On the Fourth-Order Structure Function of a Fractal Process*

Carol T. Christou and Garry M. Jacyna

Abstract Multifractal processes are key to modeling complex nonlinear systems. MITRE has applied fractal theory to agent-based combat simulations to understand complex behavior on the battlefield. The outstanding features of general fractal processes are long-range correlation and intermittency. If B is the lower band edge frequency of the high-pass signal component, the flatness function $F(B)$, defined as the ratio of the fourth-order moment to the square of the second-order moment of a stationary process, is a measure of the intermittency or burstiness of a random process at small scales. If $F(B)$ increases with no upper bound as B increases, then the process is intertmittent. In this work, we have derived an expression for the fourth-order structure function of the increments of a fractional Brownian motion (fBm) process through the use of integrals over the generalized multispectrum. It was concluded that the flatness function is independent of the lower edge of the high-pass signal component B, as expected of an fBm.

Keywords Complex systems • Agent-based simulation • Multifractal processes • Fractional Brownian motion • Intermittency • Second and fourth-order structure functions • Flatness function • Generalized multispectrum • Long-range correlation • Self-similarity and scale invariance

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C.T. Christou (✉) • G.M. Jacyna
The MITRE Corporation, 7515 Colshire Drive, McLean, VA 22102, USA
e-mail: christou@mitre.org; gjacyna@mitre.org

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1 Introduction

The use of multifractal processes for modeling complex systems has gained wide acceptance in the mathematics, engineering, and physical sciences communities, as it has grown in leaps and bounds in the past 40 years. Simple fractal processes are divided into deterministic, e.g., the Cantor set, and random fractals, such as Brownian motion and fractional Brownian motion (fBm). Fractal patterns are observed in complex dynamical systems composed of many nonlinearly interacting parts. They often exhibit properties such as long-range correlation and intermittent or bursty behavior. Long-range correlation is the tendency of a system to interact with its parts across extended spatial and temporal scales. Complex patterns emerge that cannot be explained from an analysis of individual parts. Intermittency is an indication of global change in a complex system and is related to the relative sensitivity of the system to small changes in its internal states. MITRE has applied complexity theory to agent-based simulation, where processes unfold in a highly unpredictable manner. One of these modeling areas is combat simulation, where small changes in one part of the battle space produce profound effects in another. Multifractal techniques were necessary in the development of the underlying complexity-based analysis tools.

This short chapter examines the flatness function, a measure of the intermittency of a random process, for an fBm. The focus of the development is the derivation of the fourth-order structure function, on which the flatness depends. The fourth-order structure function is derived by integration in terms of the generalized trispectrum to find the fourth-order correlation function. Section 2 contains the theoretical development, Sect. 3 the results, while the conclusions are contained in Sect. 4.

2 Theory

Important properties of fBm are self-similarity and scale invariance. It obeys the generalized scaling relation:

$$x(\lambda t) \rightarrow \text{“law”}\lambda^H x(t), \tag{1}$$

where $\rightarrow$ “law” means that the relation is a “law” or equivalence for all probability distributions and H is called the Hurst parameter. Brownian motion is a special case of this scaling law when $H = 1/2$.

Scaling properties of the correlation function are related to analogous scaling properties of the power spectrum of a random fractal process. The second-order structure function, formally defined, is the second moment of the increment process, $E(|x(t+\tau) - x(t)|^2)$. For the increments of an fBm process:

$$E[|x(t+\tau) - x(t)|^2] = \sigma^2 V_H |\tau|^{2H}, \tag{2}$$

where V_H is a function of the Hurst parameter and $E[x]$ denotes the expectation value of x. Therefore, even though fBm is not stationary, its increments are. The flatness function $F(B)$ for a zero-mean stationary random process $x(t)$ is defined as the ratio of the fourth-order moment to the square of the second-order moment [1]:

$$F(B) = E\left[x_B^4(t)\right] / E\left[x_B^2(t)\right]^2, \tag{3}$$

where B represents the lower band edge frequency of the high-pass component of the signal and $x_B(t)$ represents the signal components above that frequency. Let $X(f)$ denote the Fourier transform of the waveform $x(t)$. The high-pass component of the waveform is obtained using an ideal high-pass filter with perfect response characteristics:

