Mathematical and Physical Properties of Reliability Models in View of their Application to Modern Power System Components

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Abbreviations and list of symbols

ALT	Accelerated life test
a.s.	Almost surely
BS	Birnbaum–Saunders (distribution)
cdf	Cumulative distribution function
CLT	Central limit theorem
CV	Coefficient of variation
CRF	Conditional reliability function
DHR	Decreasing hazard rate
DRA	Direct reliability assessment
D[Y]	Standard deviation of the RV Y
E[Y]	Expectation of the RV Y
EV	Extreme value (distribution)
$F(x)$	Generic cdf
$f(x)$	Generic pdf
$f(x)$, $F(x)$	pdf and cdf of stress
$g(y)$, $G(y)$	pdf and cdf of strength

Abbreviations and list of symbols: The singular and plural of names are always spelled the same. "Log" always denotes natural logarithm. Random variables are denoted by uppercase letters.

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$G(r, \phi)$	Gamma distribution with parameters (r, ϕ)
$H(\ )$	Cumulative hrf
hrf	Hazard rate function
HRM	Hyperbolic reliability model
HV	High voltage
IDHR	First increasing, then decreasing hazard rate
IG	Inverse Gaussian (distribution)
IHR	Increasing hazard rate
IID	Independent and identically distributed (random variables)
IPM	Inverse power model
IRA	Indirect reliability assessment
IW	Inverse Weibull (distribution)
LL	Log-logistic (distribution)
LN	Lognormal (distribution)
LT	Lifetime
MRL	Mean residual life
MV	Medium voltage
$N(\alpha, \beta)$	Normal (Gaussian) random variable with mean α and standard deviation β
pdf	Probability density function
$r(s)$	Mean residual life function at age s
RF	Reliability function
$R(t)$	Reliability function at mission time t
$R(t\|s)$	Conditional reliability function at mission time t, after age s
RV	Random variable
SD, σ	Standard deviation
s-independent	Statistically independent
SP	Stochastic process
SS	Stress-strength
Var, σ^2	Variance
$W(t)$	Wear process at time t acting on a device
$W(a, b)$	Weibull model with RF: $R(x) = \exp(-ax^b)$
$W'(\alpha, \beta)$	Alternative form of Weibull model, with RF: $R(x) = \exp[-(x/\alpha)^\beta]$
$\delta(\cdot)$	Dirac delta function
X, $X(t)$	Stress (RV or SP)
Y, $Y(t)$	Strength (RV or SP)
$\Gamma(\ )$	Euler–Gamma function
$\Gamma(,)$	Incomplete gamma function
μ	Mean value (expectation)
$\Phi(z)$	Standard normal cdf
$\varphi(z)$	Standard normal pdf

1 Introduction

1.1 A Premise

In this introduction, particularly from Sect. 1.3 on, reference will be made to some mathematical probability theory which is thought to be useful for the methodological development of the sequel. Since only some particular aspects are highlighted here, it is assumed the reader being acquainted with the analytical definitions and basic properties in the theory of random variables and stochastic processes, with particular reference to the *reliability function* and related functions, such as the *hazard rate function*, the *mean residual life* (MRL) *function,* the *conditional reliability function* (CRF), which will be briefly recalled below. Suggested books in reliability theory are,[1] e.g., [3, 6, 10, 11, 16, 21, 30, 83, 88, 97, 103, 113, 123, 124, 146, 151, 152], for the above basics and also further details. It must be highlighted that the literature in this field is huge and fast growing in the large number of applicative fields, so only some of the most representative papers or books will be cited in the chapter. Some of the references are not mentioned in the text.

The need to develop new methodologies for reliability estimation of technological products is becoming increasingly important in every field of engineering. This is due to the combination of two main reasons: one is the fast extension of liberalization (which is discussed below with reference to the power systems); the other is the ever-increasing level of technological innovation, which brings about higher and higher reliability values for components, thus implying scarcity of failure data. Also, as pointed out in [68], the tendency to very short product-development times and tightened budgets imply that reliability tests must be conducted with severe time constraints, so that frequently no failures occur during such tests. These aspects are widely recognized in literature in almost all technological fields, from electrical [107] to marine [22] engineering. Therefore, the opportunity arises to adopt models and methodologies allowing the most efficient reliability estimation, based not only on the experimental data, but also on technological and physical information usually available to the engineer or analyst, as highlighted in some recent literature [42, 75, 76, 81, 158]. This information is related to wear and stresses acting on the device, e.g., overvoltages or short-circuits in the case of electrical components—and/or its typical models of aging, as expressed mathematically, e.g., by the time behavior of its hazard rate function: as a trivial example of the latter aspect, one may consider that most electronic components are scarcely affected by aging, being their failure largely “accidental”, so that a constant hazard rate function could be reasonably expected.

[1] The references at the end of the chapter are listed alphabetically, due to the length of the Bibliography, for an easier search by the reader. They are referred to in the chapter by their number.

This leads to the widespread adoption of an Exponential reliability model in such field, even in the absence of many data to support it on a statistical basis.

These two aspects (physical evolution of wear and mathematical models of aging) are of course closely related with each other, but in a way that may sometimes appear divergent from—if not contrary to—“intuition”. For instance, this chapter points out that many lifetime distributions deduced by wear models evolving as a random increasing function of time (such as the Gaussian processes leading to the Inverse Gaussian or the Birnbaum–Saunders distribution) are characterized by a decreasing hazard rate for large values of service times; instead, an increasing hazard rate function with time could be expected on purely intuitive grounds (as erroneously reported in some books). This is a property which is seldom discussed in the relevant literature, but it has found experimental evidence, e.g., in the field of electrical insulation [93, 148], and has been debated theoretically [93, 146]. Nevertheless, it is still misunderstood in some recent texts. This fact is not surprising, since many examples are found in probability theory that lead to results appearing, at first sight, illogical or paradoxical even to academics, sometimes[2] nonetheless, it highlights the need to acquire further insight into aging properties, wear mechanisms and the relations between them.

The use of “prior”[3] technological and physical information leads—as discussed throughout the chapter—to the so-called “indirect reliability assessment” (IRA). Such approach is so denoted in that it infers lifetime characteristics from the properties of the stochastic process (SP) describing the wear affecting the device, rather than using statistical fitting, which may result poor due to the limited number of data.

Such methodology based upon “prior information” may be perhaps paralleled to Bayesian estimation methodology in reliability, which—after a first popular systematic treatise in [113]—is fast developing [146, 147]. This is just an aspect of what is happening in all the fields related to inference, and caused Press in a recent important book affirm that “many believe that a paradigm shift has been taking place in the way scientific inference is carried out, away from what is sometimes referred to as classical, or frequentist, statistical inference… (toward Bayesian inference)” [135].

In a different setting, indeed, the Bayesian approach—as well known [14, 57, 113, 132, 135, 138, 146, 147]—typically uses prior information for assigning prior distributions to unknown parameters. Often, indeed, such kind of prior information is deduced by technological information, so that Bayesian estimation has found many applications in recent reliability studies, also for its proved big efficiency—with respect to “classical” statistical inference, which is

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[2] A significant citation from C.S. Pierce comes to mind: “Probability is the only branch of mathematics in which good mathematicians frequently get results which are entirely wrong”.;

[3] Here, the term “prior” means, loosely, something not closely related to observed data, but coming from pieces of information “outside the data”, in analogy with Bayesian estimation terminology.

mainly based upon data [139]—when only very few experimental data are accessible. Of course—apart from the different framework (probabilistic modeling for IRA, statistical inference for Bayesian estimation) a basic difference between the two approaches lies in the so-called (and often criticized) "subjectivity" of Bayesian statistics, which is indeed a key aspect of such methodology, often taken to its extreme limits from its more influential adherents, such as the big names of De Finetti [61] and Lindley [108]. The information which constitutes the foundation of IRA is instead generally to be considered as "objective". This is of course not the place to discuss epistemological problems, as if there exists a clear cut difference between "subjectivity" and "objectivity" (which is questionable in our opinion), but it has to be remarked that a branch of Bayesian statistics is recently making some effort toward so-called objectivity, as can be deduced even from the title of [135]: "Subjective and objective Bayesian statistics".

We agree with [30] when stating that, although the subjective meaning of probability is considered unorthodox in many empirical fields, "it is useful in reliability studies since quantification of the degree of belief is essential in using all available information when experimental data are scarce" (p. 232).

The above interpretations and discussions may be of not so much interest in view of practical engineering applications: indeed, in many applied engineering studies, Bayesian estimation is adopted for "ad hoc" reasons, without the necessity of adhering to its philosophy (even if this "non-adherence" may be questionable on theoretical grounds). As clearly reported in [71]—which is a key paper for the application of technological information to prior assessment (for the Weibull parameters, in that case)—"…when only very few (e.g., 3–5) experimental data are accessible… the controversy about whether to use Bayes or classical methods is surmounted since classical estimators, like maximum likelihood, give estimates that often appear unlikely on the basis of technical knowledge of the engineers."

Concerning IRA in itself, although its roots can be traced back in time to the classic book of Miner [128] or to [82], such studies were mainly theoretical, and continued to be such in the 1970s, as can be seen in papers such as the fundamental [77], which—although revealing itself as a seminal paper in the future—appeared indeed in a purely mathematical review ("*Annals of Probability*"), not typically read by many engineers. The first engineering applications came from the mid-1980s on, with fundamental papers such as [59, 60, 62, 70]. In particular, the original work of Erto in this field will be referenced to in the chapter, since it spans the last two decades (from the above-cited [70] to the very recent [75]) in researches leading to new physical motivations—and finding new properties—for various reliability models, such as the rarely adopted Inverse Weibull of the above papers or the new Hyperbolic Model [76]. By the way, none of these two models, which can be both derived by postulating a wear process, has an increasing hazard rate function $h(t)$ (the Inverse Weibull has a first increasing, then decreasing hrf; the Hyperbolic Model has a decreasing hrf), in accordance with what above discussed. It may not be a chance that, as above hinted at, the same author used an analogous technological approach in various applications of Bayesian inference to reliability, e.g., [69, 71, 72, 73].

Finally, among the vast bibliography it is obvious to quote [68], which apparently introduced the term "IRA". Recently, such methodology has reached high levels of mathematical sophistication by the use of advanced properties of SP (see, e.g., [81, 105, 153, 154], or some contributions in a recent book edited by Erto [74]). These approaches have even brought about new studies in theoretical mathematics and other related fields ([47], see also Sect. 3.3.12).

In the specific field of electrical engineering applications, apart quoting the same references as above, the advanced state of the art in the field of electrical insulation has to be highlighted, as witnessed, say, by [119] and in more recent times by [127]. Many more specific contributions in this field will be referred to in the appropriate Sect. 4.

In this respect, the present chapter is primarily devoted to reliability assessment of modern power system components, using some of the methods proposed by the authors of this chapter in previous papers [37, 39, 40, 42]. Indeed, the present chapter takes its origin from a previous paper published in 2008 by the same authors [42]. In addition, more models are here discussed, some of which recently developed, such as the "hyperbolic reliability model" (HRM) [76]; also some brand-new results about known models, e.g., the Inverse Weibull model [75], are hinted at. Moreover, in the chapter it is shown how popular reliability models, like Gamma, Weibull, etc., can be obtained in a straightforward way, that turns out to be particularly useful for engineering applications.

It is remarked that the present chapter is only devoted to reliability modeling of single components. So, the problem of reliability evaluation of the power system as a whole is deliberately not dealt with: excellent books such as, e.g., the ones by Billinton and Allan [16, 17] and Billinton et al. [18] are available on the subject, not to mention seminal paper such as [19]. In some cases, some of the results of the present chapter may be used perhaps to discuss the adequacy, when analyzing the whole power system, of the use of the Exponential model for any single component, a model widely—if not uniquely—adopted, for evident reasons of simplicity.

1.2 Outline of the Chapter

As discussed in [42] and mentioned above, the new deregulated market of electrical energy and the restrictions imposed by economic constraints to production times and preventive maintenance programs have stressed the need for a careful reliability analysis of electrical components. This emphasizes, on one hand, the problem of an economically optimized design of components in view of the stress levels that are expected in-service and, on the other hand, the issue of a careful conditional reliability or quantiles (percentiles) evaluation, in view of optimizing maintenance procedures. However, as previously mentioned, the deregulation also involves a greater uncertainty or lack of data in system operation and management, due to both the use of highly reliable components (characterized by a considerable level of technological innovation, and, consequently, by a high degree of reliability and large costs) and the expected very fast variability of network configurations. This makes the development of timely and economically adequate maintenance procedures more difficult.

In this framework, the aging failures of system components are a major concern and a driving factor in system planning of many utilities. Indeed, more and more system components are approaching their end-of-life stage, hence aging failures should definitely be included in power system reliability evaluation in order to avoid a severe underestimation of the system risk, as shown in [107], where ad hoc methods to incorporate aging failures in power system reliability evaluation are presented.

For all these reasons, the choice, selection, or estimation of adequate probabilistic models for the assessment of the residual (or "conditional") reliability of electrical components is the first, and often the most critical part, of any statistical investigation devoted to system reliability analysis.

The "classical" models for component reliability estimation are based on direct statistical fitting of component failure data coming from the field. This is generally accomplished in two stages:

- first, a model is selected on the basis of "goodness of fit" statistical tests, such as Kolmogorov–Smirnov test, Chi-square test, etc. [139];
- then, its parameters are estimated by well known methods such as the "Maximum likelihood" (ML) one. This so-called "direct reliability assessment" (DRA) is commonly used at the maintenance stage of devices that are already in-service. Nevertheless, such fitting may result poor due to the limited number of data for modern components, as previously mentioned.

From this respect, a help could come from the knowledge acquired over the years about the physical processes that are responsible for the degradation of materials that compose the aging electrical devices. In particular, since insulation is often the weakest part of an electrical device—particularly in medium voltage (MV) and high voltage (HV) systems—phenomenological and physical aging and life models of electrical insulation (that can be found in the vast scientific literature about this subject) can be used for achieving a so-called IRA [68]. IRA can be an effective tool for reliability evaluation of a large series of components, such as transformers, cables, motors, capacitors, etc.; it may be noteworthy that such approach is rooted on classic studies on the physics of failure, as the ones of Dasgupta and Pecht [60], conceived outside the electrical engineering field.

This is the perspective from which the problem of the selection of adequate probabilistic models for reliability assessment of power system components is analyzed in this chapter.

First, in Sect. 2, the most adopted reliability models in the literature about electrical components are synthetically reviewed and the classical DRA, i.e., reliability assessment via statistical fitting directly from in-service failure data of components, is illustrated, that is commonly used at the maintenance stage of components already at work. The properties of these models, as well as their practical consequences, are discussed, thereby arguing that direct fitting of failure data may result poor or uncertain due to the above discussed limitations. Thus, the selection—or the correct identification—of a suitable probabilistic model for power system component reliability in the field of high-reliability devices and large mission times should be better supported by probabilistic information that leads to reasonable modeling, as those coming from the study of the phenomenology and the physics of aging in the already-mentioned example of MV and HV components. The opportunity of using such kind of information for these components, pointed out in recent literature [37, 38, 59, 68], moves the treatment from "direct" to "indirect" reliability estimation. For this purpose, in Sect. 3, the main stochastic models for IRA are discussed, denoted as "stochastic wear models"—which include the "Degradation", "Stress-Strength" and "Shocks" failure models—showing how they can originate particular reliability models, thereby giving further support to the adoption of a given model (e.g., the Weibull one), beyond the simple DRA, which can be not only unsatisfactory, but even misleading.

Following the same approach with reference to electrical devices, in Sect. 4, reference models developed over the years for the estimation of insulation time-to-failure (life) and aging are illustrated, that are based mainly on experimental results coming from laboratory tests carried out on specimens. When inserted in a proper probabilistic framework, they give rise to "physical reliability models", that are usually employed for a preliminary characterization and comparison of the various materials candidate for the realization of the insulation of electrical components, as well as for the design of the insulating systems of such components. However, they can provide useful guidelines also for reliability estimation from in-service failure data and this closes the "loop" between direct and indirect reliability estimation of electrical components.

Up to this point, the discussion has a prevailing methodological aspect, referring to a vast bibliography for the numerical applications to electrical devices. Then, in the final Sect. 5—in order to better highlight the possible pitfalls brought about by DRA from an applicative and numerical point of view—it is shown, by means of numerical and graphical examples referred to typical insulation data—that seemingly similar models can possess very different lifetime percentiles, CRFs and hazard rate functions. Thus, the power system engineer must be aware of the "mathematical" consequences of the selected models, particularly in view of their aging properties. As already stated in the introduction, availability of repairable components is not considered here, only for space limitations. It is dealt with in many of the books referred above, and is thoroughly discussed in [4]. However, it is obvious that the assessment of a reliability model for a unit considered as a non-repairable unit is also the crucial starting point for its availability assessment when it is instead repairable. Moreover, anyone of the reliability features here discussed with reference to lifetimes have a natural correspondent for the RV "times to repair".

1.3 Reliability Function and Other Measures of Aging: Hazard Rate Function, CRF, MRL Function

Starting from the above considerations, the chapter tackles the overall problem of the assessment of the reliability function (RF) of a given component, considered as a non-repairable unit. It is assumed that the reader is familiar with the basic concepts of probability and statistics, such as properties of the cumulative distribution function, probability density function (pdf), and other relevant concepts and definitions, such as the moments and the quantiles of a random variable (RV), and also of the basic facts of estimation theory [7, 25, 134, 139, 141].

Denoting by T the non-negative RV "time to failure", or "lifetime" (LT), of the component and by $F(t)$ its cumulative distribution function (cdf), the RF is defined as follows:

$$R(t) = P(T > t) = 1 - F(t) \tag{1}$$

being $P(A)$ the probability of the generic random event A. The above RF is sometimes denoted also as "survival function" in literature.

The RV T is taken as continuous and distributed according to a pdf $f(t)$ such that:

$$f(t) = \mathrm{d}F/\mathrm{d}t = -\mathrm{d}R/\mathrm{d}t \tag{2}$$

and

$$R(t) = \int_t^{\infty} f(u)\mathrm{d}u. \tag{3}$$

It is remarked that the RF $R(t)$ and the cdf $F(t)$ are relevant to a time interval, and not to the end point t of the interval, as the notation seems to imply: e.g., $R(t)$ is the probability that the device operates successfully in the *whole* interval (0, t).

The expectation of the LT, denoted as MTTF (mean time to failure) can be obtained, provided that the integral exists, by:

$$E[T] = \int_0^{+\infty} R(t)\mathrm{d}t. \tag{4}$$

Basic facts about aging that are sometimes misunderstood (as discussed) are presented here, without claiming to be exhaustive (again, the reader should consult [4, 16, 21, 30, 146], and other fundamental works mentioned above).

Reliability theory is, from a purely mathematical point of view, a sort of applied probability theory devoted to the study of positive RV. However, it possesses some peculiar functions and parameters which are defined ad hoc to describe RV representing times (in particular LT, but not only), and which do not have, in practice, counterparts in other branches of probability theory.

The most popular and the most used (and sometimes even abused and misused, as we shall see) to describe the aging of the devices is the hazard rate function (hrf). Differently from the RF and the cdf, which are, as remarked above, relevant to a time interval, the hrf $h(s)$ is relevant to the instant $s \geq 0$, to be intended as the "age" of the device; it represents, in a sense, the "instantaneous failure rate" at a given point in time. Formally, if the LT, T, possesses a pfd $f(t)$—as will be tacitly assumed throughout the chapter—the hrf is defined, at any time $t \geq 0$ for which $R(t) \neq 0$, as:

$$h(t) = \frac{f(t)}{R(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}[\log(R(t))]. \tag{5}$$

The "physical" meaning of hazard rate function $h(t)$, as well as the origin of its name, lies in the following property, which is easily seen to be equivalent to the above definition:

$$\begin{aligned} h(x) &= \lim_{\Delta x \to 0^+} \frac{P\{(x < T \leq x + \Delta x)|(T > x)\}}{\Delta x} \\ &= \lim_{\Delta x \to 0^+} \frac{F(x + \Delta x) - F(x)}{\Delta x \cdot R(x)}. \end{aligned} \tag{6}$$

So, as $\Delta x \to 0^+$, the product $h(x)\Delta x$ equals the *conditional* probability that the failure occurs in the interval $(x, x + \Delta x)$, given that the device has survived until age x; i.e., such product may be interpreted as the *instantaneous* failure (conditional) probability for a device of age x.

From the above definitions, and the obvious condition $R(0^+) = 1$, it is possible to deduce the following integral relation which allows to express the interval RF, $R(t)$, in terms of the instantaneous hrf:

$$R(t) = \exp\left(-\int_0^t h(\xi)\mathrm{d}\xi\right), \quad t > 0. \tag{7}$$

Therefore, any reliability model is fully specified once either its pdf, or its RF or its hrf is given, as each pair of these three quantities is directly deducible from the remaining: e.g., starting from $h(t)$, whose parametric form can be sometimes derived from its physical meaning, the RF is obtained from (7). Then, the pdf can be attained from the following relationship:

$$f(t) = h(t)R(t). \tag{8}$$

Further, also the cdf, MTTF, etc., are easily obtainable, e.g., this is the most direct way to introduce the Weibull model (by far the most adopted in applied reliability studies) characterized by the following hrf, RF, pdf as functions of time $x > 0$, with positive parameters (a, b):

$$h(x) = abx^{b-1} \tag{9a}$$

$$R(x) = \exp\left(-ax^{b}\right) \tag{9b}$$

$$f(x) = abx^{b-1}\exp\left(-ax^{b}\right). \tag{9c}$$

In the sequel, the above model will be denoted by the symbol $W(a, b)$; it covers the popular Exponential model when $b = 1$. As well known and discussed in any textbook on the discipline, the behavior of the hrf in time may provide insight as to what is causing the failures. Indeed, a decreasing hrf suggests "infant mortality" or "wear-in", i.e., defective items fail early because of frailty, production defects, etc., and the overall hrf decreases over time as they fall out of the population. A constant hrf rate (which is peculiar of the Exponential model alone) suggests that the device fails, irrespectively of its age, because of random "accidents". An increasing hrf rate suggests that the device is subject to "wear-out", so that it is more and more likely to fail as time goes on. Experimentally, these three kinds of behavior (decreasing, constant and then increasing hrf) are, for many (but not all) products, observed to occur in succession during the whole product life, describing the so-called "bathtub curve" of the hrf. This is a very popular curve which is widely discussed in any book on the matter (see [146]), and in some detailed analytical papers, such as the article by Glaser [84].

