Chapter 2
स्फुटप्रकरणम्
True longitudes of planets

२.१ केन्द्रलक्षणं पदलक्षणं च
2.1 Definition of the anomaly and the quadrant

स्वोच्चोनो विहगः केन्द्रं तत्र राशित्रयं पदम् ।
ओजे पदे गतैष्याभ्यां बाहुकोटी समेऽन्यथा ॥ १ ॥

svoccono vihagaḥ kendraṃ tatra rāśitrayaṃ padam|
oje pade gataiṣyābhyāṃ bāhukoṭī same'nyathā || 1 ||

The *ucca* subtracted from the planet is the *kendra* (anomaly). Three *rāśis* constitute a *pada* (quadrant). In the odd quadrants, the *bāhu* and *koṭi* [are to be found] from the angle covered and to be covered [respectively]. In the even quadrants it is otherwise.

The procedure for obtaining the *madhyama-graha* i.e. the mean longitude of a planet from the *Ahargaṇa*, was explained in the previous chapter. Two corrections, namely *manda-saṃskāra* and *śīghra-saṃskāra*, have to be applied to the *madhyama-graha* to obtain the *sphuṭa-graha* or the true longitude of the planet. In these two *saṃskāras*, to be described later in this chapter, two angles, namely the *manda-kendra* (*manda* anomaly or mean anomaly) and the *śīghra-kendra* (*śīghra*-anomaly or anomaly of conjunction or solar anomaly) play important roles. In the above verse, the *kendras* and their sines and cosines (known as *bāhus* and *koṭis*) pertaining to both the *saṃskāras* are dealt with. For this, two quantities, namely the *ucca* and the *kendra*, are introduced.

Ucca and *kendra*

The *ucca* and *kendra* essentially refer to the apsis and anomaly respectively. These two terms are generally used with the adjectives *manda* and *śīghra* and appear in the two processes of correction, namely *manda-saṃskāra* and *śīghra-saṃskāra*.

The *manda-saṃskāra*[1] is a procedure to obtain the correction for the eccentricity of the planetary orbit. The terms *ucca* and *kendra* used in this context refer to the direction of the *mandocca* (apogee/aphelion of the planet) and the *manda-kendra* respectively.

Similarly, *ucca* and *kendra* used in the context of *śīghra-saṃskāra*—the process by which the geocentric longitudes of the planets are obtained from their heliocentric longitudes[2]—refer to the directions of the *śīghrocca* and *śīghra-kendra* respectively. If θ_0 refers to the longitude of the mean planet, and θ_m that of its *mandocca*, then the *manda-kendra*, θ_{mk}, is defined as

$$\theta_{mk} = \theta_0 - \theta_m. \tag{2.1}$$

If θ_{ms} is the longitude of the *manda-sphuṭa-graha*, that is, the mean longitude of the planet corrected by *manda-saṃskāra*, and θ_s that of the *śīghrocca*, then the *śīghra-kendra*, θ_{sk}, is defined as

$$\theta_{sk} = \theta_{ms} - \theta_s. \tag{2.2}$$

In the second quarter of the above verse, it is mentioned that three *rāśis* constitute a *pada*. Since *rāśi* is a 30° division on the ecliptic, by definition the term *pada* refers to a quadrant. In Fig. 2.1*a*, *APB* represents a *pada*. Before explaining the second half of the verse, it would be useful to introduce the concepts of *bāhu* and *koṭi*, which are frequently employed in this and the following chapters.

Bāhu and *Koṭi*

In Indian astronomical texts, the terms *bāhu*[3] and *koṭi* are used in association with either *cāpa* or *jyā*. The terms *cāpa* and *jyā* literally mean bow and string respectively. In this context, they refer to the arc of a circle and the chord associated with it. Sometimes instead of the term *cāpa*, *dhanus* is also used to refer to the arc of a circle.

In Fig. 2.1*a*, the arc *PAL* represents the *cāpa* and *PQL* is the *jyā* associated with the *cāpa* (arc). Though literally the term *jyā* refers to the chord *PL*, in most situations *PQ*, which is half of *PL*, is referred to as the *jyā* (Rsine) of the arc *PA*. Since *PQ* is only half of *PL*, it must actually be referred to as the *jyārdha*. However, since only *PQ* is involved in planetary computations (as will be clear later), the term *jyā* itself is used to refer to the semi-chord *PQ*, for the sake of brevity in the use of terminology. Hence the terms *bāhucāpa* and *bāhujyā* or Rsine refer to the arc *AP* and the semi-chord *PQ* in the figure, respectively. The terms *koṭicāpa* and *koṭijyā*

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[1] The significance of this is explained in detail in Appendix F. The equivalent of this correction in modern astronomy is the equation of centre.

[2] For details refer to Sections 2.26–28 and Appendix F.

[3] The literal meaning of *bāhu* is hand. Similarly, *koṭi* means side. In this context, the term *koṭi* refers to the side which is perpendicular to *bāhu*.

or Rcosine refer to the arc PB and the segment OQ (perpendicular to the chord PL), respectively.

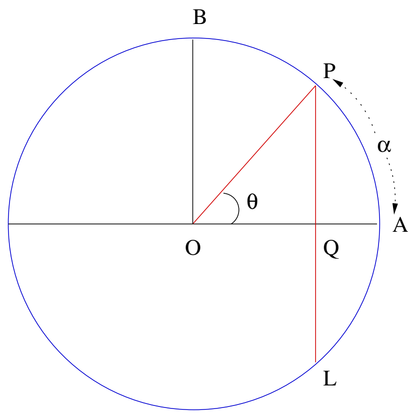

Fig. 2.1*a* *Bāhu* and *cāpa*.

Relation between the *jyās* and the sine and cosine function

Let R be the radius of the circle shown in Fig. 2.1a. Now, the quantities which are designated by the terms *bāhucāpa*, *bāhujyā*, *koṭicāpa* and *koṭijyā* are listed below:

bāhucāpa = $R\theta$ = the length of the arc AP corresponding to the angle θ.
bāhujyā = $R\sin\theta = R\times$ the *sine* of the angle θ.
koṭicāpa = $R(90-\theta)$ = the length of the arc corresponding to the angle $(90-\theta)$.
koṭijyā = $R\cos\theta = R\times$ the *cosine* of the angle θ.

In the following, we give the relationship between *sine* of an angle, θ, and the *jyā* of the corresponding arc, $\alpha = R\theta$, normally expressed in minutes. In Fig. 2.1a, let the length of the arc AP be α. Then we have the following relation between the *jyās* and the modern *sine* and *cosine* functions:

$$\begin{aligned} \textit{bāhujyā}\ \alpha &= R\sin\theta \\ \textit{koṭijyā}\ \alpha &= R\cos\theta. \end{aligned} \tag{2.3}$$

Normally the circumference of the circle is taken to be 21600 units (the number of minutes in 360°), so that an angle of $1'$ corresponds to an arc length of 1 unit. Hence the radius $R = \frac{21600}{2\pi} \approx 3437.7468$, which is approximately 3438 minutes. In Indian astronomical and mathematical texts, the radius of the circle R is referred to as the *trijyā*. This is because R is the *jyā* corresponding to the arc whose length is equal to three *rāśis* (5400′). In other words, *tri-rāśi-jyā* is shortened to *trijyā*.

Finding the *bāhu* and *koṭijyās* in different quadrants

The sine or cosine of an angle greater than 90° can always be determined in terms of an angle less than 90°. This is the essence of the second half of the verse wherein it is stated that:

- if the *kendra* is in the odd quadrant, i.e. its value lies in the range 0° – 90° or 180° – 270°, then the *bāhu* and *koṭi* are to be determined from the angles already covered and to be covered in that quadrant, respectively.
- if the *kendra* is in the even quadrant, i.e. its value lies in the range 90° – 180° or 270° – 360°, then the *bāhu* and *koṭi* are to be determined from the angles to be covered and already covered in that quadrant, respectively.

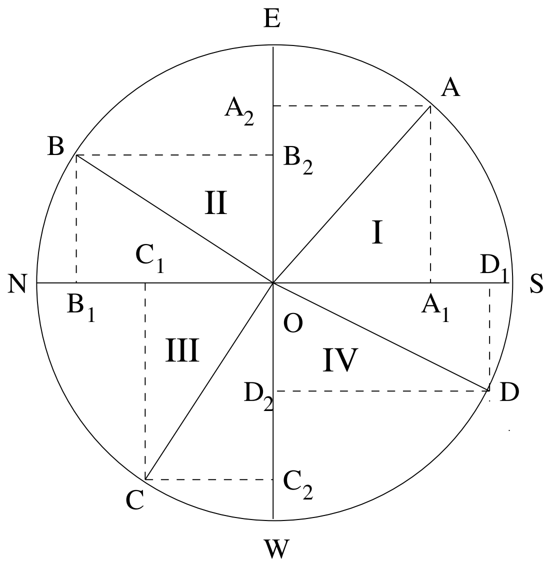

Fig. 2.1*b* *Bāhu* and *koṭi* when the *kendra* is in different quadrants.

We explain this concept further with the help of Fig. 2.1*b*. In the following we use K to denote the *kendra*. Then,

1. If K is in the first quadrant, i.e. $K = A\hat{O}A_1$, $R\sin A\hat{O}A_1 = AA_1$, $R\cos A\hat{O}A_1 = R\sin A\hat{O}A_2 = AA_2$.
2. If K is in the third quadrant, i.e. $K = C\hat{O}A_1$, $|R\sin C\hat{O}A_1| = R\sin C\hat{O}C_1 = CC_1$ and $|R\cos C\hat{O}A_1| = R\sin C\hat{O}C_2 = CC_2$.

Hence, in the above cases, the *bāhu* and *koṭi* are determined from the angles covered and to be covered respectively in the odd quadrant.

3. If K is in the second quadrant, i.e. $K = B\hat{O}A_1$, $|R\sin B\hat{O}A_1| = R\sin B\hat{O}B_1 = BB_1$, and $|R\cos B\hat{O}A_1| = R\sin B\hat{O}B_2 = BB_2$.
4. If K is in the fourth quadrant, $|R\sin D\hat{O}A_1| = R\sin D\hat{O}D_1 = DD_1$ and $|R\cos D\hat{O}A_1| = R\sin D\hat{O}D_2 = DD_2$.

Thus the *bāhu* and the *koṭi* are determined from the angles to be covered and covered respectively in the even quadrants. Here, only the procedure to find the magnitudes of the Rsines and Rcosines is given. Their signs (whether they have to be applied positively or negatively) will be stated separately in each context in which they are being employed.

The concepts of the *manda-kendra* and *śīghra-kendra* are explained in *Laghu-vivṛti* as follows:

तत्र प्रथमाध्यायोक्तप्रकारेण त्रैराशिकानीता भगणादिका ये ग्रहमध्यमाः तेभ्यो भगणानपास्य शिष्टेभ्यो राश्यादिभ्यो भागात्मकमुपदिष्टं स्वं स्वं मन्दोच्चं विशोध्य यच्छिष्यते, तदिह मन्दकेन्द्रमित्यभिधीयते। यदा पुनः मन्दफलेन स्फुटीकृतात् कुजादीनां मध्यमात् शीघ्रोच्चभूतं रविमध्यमं विशोध्यते, तदा तत्र अवशिष्टं शीघ्रकेन्द्रं भवति।

From the mean positions of the planets (*madhyama-grahas*), obtained using the rule of three described in Chapter 1, which includes an integral number of revolutions, *rāśis*, etc. [the fractional part], subtract the integral number of revolutions. From the remaining *rāśis* etc. [which represents the mean longitude of the planet] when its own *mandocca* is subtracted, the remainder obtained is said to be the *manda-kendra*. When the mean Sun, which is the *śīghrocca*, is subtracted from the *manda* corrected longitudes of Mars etc., the remainder obtained is the *śīghra-kendra*.

Note: Here it is specifically mentioned that the *śīghrocca* is the mean Sun for all the five planets while defining *śīghra-kendra*. The significance of this is explained later in sections 2.26–28 and also in Appendix F, during the discussion of *śīghra-saṃskāra* for the inner planets.

The complementarity between the sine and the cosine functions is also succinctly put forth in the commentary *Laghu-vivṛti*:

बाहुधनुर्विहीनं राशित्रयं कोटिधनुः। तद्विहीनं राशित्रयं बाहुधनुः ॥

The arc of the *bāhu* subtracted from 90° is the arc of the *koṭi*. That (arc of the *koṭi*) subtracted from 90° is the arc of the *bāhu*.

२.२ ज्याग्रहणं चापीकरणञ्च

2.2 Computation of the Rsines and the arcs

लिप्ताभ्यस्तत्त्वनेत्राप्ताः गता ज्याः शेषतः पुनः ।
गतगम्यान्तरघ्नाद्य हृतास्तत्त्वयमैः क्षिपेत् ॥ २ ॥
दोःकोटिज्ये नयेदेवं ज्याभ्यश्चापं विपर्ययात् ।

liptābhyastattvanetrāptāḥ gatā jyāḥ śeṣataḥ punaḥ |
gatagamyāntaraghnācca hṛtāstattvayamaiḥ kṣipet || 2 ||
doḥkoṭijye nayedevaṃ jyābhyaścāpaṃ viparyayāt |

By dividing the minutes [of arc] by 225, the number of *jyās* that have elapsed is obtained. Multiply the remainder by the difference between the (tabular) Rsine values of the elapsed and the next, divide by 225 and add the result to the elapsed *jyā*, to obtain the *bāhu* and *koṭi*. From the *jyās* the arcs can be obtained by the reverse process.

As already explained, in Indian astronomical and mathematical works the circumference of a circle is taken to be $360° = 21600'$. Therefore the length of the arc corresponding to each quadrant will be $5400'$. This length is divided into 24 equal segments, each segment corresponding to $225'$. In Fig. 2.2, the points P_i $(i = 1, 2, \ldots, 24)$ represent the end points of the 24 segments represented by the arcs $P_{i-1}P_i$. The set of *jyās*, $J_i = P_iN_i$, $(i = 1, 2, \ldots, 24)$ corresponding to the 24 *cāpas* P_0P_i, are explicitly stated in many texts such as *Āryabhaṭīya* and *Sūryasiddhānta*.[4] A method for obtaining more accurate values of these tabulated *jyās* will be presented in the next verse.

Let S_i represent the length of the 24 segments P_0P_i, in minutes of arc and J_i, the *jyā* corresponding to it. That is,

$$\begin{aligned} S_i &= P_0P_i = i \times 225, \\ \text{and} \qquad J_i &= P_iN_i. \qquad (i = 1, 2, \ldots, 24) \end{aligned} \tag{2.4}$$

The above verse gives an interpolation formula to find out the *jyā* corresponding to any length of arc between 0 and $5400'$ from the set of 24 *jyās* listed in Table 2.1 (page 64). Suppose S is the length of an arc in minutes that lies between S_i and S_{i+1}. That is,

$$S = S_i + r, \qquad O \leq r < 225, \tag{2.5}$$

where $S_i = i \times 225$. Since the *jyā* corresponding to the nearest arc lengths S_i and S_{i+1} on either side of S are known, the *jyā* corresponding to S is obtained by the rule of three. It is given by

$$jy\bar{a}\, S = J_i + \frac{r \times (J_{i+1} - J_i)}{225}. \tag{2.6}$$

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[4] The following verses in *Sūryasiddhānta* (II. 17–22) give the values of the 24 *jyās*:

तत्त्वाश्विनोऽङ्काब्धिकृता रूपभूमिधरर्त्तवः । खाङ्काष्टौ पञ्चशून्येशा बाणरूपगुणेन्दवः ॥
शून्यलोचनपञ्चैकाश्छिद्ररूपमुनीन्दवः । वियच्चन्द्रातिधृतयो गुणरन्ध्राम्बराश्विनः ॥
मुनिषड्यमनेत्राणि चन्द्राग्निकृतदस्रकाः । पञ्चाष्टविषयाक्षीणि कुञ्जराश्विनगाश्विनः ॥
रन्ध्रपञ्चाष्टकयमाः वस्वद्र्यङ्कयमास्तथा। कृताष्टशून्यज्वलनाः नागाद्रिशशिवह्नयः ॥
षट्पञ्चलोचनगुणाः चन्द्रनेत्राग्निवह्नयः । यमाद्रिवह्निज्वलनाः रन्ध्रशून्यार्णवाग्नयः ॥
रूपाग्निसागरगुणाः वस्वग्निकृतवह्नयः । प्रोज्झ्योत्क्रमेण व्यासार्धात्
. उत्क्रमज्यार्धपिण्डकाः ॥

The 24 *jyā* values in the above verses have been given using the *Bhūtasaṅkhyā* system of representing numbers (see Appendix A).

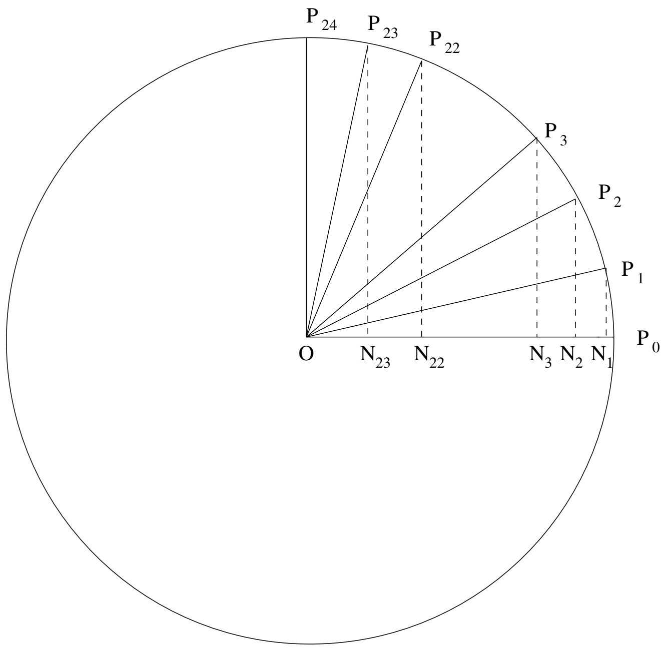

Fig. 2.2 Determination of the *jyā* corresponding to the arc lengths which are multiples of $225'$.

Illustrative example

Suppose the arc length $S = 1947$. Find the *jyā* corresponding to it.

The given arc length $S = 1947$ lies between S_8 and S_9, as $S_8 = 8 \times 225 = 1800$ and $S_9 = 2025$. Hence, S can be written as $S = S_8 + 147$. The *jyā* corresponding to arc length S is given by:

$$\mathit{jy\bar{a}}\, S = J_8 + \frac{147 \times (J_9 - J_8)}{225}.$$

For instance, we may use the values of Mādhava, quoted in *Laghu-vivṛti*, $J_8 = 1718'52''24'''$ and $J_9 = 1909'54''35'''$. Then,

$$\begin{aligned}\mathit{jy\bar{a}}\, 1947 &= 1718'52''24''' + \frac{147 \times (1909'54''35''' - 1718'52''34''')}{225} \\ &= 1843'41''02'''.\end{aligned}$$

This is the value of *jyā* (1947) obtained by the first-order interpolation, while the actual value is $1844'34''09'''$.

२.३ पठितज्यानयनम्

2.3 Computation of the tabular Rsines

विलिप्तादशकोना ज्या राश्यष्टांशधनुःकलाः ॥ ३ ॥
आद्यज्यार्धात् ततो भक्ते सार्धदेवाश्विभिस्ततः ।
त्यक्ते द्वितीयखण्डज्या द्वितीया ज्या च तद्युतिः ॥ ४ ॥
ततस्तेनैव हारेण लब्धं शोध्यं द्वितीयतः ।
खण्डात् तृतीयखण्डज्या द्वितीयस्तद्युतो गुणः ॥ ५ ॥
तृतीयः स्यात् ततश्चैवं चतुर्थाद्याः क्रमाद् गुणाः ।

viliptādaśakonā jyā rāśyaṣṭāṃśadhanuḥkalāḥ || 3 ||
ādyajyārdhāt tato bhakte sārdhadevāśvibhistataḥ |
tyakte dvitīyakhaṇḍajyā dvitīyā jyā ca tadyutiḥ || 4 ||
tatastenaiva hāreṇa labdhaṃ śodhyaṃ dvitīyataḥ |
khaṇḍāt tṛtīyakhaṇḍajyā dvitīyastadyuto guṇaḥ || 5 ||
tṛtīyaḥ syāt tataścaivaṃ caturthādyāḥ kramād guṇāḥ |

The *jyā* of one-eighth of the arc, corresponding to a *rāśi* (expressed) in minutes, is $10''$ short of that (length of the arc in minutes). The quantity obtained by dividing the first *jyārdha* by $233\frac{1}{2}$, and subtracting it from the same, is the *dvitīyakhaṇḍajyā*. This added to it (the first *jyā*) is the second *jyā*. The result obtained by dividing that (the second *jyā*) by the same divisor ($233\frac{1}{2}$) is to be subtracted from the second *khaṇḍajyā*. This is the *tṛtīyakhaṇḍajyā*. This added to the second is the third *guṇa*.[5] From that, the fourth *guṇa* etc. have to be obtained in order.

As mentioned earlier, some texts like *Āryabhaṭīya* and *Sūryasiddhānta* give the table of 24 *jyās* from which the *jyā* of any length of arc can be found, as illustrated through an example in the previous section. In the verses above, a procedure for finding more accurate values of the 24 *jyās* is described.[6] For this, two new terms, namely the *khaṇḍajyā* (Rsine difference) and the *piṇḍajyā* (whole Rsine) are introduced.

With reference to Fig. 2.2, they are defined as follows:

$$\begin{aligned} \textit{piṇḍajyā} &= P_i N_i = J_i & i &= 1,2,\ldots,24, \\ \textit{khaṇḍajyā} &= P_{i+1}N_{i+1} - P_i N_i = \Delta_i & i &= 1,2,\ldots,23. \end{aligned} \tag{2.7}$$

The term *piṇḍajyā* essentially refers to the whole or the tabulated *jyā*. They are 24 in number, represented by $J_1, J_2, \ldots, J_{24}$ and are expressed in minutes of arc. The last *piṇḍajyā*, namely $P_{24}N_{24} = P_{24}O$, is referred to as *trijyā*, and its length is equal to the radius of the circle. The difference between the successive *piṇḍajyās* are referred to as the *khaṇḍajyās*. In these verses the first *piṇḍajyā* and the procedure for generating the successive *piṇḍajyās* from that are given.

[5] The term *guṇa* has various meanings. In this verse and in verse 5a, it could be assigned the meaning rope, in which case it is the same as the word *jyā*. But in verses 6, 8 etc. of this chapter it is used to mean a multiplier (i.e. numerator).

[6] In fact, the procedure is the same as in *Āryabaṭīya*, but for the values of the first *jyā* (which is taken to be $224'50''$ instead of $225'$) and the divisor (which is taken to be $233\frac{1}{2}$ instead of $225'$).

The length of the first *piṇḍajyā* is stated to be one-eighth of a *rāśi* expressed in minutes minus 10 seconds; thus P_1N_1 (in Fig. 2.2) is equal to $224'\,50''$. This is also equal to the first *khaṇḍajyā*. Thus we have

$$jy\bar{a}\,P_0P_1 = P_1N_1 = J_1 = 224'\,50'' = \Delta_1. \tag{2.8}$$

This can be understood as follows. In Fig. 2.2,

$$P_0\hat{O}P_1 = \frac{90}{24} = 3.75^\circ = 225' = 0.65949846\,\text{radian}. \tag{2.9}$$

The first *piṇḍajyā* is often taken to be $225'$ in some earlier Indian texts like *Āryabhaṭīya* and *Sūryasiddhānta* based on the approximation,

$$R\sin\alpha \approx R\alpha = 225'. \tag{2.10}$$

In contrast to the above approximation, which of course is reasonably good for small α, the above set of verses present the value of the first *piṇḍajyā* based on a better approximation,

$$\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}. \tag{2.11}$$

In fact, it is later stated explicitly in the text (see verse 17 of this chapter) that this is the approximation that has been employed in arriving at the value of $224'\,50''$ for the first *piṇḍajyā*. Thus,

$$P_1N_1 = R\sin\alpha \approx \frac{21600}{2\pi}\left(\alpha - \frac{\alpha^3}{6}\right) = 224.8389' \approx 224'50''. \tag{2.12}$$

In the following, we give the procedure outlined in the text for obtaining the successive *khaṇḍajyās* and *piṇḍajyās*, along with the rationale behind it. The second *khaṇḍajyā* Δ_2 is defined as

$$\begin{aligned}\Delta_2 &= J_2 - J_1 \\ &= R(\sin 2\alpha - \sin\alpha),\end{aligned} \tag{2.13}$$

where $P\hat{O}P_2 = 2\alpha$. Now, $\sin 2\alpha = 2\sin\alpha\cos\alpha$. Hence,

$$\Delta_2 = R\sin\alpha(2\cos\alpha - 1). \tag{2.14}$$

Rewriting the above expression we have,

$$\Delta_2 = R\sin\alpha[1 - 2(1-\cos\alpha)]. \tag{2.15}$$

For $\alpha = 225'$, we have

$$2(1-\cos\alpha) \approx 0.004282153. \tag{2.16}$$

This is approximated in the text by

$$\frac{1}{233\frac{1}{2}} \approx 0.004282655. \tag{2.17}$$

Hence

$$\begin{aligned}
\Delta_2 &= R\sin\alpha\left(1-\frac{1}{233\frac{1}{2}}\right), \\
\text{or} \qquad \Delta_2 &= J_1 - \frac{J_1}{233\frac{1}{2}} \\
&= \Delta_1 - \frac{J_1}{233\frac{1}{2}} \\
&\approx 224'50'' - 57.77'' \\
&\approx 223'52''.
\end{aligned} \tag{2.18}$$

The second *piṇḍajyā* is given by

$$\begin{aligned}
J_2 &= J_1 + \Delta_2 \\
&= 224'50'' + 223'52'' \\
&= 448'42''.
\end{aligned} \tag{2.19}$$

The third *khaṇḍajyā* Δ_3 is defined as

$$\Delta_3 = J_3 - J_2 = R(\sin 3\alpha - \sin 2\alpha). \tag{2.20}$$

Rewriting the above expression we get

$$\begin{aligned}
\Delta_3 &= R\left[\sin(2\alpha+\alpha) - \sin 2\alpha\right] \\
&= R\left[(\sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha) - \sin 2\alpha\right] \\
&= R\left[(\sin 2\alpha\cos\alpha + (2\cos^2\alpha - 1)\sin\alpha) - \sin 2\alpha\right] \\
&= R\left[2\sin 2\alpha\cos\alpha - \sin\alpha - \sin 2\alpha\right] \\
&= R\left[\sin 2\alpha - \sin\alpha - 2\sin 2\alpha(1-\cos\alpha)\right] \\
&= \Delta_2 - J_2\, 2(1-\cos\alpha).
\end{aligned} \tag{2.21}$$

We have already noted that

$$2(1-\cos\alpha) \approx \frac{1}{233\frac{1}{2}}. \tag{2.22}$$

Hence the third *khaṇḍajyā* is given by

$$\Delta_3 = \Delta_2 - \frac{J_2}{233\frac{1}{2}}$$

$$\approx 223'52'' - 1'55''$$
$$= 221'57''. \tag{2.23}$$

Thus the third *piṇḍajyā* becomes

$$\begin{aligned} J_3 &= J_2 + \Delta_3 \\ &= 448'42'' + 221'57'' \\ &= 670'39'', \end{aligned} \tag{2.24}$$

and so on. In general, the ith *khaṇḍajyā* is given by

$$\Delta_i = \Delta_{i-1} - \frac{J_{i-1}}{233\frac{1}{2}}, \tag{2.25}$$

and the ith *piṇḍajyā* by

$$J_i = J_{i-1} + \Delta_i. \tag{2.26}$$

The iterative relation (2.25) follows from the easily verifiable relation for Δ_{i+1} given by

$$\begin{aligned} \Delta_{i+1} &= R\sin(i+1)\alpha - R\sin i\alpha \\ &= R\,[\sin i\alpha - \sin(i-1)\alpha - 2\sin i\alpha(1-\cos\alpha)], \\ &= \Delta_i - 2(1-\cos\alpha)R\sin i\alpha, \end{aligned} \tag{2.27}$$

and the above approximation (2.22) for $2(1-\cos\alpha)$. In fact, a recursion relation amounting to the above is stated a few verses later. The above iterative procedure is described in *Laghu-vivṛti* as follows:

ततो विलिप्तादशकेन विरहितात् तत्त्वनेत्रात् आद्यज्यार्धतुल्यात्, सार्धदेवाश्विभिः विभज्य यल्लिप्तादिफलं लभ्यते, तदाद्यद्वितीययोः खण्डज्ययोरन्तरं स्यात्। तदाद्यज्याखण्डतो विशोध्य शिष्टं द्वितीयखण्डज्या स्यात्। ततस्तद्युक्ता प्रथमखण्डज्या द्वितीयपिण्डज्या स्यात्। ततो द्वितीयपिण्डज्यातः पूर्वहारेणैव विभक्तं फलं द्वितीयतृतीययोः खण्डज्ययोरन्तरं स्यात्। ततः पुनः द्वितीयखण्डज्यातः विशोध्य शिष्टं तृतीयखण्डज्या स्यात्। तस्याः द्वितीयपिण्डज्यायाश्च योगः तृतीयपिण्डज्या स्यात्। अथ ततोप्युक्तप्रकारेण चतुर्थाद्याः गुणाः क्रमेण साध्याः।

Then, whatever is obtained in minutes etc. (*liptādi*) as the result when 225 diminished by 10 seconds, which is equal to the first Rsine, is divided by 233.5, will be the difference between the first and second *khaṇḍajyās*. This [result] when subtracted from the first *khaṇḍajyā* will be the second *khaṇḍajyā*. The first *khaṇḍajyā* added to this will then be the second *piṇḍajyā*. Then the result obtained by dividing the second *piṇḍajyā* by the above-mentioned divisor will be the difference between the second and third *khaṇḍajyās*. Again when this [result] is subtracted from the second *khaṇḍajyā*, [the quantity obtained] will be the third *khaṇḍajyā*. The sum of this and the second *piṇḍajyā* will be the third *piṇḍajyā*. From there on, the fourth Rsine etc. have to be obtained by the method stated above.

Laghu-vivṛti also prescribes more accurate values of the first Rsine (J_1), as well as the divisor.

