Multifractal Tubes

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Abstract Tube formulas refer to study of volumes of r neighbourhoods of sets. For sets satisfying some (possible very weak) convexity conditions, this has a long history going back to Steiner in the early Nineteenth century. However, within the past 20 years, Lapidus has initiated and pioneered a systematic study of tube formulas for fractal sets. Following this line of investigation, it is natural to ask as to what extent it is possible to develop a theory of multifractal tubes. In this survey we will explain one approach to this problem based on Olsen (Multifractal tubes, Preprint, 2011). In particular, we will propose a general framework for studying tube formulas of multifractals and, as an example, we give a complete description of the asymptotic behaviour of the multifractal tube formulas for self-similar measures satisfying the Open Set Condition.

1 Fractal Tubes

Let E be a subset of $\mathbb{R}^d$ and $r > 0$. We now write $B(E,r)$ for the open r neighbourhood of E, i.e.

$$B(E,r) = \left\{ x \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(x,E) < r \right\}.$$

Intuitively we will think of the set $B(E,r)$ as consisting of the E surrounded by a "tube" of width r. Our main interest is to compute the volume of the "tube" of width r surrounding E or equivalently computing the volume of the set $B(E,r)$ and

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subtract the volume of E. To make this formal, we define the Minkowski volume $V_r(E)$ of E by

$$V_r(E) = \frac{1}{r^d} \mathscr{L}^d(B(E,r));$$

here and below $\mathscr{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$ and the normalizing factor $\frac{1}{r^d}$ is included to make the subsequent results simpler—we note that different authors use different normalizing factors. Tube formulas refer to formulas for computing the Minkowski volume $V_r(E)$ as a function of the width r of the "tube" surrounding E. In particular, one is typically interested in the following two types of results:

1. Asymptotic behaviour: finding a formula for the asymptotic behaviour of $V_r(E)$ as $r \searrow 0$.
2. Explicit formulas: finding an explicit formulas for $V_r(E)$ valid for all small r.

For convex sets E, this problem has a rich and fascinating history starting with the work of Steiner in the early Nineteenth century. This theory reached its mature form in the 1960s where Federer [13, 14] unified the tube formulas of Steiner for convex bodies and of Weyl for smooth submanifolds, as described in [2, 21, 50], and extended these results to sets of positive reach. Federer's tube formula has since been extended in various directions by a number of researchers in integral geometry and geometric measure theory, including [18, 19, 47–49, 52, 53] and most recently (and most generally) in [25]. The books [21, 35, 48] contain extensive endnotes with further information and many other references. While the above references investigate tube formulas for sets that satisfy some (possibly very weak) convexity conditions, very recently there has been significant interest in developing a theory of tube formulas for fractal sets and a number of exciting works have appeared. Indeed, in the early 1990s, Lapidus introduced the notion of "complex dimensions" and has during the past 20 years very successfully pioneered the use of "complex dimension" to obtain explicit tube formulas for certain classes of fractal sets; this exiting theory is described in detail in Lapidus and van Frankenhuysen's intriguing books [29, 30]. In a parallel development and building on earlier work by Lalley [26–28] and Gatzouras [20] (see also [11]), Winter [51] has initiated the systematic study of curvatures of fractal sets and applied this theory to study the asymptotic behaviour of the Minkowski volume $V_r(E)$ of fractal sets E using methods from renewal theory.

The Minkowski volume $V_r(E)$ is closely related to various notions from Fractal Geometry. Indeed, using the Minkowski volume $V_r(E)$, we define the lower and upper Minkowski dimension of E by

$$\underline{\dim}_{\mathsf{M}}(E) = \liminf_{r \searrow 0} \frac{\log V_r(E)}{-\log r},$$

$$\overline{\dim}_{\mathsf{M}}(E) = \limsup_{r \searrow 0} \frac{\log V_r(E)}{-\log r}.$$

The link with Fractal Geometry is now explained as follows. Namely, box dimensions play an important role in Fractal Geometry and it is not difficult to see that the lower Minkowski dimension equals the lower box dimension and that the upper Minkowski dimension equals the upper box dimension; for the definition of the box dimensions the reader is referred to Falconer's textbook [10].

It is clearly also of interest to analyse the behaviour of the Minkowski volume $V_r(E)$ itself as $r \searrow 0$. Indeed, if, for example, $a_1, \dots, a_d, b_1, \dots, b_d$ are real numbers with $a_i \leq b_i$ for all i, and U denotes the rectangle $[a_1, b_1] \times \cdots \times [a_d, b_d]$ in $\mathbb{R}^d$, then it is clear that $\frac{1}{r^{-d}} V_r(U) \to (b_1 - a_1) \cdots (b_d - a_d) = \mathscr{L}^d(U)$. This suggests that if t is a real number, then the limit $\lim_{r \searrow 0} \frac{1}{r^{-t}} V_r(E)$ (if it exists) may be interpreted as the t-dimensional volume of E. Motivated by this, for a real number t, we therefore define the lower and upper t-dimensional Minkowski content of E by

$$\begin{aligned} \underline{M}^t(E) &= \liminf_{r \searrow 0} \frac{1}{r^{-t}} V_r(E), \\ \overline{M}^t(E) &= \limsup_{r \searrow 0} \frac{1}{r^{-t}} V_r(E). \end{aligned}$$

If $\underline{M}^t(E) = \overline{M}^t(E)$, i.e. if the limit $\lim_{r \searrow 0} \frac{1}{r^{-t}} V_r(E)$ exists, then we say the E is t Minkowski measurable, and we will denote the common value of $\underline{M}^t(E)$ and $\overline{M}^t(E)$ by $M^t(E)$, i.e. we will write

$$M^t(E) = \underline{M}^t(E) = \overline{M}^t(E).$$

Of course, a set E may not be Minkowski measurable, i.e. the limit $\lim_{r \searrow 0} \frac{1}{r^{-t}} V_r(E)$ may not exist. In this case it is natural to study the limiting behaviour of "averages" of $\frac{1}{r^{-t}} V_r(E)$. We therefore define the lower and upper average t-dimensional Minkowski content of E by

$$\begin{aligned} \underline{M}^t_{\mathsf{ave}}(E) &= \liminf_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_s(E) \frac{ds}{s}, \\ \overline{M}^t_{\mathsf{ave}}(E) &= \liminf_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_s(E) \frac{ds}{s}. \end{aligned}$$

If $\underline{M}^t_{\mathsf{ave}}(E) = \overline{M}^t_{\mathsf{ave}}(E)$, i.e. if the limit $\lim_{r \searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V_s(E) \frac{ds}{s}$ exists, then we say the E is t average Minkowski measurable, and we will denote the common value of $\underline{M}^t_{\mathsf{ave}}(E)$ and $\overline{M}^t_{\mathsf{ave}}(E)$ by $M^t_{\mathsf{ave}}(E)$, i.e. we will write

$$M^t_{\mathsf{ave}}(E) = \underline{M}^t_{\mathsf{ave}}(E) = \overline{M}^t_{\mathsf{ave}}(E).$$

While the Minkowski dimensions in many cases can be computed rigorously relatively easy, it is a notoriously difficult problem to compute the Minkowski content. In fact, it is only within the past 15 years that the Minkowski content of non-trivial examples has been computed. Indeed, using techniques from complex analysis, Lapidus and collaborators [29, 30] have computed the Minkowski content of certain self-similar subsets of the real line, and using ideas from the theory of Mercerian theorems, Falconer [11] has obtained similar results.

It is our intention to extend the notion of Minkowski volume $V_r(E)$ to multifractals and investigate the asymptotic behaviour of the corresponding multifractal Minkowski volume as $r \searrow 0$ for self-similar multifractals. In order to motivate our definitions we will now explain what the term "multifractal analysis" covers.

2 Multifractals

2.1 Multifractal Spectra

Distributions with widely varying intensity occur often in the physical sciences, for example, the spatial–temporal distribution of rainfall, the spatial distribution of oil and gas in the underground, the distribution of galaxies in the universe, the dissipation of energy in a highly turbulent fluid flow and the occupation measure on strange attractors. Such distributions are called multifractals and have recently been the focus of much attention in the physics literature.

Figure 1 shows a typical multifractal, i.e. a measure with widely varying intensity. Dark regions have high concentration of mass and light regions have low concentration of mass. For a Borel measure μ on a $\mathbb{R}^d$ and a positive number α, let us consider the set $\Delta_\mu(\alpha)$ of those points x in $\mathbb{R}^d$ for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with centre x and radius r behaves like r^α for small r, i.e. the set

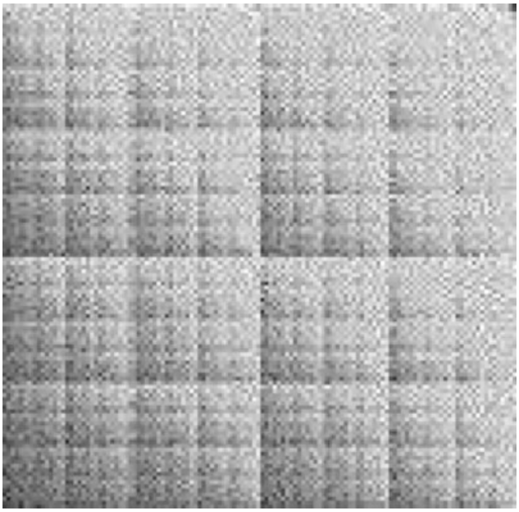

Fig. 1 A typical multifractal, i.e. a measure with widely varying intensity. *Dark regions* have high concentration of mass and *light regions* have low concentration of mass

$$\Delta_\mu(\alpha) = \left\{ x \in \operatorname{supp}\mu \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\},$$

where $\operatorname{supp}\mu$ denotes the support of the measure. If the intensity of the measure μ varies very widely, it may happen that the sets $\Delta_\mu(\alpha)$ display a fractal-like character for a range of values of α. If this is the case, then the measure is called a multifractal measure or simply a multifractal, and it is natural to study the sizes of the sets $\Delta_\mu(\alpha)$ as α varies. There are two approaches to this. We may consider the measure $\mu(\Delta_\mu(\alpha))$ of the sets $\Delta_\mu(\alpha)$ as α varies. This approach was adopted by Cutler in a series of papers [5–7] and leads to a "decomposition" of the measure into its α-dimensional components. However, typically the sets $\Delta_\mu(\alpha)$ have zero μ measure except for a few exceptional values of α. Hence, the measure $\mu(\Delta_\mu(\alpha))$ does in general not allow us to distinguish between the sets $\Delta_\mu(\alpha)$. The other approach is to find the (fractal) dimension of $\Delta_\mu(\alpha)$. In most examples of interest the set $\Delta_\mu(\alpha)$ is dense in the support of μ for all values of α for which $\Delta_\mu(\alpha)$ is non-empty, and thus

