Chapter 7
Multiresolution Analysis and Construction of Wavelets

Multiresolution analysis provides a natural framework for the understanding of wavelet bases, and for the construction of new examples. The history of the formulation of multiresolution analysis is a beautiful example of applications stimulating theoretical development.

Ingrid Daubechies

7.1 Introduction

The concept of multiresolution is intuitively related to the study of signals or images at different levels of resolution—almost like a pyramid. The resolution of a signal is a qualitative description associated with its frequency content. For a low-pass signal, the lower its frequency content (the narrower the bandwidth), the coarser is its resolution. In signal processing, a low-pass and subsampled version of a signal is usually a good coarse approximation for many real world signals. Multiresolution is especially evident in image processing and computer vision, where coarse versions of an image are often used as a first approximation in computational algorithms.

In 1986, Stéphane Mallat and Yves Meyer first formulated the idea of multiresolution analysis (MRA) in the context of wavelet analysis. This is a new and remarkable idea which deals with a general formalism for construction of an orthogonal basis of wavelets. Indeed, MRA is central to all constructions of wavelet bases. Mallat's brilliant work (1989a,b,c) has been the major source of many new developments in wavelet analysis and its wide variety of applications.

Mathematically, the fundamental idea of MRA is to represent a function (or signal) f as a limit of successive approximations, each of which is a finer version of the function f. These successive approximations correspond to different levels of resolutions. Thus, MRA is a formal approach to constructing orthogonal wavelet bases using a definite set of rules and procedures. The key feature of this analysis is to describe mathematically the process of studying signals or images at different scales. The basic principle of the MRA deals with the decomposition of the whole function space into individual subspaces $V_n \subset V_{n+1}$ so that the space V_{n+1} consists of all rescaled functions in V_n. This essentially means a decomposition of each function (or signal) into components of different scales (or frequencies) so that an

L. Debnath, F.A. Shah, *Wavelet Transforms and Their Applications*,
DOI 10.1007/978-0-8176-8418-1_7

individual component of the original function f occurs in each subspace. These components can describe finer and finer versions of the original function f. For example, a function is resolved at scales $\triangle t = 2^0, 2^{-1}, \ldots, 2^{-n}$. In audio signals, these scales are basically *octaves* which represent higher and higher frequency components. For images and, indeed, for all signals, the simultaneous existence of a multiscale may also be referred to as *multiresolution*. From the point of view of practical application, MRA is really an effective mathematical framework for hierarchical decomposition of an image (or signal) into components of different scales (or frequencies).

In general, frames have many of the properties of bases, but they lack a very important property of orthogonality. If the condition of orthogonality

$$\langle \phi_{k,\ell}, \phi_{m,n} \rangle = 0 \quad \text{for all} \quad (k,\ell) \neq (m,n) \tag{7.1.1}$$

is satisfied, the reconstruction of the function f from $\langle f, \phi_{m,n} \rangle$ is much simpler and, for any $f \in L^2(\mathbb{R})$, we have the following representation

$$f = \sum_{m,n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}, \tag{7.1.2}$$

where

$$\phi_{m,n}(x) = 2^{-m/2} \phi\left(2^{-m}x - n\right) \tag{7.1.3}$$

is an orthonormal basis of V_m.

This chapter deals with the idea of MRA with examples. Special attention is given to properties of scaling functions and orthonormal wavelet bases. This is followed by a method of constructing orthonormal bases of wavelets from MRA. Special attention is also given to the Daubechies wavelets with compact support and the Daubechies algorithm. Included are discrete wavelet transforms (DWTs) and Mallat's pyramid algorithm.

7.2 Definition of MRA and Examples

Definition 7.2.1 (Multiresolution Analysis). A *MRA* consists of a sequence $\{V_m : m \in \mathbb{Z}\}$ of embedded closed subspaces of $L^2(\mathbb{R})$ that satisfy the following conditions:

(i) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset V_{m+1} \cdots$,

(ii) $\bigcup_{m=-\infty}^{\infty} V_m$ is dense in $L^2(\mathbb{R})$, that is, $\overline{\bigcup_{m=-\infty}^{\infty} V_m} = L^2(\mathbb{R})$,

(iii) $\bigcap_{m=-\infty}^{\infty} V_m = \{0\}$,

(iv) $f(x) \in V_m$ if and only if $f(2x) \in V_{m+1}$ for all $m \in \mathbb{Z}$,

(v) there exists a function $\phi \in V_0$ such that $\{\phi_{0,n} = \phi(x-n), n \in \mathbb{Z}\}$ is an orthonormal basis for V_0, that is,

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\langle f, \phi_{0,n}\rangle|^2 \qquad \text{for all } f \in V_0.$$

The function ϕ is called the *scaling function* or *father wavelet*. If $\{V_m\}$ is a multiresolution of $L^2(\mathbb{R})$ and if V_0 is the closed subspace generated by the integer translates of a single function ϕ, then we say that ϕ generates the MRA.

Sometimes, condition (v) is relaxed by assuming that $\{\phi(x-n),\ n \in \mathbb{Z}\}$ is a *Riesz basis* for V_0, that is, for every $f \in V_0$, there exists a unique sequence $\{c_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, \phi(x-n)$$

with convergence in $L^2(\mathbb{R})$ and there exist two positive constants A and B independent of $f \in V_0$ such that

$$A \sum_{n=-\infty}^{\infty} |c_n|^2 \le \|f\|^2 \le B \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where $0 < A < B < \infty$. In this case, we have a MRA with a Riesz basis.

Note that condition (v) implies that $\{\phi(x-n),\ n \in \mathbb{Z}\}$ is a Riesz basis for V_0 with $A = B = 1$.

Since $\phi_{0,n}(x) \in V_0$ for all $n \in \mathbb{Z}$. Further, if $n \in \mathbb{Z}$, it follows from (iv) that

$$\phi_{m,n}(x) = 2^{m/2}\phi\big(2^m x - n\big),\ m \in \mathbb{Z} \tag{7.2.1}$$

is an orthonormal basis for V_m.

Consequences of Definition 7.2.1.

1. A repeated application of condition (iv) implies that $f \in V_m$ if and only if $f(2^k x) \in V_{m+k}$ for all $m, k \in \mathbb{Z}$. In other words, $f \in V_m$ if and only if $f(2^{-m}x) \in V_0$ for all $m \in \mathbb{Z}$.

   This shows that functions in V_m are obtained from those in V_0 through a scaling 2^{-m}. If the scale $m = 0$ is associated with V_0, then the scale 2^{-m} is associated with V_m, Thus, subspaces V_m are just scaled versions of the central space V_0 which is invariant under translation by integers, that is, $T_n V_0 = V_0$ for all $n \in \mathbb{Z}$.

2. It follows from Definition 7.2.1 that a MRA is completely determined by the scaling function ϕ, but not conversely. For a given $\phi \in V_0$, we first define

$$V_0 = \left\{ f(x) = \sum_{n=-\infty}^{\infty} c_n\, \phi_{0,n} = \sum_{n=-\infty}^{\infty} c_n\, \phi(x-n) : \{c_n\} \in \ell^2(\mathbb{Z}) \right\}.$$

Condition (iv) implies that V_0 has an orthonormal basis $\{\phi_{0,n}\} = \{\phi(x-n)\}$. Then, V_0 consists of all functions $f(x) = \sum_{n=-\infty}^{\infty} c_n \phi(x-n)$ with finite energy $\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$. Similarly, the space V_m has the orthonormal basis $\phi_{m,n}$ given by (7.2.1) so that $f_m(x)$ is given by

$$f_m(x) = \sum_{n=-\infty}^{\infty} c_{mn}\, \phi_{m,n}(x) \tag{7.2.2}$$

with the finite energy

$$\|f_m\|^2 = \sum_{n=-\infty}^{\infty} |c_{mn}|^2 < \infty.$$

Thus, f_m represents a typical function in the space V_m. It builds in self-invariance and scale invariance through the basis $\{\phi_{m,n}\}$.

3. Conditions (ii) and (iii) can be expressed in terms of the orthogonal projections P_m onto V_m , that is, for all $f \in L^2(\mathbb{R})$,

$$\lim_{m\to-\infty} P_m f = 0 \quad \text{and} \quad \lim_{m\to+\infty} P_m f = f. \tag{7.2.3a,b}$$

The projection $P_m f$ can be considered as an approximation of f at the scale 2^{-m}. Therefore, the successive approximations of a given function f are defined as the orthogonal projections P_m onto the space V_m:

$$P_m f = \sum_{n=-\infty}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}, \tag{7.2.4}$$

where $\phi_{m,n}(x)$ given by (7.2.1) is an orthonormal basis for V_m.

4. Since $V_0 \subset V_1$, the scaling function ϕ that leads to a basis for V_0 is also V_1. Since $\phi \in V_1$ and $\phi_{1,n}(x) = \sqrt{2}\,\phi(2x-n)$ is an orthonormal basis for V_1, ϕ can be expressed in the form

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n \, \phi_{1,n}(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \, \phi\big(2x - n\big), \tag{7.2.5}$$

where

$$c_n = \langle \phi, \phi_{1,n} \rangle \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = 1.$$

Equation (7.2.5) is called the *dilation equation.* It involves both x and $2x$ and is often referred to as the *two-scale equation* or *refinement equation* because it displays $\phi(x)$ in the refined space V_1. The space V_1 has the finer scale 2^{-1} and it contains $\phi(x)$ which has scale 1.

All of the preceding facts reveal that MRA can be described at least three ways so that we can specify

(a) the subspaces V_m,
(b) the scaling function ϕ,
(c) the coefficient c_n in the dilation equation (7.2.5).

The real importance of a MRA lies in the simple fact that it enables us to construct an orthonormal basis for $L^2(\mathbb{R})$. In order to prove this statement, we first assume that $\{V_m\}$ is a MRA. Since $V_m \subset V_{m+1}$, we define W_m as the orthogonal complement of V_m in V_{m+1} for every $m \in \mathbb{Z}$, so that we have

$$\begin{aligned} V_{m+1} &= V_m \bigoplus W_m \\ &= \Big(V_{m-1} \bigoplus W_{m-1}\Big) \bigoplus W_m \\ &= \dots \\ &= V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus \cdots \bigoplus W_m \\ &= V_0 \bigoplus \left(\bigoplus_{m=0}^{m} W_m \right) \end{aligned} \tag{7.2.6}$$

and $V_n \perp W_m$ for $n \neq m$.

Since $\bigcup_{m=-\infty}^{\infty} V_m$ is dense in $L^2(\mathbb{R})$, we may take the limit as $m \to \infty$ to obtain

$$V_0 \bigoplus \left(\bigoplus_{m=0}^{\infty} W_m \right) = L^2(\mathbb{R}).$$

Similarly, we may go in the other direction to write

$$\begin{aligned} V_0 &= V_{-1} \bigoplus W_{-1} \\ &= \left(V_{-2} \bigoplus W_{-2}\right) \bigoplus W_{-1} \\ &= \ldots \\ &= V_{-m} \bigoplus W_{-m} \bigoplus \cdots \bigoplus W_{-1}. \end{aligned}$$

We may again take the limit as $m \to \infty$. Since $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$, it follows that $V_{-m} = \{0\}$. Consequently, it turns out that

$$\bigoplus_{m=-\infty}^{\infty} W_m = L^2(\mathbb{R}). \tag{7.2.7}$$

We include here a pictorial representation of $V_1 = V_0 \bigoplus W_0$ in Fig. 7.1.

Finally, the difference between the two successive approximations $P_m f$ and $P_{m+1} f$ is given by the orthogonal projection $Q_m f$ of f onto the orthogonal complement W_m of V_m in V_{m+1} so that

$$Q_m f = P_{m+1} f - P_m f.$$

It follows from conditions (i)–(v) in Definition 7.2.1 that the spaces W_m are also scaled versions of W_0 and, for $f \in L^2(\mathbb{R})$,

$$f \in W_m \quad \text{if and only if} \quad f(2^{-m}x) \in W_0 \quad \text{for all } m \in \mathbb{Z}, \tag{7.2.8}$$

and they are translation-invariant for the discrete translations $n \in \mathbb{Z}$, that is,

$$f \in W_0 \quad \text{if and only if} \quad f(x-n) \in W_0,$$

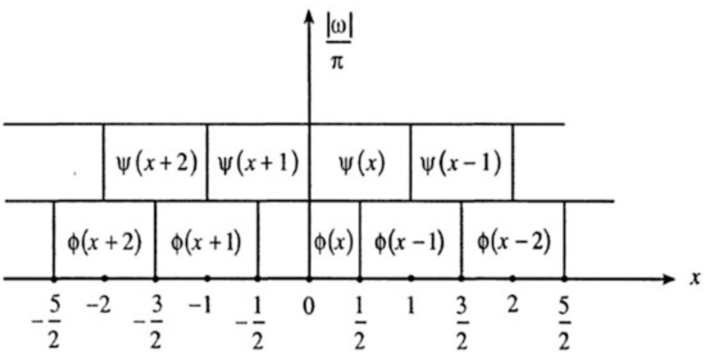

Fig. 7.1 Pictorial representation of $\mathbf{V_1} = \mathbf{V_0} \bigoplus \mathbf{W_0}$

and they are mutually orthogonal spaces generating all of $L^2(\mathbb{R})$,

$$\left.\begin{aligned} &W_m \perp W_k \quad \text{for}\, m \neq k, \\ &\bigoplus_{m\in\mathbb{Z}} W_m = L^2(\mathbb{R}) \end{aligned}\right\}. \tag{7.2.9a,b}$$

Moreover, there exists a function $\psi \in W_0$ such that $\psi_{0,n}(x) = \psi(x-n)$ constitutes an orthonormal basis for W_0. It follows from (7.2.8) that

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n), \quad \text{for } n \in \mathbb{Z} \tag{7.2.10}$$

constitute an orthonormal basis for W_m. Thus, the family $\psi_{m,n}(x)$ represents an orthonormal basis of wavelets for $L^2(\mathbb{R})$. Each $\psi_{m,n}(x)$ is represented by the point (p, s), where $p = \left(n + \frac{1}{2}\right) 2^m$ and $s = 2^m$, $(m, n \in \mathbb{Z})$ in the position-scale plane, as shown in Fig. 7.2. Since scale is the inverse of the frequency, small scales 2^m (or high frequencies 2^{-m}) are near the position axis.

Example 7.2.1 (Characteristic Function). We assume that $\phi = \chi_{[0,1]}$ is the characteristic function of the interval [0, 1]. Define spaces V_m by

$$V_m = \left\{ \sum_{k=-\infty}^{\infty} c_k\, \phi_{m,k} : \{c_k\} \in \ell^2(\mathbb{Z}) \right\},$$

where

$$\phi_{m,n}(x) = 2^{-m/2}\,\phi(2^{-m}x - n).$$

The spaces V_m satisfy all the conditions of Definition 7.2.1, and so, $\{V_m\}$ is a MRA.

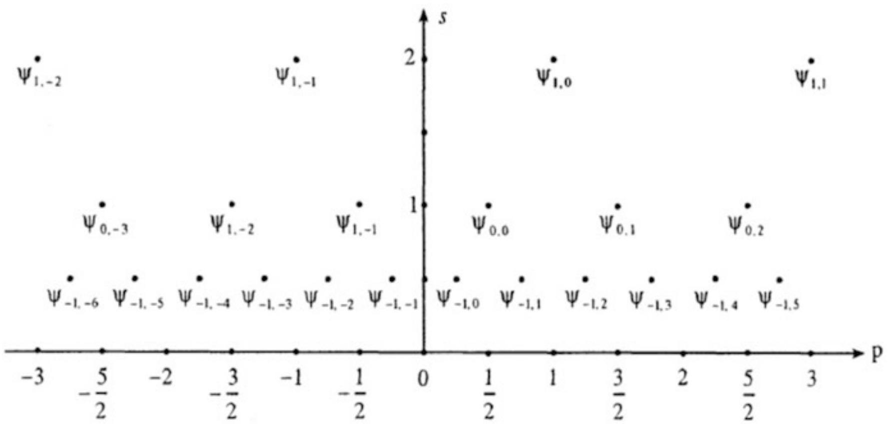

Fig. 7.2 Dyadic grid representation

Example 7.2.2 (Piecewise Constant Function). Consider the space V_m of all functions in $L^2(\mathbb{R})$ which are constant on intervals $\left[2^{-m}n, 2^{-m}(n+1)\right]$, where $n \in \mathbb{Z}$. Obviously, $V_m \subset V_{m+1}$ because any function that is constant on intervals of length 2^{-m} is automatically constant on intervals of half that length. The space V_0 contains all functions $f(x)$ in $L^2(\mathbb{R})$ that are constant on $n \leq x < n+1$. The function $f(2x)$ in V_1 is then constant on $\frac{n}{2} \leq x < \frac{n+1}{2}$. Intervals of length 2^{-m} are usually referred to as *dyadic intervals*. A sample function in spaces V_m is shown in Fig. 7.3.

Clearly, the piecewise constant function space V_m satisfies the conditions (i)–(iv) of a MRA. It is easy to guess a scaling function ϕ in V_0 which is orthogonal to its translates. The simplest choice for ϕ is the characteristic function so that $\phi(x) = \chi_{[0,1]}(x)$. Therefore, any function $f \in V_0$ can be expressed in terms of the scaling function ϕ as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, \phi(x-n).$$

Thus, the condition (v) is satisfied by the characteristic function $\chi_{[0,1]}$ as the scaling function. As we shall see later, this MRA is related to the classic Haar wavelet.

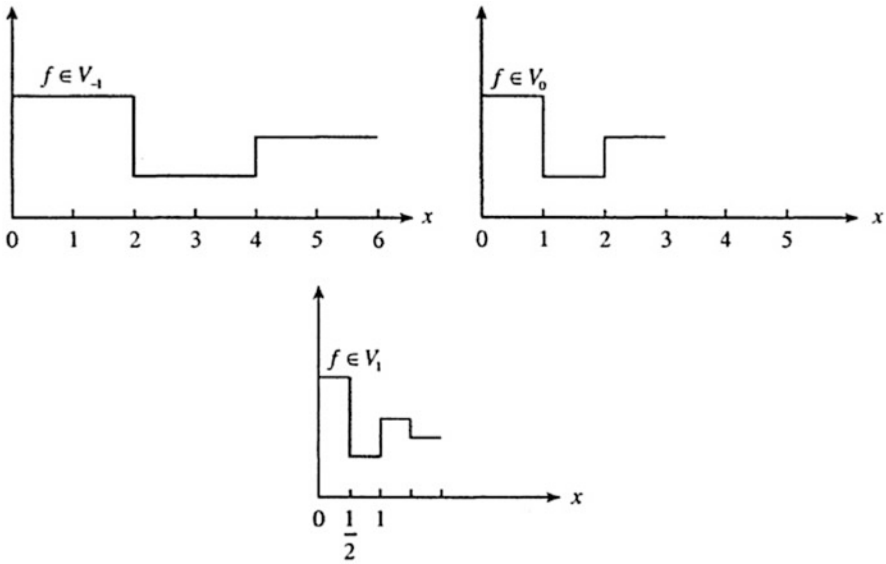

Fig. 7.3 Piecewise constant functions in V_{-1}, V_0 and V_1

7.3 Properties of Scaling Functions and Orthonormal Wavelet Bases

Theorem 7.3.1. *For any function $\phi \in L^2(\mathbb{R})$, the following conditions are equivalent.*

(a) The system $\{\phi_{0,n} \equiv \phi(x-n),\ n \in \mathbb{Z}\}$ is orthonormal.

