Chapter 5
General Theory

This chapter starts the second part of the book, where neutral type time-delay systems are studied. Issues related to the existence, uniqueness, and continuation of solutions of an initial value problem for such systems are discussed. In addition, stability concepts and basic stability results obtained with the use of the Lyapunov–Krasovskii approach, mainly in the form of necessary and sufficient conditions, are presented here.

5.1 System Description

We consider a neutral type time-delay system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] = f(t, x_t). \tag{5.1}$$

Here the functional $f(t,\varphi)$ is defined for $t \in [0,\infty)$ and $\varphi \in PC^1\left([-h,0], R^n\right)$,

$$f : [0,\infty) \times PC^1\left([-h,0], R^n\right) \longrightarrow R^n,$$

and is continuous in both arguments. The matrix D is a given $n \times n$ matrix, delay $h > 0$. The information needed to begin the computation of a particular solution of the system includes an initial time instant $t_0 \geq 0$ and an initial function $\varphi : [-h,0] \to R^n$, and it is assumed that

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-h,0]. \tag{5.2}$$

As usual, the state of the system at the time instant $t \geq t_0$ is defined as the restriction,

$$x_t : \theta \to x(t+\theta), \quad \theta \in [-h,0],$$

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of the solution $x(t)$ on the segment $[t-h,t]$. If the initial condition (t_0,φ) is indicated explicitly, then we use the notations $x(t,t_0,\varphi)$ and $x_t(t_0,\varphi)$. In the case of time-invariant systems we usually assume that $t_0 = 0$ and omit the argument t_0 in these notations.

We will use initial functions from the space $PC^1([-h,0],R^n) \subset PC([-h,0],R^n)$. Here it is assumed that a function $\varphi \in PC([-h,0],R^n)$ belongs to $PC^1([-h,0],R^n)$ if on each continuity interval $(\alpha,\beta) \in [-h,0]$ the function is continuously differentiable and the first derivative of the function, $\varphi'(\theta)$, has a finite right-hand-side limit at $\theta = \alpha$, $\varphi'(\alpha+0) = \lim_{\varepsilon\to 0}\varphi'(\alpha+|\varepsilon|)$, and a finite left-hand-side limit at $\theta = \beta$, $\varphi'(\beta-0) = \lim_{\varepsilon\to 0}\varphi'(\beta-|\varepsilon|)$. On the one hand, such a choice creates certain technical difficulties. But on the other hand, it provides several advantages in the formulations and proofs of some statements presented in the chapter. In particular, it follows from Theorem 5.1 that if $\varphi \in PC^1([-h,0],R^n)$, then $x_t(t_0,\varphi) \in PC^1([-h,0],R^n)$ for $t > t_0$.

Henceforth we assume that the following assumptions hold.

Assumption 5.1. *The difference $x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)$ is continuous and differentiable for $t \geq t_0$, except possibly for a countable number of points. This does not imply that $x(t,t_0,\varphi)$ is differentiable, or even continuous, for $t \geq t_0$.*

Assumption 5.2. *In Eq. (5.1) the right-hand-side derivative of the difference $x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)$ is assumed at the point $t = t_0$. By default, such agreement remains valid in situations where only a one-sided variation of the independent variable is allowed.*

Let $x(t)$ be a solution of the initial value problem (5.1)–(5.2); then

$$x(t) = Dx(t-h) + [\varphi(0) - D\varphi(-h)] + \int_{t_0}^{t} f(s,x_s)\mathrm{d}s, \quad t \geq t_0. \tag{5.3}$$

System (5.3) is the integral form of the initial value problem. In some sense it is more convenient to consider the integral system than the original one. For example, the choice of $PC([-h,0],R^n)$ as the space of initial functions for system (5.3) seems natural. The integral form substantially simplifies the study of discontinuity points of the solutions of system (5.1). If $\theta_1 \in [-h,0]$ is a discontinuity point of φ, then, according to Assumption 5.1, the function

$$z(t) = Dx(t-h) + [\varphi(0) - D\varphi(-h)]$$

has a jump discontinuity at $t_1 = t_0 + \theta_1 + h$ and the size of the jump at the point is such that

$$\Delta x(t_1) = D\Delta\varphi(\theta_1),$$

where $\Delta x(t_1) = x(t_1+0) - x(t_1-0)$. If $x(t)$ is defined for $t \in [t_0-h,\infty)$, then, as follows from Eq. (5.3), the solution suffers a jump discontinuity at the points $t_k = t_0+\theta_1+kh$, $k \geq 0$, and the jumps are subjected to the equation

$$\Delta x(t_{k+1}) = D\Delta x(t_k), \quad k \geq 0.$$

One of the special features of neutral type time-delay systems is the following. The discontinuity of a solution results in the discontinuity of the derivative on the left-hand side of system (5.1). Indeed, consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-Dx(t-h)] = F(x(t),x(t-h)).$$

If $\theta_1 \in [-h,0]$ is a discontinuity point of φ, then

$$\lim_{t\to t_1-0}\frac{\mathrm{d}}{\mathrm{d}t}[x(t,\varphi)-Dx(t-h,\varphi)] = F(x(t_1-0,\varphi),\varphi(\theta_1-0))$$

and

$$\lim_{t\to t_1+0}\frac{\mathrm{d}}{\mathrm{d}t}[x(t,\varphi)-Dx(t-h,\varphi)] = F(x(t_1-0,\varphi)+\Delta x(t_1,\varphi),\varphi(\theta_1-0)+\Delta\varphi(\theta_1)).$$

This means that the left-hand-side and right-hand-side derivatives at $t=t_1$ may not coincide. The following assumption makes it possible to overcome this technical difficulty.

Assumption 5.3. *It is assumed that $x(t,t_0,\varphi)$, $t \in [t_0-h,t_0+\tau]$, where $\tau > 0$, is a solution of system (5.1) if it satisfies the system almost everywhere on $[t_0,t_0+\tau]$.*

5.2 Existence Issue

We start with the following existence result.

