Chapter 8
Extensions of Multiresolution Analysis

The wavelets arrive in succession, and each wavelet eventually dies out. The wavelets all have the same basic form and shape, but the strength or impetus of each wavelet is random and uncorrelated with the strength of the other wavelets. Despite the fore-ordained death of any individual wavelet, the time-series does not die. The reason is that a new wavelet is born each day to take the place of the one that does die on any given day, the time-series is composed of many living wavelets, all of a different age, some young, others old.

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8.1 Introduction

Multiresolution analysis (MRA) is considered as the heart of wavelet theory. The concept of MRA provides an elegant tool for the construction of wavelets. An MRA is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\phi \in V_0$ such that the collection of integer translates of function ϕ, $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function ϕ is called the *scaling function* or the *father wavelet*. This classic concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. From the last decade, this elegant tool for the construction of wavelet bases have been extensively studied by several authors on the various spaces, namely, abstract Hilbert spaces, locally compact Abelian groups, Cantor dyadic groups, Vilenkin groups, local fields of positive characteristic, p-adic fields, Hyrer-groups, Lie groups, zero-dimensional groups. Notice that the technique is similar to that in the real case of $\mathbb{R}$ while the mathematical treatment needs ones conscientiousness.

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On the other hand, several new extensions of the original MRA also came into existence such as Periodic MRA, Non-stationary MRA, Generalized MRA, Frame MRA, Adaptive MRA, Projective MRA, irregular MRA, Vector-valued MRA, Nonuniform MRA (NUMRA), p-MRA on $\mathbb{R}^+$ and the list goes on.

This chapter is devoted to study the last two extensions of the classical theory of MRA listed above. In Sect. 8.2, we introduce p-MRA on a positive half-line and describe a method for constructing compactly supported orthogonal p-wavelets on $\mathbb{R}^+$ related to the generalized Walsh functions. For all integers $p, n \geq 2$, we study necessary and sufficient conditions under which the solutions of the corresponding scaling equations with p^n-numerical coefficients generate MRA in $L^2(\mathbb{R}^+)$. Further, we discuss conditions under which a compactly supported solution of the refinement equation in $L^2(\mathbb{R}^+)$ is stable and has a linearly independent system of integer shifts. In the end, we present several examples illustrating these results. In Sect. 8.3, we introduce nonuniform MRA, based on the spectral pairs, in which the translation set acting on the scaling function associated with MRA to generate the core subspace V_0 is no more a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$.

8.2 *p*-MRA on a Half-Line $\mathbb{R}^+$

We start this section with certain results on Walsh–Fourier analysis. We present a brief review of generalized Walsh functions, Walsh–Fourier transforms and its various properties.

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\text{mod}\, p), \qquad x_{-j} = [p^{1-j} x] (\text{mod}\, p), \tag{8.2.1}$$

where $x_j, x_{-j} \in \{0, 1, \ldots, p-1\}$. It is clear that for each $x \in \mathbb{R}^+$, there exist $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0 \; \forall \; j > k$.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\text{mod}\, p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \ldots, p-1\}$ and x_j, y_j are calculated by (8.2.1). As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

Note that for $p = 2$ and $j \in \mathbb{N}$, we define the numbers $x_j, x_{-j} \in \{0, 1\}$ as follows:

$$x_j = [2^j x] (\text{mod}\, 2), \qquad x_{-j} = [2^{1-j} x] (\text{mod}\, 2), \tag{8.2.2}$$

where $[\cdot]$ denotes the integral part of $x \in \mathbb{R}^+$. x_j and x_{-j} are the digits of the binary expansion

$$x = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j}. \tag{8.2.3}$$

Therefore, for fixed $x, y \in \mathbb{R}^+$, we set

$$x \oplus y = \sum_{j<0} |x_j - y_j| 2^{-j-1} + \sum_{j>0} |x_j - y_j| 2^{-j},$$

where x_j, y_j are defined in (8.2.2). By definition $x \ominus y = x \oplus y$ (because $x \oplus x = 0$).

The binary operation $\oplus$ identifies $\mathbb{R}^+$ with the group G_2 (dyadic group with addition modulo two) and is useful in the study of dyadic Hardy classes and image processing (see Farkov et al. 2011; Farkov and Rodionov 2012).

For $x \in [0, 1)$, let $r_0(x)$ is given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^{\ell}, & \text{if } x \in \left[\ell p^{-1}, (\ell+1) p^{-1}\right), \quad \ell = 1, 2, \ldots, p-1, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then, the *generalized Walsh functions* $\left\{w_m(x) : m \in \mathbb{Z}^+\right\}$ are defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \ldots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{8.2.4}$$

where x_j, y_j are given by (8.2.1).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\ x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and

$$\chi(x \oplus y, z) = \chi(x, z)\, \chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\, \overline{\chi(y, z)},$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$ (see Golubov et al. 1991).

The *Walsh–Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x, \omega)}\, dx, \tag{8.2.5}$$

where $\chi(x, \omega)$ is given by (8.2.4). The Walsh–Fourier operator $\mathscr{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathscr{F} f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh–Fourier transform are quite similar to those of the classic Fourier transform (see Golubov et al. 1991; Schipp et al. 1990). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\left\| \hat{f} \right\|_{L^2(\mathbb{R}^+)} = \| f \|_{L^2(\mathbb{R}^+)}.$$

Let $\{\omega\}$ denotes the fractional part of ω. For any $\phi \in L^2(\mathbb{R}^+)$ and $k \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\int_{\mathbb{R}^+} \phi(x)\, \overline{\phi(x \ominus k)}\, dx &= \int_{\mathbb{R}^+} \left|\hat{\phi}(\omega)\right|^2 \overline{\chi(k, \omega)}\, d\omega \\
&= \sum_{\ell=0}^{\infty} \int_{\ell}^{\ell+1} \left|\hat{\phi}(\omega)\right|^2 \overline{\chi\,(k, \{\omega\})}\, d\omega \\
&= \int_0^1 \left(\sum_{\ell \in \mathbb{Z}^+} \left|\hat{\phi}(\omega + \ell)\right|^2 \right) \overline{\chi(k, \omega)}\, d\omega.
\end{aligned} \tag{8.2.6}$$

Therefore, a necessary and sufficient condition for a system $\left\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\right\}$ to be orthonormal in $L^2(\mathbb{R}^+)$ is

$$\sum_{\ell \in \mathbb{Z}^+} \left|\hat{\phi}(\omega + \ell)\right|^2 = 1 \quad a.e. \tag{8.2.7}$$

By p-adic interval $I \subset \mathbb{R}^+$ of range n, we mean intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n}\right), \quad k \in \mathbb{Z}^+. \tag{8.2.8}$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see Golubov et al. 1991). The family $\{[0, p^{-j}) : j \in \mathbb{Z}\}$ forms a fundamental system of the p-adic topology on $\mathbb{R}^+$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ be the space of p-adic entire functions of order n, that is, the set of all functions which are constant on all p-adic intervals of range n. Thus, for every $f \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^+} f(p^{-n}k)\chi_{I_n^k}(x), \quad x \in \mathbb{R}^+. \tag{8.2.9}$$

Clearly, each Walsh function of order p^{n-1} belong to $\mathcal{E}_n(\mathbb{R}^+)$. The set $\mathcal{E}(\mathbb{R}^+)$ of p-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$, i.e.,

$$\mathcal{E}(\mathbb{R}^+) = \bigcup_{n=1}^{\infty} \mathcal{E}_n(\mathbb{R}^+).$$

It is clear that $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+)$ for $1 \le p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support.

An analog of the following proposition for p-adic entire functions on the positive half-line $\mathbb{R}^+$ was proved in Golubov et al. (1991) (Sect. 6.2).

Proposition 8.2.1. *The following properties hold:*

(i) If $f \in L^1(\mathbb{R}^+) \cap \mathcal{E}_n(\mathbb{R}^+)$, then supp $\hat{f} \subset [0, p^n)$,
(ii) If $f \in L^1(\mathbb{R}^+)$ and supp $\hat{f} \subset [0, p^n)$, then $\hat{f} \in \mathcal{E}_n(\mathbb{R}^+)$.

Similar to $\mathbb{R}$, wavelets can be constructed from a MRA on a positive half-line $\mathbb{R}^+$. For $p \ge 2$, we define a MRA on $\mathbb{R}^+$ as follows:

Definition 8.2.1. A p-MRA of $L^2(\mathbb{R}^+)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^+)$ satisfying the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$;
(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
(iv) $f \in V_j$ if and only if $f(p.) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(v) there is a function ϕ in V_0 such that the system $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ forms an orthonormal basis for V_0.

The function ϕ occurring in axiom (v) is called a *scaling function*. One also says that an p-MRA is generated by its scaling function ϕ (or ϕ generates the p-MRA). It follows immediately from axioms (iv) and (v) that

$$V_j := \overline{\text{span}}\left\{\phi_{j,k}(x) = p^{j/2}\phi\left(p^j x \ominus k\right) : k \in \mathbb{Z}^+\right\}, \quad j \in \mathbb{Z}. \tag{8.2.10}$$

According to the standard scheme (see Chap. 7) for construction of MRA-based wavelets, for each j, we define a space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j,\ j \in \mathbb{Z}$, where $W_j \perp V_j,\ j \in \mathbb{Z}$. It is not difficult to see that

$$f(\cdot) \in W_j \quad \text{if and only if} \quad f(p.) \in W_{j+1}, \quad j \in \mathbb{Z}. \tag{8.2.11}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j\geq 0} W_j \right). \tag{8.2.12}$$

As in the case of $\mathbb{R}^n$, we expect the existence of $p-1$ number of functions $\{\psi_1, \psi_2, \ldots, \psi_{p-1}\}$ to form a set of basic wavelets. In view of (8.2.11) and (8.2.12), it is clear that if $\{\psi_1, \psi_2, \ldots, \psi_{p-1}\}$ is a set of function such that the system $\{\psi_\ell(\cdot \ominus k) : 1 \leq \ell \leq p-1, k \in \mathbb{Z}^+\}$ forms an orthonormal basis for W_0, then $\{p^{j/2}\psi_\ell(p^j x \ominus k) : 1 \leq \ell \leq p-1, j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$ forms an orthonormal basis for $L^2(\mathbb{R}^+)$.

The main goal of this section is to establish necessary and sufficient conditions under which the solutions of scaling equations of the form

$$\phi(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha\, \phi(px \ominus \alpha). \tag{8.2.13}$$

generate a MRA in $L^2(\mathbb{R}^+)$. For wavelets on the real line $\mathbb{R}$, the corresponding conditions were described in Daubechies' book (1990) in Sect. 6.3.

