Chapter 6
Networked Lagrangian Systems

This chapter moves from point models primarily adopted in distributed multi-agent coordination to more realistic Lagrangian models. A class of mechanical systems including autonomous vehicles, robotic manipulators, and walking robots are Lagrangian systems. We focus on fully-actuated Lagrangian systems. We first study distributed leaderless coordination algorithms for networked Lagrangian systems. The objective is to drive a team of agents modeled by Euler–Lagrange equations to achieve desired relative deviations on their vectors of generalized coordinates with local interaction. We then study distributed coordinated regulation and distributed coordinated tracking algorithms in the presence of a leader for networked Lagrangian systems under the constraints that the leader is a neighbor of only a subset of the followers and the followers have only local interaction. In the case of coordinated regulation, the leader has a constant vector of generalized coordinates. In the case of coordinated tracking, the leader has a varying vector of generalized coordinates. In both cases, the objective is to drive the vectors of generalized coordinates of a team of followers modeled by Euler–Lagrange equations to approach that of a leader. Simulation results show the effectiveness of the proposed algorithms.

6.1 Problem Statement

In distributed multi-agent coordination problems, point models are primarily adopted due to their simplicity. However, the point models are often not realistic. Euler–Lagrange equations can be used to model a class of mechanical systems including autonomous vehicles, robotic manipulators, and walking robots. The objective of the current chapter is to study distributed *leaderless* and *leader-following coordination* problems for networked Lagrangian systems. Here we focus on fully-actuated Lagrangian systems. In the leaderless case, there does not exist a leader in the team. The objective is that a team of agents modeled by Euler–Lagrange equations achieves desired relative deviations on their vectors of generalized coordinates with local interaction. In the leader-following case, there exists a leader that specifies the

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objective for the whole team. Here the leader can be virtual or physical. In particular, we use the term *coordinated regulation* to refer to the case where the vectors of generalized coordinates of a group of followers modeled by Euler–Lagrange equations approach a constant vector of generalized coordinates of a leader with local interaction. Similar to Chap. 4, we use the term *coordinated tracking* to refer to the case where the vectors of generalized coordinates of a group of followers modeled by Euler–Lagrange equations approach a varying vector of generalized coordinates of a leader with local interaction. A coordinated regulation problem can be viewed as a special case of a coordinated tracking problem. It is worthwhile to mention that the coordinated tracking case becomes much more complex if the leader is a neighbor of only a subset of the followers.

In the leaderless case, we are motivated to derive distributed coordination algorithms when the agents have only local interaction with their neighbors and none of them has the knowledge of the group reference trajectory. The distributed feature of the algorithms makes them scalable to a large number of agents. The leaderless feature of the algorithms makes them suitable for applications where the absolute states of the agents are not what is important but rather all agents achieve relative state deviations. While there are many applications where there exists a group reference trajectory, there are also many applications where leaderless algorithms are important. Examples include rendezvous, flocking, and attitude synchronization. For example, the proposed leaderless algorithms have potential applications in automated rendezvous and docking. In addition, rigid body attitude dynamics can be written in the form of Euler–Lagrange equations. The proposed leaderless algorithms can be used for attitude synchronization of multiple rigid bodies with local interaction. Furthermore, when there is a team of networked mobile vehicles equipped with robotic arms that hold sensors (e.g., iRobot PackBot Explorer), the robotic arms on each mobile vehicle can be modeled by Euler–Lagrange equations. The proposed leaderless algorithms can be used to coordinate the robotic arms and sensors equipped on different mobile vehicles so that a team of mobile vehicles can scan an area cooperatively. We will propose and analyze three algorithms: (i) a fundamental algorithm; (ii) a nonlinear algorithm; and (iii) an algorithm that accounts for unavailability of measurements of generalized coordinate derivatives.

In the leader-following case, we are motivated to derive distributed coordinated regulation and tracking algorithms when the leader is a neighbor of only a subset of the followers and the followers have only local interaction. The presence of a leader can broaden the applications as a group objective can be encapsulated by the leader. We will consider three cases: (i) The leader has a constant vector of generalized coordinates; (ii) The leader has a constant vector of generalized coordinate derivatives; (iii) The leader has a varying vector of generalized coordinate derivatives. In the first case, we propose and analyze distributed algorithms by extending the distributed leaderless coordination algorithms. In the second case, with the aid of a distributed continuous estimator, we propose and analyze, respectively, a distributed model-dependent algorithm and a distributed model-independent algorithm accounting for parametric uncertainties. In the third case, we propose and analyze a distributed model-independent sliding-mode algorithm.

Consider a team of n agents with Euler–Lagrange equations given by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \quad i = 1, \ldots, n, \tag{6.1}$$

where $q_i \in \mathbb{R}^p$ is the vector of generalized coordinates, $M_i(q_i) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^p$ is the vector of Coriolis and centrifugal torques, $g_i(q_i)$ is the vector of gravitational torques, and $\tau_i \in \mathbb{R}^p$ is the vector of torques produced by the actuators associated with the ith agent.

Throughout the subsequent analysis, we assume that the following assumptions hold [144, 276]:

(A1) (Boundedness) For any i, there exist positive constants $k_{\underline{m}}$, $k_{\bar{m}}$, k_C, k_{C_1}, k_{C_2}, and k_g such that $M_i(q_i) - k_{\underline{m}} I_p$ is positive semidefinite, $M_i(q_i) - k_{\bar{m}} I_p$ is negative semidefinite, $\|g_i(q_i)\| \le k_g$, $\|C_i(x,y)\| \le k_C\|y\|$, and $\|C_i(x,z)w - C_i(y,v)w\| \le k_{C_1}\|z - v\|\,\|w\| + k_{C_2}\|x-y\|\,\|w\|\,\|z\|$ for all vectors $x, y, z, v, w \in \mathbb{R}^p$.
(A2) (Skew-symmetric property) $\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)$ is skew-symmetric (i.e., $y^T[\dot{M}_i(q_i) - 2C_i(q_i, \dot{q}_i)]y = 0$ for all $y \in \mathbb{R}^p$).
(A3) (Linearity in the parameters) $M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = Y_i(q_i, \dot{q}_i, \ddot{q}_i)\Theta_i$, where $Y_i(q_i, \dot{q}_i, \ddot{q}_i)$ is the regressor and Θ_i is the constant parameter vector for the ith agent.

Define $q \triangleq [q_1^T, \ldots, q_n^T]^T$ and $\dot{q} \triangleq [\dot{q}_1^T, \ldots, \dot{q}_n^T]^T$. Also define $M(q) \triangleq \text{diag}[M_1(q_1), \ldots, M_n(q_n)]$, $C(q, \dot{q}) \triangleq \text{diag}[C_1(q_1, \dot{q}_1), \ldots, C_n(q_n, \dot{q}_n)]$, and $g(q) \triangleq [g_1^T(q_1), \ldots, g_n^T(q_n)]^T$.

6.2 Distributed Leaderless Coordination for Networked Lagrangian Systems

We consider three distributed leaderless coordination algorithms for networked Lagrangian systems, namely, a fundamental algorithm, a nonlinear algorithm, and an algorithm accounting for unavailability of measurements of generalized coordinate derivatives. Define $\breve{q}_{ij} \triangleq \delta_i - \delta_j$, where $\delta_i \in \mathbb{R}^p$ is constant. Here $\breve{q}_{ij}$ denotes the constant desired relative deviation on vectors of generalized coordinates between agents i and j. The objective here is to design distributed leaderless coordination algorithms for (6.1) such that $q_i(t) - q_j(t) \to \breve{q}_{ij}$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$. Before moving on, we need the following lemma:

Lemma 6.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous odd function satisfying that $\psi(x) > 0$ if $x > 0$.*[1] *Suppose that $\varsigma_i \in \mathbb{R}^p$, $\varphi_i \in \mathbb{R}^p$, $K \in \mathbb{R}^{p\times p}$, and $D \triangleq [d_{ij}] \in \mathbb{R}^{n\times n}$. If D is symmetric, then*

[1] For a vector, $\psi(\cdot)$ is defined componentwise.

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}(\varsigma_i-\varsigma_j)^T\psi[K(\varphi_i-\varphi_j)] = \sum_{i=1}^{n}\varsigma_i^T\left\{\sum_{j=1}^{n} d_{ij}\psi[K(\varphi_i-\varphi_j)]\right\}.$$

Proof: The proof is similar to that of [248, Lemma 4.18] and is hence omitted here. ■

6.2.1 Fundamental Algorithm

In this section, we consider a fundamental coordination algorithm as

$$\tau_i = g_i(q_i) - \sum_{j=1}^{n} a_{ij}(q_i - q_j - \breve{q}_{ij}) - \sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j) - K_i\dot{q}_i, \tag{6.2}$$

where $i = 1,\ldots,n$, a_{ij} is the (i,j)th entry of the adjacency matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ associated with the undirected graph $\mathscr{G}_A \triangleq (\mathscr{V},\mathscr{E}_A)$ characterizing the interaction among the n agents for q_i, b_{ij} is the (i,j)th entry of the adjacency matrix $\mathscr{B} \in \mathbb{R}^{n\times n}$ associated with the undirected graph $\mathscr{G}_B \triangleq (\mathscr{V},\mathscr{E}_B)$ characterizing the interaction among the n agents for $\dot{q}_i$, and $K_i \in \mathbb{R}^{p\times p}$ is symmetric positive definite. Note that here $\mathscr{G}_A$ and $\mathscr{G}_B$ are allowed to be different.

Theorem 6.1. *Using* (6.2) *for* (6.1), $q_i(t) - q_j(t) \to \breve{q}_{ij}$ *and* $\dot{q}_i(t) \to \mathbf{0}_p$, $i,j = 1,\ldots,n$, *as* $t \to \infty$ *if the graph* $\mathscr{G}_A$ *is undirected connected and the graph* $\mathscr{G}_B$ *is undirected.*

Proof: Using (6.2), (6.1) can be written as

$$\frac{d}{dt}(q_i - q_j - \breve{q}_{ij}) = \dot{q}_i - \dot{q}_j,$$
$$\frac{d}{dt}\dot{q}_i = -M_i^{-1}(q_i)\left[C_i(q_i,\dot{q}_i)\dot{q}_i + \sum_{j=1}^{n} a_{ij}(q_i - q_j - \breve{q}_{ij}) + \sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j) + K_i\dot{q}_i\right]. \tag{6.3}$$

Note that the system (6.3) with states $q_i - q_j - \breve{q}_{ij}$ and $\dot{q}_i$ is nonautonomous due to the dependence of M_i and C_i on q_i. As a result, Lemma 1.31 is no long applicable for (6.3). Instead, we apply Lemma 1.36 to prove the theorem.

Let $\breve{q} \triangleq [\breve{q}_1^T,\ldots,\breve{q}_n^T]^T$ with $\breve{q}_i \triangleq q_i - \delta_i$, and $K \triangleq \mathrm{diag}(K_1,\ldots,K_n)$. Let $\mathscr{L}_A$ and $\mathscr{L}_B$ be, respectively, the Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}_A$, and $\mathscr{B}$ and hence $\mathscr{G}_B$. Note that both $\mathscr{L}_A$ and $\mathscr{L}_B$ are symmetric positive semidefinite because both $\mathscr{G}_A$ and $\mathscr{G}_B$ are undirected. Let $\tilde{q}$ be a column stack vector of all $q_i - q_j - \breve{q}_{ij}$, where $i < j$ and $a_{ij} > 0$ (i.e., agents i and j are neighbors). Define

$x \triangleq [\tilde{q}^T, \dot{q}^T]^T$. Consider the Lyapunov function candidate for (6.3) as

$$V(t,x) = \frac{1}{2}\breve{q}^T(\mathscr{L}_A \otimes I_p)\breve{q} + \frac{1}{2}\dot{q}^T M(q)\dot{q}.$$

Because the graph $\mathscr{G}_A$ is undirected, it follows from Remark 1.1 and Lemma 1.21 that $\breve{q}^T(\mathscr{L}_A \otimes I_p)\breve{q} = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}\|q_i - q_j - \breve{q}_{ij}\|^2$. It thus follows that V is positive definite and decrescent with respect to x. Note that Condition 1 in Lemma 1.36 is satisfied.

The derivative of V is given by

$$\begin{aligned}\dot{V}(t,x) &= \dot{q}^T(\mathscr{L}_A \otimes I_p)\breve{q} + \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q} \\ &= \dot{q}^T(\mathscr{L}_A \otimes I_p)\breve{q} + \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q},\end{aligned}$$

where we have used the fact that $M(q)$ is symmetric. Using (6.2), (6.1) can be written in a vector form as

$$M(q)\ddot{q} = -C(q,\dot{q})\dot{q} - (\mathscr{L}_A \otimes I_p)\breve{q} - (\mathscr{L}_B \otimes I_p)\dot{q} - K\dot{q}. \tag{6.4}$$

Note that $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric. By applying (6.4), the derivative of V can be written as

$$\dot{V}(t,x) = -\dot{q}^T(\mathscr{L}_B \otimes I_p)\dot{q} - \dot{q}^T K\dot{q} \le 0. \tag{6.5}$$

Therefore, Condition 2 in Lemma 1.36 is satisfied.

Let $W(t,x) \triangleq \dot{q}^T(\mathscr{L}_A \otimes I_p)\breve{q}$. It follows that $|W(t,x)| \le \|\dot{q}\|\,\|(\mathscr{L}_A \otimes I_p)\breve{q}\|$. Note that (6.5) implies $V[t,x(t)] \le V[0,x(0)]$, $\forall t \ge 0$, which in turn implies that $\|\tilde{q}\|$ and $\|\dot{q}\|$ are bounded. Noting that $(\mathscr{L}_A \otimes I_p)\breve{q}$ is a column stack vector of all $\sum_{j=1}^n a_{ij}(q_i - q_j - \breve{q}_{ij})$, $i = 1,\ldots,n$, it follows that $\|(\mathscr{L}_A \otimes I_p)\breve{q}\|$ is bounded. It thus follows that $|W(t,x)|$ is bounded along the solution trajectory, implying that Condition 3 in Lemma 1.36 is satisfied.

The derivative of W along the solution trajectory of (6.4) is

$$\begin{aligned}\dot{W}(t,x) =& \ddot{q}^T(\mathscr{L}_A \otimes I_p)\breve{q} + \dot{q}^T(\mathscr{L}_A \otimes I_p)\dot{q} \\ =& -\dot{q}^T C^T(q,\dot{q})M^{-1}(q)(\mathscr{L}_A \otimes I_p)\breve{q} \\ & - \breve{q}^T(\mathscr{L}_A \otimes I_p)M^{-1}(q)(\mathscr{L}_A \otimes I_p)\breve{q} \\ & - \dot{q}^T(\mathscr{L}_B \otimes I_p)M^{-1}(q)(\mathscr{L}_A \otimes I_p)\breve{q} - \dot{q}^T K M^{-1}(q)(\mathscr{L}_A \otimes I_p)\breve{q} \\ & + \dot{q}^T(\mathscr{L}_A \otimes I_p)\dot{q}.\end{aligned}$$

Note that $\|\dot{q}\|$ is bounded. It follows from Assumption (A1) that $\|M^{-1}(q)\|$ and $C(q,\dot{q})\dot{q}$ are bounded. Therefore, $\dot{W}(t,x)$ can be written as $\dot{W}(t,x) = g[\beta(t), x]$, where g is continuous in both arguments and $\beta(t)$ is continuous and bounded. On the set $\Omega \triangleq \{(\tilde{q}, \dot{q}) | \dot{V} = 0\}$, $\dot{q} = \mathbf{0}_{np}$ and $\dot{W}(t, x)$ becomes

$$\dot{W}(t, x) = -\breve{q}^T (\mathscr{L}_A \otimes I_p) M^{-1}(q) (\mathscr{L}_A \otimes I_p) \breve{q}.$$

Note that $M^{-1}(q)$ is symmetric positive definite. It follows from Assumption (A1) that

$$\breve{q}^T (\mathscr{L}_A \otimes I_p) M^{-1}(q) (\mathscr{L}_A \otimes I_p) \geq \frac{1}{k_{\bar{m}}} \left\| (\mathscr{L}_A \otimes I_p) \breve{q} \right\|^2.$$

Also note that $\|(\mathscr{L}_A \otimes I_p)\breve{q}\|^2$ is positive definite with respect to $\tilde{q}$. It follows from Lemma 1.35 that on the set Ω, there exist a class $\mathscr{K}$ function, α, such that $\|(\mathscr{L}_A \otimes I_p)\breve{q}\|^2 \geq \alpha(\|\tilde{q}\|)$. Therefore, for all $x \in \Omega$, $|\dot{W}(t, x)| \geq 1/k_{\bar{m}} \alpha(\|\tilde{q}\|)$. It follows from Lemma 1.37 that Condition 4 in Lemma 1.36 is satisfied. We conclude from Lemma 1.36 that the equilibrium of the system (6.3) (i.e., $\|\tilde{q}\| = 0$ and $\|\dot{q}\| = 0$) is uniformly asymptotically stable, which implies that $q_i(t) - q_j(t) \to \breve{q}_{ij}$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$ because $\mathscr{G}_A$ is undirected connected. ■

6.2.2 Nonlinear Algorithm

In this section, we consider a nonlinear coordination algorithm as

$$\begin{aligned} \tau_i = g_i(q_i) &- \sum_{j=1}^{n} a_{ij} \psi\left[K_q (q_i - q_j - \breve{q}_{ij})\right] \\ &- \sum_{j=1}^{n} b_{ij} \psi\left[K_{\dot{q}} (\dot{q}_i - \dot{q}_j)\right] - K_i \psi(K_{di} \dot{q}_i), \end{aligned} \tag{6.6}$$

where $i = 1, \ldots, n$, a_{ij} and b_{ij} are defined as in (6.2), K_q, $K_{\dot{q}}$, K_i, and K_{di} are p by p positive-definite diagonal matrices, and $\psi(\cdot)$ is defined in Lemma 6.1 with an additional assumption that $\psi(\cdot)$ is continuously differentiable. In the remainder of the chapter, we use a subscript (j) to denote the jth component of a vector or the jth diagonal entry of a diagonal matrix.