Theorem 5.1. *Let the functional*

$$f : [0,\infty)\times PC^1([-h,0],R^n) \longrightarrow R^n$$

satisfy the following conditions:

(i) For any $H > 0$ there exists $M(H) > 0$ such that

$$\|f(t,\varphi)\| \leq M(H), \quad (t,\varphi) \in [0,\infty)\times PC^1([-h,0],R^n), \ \|\varphi\|_h \leq H.$$

(ii) The functional $f(t,\varphi)$ is continuous with respect to both arguments.

(iii) The functional $f(t,\varphi)$ is Lipschitz with respect to the second argument, i.e., for any $H>0$ there exists a Lipschitz constant $L(H)$ such that the inequality

$$\left\| f(t,\varphi^{(1)}) - f(t,\varphi^{(2)}) \right\| \leq L(H) \left\| \varphi^{(1)} - \varphi^{(2)} \right\|_h$$

holds for $t \geq 0$, $\varphi^{(k)} \in PC^1([-h,0],R^n)$, and $\left\| \varphi^{(k)} \right\|_h \leq H$, $k=1,2$.

Then, for given $t_0 \geq 0$ and an initial function $\varphi \in PC^1([-h,0],R^n)$ there exists $\tau > 0$ such that the initial value problem (5.1)–(5.2) admits a unique solution defined on the segment $[t_0-h, t_0+\tau]$.

Proof. Given $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, we introduce the function

$$z(t) = D\varphi(t-t_0-h) + \varphi(0) - D\varphi(-h), \quad t \in [t_0, t_0+h].$$

Let us select $H>0$ such that the following inequality holds:

$$H > H_0 = \max\left\{ \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|, \sup_{t\in[t_0,t_0+h]} \|z(t)\| \right\}.$$

Now we can define the corresponding values $M = M(H)$ and $L = L(H)$; see conditions *(i)* and *(iii)* of the theorem.

Let $\tau \in (0,h)$ be such that

$$\tau L < 1 \text{ and } \tau M < H - H_0.$$

Denote by Θ the set of discontinuity points of the initial function φ, and define a piecewise continuous function $u : [t_0-h, t_0+\tau] \to R^n$ as follows:

$$u(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0],$$

and any discontinuity point $t^* \in (t_0, t_0+\tau]$ of the function is such that $t^* - t_0 - h \in \Theta$. Finally, assume that the following inequality holds:

$$\|u(t) - z(t)\| \leq (t-t_0)M, \quad t \in [t_0, t_0+\tau].$$

The preceding inequality implies that

$$\|u(t)\| \leq H_0 + \tau M < H, \quad t \in [t_0, t_0+\tau].$$

It follows from the definition of the function that

$$\|u(t)\| \leq H_0 < H, \quad t \in [t_0-h, t_0].$$

We denote by U the set of all such functions. On the set U we define an operator $\mathcal{A}$ that acts on the functions of the set

$$\mathcal{A}(u)(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h, t_0], \\ z(t) + \int\limits_{t_0}^{t} f(s, u_s)\mathrm{d}s, & t \in [t_0, t_0+\tau]. \end{cases}$$

Here $u_s : \theta \to u(s+\theta), \quad \theta \in [-h, 0]$ and $\|u_s\|_h \le H$ for $s \in [t_0, t_0+\tau]$. It is easy to verify that the theorem conditions *(i)* and *(ii)* guarantee that the transformed function, $\mathcal{A}(u)$, belongs to the same set U,

$$u \in U \Rightarrow \mathcal{A}(u) \in U.$$

Any solution $\widetilde{x}(t)$ of the initial value problem (5.1)–(5.2) defines a fixed point of the operator,

$$\widetilde{x}(t) = \mathcal{A}(\widetilde{x})(t), \quad t \in [t_0-h, t_0+\tau].$$

Observe that

$$\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) = \begin{cases} 0, & t \in [t_0-h, t_0], \\ \int\limits_{t_0}^{t} [f(s, u_s^{(1)}) - f(s, u_s^{(2)})]\mathrm{d}s, & t \in [t_0, t_0+\tau]. \end{cases}$$

Hence, for $t \in [t_0-h, t_0]$

$$\left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| = 0$$

and for $t \in [t_0, t_0+\tau]$

$$\left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| \le \int\limits_{t_0}^{t_0+\tau} \left\| f(s, u_s^{(1)}) - f(s, u_s^{(2)}) \right\| \mathrm{d}s.$$

Because $\|u_s^{(1)}\|_h \le H$ and $\|u_s^{(2)}\|_h \le H$, the Lipschitz condition *(iii)* implies that the inequality

$$\begin{aligned} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| &\le L \int\limits_{t_0}^{t_0+\tau} \left\| u_s^{(1)} - u_s^{(2)} \right\|_h \mathrm{d}s \\ &\le \tau L \sup_{s \in [t_0-h, t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\| \end{aligned}$$

holds for $t \in [t_0, t_0+\tau]$. Since the preceding inequality holds for all $t \in [t_0-h, t_0+\tau]$, we conclude that

$$\sup_{s\in[t_0-h,t_0+\tau]} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| \leq \tau L \sup_{s\in[t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|.$$

Now, because $L\tau < 1$, the operator $\mathcal{A}$ satisfies the conditions of the contraction mapping theorem, and there exists a unique fixed point of the operator $u^{(*)} \in U$. This means that

$$u^{(*)}(t) = \mathcal{A}(u^{(*)})(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h, t_0], \\ z(t) + \int\limits_{t_0}^{t} f(s, u_s^{(*)})\mathrm{d}s, & t \in [t_0, t_0+\tau], \end{cases}$$

i.e.,

$$u^{(*)}(t) - Du^{(*)}(t-h) = \varphi(0) - D\varphi(-h) + \int\limits_{t_0}^{t} f(s, u_s^{(*)})\mathrm{d}s, \quad t \in [t_0, t_0+\tau].$$

