Chapter 6
The Wavelet Transforms and Their Basic Properties

Wavelets are without doubt an exciting and intuitive concept. The concept brings with it a new way of thinking, which is absolutely essential and was entirely missing in previously existing algorithms.

Today the boundaries between mathematics and signal and image processing have faded, and mathematics has benefitted from the rediscovery of wavelets by experts from other disciplines. The detour through signal and image processing was the most direct path leading from the Haar basis to Daubechies's wavelets.

Yves Meyer

6.1 Introduction

Morlet et al. (1982a,b) modified the Gabor wavelets to study the layering of sediments in a geophysical problem of oil exploration. He recognized certain difficulties of the Gabor wavelets in the sense that the Gabor analyzing function $g_{t,\omega}(\tau) = g(\tau - t)\, e^{i\omega\tau}$ oscillates more rapidly as the frequency ω tends to infinity. This leads to significant numerical instability in the computation of the coefficients $\langle f, g_{\omega,t}\rangle$. On the other hand, $g_{\omega,t}$ oscillates very slowly at low frequencies. These difficulties led to a problem of finding a suitable reconstruction formula. In order to resolve these difficulties, Morlet first made an attempt to use analytic signals $f(t) = a(t) \exp\{i\,\phi(t)\}$ and then introduced the wavelet ψ defined by its Fourier transform

$$\hat{\psi}(\omega) = \sqrt{2\pi}\,\omega^2 \exp\left(-\frac{1}{2}\,\omega^2\right), \quad \omega > 0. \tag{6.1.1}$$

This wavelet corresponds to an analytic signal related to the second derivative $(1 - t^2) \exp\left(-\frac{1}{2}\,t^2\right)$ of the Gaussian function $\exp\left(-\frac{1}{2}\,t^2\right)$. Thus, the Morlet wavelet

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turned out to be the modulated Gaussian function. In fact, Morlet's ingenious idea was to filter the signal $f(t)$ with the aid of the filters $\hat{\psi}(a^m\omega), m \in \mathbb{Z}$ so that

$$f(t) \longrightarrow f_m(t) = \int_{-\infty}^{\infty} f(t-\tau)\, a^{-m}\, \hat{\psi}(a^{-m}\tau)\, d\tau. \tag{6.1.2}$$

Morlet's analysis showed that the quantity $\sum_{m\in\mathbb{Z}} \left|\hat{\psi}(a^{-m}\omega)\right|^2$ remained constant for sufficiently small a. It also led to stable and fast reconstruction algorithms of f from f_m even when $a = 2$. Moreover, Morlet suggested sufficiently small mesh sizes so that they allow a good reconstruction algorithm of analytic signals with coefficients

$$c_{mn} = f_m\big(n2^m\big) = \Big\langle f(t), 2^{-m}\psi\left(2^{-m}t - n\right)\Big\rangle. \tag{6.1.3}$$

Thus, Morlet's remarkable analysis led to the discovery of the wavelet transform which seems to be an efficient and effective time–frequency representation algorithm. The major difference between the Morlet wavelet representation and the Gabor wavelet is that the former has a more and more acute spatial resolution as the frequency gets higher and higher.

Based on the idea of wavelets as a family of functions constructed from translation and dilation of a single function ψ, called the *mother wavelet* (or *affine coherent states*), we define *wavelets* by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\,\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0, \tag{6.1.4}$$

where a is called a *scaling parameter* which measures the degree of compression or scale, and b is a *translation parameter* which determines the time location of the wavelet. Clearly, wavelets $\psi_{a,b}(t)$ generated by the mother wavelet ψ are somewhat similar to the Gabor wavelets $g_{t,\omega}(\tau)$ which can be considered as musical notes that oscillate at the frequency ω inside the envelope defined by $\left|g(\tau - t)\right|$ as a function of τ. If $|a| < 1$, the wavelet (6.1.4) is the compressed version (smaller support in time-domain) of the mother wavelet and corresponds mainly to higher frequencies. Thus, wavelets have time-widths adapted to their frequencies. This is the main reason for the success of the Morlet wavelets in signal processing and time–frequency signal analysis. It may be noted that the resolution of wavelets at different scales varies in the time and frequency domains as governed by the Heisenberg uncertainty principle. At large scale, the solution is coarse in the time domain and fine in the frequency domain. On the other hand, as the scale a decreases, the resolution in the time domain decreases (the time resolution becomes finer), while that in the frequency domain increases (the frequency resolution becomes coarser).

We sketch a typical mother wavelet with a compact support $[-T, T]$ in Fig. 6.1a. Different values of the parameter b represent the time localization center, and each $\psi_{a,b}(t)$ is localized around the center $t = b$. As scale parameter a varies, wavelet

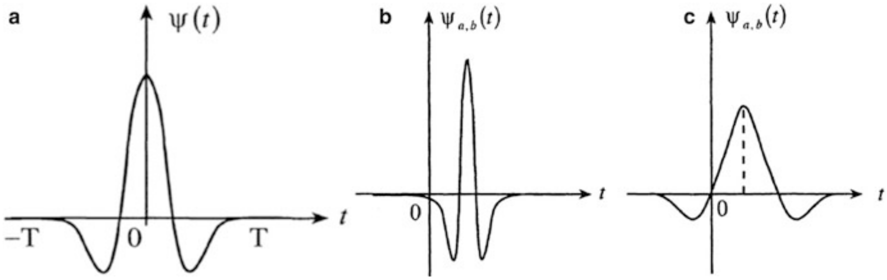

Fig. 6.1 (**a**) Typical mother wavelet. (**b**) Compressed and translated wavelet $\boldsymbol{\psi}_{\mathbf{a,b}}(\mathbf{t})$ with $\mathbf{0} < |\mathbf{a}| \ll \mathbf{1}, \mathbf{b} > \mathbf{0}$. (**c**) Magnified and translated wavelet $\boldsymbol{\psi}_{\mathbf{a,b}}(\mathbf{t})$ with $|a| \gg 1, b > 0$

$\psi_{a,b}(t)$ covers different frequency ranges. Small values of $|a|(0 < |a| \ll 1)$ result in very narrow windows and correspond to high frequencies or very fine scales $\psi_{a,b}$ as shown in Fig. 6.1b, whereas very large values of $|a|\,\big(|a| \gg 1\big)$ result in very wide windows and correspond to small frequencies or very coarse scales $\psi_{a,b}$ as shown in Fig. 6.1c. The wavelet transform (6.2.4) gives a time–frequency description of a signal f. Different shapes of the wavelets are plotted in Fig. 6.1b, c.

It follows from the preceding discussion that a typical mother wavelet physically appears as a local oscillation (or wave) in which most of the energy is localized to a narrow region in the physical space. It will be shown in Sect. 6.2 that the time resolution σ_t and the frequency resolution σ_ω proportional to the scale a and a^{-1}, respectively, and $\sigma_t\,\sigma_\omega \geq 2^{-1}$. When a decreases or increases, the frequency support of the wavelet atom is shifted toward higher or lower frequencies, respectively. Therefore, at higher frequencies, the time resolution becomes finer (better) and the frequency resolution becomes coarser (worse). On the other hand, the time resolution becomes coarser but the frequency resolution becomes finer at lower frequencies.

Morlet first called his functions "wavelets of constant shape" in order to contrast them with the analyzing functions in the short-time Fourier transform which do not have a constant shape. From a group-theoretic point of view, the wavelets $\psi_{a,b}(x)$ are in fact the result of the action of the operators $U(a,b)$ on the function ψ so that

$$\Big[U(a,b)\,\psi\Big](x) = \frac{1}{\sqrt{|a|}}\,\psi\left(\frac{x-b}{a}\right). \tag{6.1.5}$$

These operators are all unitary on the Hilbert space $L^2(\mathbb{R})$ and constitute a representation of the "$ax + b$" group

$$U(a,b)\,U(c,d) = U(ac, b + ad), \tag{6.1.6}$$

$$U(1,0) = Id, \tag{6.1.7}$$

$$U(a,b)^{-1} = U\left(\frac{1}{a}, -\frac{b}{a}\right). \tag{6.1.8}$$

This group representation is *irreducible*, that is, for any nonzero $f \in L^2(\mathbb{R})$, there exists no nontrivial g orthogonal to all the $U(a,b)f$. In other words, $U(a,b)f$ span the entire space. The multiplication of operators defines the product of pairs $(a,b),(c,d) \in \mathbb{R}/\{0\} \times \mathbb{R}$ that is; $(a,b) \circ (c,d) = (ac, b + ad)$. Like the operators $U(a,b)$, the pairs (a,b) together with the operation $\circ$ form a group. The coherent states associated with the $(ax+b)$-group, which are now known as wavelets, were first formulated by Aslaksen and Klauder (1968, 1969). The success of Morlet's numerical algorithms prompted Grossman to make a more extensive study of the Morlet wavelet transform which led to the recognition that wavelets $\psi_{a,b}(t)$ correspond to a square integrable representation of the affine group.

This chapter is devoted to wavelets and wavelet transforms with examples. The basic ideas and properties of wavelet transforms are discussed with special attention given to the use of different wavelets for resolution and synthesis of signals. This is followed by the definition and properties of discrete wavelet transforms. It is important and useful to consider discrete versions of the continuous wavelet transform due to the fact that, in many applications, especially in signal and image processing, data are represented by a finite number of values.

6.2 Continuous Wavelet Transforms and Examples

An integral transform is an operator T on a space of functions for some X which is defined by

$$\big(Tf\big)(y) = \int_x K(x,y)\, f(x)\, dx.$$

The properties of the transform depend on the function K which is called the *kernel* of the transform. For example, in the case of the Fourier transform, we have $K(x,y) = e^{-ixy}$. Note that y can be interpreted as a scaling factor. We take the exponential function $\phi(x) = e^{ix}$ and then generate a family of functions by taking scaled copies of ϕ, that is, $\phi_\alpha(x) = e^{-iax}$ for all a $\alpha \in \mathbb{R}$. The continuous wavelet transform is similar to the Fourier transform in the sense that it is based on a single function ψ and that this function is scaled. But, unlike the Fourier transform, we also shift the function, thus generating a two-parameter family of functions $\psi_{a,b}(t)$ defined by (6.1.4).

We next give formal definitions of a wavelet and a continuous wavelet transform of a function.

Definition 6.2.1 (Wavelet). A wavelet is a function $\psi \in L^2(\mathbb{R})$ which satisfies the condition

$$C_\psi \equiv \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty, \tag{6.2.1}$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$.

If $\psi \in L^2(\mathbb{R})$, then $\psi_{a,b}(t) \in L^2(\mathbb{R})$ for all a, b. For

$$\left\|\psi_{a,b}(t)\right\|^2 = |a|^{-1} \int_{-\infty}^{\infty} \left|\psi\left(\frac{t-b}{a}\right)\right|^2 dt = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \|\psi\|^2. \tag{6.2.2}$$

The Fourier transform of $\psi_{a,b}(t)$ is given by

$$\hat{\psi}_{a,b}(\omega) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\omega t}\, \psi\left(\frac{t-b}{a}\right) dt = |a|^{\frac{1}{2}} e^{-ib\omega}\, \hat{\psi}(a\omega). \tag{6.2.3}$$

Definition 6.2.2 (Continuous Wavelet Transform). If $\psi \in L^2(\mathbb{R})$, and $\psi_{a,b}(t)$ is given by (6.1.4), then the integral transformation $\mathscr{W}_\psi$ defined on $L^2(\mathbb{R})$ by

$$\mathscr{W}_\psi[f](a,b) = \langle f, \psi_{a,b}\rangle = \int_{-\infty}^{\infty} f(t)\, \overline{\psi_{a,b}(t)}\, dt \tag{6.2.4}$$

is called a *continuous wavelet transform* of $f(t)$. This definition allows us to make the following comments.