(b) $\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2k\pi)\right|^2 = 1$ *almost everywhere (a.e.).*

Proof. Obviously, the Fourier transform of $\phi_{0,n}(x) = \phi(x-n)$ is

$$\hat{\phi}_{0,n} = \exp(-in\omega)\,\hat{\phi}(\omega).$$

In view of the general Parseval relation (3.4.37) for the Fourier transform, we have

$$\begin{aligned}
\langle \phi_{0,n}, \phi_{0,m} \rangle &= \langle \phi_{0,0}, \phi_{0,m-n} \rangle \\
&= \frac{1}{2\pi} \left\langle \hat{\phi}_{0,0}, \hat{\phi}_{0,m-n} \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i(m-n)\omega\}. \left|\hat{\phi}(\omega)\right|^2 d\omega \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} \exp\{-i(m-n)\omega\} \left|\hat{\phi}(\omega)\right|^2 d\omega \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \exp\{-i(m-n)\omega\} \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 d\omega.
\end{aligned}$$

Thus, it follows from the completeness of $\{\exp(-in\omega), n \in \mathbb{Z}\}$ in $L^2(0, 2\pi)$ that

$$\langle \phi_{0,n}, \phi_{0,m} \rangle = \delta_{n,m}$$

if and only if

$$\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 = 1 \qquad \text{almost everywhere.}$$

Theorem 7.3.2. *For any two functions $\phi, \psi \in L^2(\mathbb{R})$, the sets of functions $\{\phi_{0,n} \equiv \phi(x-n), n \in \mathbb{Z}\}$ and $\{\psi_{0,m} \equiv \psi(x-m), m \in \mathbb{Z}\}$ are biorthogonal, that is,*

$$\langle \phi_{0,n}, \psi_{0,m} \rangle = 0, \qquad \textit{for all } n, m \in \mathbb{Z},$$

if and only if

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k)\, \overline{\hat{\psi}}(\omega + 2\pi k) = 0 \qquad \textit{almost everywhere.}$$

Proof. We apply arguments similar to those stated in the proof of Theorem 7.3.1 to obtain

$$\begin{aligned}
\langle \phi_{0,n}, \psi_{0,m} \rangle &= \langle \phi_{0,0}, \psi_{0,m-n} \rangle \\
&= \frac{1}{2\pi} \left\langle \hat{\phi}_{0,0}, \hat{\psi}_{0,m-n} \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i(n-m)\omega\}\, \hat{\phi}(\omega)\, \overline{\hat{\psi}}(\omega) d\omega \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} \exp\{i(n-m)\omega\}\, \hat{\phi}(\omega)\, \overline{\hat{\psi}}(\omega)\, d\omega \\
&= \frac{1}{2\pi} \int_{0}^{2\pi} \exp\{i(n-m)\omega\} \left[\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k)\, \overline{\hat{\psi}}(\omega + 2\pi k) \right] d\omega.
\end{aligned}$$

Thus,

$$\langle \phi_{0,n}, \psi_{0,m} \rangle = 0 \quad \text{for all } n \text{ and } m$$

if and only if

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(\omega + 2\pi k)\, \overline{\hat{\psi}}(\omega + 2\pi k) = 0 \qquad \text{almost everywhere.}$$

We next proceed to the construction of a mother wavelet by introducing an important generating function $\hat{m}(\omega) \in L^2[0, 2\pi]$ in the following lemma.

Lemma 7.3.1. *The Fourier transform of the scaling function* ϕ *satisfies the following conditions:*

$$\sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2 = 1 \qquad \textit{almost everywhere}, \tag{7.3.1}$$

$$\hat{\phi}(\omega) = \hat{m}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right), \tag{7.3.2}$$

where

$$\hat{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n \exp(-in\omega) \tag{7.3.3}$$

is a 2π *- periodic function and satisfies the so-called orthogonality condition*

$$\left| \hat{m}(\omega) \right|^2 + \left| \hat{m}(\omega + \pi) \right|^2 = 1 \quad \textit{a.e.} \tag{7.3.4}$$

Remark. The Fourier transform $\hat{\phi}$ of the scaling function ϕ satisfies the functional equation (7.3.2). The function $\hat{m}$ is called the *generating function* of the MRA. This function is often called the *discrete Fourier transform* of the sequence $\{c_n\}$. In signal processing, $\hat{m}(\omega)$ is called the transfer function of a *discrete filter* with impulse response $\{c_n\}$ or the *low-pass filter* associated with the scaling function ϕ.

Proof. Condition (7.3.1) follows from Theorem 7.3.1.

To establish (7.3.2), we first note that $\phi \in V_1$ and

$$\phi_{1,n}(x) = \sqrt{2}\,\phi(2x - n)$$

is an orthonormal basis for V_1. Thus, the scaling function ϕ has the following representation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n\, \phi(2x - n), \tag{7.3.5}$$

where $c_n = \langle \phi, \phi_{1,n} \rangle$ and $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$.

The Fourier transform of (7.3.5) gives

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{i\omega n}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \hat{m}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right). \tag{7.3.6}$$

This proves the functional equation (7.3.2).

To verify the orthogonality condition (7.3.4), we substitute (7.3.2) in (7.3.1) so that condition (7.3.1) becomes

$$\begin{aligned} 1 &= \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 \\ &= \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + k\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{\omega}{2} + k\pi\right)\right|^2. \end{aligned}$$

This is true for any ω and hence, replacing ω by 2ω gives

$$1 = \sum_{k=-\infty}^{\infty} \left|\hat{m}(\omega + k\pi)\right|^2 \left|\hat{\phi}(\omega + k\pi)\right|^2. \tag{7.3.7}$$

We now split the above infinite sum over k into even and odd integers and use the 2π-periodic property of the function $\hat{m}$ to obtain

$$1 = \sum_{k=-\infty}^{\infty} |\hat{m}(\omega + 2\pi k)|^2 \left|\hat{\phi}(\omega + 2\pi k)\right|^2 + \sum_{k=-\infty}^{\infty} |\hat{m}(\omega + (2k+1)\pi)|^2 \left|\hat{\phi}(\omega + (2k+1)\pi)\right|^2$$
$$= \sum_{k=-\infty}^{\infty} |\hat{m}(\omega)|^2 \left|\hat{\phi}(\omega + 2\pi k)\right|^2 + \sum_{k=-\infty}^{\infty} |\hat{m}(\omega + \pi)|^2 \left|\hat{\phi}(\omega + \pi + 2k\pi)\right|^2$$
$$= |\hat{m}(\omega)|^2 + |\hat{m}(\omega + \pi)|^2$$

by (7.3.1) used in its original form and ω replaced by $(\omega + \pi)$. This leads to the desired condition (7.3.4).

Remark. Since $\left|\hat{\phi}(0)\right| = 1 \neq 0$, $\hat{m}(0) = 1$ and $\hat{m}(\pi) = 0$. This implies that $\hat{m}$ can be considered as a low-pass filter because the transfer function passes the frequencies near $\omega = 0$ and cuts off the frequencies near $\omega = \pi$.

Lemma 7.3.2. *The function $\hat{\phi}$ can be represented by the infinite product*

$$\hat{\phi}(\omega) = \prod_{k=1}^{\infty} \hat{m}\left(\frac{\omega}{2^k}\right). \tag{7.3.8}$$

Proof. A simple iteration of (7.3.2) gives

$$\hat{\phi}(\omega) = \hat{m}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \hat{m}\left(\frac{\omega}{2}\right) \left[\hat{m}\left(\frac{\omega}{4}\right) \hat{\phi}\left(\frac{\omega}{4}\right)\right]$$

which is, by the $(k-1)$th iteration,

$$= \hat{m}\left(\frac{\omega}{2}\right) \hat{m}\left(\frac{\omega}{4}\right) \ldots \hat{m}\left(\frac{\omega}{2^k}\right) \cdot \hat{\phi}\left(\frac{\omega}{2^k}\right)$$
$$= \prod_{k=1}^{k} \hat{m}\left(\frac{\omega}{2^k}\right) \hat{\phi}\left(\frac{\omega}{2^k}\right). \tag{7.3.9}$$

Since $\hat{\phi}(0) = 1$ and $\hat{\phi}(\omega)$ is continuous, we obtain

$$\lim_{k \to \infty} \hat{\phi}\left(\frac{\omega}{2^k}\right) = \hat{\phi}(0) = 1.$$

The limit of (7.3.9) as $k \to \infty$ gives (7.3.8).

We next prove the following major technical lemma.

Lemma 7.3.3. *The Fourier transform of any function $f \in W_0$ can be expressed in the form*

$$\hat{f}(\omega) = \hat{v}(\omega) \exp\left(\frac{i\omega}{2}\right) \overline{\hat{m}\left(\frac{\omega}{2}+\pi\right)} \hat{\phi}\left(\frac{\omega}{2}\right), \tag{7.3.10}$$

where $\hat{v}(\omega)$ *is a* 2π*-periodic function and the factor* $\exp\left(\frac{i\omega}{2}\right) \overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right) \hat{\phi}\left(\frac{\omega}{2}\right)$ *is independent of* f.

Proof. Since $f \in W_0$, it follows from $V_1 = V_0 \oplus W_0$ that $f \in V_1$ and is orthogonal to V_0. Thus, it follows from $V_1 = V_0 \oplus W_0$ that $f \in V_0$ and is orthogonal to V_0. Thus, the function f can be expressed in the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, \phi_{1,n}(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n\, \phi(2x-n), \tag{7.3.11}$$

where $c_n = \langle f, \phi_{1,n} \rangle$.

We use an argument similar to that in Lemma 7.3.1 to obtain the result

$$\hat{f}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{in\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \hat{m}_f\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right), \tag{7.3.12}$$

where the function $\hat{m} f$ is given by

$$\hat{m}_f(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n \exp(-in\omega). \tag{7.3.13}$$

Evidently, $\hat{m}_f$ is a 2π-periodic function which belongs to $L^2(0, 2\pi)$. Since $f \perp V_0$, we have

$$\int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{\phi}}(\omega) \exp(in\omega)\, d\omega = 0$$

and hence,

$$\int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} \hat{f}(\omega+2\pi k)\, \overline{\hat{\phi}}(\omega+2\pi k) \right\} e^{in\omega} d\omega = 0. \tag{7.3.14}$$

Consequently,

$$\sum_{k=-\infty}^{\infty} \hat{f}(\omega+2\pi k)\, \overline{\hat{\phi}}(\omega+2\pi k) = 0. \tag{7.3.15}$$

We now substitute (7.3.12) and (7.3.2) into (7.3.15) to obtain

$$0 = \sum_{k=-\infty}^{\infty} \hat{m}_f\left(\frac{\omega}{2}+\pi k\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi k\right)\left|\hat{\phi}\left(\frac{\omega}{2}+\pi k\right)\right|^2,$$

which is, by splitting the sum into even and odd integers k and then using the 2π-periodic property of the function $\hat{m}$,

$$\begin{aligned}0 &= \sum_{k=-\infty}^{\infty} \hat{m}_f\left(\frac{\omega}{2}+2\pi k\right)\overline{\hat{m}}\left(\frac{\omega}{2}+2\pi k\right)\left|\hat{\phi}\left(\frac{\omega}{2}+2\pi k\right)\right|^2 \\ &\quad + \sum_{k=-\infty}^{\infty} \hat{m}_f\left(\frac{\omega}{2}+\pi+2\pi k\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi+2\pi k\right)\left|\hat{\phi}\left(\frac{\omega}{2}+\pi+2\pi k\right)\right|^2 \\ &= \hat{m}_f\left(\frac{\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2}\right)\sum_{k=-\infty}^{\infty}\left|\hat{\phi}\left(\frac{\omega}{2}+2\pi k\right)\right|^2 \\ &\quad + \hat{m}_f\left(\frac{\omega}{2}+\pi\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right)\sum_{k=-\infty}^{\infty}\left|\hat{\phi}\left(\frac{\omega}{2}+\pi+2\pi k\right)\right|^2,\end{aligned}$$

which is, due to orthonormality of the system $\{\phi_{0,k}(x)\}$ and (7.3.1),

$$= \left\{\hat{m}_f\left(\frac{\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2}\right) + \hat{m}_f\left(\frac{\omega}{2}+\pi\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right)\right\}\cdot 1. \tag{7.3.16}$$

Finally, replacing ω by 2ω in (7.3.16) gives

$$\hat{m}_f(\omega)\,\overline{\hat{m}}(\omega) + \hat{m}_f(\omega+\pi)\,\overline{\hat{m}}(\omega+\pi) = 0 \quad \text{a.e.} \tag{7.3.17}$$

Or, equivalently,

$$\begin{vmatrix} \hat{m}_f(\omega) & \overline{\hat{m}}(\omega+\pi) \\ -\hat{m}_f(\omega+\pi) & \overline{\hat{m}}(\omega) \end{vmatrix} = 0.$$

This can be interpreted as the linear dependence of two vectors

$$\left(\hat{m}_f(\omega), -\hat{m}_f(\omega+\pi)\right) \quad \text{and} \quad \left(\overline{\hat{m}}(\omega+\pi), \overline{\hat{m}}(\omega)\right).$$

Hence, there exists a function $\hat{\lambda}$ such that

$$\hat{m}_f(\omega) = \hat{\lambda}(\omega)\,\overline{\hat{m}}(\omega+\pi) \quad \text{a.e.} \tag{7.3.18}$$

Since both $\hat{m}$ and $\hat{m}_f$ are 2π-periodic functions, so is $\hat{\lambda}$. Further, substituting (7.3.18) into (7.3.17) gives

$$\hat{\lambda}(\omega) + \hat{\lambda}(\omega + \pi) = 0 \quad \text{a.e.} \tag{7.3.19}$$

Thus, there exists a 2π-periodic function $\hat{v}$ defined by

$$\hat{\lambda}(\omega) = \exp(i\omega)\,\hat{v}(2\omega). \tag{7.3.20}$$

Finally, a simple combination of (7.3.12), (7.3.18), and (7.3.20) gives the desired representation (7.3.10). This completes the proof of Lemma 7.3.3.

Now, we return to the main problem of constructing a mother wavelet $\psi(x)$. Suppose that there is a function ψ such that $\{\psi_{0,n} : n \in \mathbb{Z}\}$ is a basis for the space W_0. Then, every function $f \in W_0$ has a series representation

$$f(x) = \sum_{n=-\infty}^{\infty} h_n\,\psi_{0,n} = \sum_{n=-\infty}^{\infty} h_n\,\psi(x-n), \tag{7.3.21}$$

where

$$\sum_{n=-\infty}^{\infty} |h_n|^2 < \infty.$$

Application of the Fourier transform to (7.3.21) gives

$$\hat{f}(\omega) = \left(\sum_{n=-\infty}^{\infty} h_n\,e^{-in\omega}\right)\hat{\psi}(\omega) = \hat{h}(\omega)\,\hat{\psi}(\omega), \tag{7.3.22}$$

where the function $\hat{h}$ is

$$\hat{h}(\omega) = \sum_{n=-\infty}^{\infty} h_n \exp(-in\omega), \tag{7.3.23}$$

and it is a square integrable and 2π-periodic function in $[0, 2\pi]$. When (7.3.22) is compared with (7.3.10), we see that $\hat{\psi}(\omega)$ should be

$$\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right) \tag{7.3.24}$$

$$= \hat{m}_1\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right), \tag{7.3.25}$$

where the function $\hat{m}_1$ is given by

$$\hat{m}_1(\omega) = \exp(i\omega)\,\overline{\hat{m}}(\omega + \pi). \tag{7.3.26}$$

Thus, the function $\hat{m}_1(\omega)$ is called the *filter* conjugate to $\hat{m}(\omega)$ and hence, $\hat{m}$ and $\hat{m}_1$ are called *conjugate quadratic filters* in signal processing.

Finally, substituting (7.3.3) into (7.3.24) gives

$$\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2}\right) \cdot \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \overline{c_n} \exp\left\{in\left(\frac{\omega}{2} + \pi\right)\right\} \hat{\phi}\left(\frac{\omega}{2}\right)$$

$$= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \overline{c}_n \exp\left[in\pi + i(n+1)\frac{\omega}{2}\right] \hat{\phi}\left(\frac{\omega}{2}\right)$$

which is, by putting $n = -(k+1)$

$$= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \overline{c}_{-k-1}(-1)^k \exp\left(-\frac{ik\omega}{2}\right) \cdot \hat{\phi}\left(\frac{\omega}{2}\right). \tag{7.3.27}$$

Invoking the inverse Fourier transform to (7.3.27) with k replaced by n gives the mother wavelet

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1}\, \overline{c}_{-n-1} \phi(2x - n) \tag{7.3.28}$$

$$= \sqrt{2} \sum_{n=-\infty}^{\infty} d_n\, \phi(2x - n), \tag{7.3.29}$$

where the coefficients d_n are given by

$$d_n = (-1)^{n-1}\, \overline{c}_{-n-1}. \tag{7.3.30}$$

Thus, the representation (7.3.29) of a mother wavelet ψ has the same structure as that of the father wavelet ϕ given by (7.3.5).

Remarks. 1. The mother wavelet ψ associated with a given MRA is not unique because

$$d_n = (-1)^{n-1}\, \overline{c}_{2N-1-n} \tag{7.3.31}$$

defines the same mother wavelet (7.3.28) with suitably selected $N \in \mathbb{Z}$. This wavelet with coefficients d_n given by (7.3.31) has the Fourier transform

$$\hat{\psi}(\omega) = \exp\left\{(2N-1)\frac{i\omega}{2}\right\} \overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right). \tag{7.3.32}$$

The nonuniqueness property of ψ allows us to define another form of ψ, instead of (7.3.28), by

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} d_n\, \phi(2x-n), \tag{7.3.33}$$

where a slightly modified d_n is

$$d_n = (-1)^n\, \bar{c}_{1-n}. \tag{7.3.34}$$

In practice, any one of the preceding formulas for d_n can be used to find a mother wavelet.

2. The orthogonality condition (7.3.4) together with (7.3.2) and (7.3.24) implies

$$\left|\hat{\phi}(\omega)\right|^2 + \left|\hat{\psi}(\omega)\right|^2 = \left|\hat{\phi}\left(\frac{\omega}{2}\right)\right|^2. \tag{7.3.35}$$

Or, equivalently,

$$\left|\hat{\phi}(2^m\omega)\right|^2 + \left|\hat{\psi}(2^m\omega)\right|^2 = \left|\hat{\phi}(2^{m-1}\omega)\right|^2. \tag{7.3.36}$$

Summing both sides of (7.3.36) from $m = 1$ to ∞ leads to the result

$$\left|\hat{\phi}(\omega)\right|^2 = \sum_{m=1}^{\infty}\left|\hat{\psi}(2^m\omega)\right|^2. \tag{7.3.37}$$

3. If ϕ has a compact support, the series (7.3.29) for the mother wavelet ψ terminates and consequently, ψ is represented by a finite linear combination of translated versions of $\phi(2x)$.

Finally, all of the above results lead to the main theorem of this section.

Theorem 7.3.3. *If $\{V_n, n \in \mathbb{Z}\}$ is a MRA with the scaling function ϕ, then there is a mother wavelet ψ given by*

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} (-1)^{n-1}\, \bar{c}_{-n-1}\, \phi(2x-n), \tag{7.3.38}$$

where the coefficients c_n are given by

$$c_n = \langle \phi, \phi_{1,n} \rangle = \sqrt{2} \int_{-\infty}^{\infty} \phi(x)\, \overline{\phi(2x-n)}\, dx. \tag{7.3.39}$$

That is, the system $\{\psi_{m,n}(x) : m, n \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.