The functional $f(t,\varphi)$ is continuous, and $u^{(*)}(t)$ is piecewise continuous; therefore, the right-hand side of the last equality is differentiable on $[t_0, t_0+\tau]$, except at most a finite number of points, and we arrive at the conclusion that the following equality holds almost everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u^{(*)}(t) - Du^{(*)}(t-h)\right] = f(t, u_t^{(*)}), \quad t \in [t_0, t_0+\tau].$$

Because function $u^{(*)}(t)$ satisfies Eq. (5.2), it is the unique solution of the initial value problem (5.1)–(5.2). □

Remark 5.1. We can take $t_1 = t_0 + \tau$ as a new initial time instant and define the new initial function

$$\varphi^{(1)}(\theta) = u^{(*)}(t_1+\theta), \quad \theta \in [-h, 0].$$

Then the construction process can be repeated, and we extend the solution to the next segment $[t_1, t_1+\widetilde{\tau}]$. This extension process can be continued as far as the solution remains bounded.

For each solution there exists a maximal interval $[t_0, t_0+T)$ on which the solution is defined. Here we present conditions under which any solution of system (5.1) is defined on the interval $[t_0, \infty)$.

Theorem 5.2. *Let system (5.1) satisfy the conditions of Theorem 5.1. Assume additionally that $f(t,\varphi)$ satisfies the inequality*

$$\|f(t,\varphi)\| \leq \eta(\|\varphi\|_h), \quad t \geq 0, \ \varphi \in PC^1([-h,0], R^n),$$

where the function $\eta(r)$, $r \in [0,\infty)$, is continuous, nondecreasing, and such that for any $r_0 \geq 0$ the following condition holds:

$$\lim_{R\to\infty} \int_{r_0}^{R} \frac{\mathrm{d}r}{\eta(r)} = \infty.$$

Then any solution $x(t,t_0,\varphi)$ of the system is defined on $[t_0,\infty)$.

Proof. Given $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, there exists a maximal interval $[t_0,t_0+T)$ on which the corresponding solution $x(t,t_0,\varphi)$ is defined. For the sake of simplicity we denote $x(t,t_0,\varphi)$ by $x(t)$.

Denote by $[t_0,t_0+T)$ the maximal interval on which the solution is defined. Assume by contradiction that $T < \infty$, and define the smallest entire N such that $T \leq Nh$. There exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} t_k = t_0 + T$$

and

$$\lim_{k\to\infty} \|x(t_k)\| \to \infty.$$

Otherwise, by Remark 5.1, the solution can be defined on a wider segment $[t_0,t_0+T+\tau]$, $\tau > 0$.

The solution satisfies the equality

$$x(t) = Dx(t-h) + [\varphi(0) - D\varphi(-h)] + \int_{t_0}^{t} f(s,x_s)\mathrm{d}s, \quad t \in [t_0,t_0+T).$$

For a given $t \in [t_0,t_0+T)$ we define an integer k such that $t \in [t_0+(k-1)h, t_0+kh)$. Now, iterating the preceding equality $k-1$ times, we obtain that

$$x(t) = D^k x(t-kh) + \sum_{j=0}^{k-1} D^j[\varphi(0) - D\varphi(-h)] + \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} f(s,x_s)\mathrm{d}s.$$

There exist $d \geq 1$ and $\rho > 0$ such that $\left\|D^k\right\| \leq d\rho^k$ for $k \geq 0$. Thus

$$\|x(t)\| \leq \kappa \|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s,x_s)\| \mathrm{d}s, \quad t \in [t_0,t_0+T),$$

where

$$\varkappa = d\sum_{j=0}^{N-1} \rho^j, \quad \kappa = \max\left\{d, d\rho^N\right\} + (1+\rho)\varkappa.$$

For $\theta \in [-h,0]$ the following inequality holds:

$$\|x(t+\theta)\| \leq \kappa\|\varphi\|_h + \varkappa \int_{t_0}^{\max\{t+\theta,t_0\}} \|f(s,x_s)\|\,\mathrm{d}s$$
$$\leq \kappa\|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s,x_s)\|\,\mathrm{d}s;$$

hence we arrive at the inequality

$$\|x_t\|_h \leq \kappa\|\varphi\|_h + \varkappa \int_{t_0}^{t} \|f(s,x_s)\|\,\mathrm{d}s, \quad t \in [t_0, t_0+T).$$

It follows from the theorem conditions that

$$\|x_t\|_h \leq \kappa\|\varphi\|_h + \varkappa \int_{t_0}^{t} \eta(\|x_s\|_h)\mathrm{d}s, \quad t \in [t_0, t_0+T).$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \varkappa\eta(\|x_t\|_h) \leq \varkappa\eta(v(t)), \quad t \in [t_0, t_0+T).$$

This implies that

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} \leq \varkappa(t_k - t_0), \quad k = 1,2,3,\ldots.$$

On the one hand, since

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \int_{r_0}^{r_k} \frac{\mathrm{d}\xi}{\eta(\xi)},$$

where $r_0 = v(t_0) = \kappa\|\varphi\|_h \geq 0$, and

$$r_k = v(t_k) \geq \left\|x_{t_k}\right\|_h \geq \|x(t_k)\| \to \infty, \text{ as } k \to \infty,$$

we conclude that

$$\lim_{k\to\infty} \int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \infty.$$

On the other hand,

$$\lim_{k\to\infty} \varkappa(t_k - t_0) = \varkappa T;$$

therefore $T = \infty$. This contradicts our assumption that $T < \infty$. The contradiction concludes the proof of the theorem. □

5.3 Continuity of Solutions

In this section we analyze the continuity properties of the solutions of system (5.1) with respect to initial conditions as well as the system right-hand-side perturbations. These continuity properties are a direct consequence of the following theorem.