अत्र विलिप्तारूपमेव ज्याचापान्तरमभिप्रेत्य विलिप्तादशकमुक्तम्। वस्तुतः पुनः अष्टत्रिंशत् तत्पराधिकं विलिप्तानवकमेवैतत्। अत एव भागहारोऽपि न तत्र सार्धदेवाश्वितुल्यः। अपि तु द्वात्रिंशद्विलिप्ताधिकदेवाश्वितुल्य इति।

Here with the intention of specifying the difference between the Rsine and the arc in terms of *viliptās* only, it was stated to be 10 *viliptās*. Actually it is 38 *tatparās* in excess of 9 *viliptās*. It is for this reason, the divisor is also not $233\frac{1}{2}$. But 233 (minutes) and 32 *viliptās*.

What is stated above is that the first Rsine (J_1) should be taken to be $225' - 0'9''38''' = 224'50''22'''$. Similarly the divisor should be taken to be $233 + \frac{32}{60}$ instead of $233\frac{1}{2}$. The values of tabular Rsines as calculated with these more accurate values of J_1 and the divisor are more or less the same as the tabular Rsines given by Mādhava, as we show below in Table 2.1.

२.४ प्रकारान्तरेण ज्यानयनम्

2.4 Another method for obtaining the Rsines

व्यासार्धं प्रथमं नीत्वा ततो वान्यान् गुणान् नयेत् ॥ ६ ॥
त्रीशघ्नचक्रलिप्ताभ्यः व्यासोऽर्थेष्वग्निभिर्हृतः ।
तद्दलाद्यज्ययोः कृत्योः भेदान्मूलमुपान्तिमा ॥ ७ ॥
अन्त्योपान्त्यान्तरं द्विघ्नं गुणो व्यासदलं हरः ।
आद्यज्यायास्तथापि स्यात् खण्डज्यान्तरमादितः ॥ ८ ॥
ताभ्यां तु गुणहाराभ्यां द्वितीयादेरपि क्रमात् ।
उत्तरोत्तरखण्डज्याभेदाः पिण्डगुणार्धतः ॥ ९ ॥
एवं सावयवा जीवाः सम्यङ्नीत्वा पठेत् क्रमात् ।

vyāsārdhaṃ prathamaṃ nītvā tato vānyān guṇān nayet || 6 ||
trīśaghnacakraliptābhyaḥ vyāso'rtheṣvagnibhirhṛtaḥ |
taddalādyajyayoḥ kṛtyoḥ bhedānmūlamupāntimā || 7 ||
antyopāntyāntaraṃ dvighnaṃ guṇo vyāsadalaṃ haraḥ |
ādyajyāyāstathāpi syāt khaṇḍajyāntaramāditaḥ || 8 ||
tābhyāṃ tu guṇahārābhyāṃ dvitīyāderapi kramāt |
uttarottarakhaṇḍajyābhedāḥ piṇḍaguṇārdhataḥ || 9 ||
evaṃ sāvayavā jī vāḥ samyaṅnītvā paṭhet kramāt |

Or else, the *guṇās* [the values of the *jyās*] may be obtained by first obtaining the *vyāsārdha* (radius). The number of seconds of arc in a circle multiplied by 113 and divided by 355 is the diameter.[7]

The square root of the difference between the squares of half of that (diameter) and the first *jyā* is the penultimate *jyā*. The difference between the last *jyā* and the penultimate

[7] Here, a clarifying note regarding the number 355 represented using the *Bhūtasaṅkhyā* system may be useful. In the string *artheṣvagni* employed to refer to this number, the word *artha* should not be taken to refer to *puruṣārtha*, in which case it would be referring to the number 4. On the other hand, it should be taken to be referring to 5 sense organs—through the derivation "arthyate anenetyartha.h" (through which things are sought after).

one multiplied by two is the *guṇa* (multiplier) and the radius is the *hāra* (divisor). From the *ādyajyā* [multiplying it with the multiplier and dividing by the divisor], the difference between the first two *khaṇḍajyās* is obtained. With the same multiplier and divisor, and multiplying the multiplier by the second *piṇḍajyā*, the third *piṇḍajyā* etc., the difference between the successive *khaṇḍajyās* are obtained. Having thus obtained the *jyās* with their parts (seconds etc.) they may be tabulated in a sequence.

Here a procedure for generating the *jyā* table (table of Rsines) by finding the differences of the successive *khaṇḍajyās* is described. As will be seen below, this procedure merely involves the knowledge of the first *jyā* (J_1) and *trijyā*. It may be recalled that the method described in the previous section (verses 4–6a) essentially made use of the following equations for generating the successive *piṇḍajyā* values given in Table 2.1.

$$J_{i+1} = J_i + \Delta_{i+1} \qquad (0 \leq i \leq 23) \tag{2.28}$$

$$\Delta_{i+1} = \Delta_i - \frac{J_i}{233\frac{1}{2}} \qquad (1 \leq i \leq 23), \tag{2.29}$$

where Δ_i and J_i $i = 1, 2, \ldots, 24$, refer to the *khaṇḍajyās* and *piṇḍajyās* respectively. Since $\Delta_1 = J_1$, is known, all the *jyās* can be generated using the above equations recursively. Equation (2.29) can be rewritten as

$$\Delta_i - \Delta_{i+1} = \frac{J_i}{233\frac{1}{2}}. \tag{2.30}$$

In the above verses (6–9) the recursion relation which is the basis of (2.30) is stated. Here the value of the last *jyā* (J_{24} = *trijyā*), which is the same as the radius of the circle, is first stated. Since J_1 is already known, with these two *jyās* (the first and the last), the value of the penultimate *jyā* (J_{23}) is found. Then the text defines a *guṇa* or multiplier and a *hāra* or divisor, using which a recursion relation is formulated; making use of this, all the tabular differences of the *khaṇḍajyās* and hence the values of the 24 *jyās* can be obtained. This method is quite instructive and may be described as follows. It has already been noted that the circumference of the circle is taken to be 21600. The diameter of this circle is stated to be:

$$D = \frac{21600 \times 113}{355}. \tag{2.31}$$

So, essentially, $\frac{355}{113} = 3.14159$ is taken to be the approximate value of π. Using (2.3), and the notation $\alpha = 225' = 3.75°$, we have

$$\begin{aligned}
\sqrt{{J_{24}}^2 - {J_1}^2} &= R\sqrt{\sin^2 24\alpha - \sin^2 \alpha} \\
&= \sqrt{(R\sin 90)^2 - (R\sin 3.75)^2} \\
&= R\sqrt{1 - \sin^2 \alpha} \\
&= R\cos\alpha \\
&= R\sin(24\alpha - \alpha) \qquad (24\alpha = 90°)
\end{aligned}$$

$$\begin{aligned} &= R\sin 23\alpha \\ &= J_{23}, \end{aligned} \tag{2.32}$$

where R is *trijyā*. Having obtained the penultimate *jyā* from the first and the last *jyās*, the multiplier and divisor are defined. *Laghu-vivṛti* puts them in very clear terms as follows:

> तस्याः उपान्तिमज्यायाः अन्त्यज्यायाश्च व्यासार्धतुल्यायाः यदन्तरं, तद्द्विगुणितं गुणः; व्यासार्धतुल्यो हारः ।
>
> The difference between the penultimate *jyā* and the ultimate *jyā*, which is equal to the radius, multiplied by two is the multiplier. The radius is the divisor.

That is,

$$\begin{aligned} guṇa &= 2(R - R\sin 23\alpha), \\ hāra &= R. \end{aligned} \tag{2.33}$$

Now the recursion relation to obtain the sine differences or the *khaṇḍajyās* can be written as follows:

$$\begin{aligned} \Delta_i - \Delta_{i+1} &= \frac{guṇa}{hāra}\, R\sin i\,\alpha \\ &= \frac{2(R - R\sin 23\alpha)}{R} R\sin i\,\alpha. \end{aligned} \tag{2.34}$$

For instance, with $i = 1$, the above equation becomes

$$\begin{aligned} \Delta_1 - \Delta_2 &= \frac{2(R - R\sin 23\alpha)}{R} R\sin\alpha \\ &= R[2\sin\alpha - 2\sin 23\alpha\sin\alpha] \\ &= R[2\sin\alpha - (\cos 22\alpha - \cos 24\alpha)] \\ &= R[2\sin\alpha - \cos(24\alpha - 2\alpha) + 0] \\ &= R[2\sin\alpha - \sin 2\alpha]. \end{aligned} \tag{2.35}$$

From the definition of *khaṇḍajyā*, we have

$$\begin{aligned} \Delta_1 - \Delta_2 &= (J_1 - J_0) - (J_2 - J_1) \\ &= 2J_1 - J_2. \end{aligned} \tag{2.36}$$

Clearly (2.36) is the same as (2.35). In general,

$$\begin{aligned} \Delta_i - \Delta_{i+1} &= (J_i - J_{i-1}) - (J_{i+1} - J_i) \\ &= 2J_i - J_{i+1} - J_{i-1} \\ &= R\,[2\sin i\alpha - \sin(i+1)\alpha - \sin(i-1)\alpha]. \end{aligned} \tag{2.37}$$

Using $\cos(90 - \theta) = \sin\theta$, $\cos(90 + \theta) = -\sin\theta$, we get

$$\begin{aligned}\Delta_i - \Delta_{i+1} &= R\,[2\sin i\alpha - \cos(24\alpha - (i+1)\alpha) + \cos(24\alpha + (i-1)\alpha)] \\ &= R\,[2\sin i\alpha - \cos(23-i)\alpha - \cos(23+i)\alpha] \\ &= R\,[2\sin i\alpha - 2\sin 23\alpha \sin i\alpha] \\ &= \frac{2(R - R\sin 23\alpha)}{R} R\sin i\alpha, \end{aligned} \tag{2.38}$$

which is the recursion relation (2.34) for the *khaṇḍajyās* given in the text.

Commenting on the first line of the tenth verse, *evaṃ sāvayavā jīvāḥ samyaṅ-nītvā paṭhet kramāt*, Śaṅkara Vāriyar describes the accurate values of the 24 Rsines—which he attributes to Mādhava—in the following verses:

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः।
तपनो भानुसूक्तज्ञो मध्यमं विद्धि दोहनम्॥ १ ॥
धिगाज्यो नाशनं कष्टं छन्नभोगाशयाम्बिका।
मृगाहारो नरेशोऽयं वीरो रणजयोत्सुकः॥ २ ॥
मूलं विशुद्धं नालस्य गानेषु विरला नराः।
अशुद्धिगुप्ता चोरश्रीः शङ्कुकर्णो नगेश्वरः॥ ३ ॥
तनुजो गर्भजो मित्रं श्रीमानत्र सुखी सखे।
शशी रात्रौ हिमाहारो वेगज्ञः पथि सिन्धुरः॥ ४ ॥
छायालयो गजो नीलो निर्मलो नास्ति सत्कुले।
रात्रौ दर्पणमभ्राङ्गं नागस्तुङ्गनखो वली ॥ ५ ॥
धीरो युवा कथालोलः पूज्यो नारीजनैर्भगः।
कन्यागारे नागवल्ली देवो विश्वस्थली भृगुः॥ ६ ॥
तत्परादिकलान्तास्तु महाज्या माधवोदिताः।
स्वस्वपूर्वविशुद्धे तु शिष्टास्तत्खण्डमौर्विकाः॥ ७ ॥ इति ॥

Here the values of the 24 Rsines are given up to the thirds in the *Kaṭapayādi* notation. For instance, consider the first Rsine given by '*śreṣṭhaṃ nāma variṣṭhānāṃ*'. The three words here stand for 22, 50 and 224 respectively. Hence the value of the first Rsine is: $224'\ 50''\ 22'''$. The values of the other Rsines are deciphered in a similar manner. These have been arrived at by considering terms up to θ^{11} in the series expansion of $\sin\theta$ which was also derived by Mādhava:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \ldots.$$

Table of *jyās*

In Table 2.1, we reproduce the values of *jyās* corresponding to arc lengths which are multiples of $225'$, given in *Āryabhaṭīya*/*Sūryasiddhānta*, *Tantrasaṅgraha* and *Laghu-vivṛti* (considering more accurate values for the first *jyā* as well as the divisor). The values of *jyās* enunciated by Mādhava are also listed based on the verses '*śreṣṭhaṃ nāma variṣṭhānāṃ*...' cited in *Laghu-vivṛti*. In fact, the modern values presented in the last column show that the Mādhava's values are accurate up to the thirds. In *Yuktibhāṣā* it is noted that the *jyā* for any arc can be obtained without using the tabular values, by using the infinite series expansion for it.

Dhanu or *Cāpa*		Notation	Value of the *jyā* (in minutes, seconds and thirds)				
Symbol used	Length (min)	used	As in AR/SS	From TS	From LV	Given by Mādhava	Modern
S_1	225	J_1	225	224 50	224 50 21	224 50 22	224 50 21
S_2	450	J_2	449	448 42	448 42 58	448 42 58	448 42 57
S_3	675	J_3	671	670 39	670 40 16	670 40 16	670 40 16
S_4	900	J_4	890	889 44	887 45 17	889 45 15	889 45 15
S_5	1125	J_5	1105	1105 00	1105 01 41	1105 01 39	1105 01 38
S_6	1350	J_6	1315	1315 32	1315 34 11	1315 34 07	1315 34 07
S_7	1575	J_7	1520	1520 26	1520 28 41	1520 28 35	1520 28 35
S_8	1800	J_8	1719	1718 49	1718 52 32	1718 52 24	1718 52 24
S_9	2025	J_9	1910	1909 51	1909 54 46	1909 54 35	1909 54 35
S_{10}	2250	J_{10}	2093	2092 42	2092 46 19	2092 46 03	2092 46 03
S_{11}	2475	J_{11}	2267	2266 35	2266 40 10	2266 39 50	2266 39 50
S_{12}	2700	J_{12}	2431	2430 45	2430 51 40	2430 51 15	2430 51 14
S_{13}	2925	J_{13}	2585	2584 32	2585 38 37	2584 38 06	2584 38 05
S_{14}	3150	J_{14}	2728	2727 14	2727 21 31	2727 20 52	2727 20 52
S_{15}	3375	J_{15}	2859	2858 15	2858 23 42	2858 22 55	2858 22 55
S_{16}	3600	J_{16}	2978	2977 02	2977 11 30	2977 10 34	2977 10 33
S_{17}	3825	J_{17}	3084	3083 03	3083 14 23	3083 13 17	3083 13 16
S_{18}	4050	J_{18}	3177	3175 53	3176 05 07	3176 03 50	3176 03 49
S_{19}	4275	J_{19}	3256	3255 06	3255 19 50	3255 18 22	3255 18 21
S_{20}	4500	J_{20}	3321	3320 24	3320 38 11	3320 36 30	3320 36 30
S_{21}	4725	J_{21}	3372	3371 27	3371 43 24	3371 41 29	3371 41 29
S_{22}	4950	J_{22}	3409	3408 05	3408 22 20	3408 20 11	3408 20 10
S_{23}	5175	J_{23}	3431	3430 07	3430 25 35	3430 23 11	3430 23 10
S_{24}	5400	J_{24}	3438	3437 27	3437 47 29	3437 44 48	3437 44 48

Table 2.1 *Jyā* values corresponding to arc lengths which are multiples of $225'$. *Āryabhaṭīya*, *Sūryasiddhānta*, *Tantrasaṅgraha*, *Laghu-vivṛti* (with a more accurate first sine as well as the divisor) and Mādhava's values.

२.५ इष्टदोःकोटिज्यानयनम्

2.5 Obtaining the desired Rsines and Rcosines

इष्टदोःकोटिधनुषोः स्वसमीपसमीरिते ॥ १० ॥
ज्ये द्वे सावयवे न्यस्य कुर्यादूनाधिकं धनुः ।
द्विघ्नतल्लिप्तिकाप्तैकशरशैलशिखीन्दवः ॥ ११ ॥
न्यस्याच्छेदाय च मिथः तत्संस्कारविधित्सया ।
छित्वैकां प्राक् क्षिपेज्जह्यात् तद्धनुष्यधिकोनके ॥ १२ ॥
अन्यस्यामथ तां द्विघ्नां तथा स्यामिति संस्कृतिः ।

इति ते कृतसंस्कारे स्वगुणौ धनुषोस्तयोः ॥ १३ ॥
तत्राल्पीयःकृतिं[8] त्यक्त्वा पदं त्रिज्याकृतेः परः ।

iṣṭadoḥkoṭidhanuṣoḥ svasamīpasamīrite || 10 ||
jye dve sāvayave nyasya kuryādūnādhikaṃ dhanuḥ |
dvighnatalliptikāptaikaśaraśailaśikhīndavaḥ || 11 ||
nyasyācchedāya ca mithaḥ tatsaṃskāravidhitsayā |
chitvaikāṃ prāk kṣipejjahyāt taddhanuṣyadhikonake || 12 ||
anyasyāmatha tāṃ dvighnāṃ tathā syāmiti saṃskṛtiḥ |
iti te kṛtasaṃskāre svaguṇau dhanuṣostayoḥ || 13 ||
tatrālpīyaḥkṛtiṃ tyaktvā padaṃ trijyākṛteḥ paraḥ |

Having noted down the listed/tabulated values (*samīrita*) of the *dorjyās* (Rsines) and *koṭijyās* (Rcosines) corresponding to the two points which are on either side of the arc whose *dorjyā* and *koṭijyā* are desired, find the difference in the arc which may be in excess of or short of it. [The number] 13751 divided by twice this difference has to be stored [as divisor D] for dividing. This is done for mutual correction (i.e. for correcting the *dorjyā* in determining *koṭijyā* and vice versa). First divide one of them (the *dorjyā* or *koṭijyā* by D) and add or subtract this from the other (if the *dorjyā* is divided, apply it to the *koṭijyā* and if the *koṭijyā* is divided, apply it to the *dorjyā*) depending upon whether the difference is in excess or short. This result multiplied by two and operated as before (divided by D and applied to the *dorjyā* or *koṭijyā*) forms the process of correction. The correction thus carried out gives the exact value of the *dorjyā* or the *koṭijyā* of the desired arc. Of the two (*dorjyā* or *koṭijyā*) find the square of the *jyā* of the smaller one and subtract it from the square of the *trijyā*. The square root of the result gives the other (the *koṭijyā* or *dorjyā*).

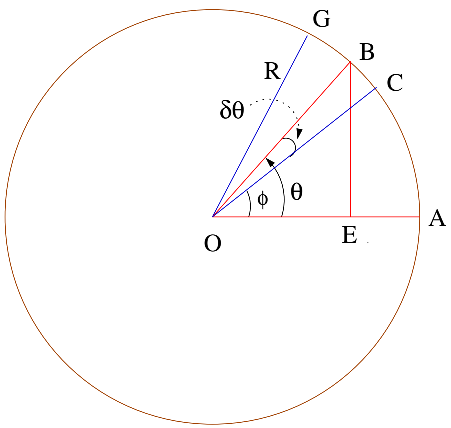

Fig. 2.3 Finding the *jyā* value corresponding to a desired arc.

In Fig. 2.3, AB is the arc whose *jyā* and *kotijyā* are desired to be found. The length of the arc $AB = R\theta$, where R is the *trijyā* and θ is the angle subtended by the arc at the centre O, expressed in radians. The *jyās* corresponding to the known arc lengths AC and AG are known from the *jyā* table (Table 2.1). The procedure for

[8] The reading in both the printed editions is: तत्राल्पीयः कृतिं। This however is grammatically incorrect. Hence we have provided the right compound form of the word above.

finding the *jyā* corresponding to the desired arc length AB from either of the two known *jyās* is described in the above verses.

It may be noted from the figure that the desired arc length $AB = R\theta$ is such that $i\alpha \leq R\theta \leq (i+1)\alpha$, for some integer $0 < i < 24$, where $\alpha = 225'$. Assume that point B is closer to C than G, i.e. $BC < BG$. Let $BC = R\delta\theta$. The problem is to find the *dorjyā* and *koṭijyā* corresponding to the arc AB, where $AB = i\alpha + R\delta\theta$.

The formulae for the two *jyās* involve an intermediate quantity (called the *hāraka*, or divisor (D) by the commentator), which is defined as:

$$D = \frac{13751}{2R\,\delta\theta}. \tag{2.39}$$

The number 13751 appearing in the numerator is essentially four times the radius R of the circle measured in minutes. In fact it is a good approximation too, as $2 \times 21600/\pi \approx 13750.98708$. Hence the above equation can be written as

$$D = \frac{4R}{2R\,\delta\theta} = \frac{2}{\delta\theta}. \tag{2.40}$$

While the *dorjyā* of an arc increases with the arc length, the *koṭijyā* decreases. Considering this, the text presents the following relations.

$$\begin{aligned}
dorjy\bar{a}(i\alpha + R\delta\theta) &= dorjy\bar{a}\, i\alpha + \frac{2}{D}\left(ko\d{t}ijy\bar{a}\, i\alpha - \frac{dorjy\bar{a}\, i\alpha}{D}\right) \\
&= dorjy\bar{a}\, i\alpha - \frac{(dorjy\bar{a}\, i\alpha)\delta\theta^2}{2} + (ko\d{t}ijy\bar{a}\, i\alpha)\,\delta\theta \\
&= dorjy\bar{a}\, i\alpha\left(1 - \frac{\delta\theta^2}{2}\right) + (ko\d{t}ijy\bar{a}\, i\alpha)\,\delta\theta, \qquad (2.41) \\
dorjy\bar{a}\,(i\alpha - R\delta\theta) &= dorjy\bar{a}\, i\alpha - \frac{2}{D}\left(ko\d{t}ijy\bar{a}\, i\alpha + \frac{dorjy\bar{a}\, i\alpha}{D}\right) \\
&= dorjy\bar{a}\, i\alpha - \frac{(dorjy\bar{a}\, i\alpha)\delta\theta^2}{2} - (ko\d{t}ijy\bar{a}\, i\alpha)\,\delta\theta \\
&= dorjy\bar{a}\, i\alpha\left(1 - \frac{R\delta\theta^2}{2}\right) - (ko\d{t}ijy\bar{a}\, i\alpha)\,\delta\theta, \qquad (2.42) \\
ko\d{t}ijy\bar{a}\,(i\alpha + R\delta\theta) &= ko\d{t}ijy\bar{a}\, i\alpha - \frac{2}{D}\left(dorjy\bar{a}\, i\alpha + \frac{ko\d{t}ijy\bar{a}\, i\alpha}{D}\right) \\
&= ko\d{t}ijy\bar{a}\, i\alpha - \frac{(ko\d{t}ijy\bar{a}\, i\alpha)\delta\theta^2}{2} - (dorjy\bar{a}\, i\alpha)\delta\theta \\
&= ko\d{t}ijy\bar{a}\, i\alpha\left(1 - \frac{R\delta\theta^2}{2}\right) - (dorjy\bar{a}\, i\alpha)\delta\theta, \qquad (2.43) \\
ko\d{t}ijy\bar{a}\,(i\alpha - R\delta\theta) &= ko\d{t}ijy\bar{a}\, i\alpha + \frac{2}{D}\left(dorjy\bar{a}\, i\alpha - \frac{ko\d{t}ijy\bar{a}\, i\alpha}{D}\right) \\
&= ko\d{t}ijy\bar{a}\, i\alpha - \frac{(ko\d{t}ijy\bar{a}\, i\alpha)\delta\theta^2}{2} + (dorjy\bar{a}\, i\alpha)\delta\theta
\end{aligned}$$

$$= ko\d{t}ijy\bar{a}\, i\alpha \left(1 - \frac{R\delta\theta^2}{2}\right) + (dorjy\bar{a}\, i\alpha)\delta\theta. \qquad (2.44)$$

In *Laghu-vivṛti* the procedure for finding the *dorjyā* and *koṭijyā* of any arc length is explained clearly as follows:

ततस्तेन हारकेण भुजाज्यां कोटिज्यां वा एकां कर्तुमिष्टां प्रथमतः विभज्य लब्धं कलादिकं फलं अन्यस्यां, भुजायाः साध्यत्वे कोटिज्यायां तस्याः साध्यत्वे भुजाज्यायां च साध्येतरज्यायां तत्सम्बन्धिनो धनुषः ऊनाधिकत्ववशात् ऋणं धनं वा कुर्यात्। अथैवं कृतां तां द्विगुणितां कृत्वा पूर्वोक्तेनैव हारकेण विभज्य लब्धं यत् फलं तत्पुनरन्यस्यां साध्यज्यायामेव तं धनुषः ऊनाधिकवशादृणं धनं वा कुर्यात्। एवं कृता भुजाज्या कोटिज्या च परस्परलब्धफलसंस्कृते स्फुटे भवतः।

By that divisor divide the *dorjyā* or *koṭijyā*, whichever is desired to be found, and this may be added to or subtracted from the other one. That is, if the *dorjyā* is desired to be found, it may be applied to the *koṭijyā* and if the *koṭijyā* is to be found it may be applied to the *dorjyā*, the application being positive or negative depending upon whether the arc $R\delta\theta$ is added to or subtracted from [$i\alpha$].

Then this quantity may be multiplied by two and divided by the same divisor. The result has to be applied to the desired *jyā* [i.e.,] if the *koṭijyā* is to be found it has to be applied to the *koṭijyā*, and if the *dorjyā* is to be found it has to be applied to the *dorjyā*, the application being positive or negative depending upon whether the arc $R\delta\theta$ is added to or subtracted from [$i\alpha$]. The *dorjyā* and *koṭijyā* thus applied to each other give the correct *jyā* of the desired arc.

If the arc length $i\alpha \pm R\delta\theta$ corresponds to an angle $\phi \pm \delta\theta$ (in radians), then equations (2.41) to (2.44) are equivalent to the following relations:

$$R\sin(\phi+\delta\theta) = R\sin\phi\left(1-\frac{\delta\theta^2}{2}\right) + (R\cos\phi)\delta\theta, \qquad (2.45)$$

$$R\sin(\phi-\delta\theta) = R\sin\phi\left(1-\frac{\delta\theta^2}{2}\right) - (R\cos\phi)\delta\theta, \qquad (2.46)$$

$$R\cos(\phi+\delta\theta) = R\cos\phi\left(1-\frac{\delta\theta^2}{2}\right) - (R\sin\phi)\delta\theta, \qquad (2.47)$$

$$R\cos(\phi-\delta\theta) = R\cos\phi\left(1-\frac{\delta\theta^2}{2}\right) + (R\sin\phi)\delta\theta. \qquad (2.48)$$

It is obvious that (2.45) to (2.48) are approximations of the standard trigonometric relations

$$R\sin(\phi+\delta\theta) = R(\sin\phi\cos\delta\theta + \cos\phi\sin\delta\theta), \qquad (2.49)$$

$$R\sin(\phi-\delta\theta) = R(\sin\phi\cos\delta\theta - \cos\phi\sin\delta\theta), \qquad (2.50)$$

$$R\cos(\phi+\delta\theta) = R(\cos\phi\cos\delta\theta - \sin\phi\sin\delta\theta), \qquad (2.51)$$

$$R\cos(\phi-\delta\theta) = R(\cos\phi\cos\delta\theta + \sin\phi\sin\delta\theta), \qquad (2.52)$$

when the approximations, $\cos\delta\theta = \left(1-\frac{\delta\theta^2}{2}\right)$ and $\sin\delta\theta = \delta\theta$, for small $\delta\theta$ are used. These also happen to be the first two terms in the Taylor series expansion of

$\sin(\phi \pm \delta\theta)$ and $\cos(\phi \pm \delta\theta)$. Śaṅkara Vāriyar, however, has given an incorrect generalisation of these to higher orders in his *Laghu-vivṛti*.

If either the *dorjyā* or the *koṭijyā* of an arc is known, the other can be determined using the following relation. Let α be the length of the arc AB (in minutes) as shown in Fig. 2.4; then,

$$dorjy\bar{a}^2 \alpha + koṭijy\bar{a}^2 \alpha = R^2, \tag{2.53}$$

which is the same as

$$\sin^2\theta + \cos^2\theta = 1. \tag{2.54}$$

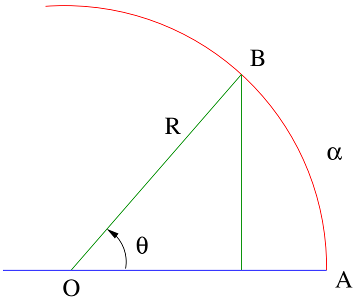

Fig. 2.4 Relation between the *dorjyā*, the *koṭijyā* and the *trijyā*.

२.६ इष्टज्यायाश्चापीकरणम्

2.6 Determining the length of the arc from the corresponding Rsine

ज्ययोरासन्नयोर्भेदभक्तस्तत्कोटियोगतः ॥ १४ ॥
छेदस्तेन हृता द्विघ्ना त्रिज्या तद्धनुरन्तरम् ॥

jyayorāsannayorbhedabhaktastatkoṭiyogataḥ || 14 ||
chedastena hṛtā dvighnā trijyā taddhanurantaram ||

The sum of the cosines divided by the difference of those two sines, which are close to each other, forms the *cheda* (divisor). Twice the *trijyā* divided by this is the difference between the corresponding arcs.

Consider Fig. 2.5*a*. P and Q are points along the circle whose distance from the point A are multiples of $\alpha = 225'$, that is $AP = i\alpha$, and $AQ = (i+1)\alpha$, where i is an integer. The *jyās* corresponding to the arcs AP and AQ are known from the table. The idea is to find the arc length (AB in minutes) corresponding to the given *jyā* (BN). Since the arc length AP is known, to determine AB we just need to find the length of the arc PB.

Let $A\hat{O}P = \theta_0$, $A\hat{O}B = \theta$ and $P\hat{O}B = \theta - \theta_0 = \delta\theta$. Then, according to the text the arc length PB is given by the following approximate formula:

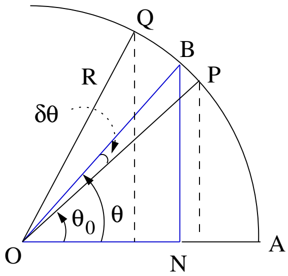

Fig. 2.5*a* Determination of the arc length corresponding to a given *jyā*.

$$PB = R\,\delta\theta \approx \frac{2R}{\left[\frac{(\cos\theta+\cos\theta_0)}{(\sin\theta-\sin\theta_0)}\right]}. \tag{2.55}$$

The rationale behind the above formula can be understood as follows. When $\delta\theta$ is small, $\sin\delta\theta \approx \delta\theta$ and $\cos\delta\theta \approx 1$. Hence, we have

$$\sin\theta = \sin(\theta_0 + \delta\theta) \approx \sin\theta_0 + \cos\theta_0\ \delta\theta \tag{2.56a}$$
$$\sin\theta_0 = \sin(\theta - \delta\theta) \approx \sin\theta - \cos\theta\ \delta\theta. \tag{2.56b}$$

The above equations may be rewritten as

$$\sin\theta - \sin\theta_0 \approx \cos\theta_0\ \delta\theta \tag{2.56c}$$
$$\sin\theta - \sin\theta_0 \approx \cos\theta\ \delta\theta, \tag{2.56d}$$

from which we have

$$2(\sin\theta - \sin\theta_0) \approx (\cos\theta + \cos\theta_0)\ \delta\theta, \tag{2.57}$$

or,

$$\delta\theta \approx \frac{2(\sin\theta - \sin\theta_0)}{(\cos\theta + \cos\theta_0)}. \tag{2.58}$$

The above equation is the same as (2.55). We now proceed to explain another method—one that is most likely to have been employed by Indian astronomers—of arriving at the above expression for $\delta\theta$ with the help of a geometrical construction (see Fig. 2.5*b*). Here J is the midpoint of the arc PB and BN, JK and PM are perpendicular to OM. As the arc PB is small, it can be approximated by a straight line and K can be taken to be the midpoint of NM.