Proof. First, we have to verify that $\{\psi_{m,n}(x) : m, n \in \mathbb{Z}\}$ is an orthonormal set. Indeed, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(x-k)\, \overline{\psi(x-\ell)}\, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-i\omega(k-\ell)\right] \left|\hat{\psi}(\omega)\right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{0}^{2\pi} \exp\left[-i\omega(k-\ell)\right] \sum_{k=-\infty}^{\infty} \left|\hat{\psi}(\omega+2\pi k)\right|^2 d\omega\end{aligned}$$

$$\sum_{k=-\infty}^{\infty} \left|\hat{\psi}(\omega + 2\pi k)\right|^2 = \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + (k+1)\pi\right)\right|^2 \left|\hat{\phi}\left(\frac{\omega}{2} + k\pi\right)\right|^2$$

which is, by splitting the sum into even and odd integers k,

$$\begin{aligned}&= \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \left|\hat{\psi}\left(\frac{\omega}{2} + k\pi\right)\right|^2 \\ &\quad + \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + (2k+2)\pi\right)\right|^2 \left|\hat{\psi}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \\ &= \left|\hat{m}\left(\frac{\omega}{2} + \pi\right)\right|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\phi}\left(\frac{\omega}{2} + 2k\pi\right)\right|^2 \\ &\quad + \left|\hat{m}\left(\frac{\omega}{2}\right)\right|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\phi}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \\ &= \left|\hat{m}\left(\frac{\omega}{2}\right)\right|^2 + \left|\hat{m}\left(\frac{\omega}{2} + \pi\right)\right|^2 = 1 \quad \text{by (7.3.4).}\end{aligned}$$

Thus, we find

$$\int_{-\infty}^{\infty} \psi(x-k)\, \overline{\psi(x-\ell)}\, dx = \delta_{k,l}.$$

This shows that $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal system. In view of Lemma 7.3.2 and our discussion preceding this theorem, to prove that it is a basis, it suffices to show that function $\hat{v}$ in (7.3.20) is square integrable over $[0, 2\pi]$. In fact,

$$\begin{aligned}\int_0^{2\pi} \left|\hat{v}(\omega)\right|^2 d\omega &= 2\int_0^{\pi} \left|\hat{\lambda}(\omega)\right|^2 d\omega \\ &= 2\int_0^{\pi} \left|\hat{\lambda}(\omega)\right|^2 \left\{\left|\hat{m}(\omega+\pi)\right|^2 + \left|\hat{m}(\omega)\right|^2\right\} d\omega, \quad \text{by (7.3.4)} \\ &= 2\int_0^{2\pi} \left|\hat{\lambda}(\omega)\right|^2 \left|\hat{m}(\omega+\pi)\right|^2 d\omega \\ &= 2\int_0^{2\pi} \left|\hat{m}_f(\omega)\right|^2 d\omega, \qquad \text{by (7.3.18)} \\ &= 2\pi \sum_{n=-\infty}^{\infty} \left|c_n\right|^2, \qquad c_n = \left\langle f, \phi_{1,n}\right\rangle \\ &= 2\pi \left\| f \right\|^2 < \infty.\end{aligned}$$

This completes the proof.

Example 7.3.1 (The Shannon Wavelet). We consider the Fourier transform $\hat{\phi}$ of a scaling function ϕ defined by

$$\hat{\phi}(\omega) = \chi_{[-\pi,\pi]}(\omega)$$

so that

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega x} d\omega = \frac{\sin \pi x}{\pi x}.$$

This is also known as the Shannon sampling function, Both $\phi(x)$ and $\hat{\phi}(\omega)$ have been introduced in Chap. 3 (see Fig. 3.12 with $\omega_0 = \pi$). Clearly, the Shannon scaling function does not have finite support. However, its Fourier transform has a finite support (band-limited) in the frequency domain and has good frequency localization, Evidently, the system

$$\phi_{0,k}(x) = \phi(x-k) = \frac{\sin \pi(x-k)}{\pi(x-k)}, \quad k \in \mathbb{Z}$$

is orthonormal because

$$\begin{aligned}\left\langle \phi_{0,k}, \phi_{0,\ell}\right\rangle &= \frac{1}{2\pi}\left\langle \hat{\phi}_{0,k}, \hat{\phi}_{0,\ell}\right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_{0,k}(\omega)\,\overline{\hat{\phi}}_{0,\ell}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-i(k-\ell)\omega\right\} d\omega = \delta_{k,\ell}.\end{aligned}$$

In general, we define, for $m = 0$,

$$V_0 = \left\{ \sum_{k=-\infty}^{\infty} c_k \frac{\sin \pi(x-k)}{\pi(x-k)} : \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty \right\},$$

and, for other $m \neq 0$, $m \in \mathbb{Z}$,

$$V_m = \left\{ \sum_{k=-\infty}^{\infty} c_k \frac{2^{m/2} \sin \pi(2^m x-k)}{\pi(2^m x-k)} : \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty \right\}.$$

It is easy to check that all conditions of Definition 7.2.1 are satisfied. We next find out the coefficients c_k defined by

$$c_k = \langle \phi, \phi_{1,n} \rangle = \sqrt{2} \int_{-\infty}^{\infty} \frac{\sin \pi(x)}{\pi x} \cdot \frac{\sin \pi(2x-k)}{\pi(2x-k)}\, dx$$

$$= \begin{cases} \dfrac{1}{\sqrt{2}}, & k = 0 \\ \dfrac{\sqrt{2}}{\pi k} \sin\left(\dfrac{\pi k}{2}\right), & k \neq 0 \end{cases}$$

Consequently, we can use the formula (7.3.38) to find the Shannon mother wavelet

$$\psi(x) = \sum_{n=-\infty}^{\infty} (-1)^{n-1} c_{-n-1}\, \phi(2x-n)$$

$$= \frac{1}{\sqrt{2}} \frac{\sin \pi(2x+1)}{\pi(2x+1)} + \frac{\sqrt{2}}{\pi} \sum_{n \neq -1} \frac{(-1)^{n-1}}{(n+1)} \cos\left(\frac{n\pi}{2}\right) \frac{\sin \pi(2x-n)}{\pi(2x-n)}.$$

Obviously, the system $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis in $L^2(\mathbb{R})$. It is known as the *Shannon system*.

Theorem 7.3.4. *If ϕ is a scaling function for a MRA and $\hat{m}(\omega)$ is the associated low-pass filter, then a function $\psi \in W_0$ is an orthonormal wavelet for $L^2(\mathbb{R})$ if and only if*

$$\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2}\right) \hat{\nu}(\omega)\, \overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right) \hat{\phi}\left(\frac{\omega}{2}\right) \tag{7.3.40}$$

for some 2π-periodic function $\hat{\nu}$ such that $|\hat{\nu}(\omega)| = 1$.

Proof. It is enough to prove that all orthonormal wavelets $\psi \in W_0$ can be represented by (7.3.40). For any $\psi \in W_0$, by Lemma 7.3.4, there must be a 2π-periodic function $\hat{\nu}$ such that

$$\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2}\right) \hat{v}(\omega)\, \overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right) \hat{\phi}\left(\frac{\omega}{2}\right).$$

If ψ is an orthonormal wavelet, then the orthonormality of $\{\psi(x-k), k \in \mathbb{Z}\}$ leads to

$$\begin{aligned} 1 &= \sum_{k=-\infty}^{\infty} \left|\hat{\psi}(\omega + 2\pi k)\right|^2 \\ &= \left|\hat{v}(\omega)\right|^2 \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + k\pi + \pi\right)\right|^2 \left|\hat{\phi}\left(\frac{\omega}{2} + \pi k\right)\right|^2 \end{aligned}$$

which is, splitting the sum into even and odd integers k,

$$\begin{aligned} &= \left|\hat{v}(\omega)^2\right| \left\{ \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + 2k\pi + \pi\right)\right|^2 \left|\hat{\phi}\left(\frac{\omega}{2} + 2k\pi\right)\right|^2 \right. \\ &\quad \left. + \sum_{k=-\infty}^{\infty} \left|\hat{m}\left(\frac{\omega}{2} + (2k+1)\pi + \pi\right)\right| \left|\hat{\phi}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \right\} \end{aligned}$$

which is, by (7.3.1) and the 2π-periodic property of $\hat{m}$,

$$\begin{aligned} &= \left|\hat{v}(\omega)\right|^2 \left\{ \left|\hat{m}\left(\frac{\omega}{2}\right)\right|^2 + \left|\hat{m}\left(\frac{\omega}{2} + \pi\right)\right|^2 \right\} \\ &= \left|\hat{v}(\omega)\right|^2, \quad \text{by (7.3.4).} \end{aligned}$$

If the scaling function ϕ of an MRA is not an orthonormal basis of V_0 but rather is a Riesz basis, we can use the following orthonormalization process to generate an orthonormal basis.

Theorem 7.3.5 (Orthonormalization Process). *If $\phi \in L^2(\mathbb{R})$ and if $\{\phi(x - n), n \in \mathbb{Z}\}$ is a Riesz basis, that is, there exists two constants $A, B > 0$ such that*

$$0 < A \leq \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 \leq B < \infty, \tag{7.3.41}$$

then $\{\tilde{\phi}(x-n), n \in \mathbb{Z}\}$ is an orthonormal basis of V_0 with

$$\hat{\tilde{\phi}}(\omega) = \frac{\hat{\phi}(\omega)}{\sqrt{\hat{\Phi}(\omega)}}, \tag{7.3.42}$$

where the function $\hat{\Phi}$ is

$$\hat{\Phi}(\omega) = \sum_{k=-\infty}^{\infty} \left|\phi(\omega + 2\pi k)\right|^2. \tag{7.3.43}$$

Proof. It follows from inequality (7.3.41) that $\tilde{\phi} \in L^2(\mathbb{R})$. It also follows from (7.3.42) that

$$\sum_{k=-\infty}^{\infty} \left|\hat{\tilde{\phi}}(\omega + 2\pi k)\right|^2 = 1.$$

We consider a 2π-periodic function $\hat{g}$ defined by

$$\hat{g}(\omega) = \frac{1}{\sqrt{\hat{\Phi}(\omega)}}$$

so that $\hat{g}$ can be expanded as a Fourier series

$$\hat{g}(\omega) = \sum_{k=-\infty}^{\infty} g_k \exp(-ik\omega).$$

The inverse Fourier transform gives g in terms of the Dirac delta function of the form

$$g(t) = \sum_{k=-\infty}^{\infty} g_k\, \delta(t-k).$$

Applying the convolution theorem to (7.3.42) gives

$$\begin{aligned}
\tilde{\phi}(x) = \phi(x) * g(x) &= \int_{-\infty}^{\infty} \phi(x-t)\, g(t)\, dt \\
&= \int_{-\infty}^{\infty} \phi(x-t) \sum_{k=-\infty}^{\infty} g_k\, \delta(t-k)\, dt \\
&= \sum_{k=-\infty}^{\infty} g_k\, \phi(x-k).
\end{aligned}$$

This shows that $\{\phi(x-n), n \in \mathbb{Z}\}$ belongs to V_0. Thus, the function $\tilde{\phi}$ satisfies condition (b) of Theorem 7.3.1. Therefore, $\{\tilde{\phi}(x-n), n \in \mathbb{Z}\}$ is an orthonormal set.

It is easy to show that the span of $\{\tilde{\phi}(x-n), n \in \mathbb{Z}\} = \tilde{V}_0$ is the same as the span of $\{\phi(x-n), n \in \mathbb{Z}\} = V_0$. Hence, the MRA is preserved under this orthonormalization process.

We describe another approach to constructing a MRA which begins with a function $\phi \in L^2(\mathbb{R})$ that satisfies the following relations

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n \phi(2x - n), \qquad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty, \tag{7.3.44}$$

and

$$0 < A \le \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 \le B < \infty, \tag{7.3.45}$$

where A and B are constants.

We define V_0 as the closed span of $\{\phi(x-n),\ n \in \mathbb{Z}\}$ and V_m as the span of $\{\phi_{m,n}(x),\ n \in \mathbb{Z}\}$. It follows from relation (7.3.45) that $\{V_m\}$ satisfies property (i) of the MRA. In order to ensure that properties (ii) and (iii) of the MRA are satisfied, we further assume that $\hat{\phi}(\omega)$ is continuous and bounded with $\hat{\phi}(0) \neq 0$.

If $\left|\hat{\phi}(\omega)\right| \le C(1 + |\omega|)^{-2^{-1}-\epsilon}$, where $\epsilon > 0$, then

$$\hat{\phi}(\omega) = \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2$$

is continuous.

This ensures that the orthonormalization process can be used. Therefore, we assume

$$\hat{\hat{\phi}}(\omega) = \frac{\hat{\phi}(\omega)}{\sqrt{\Phi(\omega)}} \quad \text{and} \quad \hat{\hat{m}}\left(\frac{\omega}{2}\right) = \frac{\hat{\hat{\phi}}(\omega)}{\hat{\hat{\phi}}\left(\frac{\omega}{2}\right)}. \tag{7.3.46a,b}$$

Using (7.3.2) in (7.3.46b) gives

$$\hat{\hat{m}}\left(\frac{\omega}{2}\right) = \left\{\frac{\hat{\Phi}\left(\frac{\omega}{2}\right)}{\hat{\Phi}(\omega)}\right\}^{\frac{1}{2}} \hat{m}\left(\frac{\omega}{2}\right). \tag{7.3.47}$$

We now recall (7.3.24) to obtain $\hat{\hat{\psi}}(\omega)$ as

$$\hat{\hat{\psi}}(\omega) = \exp\left(\frac{i\omega}{2}\right) \overline{\hat{\hat{m}}\left(\frac{\omega}{2} + \pi\right)}\, \hat{\hat{\phi}}\left(\frac{\omega}{2}\right), \tag{7.3.48}$$

which is, by (7.3.46a) and (7.3.47),

$$= \exp\left(\frac{i\omega}{2}\right)\left\{\frac{\hat{\Phi}\left(\frac{\omega}{2}+\pi\right)}{\hat{\Phi}(\omega)\hat{\Phi}\left(\frac{\omega}{2}\right)}\right\}^{\frac{1}{2}}\left\{\frac{\hat{\phi}(\omega+2\pi)\,\hat{\phi}\left(\frac{\omega}{2}\right)}{\hat{\phi}\left(\frac{\omega}{2}+\pi\right)}\right\}. \tag{7.3.49}$$

We introduce a complex function P defined by

$$P(z) = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n z^n, \quad z \in \mathbb{C}, \tag{7.3.50}$$

where $z = \exp(-i\omega)$ and $|z| = 1$.

We assume that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ so that the series defining P converges absolutely and uniformly on the unit circle in $\mathbb{C}$. Thus, P is continuous on the unit circle, $|z| = 1$.

Since $P(z) = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n\, e^{-in\omega} = \hat{m}(\omega)$, it follows that

$$\hat{m}(\omega+\pi) = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n e^{-in\omega}\cdot e^{-in\pi} = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n(-z)^n = P(-z). \tag{7.3.51}$$

Consequently, the orthogonality condition (7.3.4) is equivalent to

$$\left|P(z)\right|^2 + \left|P(-z)\right|^2 = 1. \tag{7.3.52}$$

Lemma 7.3.4. *Suppose ϕ is a function in $L^2(\mathbb{R})$ which satisfies the two-scale relation*

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n\,\phi(2x-n) \quad \textit{with} \quad \sum_{n=-\infty}^{\infty} |c_n| < \infty. \tag{7.3.53}$$

(i) If the function P defined by (7.3.50) satisfies (7.3.52) for all z on the unit circle, $|z| = 1$, and if $\hat{\phi}(0) \neq 0$, then $P(1) = 1$ and $P(-1) = 0$.
(ii) If $P(-1) = 0$, then $\hat{\phi}(n) = 0$ for all nonzero integers n.

Proof. We know that the relation

$$\hat{\phi}(\omega) = \hat{m}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = P\left(e^{\frac{-i\omega n}{2}}\right)\hat{\phi}\left(\frac{\omega}{2}\right) \tag{7.3.54}$$

holds for all $\omega \in \mathbb{R}$. Putting $\omega = 0$ leads to $P(1) = 1$. It follows from Eq. (7.3.52) with $z = 1$ that $P(-1) = 0$.

The proof of part (ii) is left to the reader as an exercise.

We close this section by describing some properties of the coefficients of the scaling function. The coefficients c_n determine all the properties of the scaling function ϕ and the wavelet function ψ. In fact, Mallat's multiresolution algorithm uses the c_n to calculate the wavelet transform without explicit knowledge of ψ. Furthermore, both ϕ and ψ can be reconstructed from the c_n and this in fact is central to Daubechies' wavelet analysis.

Lemma 7.3.5. *If c_n are coefficients of the scaling function defined by (7.3.5), then*

(i) $\displaystyle\sum_{n=-\infty}^{\infty} c_n = \sqrt{2},$ *(ii)* $\displaystyle\sum_{n=-\infty}^{\infty} (-1)^n c_n = 0,$

(iii) $\displaystyle\sum_{n=-\infty}^{\infty} c_{2n} = \frac{1}{\sqrt{2}} = \sum_{n=-\infty}^{\infty} c_{2n+1},$

(iv) $\displaystyle\sum_{n=-\infty}^{\infty} (-1)^n\, n^m\, c_n = 0$ *for* $m = 0, 1, 2, \ldots, (p-1).$

Proof. It follows from (7.3.2) and (7.3.3) that $\hat{\phi}(0) = 0$ and $\hat{m}(0) = 1$. Putting $\omega = 0$ in (7.3.3) gives (i).

Since $\hat{m}(0) = 1$, (7.3.4) implies that $\hat{m}(\pi) = 0$ which gives (ii).

Then, (iii) is a simple consequence of (i) and (ii).

To prove (iv), we recall (7.3.8) and (7.3.3) so that

$$\hat{\phi}(\omega) = \hat{m}\left(\frac{\omega}{2}\right)\hat{m}\left(\frac{\omega}{2^2}\right)\cdots$$

and

$$\hat{m}\left(\frac{\omega}{2^k}\right) = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{in\omega}{2^k}\right).$$

Clearly,

$$\hat{\phi}(2\pi) = \hat{m}(\pi)\,\hat{m}\left(\frac{\omega}{2}\right).$$

According to Strang's (1989) accuracy condition, $\hat{\phi}(\omega)$ must have zeros of the highest possible order when $\omega = 2\pi, 4\pi, 6\pi, \ldots$. Thus,

$$\hat{\phi}(2\pi) = \hat{m}(\pi)\,\hat{m}\left(\frac{\omega}{2}\right)\,\hat{m}\left(\frac{\omega}{2^2}\right)\ldots,$$

and the first factor $\hat{m}(\omega)$ will be zero of order p at $\omega = \pi$ if

$$\frac{d^m\,\hat{m}(\omega)}{d\omega^m} = 0 \quad \text{for } m = 0, 1, 2, \ldots (p-1),$$

which gives

$$\sum_{n=-\infty}^{\infty} c_n(-in)^m\, e^{-in\pi} = 0 \quad \text{for } m = 0, 1, 2, \ldots (p-1).$$

Or, equivalently,

$$\sum_{n=-\infty}^{\infty} (-1)^n\, n^m c_n = 0, \quad \text{for } m = 0, 1, 2, \ldots (p-1).$$

From the fact that the scaling function $\phi(x)$ is orthonormal to itself in any translated position, we can show that

$$\sum_{n=-\infty}^{\infty} c_n^2 = 1. \tag{7.3.55}$$

This can be seen by using $\phi(x)$ from (7.3.5) to obtain

$$\int_{-\infty}^{\infty} \phi^2(x)\, dx = 2\sum_m \sum_n c_m c_n \int_{-\infty}^{\infty} \phi(2x-m)\,\phi(2x-n)\, dx$$

where the integral on the right-hand side vanishes due to orthonormality unless $m = n$, giving

$$\begin{aligned}\int_{-\infty}^{\infty} \phi^2(x)\, dx &= 2\sum_{n=-\infty}^{\infty} c_n^2 \int_{-\infty}^{\infty} \phi^2(2x-n)\, dx \\ &= 2\sum_{n=-\infty}^{\infty} c_n^2 \cdot \frac{1}{2}\int_{-\infty}^{\infty} \phi^2(t)\, dt\end{aligned}$$

whence follows (7.3.55).