Theorem 5.3. *Assume that the right-hand side of system (5.1), $f(t,\varphi)$, satisfies the conditions of Theorem 5.1. Let $x(t,t_0,\varphi)$ be a solution of system (5.1) with the initial condition*

$$x(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0].$$

Given a perturbed system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - Dy(t-h)\right] = f(t,y_t) + g(t,y_t), \quad t \geq 0,$$

where the functional $g(t,\varphi)$ is continuous on the set $[0,\infty) \times PC^1([-h,0],R^n)$, the functional $g(t,\varphi)$ satisfies the Lipschitz condition with respect to the second argument and

$$\|g(t,\varphi)\| \leq m, \quad t \geq 0,\ \varphi \in PC^1([-h,0],R^n).$$

Let $y(t,t_0,\psi)$ be a solution of the perturbed system with the initial condition

$$y(t_0+\theta) = \psi(\theta),\ \theta \in [-h,0].$$

If both solutions are defined for $t \in [t_0, t_0+T]$, where $0 < T < \infty$, then there exist positive constants α, β, γ such that the following inequality holds:

$$\|x(t,t_0,\varphi) - y(t,t_0,\psi)\| \leq (\alpha\|\psi-\varphi\|_h + \beta m)\,\mathrm{e}^{\gamma(t-t_0)}, \quad t \in [t_0, t_0+T].$$

Proof. For the matrix D there exist $d \geq 1$ and $\rho > 0$ such that $\left\|D^k\right\| \leq d\rho^k$ for $k \geq 0$.

For the sake of simplicity we will use the following shorthand notations for the solutions $x(t,t_0,\varphi) = x(t)$ and $y(t,t_0,\psi) = y(t)$. Observe that for $t \geq t_0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - Dx(t-h)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - Dy(t-h)\right] = f(t,x_t) - f(t,y_t) - g(t,y_t).$$

Integrating the preceding equality we obtain that

$$\begin{aligned} x(t)-y(t) &= D\left[x(t-h)-y(t-h)\right] \\ &\quad + \left[\varphi(0)-D\varphi(-h)\right] - \left[\psi(0)-D\psi(-h)\right] \\ &\quad + \int_{t_0}^{t} \left[f(s,x_s)-f(s,y_s)-g(s,y_s)\right] \mathrm{d}s, \quad t \geq t_0. \end{aligned}$$

Let us first define the smallest integer N such that $T \leq hN$. Then for a given $t \in [t_0, t_0+T]$ we define an integer k such that $t \in [t_0+(k-1)h, t_0+kh)$. Now, after $k-1$ iterations we arrive at the equality

$$\begin{aligned} x(t)-y(t) &= D^k\left[x(t-kh)-y(t-kh)\right] + \sum_{j=0}^{k-1} D^j\left[\varphi(0)-\psi(0)\right] \\ &\quad - \sum_{j=0}^{k-1} D^{j+1}\left[\varphi(-h)-\psi(-h)\right] \\ &\quad + \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} \left[f(s,x_s)-f(s,y_s)-g(s,y_s)\right] \mathrm{d}s. \end{aligned} \tag{5.4}$$

Since $t-kh \in [t_0-h, t_0]$, we conclude that

$$\left\| D^k\left[x(t-kh)-y(t-kh)\right]\right\| \leq d\rho^k \left\|\varphi-\psi\right\|_h \leq \max\left\{d, d\rho^N\right\} \left\|\varphi-\psi\right\|_h.$$

It is obvious that the following two inequalities hold:

$$\left\| \sum_{j=0}^{k-1} D^j\left[\varphi(0)-\psi(0)\right]\right\| \leq \varkappa \left\|\varphi-\psi\right\|_h,$$

where

$$\varkappa = d\sum_{j=0}^{N-1} \rho^j$$

and

$$\left\| \sum_{j=0}^{k-1} D^{j+1}\left[\varphi(-h)-\psi(-h)\right]\right\| \leq \varkappa\rho \left\|\varphi-\psi\right\|_h.$$

Finally, we find that

$$\left\| \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} [f(s,x_s) - f(s,y_s)]\,\mathrm{d}s \right\| \le \varkappa \int_{t_0}^{t} \|f(s,x_s) - f(s,y_s)\|\,\mathrm{d}s$$
$$\le \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s$$

and

$$\left\| \sum_{j=0}^{k-1} D^j \int_{t_0}^{t-jh} g(s,y_s)\mathrm{d}s \right\| \le \varkappa \int_{t_0}^{t} \|g(s,y_s)\|\,\mathrm{d}s \le \varkappa m(t-t_0).$$

Here $L_1 = L(H_1)$ and

$$H_1 = \max\left\{ \sup_{t\in[t_0-h,t_0+T]} \|x(t)\|,\ \sup_{t\in[t_0-h,t_0+T]} \|y(t)\| \right\}.$$

Now equality (5.4) implies that for $t \in [t_0, t_0+T]$ the inequality

$$\|x(t) - y(t)\| \le \kappa \|\varphi - \psi\|_h + \varkappa m(t-t_0) + \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s$$

holds, where

$$\kappa = \max\left\{d, d\rho^N\right\} + \varkappa(1+\rho).$$

Applying arguments similar to that used in the proof of Theorem 5.2 we obtain that

$$\|x_t - y_t\|_h \le \kappa \|\varphi - \psi\|_h + \varkappa m(t-t_0) + \varkappa L_1 \int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s, \quad t \in [t_0, t_0+T].$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \varkappa m + \varkappa L_1 \|x_t - y_t\|_h, \quad t \in [t_0, t_0+T].$$