$$x_B(t) = \int_{|f|\geq B} X(f)\exp(2\pi i f t)\mathrm{d}f. \tag{4}$$

The flatness function is a measure of the *intermittency* or *burstiness* of a random process at small scales. If $F(B)$ increases without bound as B increases, the multifractal process is said to be intermittent. If the process $x(t)$ is not stationary, then the stationary increments $x(t+\tau) - x(t)$ are used. Here we use the stationary increments $(x_B(t+\tau) - x_B(t))$ of the generally nonstationary process $x_B(t)$:

$$F(B) = E\left[(x_B(t+\tau) - x_B(t))^4\right] / E\left[(x_B(t+\tau) - x_B(t))^2\right]^2, \tag{5}$$

where τ is the time increment. As $B\tau \to 0$, we regain the full signal, and $E\left[(x_B(t+\tau) - x_B(t))^4\right] \to E\left[(x(t+\tau) - x(t))^4\right]$. In this chapter, we examine the case of $B\tau \to 0$, for which the multifractal scaling law predicts $E\left[(x(t+\tau) - x(t))^q\right] = c(q)|\tau|^{\tau(q)+1}$, where $c(q)$ is a constant and $\tau(q)$ is the *scaling* or *generating* function [2]. In order to calculate the flatness function, one must derive expressions for both the second- and fourth-order structure functions of a process.

We assume a real fBm model for $x(t)$, which is useful in describing processes with long-term correlations, i.e., $1/\omega^n$ spectral behavior, where ω is the angular frequency. Because fBm is not a stationary process, however, it is difficult to define or interpret a power spectrum. In [3], it was shown that it is possible to construct fBm from a white-noise-type process through a stochastic integral in frequency of a stationary uncorrelated random process, in this case the time increments of fBm. A spectral representation of $x(t)$ was obtained assuming it is driven by a stationary white noise process $W(t)$, not necessarily Gaussian. Specifically, it was shown that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{e}^{i\omega t} - 1)\left(\frac{1}{i\omega}\right)^n \mathrm{d}\beta(\omega), \tag{6}$$

where $n = H + 1/2$, H being the fractal scaling parameter or Hurst coefficient, $0 < H < 1$. The quantity

$$\beta(\omega) \equiv \int_0^{\omega} F(\omega')\mathrm{d}\omega', \tag{7}$$

is a complex-valued Wiener process in frequency of orthogonal increments, where

$$F(\omega') = \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega' t} W(t)\mathrm{d}t \tag{8}$$

is the Fourier transform of the white noise process. If $W(t)$ is Gaussian, the increments are also independent. In general, if the increments in $\beta(\omega)$ are infinitesimal, it is shown in [3] that

$$E\left[\mathrm{d}\beta^*(\omega_1)\mathrm{d}\beta(\omega_2)\right] = 2\pi\gamma_2^w \delta(\omega_1 - \omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2, \tag{9}$$

where $\delta(\omega)$ is the Dirac delta function and γ_2^w is a constant related to the power of the white noise driving force. Using the spectral representation of $x(t)$, one can derive an expression for the second-order structure function, or generalized correlation

$$\begin{aligned} E[(x(t_1)-x(t))(x(t_2)-x(t))] &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\mathrm{e}^{i\omega_1 t_1} - \mathrm{e}^{i\omega_1 t}\right]\left(\frac{1}{i\omega_1}\right)^n \\ &\quad \times \left[\mathrm{e}^{-i\omega_2 t_2} - \mathrm{e}^{-i\omega_2 t}\right]\left(\frac{1}{-i\omega_2}\right)^n E\left[\mathrm{d}\beta^*(\omega_1)\mathrm{d}\beta(\omega_2)\right] \\ &\quad \times \mathrm{d}\omega_1\mathrm{d}\omega_2 \\ &= \gamma_2^w \frac{V_n}{2}\left(|t_1 - t|^{2n-1} + |t_2 - t|^{2n-1} - |t_2 - t_1|^{2n-1}\right), \\ V_n &= \left(\frac{2}{\pi}\right)\Gamma[1-2n]\sin(n\pi), \end{aligned} \tag{10}$$

where $\frac{3}{2} > n > \frac{1}{2}$ for convergence. If $t_2 = t_1 = \tau + t$, then $E\left[|(x(\tau+t) - x(t))|^2\right] = \gamma_2^w V_n |\tau|^{2n-1}$.

