Chapter 10
Time Delay

This chapter considers time delays in distributed multi-agent coordination. The time delays are inevitable in networked systems. Time-domain and frequency-domain approaches are used to study leaderless and leader-following coordination algorithms with communication and input delays under a directed interaction graph. We consider both the single-integrator and double-integrator dynamics and present stability or boundedness conditions. Several interesting phenomena are analyzed and explained. Simulation results are presented to support the theoretical results.

10.1 Problem Statement

Time delays are inevitable in networked systems due to the finite speed of information transmission and processing. The time delays are usually classified as input delays and communication delays. The input delays can be caused by information processing while the communication delays can be caused by information propagation from one agent to another. In multi-agent coordination, it is meaningful to study leaderless and leader-following coordination problems where there exist time delays. In the leaderless case, the objective is that a team of agents achieves desired relative positions with local interaction. Similar to Chap. 6, we use the term *coordinated regulation* to refer to the case where a group of followers intercepts a stationary leader with a constant position with local interaction. Similar to Chap. 4, we use the term *coordinated tracking* to refer to the case where a group of followers intercepts a dynamic leader with a varying position. Note that coordinated regulation can be viewed as a special case of coordinated tracking. In both coordinated regulation and coordinated tracking, the leader can be physical or virtual.

This chapter studies both leaderless and leader-following coordination algorithms with communication and input delays for, respectively, single-integrator dynamics and double-integrator dynamics under a directed interaction graph. We analyze stability or boundedness conditions by using time-domain and frequency-domain approaches. The contributions of the current chapter are fourfold. First, we

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assume that the interaction graph is directed and has a directed spanning tree, which is more general than the assumption that the interaction graph is undirected and connected or the interaction graph is directed and is strongly connected and balanced. Second, both communication and input delays are considered in the cases of leaderless coordination, coordinated regulation when the leader's position is constant, and coordinated tracking with full access to the leader's velocity for single-integrator dynamics while in the cases of leaderless coordination, coordinated tracking when the leader's velocity is constant, and coordinated tracking with full access to the leader's acceleration for double-integrator dynamics, which guarantees the completeness of the algorithms. Third, we show that for single-integrator dynamics the communication delay will not influence the stability of the system in the case of coordinated tracking with partial access to the leader's velocity. Fourth, as a byproduct, we find that when there exists the communication delay, the final group velocity is always dampened to zero using the leaderless coordination algorithm for double-integrator dynamics rather than a possibly nonzero constant as in the standard leaderless coordination algorithm for double-integrator dynamics in the absence of delays.

10.2 Coordination for Single-integrator Dynamics with Communication and Input Delays Under Directed Fixed Interaction

In this section, we consider the case where the agents are modeled by single-integrator dynamics given by (3.1). We assume that the agents are in a one-dimensional space for simplicity. However, all results hereafter all still valid for any high-dimensional space by use of the properties of the Kronecker product.

10.2.1 Leaderless Coordination

Define $\Delta_{ij} \overset{\triangle}{=} \delta_i - \delta_j$, where δ_i is constant. Here Δ_{ij} denotes the desired relative position deviation between agents i and j. Consider the following leaderless coordination algorithm with both communication and input delays for (3.1) as

$$u_i(t) = -\frac{1}{\sum_{j=1}^n a_{ij}} \sum_{j=1}^n a_{ij}\big[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2) - \Delta_{ij}\big], \quad i = 1, \ldots, n, \tag{10.1}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, and a_{ij}, $i, j = 1, \ldots, n$, is the (i,j)th entry of the adjacency matrix $\mathscr{A}$ associated with the directed graph $\mathscr{G} \overset{\triangle}{=} (\mathscr{V}, \mathscr{E})$ characterizing the interaction among the n agents. Let $\mathscr{L}$ be the nonsymmetric Laplacian matrix associated with $\mathscr{A}$ and hence $\mathscr{G}$. Here we assume that every agent has a neighbor, which implies that $\sum_{j=1}^{n} a_{ij} > 0$, $i = 1, \ldots, n$. The objective of (10.1) is to achieve coordination, that is, $r_i(t) - r_j(t) \to \Delta_{ij}$ as $t \to \infty$ when there exist both communication and input delays.

Using (10.1), (3.1) can be written in a vector form as

$$\dot{\breve{r}}(t) = -\breve{r}(t - \tau_1) + A\breve{r}(t - \tau_1 - \tau_2), \tag{10.2}$$

where $\breve{r} \triangleq [\breve{r}_1, \ldots, \breve{r}_n]^T$ with $\breve{r}_i \triangleq r_i - \delta_i$ and $A \triangleq [\hat{a}_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $\hat{a}_{ij} \triangleq a_{ij}/\sum_{j=1}^{n} a_{ij}$, $i, j = 1, \ldots, n$. Let $L \triangleq [\hat{\ell}_{ij}] \in \mathbb{R}^{n\times n}$ be defined as $L \triangleq I_n - A$. Compared with $\mathscr{A}$ (respectively, $\mathscr{L}$), A (respectively, L) can be viewed as another adjacency matrix (respectively, nonsymmetric Laplacian matrix) associated with $\mathscr{G}$ by choosing a different weight for each edge $(j, i) \in \mathscr{E}$. That is, the original weight a_{ij} for the edge (j, i) is replaced with a new weight $\frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}}$. When $\mathscr{G}$ has a directed spanning tree and each agent has a neighbor, it follows from Lemma 1.1 that L has a simple zero eigenvalue and all other eigenvalues have positive real parts. We have the following singular vector decomposition given as

$$W^{-1}LW = \begin{bmatrix} \widetilde{L} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 0 \end{bmatrix}. \tag{10.3}$$

Here without loss of generality, we choose the last column of W to be $\mathbf{1}_n$ by noting that $\mathbf{1}_n$ is a right eigenvector of L associated with the zero eigenvalue. Therefore, the last row of W^{-1} is $\mathbf{p}^T$, where $\mathbf{p} \in \mathbb{R}^n$ is defined in Lemma 1.1 with L playing the role of $\mathscr{L}$. It follows that when $\mathscr{G}$ has a directed spanning tree and each agent has a neighbor, all eigenvalues of $\widetilde{L}$ have positive real parts.

Define $\tilde{r} \triangleq W^{-1}\breve{r}$. Denote $\tilde{r}_{1:n-1,:}$ as the first $n-1$ rows of $\tilde{r}$ and $\tilde{r}_{n,:}$ as the last row of $\tilde{r}$. Note that $A = I_n - L$. It follows from (10.3) that $W^{-1}AW = \left[\begin{smallmatrix} I_{n-1}-\widetilde{L} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 1 \end{smallmatrix}\right]$. Define

$$\widetilde{A} \triangleq I_{n-1} - \widetilde{L}. \tag{10.4}$$

By multiplying W^{-1} on both sides of (10.2), it follows that (10.2) can be rewritten as

$$\begin{bmatrix} \dot{\tilde{r}}_{1:n-1,:}(t) \\ \dot{\tilde{r}}_{n,:}(t) \end{bmatrix} = -\begin{bmatrix} \tilde{r}_{1:n-1,:}(t-\tau_1) \\ \tilde{r}_{n,:}(t-\tau_1) \end{bmatrix} + \begin{bmatrix} \tilde{A} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{r}_{1:n-1,:}(t-\tau_1-\tau_2) \\ \tilde{r}_{n,:}(t-\tau_1-\tau_2) \end{bmatrix}. \tag{10.5}$$

Equation (10.5) can be decoupled into the following two equations:

$$\dot{\tilde{r}}_{1:n-1,:}(t) = -\tilde{r}_{1:n-1,:}(t-\tau_1) + \tilde{A}\tilde{r}_{1:n-1,:}(t-\tau_1-\tau_2), \tag{10.6a}$$

$$\dot{\tilde{r}}_{n,:}(t) = -\tilde{r}_{n,:}(t-\tau_1) + \tilde{r}_{n,:}(t-\tau_1-\tau_2). \tag{10.6b}$$

Theorem 10.1. *Suppose that the directed fixed graph $\mathscr{G}$ has a directed spanning tree and every agent has a neighbor. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, the following three conditions[1] are satisfied:*

(i) $2\tau_1 + \tau_2 < 1$.
(ii) $1 - \frac{1-e^{-s\tau_1}}{s} + \lambda_i(\widetilde{A})\frac{1-e^{-s(\tau_1+\tau_2)}}{s} \neq 0$, *for all* $s \in \mathbb{C}^+$.
(iii) *The matrix*

$$Q_{fc} \triangleq -\widetilde{L}^T P_{fc} - P_{fc}\widetilde{L} + \tau_1 S_{fc} + (\tau_1+\tau_2)H_{fc} + \tau_1 \widetilde{L}^T P_{fc} S_{fc}^{-1} P_{fc} \widetilde{L} + (\tau_1+\tau_2)\widetilde{L}^T P_{fc} \widetilde{A} H_{fc}^{-1} \widetilde{A}^T P_{fc} \widetilde{L} \tag{10.7}$$

is symmetric negative definite, where $P_{fc} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a symmetric positive-definite matrix chosen properly such that $-\widetilde{L}^T P_{fc} - P_{fc}\widetilde{L}$ is symmetric negative definite, and $S_{fc} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $H_{fc} \in \mathbb{R}^{(n-1)\times(n-1)}$ are arbitrary symmetric positive-definite matrices.

In addition, if $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, using (10.1) for (3.1)*, for all $r_i(0)$ and all $i, j = 1, \ldots, n$, $r_i(t) - r_j(t) \to \Delta_{ij}$ as $t \to \infty$. In particular, $r_i(t) \to \frac{\mathbf{p}^T \check{r}(0)}{1+\tau_2} + \delta_i$, $i = 1, \ldots, n$, as $t \to \infty$, where $\mathbf{p} \in \mathbb{R}^n$ is defined after* (10.3).

Proof: For the first statement, it is straightforward to see that there exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$ Conditions (i) and (ii) are satisfied. For Condition (iii), because $\mathscr{G}$ has a directed spanning tree and each agent has a neighbor, all eigenvalues of $\widetilde{L}$ have positive real parts. Therefore, there always exists a symmetric positive-definite matrix $P_{fc} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $-\widetilde{L}^T P_{fc} - P_{fc}\widetilde{L}$ is symmetric negative definite. It follows from (10.7) that when $\tau_1 = \tau_2 = 0$, $Q_{fc} = -\widetilde{L}^T P_{fc} - P_{fc}\widetilde{L}$. Due to the continuity of Q_{fc} with respect to τ_1 and τ_2, there must exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, Q_{fc} is symmetric negative definite.

For the second statement, we show that if $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, (10.6a) is asymptotically stable at the origin while (10.6b) is stable. It follows from Lemma 1.44 that the stability of the following system

$$\frac{d}{dt}\left[\tilde{r}_{1:n-1,:}(t) - \int_{-\tau_1}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta + \widetilde{A}\int_{-\tau_1-\tau_2}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta\right] = -\widetilde{L}\tilde{r}_{1:n-1,:}(t) \tag{10.8}$$

implies the stability of (10.6a) under Condition (ii) of the theorem. Consider the Lyapunov function candidate

[1] Note that here the three conditions are used to obtain the upper bounds $\overline{\tau_1}$ and $\overline{\tau_2}$ for allowable delays.

$$V\big[(\tilde{r}_{1:n-1,:})_t\big] = \chi^T P_{fc}\chi + \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \tilde{r}_{1:n-1,:}^T(\xi) S_{fc}\tilde{r}_{1:n-1,:}(\xi)\, d\xi\, d\theta$$
$$+ \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t} \tilde{r}_{1:n-1,:}^T(\xi) H_{fc}\tilde{r}_{1:n-1,:}(\xi)\, d\xi\, d\theta,$$

where $\chi \triangleq \tilde{r}_{1:n-1,:}(t) - \int_{-\tau_1}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta + \widetilde{A}\int_{-\tau_1-\tau_2}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta$. Taking the derivative of V along (10.8) gives

$$\dot{V}\big[(\tilde{r}_{1:n-1,:})_t\big] \le \tilde{r}_{1:n-1,:}^T(t) Q_{fc}\tilde{r}_{1:n-1,:}(t),$$

where we have used Lemma 1.23 to derive the inequality. Note that $\alpha_1\|\mathscr{D}[(\tilde{r}_{1:n-1,:})_t]\| \le V[(\tilde{r}_{1:n-1,:})_t] \le \alpha_2\|(\tilde{r}_{1:n-1,:})_t\|_c$, where

$$\mathscr{D}\big[(\tilde{r}_{1:n-1,:})_t\big] \triangleq \tilde{r}_{1:n-1,:}(t) - \int_{-\tau_1}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta + \widetilde{A}\int_{-\tau_1-\tau_2}^{0} \tilde{r}_{1:n-1,:}(t+\theta)\, d\theta,$$

$\|(\tilde{r}_{1:n-1,:})_t\|_c \triangleq \sup_{\theta\in[-\tau_1-\tau_2,0]} \|\tilde{r}_{1:n-1,:}(t+\theta)\|$, $\alpha_1 = \lambda_{\min}(P_{fc})$, and $\alpha_2 = \lambda_{\max}(P_{fc}) + \tau_1\lambda_{\max}(S_{fc}) + (\tau_1+\tau_2)\lambda_{\max}(H_{fc})$. Also note that Q_{fc} is symmetric negative definite under Condition (iii) of the theorem. It follows from Lemma 1.41 that (10.8) is asymptotically stable at the origin. Therefore, if Conditions (ii) and (iii) of the theorem are satisfied, (10.6a) is asymptotically stable at the origin.