$$\underline{\dim}_{\mathrm{B}}\Delta_\mu(\alpha) = \underline{\dim}_{\mathrm{B}}\overline{\Delta_\mu(\alpha)} = \underline{\dim}_{\mathrm{B}} \operatorname{supp}\mu$$

and

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for all values of α for which $\Delta_\mu(\alpha) \neq \varnothing$, where $\underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$ denote the lower and upper box dimension, respectively. Box dimensions are thus in general of little use in discriminating between the size of the sets $\Delta_\mu(\alpha)$. It is therefore more natural to study the Hausdorff dimension,

$$f_\mu(\alpha) = \dim \Delta_\mu(\alpha), \tag{1}$$

of the sets $\Delta_\mu(\alpha)$ as a function of α where dim denotes the Hausdorff dimension. The function in Eq. (1) and similar functions are generically known as "the multifractal spectrum of μ", "the singularity spectrum of μ" or "the spectrum of scaling indices", and one of the main problems in multifractal analysis is to study these and related functions. The function $f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [22]. The concepts underlying the above mentioned multifractal decompositions go back to two early papers by Mandelbrot [32,33] from 1972 and 1974, respectively. Mandelbrot [32,33] suggests that the bulk of intermittent dissipation of energy in a highly turbulent fluid flow occurs over a set of fractal dimension. The ideas introduced in [32, 33] were taken up by Frisch and Parisi [17] in 1985 and finally by Halsey et al. [22] in 1986. Of course, for many measures, the limit $\lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r}$ may fail to exist for all or many x, in which case we need to work with lower or upper limits as r tends to 0 and (perhaps) replace "$= \alpha$" in the definition of $\Delta_\mu(\alpha)$ with "$\leq \alpha$" or "$\geq \alpha$".

2.2 Renyi Dimensions

Based on a remarkable insight together with a clever heuristic argument Halsey et al. [22] suggest that the multifractal spectrum $f_\mu(\alpha)$ can be computed in the following way—known as the so-called "Multifractal Formalism" in the physics literature. The "Multifractal Formalism" involves the so-called Renyi dimensions which we will now define. Let μ be a Borel measure on $\mathbb{R}^d$. For $q \in \mathbb{R}$ and $r > 0$, we define the qth moment $I^q_{\mu,r}(E)$ of a subset E of $\mathbb{R}^d$ with respect to μ at scale r by

$$I^q_{\mu,r}(E) = \int_E \mu(B(x,r))^{q-1}\,\mathrm{d}\mu(x). \tag{2}$$

Next, the lower and upper Renyi dimensions of E with respect to μ are defined by

$$\underline{\dim}^q_{\mathsf{R},\mu}(E) = \liminf_{r\searrow 0} \frac{\log I^q_{\mu,r}(E)}{-\log r}, \tag{3}$$

$$\overline{\dim}^q_{\mathsf{R},\mu}(E) = \limsup_{r\searrow 0} \frac{\log I^q_{\mu,r}(E)}{-\log r}. \tag{4}$$

In particular, the Renyi dimensions of the support of μ play an important role in the statement of the "Multifractal Formalism". For this reason it is useful to denote these dimensions by separate notation, and we therefore define the lower and upper Renyi spectra $\underline{\tau}_\mu(q), \overline{\tau}_\mu(q) : \mathbb{R} \to [-\infty,\infty]$ of μ by

$$\underline{\tau}_\mu(q) = \underline{\dim}^q_{\mathsf{R},\mu}(\operatorname{supp}\mu) = \liminf_{r\searrow 0} \frac{\log I^q_{\mu,r}(\operatorname{supp}\mu)}{-\log r},$$

$$\overline{\tau}_\mu(q) = \overline{\dim}^q_{\mathsf{R},\mu}(\operatorname{supp}\mu) = \limsup_{r\searrow 0} \frac{\log I^q_{\mu,r}(\operatorname{supp}\mu)}{-\log r}.$$

2.3 The Multifractal Formalism

We can now state the "Multifractal Formalism". Loosely speaking the "Multifractal Formalism" says the the multifractal spectrum f_μ and the Renyi dimensions carry the same information. More precisely, the multifractal spectrum equals the Legendre transform of the Renyi dimensions. Before stating this formally, we remind the reader that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a real-valued function, then the Legendre transform $\varphi^* : \mathbb{R} \to [-\infty,\infty]$ of φ is defined by

$$\varphi^*(x) = \inf_y (xy + \varphi(y)).$$

The Multifractal Formalism: A Physics Folklore Theorem. *The multifractal spectrum f_μ of μ equals the Legendre transforms,*s $\underline{\tau}_\mu^*$ *and* $\overline{\tau}_\mu^*$, *of the Renyi dimensions, i.e.*

$$f_\mu(\alpha) = \underline{\tau}_\mu^*(\alpha) = \overline{\tau}_\mu^*(\alpha)$$

for all $\alpha \geq 0$.

The "Multifractal Formalism" is a truly remarkable result: it states that the locally defined multifractal spectrum f_μ can be computed in terms of the Legendre transforms of the globally defined moment scaling functions $\underline{\tau}_\mu^*$ and $\overline{\tau}_\mu^*$. There is apriori no reason to expect that the Legendre transforms of the moment scaling functions $\underline{\tau}_\mu^*$ and $\overline{\tau}_\mu^*$ should provide any information about the fractal dimension of the set of points x such that $\mu(B(x,r)) \approx r^\alpha$ for $r \approx 0$. In some sense the "Multifractal Formalism" is a genuine mystery.

During the past 20 years there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature. In the mid-1990s Cawley and Mauldin [3] and Arbeiter and Patzschke [1] verified the Multifractal Formalism for self-similar measures satisfying the open set condition (OSC), and within the last 10 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. the textbooks [12, 42] and the references therein.

3 Multifractal Tubes

3.1 Multifractal Tubes

Motivated by Lapidus and van Frankenhuysen investigations [29, 30] of tube formulas for fractal sets, it is natural to develop a theory of multifractal tube formulas for multifractal measures. In this section we will present a framework attempting to do this. As an example, we will also give a complete description of the asymptotic behaviour of the multifractal tube formulas for self-similar measures satisfying the Open Set Condition.

Multifractal tube formulas are defined as follows. First note that if $r > 0$ and E is a subset of $\mathbb{R}^d$, then the Minkowski volume $V_r(E)$ is given by

$$V_r(E) = \frac{1}{r^d}\mathscr{L}^d(B(E,r)) = \frac{1}{r^d}\int_{B(E,r)} \mathrm{d}\mathscr{L}^d(x)\,,$$

where we have rewritten the Lebesgue measure $\mathscr{L}^d(B(E,r))$ of $B(E,r)$ as the integral $\int_{B(E,r)} \mathrm{d}\mathscr{L}^d(x)$. Motivated by the Renyi dimensions (i.e. Eqs. (2) and (4)) and the above expression for $V_r(E)$, we now define the multifractal Minkowski volume as follows. Namely, let $r > 0$ and E be a subset of $\mathbb{R}^d$. For real number q and a Borel measure μ on $\mathbb{R}^d$, we now define the multifractal q Minkowski volume $V^q_{\mu,r}(E)$ of E with respect to the measure μ by

$$V^q_{\mu,r}(E) = \frac{1}{r^d} \int\limits_{B(E,r)} \mu(B(x,r))^q \, \mathrm{d}\mathscr{L}^d(x) .$$

Note, that if $q = 0$, then the q multifractal Minkowski volume $V^q_{\mu,r}(E)$ reduces to the usual Minkowski volume, i.e.

$$V^0_{\mu,r}(E) = V_r(E) .$$

The importance of the Renyi dimensions in multifractal analysis together with the formal resemblance between the multifractal Minkowski volume $V^q_{\mu,r}(E)$ and the moments $I^q_{\mu,r}(E)$ used in the definition the Renyi dimensions may be seen as a justification for calling the quantity $V^q_{\mu,r}(E)$ for the *multifractal* Minkowski volume; a further justification for this terminology will be proved below.

Using the multifractal Minkowski volume we can define multifractal Minkowski dimensions. For real number q and a Borel measure μ on $\mathbb{R}^d$, we define the lower and upper multifractal q Minkowski dimension of E, by

$$\begin{aligned} \underline{\dim}^q_{\mathsf{M},\mu}(E) &= \liminf_{r \searrow 0} \frac{\log V^q_{\mu,r}(E)}{-\log r} , \\ \overline{\dim}^q_{\mathsf{M},\mu}(E) &= \limsup_{r \searrow 0} \frac{\log V^q_{\mu,r}(E)}{-\log r} . \end{aligned}$$

Again we note the close similarity between the multifractal Minkowski dimensions and the Renyi dimensions. Indeed, the next proposition shows that this similarity is not merely a formal resemblance. In fact, for $q \geq 0$, the multifractal Minkowski dimensions and the Renyi dimensions coincide. This clearly provides further justification for calling the quantity $V^q_{\mu,r}(E)$ for the *multifractal* Minkowski volume.