Finally, we prove

$$\sum_k c_k\, c_{k+2n} = \delta_{0,n}. \tag{7.3.56}$$

We use the scaling function ϕ defined by (7.3.5) and the corresponding wavelet given by (7.3.29) with (7.3.31), that is,

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1} c_{2N-1-n}\, \phi(2x - n)$$

which is, by substituting $2N - 1 - n = k$,

$$= \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^k c_k\, \phi(2x + k - 2N + 1). \tag{7.3.57}$$

We use the fact that mother wavelet $\psi(x)$ is orthonormal to its own translate $\psi(x - n)$ so that

$$\int_{-\infty}^{\infty} \psi(x)\, \psi(x - n)\, dx = \delta_{0,n}. \tag{7.3.58}$$

Substituting (7.3.57) to the left-hand side of (7.3.58) gives

$$\int_{-\infty}^{\infty} \psi(x)\, \psi(x - n)\, dx$$
$$= 2 \sum_k \sum_m (-1)^{k+m} c_k\, c_m \int_{-\infty}^{\infty} \phi(2x + k - 2N + 1)\, \phi(2x + m - 2N + 1 - 2n)\, dx,$$

where the integral on the right-hand side is zero unless $k = m - 2n$ so that

$$\int_{-\infty}^{\infty} \psi(x)\, \psi(x - n)\, dx = 2 \sum_k (-1)^{2(k+n)} c_k\, c_{k+2n} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \phi^2(t)\, dt.$$

This means that

$$\sum_k c_k\, c_{k+2n} = 0 \quad \text{for all } n \neq 0.$$

7.4 Construction of Orthonormal Wavelets

We now use the properties of scaling functions and filters for constructing orthonormal wavelets.

Example 7.4.1 (The Haar Wavelet). Example 7.2.2 shows that spaces of piecewise constant functions constitute a MRA with the scaling function $\phi = \chi_{[0,1)}$. Moreover, ϕ satisfies the dilation equation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n\, \phi(2x - n), \tag{7.4.1}$$

where the coefficients c_n are given by

$$c_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(x)\, \phi(2x - n)\, dx. \tag{7.4.2}$$

Evaluating this integral with $\phi = \chi_{[0,1)}$ gives c_n as follows:

$$c_0 = c_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad c_n = 0 \text{ for } \quad n \neq 0, 1.$$

Consequently, the dilation equation becomes

$$\phi(x) = \phi(2x) + \phi(2x - 1). \tag{7.4.3}$$

This means that $\phi(x)$ is a linear combination of the even and odd translates of $\phi(2x)$ and satisfies a very simple *two-scale* relation (7.4.3), as shown in Fig. 7.4.

In view of (7.3.34), we obtain

$$d_0 = c_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad d_1 = -c_0 = -\frac{1}{\sqrt{2}}.$$

Thus, the Haar mother wavelet is obtained from (7.3.33) as a simple *two-scale relation*

$$\psi(x) = \phi(2x) - \phi(2x - 1) \tag{7.4.4}$$

$$= \chi_{[0,.5]}(x) - \chi_{[.5,1]}(x)$$

$$= \begin{cases} +1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases} \tag{7.4.5}$$

This two-scale relation (7.4.4) of ψ is represented in Fig. 7.5.

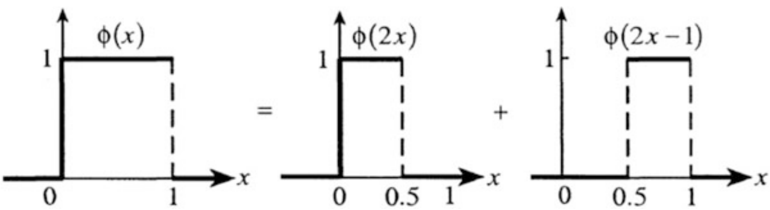

Fig. 7.4 Two-scale relation of $\boldsymbol{\phi(x) = \phi(2x) + \phi(2x - 1)}$

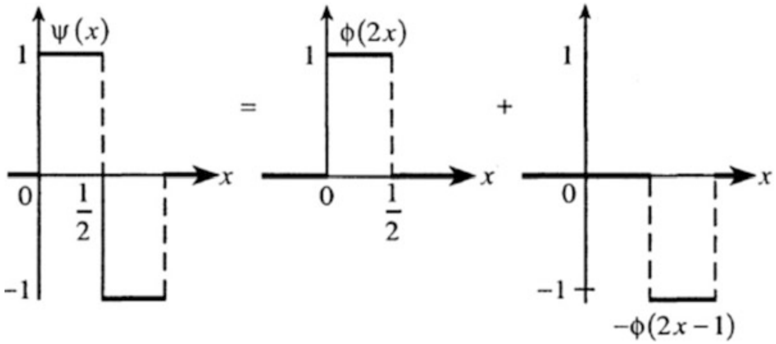

Fig. 7.5 Two-scale relation of $\psi(x) = \phi(2x) - \phi(2x-1)$

Alternatively, the Haar wavelet can be obtained from the Fourier transform of the scaling function $\phi = \chi_{[0,1]}$ so that

$$\hat{\phi}(\omega) = \hat{\chi}_{[0,1]}(\omega) = \exp\left(-\frac{i\omega}{2}\right)\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

$$= \exp\left(-\frac{i\omega}{4}\right)\cos\left(\frac{\omega}{4}\right).\exp\left(-\frac{i\omega}{4}\right)\frac{\sin\left(\frac{\omega}{4}\right)}{\left(\frac{\omega}{4}\right)}$$

$$= \hat{m}\left(\frac{\omega}{2}\right)\cdot\hat{\phi}\left(\frac{\omega}{2}\right), \tag{7.4.6}$$

where the associated filter $\hat{m}(\omega)$ and its complex conjugate are given by

$$\hat{m}(\omega) = \exp\left(-\frac{i\omega}{2}\right)\cos\left(\frac{\omega}{2}\right) = \frac{1}{2}\left(1 + e^{-i\omega}\right), \tag{7.4.7}$$

$$\overline{\hat{m}}(\omega) = \exp\left(\frac{i\omega}{2}\right)\cos\left(\frac{\omega}{2}\right) = \frac{1}{2}\left(1 + e^{i\omega}\right). \tag{7.4.8}$$

Thus, the Haar wavelet can be obtained form (7.3.24) or (7.3.40) and is given by

$$\hat{\psi}(\omega) = \hat{\nu}(\omega)\exp\left(\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

$$= \hat{\nu}(\omega)\cdot\exp\left(\frac{i\omega}{2}\right)\cdot\frac{1}{2}\left(1 - e^{\frac{i\omega}{2}}\right)\cdot\hat{\phi}\left(\frac{\omega}{2}\right)$$

where $\hat{\nu}(\omega) = -i\exp(-i\omega)$ is chosen to find the exact result (7.4.4). Using this value for $\hat{\nu}(\omega)$, we obtain

$$\hat{\psi}(\omega) = \frac{1}{2}\,\hat{\phi}\left(\frac{\omega}{2}\right) - \frac{1}{2}\exp\left(-\frac{i\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

so that the inverse Fourier transform gives the exact result (7.4.4) as

$$\psi(x) = \phi(2x) - \phi(2x-1).$$

On the other hand, using (7.3.24) also gives the Haar wavelet as

$$\begin{aligned}
\hat{\psi}(\omega) &= \exp\left(\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2}+\pi\right)\hat{\phi}\left(\frac{\omega}{2}\right)\\
&= \exp\left(\frac{i\omega}{2}\right)\cdot\frac{1}{2}\left(1-e^{\frac{i\omega}{2}}\right)\cdot\frac{\left\{1-\exp\left(\frac{-i\omega}{2}\right)\right\}}{\left(\frac{i\omega}{2}\right)} \qquad (7.4.9)\\
&= \frac{i}{\omega}\left(1-e^{\frac{i\omega}{2}}\right)^2\\
&= \frac{i}{\omega}\left(e^{\frac{i\omega}{4}}\cdot e^{-\frac{i\omega}{4}} - e^{\frac{i\omega}{4}}\cdot e^{\frac{i\omega}{4}}\right)^2\\
&= -i\exp\left(\frac{i\omega}{2}\right)\cdot\left[\frac{\sin^2\left(\frac{\omega}{4}\right)}{\left(\frac{\omega}{4}\right)}\right]\\
&= \left\{i\exp\left(-\frac{i\omega}{2}\right)\frac{\sin^2\left(\frac{\omega}{4}\right)}{\left(\frac{\omega}{4}\right)}\right\}\left\{-\exp(-i\omega)\right\}. \qquad (7.4.10)
\end{aligned}$$

This corresponds to the same Fourier transform (6.2.7) of the Haar wavelet (7.4.5) except for the factor $-\exp(-i\omega)$. This means that this factor induces a translation of the Haar wavelet to the left by one unit. Thus, we have chosen $\hat{v}(\omega) = -\exp(-i\omega)$ in (7.3.40) to find the same value (7.4.5) for the classic Haar wavelet.

Example 7.4.2 (Cardinal B-splines and Spline Wavelets). The cardinal B-splines (basis splines) consist of functions in $C^{n-1}(\mathbb{R})$ with equally spaced integer knots that coincide with polynomials of degree n on the intervals $\left[2^{-m}k, w^{-m}(k+1)\right]$. These B-splines of order n with compact support generate a linear space V_0 in $L^2(\mathbb{R})$. This leads to a MRA $\{V_m, m \in \mathbb{Z}\}$ by defining $f(x) \in V_m$ if and only if $f(2x) \in V_{m+1}$.

The cardinal B-splines $B_n(x)$ of order n are defined by the following convolution product

$$B_1(x) = \chi_{[0,1]}(x), \qquad (7.4.11)$$

$$B_n(x) = B_1(x) * B_1(x) * \cdots * B_1(x) = B_1(x) * B_{n-1}(x), \quad (n \geq 2), \tag{7.4.12}$$

where n factors are involved in the convolution product. Obviously,

$$B_n(x) = \int_{-\infty}^{\infty} B_{n-1}(x-t)\, B_1(t)\, dt = \int_0^1 B_{n-1}(x-t)\, dt = \int_{x-1}^{x} B_{n-1}(t)\, dt. \tag{7.4.13}$$

Using the formula (7.4.13), we can obtain the explicit representation of splines $B_2(x)$, $B_3(x)$, and $B_4(x)$ as follows:

$$B_2(x) = \int_{x-1}^{x} B_1(t)\, dt = \int_{x-1}^{x} \chi_{[0,1]}(t)\, dt.$$

Evidently, it turns out that

$$B_2(x) = 0 \quad \text{for} \quad x \leq 0.$$

$$B_2(x) = \int_0^x dt = x \quad \text{for} \quad 0 \leq n \leq 1, \quad (x-1 \leq 0).$$

$$B_2(x) = \int_{x-1}^{1} dt = 2-x \quad \text{for} \quad 0 \leq n \leq 2,\ (- \leq x-1 \leq 1 \leq x).$$

$$B_2(x) = 0 \quad \text{for} \quad 2 \leq x,\ (1 \leq x-1).$$

Or, equivalently,

$$B_2(x) = x\, \chi_{[0,1]}(x) + (2-x)\chi_{[1,2]}(x). \tag{7.4.14}$$

Similarly, we find

$$B_3(x) = \int_{x-1}^{x} B_2(x)\, dx.$$

More explicitly,

$$B_3(x) = 0 \quad \text{for} \quad x \leq 0.$$

$$B_3(x) = \int_0^x t\, dt = \frac{x^2}{2} \quad \text{for} \quad 0 \leq x \leq 1, \quad (x-1 \leq 0 \leq x \leq 1).$$

$$B_3(x) = \int_{x-1}^{1} t\, dt + \int_1^x (2-t)\, dt \quad \text{for} \quad 1 \leq x \leq 2, \quad (0 \leq x-1 \leq 1 \leq x \leq 2)$$

$$= \frac{1}{2}(6x - 2x^2 - 3) \quad \text{for} \quad 1 \leq x \leq 2.$$

$$B_3(x) = \int_{x-1}^{2} (2-t)\, dt = \frac{1}{2}(x-3)^2 \quad \text{for} \quad 2 \leq x \leq 3, \quad (1 \leq x-1 \leq 2 \leq x \leq 3)$$

$$B_3(x) = 0 \quad \text{for} \quad x \geq 3, \quad (2 \leq x-1).$$

Or, equivalently,

$$B_3(x) = \frac{x^2}{2}\,\chi_{[0,1]} + \frac{1}{2}\,(6x - 2x^2 - 3)\,\chi_{[1,2]} + \frac{1}{2}\,(x-3)^2\,\chi_{[2,3]}. \tag{7.4.15}$$

Finally, we have

$$B_4(x) = \int_{x-1}^{x} B_3(t)\,dt.$$

$$B_4(x) = 0 \quad \text{for} \quad x-1 \le -1 \le x \le 0.$$

$$B_4(x) = \int_0^x \left(\frac{1}{2}t^2\right) dt = \frac{1}{6}x^3 \quad \text{for} \quad -1 \le x-1 \le 0 \le x < 1.$$

$$B_4(x) = \int_{x-1}^{1} \left(\frac{1}{2}t^2\right) dt + \int_1^x \left(-\frac{3}{2} + 3t - t^2\right) dt \quad \text{for} \quad 1 \le x \le 2, (\,0 \le x-1 \le 1 \le x \le 2)$$

$$= \frac{2}{3} - 2x + 2x^2 - \frac{1}{2}x^3 \quad \text{for} \quad 1 \le x \le 2.$$

$$B_4(x) = \int_{x-1}^{2} \left(-\frac{3}{2} + 3t - t^2\right) dt + \frac{1}{2}\int_2^x (3-t)^2 dt \quad \text{for} \quad 1 \le x-1 \le 2 \le x \le 3$$

$$= \frac{1}{2}(x^3 - 2x^2 + 20x - 13) \quad \text{for} \quad 2 \le x \le 3.$$

Or, equivalently,

$$B_4(x) = \frac{1}{6}\,x^3\,\chi_{[0,1]} + \frac{1}{3}\,(2 - 6x + 6x^2 - x^3)\,\chi_{[1,2]} + \frac{1}{2}\,(x^3 - 2x^2 + 20x - 13)\,\chi_{[2,3]}. \tag{7.4.16}$$

In order to obtain the two-scale relation for the B-splines of order n, we apply the Fourier transform of (7.4.11) so that

$$\hat{B}_1(\omega) = \exp\left(-\frac{i\omega}{2}\right) \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)} = \exp\left(-\frac{i\omega}{2}\right) \operatorname{sinc}\left(\frac{\omega}{2}\right), \tag{7.4.17a}$$

$$= \frac{1}{i\omega}\left(1 - e^{-i\omega}\right) = \int_0^1 e^{-i\omega t} dt, \tag{7.4.17b}$$

where the sine function, $\operatorname{sinc}(x)$ is defined by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{7.4.18}$$

We can also express (7.4.17a) in terms of $z = \exp\left(-\frac{i\omega}{2}\right)$ as

$$\hat{B}_1(\omega) = \frac{1}{2}(1+z)\,\hat{B}_1\left(\frac{\omega}{2}\right). \tag{7.4.19}$$

Application of the convolution theorem of the Fourier transform to (7.4.12) gives

$$\hat{B}_n(\omega) = \left\{\hat{B}_1(\omega)\right\}^n = \hat{B}_1(\omega)\hat{B}_{n-1}(\omega), \tag{7.4.20}$$

$$= \left(\frac{1+z}{2}\right)^n \left\{\hat{B}_1\left(\frac{\omega}{2}\right)\right\}^n = \left(\frac{1+z}{2}\right)^n \hat{B}_n\left(\frac{\omega}{2}\right), \tag{7.4.21}$$

$$= \hat{M}_n\left(\frac{\omega}{2}\right)\hat{B}_n\left(\frac{\omega}{2}\right), \tag{7.4.22}$$

where the associated filter $\hat{M}_n$ is given by

$$\hat{M}_n\left(\frac{\omega}{2}\right) = \left(\frac{1+z}{2}\right)^n = \frac{1}{2^n}\left(1+e^{-\frac{i\omega}{2}}\right)^n = e^{-\frac{i\omega n}{2}}\left(\cos\frac{\omega}{2}\right)^n$$
$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\exp\left(-\frac{in\omega}{2}\right), \tag{7.4.23}$$

which is, by definition of $\hat{M}_n\left(\frac{\omega}{2}\right)$,

$$= \frac{1}{\sqrt{2}}\sum_{k=-\infty}^{\infty} c_{n,k}\,\exp\left(-\frac{ik\omega}{2}\right). \tag{7.4.24}$$

Obviously, the coefficients $c_{n,k}$ are given by

$$c_{n,k} = \begin{cases} \frac{\sqrt{2}}{2^n}\binom{n}{k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases} \tag{7.4.25}$$

Therefore, the spline function in the time domain is

$$B_n(x) = \sqrt{2}\sum_{k=0}^{\infty} c_{n,k}\,\phi(2x-k) = \sum_{k=0}^{n} 2^{1-n}\binom{n}{k} B_n(2x-k). \tag{7.4.26}$$

This may be referred to as the two-scale relation for the B-splines of order n.

In view of (7.4.17a), it follows that

$$\left|\hat{B}_n(\omega)\right| = \left|\operatorname{sinc}\left(\frac{\omega}{2}\right)\right|^n, \tag{7.4.27}$$

where $\operatorname{sinc}(x)$ is defined by (7.4.18). Thus, for each $n \geq 1$, $\hat{B}_n(\omega)$ is a first order *Butterworth filter* which satisfies the following conditions

$$\left|\hat{B}_n(0)\right| = 1, \left[\frac{d}{d\omega}\left|\hat{B}_n(\omega)\right|\right]_{\omega=0} = 0, \quad \text{and} \quad \left[\frac{d^2}{d\omega^2}\left|\hat{B}^2{}_n(\omega)\right|\right]_{\omega=0} \neq 0. \tag{7.4.28}$$

The graphical representation of $B_n(x)$ and their filter characteristics $\left|\hat{B}_n(\omega)\right|$ are shown in Figs. 7.6 and 7.7.

It is evident from (7.4.27) that

$$\left|\hat{B}_n(\omega + 2\pi k)\right|^2 = \frac{\sin^{2n}\left(\frac{\omega}{2} + \pi k\right)}{\left(\frac{\omega}{2} + \pi k\right)^{2n}} = \frac{\sin^{2n}\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2} + \pi k\right)^{2n}}. \tag{7.4.29}$$

We replace ω by 2ω in (7.4.29) and then sum the result over all integers k to obtain

$$\sum_{k=-\infty}^{\infty}\left|\hat{B}_n(2\omega + 2\pi k)\right|^2 = \sin^{2n}\omega \sum_{k=-\infty}^{\infty}\frac{1}{(\omega + \pi k)^{2n}}. \tag{7.4.30}$$

It is well known in complex analysis that

$$\sum_{k=-\infty}^{\infty}\frac{1}{(\omega + \pi k)} = \cot\omega, \tag{7.4.31}$$

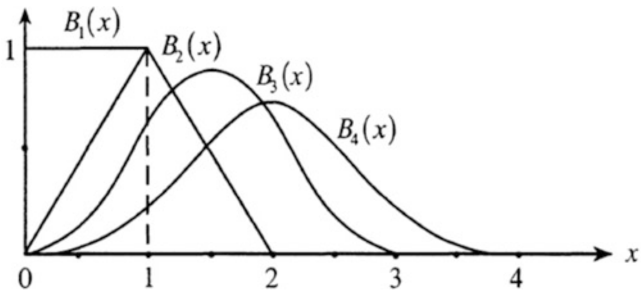

Fig. 7.6 Cardinal B-spline functions

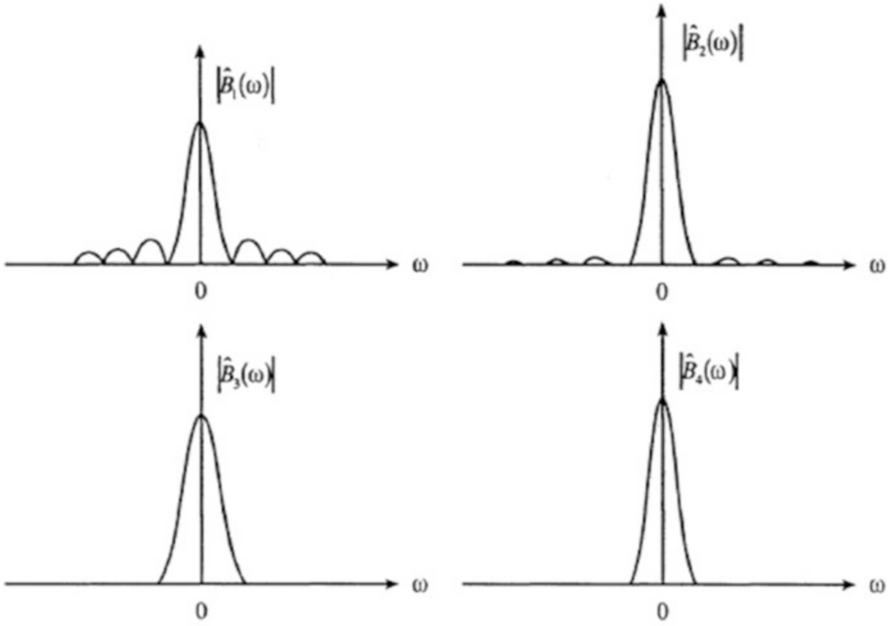

Fig. 7.7 Fourier transforms of cardinal B-spline functions

which leads to the following result after differentiating $(2n-1)$ times

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega+\pi k)^{2n}} = -\frac{1}{(2n-1)!}\,\frac{d^{2n-1}}{d\omega^{2n-1}}(\cot\omega). \tag{7.4.32}$$

Substituting this result in (7.4.30) yields

$$\sum_{k=-\infty}^{\infty} \left|\hat{B}_n(2\omega+2\pi k)\right|^2 = -\frac{\sin^{2n}(\omega)}{(2n-1)!}\,\frac{d^{2n-1}}{d\omega^{2n-1}}(\cot\omega). \tag{7.4.33}$$

These results are used to find the Franklin wavelets and the Battle–Lemarié wavelets.