Theorem 6.2. *Using* (6.6) *for* (6.1), $q_i(t) - q_j(t) \to \breve{q}_{ij}$ *and* $\dot{q}_i(t) \to \mathbf{0}_p$, $i, j = 1, \ldots, n$, *as* $t \to \infty$ *if the graph* $\mathscr{G}_A$ *is undirected connected and the graph* $\mathscr{G}_B$ *is undirected.*

Proof: Similar to the proof of Theorem 6.1, using (6.6), (6.1) can be written as a nonautonomous system with states $q_i - q_j - \breve{q}_{ij}$ and $\dot{q}_i$. We apply Lemma 1.36 to prove the theorem. Let $\tilde{q}$ and x be defined as in the proof of Theorem 6.1. Consider the Lyapunov function candidate

$$V(t,x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \sum_{\ell=1}^{p} \int_0^{q_{i(\ell)}(t)-q_{j(\ell)}(t)-\breve{q}_{ij(\ell)}} \psi[K_{q(\ell)}\tau]\, d\tau$$
$$+ \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\dot{q}_i.$$

Note that V is positive definite and decrescent with respect to x. Therefore, Condition 1 in Lemma 1.36 is satisfied.

The derivative of V is given by

$$\dot{V}(t,x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\dot{q}_i - \dot{q}_j)^T \psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big]$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\big[\ddot{q}_i^T M_i(q_i)\dot{q}_i + \dot{q}_i^T \dot{M}_i(q_i)\dot{q}_i + \dot{q}_i^T M_i(q_i)\ddot{q}_i\big].$$

Using (6.6), (6.1) can be written as

$$M_i(q_i)\ddot{q}_i = -C_i(q_i,\dot{q}_i)\dot{q}_i - \sum_{j=1}^{n} a_{ij}\psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big]$$
$$- \sum_{j=1}^{n} b_{ij}\psi\big[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)\big] - K_i\psi(K_{di}\dot{q}_i). \tag{6.7}$$

Note that $\mathscr{A}$ is symmetric because the graph $\mathscr{G}_A$ is undirected. It follows from Lemma 6.1 that

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\dot{q}_i - \dot{q}_j)^T \psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big]$$
$$= \sum_{i=1}^{n} \dot{q}_i^T \left\{\sum_{j=1}^{n} a_{ij}\psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big]\right\}.$$

Also note that $M_i(q_i)$ is symmetric and that $\dot{M}_i(q_i) - 2C_i(q_i,\dot{q}_i)$ is skew symmetric. By applying (6.7), it follows that

$$\dot{V}(t,x) = -\sum_{i=1}^{n} \dot{q}_i^T \left\{\sum_{j=1}^{n} b_{ij}\psi\big[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)\big] + K_i\psi(K_{di}\dot{q}_i)\right\}.$$

Note that $\mathscr{B}$ is symmetric because the graph $\mathscr{G}_B$ is undirected. By applying Lemma 6.1 again, it follows that the derivative of V becomes

$$\dot{V}(t,x) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j)^T \psi\big[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)\big] - \sum_{i=1}^{n} \dot{q}_i^T K_i\psi(K_{di}\dot{q}_i).$$

Given a vector z and two positive-definite diagonal matrices K_1 and K_2, z and $K_1\psi(K_2 z)$ have the same signs for each component. Therefore, it follows that $\dot{V}(t,x) \le 0$, which implies that Condition 2 in Lemma 1.36 is satisfied.

Let $W(t,x) \triangleq \sum_{i=1}^n \dot{q}_i^T \chi_i$, where

$$\chi_i \triangleq \sum_{j=1}^n a_{ij}\psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big].$$

Note that $\dot{V}(t,x) \le 0$ implies $V[t,x(t)] \le V[0,x(0)]$, $\forall t \ge 0$, which in turn implies that $\tilde{q}$ and $\dot{q}$ are bounded. It thus follows that $\|\chi_i\|$ is also bounded. Similar to the proof of Theorem 6.1, it follows that $|W(t,x)|$ is bounded along the solution trajectory, implying that Condition 3 in Lemma 1.36 is satisfied.

The derivative of $W(t,x)$ along the solution trajectory of (6.7) is

$$\begin{aligned}\dot{W}(t,x) = &-\sum_{i=1}^n \dot{q}_i^T C_i^T(q_i,\dot{q}_i)M_i^{-1}(q_i)\chi_i \\ &-\sum_{i=1}^n \left\{\sum_{j=1}^n b_{ij}\psi\big[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)\big]\right\}^T M_i^{-1}(q_i)\chi_i \\ &-\sum_{i=1}^n \chi_i^T M_i^{-1}(q_i)\chi_i - \sum_{i=1}^n \big[K_i\psi(K_{di}\dot{q}_i)\big]^T M_i^{-1}(q_i)\chi_i + \sum_{i=1}^n \dot{q}_i^T \dot{\chi}_i.\end{aligned}$$

A similar argument to that in the proof of Theorem 6.1 shows that $\dot{W}(t,x)$ can be written as $\dot{W}(t,x) = g[\beta(t),x]$, where g is continuous in both arguments and $\beta(t)$ is continuous and bounded. On the set $\{(\tilde{q},\dot{q})|\dot{V}=0\}$, $\dot{q} = \mathbf{0}_{np}$ and $\dot{W}(t,x)$ becomes

$$\dot{W}(t,x) = -\sum_{i=1}^n \chi_i^T M_i^{-1}(q_i)\chi_i.$$

If $\sum_{i=1}^n \chi_i^T\chi_i$ is positive definite with respect to $\tilde{q}$, then a similar argument to that in the proof of Theorem 6.1 implies that Condition 4 in Lemma 1.36 is satisfied. Because $\sum_{i=1}^n \chi_i^T\chi_i \ge 0$, equivalently we only need to show that $\sum_{i=1}^n \chi_i^T\chi_i = 0$ implies $q_i - q_j - \breve{q}_{ij} = \mathbf{0}_p$ for all $a_{ij} > 0$. Suppose that $\sum_{i=1}^n \chi_i^T\chi_i = 0$, which implies $\chi_i = \sum_{j=1}^n a_{ij}\psi[K_q(q_i - q_j - \breve{q}_{ij})] = \mathbf{0}_p$. It thus follows that $\sum_{i=1}^n q_i^T\{\sum_{j=1}^n a_{ij}\psi[K_q(q_i - q_j - \breve{q}_{ij})]\} = 0$, which implies from Lemma 6.1 that $\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(q_i - q_j - \breve{q}_{ij})^T\psi[K_q(q_i - q_j - \breve{q}_{ij})] = 0$. Note that $\mathscr{G}_A$ is undirected and $q_i - q_j - \breve{q}_{ij}$ and $\psi[K_q(q_i - q_j - \breve{q}_{ij})]$ have the same signs for each component. It follows that $q_i - q_j - \breve{q}_{ij} = \mathbf{0}_p$ for all $a_{ij} > 0$ when $\sum_{i=1}^n \chi_i^T\chi_i = 0$. Combining the above arguments, we conclude from Lemma 1.36 that the equilibrium $\|\tilde{q}\| = 0$ and $\|\dot{q}\| = 0$ is uniformly asymptotically stable, which implies that $q_i(t) - q_j(t) \to \breve{q}_{ij}$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$ because $\mathscr{G}_A$ is undirected connected. ■

6.2.3 Algorithm Accounting for Unavailability of Measurements of Generalized Coordinate Derivatives

Note that (6.2) and (6.6) require measurements of $\dot{q}_i$ and $\dot{q}_i - \dot{q}_j$, where $b_{ij} > 0$. In this section, we consider a coordination algorithm that removes the requirement for the measurements of $\dot{q}_i$ and $\dot{q}_i - \dot{q}_j$ as

$$\dot{\hat{x}}_i = F\hat{x}_i + \sum_{j=1}^{n} b_{ij}(q_i - q_j - \breve{q}_{ij}) + \kappa\breve{q}_i, \tag{6.8a}$$

$$y_i = P\left[F\hat{x}_i + \sum_{j=1}^{n} b_{ij}(q_i - q_j - \breve{q}_{ij}) + \kappa\breve{q}_i\right], \tag{6.8b}$$

$$\tau_i = g_i(q_i) - \sum_{j=1}^{n} a_{ij}\psi\big[K_q(q_i - q_j - \breve{q}_{ij})\big] - y_i, \tag{6.8c}$$

where $i = 1, \dots, n$, $F \in \mathbb{R}^{p\times p}$ is Hurwitz, κ is a positive scalar, a_{ij} is the (i,j)th entry of the adjacency matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ associated with the undirected graph $\mathscr{G}_A \triangleq (\mathscr{V}, \mathscr{E}_A)$ characterizing the interaction among the n agents for q_i in (6.8c), b_{ij} is the (i,j)th entry of the adjacency matrix $\mathscr{B} \in \mathbb{R}^{n\times n}$ associated with the undirected graph $\mathscr{G}_B \triangleq (\mathscr{V}, \mathscr{E}_B)$ characterizing the interaction among the n agents for q_i in (6.8a), ψ is defined in (6.6), and $P \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite solution to the Lyapunov equation $F^TP + PF = -Q$ with $Q \in \mathbb{R}^{p\times p}$ being symmetric positive definite.

Theorem 6.3. *Using (6.8) for (6.1), $q_i(t) - q_j(t) \to \breve{q}_{ij}$ and $\dot{q}_i(t) \to \mathbf{0}_p$, $i, j = 1, \dots, n$, as $t \to \infty$ if the graph $\mathscr{G}_A$ is undirected connected and the graph $\mathscr{G}_B$ is undirected.*

Proof: Similar to the proofs of Theorems 6.1 and 6.2, we apply Lemma 1.36 to prove the theorem. Let $\hat{x} \triangleq [\hat{x}_1^T, \dots, \hat{x}_n^T]^T$. Let $\tilde{q}$ be defined as in the proof of Theorem 6.2. Let $x \triangleq [\tilde{q}^T, \dot{q}^T, \dot{\hat{x}}^T]^T$. Consider the Lyapunov function candidate

$$\begin{aligned} V(t,x) = {} & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \sum_{\ell=1}^{p} \int_0^{q_{i(\ell)}(t) - q_{j(\ell)}(t) - \breve{q}_{ij(\ell)}} \psi[K_{q(\ell)}\tau]\, d\tau \\ & + \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\dot{q}_i + \frac{1}{2}\dot{\hat{x}}^T(S\otimes I_p)^{-1}(I_n\otimes P)\dot{\hat{x}}, \end{aligned}$$

where $S \triangleq \mathscr{L}_B + \kappa I_n$ with $\mathscr{L}_B$ being the Laplacian matrix associated with $\mathscr{B}$ and hence $\mathscr{G}_B$. Note that $\mathscr{L}_B$ is symmetric positive semidefinite because the graph $\mathscr{G}_B$ is undirected. It thus follows that S is symmetric positive definite, so is S^{-1}. From Lemma 1.21, note that $(S\otimes I_p)^{-1} = (S^{-1}\otimes I_p)$. Also note from Lemma 1.21 that $(S^{-1}\otimes I_p)(I_n\otimes P) = S^{-1}I_n \otimes I_pP = I_nS^{-1}\otimes PI_p = (I_n\otimes P)(S^{-1}\otimes I_p)$.

That is, $(S \otimes I_p)^{-1}$ and $I_n \otimes P$ commute. Similarly, it is straightforward to show that $(S \otimes I_p)^{-1}$ and $I_n \otimes F^T$ also commute. Note that $S^{-1}I_n \otimes I_pP$ is symmetric positive definite, so is $(S^{-1} \otimes I_p)(I_n \otimes P)$. It follows that V is positive definite and decrescent with respect to x. Therefore, Condition 1 in Lemma 1.36 is satisfied.

Following the proof of Theorem 6.2, we derive the derivative of V as

$$\begin{aligned}\dot{V}(t,x) &= -\sum_{i=1}^{n} \dot{q}_i^T y_i + \frac{1}{2}\dot{\hat{x}}^T\big(I_n \otimes F^T\big)(S \otimes I_p)^{-1}(I_n \otimes P)\dot{\hat{x}} \\ &\quad + \frac{1}{2}\dot{q}^T(S \otimes I_p)^T(S \otimes I_p)^{-1}(I_n \otimes P)\dot{\hat{x}} \\ &\quad + \frac{1}{2}\dot{\hat{x}}^T(S \otimes I_p)^{-1}(I_n \otimes P)(I_n \otimes F)\dot{\hat{x}} \\ &\quad + \frac{1}{2}\dot{\hat{x}}^T(S \otimes I_p)^{-1}(I_n \otimes P)(S \otimes I_p)\dot{q} \\ &= -\sum_{i=1}^{n} \dot{q}_i^T y_i + \frac{1}{2}\dot{\hat{x}}^T(S \otimes I_p)^{-1}\big[I_n \otimes \big(F^TP + PF\big)\big]\dot{\hat{x}} + \dot{q}^T(I_n \otimes P)\dot{\hat{x}} \\ &= -\frac{1}{2}\dot{\hat{x}}^T(S \otimes I_p)^{-1}(I_n \otimes Q)\dot{\hat{x}} \le 0,\end{aligned}$$

where we have used the fact that

$$\ddot{\hat{x}} = (I_n \otimes F)\dot{\hat{x}} + (S \otimes I_p)\dot{q}, \tag{6.9}$$

$(S \otimes I_p)^{-1}$ and $I_n \otimes F^T$ commute, $(S \otimes I_p)^{-1}$ and $I_n \otimes P$ commute, $S \otimes I_p = (S \otimes I_p)^T$, $y = (I_n \otimes P)\dot{\hat{x}}$ with $y = [y_1^T, \dots, y_n^T]^T$, and $(S \otimes I_p)^{-1}(I_n \otimes Q) = S^{-1}I_n \otimes QI_p$ is symmetric positive definite. Therefore, Condition 2 in Lemma 1.36 is satisfied.

Let $W(t,x)$ and χ_i be defined as in the proof of Theorem 6.2. Similar to the proof of Theorem 6.2, it follows that $|W(t,x)|$ is bounded along the solution trajectory, implying that Condition 3 in Lemma 1.36 is satisfied.