Here, we would like to derive the corresponding expression for the fourth-order structure function of an fBm process. We begin by first finding the fourth-order correlation

$$E\left[\mathrm{d}\beta^*(\omega_4)\mathrm{d}\beta(\omega_1)\mathrm{d}\beta(\omega_2)\mathrm{d}\beta(\omega_3)\right]$$

of the differential increment $\mathrm{d}\beta(\omega) = F(\omega)\mathrm{d}\omega$. The fourth-order correlation function of the white noise process was derived in [4] as

$$E\Big[W(t_1)W(t_2)W(t_3)W(t_4)\Big] = \gamma_4^w \delta(t_1 - t_4)\delta(t_2 - t_4)\delta(t_3 - t_4), \tag{11}$$

where γ_4^w is a constant. Therefore,

$$\begin{aligned} &E\left[\mathrm{d}\beta^*(\omega_4)\mathrm{d}\beta(\omega_1)\mathrm{d}\beta(\omega_2)\mathrm{d}\beta(\omega_3)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i\omega_4 t_4}\mathrm{e}^{-i\omega_1 t_1}\mathrm{e}^{-i\omega_2 t_2}\mathrm{e}^{-i\omega_3 t_3} \end{aligned}$$

$$\times E\left[W(t_1)W(t_2)W(t_3)W(t_4)\right]\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4$$
$$\times\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4 = \gamma_4^w\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{i\omega_4 t_4}\mathrm{e}^{-i\omega_1 t_1}$$
$$\times\mathrm{e}^{-i\omega_2 t_2}\mathrm{e}^{-i\omega_3 t_3}\delta(t_1-t_4)\delta(t_2-t_4)\delta(t_3-t_4)$$
$$\times\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4$$
$$=\gamma_4^w\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\omega_4-\omega_1-\omega_2-\omega_3)t_4}$$
$$\times\mathrm{d}t_4\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4$$
$$=\gamma_4^w(2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\omega_4-\omega_1-\omega_2-\omega_3)$$
$$\times\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4. \tag{12}$$

By analogy to Eq. (10) one may derive the general correlation of four arbitrary increments in $x(t)$:

$$E\Big[(x(t_1)-x(t))(x(t_2)-x(t))\times(x(t_3)-x(t))(x(t_4)-x(t))\Big]$$
$$=\left(\frac{1}{2\pi}\right)^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathrm{e}^{i\omega_1 t_1}-\mathrm{e}^{i\omega_1 t}\right]\left(\frac{1}{i\omega_1}\right)^n$$
$$\times\left[\mathrm{e}^{i\omega_2 t_2}-\mathrm{e}^{i\omega_2 t}\right]\left(\frac{1}{i\omega_2}\right)^n\left[\mathrm{e}^{i\omega_3 t_3}-\mathrm{e}^{i\omega_3 t}\right]\left(\frac{1}{i\omega_3}\right)^n$$
$$\times\left[\mathrm{e}^{-i\omega_4 t_4}-\mathrm{e}^{-i\omega_4 t}\right]\left(\frac{1}{-i\omega_4}\right)^n$$
$$\times E\left[\mathrm{d}\beta(\omega_1)\mathrm{d}\beta(\omega_2)\mathrm{d}\beta(\omega_3)\mathrm{d}\beta^*(\omega_4)\right]$$
$$=\left(\frac{1}{2\pi}\right)^4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathrm{e}^{i\omega_1 t_1}-\mathrm{e}^{i\omega_1 t}\right]\left(\frac{1}{i\omega_1}\right)^n$$
$$\times\left[\mathrm{e}^{i\omega_2 t_2}-\mathrm{e}^{i\omega_2 t}\right]\left(\frac{1}{i\omega_2}\right)^n\left[\mathrm{e}^{i\omega_3 t_3}-\mathrm{e}^{i\omega_3 t}\right]\left(\frac{1}{i\omega_3}\right)^n$$
$$\times\left[\mathrm{e}^{-i\omega_4 t_4}-\mathrm{e}^{-i\omega_4 t}\right]\left(\frac{1}{-i\omega_4}\right)^n(2\pi)\gamma_4^w$$
$$\times\delta(\omega_4-\omega_1-\omega_2-\omega_3)\mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4$$
$$=\left(\frac{1}{2\pi}\right)^3\gamma_4^w\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathrm{e}^{i\omega_1 t_1}-\mathrm{e}^{i\omega_1 t}\right]$$
$$\times\left[\mathrm{e}^{i\omega_2 t_2}-\mathrm{e}^{i\omega_2 t}\right]\left[\mathrm{e}^{i\omega_3 t_3}-\mathrm{e}^{i\omega_3 t}\right]$$