When $n=1$, (7.4.32) gives another useful identity

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega+2\pi k)^2} = \frac{1}{4}\operatorname{cosec}^2\left(\frac{\omega}{2}\right). \tag{7.4.34}$$

Summing (7.4.29) over all integers k and using (7.4.34) leads to

$$\sum_{k=-\infty}^{\infty} \left|\hat{B}_1(\omega + 2\pi k)\right|^2 = 4\sin^2\left(\frac{\omega}{2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^2} = 1. \tag{7.4.35}$$

This shows that the first order B-spline $B_1(x)$ defined by (7.4.11) is a scaling function that generates the classic Haar wavelet.

Example 7.4.3 (The Franklin Wavelet). The Franklin wavelet is generated by the second order ($n = 2$) splines.

Differentiating (7.4.34) $(2n - 2)$ times gives the result

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^{2n}} = \frac{1}{4(2n-1)!} \frac{d^{2n-2}}{d\omega^{2n-2}} \left[\operatorname{cosec}^2\left(\frac{\omega}{2}\right)\right]. \tag{7.4.36}$$

When $n = 2$, (7.4.36) yields the identity

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^4} = \frac{1}{\left(2\sin\frac{\omega}{2}\right)^4} \cdot \left\{1 - \frac{2}{3}\sin^2\left(\frac{\omega}{2}\right)\right\}. \tag{7.4.37}$$

For $n = 2$, we sum (7.4.29) over all integers k so that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left|\hat{B}_2(\omega + 2\pi k)\right|^2 &= 16\sin^4\left(\frac{\omega}{2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^4} \\ &= \left\{1 - \frac{2}{3}\sin^2\frac{\omega}{2}\right\}. \end{aligned} \tag{7.4.38}$$

Or, equivalently,

$$\left[\left\{1 - \frac{2}{3}\sin^2\frac{\omega}{2}\right\}^{-\frac{1}{2}}\right]^2 \sum_{k=-\infty}^{\infty} \left|\hat{B}_2(\omega + 2\pi k)\right|^2 = 1. \tag{7.4.39}$$

Thus, the condition of orthonormality (7.4.33) ensures that the scaling function ϕ has the Fourier transform

$$\hat{\phi}(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2 \left(1 - \frac{2}{3}\sin^2\frac{\omega}{2}\right)^{-\frac{1}{2}}. \tag{7.4.40}$$

Thus, the filter associated with this scaling function ϕ is obtained from (7.3.2) so that

$$\hat{m}(\omega) = \frac{\hat{\phi}(2\omega)}{\hat{\phi}(\omega)} = \left(\frac{\sin\omega}{2\sin\frac{\omega}{2}}\right)^2 \left[\frac{1-\frac{2}{3}\sin^2\frac{\omega}{2}}{1-\frac{2}{3}\sin^2\omega}\right]^{\frac{1}{2}}$$

$$= \cos^2\left(\frac{\omega}{2}\right)\left[\frac{1-\frac{2}{3}\sin^2\frac{\omega}{2}}{1-\frac{2}{3}\sin^2\omega}\right]^{\frac{1}{2}}. \tag{7.4.41}$$

Finally, the Fourier transform of the orthonormal wavelet ψ is obtained from (7.3.24) so that

$$\hat{\psi}(2\omega) = \hat{m}_1(\omega)\,\hat{\phi}(\omega) = e^{i\omega}\,\overline{\hat{m}}(\omega+\pi)\,\hat{\phi}(\omega) \tag{7.4.42}$$

$$= e^{i\omega}\left[\frac{1-\frac{2}{3}\sin^2\frac{\omega}{2}}{1-\frac{2}{3}\sin^2\omega}\right]^{\frac{1}{2}}\left(\frac{\sin^2\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2\left\{1-\frac{2}{3}\sin^2\frac{\omega}{2}\right\}^{-\frac{1}{2}}$$

$$= e^{i\omega}\,\frac{\sin^4\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2}\left[\frac{1-\frac{2}{3}\cos^2\frac{\omega}{4}}{\left(1-\frac{2}{3}\sin^2\frac{\omega}{2}\right)\left(1-\frac{2}{3}\sin^2\frac{\omega}{4}\right)}\right]^{\frac{1}{2}}. \tag{7.4.43}$$

This is known as the *Franklin wavelet* generated by the second order spline function $B_2(x)$. The scaling function ϕ for the Franklin wavelet, the magnitude of its Fourier transform, $\left|\hat{\phi}(\omega)\right|$, the Franklin wavelet ψ, and the magnitude of its Fourier transform $\left|\hat{\psi}(\omega)\right|$ are shown in Figs. 7.8 and 7.9.

Example 7.4.4 (The Battle–Lemarié Wavelet). The Fourier transform $\hat{\phi}(\omega)$ associated with the nth order spline function $B_n(x)$ is

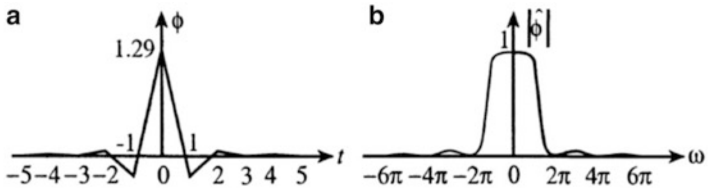

Fig. 7.8 (**a**) Scaling function of the Franklin wavelet ϕ. (**b**) The Fourier transform $|\hat{\phi}|$

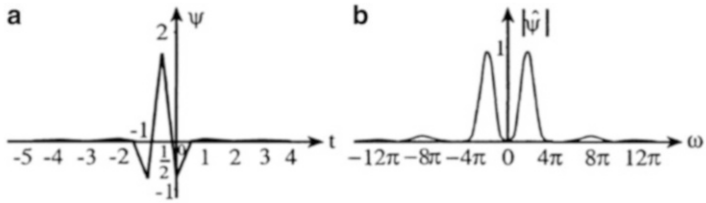

Fig. 7.9 (**a**) The Franklin wavelet ψ. (**b**) The Fourier transform $|\hat{\psi}|$

$$\hat{\phi}(\omega) = \frac{\hat{B}_n(\omega)}{\left\{\sum_{k=-\infty}^{\infty} \left|\hat{B}_n(\omega + 2k\pi)\right|^2\right\}^{\frac{1}{2}}}, \tag{7.4.44}$$

where $\hat{B}_n(\omega)$ is given by (7.4.20) and

$$\left|\hat{B}_n(\omega + 2k\pi)\right|^2 = \left\{\frac{\sin\left(\frac{\omega}{2} + k\pi\right)}{\left(\frac{\omega + 2k\pi}{2}\right)}\right\}^{2n}, \tag{7.4.45}$$

and

$$\left\{\sum_{k=-\infty}^{\infty} \left|\hat{B}_n(\omega + 2k\pi)\right|^2\right\}^{\frac{1}{2}} = \frac{2^n \sin^n\left(\frac{\omega}{2}\right)}{\sqrt{\hat{S}_{2n}(\omega)}}, \tag{7.4.46}$$

with

$$\hat{S}_{2n}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2k\pi)^{2n}}. \tag{7.4.47}$$

Consequently, (7.4.44) can be expressed in the form

$$\hat{\phi}(\omega) = \frac{\left(-\frac{i\varepsilon\omega}{2}\right)}{\omega^n \sqrt{\hat{S}_{2n}(\omega)}}, \tag{7.4.48}$$

where $\varepsilon = 1$ when n is odd or $\varepsilon = 0$ when n is even, and $\hat{S}_{2n}(\omega)$ can be computed by using the formula (7.4.36).

In particular, when $n = 4$, corresponding to the cubic spline of order four, $\hat{\phi}(\omega)$ is calculated from (7.4.48) by inserting

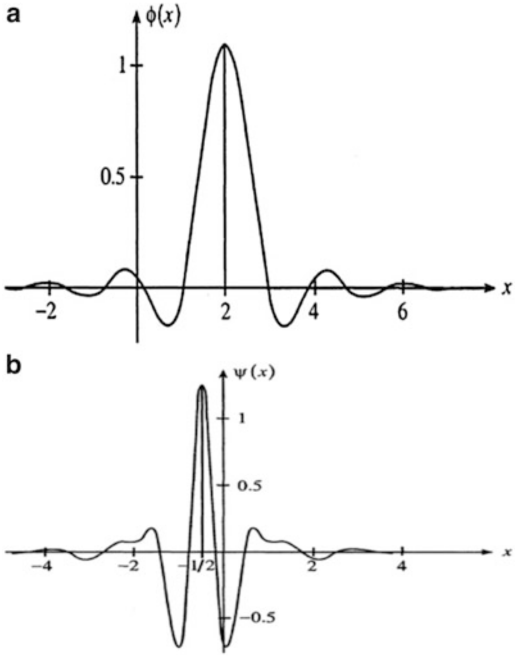

Fig. 7.10 (**a**) The Battle–Lemarié scaling function. (**b**) The Battle–Lemarié wavelet

$$\hat{S}_8(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2k\pi)^8} = \frac{\hat{N}_1(\omega) + \hat{N}_2(\omega)}{(105)\left(2\sin\frac{\omega}{2}\right)^8}, \tag{7.4.49}$$

where

$$\hat{N}_1(\omega) = 5 + 30\cos^2\left(\frac{\omega}{2}\right) + 30\left(\sin\frac{\omega}{2}\cos\frac{\omega}{2}\right)^2, \tag{7.4.50}$$

and

$$\hat{N}_2(\omega) = 70\cos^4\left(\frac{\omega}{2}\right) + 2\sin^4\left(\frac{\omega}{2}\right)\cos^2\left(\frac{\omega}{4}\right) + \frac{2}{3}\sin^6\left(\frac{\omega}{2}\right). \tag{7.4.51}$$

Finally, the Fourier transform of the Battle–Lemarié wavelet ψ can be found by using the same formulas stated in Example 7.4.2. The Battle–Lemarié scaling function ϕ and the Battle–Lemarié wavelet ψ are displayed in Fig. 7.10a, b.

The rest of this section is devoted to the construction of one of the compactly supported orthonormal wavelets first discovered by Daubechies (1988a,b). We suppose that the scaling function ϕ satisfies the two-scale relation

$$\phi(x) = \sum_{n=0}^{1} c_n\,\phi(2x-n) = c_0\,\phi(2x) + c_1\,\phi(2x-n) \tag{7.4.52}$$

for almost all $x \in \mathbb{R}$. We want $\{\phi(x-n) : n \in \mathbb{Z}\}$ to be an orthonormal set, and thus, we impose the necessary condition on the function P

$$\left|P(z)\right|^2 + \left|P(-z)\right|^2 = 1, \quad \left(z \in \mathbb{C}, |z| = 1\right).$$

We also assume $\hat{\phi}(0) = 1$. Then, $P(1) = 1$ and $P(-1) = 0$ by Lemma 7.3.4. Thus, P contains $(1+z)$ as a factor. Since P is a linear polynomial, we construct P with the form

$$P(z) = \frac{(1+z)}{2}\,S(z). \tag{7.4.53}$$

This form ensures that $P(-1) = 0$. The relation $P(1) = 1$ holds if and only if $S(1) = 1$. Indeed, the assumption on P is a particular case of a general procedure where we assume the form

$$P(z) = \left(\frac{1+z}{2}\right)^N\,S(z), \tag{7.4.54}$$

where N is a positive integer to be selected appropriately.

Writing

$$P(z) = \frac{1}{2}\,(1+z)(p_0 + p_1 z)$$

and using $P(1) = 1$ gives

$$p_0 + p_1 = 1. \tag{7.4.55}$$

The result

$$\left|P(i)\right|^2 + \left|P(-i)\right|^2 = 1 \tag{7.4.56}$$

leads to another equation for p_0 and p_1

$$\begin{aligned} 1 &= \frac{1}{4}\left|(p_0 - p_1) + i\,(p_0 + p_1)\right|^2 + \left|(p_0 - p_1) - i\,(p_0 + p_1)\right|^2 \\ &= p_0^2 + p_1^2. \end{aligned} \tag{7.4.57}$$

Solving (7.4.55) and (7.4.57) gives either $p_0 = 1$, $p_1 = 0$ or vice versa. However, the values $p_0 = 1$ and $p_1 = 0$ yield

$$P(z) = \frac{1}{2}\,(1+z).$$

Equating this value of P with its definition (7.3.51) leads to $c_0 = 1$ and $c_1 = 1$. Thus, the scaling function (7.4.52) becomes

$$\phi(x) = \phi(2x) + \phi(2x - 1).$$

This corresponds to the Haar wavelet.

With $N = 2$, we obtain, from (7.4.54),

$$P(z) = \left(\frac{1+z}{2}\right)^2 S(z) = \left(\frac{1+z}{2}\right)^2 (p_0 + p_1 z), \tag{7.4.58}$$

where p_0 and p_1 are determined from $P(1) = 1$ and (7.4.56). It turns out that

$$p_0 + p_1 = 1, \tag{7.4.59}$$

$$p_0^2 + p_1^2 = 2. \tag{7.4.60}$$

Solving these two equations yields either

$$p_0 = \frac{1}{2}\left(1 + \sqrt{3}\right) \quad \text{and} \quad p_1 = \frac{1}{2}\left(1 - \sqrt{3}\right)$$

or vice versa. Consequently, it turns out that

$$P(z) = \frac{1}{4}\left[p_0 + (2p_0 + p_1)z + (p_0 + 2p_1)z^2 + p_1 z^3\right]. \tag{7.4.61}$$

Equating result (7.4.61) with

$$P(z) = \frac{1}{2}\sum_{n=0}^{3} c_n z^n = \frac{1}{2}\left(c_0 + c_1 z + c_2 z^2 + c_3 z^3\right)$$

gives the values for the coefficients

$$c_0 = \frac{1}{4}\left(1 + \sqrt{3}\right), c_1 = \frac{1}{4}\left(3 + \sqrt{3}\right), c_2 = \frac{1}{4}\left(3 - \sqrt{3}\right), c_3 = \frac{1}{4}\left(1 - \sqrt{3}\right). \tag{7.4.62}$$

Consequently, the scaling function becomes

$$\begin{aligned}\phi(x) =& \frac{1}{4}\left(1 + \sqrt{3}\right)\phi(2x) + \frac{1}{4}\left(3 + \sqrt{3}\right)\phi(2x - 1) + \frac{1}{4}\left(3 - \sqrt{3}\right)\phi(2x - 2) \\ &+ \frac{1}{4}\left(1 - \sqrt{3}\right)\phi(2x - 3).\end{aligned} \tag{7.4.63}$$

Or equivalently,

$$\phi(x) = c_0\,\phi(2x)+(1-c_3)\,\phi(2x-1)+(1-c_0)\,\phi(2x-2)+c_3\,\phi(2x-3). \qquad (7.4.64)$$

In the preceding calculation, the factor $\sqrt{2}$ is dropped in the formula (7.3.53) for the scaling function ϕ and hence, we have to drop the factor $\sqrt{2}$ in the wavelet formula (7.3.33) so that $\psi(x)$ takes the form

$$\psi(x) = d_0\,\phi(2x) + d_1\,\phi(2x-1) + d_{-1}\,\phi(2x+1) + d_{-2}\,\phi(2x+2), \qquad (7.4.65)$$

where $d_n = (-1)^n\, c_{1-n}$ is used to find $d_0 = c_1, d_1 = -c_0, d_{-1} = -c_2, d_{-2} = c_3$. Consequently, the final form of $\psi(x)$ becomes

$$\psi(x) = \frac{1}{4}\left(1-\sqrt{3}\right)\phi(2x+2) - \frac{1}{4}\left(3-\sqrt{3}\right)\phi\,(2x+1) + \frac{1}{4}\left(3+\sqrt{3}\right)\phi(2x) - \frac{1}{4}\left(1+\sqrt{3}\right)\phi(2x-1). \qquad (7.4.66)$$

This is called the *Daubechies wavelet*. Daubechies (1992) has shown that in this family of examples the size of the support of ϕ, ψ is determined by the desired regularity. It turns out that this is a general feature and that a linear relationship between these two quantities support width and regularity, is the best. Daubechies (1992) also proved the following theorem.

Theorem 7.4.1. *If* $\phi \in C^m$, *support* $\phi \subset [0, N]$, *and* $\phi(x) = \sum_{n=0}^{N} c_n\,\phi(2x-n)$, *then* $N \geq m+2$.

For proof of this theorem, the reader is referred to Daubechies (1992).

7.5 Daubechies' Wavelets and Algorithms

Daubechies (1988a,b, 1992) first developed the theory and construction of orthonormal wavelets with compact support. Wavelets with compact support have many interesting properties. They can be constructed to have a given number of derivatives and to have a given number of vanishing moments.

We assume that the scaling function ϕ satisfies the dilation equation

$$\phi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} c_n\,\phi(2x-n), \qquad (7.5.1)$$

where $c_n = \langle \phi, \phi_{1,n}\rangle$ and $\sum_{n=-\infty}^{\infty} \left|c_n\right|^2 < \infty$.

If the scaling function ϕ has compact support, then only a finite number of c_n have nonzero values. The associated generating function $\hat{m}$,

$$\hat{m}(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} c_n \exp(-i\omega n) \tag{7.5.2}$$

is a trigonometric polynomial and it satisfies the identity (7.3.4) with special values $\hat{m}(0) = 1$ and $\hat{m}(\pi) = 0$. If coefficients c_n are real, then the corresponding scaling function as well as the mother wavelet ψ will also be real-valued. The mother wavelet ψ corresponding to ϕ is given by the formula (7.3.24) with $\left|\hat{\phi}(0)\right| = 1$. The Fourier transform $\hat{\psi}(\omega)$ of order N is N-times continuously differentiable and it satisfies the moment condition (6.2.16), that is,

$$\hat{\psi}^{(k)}(0) = 0 \quad \text{for} \quad k = 0, 1, \ldots, m. \tag{7.5.3}$$

It follows that $\psi \in C^m$ implies that $\hat{m}_0$ has a zero at $\omega = \pi$ of order $(m+1)$. In other words,

$$\hat{m}_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^{m+1} \hat{L}(\omega), \tag{7.5.4}$$

where $\hat{L}$ is a trigonometric polynomial.