Direct integration of this inequality leads to the desired result

$$\begin{aligned} \|x(t,t_0,\varphi) - y(t,t_0,\psi)\| &\le \|x_t(t_0,\varphi) - y_t(t_0,\psi)\|_h \\ &\le \kappa \|\psi - \varphi\|_h \mathrm{e}^{\varkappa L_1(t-t_0)} + \frac{m}{L}\mathrm{e}^{\varkappa L_1(t-t_0)} \\ &\le (\alpha \|\psi - \varphi\|_h + \beta m)\,\mathrm{e}^{\gamma(t-t_0)}, \quad t \in [t_0, t_0+T], \end{aligned}$$

where $\alpha = \kappa$, $\beta = L_1^{-1}$, and $\gamma = \varkappa L_1$. □

Corollary 5.1. *Let $g(t,\varphi) \equiv 0$, then $m = 0$, and both $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ are solutions of system (5.1). Assume that these solutions are defined for $t \in [t_0, t_0+T]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\psi - \varphi\|_h < \delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi) - x(t,t_0,\psi)\| < \varepsilon, \quad t \in [t_0, t_0+T].$$

In other words, $x(t,t_0,\varphi)$ depends continuously on the initial function φ.

Proof. The statement follows directly from Theorem 5.3 if we set $\delta = \varepsilon\alpha^{-1}e^{-\gamma T}$.□

Corollary 5.2. *Let $\psi(\theta) = \varphi(\theta)$, $\theta \in [-h, 0]$; then the solutions $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ have the same initial function. Assume that these solutions are defined for $t \in [t_0, t_0+T]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m < \delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi) - y(t,t_0,\varphi)\| < \varepsilon, \quad t \in [t_0, t_0+T].$$

This means that the solutions depend continuously on the right-hand side of system (5.1).

Proof. The statement follows directly from Theorem 5.3 if we set $\delta = \varepsilon\beta^{-1}e^{-\gamma T}$.□

5.4 Stability Concepts

In the rest of the chapter we assume that system (5.1) satisfies the conditions of Theorem 5.1 and additionally that it admits the trivial solution, i.e., the following identity holds:

$$f(t,0_h) \equiv 0, \text{ for } t \geq 0.$$

Definition 5.1. The trivial solution of system (5.1) is said to be stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon,t_0) > 0$ such that for every initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \delta(\varepsilon,t_0)$, the following inequality holds:

$$\|x(t,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

If $\delta(\varepsilon,t_0)$ can be chosen independently of t_0, then the trivial solution is said to be uniformly stable.

Definition 5.2. The trivial solution of system (5.1) is said to be asymptotically stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\Delta(\varepsilon,t_0) > 0$ such that for every initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon,t_0)$, the following conditions hold.

1. $\|x(t,t_0,\varphi)\| < \varepsilon$, for $t \geq t_0$.
2. $x(t,t_0,\varphi) \to 0$, as $t - t_0 \to \infty$.
   If $\Delta(\varepsilon, t_0)$ can be chosen independently of t_0 and there exists $H_1 > 0$ such that $x(t,t_0,\varphi) \to 0$, because $t - t_0 \longrightarrow \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$, then the trivial solution is said to be uniformly asymptotically stable.

Definition 5.3. The trivial solution of system (5.1) is said to be exponentially stable if there exist $\Delta_0 > 0$, $\sigma > 0$, and $\gamma \geq 1$ such that for every $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \Delta_0$, the following inequality holds:

$$\|x(t,t_0,\varphi)\| \leq \gamma \mathrm{e}^{-\sigma(t-t_0)} \|\varphi\|_h, \quad t \geq t_0.$$

As mentioned in Sect. 5.1, if an initial function φ admits a jump point θ_1, then the corresponding solution, $x(t,t_0,\varphi)$, has jump discontinuity at the points $t_k = t_0 + \theta_1 + kh$, $k \geq 1$, and the jumps at these points satisfy the jump equation

$$\Delta x(t_{k+1}) = D\Delta x(t_k), \quad k \geq 1.$$

As a consequence, we observe that system (5.1) cannot be stable if the matrix D admits an eigenvalue with magnitude greater than one. Otherwise, for any $\delta > 0$ there exists an initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \delta$, such that the corresponding solution $x(t,t_0,\varphi)$ has a sequence of jumps, and the size of the jumps tends to infinity. This observation motivates the following assumption.

Assumption 5.4. *In the rest of the chapter we assume that matrix D is Schur stable, i.e., the spectrum of the matrix lies in the open unit disc of the complex plane.*

5.5 Lyapunov–Krasovskii Approach

We will use the following concept of positive-definite functionals for system (5.1).

Definition 5.4. The functional $v(t,\varphi)$ is said to be positive definite if there exists $H > 0$ such that the following conditions are satisfied:

1. The functional $v(t,\varphi)$ is defined for $t \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.
2. $v(t,0_h) = 0$, $t \geq 0$.
3. There exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0) - D\varphi(-h)) \leq v(t,\varphi),$$
$$t \geq 0, \ \varphi \in PC^1([-h,0],R^n), \text{ with } \|\varphi\|_h \leq H.$$

4. For any given $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in φ at the point 0_h, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\varphi\|_h < \delta$ implies

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) < \varepsilon.$$

We are now ready to present some basic results of the Lyapunov–Krasovskii approach.

Theorem 5.4. *The trivial solution of system (5.1) is stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that along the solutions of the system $v(t,x_t)$, as a function of t, does not increase.*

Proof. Sufficiency: Since the matrix D is Schur stable, there exist $d \geq 1$and $\rho \in (0,1)$ such that the inequality $\left\|D^k\right\| \leq d\rho^k$ holds for $k \geq 0$. The positive definiteness of the functional $v(t,\varphi)$ implies that there exists a positive-definite function $v_1(x)$ satisfying Definition 5.4. Let $H > 0$ be that of Definition 5.4.