The generalized Walsh polynomial m_0 of the form

$$m_0(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha\, \overline{w_\alpha(\omega)}. \tag{8.2.14}$$

is called the *mask* or *solution* of the refinement equation (8.2.13). It is clear that m_0 is a p-adic step function as the Walsh functions w_α are constant on p-adic intervals I_n^s, for $0 \leq \alpha, s < p^n$. Moreover, if $b_s = m_0\big(sp^{-n}\big)$ are the values of m_0 on p-adic intervals, i.e.,

$$b_s = m_0\big(sp^{-n}\big) = \sum_{\alpha=0}^{p^n-1} a_\alpha\, \overline{w_\alpha\big(sp^{-n}\big)}, \quad 0 \leq s \leq p^n - 1. \tag{8.2.15}$$

Then, the coefficients $a_\alpha,\ 0 \leq \alpha \leq p^n - 1$ of Eq. (8.2.13) can be computed by means of the direct Vilenkin–Chrestenson transforms as

$$a_\alpha = p^{-n} \sum_{s=0}^{p^n-1} b_s\, w_\alpha\big(sp^{-n}\big), \quad 0 \le \alpha \le p^n - 1, \tag{8.2.16}$$

and conversely. Thus, the choice of the values of the mask (8.2.14) on p-adic intervals simultaneously defines the coefficients of Eq. (8.2.13) which is satisfied by the corresponding function ϕ.

Theorem 8.2.1. *If $\phi \in L^2(\mathbb{R}^+)$ is a compactly supported solution of Eq.* (8.2.13) *such that $\hat{\phi}(0) = 1$. Then*

$$\sum_{\alpha=0}^{p^n-1} a_\alpha = 1 \quad \textit{and} \quad \textit{supp}\, \phi \subset [0, p^{n-1}]. \tag{8.2.17}$$

This solution is unique, is given by the formula

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m_0\left(\frac{\omega}{p^j}\right) \tag{8.2.18}$$

and possesses the following properties:

(i) $\hat{\phi}(k) = 0$ *for all $k \in \mathbb{N}$ (the modified Strang-Fix condition);*
(ii) $\sum_{k\in\mathbb{Z}^+} \phi(x \oplus k) = 1$ *for almost all $x \in \mathbb{R}^+$.*

Proof. The Walsh–Fourier transform of (8.2.13) yields

$$\hat{\phi}(\omega) = m_0\left(p^{-1}\omega\right) \hat{\phi}\left(p^{-1}\omega\right). \tag{8.2.19}$$

Observe that $w_\alpha(0) = \hat{\phi}(0) = 1$. Hence, substituting $\omega = 0$ in (8.2.13) and (8.2.14), we obtain $m_0(0) = 1$, therefore $\sum_{\alpha=0}^{p^n-1} a_\alpha = 1$. Further, let s be the greatest integer such that ϕ does not vanish on a positive-measure subset of the interval $[s-1, s)$, i.e.,

$$\mu\big\{x \in [s-1, s) : \phi(x) \neq 0\big\} > 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^+$. Assume that $s \ge p^{n-1} + 1$ and consider an arbitrary p-adic irrational $x \in [s-1, s)$ of the form

$$x = [x] + \{x\} = \sum_{j=1}^{k} x_{-j}\, p^{j-1} + \sum_{j=1}^{k} x_j\, p^{-j} \tag{8.2.20}$$

where $\{x\} > 0, x_{-k} \neq 0, k \ge n$. For any $\alpha \in \{0, 1, \ldots, p^n - 1\}$, the element $y^{(\alpha)} = px \ominus \alpha$ is of the form

$$y^{(\alpha)} = \sum_{j=1}^{k+1} y_{-j}^{(\alpha)} p^{j-1} + \sum_{j=1}^{k} y_j^{(\alpha)} p^{-j},$$

where $y_{-k-1}^{(\alpha)} = x_{-k} \neq 0$ and, among the digits $y_j^{(\alpha)}$, $j \geq 0$, there are some nonzero ones. Therefore,

$$px \ominus \alpha > p^n \quad \text{for a.e. } x \in [s-1, s). \tag{8.2.21}$$

If $s \leq p^n$, then inequality (8.2.21) implies $\phi(px \ominus \alpha) = 0$ for a.e. $x \in [s-1, s)$. But, in that case, by (8.2.13), we have $\phi(x) = 0$ for a.e. $x \in [s-1, s)$, which contradicts the choice of s. Therefore, $s \geq p^n + 1$. Using this inequality, for any $\alpha \in \{0, 1, \ldots, p^n - 1\}$ from (8.2.20), we obtain

$$px \ominus \alpha > p(s-1) - (p^n - 1) \geq 2(s-1) - (s-2) = s.$$

Hence, just as above, it follows that $\phi(x) = 0$ for a.e. $x \in [s-1, s)$. Therefore, $s \leq p^{n-1}$ and $\operatorname{supp}\phi \subset [0, p^{n-1}]$.

We now claim that the Walsh–Fourier transform $\hat{\phi}$ satisfies (8.2.18). Since ϕ is compactly supported and belongs $L^2(\mathbb{R}^+)$, then it also belongs to $L^1(\mathbb{R}^+)$. Since $\operatorname{supp}\phi \subset [0, p^{n-1})$, it follows that $\hat{\phi} \in \mathcal{E}_{n-1}(\mathbb{R}^+)$. From the condition $\hat{\phi}(0) = 1$, we see that $\hat{\phi}(\omega) = 1$ for all $\omega \in [0, p^{1-n})$. On the other hand, $m_0(\omega) = 1$ for all $\omega \in [0, p^{1-n})$. Hence, for any natural number ℓ, we can write

$$\hat{\phi}(\omega) = \hat{\phi}\left(p^{-\ell-n}\omega\right) \prod_{j=1}^{\ell+n} m_0\left(\frac{\omega}{p^j}\right) = \prod_{j=1}^{\infty} m_0\left(\frac{\omega}{p^j}\right), \quad \omega \in [0, p^{\ell}),$$

which completes the proof of (8.2.18) and of the uniqueness of ϕ.

We observe that for each $k \in \mathbb{N}$, we have

$$\hat{\phi}(k) = \hat{\phi}(k) \prod_{s=0}^{j-1} m_0(p^s k) = \hat{\phi}\left(p^j k\right) \to 0$$

as $j \to \infty$ (because $\phi \in L^1(\mathbb{R}^+)$ and $m_0(p^s k) = 1$ by the equality $m_0(0) = 1$ and the periodicity of m_0). This means that $\hat{\phi}(k) = 0$ for all $k \in \mathbb{N}$.

By Poisson's summation formula, we obtain

$$\sum_{k \in \mathbb{Z}^+} \phi(x \oplus k) = \sum_{k \in \mathbb{Z}^+} \hat{\phi}(k) w_k(x).$$

where the equality holds almost everywhere in Lebesgue measure. Since $\hat{\phi}(k) = \delta_{0,k}$, it follows that

$$\sum_{k\in\mathbb{Z}^+} \phi(x \oplus k) = 1 \quad \text{for a.e. } x \in \mathbb{R}^+.$$

The proof of the theorem is now complete.

Assume that Eq. (8.2.13) has a compactly supported L^2-solution ϕ satisfying the condition $\hat{\phi}(0) = 1$ and the system $\{\phi(x \ominus k) : k \in \mathbb{Z}^+\}$ is orthonormal in $L^2(\mathbb{R}^+)$, then

$$m_0(0) = 1 \quad \text{and} \quad \sum_{\ell=0}^{p-1} \left|m_0\left(\omega \oplus \ell/p\right)\right|^2 = 1 \qquad \text{for all } \omega \in [0, 1/p). \tag{8.2.22}$$

Therefore, Eqs. (8.2.18) and (8.2.22) implies that the equalities

$$b_0 = 1, \quad |b_j|^2 + |b_{j+p^{n-1}}|^2 + \cdots + |b_{j+(p-1)p^{n-1}}|^2 = 1, \qquad 0 \le j \le p^{n-1} - 1, \tag{8.2.23}$$

are necessary (but not sufficient; see Example 8.2.4) for the system $\{\phi(x \ominus k) : k \in \mathbb{Z}^+\}$ to be orthonormal in $L^2(\mathbb{R}^+)$. Under what additional conditions does the function ϕ generate a p-MRA in $L^2(\mathbb{R}^+)$? The answer to this question is given below in Theorem 8.2.2.

Before we state Theorem 8.2.2, we start here with some definitions:

Definition 8.2.2. A function $f : \mathbb{R}^+ \to \mathbb{C}$ is said to be *W-continuous at a point* $x \in \mathbb{R}^+$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$. Therefore, for each $0 \le j, k < p^n$, the Walsh function $w_j(x)$ is piecewise constant and hence W-continuous. Thus $w_j(x) = 1$ for $x \in I_n^0$.

Definition 8.2.3. A subset $E \subset \mathbb{R}^+$ is said to be *W-compact* if it is compact in the p-adic topology. It is easy to see that the union of a finite family of p-adic intervals is W-compact. Moreover, a W-compact set E is said to be *congruent to* $[0, 1)$ *modulo* $\mathbb{R}^+$ if its Lebesgue measure is 1 and, for each $x \in [0, 1)$, there is an element $k \in \mathbb{Z}^+$ such that $x \oplus k \in E$.

Definition 8.2.4. If m_0 is mask of the refinable equation (8.2.13). Then, m_0 is said to satisfy the *modified Cohen condition* if, there exists a W-compact subset E of $\mathbb{R}^+$ such that

(i) E is congruent to $[0, 1)$ modulo $\mathbb{Z}^+$ and containing a neighborhood of the zero element,
(ii) the following inequality holds:

$$\inf_{j\in\mathbb{N}} \inf_{\omega\in E} \left|m_0\left(p^{-j}\omega\right)\right| > 0. \tag{8.2.24}$$

In view of the condition $m_0(0) = 1$ and the compactness of the set E, there exists a number j_0 such that $m_0\left(p^{-j}\omega\right) = 1$ for all $j > j_0, \omega \in E$. Therefore, inequality (8.3.24) holds if the polynomial m_0 does not vanish on the sets $E/p, \dots, E/p^{-j_0}$. Note that we can always choose $j_0 \leq p^n$, because the mask m_0 is periodic and totally defined by the values (8.2.15).

For an arbitrary set $B \subset [0, 1)$, we set

$$T_p B = \bigcup_{\ell=0}^{p-1} \left\{ \frac{\ell}{p} + \frac{\omega}{p} : \omega \in B \right\}.$$

Definition 8.2.5. A set B is said to be *blocked* (for the mask m_0) if it can be expressed as the union of p-adic intervals of range $n-1$, does not contain the interval $[0, p^{-n+1})$, and possesses the property $T_p B \subset B \cup \operatorname{Null} m_0$, where $\operatorname{Null} m_0$ is the set of all zeros of the mask m_0 on $[0, 1)$. It is clear that each mask can have only a finite number of blocked sets.