$$\times \left[\mathrm{e}^{-i(\omega_1+\omega_2+\omega_3)t_4} - \mathrm{e}^{-i(\omega_1+\omega_2+\omega_3)t} \right] \left(\frac{1}{i\omega_1} \right)^n \left(\frac{1}{i\omega_2} \right)^n$$

$$\times \left(\frac{1}{i\omega_3} \right)^n \left(\frac{1}{-i(\omega_1+\omega_2+\omega_3)} \right)^n \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3, \quad (13)$$

which is the expression for the fourth-order correlation in terms of a "generalized trispectrum" [5,6]:

$$\Phi_T(\omega_1,\omega_2,\omega_3) \sim \frac{1}{\omega_1^n \omega_2^n \omega_3^n (-\omega_1-\omega_2-\omega_3)^n}. \quad (14)$$

Now, if $t_1 = t_2 = t_3 = t_4 = t + \tau$, Eq. (13) is precisely the fourth-order structure function of $x(t)$. It can easily be seen that the phase factors in t will then cancel out and only phase factors in τ will appear.

Transforming to the variables $\gamma = \left[\omega_1 + \frac{\omega_2+\omega_3}{2}\right]$, $\alpha = [\omega_2 + \omega_3]$, $\beta = \left[\frac{\omega_3}{2} - \frac{\omega_2}{2}\right]$, Eq. (13) becomes

$$E\left[|(x(t+\tau)-x(t))|^4\right]) = \left(\frac{1}{2\pi}\right)^3 4\gamma_4^w \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\gamma\tau)\right)}{\left(\frac{\alpha^2}{4}-\gamma^2\right)^n}\mathrm{d}\gamma$$

$$\int_{-\infty}^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\beta\tau)\right)}{\left(\frac{\alpha^2}{4}-\beta^2\right)^n}\mathrm{d}\beta\mathrm{d}\alpha. \quad (15)$$

We see that the integrals over β and γ are identical, have even integrands, and for arbitrary n will contain real and imaginary parts. A simultaneous change of sign in both denominators implies that

$$\left[\int_0^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\beta\tau)\right)}{\left(\frac{\alpha^2}{4}-\beta^2\right)^n}\mathrm{d}\beta\right]^2 = \left[\int_0^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\beta\tau)\right)}{\left(\beta^2-\frac{\alpha^2}{4}\right)^n}\mathrm{d}\beta\right]^2. \quad (16)$$

Equation (16) in turn implies that

$$\int_0^{a/2} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\beta\tau)\right)}{\left(\frac{\alpha^2}{4}-\beta^2\right)^n}\mathrm{d}\beta = \int_{a/2}^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau)-\cos(\beta\tau)\right)}{\left(\beta^2-\frac{\alpha^2}{4}\right)^n}\mathrm{d}\beta. \quad (17)$$

Now, for $\alpha > 0$, the values of these integrals are

$$
\begin{aligned}
&\int_0^{a/2} \frac{\left(\cos(\frac{\alpha}{2}\tau) - \cos(\beta\tau)\right)}{\left(\frac{\alpha^2}{4} - \beta^2\right)^n} \mathrm{d}\beta \\
&= -\left(\alpha^{\frac{1}{2}-2n}\sqrt{\pi}\Gamma[1-n]\left(-\left(4^n\sqrt{\alpha}\cos\left(\frac{a}{2}\right)\right)\right.\right. \\
&\quad \left.\left. +2a^n J_{\frac{1}{2}-n}\left(\frac{a}{2}\right)\Gamma\left[\frac{3}{2}-n\right]\right)\right)\left(\frac{1}{4\Gamma\left[\frac{3}{2}-n\right]}\right); \\
&\int_{a/2}^{\infty} \frac{\left(\cos(\frac{\alpha}{2}\tau) - \cos(\beta\tau)\right)}{\left(\beta^2 - \frac{\alpha^2}{4}\right)^n} \mathrm{d}\beta \\
&= \left(\alpha^{\frac{1}{2}-2n}\Gamma[1-n]\left(4^n\sqrt{\alpha}\cos\left(\frac{a}{2}\right)\Gamma\left[-\frac{1}{2}+n\right]\right.\right. \\
&\quad \left.\left. +2a^n\pi\left(J_{\frac{1}{2}-n}\left(\frac{a}{2}\right)\sec(n\pi) - J_{n-\frac{1}{2}}\left(\frac{a}{2}\right)\tan(n\pi)\right)\right)\right)\left(\frac{1}{4\sqrt{\pi}}\right);
\end{aligned} \tag{18}
$$

