0}^{k} G_i = \prod_{i=0}^{k} (R_i + S). \tag{8.65}$$

It follows from the binomial expansion that $\prod_{i=0}^{k} G_i = \sum_{j=1}^{2^{k+1}} \widehat{G}_j$, where $\widehat{G}_j$ is the product of $k+1$ matrices by choosing either R_i or S in $(R_i + S)$ for $i = 0, \ldots, k$. As $k \to \infty$, $\widehat{G}_j$ takes the following three forms:

Case I. *$\widehat{G}_j$ is constructed from an infinite number of S and a finite number of R_i.* In this case, it follows that as $k \to \infty$, $\|\widehat{G}_j\|_\infty \le (\prod_{i=0}^{m} \|R_{\ell_i}\|_\infty)\|S\|_\infty^\infty = \|S\|_\infty^\infty = 0$, where we have used the fact that $\|R_{\ell_i}\|_\infty = 1$ because R_{ℓ_i} is a row-stochastic matrix and $\|S\|_\infty < 1$ as shown in Condition 2. Therefore, $\widehat{G}_j$ approaches $0_{2n\times 2n}$ as $k \to \infty$.

Case II. *$\widehat{G}_j$ is constructed from an infinite number of S and an infinite number of R_i.* A similar analysis to that in Case I shows that $\widehat{G}_j$ approaches $0_{2n\times 2n}$ as $k \to \infty$.

Case III. *$\widehat{G}_j$ is constructed from a finite number of S and an infinite number of R_i.* In this case, as $k \to \infty$, $\widehat{G}_j$ can be written as

$$\widehat{G}_j = M \underbrace{\prod_j R_{\ell_j}}_{J} N,$$

where J is the product of an infinite number of $R_{\ell_j}, j = 0, 1, \dots,$ and both M and N are products of a finite number of matrices by choosing either R_i, $i \neq \ell_j$, or S from $(R_i + S)$.[7] It follows from Lemma 8.15 that $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right] \otimes \{(1-T)I_n - \frac{T^2}{2}\mathscr{L}[k]\}$ has a directed spanning tree if the directed graph of $(1-T)I_n - \frac{T^2}{2}\mathscr{L}[k]$ has a directed spanning tree. Note that the directed graph of R_k is the same as that of $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right] \otimes (1-T)I_n - \frac{T^2}{2}\mathscr{L}[k]$ because the directed graphs of all four matrices in Condition 1 of Lemma 8.17 are the same. Because $\mathscr{G}[k]$ has a directed spanning tree, so does $(1-T)I_n - \frac{T^2}{2}\mathscr{L}[k]$, which further implies that the directed graph of R_k also has a directed spanning tree. Also note that R_k, $k = 0, 1, \dots,$ are row-stochastic matrices with positive diagonal entries. It then follows from Lemma 1.9 that R_{ℓ_j} is SIA. Therefore, it follows from Lemma 1.12 that all rows of J become the same as $k \to \infty$. By writing

$$J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}, \tag{8.66}$$

where $J_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, 4$, it follows from the fact that all rows of J become the same as $k \to \infty$ that all rows of J_i, $i = 1, \dots, 4$, also become the same as $k \to \infty$. It then follows that $R_i J = \left[\begin{smallmatrix} (1-T)I_n - \frac{T^2}{2}\mathscr{L}[i] & TI_n - \frac{T^2}{2}\mathscr{L}[i] \\ \sqrt{T}I_n - T\mathscr{L}[i] & (1-\sqrt{T})I_n - \alpha T\mathscr{L}[i]\end{smallmatrix}\right] J$ approaches $\left[\begin{smallmatrix} (1-T)I_n & TI_n \\ \sqrt{T}I_n & (1-\sqrt{T})I_n \end{smallmatrix}\right] J$ as $k \to \infty$, where we have used the fact that $\mathscr{L}[i]J_\ell$ approaches $0_{n\times n}$, $\ell = 1, \dots, 4$, as $k \to \infty$. By separating $R_i J$ into four $n \times n$ submatrices as that of J in (8.66), all rows of each of the four $n \times n$ submatrices become the same as $k \to \infty$. The same property also applies to the matrix products JR_i, SJ, and JS. A similar analysis shows that the same property also holds for the matrix product formed by pre-multiplying or post-multiplying J by a finite number of R_i and/or S. Therefore, by separating $\widehat{G}_j$ into four $n \times n$ submatrices as those of J in (8.66), it follows that all rows of each of the four $n \times n$ submatrices become the same as $k \to \infty$. Combining the previous arguments shows that as $k \to \infty$, all rows of $\widetilde{G}_{ki}$, $i = 1, \dots, 4$, become the same. ■

[7] Here M and N are I_{2n} if neither R_i nor S is chosen.