For a given $\varepsilon \in (0,H)$ we first set

$$\varepsilon_1 = \frac{1-\rho}{d}\varepsilon > 0$$

and then introduce the positive value

$$\lambda(\varepsilon_1) = \min_{\|x\|=\varepsilon_1} v_1(x). \tag{5.5}$$

Since for a given $t_0 \geq 0$ functional $v(t_0,\varphi)$ is continuous in φ at the point 0_h, there exists $\delta_1(\varepsilon,t_0) > 0$ such that $v(t_0,\varphi) < \lambda(\varepsilon_1)$ for any $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \delta_1(\varepsilon,t_0)$.

It is clear that $\delta_1(\varepsilon,t_0) \leq \varepsilon_1$; otherwise we can present an initial function $\varphi \in PC^1([-h,0],R^n)$ such that $\|\varphi\|_h < \delta_1(\varepsilon,t_0)$ and $\|\varphi(0) - D\varphi(-h)\| = \varepsilon_1$. On the one hand, for this initial function we have $v_1(\varphi(0) - D\varphi(-h)) \geq \lambda(\varepsilon_1)$. On the other hand, $v_1(\varphi(0) - D\varphi(-h)) \leq v(t_0,\varphi) < \lambda(\varepsilon_1)$. The contradiction proves the desired inequality.

Now we define the positive value

$$\delta(\varepsilon,t_0) = \frac{\delta_1(\varepsilon,t_0)}{1+d\rho}.$$

Let $\varphi \in PC^1([-h,0],R^n)$ with $\|\varphi\|_h < \delta(\varepsilon,t_0)$. Then the theorem condition implies that

$$\begin{aligned} v_1(x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)) &\leq v(t,x_t(t_0,\varphi)) \\ &\leq v(t_0,\varphi) < \lambda(\varepsilon_1), \quad t \geq t_0. \end{aligned} \tag{5.6}$$

We prove that

$$\|x(t,t_0,\varphi)-Dx(t-h,t_0,\varphi)\|<\varepsilon_1,\quad t\ge t_0.$$

Assume by contradiction that there exists a time instant $t_1 \ge t_0$ for which

$$\|x(t_1,t_0,\varphi)-Dx(t_1-h,t_0,\varphi)\|\ge\varepsilon_1.$$

Since

$$\begin{aligned}\|x(t_0,t_0,\varphi)-Dx(t_0-h,t_0,\varphi)\| &= \|\varphi(0)-D\varphi(-h)\|\\ &\le (1+d\rho)\,\|\varphi\|_h<\delta_1(\varepsilon,t_0)\le\varepsilon_1\end{aligned}$$

and $\|x(t,t_0,\varphi)-Dx(t-h,t_0,\varphi)\|$ is a continuous function of t, there exists $t^* \in [t_0,t_1]$ such that

$$\|x(t^*,t_0,\varphi)-Dx(t^*-h,t_0,\varphi)\|=\varepsilon_1.$$

On the one hand, it follows from Eq. (5.5) that

$$v_1(x(t^*,t_0,\varphi)-Dx(t^*-h,t_0,\varphi))\ge\lambda(\varepsilon_1).$$

On the other hand, Eq. (5.6) provides the opposite inequality

$$v_1(x(t^*,t_0,\varphi)-Dx(t^*-h,t_0,\varphi))<\lambda(\varepsilon_1).$$

The contradiction proves that our assumption is wrong, and

$$\|x(t,t_0,\varphi)-Dx(t-h,t_0,\varphi)\|<\varepsilon_1,\quad t\ge t_0.$$

The preceding inequality means that

$$x(t,t_0,\varphi)=Dx(t-h,t_0,\varphi)+\xi(t),\quad t\ge t_0, \tag{5.7}$$

where $\xi(t)$ is such that $\|\xi(t)\|<\varepsilon_1$, $t\ge t_0$.

For a given $t\ge t_0$ we define the entire number k such that $t\in[t_0+(k-1)h,t_0+kh)$. Iterating equality (5.7) $k-1$ times we obtain that

$$x(t,t_0,\varphi)=D^kx(t-kh,t_0,\varphi)+\sum_{j=0}^{k-1}D^j\xi(t-jh).$$

Since $t-kh\in[t_0-h,t_0]$,

$$\|x(t-kh,t_0,\varphi)\|\le\|\varphi\|_h<\delta(\varepsilon,t_0)\le\varepsilon_1,$$

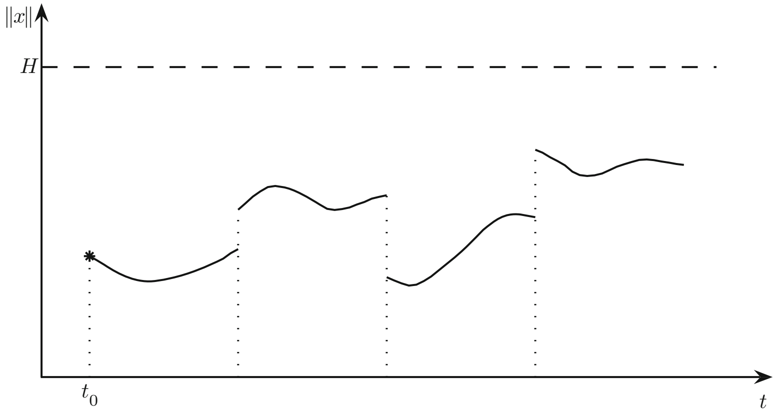

Fig. 5.1 Value of $\|x(t,t_0,\varphi\|$, the first case

and we arrive at the following inequality:

$$\|x(t,t_0,\varphi)\| \leq \left\|D^k\right\| \|x(t-kh,t_0,\varphi)\| + \sum_{j=0}^{k-1} \left\|D^j\right\| \|\xi(t-jh)\|$$

$$< d\rho^k\delta(\varepsilon,t_0) + \sum_{j=0}^{k-1} d\rho^j\varepsilon_1 < \frac{d}{1-\rho}\varepsilon_1 = \varepsilon, \quad t \geq t_0.$$

This means that $\delta(\varepsilon,t_0)$ satisfies Definition 5.1, and the trivial solution of Eq. (5.1) is stable.