In addition to the orthogonality condition (7.3.4), we assume

$$\hat{m}_0(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^{N} \hat{L}(\omega), \tag{7.5.5}$$

where $\hat{L}(\omega)$ is 2π-periodic and $\hat{L} \in C^{N-1}$. Evidently,

$$\begin{aligned} \left|\hat{m}_0(\omega)\right|^2 &= \hat{m}_0(\omega)\, \hat{m}_0(-\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^{N} \left(\frac{1 + e^{i\omega}}{2}\right)^{N} \hat{L}(\omega)\hat{L}(-\omega) \\ &= \left(\cos^2 \frac{\omega}{2}\right)^{N} \left|\hat{L}(\omega)\right|^2, \end{aligned} \tag{7.5.6}$$

where $\left|\hat{L}(\omega)\right|^2$ is a polynomial in $\cos\omega$, that is,

$$\left|\hat{L}(\omega)\right|^2 = Q(\cos\omega).$$

Since $\cos\omega = 1 - 2\sin^2\left(\frac{\omega}{2}\right)$, it is convenient to introduce $x = \sin^2\left(\frac{\omega}{2}\right)$ so that (7.5.6) reduces to the form

$$\left|\hat{m}_0(\omega)\right|^2 = \left(\cos^2 \frac{\omega}{2}\right)^{N} Q(1 - 2x) = (1 - x)^N P(x), \tag{7.5.7}$$

where $P(x)$ is a polynomial in x.

We next use the fact that

$$\cos^2\left(\frac{\omega+\pi}{2}\right) = \sin^2\left(\frac{\omega}{2}\right) = x$$

and

$$\begin{aligned} \left|\hat{L}(\omega+\pi)\right|^2 &= Q(-\cos\omega) = Q(2x-1) \\ &= Q\big(1-2(1-x)\big) = P(1-x) \end{aligned} \tag{7.5.8}$$

to express the identity (7.3.4) in terms of x so that (7.3.4) becomes

$$(1-x)^N P(x) + x^N P(1-x) = 1. \tag{7.5.9}$$

Since $(1-x)^N$ and x^N are two polynomials of degree N which are relatively prime, then, by Bezout's theorem (see Daubechies 1992), there exists a unique polynomial P_N of degree $\le N-1$ such that (7.5.9) holds. An explicit solution for $P_N(x)$ is given by

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} x^k, \tag{7.5.10}$$

which is positive for $0 < x < 1$ so that $P_N(x)$ is at least a possible candidate for $\left|\hat{L}(\omega)\right|^2$. There also exist higher degree polynomial solutions $P_N(x)$ of (7.5.9) which can be written as

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N+k-1}{k} x^k + x^N R\left(x-\frac{1}{2}\right), \tag{7.5.11}$$

where R is an odd polynomial.

Since $P_N(x)$ is a possible candidate for $\left|\hat{L}(\omega)\right|^2$ and

$$\hat{L}(\omega)\hat{L}(-\omega) = \left|\hat{L}^2(\omega)\right|^2 = Q(\cos\omega) = Q(1-2x) = P_N(x), \tag{7.5.12}$$

the next problem is how to find out $\hat{L}(\omega)$. This can be done by the following lemma:

Lemma 7.5.1 (Riesz Spectral Factorization). *If*

$$\hat{A}(\omega) = \sum_{k=0}^{n} a_k \cos^k \omega, \tag{7.5.13}$$

where $a_k \in \mathbb{R}$ *and* $a_n \neq 0$*, and if* $\hat{A}(\omega) \geq 0$ *for real* ω *with* $\hat{A}(0) = 0$*, then there exists a trigonometric polynomial*

$$\hat{L}(\omega) = \sum_{k=0}^{n} b_k\, e^{-ik\omega} \tag{7.5.14}$$

with real coefficients b_k *with* $\hat{L}(0) = 1$ *such that*

$$\hat{A}(\omega) = \hat{L}(\omega)\hat{L}(-\omega) = \left|\hat{L}(\omega)\right|^2 \tag{7.5.15}$$

is identically satisfied for ω*.*

We refer to Daubechies (1992) for a proof of the Riesz lemma 7.5.1. We also point out that the factorization of $\hat{A}(\omega)$ given in (7.5.15) is not unique.

For a given N, if we select $P = P_N$, then $\hat{A}(\omega)$ becomes a polynomial of degree $N-1$ in $\cos\omega$ and $\hat{L}(\omega)$ is a polynomial of degree $(N-1)$ in $\exp(-i\omega)$. Therefore, the generating function $\hat{m}_0(\omega)$ given by (7.5.5) is of degree $(2N-1)$ in $\exp(-i\omega)$. The interval $[0, 2N-1]$ becomes the support of the corresponding scaling function ${}_N\phi$. The mother wavelet ${}_N\psi$ obtained from ${}_N\phi$ is called the *Daubechies wavelet.*

Example 7.5.1 (The Haar Wavelet). For $N = 1$, it follows from (7.5.10) that $P_1(x) \equiv 1$, and this in turn leads to the fact that $Q(\cos\omega) = 1, \hat{L}(\omega) = 1$ so that the generating function is

$$\hat{m}_0(\omega) = \frac{1}{2}\left(1 + e^{-i\omega}\right). \tag{7.5.16}$$

This corresponds to the generating function (7.4.7) for the Haar wavelet

Example 7.5.2 (The Daubechies Wavelet). For $N = 2$, it follows from (7.5.10) that

$$P_2(x) = \sum_{k=0}^{1} \binom{k+1}{k} x^k = 1 + 2x$$

and hence (7.5.12) gives

$$\left|\hat{L}^2(\omega)\right|^2 = P_2(x) = P_2\left(\sin^2\frac{\omega}{2}\right) = 1 + 2\sin^2\frac{\omega}{2} = (2 - \cos\omega).$$

Using (7.5.14) in Lemma 7.5.1, we obtain that $\hat{L}(\omega)$ is a polynomial of degree $N - 1 = 1$ and

$$\hat{L}(\omega)\,\hat{L}(-\omega) = 2 - \frac{1}{2}\,\left(e^{i\omega} + e^{-i\omega}\right).$$

It follows from (7.5.14) that

$$\left(b_0 + b_1 e^{-i\omega}\right)\left(b_0 + b_1 e^{i\omega}\right) = 2 - \frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right). \tag{7.5.17}$$

Equating the coefficients in this identity gives

$$b_0^2 + b_1^2 = 1 \quad \text{and} \quad 2b_0 b_1 = -1. \tag{7.5.18}$$

These equations admit solutions as

$$b_0 = \frac{1}{2}\left(1 + \sqrt{3}\right) \quad \text{and} \quad b_1 = \frac{1}{2}\left(1 - \sqrt{3}\right). \tag{7.5.19}$$

Consequently, the generating function (7.5.5) takes the form

$$\begin{aligned}\hat{m}_0(\omega) &= \left(\frac{1 + e^{-i\omega}}{2}\right)^2 \left(b_0 + b_1 e^{-i\omega}\right) \\ &= \frac{1}{8}\Big[\left(1 + \sqrt{3}\right) + \left(3 + \sqrt{3}\right) e^{-i\omega} + \left(3 - \sqrt{3}\right) e^{-2i\omega} + \left(1 - \sqrt{3}\right) e^{-3i\omega}\Big]\end{aligned} \tag{7.5.20}$$

with $\hat{m}_0(0) = 1$.

Comparing coefficients of (7.5.20) with (7.3.3) gives c_n as

$$\left.\begin{aligned} c_0 &= \frac{1}{4\sqrt{2}}\left(1 + \sqrt{3}\right), \; c_1 = \frac{1}{4\sqrt{2}}\left(3 + \sqrt{3}\right) \\ c_2 &= \frac{1}{4\sqrt{2}}\left(3 - \sqrt{3}\right), \; c_3 = \frac{1}{4\sqrt{2}}\left(1 - \sqrt{3}\right) \end{aligned}\right\}. \tag{7.5.21}$$

Consequently, the Daubechies scaling function ${}_2\phi(x)$ takes the form, dropping the subscript,

$$\phi(x) = \sqrt{2}\Big[c_0\,\phi(2x) + c_1\,\phi(2x-1) + c_2\,\phi(2x-2) + c_3\,\phi(2x-3)\Big]. \tag{7.5.22}$$

Using (7.3.31) with $N = 2$, we obtain the Daubechies wavelet ${}_2\psi(x)$, dropping the subscript,

$$\begin{aligned}\psi(x) &= \sqrt{2}\Big[d_0\,\phi(2x) + d_1\,\phi(2x-1) + d_2\,\phi(2x-2) + d_3\,\phi(2x-3)\Big] \\ &= \sqrt{2}\Big[-c_3\,\phi(2x) + c_2\,\phi(2x-1) - c_1\,\phi(2x-2) + c_0\,\phi(2x-3)\Big],\end{aligned} \tag{7.5.23}$$

where the coefficients in (7.5.23) are the same as for the scaling function $\phi(x)$, but in reverse order and with alternate terms having their signs changed from plus to minus.

On the other hand, the use of (7.3.29) with (7.3.34) also gives the Daubechies wavelet ${}_2\psi(x)$ in the form

$$ {}_2\psi(x) = \sqrt{2}\Big[-c_0\,\phi(2x-1)+c_1\,\phi(2x)-c_2\,\phi(2x+1)+c_3\,\phi(2x+2)\Big]. \tag{7.5.24}$$

The wavelet has the same coefficients as ψ given in (7.5.23) except that the wavelet is reversed in sign and runs from $x = -1$ to 2 instead of starting from $x = 0$. It is often referred to as the Daubechies $D4$ *wavelet* since it is generated by four coefficients.

However, in general, c's (some positive and some negative) in (7.5.22) are numerical constants. Except for a very simple case, it is not easy to solve (7.5.22) directly to find the scaling function $\phi(x)$. The simplest approach is to set up an iterative algorithm in which each new approximation $\phi_m(x)$ is computed from the previous approximation $\phi_{m-1}(x)$ by the scheme

$$\phi_m(x) = \sqrt{2}\Big[c_0\,\phi_{m-1}(2x) + c_1\,\phi_{m-1}(2x-1) + c_2\,\phi_{m-1}(2x-2) + c_3\,\phi_{m-1}(2x-3)\Big]. \tag{7.5.25}$$

This iteration process can be continued until $\phi_m(x)$ becomes indistinguishable from $\phi_{m-1}(x)$. This iterative algorithm is briefly described below starting from the characteristic function

$$\chi_{[0,1]}(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases} \tag{7.5.26}$$

After one iteration the characteristic function over $0 \le x < 1$ assumes the shape of a staircase function over the interval $0 \le x < 2$. In order to describe the algorithm, we select the set of four coefficients c_0, c_1, c_2, c_3 given in (7.5.21), deleting the factor $\frac{1}{\sqrt{2}}$ in each coefficient so that it produces the Daubechies scaling function $\phi(x)$ given by (7.5.22) and the orthonormal Daubechies wavelet $\psi(x)$ (or $D4$ wavelet) given by (7.5.23) without the factor $\sqrt{2}$.

We represent the characteristic function by the ordinate 1 at $x = 0$. The first iteration generates a new set of four ordinates c_0, c_1, c_2, c_3 at $x = 0.0, 0.5, 1.0, 1.5$. The second iteration with ordinate c_0 at $x = 0$ produces a new set of another four ordinates $c_0^2, c_0\,c_1, c_1\,c_2, c_1\,c_3$ at $x = 0.00, 0.25, 0.75,$ and so on. After completing the second iteration process, there are ten new ordinates $c_0^2, c_0\,c_1, c_0\,c_1 + c_1\,c_0,\ \ c_0\,c_3 + c_1^2,\ \ c_1\,c_2 + c_2\,c_0, c_1\,c_3 + c_2\,c_1,\ \ c_2^2 + c_3\,c_0,\ \ c_2\,c_3 + c_3\,c_1\ \ c_3\,c_2, c_3^2$ at $x = 0.25, 0.50, 0.75, 1.00, \ldots, 2.25$. This iteration process can be described by the matrix scheme

$$\left[{}_2\phi\right] = \begin{bmatrix} c_0 & & & \\ c_1 & & & \\ c_2 & c_0 & & \\ c_3 & c_1 & & \\ & c_2 & c_0 & \\ & c_3 & c_1 & \\ & & c_2 & c_0 \\ & & c_3 & c_1 \\ & & & c_2 \\ & & & c_3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} [1] = M_2 M_1 [1], \tag{7.5.27}$$

where M_n represents the matrix of the order $\left(2^{n+1} + 2^n - 2\right) \times \left(2^n + 2^{n-1} - 2\right)$ in which each column has a submatrix of the coefficients c_0, c_1, c_2, c_3 located two places below the submatrix to its left.

We also use the same matrix scheme for developing the Daubechies wavelet ${}_2\psi(x)$ from ${}_2\phi(x)$ which is given by (7.5.22) without the factor $\sqrt{2}$. For simplicity, we assume that only one iteration process gives the final ${}_2\phi(x)$, so this can be described by four ordinates c_0, c_1, c_2, c_3 at $x = 0.0, 0.50, 1.0, 1.50$. In view of (7.5.23), these four ordinates produce ten new ordinates spaced 0.25 apart. The term $-c_3\phi(2x)$ in (7.5.23) gives $-c_3\,c_0, -c_3\,c_1, -c_3\,c_2. - c_3^2$; the term $c_2\,\phi(2x-1)$ gives $c_2\,c_0, c_2\,c_1, c_2^2, c_2\,c_3$ shifted two places to the right and so on for the other terms, so that the new ten ordinates for the wavelet are given by $-c_3\,c_0, -c_3\,c_1, -c_3\,c_2 + c_2\,c_0, -c_3^2 + c_2\,c_1, c_2^2 - c_1\,c_0, c_2\,c_3 - c_1^2, -c_1\,c_2 + c_0^2, -c_1\,c_3 + c_0\,c_1, c_0\,c_2, c_0\,c_3$. These ordinates are generated by the matrix scheme

$$\left[{}_2\psi\right] = \begin{bmatrix} -c_3 & & & \\ 0 & -c_3 & & \\ c_2 & 0 & -c_3 & \\ 0 & c_2 & 0 & -c_3 \\ -c_1 & 0 & c_2 & 0 \\ 0 & -c_1 & 0 & c_2 \\ c_0 & 0 & -c_1 & 0 \\ & c_0 & 0 & -c_1 \\ & & c_0 & 0 \\ & & & c_0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} [1]. \tag{7.5.28}$$

Or, alternatively, by the matrix scheme

$$\left[{}_2\psi\right] = \begin{bmatrix} c_0 & & & \\ c_1 & & & \\ c_2 & c_0 & & \\ c_3 & c_1 & & \\ & c_2 & c_0 & \\ & c_3 & c_1 & \\ & & c_2 & c_0 \\ & & c_3 & c_1 \\ & & & c_2 \\ & & & c_3 \end{bmatrix} \begin{bmatrix} -c_3 \\ c_2 \\ -c_1 \\ c_0 \end{bmatrix} [1]. \tag{7.5.29}$$

Making reference to Newland (1993b), it can be verified that ${}_3\psi(x)$ can be described by the matrix scheme

$$\left[{}_3\psi\right] = M_3 M_2 \begin{bmatrix} -c_3 \\ c_2 \\ -c_1 \\ c_0 \end{bmatrix} [1], \tag{7.5.30}$$

where the matrix M_3 is of order 22×10 with ten submatrices $[c_0\ c_1\ c_2\ c_3]^T$, each organized two places below its left-hand neighboring matrix.

The matrix scheme (7.5.30) is used to generate wavelets in the inverse DWT. All subsequent steps of the iteration use the matrices M_r consisting of submatrices $[c_0\ c_1\ c_2\ c_3]^T$ staggered vertically two places each. After eight steps leading to 766 ordinates as before, the resulting wavelet is very close to that in Fig. 7.11a.

In order to analyze or synthesize a part of a signal by wavelets, Daubechies (1992) considered the scaling function ϕ defined by (7.5.22) as a *building block* so that

$$\phi(x) = 0 \quad \text{when } x \le 0 \text{ or } x \ge 3. \tag{7.5.31}$$

Daubechies (1992) proved that the scaling function ϕ does not admit any simple algebraic relation in terms of elementary or special functions. She also demonstrated that ϕ satisfies several algebraic relations that play a major role in computational analysis.

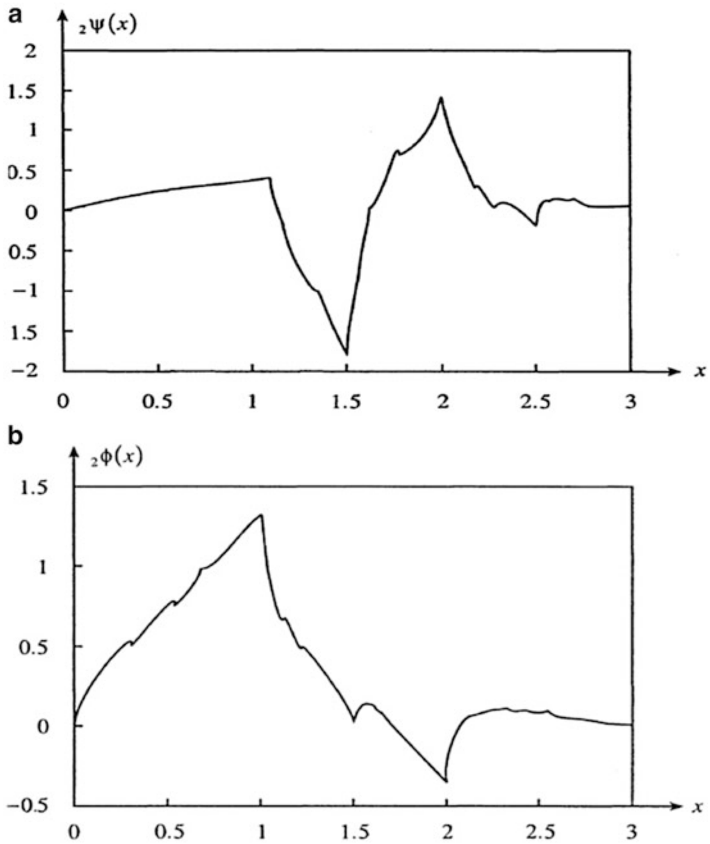

Fig. 7.11 (**a**) The Daubechies wavelet $_2\psi(\mathbf{x})$. (**b**) The Daubechies scaling function $_2\phi(\mathbf{x})$

Replacing x by $\frac{x}{2}$ in (7.5.22) gives

$$\phi\left(\frac{x}{2}\right) = \sqrt{2}\sum_{k=0}^{3} c_k \phi(x-k) \tag{7.5.32}$$

which can be found exactly if $\phi(x), \phi(x-1), \phi(x-2), \phi(x-3)$ are all known. Suppose that we can find $\phi(0), \phi(1), \phi(3)$. It is known that $\phi(-1), \phi(4)$, etc. are all zero. Then, by using (7.5.32), we can calculate

$$\phi\left(\frac{1}{2}\right),\ \phi\left(\frac{3}{2}\right),\ \phi\left(\frac{5}{2}\right).$$

Again, by using (7.5.32) and these new values, we can calculate

$$\phi\left(\frac{1}{4}\right),\ \phi\left(\frac{3}{4}\right),\ \phi\left(\frac{5}{4}\right),\ \phi\left(\frac{7}{4}\right),\ \phi\left(\frac{9}{4}\right),\ \phi\left(\frac{11}{4}\right),$$

and so on.

In order to carry out this recursive process, we set initial values

$$\phi(0) = 0, \quad \phi(1) = \frac{1}{2}\left(1+\sqrt{3}\right), \quad \phi(2) = \frac{1}{2}\left(1-\sqrt{3}\right), \quad \phi(3) = 0. \tag{7.5.33}$$

For example, for $x = 1$, we obtain from (7.5.32) that

$$\phi\left(\frac{1}{2}\right) = \sqrt{2}\left[c_0\,\phi(1) + c_1\,\phi(0) + c_2\,\phi(-1) + c_3\,\phi(-2)\right]$$

which is, by (7.5.21) and (7.5.31),

$$= \sqrt{2}\,c_0\,\phi(1) = \frac{\left(1+\sqrt{3}\right)^2}{8} = \frac{1}{4}\left(2+\sqrt{3}\right).$$

Similarly, we can calculate $\phi\left(\frac{3}{2}\right)$, $\phi\left(\frac{5}{2}\right)$ so that

$$x = \frac{1}{2}, \quad \frac{3}{2}, \quad \frac{5}{2},$$

$$\phi(x) = \frac{1}{4}\left(2+\sqrt{3}\right), \quad 0, \quad \frac{1}{4}\left(2-\sqrt{3}\right),$$

and $\phi(x \geq 3) = 0$.