Theorem 8.13. *Suppose that the directed graph $\mathscr{G}[k]$, $k = 0, 1, \ldots,$ has a directed spanning tree. Using (8.21) for (8.19), $r_i[k] - r_j[k] \to \Delta_{ij}[k]$ and $v_i[k] - v_j[k] \to 0$ as $k \to \infty$ when the positive α and T are chosen such that Conditions 1 and 2 in Lemma 8.17 are satisfied.*

Proof: Note that the solution of (8.62) can be written as

$$\begin{bmatrix} \tilde{r}[k+1] \\ v[k+1] \end{bmatrix} = \prod_{i=0}^{k} G_i \begin{bmatrix} \tilde{r}[0] \\ v[0] \end{bmatrix}. \tag{8.67}$$

When the directed graph $\mathscr{G}[k]$, $k = 0, 1, \ldots,$ has a directed spanning tree, and Conditions 1 and 2 in Lemma 8.17 are satisfied, it follows that all rows of $\widetilde{G}_{ki}$, $i = 1, \ldots, 4$, become the same as $k \to \infty$. It thus follows from (8.64) and (8.67) that $\tilde{r}_i[k] - \tilde{r}_j[k] \to 0$ and $v_i[k] - v_j[k] \to 0$ as $k \to \infty$, which implies that $r_i[k] - r_j[k] \to \Delta_{ij}$ and $v_i[k] - v_j[k] \to 0$ as $k \to \infty$. ∎

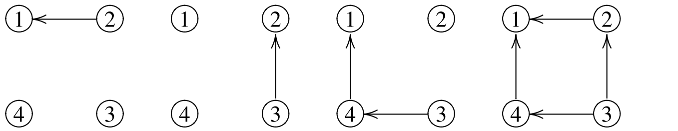
Remark 8.14 Note that Theorem 8.12 requires that the interaction graph have a directed spanning tree jointly to guarantee coordination while Theorem 8.13 requires that the interaction graph have a directed spanning tree at each time interval to guarantee coordination. The different connectivity requirement for Theorems 8.12 and 8.13 is caused by different damping terms. For the coordination algorithm with an absolute damping term, when the sampling period and the damping gain are chosen properly, all agents always have a zero final velocity irrespective of the interaction graph. However, for the coordination algorithm with a relative damping term, the agents in general do not have a zero final velocity. From this point of view, it is not surprising to see that the connectivity requirement in Theorem 8.13 corresponding to the relative damping case is more stringent than that in Theorem 8.12 corresponding to the absolute damping case.

Remark 8.15 In Theorem 8.12 (respectively, Theorem 8.13), it is assumed that the sampling period is uniform. When the sampling periods are non-uniform, we can always find corresponding damping gains such that the conditions in Theorem 8.12 (respectively, Theorem 8.13) are satisfied. Therefore, similar results can be obtained in the presence of non-uniform sampling periods if the conditions in Theorem 8.12 (respectively, Theorem 8.13) are satisfied.

8.3.3 Simulation

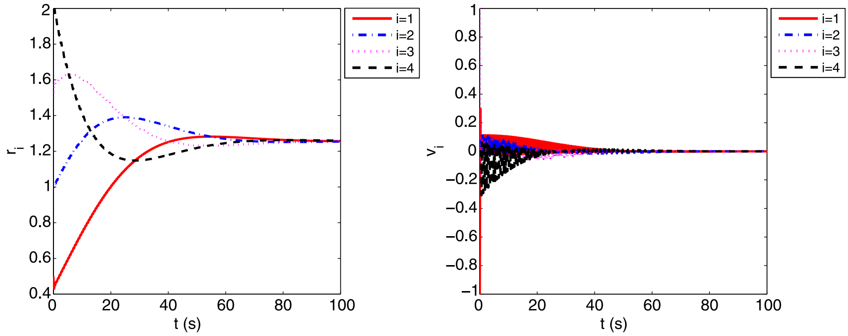
In this subsection, we present simulation results to illustrate the theoretical results derived in Sects. 8.3.1 and 8.3.2. For both coordination algorithms (8.20) and (8.21), we consider a team of four agents. Here for simplicity, we have chosen $\delta_i = 0$, $i = 1, \ldots, 4$, which implies that $\Delta_{ij} = 0$.