The derivative of $W(t,x)$ along the solution trajectory of closed-loop system (6.1) using (6.8) is

$$\begin{aligned}\dot{W}(t,x) = &-\sum_{i=1}^{n} \dot{q}_i^T C_i^T(q_i,\dot{q}_i)M_i^{-1}(q_i)\chi_i - \sum_{i=1}^{n} \chi_i^T M_i^{-1}(q_i)\chi_i \\ &-\sum_{i=1}^{n} y_i^T M_i^{-1}(q_i)\chi_i + \sum_{i=1}^{n} \dot{q}_i^T \dot{\chi}_i.\end{aligned}$$

Note that $\dot{V} = 0$ implies $\dot{\hat{x}} = \mathbf{0}_{np}$, which in turn implies that $(S \otimes I_p)\dot{q} = \mathbf{0}_{np}$ according to (6.9) and $y_i = \mathbf{0}_p$ by noting that $y_i = P\dot{\hat{x}}_i$ according to (6.8b). Because $S \otimes I_p$ is symmetric positive definite, it follows that $\dot{q} = \mathbf{0}_{np}$. On the set $\{(\tilde{q}, \dot{q}, \dot{\hat{x}}) | \dot{V} = 0\}$, $\dot{q} = \mathbf{0}_{np}$, $\dot{\hat{x}} = \mathbf{0}_{np}$, and $\dot{W}(t,x)$ becomes

$$\dot{W}(t,x) = -\sum_{i=1}^{n} \chi_i^T M_i^{-1}(q_i)\chi_i.$$

Therefore, the rest of the proof is similar to that of Theorem 6.2. We conclude that $q_i(t) - q_j(t) \to \breve{q}_{ij}$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$. ■

Remark 6.4 Note that without the terms $-\sum_{j=1}^{n} b_{ij}(\dot{q}_i - \dot{q}_j)$ in (6.2), $-\sum_{j=1}^{n} b_{ij}\psi[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)]$ in (6.6), and $\sum_{j=1}^{n} b_{ij}(q_i - q_j - \breve{q}_{ij})$ in (6.8a), or equivalently $b_{ij} \equiv 0$, Theorems 6.1, 6.2, and 6.3 are still valid as long as the graph $\mathscr{G}_A$ is undirected connected. However, these terms introduce relative damping between neighboring agents.

6.2.4 Simulation

In this section, we simulate a scenario where six two-link revolute joint arms are coordinated through local interaction using, respectively, the algorithms (6.2), (6.6), and (6.8). For simplicity, we assume that each arm is identical. The Euler–Lagrange equation of each two-link revolute joint arm is given in [276, pp. 259–262]. In particular, the inertia matrix, the vector of Coriolis and centrifugal torques, and the vector of gravitational torques are given as

$$\begin{aligned}
M_i(q_i) &= \begin{bmatrix} \Theta_{i(1)} + 2\Theta_{i(2)}\cos[q_{i(2)}] & \Theta_{i(3)} + \Theta_{i(2)}\cos[q_{i(2)}] \\ \Theta_{i(3)} + \Theta_{i(2)}\cos[q_{i(2)}] & \Theta_{i(3)} \end{bmatrix}, \\
C_i(q_i,\dot{q}_i) &= \begin{bmatrix} -\Theta_{i(2)}\sin[q_{i(2)}]\dot{q}_{i(2)} & -\Theta_{i(2)}\sin[q_{i(2)}][\dot{q}_{i(1)} + \dot{q}_{i(2)}] \\ \Theta_{i(2)}\sin[q_{i(2)}]\dot{q}_{i(1)} & 0 \end{bmatrix}, \\
g_i(q_i) &= \begin{bmatrix} \Theta_{i(4)} g\cos[q_{i(1)}] + \Theta_{i(5)} g\cos[q_{i(1)} + q_{i(2)}] \\ \Theta_{i(5)} g\cos[q_{i(1)} + q_{i(2)}] \end{bmatrix},
\end{aligned}$$

where $q_i \triangleq [q_{i(1)}, q_{i(2)}]^T$, $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity, $\Theta_i \triangleq [\Theta_{i(1)}, \Theta_{i(2)}, \Theta_{i(3)}, \Theta_{i(4)}, \Theta_{i(5)}] = [m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + J_1 + J_2, m_2 l_1 l_{c2}, m_2 l_{c2}^2 + J_2, m_1 l_{c1} + m_2 l_1, m_2 l_{c2}]$. Here the masses of links 1 and 2 are, respectively, $m_1 = 1$ kg and $m_2 = 0.8$ kg, the lengths of links 1 and 2 are, respectively, $l_1 = 0.8$ m and $l_2 = 0.6$ m, the distances from the previous joint to the center of mass of links 1 and 2 are, respectively, $l_{c1} = 0.4$ m and $l_{c2} = 0.3$ m, and the moments of inertia of links 1 and 2 are, respectively, $J_1 = 0.0533\ \mathrm{kg\,m^2}$ and $J_2 = 0.024\ \mathrm{kg\,m^2}$.

For simplicity, we assume that the graphs $\mathscr{G}_A$ and $\mathscr{G}_B$ are identical. Figure 6.1 shows $\mathscr{G}_A$ (equivalently, $\mathscr{G}_B$) for the six two-link revolute joint arms. Table 6.1 shows the control parameters for each algorithm. In simulation, we let $q_i(0) = [\frac{\pi}{7}i, \frac{\pi}{8}i]^T$ rad and $\dot{q}(0) = [0.1i - 0.4, -0.1i + 0.5]^T$ rad/s, where $i = 1, \ldots, 6$.

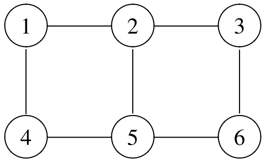

Fig. 6.1 Graph $\mathscr{G}_A$ (equivalently, $\mathscr{G}_B$) for the six two-link revolute joint arms. An *edge* between i and j denotes that agents i and j are neighbors. The *graph* is undirected connected

Table 6.1 Control parameters for each algorithm

Algorithm (6.2):
$K_i = I_2$, $a_{ij} = b_{ij} = 1$ if $(i, j) \in \mathscr{E}_A$ (or $\mathscr{E}_B$), $\check{q}_{ij} = \mathbf{0}_2$
Algorithm (6.6):
$K_q = K_{\dot{q}} = K_i = K_{di} = I_2$, $a_{ij} = b_{ij} = 1$ if $(i, j) \in \mathscr{E}_A$(or $\mathscr{E}_B$), $\check{q}_{ij} = \mathbf{0}_2$
Algorithm (6.8):
$\varGamma = -4I_2$, $\kappa = 0.2$, $P = I_2$, $K_q = 0.6I_2$, $a_{ij} = b_{ij} = 2$ if $(i, j) \in \mathscr{E}_A$ (or $\mathscr{E}_B$), $\check{q}_{ij} = \mathbf{0}_2$

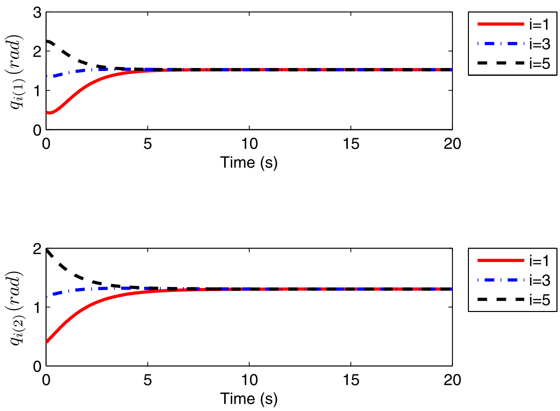

Fig. 6.2 Joint angles of arms 1, 3, and 5 using (6.2)

Figures 6.2, 6.3, and 6.4 show, respectively, the joint angles, their derivatives, and the control torques of arms 1, 3, and 5 using (6.2). Note that the joint angles of all arms achieve coordination while their derivatives converge to zero. Figures 6.5, 6.6, and 6.7 show, respectively, the joint angles, their derivatives, and the control torques of arms 1, 3, and 5 using (6.6), where $\psi(\cdot)$ is chosen as $\tanh(\cdot)$. Note that the joint angles of all arms achieve coordination while their derivatives converge to zero. Figures 6.8, 6.9, and 6.10 show, respectively, the joint angles, their derivatives, and the control torques of arms 1, 3, and 5 using (6.8). The initial conditions $\hat{x}_i(0)$ are chosen randomly. Note that the joint angles of all arms achieve coordination while their derivatives converge to zero even without measurements of absolute and relative joint angle derivatives.

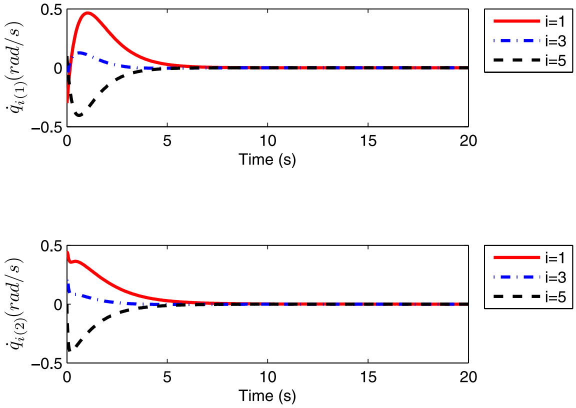

Fig. 6.3 Joint angle derivatives of arms 1, 3, and 5 using (6.2)

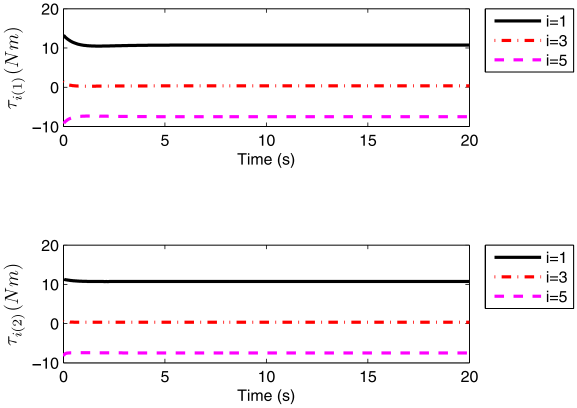

Fig. 6.4 Control torques of arms 1, 3, and 5 using (6.2)

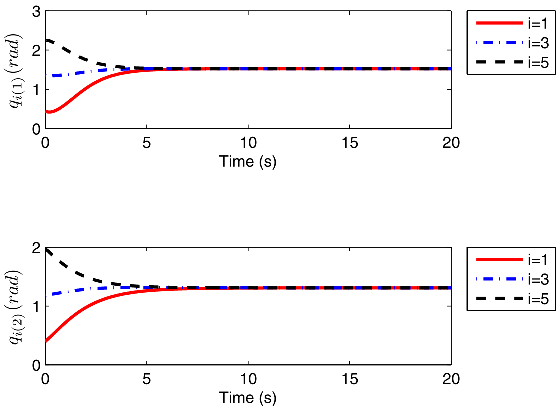

Fig. 6.5 Joint angles of arms 1, 3, and 5 using (6.6)

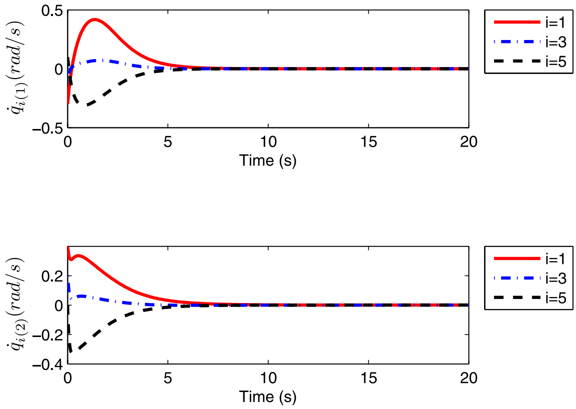

Fig. 6.6 Joint angle derivatives of arms 1, 3, and 5 using (6.6)

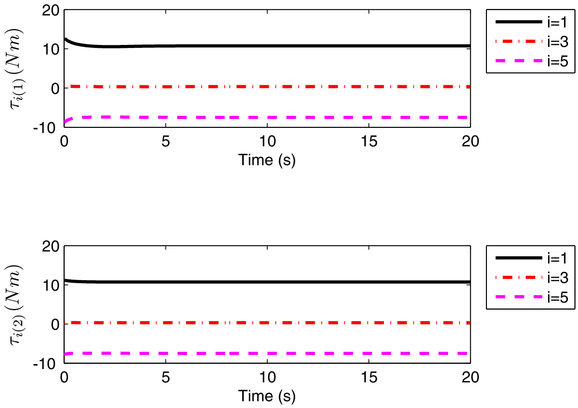

Fig. 6.7 Control torques of arms 1, 3, and 5 using (6.6)

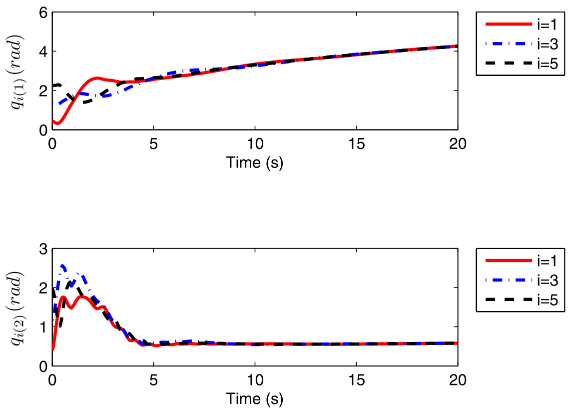

Fig. 6.8 Joint angles of arms 1, 3, and 5 using (6.8)

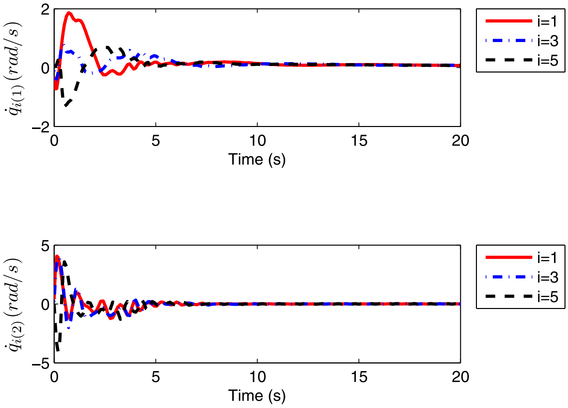

Fig. 6.9 Joint angle derivatives of arms 1, 3, and 5 using (6.8)

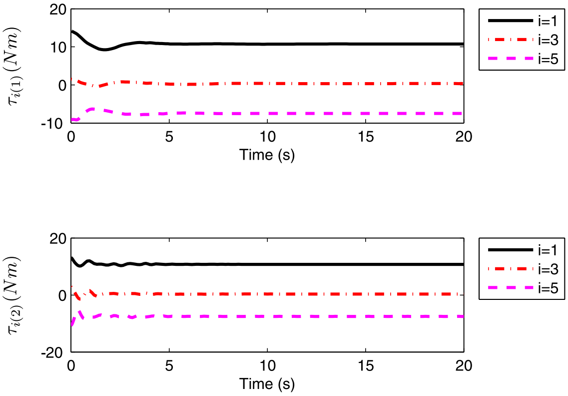

Fig. 6.10 Control torques of arms 1, 3, and 5 using (6.8)

6.3 Distributed Coordinated Regulation and Tracking for Networked Lagrangian Systems

Suppose that there exist n followers, labeled as agents 1 to n, and a leader, labeled as agent 0, in the team. Let $q_0 \in \mathbb{R}^p$ and $\dot{q}_0 \in \mathbb{R}^p$ denote, respectively, the leader's vector of generalized coordinates and vector of generalized coordinate derivatives.

Suppose that in addition to n followers, labeled as agents or followers 1 to n, there exists a leader, labeled as agent 0, in the team. Let $q_0 \in \mathbb{R}^p$ and $\dot{q}_0 \in \mathbb{R}^p$ denote, respectively, the leader's vector of generalized coordinates and vector of generalized coordinate derivatives.

6.3.1 Coordinated Regulation when the Leader's Vector of Generalized Coordinates is Constant

In this subsection, we assume that q_0 is constant (and hence $\dot{q}_0 = \mathbf{0}_p$). The objective here is to design distributed coordinated regulation algorithms for (6.1) such that $q_i(t) \to q_0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$.

We first consider a fundamental coordinated regulation algorithm as

$$\tau_i = g_i(q_i) - \sum_{j=0}^{n} a_{ij}(q_i - q_j) - \sum_{j=0}^{n} b_{ij}(\dot{q}_i - \dot{q}_j), \tag{6.10}$$

where $i = 1, \ldots, n$, $a_{ij}, i, j = 1, \ldots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ associated with the graph $\mathscr{G}_A \triangleq (\mathscr{V}, \mathscr{E}_A)$ characterizing the interaction among the n followers for q_i, $b_{ij}, i, j = 1, \ldots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{B} \in \mathbb{R}^{n\times n}$ associated with the graph $\mathscr{G}_B \triangleq (\mathscr{V}, \mathscr{E}_B)$ characterizing the interaction among the n followers for $\dot{q}_i$, $a_{i0} > 0$ (respectively, $b_{i0} > 0$), $i = 1, \ldots, n$, if follower i has access to the vector of generalized coordinates of the leader (respectively, the vector of generalized coordinate derivatives of the leader) and $a_{i0} = 0$ (respectively, $b_{i0} = 0$) otherwise. Note that here $\dot{q}_0 = \mathbf{0}_p$. Also note that $\mathscr{G}_A$ and $\mathscr{G}_B$ are allowed to be different.