Then it can be easily seen from the figure that

$$BD = R(\sin\theta - \sin\theta_0)$$
$$\text{and} \qquad OK = \frac{R(\cos\theta + \cos\theta_0)}{2}. \tag{2.59}$$

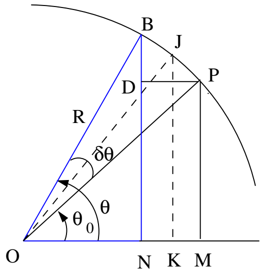

Fig. 2.5*b* Geometrical construction to determine the arc length corresponding to a given *jyā*.

Considering the similar triangles PBD and JOK, we have the relation

$$\frac{PB}{JO} = \frac{BD}{OK} \qquad \text{or} \qquad PB = JO \times \frac{BD}{OK}. \tag{2.60}$$

Using (2.59) in the above, we get

$$PB = \frac{2R}{\left[\frac{(\cos\theta + \cos\theta_0)}{(\sin\theta - \sin\theta_0)}\right]}, \tag{2.61}$$

which is the same as (2.55) given in the text.

The above verse is explained in *Laghu-vivṛti* as follows.

तत्र चापसन्धिपठितयोः निरन्तरयोर्जीवयोः यस्याः (जीवायाः) इष्टज्यां प्रत्यासन्न-तरत्वं तस्याः इष्टज्यायाश्च यो भेदः तेन तयोः कोटिज्ययोः योगं विभजेत्। तत्र लब्धः छेदो नाम। ततस्तेन छेदेन द्विगुणितां त्रिज्यां विभजेत्। तत्र लब्धं इष्टज्या-तदासन्न-चापसन्धिज्ययोर्धनुषोः अन्तरम्; इष्टज्या-तदासन्न-चापसन्धिज्ययोरन्तरोत्थस्य ज्या-भागस्य धनुरित्यर्थः।

Of the two points whose *jyā* values are listed in the table, find the one which is closer to the desired *jyā* [whose arc value is to be found]. Then find the difference between these two *jyās* (*dorjyās*), and divide the sum of the *koṭijyās* by this difference. The result is called the *cheda*. Divide twice the *trijyā* by this *cheda*. The result obtained is the difference between the arcs lying between the desired *jyā*, and the *jyā* closest to it (as found from the table); that is, it gives the length of the arc corresponding to the difference in the *dorjyās*.

२.७ सूक्ष्मज्यानयनम्

2.7 Finding more accurate values of the desired Rsine

इति ज्याचापयोः कार्यं ग्रहणं माधवोदितम् ।
विधान्तरं च तेनोक्तं तयोः सूक्ष्मत्वमिच्छताम् ॥ १५ ॥
जीवे परस्परनिजेतरमौर्विकाभ्यां अभ्यस्यविस्तृतिदलेन विभज्यमाने।
अन्योन्ययोगविरहानुगुणे भवेतां यद्वा स्वलम्बकृतिभेदपदीकृते द्वे॥ १६ ॥

iti jyācāpayoḥ kāryaṃ grahaṇaṃ mādhavoditam|
vidhāntaraṃ ca tenoktaṃ tayoḥ sūkṣmatvamicchatām || 15 ||
jīve parasparanijetaramaurvikābhyāṃ abhyasyavistṛtidalena vibhajyamāne|
anyonyayogavirahānuguṇe bhavetāṃ yadvā svalambakṛtibhedapadīkṛte dve||16||

The [above] procedure for obtaining the *jyā* and *cāpa* has thus been explained by Mādhava. He has also given another method for those desirous of obtaining accurate values. Multiply each *jyā* (*dorjyā* of an arc length) by the other *jyā* (of another arc length) and divide them by the *trijyā*. Their sum or difference becomes (the *jyā*) of the sum or difference of the arcs. Or else, the square root of the difference of their own squares and that of the *lamba* [may be added and subtracted for getting the *jyā* of the sum or difference of the arcs].