In order to better understand the meaning of the above possible behaviors of the hrf, and considering that the hrf is not a probability (and neither a conditional probability),[4] it is perhaps preferable to introduce the "conditional reliability function" (CRF), which is a function of two-time variables defined as follows[5]:

$$R(t|s) = P\{T > s + t|T > s\} = \frac{R(t+s)}{R(s)}\,(t, s \geq 0). \tag{10}$$

The above CRF, denoted as the *CRF* for a mission time t of a device of age s, equals the *conditional* probability that, the device having survived until age s, it will survive at least until time $(s + t)$, i.e., its age will be increased at least of t time units after age s. This is why it could be also denoted as "residual reliability function".[6] It is obvious that the above CRF, with respect to the time argument t, must behave as a RF, e.g., it must satisfy (at any age s): $R(0^{+}|s) = 1$,

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4 It should be clear, by the way, that the hrf must be positive, but not necessarily less than 1: it can even diverge, as happens for models possessing a pdf which vanishes at some finite point in time, as the Uniform model. It has little to do with a pdf, too: e.g., its integral over the whole interval $(0, \infty)$ must be ∞, since $R(\infty) = 0$, etc.

5 The notation $R(t|s)$ is purely symbolic, being used for suggesting the conditional aspect of the RF, and should not be confused with the *conditional* probability $P(A|B)$, the main difference being that (r, s) are deterministic numbers, while (A, B) are random events.

6 This latter would probably be a better name: here we use the term "conditional" instead of residual since it is more adopted in literature.

$R(\infty|s) = 0$, and be decreasing with t. Its behavior with age s might appear less obvious. For instance, one could naively expect that, since age should weaken every object (an indubitable fact), $R(t|s)$ is a decreasing function of s. However, it is not always so; indeed, such reasoning would ignore the fact that the CRF is a *conditional* probability, and that such conditioning may significantly change our information: e.g., sometimes, knowing that a device has survived until age s may render us more confident in its "future" survival than we could be without that information, so that the device *appears* to "strengthen" with age, and the CRF may increase with s. For instance, the CRF $R(t|s)$ increases with s—which is closely related, as remarked, to a decreasing hrf—during the abovementioned "infant mortality" period in the early life of a product, or in "accelerated life tests" (ALT), or when the LT is generated from mixtures ([10], p. 55; such property is also recalled here at the end of Sect. 3). The idiomatic expression "the device appears to strengthen with age" should be interpreted cautiously, in terms of "change of information" rather than effective strengthening of the object, a fact which of course seldom occurs in practice.[7] The statement, if not correctly interpreted (with due emphasis on the word "appears"), may seem to conflict with the obvious property that the RF *must* always decrease with time. This concept will be discussed again later.

In many cases, devices weaken (again, in a "conditional" way) with age: this is generally thought to be typical, e.g., of mechanical devices subjected to increasing wear as they work, and in practice for *all devices* (also electronic ones)—if they should be left to operate indefinitely—when their age is large enough. However, some already hinted at examples of aging related to wear (as will be also discussed in Sect. 2) should render us careful also with this observation. There is no doubt, instead, for what concerns the human beings and living organisms: their hrf $h(s)$ increases and their CFR $R(t|s)$ decreases with age s. Soon it will be recalled that the two properties are indeed equivalent.

Another measure of aging is the *MRL*. It appears to be very useful, although not so popular (strangely, since it has a clearer physical meaning than the hrf). The MRL, $r(s)$, is a function of time (age) s representing the expected residual lifetime of a device that has reached age s. So, it is a "conditional expectation", i.e., the mean value of the "residual" LT at age s—namely, the difference $(T - s)$—conditional to the event $(T > s)$. So, $r(s)$ is defined as:

$$r(s) = E[(T - s)|T > s] \tag{11}$$

and it is computable in terms of the CRF as follows:

[7] When discussing aging and hrf and CRF properties, in the authoritative [10] it is observed that "certain materials increase in strength as they are work-hardened" (p. 55). This may be true, but it is unlikely that it holds for very long time intervals: wear-out should ultimately prevail for any device, corresponding to a CRF $R(t|s)$ decreasing with s, for s large enough. Anyway, it is possible that in practice the device is maintained or retired before wearing-out, so that the ultimate, decreasing part of the CRF is not observed.

$$r(s) = \int_0^\infty R(t|s)\mathrm{d}t = \int_0^\infty \frac{R(t+s)}{R(s)}\mathrm{d}t. \tag{12}$$

It appears that $r(s)$ is increasing, constant or decreasing with age s in the same way as the CRF. Also, it can be easily shown that $r(s)$ uniquely specifies the reliability model, in that RF, pdf and hrf can be uniquely expressed in terms of $r(s)$ [30].

1.4 On the Relation Between Hazard Rate and CRF

Previously, some discussions on the physical meaning of the hrf behavior in time, deriving from the observation of the "bathtub curve", have been intuitively explained in terms of the CRF, $R(t|s)$, which possesses an easier interpretation being a probability (differently from the hrf). The CRF behavior versus age s is indeed univocally related to the hrf behavior. Using the above relations between the RF and the hrf, it is not difficult to show, for any LT distribution for which the hrf is defined, the validity of the following equivalences (see, e.g., [10, 30]), whose statements are assumed to hold for each value of $s > 0$, and for any given mission time t, which is to be intended as a constant in the right-hand side of the equivalences:

1. hrf $h(s)$ increasing with $s \Longleftrightarrow$ CRF $R(t|s)$ decreasing with s (at any given time t).
2. hrf $h(s)$ decreasing with $s \Longleftrightarrow$ CRF $R(t|s)$ increasing with s ("").
3. hrf $h(s)$ constant with $s \Longleftrightarrow$ CRF $R(t|s)$ constant with s ("").

As well known, property (c) uniquely characterizes the Exponential model, and assesses its being "memoryless". Property (a) explains why the hrf increases for devices subjected to wear; property (b) explains why the hrf decreases for devices subjected to "infant mortality" (or for devices strengthening with age). From the above properties, the abovementioned bathtub curve, as well as any hrf behavior, can be easily interpreted in terms of CRF properties, an approach which, surprisingly, is seldom found in the relevant applied literature: in most cases the CRF is not even mentioned, or finds much less space than the hrf when introducing the study of aging. For brevity, it is instead reasonable, having interpreted the hrf behavior with the help of the CRF, to use then the hrf $h(t)$ alone. Indeed, it is a function of a single time variable (just as the MRL), it is more easy to be represented graphically, and various procedures for its statistical estimation have been devised. So, also here in the sequel, only the hrf expression, if available, will be reported when discussing the various models. Nonetheless, a look at the CRF curves of the various models—as done in Sect. 5 (see Figs. 6, 7)—could be very helpful for a better understanding of their aging properties.

For the abovementioned properties of the MRL: in case (a) the MRL $r(s)$ is decreasing with s; in case (c) the MRL $r(s)$ is constant with s (so, it is equal to the MTTF; again, such independence from s uniquely characterizes the Exponential model); in case (b) the MRL $r(s)$ is increasing with s. The converse implications are not trivial, and require further assumptions [30].

At this point, it is useful, as done in most books, to operate a classification of reliability models in terms of aging, based upon hrf or CRF properties. First, we notice that we used above the symbol s to denote "age" (a past time), the symbol t to denote a mission time (a future time), so that the symbol s has the same meaning in the hrf $h(s)$ and in the CFR $R(t|s)$; although using the same symbol t to denote time in $h(t)$, $R(t)$ and $R(t|s)$ (as done in most books) would not be an error, nevertheless it may easily induce the reader into confusion for what concerns the relation among these quantities. As long as only the mathematical properties of the hrf are of interest, in Sect. 2 and in the sequel, we shall use $h(t)$ as it is customary in literature.

1.5 A Classification of Reliability Models: IHR, IDR, IDHR Models

A reliability model (or, for brevity, the LT described by such model) is defined as:

- "increasing hazard rate model" (IHR), if its hrf $h(s)$ is an increasing *function* of $s > 0$ (over the whole domain of the hrf, generally $0 < s < \infty$); example: Weibull $W(a, b)$ with $b > 1$.
- "decreasing hazard rate model" (DHR), if its hrf $h(s)$ is a decreasing *function* of $s > 0$ (*idem*); example: Weibull $W(a, b)$ with $b < 1$.
- "increasing, then decreasing hazard rate model" (IDHR), if its hrf $h(s)$ firstly increases with $s > 0$, from $s = 0$ to a given point s^*, then it *decreases* with s; examples: Lognormal and Inverse Weibull models.

The above models are recalled in more detail in Sect. 2. In case of IDHR models, also the denominations: "unimodal hrf", or "reverse bathtub-shaped hrf" are used. As reported in [93], and also illustrated for some of the basic models here presented, the LT is often represented by such models for the situations where the failure is mainly caused by fatigue.

In view of the above-recalled relations between hrf and CRF, the above classification may be equivalently formulated in terms of the CRF and some authors (e.g., [10, 30]) prefer this latter formulation, because it is more general, as it does not require the existence of a pdf (such mathematical details are omitted in this chapter, as all models here considered possess a pdf).

There are a lot of more possible classifications, e.g., in terms of "Average hazard rate", MRL, etc. [10, 30]. We only emphasize here that, as above remarked, a IHR (DHR) model is characterized by a decreasing (increasing) MRL function of time.

Of course, those reported in the above classification are not the only possible behaviors of the hrf, but they are the most useful for the description of aging for the simple analytical models which will be considered from Sect. 2 onwards; these are indeed "models", so that they must not be expected to express the true hrf of the device for its whole life, but only to represent a reasonable approximation to it, for the actual time interval during which the device operates. As an example, it should be clear that the Exponential model cannot be truly valid in real world: it is only an ideal model, in a sense it is "the most ideal" of all models, since no device can be completely memoryless : as remarked above, wear-out—from a certain point in time—must occur for every device. But, if wear appears very late with respect to the time interval (say, 20 years) for which the device is used (before its withdrawal, e.g., for technological innovation), so that the hrf can be considered roughly constant in that interval, the Exponential model can provide, also in view of its simplicity, a good approximation to the "true", unknown, model.

One of the hrf behaviors not considered in the above classification is the famous "bathtub curve", which is, in fact, an experimental curve, that matches none of the models here considered in Sects. 2 and 3; it is in fact seldom found in operating devices, if they are, as generally happens, subjected to "burn-in". Moreover, it has been already pointed out that in practice there is no need of a model which describes the device reliability for all its "theoretical" life. Finally, models with bathtub-shaped hrf can be built analytically, but are either rather complex or difficult to estimate. For instance, two models capable of representing such behavior, reported in [103, p. 47] and practically almost never used, are reported—as functions of age $x > 0$—here:

$$h(x) = \frac{\alpha}{x+\beta} + \gamma x, \quad \alpha, \beta, \gamma > 0 \tag{13}$$

$$h(x) = \frac{\beta x^{(\beta-1)}}{\alpha^{\beta}} \exp\left[\left(\frac{x}{\alpha}\right)^{\beta}\right], \quad \alpha, \beta > 0. \tag{14}$$

1.6 Final Remarks: Some Popular Misconceptions in Applied Reliability Studies

A few final remarks on some common misconceptions, or "pitfalls", related to erroneous interpretations of the hrf are deemed to be useful here, since often also authoritative bibliography seems to "slip" on this concept. Two of such pitfalls are discussed here, which are both related in a sense to the existence of models with a decreasing hrf, which, as opportunely pointed out in [146, p. 69], appears to be a subtle concept, perhaps only fully understandable (we agree with Singpurwalla) from a subjective (or even "psychological") probabilistic reasoning, even if also a sound "objective" explanation is available [141]. A recent paper on some common misconceptions about the modeling of repairable components (a problem not dealt here), with reference to power system applications, is [160]. In this paper, some basic points already in raised in the fundamental book [4] are summarized and further discussed.

1.6.1 The Pitfall of the "Average Hazard Rate" for Two-state (or Multi-State) Reliability Models

Let a device be potentially subjected, in a given time interval, to (only) one of two operating conditions, say "normal condition" (NC) or "adverse condition" (AC), depending on chance. This may be the case of a overhead transmission or distribution line subjected to normal or adverse weather conditions.[8] Let each of the two conditions correspond to a different value of an assumed-as-constant hazard rate, and let:

$$v = \text{hazard rate value in NC};\ \alpha = \text{hazard rate value in AC}; \tag{15a}$$

$$p = \text{probability of NC};\ q = 1 - p = \text{probability of AC}. \tag{15b}$$

A value $\alpha > v$ is of course expected (often, the ratio α/v can be very high in practice, e.g., 50, for overhead lines), hence this relationship will be assumed to hold. Of course, any situation of a "binary" hrf can be dealt with in such a way, by calling "normal condition" ("adverse condition") the one with the lower (higher) hazard rate value.

Under these hypotheses, the following "average value" assignment of the hrf (still assumed to be constant) to the device is often found (especially in power system studies, a recent example being [90]):

$$h = pv + q\alpha, \tag{16}$$

i.e., a weighted average of the two values v and α. So, the RF model would become:

$$R(t) = \exp(-ht). \tag{17}$$

The above hrf is denoted here as the "average hazard rate". Equation 16 seems to be very reasonable, since the random events "NC" and "AC" are incompatible and exhaustive, and it is indeed used also by some utilities. It is in fact wrong, as well as (17), if the problem is dealt with by the basic tools of elementary probability. Although it may seem paradoxical, the true result is that the overall HR is a

[8] The extension of the following reasoning to three-state or multi-state models is straightforward (e.g., a three-state model occurs in power distribution studies when also "extremely adverse" weather conditions, or similar, are considered [156].

decreasing function of time, of which the value h in (16) is only the initial value [it can be shown that (16) may be—but only in some cases—a good approximation for the hrf for very short time intervals, but the problem is that it is always presented as a true value].

The point is that the total probability theorem [which is evidently the basis of (15)] cannot be applied to the hrf, which is not a probability! (see also the following Sect. 1.6.2). It can be instead be applied, e.g., to the RF $R(t) = P(T > t)$, so that its value is not given by (16), but by:

$$R(t) = p \cdot \exp(-\nu t) + q \cdot \exp(-\alpha t), \tag{18}$$

which is quite different.[9] By derivation of the above RF with respect to time t, using (5), it is easy to get the right expression of the hrf, which is readily shown to be not a constant at all. Indeed, the hrf is a decreasing function of time. In Sect. 3.3.11 (devoted to "mixture models") also this apparent paradox will be discussed, namely that—even though the individual hrf are constant over time—the "overall" model has a decreasing hrf. The paradox can be fully justified by subjective or Bayesian reasoning, and some very interesting papers or books, such as [10, 136, 146], discuss it. Here, we only observe, as in [141], that—if one should not know the true condition under which the device operates—the larger the observed lifetime t of the device, the more likely it is that the item is subjected to NC (rather than to AC), i.e., the conditions corresponding to a lower hazard rate value. Thus, the older the device, the less likely it is to fail, so that the above "mixture" of constant hazard rates gives rise to a DHR model. A brilliant way to show this, as in [141], is using Bayes' theorem for obtaining the following conditional probability, for any given time t:

$$\begin{aligned} P(\mathrm{NC}|T > t) &= P[(\mathrm{NC}) \cap (T > t)]/P(T > t) \\ &= p \cdot \exp(-\nu t)/[p \cdot \exp(-\nu t) + \mathrm{q} \cdot \exp(-\alpha t)], \end{aligned} \tag{19}$$

which is indeed, assuming $\alpha > \nu$, increasing in t [maybe the easiest way to show this is considering the reciprocal of (19), which is clearly decreasing in t]. Of course, $P(\mathrm{AC}|T > t)$, the conditional probability of the complementary event, is—for the same reason—decreasing in t.

It should be remarked that using (16) means using an overestimation of the hrf, this implying an underestimation of system performances.

From *a* practical point of view, such discussion shows that the development of time-varying models for the hrf is highly opportune, as those proposed by Wang and Billinton in [156] for incorporating the effects of weather conditions and restoration resources in reliability evaluation of distribution systems.

For the purpose of the present chapter, however, the main point to be highlighted above is *not* the right expression or behavior of the hrf, but the mistake one

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[9] Mistaking (18) for (17) is in practice equivalent to mistaking—as to the computation of the expectation of a function ϕ of a RV X– the expectation $E[\phi(X)]$ with $\phi(E[X])$, which is a trivial error, if ϕ is not a linear function.

can make by treating the hrf as it were a probability. The same can be pointed out as far as the following topic is concerned.

1.6.2 Does a DHR Model Imply a "Strengthening" of the Device?

From one of the above relations, here reported again:

$$R(t) = \exp\left(-\int_0^t h(\xi)\mathrm{d}\xi\right), \quad t > 0, \tag{20}$$

it should be clear that a DHR model in no way implies a "strengthening" of the device under study, as already discussed above. The RF $R(t)$ always decreases with time, and this is assured by its very definition, or—looking at the above integral relation between RF and hrf—by the hrf being positive, with no regard to its behavior in time. So, it should be remarked that phrases like "the reliability of a DHR system improves with age", which are sometimes reported also in books, are at least ambiguous: in the function $R(t)$ only one time-argument appears, which cannot be arbitrarily deemed to be a "mission time" or an "age" as one likes. The truth is that $R(t)$ decreases with t, so it can never improve with "age" t. For what above discussed, a right way—maybe the only way—to express the peculiarity of a DHR model is that, for such model: "the conditional reliability $R(t|s)$ of a DHR system improves with age s, for any given mission time t". This "conditional" aspect (which is present also in the hrf itself) is sometimes forgotten, e.g., when it is stated, as in some books, that "the hrf expresses the probability that the device fails after reaching age t". It has to be noticed that, for approaching such meaning, the hrf—which, by the way, has a dimension of (1/time)—should be at least multiplied by a time interval Δt, *and even so it cannot be claimed to be a probability* (for instance, no-one can assure that this product is less than 1, also in view of the fact that the hrf can be infinite). Nor the hrf can be resembled to a pdf: as recalled in a note, its integral over the whole interval $(0, \infty)$ must be ∞, not 1 as a pdf.

Returning to DHR models (which are the ones—but not the only ones—capable of possessing an infinite hrf, as we shall see in a few lines) one could also think that a DHR (or a constant hrf) model is characterized by a RF having a slower decreasing attitude in time, with respect to a IHR model and assuming as fixed some parameters (e.g., MTTF and/or median), so that it is "better" than a IHR model in some way. Even if it can seem trivial, it may be not useless to remark that this is generally false, or it may be true only for some large enough mission times. Still from the above relation, one can only deduce that, for what concerns the comparison of 2 hrf, if $h_1(x) > h_2(x)$ in a whole interval $(0 < x < c)$, then—for the corresponding RF—$R_1(t) < R_2(t)$ for every t in $(0 < t < c)$. So, it should come as no surprise that a IHR model as a Weibull one with $b > 1$, e.g., $h(x) = ax$ $(a > 0)$, is largely "better" than a constant hrf model, since for the first we have $h(0) = 0$

and a smaller hrf values in a whole interval containing the time origin. In other words, a memoryless model is by no means a "good" or desirable model in the early age of the device, since "chance failures" are of course no better than no (or "very unlikely") failures. Even more pronounced, of course, would be a comparison between the above IHR model and a DHR model such as a Weibull one with $b < 1$ [e.g., $h(x) = a/\sqrt{x}$ $(a > 0)$], to which an infinite value of the hazard rate and a rapidly decreasing RF corresponds in early times. The IHR model $h(x) = ax$ is much better in this early period (which may also be the most crucial in terms of warranties), and possibly also better on the whole (depending, e.g., on the time interval for which the device will be used in practice).

2 "Direct" Reliability Assessment: A Review of Reliability Models

2.1 A Premise on Reliability Models, with Hints at Electrical Applications

A significant set of reliability models, which should cover almost all kinds of practical applications, are briefly reviewed in the present section, in alphabetical order.

For the purpose of the present approach, the way by which these (or other) models can be deduced from wear processes is important. This aspect is omitted here, but is tackled in the next section, so that a few of these models will be met again there.

As defined above, DRA concerns reliability analysis of components on the basis of failure data coming from devices in-service. Performing a DRA requires that the most adequate probability distribution for the reliability analysis (to be chosen from a family of commonly employed distributions for such components) is selected on the basis of data fitting, previous experience, literature or expert judgment, or better on the basis of a combination of all these aspects. Such distribution should both exhibit a good fitting to the data (proven by a proper statistical fitting test) and possess a relatively simple form, with no more than two or three parameters to be estimated from data. Only two-parameter models will be considered here, since they are by far the most adopted, together with the single parameter Exponential model, this latter being so popular that it is here considered only as a particular case of other models, namely the Gamma, Weibull and the less known HRM. The most adopted reliability models—in particular, for electrical components—are by far the *Gamma, Normal, Lognormal* (LN), and *Weibull* models. However, also some other LT distributions are worth being considered, such as the *Inverse Gaussian* (IG) distribution, the Inverse Weibull (IW) distribution, the *Birnbaum–Saunders* (BS) distribution, the *Log-logistic* (LL) distribution, and more. They have found recently some significant application for electrical components reliability, so they are briefly reviewed here, too. A particular and very significant case is that of the Weibull model: it is by far the most adopted in the field of electrical insulation, as illustrated with some detail in Sects. 3 and 4. Such model, together with some related "physical law" of aging—e.g., the *Inverse power model* (IPM)—has kept proving over the years one the most adequate for the statistical fitting of insulation lifetime data, and the fact that it possesses some physical background or motivation is a very desirable property in the present discussion.

Neither the statistical fitting nor the parameter estimation is addressed in this chapter. Relevant references well cover these topics: exhaustive treatises on the subject, in its most general form, are popular books such as [25, 96, 98, 149]. Books specifically devoted to (classical) statistical estimations for LT models are, e.g., [55, 97, 103, 124, 130]. For what concerns Bayesian inference, the key reference is [113], but the approach is also significantly present in many other books, e.g., in [30, 124, 146, 152]. Moreover, the reader should be aware that most of the references on the presented models (appearing in the reference list reported at the end of this chapter) often present also estimation methods for the model parameters. Only a final small hint at estimation, very interesting for our purposes: it is well worth highlighting that the use of Weibull model for insulation applications has stimulated many peculiar estimation methods, both in classic [52] and Bayesian statistics [39]. An excursus on basic reliability models and their key features (excluding deduction from wear models) follows from the next Sect. 2.2 to the end of present Sect. 2. For all the models here presented, the possible derivations from wear processes are reviewed in Sect. 3. Further details on the mathematical features here briefly reviewed can be found in the books cited at the beginning of this chapter, and in the numerous references at the end of this chapter; in particular, aside from the monographic books on the single models (e.g., [34] on the Inverse Gaussian distribution, or [53] on the Lognormal distribution), very detailed accounts on all the models are present in the authoritative volumes of [96], while a brief but complete review is reported in [124], where also many graphs, here omitted for the sake of brevity, are reported illustrating the cdf, pdf, hrf, etc., of the various models.