First, the kernel $\psi_{a,b}(t)$ in (6.2.4) plays the same role as the kernel $\exp(-i\omega t)$ in the Fourier transform. However, unlike the Fourier transformation, the continuous wavelet transform is not a *single* transform but any transform obtained in this way. Like the Fourier transformation, the continuous wavelet transformation is linear. Second, as a function of b for a fixed scaling parameter a, $\mathscr{W}_\psi[f](a,b)$ represents the detailed information contained in the signal $f(t)$ at the scale a. In fact, this interpretation motivated Morlet et al. (1982a,b) to introduce the translated and scaled versions of a single function for the analysis of seismic waves.

Using the Parseval relation of the Fourier transform, it also follows from (6.2.4) that

$$\begin{aligned}\mathscr{W}_\psi[f](a,b) &= \langle f, \psi_{a,b}\rangle = \frac{1}{2\pi}\left\langle \hat{f}, \hat{\psi}_{a,b}\right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\sqrt{|a|}\, \hat{f}(\omega)\, \overline{\hat{\psi}}(a\omega)\right\} e^{ib\omega} d\omega, \qquad \text{by (6.2.3).}\end{aligned}$$

This means that

$$\mathscr{F}\left\{\mathscr{W}_\psi[f](a,b)\right\} = \int_{-\infty}^{\infty} e^{-ib\omega}\, \mathscr{W}_\psi[f](a,b)\, db = \sqrt{|a|}\, \hat{f}(\omega)\, \overline{\hat{\psi}}(a\omega). \tag{6.2.5}$$

Example 6.2.1 (The Haar Wavelet). The Haar wavelet (Haar 1910) is one of the classic examples. It is defined by

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{6.2.6}$$

The Haar wavelet has compact support. It is obvious that

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0, \quad \int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1.$$

This wavelet is very well localized in the time domain, but it is not continuous. Its Fourier transform $\hat{\psi}(\omega)$ is calculated as follows:

$$\begin{aligned} \hat{\psi}(\omega) &= \int_0^{\frac{1}{2}} e^{-i\omega t} dt - \int_{\frac{1}{2}}^{1} e^{-i\omega t} dt \\ &= \frac{1}{(-i\omega)} \left\{ \left[e^{-i\omega t}\right]_0^{\frac{1}{2}} - \left[e^{-i\omega t}\right]_{\frac{1}{2}}^{1} \right\} \\ &= \left(\frac{i}{\omega}\right) \left(2e^{-\frac{i\omega}{2}} - 1 - e^{-i\omega}\right) \\ &= \frac{\sin^2 \frac{\omega}{4}}{\left(\frac{\omega}{4}\right)} \exp\left[\frac{i}{2}(\pi - \omega)\right] \\ &= i \exp\left(-\frac{i\omega}{2}\right) \frac{\sin^2\left(\frac{\omega}{4}\right)}{\left(\frac{\omega}{4}\right)} \end{aligned} \tag{6.2.7}$$

and

$$\int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega = 16 \int_{-\infty}^{\infty} |\omega|^{-3} \left|\sin \frac{\omega}{4}\right|^4 d\omega < \infty. \tag{6.2.8}$$

Both $\psi(t)$ and $\hat{\psi}(\omega)$ are plotted in Fig. 6.2. These figures indicate that the Haar wavelet has good time localization but poor frequency localization. The function $\left|\hat{\psi}(\omega)\right|$ is even, attains its maximum at the frequency $\omega_0 \sim 4.662$, and decays slowly as ω^{-1} as $\omega \to \infty$, which means that it does not have compact support in the frequency domain. Indeed, the discontinuity of ψ causes a slow decay of $\hat{\psi}$ as $\omega \to \infty$. Its discontinuous nature is a serious weakness in many applications.

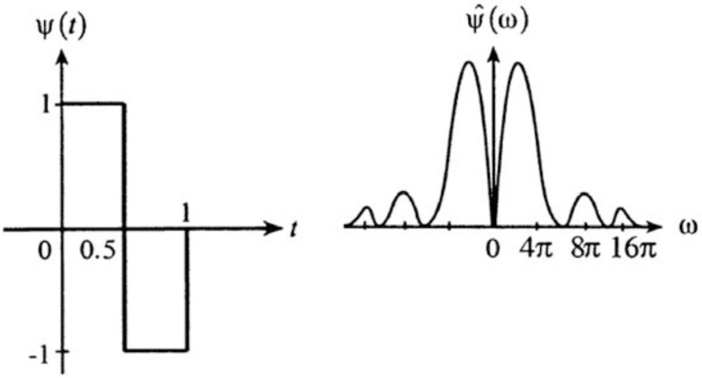

Fig. 6.2 The Haar wavelet and its Fourier transform

However, the Haar wavelet is one of the most fundamental examples that illustrate major features of the general wavelet theory.

Theorem 6.2.1. *If ψ is a wavelet and ϕ is a bounded integrable function, then the convolution function $\psi * \phi$ is a wavelet.*

Proof. Since

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\psi * \phi(x)\right|^2 dx &= \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \psi(x-u)\,\phi(u)\,du\right|^2 dx \\
&\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\psi(x-u)\right|\left|\phi(u)\right| du\right)^2 dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\psi(x-u)\right|\left|\phi(u)\right|^{\frac{1}{2}}\left|\phi(u)\right|^{\frac{1}{2}} du\right)^2 dx \\
&\le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left|\psi(x-u)\right|^2\left|\phi(u)\right| du \int_{-\infty}^{\infty} \left|\phi(u)\right| du\right) dx \\
&\le \int_{-\infty}^{\infty} \left|\phi(u)\right| du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\psi(x-u)\right|^2\left|\phi(u)\right| dx\,du \\
&= \left(\int_{-\infty}^{\infty} \left|\phi(u)\right| du\right)^2 \int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx < \infty,
\end{aligned}$$

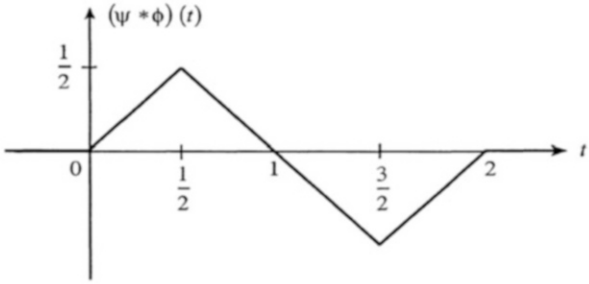

Fig. 6.3 The wavelet $(\boldsymbol{\psi} * \boldsymbol{\phi})(\mathbf{t})$

we have $\psi * \phi \in L^2(\mathbb{R})$. Moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\left|\mathscr{F}\{\psi * \phi\}\right|^2}{|\omega|} d\omega &= \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\,\hat{\phi}(\omega)\right|^2}{|\omega|} d\omega \\ &= \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} \left|\hat{\phi}(\omega)\right|^2 d\omega \\ &\leq \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega \,\sup\left|\hat{\phi}(\omega)\right|^2 < \infty.\end{aligned}$$

Thus, the convolution function $\psi * \phi$ is a wavelet.

Example 6.2.2. This example illustrates how to generate other wavelets by using Theorem 6.2.1. For example, if we take the Haar wavelet and convolute it with the following function

$$\phi(t) = \begin{cases} 0, \; t < 0, \\ 1, \; 0 \leq t \leq 1 \;, \\ 0, \; t \geq 1 \end{cases} \tag{6.2.9}$$

we obtain a simple wavelet, as shown in Fig. 6.3.

Example 6.2.3. The convolution of the Haar wavelet with $\phi(t) = \exp\left(-t^2\right)$ generates a smooth wavelet, as shown in Fig. 6.4.

In order for the wavelets to be useful analyzing functions, the mother wavelet must have certain properties. One such property is defined by the condition (6.1.4) which guarantees the existence of the inversion formula for the continuous wavelet transform. Condition (6.1.4) is usually referred to as the *admissibility condition* for the mother wavelet. If $\psi \in L^1(\mathbb{R})$, then its Fourier transform $\hat{\psi}$ is continuous. Since $\hat{\psi}$ is continuous, C_ψ can be finite only if $\hat{\psi}(0) = 0$ or, equivalently,

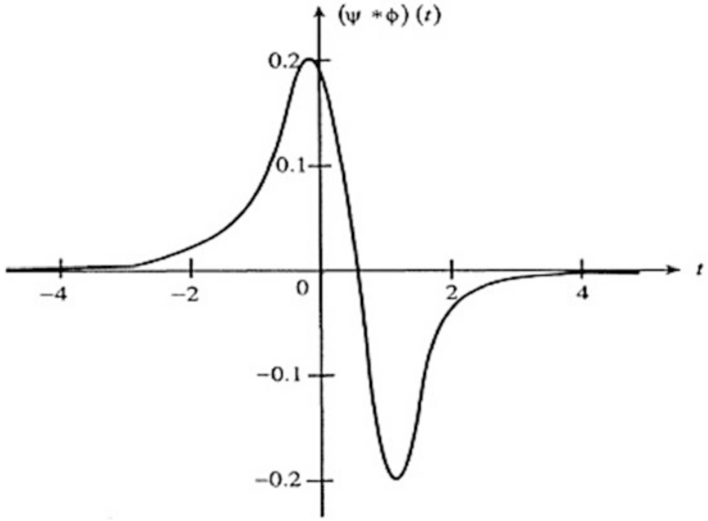

Fig. 6.4 The wavelet $(\boldsymbol{\psi} * \boldsymbol{\phi})(\mathbf{t})$

$\int_{-\infty}^{\infty} \psi(t)\, dt = 0$. This means that ψ must be an oscillatory function with zero mean. Condition (6.2.1) also imposes a restriction on the rate of decay of $\left|\hat{\psi}(\omega)\right|^2$ and is required in finding the inverse of the continuous wavelet transform.

In addition to the admissibility condition, there are other properties that may be useful in particular applications. For example, we may want to require that ψ be n times continuously differentiable or infinitely differentiable. If the Haar wavelet is convoluted $(n + 1)$ times with the function ϕ given in Example 6.2.2, then the resulting function $\psi * \phi * \cdots * \phi$ is an n times differentiable wavelet. The function in Fig. 6.4 is an infinitely differentiable wavelet. The so-called Mexican hat wavelet is another example of an infinitely differentiable (or smooth) wavelet.

Example 6.2.4 (The Mexican Hat Wavelet). The Mexican hat wavelet is defined by the second derivative of a Gaussian function as

$$\psi(t) = \left(1 - t^2\right) \exp\left(-\frac{t^2}{2}\right) = -\frac{d^2}{dt^2} \exp\left(\frac{-t^2}{2}\right) = \psi_{1,0}(t), \tag{6.2.10}$$

$$\hat{\psi}(\omega) = \hat{\psi}_{1,0}(\omega) = \sqrt{2\pi}\, \omega^2 \exp\left(-\frac{\omega^2}{2}\right). \tag{6.2.11}$$

In contrast to the Haar wavelet, the Mexican hat wavelet is a C^{∞}-function. It has two vanishing moments. The Mexican hat wavelet $\psi_{1,0}(t)$ and its Fourier transform are shown in Fig. 6.5a, b. This wavelet has excellent localization in time and frequency domains and clearly satisfies the admissibility condition.