For (10.6b), we apply the Nyquist stability criterion to find its stability condition. After Laplace transformation, (10.6b) can be written as

$$s\tilde{r}_{n,:}(s) - \tilde{r}_{n,:}(0) = -e^{-\tau_1 s}\tilde{r}_{n,:}(s) + e^{-(\tau_1+\tau_2)s}\tilde{r}_{n,:}(s),$$

which implies that $\tilde{r}_{n,:}(s) = \frac{\tilde{r}_{n,:}(0)}{s+e^{-\tau_1 s}-e^{-(\tau_1+\tau_2)s}}$. Therefore, the stability of (10.3b) is determined by the distribution of the roots of

$$s = -e^{-\tau_1 s} + e^{-(\tau_1+\tau_2)s}. \tag{10.9}$$

Note that $s = 0$ is a root of (10.9). To study the other roots, define $f(s) \triangleq [e^{-\tau_1 s} - e^{-(\tau_1+\tau_2)s}]/s$. According to the Nyquist stability criterion, if the trajectory of $f(\iota\omega)$, $\forall\omega \in (-\infty,\infty)$, does not enclose the point $(-1,0)$, then the other roots of (10.9) are stable. One sufficient condition is that $\mathrm{Re}[f(\iota\omega)] > -1$, $\forall\omega \in (-\infty,\infty)$. Note that $\mathrm{Re}[f(\iota\omega)] = \frac{\sin[(\tau_1+\tau_2)\omega]}{\omega} - \frac{\sin(\tau_1\omega)}{\omega} \ge -(\tau_1+\tau_2) - \tau_1 = -(2\tau_1+\tau_2)$. Therefore, it follows that (10.6b) is marginally stable at the origin under Condition (i) of the theorem.

Note that $\lim_{t\to\infty}\tilde{r}_{1:n-1,:}(t) = \mathbf{0}_{n-1}$. Also note that

$$\lim_{t\to\infty}\tilde{r}_{n,:}(t) = \lim_{s\to 0} s\tilde{r}_{n,:}(s) = \frac{s\tilde{r}_{n,:}(0)}{s+e^{-\tau_1 s}-e^{-(\tau_1+\tau_2)s}} = \frac{\tilde{r}_{n,:}(0)}{1+\tau_2}.$$

Because $\breve{r} = W\tilde{r}$ and the last column of W is $\mathbf{1}_n$, it follows that $\lim_{t\to\infty} \breve{r}(t) = \lim_{t\to\infty} W\tilde{r}(t) = \lim_{t\to\infty} \mathbf{1}_n\tilde{r}_{n,:}(t) = \frac{\mathbf{1}_n\tilde{r}_{n,:}(0)}{1+\tau_2}$, which implies that $\breve{r}_i(t) - \breve{r}_j(t) \to 0$ as $t \to \infty$, that is, $r_i(t) - r_j(t) \to \Delta_{ij}$ as $t \to \infty$. Because the last row of W^{-1} is $\mathbf{p}$, it follows that $\tilde{r}_{n,:}(0) = \mathbf{p}^T\breve{r}(0)$. Therefore, it follows that $r_i(t) \to \frac{\mathbf{p}^T\breve{r}(0)}{1+\tau_2} + \delta_i$ as $t \to \infty$. ■

Remark 10.2 Note that the additional dynamics caused by the model transformation from (10.6a) to (10.8) can be characterized by the solutions of the following complex equation [210]

$$\det\left[I_{n-1} - I_{n-1}\frac{1-e^{-s\tau_1}}{s} + \widetilde{A}\frac{1-e^{-s(\tau_1+\tau_2)}}{s}\right] = 0, \quad s \in \mathbb{C}.$$

Thus, if $\tau_1 + (\tau_1 + \tau_2)\|\widetilde{A}\| < 1$, there are no additional eigenvalues induced by the model transformation from (10.6a) to (10.8), which implies that the condition $\tau_1 + (\tau_1 + \tau_2)\|\widetilde{A}\| < 1$ can be used to replace Condition (ii) in Theorem 10.1.

Remark 10.3 If we let $S_{fc} = H_{fc} = I_{n-1}$ in (10.7), Condition (iii) in Theorem 10.1 can be written as

$$\overline{\tau_1} + \overline{\tau_2} < \frac{\lambda_{\min}(\widetilde{L}^T P_{fc} + P_{fc}\widetilde{L})}{2 + \|\widetilde{L}^T P_{fc}\|^2 + \|\widetilde{L}^T P_{fc}\widetilde{A}\|^2}.$$

10.2.2 Coordinated Regulation when the Leader's Position is Constant

In this subsection, we assume that in addition to n followers, labeled as agents or followers 1 to n, there exists a leader, labeled as agent 0, with position r_0. We assume that r_0 is constant. Let $\mathscr{G} \triangleq (\mathscr{V}, \mathscr{E})$ be the directed graph characterizing the interaction among the n followers. Let $\overline{\mathscr{G}} \triangleq (\overline{\mathscr{V}}, \overline{\mathscr{E}})$ be the directed graph characterizing the interaction among the leader and the followers corresponding to $\mathscr{G}$.

Consider the following coordinated regulation algorithm with both communication and input delays for the n followers with single-integrator dynamics given by (3.1) as

$$u_i(t) = -\frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij}\left[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2)\right], \quad i = 1, \dots, n, \tag{10.10}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i, j = 1, \dots, n$, is the (i, j)th entry of the adjacency matrix $\mathscr{A}$ associated with $\mathscr{G}$, and $a_{i0} > 0$ if the leader is a neighbor of agent i and $a_{i0} = 0$ otherwise. Note that in $\overline{\mathscr{G}}$ if the leader has directed paths to all followers 1 to n, it follows that $\sum_{j=0}^n a_{ij} > 0$,

$i = 1, \ldots, n$. The objective of (10.10) is to guarantee coordinated regulation, i.e., $r_i(t) \to r_0$ as $t \to \infty$.

Define $\bar{r}_i \triangleq r_i - r_0$ and $\bar{r} \triangleq [\bar{r}_1, \ldots, \bar{r}_n]^T$. Define $\overline{A} \triangleq [\bar{a}_{ij}] \in \mathbb{R}^{n\times n}$ as $\bar{a}_{ij} \triangleq a_{ij}/\sum_{j=0}^{n} a_{ij}$. Using (10.10), (3.1) can be written in a vector form as

$$\dot{\bar{r}}(t) = -\bar{r}(t-\tau_1) + \overline{A}\bar{r}(t-\tau_1-\tau_2), \tag{10.11}$$

where we have used the fact that r_0 is constant. Before moving on, we need the following lemma regarding $(I_n - \overline{A})$.

Lemma 10.1. *All eigenvalues of $I_n - \overline{A}$ have positive real parts if in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n.*

Proof: The lemma follows from Lemma 8.1 by noting that all eigenvalues of $\overline{A}$ are within the unit circle if the leader has directed paths to all followers. ■

Theorem 10.4. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, the following two conditions are satisfied*:

(i) $1 - \frac{1-e^{-s\tau_1}}{s} + \lambda_i(\overline{A})\frac{1-e^{-s(\tau_1+\tau_2)}}{s} \neq 0$, $\forall s \in \mathbb{C}^+$.
(ii) *The matrix*

$$\begin{aligned} Q_{fr} \triangleq\, & (\overline{A} - I_n)^T P_{fr} + P_{fr}(\overline{A} - I_n) + \tau_1 S_{fr} + (\tau_1+\tau_2)H_{fr} \\ & + \tau_1\big[(\overline{A} - I_n)^T P_{fr} S_{fr}^{-1} P_{fr}(\overline{A} - I_n)\big] \\ & + (\tau_1+\tau_2)\big[(\overline{A} - I_n)^T P_{fr}\overline{A} H_{fr}^{-1}\overline{A}^T P_{fr}(\overline{A} - I_n)\big] \end{aligned}$$

is symmetric negative definite, where $P_{fr} \in \mathbb{R}^{n\times n}$ is a symmetric positive-definite matrix chosen properly such that $(\overline{A} - I_n)^T P_{fr} + P_{fr}(\overline{A} - I_n)$ is symmetric negative definite, and $S_{fr} \in \mathbb{R}^{n\times n}$ and $H_{fr} \in \mathbb{R}^{n\times n}$ are arbitrary symmetric positive-definite matrices.

In addition, if $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, using (10.10) *for* (3.1), *for all $r_i(0)$, $i = 1, \ldots, n$, $r_i(t) \to r_0$ as $t \to \infty$.*

Proof: For the first statement, it is straightforward to see that there exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, Condition (i) is satisfied. Because in $\overline{\mathscr{G}}$ the leader has directed paths to all followers, it follows from Lemma 10.1 that all eigenvalues of $I_n - \overline{A}$ have positive real parts. A similar analysis to that in Theorem 10.1 shows that there exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, Condition (ii) is satisfied.

For the second statement, it follows from Lemma 1.44 that the stability of the following system

$$\frac{d}{dt}\left[\bar{r}(t) - \int_{-\tau_1}^{0} \bar{r}(t+\theta)\,d\theta + \overline{A}\int_{-\tau_1-\tau_2}^{0} \bar{r}(t+\theta)\,d\theta\right] = (\overline{A} - I_n)\bar{r}(t) \tag{10.12}$$

implies the stability of (10.11) under Condition (i) of the theorem. Consider the Lyapunov function candidate

$$V(\bar{r}_t) = \chi^T P_{fr}\chi + \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \bar{r}^T(\xi) S_{fr}\bar{r}(\xi)\, d\xi\, d\theta + \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t} \bar{r}^T(\xi) H_{fr}\bar{r}(\xi)\, d\xi\, d\theta,$$

where $\chi \triangleq \bar{r}(t) - \int_{-\tau_1}^{0} \bar{r}(t+\theta)\, d\theta + \overline{A}\int_{-\tau_1-\tau_2}^{0} \bar{r}(t+\theta)\, d\theta$. Taking the derivative of V along (10.12) gives

$$\dot{V}(\bar{r}_t) \le \bar{r}^T(t) Q_{fr}\bar{r}(t),$$

where we have used Lemma 1.23 to derive the inequality. Thus, a similar analysis to that in the proof of Theorem 10.1 shows that if the two conditions of the theorem are satisfied, (10.11) is asymptotically stable at the origin. ■

Remark 10.5 Although the approaches used in the leaderless coordination case and the coordinated regulation case are similar, the control objectives are different. In the leaderless coordination case, the final positions of each agent are determined by the interaction graph and the time delays rather than being prespecified. However, in the coordinated regulation case, there exists a leader that prespecifies the final position, and the control objective is to guarantee that the final positions of all followers approach the position of the leader. Also the result in the case of coordinated regulation can be generalized to general weights while in the case of leaderless coordination special weights are required (i.e., $\sum_{j=1}^{n} \hat{a}_{ij} = 1$). In addition, note that Remarks 10.2 and 10.3 are still valid in the coordinated regulation case.

10.2.3 Coordinated Tracking with Full Access to the Leader's Velocity

In this subsection, we consider the case where the leader's position r_0 is varying. We assume that $|\dot{r}_0| < \delta_v$ and $|\ddot{r}_0| < \delta_a$, where δ_v and δ_a are positive constants. We also assume that all followers have access to $\dot{r}_0$.