The procedures for obtaining the Rsine and the arc described in the previous verses are attributed to Mādhava. Verse 16 essentially gives the $\sin(A+B)$ formula. This formula too is attributed to Mādhava and is explained in the commentary as follows:

``` योगवियोगयोग्ये द्वे अपि अर्धज्ये परस्परस्य निजेतरज्याभ्यां स्वभुजाज्यां अन्यस्याः कोटया अन्यभुजाज्यां स्वकोटया च गुणयेत्। यदि स्वयं कोटिज्या, तर्हि तां अन्यस्य भुजाज्यया अन्यकोटिज्यां च स्वभुजया गुणयेत्। एवं कृतयोः द्वयोर्योगो वियोगो वा अभीष्टः कार्यः। ततो विस्तृतिदलेन विभजेत्। विभज्यमाने इति शानचा विभजनात् प्रागेव योगवियोगौ कर्तव्यौ इति दर्शयति। एवं कृतयोः योगो वियोगो वा स्फुटो भवति।

अथवा द्वयोर्वर्गतः पृथगेकस्यैव तयोः साधारणस्य लम्बस्य वर्गमपनीय मूलीकृते उभे योगविरहानुगुणे भवेताम्। लम्बानयनं पुनः उभयोर्जीवयोः संवर्गतः त्रिज्यया हरणेन कर्तव्यम्।

…The *dorjyās* (Rsines) [of the arcs $\alpha$ and $\beta$] whose sum or difference is desired to be found have to be multiplied mutually with the other *jyā*. That is, the *dorjyā* of one ($\alpha$) has to be multiplied by the *koṭijyā* of the other ($\beta$) and the *koṭijyā* of the one ($\alpha$) has to be multiplied by the *dorjyā* of the other ($\beta$). The sum or difference of these two quantities has to be found as desired. Then it has to be divided by the *trijyā*. Here by using the suffix, *'śānac'* in the word *vibhajyamane* [the author] indicates that the addition or subtraction has to be done before division [by the *trijyā*]. This gives the correct value of the *dorjyā* of the sum or difference of the two arcs.

Alternatively, after subtracting the square of the *lamba/lambana* separately from the squares of the two *dorjyās* and taking the square root, the two quantities (thus obtained) become suitable for addition or subtraction. The *lamba* has to be obtained by multiplying the two *dorjyās* and dividing by the *trijyā*.

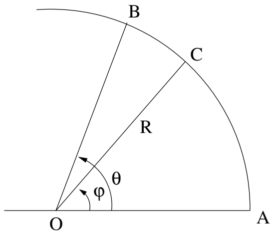


**Fig. 2.6*a*** Determination of the *jyā* corresponding to the sum or difference of two arcs.
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Let α and β be the two arc lengths corresponding to the two angles θ and ϕ as shown in Fig. 2.6a. That is, $AB = \alpha$ and $AC = \beta$ respectively. Nīlakaṇṭha gives the following two formulae for finding the *jyā* of the sum or difference of these arc lengths.

$$dorjy\bar{a}\,(\alpha \pm \beta) = \frac{dorjy\bar{a}\,\alpha\; koṭijy\bar{a}\,\beta \pm koṭijy\bar{a}\,\alpha\; dorjy\bar{a}\,\beta}{trijy\bar{a}} \tag{2.62a}$$

$$dorjy\bar{a}\,(\alpha \pm \beta) = \sqrt{dorjy\bar{a}^2\alpha - lamba^2} \pm \sqrt{dorjy\bar{a}^2\beta - lamba^2}, \tag{2.62b}$$

where *lamba* in the above equation is defined by

$$lamba = \frac{dorjy\bar{a}\,\alpha\; dorjy\bar{a}\,\beta}{trijy\bar{a}}. \tag{2.63}$$

In terms of the angles θ and ϕ, *lamba* can be expressed as

$$lamba = \frac{R\sin\theta\; R\sin\phi}{R}. \tag{2.64}$$

The term *lamba* generally means a vertical line or a plumb-line. The expression for the *lamba* given above can be understood using a geometrical construction. For this consider two angles θ and ϕ such that $\theta > \phi$, as shown in Fig. 2.6b. Find $\sin\theta$ and $\sin\phi$. Draw lines XY and OZ perpendicular to each other as indicated in the figure. Now we consider a segment of length $R\sin\phi$ and place it inclined to OZ such that the segment BN makes an angle θ with BO.

Then draw a line NC such that $O\hat{C}N = \phi$. By construction, $B\hat{N}C = \theta - \phi$. Draw a perpendicular from B which meets the line NC at D. From the triangle NBD,

$$\sin(\theta - \phi) = \frac{BD}{R\sin\phi} \tag{2.65a}$$

Also in the triangle BCD,

$$\sin\phi = \frac{BD}{BC}. \tag{2.65b}$$

From (2.65a) and (2.65b)

$$BC = R\sin(\theta - \phi). \tag{2.65c}$$

Now, applying the sine rule to the triangle NBC, we get the following relation

$$\frac{NB}{\sin\phi} = \frac{BC}{\sin(\theta - \phi)} = \frac{NC}{\sin(180 - \theta)} \tag{2.66}$$

Since $NB = R\sin\phi$ (by construction) and $BC = R\sin(\theta - \phi)$ (see (2.65c)), from the above equation, the third side NC of the triangle must be equal to $R\sin(180 - \theta)$. That is $NC = R\sin(180 - \theta) = R\sin\theta$. Now it can be easily seen that NO in Fig. 2.6b represents the expression for the *lamba* given above.

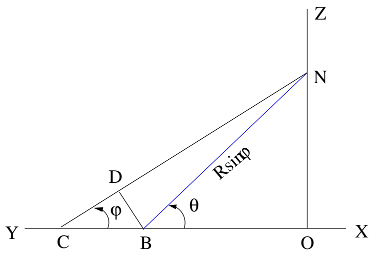

Fig. 2.6*b* Geometrical construction to understand the expression for the *lamba* given in Chapter 2, verse 16.

Using (2.3), and the above expression for the *lamba*, (2.62*a*) and (2.62*b*) reduce to the following equations respectively,

$$R\,\sin(\theta\pm\phi) = \frac{R\sin\theta\; R\cos\phi \pm R\cos\theta\; R\sin\phi}{R} \qquad (2.67a)$$

$$R\,\sin(\theta\pm\phi) = R\sin\theta\cos\phi \pm R\cos\theta\sin\phi, \qquad (2.67b)$$

which are the same as the standard formula used in planar trigonometry,

$$\sin(\theta\pm\phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi. \qquad (2.68)$$

२.८ अल्पचापज्यानयनम्
2.8 Computation of the Rsine value of a small arc

शिष्टचापघनषष्ठभागतो विस्तरार्धकृतिभक्तवर्जितम् ।
शिष्टचापमिह शिञ्जिनी भवेत् स्पष्टता भवति चाल्पतावशात् ॥ १७ ॥

śiṣṭacāpaghanaṣaṣṭhabhāgato vistarārdhakṛtibhaktavarjitam |
śiṣṭacāpamiha śiñjinī bhavet spaṣṭatā bhavati cālpatāvaśāt || 17 ||

Divide one-sixth of the cube of the remaining arc by the square of the *trijyā*. This quantity when subtracted from the remaining arc becomes the *śiñjinī* (the *dorjyā* corresponding to the remaining arc). The value is accurate because of the smallness [of the arc].

In the above verse, Nīlakaṇṭha gives the approximation for the sine of an angle when it is small. If $\alpha = R\delta\theta$ is the length of a small arc along the circle, corresponding to an angle $\delta\theta$, then the above verse gives the following expression for its *dorjyā*:

$$\textit{dorjyā}\;\alpha = \alpha - \frac{\alpha^3}{6\;\textit{trijyā}^2}. \qquad (2.69)$$

The above expression is equivalent to

$$R\sin\delta\theta = R\,\delta\theta - \frac{(R\,\delta\theta)^3}{6\,R^2}$$

$$\text{or} \qquad \sin\delta\theta = \delta\theta - \frac{(\delta\theta)^3}{6}. \tag{2.70}$$

Thus we find that $\sin\delta\theta$ is approximated by the first two terms in the series expansion for it. This gives fairly accurate results when $\delta\theta$ is small. That (2.70) is valid and yields accurate results only when the arc is small is clearly emphasised in *Laghu-vivṛti* as follows:

एवं कृतायास्तस्याः चापाल्पतावशादेव स्पष्टता भवति ।

The accuracy of this operation is due solely to the smallness of the arc.

२.९ इष्टज्यानयनम्

2.9 Computation of the desired Rsine

ऊनाधिकधनुर्ज्यां च नीत्वैवं पठितां न्यसेत् ।
ऊनाधिकधनुःकोटिजीवया तां समीपजाम् ॥ १८ ॥
निहत्य पठितां तस्याः कोटया शिष्टगुणाञ्च तम् ।
तद्योगं वाथ विश्लेषं हरेद् व्यासदलेन तु ॥ १९ ॥
इष्टज्या भवति स्पष्टा तत्फलं स्यात् कलादिकम् ।
न्यायेनानेन कोटयाश्च मौर्व्याः कार्या सुसूक्ष्मता ॥ २० ॥

ūnādhikadhanurjyāṃ ca nītvaivaṃ paṭhitāṃ nyaset |
ūnādhikadhanuḥkoṭijīvayā tāṃ samīpajām || 18 ||
nihatya paṭhitāṃ tasyāḥ koṭyā śiṣṭaguṇañca tam |
tadyogaṃ vātha viśleṣaṃ hared vyāsadalena tu || 19 ||
iṣṭajyā bhavati spaṣṭā tatphalaṃ syāt kalādikam |
nyāyenānena koṭyāśca maurvyāḥ kāryā susūkṣmatā || 20 ||

Having also obtained the *dorjyā* of the arc which is in excess or deficit [from a multiple of 225 minutes], as described above (in the previous verse), keep it separately.

Multiply the nearest *dorjyā* [obtained from the tabulated Rsines] by the *koṭijyā* of the arc which is in excess or deficit. Also multiply the *koṭijyā* by the *dorjyā* of the arc which is in excess or deficit. The sum or difference of these two has to be divided by the radius (*trijyā*).

The desired *jyā* (*dorjyā*) can thus be found accurately. By the same procedure, the *koṭijyā* of any desired arc may be found accurately.

The above verses give the formulae for finding the *dorjyā* and *koṭijyā* of an arc of any desired length. To find this using the procedure given in Indian astronomical texts, the desired arc length is expressed as a sum of two arcs, say $\alpha + \delta\alpha$ where α is an integral multiple of 225 and $0 < \delta\alpha < 225$. The formulae given in the above verses are:

$$dorjy\bar{a}\,(\alpha \pm \delta\alpha) = \frac{dorjy\bar{a}\,\alpha\ ko\d{t}ijy\bar{a}\,\delta\alpha \pm ko\d{t}ijy\bar{a}\,\alpha\ dorjy\bar{a}\,\delta\alpha}{trijy\bar{a}} \tag{2.71}$$

$$ko\d{t}ijy\bar{a}\,(\alpha \pm \delta\alpha) = \frac{ko\d{t}ijy\bar{a}\,\alpha\ ko\d{t}ijy\bar{a}\,\delta\alpha \mp dorjy\bar{a}\,\alpha\ dorjy\bar{a}\,\delta\alpha}{trijy\bar{a}}, \tag{2.72}$$

which have already been commented upon. Since $\delta\alpha$ is always small (less than $225'$ or $3.75°$), here it is suggested that the approximation (2.70) given in previous verse—which gives $\sin\delta\alpha$ correct to $O(\delta\alpha^3)$—may be used for determining the *dorjyā* $\delta\alpha$ in the above relation. Once the *dorjyā* is known, the corresponding *koṭijyā* may be found from the former using (2.53).

२.१० रविस्फुटः

2.10 True longitude of the Sun

त्र्यभ्यस्तबाहुकोटिभ्यां अशीत्याप्ते फले उभे ।
चापितं दोःफलं कार्यं स्वर्णं सूर्यस्य मध्यमे ॥ २१ ॥
केन्द्रोर्ध्वार्धे च पूर्वार्धे तत्कालार्कः स्फुटः स च ।
मध्यसावनसिद्धोऽतः कार्यः स्यादुदये पुनः ॥ २२ ॥

tryabhyastabāhukoṭibhyāṃ aśītyāpte phale ubhe |
cāpitaṃ doḥphalaṃ kāryaṃ svarṇaṃ sūryasya madhyame || 21 ||
kendrordhvārdhe ca pūrvārdhe tatkālārkaḥ sphuṭaḥ sa ca |
madhyasāvanasiddho'taḥ kāryaḥ syādudaye punaḥ || 22 ||

The *dorjyā* and *koṭijyā* [of the *manda-kendra* of the Sun] multiplied by 3 and divided by 80 form the *doḥphala* and *koṭiphala*. The arc corresponding to the *doḥphala* has to be applied to the longitude of the mean Sun positively or negatively depending upon whether the *manda-kendra* is within the six signs beginning with *Tulā* (Libra) or within the six signs beginning with *Meṣa* (Aries). The longitude thus obtained is the true longitude. Since this longitude corresponds to the true longitude at the mean sunrise, it has to be further corrected for the true sunrise.

These verses present an explicit expression for the *manda-phala* of the Sun. *Manda-phala* is a correction that needs to be applied to the mean longitude of the planet, called the *madhyama/madhyama-graha*, to obtain the *manda-sphuṭa-graha*. The significance of the *manda-phala*, whose equivalent in modern astronomy is known as the equation of centre, is explained in Appendix F.

If θ_0 be the mean longitude of the planet (here the Sun) at the mean sunrise, then the true longitude θ of the Sun at the mean sunrise is given by $\theta = \theta_0 \pm \Delta\theta$. The correction to the *madhyama* known as the *manda-phala*, $\Delta\theta$, (referred to as the arc of the *doḥphala* in verse 21) is given by

$$\textit{manda-phala} = \textit{cāpa}\left(\frac{3}{80}\textit{manda-kendrajyā}\right). \tag{2.73}$$

The term *manda-kendrajyā* in the above expression stands for the Rsine of the *manda-kendra* or mean anomaly which refers to the difference between the longitude of the mean planet and the *mandocca* (apogee). We denote it as $\theta_0 - \theta_m$, where θ_0 is the longitude of the mean planet and θ_m that of the *mandocca*. Now, the above equation translates to

$$\Delta\theta = \sin^{-1}\left(\frac{3}{80}|\sin(\theta_0 - \theta_m)|\right). \quad (2.74)$$

Here $\frac{3}{80}$ represents the ratio of the radii of the epicycle and the *manda-karṇa-vṛtta*, or '*manda*-hypotenuse circle', whose significance is explained in Appendix F. When the *manda-kendra* is within the six signs beginning with *Meṣa*, that is, $0 \leq (\theta_0 - \theta_m) \leq 180°$, it is stated that the *manda* correction has to be applied negatively. On the other hand, when it is within the six signs beginning with *Tulā*, that is $180° \leq (\theta_0 - \theta_m) \leq 360°$, the correction is to be applied positively. Thus, the true longitude of the Sun is given by

$$\theta = \theta_0 - \sin^{-1}\left(\frac{3}{80}\sin(\theta_0 - \theta_m)\right). \quad (2.75)$$

२.११ चरप्राणाः

2.11 *Prāṇās* of the ascensional difference

संस्कृतायनभागादेः दोर्ज्या कार्या रवेस्ततः ।
चतुर्विंशतिभागज्याहतायास्त्रिज्यया हृतः ॥ २३ ॥
अपक्रमगुणोऽर्कस्य तात्कालिक इह स्फुटः ।
तत्त्रिज्याकृतिविश्लेषात् मूलं द्युज्याथ कोटिका ॥ २४ ॥
दोर्ज्यापक्रमकृत्योश्च भेदान्मूलमथापि वा ।
अन्त्यद्युज्याहता दोर्ज्या त्रिज्याभक्तेष्टकोटिका ॥ २५ ॥
त्रिज्याघ्नेष्टद्युजीवाप्ता चापितार्कभुजासवः ।
दोःप्राणलिप्तिकाभेदमविनष्टं तु पालयेत् ॥ २६ ॥
विषुवद्भाहता क्रान्तिः सूर्याप्ता क्षितिमौर्विका ।
त्रिज्याघ्नेष्टद्युजीवाप्ता चापिता स्युश्चरासवः ॥ २७ ॥

saṃskṛtāyanabhāgādeḥ dorjyā kāryā ravestataḥ |
caturviṃśatibhāgajyāhatāyāstrijyayā hṛtaḥ || 23 ||
apakramaguṇo'rkasya tātkālika iha sphuṭaḥ |
tattrijyākṛtiviśleṣāt mūlaṃ dyujyātha koṭikā || 24 ||
dorjyāpakramakṛtyośca bhedānmūlamathāpi vā |
antyadyujyāhatā dorjyā trijyābhakteṣṭakoṭikā || 25 ||
trijyāghneṣṭadyujīvāptā cāpitārkabhujāsavaḥ |
doḥprāṇaliptikābhedamavinaṣṭaṃ tu pālayet || 26 ||
viṣuvadbhāhatā krāntiḥ sūryāptā kṣitimaurvikā |
trijyāghneṣṭadyujīvāptā cāpitā syuścarāsavaḥ || 27 ||

The Rsine of the longitude of the Sun (*dorjyā*) corrected for the precession of the equinox (the *saṃskṛtāyana*) has to be determined. This, when multiplied by $R\sin 24°$ and divided by the *trijyā*, gives the Rsine of the true declination of the Sun (the *apakramajyā*) at that instant of time. The square root of the difference of the squares of that and the *trijyā* is the *dyujyā*.

Then the *koṭikā* is obtained by finding the square root of the difference between the squares of the *dorjyā* and the *apakramajyā*. The *koṭikā* is also given by the product of the *antyadyujyā* ($R\cos 24$) and the *dorjyā* divided by the *trijyā*. This (the *koṭikā*) is multiplied by the *trijyā* and divided by the *dyujyā*. The arc of this is the right ascension of the Sun (the *arkabhujāsava*). The difference between the longitude and the right ascension in minutes is to be preserved such that it is not lost.

The equinoctial midday shadow (the *viṣuvadbhā*) multiplied by the Rsine of the declination (*krānti*) and divided by 12 is the *kṣitimaurvikā*. This is to be multiplied by the *trijyā* and divided by the desired *dyujyā*. The arc of that gives the ascensional difference in *prāṇas* (the *carāsava*).

While most of the quantities related to the diurnal motion of the Sun are discussed in the third chapter, some of those that are related to the determination of the true longitude of the Sun at true or actual sunrise for a given location are described here. Before explaining the above verses, it would be convenient to list the quantities defined here as follows:

Quantity	Its physical significance	Notation
apakramajyā	the Rsine of declination of the Sun	$R\sin\delta$
dyujyā	the radius of the diurnal circle of the Sun	$R\cos\delta$
arkabhujāsava	the right ascension of the Sun	α
carāsus	the ascensional difference	$\Delta\alpha$

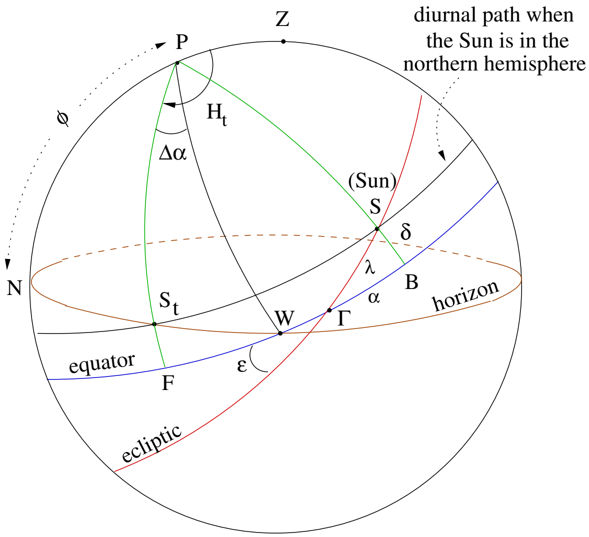

Fig. 2.7 Determination of the declination and right ascension of the Sun on any particular day.

In Indian astronomy texts, it is the *nirayaṇa* longitude or the longitude measured from a fixed star which is calculated. *Ayanāṃśa*, which is the amount of precession, has to be added to the *nirayaṇa* longitude to obtain the *sāyana* or tropical longitude λ. In Fig. 2.7, the celestial sphere is depicted for an observer at latitude ϕ, on a day when the Sun's declination is δ. Let λ and α be the Sun's (tropical) longitude and right ascension on that day.[9] The Rsine of the declination of the Sun, the *apakramajyā*, is given by

$$\begin{aligned} apakramajy\bar{a} &= \frac{dorjy\bar{a} \times caturvi\dot{m}\acute{s}atibh\bar{a}gajy\bar{a}}{trijy\bar{a}} \\ \text{or} \qquad R\sin\delta &= \frac{R\sin\lambda \times R\sin 24^\circ}{R}. \end{aligned} \tag{2.76}$$

This is the formula for declination,

$$\sin\delta = \sin\lambda \sin\varepsilon, \tag{2.77}$$

which can be easily verified by considering the spherical triangle ΓSB in Fig. 2.7 and applying the sine formula. Here ε represents obliquity of the ecliptic whose value is taken to be 24° in most of the Indian astronomy texts. The *dyujyā* is the radius of the diurnal circle of the Sun, $R\cos\delta$, and it is given as

$$\begin{aligned} dyujy\bar{a} &= \sqrt{trijy\bar{a}^2 - apakramajy\bar{a}^2} \\ \text{or} \qquad R\cos\delta &= \sqrt{R^2 - R^2\sin^2\delta}. \end{aligned} \tag{2.78}$$

Now a quantity, the *koṭikā*, is defined by the following two equivalent expressions:

$$\begin{aligned} ko\d{t}ik\bar{a} &= \sqrt{R^2\sin^2\lambda - R^2\sin^2\delta} \\ ko\d{t}ik\bar{a} &= \frac{R\cos\varepsilon\, R\sin\lambda}{R}. \end{aligned} \tag{2.79}$$

The second of these follows from the first by substituting the expression for $R\sin\delta$ given in (2.77). The *arkabhujāsava* is the right ascension of the Sun and is the arc ΓB, which is given as:

$$arkabhuj\bar{a}sava = \alpha = c\bar{a}pa\left(\frac{ko\d{t}ik\bar{a} \times trijy\bar{a}}{dyujy\bar{a}}\right). \tag{2.80}$$

Substituting the expressions for the *koṭikā* and the *dyujyā* in the above, we have

$$\alpha = R\sin^{-1}\left(\frac{R\cos\varepsilon\, R\sin\lambda}{R\cos\delta}\right) \tag{2.81}$$

or

$$R\sin\alpha = \left(\frac{R\cos\varepsilon\, R\sin\lambda}{R\cos\delta}\right). \tag{2.82}$$

This relation follows from the sine formula applied to the spherical triangle $P\Gamma S$, where the spherical angle $P\hat{\Gamma}S = 90 - \varepsilon$, the spherical angle $\Gamma\hat{P}S = \alpha$, arc $\Gamma S = \lambda$ and arc $PS = 90 - \delta$. Then

[9] The reader is referred to Appendix C on coordinate systems for details of these quantities.

$$\frac{\sin\lambda}{\sin\alpha} = \frac{\sin(90-\delta)}{\sin(90-\varepsilon)} = \frac{\cos\delta}{\cos\varepsilon}, \tag{2.83}$$

which is the same as the above.

As the axis of rotation of the Earth is perpendicular to the equator, the rotation angle measured along the equator is related to time and can be expressed in *prāṇas*. One *prāṇa* corresponds to one minute of arc along the equator. Since the right ascension is an arc measured along the equator, α is expressed in *prāṇas*.

The difference between the longitude of the Sun $\odot$ and its right ascension α figures in the equation of time described in the next set of verses (see also Appendix C). Hence $\alpha - \odot$, which is called the *prāṇaliptā* or *prāṇakalāntara*[10] is to be stored. This is the correction due to the obliquity of the ecliptic. This is explained in *Laghu-vivṛti* as follows:

> ...तत्र लब्धा सैव इष्टकोटिज्या। तां इष्टकोटिज्यां त्रिज्यया निहत्य इष्टद्युज्यया विभज्य लब्धं फलं चापीकुर्यात्। तच्च अर्कभुजासवो भवन्ति। तेषां अर्कभुजासूनां तत्कलानां च यदन्तरं तत् प्राणकलान्तरं नाम। तद्विनियोगमुत्तरत्र वक्ष्यामः। अत उक्तं अविनष्टं तु पालयेत् इति॥
>
> What is obtained thus is the *iṣṭakoṭijyā*. That has to be multiplied by the *trijyā* and divided by the *iṣṭadyujyā*. The arc of the result obtained has to be found and that is known as the *arkabhujāsava*. The difference between the *arkabhujāsava* and the Sun's longitude measured in minutes is known as the *prāṇakalāntara*.[11] The utility of this will be stated later (verse 31). Hence it is stated that this has to be preserved such that it is not lost.

The great circle passing through EPW is known as the 6 o'clock circle, as the hour angle of any object lying on that circle corresponds to six hours. For an equatorial observer, whose latitude is zero, the horizon itself is the 6 o'clock circle and the Sun always rises on it. When the latitude of a place is not zero, the Sun does not rise on the 6 o'clock circle. In Fig. 2.7,

$$H_t = Z\hat{P}S_t = Z\hat{P}W + W\hat{P}S_t = 90^\circ + \Delta\alpha \tag{2.84a}$$

is the hour angle at sunset. It is greater than 90° when the Sun's declination is north and would be less than 90° when the declination is south. From the spherical triangle PZS_t, using the cosine formula it can be shown that

$$\begin{aligned} \cos H_t &= -\tan\phi\tan\delta \\ \text{or} \qquad \sin\Delta\alpha &= \tan\phi\tan\delta. \end{aligned} \tag{2.84b}$$

H_t expressed in minutes is the time interval in *prāṇā*s between the meridian transit of the Sun and sunset. When $\delta = 0$, $H_t = 90^\circ = 5400$ *prāṇas* (6 hours). $\Delta\alpha$

---

[10] The terms *prāṇa* and *kalā* here refer to the right ascension and Sun's longitude expressed in minutes respectively. Hence the *prāṇakalāntara* is $\alpha - \odot$.

[11] It must be noted that Śaṅkara Vāriyar uses the term *prāṇakalāntara* instead of *prāṇaliptikā*. Nīlakaṇṭha himself has used the term *prāṇakalāntara* later in verse 31, where he discusses the application of the *prāṇakalāntara*.

is difference between H_t and 6 hours or 5400 *prāṇas*, and is termed the *carāsava* or the ascensional difference. It is clear that it is also the difference between sunrise and transit of the Sun across the 6 o'clock circle.

Expression for the *carāsus*

For giving the expression for the *carāsus* (ascensional difference), an intermediate quantity called the *kṣiti-maurvikā* (*kṣitijyā*, earth-sine) is defined as follows:

$$kṣitimaurvikā = \frac{viṣuvadbhā \times krānti}{12}. \tag{2.85}$$

The *viṣuvadbhā* refers to the equinoctial shadow of a stick of length 12 units. It will be shown in the next chapter that the equinoctial shadow for an observer at latitude ϕ is $12 \tan\phi$. The term *krānti* is the same as *apakramajyā* given earlier in (2.76). The expression for the *carāsus* is given by

$$carāsus = cāpa\left(\frac{kṣitimaurvikā \times trijyā}{dyujyā}\right). \tag{2.86}$$

Substituting for *kṣitimaurvikā* and *dyujyā* in the above expression we have

$$\Delta\alpha = (R\sin)^{-1}\left(\frac{R\tan\phi \; R\sin\delta}{R\cos\delta}\right), \tag{2.87}$$

which is the same as (2.84*b*). At the equator, where $\phi = 0$, $\Delta\alpha = 0$. Hence, the sunrise or sunset is exactly 6 hours before or after meridian transit. Since the *carāsus* ($\Delta\alpha$) is the interval between the sunrise at a given latitude and that at the equator, the knowledge of it is essential for finding the exact sunrise and sunset times at the observer's location. It is also needed for finding the longitude of planets at sunrise at any non-zero latitude.

२.१२ स्वदेशसूर्योदयकाले ग्रहाः

2.12 Longitude of the planets at sunrise at the observer's location

लिप्ताप्राणान्तरं भानोः दोःफलं च चरासवः ।
स्वर्णसाम्येन संयोज्या भिन्नेन तु वियोजयेत् ॥ २८ ॥
भानुमध्यमभुक्तिघ्नं चक्रलिप्ताहृतं फलम् ।
भानुमध्ये तु संस्कार्यं स्फुटभुक्त्याहतं स्फुटे ॥ २९ ॥
उदक्स्थेऽर्के चरप्राणाः शोध्याः स्वं याम्यगोलके ।
व्यस्तमस्ते तु संस्कार्या न मध्याह्नार्धरात्रयोः ॥ ३० ॥
युग्मौजपदयोः स्वर्णं रवौ प्राणकलान्तरम् ।
दोःफलं पूर्ववत् कार्यं रवेरेभिर्द्युचारिणाम् ॥ ३१ ॥

मध्यभुक्तिं स्फुटां वापि हत्वा चक्रकलाहृतम् ।
स्वर्णं कार्यं यथोक्तं तत् व्यस्तं वक्रगतौ स्फुटे ॥ ३२ ॥

liptāprāṇāntaraṃ bhānoḥ doḥphalaṃ ca carāsavaḥ |
svarṇasāmyena saṃyojyā bhinnena tu viyojayet || 28 ||
bhānumadhyamabhuktighnaṃ cakraliptāhṛtaṃ phalam |
bhānumadhye tu saṃskāryaṃ sphuṭabhuktyāhataṃ sphuṭe || 29 ||
udaksthe'rke caraprāṇāḥ śodhyāḥ svaṃ yāmyagolake |
vyastamaste tu saṃskāryā na madhyāhnārdharātrayoḥ || 30 ||
yugmaujapadayoḥ svarṇaṃ ravau prāṇakalāntaram |
doḥphalaṃ pūrvavat kāryaṃ raverebhirdyucāriṇām || 31 ||
madhyabhuktiṃ sphuṭāṃ vāpi hatvā cakrakalāhṛtam |
svarṇaṃ kāryaṃ yathoktaṃ tat vyastaṃ vakragatau sphuṭe || 32 ||

The *prāṇakalāntara*, *doḥphala* (equation of centre) and *carāsus*, all in minutes, have each to be added or subtracted depending upon their signs. This quantity multiplied by the mean daily motion of the Sun and divided by 21600 has to be applied to the mean Sun, and the same has to be multiplied by the true daily motion of the Sun and applied to the true Sun [to get the longitudes of the mean and the true Sun respectively at the true sunrise at any given location].

When the Sun is to the north (has northern declination), then the *carāsus* have to be applied negatively and when it is to the south they have to be applied positively [this sign convention is to be adopted when the longitude is to be determined at sunrise]. The *carāsus* have to be applied in the reverse order at the sunset. They need not be applied [for determining the longitude] at midday or midnight.

The *prāṇakalāntara* has to be applied positively and negatively in the even and odd quadrants respectively. The *doḥphala* has to be applied as discussed earlier. With these quantities (namely *prāṇakalāntara*, *doḥphala* and *carāsus*), which are related to the Sun, the mean or true daily motions of the planets are to be multiplied and divided by 21600. These have to be applied positively or negatively as mentioned earlier [when the planet is in direct motion] and the application has to be done in the reverse order when the planet is in retrograde motion [to get the mean and true planets at true sunrise].

In the above verses, Nīlakaṇṭha gives the procedure for obtaining the mean or true longitudes of the planets at the true sunrise at the observer's location. The longitudes obtained from the *Ahargaṇa* give the mean and true positions of the planets at the mean sunrise, i.e. when the mean Sun is on the 6 o'clock circle, at the observer's location. To get the positions of the planets at the true sunrise, i.e. when the true Sun is on the observer's horizon, corrections have to be applied.

Of the two corrections that need to be applied, one is due to the fact that at sunrise the Sun is on the horizon and not on the 6 o'clock circle. The time difference between the sunrise and the instant when it is on the 6 o'clock circle (the *carāsus*) has been discussed earlier. Now, when the Sun has a northerly declination, sunrise is earlier than its transit across the 6 o'clock circle and *carāsavas* have to be applied negatively. Similarly, when the Sun has a southerly declination, sunrise is after its transit across the 6 o'clock circle and the *carāsus* have to be applied positively. The other two corrections are due to the fact that there is a time difference between the transits of the mean Sun and the true Sun across the meridian or the 6 o'clock circle. In fact, we shall see below that the expression for the sum of these two corrections given in the text is the same as the equation of time in modern astronomy (for more details refer to Appendix C).

Equation of time