([7], p. 427, Eqs. 8 and 9), where $J_\nu(x)$ is the Bessel function of order ν. Simple plots of these results versus α easily demonstrate that the two expressions are in fact not the same. Therefore α cannot be positive but must equal zero in order for the above integrals to exist. For α=0 Eq. (15) then becomes

$$
\begin{aligned}
&E\left[|(x(t+\tau) - x(t))|^4\right] \\
&= \left(\frac{1}{2\pi}\right)^3 16\gamma_4^w |\tau|^{m-3} \int_{-\infty}^{\infty} 2\pi\delta(\alpha) \\
&\quad \int_0^{\infty} \frac{\left(\cos(\frac{\alpha}{2}) - \cos(\gamma)\right)}{\left(\frac{\alpha^2}{4} - \gamma^2\right)^n} \mathrm{d}\gamma \int_0^{\infty} \frac{\left(\cos(\frac{\alpha}{2}) - \cos(\beta)\right)}{\left(\frac{\alpha^2}{4} - \beta^2\right)^n} \mathrm{d}\beta\mathrm{d}\alpha \\
&= \left(\frac{1}{2\pi}\right)^2 16\gamma_4^w |\tau|^{m-3}\left[\int_0^{\infty} \frac{(1-\cos(\beta))}{(\beta^2)^n}\mathrm{d}\beta\right]^2 \\
&= \left(\frac{1}{2\pi}\right)^2 16\gamma_4^w |\tau|^{m-3}\left[\Gamma[1-2n]\sin(n\pi)\right]^2,
\end{aligned} \tag{19}
$$

where we have divided the variables α, β, γ by a scale factor τ, and $m = 4n+1$.[1] The integral $\int_0^{\infty} \frac{(1-\cos(\beta))}{(\beta^2)^n}\mathrm{d}\beta$ will converge at infinity if $-2n+1 < 0$ or $n > 1/2$

[1] $\delta(\frac{\alpha}{\tau}) = \tau\delta(\alpha)$.

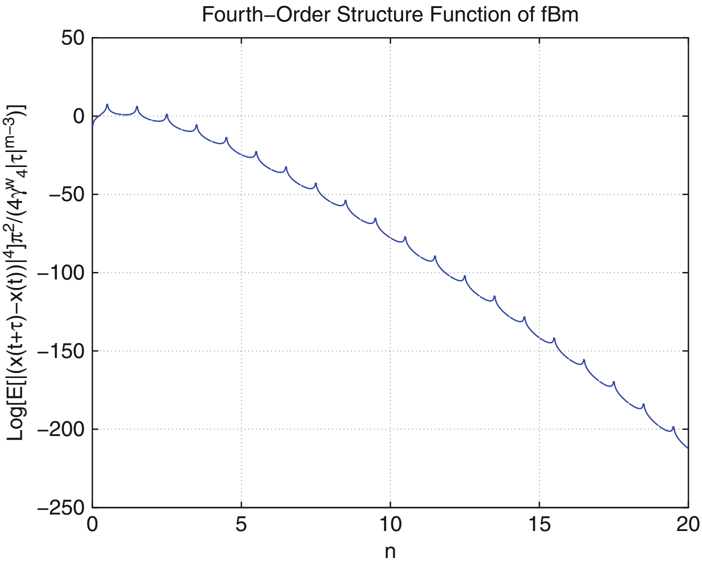

Fig. 1 Behavior of fourth-order correlation versus n

since the numerator remains finite. Near $\beta = 0$, the numerator behaves as $\sim \beta^2$; therefore the integral remains finite if $-2n+3 > 0$ or $n < 3/2$. This implies that $7 > m > 3$ for convergence. The behavior of the fourth-order structure function in Eq. (19) is shown as a function of n in Figs. 1 and 2. Figure 2 is a detailed view of the larger-scale behavior in Fig. 1. The physical region for the fBm process lies between $n = 1/2$ and $n = 3/2$, as it does in the case of the second-order structure function in Eq. (10). It is interesting to note that the large-scale behavior of the logarithm of $E\left[|(x(t+\tau)-x(t)|^4\right]$ versus n resembles a hyperbola.