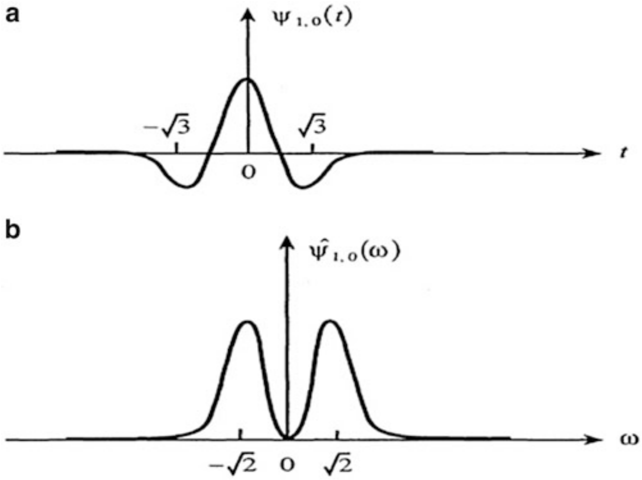

Fig. 6.5 (**a**) The Mexican hat wavelet $\boldsymbol{\psi}_{\mathbf{1,0}}(\mathbf{t})$ and (**b**) its Fourier transform $\hat{\boldsymbol{\psi}}_{\mathbf{1,0}}(\boldsymbol{\omega})$.

Two other wavelets, $\psi_{\frac{3}{2},-2}$ and $\psi_{\frac{1}{4},\sqrt{2}}$, from the mother wavelet (6.2.10) can be obtained. These three wavelets, $\psi_{1,0}(t)$, $\psi_{\frac{3}{2},-2}(t)$, and $\psi_{\frac{1}{4},\sqrt{2}}(t)$, are shown in Fig. 6.6(i), (ii), and (iii), respectively.

Example 6.2.5 (The Morlet Wavelet). The Morlet wavelet is defined by

$$\psi(t) = \exp\left(i\omega_0 t - \frac{t^2}{2}\right), \tag{6.2.12}$$

$$\hat{\psi}(\omega) = \sqrt{2\pi}\,\exp\left[-\frac{1}{2}(\omega-\omega_0)^2\right]. \tag{6.2.13}$$

The Morlet wavelet and its Fourier transform are plotted in Fig. 6.7.

Another desirable property of wavelets is the so-called localization property. We want ψ to be well localized in both time and frequency. In other words, ψ and its derivatives must decay very rapidly. For frequency localization, $\hat{\psi}(\omega)$ must decay sufficiently rapidly as $\omega \to \infty$ and $\hat{\psi}(\omega)$ should be flat in the neighborhood of $\omega = 0$. The flatness at $\omega = 0$ is associated with the number of vanishing moments of ψ. The kth moment of ψ is defined by

$$m_k = \int_{-\infty}^{\infty} t^k\,\psi(t)\,dt. \tag{6.2.14}$$

A wavelet is said to have n vanishing moments if

$$\int_{-\infty}^{\infty} t^k\,\psi(t)\,dt = 0 \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{6.2.15}$$

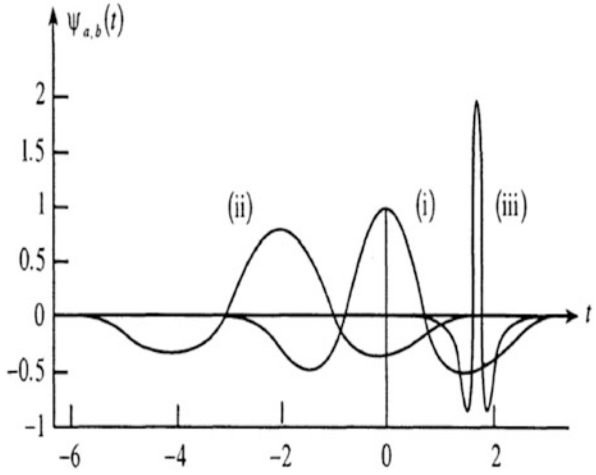

Fig. 6.6 Three wavelets $\psi_{1,0}(t)$, $\psi_{\frac{3}{2},-2}(t)$ and $\psi_{\frac{1}{4},\sqrt{2}}(t)$

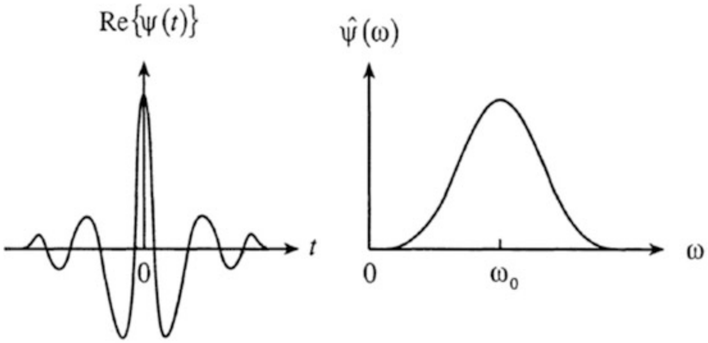

Fig. 6.7 The Morlet wavelet and its Fourier transform

Or, equivalently,

$$\left[\frac{d^k\,\hat{\psi}(\omega)}{d\omega^k}\right]_{\omega=0} = 0 \qquad \text{for} \quad k = 0, 1, \ldots, n. \tag{6.2.16}$$

Wavelets with a larger number of vanishing moments result in more flatness when frequency ω is small.

The smoothness and localization properties of wavelet ψ combined with the admissibility condition (6.2.1) suggest that

(i) Wavelets are bandpass filters; that is, the frequency response decays sufficiently rapidly as $\omega \to \infty$ and is zero as $\omega \to 0$.
(ii) $\psi(t)$ is the impulse response of the filter which again decays very rapidly as t increases, and it is an oscillatory function with mean zero. Usually, wavelets are assumed to be absolutely square integrable functions, that is, $\psi \in L^2(\mathbb{R})$.

In quantum mechanics, quantities such as $\left|\psi(t)\right|^2$ and $\left|\hat{\psi}(\omega)\right|^2$ are interpreted as the probability density functions in the time and frequency domains respectively, with mean values defined by

$$\langle t\rangle = \int_{-\infty}^{\infty} t\left|\psi(t)\right|^2 dt \quad \text{and} \quad \langle\omega\rangle = \frac{1}{2\pi}\int_0^{\infty} \omega\left|\hat{\psi}(\omega)\right|^2 d\omega. \tag{6.2.17a,b}$$

The *time resolution* (or the *time spread*) and the *frequency resolution* (or the *frequency spread*) associated with a mother wavelet ψ around the mean values are defined by

$$\sigma_t^2 = \int_{-\infty}^{\infty} \big(t - \langle t\rangle\big)^2\left|\psi(t)\right|^2 dt, \tag{6.2.18}$$

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_0^{\infty} \big(\omega - \langle\omega\rangle\big)^2\left|\hat{\psi}(\omega)\right|^2 d\omega. \tag{6.2.19}$$

Thus, for any $\psi \in L^2(\mathbb{R})$, the time and frequency resolutions of the mother wavelet are governed by the Heisenberg uncertainty principle, that is, $\sigma_t\,\sigma_\omega \geq \frac{1}{2}$.

It is easy to verify that the time–frequency resolution of a wavelet $\psi_{a,b}$ depends on the time–frequency spread of the mother wavelet. We define the energy spread of $\psi_{a,b}$ around b by

$$\begin{aligned}\sigma_{t,a,b}^2 &= \int_{-\infty}^{\infty} (t-b)^2\left|\psi_{a,b}(t)\right|^2 dt, \quad (t-b=x)\\ &= a^2\int_{-\infty}^{\infty} x^2\left|\psi(x)\right|^2 dx = a^2\sigma_t^2,\end{aligned} \tag{6.2.20}$$

where σ_t^2 is defined by (6.2.18) around the zero mean. Clearly, the wavelets have good time resolution for small values of a which correspond to high frequencies or small scales. Scale can be defined as the inverse of frequency.

On the other hand, the Fourier transform $\hat{\psi}_{a,b}(\omega)$ of $\psi_{a,b}(t)$ is given by (6.2.3), so its mean value is $\frac{1}{a}\langle\omega\rangle$. The energy spread of $\hat{\psi}_{a,b}(\omega)$ around $\frac{1}{a}\langle\omega\rangle$ is defined by

$$\sigma^2_{\omega,a,b} = \frac{1}{2\pi}\int_0^\infty \left(\omega - \frac{1}{a}\langle\omega\rangle\right)^2 \left|\hat{\psi}_{a,b}(\omega)\right|^2 d\omega, \quad (a\omega = x)$$
$$= \frac{1}{2\pi}\int_0^\infty \frac{1}{a^2}\big(x - \langle\omega\rangle\big)^2 \left|\hat{\psi}(x)\right|^2 dx = \frac{1}{a^2}\sigma^2_\omega. \tag{6.2.21}$$

This reveals that wavelets have good frequency resolution for large values of the scale a.

Thus, the time–frequency resolution of wavelets $\psi_{a,b}$ is independent of the time location but depends only on the scale a. The energy spread of the wavelet $\psi_{a,b}$ corresponds to a Heisenberg time–frequency rectangle at $\left(b, \frac{1}{a}\langle\omega\rangle\right)$ of sides $a\,\sigma_t$ along the time axis and $\frac{1}{a}\sigma_\omega$ along the frequency axis. The area of a the rectangle is equal to $\sigma_t\,\sigma_\omega$ for all scales and is governed by the Heisenberg uncertainty principle, that is, $\sigma_{t,a,b}\,\sigma_{\omega,a,b} = (a\sigma_1)(a^{-1}\sigma_\omega) = \sigma_t\,\sigma_\omega \geq \frac{1}{2}$.

We close this section by introducing a scaled version of a mother wavelet in the form

$$\psi_a(t) = |a|^{-p}\,\psi\left(\frac{t}{a}\right), \tag{6.2.22}$$

where p is a fixed but arbitrary nonnegative parameter. In particular, when $p = \frac{1}{2}$ the translated version of $\psi_a(t)$ defined by (6.2.22) reduces to wavelets (6.1.4).

Clearly, if $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$, then the Fourier transform of the dilated version of $\psi(t)$ is given by

$$\mathscr{F}\left\{D_a\psi(t)\right\} = \mathscr{F}\left\{\frac{1}{\sqrt{a}}\,\psi\left(\frac{t}{a}\right)\right\} = D_{\frac{1}{a}}\hat{f}(\omega) = \sqrt{a}\,\hat{f}(a\omega), \tag{6.2.23}$$

where $a > 0$. Thus, a contraction in one domain is accompanied by a magnification in the other but in a non-uniform manner over the time–frequency plane. A typical wavelet and its dilations are sketched in Fig. 6.8a–c together with the corresponding Fourier transforms.