Definition 8.2.6. A function $f \in L^2(\mathbb{R}^+)$ is said to be *stable* if there exist positive constants c_1 and c_2 such that

$$c_1 \left(\sum_{k \in \mathbb{Z}^+} |a_k|^2 \right)^{1/2} \leq \left\| \sum_{k \in \mathbb{Z}^+} a_k f(\cdot \ominus k) \right\| \leq c_2 \left(\sum_{k \in \mathbb{Z}^+} |a_k|^2 \right)^{1/2}$$

for each sequence $\{a_k\} \in \ell^2$. In other words, a function f is stable in $L^2(\mathbb{R}^+)$ if the functions $f(\cdot \ominus k), k \in \mathbb{Z}^+$, form a Riesz system in $L^2(\mathbb{R}^+)$. Further, we say that a function $f : \mathbb{R}^+ \to \mathbb{C}$ has a *periodic zero* at a point $x \in \mathbb{R}^+$ if $f(x \oplus k) = 0$ for all $k \in \mathbb{Z}^+$.

The following proposition is proved in Farkov (2005a,b).

Proposition 8.2.2. *For a compactly supported function $f \in L^2(\mathbb{R}^+)$ the following statements are equivalent:*

(i) the function f is stable in $L^2(\mathbb{R}^+)$;
(ii) the system $\{f(x \ominus k) : k \in \mathbb{Z}^+\}$ is linearly independent in $L^2(\mathbb{R}^+)$;
(iii) the Walsh–Fourier transform of the function f has no periodic zeros.

Besides, it has been established that the compactly supported L^2-solution ϕ of Eq. (8.2.13) satisfying the condition $\hat{\phi}(0) = 1$ is not stable if and only if the mask of (8.2.13) has a blocked set. The following assertion is also valid.

Theorem 8.2.2. *Suppose that Eq. (8.2.13) possesses a compactly supported L^2-solution ϕ such that its mask m_0 satisfies conditions (8.2.23) and $\hat{\phi}(0) = 1$. Then the following three assertions are equivalent:*

(i) the function ϕ generates a p-MRA in $L^2(\mathbb{R}^+)$;
(ii) the mask m_0 satisfies the modified Cohen condition;
(iii) the mask m_0 has no blocked sets.

We split the proof of Theorem 8.2.2 into several lemmas.

Lemma 8.2.1. *Suppose that the mask m_0 of refinement equation (8.2.13) satisfies the equalities of (8.2.22). Then, the equation has a solution $\phi \in L^2(\mathbb{R}^+)$ and moreover, $\|\phi\|_2 \le 1$.*

Proof. We define a function $\hat{\phi}(\omega)$ by equality (8.2.18) and prove that it belongs to $L^2(\mathbb{R}^+)$. In this case its inverse Walsh–Fourier transform ϕ also belongs to $L^2(\mathbb{R}^+)$ and obviously satisfies (8.2.13). We have

$$\left|\hat{\phi}(\omega)\right|^2 = \prod_{j=1}^{\infty} \left|m_0\left(p^{-j}\omega\right)\right|^2.$$

Since $|m_0(\omega)| < 1$ for all $\omega \in \mathbb{R}^+$, it follows that for each $s \in \mathbb{N}$,

$$\left|\hat{\phi}(\omega)\right|^2 \le \prod_{j=1}^{s} \left|m_0\left(p^{-j}\omega\right)\right|^2, \qquad \omega \in \mathbb{R}^+.$$

Consequently,

$$\int_0^{p^\ell} \left|\hat{\phi}(\omega)\right|^2 d\omega \le \int_0^{p^\ell} \prod_{j=1}^{s} \left|m_0\left(p^{-j}\omega\right)\right|^2 d\omega = 2^s \int_0^1 \prod_{j=0}^{s-1} \left|m_0\left(p^{j}\omega\right)\right|^2 d\omega. \tag{8.2.25}$$

The function $|m_0(\omega)|^2$ is 1-periodic and piecewise constant with step p^{-n}, therefore it is a Walsh polynomial of order $p^n - 1$:

$$\left|m_0(\omega)\right|^2 = \sum_{\alpha=0}^{p^n-1} c_\alpha w_\alpha(\omega), \tag{8.2.26}$$

where the coefficients c_α may be expressed via a_α. Now, we substitute (8.2.26) into the second equality of (8.2.22) and observe that if α is multiply to p, then $\sum_{\ell=0}^{p-1} w_\alpha(\ell/p) = p$, and this sum is equal to 0 for the rest α. As a result, we obtain $c_0 = 1/p$ and $c_\alpha = 0$ for nonzero α, which are multiply to p. Hence,

$$\left|m_0(\omega)\right|^2 = \frac{1}{p} + \sum_{\alpha=0}^{p^n-1} \sum_{\ell=1}^{p-1} c_{p\alpha+\ell}\, w_{p\alpha+\ell}(\omega).$$

This gives

$$\prod_{j=0}^{s-1} \left|m_0\left(p^{j}\omega\right)\right|^2 = p^{-s} + \sum_{\beta=1}^{\sigma(s)} b_\beta\, w_\beta(\omega), \quad \sigma(s) \le sp^{n-1}(p-1),$$

where each coefficient b_β equals to the product of some coefficients $c_{p\alpha+\ell}, \ell = 1, \ldots, p-1$.

Since

$$\int_0^1 w_\beta(\omega)\, d\omega = 0, \quad \text{for all } \beta \in \mathbb{N},$$

it follows that

$$\int_0^1 \prod_{j=0}^{s-1} \left|m_0\left(p^j \omega\right)\right|^2 = p^{-s}.$$

Substituting this into (8.2.25), we deduce

$$\int_0^{p^\ell} \left|\hat{\phi}(\omega)\right|^2 d\omega \leq 1.$$

Passing to the limit as $\ell \to +\infty$ and using the Parseval's relation, we arrive at $\|\phi\|_2 \leq 1$. The proof of the lemma is complete.

Lemma 8.2.2. *Let* $\left\{V_j\right\}_{j\in\mathbb{Z}}$ *be the family of subspaces defined by* (8.2.10) *with given* $\phi \in L^2(\mathbb{R}^+)$. *If* $\left\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\right\}$ *is an orthonormal basis in* V_0, *then* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof. Let P_j be the orthogonal projection of $L^2(\mathbb{R}^+)$ onto V_j given by the formula

$$P_j f = \sum_{k\in\mathbb{Z}^+} \langle f, \phi_{j,k}\rangle \phi_{j,k}, \quad f \in L^2(\mathbb{R}^+). \tag{8.2.27}$$

Suppose that $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Given an $\varepsilon > 0$ and a continuous function g which is compactly supported in some interval $[0, R]$, $R > 0$ and satisfies $\|f - g\|_2 < \varepsilon$. Then we have

$$\left\|f - P_j g\right\|_2 \leq \left\|P_j(f-g)\right\|_2 \leq \left\|f-g\right\|_2 < \varepsilon$$

so that

$$\left\|f\right\|_2 \leq \left\|P_j g\right\|_2 + \varepsilon.$$

Using the fact that the collection $\left\{p^{j/2}\phi(p^j x \ominus k) : k \in \mathbb{Z}^+\right\}$ is an orthonormal bases for V_j, we have

$$\begin{aligned}\|P_j g\|_2^2 &= \sum_{k\in\mathbb{Z}^+} \left|\langle P_j g, \phi_{j,k}\rangle\right|^2 \\ &= p^j \sum_{k\in\mathbb{Z}^+} \left|\int_0^R g(x)\,\overline{\phi\big(p^j x \ominus k\big)}\,dx\right|^2 \\ &\le p^j \|g\|_\infty^2 R \sum_{k\in\mathbb{Z}^+} \int_0^R \left|\phi\big(p^j x \ominus k\big)\right|^2 dx,\end{aligned}$$

where $\|g\|_\infty$ denotes the supremum norm of g. If j is chosen small enough so that $Rp^j \le 1$, then

$$\begin{aligned}\|P_j g\|_2^2 &\le \|g\|_\infty^2 \int_{S_{R,j}} |\phi(x)|^2 dx \\ &= \|g\|_\infty^2 \int_{\mathbb{R}^+} \mathbf{1}_{S_{R,j}}(x)\left|\phi(x)\right|^2 dx,\end{aligned} \tag{8.2.28}$$

where $S_{R,j} = \bigcup_{k\in\mathbb{Z}^+}\left\{y \ominus k : y \in [0, Rp^j)\right\}$ and $\mathbf{1}_{S_{R,j}}$ denotes the characteristic function of $S_{R,j}$.

It can be easily checked that

$$\lim_{j\to-\infty} \mathbf{1}_{S_{R,j}}(x) = 0 \quad \text{for all } x \notin \mathbb{Z}^+.$$

Thus, from Eq. (8.2.28) by using the dominated convergence theorem, we get

$$\lim_{j\to-\infty} \|P_j g\|_2 = 0.$$

Therefore, we conclude that $\|f\|_2 < \varepsilon$ and since ε is arbitrary, $f = 0$ and thus $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Lemma 8.2.3. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be the family of subspaces defined by (8.2.10) with given $\phi \in L^2(\mathbb{R}^+)$. If $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ is an orthonormal basis in V_0 and assume that $\hat{\phi}(\omega)$ is bounded for all ω and continuous near $\omega = 0$ with $|\hat{\phi}(0)| = 1$, then $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^+)$.*

Proof. Let $f \in \left(\bigcup_{j\in\mathbb{Z}} V_j\right)^\perp$ and $\varepsilon > 0$. We choose $g \in L^1(\mathbb{R}^+) \cap \mathcal{E}(\mathbb{R}^+)$ such that $\|f - g\|_2 < \varepsilon$. For every $j \in \mathbb{Z}^+$, we see from (8.2.27) that

$$\|P_j f\|_2^2 = \langle P_j f, P_j f\rangle = \langle f, P_j f\rangle = 0$$

and

$$\|P_j g\|_2 = \|P_j (f - g)\|_2 \le \|f - g\|_2 < \varepsilon. \tag{8.2.29}$$

Fix a number $j \in \mathbb{N}$ such that $\operatorname{supp}\hat{g} \subset [0, p^j)$ and $p^{-j}\omega \in [0, p^{-n+1})$ for all $\omega \in \operatorname{supp}\hat{g}$. Since the system $\{p^{-j/2}\chi(p^{-j}k, .) : k \in \mathbb{Z}^+\}$ is orthonormal and complete in $L^2[0, p^j]$, we see that the function $h(\omega) = \hat{g}(\omega)\hat{\phi}(p^{-j}\omega)$ satisfies

$$p^{-j}\int_0^{p^j} |h(\omega)|^2 d\omega = \sum_{k\in\mathbb{Z}^+} |c_k(h)|^2, \tag{8.2.30}$$

where

$$c_k(h) = p^{-j/2}\int_0^{p^j} h(\omega)\,\overline{\chi(p^{-j}k, \omega)}\,d\omega.$$