Consider the coordinated tracking algorithm with both communication and input delays for the n followers with single-integrator dynamics given by (3.1) as

$$u_i(t) = \dot{r}_0(t-\tau_1-\tau_2) - \frac{1}{\sum_{j=0}^{n} a_{ij}} \sum_{j=0}^{n} a_{ij}\big[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2)\big], \quad i = 1,\dots,n, \tag{10.13}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, and a_{ij}, $i = 1,\dots,n$, $j = 0,\dots,n$, is defined as in (10.10). Using (10.13), (3.1) can be written in a vector form as

$$\dot{\bar{r}} = -\bar{r}(t-\tau_1) + \overline{A}\bar{r}(t-\tau_1-\tau_2) + R_{ft}, \tag{10.14}$$

where $\bar{r}$ and $\overline{A}$ are defined as in Sect. 10.2.2, and $R_{ft} \stackrel{\triangle}{=} \mathbf{1}_n[\dot{r}_0(t-\tau_1-\tau_2) - \dot{r}_0(t) + r_0(t-\tau_1-\tau_2) - r_0(t-\tau_1)]$. By using (1.10), it follows that $R_{ft} = -\mathbf{1}_n \int_{-\tau_1-\tau_2}^{0} \ddot{r}_0(t+\theta)\,d\theta - \mathbf{1}_n \int_{-\tau_1-\tau_2}^{-\tau_1} \dot{r}_0(t+\theta)\,d\theta$.

Theorem 10.6. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0,\overline{\tau_1}]$ and $\tau_2 \in [0,\overline{\tau_2}]$,*

$$\begin{aligned} Q_{ft} \stackrel{\triangle}{=} &(\overline{A}-I_n)^T P_{fr} + P_{fr}(\overline{A}-I_n) + \tau_1\big(P_{fr} + P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr} + 2q_f P_{fr}\big) \\ &+ (\tau_1+\tau_2)\big(P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr} + P_{fr}\overline{A}^2 P_{fr}^{-1}\big(\overline{A}^T\big)^2 P_{fr} + 2q_f P_{fr}\big) \end{aligned}$$

is symmetric negative definite, where P_{fr} is defined in Theorem 10.4 and q_f is an arbitrary real number satisfying $q_f > 1$. In addition, if $\tau_1 \in [0,\overline{\tau_1}]$ and $\tau_2 \in [0,\overline{\tau_2}]$, using (10.13) for (3.1), for all $r_i(0)$ and all $i = 1,\ldots,n$, $|r_i(t) - r_0(t)|$ is uniformly ultimately bounded. In particular, the ultimate bound for $\|\bar{r}(t)\|$ is given by $\frac{\lambda_{\max}(P_{fr})a_f}{\lambda_{\min}(P_{fr})\kappa_f\lambda_{\min}(-Q_{ft})}$, where $a_f \stackrel{\triangle}{=} 2[(\tau_1+\tau_2)\delta_a + \tau_2\delta_v][\|P_{fr}\| + \tau_1\|P_{fr}\| + (\tau_1+\tau_2)\|P_{fr}\overline{A}\|]$, and κ_f is an arbitrary real number satisfying $0 < \kappa_f < 1$.

Proof: The proof of the first statement is similar to that in Theorem 10.4 and is hence omitted here. For the second statement, using (1.10), we transform (10.14) to the following system

$$\begin{aligned} \frac{d}{dt}\bar{r}(t) =& (\overline{A}-I_n)\bar{r}(t) + \int_{-\tau_1}^{0} \dot{\bar{r}}(t+\theta)\,d\theta - \overline{A}\int_{-\tau_1-\tau_2}^{0} \dot{\bar{r}}(t+\theta)\,d\theta + R_{ft} \\ =& (\overline{A}-I_n)\bar{r}(t) + \int_{-\tau_1}^{0} \big[\overline{A}\bar{r}(t-\tau_1-\tau_2+\theta) - \bar{r}(t-\tau_1+\theta)\big]\,d\theta \\ &+ \int_{-\tau_1}^{0} R_{ft}(t+\theta)\,d\theta \\ &+ \overline{A}\int_{-\tau_1-\tau_2}^{0} \big[\bar{r}(t-\tau_1+\theta) - \overline{A}\bar{r}(t-\tau_1-\tau_2+\theta)\big]\,d\theta \\ &- \overline{A}\int_{-\tau_1-\tau_2}^{0} R_{ft}(t+\theta)\,d\theta + R_{ft} \\ =& (\overline{A}-I_n)\bar{r}(t) - \int_{-2\tau_1}^{-\tau_1} \bar{r}(t+\theta)\,d\theta + \overline{A}\int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2} \bar{r}(t+\theta)\,d\theta \\ &+ \int_{-\tau_1}^{0} R_{ft}(t+\theta)\,d\theta + \overline{A}\int_{-2\tau_1-\tau_2}^{-\tau_1} \bar{r}(t+\theta)\,d\theta \\ &- \overline{A}^2\int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2} \bar{r}(t+\theta)\,d\theta - \overline{A}\int_{-\tau_1-\tau_2}^{0} R_{ft}(t+\theta)\,d\theta + R_{ft}. \end{aligned} \tag{10.15}$$

Consider the Lyapunov function candidate $V(\bar{r}) = \bar{r}^T(t)P_{fr}\bar{r}(t)$. Taking the derivative of $V(\bar{r})$ along (10.15) gives

$$
\begin{aligned}
\dot{V}(\bar{r}) \le\, & \bar{r}^T(t)\big[(\overline{A}-I_n)^T P_{fr} + P_{fr}(\overline{A}-I_n)\big]\bar{r}(t) + \tau_1\bar{r}^T(t)P_{fr}P_{fr}^{-1}P_{fr}\bar{r}(t) \\
& + \int_{-2\tau_1}^{-\tau_1} \bar{r}^T(t+\theta)P_{fr}\bar{r}(t+\theta)\,d\theta + \tau_1\bar{r}^T(t)P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr}\bar{r}(t) \\
& + \int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2} \bar{r}^T(t+\theta)P_{fr}\bar{r}(t+\theta)\,d\theta \\
& + 2\,\|\bar{r}\|\,\|P_{fr}\|\big[\tau_1(\tau_1+\tau_2)\delta_a + \tau_1\tau_2\delta_v\big] \\
& + (\tau_1+\tau_2)\bar{r}^T P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr}\bar{r} + \int_{-2\tau_1-\tau_2}^{-\tau_1} \bar{r}^T(t+\theta)P_{fr}\bar{r}(t+\theta)\,d\theta \\
& + (\tau_1+\tau_2)\bar{r}^T P_{fr}(\overline{A})^2 P_{fr}^{-1}\big(\overline{A}^T\big)^2 P_{fr}\bar{r} \\
& + \int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2} \bar{r}^T(t+\theta)P_{fr}\bar{r}(t+\theta)\,d\theta \\
& + 2\|\bar{r}\|\|P_{fr}\overline{A}\|\big[(\tau_1+\tau_2)(\tau_1+\tau_2)\delta_a + (\tau_1+\tau_2)\tau_2\delta_v\big] \\
& + 2\|\bar{r}\|\|P_{fr}\|\big[(\tau_1+\tau_2)\delta_a + \tau_2\delta_v\big],
\end{aligned}
$$

where we have used Lemma 1.23 and the facts that $|\dot{r}_0| < \delta_v$ and $|\ddot{r}_0| < \delta_a$ to derive the inequality. Take $p(s) = q_f s$. If $V[\bar{r}(t+\theta)] < p\{V[\bar{r}(t)]\} = q_f V[\bar{r}(t)]$ for $-2\tau_1 - 2\tau_2 \le \theta \le 0$, we have

$$
\begin{aligned}
\dot{V}(\bar{r}) \le\, & \bar{r}^T(t)\big[(\overline{A}-I_n)^T P_{fr} + P_{fr}(\overline{A}-I_n)\big]\bar{r}(t) + \tau_1\bar{r}^T(t)(P_{fr} + q_f P_{fr})\bar{r}(t) \\
& + \tau_1\bar{r}^T(t)\big(P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr} + q_f P_{fr}\big)\bar{r}(t) \\
& + (\tau_1+\tau_2)\bar{r}^T(t)\big(P_{fr}\overline{A}P_{fr}^{-1}\overline{A}^T P_{fr} + q_f P_{fr}\big)\bar{r}(t) \\
& + (\tau_1+\tau_2)\bar{r}^T(t)\big(P_{fr}\overline{A}^2 P_{fr}^{-1}\big(\overline{A}^T\big)^2 P_{fr} + q_f P_{fr}\big)\bar{r}(t) \\
& + 2\big\|\bar{r}(t)\big\|\|P_{fr}\|\big[\tau_1(\tau_1+\tau_2)\delta_a + \tau_1\tau_2\delta_v\big] \\
& + 2\big\|\bar{r}(t)\big\|\|P_{fr}\overline{A}\|\big[(\tau_1+\tau_2)(\tau_1+\tau_2)\delta_a + (\tau_1+\tau_2)\tau_2\delta_v\big] \\
& + 2\big\|\bar{r}(t)\big\|\|P_{fr}\|\big[(\tau_1+\tau_2)\delta_a + \tau_2\delta_v\big] \\
\le\, & \bar{r}(t)^T(t)Q_{ft}\bar{r}(t)(t) + a_f\big\|\bar{r}(t)\big\|.
\end{aligned}
$$

If $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, we have that $\lambda_{\min}(-Q_{ft}) > 0$. Given $0 < \kappa_f < 1$, if $\|\bar{r}(t)\| \ge \frac{a_f}{\kappa_f \lambda_{\min}(-Q_{ft})}$, we can obtain that

$$
\begin{aligned}
\dot{V}(\bar{r}) &\le -(1-\kappa_f)\lambda_{\min}(-Q_{ft})\big\|\bar{r}(t)\big\|^2 - \kappa_f\lambda_{\min}(-Q_{ft})\big\|\bar{r}(t)\big\|^2 + a_f\big\|\bar{r}(t)\big\| \\
&\le -(1-\kappa_f)\lambda_{\min}(-Q_{ft})\big\|\bar{r}(t)\big\|^2.
\end{aligned}
$$

Therefore, it follows from Lemma 1.42 that $\|\bar{r}(t)\|$ is uniformly ultimately bounded, which implies that $|r_i(t) - r_0(t)|$ is uniformly ultimately bounded. Moreover, it can be computed that the ultimate bound for $\|\bar{r}(t)\|$ is given by $\frac{\lambda_{\max}(P_{fr})a_f}{\lambda_{\min}(P_{fr})\kappa_f\lambda_{\min}(-Q_{ft})}$ by following a similar analysis to that in [145, pp. 172–174]. ■

Remark 10.7 Note that if $\tau_1 = \tau_2 = 0$, then $\lim_{t\to\infty}\|\bar{r}(t)\| = 0$. Also note when τ_1 and τ_2 are larger, the ultimate bound will also be larger. □

10.2.4 Coordinated Tracking with Partial Access to the Leader's Velocity

In this subsection, we assume that the leader's varying position r_0 and velocity $\dot{r}_0$ are available to only a subset of all followers. We assume that $|r_0|$ and $|\dot{r}_0|$ are bounded. We also assume that there exists only the communication delay.

Consider the following coordinated tracking algorithm with the communication delay for the n followers with single-integrator dynamics given by (3.1) as

$$u_i(t) = \frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij}\big\{\dot{r}_j(t-\tau_2) - \big[r_i(t) - r_j(t-\tau_2)\big]\big\}, \quad i=1,\ldots,n, \tag{10.16}$$

where τ_2 is the communication delay and a_{ij}, $i=1,\ldots,n$, $j=0,\ldots,n$, is defined as in (10.10). Using (10.16), (3.1) can be written in a vector form as

$$\dot{\bar{r}}(t) = \overline{A}\dot{\bar{r}}(t-\tau_2) - \bar{r}(t) + \overline{A}\bar{r}(t-\tau_2) + R_{fft}, \tag{10.17}$$

where $\bar{r}$ and $\overline{A}$ are defined as in Sect. 10.2.2, and $R_{fft} \triangleq [\dot{r}_0(t-\tau_2) - \dot{r}_0(t)]\mathbf{1}_n - [r_0(t) - r_0(t-\tau_2)]\mathbf{1}_n$.