If $p = 1$ in (6.2.22), the integral

$$\int_{-\infty}^{\infty} \psi_a(t)\,dt = \int_{-\infty}^{\infty} \psi(x)\,dx$$

does not depend on the scaling parameter a. On the other hand, the choice of $p = 0$ is found to be convenient for the study of orthonormal bases of wavelets. However, the specific value of p is completely irrelevant to the general theory of wavelets, so appropriate choices are used in the literature.

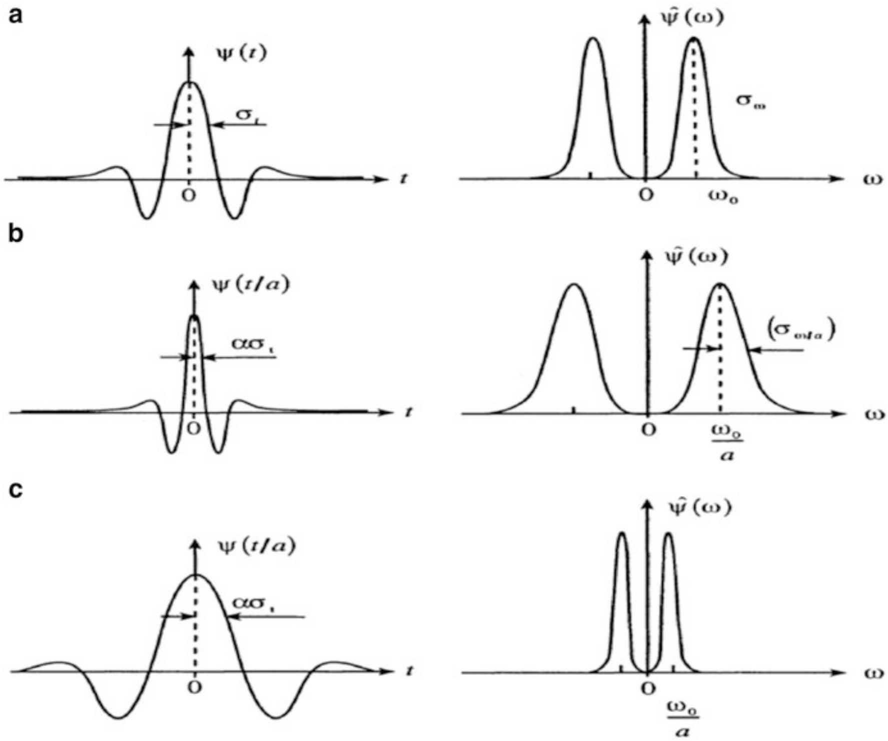

Fig. 6.8 Typical wavelet and its dilations with the corresponding Fourier transforms for (**a**) $a = 1$, (**b**) $0 < a \ll 1$, and (**c**) $a \gg 1$ in the time–frequency domains

For an arbitrary $p \geq 0$, the time localization of signals is obtained by the translated versions of $\psi_a(t)$. If $\psi(t)$ is supported on an interval of length ℓ near $t = 0$, then wavelets can be defined by the translated and scaled versions of the mother wavelet ψ as

$$\psi_{a,b}(t) = \psi_a(t-b) = |a|^{-p}\psi\left(\frac{t-b}{a}\right). \tag{6.2.24}$$

Obviously, this is supported on an interval of length $|a|\,\ell$ near $t = b$.

If we assume that $\psi \in L^2(\mathbb{R})$, then the square of the norm of $\psi_{a,b}$ is

$$\|\psi_{a,b}\|^2 = |a|^{-2p}\int_{-\infty}^{\infty}\left|\psi\left(\frac{t-b}{a}\right)\right|^2 dt = |a|^{1-2p}\|\psi\|^2. \tag{6.2.25}$$

6.3 Basic Properties of Wavelet Transforms

The following theorem gives several properties of continuous wavelet transforms.

Theorem 6.3.1. *If* ψ *and* ϕ *are wavelets and* f, g *are functions which belong to* $L^2(\mathbb{R})$*, then*

(i) (Linearity)

$$\mathscr{W}_\psi\big(\alpha f + \beta g\big)(a,b) = \alpha\big(\mathscr{W}_\psi f\big)(a,b) + \beta\big(\mathscr{W}_\psi g\big)(a,b), \tag{6.3.1}$$

where α *and* β *are any two scalars.*

(ii) (Translation)

$$\big(\mathscr{W}_\psi(T_c f)\big)(a,b) = \big(\mathscr{W}_\psi f\big)(a,b-c), \tag{6.3.2}$$

where T_c *is the translation operator defined by* $T_c f(t) = f(t-c)$.

(iii) (Dilation)

$$\Big(\mathscr{W}_\psi(D_c f)\Big)(a,b) = \frac{1}{\sqrt{c}}\big(\mathscr{W}_\psi f\big)\left(\frac{a}{c},\frac{b}{c}\right), \quad c>0, \tag{6.3.3}$$

where D_c *is a dilation operator defined by* $D_c f(t) = \dfrac{1}{c} f\left(\dfrac{t}{c}\right), \quad c>0.$

(iv) (Symmetry)

$$\big(\mathscr{W}_\psi f\big)(a,b) = \overline{\big(\mathscr{W}_f \psi\big)\left(\frac{1}{a},-\frac{b}{a}\right)}, \quad a \neq 0. \tag{6.3.4}$$

(v) (Parity)

$$\big(\mathscr{W}_{P\psi} Pf\big)(a,b) = \big(\mathscr{W}_\psi f\big)(a,-b), \tag{6.3.5}$$

where P *is the parity operator defined by* $Pf(t) = f(-t)$.

(vi) (Antilinearity)

$$\big(\mathscr{W}_{\alpha\psi+\beta\phi} f\big)(a,b)(a,b) = \overline{\alpha}\,\big(\mathscr{W}_\psi f\big)(a,b) + \overline{\beta}\,\big(\mathscr{W}_\phi f\big)(a,b), \tag{6.3.6}$$

for any scalars α, β.

(vii)

$$\big(\mathscr{W}_{T_c\psi} f\big)(a,b) = \big(\mathscr{W}_\psi f\big)(a,b+ca), \tag{6.3.7}$$

(viii)

$$\big(\mathscr{W}_{D_c\psi} f\big)(a,b) = \frac{1}{\sqrt{c}}\big(\mathscr{W}_{\psi} f\big)(ac,b), \quad c > 0. \tag{6.3.8}$$

Proofs of the above properties are straightforward and are left as exercises.

Theorem 6.3.2 (Parseval's Formula for Wavelet Transforms). *If $\psi \in L^2(\mathbb{R})$ and $\big(\mathscr{W}_{\psi} f\big)(a,b)$ is the wavelet transform of f defined by (6.2.4), then, for any functions $f, g \in L^2(\mathbb{R})$, we obtain*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\mathscr{W}_{\psi} f\Big)(a,b)\,\overline{\Big(\mathscr{W}_{\psi} g\Big)(a,b)}\,\frac{db\,da}{a^2} = C_{\psi}\langle f, g\rangle, \tag{6.3.9}$$

where

$$C_{\psi} = \int_{-\infty}^{\infty}\frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|}\,d\omega < \infty. \tag{6.3.10}$$

Proof. By Parseval's relation (3.4.37) for the Fourier transforms, we have

$$\begin{aligned}\Big(\mathscr{W}_{\psi} f\Big)(a,b) &= \int_{-\infty}^{\infty} f(t)\,|a|^{-\frac{1}{2}}\,\overline{\psi\left(\frac{t-b}{a}\right)}\,dt \\ &= \langle f, \psi_{a,b}\rangle \\ &= \frac{1}{2\pi}\left\langle \hat{f}, \hat{\psi}_{a,b}\right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,|a|^{\frac{1}{2}}\,e^{ib\omega}\,\overline{\hat{\psi}(a\omega)}\,d\omega \quad \text{by (6.2.3)}.\end{aligned} \tag{6.3.11}$$

Similarly,

$$\begin{aligned}\overline{\Big(\mathscr{W}_{\psi} g\Big)(a,b)} &= \int_{-\infty}^{\infty}\overline{g(t)}\,|a|^{-\frac{1}{2}}\,\psi\left(\frac{t-b}{a}\right)\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\hat{g}(\sigma)}\,\,|a|^{\frac{1}{2}}\,\,e^{-ib\sigma}\,\hat{\psi}(a\sigma)\,d\sigma.\end{aligned} \tag{6.3.12}$$

Substituting (6.3.11) and (6.3.12) in the left-hand side of (6.3.9) gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\mathscr{W}_{\psi} f\Big)(a,b)\,\overline{\Big(\mathscr{W}_{\psi} g\Big)(a,b)}\,\frac{db\,da}{a^2}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{db\,da}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a|\;\hat{f}(\omega)\,\overline{\hat{g}}(\sigma)\,\overline{\hat{\psi}}(a\omega)\,\hat{\psi}(a\sigma)\exp\left\{ib(\omega-\sigma)\right\}d\omega\,d\sigma,$$

which is, by interchanging the order of integration,

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}}(\sigma)\, \overline{\hat{\psi}}(a\omega)\, \hat{\psi}(a\sigma)\, d\omega\, d\sigma\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{ib(\omega-\sigma)\}\, db$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}}(\sigma)\, \overline{\hat{\psi}}(a\omega)\, \hat{\psi}(a\sigma)\, \delta(\sigma-\omega)\, d\omega\, d\sigma$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}}(\omega) \left|\hat{\psi}(a\omega)\right|^2 d\omega$$

which is, again interchanging the order of integration and putting $a\,\omega = x$,

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}}(\omega)\, d\omega \cdot \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(x)\right|^2}{|x|}\, dx$$

$$= C_\psi \cdot \frac{1}{2\pi} \left\langle \hat{f}(\omega), \hat{g}(\omega) \right\rangle.$$

Theorem 6.3.3 (Inversion Formula). *If $f \in L^2(\mathbb{R})$, then f can be reconstructed by the formula*

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathscr{W}_\psi f\right)(a,b)\, \psi_{a,b}(t)\, \frac{db\, da}{a^2}, \qquad (6.3.13)$$

where the equality holds almost everywhere.

Proof. For any $g \in L^2(\mathbb{R})$, we have, from Theorem 6.3.2,

$$C_\psi \langle f, g\rangle = \left\langle \mathscr{W}_\psi f, \mathscr{W}_\psi g \right\rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathscr{W}_\psi f\right)(a,b)\, \overline{\left(\mathscr{W}_\psi g\right)(a,b)}\, \frac{db\, da}{a^2}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathscr{W}_\psi f\right)(a,b)\, \overline{\int_{-\infty}^{\infty} g(t)\, \overline{\psi_{a,b}(t)}\, dt}\, \frac{db\, da}{a^2}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathscr{W}_\psi f\right)(a,b)\, \psi_{a,b}(t)\, \frac{db\, da}{a^2}\, \overline{g(t)}\, dt$$

$$= \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathscr{W}_\psi f\right)(a,b)\, \psi_{a,b}(t)\, \frac{db\, da}{a^2}, g \right\rangle. \qquad (6.3.14)$$

Since g is an arbitrary element of $L^2(\mathbb{R})$, the inversion formula (6.3.13) follows.

If $f = g$ in (6.3.9), then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|\left(\mathscr{W}_\psi f\right)(a,b)\right|^2 \frac{da\, db}{a^2} = C_\psi \|f\|^2 = C_\psi \int_{-\infty}^{\infty} |f(t)|^2\, dt. \qquad (6.3.15)$$

This shows that, except for the factor C_ψ, the wavelet transform is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$.