Since

$$\int_{\mathbb{R}^+} \phi(p^j x \ominus k)\,\overline{\chi(x, \omega)}\,dx = p^{-j}\hat{\phi}(p^{-j}\omega)\,\overline{\chi(p^{-j}k, \omega)},$$

we get

$$p^{-j/2}\langle g, \phi_{j,k}\rangle = p^{-j}\int_0^{p^j} h(\omega)\,\overline{\chi(p^{-j}k, \omega)}\,d\omega.$$

Thus, in view of (8.2.30), we obtain

$$\|P_j g\|_2^2 = \sum_{k\in\mathbb{Z}^+} |\langle g, \phi_{j,k}\rangle|^2 = \int_0^{p^j} \left|\hat{g}(\omega)\hat{\phi}(p^{-j}\omega)\right|^2 d\omega. \tag{8.2.31}$$

As $m_0(\omega) = 1$ on the p-adic intervals I_n^0 and $p^{-j}\omega \in [0, p^{-n+1})$ for all $\omega \in \operatorname{supp}\hat{g}$, it follows from (8.2.18) that $\hat{\phi}(p^{-j}\omega) = 1$ for all $\omega \in \operatorname{supp}\hat{g}$. Since $\operatorname{supp}\hat{g} \subset [0, p^j)$, we see from (8.2.29) and (8.2.31) that

$$\varepsilon > \|P_j g\|_2 = \|\hat{g}\|_2 = \|g\|_2.$$

Consequently,

$$\|f\|_2 < \varepsilon + \|g\|_2 < 2\varepsilon.$$

Since ε is arbitrary, therefore $f = 0$. Thus $\left(\bigcup_{j\in\mathbb{Z}} V_j\right)^{\perp} = \{0\}$ and hence the lemma is proved.

Next, we find the analogue of Cohen's condition for p-MRA on positive half-line which gives necessary and sufficient condition for the orthonormality of the system $\{\phi(x \ominus k)\}_{k\in\mathbb{Z}^+}$.

Lemma 8.2.4. *Let m_0 be a Walsh polynomial of the form* (8.2.14) *such that*

$$m_0(0) = 1 \quad \textit{and} \qquad \sum_{\ell=0}^{p-1} \left| m_0\left(\omega \oplus \ell/p\right) \right|^2 = 1 \qquad \textit{for all } \omega \in \mathbb{R}^+,$$

and let $\phi \in L^2(\mathbb{R}^+)$ be the function defined by the formula (8.2.18)*. Then the following are equivalent:*

(i) The mask m_0 satisfies the modified Cohen condition;
(ii) The system $\left\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\right\}$ is orthonormal in $L^2(\mathbb{R}^+)$.

Proof. We will start by proving (i) implies (ii). For every positive integer k, we define

$$\eta_k(\omega) = \prod_{j=1}^{k} m_0\left(\frac{\omega}{p^j}\right) \mathbf{1}_E\left(\frac{\omega}{p^j}\right), \quad \omega \in \mathbb{R}^+$$

Since $0 \in \mathrm{int}(E)$ and $m_0(\omega) = 1$ on the p-adic intervals I_n^0. Thus, it follows from (8.2.18) that

$$\lim_{k\to\infty} \eta_k(\omega) = \hat{\phi}(\omega), \quad \omega \in \mathbb{R}^+. \tag{8.2.32}$$

By our assumption (ii) and the condition $m_0(0) = 1$, there exists a number j_0 such that

$$m_0\left(\frac{\omega}{p^j}\right) = 1 \quad \text{for } j > j_0,\ \omega \in E.$$

Thus

$$\hat{\phi}(\omega) = \prod_{j=1}^{j_0} m_0\left(\frac{\omega}{p^j}\right), \quad \omega \in E.$$

Since $m_0\left(p^{-j}\omega\right) \neq 0$ on E, therefore there is a constant $c_1 > 0$ such that

$$\left| m_0\left(\frac{\omega}{p^j}\right)\right| \geq c_1 > 0 \quad \text{for } j \in \mathbb{N},\ \omega \in E,$$

and so

$$c_1^{-j_0} \left|\hat{\phi}(\omega)\right| \geq \mathbf{1}_E(\omega), \quad \omega \in \mathbb{R}^+.$$

Therefore

$$\begin{aligned}\left|\eta_k(\omega)\right| &= \prod_{j=1}^{k}\left|m_0\left(\frac{\omega}{p^j}\right)\right|\mathbf{1}_E\left(\frac{\omega}{p^j}\right)\\ &\le c_1^{-j_0}\prod_{j=1}^{k}\left|m_0\left(\frac{\omega}{p^j}\right)\right|\left|\hat{\phi}\left(\frac{\omega}{p^j}\right)\right|\end{aligned}$$

which by (8.2.18) yields

$$\left|\eta_k(\omega)\right| \le c_1^{-j_0}\left|\hat{\phi}(\omega)\right|, \quad \text{for } k\in\mathbb{N},\ \omega\in\mathbb{R}^+. \tag{8.2.33}$$

For each $k\in\mathbb{N}$, we define

$$A_k(\ell) = \int_{\mathbb{R}^+}\left|\eta_k(\omega)\right|^2\overline{\chi(\ell,\omega)}\,d\omega, \quad \ell\in\mathbb{Z}^+.$$

Setting $E_k = \left\{\omega\in\mathbb{R}^+ : p^{-k}\omega\in E\right\}$ and $\zeta = p^{-k}\omega$, we have

$$\begin{aligned}A_k(\ell) &= \int_{E_k}\prod_{j=1}^{k}\left|m_0\left(\frac{\omega}{p^j}\right)\right|^2\overline{\chi(\ell,\omega)}\,d\omega\\ &= p^k\int_E\left|m_0(\zeta)\right|^2\prod_{j=1}^{k-1}\left|m_0\left(p^j\omega\right)\right|^2\overline{\chi(\ell,p^k\omega)}\,d\zeta.\end{aligned} \tag{8.2.34}$$

Using the assumption $E\equiv[0,1)(\operatorname{mod}\mathbb{Z}^+)$, we get

$$A_k(\ell) = p^{k-1}\int_0^1\sum_{s=0}^{p-1}\left|m_0\left(\frac{\omega}{p}\oplus\frac{s}{p}\right)\right|^2\prod_{j=1}^{k-1}\left|m_0\left(p^{j-1}\omega\right)\right|^2\overline{\chi(\ell,p^{k-1}\omega)}\,d\omega,$$

and, in view of (8.2.22), we have

$$A_k(\ell) = p^{k-1}\int_0^1\prod_{j=0}^{k-2}\left|m_0\left(p^j\omega\right)\right|^2\overline{\chi(\ell,p^{k-1}\omega)}\,d\omega.$$

Hence, by (8.2.34),

$$A_k(\ell) = A_{k-1}(\ell).$$

When $k=1$, we similarly have

$$A_1(\ell) = p \int_0^1 \left|m_0(\omega)\right|^2 \overline{\chi(\ell, p\omega)}\, d\omega = \int_0^1 \overline{\chi(\ell,\omega)}\, d\omega = \delta_{0,\ell}.$$

Therefore

$$A_k(\ell) = \delta_{0,\ell}, \quad k \in \mathbb{N}, \ell \in \mathbb{Z}^+. \tag{8.2.35}$$

In particular, for all $k \in \mathbb{N}$,

$$A_k(0) = \int_{\mathbb{R}^+} \left|\eta_k(\omega)\right|^2 d\omega = 1.$$

Using (8.2.32) and Fatou's lemma, we obtain

$$\int_{\mathbb{R}^+} \left|\hat{\phi}(\omega)\right|^2 d\omega \le 1.$$

Using Lebesgues' dominated convergence theorem, we see from (8.2.32), (8.2.33) and (8.2.35) that

$$\int_{\mathbb{R}^+} \left|\hat{\phi}(\omega)\right|^2 \overline{\chi(\ell,\omega)}\, d\omega = \lim_{k\to\infty} A_k(\ell) = \delta_{0,\ell}.$$

Therefore,

$$\int_{\mathbb{R}^+} \phi(x)\, \overline{\phi(x \ominus \ell}\, dx = \delta_{0,\ell}, \quad \ell \in \mathbb{Z}^+.$$

By the Plancherel formula, it follows that the system $\left\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\right\}$ is orthonormal in $L^2(\mathbb{R}^+)$.

The converse part of this result follows on similar lines to that of Theorem 6.3.1 in Daubechies (1990).

We shall now deduce conditions for refinement equation (8.2.13) to have a stable solution. The next lemma gives a relation between stability and blocked sets.

Lemma 8.2.5. *Suppose that ϕ is a compactly supported L^2-solution of* (8.2.13) *such that $\hat{\phi}(0) = 1$. The function ϕ is not stable if and only if the mask m_0 of the refinement equation* (8.2.13) *has a blocked set.*

Proof. Applying Theorem 8.2.1 and Proposition 8.2.1, we obtain

$$\operatorname{supp}\phi \subset [0, p^{n-1}) \quad \text{and} \quad \hat{\phi} \in \mathcal{E}_{n-1}(\mathbb{R}^+).$$

Assume that the function ϕ is not stable, then by Proposition 8.2.2(iii), there exists an interval $I_{n-1}^s = I$ such that all points of the interval I are the periodic zeros of the Walsh–Fourier transform $\hat{\phi}$. Therefore, the set

$$B = \left\{ \omega \in [0, 1) : \hat{\phi}(\omega + k) = 0 \quad \text{for all } k \in \mathbb{Z}^+ \right\}$$

can be expressed as the union of some of the intervals $I_{n-1}^s, 0 \leq s \leq p^{n-1} - 1$. Since $\hat{\phi}(0) = 1$, it follows that B does not contain I_{n-1}^0. Besides, if $\omega \in B$, then, by formula (8.2.19), we have

$$m_0 \left(\frac{\omega}{p} + \frac{k}{p} \right) \hat{\phi} \left(\frac{\omega}{p} + \frac{k}{p} \right) = 0 \qquad \text{for all } k \in \mathbb{Z}^+$$

and, therefore, the elements $\omega/p + \ell/p, \ell = 0, 1, \ldots, p-1$, belong to either B or $\operatorname{Null} m_0$. Thus, if ϕ is not stable, then the set B is a blocked set for m_0.

Conversely, suppose that the mask m_0 has a blocked set B. Let us show that, in this case, each element from B is a periodic zero for $\hat{\phi}$ (and hence, by Proposition 8.2.2, the function ϕ is not stable).

Suppose, there exist an element $\omega \in B$ such that $\hat{\phi}(\omega + k) \neq 0$, for each $k \in \mathbb{Z}^+$. We choose a natural number j for which $p^{-j}(\omega + k) \in [0, p^{1-n}), k \in \mathbb{Z}^+$, and then, for each $r \in \{0, 1, \ldots, j\}$, we set

$$u_r = [p^{-r}(\omega + k)], \quad v_r = \{p^{-r}(\omega + k)\}.$$

Further, for each $r \in \{0, 1, \ldots, j-1\}$, we take $\ell_r \in \{0, 1, \ldots, p-1\}$ such that

$$u_{r+1} + v_{r_1} = \left(p^{-1} + p^{-1}\ell_r\right) + s_r$$

where $s_r \in \mathbb{Z}^+$ and $u_r/p = \ell_r/p + s_r$.