Theorem 10.8. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers* 1 *to* n. *Using* (10.16) *for* (3.1)*, for all* $r_i(0)$ *and all* $i = 1,\ldots,n$, $|r_i(t) - r_0(t)|$ *is uniformly ultimately bounded no matter how large the communication delay is.*

Proof: First, it follows from Lemma 8.1 that $\rho(\overline{A}) < 1$, which means that the neutral operator $\mathscr{D}\bar{r}_t = \bar{r}(t) - \overline{A}\bar{r}(t-\tau_2)$ is stable. Consider a Lyapunov function candidate $V(\bar{r}) = \bar{r}^T(t)\bar{r}(t)$. Taking the derivative of $V(\bar{r})$ along (10.17) gives

$$\begin{aligned}\dot{V}(\mathscr{D}\bar{r}_t) &= (\mathscr{D}\bar{r}_t)^T\big[-\bar{r}(t) + \overline{A}\bar{r}(t-\tau_2) + R_{fft}\big]\\ &= -(\mathscr{D}\bar{r}_t)^T(\mathscr{D}\bar{r}_t) + (\mathscr{D}\bar{r}_t)R_{fft}.\end{aligned}$$

In then follows that

$$\dot{V}(\mathscr{D}\bar{r}_t) \le -\|\mathscr{D}\bar{r}_t\|\big(\|\mathscr{D}\bar{r}_t\| - \|R_{fft}\|\big).$$

If $\|\mathscr{D}\bar{r}_t\| > \|R_{fft}\|$, we have $\dot{V}(\mathscr{D}\bar{r}_t) < 0$. Therefore, it follows from Lemma 1.43 that $\|\bar{r}(t)\|$ is uniformly ultimately bounded, which implies that $|r_i(t) - r_0(t)|$ is ultimately bounded. ■

Remark 10.9 From Theorem 10.8, it can be noted that the communication delay does not jeopardize the stability of the closed-loop tracking error system (10.17) in the case of coordinated tracking with partial access to the leader's velocity for single-integrator dynamics. However, with the increase of the communication delay, the tracking errors will increase as well.

Remark 10.10 In real applications, the derivatives of the neighbors' positions $\dot{r}_j(t-\tau_2)$ can be calculated by using numerical differentiation. For example, $\dot{r}_j(t-\tau_2)$ can be approximated by $[r_j(kT-\tau_2)-r_j(kT-T-\tau_2)]/T$, where k is the discrete-time index and T is the sampling period.

10.3 Coordination for Double-integrator Dynamics with Communication and Input Delays Under Directed Fixed Interaction

In this section, we consider the case where the agents are modeled by double-integrator dynamics given by (3.5). We again assume that the agents are in a one-dimensional space for simplicity. However, all results hereafter are still valid for any high-dimensional space by use of the properties of the Kronecker product.

10.3.1 Leaderless Coordination

Consider the following leaderless coordination algorithm with both communication and input delays for (3.5) as

$$\begin{aligned} u_i(t) = &-\frac{1}{\sum_{j=1}^n a_{ij}}\sum_{j=1}^n a_{ij}\big[r_i(t-\tau_1)-r_j(t-\tau_1-\tau_2)-\varDelta_{ij}\big] \\ &-\frac{\gamma_c}{\sum_{j=1}^n a_{ij}}\sum_{j=1}^n a_{ij}\big[v_i(t-\tau_1)-v_j(t-\tau_1-\tau_2)\big], \quad i=1,\ldots,n, \end{aligned} \tag{10.18}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i,j=1,\ldots,n$, is defined as in (10.1), $\varDelta_{ij}$ is defined as in Sect. 10.2.1, and γ_c is a positive gain. Here we also assume that every agent has a neighbor, which implies that $\sum_{j=1}^n a_{ij}>0$, $i=1,\ldots,n$. The objective of (10.18) is to achieve coordination, that is, $r_i(t)-r_j(t)\to\varDelta_{ij}$ and $v_i(t)-v_j(t)\to 0$ as $t\to\infty$ when there exist both communication and input delays.

Using (10.18), (3.5) can be written in a vector form as

$$\dot{\breve{r}}(t) = v(t), \tag{10.19a}$$

$$\dot{v}(t) = -\breve{r}(t-\tau_1) + A\breve{r}(t-\tau_1-\tau_2) - \gamma_c v(t-\tau_1) + \gamma_c A v(t-\tau_1-\tau_2), \tag{10.19b}$$

where $\breve{r} \triangleq [\breve{r}_1, \dots, \breve{r}_n]^T$ with $\breve{r}_i \triangleq r_i - \delta_i$, $v \triangleq [v_1, \dots, v_n]^T$, and A is defined as in (10.2). Define $\tilde{r} \triangleq W^{-1}\breve{r}$ and $\tilde{v} \triangleq W^{-1}v$, where W is defined as in (10.3). Denote $\tilde{r}_{1:n-1,:}$ and $\tilde{v}_{1:n-1,:}$ as the first $n-1$ rows of, respectively, $\tilde{r}$ and $\tilde{v}$. Denote $\tilde{r}_{n,:}$ and $\tilde{v}_{n,:}$ as the last row of, respectively, $\tilde{r}$ and $\tilde{v}$. By multiplying W^{-1} on both sides of (10.19) and after some manipulation, we obtain the following three equations

$$\dot{\tilde{z}}(t) = A_0\tilde{z}(t) + A_1\tilde{z}(t-\tau_1) + A_2\tilde{z}(t-\tau_1-\tau_2), \tag{10.20a}$$

$$\dot{\tilde{r}}_{n,:}(t) = \tilde{v}_{n,:}(t), \tag{10.20b}$$

$$\dot{\tilde{v}}_{n,:}(t) = -\tilde{r}_{n,:}(t-\tau_1) + \tilde{r}_{n,:}(t-\tau_1-\tau_2) - \gamma_c\tilde{v}_{n,:}(t-\tau_1) + \gamma_c\tilde{v}_{n,:}(t-\tau_1-\tau_2), \tag{10.20c}$$

where

$$\tilde{z} \triangleq \left[\tilde{r}^T_{1:n-1,:}, \tilde{v}^T_{1:n-1,:}\right]^T, \qquad A_0 \triangleq \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \\ 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \end{bmatrix},$$

$$A_1 \triangleq \begin{bmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\ -I_{n-1} & -\gamma_c I_{n-1} \end{bmatrix},$$

$$A_2 \triangleq \begin{bmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times(n-1)} \\ \widetilde{A} & \gamma_c\widetilde{A} \end{bmatrix},$$

and $\widetilde{A}$ is defined as in (10.4).

Theorem 10.11. *Suppose that the directed fixed graph $\mathscr{G}$ has a directed spanning tree, every agent has a neighbor, and $\gamma_c > \overline{\gamma}_c \triangleq \max_{\mu_i \neq 0} \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)|\mu_i|}}$, where μ_i is the ith eigenvalue of L defined after* (10.2)*. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, the following three conditions are satisfied*:

(i) $\gamma_c(2\tau_1 + \tau_2) + \frac{(2\tau_1+\tau_2)\tau_2}{2} < 1$.
(ii) $1 + \lambda_i(A_1)\frac{1-e^{-s\tau_1}}{s} + \lambda_i(A_2)\frac{1-e^{-s(\tau_1+\tau_2)}}{s} \neq 0$, *for all $s \in \mathbb{C}^+$.*
(iii) *The matrix*

$$\begin{aligned} Q_{\mathrm{sc}} \triangleq\; & (A_0+A_1+A_2)^T P_{\mathrm{sc}} + P_{\mathrm{sc}}(A_0+A_1+A_2) + \overline{\tau_1}S_{\mathrm{sc}} + (\overline{\tau_1}+\overline{\tau_2})H_{\mathrm{sc}} \\ & + \overline{\tau_1}\left[(A_0+A_1+A_2)^T P_{\mathrm{sc}}A_1 S_{\mathrm{sc}}^{-1}A_1^T P_{\mathrm{sc}}(A_0+A_1+A_2)\right] \\ & + (\overline{\tau_1}+\overline{\tau_2})\left[(A_0+A_1+A_2)^T P_{\mathrm{sc}}A_2 H_{\mathrm{sc}}^{-1}A_2^T P_{\mathrm{sc}}(A_0+A_1+A_2)\right] \end{aligned}$$

is symmetric negative definite, where $P_{\mathrm{sc}} \in \mathbb{R}^{(2n-2)\times(2n-2)}$ is a symmetric positive-definite matrix chosen properly such that $(A_0 + A_1 + A_2)^T P_{\mathrm{sc}} +$

$P_{\rm sc}(A_0+A_1+A_2)$ is symmetric negative definite, and $S_{\rm sc}\in\mathbb{R}^{(2n-2)\times(2n-2)}$ and $H_{\rm sc}\in\mathbb{R}^{(2n-2)\times(2n-2)}$ are arbitrary symmetric positive-definite matrices.

In addition, if $\tau_1\in[0,\overline{\tau_1}]$ and $\tau_2\in[0,\overline{\tau_2}]$, using (10.18) for (3.5), for all $r_i(0)$ and $v_i(0)$ and all $i,j=1,\dots,n$, $r_i(t)-r_j(t)\to\Delta_{ij}$ and $v_i(t)-v_j(t)\to 0$ as $t\to\infty$. In particular, $r_i(t)\to\frac{\mathbf{p}^Tv(0)}{\tau_2}+\delta_i$ and $v_i(t)\to 0$, $i=1,\dots,n$, as $t\to\infty$, where $\mathbf{p}\in\mathbb{R}^n$ is defined after (10.3).

Proof: For the first statement, it is straightforward to see that there exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1\in[0,\overline{\tau_1}]$ and $\tau_2\in[0,\overline{\tau_2}]$, Conditions (i) and (ii) are satisfied. For Condition (iii), because $\mathscr{G}$ has a directed spanning tree and each agent has a neighbor, all eigenvalues of $\widetilde{L}$ have positive real parts. Also note that $\widetilde{L}=I_{n-1}-\widetilde{A}$. It follows that

$$A_0+A_1+A_2=\begin{bmatrix}0_{(n-1)\times(n-1)} & I_{n-1}\\ -\widetilde{L} & -\gamma_c\widetilde{L}\end{bmatrix}.$$

Because all eigenvalues of $\widetilde{L}$ are also the $n-1$ nonzero eigenvalues of L and $\gamma_c>\overline{\gamma}_c$, it follows from Lemma 7.4 that all eigenvalues of $A_0+A_1+A_2$ have negative real parts. Thus there always exists a symmetric positive-definite matrix $P_{\rm sc}$ to guarantee that $(A_0+A_1+A_2)^TP_{\rm sc}+P_{\rm sc}(A_0+A_1+A_2)$ is symmetric negative definite. A similar analysis to that in Theorem 10.1 shows that there exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1\in[0,\overline{\tau_1}]$ and $\tau_2\in[0,\overline{\tau_2}]$, Condition (iii) is satisfied.