For (8.20), let $r_i[0] = 0.5i$, $i = 1, \ldots, 4$, $v_1[0] = -1$, $v_2[0] = 0$, $v_3[0] = 1$, and $v_4[0] = 0$. The directed graph $\mathscr{G}[k]$ switches from a set $\{\mathscr{G}_{(1)}, \mathscr{G}_{(2)}, \mathscr{G}_{(3)}\}$ as shown



(a) $\mathscr{G}_{(1)}$ (b) $\mathscr{G}_{(2)}$ (c) $\mathscr{G}_{(3)}$ (d) $\mathscr{G}_{(1)} \bigcup \mathscr{G}_{(2)} \bigcup \mathscr{G}_{(3)}$

Fig. 8.9 Directed graphs $\mathscr{G}_{(1)}$, $\mathscr{G}_{(2)}$, and $\mathscr{G}_{(3)}$ and their union. An *arrow* from j to i denotes that agent j is a neighbor of agent i

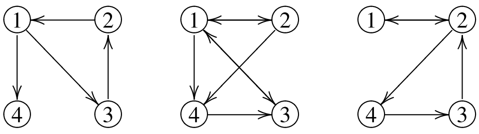


(a) Positions (b) Velocities

Fig. 8.10 Convergence result using (8.20) when $\mathscr{G}[k]$ switches from a set $\{\mathscr{G}_{(1)}, \mathscr{G}_{(2)}, \mathscr{G}_{(3)}\}$, $T = 0.2$ s, and $\alpha = 6$

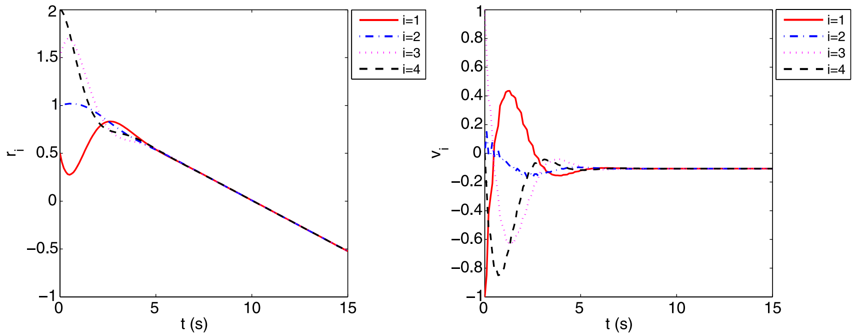
in Fig. 8.9(a)–(c). While $\mathscr{G}_{(i)}$, $i = 1, 2, 3$, does not have a directed spanning tree, their union as shown in Fig. 8.9(d) has a directed spanning tree. We let $a_{ij}[k] = 1$ if $(j, i) \in \mathscr{E}[k]$ and $a_{ij}[k] = 0$ otherwise. We choose $T = 0.2$ s and $\alpha = 6$. It can be computed that the condition in Theorem 8.12 is satisfied. Figures 8.10(a) and 8.10(b) show, respectively, the positions and velocities of the four agents using (8.20) when $\mathscr{G}[k]$ switches from $\mathscr{G}_{(1)}$ to $\mathscr{G}_{(2)}$ and then to $\mathscr{G}_{(3)}$ every sampling period and the same process then repeats. It can be seen that coordination is achieved on positions with a zero final velocity as stated in Theorem 8.12. Note that the velocities of the four agents demonstrate large oscillations as shown in Fig. 8.10(b) because $\mathscr{G}[k]$ does not have a directed spanning tree at each time sampling period and switches very fast.