6.4 The Discrete Wavelet Transforms

It has been stated in the last section that the continuous wavelet transform (6.2.4) is a two-parameter representation of a function. In many applications, especially in signal processing, data are represented by a finite number of values, so it is important and often useful to consider discrete versions of the continuous wavelet transform (6.2.4). From a mathematical point of view, a continuous representation of a function of two continuous parameters a, b in (6.2.4) can be converted into a discrete one by assuming that a and b take only integral values. It turns out that it is better to discretize it in a different way. First, we fix two positive constants a_0 and b_0 and define

$$\psi_{m,n}(x) = a_0^{-m/2}\,\psi(a_0^{-m}x - nb_0), \tag{6.4.1}$$

where both m and $n \in \mathbb{Z}$. Then, for $f \in L^2(\mathbb{R})$, we calculate the discrete wavelet coefficients $\langle f, \psi_{m,n}\rangle$. The fundamental question is whether it is possible to determine f completely by its wavelet coefficients or discrete wavelet transform which is defined by

$$\begin{aligned}\Big(\mathscr{W}_\psi f\Big)(m,n) = \big\langle f, \psi_{m,n}\big\rangle &= \int_{-\infty}^{\infty} f(t)\,\overline{\psi}_{m,n}(t)\,dt \\ &= a_0^{-\frac{m}{2}} \int_{-\infty}^{\infty} f(t)\,\overline{\psi}(a_0^{-m}t - nb_0)\,dt, \end{aligned}\tag{6.4.2}$$

where both f and ψ are continuous, $\psi_{0,0}(t) = \psi(t)$. It is noted that the discrete wavelet transform (6.4.2) can also be obtained directly from the corresponding continuous version by discretizing the parameters $a = a_0^m$ and $b = nb_0a_0^m$ (m, n are integers). The discrete wavelet transform represents a function by a countable set of wavelet coefficients, which correspond to points on a two dimensional grid or lattice of discrete points in the scale-time domain indexed by m and n. If the set $\big\{\psi_{m,n}(t)\big\}$ defined by (6.4.1) is complete in $L^2(\mathbb{R})$ for some choice of ψ, a, and b, then the set is called an *affine wavelet*. Then, we can express any $f(t) \in L^2(\mathbb{R})$ as the superposition

$$f(t) = \sum_{m,n=-\infty}^{\infty} \big\langle f, \psi_{m,n}\big\rangle\,\psi_{m,n}(t). \tag{6.4.3}$$

Such complete sets are called *frames*. They are not yet a basis. Frames do not satisfy the Parseval theorem for the Fourier series, and the expansion in terms of frames is not unique. In fact, it can be shown that

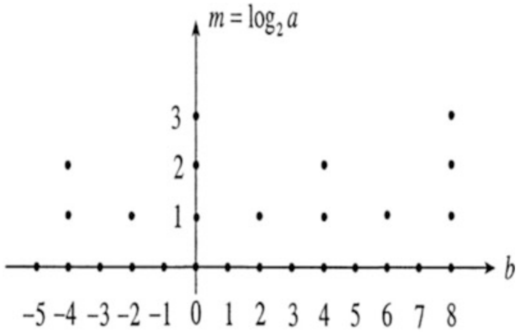

Fig. 6.9 Dyadic sampling grid for the discrete wavelet transform

$$A\|f\|^2 \leq \sum_{m,n=-\infty}^{\infty} \left|\left(f, \psi_{m,n}\right)\right|^2 \leq B\|f\|^2, \tag{6.4.4}$$

where A and B are constants. The set $\{\psi_{m,n}(t)\}$ constitutes a frame if $\psi(t)$ satisfies the admissibility condition and $0 < A < B < \infty$.

For computational efficiency, $a_0 = 2$ and $b_0 = 1$ are commonly used so that results lead to a binary dilation of 2^{-m} and a dyadic translation of $n\,2^m$.

Therefore, a practical sampling lattice is $a = 2^m$ and $b = n\,2^m$ in (6.4.1) so that

$$\psi_{m,n}(t) = 2^{-\frac{m}{2}}\,\psi(2^{-m}t - n). \tag{6.4.5}$$

With this octave time scale and dyadic translation, the sampled values of $(a, b) = (2^m, n2^m)$ are shown in Fig. 6.9, which represents the dyadic sampling grid diagram for the discrete wavelet transform. Each node corresponds to a wavelet basis function $\psi_{m,n}(t)$ with scale 2^{-m} and time shift $n\,2^{-m}$.

The answer to the preceding question is positive if the wavelets form a complete system in $L^2(\mathbb{R})$. The problem is whether there exists another function $g \in L^2(\mathbb{R})$ such that

$$\langle f, \psi_{m,n}\rangle = \langle g, \psi_{m,n}\rangle \quad \text{for all} \quad m, n \in \mathbb{Z}$$

implies $f = g$.

In practice, we expect much more than that: we want $\langle f, \psi_{m,n}\rangle$ and $\langle g, \psi_{m,n}\rangle$ to be "close" if f and g are "close." This will be guaranteed if there exists a $B > 0$ independent of f such that

$$\sum_{m,n=-\infty}^{\infty} \left|\langle f, \psi_{m,n}\rangle\right|^2 \leq B\|f\|^2. \tag{6.4.6}$$

Similarly, we want f and g to be "close" if $\langle f, \psi_{m,n}\rangle$ and $\langle g, \psi_{m,n}\rangle$ are "close." This is important because we want to be sure that when we neglect some small terms in the representation of f in terms of $\langle f, \psi_{m,n}\rangle$, the reconstructed function will not differ much from f. The representation will have this property if there exists an $A > 0$ independent of f, such that

$$A\|f\|^2 \leq \sum_{m,n=-\infty}^{\infty} |\langle f, \psi_{m,n}\rangle|^2. \tag{6.4.7}$$

These two requirements are best studied in terms of the so-called frames.

Definition 6.4.1 (Frames). A sequence $\{\phi_1, \phi_2, \dots\}$ in a Hilbert space H is called a *frame* if these exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 \leq B\|f\|^2 \tag{6.4.8}$$

for all $f \in H$. The constants A and B are called *frame* bounds. If $A = B$, then the frame is called *tight*.

If $\{\phi_n\}$ is an orthonormal basis, then it is a tight frame since $\sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 = \|f\|^2$ for all $f \in H$. The vectors $(1,0)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ form a tight frame in $\mathbb{C}^2$ which is not a basis.

As pointed out above, we want the family of functions $\psi_{m,n}$ to form a frame in $L^2(\mathbb{R})$. Obviously, the double indexing of the functions is irrelevant. The following theorem gives fairly general sufficient conditions for a sequence $\{\psi_{m,n}\}$ to constitute a frame in $L^2(\mathbb{R})$.

Theorem 6.4.1. *If ψ and a_0 are such that*

(i)

$$\inf_{1\leq|\omega|\leq a_0} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\left(a_0^m \omega\right)\right|^2 > 0,$$

(ii)

$$\sup_{1\leq|\omega|\leq a_0} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\left(a_0^m \omega\right)\right|^2 \geq 0,$$

and

(iii)

$$\sup_{\omega\in\mathbb{R}} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\left(a_0^m\,\omega\right)\right| \left|\hat{\psi}\left(a_0^m\omega + x\right)\right| \le C\left(1+|x|\right)^{-(1+\varepsilon)}$$

for some $\varepsilon > 0$ *and some constant* C*, then there exists* $\tilde{b}$ *such that* $\psi_{m,n}$ *form a frame in* $L^2(\mathbb{R})$ *for any* $b_0 \in (0, \tilde{b})$.

Proof. Suppose $f \in L^2(\mathbb{R})$. Then,

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} \left|\langle f, \psi_{m,n}\rangle\right|^2 &= \sum_{m,n=-\infty}^{\infty} \left|\int_{-\infty}^{\infty} f(x)\, a_0^{-m/2}\, \overline{\psi(a_0^{-m}x - nb_0)}\, dx\right|^2 \\ &= \sum_{m,n=-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \hat{f}(\omega)\, a_0^{m/2}\, \overline{\hat{\psi}(a_0^m\omega)}\, e^{ib_0a_0^m n\omega} d\omega\right|^2 = P\end{aligned}$$

by the general Parseval relation (see Theorem 4.11.13 of Debnath and Mikusinski 1999), basic properties of the Fourier transform (see Theorem 4.11.5 of Debnath and Mikusinski 1999), and the fact that we sum over all integers. Since, for any $s > 0$, the integral $\int_{-\infty}^{\infty} g(t)\,dt$ can be written as

$$\sum_{l=-\infty}^{\infty} \int_0^s g(t + ls)\,dt,$$

by taking $s = \dfrac{2\pi}{b_0 a_0^m}$, we obtain

$$\begin{aligned}P &= \sum_{m,n=-\infty}^{\infty} a_0^m \left|\sum_{l=-\infty}^{\infty} \int_0^s e^{2\pi i n\omega/s}\, \hat{f}(\omega + ls)\, \overline{\hat{\psi}\left(a_0^m(\omega + ls)\right)} d\omega\right|^2 \\ &= \sum_{m,n=-\infty}^{\infty} a_0^m \left|\int_0^s e^{2\pi i n\omega/s} \left(\sum_{l=-\infty}^{\infty} \hat{f}(\omega + ls)\, \overline{\hat{\psi}\left(a_0^m(\omega + ls)\right)}\right) d\omega\right|^2 \\ &= \sum_{m=-\infty}^{\infty} a_0^m s \int_0^s \left|\sum_{l=-\infty}^{\infty} \hat{f}(\omega + ls)\, \overline{\hat{\psi}\left(a_0^m(\omega + ls)\right)}\right|^2 d\omega = Q\end{aligned}$$

by Parseval's formula for trigonometric Fourier series.