2.2 Birnbaum–Saunders Model

The Birnbaum–Saunders (BS) model was introduced by Birnbaum and Saunders in 1969 [20], in relation to fatigue-affected lifetimes. It has the following cdf and pdf for $t > 0$:

$$F(t;\alpha,\beta) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right\} \tag{21}$$

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right]. \tag{22}$$

The parameter (α, β) are positive. The hrf has no simpler form than the ratio:

$$h(t) = f(t)/(1 - F(t)). \tag{23}$$

It is defined for all $t > 0$ (as happens for all models here considered, unless otherwise stated).

In a recent paper [102], it has been shown analytically that the hrf of the BS distribution is always an IDHR—or “unimodal hrf”—model for all values of the shape parameters, and the change point of the hrf can be determined as a solution of a non-linear equation. These authors have provided an approximation to this change point, and also proposed different methods for estimating the change point. After this change point, the hrf approaches a positive limit as $t \to \infty$, similar to the IG model, whose resemblance with the BS model is illustrated also later. Mean and variance of the BS model are:

$$\mu = \beta\left(0.5\alpha^2 + 1\right) \tag{24}$$

$$\sigma^2 = \beta^2\alpha^2\left[(5/4)\alpha^2 + 1\right]. \tag{25}$$

2.3 Gamma Model

The Gamma $G(r, \phi)$ model is one of the most popular in applied probability, and is characterized by the following pdf:

$$f(t;r,\phi) = \frac{\phi^r t^{(r-1)}}{\Gamma(r)}\exp(-\phi t), \quad t > 0 \tag{26}$$

were $\Gamma(x)$ is the Euler–Gamma special function, ϕ and r are positive constants representing the shape and scale parameters, respectively. The cdf is expressed through the incomplete Gamma function $\Gamma(x, y)$:

$$F(t;r,\phi) = \Gamma(r,\phi)/\Gamma(r). \tag{27}$$

In its simplest formulation, denoted as “Erlang model”, the Gamma model describes a positive RV obtained by the sum of r Exponential independent and identically distributed RV with parameter (hrf) ϕ.

For what concerns the hrf, which is not expressible analytically, it can be shown [103] that, if $r > 1$ (the most frequent case), the Gamma model implies a hrf which, starting from zero in $t = 0$, increases with time, approaching the positive limit ϕ as $t \to \infty$. If $r < 1$, the hrf diverges as $t \to 0^+$, then decreases with time, approaching the same limit ϕ as $t \to \infty$.

Mean and variance of the Gamma model are:

$$\mu = r/\phi \tag{28}$$

$$\sigma^2 = r/\phi^2. \tag{29}$$

A limit case is the Exponential one, obtained when $r = 1$, which has the constant hrf $h(t) = \phi$.

A hint at transformed Gamma RV, i.e., "Inverse Gamma" and "Generalized Gamma" RV, is given in this section.

2.4 Gaussian Model

The Gaussian or Normal model plays a fundamental role in statistical analyses because many distributions are well approximated, in view of the CLT, by the Normal probability distribution. The Normal pdf has the following expression:

$$f(t;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(t-\mu)^2\right], \tag{30}$$

where $-\infty < t < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$. As can be seen, the Normal pdf is characterized by two parameters, μ (the mean of the pdf, a real number) and σ (the SD). Its main properties (bell-shaped form, symmetric around the mean μ) are well known.

The Normal pdf and cfd can be conveniently expressed as a function of the standard Normal pdf, $\varphi(z)$, and cdf, $\Phi(z)$, that correspond to a Normal RV with zero mean and unit variance, which are thus defined as follows:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)\mathrm{d}u; \quad \phi(z) = \mathrm{d}\Phi(z)/\mathrm{d}z \tag{31}$$

so that pdf and cdf of the Gaussian model are:

$$f(t;\mu,\sigma) = \frac{\varphi((t-\mu)/\sigma)}{\sigma}; \quad F(t;\mu,\sigma) = \Phi((t-\mu)/\sigma). \tag{32}$$

It can be shown that the hrf $h(t)$ is an increasing function of time, roughly increasing linearly as t diverges. The model is not theoretically adequate, of course, for lifetimes, being defined also for $t < 0$. However, it is sometimes adopted, provided that the probability of attaining negative values is negligible (this happens in practice if $\mu > 3\sigma$). However, principally due to its scarce flexibility, discussed elsewhere in the chapter, it is almost never a good candidate as a reliability model, while it is sometimes used for "repair times" in availability studies, but not so frequently as the LN and Gamma models.

2.5 Gumbel Model

The Gumbel model here illustrated can be obtained—like the Weibull one, but under different hypotheses—as the minimum of a large (ideally infinite) set of RV, so that its pdf is also denoted as the "smallest extreme value" pdf, or also "type 1-extreme value" pdf [28, 80, 131]. The Gumbel model has the following cdf:

$$F(t; \chi, \delta) = 1 - \exp\{-\exp[(t - \chi)/\delta]\} \; (-\infty < t < +\infty). \tag{33}$$

The two parameters (χ, δ) are, respectively, real and positive. It can be seen that, theoretically, the argument of the cdf may be negative, as in the Gaussian case. By an adequate choice of χ and δ, the probability of negative values may be rendered practically zero. However, also a truncated form of the Gumbel pdf exists, restricted to positive argument values, denoted as "Gompertz model" [103]), which is also used in LT applications. The mean and SD of the model are

$$\mu = \chi - \gamma\,\delta \tag{34}$$

$$\sigma = \left(\pi/\sqrt{6}\right)\delta \tag{35}$$

being γ the Euler constant (0.5772…). The RF and pdf are easily evaluated from the above cdf, and the hrf is an increasing exponential function of time:

$$h(t; \chi, \delta) = (1/\delta)\exp[(t - \chi)/\delta] \; (-\infty < t < +\infty). \tag{36}$$

This model is widely used—especially in its truncated form—for devices largely affected by wear with increasing age (such as mechanical products, and also human beings).

Apart from lifetimes, the Gumbel model finds—for intuitive reasons—application also as a model for RV representing material "strength". Indeed, it has been used since decades as a possible alternative to the (more adopted) Weibull model, for characterizing electrical strength of insulators [87].

Also a "largest extreme value"—or "double exponential"—model exists. It is used for characterizing the maximum of a large set of RV, and has the following cdf:

$$F(t; \chi, \delta) = \exp\{-\exp[-(t - \chi)/\delta]\}(-\infty < t < +\infty). \tag{37}$$

It is very popular in engineering applications but it is seldom used for lifetimes [131, p. 40].

2.6 HRM Distribution

A new reliability model, the so-called HRM was introduced by Erto and Palumbo in 2005 [76], who showed that many failure mechanisms can produce mortality laws of Hyperbolic type: the "Deterioration", "Stress-Strength", and "Shocks" failure models are some of the above models which can lead to a HRM, as shown in the following section. The same authors illustrated also some applicative examples, with a noteworthy electrical application. Actually, the model was not completely unknown before, e.g., Lawless [103] briefly mentions it as a "generalized Pareto model". However, its properties were not fully analyzed, nor it appears to have ever been applied, before 2005.

For the purpose of the present section, a decreasing hazard rate function, approaching a value greater than zero, is the distinctive characteristic of the model. The HRM indeed takes its name from the hyperbolic form of its hrf, which is quite peculiar in the wide range of all the existing models, and can be expressed as follows (76):

$$h(t) = r + \frac{a}{t+1}, \quad a > 0,\ r > 0, \tag{38}$$

which is strictly decreasing from the early maximum value ($a + r$), to the asymptotic minimum r, with a being the limit decrement. So, as to the hrf properties, this model shows some analogies with the LN and IW ones, except that the maximum of the hrf is attained at $t = 0$ (while, in the LN and IW models, the maximum of the hrf is attained at some mission time $t > 0$). From Eq. 1, the cumulative hrf, RF, cdf, and pdf are easily derived as:

$$H(t) = \int_0^t h(t)\mathrm{d}t = r\,t + a\ln(t+1), \tag{39}$$

$$R(t) = \mathrm{e}^{-H(t)} = \frac{\exp(-rt)}{(t+1)^a}, \tag{40}$$

$$F(t) = 1 - R(t) = 1 - \frac{\exp(-rt)}{(t+1)^a}, \tag{41}$$

$$f(t) = h(t)R(t) = \left[r + \frac{a}{t+1}\right]\left[\frac{\exp(-rt)}{(t+1)^a}\right]. \tag{42}$$

It is apparent from (38) that, for t increasing infinitely, the Hyperbolic Model reduces to the Exponential model with constant hrf: $h(t) = r$ (and, so, MTTF $= 1/r$).

Erto and Palumbo [76] deduce all the non-trivial statistical properties (mean, variance, etc.) of the HRM by means of the Moment generating function $\Phi_T\{\cdot\}$ of the RV T (lifetime), i.e.:

$$\Phi_T\{x\} = E[\exp(xT)] \\ = \int_0^{\infty} \exp(xt)\left(r + \frac{a}{t+1}\right)\frac{\exp(-rt)}{(t+1)^a}\mathrm{d}t. \tag{43}$$

Denoting by Γ (·,·) the incomplete Gamma function, after some manipulations the following expressions for the mean $E[T]$ and variance $\mathrm{Var}[T]$ are obtained:

$$E[T] = \Phi'_T(0) = \frac{1}{r}[r^a \exp(r)\Gamma(-a+1, r)], \tag{44}$$

$$\begin{aligned}\mathrm{Var}[T] &= E\left[T^2\right] - E^2[T] \\ &= \frac{1}{r^2}\{2r^a \exp(r)\Gamma(-a+1, r) - 2ar^a \exp(r) \\ &\quad \times \Gamma(-a+1, r) + 2r[1 - r^a \exp(r)\Gamma(-a+1, r)] \\ &\quad - \left[r^a \exp(r)\Gamma(-a+1, r)\right]^2\},\end{aligned} \tag{45}$$

where

$$\begin{aligned}E\left[T^2\right] &= \Phi''_T(0) \\ &= \frac{1}{r^2}\{2r^a \exp(r)\Gamma(-a+1, r) - 2ar^a \exp(r) \\ &\quad \times \Gamma(-a+1, r) + 2r[1 - r^a \exp(r)\Gamma(-a+1, r)]\}.\end{aligned} \tag{46}$$

Interesting properties concerning the MRL are also illustrated in [76]; e.g., it is proven to be an increasing function of time, toward the maximum asymptotic value $1/r$ (and this is in agreement with known theoretical relations between hrf and MRL).

2.7 Inverse Gaussian Distribution

The Inverse Gaussian (IG) distribution [34], although not very popular in the field of power systems, has found many applications in theoretical reliability literature for those situations in which the LT distribution is greatly affected by early failures due to the so-called "infant mortality". The Inverse Gaussian model belongs to the IDHR family and is very similar to the LN distribution. In practice, they are in most cases undistinguishable on the basis of field data, so that it is important to understand the kind of aging process which may give rise to the IG or the LN distribution.

The IG distribution has been introduced as the first passage time of a Wiener process [34], as will be recalled in the next section. Its pdf is given by:

$$f(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right] \tag{47}$$

being t, μ, $\lambda > 0$. Using the above recalled Gaussian cdf $\Phi(x)$, the RF and hrf are given by the following expressions:

$$R(t;\mu,\lambda) = \Phi\left[\sqrt{\left(\frac{\lambda}{t}\right)}\left(1-\frac{t}{\mu}\right)\right] - \exp\left[\frac{2\lambda}{\mu}\right]\Phi\left[-\sqrt{\frac{\lambda}{t}}\left(1+\frac{t}{\mu}\right)\right] \tag{48}$$

$$h(t;\mu,\lambda) = \frac{\sqrt{\frac{\lambda}{2\pi t^3}}\exp\left[-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right]}{\Phi\left[\sqrt{\frac{\lambda}{t}}\left(1-\frac{t}{\mu}\right)\right] - \exp\left[\frac{2\lambda}{\mu}\right]\Phi\left[-\sqrt{\frac{\lambda}{t}}\left(1+\frac{t}{\mu}\right)\right]}. \tag{49}$$

Although not easily, it can be seen that the hrf first increases, reaching its maximum at a time t^* which is not analytically expressible, then approaches the positive limit $\lambda/(2\mu^2)$ as $t \to \infty$. This is a small difference with respect to the LN and LL reliability models, whose hrf goes to zero as t diverges, and a similarity with the BS model, which is indeed resembles very closely the IG model, also in its derivation (see Sect. 3).

Mean and variance are:

$$E[T] = \mu \tag{50}$$

$$\sigma^2 = \mu^3/\phi. \tag{51}$$

2.8 Inverse Weibull Distribution

The Inverse Weibull (IW) model was deduced—although often named in a different way, i.e., Frechet model—as a model for the asymptotic distribution of the maximum value from a succession of independent RV [28]. Subsequently, it was proposed with the present name when it was obtained as the distribution of the inverse (reciprocal) of a Weibull RV. Most of its properties, in particular those of its hrf, were first deduced by Erto [70], and are being developed in a forthcoming paper [75]. In [70], also the identification of the IW model within a "Stress-Strength" (SS) model has been illustrated (see Sect. 3 for details).

The pdf of a IW RV, with parameters σ and β is:

$$f(t,\alpha,\beta) = \alpha\beta(\alpha t)^{-(\beta+1)}\exp\left[-(\alpha t)^{-\beta}\right], \tag{52}$$

where $t \geq 0$, α, $\beta > 0$.

The reliability function and the hazard rate function are:

$$R(t;\alpha,\beta) = 1 - \exp\left[-(\alpha t)^{-\beta}\right] \tag{53}$$

$$h(t,\alpha,\beta) = \frac{\alpha\beta(\alpha t)^{-(\beta+1)}\exp\left[-(\alpha t)^{-\beta}\right]}{1 - \exp\left[-(\alpha t)^{-\beta}\right]}. \tag{54}$$

Also such function is of the "IDHR" family; the peak value of the hazard rate of a IW model is obtained at a mission time value belonging to an interval whose extreme points are: $T_m = [\beta/(\beta+1)]^{1/\beta}/\alpha$ (the mode of the IW distribution) and $T_n = \beta^{1/\beta}/\alpha$; the hrf is infinitesimal when $t \to \infty$. The mean (which exists only if $\beta > 1$) and the variance (which exists only if $\beta > 2$) are, setting $\theta = 1/\alpha$:

$$E[X] = \mu = \theta\,\Gamma(1 - 1/\beta) = \theta\,\Gamma(1 + 1/\beta) \tag{55}$$

$$\mathrm{Var[X]} = \theta^2\Gamma(1 - 2/\beta) - \mu^2. \tag{56}$$

2.9 Log-Logistic (LL) Distribution

The log-logistic (LL) distribution was adopted by the authors in insulation reliability studies [37, 40]. A recent application of the LL model in ALT is shown in [148], which also refers to [37, 39] for insulation reliability applications. This model is named after the fact that it characterizes a RV: $X = \exp(T)$, where T has a logistic distribution, whose cdf is:

$$F(t;\alpha,\beta) = \frac{1}{1 + \exp\left[-\frac{(t-\alpha)}{\beta}\right]} \tag{57}$$

with: $-\infty < \alpha < \infty$, $\beta > 0$, $-\infty < t < \infty$.

Thus, since the variable $T = \log(X)$ is a logistic RV, then X is a so-called log-logistic RV, which is characterized by the following cdf and pdf, in which the parameters $\lambda > 0$ and $b > 0$ are functions of (α, β) above:

$$F(x) = \frac{(\lambda x)^b}{\left[1 + (\lambda x)^b\right]},\ x > 0 \tag{58}$$

$$f(x) = \frac{b\lambda^b t^{\mathrm{b}-1}}{\left[1 + (\lambda t)^b\right]^2}. \tag{59}$$

It is often convenient to use, instead of the scale parameter λ, the parameter $c = 1/\lambda$, which is the median of X, so that:

$$F(x) = \frac{(x/c)^{b}}{\left[1 + (x/c)^{b}\right]}, \ x > 0. \tag{60}$$

It is indeed apparent that $F(c) = 0.5$, no matter the value of b. Although the LL model received some attention in survival data analysis since 1983 in a paper by Bennett [13], it is neither frequently used nor well-known in literature—apart from the popular Cox and Oakes' monograph [51]. In [37], the authors also discuss the similarity between the LL and the Weibull model, apart their hrf. In the LL model, the hazard rate function $h(x)$ is:

$$h(x) = \frac{b\lambda^{b}x^{b-1}}{[1 + (\lambda x)]^{b}}, \tag{61}$$

which is always decreasing with x if $b \leq 1$; first increasing, then decreasing with time if $b > 1$. In particular, in the latter case $h(x)$ starts from $h(0) = 0$, then reaches its maximum at $x^* = (1/\lambda)(b - 1)^{1/b}$, then $h(x)$ goes to zero as x diverges.

It must be pointed out that also another, more popular model features these properties of the hrf function, i.e., the Lognormal (LN) model. In fact, the LL distribution—as also discussed in the above references [37, 40, 51]—shares many properties with the LN distribution. The LL model is simpler analytically than the LN one, but appears to be more difficult to estimate, while methods—particularly the Maximum Likelihood (ML) one—for assessing the LN model are well established. The mean value (which only exists if $b > 1$) and the standard deviation SD (which—as the variance—only exists when $b > 2$) are given by:

$$E(X) = \frac{c\pi}{[b\sin(c\pi)]}; [X] = E[X]\mathrm{CV}[X], \tag{62}$$

where CV is the coefficient of variation, that in this case has the following expression if $b > 2$:

$$\mathrm{CV}[X] = \sqrt{[(b/\pi)\tan(\pi/b) - 1]}. \tag{63}$$

It can be shown (see [51] for some graphical illustration) that the Skewness coefficient of the LL model is positive and always larger than the corresponding Weibull one, possessing the same CV value. Thus, the LL model possesses generally larger "tails" than the Weibull one with the same central parameters and this may lead to underestimate the upper quantiles of the lifetime if a Weibull model is fitted to data generated in fact from a LL model (this can happen, as shown in [37]).

Such distribution may take its origin from a "Gamma mixture" of a Weibull RV, as will be shown in Sect. 3.

2.10 Lognormal Distribution

The Lognormal (LN) model [53] has become more and more popular in last years, also in reliability applications. The LN pdf with parameters (ξ, δ) and argument t is given by:

$$f(t;\xi,\delta) = \frac{1}{\delta t\sqrt{2\pi}} \exp\left[-\frac{1}{2\delta^2}(\ln(t)-\xi)^2\right], \tag{64}$$

where $0 \le t < \infty$, $-\infty < \xi < \infty$, $\delta > 0$, being $\xi = E[\ln(T)]$ and $\delta^2 = \mathrm{Var}[\ln(T)]$.

Denoting, as above, by $\Phi(z)$ and $\varphi(z)$ the standard Normal pdf and cdf, respectively, the LN RF and hrf are as follows, respectively:

$$R(t;\xi,\delta) = 1 - \Phi\left(\frac{\ln(t)-\xi}{\delta}\right) \tag{65}$$

$$h(t;\xi,\delta) = \frac{\varphi\left(\frac{\ln t-\xi}{\delta}\right)}{t\delta - t\delta\Phi\left(\frac{\ln t-\xi}{\delta}\right)}. \tag{66}$$

The behavior of the hrf is not easy to analyze, and was sometimes mistaken in literature, so that specific papers were devoted to it (e.g., [150]). However, it is a IDHR model: the hrf at first increases from zero, then decreases toward zero [53]. Differently from the LL model, the “change point” of the hrf cannot be evaluated analytically (as in the BS model).

The mean and the SD are given by:

$$\mu = \exp(\xi + \delta^2/2) \tag{67}$$

$$\sigma = \mu\left\{\exp(\delta^2) - 1\right\}^{\frac{1}{2}}. \tag{68}$$

2.11 Weibull Distribution (Featuring also the Exponential Model)

The Weibull model (in particular, the two-parameter Weibull model) is quite popular, probably the most popular model in reliability applications—since its birth, in 1939, for application in mechanical engineering (e.g., fatigue life of steel). Its popularity is due to two basic features: (1) its flexibility (e.g., the Gamma, Normal and the Lognormal models can be satisfactorily approximated, under many respects, by a suitable Weibull pdf; the hrf may be increasing, decreasing or constant); (2) the fact that the Weibull belongs (as it was proved in 1945 by Gnedenko) to the family of extreme-values distributions, being able to represent the failure mechanisms of “chain-like” systems that fail when the weakest link is broken [28].

The Weibull model, in the form denoted as $W(a, b)$, being a and b positive parameters, has the following hrf, RF, pdf:

$$h(x) = abx^{b-1} \exp\left(-ax^b\right) \tag{69}$$

$$R(x) = \exp\left(-ax^b\right) \tag{70}$$

$$f(x) = abx^{b-1} \exp\left(-ax^b\right). \tag{71}$$

Also an alternative parameterization, denoted as $W'(\theta, \beta)$, is often used, in which:

$$\theta = 1/a^{\kappa}; \quad \kappa = 1/b; \quad \beta = b \tag{72}$$

so that the RF is expressed by:

$$R(x) = \exp\left[-(x/\theta)^{\beta}\right]. \tag{73}$$

This latter formulation is the most adopted for expressing the mean and the variance:

$$E[X] = \theta\Gamma(1 + 1/\beta) \tag{74}$$

$$\mathrm{Var}[X] = \theta^2\,\Gamma(1 + 2/\beta) - \mu^2. \tag{75}$$

As well known, the Exponential model is a particular case of the Weibull one. Physical motivations for both of them will be discussed later.

Finally, the following relationship holds between Weibull and abovementioned Gumbel model:

$$Y = \log(X) \tag{76}$$

in which Y is a Gumbel RV and X a Weibull RV. This relationship is often useful for parameter estimation.

2.12 Caveats About Using "Popular" Reliability Models

We close this section by noting that often only some simple analytical and/or statistical considerations about probabilistic distributions are needed to select a proper reliability model, or at least to exclude some of them from subsequent analyses.