Therefore, $v_{r+1} = p^{-1}(v_r + \ell_r)$. It is readily seen that if $v_r \in B$, then $v_{r+1} \in T_p B$. Besides, the equalities

$$\hat{\phi}(\omega + k) = \hat{\phi}\left(p^{-j}\left(\omega + k\right)\right) \prod_{r=1}^{j} m_0\left(p^{-j}\left(\omega + k\right)\right) = \hat{\phi}(v_j) \prod_{r=1}^{j} m_0(v_r)$$

imply that all $v_r \neq \operatorname{Null} m_0$. Thus, if $v_r \in B$, then $v_{r+1} \in B$. Since $v_0 = \omega \in B$, this implies that all $v_j \in B$. This contradicts the fact that $v_j = p^{-j}(\omega + k) \in [0, p^{1-n})$ and $B \cap [0, p^{1-n}) = \emptyset$. This contradiction completes the proof of Lemma 8.2.5.

We now find out when solutions of refinement equation (8.2.13) generate p-MRA in $L^2(\mathbb{R}^+)$. We start with conditions for the integer translates of the solution of Eq. (8.2.13) to form an orthonormal basis of their linear span.

Lemma 8.2.6. *Suppose that ϕ is a compactly supported L^2-solution of* (8.2.13) *such that $\hat{\phi}(0) = 1$. The system $\left\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\right\}$ is orthonormal in $L^2(\mathbb{R}^+)$ if and only if the mask m_0 of* (8.2.13) *has no blocked sets and satisfies the condition*

$$\sum_{\ell=0}^{p-1} \left|m_0\left(\omega \oplus \ell/p\right)\right|^2 = 1 \qquad \textit{for all } \omega \in \mathbb{R}^+. \tag{8.2.36}$$

Proof. If the system $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ is orthonormal in $L^2(\mathbb{R}^+)$, then in view of (8.2.7) and (8.2.19), condition (8.2.36) holds, while Lemma 8.2.5 and Proposition 8.2.2 implies that there are no blocked sets.

Conversely, suppose that the mask m_0 has no blocked sets and condition (8.2.36) holds. We set

$$F(\omega) = \sum_{k\in\mathbb{Z}^+} \left|\hat{\phi}(\omega \ominus k)\right|^2. \tag{8.2.37}$$

Obviously, the function F is non-negative and 1-periodic function. By condition (8.2.7), it suffices to verify that $F(\omega) \equiv 1$. Let

$$\delta = \inf\{F(\omega) : \omega \in [0, 1)\}.$$

It follows from Theorem 8.2.1 and Proposition 8.2.1 that the function F (just as $\hat{\phi}$) is constant on the intervals $I^s_{n-1}, 0 \le s \le p^{n-1} - 1$. If F vanishes on one of these intervals, then the function $\hat{\phi}$ has a periodic zero, and hence ϕ is not stable. By Proposition 8.2.2 and Lemma 8.2.5, this contradicts the assumption that the mask m_0 has no blocked sets. Hence the number δ is positive. Besides, taking into account the modified Strang-Fix condition (see Theorem 8.2.1), we obtain $F(0) = 1$. Thus, $0 < \delta \le 1$.

Note that Eqs. (8.2.19) and (8.2.37) imply the relation

$$F(\omega) = \sum_{\ell=0}^{p-1} \left|m_0\left(\frac{\omega}{p} \ominus \frac{\ell}{p}\right)\right|^2 F\left(\frac{\omega}{p} \ominus \frac{\ell}{p}\right). \tag{8.2.38}$$

Now suppose that $M_\delta = \{F(\omega) = \delta : \omega \in [0, 1)\}$. If $0 < \delta < 1$, then (8.2.36) and (8.2.38) imply that, for any $\omega \in M_\delta$, the elements $p^{-1}\omega \ominus p^{-1}\ell, \ell = 0, 1, \ldots, p-1$, belong to either M_δ or $\operatorname{Null} m_0$. This means that the set M_δ is a blocked set, which contradicts the assumption. Thus, $F(\omega) \ge 1$ for all $\omega \in [0, 1)$. Combining this with the equalities

$$\int_0^1 F(\omega)\, d\omega = \sum_{k\in\mathbb{Z}^+} \int_k^{k+1} \left|\hat{\phi}(\omega)\right|^2 d\omega = \int_{\mathbb{R}^+} \left|\hat{\phi}(\omega)\right|^2 d\omega = \|\phi\|^2$$

we find by Lemma 8.2.1 that

$$\int_0^1 F(\omega)\, d\omega = 1.$$

Applying the inequality $F(\omega) \geq 1$ again and using the fact that the function F is constant on each $I_{n-1}^{s}, 0 \leq s \leq p^{n-1} - 1$, we find that $F(\omega) = 1$. This proves the Lemma 8.2.6 completely.

Proof of the Theorem 8.2.2. Suppose that the mask m_0 satisfies any of condition (ii) or (iii). Then it follows from Lemma 8.2.4 and Lemma 8.2.6 that the system $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ is orthonormal in $L^2(\mathbb{R}^+)$. We define the subspaces $V_j, j \in \mathbb{Z}$, by formula (8.2.10). The embedding's $V_j \subset V_{j+1}$ are a consequence of the fact that ϕ satisfies (8.2.13) while condition (iv) of the Definition 8.2.1 p-MRA is given by the orthonormality of the system $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$. The remaining two conditions (ii) and (iii) follows from the results of Lemma 8.2.2 and Lemma 8.2.3. Thus, the implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(i) are valid. The inverse implications follow directly from Lemma 8.2.4 and Lemma 8.2.6. ☐

Theorems 8.2.1 and 8.2.2 imply the following procedure for constructing orthogonal p-wavelets in $L^2(\mathbb{R}^+)$:

1. Choose numbers b_s, $0 \leq s \leq p^n - 1$ for which conditions (8.2.23) hold.
2. Using formula (8.2.16), calculate the coefficients $a_\alpha, 0 \leq \alpha \leq p^n - 1$ and verify that the mask m_0 defined by (8.2.14) has no blocked sets.
3. Find

$$m_\ell(\omega) = \sum_{\alpha=0}^{p^n-1} a_\alpha^\ell \overline{w_\alpha(\omega)}, \quad 1 \leq \ell \leq p-1,$$

   such that the matrix $\{m_\ell(\omega + k/p)\}_{\ell,k=0}^{p-1}$ is unitary.
4. Determine $\psi_1, \ldots, \psi_{p-1}$ using the formula

$$\psi_\ell(x) = p \sum_{\alpha=0}^{p^n-1} a_\alpha^\ell \, \phi(px \ominus \alpha), \quad 1 \leq \ell \leq p-1. \tag{8.2.39}$$

Let us present some examples of functions ϕ satisfying Eq. (8.2.13) and generating an p-MRA in $L^2(\mathbb{R}^+)$. Recall that $\mathbf{1}_E$ denotes the characteristic function of the set $E \subset \mathbb{R}^+$.

Example 8.2.1. If $a_0 = a_1 = \cdots = a_{p-1} = 1/p$ and all $a_0 = 0$ for $\alpha \geq p$, then the solution of Eq. (8.2.13) is the function $\phi = \mathbf{1}_{[0,p^{n-1})}$; in particular, for $n = 1$,(8.2.13) satisfies the Haar function given by $\phi = \mathbf{1}_{[0,1)}$.

Example 8.2.2. Suppose that $p = n = 2$ and

$$b_0 = 1, \quad b_1 = a, \quad b_2 = 0, \quad b_3 = b,$$

where $|a|^2 + |b|^2 = 1$. Then the function ϕ satisfies the equation

$$\phi(x) = p \sum_{k=0}^{3} a_k \phi(2x \ominus k)$$

with coefficients a_k's given by (8.2.16) as

$$a_0 = \frac{1+a+b}{4}, \quad a_1 = \frac{1+a-b}{4}, \quad a_2 = \frac{1-a-b}{4}, \quad a_3 = \frac{1-a+b}{4}.$$

For $a \neq 0$, the modified Cohen condition holds on the set $E = [0,1)$ and the corresponding solution ϕ generates a MRA in $L^2(\mathbb{R}^+)$. In particular, for $a = 1$ and $a = -1$ the Haar function: $\phi(x) = \mathbf{1}_{[0,1)}(x)$ and the displaced Haar function: $\phi(x) = \mathbf{1}_{[0,1)}(x \ominus 1)$ are obtained.

Further, if $0 < |a| < 1$, then ϕ generates MRA in $L^2(\mathbb{R}^+)$ and possesses the following self-similarity property:

$$\phi(x) = \begin{cases} \dfrac{1+a-b}{2} + b\phi(2x), & 0 \le x < 1, \\ \dfrac{1-a+b}{2} - b\phi(2x), & 1 \le x \le 2, \end{cases}$$

and is represented by a lacunary Walsh series:

$$\phi(x) = \frac{1}{2}\mathbf{1}_{[0,1)}\left(\frac{x}{2}\right)\left(1 + a\sum_{j=0}^{\infty} b^j w_{2^{j+1}-1}\left(\frac{x}{2}\right)\right), \quad x \in \mathbb{R}^+.$$

Also, in case $a = 0$, the function ϕ is defined by the formula $\phi(x) = (1/2)\mathbf{1}_{[0,1)}(x/2)$, and the system $\{\phi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ is linearly dependent $\big($because $\phi(x \ominus 1) = \phi(x)\big)$.