Since

$$\left|\sum_{l=-\infty}^{\infty} \hat{f}(\omega + ls)\,\overline{\hat{\psi}\big(a_0^m(\omega + ls)\big)}\right|^2$$
$$= \left(\sum_{l=-\infty}^{\infty} \hat{f}(\omega + ls)\,\overline{\hat{\psi}\big(a_0^m(\omega + ls)\big)}\right)\left(\sum_{k=-\infty}^{\infty} \overline{\hat{f}(\omega + ks)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)\right)$$

and

$$F(\omega) = \sum_{k=-\infty}^{\infty} \overline{\hat{f}(\omega + ks)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)$$

is a periodic function with a period of s, we have

$$\int_0^s \left(\sum_{l=-\infty}^{\infty} \hat{f}(\omega + ls)\,\overline{\hat{\psi}\big(a_0^m(\omega + ls)\big)}\right) F(\omega)\,d\omega$$
$$= \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{\psi}\big(a_0^m\omega\big)}\,F(\omega)\,d\omega$$
$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{\psi}\big(a_0^m\omega\big)}\,\overline{\hat{f}(\omega + ks)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)\,d\omega.$$

Consequently,

$$Q = \frac{2\pi}{b_0} \sum_{m,k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{f}(\omega + ks)}\,\overline{\hat{\psi}(a_0^m s)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)\,d\omega$$
$$= \frac{2\pi}{b_0} \int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \sum_{m=-\infty}^{\infty} \left|\hat{\psi}(a_0^m\omega)\right|^2 d\omega$$
$$+ \frac{2\pi}{b_0} \sum_{\substack{m,k=-\infty \\ k\neq 0}}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{f}(\omega + ks)}\,\overline{\hat{\psi}(a_0^m s)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)\,d\omega.$$

To find a bound on the second summation, we apply the Schwarz inequality:

$$\left|\left(\frac{2\pi}{b_0}\right) \sum_{\substack{m,k=-\infty \\ k\neq 0}}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{f}(\omega + ks)}\,\overline{\hat{\psi}(a_0^m s)}\,\hat{\psi}\big(a_0^m(\omega + ks)\big)\,d\omega\right|$$

$$\leq \left(\frac{2\pi}{b_0}\right) \sum_{\substack{m,k=-\infty \\ k\neq 0}}^{\infty} \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \left|\hat{\psi}(a_0^m \omega)\right| \left|\hat{\psi}\big(a_0^m(\omega + ks)\big)\right| d\omega\right)^{\frac{1}{2}}$$

$$\times \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega + ks)\right|^2 \left|\hat{\psi}(a_0^m \omega)\right| \left|\hat{\psi}\big(a_0^m(\omega + ks)\big)\right| d\omega\right)^{\frac{1}{2}} = R.$$

Then, by first changing the variables in the second factor and using Hölder's inequality (see Theorem 1.2.1 of Debnath and Mikusinski 1999), we have

$$R = \left(\frac{2\pi}{b_0}\right) \sum_{\substack{m,k=-\infty \\ k\neq 0}}^{\infty} \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \left|\hat{\psi}\left(a_0^m \omega\right)\right| \left|\hat{\psi}\big(a_0^m(\omega + ks)\big)\right| d\omega\right)^{\frac{1}{2}}$$

$$\times \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \left|\hat{\psi}\big(a_0^m(\omega - ks)\big)\right| \left|\hat{\psi}(a_0^m \omega)\right| d\omega\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2\pi}{b_0}\right) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \sum_{m=-\infty}^{\infty} \left|\hat{\phi}\left(a_0^m \omega\right)\right| \left|\hat{\psi}\big(a_0^m(\omega + ks)\big)\right| d\omega\right)^{\frac{1}{2}}$$

$$\times \left(\int_{-\infty}^{\infty} \left|\hat{f}(\omega)\right|^2 \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\big(a_0^m(\omega - ks)\big)\right| \left|\hat{\psi}(a_0^m \omega)\right| d\omega\right)^{\frac{1}{2}} = S.$$

If we denote

$$\beta(\xi) = \sup_{\omega\in\mathbb{R}} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}\big(a_0^m \omega\big)\right| \left|\hat{\psi}\big(a_0^m \omega + \xi\big)\right|,$$

then

$$S = \left(\frac{2\pi}{b_0}\right) \|f\|^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\beta\big(a_0^m ks\big)\, \beta\big(-a_0^m ks\big)\right]^{\frac{1}{2}}$$

$$= \left(\frac{2\pi}{b_0}\right) \|f\|^2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\beta\left(\frac{2\pi k}{b_0}\right) \beta\left(-\frac{2\pi k}{b_0}\right)\right]^{\frac{1}{2}}.$$

Consequently, if we denote

$$A = \left(\frac{2\pi}{b_0}\right) \left\{ \sup_{\omega\in\mathbb{R}} \sum_{m=-\infty}^{\infty} \left|\hat{\psi}(a_0^m \omega)\right|^2 - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[\beta\left(\frac{2\pi k}{b_0}\right) \beta\left(-\frac{2\pi k}{b_0}\right)\right]^{1/2} \right\}$$

and

$$B = \left(\frac{2\pi}{b_0}\right)\left\{\inf_{\omega\in\mathbb{R}}\sum_{m=-\infty}^{\infty}\left|\hat{\psi}(a_0^m\omega)\right|^2 + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi k}{b_0}\right)\beta\left(-\frac{2\pi k}{b_0}\right)\right]^{1/2}\right\},$$

we conclude

$$A\|f\|^2 \leq \sum_{n,m=-\infty}^{\infty}|\langle f,\phi_n\rangle|^2 \leq B\|f\|^2.$$

Since $\beta(\xi) \leq C\,(1+|\xi|)^{-(1+\varepsilon)}$, we find

$$\begin{aligned}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left[\beta\left(\frac{2\pi k}{b_0}\right)\beta\left(-\frac{2\pi k}{b_0}\right)\right]^{1/2} &= 2\sum_{k=1}^{\infty}\left[\beta\left(\frac{2\pi k}{b_0}\right)\beta\left(-\frac{2\pi k}{b_0}\right)\right]^{1/2}\\ &\leq 2C\sum_{k=1}^{\infty}\left(1+\frac{2\pi k}{b_0}\right)^{-(1+\epsilon)}\\ &\leq 2C\int_0^{\infty}\left(1+\frac{2\pi k}{b_0}\right)^{-(1+\epsilon)}dt\\ &= \frac{Cb_0}{\pi\epsilon}.\end{aligned}$$

Since $\left(\frac{Cb_0}{\pi\epsilon}\right) \to 0$ as $b_0 \to 0$ and $\inf_{1\leq|\omega|\leq a_0}\sum_{m=-\infty}^{\infty}\left|\hat{\psi}(a_o^m\omega)\right|^2 > 0$, there exists $\tilde{b}$ such that $A > 0$ for any $b_0 \in (0,\tilde{b})$. Moreover, since $\sup_{1\leq|\omega|\leq a_0}\sum_{m=-\infty}^{\infty}\left|\hat{\psi}(a_o^m\omega)\right|^2 < \infty$, we also have $B < \infty$ for all $b_0 \in (0,\tilde{b})$. Thus, $\psi_{m,n}$ constitute a frame for all such b_0. This completes the proof.

The major problem of this section is reconstruction of f from $\langle f,\psi_{m,n}\rangle$ and representation of f in terms of $\psi_{m,n}$ For a complete orthonormal system $\{\phi_n\}$ both questions are answered by the equality

$$f = \sum_{n=1}^{\infty}\langle f,\phi_{m,n}\rangle\,\phi_n. \tag{6.4.9}$$

However, since we do not have orthogonality, the problem is more complete for frames.

Definition 6.4.2 (Frame Operator). Let $\{\phi_1, \phi_2, \ldots\}$ be a frame in a Hilbert space H. The operator F from H into l^2 defined by

$$F\{f\} = \{\langle f, \phi_n\rangle\}$$

is called a *frame operator*.

Lemma 6.4.1. *Let F be a frame operator. Then, F is a linear, invertible, and bounded operator. Its inverse F^{-1} is also a bounded operator.*

The proof is easy and left as an exercise.

Consider the adjoint operator F^* of a frame operator F associated with frame $\{\phi_n\}$. For any $\{c_n\} \in l^2$, we have

$$\langle F^*(c_n), f\rangle = \langle (c_n), Ff\rangle = \sum_{n=1}^{\infty} c_n \langle \phi_n, f\rangle = \left\langle \sum_{n=1}^{\infty} c_n \phi_n, f \right\rangle.$$

Thus, the adjoint operator of a frame operator has the form

$$F^*(c_n) = \sum_{n=1}^{\infty} c_n\, \phi_n. \tag{6.4.10}$$

Since

$$\sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 = \|F\, f\|^2 = \langle F^* F\, f, f\rangle,$$

we note that the condition (6.4.4) can be expressed as

$$AI \le F^* F \le BI,$$

where the inequality $\le$ is to be understood in the sense defined in Sect. 4.6 (see Debnath and Mikusinski 1999).

Theorem 6.4.2. *Let $\{\phi_1, \phi_2, \phi_3, \ldots\}$ be frame bounds A and B and let F be the associated frame operator. Define*

$$\tilde{\phi_n} = (F^*\, F)^{-1} \phi_n.$$

Then, $\{\hat{\phi}_n\}$ is a frame with bounds $\dfrac{1}{B}$ and $\dfrac{1}{A}$.

Proof. By Corollary 4.5.1 as stated by Debnath and Mikusinski (1999), we have $(F^* F)^{-1} = \left((F^* F)^{-1}\right)^*$. Consequently,

$$\left\langle f, \tilde{\phi_n} \right\rangle = \left\langle f, (F^* F)^{-1} \phi_n \right\rangle = \left\langle (F^* F)^{-1} f, \phi_n \right\rangle$$

and then

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\langle f,\{\tilde{\phi_n}\}\rangle\right|^2 &= \sum_{n=1}^{\infty}\left|\left\langle\left(F^*F\right)^{-1}f,\phi_n\right\rangle\right|^2\\ &= \left\|F\left(F^*F\right)^{-1}f\right\|^2\\ &= \left\langle F\left(F^*F\right)^{-1}f, F\left(F^*F\right)^{-1}f\right\rangle\\ &= \left\langle\left(F^*F\right)^{-1}f,f\right\rangle.\end{aligned}$$

Now, since $AI \leq F^*F \leq BI$, Theorem 4.6.5 proved by Debnath and Mikusinski (1999) implies

$$\frac{1}{B}I \leq \left(F^*F\right)^{-1} \leq \frac{1}{A}I,$$

which leads to the inequality

$$\frac{1}{B}\|f\|^2 \leq \sum_{n=1}^{\infty}\left|\langle f,\{\tilde{\phi_n}\}\rangle\right|^2 \leq \frac{1}{A}\|f\|^2.$$

This proves the theorem. The sequence $\left\{\tilde{\phi_n}\right\}$ is called the *dual frame.*

Lemma 6.4.2. *Let F be the frame operator associated with the frame $\left\{\phi_1,\phi_2,\phi_3,\ldots\right\}$ and $\tilde{F}$ be the frame operator associated with the dual frame $\left\{\tilde{\phi_1},\tilde{\phi_2},\tilde{\phi_3},\ldots\right\}$. Then,*

$$\tilde{F}^*F = I = F^*\tilde{F}.$$

Proof. Since

$$F\left(F^*F\right)^{-1}f = \left\{\left\langle\left(F^*F\right)^{-1}f,\phi_n\right\rangle\right\} = \left\{\langle f,\tilde{\phi_n}\rangle\right\} = \tilde{F}f, \tag{6.4.11}$$

we have

$$\tilde{F}^*F = \left(F\left(F^*F\right)^{-1}\right)^*F = \left(F^*F\right)^{-1}F^*F = I$$

and

$$F^*\tilde{F} = F^*F\left(F^*F\right)^{-1} = I.$$

Now, we are ready to state and prove the main theorem, which answers the question of reconstructability of f from the sequence $\left\{\langle f,\phi_n\rangle\right\}$.

Theorem 6.4.3. *Let* $\{\phi_1, \phi_2, \phi_3, \ldots\}$ *constitute a frame in a Hilbert space* H*, and let* $\{\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \ldots\}$ *be the dual frame. Then, for any* $f \in H$*,*

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \tilde{\phi_n} \tag{6.4.12}$$

and

$$f = \sum_{n=1}^{\infty} \langle f, \tilde{\phi_n} \rangle \phi_n. \tag{6.4.13}$$

Proof. Let f be the frame operator associated with $\{\phi_n\}$ and let $\tilde{F}$ be the frame operator associated with the dual frame $\{\tilde{\phi_n}\}$. Since $I = \tilde{F}^* F$, for any $f \in H$, we have

$$f = \tilde{F}^* F f = \tilde{F}^* \{\langle f, \phi_n \rangle\} = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \tilde{\phi_n}$$

by (6.4.10). The proof of the other equality is similar.