For (8.21), $r_i[0]$ and $v_i[0]$ are chosen the same as for (8.20). The directed graph $\mathscr{G}[k]$ switches from a set $\{\mathscr{G}_{(4)}, \mathscr{G}_{(5)}, \mathscr{G}_{(6)}\}$ as shown in Fig. 8.11. Note that each directed graph $\mathscr{G}_{(i)}$, $i = 4, 5, 6$, has a directed spanning tree. Here again we let $a_{ij}[k] = 1$ if $(j, i) \in \mathscr{E}[k]$ and $a_{ij}[k] = 0$ otherwise. We choose $T = 0.1$ s and $\alpha = 1$. It can be computed that the condition in Theorem 8.13 is satisfied. Figures 8.12(a) and 8.12(b) show, respectively, the positions and velocities of the four agents using (8.21) when $\mathscr{G}[k]$ switches from $\mathscr{G}_{(4)}$ to $\mathscr{G}_{(5)}$ and then to $\mathscr{G}_{(6)}$ every sampling period and the same process then repeats. It can be seen that coordination is achieved on positions with a constant final velocity as stated in Theorem 8.13.



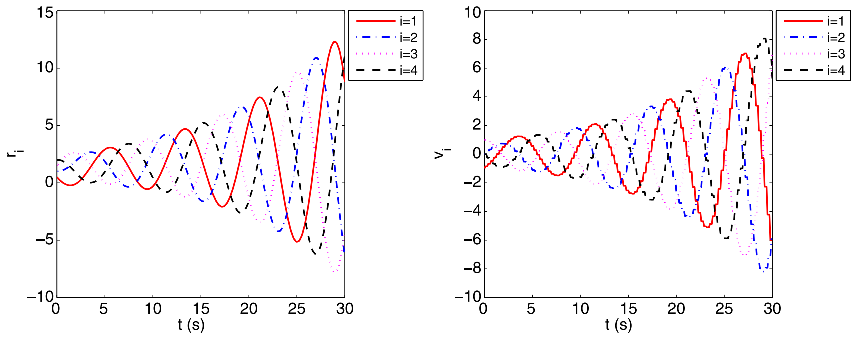
(a) $\mathscr{G}_{(4)}$ (b) $\mathscr{G}_{(5)}$ (c) $\mathscr{G}_{(6)}$

Fig. 8.11 Directed graphs $\mathscr{G}_{(4)}$, $\mathscr{G}_{(5)}$, and $\mathscr{G}_{(6)}$. An *arrow* from j to i denotes that agent j is a neighbor of agent i



(a) Positions (b) Velocities

Fig. 8.12 Convergence result using (8.21) when $\mathscr{G}[k]$ switches from a set $\{\mathscr{G}_{(4)}, \mathscr{G}_{(5)}, \mathscr{G}_{(6)}\}$, $T = 0.1$ s, and $\alpha = 1$



(a) Positions (b) Velocities

Fig. 8.13 Convergence result using (8.21) when $\mathscr{G}[k]$ switches from a set $\{\mathscr{G}_{(1)}, \mathscr{G}_{(2)}, \mathscr{G}_{(3)}\}$, $T = 0.1$ s, and $\alpha = 1$

We also show an example to illustrate that using (8.21) for (8.19), coordination is not necessarily achieved even if the interaction graph has a directed spanning tree jointly, and α and T satisfy Conditions 1 and 2 in Lemma 8.17. The initial positions and velocities, α, and T are chosen to be the same as those for Figs. 8.12(a) and 8.12(b). Figures 8.13(a) and 8.13(b) show, respectively, the positions and velocities of the four agents using (8.21) when $\mathscr{G}[k]$ switches from $\mathscr{G}_{(1)}$ to $\mathscr{G}_{(2)}$ then to $\mathscr{G}_{(3)}$ every sampling period and the same process then repeats. It can be seen that coordination is not achieved even when the interaction graph has a directed spanning tree jointly and α and T satisfy Conditions 1 and 2 in Lemma 8.17.

8.4 Notes

The results in this chapter are based mainly on [32, 33, 36, 44, 45, 249]. For further results on distributed multi-agent coordination in a sampled-data setting, see [99, 101, 116, 196, 328]. In particular, [116] shows conditions on sampled-data coordination under an undirected interaction graph through average-energy-like Lyapunov functions. Considering the fact that communication among agents might be unstable, the authors in [196] further study the case of stochastic undirected interaction. However, the stability condition derived in [196] is stringent and difficult to determine. In [99, 101], sampled-data coordination is studied for agents with double-integrator dynamics in both synchronous and asynchronous cases. In particular, the conditions are derived by using linear matrix inequalities. In [328], the mean-square consentability problem is studied for agents with double-integrator dynamics in a sampled-data setting with a stochastically switching interaction graph.