Theorem 6.5. *Using* (6.10) *for* (6.1)*,* $q_i(t) \to q_0$ *and* $\dot{q}_i(t) \to \mathbf{0}_p$*,* $i = 1, \ldots, n$*, as* $t \to \infty$ *if both* $\mathscr{G}_A$ *and* $\mathscr{G}_B$ *are undirected connected, at least one follower has access to* q_0 (*i.e., at least one* $a_{i0} > 0$)*, and at least one follower has access to* $\dot{q}_0$ (*i.e., at least one* $b_{i0} > 0$).

Proof: Let $\tilde{q}$ be a column stack vector of $q_i - q_0$, $i = 1, \ldots, n$. Let $\mathscr{L}_A$ and $\mathscr{L}_B$ be, respectively, the Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}_A$ and $\mathscr{B}$ and hence $\mathscr{G}_B$. Let $H_A \triangleq \mathscr{L}_A + \text{diag}(a_{10}, \ldots, a_{n0})$ and $H_B \triangleq \mathscr{L}_B + \text{diag}(b_{10}, \ldots, b_{n0})$.

Using (6.10), (6.1) can be written as

$$\begin{aligned}
\frac{d}{dt}\tilde{q} &= \dot{q}, \\
\frac{d}{dt}\dot{q} &= -M^{-1}(q)\big[C(q, \dot{q})\dot{q} + (H_A \otimes I_p)\tilde{q} + (H_B \otimes I_p)\dot{q}\big].
\end{aligned} \tag{6.11}$$

Note that the system (6.11) with states $\tilde{q}$ and $\dot{q}$ is autonomous because $q = \tilde{q} + \mathbf{1}_n \otimes q_0$, where q_0 is constant. As a result, Lemma 1.31 can be applied to prove the theorem.

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\tilde{q}^T(H_A \otimes I_p)\tilde{q} + \frac{1}{2}\dot{q}^T M(q)\dot{q}.$$

Because both $\mathscr{G}_A$ and $\mathscr{G}_B$ are undirected connected, at least one $a_{i0} > 0$, and at least one $b_{i0} > 0$, it follows from Lemma 1.6 that both H_A and H_B are symmetric positive definite. Therefore, V is positive definite and radially bounded with respect to $\tilde{q}$ and $\dot{q}$. The derivative of V is given by

$$\begin{aligned}\dot{V} =& \dot{\tilde{q}}^T(H_A \otimes I_p)\tilde{q} + \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q} \\ =& \dot{q}^T(H_A \otimes I_p)\tilde{q} + \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q},\end{aligned}$$

where we have used the fact that $M(q)$ is symmetric and $\dot{\tilde{q}} = \dot{q}$. Using (6.10), (6.1) can be written in a vector form as

$$M(q)\ddot{q} = -C(q,\dot{q})\dot{q} - (H_A \otimes I_p)\tilde{q} - (H_B \otimes I_p)\dot{q}. \tag{6.12}$$

Note that $\dot{M}(q) - 2C(q,\dot{q})$ is skew symmetric. By applying (6.12), the derivative of V can be written as

$$\dot{V} = -\dot{q}^T(H_B \otimes I_p)\dot{q} \leq 0.$$

On the set $\{(\tilde{q},\dot{q})|\dot{V} = 0\}$, note that $\dot{V} \equiv 0$ implies $\dot{q} \equiv \mathbf{0}_{np}$, which in turn implies $(H_A \otimes I_p)\tilde{q} \equiv \mathbf{0}_{np}$ according to (6.12). Because H_A is symmetric positive definite, it follows that $\tilde{q} \equiv \mathbf{0}_{np}$. By Lemma 1.31, it follows that $\tilde{q}(t) \to \mathbf{0}_{np}$ and $\dot{q}(t) \to \mathbf{0}_{np}$ as $t \to \infty$, which in turn implies that $q_i(t) \to q_0$ and $\dot{q}_i(t) \to \mathbf{0}_p$ as $t \to \infty$. ■

We next consider a nonlinear coordinated regulation algorithm as

$$\tau_i = g_i(q_i) - \sum_{j=0}^{n} a_{ij}\psi\big[K_q(q_i - q_j)\big] - \sum_{j=0}^{n} b_{ij}\psi\big[K_{\dot{q}}(\dot{q}_i - \dot{q}_j)\big], \tag{6.13}$$

where $i = 1,\dots,n$, a_{ij} and b_{ij}, $i = 1,\dots,n$, $j = 0,\dots,n$, are defined as in (6.10), $K_q \in \mathbb{R}^{p\times p}$ and $K_{\dot{q}} \in \mathbb{R}^{p\times p}$ are positive-definite diagonal matrices, and $\psi(\cdot)$ is defined in Lemma 6.1.

Theorem 6.6. *Using* (6.13) *for* (6.1), $q_i(t) \to q_0$ *and* $\dot{q}_i(t) \to \mathbf{0}_p$, $i = 1,\dots,n$, *as* $t \to \infty$ *if both* $\mathscr{G}_A$ *and* $\mathscr{G}_B$ *are undirected connected, at least one follower has access to* q_0 (*i.e., at least one* $a_{i0} > 0$)*, and at least one follower has access to* $\dot{q}_0$ (*i.e., at least one* $b_{i0} > 0$).

Proof: Similar to the proof of Theorem 6.5, using (6.13), (6.1) can be written as an autonomous system with states $q_i - q_0$ and $\dot{q}_i$, $i = 1,\dots,n$. Consider the Lyapunov function candidate

$$\begin{aligned}V =& \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^T M_i(q_i)\dot{q}_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\sum_{\ell=1}^{p}\int_0^{q_{i(\ell)}(t)-q_{j(\ell)}(t)} \psi[K_{q(\ell)}\tau]\,d\tau \\ &+ \sum_{i=1}^{n} a_{i0}\sum_{\ell=1}^{p}\int_0^{q_{i(\ell)}(t)-q_{0(\ell)}} \psi[K_{q(\ell)}\tau]\,d\tau.\end{aligned}$$

Note that V is positive definite and radially unbounded with respect to $q_i - q_0$ and $\dot{q}_i$, $i = 1,\dots,n$, under the condition of the theorem. The rest of the proof is similar to that of Theorem 6.2 by applying Lemma 1.31. ■

We finally consider a coordinated regulation algorithm that removes the requirement for the measurements of generalized coordinate derivatives as

$$\dot{\hat{x}}_i = \Gamma \hat{x}_i + \sum_{j=0}^{n} b_{ij}(q_i - q_j), \tag{6.14a}$$

$$y_i = P\left[\Gamma \hat{x}_i + \sum_{j=0}^{n} b_{ij}(q_i - q_j)\right], \tag{6.14b}$$

$$\tau_i = g_i(q_i) - \sum_{j=0}^{n} a_{ij}\psi\big[K_q(q_i - q_j)\big] - y_i, \tag{6.14c}$$

where $i = 1, \ldots, n$, $\Gamma \in \mathbb{R}^{p\times p}$ is Hurwitz, a_{ij} and b_{ij}, $i = 1, \ldots, n, j = 0, \ldots, n$, are defined analogously to those in (6.10), and $P \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite solution to the Lyapunov equation $\Gamma^T P + P\Gamma = -Q$ with $Q \in \mathbb{R}^{p\times p}$ being symmetric positive definite.

Theorem 6.7. *Using* (6.14) *for* (6.1)*,* $q_i(t) \to q_0$ *and* $\dot{q}_i(t) \to \mathbf{0}_p$*,* $i = 1, \ldots, n$*, as* $t \to \infty$ *if both* $\mathscr{G}_A$ *and* $\mathscr{G}_B$ *are undirected connected, at least one follower has access to* q_0 (*i.e., at least one* $a_{i0} > 0$)*, and at least one follower has access to* $\dot{q}_0$ (*i.e., at least one* $b_{i0} > 0$)*.*

Proof: Similar to the proof of Theorem 6.5, using (6.14), (6.1) can be written as an autonomous system with states $q_i - q_0$, $\dot{q}_i$, and $\dot{\hat{x}}_i$, $i = 1, \ldots, n$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^T M_i(q_i)\dot{q}_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\sum_{\ell=1}^{p}\int_0^{q_{i(\ell)}(t)-q_{j(\ell)}(t)} \psi[K_{q(\ell)}\tau]\,d\tau$$
$$+\sum_{i=1}^{n} a_{i0}\sum_{\ell=1}^{p}\int_0^{q_{i(\ell)}(t)-q_{0(\ell)}} \psi[K_{q(\ell)}\tau]\,d\tau + \frac{1}{2}\dot{\hat{x}}^T(H_B\otimes I_p)^{-1}(I_n\otimes P)\dot{\hat{x}},$$

where $\hat{x} \triangleq [\hat{x}_1^T, \ldots, \hat{x}_n^T]^T$ and $H_B \triangleq \mathscr{L}_B + \text{diag}(b_{10}, \ldots, b_{n0})$. Note that V is positive definite and radially unbounded with respect to $q_i - q_0$, $\dot{q}_i$, $i = 1, \ldots, n$, and $\dot{\hat{x}}$ under the condition of the theorem. The rest of the proof is similar to that of Theorem 6.3 by applying Lemma 1.31. ■

Remark 6.8 Let $\overline{\mathscr{G}}_A \triangleq (\overline{\mathscr{V}}, \overline{\mathscr{E}}_A)$ and $\overline{\mathscr{G}}_B \triangleq (\overline{\mathscr{V}}, \overline{\mathscr{E}}_B)$ be, respectively, the directed graph characterizing the interaction among the leader and the followers corresponding to, respectively, $\mathscr{G}_A$ and $\mathscr{G}_B$. From the proofs of Theorems 6.5, 6.6, and 6.7, it can be seen that all conclusions of the theorems still hold as long as $\mathscr{G}_A$ and $\mathscr{G}_B$ are undirected and in $\overline{\mathscr{G}}_A$ and $\overline{\mathscr{G}}_B$ the leader has directed paths to all followers or equivalently H_A and H_B are symmetric positive definite (see Lemma 1.6).

6.3.2 Coordinated Tracking when the Leader's Vector of Generalized Coordinate Derivatives is Constant

In this subsection, we assume that $\dot{q}_0$ is constant. The objective here is to design distributed coordinated tracking algorithms for (6.1) such that $q_i(t) - q_0(t) \to \mathbf{0}_p$ and $\dot{q}_i(t) \to \dot{q}_0$ as $t \to \infty$. Before moving on, we need the following lemma.

Lemma 6.2. *For differentiable vectors x, y, and $z \in \mathbb{R}^p$, under Assumption* (A1)*, $\dot{C}_i(x,y)z$ is bounded if all vectors $\dot{x}$, $\dot{y}$, y, and z are bounded.*

Proof: Let $e_j \in \mathbb{R}^p$ denote the vector with 1 as its jth component and 0 elsewhere. Let $C_{i(k,m)}(x,y)$ be the (k,m)th entry of $C_i(x,y)$. For $x \triangleq [x_1,\ldots,x_p]^T \in \mathbb{R}^p$, $y \triangleq [y_1,\ldots,y_p]^T \in \mathbb{R}^p$, and $z \in \mathbb{R}^p$, we have

$$\begin{aligned}\dot{C}_{i(k,m)}(x,y) &= \sum_{j=1}^{p}\left[\frac{\partial C_{i(k,m)}(x,y)}{\partial x_j}\dot{x}_j + \frac{\partial C_{i(k,m)}(x,y)}{\partial y_j}\dot{y}_j\right] \\ &= \lim_{\varepsilon\to 0}\sum_{j=1}^{p}\left[\frac{C_{i(k,m)}(x+\varepsilon e_j,y) - C_{i(k,m)}(x,y)}{\varepsilon}\dot{x}_j \right.\\ &\quad \left. + \frac{C_{i(k,m)}(x,y+\varepsilon e_j) - C_{i(k,m)}(x,y)}{\varepsilon}\dot{y}_j\right].\end{aligned}$$

It thus follows that

$$\begin{aligned}\left\|\dot{C}_i(x,y)z\right\| &= \left\|\begin{bmatrix}\dot{C}_{i(1,1)}(x,y) & \cdots & \dot{C}_{i(1,p)}(x,y)\\ \vdots & \ddots & \vdots \\ \dot{C}_{i(p,1)}(x,y) & \cdots & \dot{C}_{i(p,p)}(x,y)\end{bmatrix} z\right\| \\ &= \left\|\lim_{\varepsilon\to 0}\sum_{j=1}^{p}\left[\frac{C_i(x+\varepsilon e_j,y) - C_i(x,y)}{\varepsilon}\dot{x}_j z \right.\right.\\ &\quad \left.\left. + \frac{C_i(x,y+\varepsilon e_j) - C_i(x,y)}{\varepsilon}\dot{y}_j z\right]\right\| \\ &\le \lim_{\varepsilon\to 0}\sum_{j=1}^{p}\left[\frac{\|C_i(x+\varepsilon e_j,y)z - C_i(x,y)z\||\dot{x}_j|}{|\varepsilon|}\right.\\ &\quad \left. + \frac{\|C_i(x,y+\varepsilon e_j)z - C_i(x,y)z\||\dot{y}_j|}{|\varepsilon|}\right] \\ &\le \sum_{j=1}^{p}\big(k_{C_2}\|y\|\|z\||\dot{x}_j| + k_{C_1}\|z\||\dot{y}_j|\big) \\ &= k_{C_2}\|y\|\|z\|\|\dot{x}\|_1 + k_{C_1}\|z\|\|\dot{y}\|_1, \end{aligned} \tag{6.15}$$

where we have used Assumption (A1) to obtain the second inequality. From (6.15), if all vectors $\dot{x}$, $\dot{y}$, y, and z are bounded, it follows that $\dot{C}_i(x,y)z$ is bounded. ■

6.3.2.1 Model-dependent Coordinated Tracking Algorithm

In this subsection, we propose a distributed model-dependent coordinated tracking algorithm for (6.1) as

$$\tau_i = \tau_{i1} + \tau_{i2} + \tau_{i3}, \tag{6.16a}$$

$$\tau_{i1} = -\sum_{j=0}^{n} a_{ij}(q_i - q_j), \tag{6.16b}$$

$$\tau_{i2} = -\sum_{j=1}^{n} c_{ij}\big[(\dot{q}_i - \hat{v}_i) - (\dot{q}_j - \hat{v}_j)\big] - c_{i0}(\dot{q}_i - \hat{v}_i), \tag{6.16c}$$

$$\tau_{i3} = M_i(q_i)\dot{\hat{v}}_i + C_i(q_i,\dot{q}_i)\hat{v}_i + g_i(q_i), \tag{6.16d}$$

$$\dot{\hat{v}}_i = -\sum_{j=1}^{n} b_{ij}(\hat{v}_i - \hat{v}_j) - b_{i0}(\hat{v}_i - \dot{q}_0), \tag{6.16e}$$

where $i = 1, \ldots, n$, a_{ij} (respectively, b_{ij} and c_{ij}), $i, j = 1, \ldots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ (respectively, $\mathscr{B} \in \mathbb{R}^{n\times n}$ and $\mathscr{C} \in \mathbb{R}^{n\times n}$) associated with the graph $\mathscr{G}_A \triangleq (\mathscr{V}, \mathscr{E}_A)$ [respectively, $\mathscr{G}_B \triangleq (\mathscr{V}, \mathscr{E}_B)$ and $\mathscr{G}_C \triangleq (\mathscr{V}, \mathscr{E}_C)$] characterizing the interaction among the n followers for q_i (respectively, $\hat{v}_i$ and $\dot{q}_i - \hat{v}_i$), $a_{i0} > 0$ (respectively, $b_{i0} > 0$ and $c_{i0} > 0$) if in $\overline{\mathscr{G}}_A$ (respectively, $\overline{\mathscr{G}}_B$ and $\overline{\mathscr{G}}_C$) the leader is a neighbor of the follower and $a_{i0} = 0$ (respectively, $b_{i0} = 0$ and $c_{i0} = 0$) otherwise, and $\hat{v}_i$ is the ith follower's estimate of the leader's vector of generalized coordinate derivatives. Here $\overline{\mathscr{G}}_A$ (respectively, $\overline{\mathscr{G}}_B$ and $\overline{\mathscr{G}}_C$) is the directed graph characterizing the interaction among the leader and the followers corresponding to $\mathscr{G}_A$ (respectively, $\mathscr{G}_B$ and $\mathscr{G}_C$). Here (6.16b) is used to drive the vector of generalized coordinates of follower i to track those of the followers and the leader who are its neighbors, (6.16c) is used to drive the vector of generalized coordinate derivatives of follower i to track $\hat{v}_i$, (6.16d) is the compute-torque control with compensation, and (6.16e) is used to estimate the leader's vector of generalized coordinate derivatives.