Proposition 1 ([38]). *Let μ be a Borel measure on $\mathbb{R}^d$ and $E \subseteq \mathbb{R}^d$. If $q \geq 0$, then*

$$\begin{aligned} \underline{\dim}^q_{\mathsf{M},\mu}(E) &= \underline{\dim}^q_{\mathsf{R},\mu}(E) , \\ \overline{\dim}^q_{\mathsf{M},\mu}(E) &= \overline{\dim}^q_{\mathsf{R},\mu}(E) . \end{aligned}$$

In particular, if $q \geq 0$, then

$$\underline{\dim}^q_{\mathsf{M},\mu}(\operatorname{supp}\mu) = \underline{\tau}_\mu(q) ,$$

$$\overline{\dim}^{q}_{\mathsf{M},\mu}(\operatorname{supp}\mu) = \overline{\tau}_{\mu}(q).$$

Proof. This follows easily from the definitions. □

Having defined multifractal Minkowski dimensions, we also define multifractal Minkowski content and average multifractal Minkowski content. For real numbers q and t, we define the lower and upper (q,t)-dimensional multifractal Minkowski content of E with respect to μ by

$$\underline{M}^{q,t}_{\mu}(E) = \liminf_{r\searrow 0} \tfrac{1}{r^{-t}} V^{q}_{\mu,r}(E),$$
$$\overline{M}^{q,t}_{\mu}(E) = \limsup_{r\searrow 0} \tfrac{1}{r^{-t}} V^{q}_{\mu,r}(E).$$

If $\underline{M}^{q,t}_{\mu}(E) = \overline{M}^{q,t}_{\mu}(E)$, i.e. if the limit $\lim_{r\searrow 0} \frac{1}{r^{-t}} V^{q}_{\mu,r}(E)$ exists, then we say the E is (q,t) multifractal Minkowski measurable with respect to μ, and we will denote the common value of $\underline{M}^{q,t}_{\mu}(E)$ and $\overline{M}^{q,t}_{\mu}(E)$ by $M^{q,t}_{\mu}(E)$, i.e. we will write

$$M^{q,t}_{\mu}(E) = \underline{M}^{q,t}_{\mu}(E) = \overline{M}^{q,t}_{\mu}(E).$$

Of course, sets may not be multifractal Minkowski measurable, and it is therefore useful to introduce a suitable averaging procedure when computing the multifractal Minkowski content. Motivated by this we define the lower and upper (q,t)-dimensional average multifractal Minkowski content of E with respect to μ by

$$\underline{M}^{q,t}_{\mu,\mathsf{ave}}(E) = \liminf_{r\searrow 0} \tfrac{1}{-\log r} \int_r^1 \tfrac{1}{s^{-t}} V^{q}_{\mu,s}(E) \tfrac{\mathrm{d}s}{s},$$
$$\overline{M}^{q,t}_{\mu,\mathsf{ave}}(E) = \liminf_{r\searrow 0} \tfrac{1}{-\log r} \int_r^1 \tfrac{1}{s^{-t}} V^{q}_{\mu,s}(E) \tfrac{\mathrm{d}s}{s}.$$

If $\underline{M}^{q,t}_{\mu,\mathsf{ave}}(E) = \overline{M}^{q,t}_{\mu,\mathsf{ave}}(E)$, i.e. if the limit $\lim_{r\searrow 0} \frac{1}{-\log r} \int_r^1 \frac{1}{s^{-t}} V^{q}_{\mu,s}(E) \frac{\mathrm{d}s}{s}$ exists, then we say the E is (q,t) average multifractal Minkowski measurable with respect to μ, and we will denote the common value of $\underline{M}^{q,t}_{\mu,\mathsf{ave}}(E)$ and $\overline{M}^{q,t}_{\mu,\mathsf{ave}}(E)$ by $M^{q,t}_{\mu,\mathsf{ave}}(E)$, i.e. we will write

$$M^{q,t}_{\mu,\mathsf{ave}}(E) = \underline{M}^{q,t}_{\mu,\mathsf{ave}}(E) = \overline{M}^{q,t}_{\mu,\mathsf{ave}}(E).$$

3.2 *Multifractal Tubes of Self-similar Measures*

As an example, we will now compute the multifractal Minkowski content of self-similar measures. We first recall the definition of self-similar measures. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1,\dots,N$ be contracting similarities and let $(p_1,\dots,p_N)$ be a probability vector. We denote the Lipschitz constant of S_i by $r_i \in (0,1)$. Let K and μ be the self-similar set associated with the list $(S_1,\dots,S_N)$ and the self-similar measure associated with the list $(S_1,\dots,S_N,p_1,\dots,p_N)$, i.e. K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K=\bigcup_i S_i(K)\,, \tag{5}$$

and μ the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu=\sum_i p_i\mu\circ S_i^{-1}\,, \tag{6}$$

cf. [24]. We note that it is well-known that $\operatorname{supp}\mu=K$.

We will frequently assume that the list $(S_1,\dots,S_N)$ satisfies certain "disjointness" conditions, viz the OSC or the strong separation condition (SSC) defined below.

The Open Set Condition: There exists an open non-empty and bounded subset U of $\mathbb{R}^d$ with $\cup_i S_iU\subseteq U$ and $S_iU\cap S_jU=\varnothing$ for all i,j with $i\neq j$.

The Strong Separation Condition: There exists an open non-empty and bounded subset U of $\mathbb{R}^d$ with $\cup_i S_iU\subseteq U$ and $\overline{S_iU}\cap\overline{S_jU}=\varnothing$ for all i,j with $i\neq j$.

Multifractal analysis of self-similar measures has attracted an enormous interest during the past 20 years. For example, using methods from ergodic theory, Peres and Solomyak [43] have recently shown that for any self-similar measure μ, the Renyi dimensions always exist, i.e. the limit $\lim_{r\searrow 0}\frac{\log I^q_{\mu,r}(K)}{-\log r}$ always exists regardless of whether or not the OSC is satisfied provided $q\geq 0$. If in addition the OSC is satisfied, an explicit expression for the two limits $\underline{\tau}_\mu(q)=\liminf_{r\searrow 0}\frac{\log I^q_{\mu,r}(K)}{-\log r}$ and $\overline{\tau}_\mu(q)=\limsup_{r\searrow 0}\frac{\log I^q_{\mu,r}(K)}{-\log r}$ can be obtained. Indeed, Arbeiter and Patzschke [1] and Cawley and Mauldin [3] proved that if the OSC is satisfied, then

$$\begin{aligned}\underline{\tau}_\mu(q)&=\liminf_{r\searrow 0}\frac{\log I^q_r(K)}{-\log r}\\&=\beta(q)\,,\\\overline{\tau}_\mu(q)&=\limsup_{r\searrow 0}\frac{\log I^q_r(K)}{-\log r}\\&=\beta(q)\,,\end{aligned} \tag{7}$$

for $q\in\mathbb{R}$, where $\beta(q)$ is defined by

$$\sum_i p_i^q r_i^{\beta(q)}=1\,. \tag{8}$$

Arbeiter and Patzschke [1] and Cawley and Mauldin [3] also verified the Multifractal Formalism for self-similar measures satisfying the OSC. Namely, in [1, 3], it is proved that if μ is a self-similar measure satisfying the OSC, then

$$f_\mu(\alpha) = \beta^*(\alpha)$$

for all $\alpha \geq 0$; recall, that the definition of the Legendre transform φ^* of a real-valued function $\varphi : \mathbb{R} \to \mathbb{R}$ is given in Sect. 2.3. We continue this line of investigation by computing the multifractal Minkowski dimensions and multifractal Minkowski content of self-similar measures satisfying various separation conditions. Firstly, we note that the multifractal Minkowski dimensions coincide with $\beta(q)$. This is not a deep fact and is included mainly for completeness.

Theorem 1 ([38]). *Let K and μ be given by Eqs. (5) and (6). Fix $q \in \mathbb{R}$ and assume that Condition (i) or Condition (ii) below is satisfied.*

(i) The OSC is satisfied and $0 \leq q$.
(ii) The SSC is satisfied.

Then we have

$$\underline{\dim}^q_{\mathsf{M},\mu}(K) = \overline{\dim}^q_{\mathsf{M},\mu}(K) = \beta(q)$$

for all $q \in \mathbb{R}$.

Proof. As noted above, this is not a deep fact and follows from the definitions using standard arguments similar to those in [1] or Falconer's textbook [12]. □

Next, we give a complete description of the asymptotic behaviour of the multifractal tube formulas for self-similar measures satisfying the Open Set Condition. In particular, we prove that if the set $\{\log r_1^{-1}, \dots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then K is $(q, \beta(q))$ multifractal Minkowski measurable with respect to μ, and if the set $\{\log r_1^{-1}, \dots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$, then K is $(q, \beta(q))$ average multifractal Minkowski measurable with respect to μ. This is the content of Theorem 2. The proof of Theorem 2 is based on Renewal Theory and will be discussed after the statement of the theorem.

Theorem 2 ([38]). *Let K and μ be given by Eqs. (5) and (6). Fix $q \in \mathbb{R}$ and assume that Condition (i) or Condition (ii) below is satisfied.*

(i) The OSC is satisfied and $0 \leq q$.
(ii) The SSC is satisfied.

Define $\lambda_q : (0, \infty) \to \mathbb{R}$ by

$$\lambda_q(r) = V^q_{\mu,r}(K) - \sum_i p_i^q \mathbf{1}_{(0,r_i]}(r) V^q_{\mu, r_i^{-1} r}(K).$$

Then we have the following:

1. *If the set $\{\log r_1^{-1}, \dots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then*

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(K) = c_q + \varepsilon_q(r),$$

where $c_q \in \mathbb{R}$ *is the constant given by*

$$c_q = \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^1 r^{\beta(q)} \lambda_q(r) \, \frac{\mathrm{d}r}{r}$$

and $\varepsilon_q(r) \to 0$ *as* $r \searrow 0$. *In addition,* K *is* $(q, \beta(q))$ *multifractal Minkowski measurable with respect to* μ*:*

$$M^{q,\beta(q)}_{\mu}(K) = \tfrac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^1 r^{\beta(q)} \lambda_q(r) \tfrac{\mathrm{d}r}{r}\,.$$

2. *If the set* $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ *is contained in a discrete additive subgroup of* $\mathbb{R}$ *and* $\langle \log r_1^{-1}, \ldots, \log r_N^{-1} \rangle = u\mathbb{Z}$ *with* $u > 0$*, then*

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(K) = \pi_q(r) + \varepsilon_q(r),$$

where $\pi_q : (0, \infty) \to \mathbb{R}$ *is the multiplicatively periodic function with period equal to* e^u*, i.e.*

$$\pi_q(e^u r) = \pi_q(r)$$

for all $r \in (0, \infty)$*, given by*

$$\pi_q(r) = \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \sum_{n \in \mathbb{Z}, re^{nu} \leq 1} (re^{un})^{\beta(q)} \, \lambda_q(re^{un}) \, u$$

and $\varepsilon_q(r) \to 0$ *as* $r \searrow 0$. *In addition,* K *is* $(q, \beta(q))$ *average multifractal Minkowski measurable with respect to* μ *with*

$$M^{q,\beta(q)}_{\mu,\mathsf{ave}}(K) = \tfrac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^1 r^{\beta(q)} \lambda_q(r) \tfrac{\mathrm{d}r}{r}\,.$$

It is instructive to consider the special case $q = 0$. Indeed, since the multifractal Minkowski volume for $q = 0$ equals the usual Minkowski volume and since the (q,t)-dimensional multifractal Minkowski content for $q = 0$ equals the usual t-dimensional Minkowski content, the following corollary providing formulas for the asymptotic behaviour of the Minkowski volume of self-similar sets follows immediately from Theorem 2 by putting $q = 0$. This result was first obtained by Gatzouras [20] and later by Winter [51].