Necessity: Now, the trivial solution of system (5.1) is stable, and we must prove that there exists a functional $v(t,\varphi)$ that satisfies the theorem conditions.

Construction of the functional: Since the trivial solution of system (5.1) is stable, for $\varepsilon = H$ there exists $\delta(H,t_0) > 0$ such that the inequality $\|\varphi\|_h < \delta(H,t_0)$ implies that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. We define the functional $v(t,\varphi)$ as follows:

$$v(t_0,\varphi) = \begin{cases} \sup\limits_{t\geq t_0} \|x(t,t_0,\varphi) - Dx(t-h,t_0\varphi)\|, \text{ if } \|x(t,t_0,\varphi)\| < H, \text{ for } t \geq t_0, \\ (1+d\rho)H \text{ if there exists } T \geq t_0 \text{ such that } \|x(T,t_0,\varphi)\| \geq H. \end{cases} \tag{5.8}$$

These two possibilities are illustrated in Figs. 5.1 and 5.2, respectively.

We verify first that the functional is positive definite. To this end, we must verify that it satisfies the conditions of Definition 5.4.

Condition 1: Actually, Eq. (5.8) allows us to compute $v(t_0,\varphi)$ for any $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.

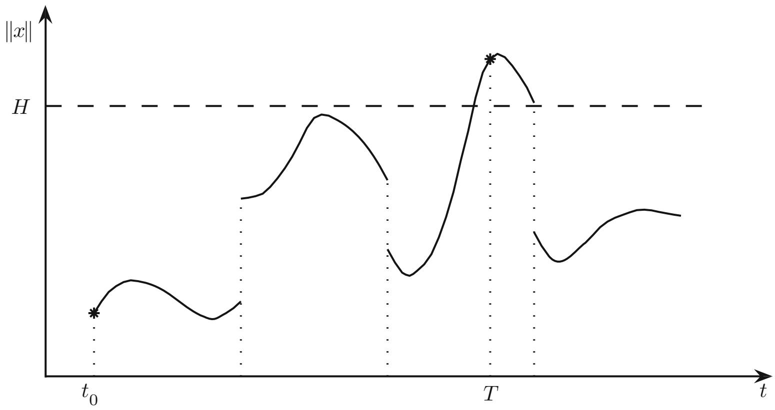

Fig. 5.2 Value of $\|x(t,t_0,\varphi\|$, the second case

Condition 2: Since for $\varphi = 0_h$ the corresponding solution is trivial, $x(t,t_0,0_h) = 0$, $t \geq t_0$, then $v(t_0,0_h) = 0$.

Condition 3: The function $v_1(x) = \|x\|$ is positive definite. In the case where $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$, we have

$$v_1(\varphi(0) - D\varphi(-h)) = \|\varphi(0) - D\varphi(-h)\| \leq \sup_{t \geq t_0} \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| = v(t_0,\varphi).$$

And in the other case where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| \geq H$, the following inequality holds:

$$v_1(\varphi(0) - D\varphi(-h)) = \|\varphi(0) - D\varphi(-h)\| \leq (1 + d\rho)H = v(t_0,\varphi).$$

Condition 4: Given $t_0 \geq 0$, the stability of the trivial solution means that for any $\varepsilon > 0$ there exists $\delta_1 = \delta(\frac{\varepsilon}{1+d\rho}, t_0) > 0$ such that $\|\varphi\|_h < \delta_1$ implies

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1+d\rho}, \quad t \geq t_0.$$

This means that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \|x(t,t_0,\varphi)\| + d\rho \|x(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

The preceding inequality demonstrates that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon.$$

This observation makes it clear that for a fixed $t_0 \geq 0$ the functional $v(t_0, \varphi)$ is continuous in φ at the point 0_h.

Now we check that functional (5.8) satisfies the theorem condition. First, we consider the case where $\|x(t, t_0, \varphi)\| < H$ for $t \geq t_0$. In this case, given two time instants t_1 and t_2 such that $t_2 > t_1 \geq t_0$, we have that

$$v(t_1, x_{t_1}(t_0, \varphi)) = \sup_{t \geq t_1} \|x(t, t_0, \varphi) - Dx(t-h, t_0, \varphi)\|$$

and

$$v(t_2, x_{t_2}(t_0, \varphi)) = \sup_{t \geq t_2} \|x(t, t_0, \varphi) - Dx(t-h, t_0, \varphi)\| .$$

Since for the second value the range of the supremum is smaller than that for the first one, we conclude that

$$v(t_2, x_{t_2}(t_0, \varphi)) \leq v(t_1, x_{t_1}(t_0, \varphi)).$$

This means that along the solution the functional $v(t, x_t(t_0, \varphi))$ does not increase as a function of t. In the second case, where there exists $T \geq t_0$ such that $\|x(T, t_0, \varphi)\| \geq H$, we have the equality

$$v(t_2, x_{t_2}(t_0, \varphi)) = v(t_1, x_{t_1}(t_0, \varphi)) = (1 + d\rho)H,$$

and, once again, the functional does not increase along the solution of system (5.1).□

Remark 5.2. On the one hand, functional (5.8) has only an academic value. Obviously, we cannot use such functionals in applications. On the other hand, it demonstrates that the Lyapunov–Krasovskii approach is universal: for any system with a stable trivial solution there are positive-definite functionals satisfying Theorem 5.4.