Before presenting our main results, we need the following lemmas.

Lemma 6.3. *If $\mathscr{G}_B$ is undirected connected, and at least one follower has access to the constant $\dot{q}_0$ (i.e., at least one $b_{i0} > 0$), using (6.16e), $\hat{v}_i(t) \to \dot{q}_0$ exponentially as $t \to \infty$.*

Proof: Let $\bar{v}_i \triangleq \hat{v}_i - \dot{q}_0$ and $\bar{v} \triangleq [\bar{v}_1^T, \ldots, \bar{v}_n^T]^T$. Note that $\ddot{q}_0 = 0$ because $\dot{q}_0$ is constant. Then (6.16e) can be written as $\dot{\bar{v}}_i = -\sum_{j=0}^{n} b_{ij}(\bar{v}_i - \bar{v}_j)$, which can be written in a vector form as

$$\dot{\bar{v}} = -(\mathscr{L}_B \otimes I_p)\bar{v} - \big[\mathrm{diag}(b_{10}, \ldots, b_{n0}) \otimes I_p\big]\bar{v} = -(H_B \otimes I_p)\bar{v}, \tag{6.17}$$

where $\mathscr{L}_B$ is the Laplacian matrix associated with $\mathscr{B}$ and hence $\mathscr{G}_B$, $H_B \triangleq \mathscr{L}_B + \mathrm{diag}(b_{10}, \ldots, b_{n0})$, and we have used Lemma 1.21 to obtain the last equality.

Because $\mathscr{G}_B$ is undirected connected and at least one $b_{i0} > 0$, we conclude from Lemma 1.6 that H_B is symmetric positive definite, which means that $\lambda_{\min}(H_B) > 0$. Consider the Lyapunov function candidate $V_0 = \frac{1}{2}\bar{v}^T\bar{v}$. The derivative of V_0 is given by

$$\dot{V}_0 = -\bar{v}^T(H_B \otimes I_p)\bar{v} \leq -\lambda_{\min}(H_B)\bar{v}^T\bar{v} = -2\lambda_{\min}(H_B)V_0,$$

After some manipulation, we can get

$$V_0(t) \leq V_0(0)e^{-2\lambda_{\min}(H_B)t}. \tag{6.18}$$

Therefore, $\bar{v} = \mathbf{0}_{np}$ is globally exponentially stable, which implies that $\hat{v}_i(t) \to \dot{q}_0$ exponentially as $t \to \infty$. ■

Lemma 6.4 ([225]). *Consider the following cascade system*

$$\dot{x} = f(t,x) + h(x,\xi),\ f(t,0) = 0, h(x,0) = 0, \tag{6.19}$$

$$\dot{\xi} = A\xi, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{6.20}$$

where $f(t,x)$ is continuously differentiable in (t,x), and $h(x,\xi)$ is locally Lipschitz in (x,ξ). When $\xi = 0$, (6.19) can be written as

$$\dot{x} = f(t,x). \tag{6.21}$$

If (6.21) has the origin as a globally asymptotically stable equilibrium, A is Hurwitz, and all solutions of (6.19) and (6.20) are bounded, then the cascade system is globally asymptotically stable at the origin.

We have the following theorem in the case of a constant $\dot{q}_0$.

Theorem 6.9. *Using (6.16) for (6.1), if $\mathscr{G}_A$, $\mathscr{G}_B$, and $\mathscr{G}_C$ are all undirected connected, at least one $a_{i0} > 0$, at least one $b_{i0} > 0$, and at least one $c_{i0} > 0$, $q_i(t) - q_0(t) \to \mathbf{0}_p$, and $\dot{q}_i(t) \to \dot{q}_0$, $i = 1, \ldots, n$, as $t \to \infty$.*

Proof: Let $\tilde{q}$ and $\hat{v}$ be, respectively, a column stack vector of $q_i - q_0$ and $\hat{v}_i$, $i = 1, \ldots, n$. Define $\tilde{v} \triangleq \dot{q} - \hat{v}$. Let $\bar{v}$ be defined in the proof of Lemma 6.3. Note that $\bar{v} = \hat{v} - \mathbf{1}_n \otimes \dot{q}_0$. Using (6.16), (6.1) can be written in a vector form as

$$M(q)\dot{\tilde{v}} = -C(q,\dot{q})\tilde{v} - (H_A \otimes I_p)\tilde{q} - (H_c \otimes I_p)\tilde{v}, \tag{6.22}$$

where $H_A \triangleq \mathscr{L}_A + \mathrm{diag}(a_{10}, \ldots, a_{n0})$ and $H_C \triangleq \mathscr{L}_C + \mathrm{diag}(c_{10}, \ldots, c_{n0})$ with $\mathscr{L}_A$ and $\mathscr{L}_C$ being, respectively, the Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}_A$ and $\mathscr{C}$ hence $\mathscr{G}_C$. Let $x_1 \triangleq \tilde{q}$, $x_2 \triangleq \tilde{v}$, $x \triangleq [x_1^T, x_2^T]^T$, and $\xi \triangleq \bar{v}$. Equations (6.22) and (6.17) can be written as

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -M^{-1}(q)[(H_A \otimes I_p)x_1 + Qx_2] \end{bmatrix}}_{f(t,x)} + \underbrace{\begin{bmatrix} \xi \\ \mathbf{0}_{np} \end{bmatrix}}_{h(x,\xi)}, \tag{6.23}$$

$$\dot{\xi} = \underbrace{-(H_B \otimes I_p)}_{A}\xi, \tag{6.24}$$

where $Q \triangleq C(q,\dot{q}) + H_C \otimes I_p$, H_B is defined in (6.17), and we have used the fact that $\dot{x}_1 = \dot{q} - \mathbf{1}_n \otimes \dot{q}_0 = \tilde{v} + \hat{v} - \mathbf{1}_n \otimes \dot{q}_0 = x_2 + \xi$. Note that $q = x_1 + \mathbf{1}_n \otimes q_0$ and that $\dot{q}$ in $C(q,\dot{q})$ and hence in Q is not treated as a state but as a function of t. Hence, (6.23) and (6.24) takes in the form of the cascade system (6.19) and (6.20), and

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -M^{-1}(q)[(H_A \otimes I_p)x_1 + Qx_2] \end{bmatrix}}_{f(t,x)} \tag{6.25}$$

takes in the form of (6.21).

First, we show that all solutions of (6.23) and (6.24) are bounded. Note that $\mathscr{G}_B$ is undirected connected and at least one $b_{i0} > 0$. From Lemma 6.3, noting that $\xi \equiv \bar{v}$, we get that the solution of (6.24) (i.e., ξ) is bounded. Consider the Lyapunov function candidate as

$$V(t,x) = \frac{1}{2}x_1^T(H_A \otimes I_p)x_1 + \frac{1}{2}x_2^T M(q)x_2. \tag{6.26}$$

Because $\mathscr{G}_A$ is undirected connected and at least one $a_{i0} > 0$, it follows from Lemma 1.6 that H_A is symmetric positive definite. Also note that $M(q)$ in symmetric positive definite. It follows from Assumption (A1) that $V(t,x)$ is positive definite. Therefore, we have

$$\begin{aligned} V &\geq \frac{1}{2}\lambda_{\min}(H_A)\|x_1\|^2 + \frac{1}{2}k_{\underline{m}}\|x_2\|^2 \\ &\geq \frac{1}{2}\min\left[\lambda_{\min}(H_A), k_{\underline{m}}\right]\|x\|^2, \end{aligned} \tag{6.27}$$

and

$$\begin{aligned} \left\|\frac{\partial V}{\partial x}\right\| &= \left\|\left\{\left[(H_A \otimes I_p)x_1\right]^T, \left[M(q)x_2\right]^T\right\}^T\right\| \\ &\leq \max\left[\lambda_{\max}(H_A), k_{\bar{m}}\right]\|\bar{x}\| \leq \gamma\sqrt{V}, \end{aligned} \tag{6.28}$$

where $\gamma \triangleq \frac{\sqrt{2}\max[\lambda_{\max}(H_A),k_{\bar{m}}]}{\sqrt{\min[\lambda_{\min}(H_A),k_{\underline{m}}]}}$, and we have used Assumption (A1) to obtain (6.27) and have used (6.27) and Assumption (A1) again to obtain (6.28).

The derivative of V along (6.25) is

$$\begin{aligned}\dot{V}_{(6.25)} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) = \dot{x}_1^T (H_A \otimes I_p) x_1 + \frac{1}{2} x_2^T \dot{M}(q) x_2 + x_2^T M(q) \dot{x}_2 \\ &= x_2^T (H_A \otimes I_p) x_1 + x_2^T \left[\frac{1}{2} \dot{M}(q) - C(q,\dot{q}) \right] x_2 \\ &\quad - x_2^T \left[(H_A \otimes I_p) x_1 + (H_C \otimes I_p) x_2 \right] = -x_2^T (H_C \otimes I_p) x_2, \end{aligned} \tag{6.29}$$

where we have used Assumption (A2) to obtain the last equality. Because $\mathscr{G}_C$ is undirected connected and at least one $c_{i0} > 0$, it follows from Lemma 1.6 that H_C is symmetric positive definite. Therefore, it follows that $\dot{V}_{(6.25)} \leq 0$. Note that

$$\left\| h(x,\xi) \right\| = \left\| \begin{bmatrix} \xi \\ \mathbf{0}_{np} \end{bmatrix} \right\| = \|\xi\|. \tag{6.30}$$

Then, the derivative of V along (6.23) can be written as

$$\begin{aligned}\dot{V}_{(6.23)} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} h(x,\xi) = \dot{V}_{(6.25)} + \frac{\partial V}{\partial x} h(x,\xi) \\ &\leq \left\| \frac{\partial V}{\partial x} \right\| \left\| h(x,\xi) \right\| \leq \gamma \|\xi\| \sqrt{V}, \end{aligned} \tag{6.31}$$

where we have used (6.28) and (6.30) to obtain the last inequality. From (6.18), noting that $\xi \equiv \bar{v}$, we can get

$$\begin{aligned}\int_0^t \left\| \xi(\tau) \right\| d\tau &\leq \left\| \xi(0) \right\| \int_0^t e^{-\lambda_{\min}(H_B)\tau} \, d\tau \\ &= \frac{\|\xi(0)\|}{\lambda_{\min}(H_B)} \left[1 - e^{-\lambda_{\min}(H_B)t} \right]. \end{aligned} \tag{6.32}$$

Note that (6.31) is equivalent to the following inequality

$$\frac{\dot{V}}{\sqrt{V}} \leq \gamma \|\xi\|. \tag{6.33}$$

Integrating both sides of (6.33) from 0 to $t > 0$ and after some manipulation, we obtain

$$\begin{aligned}\sqrt{V\big(t,x(t)\big)} &\leq \sqrt{V\big(0,x(0)\big)} + \frac{\gamma}{2} \int_0^t \left\| \xi(\tau) \right\| d\tau \\ &\leq \sqrt{V\big(0,x(0)\big)} + \frac{\gamma \|\xi(0)\|}{2\lambda_{\min}(H_B)}, \end{aligned} \tag{6.34}$$

where we have used (6.32) to get the last inequality. From (6.34), we can conclude that $V(t,x)$ is uniformly bounded along the solution of (6.23). It thus follows that the solution of (6.23) (i.e., x_1 and x_2) is bounded.

Second, we show that the system (6.25) is globally asymptotically stable at the origin. Note that from the fact that $V(t,x)$ is positive definite and the fact that

$\dot{V}_{(6.25)} \leq 0$ that x_1 and x_2 are bounded in the system (6.25). It thus follows that $\dot{x}_1$ is bounded because $\dot{x}_1 = x_2$ in (6.25). Also note that $\bar{v}$ is bounded from (6.18), $\hat{v}$ is bounded from $\hat{v} = \bar{v} + \mathbf{1}_n \otimes \dot{q}_0$ and the fact that $\dot{q}_0$ is constant, and $\dot{q}$ is bounded from $\dot{q} = x_2 + \hat{v}$. Because $M^{-1}(q)$ is bounded and $C(q, \dot{q})x_2$ is bounded if $\dot{q}$ and x_2 are bounded from Assumption (A1), we can conclude that $\dot{x}_2$ is bounded from (6.25). Thus, we have explicitly shown that all vectors x_1, x_2, $\dot{x}_1$, $\dot{x}_2$, and $\dot{q}$ are bounded. By differentiating $\dot{V}_{(6.25)}$, we can see that $\ddot{V}_{(6.25)}$ is bounded. Therefore, $\dot{V}_{(6.25)}$ is uniformly continuous in time. From Lemma 1.33, we get that $\dot{V}_{(6.25)}(t) \to 0$ as $t \to \infty$. Then from (6.29), we get that $x_2(t) \to \mathbf{0}_{np}$ as $t \to \infty$ because H_C is symmetric positive definite. The second equation in (6.25) can be written as

$$M(q)\dot{x}_2 = -(H_A \otimes I_p)x_1 - Qx_2. \tag{6.35}$$

Differentiating (6.35), we get

$$M(q)\ddot{x}_2 + \dot{M}(q)\dot{x}_2 = -(H_A \otimes I_p)\dot{x}_1 - Q\dot{x}_2 - \dot{C}(q, \dot{q})x_2, \tag{6.36}$$

where we have used the fact that $\dot{Q} = \dot{C}(q, \dot{q})$. From Assumption (A2), we can obtain that $\dot{M}(q) = C(q, \dot{q}) + C^T(q, \dot{q})$. From the proceeding boundedness statements, we can conclude that $\dot{M}(q)\dot{x}_2$ is bounded because $C(q, \dot{q})\dot{x}_2$ is bounded and $\dot{C}(q, \dot{q})x_2$ is bounded from Lemma 6.2. Then from (6.36), it follows that $\ddot{x}_2$ is bounded, which means that $\dot{x}_2$ is uniformly continuous. From Lemma 1.33, we get that $\dot{x}_2(t) \to \mathbf{0}_{np}$ as $t \to \infty$. Because both $x_2(t) \to \mathbf{0}_{np}$ and $\dot{x}_2(t) \to \mathbf{0}_{np}$ as $t \to \infty$, according to (6.35), we can obtain that $(H_A \otimes I_p)x_1(t) \to \mathbf{0}_{np}$ as $t \to \infty$. Note that H_A is symmetric positive definite. It thus follows that $x_1(t) \to \mathbf{0}_{np}$ as $t \to \infty$. Also because V given by (6.26) is radially unbounded with respect to x, it follows that (6.25) is globally asymptotically stable at the origin.

Third, it follows from Lemma 1.6 that A in (6.24) is Hurwitz. We conclude from Lemma 6.4 that the cascade system (6.23) and (6.24) is globally asymptotically stable at the origin, i.e., $x_1(t) \to \mathbf{0}_{np}$, $x_2(t) \to \mathbf{0}_{np}$ and $\xi(t) \to \mathbf{0}_{np}$ as $t \to \infty$. Note that $x_1 = q - \mathbf{1}_n \otimes q_0$. We can get $q_i(t) - q_0(t) \to \mathbf{0}_{np}$, $i = 1, \ldots, n$, as $t \to \infty$. Also note that $x_2 = \dot{q} - \mathbf{1}_n \otimes \dot{q}_0 - \xi$. We can conclude that $\dot{q}_i(t) \to \dot{q}_0$, $i = 1, \ldots, n$, as $t \to \infty$ because $x_2(t) \to \mathbf{0}_{np}$ and $\xi(t) \to \mathbf{0}_{np}$ as $t \to \infty$. ■

Remark 6.10 We here show that the conditions in Theorem 6.9 can be relaxed. In fact, the conclusion of Theorem 6.9 holds as long as H_A is symmetric positive definite, $-H_B$ is Hurwitz,[2] and H_C is symmetric positive definite. Lemma 1.6 implies that if $\mathscr{G}_A$ (respectively, $\mathscr{G}_C$) is undirected and in $\overline{\mathscr{G}}_A$ (respectively, $\overline{\mathscr{G}}_C$) the leader has directed paths to all followers, then H_A (respectively, H_C) is symmetric positive definite. Also, in $\overline{\mathscr{G}}_B$, if the leader has directed paths to all followers, then $-H_B$ is Hurwitz. Note that here $\mathscr{G}_B$ can be directed. Therefore, the connectivity conditions in Theorem 6.9 can be relaxed.