Example 8.2.3. Suppose that $p = 3, n = 2$, and

$$b_0 = 1,\ b_1 = a,\ b_2 = \alpha,\ b_3 = 0,\ b_4 = b,\ b_5 = \beta,\ b_6 = 0,\ b_7 = c,\ b_8 = \gamma,$$

where

$$|a|^2 + |b|^2 + |c|^2 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1.$$

By (8.2.16), the coefficients of Eq. (8.2.13) in the case under consideration can be calculated by the formulas

$$a_0 = \frac{1}{9}\,(1 + a + b + c + \alpha + \beta + \gamma)\,,$$

$$a_1 = \frac{1}{9}\left(1 + a + \alpha + (b + \beta)\varepsilon_3^2 + (c + \gamma)\varepsilon_3\right),$$

$$a_2 = \frac{1}{9}\left(1 + a + \alpha + (b + \beta)\varepsilon_3 + (c + \gamma)\varepsilon_3^2\right),$$

$$a_3 = \frac{1}{9}\left(1 + (a + b + c)\varepsilon_3^2 + (\alpha + \beta + \gamma)\varepsilon_3\right),$$

$$a_4 = \frac{1}{9}\left(1 + c + \beta + (a + \gamma)\varepsilon_3^2 + (b + \alpha)\varepsilon_3\right),$$

$$a_5 = \frac{1}{9}\left(1 + b + \gamma + (a + \beta)\varepsilon_3^2 + (c + \alpha)\varepsilon_3\right),$$

$$a_6 = \frac{1}{9}\left(1 + (a + b + c)\varepsilon_3 + (\alpha + \beta + \gamma)\varepsilon_3^2\right),$$

$$a_7 = \frac{1}{9}\left(1 + b + \gamma + (a + \beta)\varepsilon_3 + (c + \alpha)\varepsilon_3^2\right),$$

$$a_8 = \frac{1}{9}\left(1 + c + \beta + (a + \gamma)\varepsilon_3 + (b + \alpha)\varepsilon_3^2\right),$$

where $\varepsilon_3 = \exp(2\pi i/3)$. For the corresponding mask m_0, the blocked sets are:

1. $B_1 = \left[\frac{1}{3}, \frac{2}{3}\right)$ for $a = c = 0$,
2. $B_2 = \left[\frac{2}{3}, 1\right)$ for $\alpha = \beta = 0$,
3. $B_3 = \left[\frac{1}{3}, 1\right)$ for $a = \alpha = 0$.

Suppose that

$$\gamma(1,0) = a,\ \gamma(2,0) = \alpha,\ \gamma(1,1) = b,\ \gamma(2,1) = \beta,\ \gamma(1,2) = c,\ \gamma(2,2) = \gamma,$$

and $v_j \in \{1,2\}$, then we set

$$d_\ell = \gamma(v_0, 0) \quad \text{for } \ell = v_0;$$

$$d_\ell = \gamma(v_1, 0)\gamma(v_0, v_1) \quad \text{for } \ell = v_0 + 3v_1;$$

$$\vdots$$

$$d_\ell = \gamma(v_k, 0)\gamma(v_{k-1}, v_k)\ldots\gamma(v_0, v_1) \quad \text{for } \ell = \sum_{j=0}^{k} v_j 3^j,\ k \geq 2.$$

The solution of Eq. (8.2.13) can be expressed (see Farkov 2005a,b) as the series

$$\phi(x) = \frac{1}{3}\mathbf{1}_{[0,1)}\left(\frac{x}{3}\right)\left(1 + \sum_{\ell} d_\ell w_\ell\left(\frac{x}{3}\right)\right), \quad x \in \mathbb{R}^+. \tag{8.2.40}$$

Taking into account the expressions for the blocked sets given above and using Theorem 8.2.2, we find that the function (8.2.40) generates a MRA in $L^2(\mathbb{R}^+)$ in the following three cases:

1. $a \neq 0, \alpha \neq 0$;
2. $a = 0, \alpha \neq 0, c \neq 0$;
3. $\alpha = 0, a \neq 0, \beta \neq 0$.

Example 8.2.4. Suppose that, for some numbers $b_s, 0 \leq s \leq p^n - 1$, relations (8.2.23) hold. Applying formulas (8.2.16), we find the coefficients of the mask m_0 as defined by (8.2.14) taking the values b_s on the intervals $I_n^s, 0 \leq s \leq p^n - 1$. If, additionally, it is known that $b_j \neq 0$ for $j \in \{1, 2, \ldots, p^{n-1} - 1\}$, then Eq. (8.2.13) with the obtained coefficients a_α has a solution generating an p-MRA in $L^2(\mathbb{R}^+)$ (the modified Cohen condition holds for $E = [0, 1)$).

8.3 Nonuniform MRA

The previous concepts of MRA are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed (1998a,b) considered a generalization of Mallat's celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer relatively prime to N. They call this a *NUMRA*.

In this theory, the translation set Λ is chosen so that for some measurable set $A \subset \mathbb{R}$ with $0 < |A| < \infty$, (A, Λ) forms a spectral pair, i.e., the collection $\{A^{-1/2} e^{2\pi i \omega \cdot \lambda} \chi_A(\omega)\}_{\lambda \in \Lambda}$ forms an orthonormal basis for $L^2(A)$, where $\chi_A(\omega)$ is the characteristic function of A. The notion of spectral pairs was introduced by Fuglede (1974). The following proposition is proved in Gabardo and Nashed (1998a,b).

Proposition 8.3.1. *Let* $\Lambda = \{0, a\} + 2\mathbb{Z}$, *where* $0 < a < 2$ *and let* A *be a measurable subset of* $\mathbb{R}$ *with* $0 < |A| < \infty$. *Then* (A, Λ) *is a spectral pair if and only if there exist an integer* $N \geq 1$ *and an odd integer* r, *with* $1 \leq r \leq 2N - 1$ *and* r *and* N *relatively prime, such that* $a = r/N$, *and*

$$\sum_{j=0}^{N-1} \delta_{j/2} * \sum_{n\in\mathbb{Z}} \delta_{nN} * \chi_A = 1, \tag{8.3.1}$$

where $*$ denotes the usual convolution product of Schwartz distributions and δ_c is the Dirac measure at c.

Following is the definition of nonuniform MRA associated with the translation set Λ on $\mathbb{R}$ introduced by Gabardo and Nashed (1998a,b).

Definition 8.3.1. Let N be an integer, $N \geq 1$, and $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where r is an odd integer relatively prime to N with $1 \leq r \leq 2N-1$. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ will be called a *NUMRA* associated with Λ if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iii) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(iv) There exists a function ϕ in V_0, called the *scaling function*, such that the collection $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ is a complete orthonormal system for V_0.

It is worth noticing that, when $N = 1$, one recovers from the definition above the standard definition of a one-dimensional MRA with dilation factor equal to 2. When, $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset 2\mathbb{Z} \subset \Lambda$. However, the existence of associated wavelets with the dilation $2N$ and translation set Λ is no longer guaranteed as is the case in the standard setting.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_k \perp W_\ell \quad \text{if } k \neq \ell. \tag{8.3.2}$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{k=0}^{j-J-1} W_{j-k}, \tag{8.3.3}$$

where all these subspaces are orthogonal. By virtue of condition (ii) in the Definition 8.3.1, this implies

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j, \tag{8.3.4}$$

a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces.

Observe that the dilation factor in the NUMRA is $2N$. As in the standard case, one expects the existence of $2N-1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $2N$ form an orthonormal basis for $L^2(\mathbb{R})$.

A set of functions $\{\psi_1, \psi_1, \ldots, \psi_{2N-1}\}$ in $L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the NUMRA $\{V_j\}$ if the family of functions $\{\psi_\ell(\cdot - \lambda) : 1 \le \ell \le 2N-1, \lambda \in \Lambda\}$ forms an orthonormal basis for W_0.

In the following, our task is to find a set of wavelet functions $\{\psi_1, \psi_1, \ldots, \psi_{2N-1}\}$ in W_0 such that $\{(2N)^{j/2}\psi_\ell((2N)^j x - \lambda) : 1 \le \ell \le 2N-1, \lambda \in \Lambda\}$ constitutes an orthonormal basis of W_j. By means of NUMRA, this task can be reduce to find $\psi_\ell \in W_0$ such that $\{\psi_\ell(x-\lambda) : 1 \le \ell \le 2N-1, \lambda \in \Lambda\}$ constitutes an orthonormal basis of W_0.

Let ϕ be a scaling function of the given NUMRA. Since $\phi \in V_0 \subset V_1$, and the $\{\phi_{1,\lambda}\}_{\lambda\in\Lambda}$ is an orthonormal basis in V_1, we have

$$\phi(x) = \sum_{\lambda\in\Lambda} a_\lambda \phi_{1,\lambda}(x) = \sum_{\lambda\in\Lambda} a_\lambda (2N)^{1/2}\phi\big((2N)x - \lambda\big), \tag{8.3.5}$$

with

$$a_\lambda = \langle \phi, \phi_{1,\lambda}\rangle = \int_{\mathbb{R}} \phi(x)\,\overline{\phi_{1,\lambda}(x)}\,dx \quad \text{and} \quad \sum_{\lambda\in\Lambda} |a_\lambda|^2 < \infty. \tag{8.3.6}$$

Equation (8.3.5) can be written in frequency domain as

$$\hat{\phi}\,(2N\omega) = m_0(\omega)\,\hat{\phi}(\omega), \tag{8.3.7}$$

where $m_0(\omega) = \sum_{\lambda\in\Lambda} a_\lambda e^{-2\pi i \lambda\cdot\omega}$, is called the *symbol* of $\phi(x)$.

We denote $\psi_0 = \phi$, the scaling function, and consider $2N-1$ functions ψ_ℓ, $1 \le \ell \le 2N-1$, in W_0 as possible candidates for wavelets. Since $(1/2N)\psi_\ell(x/2N) \in V_{-1} \subset V_0$, it follows from property (iv) of Definition 8.3.1 that for each ℓ, $0 \le \ell \le 2N-1$, there exists a sequence $\{a_\lambda^\ell : \lambda \in \Lambda\}$ with $\sum_{\lambda\in\Lambda} \left|a_\lambda^\ell\right|^2 < \infty$ such that

$$\frac{1}{2N}\psi_\ell\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} a_\lambda^\ell\, \varphi(x-\lambda). \tag{8.3.8}$$

Taking Fourier transform, we get

$$\hat{\psi}_\ell\,(2N\omega) = m_\ell(\omega)\,\hat{\phi}(\omega), \tag{8.3.9}$$

where

$$m_\ell(\omega) = \sum_{\lambda\in\Lambda} a_\lambda^\ell\, e^{-2\pi i \lambda\omega}. \tag{8.3.10}$$

The functions $m_\ell, 0 \le \ell \le 2N-1$, are locally L^2 functions. In view of the specific form of Λ, we observe that

$$m_\ell(\omega) = m_\ell^1(\omega) + e^{-2\pi i r\omega/N}\, m_\ell^2(\omega), \quad 0 \le \ell \le 2N-1, \tag{8.3.11}$$

where m_ℓ^1 and m_ℓ^2 are locally L^2, $1/2$-periodic functions.

We are now in a position to establish the completeness of the system $\{\psi_\ell(x-\lambda)\}_{1\le\ell\le 2N-1,\lambda\in\Lambda}$ in V_1 and in fact, we will find two equivalent conditions to the orthonormality of the system by means of the periodic functions m_ℓ as defined in (8.3.11).