Corollary 1 ([20]). *Let* K *be given by Eqs. (5) and (6). Assume that the OSC is satisfied. Let* t *denote the common value of the box dimensions and the Hausdorff* *dimension of K, i.e. t is the unique number such that $\sum_i r_i^t = 1$ (see [12] or [24]). Define $\lambda : (0,\infty) \to \mathbb{R}$ by*

$$\lambda(r) = V_r(K) - \sum_i \mathbf{1}_{(0,r_i]}(r) V_{r_i^{-1}r}(K).$$

Then we have:

1. *If the set $\{\log r_1^{-1}, \dots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then*
$$\frac{1}{r^{-t}} V_r(K) = c + \varepsilon(r),$$
where $c \in \mathbb{R}$ is the constant given by
$$c = \frac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^{\beta(q)} \lambda(r) \frac{\mathrm{d}r}{r}$$
and $\varepsilon(r) \to 0$ as $r \searrow 0$. In addition, K is t Minkowski measurable with
$$M^t(K) = \tfrac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^t \lambda(r) \tfrac{\mathrm{d}r}{r}.$$

2. *If the set $\{\log r_1^{-1}, \dots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ and $\langle \log r_1^{-1}, \dots, \log r_N^{-1} \rangle = u\mathbb{Z}$ with $u > 0$, then*
$$\frac{1}{r^{-t}} V_r(K) = \pi(r) + \varepsilon(r),$$
where $\pi : (0,\infty) \to \mathbb{R}$ is the multiplicatively periodic function with period equal to e^u, i.e.
$$\pi(e^u r) = \pi(r)$$
for all $r \in (0,\infty)$, given by
$$\pi(r) = \frac{1}{-\sum_i r_i^t \log r_i} \sum_{n \in \mathbb{Z}, re^{nu} \le 1} (re^{un})^t \lambda(re^{un}) u$$
and $\varepsilon(r) \to 0$ as $r \searrow 0$. In addition, K is t average Minkowski measurable with
$$M^t_{\mathsf{ave}}(K) = \tfrac{1}{-\sum_i r_i^t \log r_i} \int_0^1 r^t \lambda(r) \tfrac{\mathrm{d}r}{r}.$$

Proof. Since $\beta(0) = \underline{\dim}_{\mathsf{B}}(K) = \overline{\dim}_{\mathsf{B}}(K) = \dim(K) = t$ (see [12] or [24]) and $V^0_{\mu,r}(K) = V_r(K)$, this follows from Theorem 2 by putting $q = 0$. □

3.3 How Does One Prove Theorem 2 on the Asymptotic Behaviour of Multifractal Tubes of Self-similar Measures?

How does one prove Theorem 2? The proof is based on Renewal Theory and, in particular, on a very recent renewal theorem by Levitin and Vassiliev [31]. Below we state Levitin and Vassiliev's Renewal Theorem.

Theorem 3 (Levitin and Vassiliev's Renewal Theorem [31]). *Let* $t_1,\dots,t_N>0$ *and* $p_1,\dots,p_N>0$ *with* $\sum_i p_i=1$. *Define the probability measure* P *by*

$$P=\sum_i p_i\delta_{t_i}\,.$$

Let $\lambda,\Lambda:\mathbb{R}\to\mathbb{R}$ *be real-valued functions satisfying the following conditions:*

1. *The function* λ *is piecewise continuous.*
2. *There are constants* $c,k>0$ *such that*
$$|\lambda(t)|\le ce^{-k|t|}$$
for all $t\in\mathbb{R}$.
3. *We have*
$$\Lambda(t)\to 0 \text{ as } t\to-\infty.$$
4. *We have*
$$\Lambda(t)=\int\Lambda(t-s)\,\mathrm{d}P(s)+\lambda(t)$$
for all $t\in\mathbb{R}$.

Then the following holds:

1. *The non-arithmetic case: If* $\{t_1,\dots,t_N\}$ *is not contained in a discrete additive subgroup of* $\mathbb{R}$, *then*
$$\Lambda(t)=c+\varepsilon(t)$$
for all $t\in\mathbb{R}$ *where*
$$c=\frac{1}{\int s\,\mathrm{d}P(s)}\int\lambda(s)\,\mathrm{d}s$$
and $\varepsilon(t)\to 0$ *as* $t\to\infty$. *In addition,*
$$\frac{1}{T}\int_0^T\Lambda(t)\,\mathrm{d}t\to c=\frac{1}{\int s\,\mathrm{d}P(s)}\int\lambda(s)\,\mathrm{d}s \text{ as } T\to\infty. \tag{9}$$
2. *The arithmetic case: If* $\{t_1,\dots,t_N\}$ *is contained in a discrete additive subgroup of* $\mathbb{R}$ *and* $\langle t_1,\dots,t_N\rangle=u\mathbb{Z}$ *with* $u>0$, *then*
$$\Lambda(t)=\pi(t)+\varepsilon(t)$$

for all $t \in \mathbb{R}$ where $\pi : \mathbb{R} \to \mathbb{R}$ is the periodic function with period equal to u, i.e.

$$\pi(t+u) = \pi(t)$$

for all $t \in \mathbb{R}$, given by

$$\pi(t) = \frac{1}{\int s \,\mathrm{d}P(s)} u \sum_{n\in\mathbb{Z}} \lambda(t+nu)$$

and $\varepsilon(t) \to 0$ as $t \to \infty$. In addition

$$\frac{1}{T}\int_0^T \Lambda(t)\,\mathrm{d}t \to c = \frac{1}{\int s\,\mathrm{d}P(s)} \int \lambda(s)\,\mathrm{d}s \text{ as } T \to \infty. \tag{10}$$

Proof. All statements, except Eqs. (9) and (10), follow [31]. Below we prove Eqs. (9) and (10). Indeed, Eq. (9) follows immediately and Eq. (10) is proved as follows. Namely, since π is periodic with period equal to u, we conclude that

$$\begin{aligned} \frac{1}{T}\int_0^T \Lambda(t)\,\mathrm{d}t &= \frac{1}{T}\int_0^T \pi(t)\,\mathrm{d}t + \frac{1}{T}\int_0^T \varepsilon(t)\,\mathrm{d}t \\ &\to \frac{1}{u}\int_0^u \pi(t)\,\mathrm{d}t \\ &= \frac{1}{\int t\,\mathrm{d}P(t)} \int_0^u \sum_{n\in\mathbb{Z}} \lambda(t+nu)\,\mathrm{d}t\,. \end{aligned} \tag{11}$$

Next, observe that since $|\lambda(t)| \le ce^{-k|t|}$ for all $t \in \mathbb{R}$ and $\int ce^{-k|t|}\,\mathrm{d}t < \infty$, it follows from two applications of Lebesgue's Dominated Convergence Theorem and the fact that π is periodic with period equal to u that

$$\begin{aligned} \int_0^u \sum_{n\in\mathbb{Z}} \lambda(t+nu)\,\mathrm{d}t &= \sum_{n\in\mathbb{Z}} \int_0^u \lambda(t+nu)\,\mathrm{d}t \\ &= \sum_{n\in\mathbb{Z}} \int_{nu}^{(n+1)u} \lambda(t)\,\mathrm{d}t \\ &= \sum_{n\in\mathbb{Z}} \int \mathbf{1}_{[nu,(n+1)u)}(t)\,\lambda(t)\,\mathrm{d}t \\ &= \int \sum_{n\in\mathbb{Z}} \mathbf{1}_{[nu,(n+1)u)}(t)\,\lambda(t)\,\mathrm{d}t \\ &= \int \lambda(t)\,\mathrm{d}t\,. \end{aligned} \tag{12}$$

Finally, combining Eqs. (11) and (12) shows that

$$\frac{1}{T}\int_0^T \Lambda(t)\,\mathrm{d}t \to \frac{1}{\int t\,\mathrm{d}P(t)}\int \lambda(t)\,\mathrm{d}t\,.$$

This completes the proof. □

The key difference between Levitin and Vassiliev's Renewal Theorem and the classical renewal theorem from Feller's books [15, 16] is the conclusion in the arithmetic case. While the assumptions in the classical renewal theorem are weaker, the conclusion in the arithmetic case is also weaker. More precisely, in the arithmetic case, Levitin and Vassiliev's Renewal Theorem says that the error-term $\varepsilon(t)$ tends to 0 as t tends to infinity, i.e.

$$\lim_{t\to\infty}\varepsilon(t) = 0\,,$$

whereas the classical renewal theorem only allows us to conclude that the error-term $\varepsilon(t)$ tends to 0 as t tends to infinity through "steps" of length u, i.e.

$$\lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \varepsilon(nu+s) = 0$$

for all $s\in\mathbb{R}$.

Using Levitin and Vassiliev's Renewal Theorem (Theorem 3) we can now prove Theorem 2. Below is a sketch of the proof.

Sketch of Proof of Theorem 2

In order to prove Theorem 2, we will apply Levitin and Vassiliev's Renewal Theorem to the probability measure $P = P_q$ and the functions $\lambda = \lambda_q^0$ and $\Lambda = \Lambda_q^0$ defined below. First recall that $\lambda_q : (0,\infty)\to\mathbb{R}$ is defined by

$$\lambda_q(r) = V_{\mu,r}^q(K) - \textstyle\sum_i p_i^q\,\mathbf{1}_{(0,r_i]}(r)\,V_{\mu,r_i^{-1}r}^q(K)\,,$$

and define $\Lambda_q : (0,\infty)\to\mathbb{R}$ by

$$\Lambda_q(r) = V_{\mu,r}^q(K)\,.$$

We can now define the functions $\lambda_q^0, \Lambda_q^0 : \mathbb{R}\to\mathbb{R}$. Namely, define $\lambda_q^0 : \mathbb{R}\to\mathbb{R}$ by

$$\lambda_q^0(t) = \mathbf{1}_{[0,\infty)}(t)\,e^{-t\beta(q)}\lambda_q(e^{-t})\,,$$

and define $\Lambda_q^0 : \mathbb{R}\to\mathbb{R}$ by

$$\Lambda_q^0(t) = \mathbf{1}_{[0,\infty)}(t)\,e^{-t\beta(q)}\Lambda_q(e^{-t})\,.$$

Finally, define the probability measure P_q by

$$P_q = \sum_i p_i^q r_i^{\beta(q)} \delta_{\log r_i^{-1}} .$$

The crux of the matter now is to show that the probability measure $P = P_q$ and the functions $\lambda = \lambda_q^0$ and $\Lambda = \Lambda_q^0$ satisfy conditions (1)–(4) in Levitin and Vassiliev's Renewal Theorem.