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[2] If $-H_B$ is Hurwitz, so is $-H_B \otimes I_p$. It thus follows that in (6.17) $\bar{v} = \mathbf{0}_{np}$ is globally exponentially stable even if H_B might not be symmetric.

6.3.2.2 Coordinated Tracking Algorithm Accounting for Parametric Uncertainties

In this subsection, we present a distributed coordinated tracking algorithm that accounts for unknown parametric uncertainties of the Euler–Lagrange dynamics. Before moving on, we introduce the following auxiliary variables

$$\dot{q}_{ri} \triangleq \hat{v}_i - \alpha \left[\sum_{j=0}^{n} a_{ij}(q_i - q_j) \right], \tag{6.37}$$

$$s_i \triangleq \dot{q}_i - \dot{q}_{ri} = \dot{q}_i - \hat{v}_i + \alpha \left[\sum_{j=0}^{n} a_{ij}(q_i - q_j) \right], \tag{6.38}$$

where $i = 1, \ldots, n$, α is a positive constant, $\hat{v}_i$ is the ith follower's estimate of the leader's vector of generalized coordination derivatives, $a_{ij}, i, j = 1, \ldots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{A}$ associated with the graph $\mathscr{G}_A \triangleq (\mathscr{V}, \mathscr{E}_A)$ characterizing the interaction among the n followers for q_i (and $\dot{q}_i$ as shown later on), and $a_{i0} > 0$ if the leader has access to q_0 (and $\dot{q}_0$ as shown later on) and $a_{i0} = 0$ otherwise. From Assumption (A3), we get

$$M_i(q_i)\ddot{q}_{ri} + C_i(q_i, \dot{q}_i)\dot{q}_{ri} + g_i(q_i) = Y_i(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri})\Theta_i,$$

where $i = 1, \ldots, n$, and Θ_i is the unknown constant parameter vector for the ith follower defined in Assumption (A3).

We propose the following coordinated tracking algorithm for (6.1) in the presence of parametric uncertainties

$$\tau_i = -K_i s_i - \eta(\dot{q}_i - \hat{v}_i) + Y_i \widehat{\Theta}_i, \tag{6.39a}$$

$$\dot{\hat{v}}_i = -\sum_{j=1}^{n} b_{ij}(\hat{v}_i - \hat{v}_j) - b_{i0}(\hat{v}_i - \dot{q}_0), \tag{6.39b}$$

where K_i is a symmetric positive-definite matrix, η is a positive constant, $Y_i \triangleq Y_i(q_i, \dot{q}_i, \dot{q}_{ri}, \ddot{q}_{ri})$, $\widehat{\Theta}_i$ is the estimate of Θ_i, and b_{ij}, $i = 1, \ldots, n, j = 0, \ldots, n$, is defined in (6.16e). Here $\widehat{\Theta}_i$ is updated by the following adaptation law

$$\dot{\widehat{\Theta}}_i = -\Lambda_i Y_i^T s_i, \tag{6.40}$$

where Λ_i is a symmetric positive-definite matrix. Let $\widetilde{\Theta}_i \triangleq \Theta_i - \widehat{\Theta}_i$, and $\widetilde{\Theta}$, Θ, $\widehat{\Theta}$, s, $\tilde{q}$, and $\hat{v}$ be, respectively, the column stack vector of $\widetilde{\Theta}_i$, Θ_i, $\widehat{\Theta}_i$, s_i, $\tilde{q}_i \triangleq q_i - q_0$, and $\hat{v}_i$, $i = 1, \ldots, n$. Note from (6.38) that $\dot{q}_i - \hat{v}_i = s_i - \alpha \sum_{j=0}^{n} a_{ij}(q_i - q_j)$. Hence, the closed-loop system (6.1) using (6.39a) can be written in a vector form as

$$M(q)\dot{s} = -C(q,\dot{q})s - Ks - \eta\big[s - \alpha(H_A \otimes I_p)\tilde{q}\big] - Y\widetilde{\Theta}, \tag{6.41}$$

where H_A is defined as in (6.22), $Y \triangleq \text{diag}(Y_1, \dots, Y_n)$, and $K \triangleq \text{diag}(K_1, \dots, K_n)$.

Theorem 6.11. *Using (6.39) and (6.40) for (6.1), if both $\mathcal{G}_A$ and $\mathcal{G}_B$ are undirected connected, at least one $a_{i0} > 0$, and at least one $b_{i0} > 0$, $q_i(t) - q_0(t) \to \mathbf{0}_p$ and $\dot{q}_i(t) \to \dot{q}_0$, $i = 1, \dots, n$, as $t \to \infty$ in the presence of parametric uncertainties.*

Proof: Let $x_1 \triangleq \tilde{q}$, $x_2 \triangleq s$, $x \triangleq [x_1^T, x_2^T]^T$, and $\xi \triangleq \hat{v} - \mathbf{1}_n \otimes \dot{q}_0$. Equations (6.41) and (6.39b) can be written as

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 - \alpha(H_A \otimes I_p)x_1 \\ -M^{-1}(q)\big[-\eta\alpha(H_A \otimes I_p)x_1 + Qx_2 + Y\widetilde{\Theta}\big] \end{bmatrix}}_{f(t,x)} + \underbrace{\begin{bmatrix} \xi \\ \mathbf{0}_{np} \end{bmatrix}}_{h(x,\xi)}, \tag{6.42}$$

$$\dot{\xi} = \underbrace{-(H_B \otimes I_p)}_{A}\xi, \tag{6.43}$$

where $Q \triangleq C(q,\dot{q}) + K + \eta I_n \otimes I_p$ and H_B is defined as in (6.17). Note that $q = x_1 + \mathbf{1}_n \otimes q_0$ and that $\dot{q}$ and $\widetilde{\Theta}$ in (6.42) are not treated as states, but as functions of t. Hence, (6.42) and (6.43) take in the form of the cascade system (6.19) and (6.20), and

$$\dot{x} = \begin{bmatrix} x_2 - \alpha(H_A \otimes I_p)x_1 \\ -M^{-1}(q)\big[-\eta\alpha(H_A \otimes I_p)x_1 + Qx_2 + Y\widetilde{\Theta}\big] \end{bmatrix} \tag{6.44}$$

takes in the form of (6.21).

First, we will show that all solutions of (6.42) and (6.43) are bounded. Because $\mathcal{G}_A$ (respectively, $\mathcal{G}_B$) is undirected connected and at least one $a_{i0} > 0$ (respectively, $b_{i0} > 0$), it follows from Lemma 1.6 that H_A (respectively, H_B) is symmetric positive definite. We get that the solution of (6.43) (i.e., ξ) is bounded. Consider a nonnegative scalar function as

$$V(t,x) = \frac{\eta\alpha}{2}x_1^T(H_A \otimes I_p)x_1 + \frac{1}{2}x_2^T M(q)x_2 + \frac{1}{2}\widetilde{\Theta}^T \Xi \widetilde{\Theta}, \tag{6.45}$$

where $\Xi \triangleq \text{diag}(\Lambda_1^{-1}, \dots, \Lambda_n^{-1})$ is symmetric positive definite. We have

$$\begin{aligned} V &\geq \frac{\eta\alpha}{2}\lambda_{\min}(H_A)\|x_1\|^2 + \frac{1}{2}k_{\underline{m}}\|x_2\|^2 \\ &\geq \frac{1}{2}\min\big[\eta\alpha\lambda_{\min}(H_A), k_{\underline{m}}\big]\|x\|^2, \end{aligned}$$

and

$$\left\|\frac{\partial V}{\partial x}\right\| = \left\|\left\{\eta\alpha\left[(H_A\otimes I_p)x_1\right]^T, \left[M(q)x_2\right]^T\right\}^T\right\|,$$
$$\leq \max\left[\eta\alpha\lambda_{\max}(H_A), k_{\bar{m}}\right]\|x\| \leq \gamma\sqrt{V},$$

where $\gamma \triangleq \frac{\sqrt{2}\max[\eta\alpha\lambda_{\max}(H_A),k_{\bar{m}}]}{\sqrt{\min[\eta\alpha\lambda_{\min}(H_A),k_{\underline{m}}]}}$.

The derivative of V along (6.44) is

$$\begin{aligned}\dot{V}_{(6.44)} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x)\\ &= \eta\alpha x_1^T(H_A\otimes I_p)\left[x_2 - \alpha(H_A\otimes I_p)x_1\right]\\ &\quad + \frac{1}{2}x_2^T\dot{M}(q)x_2 + x_2^T M(q)\dot{x}_2 + \widetilde{\Theta}^T\varXi\dot{\widetilde{\Theta}}\\ &= -\eta\left[x_2 - \alpha(H_A\otimes I_p)x_1\right]^T\left[x_2 - \alpha(H_A\otimes I_p)x_1\right]\\ &\quad - x_2^T K x_2,\end{aligned} \tag{6.46}$$

where we have used Assumption (A2) and the fact that $\dot{\widetilde{\Theta}} = \varXi^{-1}Y^T s$ according to (6.40) and $s \equiv x_2$ to obtain the last equality. Note that $\dot{V}_{(6.44)} \leq 0$ because $\eta > 0$ and K is symmetric positive definite. Then the derivative of V along (6.42) can be written as

$$\begin{aligned}\dot{V}_{(6.42)} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) + \frac{\partial V}{\partial x}h(x,\xi) = \dot{V}_{(6.44)} + \frac{\partial V}{\partial x}h(x,\xi)\\ &\leq \left\|\frac{\partial V}{\partial x}\right\|\left\|h(x,\xi)\right\| \leq \gamma\|\xi\|\sqrt{V}.\end{aligned} \tag{6.47}$$

Following the same steps as in the proof of Theorem 6.9, we can easily get that $V(t,x)$ is uniformly bounded along the solution of (6.42), which means that x_1, x_2 and $\widetilde{\Theta}$ are all bounded.

Second, we show that the system (6.44) is globally asymptotically stable at the origin. By following similar boundedness statements in the proof of Theorem 6.9 and applying Lemma 1.33, we conclude that $\dot{V}_{(6.44)}(t) \to 0$ as $t \to \infty$. Then from (6.46) we can get that $x_2(t) - \alpha(H_A\otimes I_p)x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$, which means that $(H_A\otimes I_p)x_1(t) \to 0$ as $t \to \infty$. Because H_A is symmetric positive definite, it follows that $x_1(t) \to 0$ as $t \to \infty$. Note that V defined by (6.45) is radially unbounded with respect to x, it follows that the system (6.44) is globally asymptotically stable at the origin.

Third, note that H_B is symmetric positive definite, which implies that A in (6.43) is Hurwitz. We conclude from Lemma 6.4 that the cascade system (6.42) and (6.43) is globally asymptotically stable at the origin, which in turn proves the theorem. ■

Remark 6.12 In the adaptive case, we need to introduce the auxiliary variables. If we just introduce the adaptation law without the auxiliary variables, and choose the Lyapunov function as (6.26) with an addition of $\frac{1}{2}\widetilde{\Theta}^T\varXi\,\widetilde{\Theta}$, then $\dot{V}_{(6.44)}$ is negative semidefinite as in (6.29). However, in this case, we cannot get that $x_1(t) \to 0$ from

$x_2(t) \to 0$ as $t \to 0$. By introducing the auxiliary variables defined in (6.37) and (6.38), we need both $\sum_{j=0}^{n} a_{ij}(q_i - q_j)$ and its derivative in the control algorithm (6.39) because they are needed to derive $\dot{q}_{ri}$ and $\ddot{q}_{ri}$ (and hence Y_i). It is therefore required that the interaction graphs associated with the followers for both q_i and $\dot{q}_i$ be the same. In addition, the connectivity condition in Theorem 6.11 can be relaxed similar to that in Remark 6.10.

6.3.3 Coordinated Tracking when the Leader's Vector of Generalized Coordinate Derivatives is Varying

In this subsection, $\dot{q}_0(t)$ is allowed to be varying. The objective here is to design a distributed model-independent sliding-mode algorithm for (6.1) such that $q_i(t) - q_0(t) \to \mathbf{0}_p$ and $\dot{q}_i(t) - \dot{q}_0(t) \to \mathbf{0}_p$ as $t \to \infty$. Define the following auxiliary variables

$$s_i \triangleq \dot{q}_i + \lambda q_i, \quad i = 0, 1, \ldots, n, \tag{6.48}$$

where λ is a positive constant. Also define the error variable between s_i and s_0 as

$$\tilde{s}_i \triangleq s_i - s_0 = \dot{q}_i - \dot{q}_0 + \lambda(q_i - q_0), \quad i = 1, \ldots, n. \tag{6.49}$$

Then (6.1) can be written as

$$M_i(q_i)\dot{s}_i + C_i(q_i, \dot{q}_i)s_i = \tau_i + \lambda M_i(q_i)\dot{q}_i + \lambda C_i(q_i, \dot{q}_i)q_i - g_i(q_i). \tag{6.50}$$

We propose the distributed coordinated tracking algorithm for (6.50) [and hence (6.1)] as

$$\begin{aligned}\tau_i = -\alpha\left[\sum_{j=0}^{n} a_{ij}(s_i - s_j)\right] - \beta\Bigg(\sum_{j=1}^{n} a_{ij}\Bigg\{\text{sgn}\left[\sum_{k=0}^{n} a_{ik}(s_i - s_k)\right] \\ - \text{sgn}\left[\sum_{k=0}^{n} a_{jk}(s_j - s_k)\right]\Bigg\} + a_{i0}\text{sgn}\left[\sum_{j=0}^{n} a_{ij}(s_i - s_j)\right]\Bigg),\end{aligned} \tag{6.51}$$

where α is a nonnegative constant, β is a positive constant, a_{ij}, $i, j = 1, \ldots, n$ is the (i, j)th entry of the adjacency matrix $\mathscr{A}$ associated with the undirected graph $\mathscr{G}_A \triangleq (\mathscr{V}, \mathscr{E}_A)$ characterizing the interaction among the followers, $a_{i0} > 0$ if the leader is a neighbor of follower i and $a_{i0} = 0$ otherwise, and $\text{sgn}(\cdot)$ is defined componentwise. Let s and $\tilde{s}$ be, respectively, the column stack vector of s_i and $\tilde{s}_i$, $i = 1, \ldots, n$. We can rewrite the closed-loop system (6.50) [and hence (6.1)] using (6.51) in a vector form as

$$M(q)\dot{\tilde{s}} + C(q,\dot{q})\tilde{s} = -\alpha(H_A \otimes I_p)\tilde{s} - \beta(H_A \otimes I_p)\,\mathrm{sgn}\big[(H_A \otimes I_p)\tilde{s}\big] + \Delta, \tag{6.52}$$

where $H_A \triangleq \mathscr{L}_A + \mathrm{diag}(a_{10}, \ldots, a_{n0})$ with $\mathscr{L}_A$ being the Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}_A$ and $\Delta \triangleq -M(q)(\mathbf{1}_n \otimes \dot{s}_0) - C(q,\dot{q})(\mathbf{1}_p \otimes s_0) + \lambda M(q)\dot{q} + \lambda C(q,\dot{q})q - g(q)$. Note that H_A is symmetric positive semidefinite because $\mathscr{G}_A$ is undirected.

Remark 6.13 Note that the algorithm (6.51) is discontinuous. Therefore, the stability analysis of the closed-loop system (6.1) using (6.51) is conducted for the Filippov solutions via the nonsmooth analysis in Sect. 1.5. Accordingly, Remarks 4.1 and 4.3 also apply here.

Before moving on, we need the following assumption on the boundedness of $\dot{q}_0$ and $\ddot{q}_0$:

(A4) Both $\dot{q}_0$ and $\ddot{q}_0$ are bounded, and in particular, $\|\mathbf{1}_n \otimes \dot{q}_0\| \le k_v$ and $\|\mathbf{1}_n \otimes \ddot{q}_0\| \le k_a$.