The 'mean Sun' is a fictitious body which is moving along the equator uniformly with the average angular velocity of the true Sun. In other words, the right ascension of the mean Sun (denoted by R.A.M.S.) increases by 360° in the same time period as the longitude of the true Sun increases by 360°. As the R.A.M.S. increases uniformly, the time interval between the successive transits of the mean Sun across the meridian or the 6 o'clock circle is constant. This is the mean civil day. All the civil time measurements are with reference to the mean Sun. The time interval between the transits of the mean Sun and the true Sun across the meridian or the 6 o'clock circle is known as the equation of time and is given by

$$\begin{aligned} E &= H.A.M.S. - H.A.\odot \\ &= R.A.\odot - R.A.M.S. \\ &= \alpha - \alpha_{M.S}, \end{aligned} \tag{2.88}$$

where $\odot$ stands for the true Sun. It will also be used to refer to the longitude of the true Sun later. Since the dynamical mean Sun moves along the ecliptic uniformly with the average angular velocity of the true Sun—and both of them are assumed to meet each other at the equinox Γ—the longitude of the dynamical mean Sun or the mean longitude of the Sun (l) is the same as the R.A.M.S. Hence the equation of time will be $E = \alpha - l$. This can be rewritten as

$$E = (\alpha - \odot) + (\odot - l). \tag{2.89}$$

The first term in the equation of time is the *prāṇakalāntara* = $\alpha - \odot$. Now $\sin\alpha = \frac{\cos\varepsilon \sin\odot}{\cos\delta}$. As $\delta < \varepsilon$, $|\sin\alpha| < |\sin\odot|$. This implies that $\alpha < \odot$ when α and $\odot$ are in the odd quadrants and $\alpha > \odot$ when α and $\odot$ are in the even quadrants. Hence the *prāṇakalāntara* has to be applied positively and negatively in the even and odd quadrants respectively. The sign of the *doḥphala* ($\odot - l$) has already been discussed earlier. It is negative in the first and second quadrants and positive in the third and fourth quadrants.

Application of corrections

The three corrections, namely the *prāṇakalāntara*, *doḥphala* and *carāsus*, have to be applied to the mean or true longitude of planets at mean sunrise at the equator (or the 6 o'clock circle) to obtain the mean or true longitude at true sunrise on the observer's horizon. The motion of a planet in one *prāṇa* is equal to its daily motion divided by 21600. The net correction would be the sum of the three quantities (taking appropriate signs into account) multiplied by the above ratio. When the planet is in retrograde motion, the longitude decreases with time. Hence, all the signs discussed above have to be reversed in such a situation.

Śaṅkara Vāriyar in his *Yukti-dīpikā* gives a graphic description of what is meant by *cara*, and how it is to be used in the determination of the duration of day and night at the observer's location (having non-zero latitude).

स्वाक्षे स्वक्षितिजे ज्ञेयावुदयास्तमयौ रवेः ।
उन्मण्डलक्षितिजयोः अन्तरालधनुश्चरम् ॥
सौम्ये पूर्वमुदेत्यर्कः पश्चात्तेनास्तमेति च ।
अतश्चरेण द्विघ्नेन दिनं तत्र तु वर्धते ॥
क्षीयते च निशा यस्मात् मिथोभिन्ने दिनक्षये ।
याम्ये पश्चादुदेत्यर्कः तेन प्रागस्तमेति च ॥
अतश्चरेण द्विघ्नेन तत्र तु क्षीयते दिनम् ।
वर्धते च निशा यस्मात् व्यस्तत्वं गोलयोर्मिथः ॥
चरप्राणगतिः स्वर्णं याम्योदग्गोलयोस्ततः ।
उदयेऽस्तमये व्यस्तं ग्रहे रव्युदयावधौ ॥
मध्यार्कप्रमितं तुल्यरूपमेव सदा दिनम् ।
तुल्यत्वात् तद्गतेर्भिन्नं स्फुटार्कप्रमितं दिनम् ॥ [12]

The rising and setting of the Sun has to be determined with respect to the horizon corresponding to the observer's own latitude. The length of the arc [of the diurnal circle] lying between the *unmaṇḍala* (6 o'clock circle) and the *kṣitija* (horizon) is referred to as the *cara*.

When the Sun has northern declination it rises earlier and sets later. Hence the duration of the day increases by twice the *cara*. Naturally the duration of the night decreases, and hence day and night have different durations. When the Sun has southern declination it rises later and sets earlier. Therefore the duration of the day decreases by twice the *cara* and that of the night increases. [While this is true for an observer in the northern hemisphere] the reverse happens in the southern hemisphere.

The *caraprāṇas* have to be applied negatively and positively when the Sun has northern and the southern declination respectively. This is true at sunrise and during sunset they have to be applied in the reverse order. Since the mean Sun moves with uniform velocity, the duration of the day will always be uniform when measured with respect to the mean Sun. But the duration will vary when measured with respect to the true Sun.

२.१३ दिनक्षपयोर्मानम्

2.13 Durations of the day and the night

अहोरात्रचतुर्भागे चरप्राणान् क्षिपेदुदक् ॥ ३३ ॥
याम्ये शोध्या दिनार्धं तत् रात्र्यर्धं व्यत्ययाद्भवेत् ।
दिनक्षपे द्विनिघ्ने ते चन्द्रादेः स्वैश्चरासुभिः ॥ ३४ ॥

ahorātracaturbhāge caraprāṇān kṣipedudak || 33 ||
yāmye śodhyā dinārdhaṃ tat rātryardhaṃ vyatyayādbhavet |
dinakṣape dvinighne te candrādeḥ svaiścarāsubhiḥ || 34 ||

In the north (when the declination of the Sun is towards north), the *caraprāṇā* has to be added to one-fourth of the *ahorātra* and in the south it has to be subtracted. This gives the

[12] {TS 1977}, p. 154.

half-duration of the day. The half-duration of the night is obtained by applying the *cara* in the reverse order. By multiplying these durations by two, the durations of the day and night are obtained respectively. For the Moon and others, the half-durations [of their own days and nights] have to be obtained from their own *carapr̄āṇas*.

While the time unit, namely a day, can be considered with respect to different planets, we first consider the Sun and the solar day. By definition, on an average, one-fourth of an *ahorātra* or mean solar day or civil day is 6 hours. To this, correction due to the *cara* has to be added or subtracted in order to find the 'actual' half-duration of the day, i.e. the time interval between sunrise and the meridian transit of the Sun. Recalling that one hour corresponds to 15°, the half-duration of the day (in hours) for an observer with latitude ϕ is given by

$$6 + \frac{(R\sin^{-1})(R\tan\phi\tan\delta)\ \text{[in deg]}}{15}, \tag{2.90}$$

where the second term is positive or negative depending upon the sign of δ, i.e. depending on whether the Sun is in the northern or southern hemisphere. δ is obtained using the relation

$$R\sin\delta = R\sin\varepsilon\sin\lambda. \tag{2.91}$$

As pointed out later, it is noted in *Laghu-vivṛti* that λ at true sunrise should be used in the calculation to obtain the first half-duration of the day. Similarly λ at true sunset should be used to obtain the second half-duration of the day. This is explained in *Laghu-vivṛti* as follows:

अहोरात्रस्य षष्टिघटिकात्मकस्य यश्चतुर्भागः पञ्चदशघटिकारूपः तस्मिन् तात्कालिकात् सायनार्कादानीतान् चरप्राणान् नाडीकृत्य क्षिपेत् यद्युदग्गोलगतः सायनस्फुटार्कः। याम्यगोलगते पुनस्तस्मिन् ततोऽहोरात्रचतुर्भागाद् विशोधयेत्। एवं कृतोऽहोरात्रचतुर्भागः तस्मिन् दिने दिनार्धं भवति। रात्र्यर्धं पुनः ततो व्यत्ययाद् भवति।उदग्गोले चरप्राणविरहितः अहोरात्रचतुर्भागः रात्र्यर्धं, याम्यगोले तु तत्सहितः इति। एवं कृतं दिनार्धं रात्र्यर्धं च द्विगुणितं कृत्स्नं दिनमानं क्षपामानं च भवति।

The *carapr̄āṇas* obtained from the *sāyana* longitude λ of the Sun, when it is in the northern hemisphere ($0 < \lambda < 180$), converted into *nāḍīs*, have to be applied positively to one-fourth of the duration of the *ahorātra*, which is 15 *ghaṭikās*, the duration of the *ahorātra* itself being 60 *ghaṭikās*. If the Sun is in the southern hemisphere ($180 < \lambda < 360$), then the *carapr̄āṇas*, converted into *nāḍīs* have to be applied negatively to one-fourth of the duration of the *ahorātra*. Thus one-fourth of the *ahorātra* being corrected by the *carapr̄āṇa* gives the half-duration of the day. The half-duration of the night is obtained by carrying out the reverse process. The half-duration of the night, which was obtained by subtracting the *carapr̄āṇa* in the northern hemisphere, is to be obtained by its addition in the southern hemisphere. The half-durations of the day and night when multiplied by two give the durations of day and night.

To get the half-durations of the day and night more accurately, a better procedure is suggested.

अत्र पुनः औदयिकात् सायनार्कात् आनीतमेव चरं दिनपूर्वार्धे कार्यम्; आस्तमिकादानीतं च अपरार्धे ।रात्रिपूर्वार्धेऽपि आस्तमिकादानीतं; अपरार्धेऽपि औदयिकादानीत-

मिति। एवं कृतयोः दिनार्धयोः क्षपार्धयोश्च योगः दिनक्षपयोः स्फुटतरं मानमिति। अथ प्राणकलान्तरमपि दिनमानार्थं कर्तव्यम्। तेन दिनरात्र्योः उभयतः सायनार्कतः यत् प्राणकलान्तरद्वितयमानीतं तयोर्विवरमपि दिनक्षपामानयोः कर्तव्यमेव, येन दिनक्षपयोर्माने स्फुटतमे स्यातामिति।

The *cara* obtained from the *sāyana* Sun at sunrise (instead of mean sunrise at the equator) has to be applied in the forenoon and the one obtained from the *sāyana* Sun at sunset in the afternoon. Similarly, the *cara* obtained from the *sāyana* Sun at the sunset and sunrise have to be applied for obtaining the duration of the first and second half of the night respectively. The duration of the day and night obtained thus (rather than those obtained from the earlier method) would be more accurate.

The *prāṇakalāntara* correction should also be implemented in finding the duration of the day. The difference in the *prāṇakalāntaras* obtained from the *sāyana* Sun at sunrise and sunset has to be applied to obtain more accurate durations of day and night.

Duration of the day of the planets

The stars are considered to be fixed objects in the sky. The sidereal day is defined as the time interval between two successive rises of the star across the horizon and is equal to the time taken by the Earth to complete one revolution around its axis. A 'planet-day' is defined in a similar manner. The time interval between two successive sunrises is the 'sun-day' or a solar day. The time interval between two successive moonrises is the 'moon-day' or lunar day.[13]. Similarly the time interval between two successive rises of any particular planet is defined to be the duration of that 'planet-day'.

This concept of the day of planets may be understood with the help of Fig. 2.8. In Fig. 2.8*a*, we have depicted a situation where a star X, the Sun S and the Moon M are all in conjunction and are just about to rise above the horizon. After exactly one sidereal day ($\approx$ 23 h 56 m) the star X will be back on the horizon. However, the Sun and Moon, due to their orbital motion eastwards, will not be back on the horizon. They would have moved in their respective orbits through distances, given by their daily motions which are approximately 1° and 13° respectively. This situation is depicted in Fig. 2.8*b* where X, S' and M' represent the star, the Sun and the Moon respectively.

It may be noted here that the Moon is shown to be on the ecliptic. Though the orbit of the Moon is slightly inclined to the ecliptic, since its orbital inclination is very small (approximately 5°), the angular distance covered by the Moon in its orbit can be taken to be roughly the angular distance covered by it on the ecliptic. After one sidereal day the star X will be again on the horizon. Only when the earth rotates through an angle equal to the difference between the right ascensions of X and S' will the Sun be on the horizon. This is taken to be the arc XS' on the ecliptic itself. (This can only be approximate.) Similarly only when it rotates through an angle XM' will the Moon be on the horizon (in the same approximation). Hence the duration of a solar day is given by

[13] This definition of lunar day should not be confused with that of a *tithi* defined earlier.

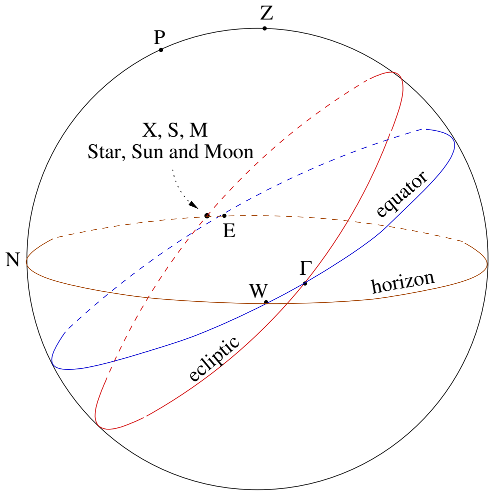

Fig. 2.8*a* The star X, Sun S and the Moon M at sunrise on a particular day.

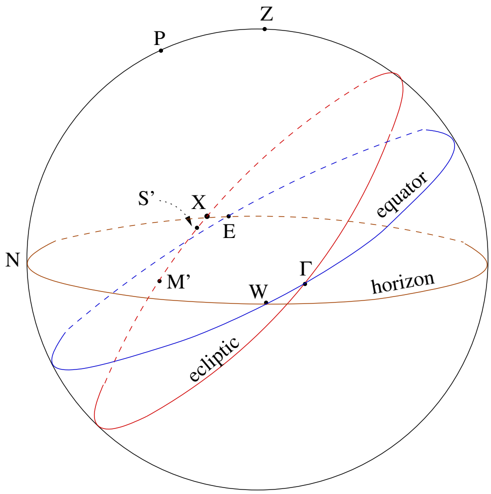

Fig. 2.8*b* The star X, Sun S' and the Moon M' exactly after one sidereal day.

$$\begin{aligned}\text{Solar day} &= \text{Sidereal day + Time taken by the earth to} \\ &\qquad\qquad\qquad \text{rotate through } XS' \\ &= 21600 + XS' \text{(in minutes of arc)} \\ &= 21600 + \text{Sun's daily motion (in } pr\bar{a}\d{n}as).\end{aligned}$$

In the above expression, the number 21600 represents the number of $pr\bar{a}\d{n}as$ (≈4 seconds) in a sidereal day, and XS' is expressed in minutes. Similarly the duration of the lunar day is given by

$$\begin{aligned}\text{Lunar day} &= \text{Sidereal day + Time taken by the earth to} \\ &\qquad\qquad\qquad\text{rotate through } XM' \\ &= 21600 + XM' \\ &= 21600 + \text{Moon's daily motion (in } pr\bar{a}\d{n}as\text{).} \qquad (2.92)\end{aligned}$$

Similarly a 'planet day' can be defined for other planets also.

For finding half of the duration for which a planet is above the horizon, its own *caraprāṇa* has to be added to or subtracted from one-fourth of its own 'planet-day'. In *Laghu-vivṛti* there is a discussion on this:

चन्द्रादेः पुनः स्वैः स्वैः चरासुभिः उक्तवत् संस्कृतात् निजाहोरात्रचतुर्भागतः दिनार्धं रात्र्यर्धं च अवगन्तव्यम्। अहोरात्रश्च तस्य निजदिनस्फुटभोगप्राणाधिकचक्रकलातुल्य-प्रमाणः। नन्वेवं कृतस्यापि इन्दोः दिनक्षपयोर्मानस्य न स्फुटत्वम्। निजदोःफलकला-दिना अधिकोनत्वसंभवात्, सत्यम् ; अत एव हि तत्र स्वदिनान्तरितयोः स्फुटयोर्विधि-वत् कृतस्वचरप्राणकलान्तरयोः अन्तरेण चक्रकला सहितेन दिनासवः क्रियन्ते।

For the Moon and others (planets) their own durations of day and night have to be obtained from the quarter of their true *ahorātra* corrected for their own *carāsus*. The duration of their day is [nearly] equal to the sum of the their daily motion in *prāṇas* plus the number of minutes in 360 degrees. Even after applying this, the duration of the day or night of the Moon [and other planets] would not be correct as it may differ [from the actual value] by its own *doḥphala*. True; it is only to take this discrepancy into account that the [true] duration of a lunar day in minutes is obtained from the difference in the true positions of the Moon [and other planets], at intervals separated by the durations of their days corrected by *caraprāṇa* etc., added to the number of minutes in 360 degrees.

Ascensional difference in the case of the Moon and other planets:

It may be recalled that (2.67) gives the expression for finding the *caraprāṇa* in the case of the Sun. For the Moon and other planets the procedure to be adopted is stated in *Laghu-vivṛti* as follows:

स्वचरप्राणानयनमपि अमुनैव इष्टक्रान्त्या विक्षेपसंस्कृतया छायागणिते प्रदर्शितम्-

क्रान्तिज्या विषुवद्भाघ्ना क्षितिज्या द्वादशोद्धृता।

व्यासार्धघ्ना द्युजीवाप्ता चापितास्युस्चरासवः॥ [14]

For planets other than Sun the procedure for obtaining their own *caraprāṇa* from the declination corrected for the latitude of the planet has been shown by [the author] himself in [his] *Chāyāgaṇita*:

The sine of the declination multiplied by the *viṣuvadbhā* and divided by twelve is the *kṣitijyā*. This has to be multiplied by the *trijyā* and divided by the *dyujyā*. The arc of this is the *carāsava*.

In Fig. 2.9, *S* represents the position of the Sun on the observer's meridian on an equinoctial day. Since the motion of the sun takes place along the equator on

[14] {CCG 1976}, p. 16.

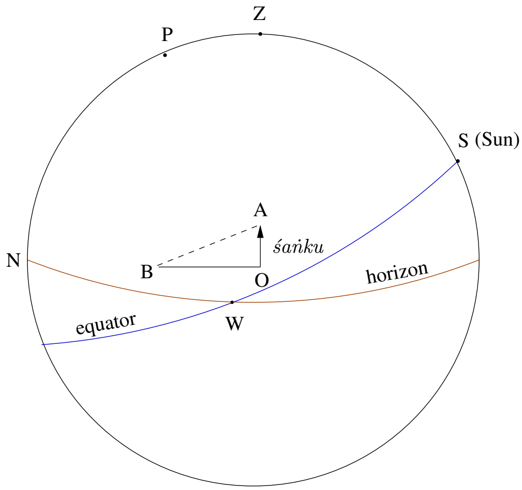

Fig. 2.9 Shadow of *śaṅku* on an equinoctial day.

an equinoctial day, the equator itself serves as the diurnal circle. The length of the shadow of a stick of length 12 units, when the Sun is on the observer's meridian on the equinoctial day, is termed the *viṣuvadbhā*. In the figure, OA represents the stick of length 12 units, referred to as a *śaṅku*. Since $ZS = \phi$, the latitude of the observer, $O\hat{A}B = \phi$. Hence,

$$\textit{viṣuvadbhā} \; = 12 \tan\phi. \tag{2.93}$$

From (2.85), *kṣitijyā* is given by $\frac{12\tan\phi \times R\sin\delta}{12}$. Also the ascensional difference *carāsava* of the planet is given by

$$\textit{carāsus} = (R\sin)^{-1} R\tan\phi \tan\delta, \tag{2.94}$$

where δ is the declination of the planet. The declination δ of a planet with longitude λ and latitude β as depicted in Fig. 2.10 is given by

$$\begin{aligned} \sin\delta &= \cos\varepsilon \sin\beta + \sin\varepsilon \cos\beta \sin\lambda \\ &= \cos\varepsilon \sin\beta + \cos\beta \sin\delta_E, \end{aligned} \tag{2.95}$$

where δ_E is the declination of an object on the ecliptic with the same longitude as the planet. That is, $\sin\delta_E = \sin\varepsilon \sin\lambda$. Thus, in the case of planet having a latitude, a correction has to be applied to δ_E to obtain the actual declination δ. From the 'planet-day' and the *carāsava* of the planet, the time interval between the rising and setting of the planet which is the duration of the 'day' for the planet can be obtained.

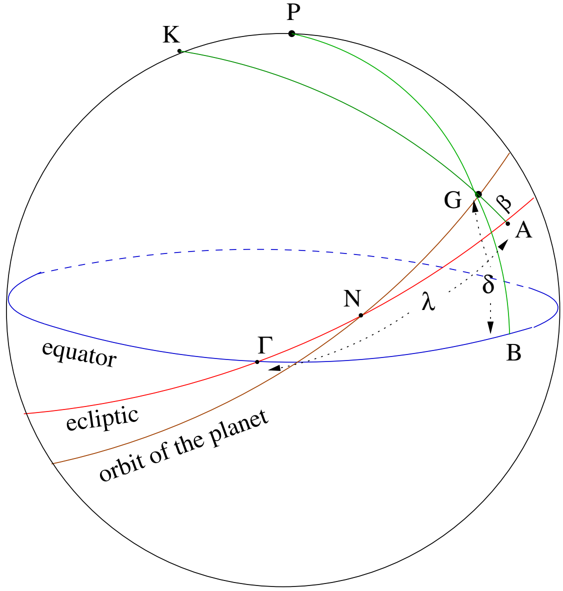

Fig. 2.10 Determination of *caraprāṇa* for planets.

२.१४ चन्द्रस्फुटीकरणम्

2.14 Obtaining the true Moon

इन्दूच्चयोः स्वदेशोत्थरव्यानीतचरादिजम् ।
संस्कारं मध्यमे कृत्वा स्फुटीकार्यो निशाकरः ॥ ३५ ॥
दोःकोटिज्ये तु सप्तघ्ने अशीत्याप्ते फले उभे ।
चापितं दोःफलं कार्यं स्वमध्ये स्फुटसिद्धये ॥ ३६ ॥

indūccayoḥ svadeśottharavyānītacarādijam|
saṃskāraṃ madhyame kṛtvā sphuṭīkāryo niśākaraḥ || 35 ||
doḥkoṭijye tu saptaghne aśītyāpte phale ubhe|
cāpitaṃ doḥphalaṃ kāryaṃ svamadhye sphuṭasiddhaye || 36 ||

The mean position of the Moon and its apogee have to be corrected by the *caraprāṇa* etc. obtained from the Sun, and then the *sphuṭa-karma* (procedure for the true longitude) has to be carried out.

The *dorjyā* and the *koṭijyā* multiplied by 7 and divided by 80 form the *doḥphala* and *koṭiphala*. The arc of the *doḥphala* has to be applied to the mean position to get the true position.

The mean positions of the planets obtained from the *Ahargaṇa* (count of days) correspond to their mean positions at the mean sunrise for an observer at Ujjayinī. To get their mean positions for other observers, corrections such as *deśāntara, cara* etc. have to be applied (see the previous section as well as Section 1.14). These are corrections to be carried out to get the mean position of the planet at the true sunrise at the observer's location. Verse 35 reemphasizes these corrections.

To get the true position of the planet at true sunrise, the equation of centre has to be applied to the mean planet at true sunrise. Verse 36 describes how this correction has to be implemented in the case of the Moon. The ratio of the mean radius of the epicycle and the radius of the deferent circle (the *trijyā*) is taken to be $\frac{7}{80}$ for the Moon. Hence according to the text the true longitude of the Moon, θ, is

$$\theta = \theta_0 - \sin^{-1}\left(\frac{7}{80}\sin(\theta_0 - \theta_m)\right),$$

where θ_0 is the mean longitude of the Moon and θ_m the longitude of the *mandocca*.

The procedure for obtaining the true longitude of the Moon is explained in the commentary as follows:

चन्द्रतदुच्चयोरपि इष्टद्युगणतः त्रैराशिकसिद्धे मध्यमे देशान्तरस्य रविदोःफलादीनां त्रयाणां च गतिं विधिवत् कृत्वा चन्द्रस्य मध्यतः तन्मन्दोच्चं विशोध्य शिष्टे तत्केन्द्रे भगणपूर्वोर्ध्वार्धगतत्वं च अवधार्य दोःकोटयोरुभयोरपि जीवे रविकेन्द्रोक्तवद्गृह्णीयात्।

तथा गृहीते दोःकोटिज्ये उभे अपि सप्तभिर्निहत्य अशीत्या विभज्य लब्धे दोःकोटिफले स्याताम्। तत्र कोटिफलस्य उपयोगं वक्ष्यामः। दोःफलं पुनश्चापीकृत्य तन्मध्यमे स्वकेन्द्रभगणपूर्वोर्ध्वार्धगतत्ववशात् ऋणं धनं वा कुर्यात्। एवं कृतश्चन्द्रमध्यमः स्वदेशस्फुटार्कोदयावधिकः स्फुटो भवति।

From the *deśāntara*, as well as the three corrections *manda-phala* etc. related to the Sun [obtaining the true sunrise time], the mean positions of the Moon and its apogee [at true sunrise time], are obtained from the *Ahargaṇa* by the rule of three. Then subtracting the apogee from the mean longitude, the *manda-kendra* of the Moon is determined. Depending upon the quadrant in which the *manda-kendra* lies, the *dorjyā* and *koṭijyā* have to be found following the procedure that was given for the Sun.

The *dorjyā* and *koṭijyā* obtained thus have to be multiplied by 7 and divided by 80 to get the *doḥphala* and *koṭiphala* respectively. The use of the *koṭiphala* will be stated later. The arc corresponding to the *doḥphala* is applied to the mean planet either positively or negatively depending upon the quadrant in which the *kendra* lies. These corrections applied to the mean Moon give its true position at the true sunrise at the observer's location.

२.१५ चरज्यादीनां चापीकरणम्

2.15 Finding the arc corresponding to *cara* etc.

ज्याचापान्तरमानीय शिष्टचापघनादिना ।
युक्त्वा ज्यायां धनुः कार्यं पठितज्याभिरेव वा ॥ ३७ ॥
त्रिखरूपाष्टभूनागरुद्रैः त्रिज्याकृतिः समा ।
एकादिघ्नया दशाप्ता या घनमूलं ततोऽपि यत् ॥ ३८ ॥
तन्मितज्यासु योज्याः स्युः एकद्व्याद्या विलिप्तिकाः ।
चरदोःफलजीवादेः एवमल्पधनुर्नयेत् ॥ ३९ ॥

jyācāpāntaramānīya śiṣṭacāpaghanādinā|
yuktvā jyāyāṃ dhanuḥ kāryaṃ paṭhitajyābhireva vā|| 37 ||
trikharūpāṣṭabhūnāgarudraiḥ trijyākṛtiḥ samā|

ekādighnyā daśāptā yā ghanamūlaṃ tato'pi yat || 38 ||
tanmitajyāsu yojyāḥ syuḥ ekadvyādyā viliptikāḥ |
caradoḥphalajīvādeḥ evamalpadhanurnayet || 39 ||

The arc corresponding to a *jyā* may be obtained either by finding the difference between the *jyā* and the arc as given in the verse [beginning] *śiṣṭacāpaghana* etc., and adding that (difference) to the *jyā*, or from the table of *jyās* listed earlier.

The square of the *trijyā* is 11818103 (in minutes). Multiply this by 1, 2 etc., divide by 10 and find the cube roots of these results. If the *jyā* (whose arc is to be found) has a measure equal to these (the above cube roots), then 1, 2, etc. seconds have to be added to them. Thus the arc of the R sine of small angles involved in the *caradoḥphala* may be obtained.

In Fig. 2.11, let PN represent the *jyā* whose corresponding arc length AP is to be determined. If R is the radius of the circle and $A\hat{O}P = \alpha$, then the length of the *jyā* corresponding to this angle is given by

$$\textit{jyā} = PN = l = R\sin\alpha. \tag{2.96}$$

When α is small we know that

$$\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}.$$

$$\text{Hence,} \qquad R\sin\alpha \approx R\alpha - \frac{(R\alpha)^3}{6R^2}. \tag{2.97}$$

Or, the difference (D) between the *cāpa* (arc) and its *jyā* (Rsine) is given by

$$D \approx R\alpha - l = \frac{(R\alpha)^3}{6R^2}. \tag{2.98}$$

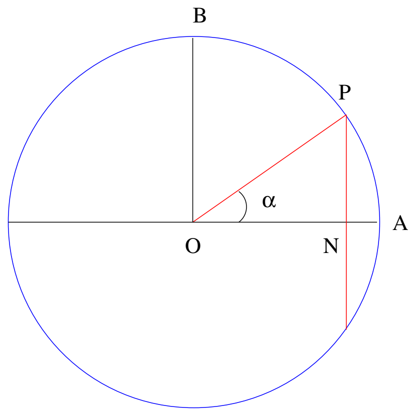

Fig. 2.11 Finding the arc length of a given *jyā* when it is very small.

An iterative procedure for obtaining the arc length corresponding to a given *jyā* is described in the above verses. This procedure is simple and also yields fairly accurate results for small angles. We may explain the procedure outlined here as follows.

As a first approximation, we take the arc length (which itself is very small) to be the *jyā* itself, i.e. $R\alpha \approx l$. Hence from (2.98) the difference between the arc length and its *jyā* becomes

$$D_1 = \frac{l^3}{6R^2}. \tag{2.99}$$

As a second approximation, we take the arc length to be $R\alpha = l + D_1$. Hence in the next approximation the difference (D_2) between the arc length and its *jyā* becomes

$$D_2 = \frac{(l+D_1)^3}{6R^2}. \tag{2.100}$$

As a third approximation, when we take $R\alpha = l + D_2$, we have

$$D_3 = \frac{(l+D_2)^3}{6R^2}. \tag{2.101}$$

In general,

$$D_i = \frac{(l+D_{i-1})^3}{6R^2}. \tag{2.102}$$

The above iteration process is continued till $D_i = D_{i-1}$, to a given level of accuracy. When this condition is satisfied, we have arrived at the required arc length corresponding to the given *jyā*, given by

$$R\alpha = l + D_i. \tag{2.103}$$

Aviśeṣakarma

The iterative procedure, known as *aviśeṣakarma*, to be employed is described in *Laghu-vivṛti* as follows:

चापीचिकीर्षितां ज्यां कात्स्न्र्येन घनीकृत्य षड्भिर्विभज्य लब्धं पुनः त्रिज्याकृत्या च विभजेत्। तत्र लब्धं लिप्तादिकं ज्याचापान्तरम्। अहार्यत्वे पुनः षष्ट्या निहत्य त्रिज्याकृत्या विभजेत्। तत्र लब्धं विलिप्तादिकं ज्याचापन्तरमिति।

नन्वत्र शिष्टचापघनेत्यादिना इष्टचापतः तज्ज्याचापान्तरं क्रियते। न पुनः इष्टज्यातः तच्चापान्तरम्। सत्यम् ; अत एव अत्र अविशेषकर्म क्रियते। तद्यथा - उक्तवदानीतम् इष्टज्याचापान्तरम् इष्टज्यायां प्रक्षिप्य पुनरपि तद्घनतः पूर्ववदानीतं ज्याचापान्तरं मुहुराद्यज्यायामेव प्रक्षिपेत् यावदविशेषः। अविशिष्टेन ज्याचापान्तरेण युक्ता इष्टज्या चापीकृता स्यादिति।

Find the cube of the given *jyā* and divide it by six. This may further be divided by the square of the *trijyā*. The result is the difference between the *jyā* and *cāpa* in minutes. If it is not divisible [if there is a fraction], then it has to be multiplied by 60 and then divided by the square of the *trijyā*. The result thus obtained will be the difference between the *jyā* and the *cāpa* in seconds.

Is it not true that, as per the procedure described in [the verse] *śiṣṭacāpaghana* ..., we find the difference between the *jyā* and *cāpa* from the given (known) *cāpa* and not from the given *jyā*? Yes, it is true. It is only because of this, that an iterative procedure (*aviśeṣakarma*) is followed here where the difference between the *jyā* and *cāpa* is to be found from the given *jyā*. It is as follows: The difference between the *jyā* and *cāpa* obtained as described earlier must be applied to the given *jyā* and from the cube of that the [next approximation to the] difference between the *jyā* and *cāpa* must be determined. This again has to be applied to the given *jyā*, and the process has to be repeated till the result becomes *aviśiṣṭa* (not different from the earlier). This difference added to the given *jyā* will be the required *cāpa*.

Finding the arc length corresponding to a given *jyā* from a look-up table

Apart from the iterative procedure described above, Nīlakaṇṭha also gives an ingenious way by which one can find out the arc length corresponding to a given *jyā*, when the *jyā* is small. Here the idea is to make use of a table of *jyās* and the differences D_i's, in order to obtain the required arc length and thereby avoid the iterative process. The procedure is as follows:

The difference between the *cāpa* and its *jyā* is given by

$$D = R\alpha - l \approx \frac{(R\alpha)^3}{6R^2} = \frac{(l)^3}{6R^2}. \tag{2.104}$$

In the above equation all the quantities are expressed in minutes. When the difference $D = 1''$, which is one-sixtieth of a minute, we obtain

$$\frac{(l)^3}{6R^2} = \frac{1}{60}. \tag{2.105}$$

This implies that when $D = 1''$ the corresponding *jyā* is given by

$$l_1 = \left(\frac{1.R^2}{10}\right)^{\frac{1}{3}}. \tag{2.106a}$$

Similarly when $D = 2''$, the corresponding *jyā* is given by

$$l_2 = \left(\frac{2.R^2}{10}\right)^{\frac{1}{3}}, \tag{2.106b}$$

and so on. In general, when $D = i''$, the corresponding *jyā* is given by

$$l_i = \left(\frac{i.R^2}{10}\right)^{\frac{1}{3}}. \tag{2.107}$$

Here, l_i's correspond to the *jyās*, when the difference between the *jyā* and the *cāpa* (D) is i''. Hence, the lengths of the *cāpas*, A_is, corresponding to the *jyās*, l_i, are given by