Condition (1) is satisfied. This is not difficult to show. Indeed, it follows by applying results from [34] that the function $f : (0,\infty) \to \mathbb{R}$ defined by $f(r) = \int_{B(K,r)} \mu(B(x,r))^q \, \mathrm{d}\mathscr{L}^d(x)$ is continuous. This clearly implies that condition (1) is satisfied.

Condition (2) is satisfied. This is the difficult part of the proof and requires a number of very delicate estimates. In particular, the proof of condition (2) is based on the three key estimates below. The proofs of Key Estimate 2 and Key Estimate 3 are both highly technical and require a number very delicate estimates. Below we state the three key estimates. However, we will not prove the estimates. Instead the reader is referred to [38]. Before we can state the key estimates we need to introduce some notation. For $i \neq j$ and $r > 0$, let

$$Q_{i,j}^q(r) = \frac{1}{r^d} \int\limits_{B(S_iK,r)\cap B(S_jK,r)} \mu(B(x,r))^q \, \mathrm{d}\mathscr{L}^d(x) .$$

Let $\Sigma = \{1,\ldots,N\}$ and write

$$\Sigma^m = \{1,\ldots,N\}^m ,$$
$$\Sigma^* = \bigcup_m \Sigma_m ,$$

i.e. Σ^m is the family of all strings $\mathbf{i} = i_1 \ldots i_m$ of length m with $i_j \in \{1,\ldots,N\}$, and Σ^* is the family of all finite strings $\mathbf{i} = i_1 \ldots i_m$ with $i_j \in \{1,\ldots,N\}$. For $\mathbf{i} \in \Sigma^m$, we write $|\mathbf{i}| = m$ for the length of $\mathbf{i}$ and for a positive integer n with $n \leq m$, we write $\mathbf{i}|n = i_1 \ldots i_n$ for the truncation of $\mathbf{i}$ to the nth place. Also, for $\mathbf{i} = i_1 \ldots i_m, \mathbf{j} = j_1 \ldots j_n \in \Sigma^*$, let $\mathbf{ij} = i_1 \ldots i_m j_1 \ldots j_n$ denote the concatenation of $\mathbf{i}$ and $\mathbf{j}$. Next, if $\mathbf{i} = i_1 \ldots i_m \in \Sigma^*$, we will write

$$\begin{aligned} S_{\mathbf{i}} &= S_{i_1} \circ \cdots \circ S_{i_m} , \\ r_{\mathbf{i}} &= r_{i_1} \cdots r_{i_m} , \\ p_{\mathbf{i}} &= p_{i_1} \cdots p_{i_m} . \end{aligned} \tag{13}$$

Also for brevity, put $r_{\min} = \min_{i=1,\ldots,N} r_i$ and $r_{\max} = \max_{i=1,\ldots,N} r_i$.

For $\mathbf{i}, \mathbf{h} \in \Sigma^*$, we write $\mathbf{i} \prec \mathbf{h}$ if and only if $\mathbf{i}$ is a substring of $\mathbf{h}$, i.e. if and only if there are strings $\mathbf{s}, \mathbf{t} \in \Sigma^*$ such that $\mathbf{h} = \mathbf{sit}$. If $(S_1,\ldots,S_n)$ satisfies the OSC, then it follows from a result by Schief [46] that there exists an open, bounded and non-empty subset U of $\mathbb{R}^d$ with $\cup_i S_i U \subseteq U$, $S_i U \cap S_j U = \varnothing$ for all i,j with $i \neq j$, and $U \cap K \neq \varnothing$. In particular, since $U \cap K \neq \varnothing$, we can choose $\mathbf{l} \in \Sigma^*$ such that

$$S_{\mathbf{l}} K \subseteq U\,, \tag{14}$$

and the compactness of $S_{\mathbf{l}}K$ now implies that $d_0 = \operatorname{dist}(S_{\mathbf{l}}K, \mathbb{R}^d \setminus U) > 0$. For brevity write $D_0 = \mathrm{d}K$. Choose a positive integer M such that $\frac{1}{r_{\max}^{M-1}} \geq 2\,\frac{D_0}{d_0}$, and put $a = \frac{1}{D_0}\,\frac{r_{\min}}{r_{\max}^{M+1}}$ and $b = \frac{1}{D_0}\,\frac{1}{r_{\min}^{M+1}}$. Finally, define $Z^q : (0,\infty) \to \mathbb{R}$ by

$$Z^q(r) = \sum_{\mathbf{h} \in \Sigma^*, |\mathbf{h}| \geq |\mathbf{l}|, ar \leq r_{\mathbf{h}} \leq br, \mathbf{l} \not\prec \mathbf{h}} p_{\mathbf{h}}^q\,.$$

The three key estimates are now:

Key Estimate 1. $|\lambda_q(r)| \leq \sum_{i \neq j} Q_{i,j}^q(r)$ *for all* $0 < r < r_{\min}$.
Key Estimate 2. *There is a constant* $c > 0$ *such that*

$$\sum_{i \neq j} Q_{i,j}^q(r) \leq \begin{cases} c\,Z^q(\tfrac{1}{2}r) & \text{for } q < 0 \text{ and all } r > 0, \\ c\,Z^q(2r) & \text{for } 0 \leq q \text{ and all } r > 0. \end{cases}$$

Key Estimate 3. *There are constants* $k > 0$ *and* $\gamma(q) \in \mathbb{R}$ *with* $\gamma(q) < \beta(q)$ *such that*

$$Z^q(r) \leq k\,r^{-\gamma(q)} \quad \text{for all } r > 0.$$

Combining the three key estimates we can now prove that condition (2) is satisfied. Indeed, choose $t_0 > 0$ such that $e^{-t} < r_{\min}$ for $t \geq t_0$. For $t \geq t_0$, we now have

$$|\lambda_q^0(t)| = \mathbf{1}_{[0,\infty)}(t)\,e^{-t\beta(q)}\,|\lambda_q(e^{-t})| \tag{15}$$

$$\leq e^{-t\beta(q)} \sum_{i \neq j} Q_{i,j}^q(e^{-t}) \qquad \text{[by Key Estimate 1]} \tag{16}$$

$$\leq \begin{cases} e^{-t\beta(q)} c\,Z^q(\tfrac{1}{2}e^{-t}) & \text{for } q < 0, \\ e^{-t\beta(q)} c\,Z^q(2e^{-t}) & \text{for } 0 \leq q \end{cases} \qquad \text{[by Key Estimate 2]} \tag{17}$$

$$\leq \begin{cases} e^{-t\beta(q)}\,c\,k\,(\tfrac{1}{2}e^{-t})^{-\gamma(q)} & \text{for } q < 0, \\ e^{-t\beta(q)}\,c\,k\,(2e^{-t})^{-\gamma(q)} & \text{for } 0 \leq q \end{cases} \qquad \text{[by Key Estimate 3]} \tag{18}$$

$$\tag{19}$$

$$= c_0\,e^{-(\beta(q)-\gamma(q))t}, \tag{20}$$

where $c_0 = c\,k \max((\tfrac{1}{2})^{-\gamma(q)}, 2^{-\gamma(q)})$.

Next, since λ_q^0 is piecewise continuous (by condition (1)), we conclude that λ_q^0 is bounded on the compact interval $[0,t_0]$, and we therefore deduce that there is a constant M_0 such that $|\lambda_q^0(t)| \le M_0$ for all $t \in [0,t_0]$. It follows from this and Eq. (20) that

$$|\lambda_q^0(t)| \le \max\left(\frac{M_0}{e^{-(\beta(q)-\gamma(q))t_0}}\,,\,c_0\right) e^{-(\beta(q)-\gamma(q))t} \tag{21}$$

for all $t \ge 0$.

Inequality Eq. (21) and the fact that $\lambda_q^0(t) = 0$ for all $t < 0$ now prove that condition (2) is satisfied.

Condition (3) is satisfied. This follows trivially from the fact that $\Lambda_q^0(t) = 0$ for all $t < 0$.

Condition (4) is satisfied. Indeed, it follows immediately from the definitions of λ_q^0, Λ_q^0 and P_q that

$$\begin{aligned}
\Lambda_q^0(t) &= \mathbf{1}_{[0,\infty)}(t)\,e^{-t\beta(q)}\Lambda_q(e^{-t})\\
&= \mathbf{1}_{[0,\infty)}(t)\,e^{-t\beta(q)}\left(\sum_i p_i^q\,\mathbf{1}_{(0,r_i]}(e^{-t})\,V^q_{\mu,r_i^{-1}e^{-t}}(K) + \lambda_q(e^{-t})\right)\\
&= \sum_i p_i^q\,e^{-t\beta(q)}\,\mathbf{1}_{(0,r_i]}(e^{-t})\,\mathbf{1}_{[0,\infty)}(t)\,V^q_{\mu,r_i^{-1}e^{-t}}(K) + \lambda_q^0(t)\\
&= \sum_i p_i^q r_i^{\beta(q)}\,\mathbf{1}_{[0,\infty)}(t-\log r_i^{-1})\,\mathbf{1}_{[0,\infty)}(t)\,e^{-\beta(q)(t-\log r_i^{-1})}V^q_{\mu,e^{-(t-\log r_i^{-1})}}(K)+\lambda_q^0(t)\\
&= \sum_i p_i^q r_i^{\beta(q)}\,\mathbf{1}_{[0,\infty)}(t-\log r_i^{-1})\,e^{-\beta(q)(t-\log r_i^{-1})}V^q_{\mu,e^{-(t-\log r_i^{-1})}}(K) + \lambda_q^0(t)\\
&= \sum_i p_i^q r_i^{\beta(q)}\Lambda_q^0(t-\log r_i^{-1}) + \lambda_q^0(t)\\
&= \int \Lambda_q^0(t-s)\,\mathrm{d}P_q(s) + \lambda_q^0(t)
\end{aligned}$$

for all $t \in \mathbb{R}$. This proves that condition (4) is satisfied.