For the second statement, we show that if $\tau_1\in[0,\overline{\tau_1}]$ and $\tau_2\in[0,\overline{\tau_2}]$, (10.20a) is asymptotically stable while (10.20b) is stable. It follows from Lemma 1.44 that the stability of the following system

$$\begin{aligned}&\frac{d}{dt}\left[\tilde{z}(t)+A_1\int_{-\tau_1}^{0}\tilde{z}(t+\theta)\,d\theta+A_2\int_{-\tau_1-\tau_2}^{0}\tilde{z}(t+\theta)\,d\theta\right]\\&\quad=(A_0+A_1+A_2)\tilde{z}(t)\end{aligned}\tag{10.21}$$

implies the stability of (10.20a) if Condition (ii) in the theorem is satisfied. Then, consider the Lyapunov function candidate

$$\begin{aligned}V(\tilde{z}_t)=&\chi^TP_{\rm sc}\chi\\&+\int_{-\tau_1}^{0}\int_{t+\theta}^{t}\tilde{z}(\xi)^TS_{\rm sc}\tilde{z}(\xi)\,d\xi\,d\theta+\int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t}\tilde{z}(\xi)^TH_{\rm sc}\tilde{z}(\xi)\,d\xi\,d\theta,\end{aligned}$$

where $\chi\stackrel{\triangle}{=}\tilde{z}(t)+A_1\int_{-\tau_1}^{0}\tilde{z}(t+\theta)\,d\theta+A_2\int_{-\tau_1-\tau_2}^{0}\tilde{z}(t+\theta)\,d\theta$. Taking the derivative of $V(\tilde{z}_t)$ along (10.21) gives

$$\dot{V}(\tilde{z}_t)\le\tilde{z}(t)^TQ_{\rm sc}\tilde{z}(t).$$

A similar analysis to that in the proof of Theorem 10.1 shows that if Conditions (ii) and (iii) are satisfied, (10.20a) is asymptotically stable at the origin. For

(10.20b) and (10.20c), we apply the Nyquist stability criterion to find the stability condition. Applying Laplace transform to (10.20b) and (10.20c), we obtain that $\tilde{r}_{n,:}(s) = \frac{s\tilde{r}_{n,:}(0)+\tilde{v}_{n,:}(0)}{s^2+(\gamma_c s+1)[e^{-\tau_1 s - e^{-(\tau_1+\tau_2)s}}]}$. Therefore, the stability of (10.20b) and (10.20c) is determined by the distribution of the roots

$$s^2 + (\gamma_c s + 1)\left[e^{-\tau_1 s - e^{-(\tau_1+\tau_2)s}}\right] = 0. \tag{10.22}$$

Note that (10.22) has two zero roots. To study the other roots, define $g(s) \triangleq (\gamma_c s + 1) \times [e^{-\tau_1 s} - e^{-(\tau_1+\tau_2)s}]/s^2$. By using the Nyquist stability criterion, it follows that the roots of (10.22) are stable if $\mathrm{Re}[g(\iota\omega)] > -1$, $\forall \omega \in (-\infty, \infty)$. Because

$$\begin{aligned}
\mathrm{Re}\big[g(\iota\omega)\big] &= \frac{-\gamma_c \sin(\tau_1\omega) + \gamma_c \sin[(\tau_1+\tau_2)\omega]}{\omega} + \frac{-\cos(\tau_1\omega) + \cos[(\tau_1+\tau_2)\omega]}{\omega^2} \\
&= \frac{-\gamma_c \sin(\tau_1\omega) + \gamma_c \sin[(\tau_1+\tau_2)\omega]}{\omega} - \frac{2\sin[\frac{(2\tau_1+\tau_2)}{2}\omega]\sin(\frac{\tau_2}{2}\omega)}{\omega^2} \\
&\geq -\gamma_c\tau_1 - \gamma_c(\tau_1+\tau_2) - \frac{(2\tau_1+\tau_2)\tau_2}{2},
\end{aligned}$$

it follows that (10.20b) and (10.20c) are marginally stable under Condition (i) of the theorem. Note that the asymptotical stability of (10.20a) implies that $\lim_{t\to\infty} \tilde{z}(t) = \mathbf{0}_{2n-2}$. Also by using the final value theorem, it follows that $\lim_{t\to\infty} \tilde{r}_{n,:}(t) = \frac{\tilde{v}_{n,:}(0)}{\tau_2}$. After similar manipulation to that in Theorem 10.1, it follows that $r_i(t) \to \frac{\mathbf{p}^T v(0)}{\tau_2} + \delta_i$ and $v_i(t) \to 0$ (and hence $r_i(t) - r_j(t) \to \Delta_{ij}$ and $v_i(t) - v_j(t) \to 0$) as $t \to \infty$. ■

Remark 10.12 Due to the existence of the communication delay, using (10.18) for (3.5), the final velocity is dampened to zero instead of a possible nonzero constant as compared with the standard consensus algorithm for double-integrator dynamics [248, Chap. 4]. Also note that if there exists only the input delay, the final velocity is a possibly nonzero constant, which is consistent with the results using the standard consensus algorithm for double-integrator dynamics in [248, Chap. 4].

Remark 10.13 Note that compared with the case for single-integrator dynamics in Sect. 10.2.1, the case for double-integrator dynamics requires more stringent conditions to guarantee coordination.

10.3.2 Coordinated Tracking when the Leader's Velocity is Constant

In this subsection, we assume that in addition to n followers, labeled as agents or followers 1 to n, there exists a leader, labeled as agent 0, with position r_0 and velocity v_0. Here we assume that v_0 is constant. Let $\mathscr{G} \triangleq (\mathscr{V}, \mathscr{E})$ and $\overline{\mathscr{G}} \triangleq (\overline{\mathscr{V}}, \overline{\mathscr{E}})$ be defined as in Sect. 10.2.2.

Consider the following coordinated tracking algorithm with both communication and input delays for the n followers with double-integrator dynamics given by (3.5) as

$$\begin{aligned} u_i(t) = &- \frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij} \big[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2) \big] \\ &- \frac{\gamma_r}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij} \big[v_i(t-\tau_1) - v_j(t-\tau_1-\tau_2) \big], \quad i=1,\dots,n, \end{aligned} \tag{10.23}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i = 1,\dots,n$, $j = 0,\dots,n$, is defined as in (10.10), and γ_r is a positive gain. Note that in $\overline{\mathscr{G}}$ if the leader has directed paths to all followers 1 to n, it follows that $\sum_{j=0}^n a_{ij} > 0$, $i = 1,\dots,n$.

Using (10.23), (3.5) can be written in a vector form as

$$\dot{\overline{\chi}}(t) = \overline{A}_0 \overline{\chi}(t) + \overline{A}_1 \overline{\chi}(t-\tau_1) + \overline{A}_2 \overline{\chi}(t-\tau_1-\tau_2) + R_{\text{sr}}, \tag{10.24}$$

where $\bar{r} \triangleq [r_1 - r_0, \dots, r_n - r_0]^T$, $\bar{v} \triangleq [v_1 - v_0, \dots, v_n - v_0]^T$, $\overline{\chi} \triangleq [\bar{r}^T, \bar{v}^T]^T$, $\overline{A}_0 \triangleq \left[\begin{smallmatrix} 0_{n\times n} & I_n \\ 0_{n\times n} & 0_{n\times n} \end{smallmatrix}\right]$, $\overline{A}_1 \triangleq \left[\begin{smallmatrix} 0_{n\times n} & 0_{n\times n} \\ -I_n & -\gamma_r I_n \end{smallmatrix}\right]$, $\overline{A}_2 \triangleq \left[\begin{smallmatrix} 0_{n\times n} & 0_{n\times n} \\ \overline{A} & \gamma_r \overline{A} \end{smallmatrix}\right]$, and $R_{\text{sr}} \triangleq \left[\begin{smallmatrix} \mathbf{0}_n \\ -\tau_2 v_0 \mathbf{1}_n \end{smallmatrix}\right]$. Note that here $\overline{A}$ is defined before (10.11) and we have used the fact that v_0 is constant. By letting $\zeta \triangleq (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^{-1} R_{\text{sr}}$ and $\widehat{\chi} \triangleq \overline{\chi} - \zeta$, we can transform (10.24) as

$$\dot{\widehat{\chi}} = \overline{A}_0 \widehat{\chi}(t) + \overline{A}_1 \widehat{\chi}(t-\tau_1) + \overline{A}_2 \widehat{\chi}(t-\tau_1-\tau_2). \tag{10.25}$$

Theorem 10.14. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n and $\gamma_r > \overline{\gamma}_r \triangleq \max_i \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)|\mu_i|}}$, where μ_i is the ith eigenvalue of $I_n - \overline{A}$. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, the following two conditions are satisfied*:

(i) $1 + \lambda_i(\overline{A}_1) \frac{1 - e^{-s\tau_1}}{s} + \lambda_i(\overline{A}_2) \frac{1 - e^{-s(\tau_1+\tau_2)}}{s} \neq 0$, *for all* $s \in \mathbb{C}^+$.
(ii) *The matrix*

$$\begin{aligned} Q_{\text{sr}} \triangleq{} & (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\text{sr}} + P_{\text{sr}} (\overline{A}_0 + \overline{A}_1 + \overline{A}_2) + \tau_1 S_{\text{sr}} + (\tau_1 + \tau_2) H_{\text{sr}} \\ &+ \tau_1 \big[(\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\text{sr}} \overline{A}_1 S_{\text{sr}}^{-1} \overline{A}_1^T P_{\text{sr}} (\overline{A}_0 + \overline{A}_1 + \overline{A}_2) \big] \\ &+ (\tau_1 + \tau_2) \big[(\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\text{sr}} \overline{A}_2 H_{\text{sr}}^{-1} \overline{A}_2^T P_{\text{sr}} (\overline{A}_0 + \overline{A}_1 + \overline{A}_2) \big] \end{aligned}$$

is symmetric negative definite, where $P_{\text{sr}} \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive-definite matrix chosen properly such that $(\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\text{sr}} + P_{\text{sr}}(\overline{A}_0 +$

$\overline{A}_1 + \overline{A}_2$) *is symmetric negative definite, and* $S_{\text{sr}} \in \mathbb{R}^{2n\times 2n}$ *and* $H_{\text{sr}} \in \mathbb{R}^{2n\times 2n}$ *are arbitrary symmetric positive-definite matrices.*

In addition, if $\tau_1 \in [0, \overline{\tau_1}]$ *and* $\tau_2 \in [0, \overline{\tau_2}]$*, using* (10.23) *for* (3.5)*,* $\bar{r}(t) \to \tau_2 v_0 (I_n - \overline{A})^{-1}\mathbf{1}_n$ *and* $\bar{v}(t) \to \mathbf{0}_n$ *as* $t \to \infty$*.*

Proof: For the first statement, it is straightforward to show that Condition (i) is satisfied. For Condition (ii), note that $\overline{A}_0+\overline{A}_1+\overline{A}_2 = \begin{bmatrix} 0_{n\times n} & I_n \\ -(I_n-\overline{A}) & -\gamma_r(I_n-\overline{A}) \end{bmatrix}$. Because in $\overline{\mathscr{G}}$ the leader has directed paths to all followers, it follows from Lemma 10.1 that all eigenvalues of $I_n - \overline{A}$ have positive real parts. Because $\gamma_r > \overline{\gamma}_r$, it thus follows from Lemma 7.4 that all eigenvalues of $\overline{A}_0 + \overline{A}_1 + \overline{A}_2$ have negative real parts. The rest of the proof is similar to that in Theorem 10.1.

For the second statement, it follows from Lemma 1.44 that the stability of the following system

$$\begin{aligned}\frac{d}{dt}\left[\widehat{\chi}(t) + \overline{A}_1 \int_{-\tau_1}^{0} \widehat{\chi}(t+\theta)\,d\theta + \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} \widehat{\chi}(t+\theta)\,d\theta\right] \\ = (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)\widehat{\chi}(t) \end{aligned} \tag{10.26}$$

implies the stability of (10.25) under Condition (i) of the theorem. Then, consider a Lyapunov function candidate

$$\begin{aligned} V(\widehat{\chi}_t) = \varphi^T P_{\text{sr}} \varphi + \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \widehat{\chi}^T(\xi) S_{\text{sr}} \widehat{\chi}(\xi)\,d\xi\,d\theta \\ + \int_{-\tau_1-\tau_2}^{0}\int_{t+\theta}^{t} \widehat{\chi}^T(\xi) H_{\text{sr}} \widehat{\chi}(\xi)\,d\xi\,d\theta, \end{aligned}$$

where $\varphi \triangleq \widehat{\chi}(t) + \overline{A}_1 \int_{-\tau_1}^{0} \widehat{\chi}(t+\theta)\,d\theta + \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} \widehat{\chi}(t+\theta)\,d\theta$. Taking the derivative of $V(\widehat{\chi}_t)$ along (10.26) gives

$$\dot{V}(\widehat{\chi}_t) \leq \widehat{\chi}^T(t) Q_{\text{sr}} \widehat{\chi}(t),$$

where we have used Lemma 1.23 to derive the inequality. A similar analysis to that in the proof of Theorem 10.1 shows that (10.25) is asymptotically stable at the origin, which implies that $\widehat{\chi}(t) \to \mathbf{0}_{2n}$ as $t \to \infty$. Note that $\zeta = [\tau_2 v_0 [(I_n - \overline{A})^{-1}\mathbf{1}_n]^T, \mathbf{0}_n^T]^T$ by computation and $\overline{\chi} = \widehat{\chi} + \zeta$. It follows that $\bar{r}(t) \to \tau_2 v_0 (I_n - \overline{A})^{-1}\mathbf{1}_n$ and $\bar{v}(t) \to \mathbf{0}_n$ as $t \to \infty$. ■

Corollary 10.1. *Suppose that the conditions in Theorem 10.14 hold. If* $v_0 = 0$*, then* $r_i(t) \to r_0$ *and* $v_i(t) \to 0$ *as* $t \to \infty$*.*

Remark 10.15 Note that different from the results in the case for single-integrator dynamics in Sect. 10.2.2, the tracking errors of the followers $r_i(t) - r_0(t)$ might not approach zero but approach possibly different constants in the case of double-integrator dynamics.