$$A_i = l_i + i. \tag{2.108}$$

In Table 2.2, the *jyā* values are listed corresponding to the integral values of the difference between the *jyā* and the arc length, as given in *Laghu-vivṛti*. These are

Difference D = *cāpa*–*jyā* in seconds	Given value of *jyā* min	sec	Textual value of *cāpa* min	sec	Computed value of *cāpa* min	sec
1	105	43	105	44	105	43.56
2	133	11	133	13	133	12.42
3	152	26	152	29	152	29.04
4	167	46	167	50	167	49.80
5	180	43	180	48	180	47.34
6	192	02	192	08	192	07.02
7	202	08	202	15	202	14.82
8	211	20	211	28	211	27.12
9	219	47	219	56	219	55.14
10	227	38	227	48	227	46.80
11	234	58	235	09	235	07.98
12	241	52	242	04	242	03.18
13	248	24	248	37	248	35.88
14	254	36	254	50	254	48.90
15	260	31	260	46	260	44.58
16	266	10	266	26	266	24.78
17	271	36	271	53	271	51.12
18	276	48	277	06	277	04.86
19	281	50	282	09	282	07.20
20	286	40	286	60	286	59.10
21	291	22	291	43	291	41.46
22	295	55	296	17	296	14.94
23	300	18	300	41	300	40.26
24	304	36	304	60	304	58.02

Table 2.2 Look-up table from which the values of arc lengths of small *jyā*s can be directly written down without performing any iteration, when the difference between the *jyā* and the *cāpa* is equal to integral number of seconds.

the l_is, $i = 1 \ldots 24$ in (2.107), which are listed in the second column. The third column gives the sum of columns 1 and 2. The fourth column gives the values of the arc length as computed by us using (2.108), which in turn involves the computation of the cube root of (2.107), for different values of i ($i = 1 \ldots 24$). In doing so, we have also used the exact value of the *trijyā* (in minutes), that is, $R = \frac{21600}{2\pi}$. Given the fact that some approximation in the *trijyā* value and the extraction of the cube root is involved in the computation of arc length, it is remarkable that the value given in the text differs at the most by $2''$ from the exactly computed value of the arc length. The idea behind listing these 24 *jyā* values is to avoid the iterative process outlined earlier, when the *jyā* value is small.

Finding the arc length from the look-up table

The procedure is explained in the commentary as follows:

अथवा एकद्व्यादिविलिप्तारूपं यज्ज्याचापान्तरं तद्विधायिनीः बह्वीः जीवाः पठित्वा तत्तुल्यासु अभीष्टज्यासु एकद्व्यादि विलिप्तारूपं ज्याचापान्तरं इष्टज्यायां प्रक्षिप्य तच्चापं कर्तव्यमिति। तत् कथमिति चेत्-

तत्र त्रिज्यायाः कृतिः त्रिखरूपाष्टभूनागरुद्रैस्तुल्यसङ्ख्या प्रसिद्धा। तत्र त्रिज्याकृतेः एकद्व्यादिनिहतायाः दशभिर्विभज्य लब्धात् फलात् घनमूलमानयेत्। तत्तत्तुल्यासु इष्टज्यासु क्रमादेकद्व्यादिविलिप्तिकाः ज्याचापान्तरत्वेन ग्राह्या इति।तथानीतज्याचापा-न्तरम् इष्टज्यायां प्रक्षिप्य चापीकरणं कार्यमिति। तद्यथा-

लवणं निन्द्यं कपिला गोपी चरराशयस्तवार्थितया ।
लघुनोद्दिष्टो राज्ञः प्रळयो धाम्नां त्रिनेत्र नरकपुरम् ॥
सवधूटीन्द्रो जलसूरद्रीहिमवान् गुरुस्त्रिशङ्कुवरः ।
वरदो वज्री तिलभूर्मेरुः कालेन तत्र नृपतिचरः ॥
तिलकं सान्द्रं धावतिसरित् न मे कुञ्जरो निवृत्तजरः ।
श्रेष्ठकळत्रममाशाधात्री धूपोऽग्नीनाम्बुतिलवनगः ॥
एवमार्यात्रयेणोक्ता मौर्विकाविकलादयः ।
चापीकरणमेताभिः सुकरं दोःफलेऽल्पके ॥
लवणादिषु जीवासु यया तुल्यं भुजाफलम् ।
तत्सङ्ख्या विकला क्षेप्याः तत्र चापप्रसिद्धये ॥ इति ॥

Or if the difference between the *jyā* and the arc length is equal to 1″, 2″, 3″ etc. then construct the table listing the *jyās* corresponding to these differences. If the *jyā* whose *cāpa* is to be determined happens to be (very close to) one of the values listed in the table, then add this difference between *jyā* and *cāpa* (1″, 2″, 3″ etc.) to the *jyā* to get the required *cāpa*. How should this be implemented?

It is well known that the square of the *trijyā* = 11818103. Multiply this by 1, 2, 3, etc., divide by 10, and take the cube roots of the resulting quantities [in minutes etc]. If the *jyā* whose *cāpa* is desired to be found happens to be one of the values [listed in the table], then it is to be understood that the corresponding difference between the *jyā* and *cāpa* is going to be only 1″, 2″, 3″, etc. The difference between the *jyā* and *cāpa*, obtained thus, may thus be added to the given *jyā* to get the desired *cāpa*. This may be done as follows.[15]

Thus the *jyās* in seconds and minutes are given in three *āryā* verses. For instance, the *lavaṇaṃ nindyam* and the *kapilā gopī* stand for $105'43''$ and $133'11''$, respectively. Finding the arc lengths from the *jyās*, when they are small, is quite simple making use of these values. If the *doḥphala* (whose arc length is to be calculated) is equal to one of the values listed, beginning with the *lavaṇa*, then the corresponding number of seconds have to be added to the *jyās* to get the corresponding *cāpa*.

In the commentary it is also stated that using the table and determining the arc lengths may not be as accurate as the result obtained by using the iterative procedure:

[15] The values of the *jyās* given in the succeeding verses *lavaṇaṃ* . . . , are listed in second column of Table 2.2.

यद्यपि सुसूक्ष्मचापीकरणोपायः पूर्वमेव प्रदर्शितः तथापि अल्पीयस्याः जीवायाः चापीकरणमेवं कर्तव्यम्, इति इहापि प्रदर्शितम्। अत उक्तम् - चरदोःफलजीवादेः एवमल्पधनुर्नयेत् - इति।

Though the procedure for obtaining more accurate values of the arc length has already been stated, for smaller *jyās* the arc lengths may be obtained by this method (from the look-up tables). That is why it is stated: The small arc length of the *cara-doḥphala* etc. should be obtained by this method.

The same idea is conveyed in *Yukti-dīpikā* in the following manner:

उक्तं चापघने षड्घ्नत्रिज्यावर्गे कलासमम् ॥
दशांशे तत्कृतेः चापज्यान्तरं विकलासमम् ।
एकादिघ्नात्ततस्त्रिज्यावर्गतो दशभिर्हृतात् ॥
घनमूलं तु यल्लब्धं तत्तुल्ये धनुषि स्थिते ।
एकद्व्याद्या विलिप्ताः स्युः चापज्याविवरोद्भवाः ॥
तदूनं चापमर्धज्या तद्युता ज्या च तद्धनुः ।
कार्योऽविशेषश्चापाप्तौ चापाल्पत्वे दृढं च तत् ॥ [16]

It has been stated implicitly (in verse 17 of the text) that the difference between the *jyā* and *cāpa* will be equal to $1'$ (one *kalā*), when the cube of the arc length is equal to six multiplied by square of the *trijyā*. The same will be equal to $1''$ (one *vikalā*) when the cube of the arc length is equal to one-tenth of the square of *trijyā*.

Now, the square of the *trijyā* divided by 10 is multiplied by 1,2,3, etc. Then the cube roots of the results are taken [and stored separately]. These correspond to the arc lengths, when the difference between the *jyā* and *cāpa* is equal to $1''$, $2''$, $3''$, etc., respectively. When differences are subtracted from the arc length we get the *jyā* and when they are added to the *jyā* we get the arc length. *Aviśeṣakarma* must be done in order to get accurate results for the *cāpa* from the *jyā* whose values are small.

In fact the accuracy of the tabulated results is of the order of 0.003%. For instance for a *cāpa* of $105'44''$, the listed *jyā* value is $105'43''$, whereas the exact Rsine value is $105'43.02''$. The percentage error is 0.0003%. This is not surprising considering the fact that for a small α the fractional error in retaining terms only up to α^3 in $\sin\alpha$ is $\frac{\alpha^5}{5!}$.

२.१६ मन्दशीघ्रकर्णानयनम्

2.16 Obtaining the *manda* and *śīghra* hypotenuses

आद्ये पदे चतुर्थे च व्यासार्धे कोटिजं फलम् ।
युक्त्वा त्यक्त्वान्ययोः तद्दोःफलवर्गैक्यजं पदम् ॥ ४० ॥
कर्णः स्यादविशेषोऽस्य कार्यो मन्दे चले न तु ।

ādye pade caturthe ca vyāsārdhe koṭijaṃ phalam |
yuktvā tyaktvānyayoḥ taddoḥphalavargaikyajaṃ padam || 40 ||
karṇaḥ syādaviśeṣo'sya kāryo mande cale na tu |

[16] {TS 1977}, p. 158.

> Having added the *koṭiphala* to the radius (*vyāsārdha*) in the first and the fourth quadrants and having subtracted [the *koṭiphala*] from it (the radius) in the other two [quadrants] let the square root of the sum of the squares of this and the *doḥphala* be obtained. This is the *karṇa* and in the *manda* process this has to be further iterated upon, but not in the *śīghra* (*cala*).

The method given in the above verse for finding the *karṇa* can be explained with the help of an epicycle model represented in Fig. 2.12*a*. Here the mean planet P_0 is assumed to be moving on the deferent circle centred around O, and the true planet P is located on the epicycle such that PP_0 is parallel to OU (the direction of the *mandocca*). $O\Gamma$ represents the direction of *Aśvini nakṣatra* (*Meṣādi* or first point of Aries).

In Fig. 2.12*a* let R and r be the radii of the deferent circle and the epicycle respectively. OU represents the direction of the *mandocca* whose longitude is given by $\Gamma\hat{O}U = \theta_m$. The longitude of the mean planet P_0 is given by $\Gamma\hat{O}P_0 = \theta_0$. θ_{ms} represents the longitude of the *manda-sphuṭa-graha*. It is easily seen that

$$U\hat{O}P_0 = P\hat{P_0}N = \theta_0 - \theta_m, \tag{2.109}$$

where $(\theta_0 - \theta_m)$ is the *manda-kendra*. The *doḥphala* and the *koṭiphala* are given by

$$doḥphala = PN = |r\sin(\theta_0 - \theta_m)| \tag{2.110}$$

and

$$koṭiphala = P_0N = |r\cos(\theta_0 - \theta_m)|. \tag{2.111}$$

Now, the *manda-karṇa* K is the distance between the planet and the centre of the deferent circle. Clearly,

$$\begin{aligned} K &= OP \\ &= \left[(ON)^2 + (PN)^2\right]^{\frac{1}{2}} \\ &= \left[(R + r\cos(\theta_0 - \theta_m))^2 + (r\sin(\theta_0 - \theta_m))^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.112}$$

Here, $r\cos(\theta_0 - \theta_m) = \pm|r\cos(\theta_0 - \theta_m)|$ is positive in the first and fourth quadrants and negative in the second and third quadrants. That is why it is stated that the *koṭiphala* has to be added to the *trijyā* in the first and fourth quadrants and subtracted from it in the second and third quadrants.

It is also stated that the *karṇa* K has to be determined iteratively in the *manda-saṃskāra* to obtain the *aviśeṣa-karṇa* (iterated hypotenuse). This is because r in (2.112) is not a constant but is itself proportional to K. That is,

$$r = \frac{r_0}{R}K, \tag{2.113}$$

where r_0 is the radius of the epicycle whose value is specified in the text. The iterative procedure to determine K and r is discussed in the next section. In the *śīghra-saṃskāra*, r is fixed for each planet, and no iterative procedure is necessary to find K.

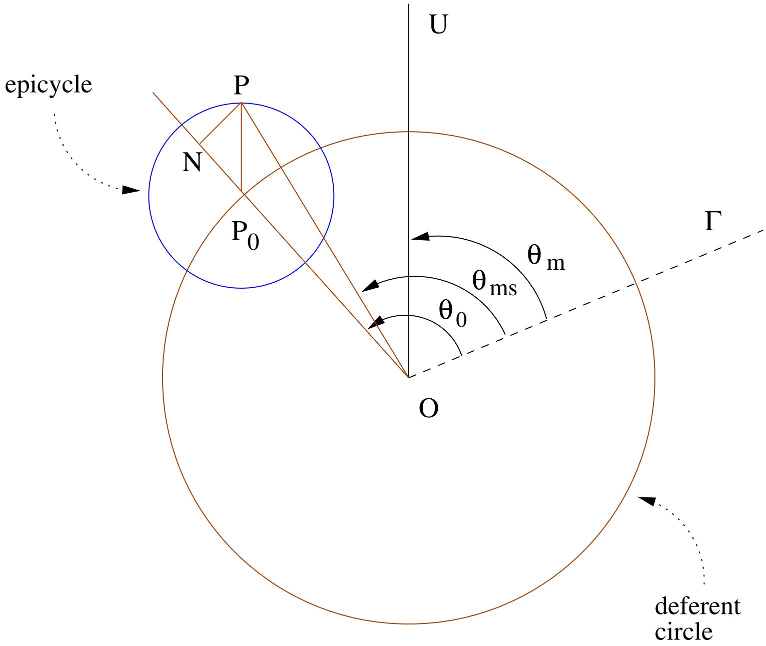

Fig. 2.12*a* Obtaining the *manda-karṇa* in the epicycle model.

In Fig. 2.12*a*, the longitude of the planet is given by $\Gamma\hat{O}P = \theta_{ms} = \theta$. Then $P\hat{O}P_0 = \theta_m - \theta$ is the difference between the mean and true planets. Now,

$$PN = OP\sin(P\hat{O}P_0) = K\sin(\theta_m - \theta). \tag{2.114}$$

PN is also given by

$$PN = PP_0\sin(P\hat{P}_0N) = r\sin(\theta_0 - \theta_m). \tag{2.115}$$

Equating the above two expressions for PN,

$$\begin{aligned} K\sin(\theta_m - \theta) &= r\sin(\theta_0 - \theta_m) \\ \text{or} \qquad \sin(\theta_m - \theta) &= \frac{r}{K}\sin(\theta_0 - \theta_m) \\ &= \frac{r_0}{R}\sin(\theta_0 - \theta_m). \end{aligned} \tag{2.116}$$

Thus the true planet θ can be obtained from the mean planet θ_0 from the above equation. It may be noted that (2.116) does not involve the *manda-karṇa* K.

While commenting on these verses, the eccentric and epicyclic models are described in *Yukti-dīpikā*. First, we give the verses explaining the eccentric model.

ग्रहोच्चसूत्रान्तरालं ग्रहवृत्तगतं भुजा ।
कोटिस्तत्केन्द्रतो दोर्ज्यामूलान्ता परिकल्प्यते ॥
केन्द्रान्तरं चान्त्यफलं स्यात् कक्ष्याग्रहवृत्तयोः ।
दोर्ज्यामूले तु कक्ष्यातो बहिरन्तर्गते क्रमात् ॥
कोटयन्त्यफलयोर्योगभेदाभ्यां स्फुटकोटिका ।
तयोर्वर्गयुतेर्मूलं कक्ष्याकेन्द्रग्रहान्तरम् ॥
कर्णः स एव विज्ञेयः प्रतिवृत्तकलामितः ।[17]

The distance of separation between the planet and the *uccasūtra* is the *dorjyā* measured with respect to the *grahavṛtta* (the circle in which the planet moves). The *koṭijyā* is equal to the distance of separation between the centre of the *grahavṛtta* and the foot of the *dorjyā* on the *uccasūtra*.

The distance of separation between the centres of the *grahavṛtta* and the *kakṣyāvṛtta* is the *antyaphala*. The *sphuṭakoṭikā* is obtained by adding or subtracting the *antyaphala* to or from the *koṭijyā* depending upon whether the foot of the *dorjyā* is outside or inside the *kakṣyāvṛtta*. The square root of the sum of the squares of the two [*dorjyā* and *sphuṭakoṭikā*] is the distance of separation between the centre of *kakṣyāvṛtta* and the planet. This has to be understood as the *karṇa* measured in terms of the *prativṛtta*.

In Fig. 2.12*b*, the circle centred around O' is called the *grahavṛtta*, or *prativṛtta* or *pratimaṇḍala* (the eccentric circle), and the one centred around O is the *kakṣyāvṛtta* (the deferent circle). OU represents the direction of the *mandocca*. These two circles, namely the *grahavṛtta* and the *kakṣyavṛtta*, have the same radius and their centres are displaced along the direction of the *mandocca* U. The dotted circle with its centre at the centre of the *kakṣyāvṛtta* is known as the *karṇamaṇḍala* or *karṇavrtta* (hypotenuse circle). The distance of separation between the centres of the *grahavṛtta* and the *kakṣyāvṛtta* is referred to as the *antyaphala*. If R is the radius of the *grahavṛtta* and $(\theta_0 - \theta_m)$ the *manda-kendra*, then the *dorjyā* and *koṭijyā* are given by

$$dorjy\bar{a} = PN = |R\sin(\theta_0 - \theta_m)| \tag{2.117}$$

and

$$ko\d{t}ijy\bar{a} = O'N = |R\cos(\theta_0 - \theta_m)|. \tag{2.118}$$

The *sphuṭakoṭikā* is defined by

$$\begin{aligned} sphu\d{t}ako\d{t}ik\bar{a} = ON &= ko\d{t}ijy\bar{a} \overset{+}{\sim} antyaphala \\ &= |R\cos(\theta_0 - \theta_m)| \overset{+}{\sim} r. \end{aligned} \tag{2.119}$$

It is stated that the '$\sim$' sign should be taken when both the edges of the *dorjyā* (points P and N in Fig. 2.12*b*) lie within the *kakṣyāvṛtta*, and '$+$' when at least one or both the edges of the *dorjyā* lie outside the *kakṣyāvṛtta*.

Actually, whether the '$+$' or the '$\sim$' sign has to be taken depends on whether P lies above or below the straight line perpendicular to OU passing through O', that is, when $(\theta_0 - \theta_m)$ is in the first/fourth quadrants or in the second/third quadrants respectively. If K represents the *karṇa* OP, then it is given by

[17] {TS 1977}, pp. 161–2.

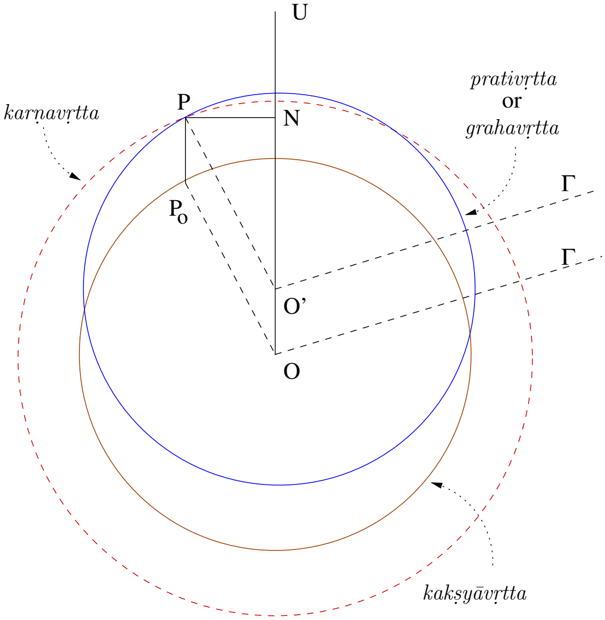

Fig. 2.12*b* Obtaining the *manda-karṇa* in the eccentric model.

$$\begin{aligned}
K &= OP \\
&= (PN^2 + ON^2)^{\frac{1}{2}} \\
&= \left[dorjy\bar{a}^2 + sphu\d{t}ako\d{t}ik\bar{a}^2\right]^{\frac{1}{2}} \\
&= \left[(R\sin(\theta_0 - \theta_m))^2 + (|R\cos(\theta_0 - \theta_m)| \mathrel{\underset{\sim}{+}} r)^2\right]^{\frac{1}{2}} \\
&= \left[(R\sin(\theta_0 - \theta_m))^2 + (R\cos(\theta_0 - \theta_m) + r)^2\right]^{\frac{1}{2}}. \qquad (2.120)
\end{aligned}$$

K can be determined using the above formula, or by using equation (2.112), which are equivalent. This is explained in the following verses of *Yukti-dīpikā*:

कक्ष्यावृत्तस्य तन्नेमिस्थोच्चनीचस्य च द्वयोः ।
केन्द्रद्वयावभेदी यो मार्गस्तस्माद् ग्रहान्तरम् ॥
दोःफलं यत्तु तन्मूलान्तरं नीचोच्चकेन्द्रतः ।
कोटीफलं तद्युतोना त्रिज्या कक्ष्याख्यवृत्तितः ॥
क्रमाद् दोःफलमूले तु बहिरन्तर्गते सति ।
सा तु दोःफलमूलस्य कक्ष्याकेन्द्रस्य चान्तरम् ॥
तत्कृतौ दोःफलकृतिं युक्त्वा कर्णः पदीकृतः ।
एवं कर्णो द्विधा साध्यः स तु मान्दो विशिष्यते ॥ [18]

[18] {TS 1977}, p. 162.

The distance of separation between the planet, and the line passing through the centre of the *kakṣyāvṛtta* and the centre of the *uccanīcavṛtta* (epicycle) which moves on the circumference of the *kakṣyāvṛtta*, is the *doḥphala*. The distance of separation between the foot of the perpendicular [of the *dorjyā*] and the centre of the *uccanīcavṛtta* is the *koṭiphala*. Depending upon whether the foot of the *doḥphala* lies outside the *kakṣyāvṛtta* or inside, the *koṭiphala* has to be added to or subtracted from the *trijyā*. This gives the distance of separation between the centre of *kakṣyāvṛtta* and the foot of the *doḥphala*. The square root of the sum of the square of this (distance of separation) and the square of the *doḥphala* is the *karṇa*. In this way the *karṇa* can be obtained in two ways and it has to be iterated in the case of the *manda-saṃskāra*.

२.१७ अविशेषकर्णानयनम्

2.17 Obtaining the iterated hypotenuse

दोःकोटिफलनिघ्नाद्ये कर्णात् त्रिज्याहृते फले ॥ ४१ ॥
ताभ्यां कर्णः पुनस्साध्यः भूयः पूर्वफलाहतात् ।
तत्तत्कर्णात् त्रिभज्याप्तफलाभ्यामविशेषयेत् ॥ ४२ ॥

doḥkoṭiphalanighnādye karṇāt trijyāhṛte phale || 41 ||
tābhyāṃ karṇaḥ punassādhyaḥ bhūyaḥ pūrvaphalāhatāt|
tattatkarṇāt tribhajyāptaphalābhyāmaviśeṣayet || 42 ||

The *doḥphala* and the *koṭiphala* [initially obtained] are multiplied by the *karṇa* [obtained from them] and divided by *trijyā*. From these resulting *phalas*, the *karṇa* has to be obtained again. Further, the previous *phalas* must be multiplied by the corresponding *karṇas* and divided by the *trijyā*, and the process has to be repeated to get the *aviśeṣa-karṇa* (the hypotenuse which does not change on iteration).

It was shown earlier (2.112) that

$$K = \left[(R + r\cos(\theta_0 - \theta_m))^2 + (r\sin(\theta_0 - \theta_m))^2\right]^{\frac{1}{2}}. \tag{2.121}$$

Here the radius of the epicycle r itself is proportional to *karṇa* K (2.113) and therefore needs to be determined along with K iteratively.

Procedure for finding the iterated hypotenuse

We explain the procedure for finding the iterated hypotenuse or *aviśeṣa-karṇa* with the help of Fig. 2.12*a*. Let R, r be the radii of the deferent circle and the epicycle respectively. $U\hat{O}P_0$ is the *manda-kendra* $(\theta_0 - \theta_m)$. The quantities $r\sin(\theta_0 - \theta_m) = PN$ and $r\cos(\theta_0 - \theta_m) = P_0N$ are referred to as the *doḥphala* and *koṭiphala* respectively. Thus, in the first approximation, r is set equal to r_0 and the *doḥphala* and *koṭiphala* are taken to be $r_0\sin(\theta_0 - \theta_m)$ and $r_0\cos(\theta_0 - \theta_m)$ respectively. Let them be denoted d_1 and k_1. The *karṇa* OP which represents the distance of the planet from the centre of the *kakṣyāvṛtta* is given by

$$K_1 = \left[(R + k_1)^2 + d_1^2\right]^{\frac{1}{2}}. \tag{2.122}$$

Here K_1 is the first approximation to the *manda-karṇa*. Then, the *doḥphala* (d_2) and *koṭiphala* (k_2) are obtained as follows:

$$d_2 = \frac{K_1 \times d_1}{R} \qquad k_2 = \frac{K_1 \times k_1}{R}. \tag{2.123}$$

The second approximation to the *manda-karṇa*, K_2, is given by

$$K_2 = \left[(R + k_2)^2 + d_2^2\right]^{\frac{1}{2}}. \tag{2.124}$$

Then, the *doḥphala* (d_3) and *koṭiphala* (k_3) are obtained as follows:

$$d_3 = \frac{K_2 \times d_1}{R} \qquad k_3 = \frac{K_2 \times k_1}{R}. \tag{2.125}$$

The third approximation to the *manda-karṇa*, K_3, is obtained by

$$K_3 = \left[(R + k_3)^2 + d_3^2\right]^{\frac{1}{2}}. \tag{2.126}$$

The above process is carried out until $K_i \approx K_{i-1}$, to the desired accuracy. When this happens, K_i is referred to as the *aviśeṣa-karṇa*. This *aviśeṣa-karṇa* is to be used in *manda-saṃskāra* to obtain the *manda-phala*.

The rationale behind the iterative process used in obtaining the *aviśeṣa-karṇa* is explained in *Yukti-dīpikā* as follows:

मान्दं नीचोच्चवृत्तं तत्कर्णवृत्तकलामितम् ।
यतस्तत्कर्णवृद्धिक्षयानुसारीदमुच्यते ॥
मन्दकर्णे स्वलिप्ताभिः प्रमिते त्रिज्यया समे ।
पठितः परिधिर्मान्दः कर्णवृत्तकलामितः ॥
ऊनाधिके ततः कर्णे प्रतिवृत्तकलामिते ।
तेन कर्णेन दोःकोटिफले ताभ्यां तु तन्नयेत् ॥
अन्योन्याश्रयता चैषां अविशेषान्निरस्यते ।
तत्तत्कर्णाद्युदोःकोटिघाताभ्यां त्रिज्ययाहृते ॥
ताभ्यां त्रिभज्यया चापि प्राग्वत् कर्णं मुहुर्नयेत्।[19]

The *manda-nīcocca-vṛtta* (*manda* epicycle) is measured in terms of *karṇavṛtta* (hypotenuse circle) because it is said to increase or decrease in accordance with the *karṇavṛtta*. The tabulated value of the circumference of the *manda* circle is in the measure of the *karṇavṛtta*, when the *manda-karṇa* is taken to be the *trijyā*. When the *karṇa* increases and decreases and this value is measured in terms of *prativṛtta*, then the *doḥ* and *koṭiphala* have to be obtained from that *karṇa*. It is from them (*doḥ* and *koṭiphala*) that (the measure of *manda-nīcocca-vṛtta*) has to be obtained. This interdependence is eliminated by doing an iteration, the *aviśeṣakarma*. Multiplying the *doḥphala* and *koṭiphala* by *karṇa* and dividing it by the *trijyā* [the new *doḥphala* and *koṭiphala* are determined]. With the *trijyā* and these, once again the *karṇa* has to be obtained as explained earlier.

Now,

$$\sqrt{d_1^2 + k_1^2} = r_0. \tag{2.127}$$

[19] {TS 1977}, pp. 162–3.

From Fig. 2.12*a* and the equivalent of (2.125) it can be seen that for any *i*,

$$\sqrt{d_i^2 + k_i^2} = \frac{K_{i-1}}{R}\sqrt{d_1^2 + k_1^2} \tag{2.128}$$

$$= \frac{K_{i-1}}{R}\, r_0. \tag{2.129}$$

After a few iterations, the successive values of the radius and the *karṇa* start converging. That is,

$$\sqrt{d_{i-1}^2 + k_{i-1}^2} \approx \sqrt{d_i^2 + k_i^2} \to r$$

and

$$K_{i-1} \approx K_i \to K.$$

Hence

$$\frac{r}{K} = \frac{r_0}{R}. \tag{2.130}$$

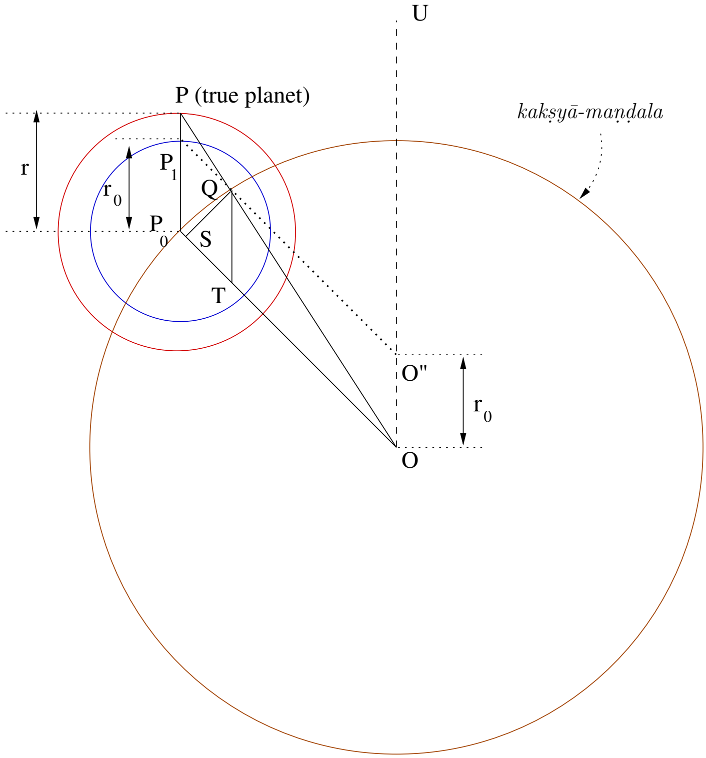

Fig. 2.12*c* Variation of the epicycle with the *karṇa* in the *manda* process and the *aviśiṣṭa-manda-karṇa*.

In Fig. 2.12*c*, P_0 is the mean planet moving in the *kakṣyāmaṇḍala* with O as the centre, and OU is the direction of the *mandocca*. Draw a circle of radius r_0 with P_0 as centre. Let P_1 be the point on this circle such that P_0P_1 is in the direction of the *mandocca* (parallel to OU). Let O'' be a point on the line OU, such that $OO'' = r_0$. Join P_1O'' and let that line meet the *kakṣyāmaṇḍala* at Q. Extend OQ and P_0P_1 so as to meet at P. The true planet is located at P. Then it can be shown that $OP = K$ and $P_0P = r$ are the actual *manda-karṇa* and the corresponding (true) radius of the epicycle as will result by the process of successive iteration.[20] Since P_1O'' is parallel to P_0O, the triangles OP_0P and $QO''O$ are similar and we have

$$\frac{r}{K} = \frac{P_0P}{OP} = \frac{O''O}{QO} = \frac{r_0}{R}. \tag{2.131}$$

The process of successive iteration to obtain K is essentially the following. In triangle OP_1P_0, with the angle $P_1\hat{P}_0O = 180^\circ - (\theta_0 - \theta_m)$, the first approximation to the *karṇa* (*sakṛt-karṇa*) $K_1 = OP_1$ and the mean epicycle radius $r_0 = P_1P_0$ are related by

$$K_1 = \sqrt{R^2 + r_0^2 + 2r_0R\cos(\theta_0 - \theta_m)}. \tag{2.132}$$

In the RHS of (2.132), we replace r_0 by the next approximation to the radius of the epicycle

$$r_1 = \frac{r_0}{R}K_1, \tag{2.133}$$

and obtain the next approximation to the *karṇa*,

$$K_2 = \sqrt{R^2 + r_1^2 + 2r_1R\cos(\theta_0 - \theta_m)}, \tag{2.134}$$

and so on. This process is iterated till K_i and K_{i+1} become indistinguishable, and that will be the *aviśiṣṭa-karṇa* (iterated hypotenuse) K,[21] which is related to the corresponding epicycle radius r as in (2.133) by

$$r = \frac{r_0}{R}K. \tag{2.135}$$

२.१८ अविशेषकर्णानयने प्रकारान्तरम्

2.18 Another method of obtaining the iterated hypotenuse

विस्तृतिदलदोःफलकृतिवियुतिपदं कोटिफलविहीनयुतम् ।
केन्द्रे मृगकर्किगते स खलु विपर्ययकृतो भवेत् कर्णः ॥ ४३ ॥

[20] {MB 1960}, pp. 111–19.

[21] The term *viśeṣa* means 'distinction'. Hence, *aviśeṣa* is 'without distinction'. Therefore the term *aviśiṣṭa-karṇa* refers to that *karṇa* obtained after doing a series of iterations such that the successive values of the *karṇa* do not differ from each other.

तेन हृता त्रिज्याकृतिः अयत्नविहितोऽविशेषकर्णः स्यात् ।
इति वा कर्णः साध्यः मान्दे सकृदेव माधवप्रोक्तः ॥ ४४ ॥

vistṛtidaladoḥphalakṛtiviyutipadaṃ koṭiphalavihīnayutam|
kendre mṛgakarkigate sa khalu viparyayakṛto bhavet karṇaḥ || 43 ||
tena hṛtā trijyākṛtiḥ ayatnavihito'viśeṣakarṇaḥ syāt|
iti vā karṇaḥ sādhyaḥ māṇde sakṛdeva mādhavaproktaḥ || 44 ||

The square of the *doḥphala* is subtracted from the square of the *trijyā* and its square root is taken. The *koṭiphala* is added to or subtracted from this depending upon whether the *kendra* (anomaly) is within 6 signs beginning from *Karki* (Cancer) or *Mṛga* (Capricorn). This gives the *viparyaya-karṇa*. The square of the *trijyā* divided by this *viparyaya-karṇa* is the *aviśeṣa-karṇa* (iterated hypotenuse) obtained without any effort [of iteration]. This is another way by which the [*aviśeṣa*]-*karṇa* in the *manda* process can be obtained as enunciated by Mādhava.

A method to determine the *manda-karṇa* without an iterative process is discussed here. This method is attributed to Mādhava of Saṅgamagrāma, the renowned mathematician and astronomer of the 14th century. A new quantity called the *viparyaya-karṇa* or *viparīta-karṇa* is introduced for this purpose. This *viparīta-karṇa* ('inverse' hypotenuse) is nothing but the radius of the *kakṣyāvṛtta* when the *manda-karṇa* is taken to be the *trijyā*, *R*.

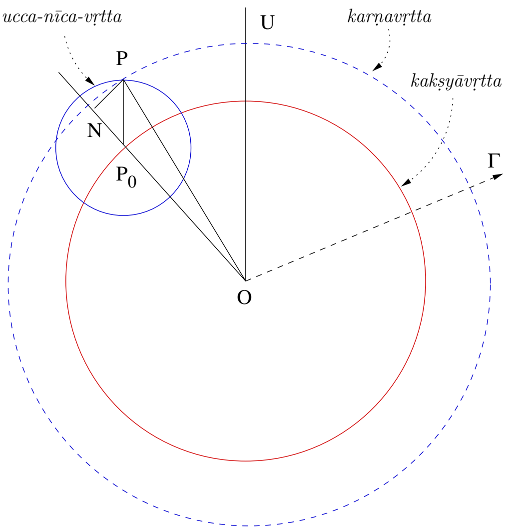

Fig. 2.13*a* Determination of the *viparīta-karṇa* when the *kendra* is in the first quadrant.

The rationale behind the formula given for *viparīta-karṇa* is outlined in the Malayalam text *Yuktibhāṣā*, and can be understood with the help of Figs. 2.13*a* and *b*. In these figures P_0 and P represent the mean and the true planet respectively. N denotes the foot of the perpendicular drawn from the true planet P to the line joining the centre of the circle and the mean planet. NP is equal to *doḥphala*. Let the radius of the *karṇavṛtta* OP be set equal to the *trijyā* R. Then the radius of the *uccanīcavṛtta* P_0P is r_0, as it is in the measurement of the *karṇavṛtta*. In this measurement, the radius of the *kakṣyāvṛtta* $OP_0 = R_v$, the *viparīta-karṇa*, and is given by

$$\begin{aligned} R_v &= ON \pm P_0N \\ &= \sqrt{R^2 - (r_0 \sin(\theta_0 - \theta_m))^2} \pm |r_0 \cos(\theta_0 - \theta_m)|. \end{aligned} \tag{2.136}$$

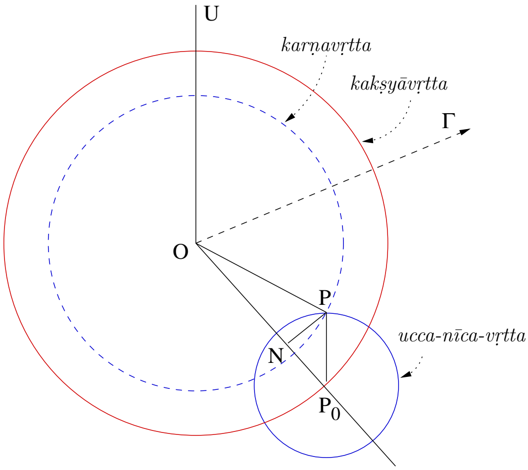

Fig. 2.13*b* Determination of the *viparīta-karṇa* when the *kendra* is in the third quadrant.

Here we should take the '$-$' sign when the *manda-kendra* is in the first and fourth quadrants $270 \leq (\theta_0 - \theta_m) < 90$ and the '$+$' sign when it is in the second and third quadrants $90 \leq (\theta_0 - \theta_m) < 270$. When the radius of the *kakṣyāvṛtta* is the *trijyā* R, the value of *manda-karṇa* is K, and when the radius of the *manda-karṇa* is R, the radius of the *kakṣyāvṛtta* is R_v. Hence

$$\frac{K}{R} = \frac{R}{R_v}$$

$$\text{or} \qquad K = \frac{R^2}{R_v}. \tag{2.137}$$

Thus the *aviśiṣṭa-manda-karṇa*, also referred to as the *aviśeṣa-karṇa*, is given by

$$\textit{aviśeṣa-karṇa} = \frac{\textit{trijyā}^{\,2}}{\textit{viparyaya-karṇa}}. \tag{2.138}$$