For instance, the use, and sometimes the abuse, of the "classical" Gaussian and Weibull models is typical in power systems literature. For instance, in [107] these two models are employed in order to represent HV cables LT data, with mean of 45 years and SD of 15 years. For such case-study, the Gaussian and Weibull models imply both RF and hrf which are very similar, and also extreme percentiles can be shown to be fairly close (see also Sect. 5).

However, the adoption of a Gaussian model is at least questionable for no less than three, very simple, reasons which are—rather surprisingly—often neglected in literature:

1. A Gaussian random variable may always assume (even if with small probability, provided that the mean value is larger than three times the standard deviation) negative values, and this fact makes such model theoretically not suitable to describe LT values.
2. The Gaussian model is not flexible (its pdf can have only one shape, the well known so-called "bell-shaped" one).
3. The Gaussian model has always a monotone hrf; it is indeed a IHR model, regardless of the parameter values (this is another aspect of the lack of flexibility of the model).

On the contrary, the choice of the Weibull model has some good theoretical reasons supporting it for application to a LT distribution, e.g., the "Extreme Value" theory, while also the Weibull model has a monotonic hrf; in particular, as well known, such model belongs to one of the three families: IHR (if $\beta > 1$), DHR (if $\beta < 1$), or constant hrf ($\beta = 1$). This kind of property may result unsatisfactory for the purpose of describing the component reliability over large LT intervals, as nowadays requested within the "life extension" programs of deregulated electric market.

In authors' opinion, such kind of motivations shows clearly that it is very useful to identify the reliability model on the basis of both *theoretical* and "physical" reasons. The next section, about the so-called "physical reliability models", illustrates the most adopted kind of physical motivations behind the identification of a reliability model. Experience in power systems operation shows indeed that, in many cases, failures are associated with "stresses", e.g., rated voltage and temperature (that are steady) as well as overvoltages, fault currents, temperature and mechanical stresses, etc. (that can occur randomly during component lifetime). Fortunately, probabilistic aging and life models about endurance ("strength") of electrical components to stresses, are available, often from "accelerated tests". By this way, it is possible to take advantage of available data on the physical processes of stress and/or strength, according to what has been called an "indirect" assessment of item's reliability—as discussed in [42].

A hint at "Inverse Gamma" and "Generalized Gamma" models

Since they are referred to in Sect. 3, and sometimes (not often) used in literature, only some hints at two kinds of "transformed" Gamma RV, i.e., the "Inverse Gamma" and "Generalized Gamma" models [96, 103] are given here. Their pdf are not difficult to express by means of the well known rule of transformations [134] and the reader may consult the references for more details.

Let X be a Gamma $G(r, \phi)$ RV, then:

$$Y = 1/X \tag{77}$$

is a so-called "Inverse Gamma" RV. Now, letting k be a positive parameter, the RV T defined as

$$T = X^k \tag{78}$$

has a "Generalized Gamma" pdf.

The Inverse Gamma model has the following pdf, with argument y:

$$f(y; r, \phi) = \frac{\phi^r}{y^{(r+1)}\Gamma(r)} \exp(-\phi/y), \quad y > 0. \tag{79}$$

The Generalized Gamma model has the following pdf, with argument t:

$$f(t; r, \phi) = \frac{b\phi^r t^{(rb-1)}}{\Gamma(r)} \exp(-\phi t^b), \quad t > 0, \tag{80}$$

in which $b = 1/k$. Such model may be very useful in some applications, for instance in selecting a proper model from data, since it implies the Gamma (obviously, for $k = 1$) and the Weibull (for $r = 1$) as particular cases; moreover, also the LN model is well approximated if r is large enough.

3 Identification of Probabilistic Life Models from Stochastic Process of Wear

3.1 Outline of the Section: Inferring Probabilistic Life Models From the Stochastic Process of Wear

This section and the successive ones are devoted to the IRA, i.e., to the lifetime model assessment deduced or inferred from the knowledge of the probabilistic laws of the stochastic processes of degradation and stresses which unavoidably affect any device. Often, also the device "strength", i.e., the maximum stress amplitude that the device is able to withstand before failing, is a RV or, in general, a SP, due to the unavoidable randomness intrinsic in its aging, because of uncontrollable variations from item to item in the manufacturing processes, to randomness of environmental conditions, etc. (these aspects will be dealt with more detail in relation to specific applications of the next section). The combination of stress and strength is generically denoted under the name of "wear" in the following. The distinction between "continuous" and "discontinuous" wear or failure processes, although it may be useful sometimes (the "stress" or "shock" processes should be framed into the latter, according to some publications), is not maintained here, also because it is very difficult to define and distinguish clearly the two kinds of processes, which, in fact, are superimposed in practice. As already discussed, by means of the probabilistic knowledge of wear the model assessment can be performed by means of lifetime data analysis, as in DRA. DRA should be, of course, always performed, but it cannot be at all discriminatory among several "similar" models (e.g., the LN and the Gamma one) when only a few data (say, less than 20) are available, as often occur in practice. Needless to say, the IRA alone cannot be claimed to be the solution to the problem of model assessment as well, a problem that is always very critical within the reliability analysis of modern technology products, for which no definitive solution can be easily found.

Indeed, it must be highlighted that IRA requires that the wear process be known by prior information and/or be somewhat measurable, directly or indirectly (e.g., by measuring its effects on lifetime reduction of similar devices).

This is not always the case in practical applications, of course, so that, in general, an adequate and feasible lifetime model assessment should be better performed by a reasonable combination of both direct and IRA. Often this will not bring to definite conclusions, but sometimes even the exclusion of some models in favor of a restricted choice can be a useful result: this happens, for example, when the allowable models are in practice very similar (as often happens, e.g., for the LN and the IG one: see the final example of Sect. 5), so that "mistaking" one for the other does not bring about remarkable errors (never forgetting the obvious principle that the "only true" model does not exist).

However, the above possible limitations of IRA might be overcome for what concerns the application of the chapter—illustrated in the following sections—which are devoted mainly to electrical components (and, in particular, to electrical insulation): indeed in this field the above requirement of the "measurability" of stress and wear is mostly satisfied, also with the help of extensive experimental surveys conducted by means of ALT. After many decades of experience, such tests allowed to validate well-established models that relate lifetime and applied stress (voltage, temperature, etc.), such as the popular IPM [35, 51, 97]. In the next Sect. 3.2, general *Stress-Strength* (SS) models [99] are reviewed. While describing such models, some reliability distributions (as those already reviewed in previous Sect. 2) are directly obtained. Then, in Sect. 3.3, a complete list of all the models of Sect. 2 together with their possible generative mechanisms based upon degradation is illustrated: the list is by no means meant to be exhaustive, but it only serves as a reference and for illustrating a methodology. Some of the "dynamic" models here reported have been deduced following the same approach as in a recent book by Singpurwalla [146]. Recent accounts of SS and fatigue damage models, particularly devoted to mechanical engineering applications (which were the origin of such models) can be found in [22, 29, 158]. In [90], a generalized Stress-Strength model is considered with reference to stochastic loading and strength aging degradation in a more general way with respect to those dealt with here.

For the sake of brevity and simplicity, only a hint is made here at more complex wear models such as those based on the advanced theory of SP, such as Wiener diffusion processes [157], Gamma processes [1, 22, 45, 145, 153–155], Markov and semi-Markov models of deterioration [44, 46]: most of such models were derived in the framework of Structural Reliability, and are still seldom adopted for electrical devices. Advanced models for comparing DRA and IRA are exposed with emphasis to the statistical estimation point of view in [110, 111], which assume a measurable degradation process, also taking into account the measurement error: it is shown that IRA is, under the assumed degradation model, more efficient for estimating extreme quantiles of the LT distribution. In the same field, a new formulation of degradation modeling with random coefficients models has been recently proposed by [81], using the so-called *Bernstein distribution* in a sensor-based prognostics framework, with the purpose of predicting residual life distributions.

3.2 Stress-Strength Models

A classical family of models for "physical" reliability evaluation alternative to probabilistic life models is made of the so-called probabilistic "Stress-Strength" (SS) models, based on the characterization of the stochastic process describing the wear caused by random stresses [94, 99].

3.2.1 Static and "Quasi Static" Stress-Strength Models

Let us denote by X and Y the two random variables (RV):

- X: the "Stress".
- Y: the "Strength".

For instance, in the application to components insulation (see following section), the random variable X ("Stress") is the peak value of stress (voltage surge); the RV Y ("Strength") is the insulation electric strength. It is apparent that both Strength and Stress are, in general, affected by randomness.

Then, in its simplest, "static" form, the SS model is based on the following expression of the reliability function (RF), i.e.:

$$R = P(X < Y). \tag{81}$$

The model is "static" in that the mission time t does not explicitly appear and only RV (X and Y) are used instead of SP, as would be more appropriate (see Sect. 3.2.3). This means that Strength and Stress are assumed as constant, although random, in the time interval to which the RF is referred. So, the pdf of X and Y are assumed as time-independent, or the mission time pre-determined; the more realistic "dynamic" version of SS models is discussed later.

Denoting with $f(y)$ ($F(y)$) the pdf (cdf) of Y, and with $g(x)$ ($G(x)$) the pdf (cdf) of X, the RF of the device is given—under the reasonable hypothesis that the RV X and Y are statistically independent—by:

$$R = \int_0^\infty g(x)P(X < Y|X = x)\mathrm{d}x = \int_0^\infty g(x)(1 - F(x))\mathrm{d}x. \quad (82)$$

In the above equations, time does not appear (at least in explicit form). Various models derived from (82) are illustrated in the above and related references. The analytical solution of (82) exists in a few cases, among which the "Weibull case" for X and Y is illustrated as an example here below.

3.2.2 Example: A Weibull Stress-Strength Model Leading to a Log-Logistic Distribution

We use here as "quasi static SS model", i.e., a simple dynamic generalization of a static SS model, obtained by letting some parameter vary with time. Let X and Y be two Weibull RV with equal shape parameter β, and with scale parameters θ for the Strength X, and α for the Stress Y, i.e., let the cdf of X and Y be given, respectively, by:

$$G(x) = 1 - \exp\left[-(\mathrm{x}/\theta)^\beta\right]; \quad F(y) = 1 - \exp\left[-(\mathrm{y}/\alpha)^\beta\right]. \quad (83a)$$

As for time dependence, it is reasonable to consider- as proposed in [37]—the following "Inverse power" characterization of the Strength scale parameter α with time t, in which k and m are positive constants:

$$\alpha = \alpha(t) = k/t^m. \quad (83b)$$

Indeed, since the expectation of Y is proportional to α—it is recalled that $\mu = \alpha\Gamma(1 + 1/\beta)$—relationship (83b) implies that Y decreases with time t as a power function of t, a popular model in LT analyses, which will be met throughout the chapter. Then, after easy computations shown in [37, 41], the following log-logistic (LL) model [51, 96] is obtained:

$$R(t) = 1\Big/\left[1 + (\lambda t)^b\right], \quad (84)$$

where $b = m\beta; \lambda = (\theta/k)^{1/m}$.

The LL model belongs to the IDHR or (less frequently) to the DHR family of reliability models, depending on the value of the shape parameter b. Indeed, its hazard rate function $h(t)$ has the following expression:

$$h(t) = b\lambda^b t^{b-1}\Big/\left[1 + (\lambda t)^b\right], \quad (85)$$

which is always decreasing with time if $b \leq 1$; first increasing, then decreasing with time if $b > 1$ (such properties were already discussed in more detail in Sect. 2.8, after Eq. 61).

In fact, the LL distribution—as also discussed in the above references—shares many properties, such as being IDHR, with the LN distribution. Although appearing against intuition, the IDHR (or DHR) property has been sometimes observed, as already recalled, for some electrical components, and has often been motivated in theoretical reliability literature in relation with random heterogeneity of materials or subjective probabilistic reasoning [10, 79, 109, 146].

3.2.3 Dynamic "Stress-Strength" Models

The static model is, of course, of limited application, for at least two reasons implying a time variation of the pdf of stress and/or strength:

1. The stress X is always best described by a *stochastic process* in time [134, 151], $X(t)$, since many random variables (fault time occurrence, duration, amplitude, location, etc.), most of which are time-dependent, are involved in its definition.
2. The strength $Y(t)$ is generally decreasing in time due to aging effects. Being a SP, the fact that $Y(t)$ is decreasing in time must be defined on a probabilistic basis, as discussed below.

For a typical example of item (1), one can imagine the stress process as constituted by a succession of random "shocks" events which occur at random times: $T_1, T_2, \ldots, T_n$. This is denoted as a "Shock type" stress, which is the most common one in the case of electrical systems (examples: overvoltages, short circuit currents).

However, although quite common, the above "Shock type" stress process is not the most general, as there always exists, in practice, an "ordinary" stress which is continuous in time (i.e., caused by weather conditions, or also by nominal voltage, etc.), upon which the Shock type Stress is "superimposed". A general view of dynamic Stress-Strength Models allowing for the description of Stress or Strength processes by means of continuous SP is given in the following.

Let us define the following stochastic processes:

- $X(t)$ = "Stress" Process.
- $Y(t)$ = "Strength" Process.

Then the RF over the interval (0, t) is given by:

$$R(t) = P[X(s) < Y(s), \quad \forall\, s \in (0, t)]. \tag{86}$$

Accordingly, the LT of the component—i.e., the RV here denoted by T—is given by the first time instant at which the Strength is greater than the Stress:

$$T = \inf\{t : t > 0,\, X(t) > Y(t)\}. \tag{87}$$

Some simple examples of Stress and Strength processes are illustrated below. For instance, being of course the Strength process $Y(t)$ closely related to the aging of the device (possibly due also to the wear cumulated up to time t because of all previous shocks), it may be considered as a continuous process which is generally decreasing in time, in a stochastic sense; i.e., denoting by (a, b) generic time instants:

$$a < b \rightarrow P[Y(a) > y] > P[Y(b) > y], \quad \forall y > 0. \tag{88}$$

Equivalently, by introducing the cdf of $Y(t)$:

$$F(y;t) = P[Y(t) < y)], \quad y > 0;\ t > 0. \tag{89}$$

Equation 88 may be written as follows:

$$a < b \rightarrow F(y;a) < F(y;b), \quad \forall y > 0. \tag{90a}$$

Sometimes, the milder condition may be imposed that $Y(t)$ is decreasing in the mean value sense, i.e.:

$$a < b \rightarrow E[Y(a)] < E[Y(b)]. \tag{90b}$$

It is easy to show that (90a) implies (90b), but the converse is not generally true except in some cases, as in the following example Sect. 3.2.4.

A stronger condition of decreasing Stress may be the a.s. (almost sure, i.e., with probability = 1) one, i.e.:

$$a < b \rightarrow Y(a) > Y(b), \text{ a.s.} \tag{90c}$$

Of course, this implies both (90a) and (90b).

3.2.4 Example: A Weibull Strength Model with IPM Time Variation

Based on already recalled extreme-value theory, the strength Y of a material can be characterized by a Weibull distribution. Moreover, let us assume that the time dependence of strength is contained in the scale parameter α, so that the time-dependent cdf of the Stress is expressed by:

$$F(y,t) = 1 - \exp\left[-\left(\frac{y}{\alpha(t)}\right)^{\beta}\right], \quad y > 0,\ t > 0. \tag{91}$$

Moreover, let $\alpha = \alpha(t)$ be decreasing in time, e.g.: $\alpha(t) = k/t^m$, as in the IPM model. In this case, it is easy to see that both (90a) and (90b), hold; the first is immediate, the second comes from the mean value expression:

$$E[Y(t)] = \alpha(t)\Gamma(1 + 1/bgr;). \tag{92}$$

In the above Weibull example, the "a.s. decreasing Y" property is not assured.

Similar properties—"mutatis mutandis"—may be adopted for the Stress process $X(t)$ which may be considered as a continuous process, generally increasing (in a stochastic sense) in time.

3.2.5 Dynamic Stress-Strength Models with "Shock Type" Stress: A Cumulative-Damage Model

As previously pointed out, stresses are often caused by repeated "shocks" (this is indeed the case of overvoltages, fault currents, etc.), whose succession constitutes a typical example of a stochastic process. Their effect may be cumulated or not, depending on the kind of component (e.g., in the case of an insulation, this may also depend on whether it is self-healing or not). So, let us consider a stress process as constituted by a succession of random "shocks" events which occur at random times: T_1, T_2, …, T_n,… This SP, denoted as a "Shock type" stress, can be considered as a "point process" [134, 151] of random variables Z_j ($j = 1$, …, n,…) occurring at the random instants T_j ($k = 1$, …, n, …): i.e., the RV Z_j represents the stress amplitude associated with the shock event occurring at time T_j. In practice, the stress "process", viewed as a continuous function of time t, $W(t)$, is always zero except for "spikes" (of negligible duration) with amplitude Z_k at times T_k. Moreover, also the number of events occurring in a given interval (0, t) is random. So, let us denote by $N(t)$ the following stochastic process: $N(t)$ = number of stresses occurring in the interval (0, t).

Due to the fact that the stresses (overvoltages) are purely accidental, a reasonable hypothesis is that the process $N(t)$ can be described by a (homogeneous) Poisson process [43, 134, 140], so that its probability distribution is expressed by:

$$p(k,t) \equiv P[N(t) = k] = \frac{(\phi t)^k}{k!}\exp(-\phi t) \quad k = 0, 1, \ldots, \infty, \tag{93}$$

where ϕ is the mean frequency of occurrence of the event (i.e., the mean number of shocks per unit time). The above Poisson model has always found many applications for describing the fault process in the case of power systems [3, 5], and here—for brevity—it will be the only one considered. Extension of SS theory to non-homogeneous Poisson processes is dealt with, e.g., in [89]. Let us suppose, as a typical case which finds many applications in literature (see, e.g., [2] for an application to HV circuit breakers), a "cumulative wear process" and denoting by Z_K the stress amplitude at time T_K, the total wear acting at the end of the interval (0, t) on the component is given by the SP:

$$W(t) = \sum_{k=0}^{N(t)} Z_k, \quad \text{if } N(t) > 0 \tag{94a}$$

$$W(t) = 0, \quad \text{if } N(t) = 0. \tag{94b}$$

Of course, Z_K is generally a RV, since its value cannot be predicted. So, the wear process $W(t)$ is characterized as a "Compound Poisson process" [134].

Let us suppose, for the moment, that the strength Y is not a RV, but it is a constant y (time-independent). Since $W(t)$ is an (almost surely) increasing function of time—provided that the RV Z_K are non-negative, as reasonable—and the fault occurs as soon as the total wear $W(t)$ is greater than the strength y, the device reliability function is, in term of the wear cdf:

$$R(t) = P(W(t) < y) = F_w(y,\, t), \quad t > 0. \tag{95}$$

The probability distribution—and so the RF of (95)—of the process $W(t)$ of (93), (94) may be deduced as follows [134, 140], assuming as reasonable that $N(t)$ and the Z_K are s-independent: let Q_n be the whole damage conditioned to the occurrence of a deterministic number, n, of stresses:

$$Q_n = \sum_{k=1}^{n} Z_k. \tag{96}$$

Denoting by $F_{Q_n}(w)$ the cdf of Q_n, the cdf of $W(t)$, i.e., the probability of the event $W(t) < w$, is given—according to the total probability theorem—by:

$$F_W(w,t) = \mathrm{e}^{-\phi t} + \sum_{n=1}^{\infty} F_{Q_n}(w) p(n,t). \tag{97}$$

It is remarked that the above function depends both on wear magnitude, w, and on the time instant t. Both w and t are positive. Only in some cases, the pdf of $W(t)$ can be expressed in analytical way (if not in a closed form), and this happens only if it is assumed that the RV Z_k are identically distributed and independent of the process $N(t)$; for example, if the Z_K RV are exponential, then $W(t)$ has a Bessel distribution. However, if the variables Z_k are independent, $W(t)$ is a stochastic process with independent increments; as t increases—according the Central Limit Theorem—it approaches a Gaussian process [134]; the process mean and variance can be obtained as follows, assuming—as above said—that the Z_K RV are independent, with equal mean and variance:

$$E[Z_K] = \mu_z,\ \forall k \quad V[Z_K] = \sigma_z^2,\ \forall k. \tag{98}$$

Then, it is easy to show that the mean value of $W(t)$ at time t is equal to:

$$E[W(t)] = \mu_z \phi t \tag{99}$$

and the variance and the auto-covariance function C_W [134] of the process $W(t)$ are given—denoting by (t, t_1, t_2) generic time instants—by, respectively:

$$\mathrm{Var}[W(t)] = \left(\mu_z^2 + \sigma_z^2\right)\phi t \tag{100}$$

$$C_W(t_1,t_2) = \varphi\left(\mu_z^2 + \sigma_z^2\right)\min(t_1,t_2). \tag{101}$$

For the purpose of the RF evaluation, the fact that $W(t)$ approaches—according to the Central Limit Theorem—a Gaussian process implies that the LT distribution may be represented, at least approximately, by a "Birnbaum–Saunders" distribution for the time to failure T, as will be highlighted in Sect. 3.3. Such model, as recalled, is again IDHR. A more complete discussion of aging properties, for such cumulative models deriving from shocks, can be found in the classical book [10], characterized by a superb level of mathematics.

The extension of (95) to the case when the strength Y is a RV, with (time-independent) pdf $g(y)$, is straightforward, still using the total probability theorem:

$$R(t) = P(W(t) < Y) = \int_{o}^{\infty} F_w(y, t)g(y)\mathrm{d}y. \tag{102}$$

In this case, the LT distribution is—under the above hypotheses—again IDHR, being so the time-dependent integrand of (102).

3.2.6 Dynamic Stress-Strength Models with "Shock type" Stress: A memoryless Dynamic Stress-Strength Model

With reference to the general expression (81) of the RF over the *interval* (0, t), let us assume that:

- $X(t)$ is a SP which can be described as a "Shock type" stress.
- The shocks occurring at the time instants T_j.

The device fails only because of the occurrence of a stress, i.e., at the time $t = T_j$ when stress amplitude is greater than Strength $Y(t) = Y(T_j)$; of course, such failure time is a RV.

It is observed that, in order that the device does not fail in the *whole* interval (0, t), then every Stress within the given interval must be smaller than the relevant Strength, i.e., ($X_j < Y_j$) must be verified for every index $j = 1, \ldots, N(t)$, where $X_k = X(T_k)$, and $Y_k = X(T_k)$, T_k being the RV "time of k-th stress occurrence".