A similar calculation gives the values of ϕ at multiples of $\frac{1}{4}$ as given below:

$$x = \frac{1}{4}, \quad \frac{3}{4}, \quad \frac{5}{4}, \quad \frac{7}{4}, \quad \frac{9}{4},$$

$$\phi(x) = \frac{5+3\sqrt{3}}{16}, \quad \frac{9+5\sqrt{3}}{16}, \quad \frac{2\left(1+\sqrt{3}\right)}{16}, \quad \frac{2\left(1-\sqrt{3}\right)}{16}, \quad \frac{9-5\sqrt{3}}{16}.$$

The Daubechies wavelet $\psi(x)$ is given by (7.5.24). In view of (7.5.31), it turns out that $\psi(x) = 0$ if $2x + 2 \leq 0$ or $2x - 1 \geq 3$, that is, $\psi(x) = 0$ for $x \leq -1$ or $x \geq 2$. Hence, ψ can be computed from (7.5.24) with (7.5.21) and (7.5.33). For example,

$$\begin{aligned}\psi(0) &= \sqrt{2}\left[c_3\,\phi(2) - c_2\,\phi(1) + c_1\,\phi(0) - c_0\,\phi(-1)\right]\\ &= \sqrt{2}\left[c_3\,\phi(2) - c_3\,\phi(1)\right] = \left(\frac{1-\sqrt{3}}{4}\right)\left(\frac{1-\sqrt{3}}{2}\right) - \left(\frac{3-\sqrt{3}}{4}\right)\left(\frac{1+\sqrt{3}}{2}\right)\\ &= \frac{1}{2}\left(1-\sqrt{3}\right).\end{aligned}$$

Consequently, $\psi(x)$ at $x = -1, -\frac{1}{2}, 0, 1, \frac{3}{2}$ is given as follows:

$$x = -1, \quad -\frac{1}{2}, \quad 0, \quad 1, \quad \frac{3}{2},$$

$$\psi(x) = 0, \quad -\frac{1}{4}, \quad \frac{1}{2}\left(1-\sqrt{3}\right), \quad -\frac{1}{2}\left(1+\sqrt{3}\right), \quad -\frac{1}{4}.$$

Both Daubechies' scaling function ϕ and Daubechies' wavelet ψ for $N = 2$ are shown in Fig. 7.11a, b, respectively.

In view of its fractal shape, the Daubechies wavelet ${}_2\psi(x)$ given in Fig. 7.11a has received tremendous attention so that it can serve as a basis for signal analysis. According to Strang's (1989) analysis, a wavelet expansion based on the $D4$ wavelet represents a linear function $f(x) = ax$ exactly, where a is a constant. Six wavelet coefficients are needed to represent $f(x) = ax + bx^2$, where a and b are constants. In general, more wavelet coefficients are necessary to represent a polynomial with terms like x^n. Figure 7.12a, b exhibits wavelets with $N = 3, 5, 7$, and 10 coefficients. The range of these wavelets is always $(2N - 1)$ unit intervals so that more wavelet coefficients generate longer wavelets. As N increases, wavelets lose their irregular shape and become increasingly smooth with a Gaussian harmonic waveform. For $N = 10$, the frequency of the waveform is not constant and some minor irregularities still persist on the right. Each of the wavelets in Fig. 7.12a, b represents the basis for a family of wavelets of different levels and different locations along the x-axis. The only difference is that a wavelet with $2N$ coefficients occupies $(2N - 1)$ unit intervals with the exception of the Haar wavelet which occupies one interval. Wavelets at each level overlap one another and the amount of overlap depends on the number of wavelet coefficients involved.

The recursive method just described above yields the values of the building block $\phi(x)$ and the wavelet $\psi(x)$ only at integral multiples of positive or negative powers of 2. These values are sufficient for equally spaced samples from a signal. Due to the importance of such powers of 2, the idea of a dyadic number and related notation and terminology seem to be useful in wavelet algorithms.

Definition 7.5.1 (Dyadic Number). A number m is called a *dyadic number* if and only if it is an integral multiple of an integral power of 2.

We denote the set of all dyadic numbers by $\mathbb{D}$ and the set of all integral multiples by $\mathbb{D}_n$ for $n \in \mathbb{N}$. A dyadic number has a finite binary expansion, and a dyadic number in $\mathbb{D}_n$ has a binary expansion with at most n binary digits past the binary point.

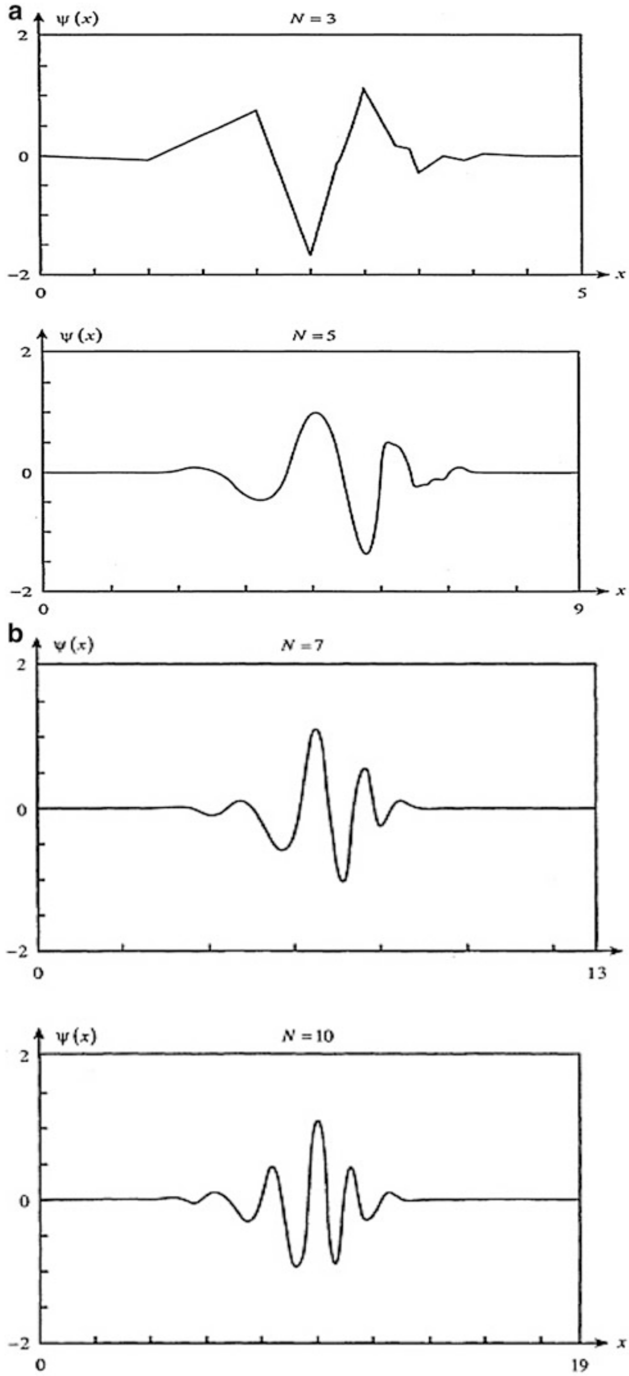

Fig. 7.12 (**a**) Wavelets for $N = 3, 5$ drawn using the Daubechies algorithm. (**b**) Wavelets for $N = 7, 10$ drawn using the Daubechies algorithm

Definition 7.5.2. The set of all linear combinations of 1 and $\sqrt{3}$ with dyadic coefficients $p, q \in \mathbb{D}$ is denoted by $\mathbb{D}\left[\sqrt{3}\right]$ so that

$$\mathbb{D}\left[\sqrt{3}\right] = \left\{p + q\sqrt{3} : p, q \in \mathbb{D}\right\}.$$

For every integer n, we consider combinations with coefficients in $\mathbb{D}_n$ so that

$$\mathbb{D}_n\left[\sqrt{3}\right] = \left\{p + q\sqrt{3} : p, q \in \mathbb{D}_n\right\}.$$

We define the *conjugate* $\overline{m}$ of m by

$$\overline{\left(p + q\sqrt{3}\right)} = \left(p - q\sqrt{3}\right).$$

The set $\mathbb{D}\left[\sqrt{3}\right]$ is an *integer ring* under ordinary addition and multiplication. In terms of two quantities

$$a = \frac{1}{4}\left(1 + \sqrt{3}\right) \quad \text{and} \quad \overline{a} = \frac{1}{4}\left(1 - \sqrt{3}\right), \tag{7.5.34}$$

the scaling function ${}_2\phi$ can be written as

$${}_2\phi(x) = \sqrt{2}\sum_{k=0}^{2N-1} c_k\, \phi(2x - k), \quad (N = 2) \tag{7.5.35a}$$

$$= a\, \phi(2x) + (1 - \overline{a})\, \phi(2x - 1) + (1 + a)\, \phi(2x - 2) + \overline{a}\, \phi(2x - 3). \tag{7.5.35b}$$

If $0 \le m \le 2N - 1$, (7.5.35) can be rewritten as

$$\phi(m) = \sqrt{2}\sum_{k=0}^{2N-1} c_{2m-k}\, \phi(k). \tag{7.5.36}$$

This system of equations can be written in the matrix form

$$\begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 1 - a & 1 - \overline{a} & a & 0 \\ 0 & \overline{a} & 1 - a & 1 - \overline{a} \\ 0 & 0 & 0 & \overline{a} \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \phi(3) \end{bmatrix}. \tag{7.5.37}$$

This system (7.5.37) has exactly one solution,

$$\phi(0) = 0, \quad \phi(1) = 2a, \quad \phi(2) = 2\overline{a}, \quad \phi(3) = 0. \tag{7.5.38}$$

We set $\phi(k) = 0$ for all remaining values of $k \in \mathbb{Z}$. Then, ϕ can recursively be calculated for all of $\mathbb{D}$ by (7.5.35b).

Finally, we conclude this section by including the Daubechies scaling function ${}_3\phi(x)$ and the Daubechies wavelet ${}_3\psi(x)$ for $N = 3$. In this case, (7.5.10) gives

$$P(x) = P_3(x) = 1 + 3x + 6x^2, \tag{7.5.39}$$

where

$$x = \sin^2 \frac{\omega}{2} = \frac{1}{4}\left(-e^{-i\omega} + 2 - e^{i\omega}\right) \quad \text{and}$$
$$x^2 = \frac{1}{16}\left(e^{-2i\omega} + 4 + e^{2i\omega} - 4e^{-i\omega} - 4e^{i\omega} + 2\right).$$

Consequently, (7.5.12) gives the result

$$\left|\hat{L}(\omega)\right|^2 = \frac{3}{8}e^{-2i\omega} - \frac{9}{4}e^{-i\omega} + \frac{19}{4} - \frac{19}{4}e^{i\omega} + \frac{3}{8}e^{2i\omega}. \tag{7.5.40}$$

In this case,

$$A(\omega) = b_0 + b_1 e^{-i\omega} + b_2 e^{-2i\omega}, \tag{7.5.41}$$

so that

$$\begin{aligned}\left|\hat{L}(\omega)\right|^2 &= A(\omega)A(-\omega) = \left(b_0 + b_1e^{-i\omega} + b_2e^{-2i\omega}\right)\left(b_0 + b_1e^{i\omega} + b_2e^{2i\omega}\right)\\ &= \left(b_0^2 + b_1^2 + b_2^2\right) + e^{-i\omega}\left(b_0 b_1 + b_2 b_1\right) + e^{i\omega}\left(b_0 b_1 + b_1 b_2\right) + b_0 b_2 e^{2i\omega} + b_0 b_2 e^{-2i\omega}.\end{aligned} \tag{7.5.42}$$

Equating the coefficients in (7.5.40) and (7.5.42) gives

$$b_0^2 + b_1^2 + b_2^2 = \frac{19}{4}, \quad b_1 b_0 + b_2 b_1 = -\frac{9}{4}, \quad b_2 b_0 = \frac{3}{8}. \tag{7.5.43}$$

In view of the fact that $\left|\hat{L}(0)\right|^2 = 1$ and $P(0) = 1$, the Riesz lemma 7.5.1 ensures that there are real solutions (b_0, b_1, b_2) that satisfy the additional requirement $b_0 + b_1 + b_2 = 1$. Eliminating b_1 from this equation and the second equation in (7.5.43) gives

$$b_1^2 - b_1 - \frac{9}{4} = 0$$

so that

$$b_1 = \frac{1}{2}\left(1 \pm \sqrt{10}\right). \tag{7.5.44}$$

Consequently,

$$b_0 + b_2 = \frac{1}{2}\left(1 \mp \sqrt{10}\right). \tag{7.5.45}$$

The plus and the minus signs in these equations result in complex roots for b_0 and b_2 . This means that the real root for b_1 corresponds to the minus sign in (7.5.44) so that

$$b_1 = \frac{1}{2}\left(1 - \sqrt{10}\right). \tag{7.5.46}$$

Obviously,

$$b_0 + b_2 = \frac{1}{2}\left(1 + \sqrt{10}\right) \quad \text{and} \quad b_0\, b_2 = \frac{3}{8}$$

lead to the fact that b_0 and b_2 satisfy

$$x^2 - \frac{1}{2}\left(1 + \sqrt{10}\right) x + \frac{3}{8} = 0. \tag{7.5.47}$$

Thus,

$$(b_0, b_2) = \frac{1}{4}\left[\left(1 + \sqrt{10}\right) \pm \sqrt{5 + 2\sqrt{10}}\right]. \tag{7.5.48}$$

Consequently, $A(\omega)$ is explicitly known, and hence $\hat{m}(\omega)$ becomes

$$\hat{m}(\omega) = \frac{1}{8}\Big[b_0 + (3b_0 + b_1)e^{-i\omega} + (3b_0 + 3b_1 + b_2)e^{-2i\omega} + (b_0 + 3b_1 + 3b_2)e^{-3i\omega} + (b_1 + 3b_2)e^{-4i\omega} + b_2 e^{-5i\omega}\Big], \tag{7.5.49}$$

which is equal to (7.3.3). Equating the coefficients of (7.3.3) and (7.5.49) gives all six c_k's as

$$c_0 = \frac{\sqrt{2}}{8}\, b_0 = \frac{\sqrt{2}}{32}\left[\left(1 + \sqrt{10}\right) + \sqrt{5 + 2\sqrt{10}}\right],$$

$$c_1 = \frac{\sqrt{2}}{8}(3\,b_0 + b_1) = \frac{\sqrt{2}}{32}\left[\left(5+\sqrt{10}\right) + 3\sqrt{5+2\sqrt{10}}\,\right],$$
$$c_2 = \frac{\sqrt{2}}{8}(3\,b_0 + 3\,b_1 + b_2) = \frac{\sqrt{2}}{32}\left[\left(5-\sqrt{10}\right) + \sqrt{5+2\sqrt{10}}\,\right],$$
$$c_3 = \frac{\sqrt{2}}{8}(b_0 + 3\,b_1 + 3\,b_2) = \frac{\sqrt{2}}{32}\left[\left(5-\sqrt{10}\right) - \sqrt{5+2\sqrt{10}}\,\right],$$
$$c_4 = \frac{\sqrt{2}}{8}(b_1 + 3\,b_2) = \frac{\sqrt{2}}{32}\left[\left(5+\sqrt{10}\right) - 3\sqrt{5+2\sqrt{10}}\,\right],$$
$$c_5 = \frac{\sqrt{2}}{8}\,b_2 = \frac{\sqrt{2}}{32}\left[\left(1+\sqrt{10}\right) - \sqrt{5+2\sqrt{10}}\,\right],$$

Evidently, the Daubechies scaling function ${}_3\phi(x)$ and the Daubechies wavelet ${}_3\psi(x)$ (or simply $D6$ *wavelet*) can be rewritten as

$$ {}_3\phi(x) = \sqrt{2}\sum_{k=0}^{5} c_k\,\phi(2x-k). \tag{7.5.50}$$

$$ {}_3\psi(x) = \sqrt{2}\sum_{k=0}^{5} d_k\,\phi(2x-k). \tag{7.5.51}$$

where c_k and d_k are explicitly known. Figure 7.13a, b exhibits the scaling function ${}_3\phi(x)$ and the wavelet ${}_3\psi(x)$.

With a given even number of wavelet coefficients c_k, $k = 0, 1, \ldots, 2N-1$, we can define the scaling function ϕ by

$$\phi(x) = \sqrt{2}\sum_{k=0}^{2N-1} c_k\,\phi(2x-k) \tag{7.5.52}$$

and the corresponding wavelet by

$$\psi(x) = \sqrt{2}\sum_{k=0}^{2N-1} (-1)^k\,c_k\,\phi(2x+k-2N+1), \tag{7.5.53}$$

where the coefficients c_k satisfy the following conditions

$$\sum_{k=0}^{2N-1} c_k = \sqrt{2}, \qquad \sum_{k=0}^{2N-1} (-1)^k\,k^m\,c_k = 0, \tag{7.5.54a,b}$$

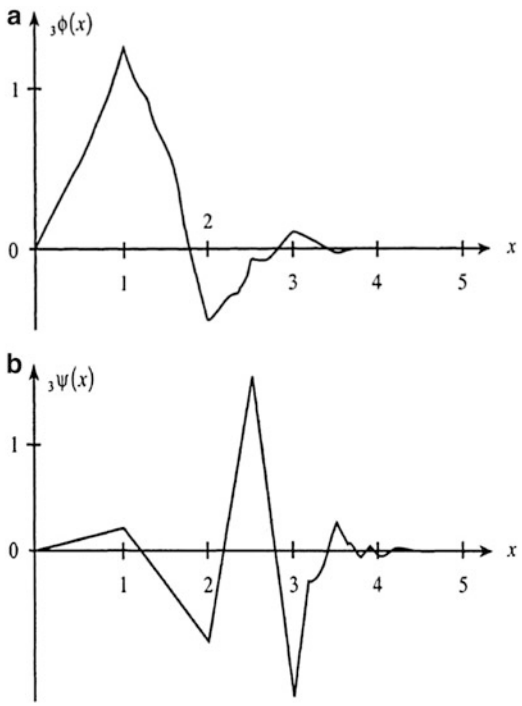

Fig. 7.13 (**a**) The Daubechies scaling function $_3\boldsymbol{\phi}(\mathbf{x})$ for $N = 3$. (**b**) The Daubechies wavelet $_3\boldsymbol{\psi}(\mathbf{x})$ for $N = 3$

where $m = 0, 1, 2, \ldots, N-1$ and

$$\sum_{k=0}^{2N-1} c_k\, c_{k+2m} = 0, \quad m \neq 0, \tag{7.5.55}$$

where $m = 0, 1, 2, \ldots, N-1$ and

$$\sum_{k=0}^{2N-1} c_k^2 = 1. \tag{7.5.56}$$

When $N = 1$, two coefficients, c_0 and c_1, satisfy the following equations:

$$c_0 + c_1 = \sqrt{2}, \quad c_0 - c_1 = 0, \quad c_0^2 + c_1^2 = 1$$

which admit solutions

$$c_0 = c_1 = \frac{1}{\sqrt{2}}.$$

They give the classic Haar scaling function and the Haar wavelet.

When $N = 2$, four coefficients c_0, c_1, c_2, c_3 satisfy the following equations:

$$c_0 + c_1 + c_2 + c_3 = \sqrt{2}, \quad c_0 - c_1 + c_2 - c_3 = 0,$$

$$c_0\, c_2 + c_1\, c_3 = 0, \qquad c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1.$$

These give solutions

$$c_0 = \frac{1}{4\sqrt{2}}\left(1+\sqrt{3}\right), \quad c_1 = \frac{1}{4\sqrt{2}}\left(3+\sqrt{3}\right),$$
$$c_2 = \frac{1}{4\sqrt{2}}\left(3-\sqrt{3}\right), \quad c_3 = \frac{1}{4\sqrt{2}}\left(1-\sqrt{3}\right).$$

These coefficients constitute the Daubechies scaling function (7.5.22) and the Daubechies $D4$ wavelet (7.5.23) or (7.5.24).