10.3.3 Coordinated Tracking with Full Access to the Leader's Acceleration

In this subsection, we consider the case where the leader's position r_0 and velocity v_0 are varying. We assume that all followers have access to $\dot{v}_0$. We also assume that $|v_0| < \delta_v$, $|\dot{v}_0| < \delta_a$, and $|\ddot{v}_0| < \delta_{\dot{a}}$, where δ_v, δ_a and $\delta_{\dot{a}}$ are positive constants.

Consider the following coordinated tracking algorithm with both communication and input delays for the n followers with double-integrator dynamics given by (3.5) as

$$\begin{aligned} u_i(t) = {} & \dot{v}_0(t-\tau_1-\tau_2) - \frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij} \big\{ \big[r_i(t-\tau_1) - r_j(t-\tau_1-\tau_2) \big] \\ & + \gamma_t \big[v_i(t-\tau_1) - v_j(t-\tau_1-\tau_2) \big] \big\}, \quad i = 1, \ldots, n, \end{aligned} \tag{10.27}$$

where τ_1 and τ_2 are, respectively, the input and communication delays, a_{ij}, $i = 1, \ldots, n$, $j = 0, \ldots, n$, is defined as in (10.10), and γ_t is a positive gain. Using (10.27), (3.5) can be written in a vector form as

$$\dot{\overline{\chi}}(t) = \overline{A}_0 \overline{\chi}(t) + \overline{A}_1 \overline{\chi}(t-\tau_1) + \overline{A}_2 \overline{\chi}(t-\tau_1-\tau_2) + R_{\mathrm{st}}, \tag{10.28}$$

where $\overline{\chi}$ is defined as in (10.24), $\overline{A}_0$, $\overline{A}_1$, and $\overline{A}_2$ are defined as in (10.24) with γ_r replaced with γ_t, $R_{\mathrm{st}} \triangleq \left[\begin{smallmatrix} \mathbf{0}_n \\ R_1 \end{smallmatrix} \right]$, and $R_1 \triangleq \mathbf{1}_n [\dot{v}_0(t-\tau_1-\tau_2) - \dot{v}_0(t) + r_0(t-\tau_1-\tau_2) - r_0(t-\tau_1) + \gamma_t v_0(t-\tau_1-\tau_2) - v_0(t-\tau_1)]$. By using (1.10), it follows that $R_1 = -\mathbf{1}_n \int_{-\tau_1-\tau_2}^{0} \ddot{v}_0(t+\theta)\, d\theta - \mathbf{1}_n \int_{-\tau_1-\tau_2}^{-\tau_1} v_0(t+\theta)\, d\theta - \gamma_t \mathbf{1}_n \int_{-\tau_1-\tau_2}^{-\tau_1} \dot{v}_0(t+\theta)\, d\theta$.

Theorem 10.16. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers 1 to n and $\gamma_t > \overline{\gamma}_r$, where $\overline{\gamma}_r$ is defined in Theorem 10.14. There exist positive $\overline{\tau_1}$ and $\overline{\tau_2}$ such that for $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$,*

$$\begin{aligned} Q_{\mathrm{st}} \triangleq {} & (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\mathrm{sr}} + P_{\mathrm{sr}} (\overline{A}_0 + \overline{A}_1 + \overline{A}_2) \\ & + \tau_1 \big(P_{\mathrm{sr}} \overline{A}_1 \overline{A}_0 P_{\mathrm{sr}}^{-1} \overline{A}_0^T \overline{A}_1^T P_{\mathrm{sr}} + P_{\mathrm{sr}} (\overline{A}_1)^2 P_{\mathrm{sr}}^{-1} \big(\overline{A}_1^T \big)^2 P_{\mathrm{sr}} \\ & + P_{\mathrm{sr}} \overline{A}_1 \overline{A}_2 P_{\mathrm{sr}}^{-1} \overline{A}_2^T \overline{A}_1^T P_{\mathrm{sr}} + 3 q_s P_{\mathrm{sr}} \big) \\ & + (\tau_1 + \tau_2) \big(P_{\mathrm{sr}} \overline{A}_2 \overline{A}_0 P_{\mathrm{sr}}^{-1} \overline{A}_0^T \overline{A}_2^T P_{\mathrm{sr}} + P_{\mathrm{sr}} \overline{A}_2 \overline{A}_1 P_{\mathrm{sr}}^{-1} \overline{A}_1^T \overline{A}_2^T P_{\mathrm{sr}} \\ & + P_{\mathrm{sr}} \big(\overline{A}_2 \big)^2 P_{\mathrm{sr}}^{-1} \big(\overline{A}_2^T \big)^2 P_{\mathrm{sr}} + 3 q_s P_{\mathrm{sr}} \big) \end{aligned}$$

is symmetric negative definite, where P_{sr} is defined in Theorem 10.14 and q_s is an arbitrary real number satisfying $q_s > 1$. In addition, if $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, using (10.27) for (3.5), for all $r_i(0)$ and $v_i(0)$ and all $i = 1, \ldots, n$, $|r_i(t) - r_0(t)|$ and $|v_i(t) - v_0(t)|$ are uniformly ultimately bounded. In particular, the ultimate bound for $\|\overline{\chi}(t)\|$ is given by $\frac{\lambda_{\max}(P_{\mathrm{sr}}) a_s}{\lambda_{\min}(P_{\mathrm{sr}}) \kappa_s \lambda_{\min}(-Q_{\mathrm{st}})}$, where $a_s \triangleq 2[\|P_{\mathrm{sr}}\| + \|P_{\mathrm{sr}} \overline{A}_1\| \tau_1 + \|P_{\mathrm{sr}} \overline{A}_2\| (\tau_1 + \tau_2)][(\tau_1 + \tau_2)\delta_{\dot{a}} + \tau_2 \delta_v + \gamma_t \tau_2 \delta_a]$, and κ_s is an arbitrary real number satisfying $0 < \kappa_s < 1$.

Proof: The proof for the first statement is similar to that in Theorem 10.14 and is hence omitted here. For the second statement, using (1.10), we transform (10.28) to the following system

$$\begin{aligned}
\frac{d}{dt}\overline{\chi}(t) =& (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)\overline{\chi}(t) - \overline{A}_1 \int_{-\tau_1}^{0} \dot{\overline{\chi}}(t+\theta)\, d\theta \\
& - \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} \dot{\overline{\chi}}(t+\theta)\, d\theta + R_{\text{st}} \\
=& (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)\overline{\chi}(t) \\
& - \overline{A}_1 \int_{-\tau_1}^{0} \big[\overline{A}_0\overline{\chi}(t+\theta) \\
& + \overline{A}_1\overline{\chi}(t-\tau_1+\theta) + \overline{A}_2\overline{\chi}(t-\tau_1-\tau_2+\theta)\big]\, d\theta \\
& - \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} \big[\overline{A}_0\overline{\chi}(t+\theta) + \overline{A}_1\overline{\chi}(t-\tau_1+\theta) \\
& + \overline{A}_2\overline{\chi}(t-\tau_1-\tau_2+\theta)\big]\, d\theta \\
& - \overline{A}_1 \int_{-\tau_1}^{0} R_{\text{st}}(t+\theta)\, d\theta - \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} R_{\text{st}}(t+\theta)\, d\theta + R_{\text{st}} \\
=& (\overline{A}_0 + \overline{A}_1 + \overline{A}_2)\overline{\chi}(t) - \overline{A}_1\overline{A}_0 \int_{-\tau_1}^{0} \overline{\chi}(t+\theta)\, d\theta - \overline{A}_1^2 \int_{-2\tau_1}^{-\tau_1} \overline{\chi}(t+\theta)\, d\theta \\
& - \overline{A}_1\overline{A}_2 \int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2} \overline{\chi}(t+\theta)\, d\theta - \overline{A}_2\overline{A}_0 \int_{-\tau_1-\tau_2}^{0} \overline{\chi}(t+\theta)\, d\theta \\
& - \overline{A}_2\overline{A}_1 \int_{-2\tau_1-\tau_2}^{-\tau_1} \overline{\chi}(t+\theta)\, d\theta - \overline{A}_2^2 \int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2} \overline{\chi}(t+\theta)\, d\theta \\
& - \overline{A}_1 \int_{-\tau_1}^{0} R_{\text{st}}(t+\theta)\, d\theta - \overline{A}_2 \int_{-\tau_1-\tau_2}^{0} R_{\text{st}}(t+\theta)\, d\theta + R_{\text{st}}.
\end{aligned}$$

Consider the Lyapunov function candidate $V(\overline{\chi}) = \overline{\chi}^T(t)P_{\text{sr}}\overline{\chi}(t)$. Taking the derivative of $V(\overline{\chi})$ along (10.28) gives

$$\begin{aligned}
\dot{V}(\overline{\chi}) \le& \overline{\chi}^T \big[(\overline{A}_0 + \overline{A}_1 + \overline{A}_2)^T P_{\text{sr}} + P_{\text{sr}}(\overline{A}_0 + \overline{A}_1 + \overline{A}_2)\big]\overline{\chi} \\
& + \tau_1\overline{\chi}^T P_{\text{sr}}\overline{A}_1\overline{A}_0 P_{\text{sr}}^{-1}\overline{A}_0^T\overline{A}_1^T P_{\text{sr}}\overline{\chi} \\
& + \int_{-\tau_1}^{0} \overline{\chi}^T(t+\theta)P_{\text{sr}}\overline{\chi}(t+\theta)\, d\theta + \tau_1\overline{\chi}^T P_{\text{sr}}(\overline{A}_1)^2 P_{\text{sr}}^{-1}\big(\overline{A}_1^T\big)^2 P_{\text{sr}}\overline{\chi} \\
& + \int_{-2\tau_1}^{-\tau_1} \overline{\chi}^T(t+\theta)P_{\text{sr}}\overline{\chi}(t+\theta)\, d\theta + \tau_1\overline{\chi}^T P_{\text{sr}}\overline{A}_1\overline{A}_2 P_{\text{sr}}^{-1}\overline{A}_2^T\overline{A}_1^T P_{\text{sr}}\overline{\chi} \\
& + \int_{-2\tau_1-\tau_2}^{-\tau_1-\tau_2} \overline{\chi}^T(t+\theta)P_{\text{sr}}\overline{\chi}(t+\theta)\, d\theta \\
& + (\tau_1+\tau_2)\overline{\chi}^T P_{\text{sr}}\overline{A}_2\overline{A}_0 P_{\text{sr}}^{-1}\overline{A}_0^T\overline{A}_2^T P_{\text{sr}}\overline{\chi}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau_1-\tau_2}^{0} \overline{\chi}^T(t+\theta)P_{\mathrm{sr}}\overline{\chi}(t+\theta)\,d\theta \\
&+(\tau_1+\tau_2)\overline{\chi}^T P_{\mathrm{sr}}\overline{A}_2\overline{A}_1 P_{\mathrm{sr}}^{-1}\overline{A}_1^T\overline{A}_2^T P_{\mathrm{sr}}\overline{\chi} \\
&+\int_{-2\tau_1-\tau_2}^{-\tau_1} \overline{\chi}^T(t+\theta)P_{\mathrm{sr}}\overline{\chi}(t+\theta)\,d\theta \\
&+(\tau_1+\tau_2)\overline{\chi}^T P(\overline{A}_2)^2 P_{\mathrm{sr}}^{-1}\big(\overline{A}_2^T\big)^2 P_{\mathrm{sr}}\overline{\chi} \\
&+\int_{-2\tau_1-2\tau_2}^{-\tau_1-\tau_2} \overline{\chi}^T(t+\theta)P_{\mathrm{sr}}\overline{\chi}(t+\theta)\,d\theta \\
&+2\|\overline{\chi}\|\|P_{\mathrm{sr}}\|\big[(\tau_1+\tau_2)\delta_{\dot{a}}+\tau_2\delta_v+\gamma_t\tau_2\delta_a\big] \\
&+2\|\overline{\chi}\|\|P_{\mathrm{sr}}\overline{A}_1\|\tau_1\big[(\tau_1+\tau_2)\delta_{\dot{a}}+\tau_2\delta_v+\gamma_t\tau_2\delta_a\big] \\
&+2\|\overline{\chi}\|\|P_{\mathrm{sr}}\overline{A}_2\|(\tau_1+\tau_2)\big[(\tau_1+\tau_2)\delta_{\dot{a}}+\tau_2\delta_v+\gamma_t\tau_2\delta_a\big],
\end{aligned}
$$

where we have used Lemma 1.23 and the facts that $|v_0| < \delta_v$, $|\dot{v}_0| < \delta_a$, and $|\ddot{v}_0| < \delta_{\dot{a}}$ to derive the inequality. Take $p(s) = q_s s$. If $V[\overline{\chi}(t+\theta)] < p\{V[\overline{\chi}(t)]\} = q_s V[\overline{\chi}(t)]$ for $-2\tau_1 - 2\tau_2 \leq \theta \leq 0$, by following a similar analysis to that in the proof of Theorem 10.6, we have

$$
\dot{V}(\overline{\chi}) \leq \overline{\chi}^T(t)Q_{\mathrm{st}}\overline{\chi}(t) + a_s\big\|\overline{\chi}(t)\big\|.
$$