Since conditions (1)–(4) are satisfied, Levitin and Vassiliev's Renewal Theorem can now be applied to the probability measure $P = P_q$ and the functions $\lambda = \lambda_q^0$ and $\Lambda = \Lambda_q^0$. We divide the proof into two cases.

Case 1. If $\{\log r_1^{-1},\ldots,\log r_N^{-1}\}$ *is not contained in a discrete additive subgroup of* $\mathbb{R}$. If $\{\log r_1^{-1},\ldots,\log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then Levitin and Vassiliev's Renewal Theorem implies that

$$\Lambda_q^0(t) = c_q + \varepsilon_q^0(t),$$

where $c_q \in \mathbb{R}$ is the constant given by

$$\begin{aligned} c_q &= \frac{1}{\int s\,\mathrm{d}P_q(s)} \int \lambda_q^0(s)\,\mathrm{d}s \\ &= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^\infty e^{-s\beta(q)} \lambda_q(e^{-s})\,\mathrm{d}s \\ &= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \int_0^1 r^{\beta(q)} \lambda_q(r)\,\frac{\mathrm{d}r}{r} \end{aligned}$$

and

$$\varepsilon_q^0(t) \to 0 \text{ as } t \to \infty.$$

In particular, we have

$$r^{\beta(q)} V_{\mu,r}^q(K) = \Lambda_q^0(\log \tfrac{1}{r}) = c_q + \varepsilon_q(r), \tag{22}$$

where $\varepsilon_q(r) = \varepsilon_q^0(\log \frac{1}{r}) \to 0$ as $r \searrow 0$.

Finally, it follows from Eq. (22) that

$$r^{\beta(q)} V_{\mu,r}^q(K) \to c_q \text{ as } r \searrow 0.$$

This completes the proof of Theorem 2 in Case 1.

Case 2. If $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ *is contained in a discrete additive subgroup of* $\mathbb{R}$. If $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$ and $\langle t_1, \ldots, t_N \rangle = u\mathbb{Z}$ with $u > 0$, then Levitin and Vassiliev's Renewal Theorem implies that

$$\Lambda_q^0(t) = \pi_q^0(r) + \varepsilon_q^0(t),$$

where $\pi_q^0 : \mathbb{R} \to \mathbb{R}$ is the function given by

$$\begin{aligned} \pi_q^0(t) &= \frac{1}{\int s\,\mathrm{d}P_q(s)}\, u \sum_{n\in\mathbb{Z}} \lambda_q^0(t+nu) \\ &= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i}\, u \sum_{n\in\mathbb{Z}} \lambda_q^0(t+nu) \end{aligned}$$

and

$$\varepsilon_q^0(t) \to 0 \text{ as } t \to \infty.$$

Moreover, we have

$$\pi_q^0(t+u) = \pi_q^0(t)$$

for all $t \in \mathbb{R}$, i.e. π_q^0 is additively periodic with period equal to u. In particular, we have

$$r^{\beta(q)} V_{\mu,r}^q(K) = \Lambda_q^0(\log \tfrac{1}{r}) = \pi_q(r) + \varepsilon_q(r),$$

where $\pi_q : \mathbb{R} \to \mathbb{R}$ is the function given by

$$\begin{aligned}
\pi_q(r) &= \pi_q^0(\log \tfrac{1}{r}) \\
&= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \, u \sum_{n\in\mathbb{Z}} \lambda_q^0(\log \tfrac{1}{r} + nu) \\
&= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \, u \sum_{n\in\mathbb{Z}} \mathbf{1}_{[0,\infty)}(\log \tfrac{1}{r} + nu)\, e^{-\beta(q)(\log \frac{1}{r} + nu)}\, \lambda_q(e^{-(\log \frac{1}{r} + nu)}) \\
&= \frac{1}{-\sum_i p_i^q r_i^{\beta(q)} \log r_i} \, u \sum_{n\in\mathbb{Z}, re^{nu} \le 1} (re^{nu})^{\beta(q)}\, \lambda_q(re^{nu})
\end{aligned}$$

and $\varepsilon_q(r) = \varepsilon_q^0(\log \frac{1}{r}) \to 0$ as $r \searrow 0$. Moreover, since π_q^0 is additively periodic with period equal to u, we have

$$\pi_q(e^u r) = \pi_q^0(\log \tfrac{1}{e^u r}) = \pi_q^0(\log \tfrac{1}{r} - u) = \pi_q^0(\log \tfrac{1}{r}) = \pi_q(r)$$

for all $r > 0$, i.e. π_q is multiplicatively periodic with period equal to e^u.

Finally it follows from Levitin and Vassiliev's Renewal Theorem that

$$\frac{1}{T} \int_0^T \Lambda_q^0(t)\,\mathrm{d}t \to c_q \text{ as } T \to \infty.$$

However, since

$$\begin{aligned}
\frac{1}{T} \int_0^T \Lambda_q^0(t)\,\mathrm{d}t &= \frac{1}{T} \int_0^T e^{-t\beta(q)}\, V_{\mu,e^{-t}}^q(K)\,\mathrm{d}t \\
&= \frac{1}{-\log e^{-T}} \int_{e^{-T}}^1 s^{\beta(q)}\, V_{\mu,s}^q(K)\, \frac{\mathrm{d}s}{s},
\end{aligned}$$

we now conclude that

$$\frac{1}{-\log r} \int_r^1 s^{\beta(q)}\, V_{\mu,s}^q(K)\, \frac{\mathrm{d}s}{s} \to c_q \text{ as } r \searrow 0.$$

This completes the proof of Theorem 2 in Case 2. □

4 Multifractal Tube Measures

4.1 Multifractal Tube Measures

The statement in Theorem 2 is a global one: it provides information about the limiting behaviour of the suitably normalized multifractal Minkowski volume

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(K)$$

of the entire support K of μ as $r \searrow 0$. However, it is equally natural to ask for local versions of Theorem 2 describing the limiting behaviour of the normalized multifractal Minkowski volume

$$\frac{1}{r^{-\beta(q)}} V^q_{\mu,r}(E)$$

of (well-behaved) subsets E of the support of μ as $r \searrow 0$. In order to address this question, we now introduce multifractal tube measures. A further motivation for introducing multifractal tube measures comes from convex geometry and will be discussed below.

The multifractal tube measures are defined as follows. Fix a Borel measure μ on $\mathbb{R}^d$ and $r > 0$. For a real number q, we define the multifractal Minkowski tube measure $\mathscr{T}^q_{\mu,r}$ by

$$\mathscr{T}^q_{\mu,r}(E) = \frac{1}{r^d} \int_{E \cap B(\operatorname{supp}\mu, r)} \mu(B(x,r))^q \, \mathrm{d}\mathscr{L}^d(x)$$

for Borel subsets E of $\mathbb{R}^d$. Of course, the measures $\mathscr{T}^q_{\mu,r}$ will, in general, not converge weakly as $r \searrow 0$ (indeed, this is clear since Theorem 2 shows that, in general, $\mathscr{T}^q_{\mu,r}(\mathbb{R}^d) = V^q_{\mu,r}(K)$ does not converge as $r \searrow 0$). Hence in order to ensure weak convergence of $\mathscr{T}^q_{\mu,r}$ as $r \searrow 0$ it is necessary to normalize the measures $\mathscr{T}^q_{\mu,r}$. There are two natural ways to normalized. Firstly we can normalize by volume. More precisely, we define the volume normalized multifractal tube measure $\mathscr{V}^q_{\mu,r}$ by

$$\mathscr{V}^q_{\mu,r} = \frac{1}{\mathscr{T}^q_{\mu,r}(\mathbb{R}^d)} \mathscr{T}^q_{\mu,r} .$$

Secondly, we can normalize by scaling. More precisely, we defined the lower and upper scaling normalized multifractal tube measures $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ by

$$\underline{\mathscr{S}}^q_{\mu,r} = \frac{1}{r^{-\underline{\dim}^q_{\mathsf{M},\mu}(\operatorname{supp}\mu)}} \mathscr{T}^q_{\mu,r} ,$$

$$\overline{\mathscr{S}}^q_{\mu,r} = \frac{1}{r^{-\overline{\dim}^q_{\mathsf{M},\mu}(\operatorname{supp}\mu)}} \mathscr{T}^q_{\mu,r} .$$

It is instructive to consider the particular case $q = 0$. To discuss this case we first make the following definition. Namely, if U is a closed subset of $\mathbb{R}^d$ and $r > 0$, the parallel volume measure $\mathsf{V}_{U,r}$ of U is defined by

$$\mathsf{V}_{U,r}(E) = \frac{\mathscr{L}^d(E \cap B(U,r))}{\mathscr{L}^d(B(U,r))},$$

see, for example, the texts [21,35,48]. We now note that if $q = 0$ and μ is any Borel measure with $\operatorname{supp}\mu = U$, then the volume normalized multifractal tube measure $\mathscr{V}^q_{\mu,r}$ simplifies to

$$\begin{aligned} \mathscr{V}^0_{\mu,r}(E) &= \frac{\mathscr{L}^d(E \cap B(\operatorname{supp}\mu, r))}{\mathscr{L}^d(B(\operatorname{supp}\mu, r))} \\ &= \frac{\mathscr{L}^d(E \cap B(U,r))}{\mathscr{L}^d(B(U,r))} \\ &= \mathsf{V}_{U,r}(E). \end{aligned} \tag{23}$$

This observation provides a further motivation for introducing multifractal tube measures. Namely, the measure $\mathscr{V}^0_{\mu,r}(E) = \mathsf{V}_{U,r}(E)$ is closely related to the notion of curvature measures in convex geometry. Curvature measures were introduced in the 1950s and are now recognized as a very powerful tool for analysing geometric properties of convex sets; see [21, 35, 48]. Indeed, if U is a closed convex subset of $\mathbb{R}^d$ with non-empty interior and $l = 0, 1, 2, \ldots, d$, then it is possible to define the lth order curvature measure V^l_U associated with U. Each curvature measure V^l_U is defined as the weak limit $\mathsf{V}^l_U = \lim_{r \searrow 0} \mathsf{V}^l_{U,r}$ of a certain family $(\mathsf{V}^l_{U,r})_{r>0}$ of measures. While we will not provide the reader with the definition of the measures $\mathsf{V}^l_{U,r}$ for a general integer $l = 0, 1, 2, \ldots, d$ (instead the interested reader can find the definition in previously mentioned texts [21,35,48]), we do note that if $l = d$, then $\mathsf{V}^d_{U,r} = \mathsf{V}_{U,r}$. In particular, the d-th order curvature measure V^d_U is defined by

$$\begin{aligned} \mathsf{V}^d_U &= \lim_{r \searrow 0} \mathsf{V}^d_{U,r} \\ &= \lim_{r \searrow 0} \mathsf{V}_{U,r}, \end{aligned}$$

where lim denotes the limit with respect to the weak topology. This and the fact that $\mathscr{V}^0_{\mu,r} = \mathsf{V}_{U,r}$ show that the weak limit

$$\lim_{r \searrow 0} \mathscr{V}^q_{\mu,r}$$

(if it exists) may be viewed as a dth order multifractal curvature measure and the study of multifractal tube measures can therefore be seen as a first attempt to create a theory of multifractal curvatures.