The RF can be obtained first by conditioning on the event $E_n = [N(t) = n]$:

$$R(t|\, E_n) = P[(X_1 < Y_1) \cap (X_2 < Y_2) \cap \cdots \cap (X_n < Y_n)|E_n]. \tag{103}$$

Note that $R(t|E_n)$ is indicated as R_n in what follows.

Then, once the functions R_n have been computed, the RF $R(t)$ can be obtained applying the total probability theorem as a function of the R_ns and of the distribution of the point process $N(t)$:

$$R(t) = \sum_{n=0}^{\infty} R_n(t)p(n, t), \quad t > 0, \tag{104}$$

where $p(n, t) = P[N(t) = n]$. In the following, the above introduced Poisson law will be used, but the methodology is not dependent on such assumption.

In the model here hypothesized, both stress and stress are time-independent. In this case, assuming also that the RV X_j ($j = 1,\ldots n,\ldots$) and Y_j ($j = 1,\ldots n,\ldots$) are statistically independent of each other and of $N(t)$, then:

$$R_n = R(t|E_n) = P[(X_1 < Y_1) \cap (X_2 < Y_2) \cap \cdots \cap (X_n < Y_n)|E_n)] = \prod_{k=1}^{n} P(X_k < Y_k). \tag{105}$$

So letting $\prod_{k=1}^{n} P(X_k < Y_k) = r_n$, the RF is given by:

$$R(t) = \sum_{n=0}^{\infty} r_n(t)p(n,t), \quad t > 0. \tag{106}$$

A simple case where the RF is analytically computable is when:

- the X_j are IID with common cdf $G(x) = F_X(x) = P(X_j < x), \quad \forall j = 1, 2, \ldots n, \ldots$ (independent of time) and pdf $g(x)$;
- the Y_j are IID with common cdf $F(y) = F_y(y) = P(Y_j < y), \quad \forall j = 1, 2, \ldots n, \ldots,$ (independent of time), and pdf $f(y)$;

Then:

$$r_n = r^n, \tag{107}$$

where using the same approach as in (82), and denoting by X a generic one of the X_j RV (and the same for Y and Y_j):

$$r = P(X < Y) = \int_0^{\infty} g(x)(1 - F(x))\mathrm{d}x. \tag{108}$$

Under the above hypotheses, the wear process can be defined as a "memory-less" one, since wear at age t does not depend on previously occurred shocks. It is possible that such model applies to self-healing insulating materials.

According to (106), by resorting to well known properties of series expansion of the exponential function appearing in the assumed Poisson law $p(n, t)$, the result is straightforward:

$$R(t) = \exp[-\phi t(1 - r)] = \exp(-\phi q t) \tag{109}$$

having defined q as the elementary failure probability $q = 1 - r = P(X > Y) =$ probability that the generic stress X_j is greater than the generic strength Y_j.

In the case that the Stress is a constant, y, then:

$$q = 1 - F_X(y) \rightarrow R(t) = \exp[-\phi t(1 - F_X(y))]. \tag{110}$$

The above RF is clearly an Exponential one, i.e., it may be expressed as:

$$R(t) = \exp(-\lambda t) \tag{111}$$

with parameter λ(hazard rate) $= \phi q =$ (mean stress occurrence) $\times$ (elementary failure probability).

As an example, a particular case of interest is that in which the generic Strength Xj has—as already used above—a Weibull cdf:

$$F_X(x) = 1 - \exp\left(-ax^b\right), \quad \text{for}\, x > 0 \tag{112}$$

in this case, the RF of (110) has the following expression:

$$R(t) = \exp\left[-(\phi t)\exp\left(-ay^b\right)\right] \tag{113}$$

or alternatively, by setting $\tau = 1/a^{1/b}$:

$$R(t) = \exp\left[-(\phi t)\exp\left(-(y/\tau)^b\right)\right]. \tag{114}$$

The above RF may be expressed as a function of the mean value m of the Strength X, given by $m = E[X] = \tau\Gamma(1 + 1/b)$.

DRA and IRA are compared for this model, also in view of estimation, in [38].

In conclusion, it is highlighted that the illustrated SS model is, in this case, characterized by a constant hazard rate. This fact—together with the IDHR property of many models previously examined (LN, IG, IW, LL)—may appear at a first sight conflicting with intuition, again, since one could expect that the presence of stresses, whose number surely increases with time, implies an increasing hazard rate. This confirms that a careful analysis of the hypotheses which yield the reliability model may lead to non-trivial conclusions, difficult to be anticipated by pure intuition. By the way, by means of the above SS model a further (and seldom reported in literature) justification for the LT being an Exponential random variable is obtained, which is to be added to those leading to the Weibull or the Gamma model with shape parameter 1, examined in the following.

3.3 Identification of Main Probabilistic Lifetime Models by IRA

In this section, the same list of reliability models of Sect. 2 is considered, with their generative mechanisms from wear processes. It is recalled that the identification of the Exponential and the log-logistic model have been already deduced previously (Sect. 3.2), so that they will be considered again here with different motivations; also the Birnbaum–Saunders distribution has been hinted at before (see the comment following Eq. 101).

3.3.1 Birnbaum–Saunders Model

The BS model [62, 63, 96]—sometimes denoted as “fatigue life” model—was originally derived on the basis of a “discrete” stress process, accounting for accumulating cracks on a material, which can cause its failure when a given “critical dimension”, y, is overcome. Let the material be subjected to repeated cycles of a common stress, the single stress amplitudes being s-independent Gaussian RV; so, using an approach similar to that used in Sect. 3.2.5, assuming a "cumulative wear process", and denoting by Z_K the stress amplitude at k-th cycle, the total stress after n cycles is of course

$$W_n = \sum_{k=0}^{n} Z_k, \quad n = 1, 2, \ldots, \infty. \tag{115}$$

Then, the original BS model was obtained by observing that the "discrete" failure time N (i.e., the number of cycles after which the failure occurs) has the following cdf (defined for discrete values of n in the set of natural numbers):

$$P(N \leq n) = P(W_n > y) \tag{116}$$

(it can be easily deduced, e.g., that the random events: $(N > n)$ and $(W_n < y)$ are equivalent; then, considering the complementary events, the above relation is obtained).

Assuming that the Z_K RV are IID Gaussian RV, with $N(a, b)$ distribution ($a > 0$ for obvious reasons), or that they are in a high number so that the central limit theorem holds, then W_n is a $N(an, b\sqrt{n})$ RV, so that:

$$P(N \leq n) = 1 - P(W_n \leq y) = 1 - \Phi\left[(y - an)/\mathrm{b}\sqrt{\mathrm{n}}\right] = \Phi\left[(an - y)/b\sqrt{n}\right], \tag{117}$$

where the known property of the standard Normal cdf: $\Phi(-x) = 1 - \Phi(x)$ has been used. It is remarked that the symbol n above denotes the "time" argument of the cdf. Now, in the original deduction of the model, discrete time n was "transformed" into continuous time t (this is somewhat incorrect, but can be roughly justified by letting the cracks occurring at a constant rate r in time, so that $n = rt$), then the following cdf can be easily obtained—with an understandable meaning of the positive constants (α, β):

$$F_T(t; \alpha, \beta) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right]\right\} \tag{118}$$

which is indeed a BS cdf.

The limits of the above derivation of the BS model were highlighted, e.g., by [15], where it is correctly observed that a Gaussian RV can assume—even if with small probability, by an adequate choice of the relevant constants—values less than 0: so, the above stress model does not guarantee that W_n is an increasing function of time n (indeed, a "negative-amplitude" crack can occur every now and then), thereby resulting in a non-realistic model for accumulated stress. In [15], the authors, by comparing the BS with the similar IG model (to be dealt with afterwards), find the IG model superior in that it is free of the above limitations and directly formulated in continuous time.

However, as also reported in [96], Desmond [62] derived a more general form of the BS model relaxing the hypothesis of Gaussian crack amplitude and of s-independence. He showed that the BS pdf can be obtained by a mixture of two appropriate IG pdf, too.

However, the BS model remains less attractive and less adopted than the IG one, mainly because well-established estimation methods for the IG exist, while only some "ad hoc" methods are available for the BS model.

3.3.2 Gamma Model (featuring also the Exponential model)

The Gamma $G(r, \phi)$ pdf is given by:

$$f(t; r, \phi) = \frac{\phi^r t^{(r-1)}}{\Gamma(r)} \exp(-\phi t), \quad t > 0. \tag{119}$$

Two main methods of deducing the Gamma model from wear processes are presented here below at points (a) and (b), followed by the particular Exponential model at point (c). Two (minor) transformed forms generated by the model, denoted "Inverse Gamma model" and "Generalized Inverse Gamma model", are only hinted at for brevity afterwards, when dealing with the LN model.

1. It is well known [96] that a Gamma RV $G(r, \phi)$, if r is a positive integer, can be obtained as the sum of r Exponential statistically independent and identically distributed RV with parameter ϕ, which is also the value of their common, constant, hrf. Although this motivation for its use appears to be rarely adopted (apart from the particular case of an "Exponential unit" with $r - 1$ stand-by redundant identical units), it can be advocated when the LT of the devices passes through a series of s-independent "stages", each one lasting a time interval which is distributed according the same Exponential RV. This may be the case, e.g., when the device, starting from an initial "good" state, reaches the failed state after a few "partial failure" states. In practice, however, it is difficult to imagine a situation in which all the stages have the same pdf (thus the same value of parameter ϕ) and are s-independent. On the other hand, if the s-independence subsists, and the single-stage RV are Exponential RV with different values of the parameter ϕ, the pdf of their sum has a closed form expression [96], which can be approximated in many cases by a Gamma pdf, even when the number of stages, r, is unknown (as realistic). In the general case of unknown single pdf, with r high enough (say, $r > 5$), the CLT may be a valid reason for approximating the LT by a Gaussian RV. Indeed, it is true that the Gamma model $G(r, \phi)$ is closely approximated by the Gaussian one for r high enough.
2. Although the following motivation appears to be rarely, if ever, highlighted in literature (maybe for its triviality), the Gamma model may also be deduced by means of a "cumulative wear process", i.e., the SP denoted as $W(t)$ in Eq. 94a, 94b,

as briefly accounted in the following. Let us suppose that the stress amplitudes Z_K at time T_K—appearing in (94a)—are not RV, but possess the same constant (deterministic) value. With no loss of generality—by choosing a proper stress amplitude unit—it may be supposed that $Z_K = 1$ for each k. Thus, the SP $W(t)$ simply equals the Poisson process $N(t)$. Then, assuming—as in (95)—that the strength Y has also a constant value $y > 0$, the device reliability function is given—from (95), using the Poisson distribution of (93), and denoting by r the largest integer value smaller than y—by:

$$R(t) = P(N(t) < y) = \sum_{k=0}^{r} \frac{(\phi t)^k}{k!} \exp(-\phi t). \tag{120}$$

This is a well known formulation—denoted as "Erlang"—of the Gamma RF [96]. It is easy to see indeed that also in this case, the time to failure is the sum of r Exponential independent and identically distributed RV with parameter (hrf) ϕ, where ϕ and r are positive constants representing the shape and scale parameters, respectively.
In view of relaxing some of the hypotheses leading to the above model, it is interesting to remark that the Gamma model has proven to be a satisfactory approximation to the true model even in the case of random stress (Z_K): this has been shown numerically, in some cases, in [2], but cannot be, at present, claimed to be always true.

3. (*Exponential model*) The above points also apply to the Exponential RV, when the particular case $r = 1$ is considered. Moreover, at least two additional deductions of the Exponential model, which appear more fundamental, must be considered. The first is the classic one of the "memoryless" property, already recalled, by which such model is the only one allowable for devices whose failure is only due to accidents, with no regard to age. Indeed, the Exponential model is the only one possessing a constant hrf, and so a CRF $R(t|s)$ independent from age s. The second deduction has been obtained as a consequence of the particular Dynamic Stress-Strength Models with "Shock type" Stress of Sect. 3.2.6. Moreover, the Exponential model can be obtained as a particular case of the Weibull model with shape parameter 1, which will be considered afterwards.

3.3.3 Gaussian Model

Despite its high popularity, it is difficult to justify the adoption of the Gaussian model as a lifetime model, first of all because it is not restricted to positive values. In addition, such model can be theoretically obtained only—by virtue of the CLT—when the LT can be expressed as the sum of many s-independent RV. This is why the peculiar cases already illustrated for the Gamma model $G(r, \phi)$—with r high enough—can be often approximated also by the Gaussian one, but actually there are only drawbacks in doing so, for theoretical and practical (statistical) reasons already presented, i.e., the non-positivity, the inadequacy in representing skewed pdf (which are by far the most part of LT pdf occurring in the real world), and, in general, the scarce flexibility of the model, also with reference to the hrf (which can assume only a roughly linear form in time). It is also to be remarked that other, more flexible models—such as the Gamma $G(r, \phi)$, the LN [$LN(\alpha, \beta)$], and the Weibull $W(a, b)$—can also closely approximate symmetrical pdf, by an opportune choice of their shape parameter values (the first if $r > 10$, the second if $\beta \rightarrow 0$, the third if $b \approx 3.6$). The CLT motivations can support the use of the Gaussian model as a repair time model in availability studies that are outside the scope of this chapter. Obviously, the Gaussian model fully maintains its well-established supremacy—also in reliability applications—for what concerns inference studies (e.g., for obtaining the distribution of statistical estimators, confidence intervals, etc.).

3.3.4 Gumbel Model

It is recalled that the Gumbel model is the "smallest extreme value" model, and this constitutes also a clear motivation for its use, even if it might be in many cases better advocated for systems rather than for components. From a statistical modeling point of view, the fact that the hrf can have only one shape (the increasing exponential one) is a limit of the model; at the same time, this may make it useful for devices largely affected by wear with increasing age (such as mechanical products; for the same reasons, also human being LT are often characterized by this or similar models).

Instead of LT values, a physical reason for adopting the Gumbel model can be found in the case of strength values. This may be the case when dealing with dielectric strength of electrical insulation; indeed, the value of breakdown voltage of a large-size insulation system may be considered as the minimum between the values of breakdown voltage of smaller elements. Very significant applications in this field, also comparing the Weibull and the Gumbel models (that can be used both as a *smallest extreme value model*) are in [66, 87], the first also containing physical properties which can justify the models.

3.3.5 "Hyperbolic Reliability Model"

The HRM is characterized by the hrf:

$$h(t) = r + \frac{a}{t+1}, \quad a > 0, r > 0. \tag{121}$$

Erto and Palumbo [76] showed that the HRM identification can be performed observing al least three mechanisms of failure leading to this law of mortality, which they shown to be: (1) a "Deterioration" mechanism of failure; (2) a "Stress-

Strength" mechanism of failure; (3) a "Shocks" mechanism of failure. For sake of brevity, referring to the above paper for the first and second, only the third mechanism is illustrated here. Let a device be subjected to a random succession of shocks, which can potentially cause system failure; let the shocks occur following, as discussed previously, a Poisson law:

$$\Pr\{N_s = n_s\} = \frac{(\delta t)^{n_s}}{n_s!}\exp(-\delta t), \quad \delta > 0. \tag{122}$$

Now, if the survival probability for each shock depends on the run time (but not on the number of previously suffered shocks) following the law:

$$S_v(t) = \rho - \frac{\tau}{t+1}, \quad 0<\rho<1,\ \tau > 0, \tag{123}$$

then the probability of failure in a time interval Δt is:

$$\begin{aligned} F(t+\Delta t) - F(t) = R(t)\sum_{i=1}^{\infty}\Bigg\{&\frac{(\delta\Delta t)^i}{i!} \\ &\times \exp(-\delta\Delta t)\left[1 - S_v^i(t)\right]\Bigg\} \end{aligned} \tag{124}$$

from which

$$h(t) = \lim_{\Delta t\to 0}\frac{F(t+\Delta t) - F(t)}{R(t)\Delta t} = \delta[1 - S_v(t)], \tag{125}$$

and substituting $S_v(t)$ with its time-dependent function

$$h(t) = \delta(1-\rho) + \frac{\delta\,\tau}{t+1}. \tag{126}$$

Renaming the two products

$$\delta(1-\rho) = r, \quad \delta\tau = a, \tag{127}$$

the hrf in Eq. 121 is obtained.

As previously hinted at, an electrical application—relevant to the times-to-breakdown of an insulating fluid, working at constant voltage equal to 32 kV—was successfully illustrated by means of the HRM in [76].

3.3.6 Inverse Gaussian Model

Also, the IG model can be obtained from a "Stress-Strength" (SS) model arising from a Wiener Stress process and a deterministic Strength. Indeed, as hinted at previously, the IG distribution [34] has been introduced as the first passage time of a Wiener process [134], i.e., a Gaussian Stochastic Process with independent increments. Let us hypothesize that the SP $W(t)$ describing wear is a Wiener process with "drift" $\mu > 0$ and "diffusion constant" $v > 0$. Then, $W(t)$ satisfies the differential equation:

$$-\mathrm{d}W/\mathrm{d}t = \mu + G(t), \tag{128}$$

where μ is a positive constant and $G(t)$ is a Gaussian SP with the following mean and covariance functions [134]:

$$E[G(t)] = 0,\ \mathrm{Cov}[G(t)G(t-s)] = v\delta(s) \quad (v > 0). \tag{129}$$

If the wear process is defined by a stress characterized by the above Wiener process and a deterministic strength b, the associated LT RV is defined by the first instant, T, in which the SP $W(t)$ crosses the barrier b, i.e.:

$$T = \inf\{t : t > 0,\ W(t) = b\}. \tag{130}$$

As shown in [34], the pdf of T—by a proper choice of the parameters μ and λ (both having the dimensions of time), and letting, with no loss of generality, $b = 1$—is given by:

$$f(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left[-\frac{\lambda}{2\mu^2 t}(t-\mu)^2\right], \quad t,\mu,\lambda > 0 \tag{131}$$

i.e., the above-mentioned IG pdf as in Sect. 2.

It has been already recalled in Sect. 2 that the IG and the BS models are very similar, and this may be also justified by the above similar derivations from Gaussian wear processes; it is again highlighted that they are both IDHR.

3.3.7 Inverse Weibull Model

The reliability function of the IW model is:

$$R(t;\alpha,\beta) = 1 - \exp\left[-(\alpha t)^{-\beta}\right]. \tag{132}$$

Apart from deriving the abovementioned peculiar properties of aging of this IDHR model, Erto [70] also showed that it can be originated by reasonable Stress-Strength models and deduced at least two possible "physical" derivation of the IW model, shown in the following.

1. (*Stress is a Weibull RV and strength is a deterministic function of time*) Let the distribution of the RV stress, X—assumed as time-independent—be a RV distributed according a Weibull $W'(u, v)$ law, thereby having cdf:

$$F_X(x) = P(X \le x) = 1 - \exp\{-[x/u]^v\} \quad (b > 0). \tag{133}$$

Let the strength $y(t)$ be a deterministic decreasing function of time, described by a process with an "inverse power" aging law as:

$$y = y(t) = k/t^h, \quad h, k > 0 \tag{134}$$

(a lowercase letter is used for strength y since it is not a RV here; h and k are deterministic constants, even if unknown in practice); since X has the above $W'(u, v)$ cdf, the RF at time t is:

$$\begin{aligned} R(t) &= P[y(t) > X] = F_X[y(t)] = 1 - \exp\left\{-\left[k/\left(ut^h\right)\right]^v\right\} \\ &= 1 - \exp\left\{-[1/(\alpha t)]^\beta\right\}. \end{aligned} \tag{135}$$

This is indeed an IW model with parameters σ and β both positive, assuming $\beta = hv$ and $\sigma = (u/k)^{1/h}$.

2. Strength is a deterministic constant, Stress is a Weibull SP, increasing in time) In the same framework as above, let the strength y be a deterministic constant, and let the stress $X = X(t)$ be a random function (SP) of time, with Weibull $W(a, b)$ cdf—$F_X(x, t)$—and time-dependent scale parameter $a = a(t)$:

$$F_X(x, t) = P[X(t) \le x] = 1 - \exp\left\{-[x/a(t)]^b\right\}(b > 0). \tag{136}$$

Let us suppose that $a(t)$ is an increasing power function of time:

$$a(t) = kt^m \quad (k, m > 0). \tag{137}$$

This second hypothesis implies, as reasonable, that the mean value of stress increases with time, since, under the Weibull model:

$$E[X(t)] = a(t)\Gamma(1 + 1/b) = kt^m\Gamma(1 + 1/b) \tag{138}$$

with b constant. Thus, the reliability function at time t is:

$$\begin{aligned} R(t) &= P[X(t) \le y] = 1 - \exp\left\{-[y/a(t)]^b\right\} = 1 - \exp\left\{-[y/kt^m]^b\right\} \\ &= 1 - \exp\left\{-[1/(\alpha t)]^\beta\right\} \end{aligned} \tag{139}$$

in which the positive constants, α and β, have a clear meaning as functions of (b, k, m, y). Again, T is an IW RV.

A final remark. Since all the above deductions appear to be coherent with the known, measurable, properties of stress and strength of electrical insulation, in our opinion they should stimulate new applications of this seldom adopted model. It should also be remarked that they can lead to further justification of the already recalled IDHR property observed sometimes, also in data obtained by ALT, for such materials [93, 148].

3.3.8 Log-Logistic (LL) Model (with a Hint at Burr Model)

Two methods of generating the LL model are hinted at here. Indeed, the LL model has been already been deduced—as in [37]—in Sect. 3.2.2 in the framework of a "quasi static" Weibull Stress-Strength model. Moreover, it will be shown in the last part of this section that it can be motivated by a mixture model deriving from a Weibull $W(A, b)$ RV, with random scale A. So, the model is characterized by a conditional RF:

$$R(t|A) = \exp\left(-At^b\right) \tag{140}$$

with random scale parameter, A, distributed according an Exponential distribution. As it will be shown when dealing with mixtures (Sect. 3.3.11), if A is an Exponential RV with mean s, then, applying the total probability theorem, the unconditional RF $R(t)$ deduced by the conditional RF above has indeed the following expression, which is clearly seen to be coincident with a LL RF:

$$R(t) = \frac{1}{1 + st^b}. \tag{141}$$

Such model can be considered also as a particular case of the "Burr" model (Sect. 3.3.11), and is still IDHR or DHR. Finally, it is observed that the LL model appears to be very similar to the IW model, for which indeed an analogous "Stress-Strength model" motivation has been presented just above. More evident, sometimes impressive, is the analogy with the LN model (including the IDHR property with hrf decreasing toward zero). This analogy arises from the strong similarity between the Gaussian and the Logistic model which give rise to the LN and LL model, respectively, through the exponential function $y = \exp(x)$. In [37], a very good approximation of the LL model was obtained, which worked satisfactorily for the RF, the pdf and the hrf (while it is well known that it is very difficult to approximate all these functions at the same time). This was also shown by many graphs reported in the abovementioned paper, but should be tested with a wider range of parameter values. The above approximation is obtained by the simple method of equating the first and second moment of the logarithms of the LL and LN RV, which of course uniquely determine the parameter values of the two models (it is not practical to equate the simple moments of the two models, since the LL does not possess the first moment if $b < 1$, the second moment if $b < 2$).