Since r_0 is a known quantity, for any given value of $(\theta_0 - \theta_m)$ R_v can be determined from (2.136). Once R_v is known, using (2.137) the *aviśiṣṭa-manda-karṇa*, K, can be found in one step without resorting to the tedious iterative process described in the previous section for its computation.

The formula for the *viparīta-karṇa* in (2.136) can also be understood from the geometrical construction in Fig. 2.12*c*. As the triangles OP_0P and OTQ are similar,

$$\frac{OT}{OQ} = \frac{OP_0}{OP}$$
$$\text{or} \qquad OT = \frac{R^2}{K}, \tag{2.139}$$

as $OQ = OP_0 = R$. Hence the *viparyaya-karṇa* $R_v = OT$. Also,

$$Q\hat{T}S = U\hat{O}P_0 = \theta_0 - \theta_m. \tag{2.140}$$

Hence, $QS = r_0 \sin(\theta_0 - \theta_m)$ and $ST = r_0 \cos(\theta_0 - \theta_m)$. Now

$$\begin{aligned} OT &= OS - ST \\ &= \sqrt{OQ^2 - SQ^2} - ST \\ &= \sqrt{R^2 - r_0^2 \sin^2(\theta_0 - \theta_m)} - r_0 \cos(\theta_0 - \theta_m), \end{aligned} \tag{2.141}$$

which is the same as (2.136).

२.१९ अविशेषकर्णेन अर्कस्फुटीकरणम्
2.19 Correcting the Sun using the iterated hypotenuse

त्रिज्याघ्नो दोर्गुणः कर्णभक्तः स्फुटभुजागुणः ।
तद्धनुः संस्कृतं स्वोच्चं नीचं वा युक्तितः स्फुटम् ॥ ४५ ॥

trijyāghno dorguṇaḥ karṇabhaktaḥ sphuṭabhujāguṇaḥ |
taddhanuḥ saṃskṛtaṃ svoccaṃ nīcaṃ vā yuktitaḥ sphuṭam || 45 ||

The true *dorjyā* is [equal to] the *dorjyā* multiplied by the *trijyā* and divided by the *karṇa*. The arc of this appropriately applied to the *ucca* or *nīca* gives the true position [of the planet].

This can be explained from Fig. 2.14*a*. Let $\phi = P\hat{O}U$ be the difference $(\theta - \theta_m)$ between the *manda-sphuṭa* and the *ucca*. Now

$$PN = P_0N_0,$$
$$\text{or} \qquad K\sin\phi = R\sin(\theta_0 - \theta_m). \tag{2.142a}$$

Hence

$$R\sin\phi = R\sin(\theta_0 - \theta_m)\frac{R}{K},$$
$$\text{or} \qquad \phi = (R\sin^{-1})\left[R\sin(\theta_0 - \theta_m)\frac{R}{K}\right]. \tag{2.142b}$$

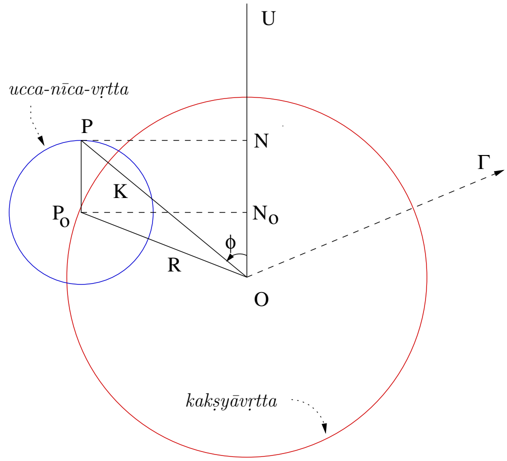

Fig. 2.14*a* The true position of the planet from the *ucca* and *nīca*.

Then the true planet ($\varGamma\hat{O}P$) is obtained as

$$\begin{aligned}\varGamma\hat{O}P &= \varGamma\hat{O}U + \phi \\ &= ucca + \phi. \end{aligned} \qquad (2.143)$$

२.२० रविस्फुटात् तन्मध्यमानयनम्
2.20 Obtaining the mean Sun from the true Sun

अर्कस्फुटेनानयनं प्रकुर्यात् स्वमध्यमस्यात्र वितुङ्गभानोः ।
भुजागुणं कोटिगुणं च कृत्वा मृगादिकेन्द्रेऽन्त्यफलाख्यकोटयोः ॥ ४६ ॥
भेदः कुलीरादिगते तु योगः तद्वर्गयुक्तात् भुजवर्गतो यत् ।
पदं विपर्यासकृतः स कर्णः त्रिज्याकृतेस्तद्विहृतस्तु कर्णः ॥ ४७ ॥
तेनाहतामुच्चविहीनभानोः जीवां भजेद् व्यासदलेन लब्धम् ।
स्वोच्चे क्षिपेच्चापितमाद्यपादे चक्रार्धतः शुद्धमपि द्वितीये ॥ ४८ ॥
चक्रार्धयुक्तं तु तृतीयपादे संशोधितं मण्डलतश्चतुर्थे ।
एवं कृतं सूक्ष्मतरं हि मध्यं पूर्वं पदं यावदिहाधिकं स्यात् ॥ ४९ ॥
अन्त्यात् फलात् कोटिगुणं चतुर्थं त्वारभ्यते यद्यधिकात्र कोटिः ।
सर्वत्र विष्कम्भदलं श्रुतौ वा व्यासार्धके स्याद्विपरीतकर्णः ॥ ५० ॥

arkasphuṭenānayanaṃ prakuryāt svamadhyamasyātra vituṅgabhānoḥ|
bhujāguṇaṃ koṭiguṇaṃ ca kṛtvā mṛgādikendre'ntyaphalākhyakoṭyoḥ|| 46 ||
bhedaḥ kulīrādigate tu yogaḥ tadvargayuktāt bhujavargato yat|
padaṃ viparyāsakṛtaḥ sa karṇaḥ trijyākṛtestadvihṛtastu karṇaḥ || 47 ||
tenāhatāmuccavihīnabhānoḥ jīvāṃ bhajed vyāsadalena labdham|
svocce kṣipeccāpitamādyapāde cakrārdhataḥ śuddhamapi dvitīye|| 48 ||
cakrārdhayuktaṃ tu tṛtīyapāde saṃśodhitaṃ maṇḍalataścaturthe|

evaṃ kṛtaṃ sūkṣmataraṃ hi madhyaṃ pūrvaṃ padaṃ yāvadihādhikaṃ syāt|| 49 ||
antyāt phalāt koṭiguṇaṃ caturthaṃ tvārabhyate yadyadhikātra koṭiḥ|
sarvatra viṣkambhadalaṃ śrutau vā vyāsārdhake syādviparītakarṇaḥ|| 50 ||

The mean position of the Sun has to be obtained from the true position [as follows]. Having subtracted the longitude of the apogee from the true Sun, the *dorjyā* and *koṭijyā* are obtained. When the *manda-kendra* lies within the six signs beginning from *Mṛga*, the difference between the *antyaphala* and the *koṭijyā* has to be taken, and when it is within the six signs beginning from *Karka*, their sum has to be taken. The square root of the sum of the square of this and the square of the *dorjyā* is the *viparīta-karṇa*. The square of the *trijyā* divided by this *viparīta-karṇa* is the *karṇa*.

This (*karṇa*) is multiplied by the *dorjyā* obtained by subtracting the longitude of the apogee from the Sun, and divided by the *trijyā*. The arc of the result has to be applied positively to the longitude of the *mandocca* when the *manda-kendra* is in the first quadrant. 180 minus the arc, 180 (*cakrārdha*) plus the arc and 360 minus the arc have to be applied to the *mandocca* when the *manda-kendra* lies in the second, third and fourth quadrants respectively. The mean longitude obtained thus is accurate. In the first quadrant the *koṭijyā* is greater than the *antyaphala*. [Similarly] the fourth quadrant is said to commence when the *koṭiphala* becomes greater than the *antyaphala*. Always the *karṇa* bears the same relation to the *trijyā* as the *trijyā* to the *viparīta-karṇa* (inverse hypotenuse).

Normally the texts present the procedure for determining the true position of a planet from its mean position. The above set of verses present a procedure for solving the inverse problem, namely finding the mean Sun from its true position. We explain this procedure with the help of Fig. 2.14*b*. Here, the longitudes of the mean Sun, the true Sun and the *ucca* (apogee) are given by

$$\begin{aligned} \theta_0 &= \Gamma\hat{O}P_0 = P\hat{O'}P \\ \theta &= \Gamma\hat{O}P \\ \text{and} \qquad \theta_m &= \Gamma\hat{O}U = \Gamma\hat{O'}U, \end{aligned} \tag{2.144}$$

respectively. Further,

$$\begin{aligned} \theta - \theta_m &= N\hat{O}P \\ \theta_0 - \theta_m &= N\hat{O'}P = N\hat{O}P_0. \end{aligned} \tag{2.145}$$

Also, the *aviśiṣṭa-manda-karṇa* (iterated *manda* hypotenuse) $K = OP$ and the *vyāsārdha* $R = OP_0 = O'P$. The true epicycle radius $r = OO'$.

The word *antyaphala* used in the above verse has a special significance whose relation with the *manda-karṇa* may precisely be expressed as follows:

$$antyaphala = r_0 = \frac{r_0}{r}.r = \frac{R}{K}.r = \frac{R}{K}.OO'. \tag{2.146}$$

Now,

$$\begin{aligned} dorjy\bar{a} = R\sin N\hat{O}P &= \frac{R}{K}.K\sin N\hat{O}P \\ &= \frac{R}{K}.K\sin(\theta-\theta_m) = \frac{R}{K}.PN \end{aligned}$$

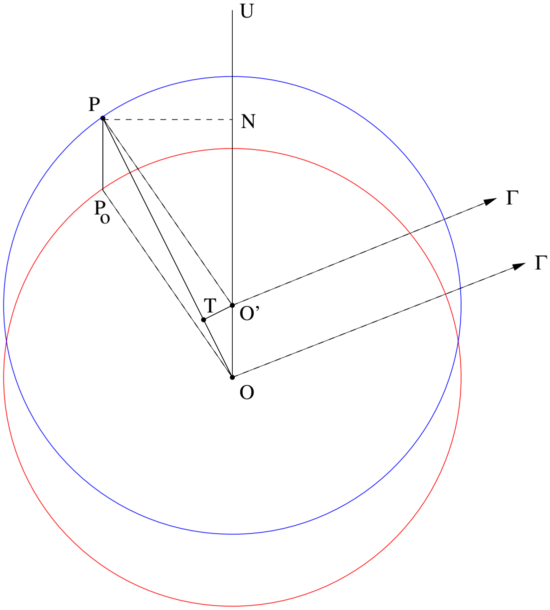

Fig. 2.14*b* Obtaining the *madhyama* (mean position) from the *sphuṭa* (true position).

$$
\begin{aligned}
kotijy\bar{a} = R\cos N\hat{O}P &= \frac{R}{K}.K\cos N\hat{O}P \\
&= \frac{R}{K}.K\cos(\theta - \theta_m) = \frac{R}{K}.ON.
\end{aligned} \tag{2.147}
$$

Hence the difference between the *koṭijyā* and the *antyaphala* is given by

$$
\begin{aligned}
kotijy\bar{a} - antyaphala &= R\cos(\theta - \theta_m) - r_0 \\
&= \frac{R}{K}(ON - OO') \\
&= \frac{R}{K}.O'N.
\end{aligned} \tag{2.148}
$$

Therefore,

$$
\begin{aligned}
\sqrt{(kotijy\bar{a} - antyaphala)^2 + (dorjy\bar{a})^2} &= \frac{R}{K}\sqrt{O'N^2 + PN^2} \\
&= \frac{R}{K}.O'P \\
&= \frac{R^2}{K}.
\end{aligned} \tag{2.149}
$$

The expression obtained above is the same as the *viparīta-karṇa* R_v appearing in (2.136). Now, using (2.147) and (2.148), this may be expressed as

$$\mathrm{R}_v = \sqrt{(R\cos(\theta-\theta_m)-r_0)^2 + R^2\sin^2(\theta-\theta_m)}. \tag{2.150}$$

Since the positions of the *ucca* and the true planet are known, R_v can be determined. Also, the *manda-karṇa* $K = \frac{R^2}{R_v}$ can be determined from $\theta - \theta_m$. Now

$$\begin{aligned} PN &= K\sin(\theta-\theta_m) \\ &= O'P\sin(N\hat{O}'P) \\ &= R\sin(\theta_0-\theta_m). \end{aligned} \tag{2.151}$$

Hence

$$\begin{aligned} \textit{madhyama} - \textit{ucca} &= \theta_0 - \theta_m \\ &= R\sin^{-1}\left[R\sin(\theta-\theta_m)\frac{K}{R}\right]. \end{aligned} \tag{2.152}$$

From this *madhyama* − *ucca* is obtained. When this is added to the *ucca*, the *madhyama* is obtained. When *sphuṭa* − *ucca* is positive, $O'N$ = *koṭijyā* − *antyaphala*. In the above, it is $R\sin$ (*madhyama* − *ucca*) which is found first in terms of $R\sin$(*sphuṭa* − *ucca*). The quadrant in which (*madhyama* − *ucca*) lies can be determined without any ambiguity from the geometry.

When it is in the second or third quadrants, $R\cos$(*sphuṭa* − *ucca*) is negative and $O'N$ = *koṭijyā* + *antyaphala*. Of course, in all cases, the formula for R_v given above is valid. Now, when the true planet is to be found from the mean planet, it is not necessary to calculate the *manda-karṇa* K. However in the reverse case, when the mean planet is to be found from the true planet, it becomes necessary to first calculate K.

An elaborate explanation for the above verses is to be found in in *Yukti-dīpikā*.

कर्णवृत्ते स्फुटोच्चान्तरालज्या स्वकलामिता ।
तदुच्चसूत्रसंपातात् कोटिस्तत्केन्द्रगामिनी ॥
तदन्त्यफलयोर्योगो विश्लेषो वा यथोचितम् ।
प्रतिमण्डलकेन्द्रस्य दोर्ज्यामूलस्य चान्तरम् ॥
तद्दोर्वर्गयुतेर्मूलं प्रतिमण्डलकेन्द्रतः ।
ग्रहावधिव्यासदलं कर्णवृत्तकलामितम् ॥
व्यासार्धं प्रतिवृत्तस्य त्रिज्यैव स्वकलामिता ।
तदेव व्यस्तकर्णः स्यात् कर्णवृत्तकलामितम् ॥
त्रिज्यातुल्ये व्यस्तकर्णे कर्णो न्यूनाधिकस्ततः ।
ततस्त्रैराशिकेनात्र मन्दकर्णः स्फुटो भवेत् ॥
उच्चोनस्फुटतो दोर्ज्यां मन्दकर्णहतां हरेत् ।
त्रिज्यया तद्गुणां दोर्ज्यां व्यस्तकर्णेन वा हरेत् ॥
लब्धचापं धनर्णं स्यात् स्वोच्चे मध्यमसिद्धये ।

यादृशो नियमस्तत्र व्यस्तकर्णत्रिजीवयोः ॥
तादृशो नियमो वेद्यः त्रिजीवामन्दकर्णयोः ।
कर्णस्त्रैराशिकेनातः व्यस्तकर्णाद्विधीयते ॥
मध्यमात् स्फुटसंसिद्धिः दोःफलाद्यदि केवलात् ।
मध्यसिद्धिः स्फुटात् तस्मात् कर्णघ्नात् त्रिज्यया हृतात् ॥
अथ स्फुटोच्चान्तरदोर्गुणं श्रुतिहतं हरेत् ।
त्रिज्यया लब्धचापेन कृते स्वोच्चे स्वमध्यमः ॥
तदेव चापितं स्वोच्चे चक्रार्धं तेन वर्जितम् ।
चक्रार्धयुक्तं चक्राच्च त्यक्तं पदवशात् क्षिपेत् ॥ [22]

In the *karṇavṛtta* the *jyā* of the difference between the longitude of the true planet and its *mandocca* corresponds to the *dorjyā* in its own measure. The distance of separation between the point of intersection (N in the Fig. 2.14b) of the *jyā* with the *uccasūtra* (the apsis line) and the centre of the *karṇavṛtta* (O) corresponds to the *koṭijyā* (ON). The sum or difference of the *antyaphala* (OO') with this *koṭijyā*, as the case may be, gives the distance of separation between the centre of the *pratimaṇḍala* and the foot of the *dorjyā* (N). The square root of the sum of the squares of this ($O'N$) and the *dorjyā* (PN) gives the distance between the centre of *pratimaṇḍala* and the planet. This is the radius of the *pratimaṇḍala* in the measure of the *karṇavṛtta*. The radius of the *prativṛtta* with respect to its own measure is the *trijyā*. This (*trijyā*) will be the *vyasta-karṇa* (inverse-hypotenuse) in the measure of the *karṇavṛtta*. When the *vyasta-karṇa* is set equal to the *trijyā*, then the actual *karṇa* will be smaller or larger than that. Thus by the rule of three the true *manda-karṇa* is obtained.

The *dorjyā* obtained by subtracting the *mandocca* from the true Sun is multiplied by the *manda-karṇa* and divided by the *trijyā*. Or the *trijyā* multiplied by the *dorjyā* is divided by *vyasta-karṇa*. The arc of this is applied positively or negatively to the *mandocca* to get the mean Sun. It is to be understood that whatever is the relation between the *vyasta-karṇa* and the *trijyā*, the same relation is valid between the *trijyā* and the *manda-karṇa*. This is the reason why the *manda-karṇa* is obtained from the *vyasta-karṇa* by the rule of three.

As the true position of the planet is obtained from the mean position just by finding the *doḥphala*, the mean position is obtained from the true position by multiplying [the *dorjyā*] by the *manda-karṇa* and dividing by the *trijyā*. Then the *dorjyā* obtained by subtracting the *mandocca* from the true Sun is multiplied by the *manda-karṇa* and divided by the *trijyā*. The arc applied to the *mandocca* of the Sun will give the position of the mean Sun. Depending upon the quadrant, the same arc has to be applied to the *mandocca* after subtracting it from 180°, or adding 180° to it or subtracting it from 360°.

The procedure stated here is a slight variant of the one described earlier. Here, PN, ON and OO' are the *dorjyā*, the *koṭijyā* and the *antyaphala* respectively in the measure of the *karṇavṛtta* and are equal to $R\sin(\theta-\theta_m)$, $R\cos(\theta-\theta_m)$ and r_0 in the same measure. In this measure, the radius of the *pratimaṇḍala*, $O'P$, is the *vyasta-karṇa* or *viparita-karṇa*, R_v, given in (2.136). Then the *manda-karṇa*, K, in the measure of the *pratimaṇḍala* (when the radius is R, as usual) is determined from

$$\frac{K}{R}=\frac{R}{R_v}, \tag{2.153}$$

and $madhyama - ucca$ is obtained as earlier.

[22] {TS 1977}, pp. 165–6.

२.२१ स्फुटान्मध्यमानयने प्रकारान्तरम्

2.21 Another method for getting the mean planet from the true planet

अर्केन्द्वोः स्फुटतो मृदूच्चरहितात् दोःकोटिजाते फले
नीत्वा कर्किमृगादितो विनिमयेनानीय कर्णं सकृत् ।
त्रिज्या दोःफलघाततः श्रुतिहृतं चापीकृतं तत् स्फुटे
केन्द्रे मेषतुलादिगे धनमृणं तन्मध्यसंसिद्धये ॥ ५१ ॥

arkendvoḥ sphuṭato mṛdūccarahitāt doḥkoṭijāte phale
nītvā karkimṛgādito vinimayenānīya karṇaṃ sakṛt |
trijyā doḥphalaghātataḥ śrutihṛtaṃ cāpīkṛtaṃ tat sphuṭe
kendre meṣatulādige dhanamṛṇaṃ tanmadhyasaṃsiddhaye|| 51 ||

Subtracting the longitude of their own *mandoccas* from the true positions of the Sun and the Moon, obtain their *doḥphala* and *koṭiphala*. Find the *sakṛt karṇa* (one-step hypotenuse) once by interchanging the sign [in the cosine term] depending upon whether the *kendra* is within the six signs beginning with *Karki* or *Mṛga*. Multiplying the *doḥphala* and *trijyā*, and dividing this product by the *karṇa* [here referred to as *śruti*], the arc of the result is applied to the true planet to obtain the mean planet. This arc has to be applied positively and negatively depending upon whether the *kendra* lies within the six signs beginning with *Meṣa* or *Tulā* respectively.

Now,

$$\begin{aligned} \textit{bāhuphala} &= r_0 \sin(\theta - \theta_m) \\ \textit{koṭiphala} &= r_0 \cos(\theta - \theta_m). \end{aligned} \tag{2.154}$$

Taking the one-step *karṇa* (*sakṛtkarṇa*) with the opposite sign in the *koṭiphala*, we have

$$\textit{karṇa} = [(R - r_0 \cos(\theta - \theta_m))^2 + (r_0 \sin(\theta - \theta_m))^2]^{\frac{1}{2}}. \tag{2.155}$$

This is the same as the *viparīta-karṇa* R_v given by (2.150). In Fig. 2.14*b*, draw $O'T$ perpendicular to OP. Then in triangle $O'PT$,

$$\begin{aligned} O'T &= O'P \sin(O'\hat{P}T) \\ &= O'P \sin(P\hat{O}P_0) \\ &= R \sin(\theta_0 - \theta). \end{aligned} \tag{2.156}$$

$$\text{Also} \qquad O'T = r \sin(\theta - \theta_m). \tag{2.157}$$

Equating the above two expressions for $O'T$,

$$\begin{aligned} R \sin(\theta_0 - \theta) &= r \sin(\theta - \theta_0) \\ \text{or} \qquad R \sin(\theta_0 - \theta) &= r_0 \sin(\theta - \theta_0) \frac{R}{R_v}, \end{aligned} \tag{2.158}$$

where we have used (2.135) and (2.153). Hence,

$$\theta_0 - \theta = (R\sin)^{-1}\left[r_0 \sin(\theta - \theta_0)\frac{R}{R_v}\right]. \tag{2.159}$$

Thus the mean planet θ_0 can be obtained by adding the above difference to the true planet θ. $\theta_0 - \theta$ is positive when the *kendra* (anomaly) $\theta - \theta_m$ is within the six signs beginning with *Meṣa*, i.e. $0° \leq \theta - \theta_m \leq 180°$, and negative when the *kendra* is within the six signs beginning with *Tulā*, i.e. $180° \leq \theta - \theta_m \leq 360°$.

२.२२ मन्दकर्णानयने प्रकारान्तरम्

2.22 Another method for getting the *manda*-hypotenuse

मध्यतः स्फुटतश्चोच्चं उज्झित्वा तद्भुजे उभे ।
गृहीत्वाद्या तयोस्त्रिज्या हतान्याप्ता श्रुतिस्फुटा ॥ ५२ ॥

madhyataḥ sphuṭataścoccaṃ ujjhitvā tadbhuje ubhe |
gṛhītvādyā tayostrijyā hatānyāptā śrutisphuṭā || 52 ||

Subtracting the *mandocca* from the mean and the true positions separately, obtain the two *dorjyās*. Of these, the former multiplied by the *trijyā* and divided by the latter gives the exact value of *śrutisphutā* (*aviśiṣṭa-manda-karṇa*).

In Fig. 2.14*b*, $R\sin(\theta_0 - \theta_m)$ and $R\sin(\theta - \theta_m)$ are the *dorjyās* corresponding to the *manda-kendras* of the mean and true planet respectively. It is noted from the figure that

$$\begin{aligned} PN &= K\sin(\theta - \theta_m) \\ &= R\sin(\theta_0 - \theta_m). \end{aligned} \tag{2.160}$$

Hence,

$$\begin{aligned} K &= R \times \frac{R\sin(\theta_0 - \theta_m)}{R\sin(\theta - \theta_m)} \\ \text{or} \qquad \textit{śruti} &= \frac{\textit{trijyā} \times \textit{ādyā}}{\textit{anyā}}. \end{aligned} \tag{2.161}$$

where *ādyā* and *anyā* refer to $R\sin(\theta_0 - \theta_m)$ and $R\sin(\theta - \theta_m)$ respectively, and the *aviśiṣṭa-manda-karṇa* is termed the *śrutisphuṭā* here.

२.२३ ग्रहतात्कालिकगतिः

2.23 Instantaneous velocity of a planet

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत् ।
तत्र कोटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते ॥ ५३ ॥

तद्विशोध्य मृगादिके गतेः क्षिप्यतामिह तु कर्कटादिके ।
तद्भवेत्स्फुटतरा गतिर्विधोरस्य तत्समयजा रवेरपि ॥ ५४ ॥

candrabāhuphalavargaśodhitatrijyakākṛtipadena saṃharet|
tatra koṭiphalaliptikāhatāṃ kendrabhuktirihayacca labhyate|| 53 ||
tadviśodhya mṛgādike gateḥ kṣipyatāmiha tu karkaṭādike|
tadbhavetsphuṭatarā gatirvidhorasya tatsamayajā raverapi|| 54 ||

Let the product of the *koṭiphala* (in minutes) and the daily motion of the *kendra* be divided by the square root of the square of the *bāhuphala* of the Moon subtracted from the square root of the *trijyā*. The quantity thus obtained has to be subtracted from the daily motion [of the Moon] if [the *kendra* lies within the six signs] beginning from *Makara* and is to be added to the daily motion if [the *kendra* lies within the six signs] beginning from *Karkaṭaka*. This will be a far more accurate (*sphuṭatarā*) value of the instantaneous velocity (*tatsamayajā gati*) of the Moon. For the Sun also [the instantaneous velocity can be obtained similarly].

The *bāhuphala* (or *doḥphala*) and *koṭiphala* are given by

$$\begin{aligned} bāhuphala &= r_0 \sin(\theta_0 - \theta_m) \\ \text{and} \qquad koṭiphala &= r_0 \cos(\theta_0 - \theta_m), \end{aligned} \tag{2.162}$$

where $\theta_0 - \theta_m$ is the *manda-kendra*; θ_0 and θ_m represent the longitude of the Moon and its *mandocca* respectively (see Fig. 2.12*a*). The term *kendrabhukti* refers to the daily motion of the *kendra* given by

$$kendrabhukti = \frac{\Delta(\theta_0 - \theta_m)}{\Delta t}, \tag{2.163}$$

where Δt refers to the time interval of one day and $\Delta(\theta_0 - \theta_m)$ represents the difference in the daily motion of the Moon and its *mandocca*. As the mean longitude and *mandocca* increase uniformly with time,

$$\frac{d}{dt}(\theta_0 - \theta_m) = \frac{\Delta}{\Delta t}(\theta_0 - \theta_m), \tag{2.164}$$

is a constant. It is stated here that a correction term has to be added to the above *kendrabhukti* to obtain a more accurate value of the rate of motion of the *kendra*. The correction factor is stated to be

$$\frac{koṭiphala \times kendrabhukti}{\sqrt{(trijyā^2 - bāhuphala^2)}} = -\frac{r_0 \cos(\theta_0 - \theta_m)\frac{\Delta(\theta_0 - \theta_m)}{\Delta t}}{\sqrt{R^2 - r_0^2 \sin^2(\theta_0 - \theta_m)}}. \tag{2.165}$$

Further, it is mentioned that the correction term is to be subtracted from the *kendrabhukti* when $\theta_0 - \theta_m$ is in the first and fourth quadrants (*Mṛgādi*) and it is to be added when it is in the second and third quadrants (*Karkādi*). This accounts for the negative sign in the RHS of the above equation (2.165).

Now the *manda-kendra* of the Moon's true longitude is given by

$$\theta - \theta_m = (\theta_0 - \Delta\theta) - \theta_m,$$

where the *manda* correction $\Delta\theta$ is given by

$$\Delta\theta = \sin^{-1}\left(\frac{r_0}{R}\sin(\theta_0 - \theta_m)\right), \tag{2.166}$$

as explained earlier. Hence,

$$\theta = \theta_0 - \sin^{-1}\left(\frac{r_0}{R}\sin(\theta_0 - \theta_m)\right). \tag{2.167}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\theta &= \frac{d\theta_0}{dt} - \frac{d}{dt}\sin^{-1}\left(\frac{r_0}{R}\sin(\theta_0 - \theta_m)\right) \\ &= \frac{d\theta_0}{dt} - \frac{r_0\cos(\theta_0 - \theta_m)\frac{d(\theta_0 - \theta_m)}{dt}}{\sqrt{R^2 - r_0^2\sin^2(\theta_0 - \theta_m)}}. \end{aligned} \tag{2.168a}$$

It may be mentioned here that in the case of all the planets, except the Moon, the rate of change of the *mandocca* is extremely small and can be neglected. That is, $\frac{d\theta_m}{dt} \approx 0$. Then the above equation reduces to

$$\frac{d}{dt}\theta = \frac{d\theta_0}{dt}\left(1 - \frac{r_0\cos(\theta_0 - \theta_m)}{\sqrt{R^2 - r_0^2\sin^2(\theta_0 - \theta_m)}}\right). \tag{2.168b}$$

Note:

1. It is remarkable that the author in this verse gives the correct form for the derivative of the inverse sine function. In his *Jyotirmīmāṃsa*, Nīlakaṇṭha mentions that this verse is due to his teacher *Dāmodara*.
2. The differentials of the sine and cosine functions were used in Indian astronomy at least from the time of Mañjulācārya in his *Laghu-mānasā*. Bhāskara II clearly makes use of them in his *Siddhāntaśiromaṇi*.
3. The significance of this verse lies in the fact that it is for the first time that the derivative of the arcsine function is being considered here in the context of discussing the *tātkālika-gati* or instantaneous rate of motion of the planet.

२.२४ नक्षत्रतिथ्यानयनम्

2.24 Finding *nakṣatra* and *tithi*

लिप्तीकृतो निशानाथः शतैर्भाज्योष्टभिः फलम् ।
अश्विन्यादीनि भानि स्युः षष्ट्या हत्वा गतागते ॥ ५५ ॥
गतगन्तव्यनाड्यः स्युः स्फुटभुक्त्योदयावधेः ।

अर्कहीनो निशानाथः लिप्तीकृत्य विभज्यते ॥ ५६ ॥
शून्याश्विपर्वतैर्लब्धाः तिथयो या गताः क्रमात् ।
भुक्त्यन्तरेण नाड्यः स्युः षष्ट्या हत्वा गतागते ॥ ५७ ॥
तिथ्यर्धहारलब्धानि करणानि बबादितः ।
विरूपाणि सिते पक्षे सरूपाण्यसिते विदुः ॥ ५८ ॥
विष्कम्भाद्या रवीन्द्वैक्यात् योगाश्चाष्टशतीहृताः ।
भुक्तियुक्त्या गतैष्याभ्यां षष्टिघ्नाभ्यां च नाडिकाः ॥ ५९ ॥

liptīkṛto niśānāthaḥ śatairbhājyoṣṭabhiḥ phalam |
aśvinyādīni bhāni syuḥ ṣaṣṭyā hatvā gatāgate || 55 ||
gatagantavyanāḍyaḥ syuḥ sphuṭabhuktyodayāvadheḥ |
arkahīno niśānāthaḥ liptīkṛtya vibhajyate || 56 ||
śūnyāśviparvatairlabdhāḥ tithayo yā gatāḥ kramāt |
bhuktyantareṇa nāḍyaḥ syuḥ ṣaṣṭyā hatvā gatāgate || 57 ||
tithyardhahāralabdhāni karaṇāni babāditaḥ |
virūpāṇi site pakṣe sarūpāṇyasite viduḥ || 58 ||
viṣkambhādyā ravīndvaikyāt yogāścāṣṭaśatīhṛtāḥ |
bhuktiyuktyā gataiṣyābhyāṃ ṣaṣṭighnābhyāṃ ca nāḍikāḥ || 59 ||

The longitude of the lord of the night (the Moon) in minutes is divided by 800. The quotient gives the number of *nakṣatras* that have elapsed beginning from the *Aśvini nakṣatra*. The remainder [which corresponds to the minutes covered by the Moon in the present *nakṣatra*] and the one which has to be covered multiplied by 60 and divided by the daily motion of the Moon [in minutes] at sunrise gives the *nāḍikās* that have elapsed and are yet to elapse in the present *nakṣatra*. The longitude of the Sun subtracted from that of the Moon, in minutes, is divided by $720'$. The quotient gives the number of *tithis* elapsed. The remainder and the quantity obtained by subtracting the remainder from $720'$, multiplied by 60 and divided by the difference in the daily motion of the Sun and the Moon, gives the number of *ghaṭikās* that have elapsed and are yet to elapse in the present *tithi*.

The same (difference in longitude between the Sun and the Moon) divided by half the divisor used in the *tithi* calculation gives the number of *karaṇas* elapsed, starting with *bava*. In the bright fortnight the *karaṇas* are without form and in the dark fortnight with form. The sum of the longitudes of the Sun and the Moon [in minutes] divided by 800 gives the *yogas*, starting with the *viṣkambha*. The remainder and the quantity obtained by subtracting the remainder from 800, multiplied by 60 and divided by the sum of the daily motion of the Sun and the Moon, gives the number of *ghaṭikās* that have elapsed and are yet to elapse in the present *yoga*.

The ecliptic is divided in to 27 equal parts called *nakṣatras* beginning with *Aśvinī* and ending with *Revatī*. Hence each *nakṣatra* corresponds to $\frac{21600}{27} =$ 800 minutes, along the ecliptic. The *nakṣatra* at any instant refers to the particular portion of the ecliptic in which the Moon is situated. Clearly, when the longitude of the Moon in minutes is divided by 800 the quotient gives the number of *nakṣatras* which have elapsed and the remainder corresponds to the minutes covered by the Moon in the present *nakṣatra*. When this is divided by the daily motion of the Moon in minutes at that time (taken to be the value at sunrise) and multiplied by 60, the result gives the *ghaṭikas* that have elapsed in the present *nakṣatra*. Similarly the *ghaṭikās* yet to elapse in the present *nakṣatra* can be calculated.

A *tithi* is the (variable) unit of time during which the difference between the longitudes of the Moon and the Sun increases by 12° or $720'$. Hence there are 30 *tithis* during a lunar month. Hence, when the difference in longitudes of Moon and the Sun in minutes is divided by 720′, the quotient gives the number of *tithis* elapsed in that month. The number of *ghaṭikās* (*nāḍikās*) which have elapsed and are yet to elapse in the present *tithi* are calculated in the manner indicated.

A *karaṇa* is half a *tithi* by definition and there are 60 *karaṇas* in a lunar month. The number of *karaṇas* that have elapsed can be calculated in the same manner as the number of *tithis*, except that the divisor is 360′ instead of 720′.

There are two types of *karaṇas*, namely *cala* (movable) and *sthira* (fixed). In this context these terms are used to mean repeating and non-repeating *karaṇas*. Of the 11 *karaṇas*, 7 are repeating and 4 are non-repeating. The 7 *cala-karaṇas* (moving *karaṇas*) have 8 cycles, thus forming 56 *karaṇas*. The 4 *sthira-karaṇas* (fixed *karaṇas*) occur just once each in a lunar month. The moving and fixed *karaṇas* together make up 60 *karaṇas* in a lunar month.

The names of the *karaṇas* and the pattern in which the *cala* and *sthirakaraṇas* occur are given in the following verses, quoted in *Laghu-vivṛti*:

बबबालवकौलवतैतिलगजवणिजाख्यविष्टिनामानि ।
सितपक्षस्यापरार्धात्[23] परिवर्तन्तेऽष्ट कृत्वोऽतः ॥
कृष्णचतुर्दश्यन्ते शकुनिः पर्वणि चतुष्पदः ।
प्रथमे तिथ्यर्धेऽन्त्ये नागः किंस्तुघ्नः प्रतिपदाद्यर्धे ॥

The *karaṇas* named *baba, bālava, kaulava, taitila, gaja, vaṇija* and *viṣṭi* repeat themselves eight times from the later half of the first *tithi*, *prathama*, of the bright fortnight. *Śakuni* occurs in the later half of the *caturdaśī* of the dark fortnight, *catuṣpada* and *nāga* in the first and second halves of [the following] *amavāsya* and *kiṃstughna* in the first half of the *prathamā* of the bright fortnight.