It is, of course, also possible to define versions of the parallel volume measure analogous to $\underline{\mathscr{L}}^q_{\mu,r}$ and $\underline{\mathscr{L}}^q_{\mu,r}$. Indeed, if U is a closed subset of $\mathbb{R}^d$ and $r>0$, we define the lower and upper scaling parallel volume measures $\underline{\mathsf{S}}_{U,r}$ and $\overline{\mathsf{S}}_{U,r}$ of U by

$$\underline{\mathsf{S}}_{U,r}(E) = \frac{1}{r^{-\underline{\dim}_{\mathsf{M}}(U)+d}}\,\mathscr{L}^d(E\cap B(U,r))\,,$$

$$\overline{\mathsf{S}}_{U,r}(E) = \frac{1}{r^{-\overline{\dim}_{\mathsf{M}}(U)+d}}\,\mathscr{L}^d(E\cap B(U,r))\,;$$

recall that $\underline{\dim}_{\mathsf{M}}$ and $\overline{\dim}_{\mathsf{M}}$ denote the lower and upper Minkowski dimension, respectively. As above, we note that if $q=0$ and μ is any probability measure with $\operatorname{supp}\mu = U$, then the scaling normalized multifractal tube measures $\underline{\mathscr{L}}^q_{\mu,r}$ and $\underline{\mathscr{L}}^q_{\mu,r}$ simplify to

$$\underline{\mathscr{L}}^0_{\mu,r}(E) = \underline{\mathsf{S}}_{U,r}(E)\,, \tag{24}$$

$$\overline{\mathscr{S}}^0_{\mu,r}(E) = \overline{\mathsf{S}}_{U,r}(E)\,. \tag{25}$$

4.2 *Multifractal Tube Measures of Self-similar Measures*

For self-similar measures μ satisfying the OSC, we will now investigate the existence of the weak limits of the multifractal tube measures $\mathscr{V}^q_{\mu,r}$, $\underline{\mathscr{L}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ as $r \searrow 0$. In fact, in many cases, these limits exist and equal (the suitably) normalized multifractal Hausdorff measure restricted to the support of μ.

We start by recalling the definition of the multifractal Hausdorff measure. In an attempt to develop a general theoretical framework for studying the multifractal structure of Borel measures, Olsen [36], Pesin [41] and Peyrière [44] introduced a family of measures $\{\mathscr{H}^{q,t}_\mu \mid q,t\in\mathbb{R}\}$ based on certain generalizations of the Hausdorff measure. The measures $\mathscr{H}^{q,t}_\mu$ have subsequently been investigated further by a large number of authors, including [4, 8, 9, 23, 37, 39, 40, 45]. Let $E\subseteq\mathbb{R}^d$ and $\delta>0$. A countable family $\mathscr{B}=(B(x_i,r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a centred δ-covering of E if $E\subseteq\cup_i B(x_i,r_i)$, $x_i\in E$ and $0<r_i<\delta$ for all i. For $E\subseteq\mathbb{R}^d$, $q,t\in\mathbb{R}$ and $\delta>0$ write

$$\begin{aligned}
\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E) &= \inf\left\{\sum_i \mu(B(x_i,r_i))^q(2r_i)^t \,\middle|\, (B(x_i,r_i))_i \text{ is a centred } \delta\text{-covering of } E\right\},\\
\overline{\mathscr{H}}^{q,t}_{\mu}(E) &= \sup_{\delta>0}\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E)\,,\\
\mathscr{H}^{q,t}_{\mu}(E) &= \sup_{F\subseteq E}\overline{\mathscr{H}}^{q,t}_{\mu}(F)\,.
\end{aligned}$$

It follows from [36] that $\mathscr{H}_\mu^{q,t}$ is a measure on the family of Borel subsets of $\mathbb{R}^d$. The measure $\mathscr{H}_\mu^{q,t}$ is, of course, a multifractal generalization of the centred Hausdorff measure. In fact, it is easily seen that if $t \geq 0$, then $2^{-t}\mathscr{H}_\mu^{0,t} \leq \mathscr{H}^t \leq \mathscr{H}_\mu^{0,t}$ where $\mathscr{H}^t$ denotes the t-dimensional Hausdorff measure. It is also easily seen that the measure $\mathscr{H}_\mu^{q,t}$ in the usual way assign a dimension to each subset E of $\mathbb{R}^d$ (cf. [36]): there exists a unique number $\dim_\mu^q(E) \in [-\infty, \infty]$ such that

$$\mathscr{H}_\mu^{q,t}(E) = \begin{cases} \infty & \text{for} \quad t < \dim_\mu^q(E) \\ 0 & \text{for} \quad \dim_\mu^q(E) < t \end{cases}.$$

The number $\dim_\mu^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E. In fact, it follows immediately from the definitions that $\dim(E) = \dim_\mu^0(E)$. One of the main importances of the multifractal Hausdorff measure $\mathscr{H}_\mu^{q,t}$ is its connection with the multifractal spectrum of μ. Indeed, if we define the dimension function $b_\mu : \mathbb{R} \to [-\infty, \infty]$ by

$$b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu),$$

then it follows from [36] that the multifractal spectrum f_μ of μ (cf. Eq. (1)) is bounded above by the Legendre transform b_μ^* of b_μ, i.e.

$$f_\mu(\alpha) \leq b_\mu^*(\alpha)$$

for all $\alpha \geq 0$, cf. [36]; recall, that the definition of the Legendre transform φ^* of a real-valued function $\varphi : \mathbb{R} \to \mathbb{R}$ is given in Sect. 2.3. This inequality may be viewed as a rigorous version of the "Multifractal Formalism". Furthermore, for many natural families of measure we have $f_\mu(\alpha) = b_\mu^*(\alpha)$ for all $\alpha \geq 0$, cf. [4, 8, 9, 36, 37].

We can now explicitly identify the weak limits of the multifractal tube measures $\mathscr{V}_{\mu,r}^q$, $\underline{\mathscr{S}}_{\mu,r}^q$ and $\overline{\mathscr{S}}_{\mu,r}^q$ as $r \searrow 0$ for self-similar measures μ. The first result shows that the weak limit of $\mathscr{V}_{\mu,r}^q$ (as $r \searrow 0$) always exists and equals the normalized multifractal Hausdorff measure.

Theorem 4 ([38]). *Let K and μ be given by Eqs. (5) and (6). Fix $q \in \mathbb{R}$ and assume that Condition (i) or Condition (ii) below is satisfied.*

(i) The OSC is satisfied and $0 \leq q$.
(ii) The SSC is satisfied.

Then we have

$$\mathscr{V}_{\mu,r}^q \to \frac{1}{\mathscr{H}_\mu^{q,\beta(q)}(K)} \mathscr{H}_\mu^{q,\beta(q)} \llcorner K \qquad \textit{weakly.}$$

Next, we study the limiting behaviour of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ as $r \searrow 0$ for self-similar measures μ. Contrary to Theorem 4, the weak limits of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ as $r \searrow 0$ may not exist. Indeed, if the set $\{\log r_1^{-1},\dots,\log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$, then the weak limits of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ as $r \searrow 0$ do not necessarily exist; however, the weak limits of certain averages of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ exist and equal a multiple of the normalized multifractal Hausdorff measure. On the other hand, if the set $\{\log r_1^{-1},\dots,\log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then the weak limits of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$ as $r \searrow 0$ always exist and, as above, they equal a multiple of the normalized multifractal Hausdorff measure.

Theorem 5 ([38]). *Let K and μ be given by Eqs. (5) and (6). Fix $q \in \mathbb{R}$ and assume that Condition (i) or Condition (ii) below is satisfied.*

(i) The OSC is satisfied and $0 \le q$.
(ii) The SSC is satisfied.

Then the following holds:

(1) We have

$$\underline{\mathscr{S}}^q_{\mu,r} = \overline{\mathscr{S}}^q_{\mu,r} = \frac{1}{r^{-\beta(q)}}\ \mathscr{I}^q_{\mu,r}\,.$$

Write $\mathscr{S}^q_{\mu,r}$ for the common value of $\underline{\mathscr{S}}^q_{\mu,r}$ and $\overline{\mathscr{S}}^q_{\mu,r}$, i.e. write

$$\mathscr{S}^q_{\mu,r} = \frac{1}{r^{-\beta(q)}}\ \mathscr{I}^q_{\mu,r}\,.$$

Also, write

$$\mathscr{S}^q_{\mu,r,\mathsf{ave}} = \frac{1}{-\log r}\int_r^1 \frac{1}{s^{-\beta(q)}}\ \mathscr{I}^q_{\mu,s}\,\frac{ds}{s}\,.$$

(2) If the set $\{\log r_1^{-1},\dots,\log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then

$$\mathscr{S}^q_{\mu,r} \to M^{q,\beta(q)}_{\mu}(K)\ \tfrac{1}{\mathscr{H}^{q,\beta(q)}_{\mu}(K)}\ \mathscr{H}^{q,\beta(q)}_{\mu}\llcorner K \qquad \textit{weakly,}$$

$$\mathscr{S}^q_{\mu,r,\mathsf{ave}} \to M^{q,\beta(q)}_{\mu,\mathsf{ave}}(K)\ \tfrac{1}{\mathscr{H}^{q,\beta(q)}_{\mu}(K)}\ \mathscr{H}^{q,\beta(q)}_{\mu}\llcorner K \qquad \textit{weakly;}$$

recall that K is $(q,\beta(q))$ multifractal Minkowski measurable with respect to μ and $(q,\beta(q))$ average multifractal Minkowski measurable with respect to μ by Theorem 2 and the multifractal Minkowski content $M^{q,\beta(q)}_{\mu}(K)$ and the average multifractal Minkowski content $M^{q,\beta(q)}_{\mu,\mathsf{ave}}(K)$ are therefore well defined.