In some applications, when both models fitted well data, the LL has been sometimes preferred to the LN for its simpler analytical expression (this is evident especially for the hrf); on the other hand, the LN has no restriction on the parameters for what concerns the existence of its moments, and possesses more desirable properties from the parameter estimation point of view. Indeed, ML and moment estimates are well established and readily available in the LN model, while their deduction is cumbersome for the LL model; for the latter, even the moment estimates can be problematic in view of the above restrictions on the theoretical moments.

3.3.9 Lognormal Model (with Some Reference to the "Inverse Gamma" and "Inverse Generalized Gamma" Model)

Many ways for deducing the LN model on theoretical grounds from wear processes have been illustrated in literature, while—on the "practical" side—many applied papers have shown its great capacity to fit experimental data from very different fields (e.g., speaking of duration data, from the time to first marriage of a person, to the microelectronics lifetime model). In fact, reliability theory is one of the few fields in which the use of the LN model largely surmounts that of the Gaussian model (from which the LN one was derived). Often, the LN model is derived as a proper model for wear itself rather than for lifetimes (as also briefly shown below). Its applications to LT distributions (as witnessed by [53]) are however numerous, due especially to its flexibility: the Lognormal pdf is indeed capable of assuming a large variety of shapes with positive skewness index, which allows for typical large "right tails". In particular, Cox and Oakes [51] show that, among the most popular models, the LN possesses one of the most high "skewness coefficient" values for a given coefficient of variation (CV) value. Furthermore, two properties that appear to be scarcely recalled improve its flexibility: (a) as already hinted at, if the β (shape) coefficient of the LN(α, β) model is small enough (in practice, $\beta < 0.3$) the LN pdf tends to become symmetrical and may satisfactorily approximate even a Gaussian model with the same mean [*this fact can be proved analytically, using the series expansion of* $y = exp(x)$ *for* $x \to 0$]; (b) the CV, v, can assume a wide range of values: in particular—if $\beta = 0.8325$—the value $v = 1$ is obtained, as for the Exponential model, to which the LN appears in this case to be very similar, and often indistinguishable from it (this may explain also its abovementioned applicability to microelectronics).

Also the "decreasing hazard rate" property of the Lognormal distribution for large values of time is a desirable property, for instance in ALT and insulation applications. The hrf properties of the LN (which is, it is recalled, a IDHR model), its high variability and the presence of large right tails are perhaps the main reasons for its being the most applied for repair times, as shown in power systems literature [24, 159]. The above properties all account for the possibility of relatively large times compared to their expected values: thus, the LN assumption may also be justified by a "conservative" approach which seems very appropriate for repair times (and, in general, for "waiting times") when the exact distribution is unknown; such kind of properties, however, are still more evident for the less known and less adopted LL model, which has indeed a higher "skewness coefficient" value for a given CV value, as shown in [51].

All the above reasons and the rapidly growing literature on the model have validated the authoritative forecast of two scholars such as Johnson and Kotz, who in 1970 already stated that "it is quite likely that the LN distribution will be one of the most widely applied in practical statistical work in the near future", as reported in the preface to [53].

Only two significant models which may give origin to the LN model for LT are presented here; by the way, they also can give rise—under different hypotheses—

to the so-called “Inverse Gamma” and “Inverse Generalized Gamma” models, as shown at the end of this section. A LN model for wear can also be the origin of a BS model for the corresponding lifetime distribution, as shown in the end of the present section.

Linear function Stress Process leading to the Lognormal model. In many cases, it may be reasonable to express, or approximate over some time interval, Stress (or Strength) by means of linear (random) functions of time (see [40, p. 103] for the physical deduction of such models, with some examples beyond those here presented, as that of “alpha” pdf, including the effect of temperature as a stress parameter in the *Arrhenius model*).

A linear Stress process may be written as:

$$X(t) = Bt \tag{142}$$

being B, in the general case, a RV, with $B > 0$ almost surely (a.s.), so that the Stress is a.s. increasing in time (for this reason a Gaussian model, e.g., for the RV B is not opportune).

As a simple example of a SS model with linear Stress process and a LN Strength, let us consider the Strength model of last equation with B Lognormal-distributed, and let Y also be Lognormal-distributed, and B and Y be independent. Then, of course, the LT T is such that:

$$BT = Y \rightarrow T = Y/B. \tag{143}$$

Then, since the ratio of two independent LN RV is also a LN RV (this is indeed the same property as that the difference of two independent Gaussian random variables is also Gaussian), T is a LN RV, with parameters which can be obtained very simply.

If the RV Y is deterministic (i.e.: $Y = y$, constant), then T is again a LN RV.

Power Function Stress Process leading to the Lognormal Model. Quite similar results are obtained if stress $X(t)$ is a “power function” of time such as:

$$X(t) = Bt^c. \tag{144}$$

Let again B a Lognormal-distributed RV: also in this case, with a LN strength Y, then the LT is again LN. This can be seen very easily recalling that, if T is a LN RV, then also T^h is a LN RV, whatsoever the real value of the exponent h.

A hint at “Inverse Gamma” and “Inverse Generalized Gamma” Model. The chance is taken here of giving at least a brief mention of two models, which are defined in reliability or survival literature (e.g., in [103]), but are very seldom used. A little diversion from the LN model will be allowed here, only because the two models both appear similar to the LN one, also their deduction being similar. With reference to the linear Stress process of point (1): $X(t) = Bt$ with fixed strength y, let us suppose that B is not LN, but Gamma-distributed (another reasonable hypothesis, since this is an adequate model for positive quantities, and also flexible). Then the LT, which is expressed as above:

$$T = y/B \tag{145}$$

is a so-called “Inverse Gamma” RV, already described in Sect. 2) and used until now mainly as prior pdf in Bayesian estimation, e.g., for the MTTF parameter in the Exponential model [113]. It is curious to notice that, if B is a $G(r,)$ Gamma RV with $r = 0.5$, then T follows a particular form of the *Inverse Gaussian* model.

Finally, let us assume that the stress $X(t)$ is a “power function” of time of the kind:

$$X(t) = Bt^c \tag{146}$$

and that B is again a Gamma RV. Under such hypotheses—with fixed strength y—the LT is given by:

$$T = (y/B)^d \tag{147}$$

with $d = 1/c$, and B^d has a “Generalized Gamma” pdf (already described in Sect. 2). Then T has a so-called “Inverse Generalized Gamma” pdf, which seems to be never used before as a LT pdf. It was used by the authors as a prior pdf in Bayesian estimation for some insulation reliability applications [39], and was also used in relation with heterogeneity studies in statistics with applications to economy (e.g., for the durations of unemployment spells).

3.3.10 Weibull Model (Also Featuring the IPM)

It is not difficult to deduce the Weibull model from wear process considerations, also using its property of being a particular Extreme Value Distribution for Minima. Here, the following simple derivation is proposed, in the framework of Stress-Strength models, using the EV property not for LT itself, but for dielectric Strength (as in [35]). By the way, the same procedure, as it will be seen, may yield a theoretical justification for the well-known IPM, often used especially in ALT on insulation, as discussed in [42, 131]; it is one of the most popular models—among the many proposed since the seminal paper (devoted to survival analysis) by Cox [50]—taking into account the effect of “covariates” (here, the stress, intended as the applied voltage) on the lifetimes.

Let the Strength of the object (e.g., the breakdown voltage for insulation) be a RV Y, and z be the applied constant (in time) stress (e.g., the applied voltage peak value).

The random nature of device strength at time t is to be addressed to the lack of homogeneity of the device material. In order to emphasize this fact in probabilistic terms, let us suppose that the device can be considered (as reasonable for large-size insulation) as a system constituted of a number n of homogeneous elemental components. Denoting as Y_i the strength of the ith elemental component ($i = 1, \ldots, n$), the strength Y of the device can be then expressed as a function of the RVs Yi as it follows (omitting for the moment any relevant time dependence):

$$Y = \min(Y_1, Y_2, \ldots, Y_n) \rightarrow P(Y > z) = P(Y_1 > z, Y_2 > z, \ldots, Y_n > z). \quad (148)$$

Such relationship holds in every fixed time instant *t,* and implies that the device strength is determined by the strength of its "weakest link", or rather of the least strong elemental component. In what follows, it is supposed that the RVs Y_i are statistically independent; in general, however, they will not be identically distributed. Of course, for such model to be realistic, number n has to be high (ideally, infinite): this allows to get, under very general conditions, a probabilistic characterization of Y resorting to the asymptotic theory of the extreme values [28]. Under general hypotheses, the limit cdf of Y is of the Weibull type, and it can be expressed in the form:

$$F_Y(r) = 1 - \exp(-m \cdot r^k), \quad r \geq 0;\ m > 0,\ k > 0. \quad (149)$$

Parameters m and k are real and positive, and depend on the parameters of the component distributions. Thus, in practice they must be estimated experimentally, since in general the Y_i RVs are not observable. For instance, relationship (149) is verified if the component cdf are of the Gamma, Exponential, Beta, Weibull, Pareto, etc., type, that is for most of the distributions that have a pdf "limited to the left". Since a strength-type RV is intrinsically non-negative (as is in fact the case of dielectric strength), and typically it has a domain (a, ∞), the cdf of the Y_i RVs are of such type, too.

Let us now consider the time dependence of the distribution of the process Y_t. Such dependence will be clearly expressed by the time variation of the parameters m (scale parameter) and/or k (shape parameter) with time. The expectation of the above Weibull model, $W(m, k)$, for the RV Y is expressed as:

$$\mu = \Theta^c \cdot \Gamma(1 + c), \quad \text{where: } \Theta \equiv 1/m, \quad c \equiv 1/k \quad (150)$$

and $\Gamma(x)$ is the Euler-Gamma function.

From (150) it is not difficult to deduce that the mean μ is a decreasing function of m, while it is not a monotonous function of k. Since it has been assumed previously that the strength is decreasing with time, its average value will be decreasing with time as well. The simplest way for accounting of this behavior—on the basis of what previously said about the μ versus m relationship—is to assume that the scale parameter m is an increasing function of time. Moreover, if (149) has to match the properties of a cumulative distribution function, it can be immediately noticed that the strength Y_t has to be infinite in $t = 0$ and zero in $t = \infty$. This gives rise to the following conditions on the time function $m(t)$:

$$m(0) = 0; \quad m(\infty) = \infty. \quad (151)$$

Such conditions are satisfied by the following simple model:

$$m = m(t) = m_0 t^b, \quad m_0 > 0,\ b > 0, \quad (152)$$

where m_0 and b are proper constants to be determined. If such model holds, the time behavior of the mean value of strength in time is expressed by a function of the type:

$$\mu = \mu(t) = \mu_0/t^p, \quad \mu_0 > 0, \quad p > 0 \tag{153}$$

[in particular, $\mu_0 = (1 + c)/(m_0)^c$; $p = bc$].

By introducing (152) into (149), the distribution function of process Y_t (for $r > 0$ and $t > 0$) is obtained:

$$F_Y(r;t) \equiv P(Y_t \leq r) = 1 - \exp\left[-\left(m_0 t^b r^k\right)\right] \tag{154}$$

being m_0, b, $k > 0$.

Therefore—being z the constant stress applied to the device, the reliability function of the device is obtained, using the above relations:

$$R(t;z) = P(Y_t > z) = 1 - F_Y(z;t) = \exp\left[-\left(m_0 t^b z^k\right)\right]. \tag{155}$$

Equation 155 shows that the reliability function of the device is of the Weibull type; in particular, the RV T follows a Weibull distribution with shape parameter b [defined according to (152)] and scale parameter dependent on stress z, namely: $T \sim W(a, b)$, being:

$$a = a(z) = m_0 z^k, \tag{156}$$

where b, m_0, k are constants >0.

Parameter z is known for hypothesis (constant stress, e.g., applied voltage), whereas constants b, m_0 and k must be evaluated from available experimental data.

The above method leads to the same results if it is assumed that also the (deterministic) stress z varies with time, if this happens by means of a "power function" such as:

$$z(t) = z_0 t^q. \tag{157}$$

The statistical relationship between LT and stress z, expressed by (156) and popular in experimental applications, is the above-mentioned IPM. This denomination comes from the fact that, according to such model, the mean lifetime varies as a (positive) power of the inverse of stress; the same holds also for the percentiles of T. Indeed:

$$E[T] = (1/a)^C \cdot \Gamma(1 + c) = \alpha/z^h \tag{158}$$

being

$$c \equiv 1/b > 0;\ \alpha = (1/m_0)^c\, \Gamma(1 + c) > 0;\ h = c\,k > 0. \tag{159}$$

An analogous relationship holds for the LT percentiles. Since $h > 0$, the above relationships clearly highlight the decrease of the expected duration with the increase of stress z, as well as relationship (155) points out the reduction of the probability of survival as z increases.

Finally, it must be underlined that the IPM here introduced constitutes a particular “proportional hazard model”, as well as an “accelerated time model” [51], both widely applied in LT analysis, and validated by numerous statistical tests.

3.3.11 A Hint at Other Models: Mixture Models, Featuring Burr and Ll Models

It is obvious that the LT models are potentially infinite, as the possible ways of constructing them by means of opportune wear models; e.g., even a very simple wear model such as $W(t) = A + B_t$, by choosing among the infinite couples of RV A and B, and allowing for possible randomness of strength, can give rise to an enormous variety of RF.

However, it should kept in mind that the problem of finding the “true” model is insoluble and probably not very interesting from a practical point of view, as long as the resulting models may often not be very different from already established models, with the possible drawback of being characterized by too many parameters to be useful for real applications, in view of the need to estimate those parameters from few data.

So, also for reasons of space, only a peculiar family of models deduced from the combination of two models is here sketched., i.e., the “Mixture models”, based upon a random hazard rate.

A large variety of models can be deduced indeed by allowing some parameter of the LT pdf vary randomly among items, accounting for heterogeneity of material or production process, or random variability of environment in which the items operate. Theoretical studies on random hazard rate functions may be found in [85, 143, 146]. A popular model was introduced, among others, by [109], i.e., a model characterized by a “Proportional Hazard Model” [51] with random factor Z accounting for the above randomness, so that for a given value of the RV Z, the random hrf is written as:

$$h(t|Z) = Zh_b(t), \tag{160}$$

in which $h(t|Z)$ denotes a “conditional” hrf, given the positive random factor Z, and $h_b(t)$ is the “baseline”, deterministic hrf. In particular, an analytical model is proposed by [109], based on a conditional Weibull hazard rate with shape parameter b:

$$h(t|Z) = Zbt^{b-1}, \quad (t > 0, b > 0). \tag{161}$$

For the random scale parameter characterization, positive RV such as the Lognormal, Gamma and Inverse Gaussian distributions may be considered with reasonable motivations [35, 36]. It is easy to show that, if Z has pdf $g(z)$, the (unconditional) reliability function is given by the so-called [10] "mixture" of the RF according to the pdf $g(z)$:

$$R(t) = \int_0^\infty \exp\left[-zt^b\right] g(z)\mathrm{d}z \tag{162}$$

which is the Laplace Transform of the $g(z)$, evaluated in $s = t^b$. Differentiation of the log of $R(t)$ leads to the unconditional hrf. In the mixture models denoted as "Weibull-Gamma" (meaning that Z has a Gamma distribution) and "Weibull-Inverse Gaussian" (meaning that Z has an Inverse Gaussian distribution) analytical results exist [36]. In particular, let us assume that Z is Gamma-distributed, with pdf:

$$g(z; r, s) = \frac{z^{r-1}}{s^r \Gamma(r)} \exp\left[-\frac{z}{s}\right] \quad z > 0. \tag{163}$$

Then, it is easy to show, applying above relations, that the unconditional reliability function and hazard rate are, respectively, given by:

$$R(t) = \frac{1}{(1 + st^b)^r} \tag{164}$$

$$h(t) = \frac{rsbt^{b-1}}{(1 + st^b)}. \tag{165}$$

The above RF and hrf belong to a particular form of the so-called "Burr model" [96], already hinted at in Sect. 3.3.8. It is noticeable that such model coincides with the already illustrated Log-logistic one when $r = 1$, as anticipated in Sect. 3.3.8.

It is, thus, remarked that in this chapter two peculiar (and completely different) ways have been shown for deducing a LL model from a Weibull model: the first was the SS model of Sect. 3.2 (example 3.2.2), this second is the one of mixture models.

As already discussed, the LL hazard rate function $h(t)$ possesses the following properties:

- if $b \leq 1$, $h(t)$ is decreasing in t, with limit value 0 (DHR model);
- if $b > 1$, $h(t)$ first increases—starting from $h(0) = 0$—then decreases toward 0 (IDHR model).

The particular case $b = 1$ (Exponential-Gamma mixture model) is also denoted as "Lomax" or "Pareto of the second kind" [96, 109].

Noticeably, the above case $b = 1$ shows also a paradox discussed in literature [10, 35, 79, 125, 136, 142], even if often neglected. It was already met, for the discrete case, in Sect. 1.6. The (apparent) paradox is that the random variability of the environment (Z factor) gives rise to a decreasing hrf (DHR), even though the individual hrf is constant over time. The paradox can be fully justified by subjective probability reasoning [10, 136, 146]; it has also been explained in [125] with new motivations.

Still more remarkable, perhaps, is the fact that, if $b > 1$, the individual hrf are increasing (IHR model), while the overall (unconditional) hrf is decreasing for large t (IDHR model).

The above result also shows that DHR models remain, instead, DHR even after any mixture, and also this fact is a general property [10].

The above and many more properties of mixtures of LT have been thoroughly analyzed also in Mathematical Demography [86]. Recently, Singpurwalla (see [146], chapter 7) provided new insight in the study of life distributions derived from the characterization of the hrf as a stochastic process, motivated by the randomness of the dynamic variability of environment.

For completeness, it must be added (even if it is well beyond the topics of the present contribution) that the study of lifetimes influenced by a common random environment provided further methods to explore the statistical dependence between lifetimes of different components of a system [48, 54, 109]. This is a crucial topic which arises naturally in reliability analysis of electrical power systems, since—for obvious "physical" reasons—the components of such systems can seldom be considered as really s-independent.

Finally, returning to the main point, it is noticeable that the LL model may be obtained by a completely different approach from those presented previously.

3.3.12 A Remark on Non-Reliability Applications

Finally, it is noteworthy to remark—not only from an academic point of view—that all the above-mentioned studies on wear process, diffusion processes, Brownian motion, barrier crossings, etc., which were originated by motivations coming from reliability theory, have stimulated new important studies in the theory of stochastic processes, as witnessed by a recent paper by Cinlar [47], which adds significantly to all his previously cited papers and books. Interesting relations between reliability and biology (bio-mathematics) may be found, e.g., in the study of Ricciardi [137] on diffusion processes, which was referred to by Ebrahimi [68] in his key paper on IRA. Ebrahimi used indeed a Lognormal "Ito diffusion processes", typical of biology, as the basis for his methodology. Other noticeable applications in Economy are discussed in [101, 146] among the others.

Law of proportionate effect giving origin to the BS or LN model. A well-known physical model for wear—first analyzed in mechanical engineering in relation to fatigue crack growth process [53, 59]—considers a stress process acting on a device at discrete times $T_{k,}$ producing a succession of "cracks" on the material, which can ultimately yield its failure when the crack size exceeds the device strength. It is assumed a reasonable multiplicative effect—according to a model denoted as "law of proportionate effect"—on the component, such that the “crack size” after the $(k+1)$th stress, W_{k+1}, is proportional to the previous crack size W_k, according to the relation:

$$W_{k+1} = W_k(1 + Z_k), \quad k = 0, 1, \ldots \tag{166}$$

being Z_k a succession of non-negative, independent RV. It is easy to show that, as time (thus index k) diverges, the succession of RV W_k converges to a LN RV, according to the Central Limit Theorem applied to the logarithms of W_k. For instance, assuming for simplicity (but these hypotheses can be relaxed) that the RV $Y_k = \log(1 + Z_k)$ are independent and identically distributed (IID) random variables with common mean μ and common standard deviation σ, then the RV $L_k = \log(W_k/W_0)$ is—if k is large enough—approximately Gaussian, and its mean and variance are, respectively, $k\mu$ and $k\sigma^2$.

So, if the device strength, y, is deterministic, and the failure occurs as soon as the wear process overcomes y, it is easy to see—by introducing the log of both wear and strength—that a BS model is obtained for the LT [103, 112]. A similar model holds if strength is a LN RV (s-independent from stress).

If the failure, under this model, is not due to the “strength overcrossing”, but other hypotheses can be assumed for failure mechanism (as in [30], p. 33), then the LT may be instead described itself by a LN model.

4 Probabilistic Life Models for Electrical Insulation

4.1 *Basic aspects and main models for electrical insulation reliability*

As previously pointed out, the time-to-failure (life)[10] of an insulation, or of a component of which the insulation is a part, is a RV. Thus, it is always associated with the relevant failure probability, namely the probability of failing under the action of applied stresses, or, conversely, with the corresponding reliability, namely the probability of withstanding the applied stresses and surviving. In fact, aging and failure processes are regulated by stochastic laws, as demonstrated by the fact that identical specimens manufactured with the same material, subjected to the same levels of stresses, exhibit different failure times, because of the intrinsic inhomogeneities of the materials, the uncertainties in the manufacturing processes, the imperfect control of the test conditions and so on [127, 144].

In the particular case of electrical breakdown of solid insulation subjected to applied voltage, a huge series of experimental tests have been conducted through

[10] The term “life” is used throughout the present Sect. 4 for indicating the generic percentile of the distribution of times-to-failure of an insulation, according to a very common practice in electrical insulation literature since the very early times till now (see, e.g., [58, 65, 118, 127, 144]).

many decades. The results of such tests, combined with sound mathematical reasons related to the "Extreme Value" theory (which became more and more popular since the fifties until our days), and the advances in physical knowledge of the mechanisms of wear and stresses acting on insulation, have shown that the probability distribution that has turned out to be the best for reproducing the relationship between failure probability and life is the above-described Weibull distribution [129, 130]. This model, indeed, seems to realize the already-mentioned desirable combination of DRA and IRA, in a process which is of course dynamic and which, by means of modern literature [41, 87], is gaining further contributions and investigations.