Śaṅkara Vāriyar also quotes the following verses which give the different names of both moving and fixed *karaṇas*. The moving *karaṇas* are: *siṃha, vyāghra, varāha, khara, ibha, paśu* and *viṣṭi*. The fixed *karaṇas* are: *pakṣī, catuṣpāt*, *nāga* and *kiṃstughna*.

तथा शुक्लप्रतिपदाद्यन्त्यार्धात् करणानि मुहुर्मुहुः ।
सिंहो व्याघ्रो वराहश्च खरेभपशुविष्टयः ॥
पक्षी चतुष्पान्नागश्च किंस्तुघ्नश्चान्ततः स्थिराः ।

The *yogas* involve the sum of the longitudes of the Sun and the Moon. There are 27 *yogās* in a 360° (21600’) cycle, each *yoga* corresponding to 800′. The number of *yogās* that have elapsed and the minutes or *ghatikās* that have elapsed and are yet to elapse in the present *yoga* are calculated in the same manner as in the case of the *tithis*, except that the sum of the longitudes of the Sun and Moon and the sum of their daily motion are involved here. In *Laghu-vivṛti*, the names of the *yogas* are listed in the following verses:

चन्द्रार्कस्फुटसंयोगे दृष्टा दस्रादिकारकाः ।
विष्कम्भः प्रीतिरायुष्मान् सौभाग्यः शोभनस्तथा ॥

[23] The incorrect reading सितपक्षत्यपरार्धात् in the printed edition ({TS 1958}, p. 40) has been modified as above.

अतिगण्डः सुकर्मा च धृतिः शूलं तथैव च ।
गण्डो वृद्धिर्ध्रुवश्चैव व्याघातो हर्षणस्तथा ॥
वज्रस्सिद्धिर्व्यतीपातो वरीयान् परिघः शिवः ।
सिद्धः साध्यः शुभः शुभ्रः ब्राह्मो माहेन्द्रवैधृतौ ॥

When the longitude of the Sun and the Moon are added the *dasrādikārakās* are seen. They are: *viṣkambha, prīti, ayuṣmān, saubhāgya, śobhana, atigaṇḍa, sukarmā, dhṛti, śūla, gaṇḍa, vṛddhi, dhruva, vyāghāta, harṣaṇa, vajra, siddhi, vyatīpāta, varīyān, parigha, śiva, siddha, sādhya, śubha, śubhra, brāhma, māhendra and vaidhṛti.*

२.२५ ग्रहसंस्कारप्रकारः

2.25 The scheme of correction for the planets

मान्दं शैघ्रं पुनर्मान्दं शैघ्रं चत्वार्यनुक्रमात् ।
कुजगुर्वर्कजानां हि कर्माण्युक्तानि सूरिभिः ॥ ६० ॥

māndaṃ śaighraṃ punarmāndaṃ śaighraṃ catvāryanukramāt|
kujagurvarkajānāṃ hi karmāṇyuktāni sūribhiḥ || 60 ||

The earlier *ācaryās* have stated that *manda, śīghra*, and again *manda* and *śīghra* are the four corrections which have to be applied in sequence to the planets Mars, Jupiter and Saturn [to obtain the true longitudes of the planets from their mean longitudes].

Though there are essentially only two corrections, namely *manda* and *śīghra* for the actual planets, that is Mercury, Venus, Mars, Jupiter and Saturn, the actual computation of their longitude involves a four-step procedure in most Indian texts. Nīlakaṇṭha, as we shall see below, prescribes this four-step process only in the case of the exterior planets, Mars, Jupiter and Saturn. The actual procedure prescribed in *Tantrasaṅgraha* is described in the next few verses.

२.२६ कुजगुरुमन्दस्फुटीकरणम्

2.26 The correction for Mars, Jupiter and Saturn

दोःकोटिज्याष्टमांशौ स्वखाब्ध्यंशोनौ शनेः फले ।
दोर्ज्या त्रिज्याप्तसप्तैक्यं गुणो मान्दे कुजेज्ययोः ॥ ६१ ॥
नवाघ्नयो द्व्यशीतिश्च हारौ दोःकोटिजीवयोः ।
पृथक्स्थे मध्यमे कार्यं दोःफलस्य धनुर्दलम् ॥ ६२ ॥
रविमध्यं विशोध्यास्मात् पृथक्स्थात् बाहुकोटिके ।
आनीय बाहुजीवायाः त्रिज्याप्तं गुरुमन्दयोः ॥ ६३ ॥
षोडशभ्यो नवभ्यश्च कुजस्यापि स्वदोर्गुणात् ।
त्रिज्याप्तं द्विगुणं शोध्यं त्रीषुभ्यः शिष्यते गुणः ॥ ६४ ॥
अशीतिरेव तेषां हि हारस्ताभ्यां फले उभे ।
आनीय पूर्ववत् कर्णं सकृत्कृत्वाथ दोःफलम् ॥ ६५ ॥

त्रिज्याघ्नं कर्णभक्तं यत् तद्धनुर्दलमेव च ।
मध्यमे कृतमान्दे तु संस्कृत्यातो विशोधयेत् ॥ ६६ ॥
मन्दोच्चं तत्फलं कृत्स्नं कुर्यात् केवलमध्यमे ।
तस्मात् पृथक्कृताच्छैघ्रं प्राग्वदानीय चापितम् ॥ ६७ ॥
कृतमान्दे तु कर्तव्यं सकलं स्यात् स्फुटः स च ।

doḥkoṭijyāṣṭamāṃśau khakhābdhyaṃśonau śaneḥ phale |
dorjyā trijyāptasaptaikyaṃ guṇo mānde kujeḍyayoḥ || 61 ||
navāgnayo dvyaśītiśca hārau doḥkoṭijīvayoḥ |
pṛthaksthe madhyame kāryaṃ doḥphalasya dhanurdalam || 62 ||
ravimadhyaṃ viśodhyāsmāt pṛthaksthāt bāhukoṭike |
ānīya bāhujīvāyāḥ trijyāptaṃ gurumandayoḥ || 63 ||
ṣoḍaśabhyo navabhyaśca kujasyāpi svadorguṇāt |
trijyāptaṃ dviguṇaṃ śodhyaṃ trīṣubhyaḥ śiṣyate guṇaḥ || 64 ||
aśītireva teṣāṃ hi hārastābhyāṃ phale ubhe |
ānīya pūrvavat karṇaṃ sakṛtkṛtvātha doḥphalam || 65 ||
trijyāghnaṃ karṇabhaktaṃ yat taddhanurdalameva ca |
madhyame kṛtamānde tu saṃskṛtyāto viśodhayet || 66 ||
mandoccaṃ tatphalaṃ kṛtsnaṃ kuryāt kevalamadhyame |
tasmāt pṛthakkṛtācchaighraṃ prāgvadānīya cāpitam || 67 ||
kṛtamānde tu kartavyaṃ sakalaṃ syāt sphuṭaḥ sa ca |

One-eighth of the *dorjyā* and *koṭijyā* (sine and cosine of the *manda-kendra*), diminished by one-fortieth of the same, form the *doḥphala* and *koṭiphala* in the case of Saturn. The *dorjyā* divided by the *trijyā* and added to 7, forms the *guṇa* (multiplier) for Mars and Jupiter. 39 and 82 are the *hāra* (divisor) for Mars and Jupiter respectively. Half of the arc of the *doḥphala* has to be applied to the mean longitude of the planet (P_0) to get the first corrected longitude (P_1).

Subtracting the longitude of the Sun (the *śīghrocca*) from this (P_1), the *dorjyā* and *koṭijyā* are obtained. Dividing the *dorjyā* by the *trijyā* and subtracting from 16 and 9, we get the multipliers for Jupiter and Mars respectively. The same (*dorjyā*) multiplied by 2 and subtracted from 53 forms the multiplier for Mars.

80 is the divisor for all of them (in the *śīghra-saṃskāra*). From them (the multiplier and divisor of all the three planets) after obtaining the *doḥphala* and *koṭiphala*, and the *sakṛtkarṇa* (once calculated hypotenuse), half of the *doḥphala* multiplied by the *trijyā* and divided by the *karṇa* is applied to the first corrected longitude (P_1). (The longitude thus obtained is, say, P_2.) From this (P_2), let the *mandocca* be subtracted and the full *manda-phala* be obtained; let that be applied to the original mean planet (P_0 to get say P_3). From that (P_3) let *śīghra-phala* be obtained as before, and let this be applied fully to the *manda*-corrected planet (P_3). The longitude obtained thus is the *sphuṭa* (the true longitude of the planet).

A detailed and comprehensive discussion of the planetary model, and the geometrical picture implied by it in the traditional scheme, as well as the modification introduced by Nīlakaṇṭha, can be found in Appendix F. Here and in the following sections we confine our explanation mainly to the computational scheme described in the verses of the text.

The computation of the *manda-sphuṭa* has already been described in the earlier verses in this chapter. Let θ_0, θ_m, θ_{ms} be the mean longitude and the longitudes of the *mandocca* and the *manda-sphuṭa* respectively. Also let R, r and K be radii of the deferent circle (*trijyā*), the epicycle and the *manda-karṇa-vṛtta* respectively. r is proportional to K and $\frac{r}{K} = \frac{r_0}{R}$ where, r_0 is the tabulated value of the radius of the epicycle. Then $\theta_{ms} - \theta_0$ is found from

$$\begin{aligned} K\sin(\theta_{ms} - \theta_0) &= -r\sin(\theta_0 - \theta_m) \\ \text{or} \qquad R\sin(\theta_{ms} - \theta_0) &= -\frac{r}{K} R\sin(\theta_0 - \theta_m) \\ &= -\frac{r_0}{R} R\sin(\theta_0 - \theta_m). \end{aligned} \tag{2.169}$$

$R\sin(\theta_0 - \theta_m)$ is the *dorjyā*, $r_0 \sin(\theta_0 - \theta_m)$ is the *doḥphala* and $\theta_0 \sim \theta_{ms}$ is the 'arc' of the *dophala*. In the above verses $\frac{r_0}{R}$ for Saturn, Mars and Jupiter are specified to be

$$\frac{r_0}{R}\,(\text{Saturn}) = \frac{1}{8} - \frac{1}{320} = \frac{39}{320} \tag{2.170}$$

$$\frac{r_0}{R}\,(\text{Mars}) = \frac{7 + |\sin(\theta_0 - \theta_m)|}{39} \tag{2.171}$$

$$\text{and} \qquad \frac{r_0}{R}\,(\text{Jupiter}) = \frac{7 + |\sin(\theta_0 - \theta_m)|}{82}. \tag{2.172}$$

Note that r_0 is not constant for Mars and Jupiter, but varies with the *manda-kendra*, $\theta_0 - \theta_m$. When $\theta_{ms} - \theta_0$, found from the above equation, is added to θ_0, we obtain the *manda-sphuṭa-graha* (*manda*-corrected planet) θ_{ms}. The true geocentric longitude of the exterior planets is obtained from the *manda-sphuṭa* θ_{ms} as follows.

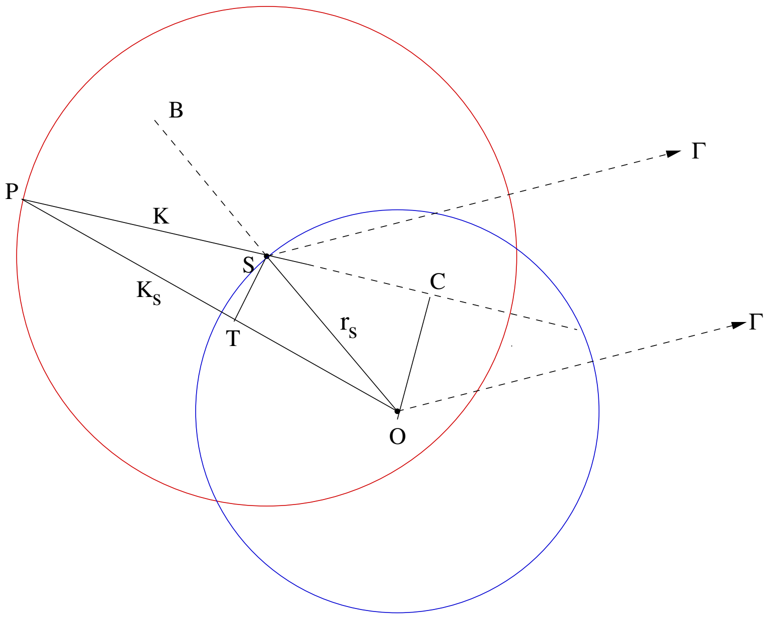

Fig. 2.15 Obtaining the *sphuṭa-graha* (geocentric longitude) from the *manda-sphuṭa-graha* (true heliocentric longitude) in the case of exterior planets .

In Fig. 2.15 the *śīghra-nīcocca-vṛtta* or *śīghra-vṛtta* or *śīghra*-circle is a circle with the *bhagolamadhya* (the centre of the Earth) as the centre at O. The radius of this circle is the *śīgrāntyaphala* r_s. The *śīghrocca* S, which is the mean Sun, is located on this circle. The planet P is located on the *manda-karṇa-vṛtta* of radius K with S as the centre, such that $\theta_{ms} = \Gamma\hat{S}P$ is the *manda-sphuṭa-graha*. Then the *śīghra-sphuṭa* (*śīghra*-corrected planet) is found in the same manner from the *manda-sphuṭa* as the *manda-sphuṭa* is found for the mean planet, the *madhyama-graha*.

Let θ_s be the longitude of the *śīghrocca*. That is, $\theta_s = \Gamma\hat{O}S$. Also from the figure,

$$\begin{aligned} \textit{śīghrocca}\ \theta_s &= \Gamma\hat{S}B \\ \textit{manda-sphuṭa}\ \theta_{ms} &= \Gamma\hat{S}P \\ \textit{śīghra-sphuṭa}\ \theta &= \Gamma\hat{O}P. \end{aligned} \tag{2.173}$$

Therefore

$$O\hat{S}C = P\hat{S}B = \theta_{ms} - \theta_s. \tag{2.174}$$

Further,

$$\begin{aligned} \textit{śīghrābhujaphala}\ OC &= r_s \sin(O\hat{S}C) \\ &= r_s \sin(\theta_{ms} - \theta_s) \\ \textit{śīghrakoṭiphala}\ SC &= r_s \cos(\theta_{ms} - \theta_s). \end{aligned} \tag{2.175}$$

Hence the *śīghra-karṇa* (*śīghra*-hypotenuse)

$$K_s = OP = \sqrt{(K + r_s \cos(\theta_{ms} - \theta_s))^2 + r_s^2 \sin^2(\theta_{ms} - \theta_s)}. \tag{2.176}$$

It can be easily seen that

$$O\hat{P}C = \theta_{ms} - \theta. \tag{2.177}$$

Also from the triangle POC,

$$OP \sin O\hat{P}C = OC. \tag{2.178}$$

Now using (2.175) to (2.177) in the above equation we have

$$\begin{aligned} K_s \sin(\theta_{ms} - \theta) &= r_s \sin(\theta_{ms} - \theta_s) \\ \text{or} \qquad R \sin(\theta_{ms} - \theta) &= \frac{R}{K_s} r_s \sin(\theta_{ms} - \theta_s). \end{aligned} \tag{2.179}$$

The arc corresponding to $\theta_{ms} - \theta$ is found from this. Subtracting $\theta_{ms} - \theta$ from the *manda-sphuṭa* θ_{ms}, we obtain the *śīghra-sphuṭa* θ. Here θ_{ms} is the true longitude of the planet with respect to S, which is taken to be the mean Sun. Hence θ_{ms} is essentially the true heliocentric longitude of the planet. So the true geocentric longitude θ is obtained from the true heliocentric longitude θ_{ms} using the above procedure. Now,

$$\text{\textit{śīghra-kendradorjyā}} = R\sin(\theta_{ms} - \theta_s) \tag{2.180}$$

$$\text{\textit{sīghrabhujāphala}},\ r_s \sin(\theta_{ms} - \theta_s) = \frac{r_s}{R} R\sin(\theta_{ms} - \theta_s), \tag{2.181}$$

where *śīghra-kendradorjyā* is the Rsine of the *śīghra*-anomaly (anomaly of conjunction). In the *śīghra-saṃskāra*, the value of r_s is given in the text. Unlike in the calculation of the *manda-sphuṭa*, where the *manda-karṇa* K does not appear, here the *śīghra-karṇa* does appear in the computation of the *śīghra-sphuṭa*.

The values of $\frac{r_s}{R}$ for Mars, Jupiter and Saturn are given in the above verses as follows:

$$\frac{r_s}{R}\,(\text{Mars}) = \frac{53 - 2|\sin(\theta_{ms} - \theta_s)|}{80}, \tag{2.182}$$

$$\frac{r_s}{R}\,(\text{Jupiter}) = \frac{16 - |\sin(\theta_{ms} - \theta_s)|}{80}, \tag{2.183}$$

$$\frac{r_s}{R}\,(\text{Saturn}) = \frac{9 - |\sin(\theta_{ms} - \theta_s)|}{80}. \tag{2.184}$$

Planet	Range of ratio $\frac{r_s}{R}$	Average value (modern)
Mars	0.637–0.662	0.656
Jupiter	0.187–0.200	0.192
Saturn	0.100–0.115	0.105

Table 2.3 The range of variation in the ratio of the Earth–Sun to the planet–Sun distances for the exterior planets.

The range of variation of $\frac{r_s}{R}$ as obtained from the above equations along with the average value of the ratio of the Earth–Sun and planet–Sun distances as per modern astronomy are listed in Table 2.3. In Fig. 2.15,

$$\frac{\text{Earth–mean Sun distance}}{\text{planet–mean Sun distance}} = \frac{r_s}{K}, \tag{2.185}$$

where K varies depending upon the *manda-sphuṭa-graha* or the true heliocentric longitude. Taking the mean value of K to be R, the ratio would be $\frac{r_s}{R}$, which still depends upon $(\theta_{ms} - \theta_s)$. Even then, $\frac{r_s}{R}$ is always close to the average value of the ratio of the Earth–Sun and planet–Sun distances for each planet according to modern astronomy.

Āryabhaṭīya-bhāṣya and *Yuktibhāṣā* discuss the geometrical picture in detail. However they do not mention that $\frac{r_s}{R}$ is the ratio of the physical Earth–Sun to planet–Sun distances. There is an important later work of Nīlakaṇṭha, namely *Grahasphuṭānayane vikṣepavāsanā*, which indeed mentions this explicitly. This is discussed in detail in Appendix F.

The procedure for obtaining the *śīghra-sphuṭa* of these three planets, given in the above verses, is not a straightforward, two step process of (i) obtaining the *manda-*

sphuṭa first from the mean planet and then (ii) obtaining the *śīghra-sphuṭa* from the *manda-sphuṭa*. Instead, the following four-step procedure is prescribed:

1. Obtain the *manda-phala* from the mean planet θ_0. Apply half of this *manda-phala* to θ_0 to obtain the first corrected planet P_1.
2. Find the *śīghra-sphuṭa* taking P_1 as the *manda-sphuṭa*, using (2.179). Here it is understood that in the calculation of the *śīghra-karṇa*, the *manda-karṇa* is replaced by the *trijyā* R, so only the *śīghra-kendra* $(\theta_{ms} - \theta_s)$ and the value of r_s (which depends upon the *śīghra-kendra*) figure in the calculation of this *śīghra-sphuṭa*. This is the second corrected planet P_2.
3. Treating P_2 as the mean planet, the *manda-phala* is calculated with $P_2 - \theta_m$ as the anomaly. Apply the full *manda-phala* to θ_0. The resulting quantity is the third corrected planet P_3.
4. Treating P_3 as the *manda-sphuṭa*, the *śīghra-sphuṭa* P is calculated again using R instead of K in the calculation of the *śīghra-karṇa* K_s.

In fact, this four-step procedure to compute the true geocentric longitude is the standard one prescribed in many Indian texts. *Yuktibhāṣā* attempts to provide the rationale for this, though the arguments given there are not entirely clear. However the motivation for this procedure is clear enough and is as follows.

Now, the *manda* correction can be read off from a table, given the mean epicycle radius and the *manda-kendra*. But this is not so in the case of the *śīghra* correction, for the *śīghra-phala* depends not only on the *śīghra-kendra* but also on the *śīghra-karṇa*, which depends on the *manda-karṇa* (the distance SP in Fig. 2.15), which in turn is dependant on the *manda-kendra*. Hence, given the radius of the *śīghra-vṛtta*, the *śīghra-phala* cannot be read off from a table as a function of the *śīghra-kendra* alone, as it depends also on the *manda-karṇa* and hence on the *manda-kendra*. *Yuktibhāṣā* seems to argue that the four-step process is an attempt to stimulate, to some extent, the effect of the *manda-karṇa* in the *śīghra-phala*. Thus, in steps two and four above, the *śīghra-phala* is calculated using the *trijyā* instead of the *aviṣiṣṭa-manda-karṇa*.

२.२७ बुधस्फुटीकरणम्

2.27 The correction for Mercury

बुधमध्यात् स्वमन्दोच्चं त्यक्त्वा दोःकोटिजीवयोः ॥ ६८ ॥
षडंशाभ्यां फलाभ्यां तु कर्णः कार्योऽविशेषतः ।
दोःफलं केवलं स्वर्णं केन्द्रे जूकक्रियादिगे ॥ ६९ ॥
एवं कृतं हि तन्मध्यं स्फुटमध्यं बुधस्य तु ।
रविमध्यं ततः शोध्यं दोःकोटिज्ये ततो नयेत् ॥ ७० ॥
दोर्ज्या द्विघ्ना त्रिभज्याप्ता शोध्यैकत्रिंशतो गुणः ।
मन्दकर्णहतः सोऽपि त्रिज्याप्तः स्यात् स्फुटो गुणः ॥ ७१ ॥
तद्धते बाहुकोटिज्ये खाहिभक्ते फले उभे ।

ताभ्यां कर्णं सकृन्नीत्वा त्रिज्याघ्नं दोःफलं हरेत् ॥ ७२ ॥
कर्णेनाप्तस्य यच्चापं कृत्स्नं तद्भानुमध्यमे ।
क्रमेण प्रक्षिपेज्जह्यात् केन्द्रे मेषतुलादिगे ॥ ७३ ॥
एवं शीघ्रफलेनैव संस्कृतं रविमध्यमम् ।
बुधः स्यात् स स्फुटः शुक्रोऽप्येवमेव स्फुटो भवेत् ॥ ७४ ॥

budhamadhyāt svamandoccaṃ tyaktvā doḥkoṭijīvayoḥ || 68 ||
ṣaḍaṃśābhyāṃ phalābhyāṃ tu karṇaḥ kāryo'viśeṣataḥ |
doḥphalaṃ kevalaṃ svarṇaṃ kendre jūkakriyādige || 69 ||
evaṃ kṛtaṃ hi tanmadhyaṃ sphuṭamadhyaṃ budhasya tu |
ravimadhyaṃ tataḥ śodhyaṃ doḥkoṭijye tato nayet || 70 ||
dorjyā dvighnā tribhajyāptā śodhyaikatriṃśato guṇaḥ |
mandakarṇahataḥ so'pi trijyāptaḥ syāt sphuṭo guṇaḥ || 71 ||
taddhate bāhukoṭijye khāhibhakte phale ubhe |
tābhyāṃ karṇaṃ sakṛnnītvā trijyāghnaṃ doḥphalaṃ haret|| 72 ||
karṇenāptasya yaccāpaṃ kṛtsnaṃ tadbhānumadhyame |
krameṇa prakṣipejjahyāt kendre meṣatulādige || 73 ||
evaṃ śīghraphalenaiva saṃskṛtaṃ ravimadhyamam |
budhaḥ syāt sa sphuṭaḥ śukro'pyevameva sphuṭo bhavet || 74 ||

From the *madhyamagraha* of Mercury, subtracting the *mandocca*, the *dorjyā* and *koṭijyā* are obtained. From one-sixth of these values, the *aviṣiṣṭa-manda-karṇa* is found iteratively. The *doḥphala* has to be added to or subtracted from the *madhyamagraha*, depending on whether the *manda-kendra* lies within 6 signs of *Meṣa* or *Tulā*. The value thus obtained is the *manda-sphuṭa-graha* of Mercury (say P_1).

Then subtracting the mean Sun (which is the *śīghrocca*) from this (P_1), obtain the *dorjyā* and *koṭijyā* (corresponding to the *śīghra-kendra*). The *dorjyā* multiplied by 2, divided by *trijyā* and subtracted from 31 forms the multiplier. This multiplier multiplied by the *aviṣiṣṭa-manda-karṇa* and divided by the *trijyā* forms the *sphuṭaguṇa* (true multiplier).

The *dorjyā* and *koṭijyā*, multiplied by the *sphuṭaguṇa* and divided by 80, form the *doḥphala* and *koṭiphala* respectively. From these two (the *doḥphala* and *koṭiphala*), obtain the *śīghra-karṇa* once (not iteratively) and divide the product of the *trijyā* and *doḥphala* by this *śīghra-karṇa*. The arc of this result is fully applied to the mean Sun. It is either added or subtracted depending upon whether the *śīghra-kendra* lies within 6 signs of *Meṣa* or *Tulā*. The mean Sun corrected by this *śīghra-phala* gives the true geocentric longitude of Mercury. The true geocentric longitude of Venus is obtained in a similar manner.

Unlike a four-step procedure employed for the exterior planets to obtain the *sphuṭa-graha* (true planet), in the case of interior planets only a two-step procedure is prescribed. First the *manda-sphuṭa-graha* (*manda*-corrected planet) is obtained from the *madhyama-graha* (mean planet) through *manda-saṃskāra* (*manda*-correction), that is, the equation of centre, and then the *sphuṭa-graha* is obtained through the *śīghra-saṃskāra* (*śīghra* correction).

The *manda-sphuṭagraha* of Mercury is obtained from the mean heliocentric planet following the same procedure as for the exterior planets. Here $\frac{r_0}{R}$ is specified as $\frac{1}{6}$, where r_0 is the mean radius of the epicycle. The *aviśiṣṭa-manda-karṇa* K is also calculated as described earlier. The procedure for obtaining the true geocentric longitude of Mercury from the *manda-sphuṭa-graha* as described in these verses can be understood from Fig. 2.16 (see also Appendix F).

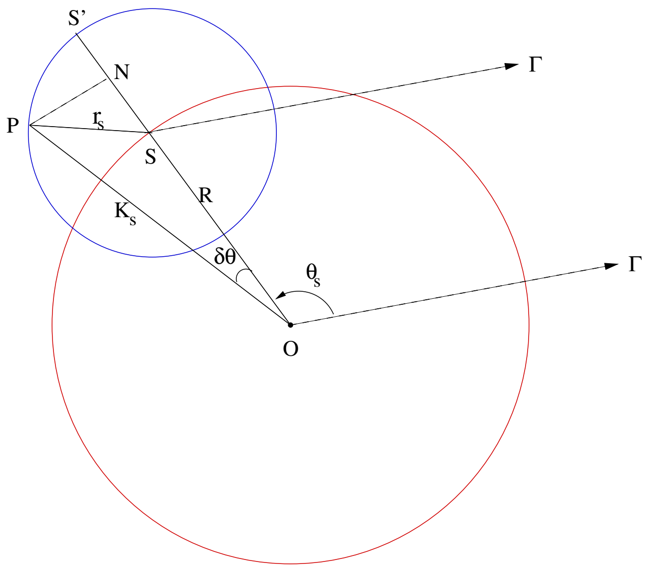

Fig. 2.16 Obtaining the *sphuṭa-graha* (the geocentric longitude) from the *manda-sphuṭa-graha* (the true heliocentric longitude) in the case of interior planets.

The mean Sun S is located on a circle of radius R with the centre of the Earth as the centre. Its longitude $\theta_s = \Gamma\hat{O}S$. Draw a circle of radius r_s around S. Mercury is located on this point such that its longitude is the *manda-sphuṭa-graha* $\theta_{ms} = \Gamma\hat{S}P$ with respect to S. Then $\theta = \Gamma\hat{O}P$ is the true geocentric longitude of Mercury called the *sphuṭa-graha* or simply the *sphuṭa*. Now,

$$\begin{aligned} \textit{śīghra-kendra} &= \theta_{ms} - \theta_s \\ &= \Gamma\hat{S}P - \Gamma\hat{O}S \\ &= \Gamma\hat{S}P - \Gamma\hat{S}S' \\ &= S'\hat{S}P. \end{aligned} \tag{2.186}$$

The radius of the epicycle r_s is given by

$$\frac{r_s}{R} = \frac{31 - 2|\sin\theta_{ms} - \theta_s|}{80} \times \frac{K}{R}, \tag{2.187}$$

and the *śīghra-karṇa* K_s is obtained from

$$\begin{aligned} K_s = OP &= \sqrt{ON^2 + PN^2} \\ &= \sqrt{(R + r_s\cos(\theta_{ms} - \theta_s))^2 + (r_s\sin(\theta_{ms} - \theta_s))^2}. \end{aligned} \tag{2.188}$$

The *śīghra* correction $S'OP = \delta\theta$ is found from the relation

$$
\begin{aligned}
OP \sin \delta\theta &= PN \\
&= r_s \sin(\theta_{ms} - \theta_s) \\
\text{or} \qquad K_s \sin \delta\theta &= r_s \sin(\theta_{ms} - \theta_s) \\
\text{or} \qquad R \sin \delta\theta &= r_s \sin(\theta_{ms} - \theta_s) \frac{R}{K_s},
\end{aligned} \tag{2.189}
$$

where

$$
\begin{aligned}
do\d{h}phala &= r_s \sin(\theta_{ms} - \theta_s) \\
&= \frac{r_s}{R} R \sin(\theta_{ms} - \theta_s) \\
&= R \sin(\theta_{ms} - \theta_s) \times \left[(31 - 2|\sin(\theta_{ms} - \theta_s)|) \times \frac{K}{R} \right] \times \frac{1}{80} \\
&= dorjya \times sphu\d{t}agu\d{n}a \times \frac{1}{80}.
\end{aligned} \tag{2.190}
$$

Similarly,

$$
\begin{aligned}
ko\d{t}iphala &= r_s \cos(\theta_{ms} - \theta_s) \\
&= ko\d{t}ijy\bar{a} \times sphu\d{t}agu\d{n}a \times \frac{1}{80}.
\end{aligned} \tag{2.191}
$$

Adding the arc $\delta\theta$ obtained thus to the longitude of the mean Sun θ_s, we obtain the true geocentric longitude of Mercury, $\theta = \Gamma \hat{O} P = \theta_s + \delta\theta$.

In the earlier Indian texts, as was the case also in the Greco-European tradition up to Kepler, the equation of centre of the interior planet used to be applied wrongly to the mean Sun, which was taken as the mean planet in the case of interior planets. It is in *Tantrasaṅgraha* that the equation of centre is correctly applied to the mean heliocentric planet to obtain the true heliocentric planet, for the first time in the history of astronomy. We have already commented on this major modification that has been introduced for the interior planets in *Tantrasaṅgraha*, wherein the mean heliocentric planet is taken as the mean planet and the specified revolution number is noted as its own (*svaparyayāḥ*), and the mean Sun is taken as the *śīghrocca* for all the planets.

Now, ignoring the correction due to the eccentricity, the ratio of the Mercury–Sun to the Earth–Sun distance may be compared with the ratio $\frac{r_s}{R}$ given in (2.187):

$$
\frac{\text{Mercury–Sun distance}}{\text{Earth–Sun distance}} = \frac{31 - 2|\sin(\theta_{ms} - \theta_s)|}{80}. \tag{2.192}
$$

It may be noted that this ratio varies between $\frac{29}{80} = 0.362$ and $\frac{31}{80} = 0.387$, as compared with the average modern value of 0.387. The factor $\frac{K}{R}$ in $\frac{r_s}{R}$ in (2.187) takes into account the eccentricity of the planetary orbit.

Finally it may be mentioned that here, in calculating the true position of Mercury, only a two-step procedure is prescribed. The *śīghra-phala*, however, depends on the *manda-karṇa* and hence the *manda-kendra* also. Further, it is the iterated *manda-karṇa* that is involved in this calculation. A similar procedure is advocated for obtaining the true position of Venus.

२.२८ शुक्रस्फुटीकरणम्
2.28 The correction for Venus

मन्दकेन्द्रभुजा जीवा खजिनांशेन संयुता ।
मनवस्तस्य हारः स्यात् तद्भक्ते बाहुकोटिके ॥ ७५ ॥
स्यातां मन्दफले तस्य दोःफलं च स्वमध्यमे ।
कृत्वाऽविशेषकर्णं च क्रियतां शीघ्रकर्म च ॥ ७६ ॥
द्विघ्ना दोर्ज्या त्रिभज्याप्ता शोध्यास्यैकोनषष्टितः ।
गुणः सोऽपि स्फुटीकार्यः मन्दकर्णेन पूर्ववत् ॥ ७७ ॥
गुणः स मन्दकर्णघ्नः त्रिज्याप्तस्तस्य च स्फुटः ।
अशीत्याप्ते भुजाकोटी तद्घ्ने शीघ्रफले भृगोः ॥ ७८ ॥
दोःफलं त्रिज्यया हत्वा शीघ्रकर्णहृतं भृगोः ।
चापितं भास्वतो मध्ये संस्कुर्यात् स स्फुटः सितः ॥ ७९ ॥

mandakendrabhujā jīvā khajināṃśena saṃyutā |
manavastasya hāraḥ syāt tadbhakte bāhukoṭike || 75 ||
syātāṃ mandaphale tasya doḥphalaṃ ca svamadhyame |
kṛtvā'viśeṣakarṇaṃ ca kriyatāṃ śīghrakarma ca || 76 ||
dvighnā dorjyā tribhajyāptā śodhyāsyaikonaṣaṣṭitaḥ |
guṇaḥ so'pi sphuṭīkāryaḥ mandakarṇena pūrvavat || 77 ||
guṇaḥ sa mandakarṇaghnaḥ trijyāptastasya ca sphuṭaḥ |
aśītyāpte bhujākoṭī tadghne śīghraphale bhṛgoḥ || 78 ||
doḥphalaṃ trijyayā hatvā śīghrakarṇahṛtaṃ bhṛgoḥ |
cāpitaṃ bhāsvato madhye saṃskuryāt sa sphuṭaḥ sitaḥ || 79 ||

The 240th part of the Rsine of the *manda-kendra* added to 14 (forms the divisor). The *dorjyā* and the *koṭijyā* divided by this divisor form the *doḥphala* and *koṭiphala* in the *manda-saṃskāra*. After adding the arc of the *doḥphala* to the *madhyama-graha*, let the *aviṣiṣṭa-manda-karṇa* be found and *śīghra-saṃskāra* be carried out as set forth below.

The *dorjyā* (corresponding to the *śīghra-kendra*) multiplied by two, divided by the *trijyā*, and subtracted from 59, forms the multiplier. This multiplied by the *aviṣiṣṭa-manda-karṇa* and divided by *trijyā* forms the *sphuṭaguṇa*. The *dorjyā* and *koṭijyā* multiplied by the *sphuṭaguṇa* and divided by 80 are the *doḥphala* and *koṭiphala*. The arc of the *doḥphala* multiplied by the *trijyā* and divided by the *śīghra-karṇa* should be applied to the mean Sun. This gives the true longitude of the Venus.

The procedure for calculating the geocentric longitude of Venus is the same as for that of Mercury. The *manda-sphuṭagraha* is calculated taking the ratio of the epicycle to the deferent[24] to be

[24] It is interesting to note that the expression for the denominator given here, namely $14 + \frac{R|\sin(\theta_0 - \theta_m)|}{240}$, is such that the second term can be as large as the first one.

$$\frac{r_0}{R} = \frac{1}{14 + \frac{R|\sin(\theta_0 - \theta_m)|}{240}}. \tag{2.193}$$

The *śīghra-saṃskāra* is identical with that for Mercury, as shown in Fig. 2.16. In the same way as in (2.192), here we can set

$$\begin{aligned} \frac{\text{Venus–Sun distance}}{\text{Earth–Sun distance}} &= \frac{r_s}{R} \\ &= \frac{59 - 2|\sin(\theta_{ms} - \theta_s)|}{80} \times \frac{K}{R}. \end{aligned} \tag{2.194}$$

Ignoring the correction for eccentricity (taking $K = R$), we find that $\frac{r_s}{R}$ varies between $\frac{57}{80} = .712$ and $\frac{59}{80} = .737$, as compared with the average modern value of .723.

२.२९ ग्रहाणां दिनभुक्तिः
2.29 The daily motion of the planets

श्वस्तनेऽद्यतनाच्छुद्धे वक्रभोगोऽवशिष्यते ।
विपरीतविशेषोत्थचारभोगस्तयोः स्फुटः ॥ ८० ॥

śvastane'dyatanācchuddhe vakrabhogo'vaśiṣyate |
viparītaviśeṣotthacārabhogastayoḥ sphuṭaḥ || 80 ||

The longitude of the planet found for tomorrow is subtracted from the longitude of the planet today. The result [if positive] is the retrograde daily motion of the planet; if otherwise, the result gives the direct daily motion of the planet.

In this verse, essentially, the definition of direct/retrograde motion is given. By *bhoga* is meant daily motion, the angular distance travelled by the planet in one day as observed by an observer on the surface of the Earth.