(3) If the set $\{\log r_1^{-1},\dots,\log r_N^{-1}\}$ *is contained in a discrete additive subgroup of* $\mathbb{R}$*, then*

$$\mathscr{S}^{q}_{\mu,r,\mathsf{ave}} \to M^{q,\beta(q)}_{\mu,\mathsf{ave}}(K)\,\frac{1}{\mathscr{H}^{q,\beta(q)}_{\mu}(K)}\,\mathscr{H}^{q,\beta(q)}_{\mu}\llcorner K \qquad \textit{weakly;}$$

recall that K *is* $(q,\beta(q))$ *average multifractal Minkowski measurable with respect to* μ *by Theorem 2 and the average multifractal Minkowski content* $M^{q,\beta(q)}_{\mu,\mathsf{ave}}(K)$ *is therefore well defined.*

As with Theorem 2, it is instructive to consider the special case $q=0$. Indeed, we note (cf. Eq. (23)) that

$$\begin{aligned}\mathscr{V}^{0}_{\mu,r}(E) &= \frac{\mathscr{L}^d(E\cap B(K,r))}{\mathscr{L}^d(B(K,r))}\\ &= \mathsf{V}_{K,r}(E),\end{aligned}$$

i.e. $\mathscr{V}^{0}_{\mu,r}$ equals the normalized parallel body measure $\mathsf{V}_{K,r}$. Also, writing t for the common value of the box dimensions and Hausdorff dimension of K, we note [see Eq. (25)] that

$$\begin{aligned}\underline{\mathscr{S}}^{0}_{\mu,r}(E) = \overline{\mathscr{S}}^{0}_{\mu,r}(E) &= \frac{1}{r^{-t+d}}\,\mathscr{L}^d(E\cap B(K,r))\\ &= \underline{\mathsf{S}}_{K,r}(E) = \overline{\mathsf{S}}_{K,r}(E)\,,\end{aligned}$$

i.e. $\underline{\mathscr{S}}^{0}_{\mu,r}$ and $\overline{\mathscr{S}}^{0}_{\mu,r}$ equal the scaling parallel body measures $\underline{\mathsf{S}}_{K,r}$ and $\overline{\mathsf{S}}_{K,r}$. The following corollaries therefore follow immediately from Theorem 2, Theorems 1 and 2 by putting $q=0$. These results were first obtained by Winter in his doctoral dissertation [51].

Corollary 2 ([51]). *Let* K *be given by Eq. (5). Assume that the OSC is satisfied. Let* t *denote the common value of the box dimensions and the Hausdorff dimension of* K*, i.e.* t *is the unique number such that* $\sum_i r_i^t = 1$*. For* $r>0$*, the normalized parallel body measure* $\mathsf{V}_{K,r}$ *is given by*

$$\mathsf{V}_{K,r}(E) = \frac{1}{\mathscr{L}^d(B(K,r))}\,\mathscr{L}^d(E\cap B(K,r))\,.$$

Then we have

$$\mathsf{V}_{K,r} \to \frac{1}{\mathscr{H}^t(K)}\,\mathscr{H}^t\llcorner K \qquad \textit{weakly.}$$

Proof. Since $\mathscr{V}^{0}_{\mu,r} = \mathsf{V}_{K,r}$, this follows from Theorem 4 by putting $q=0$. □

Corollary 3 ([51]). *Let K be given by Eq. (5). Assume that the OSC is satisfied. Let t denote for the common value of the box dimensions and the Hausdorff dimension of K, i.e. t is the unique number such that* $\sum_i r_i^t = 1$.

(1) We have

$$\underline{\mathsf{S}}_{K,r}(E) = \overline{\mathsf{S}}_{K,r}(E) = \frac{1}{r^{-t+d}}\mathscr{L}^d(E \cap B(K,r))\,.$$

Write $\mathsf{S}_{K,r}$ *for the common value of* $\underline{\mathsf{S}}_{K,r}$ *and* $\overline{\mathsf{S}}_{K,r}$*, i.e. write*

$$\mathsf{S}_{K,r}(E) = \frac{1}{r^{-t+d}}\mathscr{L}^d(E \cap B(K,r))\,.$$

Also, write

$$\mathsf{S}_{K,r,\mathsf{ave}}(E) = \frac{1}{-\log r}\int_r^1 \frac{1}{s^{-t+d}}\mathscr{L}^d(E \cap B(K,s))\,\frac{\mathrm{d}s}{s}\,.$$

(2) If the set $\{\log r_1^{-1},\ldots,\log r_N^{-1}\}$ *is not contained in a discrete additive subgroup of* $\mathbb{R}$*, then*

$$\mathsf{S}_{K,r} \to M^t(K)\,\frac{1}{\mathscr{H}^t(K)}\,\mathscr{H}^t\llcorner K \qquad \textit{weakly},$$

$$\mathsf{S}_{K,r,\mathsf{ave}} \to M^t_{\mathsf{ave}}(K)\,\frac{1}{\mathscr{H}^t(K)}\,\mathscr{H}^t\llcorner K \qquad \textit{weakly;}$$

recall that K is t Minkowski measurable and t average Minkowski measurable by Corollary 1 and the Minkowski content $M^t(K)$ *and the average Minkowski content* $M^t_{\mathsf{ave}}(K)$ *are therefore well defined.*

(3) If the set $\{\log r_1^{-1},\ldots,\log r_N^{-1}\}$ *is contained in a discrete additive subgroup of* $\mathbb{R}$ *then*

$$\mathsf{S}_{K,r,\mathsf{ave}} \to M^t_{\mathsf{ave}}(K)\,\frac{1}{\mathscr{H}^t(K)}\,\mathscr{H}^t\llcorner K \qquad \textit{weakly;}$$

recall that K is t average Minkowski measurable by Corollary 1 and the average multifractal Minkowski content $M^t_{\mathsf{ave}}(K)$ *is therefore well defined.*

Proof. Since $\underline{\mathscr{S}}^0_{\mu,r} = \overline{\mathscr{S}}^0_{\mu,r} = \mathsf{S}_{K,r}$, this follows from Theorem 5 by putting $q = 0$. □

In Sect. 4.1 it was suggested that one motivation for introducing the multifractal tube measures $\mathscr{V}^q_{\mu,r}$ is that the limiting behaviour of $\mathscr{V}^q_{\mu,r}$ may be viewed as providing a local version of Theorem 2. Namely, Theorem 2 describes the limiting behaviour of $\frac{1}{r^{-\beta(q)}}V^q_{\mu,r}(K)$ as $r \searrow 0$, whereas Theorem 4 provides information about the the limiting behaviour of $\frac{1}{r^{-\beta(q)}}V^q_{\mu,r}(E)$ as $r \searrow 0$ for "well-behaved" subsets E of K. The viewpoint is made precise in the next corollary. Below we use the following notation, namely, if X is a metric space and $E \subseteq X$, then we will denote the the boundary of E in X by ∂E.

Corollary 4. *Let K and μ be given by Eqs. (5) and (6). Fix $q \in \mathbb{R}$ and assume that Condition (i) or Condition (ii) below is satisfied.*

(i) The OSC is satisfied and $0 \leq q$.
(ii) The SSC is satisfied.

Let $E \subseteq \mathbb{R}^d$ be a Borel set with:

1. $\mathscr{H}_{\mu}^{q,\beta(q)}(E \cap K) > 0$
2. $\mathscr{H}_{\mu}^{q,\beta(q)}(\partial E \cap K) = 0$
3. $E \cap B(K,r) = B(E \cap K, r)$ *for r small enough*

(Observe that, for example, the set $E = \mathbb{R}^d$ satisfies the above conditions, and if $K = L \cup M$ with $\operatorname{dist}(L,M) > 0$ and $\mathscr{H}_{\mu}^{q,\beta(q)}(L) > 0$ and $0 < \delta < \operatorname{dist}(L,M)$, then the set $E = B(L,\delta)$ satisfies the above conditions.)

Then we have the following:

1. *If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is not contained in a discrete additive subgroup of $\mathbb{R}$, then $E \cap K$ is $(q,\beta(q))$ multifractal Minkowski measurable with respect to μ with*

$$M_{\mu}^{q,\beta(q)}(E \cap K) = M_{\mu}^{q,\beta(q)}(K)\, \frac{\mathscr{H}_{\mu}^{q,\beta(q)}(E \cap K)}{\mathscr{H}_{\mu}^{q,\beta(q)}(K)};$$

recall that K is $(q,\beta(q))$ average multifractal Minkowski measurable with respect to μ by Theorem 2 and the multifractal Minkowski content $M_{\mu}^{q,\beta(q)}(K)$ is therefore well defined.

2. *If the set $\{\log r_1^{-1}, \ldots, \log r_N^{-1}\}$ is contained in a discrete additive subgroup of $\mathbb{R}$, then $E \cap K$ is $(q,\beta(q))$ average multifractal Minkowski measurable with respect to μ with*

$$M_{\mu,\mathsf{ave}}^{q,\beta(q)}(E \cap K) = M_{\mu,\mathsf{ave}}^{q,\beta(q)}(K)\, \frac{\mathscr{H}_{\mu}^{q,\beta(q)}(E \cap K)}{\mathscr{H}_{\mu}^{q,\beta(q)}(K)};$$

recall that K is $(q,\beta(q))$ average multifractal Minkowski measurable with respect to μ by Theorem 2 and the average multifractal Minkowski content $M_{\mu,\mathsf{ave}}^{q,\beta(q)}(K)$ is therefore well defined.

Proof. This follows immediately from Theorem 5 since the condition $E \cap B(K,r) = B(E \cap K, r)$ implies that

$$\begin{aligned}\mathscr{I}_{\mu,r}^{q}(E) &= \frac{1}{r^d}\int_{E \cap B(K,r)} \mu(B(x,r))^q \,\mathrm{d}\mathscr{L}^d(x) = \frac{1}{r^d}\int_{B(E \cap K,r)} \mu(B(x,r))^q \,\mathrm{d}\mathscr{L}^d(x)\\ &= V_{\mu,r}^{q}(E \cap K).\end{aligned}$$

□

Note that Corollary 4 is a genuine extension of Theorem 2: namely, if we let $E = K$ in Corollary 4, then Corollary 4 simplifies to Theorem 2.

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