Under the Weibull distribution, the cumulative failure probability (cdf), F, versus time t, is expressed—using for convenience the $W'(\alpha, \beta)$ parametric form—as follows:

$$F(t) = P(T \leq t) = 1 - \exp\left[-(t/\alpha)^{\beta},\right] \tag{167}$$

where α is the scale parameter and β is the shape parameter of the distribution: β is linked to the dispersion of the times-to-breakdown, while α coincides with the breakdown time at 63.2% failure probability, i.e., with the 63.2th percentile of breakdown times. It must be highlighted that both parameters are function of the stresses applied to the insulation, though the dependence of β is usually weaker and is neglected in practice [119, 127]. Hence, by indicating with $S_1, S_2, \ldots, S_N$ the values of the N stresses applied to the insulation (assumed as constant with time) α can be written as $\alpha = \alpha(S_1, S_2, \ldots, S_N)$. As a consequence, equation (167) can be rewritten as follows:

$$F(t; S_1, S_2, \ldots, S_N) = 1 - \exp\left\{-\left[\frac{t}{\alpha(S_1, S_2, \ldots, S_N)}\right]^{\beta}\right\}. \tag{168}$$

From (168), by virtue of the meaning of α, and denoting by t_F the 100 Fth percentile of time-to-breakdown, the so-called "probabilistic life model" of the considered insulating system can be derived, namely a relationship between life, stress levels and failure probability (or, conversely, reliability) [126]. In order to do that, Eq. 168 should be expressed in terms of t_F:

$$t_F(S_1, S_2, \ldots, S_N) = [-\ln(1 - F)]^{1/\beta} \alpha(S_1, S_2, \ldots, S_N) \tag{169}$$

that is the expression of the probabilistic life model implicit in terms of the stresses. It can be noticed that it enables the derivation, for any value of stresses S_1, $S_2, \ldots, S_N$, of the relevant 100 Fth percentile of breakdown time, $t_F(S_1, S_2, \ldots, S_N)$.

Note that, for its statistical significance, the scale parameter $\alpha = \alpha(S_1, S_2, \ldots, S_N)$ of the Weibull distribution is commonly chosen as the reference percentile of the distribution of failure times coming from breakdown tests on electrical insulation. Any other failure time percentile can thus be derived from $\alpha = \alpha(S_1, S_2, \ldots, S_N)$ and β resorting to Eq. 169.

Probabilistic life models are the fundamental tool for carrying out a reliability analysis on the basis of laboratory test results only, before the component is put in-service. Indeed, under the hypothesis that stress levels are constant and fixed, probabilistic life models enable the estimation of key reliability parameters and functions, such as the Mean Time To Failure (MTTF), the hazard function and the reliability function of the insulation, thus of the electrical device of which the insulation is the weakest part (and ultimately of the power system which the device belongs to). In fact, from Eq. 168, the reliability function at mission time t can be evaluated trivially via the following RF $R(t;\ S_1,\ S_2,\ \ldots,\ S_N)$:

$$\begin{aligned} R(t;S_1,S_2,\ldots,S_N) &= 1 - F(t;S_1,S_2,\ldots,S_N) \\ &= \exp\left\{-\left[\frac{t}{\alpha(S_1,S_2,\ldots,S_N)}\right]^{\beta}\right\}. \end{aligned} \tag{170}$$

Thus, failure rate at the same time t can be estimated via the following hazard function:

$$\begin{aligned} h(t;S_1,S_2,\ldots,S_N) &= \frac{\beta}{\alpha(S_1,S_2,\ldots,S_N)}\left[\frac{t}{\alpha(S_1,S_2,\ldots,S_N)}\right]^{\beta-1} \\ &= \frac{\beta t^{\beta-1}}{[\alpha(S_1,S_2,\ldots,S_N)]^{\beta}}. \end{aligned} \tag{171}$$

Actually, in Eqs. 168–171 the functional dependence of the relevant reliability model versus applied stresses is not fully assessed, until the functional dependence of α on S_1, S_2, …, S_N is not fully assessed; this is needed for the estimation of reliability and related quantities. Such functional dependence can be explained provided that the life model holding for the considered insulation (or component) has been singled out.

4.2 Insulation life models

Life modeling of electrical insulation has the goal of determining the most appropriate mathematical relationship (model) between the time-to-failure (life) of a given insulation and the levels of the various stresses applied to such insulation [119, 127, 144]. Therefore, referring to the 63.2% failure probability a life model in its most general form can be expressed as follows:

$$\alpha(S_1,S_2,\ldots,S_N) = f(S_1,S_2,\ldots,S_N;p_1,p_2,\ldots,p_M), \tag{172}$$

where p_1, p_2, …, p_{M} are the model parameters and $f(S_1,\ S_2,\ \ldots,\ S_N;\ p_1,\ p_2,\ \ldots,\ p_M)$ is a proper mathematical function of model parameters and applied stresses. Moreover, the functional dependence of the model on applied stresses and model parameters is generally such that relationship (172) can be recast trivially in the following form:

$$\begin{aligned}\alpha(S_1, S_2, \ldots, S_N) &= f\left(S_{1,0}, S_{2,0}, \ldots, S_{N,0}; p_1, p_2, \ldots, p_M\right) \\ &\quad \times \frac{f(S_1, S_2, \ldots, S_N; p_1, p_2, \ldots, p_M)}{f\left(S_{1,0}, S_{2,0}, \ldots, S_{N,0}; p_1, p_2, \ldots, p_M\right)} \\ &= \alpha_0 f'(S_1, S_2, \ldots, S_N; p_2, \ldots, p_M)\end{aligned} \tag{173}$$

where $\alpha_0 = \alpha(S_{1,0}, S_{2,0}, \ldots, S_{N,0}) = p_1$ is the 63.2th failure time percentile at reference values of stresses $S_{1,0}$, $S_{2,0}$, …, $S_{N,0}$ and is sometimes referred to as "scale parameter" of the life model [118]; $f'(S_1, S_2, \ldots, S_N; p_2, \ldots, p_M) = f(S_1, S_2, \ldots, S_N; p_1, p_2, \ldots, p_M)/f(S_{1,0}, S_{2,0}, \ldots, S_{N,0}; p_1, p_2, \ldots, p_M)$ is a dimensionless function that encompasses the whole dependence of the model on applied stresses.

Although the 63.2th percentile is usually chosen as the reference one of the failure-time distribution obtained from breakdown tests on electrical insulation (see Sect. 4.1), other life percentiles than the 63.2th could be considered (see Eq. 169) and this would affect the value of α_0 in (173) [119, 127, 144].

As it can be argued from (172) to (173), a life model valid for a certain insulation provides life estimates for that insulation (or for the component which the insulation belongs to) at selected levels of the applied stresses, on condition that the values of the model parameters are known.

Insulation life models are commonly used first of all for characterizing and comparing the endurance properties of various materials candidate for the realization of the insulation of electrical components. For a given insulating material, the life model parameters are usually derived via laboratory tests performed on small-size specimens, thereby achieving considerable time and cost savings with respect to tests on full-size insulation systems.

Secondly, since insulation is often the weakest part of an electrical device (as highlighted in Sect. 4.1) insulation life models can be employed also for inferring the service life of power components. However, this requires an extrapolation of test results and relevant model parameter values to the full-size insulation system of the considered power component; this introduces a degree of uncertainty in life estimation of the power component itself.

The extrapolation can be performed, e.g., via the statistical "enlargement law" [114–116, 129]. This law provides the relationship between full-size insulation life, t_D (at design values of applied stresses, $S_{1,D}$, $S_{2,D}$, …, $S_{N,D}$, and failure probability, P_D) and test-size insulation life, $\alpha(S_{1,D}, S_{2,D}, \ldots, S_{N,D})$ (at the same values of $S_{1,D}$, $S_{2,D}$, …, $S_{N,D}$, but at failure probability 63.2%, that is usually the reference probability for test result processing, as pointed out above), namely:

$$t_D = \alpha(S_{1,D}, S_{2,D}, \ldots, S_{N,D})[\ln(1 - P_D)/D]^{1/\beta}, \tag{174}$$

where D is the so-called enlargement factor. As an example, when dealing with power cables, for test minicables of length l_T, conductor radius r_T, outer insulation radius R_T, and power cables of length l_D, conductor radius r_D, outer insulation radius R_D, D can be written as [129]:

$$D = (l_D/l_T)(r_D/r_T)^2\left[1-(r_D/R_D)^{\beta_E-2}\right]\Big/\left[1-(r_T/R_T)^{\beta_E-2}\right] \tag{175}$$

where β_{E} is the shape parameter of the Weibull probability distribution of dielectric strength for both mini cables and full-size cables (β_{E} is not affected by the scaling process) [129].

On the basis of the enlargement law (174), Eq. 173 can be rewritten as follows:

$$t_D = \alpha_0 f'\left(S_{1,D}, S_{2,D}, \ldots, S_{N,D}; p_2, \ldots, p_M\right)[-\ln(1-P_D)/D]^{1/\beta}. \tag{176}$$

During the last three decades, the understanding of aging mechanisms for insulating materials subjected to different types of service stresses has grown continuously, leading to significant achievements. Such achievements go from the development of "phenomenological" life models—able to fit failure time data for various stresses (singly or simultaneously applied) and useful for deriving parameters for material evaluation (see e.g., [78, 144])—to "physical" models—that describe different physical–chemical mechanisms responsible for insulation degradation under different types and/or ranges of applied stress (see e.g., [23, 64, 65]). All this information provides a considerable help at the design stage of a full-size insulation, thus of an electrical component on the whole.

The introduction of normative references in IEC and IEEE publications (see e.g., [91, 92]) regarding thermal, electrical, mechanical, environmental and multiple stresses support the maturity and the progresses obtained on this topic. Most of the newly acquired knowledge is associated with the diffusion of polymeric insulation and the requirement of increasing design stresses, in order to reach more compact devices and reduce costs without affecting insulation system reliability, as required by the deregulated electricity market (see above).

In the following part of this section, a few fundamental life models available in the literature and employed for time-to-failure estimation of insulating materials and systems are presented. Focus is made, for the sake of brevity, on electrical and thermal stress, these being the stresses which mostly age and cause failure of the insulation of electrical devices. When such models are inserted in the probabilistic framework outlined at Sect. 4.1, reliability can be evaluated first of all in aprioristic terms with respect to real operating conditions of insulating materials and systems—that usually include time-varying stresses—i.e., on the basis of "rated" or "design" stress levels that are assumed as constant. This kind of indirect reliability evaluation, illustrated in what follows of this section, provides however a fundamental indication for the design of the insulating systems of power components such as cables, capacitors, transformers and motors.

On the other hand, since power devices in their actual service conditions are mostly—if not ever—subjected to time-varying stress levels, the need for a "closer-to-real-world" reliability evaluation for power components arises. For this reason, some recent theoretical developments for IRA under time-varying stresses—relevant to daily load cycles and based on the "good-old" cumulative-damage law of Miner [128]—are illustrated in Sect. 4.3.

4.2.1 Life Modeling Under Electro-Thermal Stress

The problem of insulation life modeling under the combination of electrical and thermal stress (i.e., when both voltage and temperature are applied), referred to as electro-thermal stress, was solved by combining two single stress-life models that hold, respectively, when only either temperature or voltage is applied [78, 144]. It is therefore convenient to review briefly these single stress-life models.

When only a constant temperature (thermal stress) is applied to insulation, electrical breakdown does not occur, of course. Thus, failure is said to take place conventionally when a selected diagnostic property, that has to be monotonous in time and correlated to thermal degradation (e.g., dielectric strength, yield strength, weight, density, etc.), reaches a proper end point, i.e., a fixed limit beyond which the insulation is no more able to perform satisfactorily [92, 144]. The first studies of insulation aging regarded mainly endurance to thermal stress. A fundamental approach was Dakin's theory, dating back to 1948 [58], according to which temperature speeds up the rate of thermally activated degradation reactions (e.g., oxidation, cross-linking, etc.), thereby accelerating the chemical aging of insulation. As a consequence, it can be shown that the logarithm of the time-to-end point, i.e., "thermal life", is inversely proportional to absolute temperature, Θ [92], and that the 63.2th percentile of thermal life, $\alpha(\Theta)$ can be expressed as [127, 144]:

$$\alpha(\Theta) = \alpha_0 \exp[-B(1/\Theta_0 - 1/\Theta)], \tag{177}$$

where Θ_0 is a reference value of absolute temperature, α_0 is the 63.2th percentile of thermal life at such reference temperature, $B = \Delta W/k_B$ is a constant typical of the material, ΔW being the activation energy of the main thermal degradation reaction involved and $k_B = 1.38 \times 10^{-23}\ J/K$ the Boltzmann constant. Equation 177 is referred to as the Arrhenius model.

For the Arrhenius model it is common practice to choose as the reference temperature a value higher than the operating temperature of the insulation, e.g., the value that corresponds to a mean life of 20,000 h (such value in °C is referred to as the "Temperature Index", TI) [92]. By this way, α_0 is known and fixed a priori, and the Arrhenius model is characterized by two parameters, namely Θ_0 and B. The model is usually represented in the so-called Arrhenius graph, having coordinates log(life) versus $-1/\Theta$, thereby giving rise to a straight line of slope B, that enables the extrapolation from test to service temperatures. In particular, B is very important: an insulation that exhibits a higher value of B (for the same value of α_0 and Θ_0, being Θ_0 higher than the service temperature) features a longer life at temperatures lower than Θ_0, i.e., down to the service temperature of the insulation. Thus, the higher is B, the better is the insulation [127, 144] (see Fig. 1a).

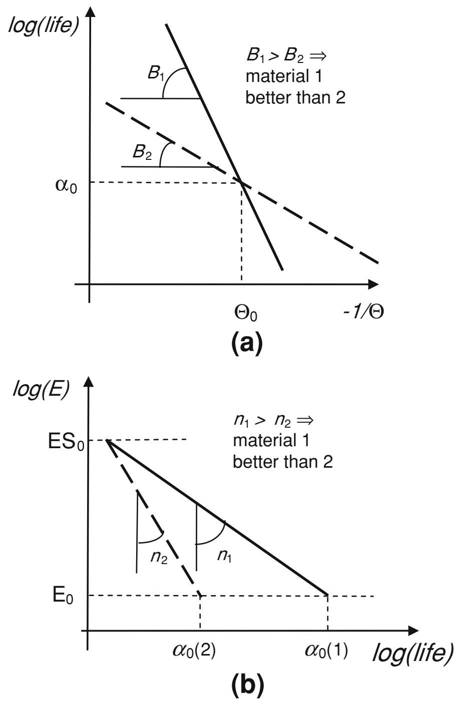


Fig. 1 **a** Arrhenius Model in log(life) versus –1/Θ coordinates (Arrhenius graph). **b** Inverse Power Model in log(*E*) versus log(life) coordinates

International Standards have been established for evaluating thermal endurance capabilities of insulating materials via indices such as the above-mentioned TI and the HIC (Halving Interval in Celsius, the temperature difference giving rise to halving of life, starting from the temperature of TI) [92].

When a fast-increasing voltage is applied to an insulation system, electrical breakdown (i.e., the discharge of the whole insulation thickness) takes place as the applied voltage exceeds a value typical of the considered insulation, the breakdown voltage; the relevant electric field is referred to as dielectric strength and depends on several quantities [127, 144].

When a constant (in the rms sense) voltage only is applied to an insulation system, the so-called IPM and Exponential model (EM) are mostly used for expressing the relationship between applied voltage and time to breakdown (electrical life). According to the IPM and the EM, the 63.2th percentile of electrical life, $\alpha(E)$, can be expressed, respectively, as follows:

$$\alpha(E) = \alpha_0 (E/E_0)^{-n} \quad \text{(IPM)} \tag{178}$$

$$\alpha(E) = \alpha_0 \exp(-h(E - E_0)) \quad \text{(EM)} \tag{179}$$

where E is the magnitude of electric field, also referred to as "electrical stress" (proportional to the applied voltage via trivial geometrical factors), E_0 is the value of electric field under which the aging produced by the electrical stress (i.e., the electrical aging) is negligible, α_0 is the value of the 63.2th percentile of time-to-breakdown corresponding to E_0 (i.e., $\alpha(E = E_0) = \alpha_0$).

Equations 178 and 179 provide straight lines in log–log and semi-log coordinate systems, respectively, with slopes $-1/n$ and $-1/h$, if—as usual—E is in ordinate and L in abscissa. Coefficient n (or h) is called Voltage Endurance Coefficient (VEC). The VEC is a fundamental parameter for insulation characterization and design (together with dielectric strength): indeed, for the same values of initial dielectric strength ES_0, an insulation that exhibits a higher value of n features a longer life for electric field values below ES_0. Hence, the larger the VEC, the better the insulation endurance, i.e., its ability to endure electrical stress (see Fig. 1b).

The IPM and the EM have essentially an empirical background, because most of the ALT data can be fitted by straight lines in log–log or semilog plots (the linearization of the stress-life relationship is needed to extract coefficients for material characterization, as well as to derive design field estimation through extrapolation from the results of ALT, carried out at stresses considerably larger than the service one). However, both models can acquire a theoretical background; in particular, the IPM was associated with a statistical approach based on the Weibull distribution (as illustrated in Sect. 3.3.10 above) and was applied to power cable insulation [130, 144].

By combining the Arrhenius thermal model (Eq. 177) and the IPM (Eq. 178), the following "electro-thermal life model" was obtained [144]:

$$\alpha(E, \Theta) = \alpha_0 (E/E_0)^{-(n_0 - bcT)} \exp(-BcT), \tag{180}$$

where $\alpha(E, \Theta)$ is the 63.2th percentile of electro-thermal life, cT is the so-called "conventional thermal stress", defined as $cT = 1/\Theta_0 - 1/\Theta$. Model (180), that features four parameters (i.e., B and n_0 for thermal and electrical endurance, respectively, b for the extent of stress synergism, α_0 as the scale parameter) was fitted satisfactorily to several sets of data, relevant to different materials [119, 127, 144].

From the above considerations—and by comparing Eqs. 172, 173 with 180—it can be concluded that, in the presence of electrical and thermal stresses only, $\alpha(S_1, S_2, \ldots, S_N)$ reduces to $\alpha(E, \Theta)$, that can be expressed through Eq. 180. Hence, by inserting Eq. 180 into Eq. 169 one gets the following probabilistic electro-thermal life model:

$$t_F(E, \Theta) = [-\ln(1 - F)]^{1/\beta} \alpha_0 (E/E_0)^{-(n_0 - b\,cT)} \exp(-BcT) \tag{181}$$

while from Eqs. 168, 170, 171, respectively, one obtains the relevant cumulative failure probability, hazard and reliability functions:

$$F(t; E, \Theta) = 1 - \exp\left\{ -\left[\frac{t}{\alpha_0 (E/E_0)^{-(n_0 - b\,cT)} \exp(-BcT)} \right]^{\beta} \right\} \tag{182}$$

$$h(t;E,\Theta) = \frac{\beta t^{\beta-1}}{[\alpha(E,T)]^{\beta}} = \frac{\beta t^{\beta-1}}{\left[\alpha_0 (E/E_0)^{-(n_0-b\,cT)} \exp(-BcT)\right]^{\beta}} \tag{183}$$

$$R(t;E,\Theta) = \exp\left\{ -\left[\frac{t}{\alpha_0 (E/E_0)^{-(n_0-b\,cT)} \exp(-BcT)} \right]^{\beta} \right\}. \tag{184}$$

Equation 184 represents a "physical reliability model" that can be used for reliability estimation in the case of a typical solid insulation for MV and HV subjected to electrical and thermal stress. It can be argued that it is a Weibull reliability function, thus it is characterized by the relevant mathematical properties illustrated at previous section.

4.2.2 Life Modeling in Distorted Regime

The electro-thermal life model of Eq. 180 holds for constant temperature and either dc electrical field or sinusoidal field at industrial (or moderately higher) frequency [119, 127, 144]. However, the ever-increasing diffusion of power electronics and non-linear loads in power systems involves a consequent increase in the level of harmonic distortion (both in current and in voltage). Current distortion gives rise to an increase in Joule losses in conducting parts, thereby raising insulation temperature and accelerating thermal degradation. Voltage distortion may increase the peak and/or the rms voltage and the rate of voltage rise with respect to sinusoidal conditions; this can accelerate also the electrical aging of components under distorted voltage [120, 121, 122]. Experimental tests and theoretical studies, carried out on different insulation systems subjected to various distorted current and voltage waveforms, showed that insulation life under distorted regime can be reduced with respect to life at rated sinusoidal voltage and temperature mainly due to the temperature increase produced by harmonic currents in conductors and electrical stress increase due to distorted voltage waveform [32, 122]. The parameters that weigh the severity of the distorted voltage waveform with respect to the nominal sinusoidal voltage are the *peak factor*, K_p, the rms *factor*, K_{rms}, and the *shape factor*, K_f, defined as [32]:

$$K_p = V_p / V_{1p,n} \tag{185}$$

$$K_{\mathrm{rms}} = V / V_{1,n} \tag{186}$$

$$K_f = \sqrt{\sum_{h=1}^{H} h^2 \left(V_h / V_{1,n}\right)^2}, \tag{187}$$

where V_p and $V_{1p,n}$ are peak values of distorted voltage and of rated sinusoidal voltage, respectively, V and $V_{1,n}$ are rms values of distorted voltage and of rated sinusoidal voltage, V_h is rms value of the h-order harmonic and H is the maximum order of harmonics occurring in the system. In the nominal sinusoidal regime, $K_p = K_{\mathrm{rms}} = K_f = 1$, while in distorted regime K_p, K_{rms} and K_f can exceed unity, thereby causing an acceleration of electrical degradation.