If $\tau_1 \in [0, \overline{\tau_1}]$ and $\tau_2 \in [0, \overline{\tau_2}]$, we have that $\lambda_{\min}(-Q_{\mathrm{st}}) > 0$. Given $0 < \kappa_s < 1$, if $\|\overline{\chi}(t)\| \geq \frac{a_s}{\kappa_s\lambda_{\min}(-Q_{\mathrm{st}})}$, we can obtain

$$
\begin{aligned}
\dot{V}(\overline{\chi}) &\leq -(1-\kappa_s)\lambda_{\min}(-Q_{\mathrm{st}})\big\|\overline{\chi}(t)\big\|^2 - \kappa_s\lambda_{\min}(-Q_{\mathrm{st}})\big\|\overline{\chi}(t)\big\|^2 + a_s\big\|\overline{\chi}(t)\big\| \\
&\leq -(1-\kappa_s)\lambda_{\min}(-Q_{\mathrm{st}})\big\|\overline{\chi}(t)\big\|^2.
\end{aligned}
$$

Therefore, it follows from Lemma 1.42 that $\|\overline{\chi}(t)\|$ is uniformly ultimately bounded, which implies that $|r_i(t) - r_0(t)|$ and $|v_i(t) - v_0(t)|$ are uniformly ultimately bounded. Moreover, it can be computed that the ultimate bound for $\|\overline{\chi}(t)\|$ is given by $\frac{\lambda_{\max}(P_{\mathrm{sr}})a_s}{\lambda_{\min}(P_{\mathrm{sr}})\kappa_s\lambda_{\min}(-Q_{\mathrm{st}})}$ by following a similar analysis to that in [145, pp. 172–174]. ■

10.3.4 Coordinated Tracking with Partial Access to the Leader's Acceleration

In this subsection, we assume the leader's varying position r_0, velocity v_0, and acceleration $\dot{v}_0$ are available to only a subset of all followers. We assume that $|r_0|$, $|v_0|$, and $|\dot{v}_0|$ are bounded. We also assume that there exists only the communication delay.

Consider the following coordinated tracking algorithm for the n followers with double-integrator dynamics given by (3.5) as

$$u_i(t) = \frac{1}{\sum_{j=0}^n a_{ij}} \sum_{j=0}^n a_{ij} \{ \dot{v}_j(t-\tau_2) - [r_i(t) - r_j(t-\tau_2)] - \gamma_{ft}[v_i(t) - v_j(t-\tau_2)] \}, \quad i = 1, \ldots, n, \tag{10.29}$$

where τ_2 is the communication delay, a_{ij}, $i = 1, \ldots, n$, $j = 0, \ldots, n$, is defined as in (10.10), and γ_{ft} is a positive gain. Using (10.29), (3.5) can be written in a vector form as

$$\dot{\overline{\chi}}(t) = D_f \dot{\overline{\chi}}(t-\tau_2) + \overline{A}_{f0}\overline{\chi} + \overline{A}_{f1}\overline{\chi}(t-\tau_2) + R_{\text{sft}}, \tag{10.30}$$

where $\overline{\chi}$ is defined as in (10.24), $D_f \triangleq \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \overline{A} \end{bmatrix}$, $\overline{A}_{f0} \triangleq \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & -\gamma_{ft} I_n \end{bmatrix}$, $\overline{A}_{f1} \triangleq \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ \overline{A} & \gamma_{ft}\overline{A} \end{bmatrix}$, $R_{\text{sft}} \triangleq \begin{bmatrix} \mathbf{0}_n \\ R_2 \end{bmatrix}$, and $R_2 \triangleq [\dot{v}_0(t-\tau_2) - \dot{v}_0(t)]\mathbf{1}_n - [r_0(t) - r_0(t-\tau_2)]\mathbf{1}_n - \gamma_{ft}[v_0(t) - v_0(t-\tau_2)]\mathbf{1}_n$. Note that here $\overline{A}$ is defined before (10.11).

Theorem 10.17. *Suppose that in $\overline{\mathscr{G}}$ the leader has directed paths to all followers* 1 *to n, and $\gamma_{ft} > \overline{\gamma}_r$, where $\overline{\gamma}_r$ is defined in Theorem* 10.14. *Using* (10.29) *for* (3.5)*, for all $r_i(0)$ and $v_i(0)$ and all $i = 1, \ldots, n$, $|r_i(t) - r_0(t)|$ and $|v_i(t) - v_0(t)|$ are uniformly ultimately bounded if*

$$\lambda > 2q_{\text{sf}} \sqrt{\frac{\lambda_{\max}(P_{\text{sr}})}{\lambda_{\min}(P_{\text{sr}})}} \left\| P_{\text{sr}}(\overline{A}_{f0} D_f + \overline{A}_{f1}) \right\| + 2\|P_{\text{sr}}\overline{A}_{f1}\|, \tag{10.31}$$

where $\lambda \triangleq \lambda_{\min}[-(\overline{A}_{f0} + \overline{A}_{f1})^T P_{\text{sr}} - P_{\text{sr}}(\overline{A}_{f0} + \overline{A}_{f1})]$,[2] *$P_{\text{sr}} \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive-definite matrix chosen properly such that $(\overline{A}_{f0} + \overline{A}_{f1})^T P_{\text{sr}} + P_{\text{sr}}(\overline{A}_{f0} + \overline{A}_{f1})$ is symmetric negative definite, and q_{sf} is an arbitrary real number satisfying $q_{\text{sf}} > 1$.*

Proof: First, it follows from Lemma 8.1 that $\rho(\overline{A}) < 1$, which implies that $\rho(D_f) < 1$. Therefore, the neutral operator $\mathscr{D}\overline{\chi}_t = \overline{\chi} - D_f\overline{\chi}(t-\tau_2)$ is stable. Consider a Lyapunov function candidate $V(\overline{\chi}) = \overline{\chi}^T(t) P_{\text{sr}} \overline{\chi}(t)$. Taking the derivative of $V(\overline{\chi})$ along (10.30) gives

$$\begin{aligned} \dot{V}(\mathscr{D}\overline{\chi}_t) &= 2(\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}} \left[\overline{A}_{f0}\overline{\chi} + \overline{A}_{f1}\overline{\chi}(t-\tau_2) + R_{\text{sft}} \right] \\ &= 2(\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}} \left[\overline{A}_{f0}(\mathscr{D}\overline{\chi}_t) + \overline{A}_{f0} D_f \overline{\chi}(t-\tau_2) + \overline{A}_{f1}\overline{\chi}(t-\tau_2) + R_{\text{sft}} \right] \end{aligned}$$

[2] Note that $\overline{A}_{f0} + \overline{A}_{f1} = \begin{bmatrix} 0_{n\times n} & I_n \\ -(I_n - \overline{A}) & -\gamma_{ft}(I_n - \overline{A}) \end{bmatrix}$. Similar to the proof of the first statement in Theorem 10.14, it follows that all eigenvalues of $\overline{A}_{f0} + \overline{A}_{f1}$ have negative real parts under the condition of the theorem.

$$
\begin{aligned}
&= (\mathscr{D}\overline{\chi}_t)^T \big[(\overline{A}_{f0} + \overline{A}_{f1})^T P_{\text{sr}} + P_{\text{sr}}(\overline{A}_{f0} + \overline{A}_{f1})\big](\mathscr{D}\overline{\chi}_t) \\
&\quad + 2(\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}}(\overline{A}_{f0} D_f + \overline{A}_{f1})\overline{\chi}(t-\tau_2) \\
&\quad - 2(\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}}\overline{A}_{f1}(\mathscr{D}\overline{\chi}_t) + 2(\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}} R_{\text{sft}}.
\end{aligned}
$$

Let $p(s) = q_{\text{sf}}^2 s$. If $V[\overline{\chi}(\theta)] < p[V(\mathscr{D}\overline{\chi}_t)]$ for $t - \tau_2 \le \xi \le t$, it is equivalent that $\overline{\chi}^T(\theta) P_{\text{sr}} \overline{\chi}(\theta) < q_{\text{sf}}^2 (\mathscr{D}\overline{\chi}_t)^T P_{\text{sr}} (\mathscr{D}\overline{\chi}_t)$ for $t - \tau_2 \le \theta \le t$. Therefore, we have $\|\overline{\chi}(t-\tau_2)\| < q_{\text{sf}} \sqrt{\frac{\lambda_{\max}(P_{\text{sr}})}{\lambda_{\min}(P_{\text{sr}})}} \|\mathscr{D}\overline{\chi}_t\|$. Thus, it follows that

$$
\begin{aligned}
\dot{V}(\mathscr{D}\overline{\chi}_t) \le & -\lambda \|\mathscr{D}\overline{\chi}_t\|^2 + 2q_{\text{sf}} \sqrt{\frac{\lambda_{\max}(P_{\text{sr}})}{\lambda_{\min}(P_{\text{sr}})}} \big\|P_{\text{sr}}(\overline{A}_{f0} D_f + \overline{A}_{f1})\big\| \|\mathscr{D}\overline{\chi}_t\|^2 \\
& + 2\|P_{\text{sr}}\overline{A}_{f1}\| \|\mathscr{D}\overline{\chi}_t\|^2 + 2\|P_{\text{sr}} R_{\text{sft}}\| \|\mathscr{D}\overline{\chi}_t\|.
\end{aligned}
$$

Therefore, if (10.31) holds, it follows from Lemma 1.43 that $\|\overline{\chi}(t)\|$ is uniformly ultimately bounded, which implies that $|r_i(t) - r_0(t)|$ and $|v_i(t) - v_0(t)|$ are uniformly ultimately bounded. ■

Remark 10.18 Note that different from the case for single-integrator dynamics where the tracking errors are bounded no matter how large the communication delay is, a certain delay independent condition (10.31) has to be satisfied beforehand to ensure the tracking errors are uniformly ultimate bounded in the case of double-integrator dynamics.

10.4 Simulation

In this section, we present simulation results to validate the theoretical results in Sects. 10.2 and 10.3. For the leaderless coordination problem, we consider a team of six agents with the adjacency matrix $\mathscr{A}$ chosen as

$$
\mathscr{A} = \begin{bmatrix}
0 & 5 & 0 & 2.5 & 0 & 2.5 \\
8 & 0 & 1 & 0 & 1 & 0 \\
0 & 2 & 0 & 2 & 3 & 3 \\
1 & 0 & 1 & 0 & 8 & 0 \\
0 & 1.2 & 0 & 1.8 & 0 & 7 \\
5 & 1 & 0 & 2 & 2 & 0
\end{bmatrix}.
$$

For the leader-following coordination problem, we consider a team consisting of six followers and one leader. The adjacency matrix $\mathscr{A}$ associated with the six followers is defined as

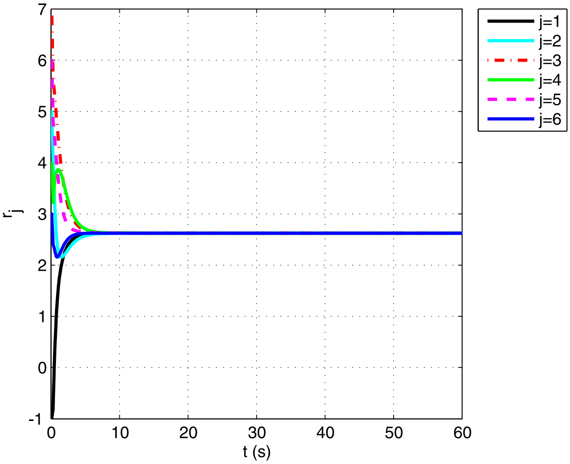

Fig. 10.1 Agents' positions using (10.1)

$$\mathscr{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 8 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1.2 & 0 & 1.8 & 0 & 7 \\ 5 & 1 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

We also let $a_{10} = 1$, $a_{30} = 4$, $a_{40} = 8$, and $a_{i0} = 0$ otherwise.