Using the definition of mother wavelet (6.1.4), we can introduce a family Ψ of vectors $\psi_{a,b} \in L^2$ by

$$\Psi = \left\{ \psi_{a,b} \middle| \langle a, b \rangle \in \mathbb{R}^2 \right| \right\}. \tag{6.4.14}$$

We can then define a frame operator T which transforms a time signal $f \in L^2$ into a function Tf so that

$$Tf(a,b) = \langle f, \psi_{a,b} \rangle = \mathscr{W}[f](a,b). \tag{6.4.15}$$

Thus, the wavelet transform can be interpreted as the frame operator T corresponding to the family Ψ. In view of the measure $d\mu$ defined in the (a, b) plane by

$$d\mu = d\mu(a,b) = \frac{1}{|a|^2}\, da\, db, \tag{6.4.16}$$

we interpret the integral in (6.3.9) as the inner product in a Hilbert space $H = L^2(\mathbb{R}^2, d\mu)$ so that (6.3.9) can be expressed in terms of the norm as

$$\|\mathscr{W} f\|^2 = C_\psi \|f\|^2 \tag{6.4.17}$$

for all $f \in L^2$ and C_ψ is defined by (6.3.10). Thus, (6.4.17) can be interpreted in terms of frame. The family Ψ represents a tight frame for any mother wavelet with frame constant C_ψ.

6.5 Orthonormal Wavelets

Since the discovery of wavelets, orthonormal wavelets with good time–frequency localization are found to play an important role in wavelet theory and have a great variety of applications. In general, the theory of wavelets begins with a single function $\psi \in L^2(\mathbb{R})$, and a family of functions $\psi_{m,n}$ is generated from this single function ψ by the operation of binary dilations (that is, dilation by 2^m) and dyadic translation of $n2^{-m}$ so that

$$\begin{aligned}\psi_{m,n}(x) &= 2^{m/2}\psi\left(2^m\left(x-\frac{n}{2^m}\right)\right), \quad m,n \in \mathbb{Z}\\ &= 2^{m/2}\psi(2^m x - n), \end{aligned}\tag{6.5.1}$$

where the factor $2^{m/2}$ is introduced to ensure orthonormality.

A situation of interest in applications is to deal with an orthonormal family $\{\psi_{m,n}\}$, that is,

$$\langle \psi_{m,n}, \psi_{k,\ell}\rangle = \int_{-\infty}^{\infty} \psi_{m,n}(x)\, \psi_{k,\ell}(x)\, dx = \delta_{m,k}\, \delta_{n,\ell}, \tag{6.5.2}$$

where $m,n,k,\ell \in \mathbb{Z}$.

To show how the inner products behave in this formalism, we prove the following lemma.

Lemma 6.5.1. *If ψ and $\phi \in L^2(\mathbb{R})$, then*

$$\langle \psi_{m,k}, \phi_{m,\ell}\rangle = \langle \psi_{n,k}, \phi_{n,\ell}\rangle, \tag{6.5.3}$$

for all $m,n,k,\ell \in \mathbb{Z}$

Proof. we have

$$\langle \psi_{m,k}, \phi_{m,\ell}\rangle = \int_{-\infty}^{\infty} 2^m \psi\big(2^m x - k\big)\phi\big(2^m x - \ell\big)\, dx$$

which is, by letting $2^m x = 2^n t$,

$$\begin{aligned}&= \int_{-\infty}^{\infty} 2^n \psi\big(2^n t - k\big)\, \phi\big(2^n t - \ell\big)\, dt\\ &= \langle \psi_{n,k}, \phi_{n,\ell}\rangle.\end{aligned}$$

Moreover,

$$\|\psi_{m,n}\| = \|\psi\|.$$

Definition 6.5.1 (Orthonormal Wavelet). A wavelet $\psi \in L^2(\mathbb{R})$ is called *orthonormal* if the family of functions $\psi_{m,n}$ generated from ψ by (6.5.1) is orthonormal.

As in the classical Fourier series, the wavelet series for a function $f \in L^2(\mathbb{R})$ based on a given orthonormal wavelet ψ is given by

$$f(x) = \sum_{m,n=-\infty}^{\infty} c_{m,n}\, \psi_{m,n}(x), \tag{6.5.4}$$

where the wavelet coefficients $c_{m,n}$ are given by

$$c_{m,n} = \langle f, \psi_{m,n} \rangle \tag{6.5.5}$$

and the double wavelet series (6.5.4) converges to the function f in the L^2-norm.

The simplest example of an orthonormal wavelet is the classic Haar wavelet (6.2.6). To prove this fact, we note that the norm of ψ defined by (6.2.6) is one and the same for $\psi_{m,n}$ defined by (6.5.1). We have

$$\langle \psi_{m,n}, \psi_{k,\ell} \rangle = \int_{-\infty}^{\infty} 2^{m/2}\, \psi(2^m x - n) 2^{k/2}\, \psi(2^k x - \ell)\, dx$$

which is, by the change of variables $2^m x - n = t$,

$$= 2^{k/2}\, 2^{-m/2} \int_{-\infty}^{\infty} \psi(t)\, \psi\big(2^{k-m}(t+n) - \ell\big)\, dt. \tag{6.5.6}$$

For $m = k$, this result gives

$$\langle \psi_{m,n}, \psi_{m,\ell} \rangle = \int_{-\infty}^{\infty} \psi(t)\, \psi(t + n - \ell)\, dt = \delta_{0,n-\ell} = \delta_{n,\ell}, \tag{6.5.7}$$

where $\psi(t) \neq 0$ in $0 \leq t \leq 1$ and $\psi(t - \overline{\ell - n}) \neq 0$ in $\ell - n \leq t < 1 + \ell - n$, and these intervals are disjoint from each other unless $n = \ell$.

We now consider the case $m \neq k$. In view of symmetry, it suffices to consider the case $m > k$. Putting $r = m - k > 0$ in (6.5.6), we can complete the proof by showing that, for $k \neq m$,

$$\langle \psi_{m,n}, \psi_{k,l} \rangle = 2^{r/2} \int_{-\infty}^{\infty} \psi(t)\, \psi(2^r t + s)\, dt = 0, \tag{6.5.8}$$

where $s = 2^r n - \ell \in \mathbb{Z}$.

In view of the definition of the Haar wavelet ψ, we must prove that the integral in (6.5.8) vanishes for $k \neq m$. In other words, it suffices to show

$$\int_0^{\frac{1}{2}} \psi\left(2^r t + s\right) dt - \int_{\frac{1}{2}}^1 \psi\left(2^r t + s\right) dt = 0.$$

Invoking a simple change of variables, $2^r t + s = x$, we find

$$\int_s^a \psi(x)\, dx - \int_a^b \psi(x)\, dx = 0, \tag{6.5.9}$$

where $a = s + 2^{r-1}$ and $b = s + 2^r$.

A simple argument reveals that $[s, a]$ contains the support [0,1] of ψ so that the first integral in (6.5.9) is identically zero. Similarly, the second integral is also zero. This completes the proof that the Haar wavelet ψ is orthonormal.

Example 6.5.1 (Discrete Haar Wavelet). The discrete Haar wavelet is defined by

$$\begin{aligned}\psi_{m,n}(t) &= 2^{-m/2}\psi\left(2^{-m}t - n\right) \\ &= \begin{cases} 1, & 2^m n \le t < 2^m n + 2^{m-1} \\ -1, & 2^m n + 2^{m-1} \le t < 2^m n + 2^m \\ 0, & \text{otherwise} \end{cases},\end{aligned} \tag{6.5.10}$$

where ψ is the Haar wavelet defined by (6.2.6).

Since $\left\{\psi_{m,n}(t)\right\}$ is an orthonormal set, any function $f \in L^2(\mathbb{R})$ can be expanded in the wavelet series in the form

$$f(t) = \sum_{m,n=-\infty}^{\infty} \left\langle f, \psi_{m,n}\right\rangle \psi_{m,n}, \tag{6.5.11}$$

where the coefficients $\langle f, \psi_{m,n}\rangle$ satisfy (6.4.4) with $A = B = 1$. To prove this, we assume

$$f(t) = \begin{cases} a, \ 0 \le t < \dfrac{1}{2} \\ b, \ \dfrac{1}{2} \le t < 1 \\ 0, \ \text{otherwise} \end{cases} \tag{6.5.12}$$

Evidently. it follows that

$$\left\langle f, \psi_{m,n}\right\rangle = 0, \quad \text{for } m < 0 \quad \text{or } n \neq 0,$$

and

$$\langle f, \psi_{0,0}\rangle = \frac{1}{2}(a-b), \tag{6.5.13}$$

$$\langle f, \psi_{1,0}\rangle = \frac{1}{\sqrt{2}}\left(\frac{a}{2}+\frac{b}{2}\right), \tag{6.5.14}$$

$$\langle f, \psi_{2,0}\rangle = \frac{1}{2}\left(\frac{a}{2}+\frac{b}{2}\right), \tag{6.5.15}$$

$$\cdots$$

$$\langle f, \psi_{m,0}\rangle = 2^{-\frac{m}{2}}\left(\frac{a}{2}+\frac{b}{2}\right). \tag{6.5.16}$$

Consequently.

$$\langle f, \psi_{m,0}\rangle\, \psi_{0,0}(t) = \begin{cases} \frac{1}{2}(a-b), & 0 \le t < \frac{1}{2} \\ -\frac{1}{2}(a-b), & \frac{1}{2} \le t < 1, \end{cases} \tag{6.5.17}$$

and for $m \ge 1$,

$$\langle f, \psi_{m,0}\rangle\, \psi_{m,0}(t) = 2^{-m}\left(\frac{a}{b}+\frac{b}{2}\right), \quad 0 \le t \le 1. \tag{6.5.18}$$

Finally, it turns out that

$$\sum_{m=0}^{\infty}\langle f, \psi_{m,0}\rangle\, \psi_{m,0}(t) = \begin{cases} \frac{1}{2}(a-b) + \frac{1}{2}(a+b)\sum_{m=1}^{\infty} 2^{-m}, & 0 \le t < \frac{1}{2} \\ -\frac{1}{2}(a-b) + \frac{1}{2}(a+b)\sum_{m=1}^{\infty} 2^{-m}, & \frac{1}{2} \le t < 1. \end{cases} \tag{6.5.19}$$

Since $\sum_{m=0}^{\infty} 2^{-m} = 1$, result (6.5.19) reduces to

$$\sum_{m=0}^{\infty}\langle f, \psi_{m,0}\rangle\psi_{m,0}(t) = \begin{cases} a, & 0 \le t < \frac{1}{2} \\ b, & \frac{1}{2} \le t < 1 \end{cases} \tag{6.5.20}$$

which confirms (6.5.12).

Moreover, it follows from (6.5.13) and (6.5.16) that

$$\sum_{m=0}^{\infty} \left|\langle f, \psi_{m,0}\rangle\right|^2 = \left|\langle f, \psi_{0,0}\rangle\right|^2 + \sum_{m=1}^{\infty} \left|\langle f, \psi_{m,0}\rangle\right|^2$$
$$= \left(\frac{a}{2} - \frac{b}{2}\right)^2 + \left(\frac{a}{2} + \frac{b}{2}\right)^2$$
$$= \frac{1}{2}\left(a^2 + b^2\right) = \int_0^1 f^2(t)\,dt. \tag{6.5.21}$$

This verifies (6.4.4).