It was shown in [32, 122] that the following electro-thermal life model holds under distorted regime:

$$\alpha_{NS}(E_S, \Theta_S) = \alpha_S(E_S, \Theta_S)\exp(-B\Delta cT_{\mathrm{arm}})K_p^{-n_p}K_f^{-n_f}K_{\mathrm{rms}}^{-n_r}, \tag{188}$$

where $\alpha_{NS}(E_S, \Theta_S)$ is the 63.2th percentile of insulation life in a distorted regime characterized by a rated sinusoidal rms electric field E_S and a rated temperature Θ_S, $\alpha_S(E_S, \Theta_S)$ is the 63.2th percentile of insulation life in rated sinusoidal conditions, n_p, n_f and n_r are exponents that account for the aging acceleration effect of factors K_p, K_f and K_{rms}, respectively, B is the above constant introduced when dealing with the Arrhenius model and ΔcT_{arm} is a quantity depending on the temperature rise due to harmonics, $\Delta\Theta_{\mathrm{arm}}$, defined as:

$$\Delta cT_{arm} = 1/\Theta_S - 1/(\Theta_S + \Delta\Theta_{arm}) \tag{189}$$

being $\Theta = \Theta_S + \Delta\Theta_{\mathrm{arm}}$ insulation temperature in distorted regime. By means of (188), life in distorted regime can be related directly to life in nominal sinusoidal conditions, pointing out the (possible) life reduction caused by current and voltage harmonics.

Therefore, by assuming that the insulation system works in non-sinusoidal regime and that the life model of Eq. 188 holds, then $\alpha(S_1, S_2, \ldots, S_N)$ reduces to $\alpha_{NS}(E_S, \Theta_S)$, that can be expressed through Eq. 188. Hence, by inserting Eq. 188 into 169 one gets the following probabilistic electro-thermal life model for distorted regime (in the presence of a rated fundamental sinusoidal component of electric field having rms value E_S and of a rated temperature Θ_S):

$$t_{F,NS}(E_S, \Theta_S) = [-\ln(1-F)]^{1/\beta}\alpha_S(E_S, \Theta_S)\exp(-B\Delta cT_{\mathrm{arm}})K_p^{-n_p}K_f^{-n_f}K_{\mathrm{rms}}^{-n_r}. \tag{190}$$

By substituting Eq. 188 into Eqs. 168, 170, 171, respectively, one obtains also the relevant cumulative failure probability, hazard and reliability functions, respectively:

$$F(t; E_S, \Theta_S) = 1 - \exp\left\{-\left[\frac{t}{\alpha_S(E_S, \Theta_S)\exp(-B\Delta cT_{\mathrm{arm}})K_p^{-n_p}K_f^{-n_f}K_{\mathrm{rms}}^{-n_r}}\right]^{\beta}\right\} \tag{191}$$

$$R(t; E_S, \Theta_S) = \exp\left\{-\left[\frac{t}{\alpha_S(E_S, \Theta_S)\exp(-B\Delta cT_{\mathrm{arm}})K_p^{-n_p}K_f^{-n_f}K_{\mathrm{rms}}^{-n_r}}\right]^{\beta}\right\} \tag{192}$$

$$h(t;E_S,\Theta_S)=\frac{\beta}{\alpha_{NS}(E_S,\Theta_S)}\left[\frac{t}{\alpha_{NS}(E_S,\Theta_S)}\right]^{\beta-1}$$
$$=\frac{\beta t^{\beta-1}}{\left[\alpha_S(E_S,\Theta_S)\exp(-B\Delta cT_{\mathrm{arm}})K_p^{-n_p}K_f^{-n_f}K_{\mathrm{rms}}^{-n_r}\right]^{\beta}}. \tag{193}$$

Equation 192 represents a "physical reliability model" that can be used for reliability estimation in the case of a typical solid insulation subjected to distorted voltage. It can be argued that the reliability function follows again the Weibull distribution, as in the sinusoidal case, given that β is constant with applied stresses, as pointed out when dealing with Eq. 167.

In [121], the trend of failure rate versus aging time for a MV power cable feeding a traction system with AC/DC 12-pulse converters, subjected to harmonic voltages characteristic of the converters (i.e., essentially the 11th and 13th) at the limits stated by Standard EN 50160, is reported. It is shown that, although the level of voltage distortion complies with standard limits, nevertheless the reliability of the cable can be severely affected by voltage harmonics. On the other hand, this effect can be more than counterbalanced by a temperature significantly lower than rated temperature, as it is mostly the case for components of the supply system of electrical traction systems subjected to voltage and current harmonics.

4.3 *Life Modeling Under Time-Varying Stress: The Case of Load Cycles*

Power components in their real service conditions are mostly—if not ever—subjected to time-varying stresses. This involves the need for a "closer-to-real-world" reliability evaluation for power components, i.e., a reliability evaluation that accounts for the time variation of operating stress. Generally speaking, this problem is quite cumbersome.

However, it can be argued that a big deal of electrical devices—e.g., all the components of power transmission and distribution grids—exhibit every day more or less the same rms current and voltage values at the same hours, at least during working days of a given period of the year, under typical operating conditions of the users [118]. Thus, apart from the statistical fluctuations due to the random time-varying nature of the supplied loads and the deterministic fluctuations associated with the weekly and/or seasonal characteristics of the loads, such components are subjected to daily load cycles. Moreover, applied rms voltage is approximately constant with time—apart a generalized voltage increase when load decreases and vice versa; such variations, however, under normal operating conditions are within $\pm 10\%$ of rated voltage of components/systems. Thus, time varying stresses are mostly associated with current variations in the form of daily current/load cycles.

As a consequence, component life (i.e., the generic 100 Fth percentile of time-to-failure, t_F, see footnote at Sect. 4.1) can be divided approximately into K equal intervals. These intervals have all the same duration, $t_{\mathrm{cycle}} = 1$ day, and the same relationship between rms load current, I, and cycling time, $t \in [0, t_{\mathrm{cycle}}]$. Then, t_F can be expressed as a simple function of K, referred to as the number of "cycles-to-failure", namely:

$$t_F = K\, t_{\mathrm{cycle}}. \tag{194}$$

Daily load cycles can be thought of as a sequence of N equally lasting current steps of height I_i (rms current value) and duration $\Delta t_i = t_{\mathrm{cycle}}/N$ ($i = 1, \ldots, N$), as that sketched in Fig. 2 for $N = 6$ (a simplification of a typical load cycle of HVAC cables). Let us refer to such cycles as "stepwise-constant" daily cycles. Stepwise-constant daily cycles can reproduce satisfactorily every daily load cycle, on the condition that a sufficiently high number of steps N is taken. Thus, only daily load cycles of the stepwise-constant kind will be treated here.

Assuming as above that the weakest part of a power device is its insulation, the predominant stresses acting on insulation in-service commonly arise from the *electric field* associated with voltage (electric stress) and the *temperature* associated with Joule losses in conducting elements plus dielectric losses in the insulation (thermal stress). Therefore, in general, the maximum stresses applied to a power device are maximum temperature and electric field in the insulation. In this framework, the life of a power device subjected to load cycles is assumed here to end when its insulation fails because of the degradation caused by the maximum stresses, that act all over its life as a consequence of a fixed stepwise-constant daily load cycle.

As argued above, applied voltage is approximately constant with time. Hence, maximum electric field can be hypothesized as steady and equal to its design value, E_n. On the contrary, maximum temperature varies during each ith interval Δt_i of the load cycle, due to the relevant variation of rms current. The temperature rate-of-change depends on the difference between power losses in the present

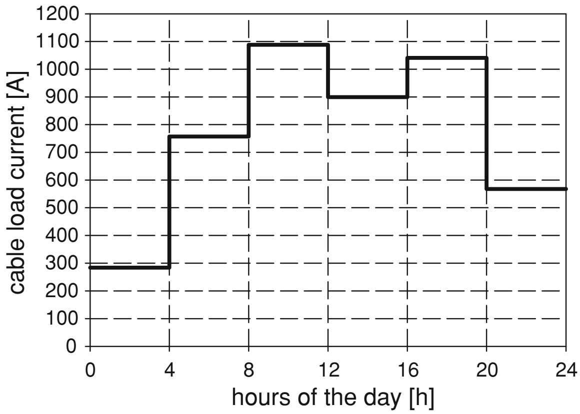


Fig. 2 Stepwise-con stant daily load cycle for a HVAC cable ($N = 6$ equally lasting current steps)

step—proportional to I_i^2—and in the previous step—proportional to I_{i-1}^2—as well as on heat storage and exchange properties of the layers that constitute the insulation and its outer environment. The combination of these effects gives rise to a thermal transient during which temperature varies starting from an initial value $T_{i,0}$ (the temperature at the beginning of Δt_i) and tending toward a steady value, $T_{i,\infty}$ (the regime temperature corresponding to a constant rms current I_i). The transient temperature within each Δt_i can be derived by means of an *ad hoc* transient thermal model (see, e.g., [117, 118] for power cables).

Every step Δt_i of the cycle can be then split into infinitesimal intervals, which range from a generic time t to a subsequent time $t + \mathrm{d}t$. Thus, each infinitesimal interval corresponds to one single value of transient temperature, $T_i(t)$. Hence, the fraction of life lost by component insulation during a given dt within Δt_i, denoted as dLF, can be written as follows:

$$\mathrm{dLF} = \mathrm{dLF}[E_n, T_i(t)] = \frac{\mathrm{d}t}{t_F[E_n, T_i(t)]}, \tag{195}$$

where $t_F[E_n, T_i(t)]$ is insulation life at constant values of maximum electric field and temperature, E_n and $T_i(t)$, respectively. $t_F[E_n, T_i(t)]$ must be evaluated via an electro-thermal life model valid for the insulation of the examined component [117, 118], e.g., Eq. 180 for an insulation subjected to temperature plus sinusoidal voltage.

According to Miner's cumulative-damage theory [128], the sum of all life fractions lost (referred to as "loss-of-life fractions" from now on) should yield 1 at failure. Therefore, cable life can be estimated by applying (195) to every infinitesimal interval dt of each step Δt_i of the cycle and setting the sum of all the relevant loss-of-life fractions at failure—in fact an integral—equal to 1. Thus, by defining the loss-of-life fraction relevant to the ith step of the cycle, LF_i, as:

$$\mathrm{LF}_i = \int_0^{\Delta t_i} \mathrm{dLF}[E_n, T_i(t)] = \int_0^{\Delta t_i} \frac{\mathrm{d}t}{t_F[E_n, T_i(t)]} \tag{196}$$

a relationship that contains t_F, i.e., cable life under the considered stepwise-constant load cycle, and K, i.e., the number of cycles-to-failure (see 194) is achieved, namely [128]:

$$\int_0^{L} \mathrm{dLF} = K \sum_{i=1}^{N} \left\{ \int_0^{\Delta t_j} \mathrm{dLF}[E_n, T_i(t)] \right\} = K \sum_{i=1}^{N} \mathrm{LF}_i = 1. \tag{197}$$

Now, K can be attained simply from (197):

$$K = \left[\sum_{i=1}^{N} \mathrm{LF}_i \right]^{-1} \tag{198}$$

and life t_F can be inferred directly from (194), (198), by computing the various life fractions LF_i ($i = 1, \ldots, N$) via (196). As this latter equation shows, such computation is easy for stepwise-constant daily load cycles, provided that a life model valid for cable insulation is available (see Sect. 4.2.1) and time functions $T_{\mathrm{i}}(t)$ can be calculated [117, 118].

Percent life variation (possibly extension) of the power component under load cycling, i.e., the percent variation of the generic 100 Fth percentile of component time-to-failure under load cycling with respect to rated 100 Fth percentile of time-to-failure at rated voltage and temperature, $t_{F,n}$, can be evaluated by means of the following quantity:

$$\Delta t_{F,100} = 100\left(t_F - t_{F,n}\right)/t_{F,n}. \tag{199}$$

The definition of $\Delta t_{F,100}$ accounts for the sign of the difference between estimated life and rated life. Indeed, such difference is >0 in the case of life extension, <0 in the case of life reduction with respect to rated life.

Note that $\Delta t_{F,100}$ of (199) is based on t_F, which is derived through (198) by considering thermal transients and electric stress plus electro-thermal synergism.

5 Model Selection in View of Aging Properties: A Numerical Illustration

In order that the whole previous discussion does not appear to be purely academic, it seems opportune to illustrate in this final section the advantages which can be brought about by IRA, or else the drawbacks of a mere DRA, by means of practical numerical examples relevant to real life data.

We start from observing that, for an adequate selection of the reliability model, one must be aware that two or more models may often be—or appear to be—similar. This is often the case, e.g., for the LN and Weibull models, as discussed in literature [49, 51, 67, 104], especially as long as one is interested in the estimates of "central" moments or percentiles (e.g., the median value). However, the same is not true for lower LT percentiles, or hazard rate functions (hrf), which are of main practical interest for characterizing the aging properties of the device [103, 124]. Here, some results are presented that were partly illustrated also by the authors of this chapter in [42]. Let us consider for example (as indicated in Table 1) the

Table 1 Values of theoretical 1st, 5th and 50th percentiles, in years, under five different reliability models with the same mean (45 years) and SD (15 years)

Model	Normal	Weibull	Gamma	Lognormal	Inverse Gaussian
Percentile $T_{0.01}$ (years)	10.10	12.65	17.54	20.06	20.05
Percentile $T_{0.05}$ (years)	20.33	20.61	23.48	25.03	25.05
Median = $T_{0.5}$ (years)	45.00	44.92	43.34	42.69	42.65

theoretical value of the 1st and the 5th LT percentile $T_{0.01}$ and $T_{0.05}$—together with the median $T_{0.50}$—for five different reliability models having the same mean (45 years) and SD (15 years), chosen as the typical values reported in [106] for electrical insulation components. Here, it is denoted by T_p a LT value such that: $F(T_p) = p$, being $F(t) = 1 - R(t)$ the cdf of the LT, T. So, for instance, the percentile $t^* = T_{0.01}$ is the life span such that only 1% of the devices fails before t^*.

It can be seen, looking at the values in the rows of Table 1, that very strong differences are obtained for the above percentiles, especially between the above discussed LN (and also IG) and Weibull models, or LN and Normal models. The differences are larger (up to 100%!) for lower percentiles, while they are reasonably small for the median values, as anticipated.

Moreover, very different behaviors of the hrf are obtained, as shown in Fig. 3 for three of the above models (i.e., Gamma, LN and Weibull). Such differences become noticeable particularly after mission times near to the MTTF value (45 years), namely an age that is being approached by many power system components currently in-service.

On the contrary (as noticed in Sect. 2.2) the Gaussian and Weibull models with same mean and SD possess generally extreme percentiles values which are very similar, and the same is true for both RF and hrf. However, they share also the drawbacks discussed in the same Sect. 2.12, among which we mention here again the lack of flexibility with respect to the hrf behavior description.

Also, the LN and IG models are very similar, but this will be discussed later.

On the other hand, due to the popularity of the Weibull model it is interesting to remark two undesirable consequences of the "wrong" assumption of a Weibull model, when in fact a LN model is true:

1. The wrong assumption may cause unnecessary maintenance actions. Indeed, it is known that typical age maintenance programs are opportune if and only if the hrf is increasing in time [9, 83]: on the contrary, often the LN model (which is IDHR) is mistaken for a IHR Weibull one.
2. The wrong assumption implies an improper under-estimation of the RF in the lower tail, as confirmed by the above results on the 1st LT percentile $T_{0.01}$

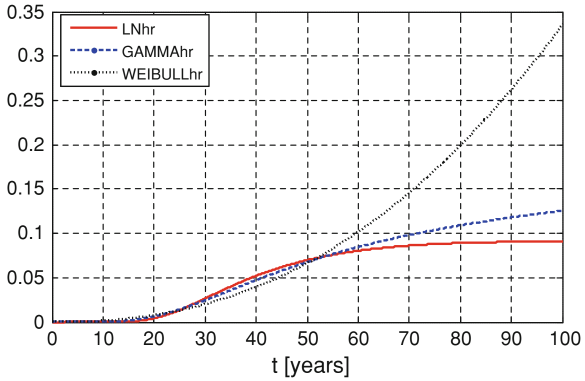


Fig. 3 Hrf curves for the Gamma, LN and Weibull reliability models, with same mean (45 years) and SD (15 years)

(in particular, the Weibull model provides for such parameter an estimate which is about one half of the "true" value). Of course, this under-estimation may bring about unnecessary costs in view of maintenance actions.

As previously pointed out, such results—which are here discussed only in the framework of probabilistic models—imply, in view of the statistical assessment of a reliability model that, in practice, two or more different models may be often undistinguishable, but in the presence of many data. Indeed, as well known from statistical theory, many data are needed to efficiently estimate extreme percentiles, but these are of course rarely available in this field, so that the model identification is an unavoidable, difficult and very critical first step in any reliability analysis. In view of data scarcity, the most useful way to perform such step is (when feasible, as often happens in engineering applications) to use technological and probabilistic information about wear and aging which unavoidably affect every power system component.

On the other hand, fortunately some of the above models are very similar to each other under many respects, especially—among those here considered—the LN model and the IG model, as shown in Figs. 4 and 5, respectively, for the pdf and hrf functions corresponding to the same values of mean (45 years) and SD (15 years) as above. This happens also, often (but not always), as far as the LN and LL models are concerned, as discussed in previous sections and in [37], too.

This kind of property may be exploited when the exact distribution is unknown, but—as sometimes happens in the field of electrical insulation (see Sect. 2)—the researcher knows that the aging mechanism gives rise to a decreasing hazard rate for large mission times. For instance, often the LN distribution is used in such cases because its parameters can be statistically estimated more easily with respect to other models possessing similar characteristics [51].

While, above, differences in the hrf curves for some models were illustrated (see Fig. 3), it is to be remarked that the differences in the CRF curves may be

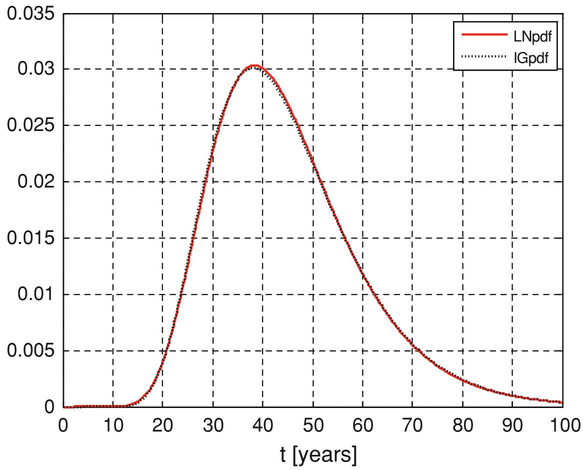


Fig. 4 Pdf of LN model and IG model, with same mean (45 years) and SD (15 years)

Fig. 5 Hrf of LN model and IG model, with same mean (45 years) and SD (15 years)

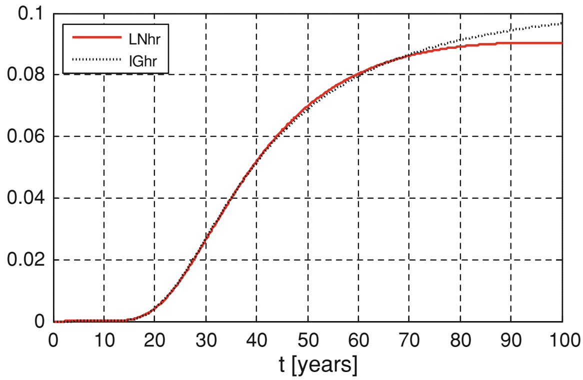
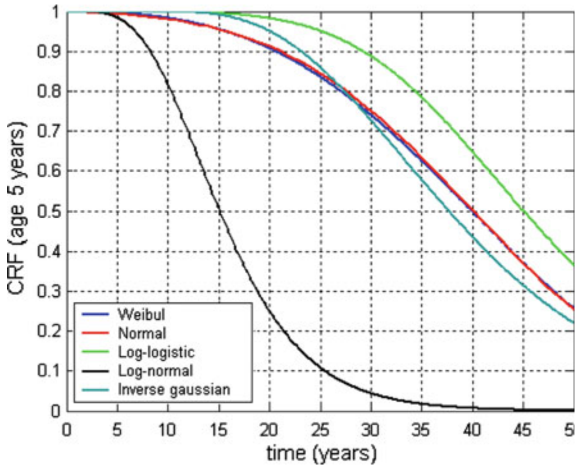


Fig. 6 CRF, relevant to an age of 5 years, for the Weibull, Normal, Log-logistic, Lognormal, Inverse Gaussian distributions with same mean (45 years) and standard deviation (15 years)



even more pronounced. This is shown, e.g., in next Figs. 6 and 7 showing the CRF $R(t|s)$ versus t, relevant, respectively, to an age of $s = 5$ and $s = 35$ years, for the Weibull, normal, Log-logistic, Lognormal, Inverse Gaussian distributions with same mean (45 years) and standard deviation (15 years), in order to appreciate the differences that exist among the different models.

It seems that such comparison is very interesting, also because the CRF is seldom reported, although possessing, in the authors' opinion, a deeper "physical" meaning than the hrf (see Sect. 1.4). Its importance in view of proper maintenance actions is understandable, and this is a key point in the necessity of an adequate reliability model selection.

In the above discussion—as in the whole chapter—the problem of finding statistical estimates of the relevant quantities (here, the percentiles and the hrf) has not been addressed. In practice, the discussed differences between "similar" models may be even emphasized when the data are scarce, as pointed out and illustrated in [104]. In these kind of analyses, characterized by lack of data and

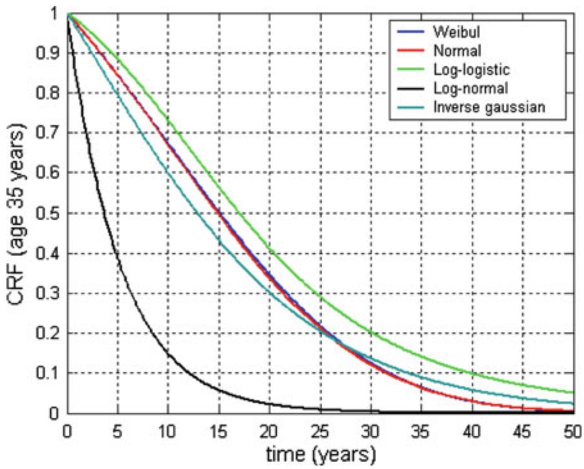


Fig. 7 CRF, relevant to an age of 35 years, for the Weibull, Normal, Log-logistic, Lognormal, Inverse Gaussian distributions with same mean (45 years) and standard deviation (15 years)

prior technological information, the authors believe that the Bayesian estimation methodology [113, 146, 147] can be the most adequate tool, as discussed and shown in [12, 39, 72, 73] with reference to electrical device reliability analyses.

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