For single-integrator dynamics, we choose $r(0) = [-1, 5, 7, 4, 6, 3]^T$, where r is a column stack vector of all r_i, $i = 1, \ldots, 6$. For (10.1), we let $\Delta_{ij} = 0$ for simplicity. For (10.10), we let $r_0 = 3.5$. For (10.13), we let $r_0(t) = 3.5 - 4\cos(\frac{t}{4})$. For all (10.1), (10.10), and (10.13), the input delay and the communication delay are chosen as, respectively, $\tau_1 = 0.1$ s and $\tau_2 = 0.2$ s. For (10.16), we let $r_0(t) = 3.5 - 4\cos(\frac{t}{4})$ and the communication delay be $\tau_2 = 0.2$ s.

Figures 10.1, 10.2, 10.3, and 10.4 show the positions of the agents using, respectively, (10.1), (10.10), (10.13), and (10.16). It can be seen from Figs. 10.1 and 10.2 that the agents achieve, respectively, leaderless coordination and coordinated regulation. In the case of coordinated tracking, Figs. 10.3 and 10.4 show that the tracking errors are uniformly ultimately bounded due to the existence of the delays and the fact that the leader is dynamic.

For double-integrator dynamics, we choose $r(0) = [-0.4, 0.5, 0.7, 0.4, 1.2, 0.3]^T$ and $v(0) = [-0.1, 0.2, 0.7, 0.4, -0.1, 0.3]^T$. For (10.18), we let $\Delta_{ij} = 0$ for simplicity. For (10.23), we consider two subcases. In one subcase, we let $r_0 = -0.2$ and $v_0 = 0$. In the other subcase, we let $r_0(t) = -0.2 + 0.1t$ and $v_0 = 0.1$. For (10.27), we let $r_0(t) = -0.2 + 0.3t - 1.6\sin(\frac{t}{4})$ and $v_0(t) = 0.3 - 0.4\cos(\frac{t}{4})$. For all (10.18), (10.23), and (10.27), we choose $\tau_1 = 0.3$ s and $\tau_2 = 0.1$ s. For (10.29), we let $r_0(t) = -0.2 + 0.3t - 1.6\sin(\frac{t}{4})$, $v_0(t) = 0.3 - 0.4\cos(\frac{t}{4})$, and $\tau_2 = 0.1$ s.

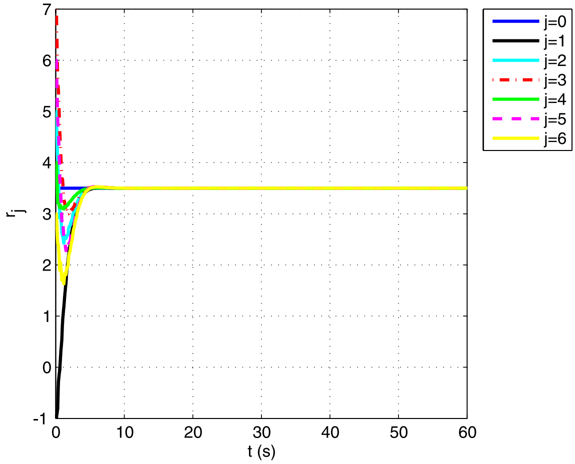

Fig. 10.2 Agents' positions using (10.10)

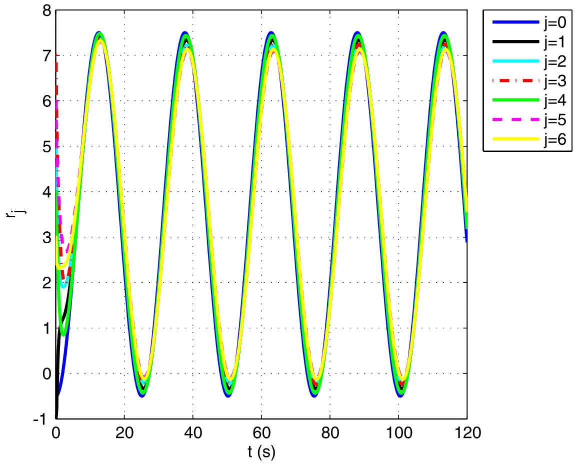

Fig. 10.3 Agents' positions using (10.13)

Figure 10.5 the positions and velocities of the agents using (10.18). It is interesting to notice that unlike the result using the standard consensus algorithm with relative damping for double-integrator dynamics [248, Chap. 4], the final velocities of the agents are always dampened to zero rather than a possibly nonzero constant. Figures 10.6 and 10.7 show the positions and velocities using (10.23) with, respectively, $v_0 = 0$ and $v_0 = 0.1$. It is worth noticing that when v_0 is a nonzero constant (respectively, zero), the tracking errors $r_i(t) - r_0(t)$ approach constant values (respectively, zero). Figures 10.8 and 10.9 show the positions and velocities using, respectively, (10.27) and (10.29). The tracking errors are uniformly ultimately bounded due to the existence of the delays and the fact that the leader is dynamic.

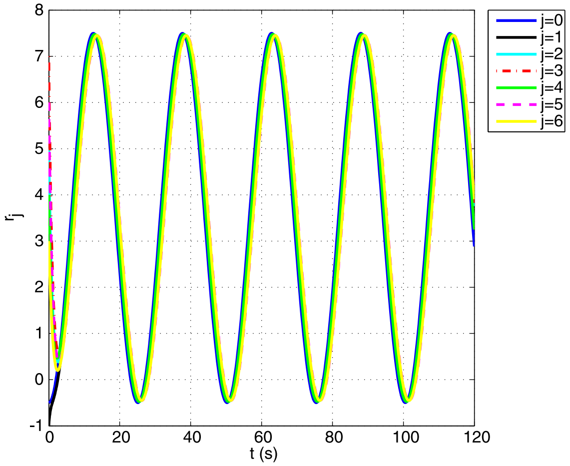

Fig. 10.4 Agents' positions using (10.16)

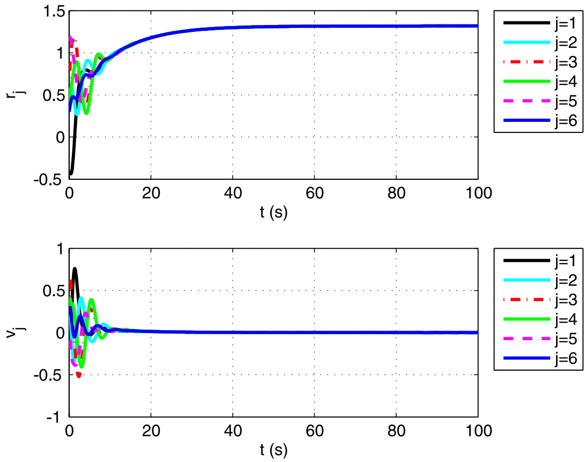

Fig. 10.5 Agents' positions using (10.18)

10.5 Notes

The results in this chapter are based mainly on [190, 191]. For further results on distributed multi-agent coordination with time delays, see [126, 174, 192, 199, 206, 214, 228, 264, 283, 285, 290, 291, 316, 322, 323]. A frequency-domain approach is used in [214] to find the stability conditions for a leaderless coordination algorithm with input delays. A time-domain approach based on Lyapunov–Krasovaskii theorem is adopted in [174] to obtain the stability conditions for a similar leaderless coordination algorithm with uniform input delays under a strongly connected and balanced interaction graph. Besides leaderless coordination algorithms, leader-

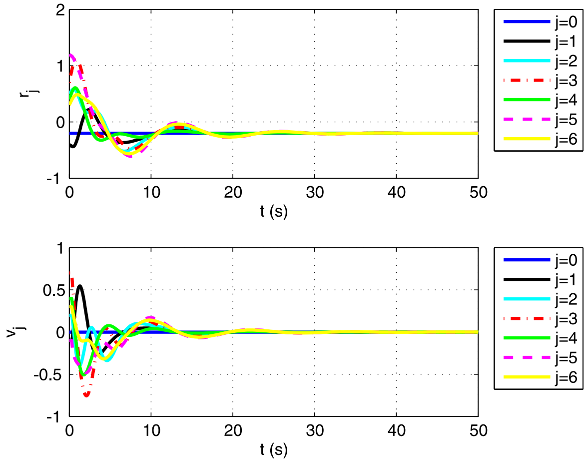

Fig. 10.6 Agents' positions using (10.23) with $v_0 = 0$

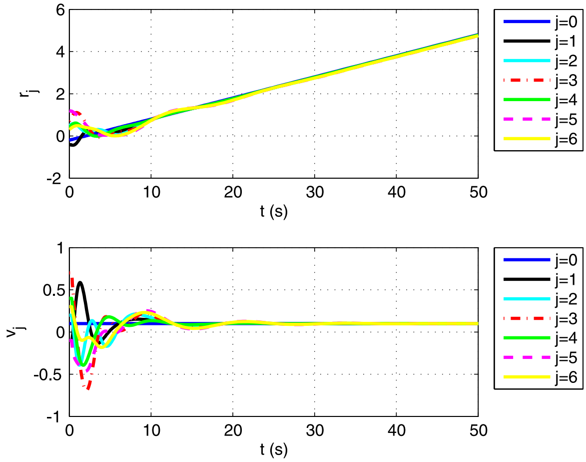

Fig. 10.7 Agents' positions using (10.23) with $v_0 = 0.1$

following coordination algorithms with input delays are also studied. By combining the results in [174] and [120], the authors in [228] propose a first-order coordinated tracking algorithm with input delays, where an estimator is used to estimate the leader's velocity. In [174, 214, 228], the interaction graph is assumed to be either undirected or strongly connected and balanced. The extension to the case where the interaction graph has a directed spanning tree and the input delays are non-uniform is provided in [291], where a frequency-domain approach is adopted to find conditions to achieve leaderless coordination. Except for input delays, the influence of communication delays on coordination algorithms is also studied. It is shown in [199] shows that communication delays will not jeopardize the stability

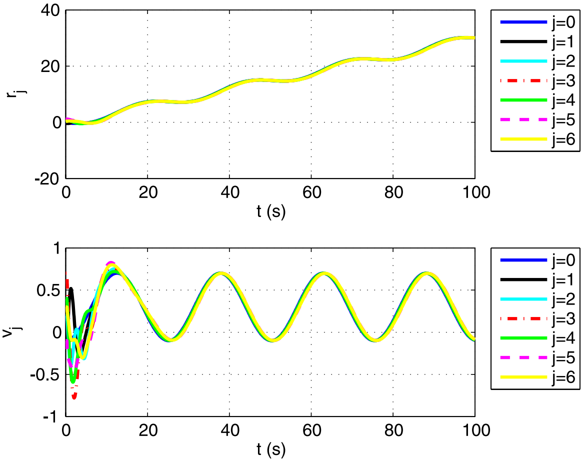

Fig. 10.8 Agents' positions using (10.27)

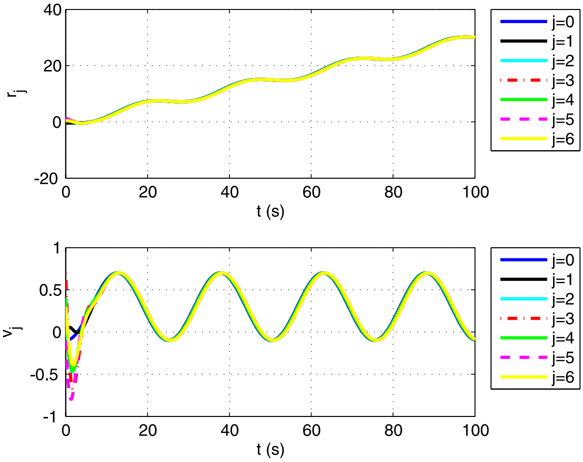

Fig. 10.9 Agents' positions using (10.29)

of a first-order leaderless coordination algorithm under a directed interaction graph. A similar algorithm is discussed in [264], where the effect of the initial conditions is highlighted. A second-order coordinated regulation algorithm with non-uniform communication delays is studied in [206], where a damping term is used to regulate the velocities of all agents to zero and the interaction graph is assumed to be undirected. The case with both communication and input delays is studied in [290], where a first-order leaderless coordination algorithm is studied in a discrete-time setting by a frequency-domain approach.