Example 6.5.2 (The Discrete Shannon Wavelet). The Shannon function ψ whose Fourier transform satisfies

$$\hat{\psi}(\omega) = \hat{\chi}_I(\omega), \tag{6.5.22}$$

where $I = [-2\pi, -\pi] \cup [\pi, 2\pi]$, is called the *Shannon wavelet*. Thus, this wavelet $\psi(t)$ can directly be obtained from the inverse Fourier transform of $\hat{\psi}(\omega)$ so that

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\hat{\psi}(\omega)\,d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-2\pi}^{-\pi} e^{i\omega t}\,d\omega + \int_{\pi}^{2\pi} e^{i\omega t}\,d\omega\right]$$
$$= \frac{1}{\pi t}\left(\sin 2\pi t - \sin \pi t\right) = \frac{\sin\left(\frac{\pi t}{2}\right)}{\left(\frac{\pi t}{2}\right)}\cos\left(\frac{3\pi t}{2}\right). \tag{6.5.23}$$

This function ψ is orthonormal to its translates by integers. This follows from Parseval's relation

$$\langle \psi(t), \psi(t-n)\rangle = \frac{1}{2\pi}\langle \hat{\psi}, e^{-in\omega}\hat{\psi}\rangle$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}(\omega)\,e^{in\omega}\,\overline{\hat{\psi}}(\omega)\,d\omega$$
$$= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} e^{in\omega}d\omega = \delta_{0,n}.$$

It can easily be verified that the wavelet basis is now given by

$$\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m}t - n - \frac{1}{2}\right), \quad m, n \in \mathbb{Z},$$

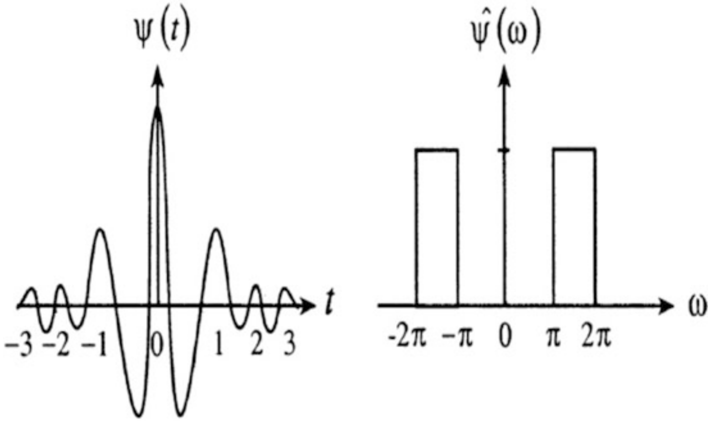

Fig. 6.10 The Shannon wavelet and its Fourier transform

where $\psi\left(t-n-\frac{1}{2}\right)$, $n \in \mathbb{Z}$ is an orthonormal basis for ω_0 and $\psi_{m,n}(t)$, $n \in \mathbb{Z}$ is a basis for functions supported on the interval

$$\left[-2^{-n+1}\pi, -2^{-m}\pi\right] \cup \left[2^{-m}\pi, 2^{-m+1}\pi\right].$$

Since m may be an arbitrarily large integer, we have a basis for $L^2(\mathbb{R})$ functions. Both $\psi(t)$ and $\hat{\psi}(\omega)$ are shown in Fig. 6.10.

It may be observed that the Shannon wavelet is not well localized (noncompact) in the time domain and decays as fast as t^{-1}, and hence, it has poor time localization. However, its Fourier transform is band-limited (compact support) and hence has good frequency localization. These features exhibit a striking contrast with the Haar wavelet.

With the dyadic sampling lattice $a = 2^m$ and $b = 2^m n$, the discrete Shannon wavelet is given by

$$\psi_{m,n}(t) = 2^{-\frac{m}{2}} \frac{\sin\left\{\frac{\pi}{2}\left(2^{-m}t - n\right)\right\}}{\frac{\pi}{2}\left(2^{-m}t - n\right)} \cos\left\{\frac{3\pi}{2}\left(2^{-m}t - n\right)\right\}. \tag{6.5.24}$$

Its Fourier transform is

$$\hat{\psi}_{m,n}(\omega) = \begin{cases} 2^{m/2}\exp\left(-i\omega n 2^m\right), & 2^{-m}\pi < |\omega| < 2^{-m+1}.\pi \\ 0, & \text{otherwise} \end{cases} \tag{6.5.25}$$

Evidently, $\hat{\psi}_{m,n}(\omega)$ and $\hat{\psi}_{k,\ell}(\omega)$ do not overlap for $m \neq k$. Hence, by the Parseval relation (3.4.37), it turns out that, for $m \neq k$,

$$\left\langle \psi_{m,n}, \psi_{k,\ell} \right\rangle = \frac{1}{2\pi} \left\langle \hat{\psi}_{m,n}, \hat{\psi}_{k,\ell} \right\rangle = 0. \tag{6.5.26}$$

For $m = k$, we have

$$\begin{aligned} \left\langle \psi_{m,n}, \psi_{k,\ell} \right\rangle &= \frac{1}{2\pi} \left\langle \hat{\psi}_{m,n}, \hat{\psi}_{k,\ell} \right\rangle \\ &= \frac{1}{2\pi}\, 2^{-m} \int_{-\infty}^{\infty} \exp\left\{ -i\omega 2^{-m}(n-\ell) \right\} \left| \hat{\psi}(2^{-m}\omega) \right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{ -i\sigma(n-\ell) \right\} d\sigma = \delta_{n,\ell}. \end{aligned} \tag{6.5.27}$$

This shows that $\{\psi_{m,n}(t)\}$ is an orthonormal set.

6.6 Exercises

1. Discuss the scaled and translated versions of the mother wavelet $\psi(t) = t \exp\left(-t^2\right)$.

2. Show that the Fourier transform of the normalized Mexican hat wavelet

$$\psi(t) = \frac{2}{\pi^{\frac{1}{4}}\sqrt{3a}} \left(1 - \frac{t^2}{a^2}\right) \exp\left(-\frac{t^2}{2a^2}\right)$$

is

$$\hat{\psi}(\omega) = \sqrt{\frac{8}{3}}\, a^{5/2}\, \pi^{1/4}\, \omega^2 \exp\left(-\frac{a^2\omega^2}{2}\right).$$

3. Show that the continuous wavelet transform can be expressed as a convolution, that is,

$$\mathscr{W}_{\psi}[f](a,b) = \left(f * \psi_a\right)(b),$$

where

$$\psi_a(t) = \frac{1}{\sqrt{a}}\, \overline{\psi}\left(-\frac{t}{a}\right).$$

What is the physical significance of the convolution?

4. If f is a homogeneous function of degree n, show that

$$\left(\mathscr{W}_\psi f\right)(\lambda a, \lambda b) = \lambda^{n+\frac{1}{2}}\left(\mathscr{W}_\psi f\right)(a, b).$$

5. Prove that the vectors $(1,0)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ form a tight frame in $\mathbb{C}$.

6. If $\{\phi_n\}$ is a tight frame in a Hilbert space H with frame bound A, show that

$$A\langle f, g\rangle = \sum_{n=1}^{\infty}\langle f, \phi_n\rangle\langle\phi_n, g\rangle$$

for all $f, g \in H$.

7. If $\{\phi_n\}$ is a tight frame in a Hilbert space H with frame bound 1, show that $\{\phi_n\}$ is an orthonormal basis in H.

8. Show that

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \cdot \frac{\sin \pi(2x-n)}{\pi(2x-n)}\, dx = \frac{1}{2\pi n} \sin\left(\frac{n\pi}{2}\right).$$

9. Show that the Fourier transform of one-cycle of the sine function

$$f(t) = \sin t, \qquad |t| < \pi;$$

is

$$\hat{f}(\omega) = \frac{2i}{(\omega^2 - 1)} \sin \pi\omega.$$

10. For the Shannon wavelet

$$\psi(t) = \frac{\sin\left(\frac{\pi t}{2}\right)}{\left(\frac{\pi t}{2}\right)} \cos\left(\frac{3\pi t}{2}\right),$$

show that its Fourier transform is

$$\hat{\psi}(\omega) = \begin{cases} 1, & \pi < |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

11. Show that the Fourier transform of the wavetrain

$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{t^2}{2\sigma^2}\right) \cos \omega_0 t$$

is

$$\hat{f}(\omega) = \frac{1}{2}\left[\exp\left\{-\frac{\sigma^2}{2}(\omega - \omega_0)^2\right\} + \exp\left\{-\frac{\sigma^2}{2}(\omega + \omega_0)^2\right\}\right].$$

Explain the physical features of $\hat{f}(\omega)$.

12. Show that the Fourier transform of

$$f(t) = \frac{1}{\sqrt{2\pi}} \chi_a(t)\, e^{i\omega_0 t}$$

is

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin\left[a(\omega - \omega_0)\right]}{(\omega - \omega_0)}.$$

Explain the features of $\hat{f}(\omega)$.

13. If

$$\psi\left(\frac{t-b}{a}\right) = \begin{cases} 1, & b \le t < b + \frac{a}{2} \\ -1, & b + \frac{a}{2} \le t < b + a \\ 0, & \text{otherwise} \end{cases}$$

where $a > 0$, show that

$$\mathscr{W}_\psi[f](a,b) = \frac{1}{\sqrt{a}} \int_b^{b+\frac{a}{2}} \left[f(t) - f\left(t + \frac{a}{2}\right)\right] dt.$$

14. Suppose ψ_1 and ψ_2 are two wavelets and the integral

$$\int_{-\infty}^{\infty} \frac{\overline{\hat{\psi}_1(\omega)}\, \hat{\psi}_2(\omega)}{|\omega|}\, d\omega = C_{\psi_1\psi_2} < \infty.$$

If $\mathscr{W}_{\psi_1}[f](a,b)$ and $\mathscr{W}_{\psi_2}[f](a,b)$ denote wavelet transforms, show that

$$\left\langle \mathscr{W}_{\psi_1} f, \mathscr{W}_{\psi_2} g \right\rangle = C_{\psi_2\psi_2} \langle f, g \rangle,$$

where $f, g \in L^2(\mathbb{R})$.

15. The Meyer wavelet ψ is defined by its Fourier transform

$$\hat{\psi}(\omega) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \exp\left(\dfrac{i\omega}{2}\right) \sin\left\{\dfrac{\pi}{2} v\left(\dfrac{3}{2\pi}|\omega| - 1\right)\right\}, & \dfrac{2\pi}{3} \le |\omega| \le \dfrac{4\pi}{3} \\ \dfrac{1}{\sqrt{2\pi}} \exp\left(\dfrac{i\omega}{2}\right) \cos\left\{\dfrac{\pi}{2} v\left(\dfrac{3}{2\pi}|\omega| - 1\right)\right\}, & \dfrac{4\pi}{3} \le |\omega| \le \dfrac{8\pi}{3} \end{cases},$$

where v is a C^k or C^∞ function satisfying

$$v(x) = \begin{cases} 0 \text{ if } & x \le 0 \\ 1 \text{ if } & x \ge 1 \end{cases}$$

and the property

$$v(x) + v(1 - x) = 1.$$

Show that $\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m}t - n\right)$ constitutes an orthonormal basis.