Remark 6.14 We do not restrict q_0 to be bounded in (A4). Most desired trajectories have the properties of (A4), so (A4) is a reasonable assumption. In the control algorithm (6.51), there is no need to know the value of $\ddot{q}_0$.

Next, we show the boundedness of Δ in (6.52). From Assumption (A1), it follows that $\|g(q)\| \le \sqrt{n}k_g$. Following Assumptions (A1) and (A4), we have

$$\begin{aligned}\|\Delta\| &= \big\|-M(q)(\mathbf{1}_n \otimes \ddot{q}_0) - C(q,\dot{q})(\mathbf{1}_n \otimes \dot{q}_0) + \lambda M(q)\dot{\tilde{q}} + \lambda C(q,\dot{q})\tilde{q} - g(q)\big\| \\ &\le k_{\overline{m}}k_a + k_C k_v\|\dot{q}\| + \lambda k_{\overline{m}}\|\dot{\tilde{q}}\| + \lambda k_C\|\dot{q}\|\|\tilde{q}\| + \sqrt{n}k_g,\end{aligned} \tag{6.53}$$

where $\tilde{q} \triangleq q - \mathbf{1}_n \otimes q_0$. Note that (6.49) can be written in a vector form as $\tilde{s} = \dot{\tilde{q}} + \lambda\tilde{q}$. Multiplying $e^{\lambda\tau}$ on both sides and integrating from 0 to t, we have

$$\tilde{q}(t) = e^{-\lambda t}\left[\tilde{q}(0) + \int_0^t e^{\lambda\tau}\tilde{s}(\tau)\,d\tau\right]. \tag{6.54}$$

Lemma 6.5. *Define a norm-like function $\|x\|_M \triangleq \sqrt{x^T M(q)x}$, where $x \in \mathbb{R}^{np}$. Then $\sqrt{k_{\underline{m}}}\,\|x\| \le \|x\|_M \le \sqrt{k_{\overline{m}}}\,\|x\|$ for all $t \ge 0$.*

Proof: Note that from Assumption (A1), $k_{\underline{m}}z^T z \le z^T M_i(q_i)z \le k_{\overline{m}}z^T z$ for $z \in \mathbb{R}^p$, $i = 1, \ldots, n$. It thus follows that $k_{\underline{m}}x^T x \le x^T M(q)x \le k_{\overline{m}}x^T x$, which means that $\sqrt{k_{\underline{m}}}\|x\| \le \|x\|_M \le \sqrt{k_{\overline{m}}}\|x\|$. ■

From (6.54), we have

$$\big\|\tilde{q}(t)\big\| \le e^{-\lambda t}\big\|\tilde{q}(0)\big\| + \frac{\sup_{0\le\tau\le t}\|\tilde{s}(\tau)\|}{\lambda}\big(1 - e^{-\lambda t}\big). \tag{6.55}$$

From Lemma 6.5, we can get $\|\tilde{s}(t)\| \le \|\tilde{s}(t)\|_M/\sqrt{k_{\underline{m}}}$ for all $t \ge 0$. It thus follows that $\sup_{0\le\tau\le t}\|\tilde{s}(\tau)\| \le \sup_{0\le\tau\le t}\|\tilde{s}(\tau)\|_M/\sqrt{k_{\underline{m}}}$. Define

$$\phi(t) \triangleq \sup_{0\le\tau\le t} \|\tilde{s}(\tau)\|_M. \tag{6.56}$$

It follows from (6.55) and (6.56) that

$$\|\tilde{q}(t)\| \le \|\tilde{q}(0)\| + \frac{\sup_{0\le\tau\le t}\|\tilde{s}(\tau)\|}{\lambda} \le \|\tilde{q}(0)\| + \frac{\phi(t)}{\lambda\sqrt{k_{\underline{m}}}} \triangleq k_e. \tag{6.57}$$

Note from the definition of $\tilde{s}$ that $\tilde{s} = \dot{\tilde{q}} + \lambda\tilde{q}$. It thus follows that

$$\|\dot{\tilde{q}}(t)\| = \|\tilde{s}(t) - \lambda q_e(t)\| \le \frac{\phi(t)}{\sqrt{k_{\underline{m}}}} + \lambda k_e \triangleq k_{\dot{e}} \tag{6.58}$$

and

$$\|\dot{q}(t)\| = \|\dot{\tilde{q}}(t) + \mathbf{1}_n \otimes \dot{q}_0(t)\| \le k_{\dot{e}} + k_v. \tag{6.59}$$

Substituting (6.57)–(6.59) into (6.53), it follows that

$$\begin{aligned}\|\Delta\| &\le k_{\bar{m}}(k_a + \lambda k_{\dot{e}}) + k_C(k_v + \lambda k_e)(k_{\dot{e}} + k_v) + \sqrt{n}k_g \\ &= a\phi^2(t) + b\phi(t) + c \triangleq \gamma(t),\end{aligned} \tag{6.60}$$

where

$$\begin{aligned} a &\triangleq 2k_C/k_{\underline{m}}, \\ b &\triangleq \big(3k_Ck_v + 2\lambda k_{\bar{m}} + 3\lambda k_C\|\tilde{q}(0)\|\big)/\sqrt{k_{\underline{m}}}, \\ c &\triangleq k_{\bar{m}}k_a + \sqrt{n}k_g + k_Ck_v^2 + \big(2\lambda k_Ck_v + \lambda^2 k_{\bar{m}}\big)\|\tilde{q}(0)\| + \lambda^2 k_C\|\tilde{q}(0)\|^2. \end{aligned}$$

Noting that a, b, c are positive constants, and $\phi(t) \ge 0$, we get that $\gamma(t)$ is monotonically increasing on $[0,\infty)$ because $\phi(t)$ is monotonically increasing on $[0,\infty)$. If there exists some bounded disturbance in (6.1), with an addition of a constant in c, the following results still hold. Thus, the coordinated tracking algorithm (6.51) is robust to bounded disturbance.

Consider the following Lyapunov function candidate for (6.52) as

$$V = \frac{1}{2}\tilde{s}^T M(q)\tilde{s}. \tag{6.61}$$

It follows that

$$\begin{aligned}\max \widetilde{L}_F V = \dot{V} &= \tilde{s}^T M(q)\dot{\tilde{s}} + \frac{1}{2}\tilde{s}^T\dot{M}(q)\tilde{s} \\ &= \tilde{s}^T\big\{-\alpha(H_A\otimes I_p)\tilde{s} - \beta(H_A\otimes I_p)\mathrm{sgn}\big[(H_A\otimes I_p)\tilde{s}\big] + \Delta\big\} \\ &= -\alpha\tilde{s}^T(H_A\otimes I_p)\tilde{s} - \beta\big\|(H_A\otimes I_p)\tilde{s}\big\|_1 + \Delta^T\tilde{s}\end{aligned} \tag{6.62}$$

$$\leq -\alpha\tilde{s}^T(H_A\otimes I_p)\tilde{s} - \beta\big\|(H_A\otimes I_p)\tilde{s}\big\| + \|\Delta\|\|\tilde{s}\|$$
$$\leq -\alpha\tilde{s}^T(H_A\otimes I_p)\tilde{s} - \big[\beta\lambda_{\min}(H_A)-\gamma(t)\big]\|\tilde{s}\|, \tag{6.63}$$

where we have used Assumption (A2) to obtain the second equality, the fact that $\|x\| \leq \|x\|_1$ to obtain the first inequality, and the fact that $\|H_A x\| \geq \lambda_{\min}(H_A)x$, $\forall x\in\mathbb{R}^n$, and $\|\Delta\|\leq\gamma(t)$ from (6.60) to obtain the last inequality.

If we can choose β such that $\beta\lambda_{\min}(H_A)-\gamma(t)>0$,[3] then we can show that $\dot{V}$ is negative definite. However, $\gamma(t)$ is a time-varying function involving $\tilde{s}(t)$, which implies that we need to know all $\tilde{s}_i(t)$, $i=1,\dots,n$, at each time to find a proper β. Unfortunately, it is not possible to do so because the leader is the neighbor of only a subset of the followers. So we need the following lemmas.

Lemma 6.6. *If $\beta\lambda_{\min}(H_A)-\gamma(t_1)>0$ and $\|\tilde{s}(t_1)\|=0$ at some time $t_1\geq 0$, then $\|\tilde{s}(t)\|\equiv 0$ for all $t\geq t_1$.*

Proof: Treat t_1 as the initial time. Let $\phi(t)$ be defined as in (6.56) with $0\leq\tau\leq t$ replaced with $t_1\leq\tau\leq t$, and let $\gamma(t)$ be defined as in (6.60) with $\tilde{q}(0)$ in the variables b and c replaced with $\tilde{q}(t_1)$. Note that $\|\tilde{s}(t)\|$ is continuous, which implies that $\phi(t)$ and $\gamma(t)$ are also continuous. Because $\beta\lambda_{\min}(H_A)-\gamma(t_1)>0$, there exists a neighborhood Ω of t_1 such that $\beta\lambda_{\min}(H_A)-\gamma(t)>0$ when $t\in\Omega_1\triangleq\Omega\cap\{t>t_1\}$. For $t\in\Omega_1$, from (6.63), we can get that $\dot{V}(t)\leq-[\beta\lambda_{\min}(H_A)-\gamma(t)]\|\tilde{s}\|\leq 0$. Also note that $\|\tilde{s}(t_1)\|=0$, which means that $V(t_1)=0$ and $\dot{V}(t_1)=0$ from (6.62). Because $V(t)\geq 0$ for all $t\geq 0$, we can conclude that $V(t)=0$, i.e., $\|\tilde{s}(t)\|=0$, for $t\in\Omega_1$.

We then prove the lemma by contradiction. Suppose that there exists $t_2>t_1$ such that $\|\tilde{s}(t_2)\|\neq 0$ (and hence $\|\tilde{s}(t_2)\|>0$). From the continuity of $\|\tilde{s}(t)\|$, there exists $t_3\in(t_1,t_2)$ and a neighborhood Ω_2 of t_3 such that $\|\tilde{s}(t)\|=0$ for $t\in[t_1,t_3]$ and $\|\tilde{s}(t)\|>0$ for $t\in\Omega_3\triangleq\Omega_2\cap(t_3,t_2)$. From the definition of $\gamma(t)$, we can get that $\beta\lambda_{\min}(H_A)-\gamma(t_3)=\beta\lambda_{\min}(H_A)-\gamma(t_1)>0$. From the continuity of $\gamma(t)$, there exists a neighborhood Ω_4 of t_3 such that $\beta\lambda_{\min}(H_A)-\gamma(t)>0$ for $t\in\Omega_5\triangleq\Omega_4\cap(t_3,t_2)$, which means that $\dot{V}\leq-[\beta\lambda_{\min}(H_A)-\gamma(t)]\|\tilde{s}\|\leq 0$ for $t\in\Omega_5$. Also note that $V(t_3)=0$, $\dot{V}(t_3)=0$, and $V(t)\geq 0$ for all $t\geq 0$, we can get that $V(t)=0$, i.e., $\|\tilde{s}(t)\|=0$, for $t\in\Omega_5$. Note that both Ω_2 and Ω_4 are neighborhoods of t_3. We can get that $\Omega_3\cap\Omega_5\neq\varnothing$. Thus, we have that $\|\tilde{s}(t)\|>0$ for $t\in\Omega_3$ and $\|\tilde{s}(t)\|=0$ for $t\in\Omega_5$, which results in a contradiction for $t\in\Omega_3\cap\Omega_5$. ∎

Lemma 6.7. *If β is chosen such that $\beta\lambda_{\min}(H_A)-\gamma(0)>0$, then $\beta\lambda_{\min}(H_A)-\gamma(t)=\beta\lambda_{\min}(H_A)-\gamma(0)>0$ for all $t\geq 0$, or there exists $\bar{t}\geq 0$ such that $\|\tilde{s}(t)\|\equiv 0$ for all $t\geq\bar{t}$.*

Proof: From the definition of $\phi(t)$ in (6.56) and $\gamma(t)$ in (6.60), for all $t\geq 0$, we have that $\phi(t)\geq\phi(0)$ and $\gamma(t)\geq\gamma(0)$. Thus, if $\gamma(t)=\gamma(0)$ for all $t\geq 0$, we can

[3] Of course, in this case, $\lambda_{\min}(H_A)$ must be positive, implying that H_A must be symmetric positive definite rather than just symmetric positive semidefinite.

conclude our proof. If β is chosen such that $\beta\lambda_{\min}(H_A) - \gamma(0) > 0$, from (6.63), $\dot{V}(0) \leq 0$. If $\dot{V}(0) = 0$, i.e., $\|\tilde{s}(0)\| = 0$, we have that $\|\tilde{s}(t)\| = 0$ for all $t \geq \bar{t} \stackrel{\triangle}{=} 0$ from Lemma 6.6. This conclude our proof.

Because $\tilde{s}(t)$ is continuous, so is V defined in (6.61). If $\dot{V}(0) < 0$, there must exist a neighborhood Ω of 0 such that $V(t) < V(0)$ when $t \in \Omega_1 \stackrel{\triangle}{=} \Omega \cap \{t > 0\}$. If there exits $t_1 > 0$ such that $\gamma(t_1) > \gamma(0)$, from (6.60) and the monotonic property of $\gamma(t)$, we have that $\phi(t_1) > \phi(0)$. From the definition of $\phi(t)$ in (6.56), there must exist $t_2 \in (0, t_1)$ such that $\|\tilde{s}(t_2)\|_M > \|\tilde{s}(0)\|_M$. Without loss of generality, suppose that t_2 is in the interval where $\|\tilde{s}(t)\|_M$ becomes larger than $\|\tilde{s}(0)\|_M$ for the first time. Note that $V(t) = \frac{1}{2}\tilde{s}^T(t)M(q)\tilde{s}(t) = \frac{1}{2}\|\tilde{s}(t)\|_M^2$. Because $V(t) < V(0)$ for all $t \in \Omega_1$, which implies that $\|\tilde{s}(t)\|_M < \|\tilde{s}(0)\|_M$ for all $t \in \Omega_1$. Also note that $\|\tilde{s}(t)\|_M$ is continuous and $\|\tilde{s}(t_2)\|_M > \|\tilde{s}(0)\|_M$. There must exist $t_3 \in (0, t_2)$ such that $\|\tilde{s}(t_3)\|_M = \|\tilde{s}(0)\|_M$ and $\|\tilde{s}(t)\|_M < \|\tilde{s}(0)\|_M$ for all $t \in (0, t_3)$, which means that $V(t_3) = V(0)$ and $V(t) < V(0)$ for all $t \in (0, t_3)$. From the mean value theorem, there is a point $t_4 \in (0, t_3)$ at which $\dot{V}(t_4) = 0$. But on the other hand, $\|\tilde{s}(t_4)\|_M < \|\tilde{s}(0)\|_M$ because $t_4 \in (0, t_3)$. Because for all $t \in (0, t_3)$, $\|\tilde{s}(t)\|_M < \|\tilde{s}(0)\|_M$, it follows that $\phi(t_4) = \phi(0)$, which means that $\gamma(t_4) = \gamma(0)$. We can conclude that $\beta\lambda_{\min}(H_A) - \gamma(t_4) = \beta\lambda_{\min}(H_A) - \gamma(0) > 0$. From (6.63), we have that $\dot{V}(t_4) \leq 0$. On the one hand, if $\dot{V}(t_4) < 0$, there is a contradiction because we have already shown that $\dot{V}(t_4) = 0$, which implies that there does not exist $t_1 > 0$ such that $\gamma(t_1) > \gamma(0)$. Note further that $\gamma(t) \geq \gamma(0)$ for all $t \geq 0$, which means that $\gamma(t) = \gamma(0)$ for all $t \geq 0$, i.e., $\beta\lambda_{\min}(H_A) - \gamma(t) = \beta\lambda_{\min}(H_A) - \gamma(0) > 0$. This conclude our proof. On the other hand, if $\dot{V}(t_4) = 0$, noting that $\dot{V}(t_4) \leq -[\beta\lambda_{\min}(H_A) - \gamma(t_4)]\|\tilde{s}\| \leq 0$ because $\beta\lambda_{\min}(H_A) - \gamma(t_4) > 0$, we can get that $\|\tilde{s}(t_4)\| = 0$. Thus, it follows from Lemma 6.6 that $\|\tilde{s}(t)\| \equiv 0$ for any $t > \bar{t} \stackrel{\triangle}{=} t_4$. This also conclude our proof. ■