7.6 Discrete Wavelet Transforms and Mallat's Pyramid Algorithm

In harmonic analysis, a signal is decomposed into harmonic functions of different frequencies, whereas in wavelet analysis a signal is decomposed into wavelets of different scales (or levels) along the x-axis. Any arbitrary signal $f(x)$ can be decomposed into wavelet components at different scales as

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{m,k}\, \psi\left(2^m x - k\right), \tag{7.6.1}$$

where $c_{m,k}$ are wavelet coefficients.

It is well known that the Haar wavelet is the simplest orthonormal wavelet defined in Example 6.2.1. This wavelet is a member of a family of similar-shaped wavelets of different horizontal scales, each located at a different position of the x-axis. Obviously, there are two half-length wavelets represented by $\psi(2x)$ and $\psi(2x-1)$ and four quarter-length wavelets represented by $\psi(4x)$, $\psi(4x-1)$, $\psi(4x-2)$, and $\psi(4x-3)$, as shown in Fig. 7.14a–c.

The position and scale of each wavelet can be obtained from its argument. For instance, $\psi(2x-1)$ is the same as $\psi(2x)$ except that it is compressed into half the horizontal length and starts at $x = 2^{-1}$ instead of at $x = 0$. The level of the wavelet is determined by how many wavelets fit into the unit interval $0 \le x < 1$. At level 0, there is $2^0 = 1$ wavelet (the Haar wavelet) in each unit interval, as shown in Fig. 7.14a. At level 1, there are $2^1 = 2$ wavelets in the unit interval (see Fig. 7.14b). At level 2, there are $2^2 = 4$ wavelets in the unit interval, as shown in Fig. 7.14c, and so on. On the other hand, at level -1, there is $2^{-1} = \frac{1}{2}$ a wavelet in the unit interval and at level -2 there is $2^{-2} = \frac{1}{4}$ of a wavelet in the unit interval, and so on.

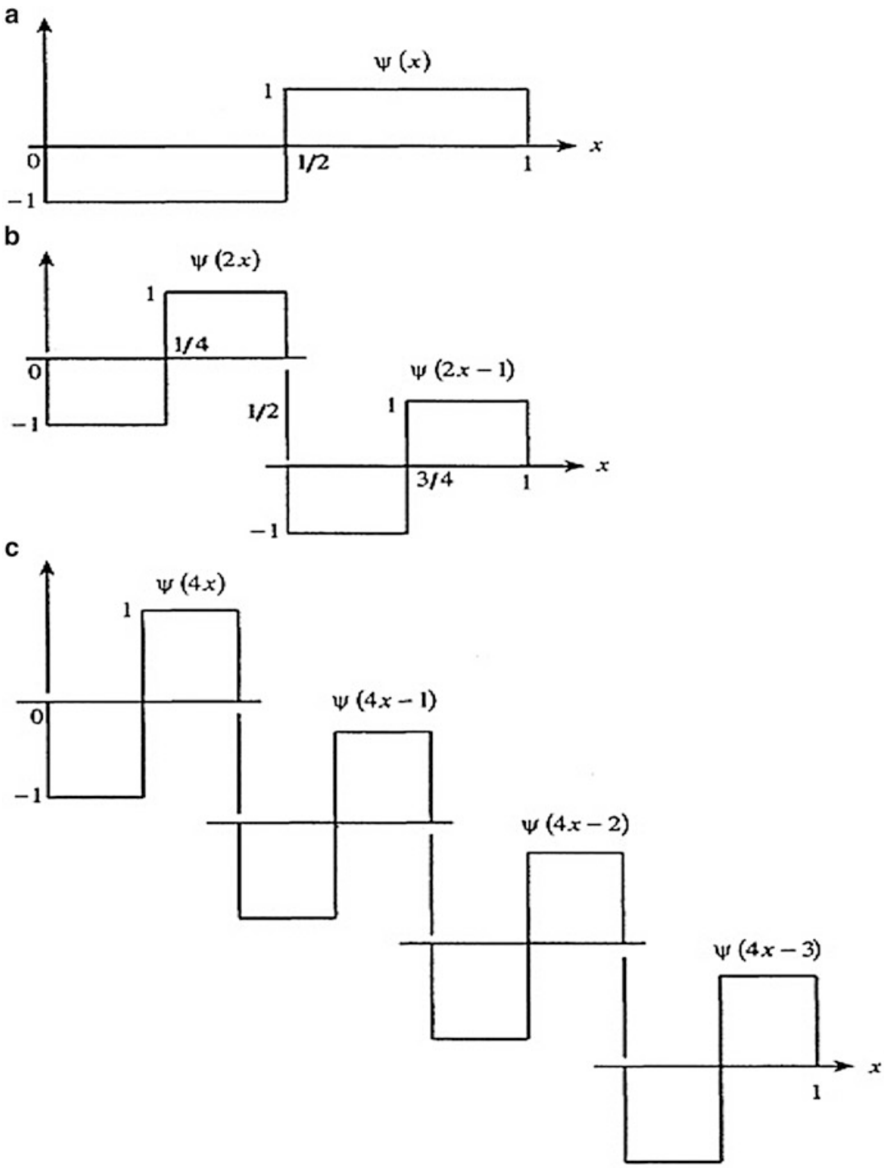

Fig. 7.14 (**a**–**c**) The Haar wavelet at levels $m = 0, 1, 2$

It is shown in Fig. 7.15 that, for all levels less than zero ($m < 0$ or $m \leq -1$), the contribution is constant over each unit interval. Evidently, the sum of the contributions from all of these levels is also constant. It is known that the scaling function $\phi(x)$ for the Haar wavelet is also constant so that $\phi(x) = 1$ for $0 \leq x < 1$. Consequently, the representation (7.6.1) becomes

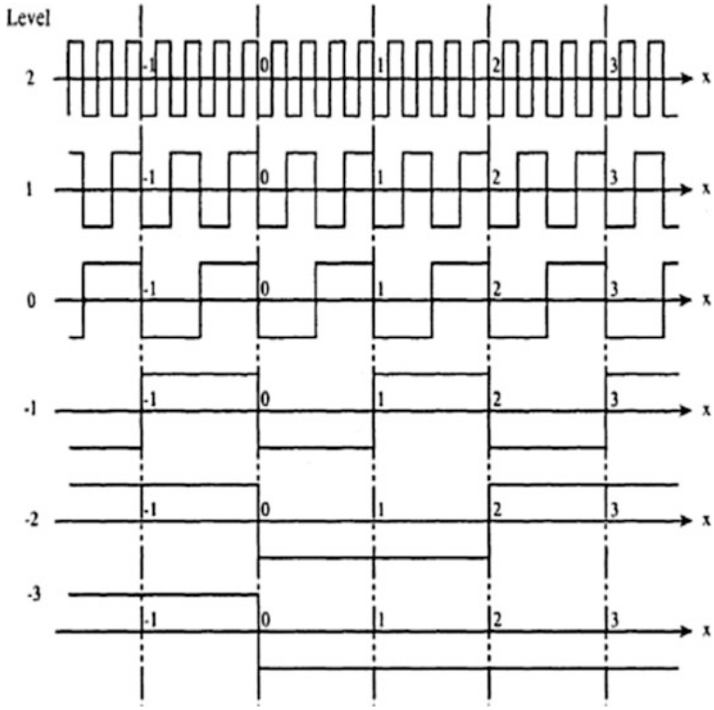

Fig. 7.15 (**a**–**c**) The Haar wavelet at levels $m = 0, 1, 2$

$$f(x) = \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} c_{m,k}\,\psi\left(2^m x - k\right) + \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} c_{m,k}\,\psi\left(2^m x - k\right)$$

$$= \sum_{k=-\infty}^{\infty} c_{\phi,k}\,\phi(x-k) + \sum_{m=0}^{\infty}\sum_{m=-\infty}^{\infty} c_{m,k}\,\psi\left(2^m x - k\right). \tag{7.6.2}$$

Under very general conditions on f and ψ, the wavelet series (7.6.1) and (7.6.2) converge so that they represent a practical basis for signal analysis.

In order to develop a DWT analysis, it is convenient to define $f(x)$ in the unit interval $0 \le x < 1$. If time t is an independent variable for a signal over duration T, then $x = (t/T)$ and $0 \le x < 1$ where x is a dimensionless variable. We assume that $f(x),\ 0 \le x < 1$, is one period of a periodic signal so that the signal is exactly repeated in adjacent unit intervals to yield

$$F(x) = \sum_k f(x-k), \tag{7.6.3}$$

where $f(x)$ is zero outside the interval $0 \le x < 1$.

We consider the Daubechies wavelet $D4$, $\psi(x)$, which occupies three unit intervals $0 \leq x < 3$. In the unit interval $0 \leq x < 1$, $f(x)$ will have contributions from the first third of $\psi(x)$, the middle third of $\psi(x+1)$, and the last third of $\psi(x+2)$. When any wavelet that begins in the interval $0 \leq x < 1$ runs off the end $x = 1$, it may be assumed to be wrapped around the interval several times if there are many coefficients, so that the wavelet extends over many intervals. With this assumption, the wavelet representation (7.6.2) of $f(x)$ in $0 \leq x < 1$ can be expressed as

$$f(x) = a_0\,\phi(x) + a_1\,\psi(x) + \left[a_2\; a_3\right]\begin{bmatrix}\psi(2x)\\ \psi(2x-1)\end{bmatrix} + \left[a_4\; a_5\; a_6\; a_7\right]\begin{bmatrix}\psi(4x)\\ \psi(4x-1)\\ \psi(4x-2)\\ \psi(4x-3)\end{bmatrix}$$
$$+ \cdots + a_{2^m+k}\,\psi\big(2^m x - k\big) + \cdots, \tag{7.6.4}$$

where the coefficients $a_1, a_2, a_3, \ldots$ represent the amplitudes of each of the wavelets after wrapping to one cycle of the periodic function (7.6.3) in $0 \leq x < 1$. Due to the wrapping process, the scaling function $\phi(x)$ always becomes a constant. The second term at $a_1\psi(x)$ is a wavelet of scale zero, the third and fourth terms $a_2\psi(x)$ and $a_3\psi(2x-1)$ are wavelets of scale one, and the second is translated $\triangle x = 2^{-1}$ with respect to the first. The next four terms represent wavelets of scale two and so on for wavelets of increasingly higher scale. The higher the scale, the finer the detail; so there are more coefficients involved. At scale m, there are 2^m wavelets, each spaced $\triangle x = 2^{-m}$ apart along the x-axis.

In view of orthonormal properties, the coefficients can be obtained from

$$\begin{aligned}\int \psi\big(2^m x - k\big)\, f(x)\, dx &= a_{2^m+k} \int \psi^2\big(2^m x - k\big)\, dx \\ &= \frac{1}{2^k}\, a_{2^m+k} \int \psi^2(x)\, dx\end{aligned}$$

and

$$a_{2^m+k} = 2^k \int f(x)\psi\big(2^m x - k\big)\, dx \tag{7.6.5}$$

because $\int \psi^2(x)\, dx = 1$.

In view of the fact that

$$\int_{-\infty}^{\infty} \phi^2(x)\, dx = 1,$$

it follows that the coefficient a_0 is given by

$$a_0 = \int f(x)\, \phi(x)\, dx. \tag{7.6.6}$$

Usually, the limits of integration in the orthogonality conditions are from $-\infty$ to $+\infty$, but the integrand in each case is only nonzero for the finite length of the shortest wavelet or scaling function involved. The limits of integration on (7.6.5) and (7.6.6) may extend over several intervals, provided the wavelets and scaling functions are not wrapped. Since $f(x)$ is one cycle of a periodic function, which repeats itself in adjacent intervals, all contributions to the integrals from outside the unit interval $(0 \le x < 1)$ are included by integrating from $x = 0$ to $x = 1$ for the wrapped functions. Consequently, results (7.6.5) and (7.6.6) can be expressed as

$$a_{2^m+k} = 2^m \int_0^1 f(x)\, \psi\big(2^m x - k\big)\, dx \tag{7.6.7}$$

and

$$a_0 = \int_0^1 f(x)\, \phi(x)\, dx, \tag{7.6.8}$$

where $\phi(x)$ and $\psi(2^m x - k)$ involved in (7.6.7) and (7.6.8) are wrapped around the unit interval $(0 \le x < 1)$ as many times as needed to ensure that their whole length is included in $(0 \le x < 1)$.

The DWT is an algorithm for computing (7.6.7) and (7.6.8) when a signal $f(x)$ is sampled at equally spaced intervals over $0 \le x < 1$. We assume that $f(x)$ is a periodic signal with period one and that the scaling and wavelet functions wrap around the interval $0 \le x < 1$. The integrals (7.6.7) and (7.6.8) can be computed to the desired accuracy by using $\phi(x)$ and $\psi(2^m x - k)$. However, a special feature of the DWT algorithm is that (7.6.7) and (7.6.8) can be computed without generating $\phi(x)$ and $\psi(2^m x - k)$ explicitly. The DWT algorithm was first introduced by Mallat (1989b) and hence is known as *Mallat's pyramid algorithm* (or *Mallat's tree algorithm*). For a detailed information on this algorithm, the reader is also referred to Newland (1993a,b).

7.7 Exercises

1. Show that the two-scale equation associated with the linear spline function

$$B_1(t) = \begin{cases} 1 - |t|, & 0 < |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

is

$$B_1(t) = \frac{1}{2} B_1(2t+1) + B_1(2t) + \frac{1}{2} B_1(2t-1).$$

Hence, show that

$$\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 = 1 - \frac{2}{3}\sin^2\frac{\omega}{2}.$$

2. Use the Fourier transform formula (7.4.43) for $\hat{\psi}(\omega)$ of the Franklin wavelet ψ to show that ψ satisfies the following properties:

(a) $\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)\,dt = 0,$

(b) $\int_{-\infty}^{\infty} t\,\psi(t)\,dt = 0,$

(c) ψ is symmetric with respect to $t = -\frac{1}{2}$.

3. From an expression (7.4.41) for the filter, show that

$$\hat{m}(\omega) = \frac{\left(2 + 3\cos\omega + \cos^2\omega\right)}{\left(1 + 2\cos^2\omega\right)}$$

and hence deduce

$$\hat{\psi}(2\omega) = \exp(-i\omega)\left[\frac{2 - \cos\omega + \cos^2\omega}{1 + 2\cos^2\omega}\right]\hat{\phi}(\omega).$$

4. Using result (7.4.20), prove that

$$\frac{\hat{B}_n(\omega)}{\hat{B}_n\left(\frac{\omega}{2}\right)} = \left(\frac{1 + e^{-\frac{i\omega}{2}}}{2}\right)^n.$$

Hence, derive the following:

(a) $\hat{B}_n(\omega) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\exp\left\{-\frac{ik\omega}{2}\right\}\hat{B}_n\left(\frac{\omega}{2}\right),$

(b) $B_n(t) = \frac{1}{2^{n-1}}\sum_{k=0}^{n}\binom{n}{k} B_n(2t-k).$

5. Obtain a solution of (7.5.22) for the following cases:

(a) $c_0 = c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = c_3 = 0,$

(b) $c_0 = c_2 = \frac{1}{2\sqrt{2}}, \quad c_1 = \frac{1}{\sqrt{2}}, \quad c_3 = 0,$

(c) $c_0 = \sqrt{2}, \quad c_1 = c_2 = c_3 = 0.$

6. If the generating function is defined by (7.3.3), then show that

(a) $\sum_{n=-\infty}^{\infty} c_n = \sqrt{2},$

(b) $\sum_{n=-\infty}^{\infty} c_{2n} = \sum_{n=-\infty}^{\infty} c_{2n+1} = \frac{1}{\sqrt{2}}.$

7. Using the Strang (1989) accuracy condition that $\hat{\phi}(\omega)$ must have zeros of n when $\omega = 2\pi, 4\pi, 6\pi, \ldots$, show that

$$\sum_{k=-\infty}^{\infty} (-1)^k k^m c_k = 0, \quad m = 0, 1, 2, \ldots, (n-1).$$

8. Show that

(a) $\sum_{k=-\infty}^{\infty} c_k^2 = 1,$

(b) $\sum c_k c_{k+2m} = 0, \quad m \neq 0$, where c_k are coefficients of the scaling function defined by (7.3.5).

(c) Derive the result in (b) from the result in Exercise 5.

9. Given six wavelet coefficients c_k $(N = 6)$, write down six equations from (7.5.50a,b)–(7.5.52). Show that these six equations generate the Daubechies scaling function (7.5.50) and the Daubechies $D6$ wavelet (7.5.51).

10. Using the properties of m and $\hat{m}_1$ prove that

(a) $\hat{\phi}\left(\frac{\omega}{2}\right) = \left[\overline{\hat{m}}\left(\frac{\omega}{2}\right) + \overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right)\right]\hat{\phi}(\omega) + \left[\overline{\hat{m}_1}\left(\frac{\omega}{2}\right) + \overline{\hat{m}_1}\left(\frac{\omega}{2} + \pi\right)\right]\hat{\psi}(\omega)$

(b)

$$\exp\left(-\frac{i\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = \left[\exp\left(-\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2}\right) - \exp\left(-\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right)\right]\hat{\phi}(\omega)$$
$$+ \left[\exp\left(-\frac{i\omega}{2}\right)\overline{\hat{m}}_1\left(\frac{\omega}{2}\right) - \exp\left(-\frac{i\omega}{2}\right)\overline{\hat{m}}\left(\frac{\omega}{2} + \pi\right)\right]\hat{\psi}(\omega).$$

11. If $\hat{m}(\omega) = \frac{1}{2}\left(1 + e^{-i\omega}\right)\left(1 - e^{-i\omega} + e^{-2i\omega}\right) = e^{-\frac{3i\omega}{2}} \cos\left(\frac{3\omega}{2}\right)$, show that it satisfies the condition (7.3.4) and $\hat{m}(0) = 1$. Hence, derive the following results

(a) $\hat{\phi}(\omega) = \exp\left(-\frac{3i\omega}{2}\right) \dfrac{\sin\left(\frac{3\omega}{2}\right)}{\left(\frac{3\omega}{2}\right)}$,

(b) $\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 = \frac{1}{9}\left(3 + 4\cos\omega + 2\cos 2\omega\right)$,

(c) $\phi(x) = \begin{cases} \frac{1}{3}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$,

(d) $c_n = \int_{-\infty}^{\infty} \phi(x)\, \overline{\phi(x-n)}\, dx = \frac{1}{3}\int_0^3 \phi(x-n)\, dx = \frac{1}{3}\int_n^{n+3} \phi(x)\, dx.$

12. Show that, for any $x \in [0, 1]$,

(a)

$$\sum_{k=-\infty}^{\infty} \phi(x-k) = 1 \quad \text{and} \quad \text{(b)} \quad \sum_{k=-\infty}^{\infty} (c+k)\, \phi(x-k) = x,$$

where $c = \frac{1}{2}\left(3 - \sqrt{3}\right)$.

Hence, using (a) and (b), show that

(c) $2\phi(x) + \phi(x+1) = x + 2 - c$,
(d) $\phi(x+1) + 2\phi(x+2) = c - x$,
(e) $\phi(x) - \phi(x+2) = x + c + \left(\sqrt{3} - 2\right)$.

13. Use (7.3.31) and (7.4.64) to show that

$$\psi(x) = -c_0\, \phi(2x) + (1 - c_0)\, \phi(2x-1) - (1 - c_3)\, \phi(2x-2) + c_0\, \phi(2x-3).$$

14. Using (7.4.64), prove that $\psi(x)$ defined in Exercise 13 satisfies the following properties:

(a) $\operatorname{supp} \psi(x) \subset [0, 3]$,

(b) $\int_{-\infty}^{\infty} \psi(x)\, \psi(x-k)\, dx = \begin{cases} 0, & k \ne 0 \\ 1, & k = 0 \end{cases}$,

(c) $\int_{-\infty}^{\infty} \psi(x-k)\, \psi(x)\, dx = 0 \quad \text{for all } k \in \mathbb{Z}.$