Lemma 8.3.1. *Let ϕ be a scaling function of the given NUMRA as in Definition 8.3.1. Suppose that there exist $2N-1$ functions ψ_ℓ, $1 \le \ell \le 2N-1$, in V_1 such that the family of functions $\{\psi_\ell(x-\lambda)\}_{0\le\ell\le 2N-1,\lambda\in\Lambda}$ forms an orthonormal system in V_1. Then the system is complete in V_1.*

Proof. By the orthonormality of $\psi_\ell \in L^2(\mathbb{R}), 0 \le \ell \le 2N-1$, we have in the time domain

$$\left\langle \psi_k(x-\lambda), \psi_\ell(x-\sigma)\right\rangle = \int_{\mathbb{R}} \psi_k(x-\lambda)\,\overline{\psi_\ell(x-\sigma)}\,dx = \delta_{k,\ell}\delta_{\lambda,\sigma},$$

where $\lambda,\sigma \in \Lambda$ and $k,\ell \in \{0,1,2,\ldots,2N-1\}$. Equivalently, in the frequency domain, we have

$$\delta_{k,\ell}\delta_{\lambda,\sigma} = \int_{\mathbb{R}} \hat{\psi}_k(\omega)\,\overline{\hat{\psi}_\ell(\omega)}\,e^{-2\pi i\omega(\lambda-\sigma)}d\omega.$$

Taking $\lambda = 2m, \sigma = 2n$ where $m,n \in \mathbb{Z}$, we have

$$\begin{aligned}\delta_{k,\ell}\delta_{m,n} &= \int_{\mathbb{R}} \hat{\psi}_k(\omega)\,\overline{\hat{\psi}_\ell(\omega)}\,e^{-2\pi i\omega 2(m-n)}d\omega \\ &= \int_{[0,N)} e^{-4\pi i\omega(m-n)} \sum_{j\in\mathbb{Z}} \hat{\psi}_k(\omega+Nj)\,\overline{\hat{\psi}_\ell(\omega+Nj)}\,d\omega.\end{aligned}$$

Let

$$h_{k,\ell}(\omega) = \sum_{j\in\mathbb{Z}} \hat{\psi}_k(\omega+Nj)\,\overline{\hat{\psi}_\ell(\omega+Nj)}.$$

Then, we have

$$\begin{aligned}\delta_{k,\ell}\delta_{m,n} &= \int_{[0,N)} e^{-4\pi i\omega(m-n)} h_{k,\ell}(\omega)\, d\omega \\ &= \int_{[0,1/2)} e^{-4\pi i\omega(m-n)} \left[\sum_{p=0}^{2N-1} h_{k,\ell}\left(\omega + \frac{p}{2}\right)\right] d\omega,\end{aligned}$$

and

$$\sum_{p=0}^{2N-1} h_{k,\ell}\left(\omega + \frac{p}{2}\right) = 2\delta_{k,\ell}. \tag{8.3.12}$$

Also on taking $\lambda = \dfrac{r}{N} + 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}0 &= \int_{\mathbb{R}} e^{-4\pi i\omega(m-n)}\, e^{-2\pi i\omega\cdot r/N}\, \hat{\psi}_k(\omega)\, \overline{\hat{\psi}_\ell(\omega)}\, d\omega \\ &= \int_{[0,N)} e^{-4\pi i\omega(m-n)}\, e^{-2\pi i\omega\cdot r/N} \sum_{j\in\mathbb{Z}} \hat{\psi}_k(\omega + Nj)\, \overline{\hat{\psi}_\ell(\omega + Nj)}\, d\omega \\ &= \int_{[0,N)} e^{-4\pi i\omega(m-n)}\, e^{-2\pi i\omega\cdot r/N} h_{k,\ell}(\omega)\, d\omega \\ &= \int_{[0,1/2)} e^{-4\pi i\omega(m-n)}\, e^{-2\pi i\omega\cdot r/N} \left[\sum_{p=0}^{2N-1} e^{-\pi i pr/N}\, h_{k,\ell}\left(\omega + \frac{p}{2}\right)\right] d\omega.\end{aligned}$$

Thus, we conclude that

$$\sum_{p=0}^{2N-1} \alpha^p\, h_{k,\ell}\left(\omega + \frac{p}{2}\right) = 0, \qquad \text{where } \alpha = e^{-\pi i r/N}. \tag{8.3.13}$$

Now we will express the conditions (8.3.12) and (8.3.13) in terms of m_ℓ as follows:

$$
\begin{aligned}
h_{k,\ell}(2N\omega) &= \sum_{j\in\mathbb{Z}} \hat{\psi}_k\left(2N\left(\omega+\frac{j}{2}\right)\right)\overline{\hat{\psi}_\ell\left(2N\left(\omega+\frac{j}{2}\right)\right)} \\
&= \sum_{j\in\mathbb{Z}} m_k\left(\omega+\frac{j}{2}\right)\hat{\phi}\left(\omega+\frac{j}{2}\right)\overline{m_\ell\left(\omega+\frac{j}{2}\right)}\,\overline{\hat{\phi}\left(\omega+\frac{j}{2}\right)} \\
&= \sum_{j\in\mathbb{Z}} m_k\left(\omega+\frac{j}{2}\right)\overline{m_\ell\left(\omega+\frac{j}{2}\right)}\left|\hat{\phi}\left(\omega+\frac{j}{2}\right)\right|^2 \\
&= \left[m_k^1(\omega)\,\overline{m_\ell^1(\omega)} + m_k^2(\omega)\overline{m_\ell^2(\omega)}\right]\sum_{j\in\mathbb{Z}}\left|\hat{\phi}\left(\omega+\frac{j}{2}\right)\right|^2 \\
&\quad + \left[m_k^1(\omega)\,\overline{m_\ell^2(\omega)}\sum_{j\in\mathbb{Z}} e^{2\pi i(\omega+j/2)r/N}\left|\hat{\phi}\left(\omega+\frac{j}{2}\right)\right|^2\right] \\
&\quad + \left[m_k^2(\omega)\,\overline{m_\ell^2(\omega)}\sum_{j\in\mathbb{Z}} e^{-2\pi i(\omega+j/2)r/N}\left|\hat{\phi}\left(\omega+\frac{j}{2}\right)\right|^2\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
h_{k,\ell}(2N\omega) &= \left[m_k^1(\omega)\overline{m_\ell^1(\omega)} + m_k^2(\omega)\overline{m_\ell^2(\omega)}\right]\sum_{j=0}^{2N-1} h_{0,0}\left(\omega+\frac{j}{2}\right) \\
&\quad + \left[m_k^1(\omega)\overline{m_\ell^2(\omega)}\,e^{2\pi i\omega r/N}\sum_{j=0}^{2N-1}\alpha^{-j}h_{0,0}\left(\omega+\frac{j}{2}\right)\right] \\
&\quad + \left[m_k^2(\omega)\overline{m_\ell^2(\omega)}\,e^{-2\pi i\omega r/N}\sum_{j=0}^{2N-1}\alpha^{j}h_{0,0}\left(\omega+\frac{j}{2}\right)\right] \\
&= 2\left[m_k^1(\omega)\overline{m_\ell^1(\omega)} + m_k^2(\omega)\overline{m_\ell^2(\omega)}\right].
\end{aligned}
$$

By using the last identity and Eqs. (8.3.12) and (8.3.13), we obtain

$$
\sum_{p=0}^{2N-1}\left[m_k^1\left(\omega+\frac{p}{4N}\right)\overline{m_\ell^1\left(\omega+\frac{p}{4N}\right)} + m_k^2\left(\omega+\frac{p}{4N}\right)\overline{m_\ell^2\left(\omega+\frac{p}{4N}\right)}\right] = \delta_{k,\ell}, \tag{8.3.14}
$$

and

$$\sum_{p=0}^{2N-1} \alpha^p \left[m_k^1\left(\omega + \frac{p}{4N}\right) \overline{m_\ell^1\left(\omega + \frac{p}{4N}\right)} + m_k^2\left(\omega + \frac{p}{4N}\right) \overline{m_\ell^2\left(\omega + \frac{p}{4N}\right)} \right] = 0, \tag{8.3.15}$$

for $0 \le k, \ell \le 2N-1$, where $\alpha = e^{-\pi i r/N}$.

Both of these conditions together are equivalent to the orthonormality of the system $\{\psi_\ell(x-\lambda) : 0 \le \ell \le 2N-1, \lambda \in \Lambda\}$. The completeness of this system in V_1 is equivalent to the completeness of the system $\left\{\frac{1}{2N}\psi_\ell\big((x/2N)-\lambda\big) : 0 \le \ell \le 2N-1, \lambda \in \Lambda\right\}$ in V_0. For a given arbitrary function $f \in V_0$, by assumption, there exist a unique function $m(\omega)$ of the form $\sum_{\lambda\in\Lambda} b_\lambda e^{-2\pi i\lambda\omega}$, where $\sum_{\lambda\in\Lambda} |b_\lambda|^2 < \infty$ such that $\hat{f}(\omega) = m(\omega)\,\hat{\phi}(\omega)$. Therefore, in order to prove the claim, it is enough to show that the system of functions

$$\mathcal{P} = \left\{e^{-4\pi i N\omega\lambda} m_\ell(\omega)\,\chi_A(\omega) : 0 \le \ell \le 2N-1, \lambda \in \Lambda\right\}$$

is complete in $L^2(A)$, where $A \subset \mathbb{R}$ with $0 < |A| < \infty$. Since the collection $\left\{e^{2\pi i\omega\lambda}\,\chi_A(\omega)\right\}_{\lambda\in\Lambda}$ is an orthonormal basis for $L^2(A)$, therefore there exist locally L^2 functions g_1 and g_2 such that

$$g(\omega) = \left[g_1(\omega) + e^{-2\pi i\omega r/N} g_2(\omega)\right]\chi_A(\omega).$$

Assuming that g is orthogonal to all functions in $\mathcal{P}$, we then have for any $\lambda \in \Lambda$ and $\ell \in \{0, 1, \ldots, 2N-1\}$, that

$$\begin{aligned} 0 &= \int_A e^{-4\pi i N\omega\lambda} m_\ell(\omega)\,\overline{g(\omega)}\,d\omega \\ &= \int_{[0,1/2)} e^{-4\pi i N\omega\lambda} \left[m_\ell(\omega)\,\overline{g(\omega)} + m_\ell(\omega + N/2)\,\overline{g(\omega + N/2)}\right] d\omega \\ &= \int_{[0,1/2)} e^{-4\pi i N\omega\lambda} \left[m_\ell^1(\omega)\,\overline{g_1(\omega)} + m_\ell^2(\omega)\,\overline{g_2(\omega)}\right] d\omega. \end{aligned} \tag{8.3.16}$$

Taking $\lambda = 2m$, where $m \in \mathbb{Z}$ and defining

$$w_\ell(\omega) = m_\ell^1(\omega)\,\overline{g_1(\omega)} + m_\ell^2(\omega)\,\overline{g_2(\omega)}, \quad 0 \le \ell \le 2N-1,$$

we obtain

$$\begin{aligned} 0 &= \int_{[0,1/2)} e^{-2\pi i\omega(4N)m} w_\ell(\omega)\,d\omega \\ &= \int_{[0,1/4N)} e^{-2\pi i\omega(4N)m} \sum_{j=0}^{2N-1} w_\ell\left(\omega + \frac{j}{4N}\right) d\omega. \end{aligned}$$

Since this equality holds for all $m \in \mathbb{Z}$, therefore

$$\sum_{j=0}^{2N-1} w_\ell \left(\omega + \frac{j}{4N}\right) = 0 \quad \text{for a.e. } \omega \tag{8.3.17}$$

Similarly, on taking $\lambda = 2m + r/N$, where $m \in \mathbb{Z}$, we obtain

$$\begin{aligned} 0 &= \int_{[0,1/2)} e^{-2\pi i \omega(4N)m} e^{-2\pi i 2r\omega} w_\ell(\omega)\, d\omega \\ &= \int_{[0,1/4N)} e^{-2\pi i \omega(4N)m} e^{-2\pi i 2r\omega} \sum_{j=0}^{2N-1} \alpha^j w_\ell \left(\omega + \frac{j}{4N}\right) d\omega. \end{aligned}$$

Hence, we deduce that

$$\sum_{j=0}^{2N-1} \alpha^j w_\ell \left(\omega + \frac{j}{4N}\right) = 0 \quad \text{for a.e. } \omega,$$

which proves our claim.