Theorem 6.15. *Suppose that $\mathscr{G}_A$ is undirected connected and the leader is a neighbor of at least one follower (i.e., at least one $a_{i0} > 0$). Using (6.51) for (6.1), $q_i(t) - q_0(t) \to \mathbf{0}_p$ and $\dot{q}_i(t) - \dot{q}_0(t) \to \mathbf{0}_p$, $i = 1, \ldots, n$, exponentially as $t \to \infty$ if β is chosen such that $\beta > \gamma(0)/\lambda_{\min}(H_A)$.*

Proof: Because $\mathscr{G}_A$ is undirected connected and at least one $a_{i0} > 0$, it follows from Lemma 1.6 that H_A is symmetric positive definite. Because $\beta > \gamma(0)/\lambda_{\min}(H_A)$, it follows from Lemma 6.7 that either $\beta\lambda_{\min}(H_A) - \gamma(t) = \beta\lambda_{\min}(H_A) - \gamma(0) > 0$ or there exists $\bar{t} \geq 0$ such that $\|\tilde{s}(t)\| \equiv 0$ for all $t \geq \bar{t}$. In the first case, consider the Lyapunov function candidate given by (6.61). Noting that $V = \frac{1}{2}\|\tilde{s}\|_M^2$, it follows from (6.63) that

$$\dot{V} \leq -\big[\beta\lambda_{\min}(H_A) - \gamma(t)\big]\|\tilde{s}\| \leq -\big[\beta\lambda_{\min}(H_A) - \gamma(0)\big]\frac{\|\tilde{s}\|_M}{\sqrt{k_{\bar{m}}}} = -\eta\sqrt{V},$$

where $\eta \stackrel{\triangle}{=} \sqrt{2/k_{\bar{m}}}[\beta\lambda_{\min}(H_A) - \gamma(0)]$, and we have used the fact that $\|s\| \geq \|s\|_M/\sqrt{k_{\bar{m}}}$ from Lemma 6.5 to obtain the second inequality. After some manipulation, we get

$$2\sqrt{V(t)} \le 2\sqrt{V(0)} - \eta t.$$

Therefore, we have $V(t) \equiv 0$ and equivalently $\|\tilde{s}(t)\| \equiv 0$ when $t \ge \frac{2\sqrt{V(0)}}{\eta}$. In the second case, there exists $\bar{t} \ge 0$ such that $\tilde{s}(t) \equiv 0$ when $t \ge \bar{t}$. Combining both cases, we can get that $\|\tilde{s}(t)\| \equiv 0$ when $t \ge \bar{t}_1 \triangleq \max\{\frac{2\sqrt{V(0)}}{\eta}, \bar{t}\}$, which implies $\dot{\tilde{q}}(t) + \lambda\tilde{q}(t) \equiv \mathbf{0}_{np}$, $i = 1, \dots, n$, when $t \ge \bar{t}_1$. Noting that the solution of $\dot{\tilde{q}}(t) + \lambda\tilde{q}(t) \equiv \mathbf{0}_{np}$ is $\tilde{q}(t) = e^{-\lambda(t-\bar{t}_1)}\tilde{q}(\bar{t}_1)$ and $\dot{\tilde{q}}(t) = -\lambda e^{-\lambda(t-\bar{t}_1)}\tilde{q}(\bar{t}_1)$, we can conclude that $q_i(t) - q_0(t) \to \mathbf{0}_p$ and $\dot{q}_i(t) - \dot{q}_0(t) \to \mathbf{0}_p$ exponentially as $t \to \infty$. ■

Remark 6.16 From Lemma 6.7, β can be chosen according to $\gamma(0)$, which means that the initial values $\|\tilde{q}(0)\|$ and $\|\tilde{s}(0)\|$ should be known by each follower to compute β even if the leader is a neighbor of only a subset of the followers. However, because only the initial value is needed, it is reasonable. Also note that the lower bound of β might be conservative. In reality, a smaller value might be chosen. Moreover, β can be tuned according to the performance of the whole system in practice, so the accurate knowledge of $\|\tilde{q}(0)\|$ and $\|\tilde{s}(0)\|$ might not be needed.

Remark 6.17 Note that the algorithm (6.51) is model-independent. The bound of $\|\varDelta\|$ in (6.53) is dependent on the bound of $\|g(q)\|$. In practice, one might know the nominal dynamics of $g_i(q_i)$, denoted as $g_i^0(q_i)$. Assume that $\|g_i(q_i)-g_i^0(q_i)\| \le k_{\tilde{g}}$, where $k_{\tilde{g}}$ is a known positive constant generally smaller than k_g. If we choose the control algorithm as $\check{\tau}_i = \tau_i + g_i^0(q_i)$, then k_g in (6.53) can be replaced with a smaller parameter $k_{\tilde{g}}$. In addition, by doing so, it is no longer required that $\|g_i(q_i)\|$ is bounded. That is, Assumption (A1) can be relaxed.

Remark 6.18 From the proof in Theorem 6.15, the error vector $\tilde{s}$ will first decrease to zero in finite time. Then, $q_i - q_0$ and $\dot{q}_i - \dot{q}_0$ converge zero exponentially fast with an exponential convergence rate λ. In addition, similar to that Remark 6.10, the condition $\mathscr{G}_A$ is undirected connected and at least one $a_{i0} > 0$ in Theorem 6.15 can be relaxed to be the condition that H_A is symmetric positive definite, which in turn implies a weaker connectivity condition.

Remark 6.19 Note that from the algorithm (6.51), only a subset of the followers needs to have access to q_0 and $\dot{q}_0$, and $\ddot{q}_0$ is not needed. It should be noted that (6.51) requires the availability of information (vectors of generalized coordinates and their derivatives) from both the one-hop and two-hop neighbors due to the challenge involved in distributed coordinated tracking of a leader with a varying vector of generalized coordinated derivatives with local interaction. However, only the sign of the information of the two-hop neighbors is required.

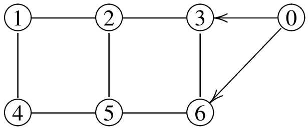

Fig. 6.11 The interaction graph associated with the leader and the six followers. An *edge* between i and j means that followers i and j are neighbors of each other while an *arrow* from 0 to i means that the leader is a neighbor of follower i

6.3.4 Simulation

In this subsection, we simulate a scenario where six two-link revolute joint arms (followers) track a leader with local interaction using, respectively, (6.16), (6.39) and (6.51). The models and the parameters of the followers are given as in Sect. 6.2.4. From the model parameter values in Sect. 6.2.4, we obtain that $k_{\underline{m}} = 0.0256$, $k_{\bar{m}} = 1.2757$, and $k_C = 0.09$. We also assume that the nominal dynamics $g_i^0(q_i)$ is set off from the real dynamics $g_i(q_i)$ by 10%.

We assume that $\mathscr{G}_A$, $\mathscr{G}_B$, and $\mathscr{G}_C$ associated with the followers (also $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$) are identical for simplicity. Figure 6.11 shows the interaction graph associated with the leader and the six followers. In our simulations, we choose $a_{ij} = 1$, $i = 1, \ldots, 6, j = 0, \ldots, 6$, if agent j is a neighbor of agent i, and $a_{ij} = 0$ otherwise. We let $q_i(0) = [\frac{\pi}{7}i, \frac{\pi}{8}i]^T$rad and $\dot{q}_i(0) = [0.05i - 0.2, -0.05i + 0.2]^T$rad/s, where $i = 1, \ldots, 6$. For the algorithms (6.16) and (6.39), the vector of joint angles of the leader are chosen as $q_0(t) = [0.04t, 0.05t]^T$rad, and the vector of joint angle derivatives of the leader is hence $\dot{q}_0 = [0.04, 0.05]^T$rad/s. The control parameters in (6.39) are chosen as $K_i = I_2$, $\alpha = 0.5$, $\eta = 0.5$, and $\Lambda_i = 0.2I_2$. For the algorithm (6.51), the vector of joint angles of the leader is chosen as $q_0(t) = [\cos(\frac{2\pi}{60}t), \sin(\frac{2\pi}{60}t)]^T$rad, the vector of joint angle derivatives of the leader is hence $\dot{q}_0(t) = \frac{2\pi}{60}[-\sin(\frac{2\pi}{60}t), \cos(\frac{2\pi}{60}t)]^T$rad/s, and the control parameters are chosen as $\alpha = 2$, $\lambda = 0.5$, and $\beta = 7.5$.

Figure 6.12 shows the differences between the joint angles of arms 1, 3 and 5 and the leader using (6.16). Figure 6.13 shows the differences between the joint angle derivatives of arms 1, 3 and 5 and the leader using (6.16). Note that all followers' joint angles approach those of the leader and all followers' joint angle derivatives also approach those of the leader.

Figure 6.14 shows the differences between the joint angles of arms 1, 3 and 5 and the leader using (6.39). Figure 6.15 shows the differences between the joint angle derivatives of arms 1, 3 and 5 and the leader using (6.39). Again, note that all followers' joint angles approach those of the leader and all followers' joint angle derivatives also approach those of the leader.

Figure 6.16 shows the differences between the joint angles of arms 1, 3, and 5 and the leader. Figure 6.17 shows the differences between the joint angle derivatives of arms 1, 3, and 5 and the leader using (6.51) by introducing the nominal dynamics $g_i^0(q_i)$ as a compensation term in (6.51). Note that all followers' joint angles ap-

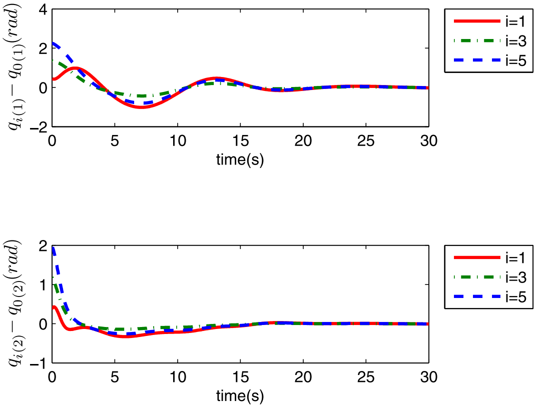

Fig. 6.12 Differences between the joint angles of arms 1, 3, and 5 and the leader using (6.16)

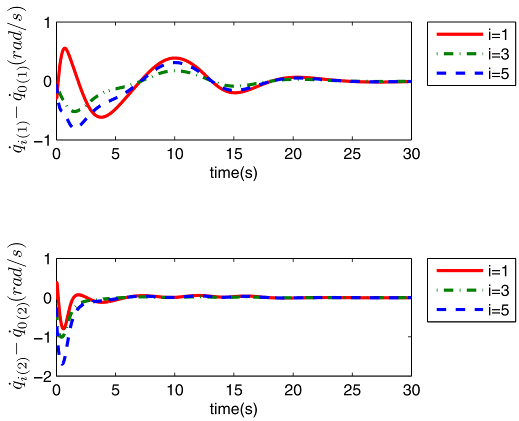

Fig. 6.13 Differences between the joint angle derivatives of arms 1, 3, and 5 and the leader using (6.16)

proach those of the leader and all followers' joint angle derivatives also approach those of the leader.

6.4 Notes

The results in Sect. 6.2 are based mainly on [245]. The results in Sect. 6.3 are mainly based on [189]. For further results on coordination of networked Lagrangian systems, see [52, 59, 62, 64, 119, 254, 275, 284]. In particular, [254] studies position synchronization of robotic manipulators when only position measurements are available under a fully connected interaction graph. In [59], output synchronization is studied for general passive systems under a passivity-based framework, which unifies several existing results on consensus or synchronization in the literature.

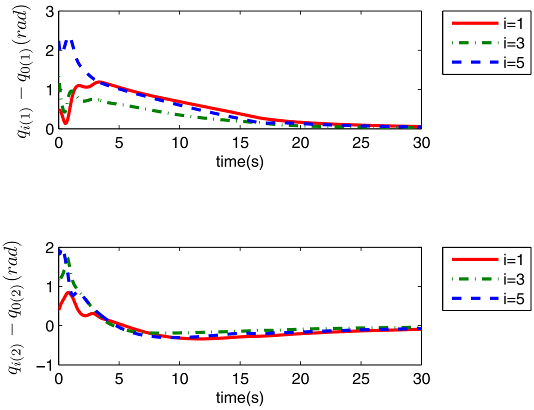

Fig. 6.14 Differences between the joint angles of arms 1, 3, and 5 and the leader using (6.39)

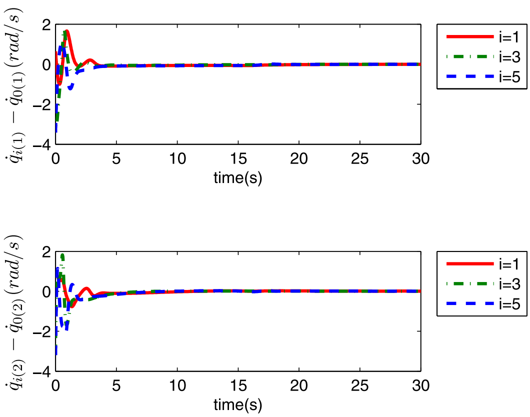

Fig. 6.15 Differences between the joint angle derivatives of arms 1, 3, and 5 and the leader using (6.39)

To use the passivity property, the control law on synchronization of networked Lagrangian systems derived in [59] requires the knowledge of the inertial matrix and the Coriolis and centrifugal torques. In [62], a controller based on potential functions is proposed for networked Lagrangian systems to achieve leaderless flocking (i.e., velocity synchronization and collision avoidance). Communication delays and switching interaction graphs are also considered. In [284], position synchronization of multi-axis motions is addressed via a cross-coupling technique. In [275], output synchronization of networked Lagrangian systems is studied under both fixed and switching interaction graphs in the presence of communication delays. The contraction analysis is used in [64] to study coordinated tracking for multiple robotic manipulators. Utilizing potential functions, [52] designs a control scheme that can force multiple robots modeled by Euler–Lagrange equations to move as a group inside a desired region while maintaining a minimum distance among themselves.

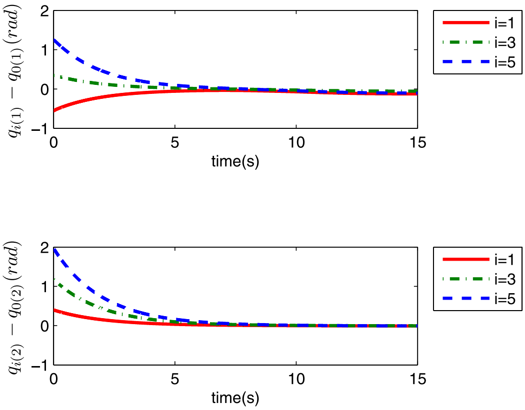

Fig. 6.16 Differences between the joint angles of arms 1, 3, and 5 and the leader using (6.51) with a compensation term $g_i^0(q_i)$

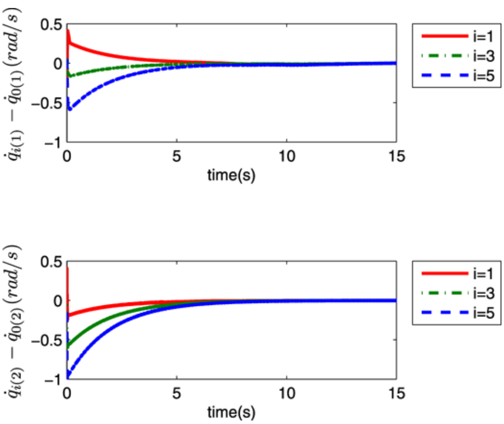

Fig. 6.17 Differences between the joint angle derivatives of arms 1, 3, and 5 and the leader using (6.51) with a compensation term $g_i^0(q_i)$

The robots can achieve velocity synchronization finally. Despite the fact that tracking of a leader or a reference is considered in [52, 64, 275, 284], it is assumed that the leader is a neighbor of all followers or all followers have access to the reference. Unfortunately, this assumption is rather restrictive and not realistic. In [119], the problem of position synchronization of networked Lagrangian systems is studied with communication constraints caused by delays and limited data rates, where the leader modeled by Euler–Lagrange equations is a neighbor of only a subset of the followers and the close-loop system is shown to be input-to-state stable. In the absence of network effects, while the result in [119] can guarantee distributed coordinated regulation where the leader has a constant vector of generalized coordinates, the result is not applicable to ensure distributed coordinated tracking where the leader has a varying vector of generalized coordinates and the leader is a neighbor of only a subset of the followers.