3 Results

Let us evaluate the integral in Eq. (15) directly for $n = 1$. Since

$$\int_{-\infty}^{\infty} \frac{\left(\cos(\frac{\alpha}{2}) - \cos(\gamma)\right)}{\left(\frac{\alpha^2}{4} - \gamma^2\right)} \mathrm{d}\gamma = 2\frac{\pi}{\alpha}\sin\left(\frac{\alpha}{2}\right), \tag{20}$$

([7], p. 407, Eq. 9), we have

$$\int_{-\infty}^{\infty} 4\pi^2 \frac{\sin^2\left(\frac{\alpha}{2}\right)}{\alpha^2} \mathrm{d}\alpha = 8\pi^2\frac{\pi}{4} = 2\pi^3, \tag{21}$$

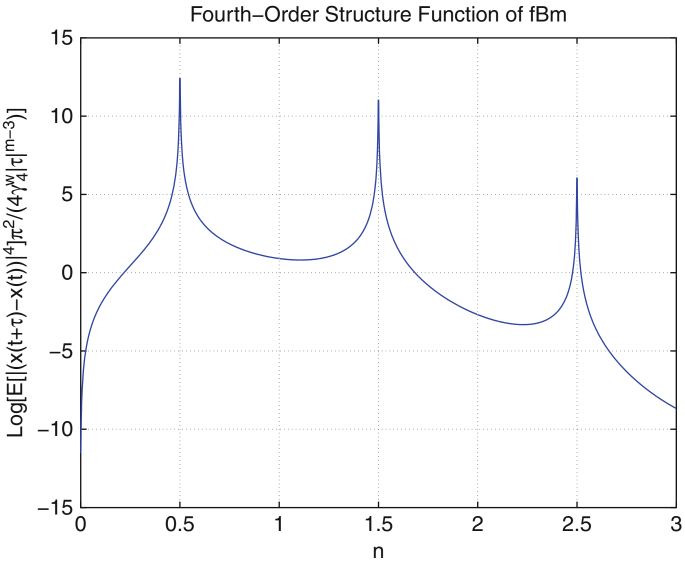

Fig. 2 Behavior of fourth-order correlation versus n near physical region

([7], p. 414, Eq. 3). Alternatively, inserting the delta function in Eq. (15) and setting $\alpha = 0$, we have

$$\int_{-\infty}^{\infty} \frac{(1-\cos(\gamma))}{(\gamma^2)} \mathrm{d}\gamma = 2\frac{\pi}{2}, \tag{22}$$

which when squared and combined with the factor 2π from the delta function gives us $2\pi^3$, the same answer as above. Thus, the fourth-order structure function for fBm is nonzero only along the normal submanifold $\alpha = 0$ of the three-dimensional frequency space.

Dividing Eq. (19) by Eq. (10) squared for $t_1 = t_2 = \tau + t$, we obtain for the flatness function the quantity $\gamma_4^w/(\gamma_2^w)^2$, which, as expected, is a constant for the increments of an fBm. This then implies that the fBm is also not intermittent.

4 Conclusions

In trying to model complex behavior on the battlefield through agent-based simulation, it became necessary to use the theory of complexity, in particular that of multifractal processes. Generalized fractal phenomena are characterized by long-range correlation and intermittency. This work examined the properties of an fBM process, whose increments are stationary. Specifically, in order to determine the intermittency characteristics, it was necessary to calculate the flatness function, the ratio of the fourth-order structure function to the square of the second-order structure function of the increments of an fBm. The focus was on the derivation of the fourth-order structure function through the use of the generalized trispectrum of the process. It was found that the flatness function is independent of the lower edge frequency of the high-pass signal component, which leads to the conclusion that an fBm is not intermittent.

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