If $\psi_0, \psi_1, \ldots, \psi_{2N-1} \in V_1$ are as in Lemma 8.3.1, one can obtain from them an orthonormal basis for $L^2(\mathbb{R})$ by following the standard procedure for construction of wavelets from a given MRA (see Chap. 7). It can be easily checked that for every $j \in \mathbb{Z}$, the collection $\left\{(2N)^{j/2}\psi_\ell\big((2N)^j x - \lambda\big) : 0 \le \ell \le 2N-1, \lambda \in \Lambda\right\}$ is a complete orthonormal system for V_{j+1}. Therefore, it follows immediately from (8.3.4) that the collection $\left\{(2N)^{j/2}\psi_\ell\big((2N)^j x - \lambda\big) : 1 \le \ell \le 2N-1,\right.$ $\left.\lambda \in \Lambda\right\}$ forms a complete orthonormal system for $L^2(\mathbb{R})$.

The following theorem proves the necessary and sufficient condition for the existence of associated set of wavelets to NUMRA.

Theorem 8.3.1. *Consider a NUMRA with associated parameters N and r as in Definition 8.3.1, such that the corresponding space V_0 has an orthonormal system of the form $\{\phi(x-\lambda) : \lambda \in \Lambda\}$, where $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ and $\hat{\phi}$ satisfies the two scale relation*

$$\hat{\phi}\,(2N\omega) = m_0(\omega)\,\hat{\phi}(\omega), \tag{8.3.18}$$

where m_0 is of the form

$$m_0(\omega) = m_0^1(\omega) + e^{-2\pi i \omega r/N} m_0^2(\omega), \tag{8.3.19}$$

for some locally L^2 functions m_0^1 and m_0^2. Define M_0 as

$$M_0(\omega) = \left|m_0^1(\omega)\right|^2 + \left|m_0^2(\omega)\right|^2. \tag{8.3.20}$$

Then a necessary and sufficient condition for the existence of associated wavelets $\psi_1, \ldots, \psi_{2N-1}$ *is that* M_0 *satisfies the identity*

$$M_0\left(\omega + \frac{1}{4}\right) = M_0(\omega). \tag{8.3.21}$$

Proof. The orthonormality of the collection of functions $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ which satisfies (8.3.18), implies the following identities as shown in the proof of Lemma 8.3.1

$$\sum_{p=0}^{2N-1}\left[\left|m_0^1\left(\omega + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\omega + \frac{p}{4N}\right)\right|^2\right] = 1, \tag{8.3.22}$$

and

$$\sum_{p=0}^{2N-1}\alpha^p\left[\left|m_0^1\left(\omega + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\omega + \frac{p}{4N}\right)\right|^2\right] = 0, \tag{8.3.23}$$

where $\alpha = e^{-\pi i r/N}$. Similarly, if $\{\psi_\ell\}_{\ell=1,\ldots,2N-1}$ is a set of wavelets associated with the given NUMRA then it satisfies the relation (8.3.9) and the orthonormality of the collection $\{\psi_\ell\}_{\ell=0,1,\ldots,2N-1}$ in V_1 is equivalent to the identities

$$\sum_{p=0}^{2N-1}\left[m_k^1\left(\omega + \frac{p}{4N}\right)\overline{m_\ell^1\left(\omega + \frac{p}{4N}\right)} + m_k^2\left(\omega + \frac{p}{4N}\right)\overline{m_\ell^2\left(\omega + \frac{p}{4N}\right)}\right] = \delta_{k,\ell}, \tag{8.3.24}$$

and

$$\sum_{p=0}^{2N-1}\alpha^p\left[m_k^1\left(\omega + \frac{p}{4N}\right)\overline{m_\ell^1\left(\omega + \frac{p}{4N}\right)} + m_k^2\left(\omega + \frac{p}{4N}\right)\overline{m_\ell^2\left(\omega + \frac{p}{4N}\right)}\right] = 0, \tag{8.3.25}$$

for $0 \le k, \ell \le 2N-1$.

If $\omega \in [0, 1/4N]$ is fixed and $a_\ell(p) = m_\ell^1\left(\omega + \frac{p}{4N}\right)$, $b_\ell(p) = m_\ell^2\left(\omega + \frac{p}{4N}\right)$ are vectors in $\mathbb{C}^{2N}$ for $p = 0, 1, \ldots, 2N-1$, where $0 \le \ell \le 2N-1$, then the solvability of system of Eqs. (8.3.24) and (8.3.25) is equivalent to

$$M_0\left(\omega + \frac{(p+N)}{4N}\right) = M_0\left(\omega + \frac{p}{4N}\right), \quad \omega \in [0, 1/4N],\ p = 0, 1, \ldots, 2N-1,$$

which is equivalent to (8.3.21). For the proof of this result, the reader is refer to Gabardo and Nashed (1998a,b).

We note here that the function M_0 in the above theorem can also be written in terms of the filter m_0 as

$$M_0(\omega) = \frac{\left[\left|m_0\left(\omega + \frac{N}{2}\right)\right|^2 + |m_0(\omega)|^2\right]}{2}.$$

When $N = 1$, we have $r = 1$ and $\alpha = -1$ so that Eqs. (8.3.22) and (8.3.23) reduces to $M_0(\omega) = 1/2$, or the more familiar quadrature mirror filter condition from wavelet analysis $|m_0(\omega + 1/2)|^2 + |m_0(\omega)|^2 = 1$, and, in particular, M_0 is automatically $1/4$-periodic. When $N = 2$, we must have $r = 1$ or 3, so that $\alpha = \pm i$. In that case, the $1/4$-periodicity of M_0 follows again automatically from (8.3.22) and (8.3.23). When $N \geq 3$, we note that the conditions (8.3.22) and (8.3.23) do not imply the $1/4$-periodicity of the function M_0 (see Gabardo and Nashed 1998a,b).

Example 8.3.1 (Haar NUMRA). If we take $r = 1$, then $\Lambda = \{0, 1/N\} + 2\mathbb{Z}$ and choosing $\phi = \chi_{A_N}$, where

$$A_N = \bigcup_{j=0}^{N-1} \left[\frac{(2j)}{N}, \frac{(2j+1)}{N}\right),$$

we have

$$\phi = \chi_{[0,1/N)} * \sum_{j=0}^{N-1} \delta_{2j/N}.$$

We now define V_0 as the closed linear span of $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$, i.e., $V_0 = \overline{\text{span}}\,\{\phi(x-\lambda) : \lambda \in \Lambda\}$ and V_j, for each integer j, by the relation $f(x) \in V_j$ if and only if $f\left(x/(2N)^j\right) \in V_0$. Then, the condition (i) of the Definition 8.3.1 is verified by fact that

$$\begin{aligned}\frac{1}{2N}\,\phi\left(\frac{x}{2N}\right) &= \chi_{[0,2)} * \frac{1}{2N}\sum_{j=0}^{N-1} \delta_{4j}\\ &= \left(\delta_0 + \delta_{1/N}\right) * \phi * \frac{1}{2N}\sum_{j=0}^{N-1} \delta_{4j}.\end{aligned} \tag{8.3.26}$$

Equation (8.3.25) can be written in the frequency domain as

$$\hat{\phi}(2N\omega) = m_0(\omega)\,\hat{\phi}(\omega), \tag{8.3.27}$$

where

$$m_0(\omega) = \frac{1}{2N}\left(1 + e^{-2\pi i\omega/N}\right)\left[\sum_{k=0}^{N-1} e^{-8\pi i\omega k}\right].$$

Furthermore, we have

$$m_0^1(\omega) = m_0^2(\omega) = \frac{1}{2N}\sum_{k=0}^{N-1} e^{-8\pi i\omega k}. \tag{8.3.28}$$

Here, both the functions m_0^1 and m_0^2 are $1/4$-periodic and so is M_0. Therefore, Theorem 8.3.1 can be applied to show the existence of the associated wavelets. Hence, when $N = 1$, $\phi = \chi_{[0,1)}$ and $m_0^1(\omega) = m_0^2(\omega) = 1/2$, then the corresponding wavelet ψ_1 is given by the identity

$$\hat{\psi}_1(2\omega) = \frac{e^{-2\pi i\omega} - 1}{2}\,\hat{\phi}(\omega).$$

Or, equivalently

$$\psi_1 = -\chi_{[0,1/2)} + \chi_{[1/2,1)},$$

which is the classical *Haar wavelet*. For $N = 2$, the periodic function m_0^1 and m_0^2 are given by

$$m_0^1(\omega) = m_0^2(\omega) = \frac{e^{-4\pi i\omega}\cos(4\pi\omega)}{2},$$

and thus $M_0(\omega) = \cos^2(4\pi i\omega)/2$. In this case, the associated wavelets can easily be computed using the relation $\hat{\psi}_\ell(4\omega) = m_\ell(\omega)\,\hat{\phi}(\omega),\ \ell = 1, 2, 3$. Therefore, we have

$$\begin{aligned}
\psi_1 &= \chi_{[0,1/2)} - \chi_{[1,3/2)},\\
\psi_2 &= -\chi_{[-8/8,-7/8)} + \chi_{[-7/8,-6/8)} - \chi_{[-6/8,-5/8)} + \chi_{[-5/8,-4/8)}\\
&\quad -\chi_{[0,1/8)} + \chi_{[1/8,2/8)} - \chi_{[2/8,3/8)} + \chi_{[3/8,4/8)},\\
\psi_3 &= -\chi_{[-8/8,-7/8)} + \chi_{[-7/8,-6/8)} - \chi_{[-6/8,-5/8)} + \chi_{[-5/8,-4/8)}\\
&\quad +\chi_{[0,1/8)} - \chi_{[1/8,2/8)} + \chi_{[2/8,3/8)} - \chi_{[3/5,4/8)}.
\end{aligned}$$