Theorem 5.5. *The trivial solution of system (5.1) is uniformly stable if and only if there exists a positive-definite functional $v(t, \varphi)$ such that the following conditions are satisfied:*

1. *The value of the functional along the solutions of the system, $v(t, x_t)$, as a function of t does not increase.*
2. *The functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$.*

Proof. Sufficiency: We use notations from the proof of the sufficiency part of Theorem 5.4. Now the functional $v(t, \varphi)$ is continuous in φ at the point 0_h, uniformly for $t \geq 0$, so there exists a positive value $\delta_1(\varepsilon)$ such that the inequality $v(t_0, \varphi) < \lambda(\varepsilon_1)$ holds for any $t_0 \geq 0$ and $\varphi \in PC^1([-h, 0], R^n)$, with $\|\varphi\|_h < \delta_1(\varepsilon)$. Therefore, the value

$$\delta(\varepsilon) = \frac{\delta_1(\varepsilon)}{1 + d\rho}$$

does not depend on t_0. The remainder of the sufficiency part of the proof coincides with that of Theorem 5.4.

Necessity: The uniform stability of the trivial solution of system (5.1) implies that δ can be chosen independently of t_0, $\delta = \delta(\varepsilon)$. We show that functional (5.8) satisfies the second condition of the theorem. Let us select for a given $\varepsilon > 0 (\varepsilon < H)$ the value

$$\delta_1 = \delta\left(\frac{\varepsilon}{1 + d\rho}\right).$$

Then, for any $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h < \delta_1$ and $t_0 \geq 0$, we have that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1 + d\rho}, \text{ for } t \geq t_0.$$

This means that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \|x(t,t_0,\varphi)\| + d\rho \|x(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0.$$

The preceding inequality demonstrates that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon, \quad t_0 \geq 0.$$

In other words, functional (5.8) is continuous in φ at the point 0_h, uniformly with respect to $t_0 \geq 0$. □

Corollary 5.3. *Let the condition of Theorem 5.4 be fulfilled, and let the functional $v(t,\varphi)$ admit an upper estimate of the form*

$$v(t,\varphi) \leq v_2(\varphi), \quad t \geq 0, \ \varphi \in PC^1([-h,0], R^n), \text{ with } \|\varphi\|_h \leq H,$$

with a positive-definite functional $v_2(\varphi)$; then the trivial solution of system (5.1) is uniformly stable.

Theorem 5.6. *The trivial solution of system (5.1) is asymptotically stable if and only if the following conditions hold.*

1. *There exists a positive-definite functional $v(t,\varphi)$, defined for $t \geq 0$ and $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h \leq H$ such that along the solutions of the system $v(t,x_t)$, as a function of t, does not increase.*
2. *For any $t_0 \geq 0$ there exists a positive value $\mu(t_0)$ such that if $\varphi \in PC^1([-h,0], R^n)$ and $\|\varphi\|_h < \mu(t_0)$, then $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$.*

Proof. Sufficiency: Since the matrix D is Schur stable, there exists $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\|D^k\| \leq d\rho^k$ holds for $k \geq 0$. The first condition of the theorem implies the stability of the trivial solution of system (5.1); see Theorem 5.4. Thus, for any $\varepsilon \in (0,H)$ and $t_0 \geq 0$ there exists $\delta(\varepsilon,t_0) > 0$ such that if $\|\varphi\|_h < \delta(\varepsilon,t_0)$, then $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. Let us define the value

$$\Delta(\varepsilon,t_0) = \min\left\{\delta\left(\varepsilon,t_0\right),\mu\left(t_0\right)\right\}.$$

Now, given an initial function $\varphi \in PC^1\left([-h,0],R^n\right)$ such that $\|\varphi\|_h < \Delta(\varepsilon,t_0)$, we will demonstrate that $x(t,t_0,\varphi) \to 0$ as $t-t_0 \to \infty$. The functional $v(t,\varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$such that

$$v_1(\varphi(0) - D\varphi(-h)) \leq v(t,\varphi)$$

for $t \geq 0$ and $\varphi \in PC^1\left([-h,0],R^n\right)$, with $\|\varphi\|_h \leq H$. For a given $\varepsilon_1 > 0 (\varepsilon_1 < \varepsilon)$ we set

$$\varepsilon_2 = \frac{1-\rho}{2d}\varepsilon_1 > 0$$

and define the positive value

$$\alpha = \min_{\varepsilon_2 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that $v(t,x_t(t_0,\varphi)) < \alpha$ for $t \geq t_0 + T$ and

$$v_1(x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)) \leq v(t,x_t(t_0,\varphi)) < \alpha, \quad t-t_0 \geq T,$$

so we must conclude that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_2, \quad t-t_0 \geq T.$$

This means that

$$x(t,t_0,\varphi) = Dx(t-h,t_0,\varphi) + \nu(t), \quad t-t_0 \geq T, \tag{5.9}$$

where

$$\|\nu(t)\| < \varepsilon_2, \quad t-t_0 \geq T.$$

For a given $t \geq t_0 + T$ we define the integer number k such that $t \in [t_0 + T + (k-1)h, t_0 + T + kh)$. Then, iterating equality (5.9) $(k-1)$ times, we obtain that

$$x(t,t_0,\varphi) = \sum_{j=0}^{k-1} D^j \nu(t-jh) + D^k x(t-kh,t_0,\varphi)$$

and

$$\|x(t,t_0,\varphi)\| \le \sum_{j=0}^{k-1} d\rho^j \varepsilon_2 + d\rho^k \varepsilon < \frac{d}{1-\rho}\varepsilon_2 + d\rho^k\varepsilon \le \frac{1}{2}\varepsilon_1 + d\rho^k\varepsilon, \quad t-t_0 \ge T.$$

Since $\rho^k \to 0$ as $k \to \infty$, then, starting from some k_0, the following inequality holds:

$$d\rho^k\varepsilon < \frac{1}{2}\varepsilon_1, \quad k \ge k_0.$$

This means that $\|x(t,t_0,\varphi)\| < \varepsilon_1$ for $t \ge t_0 + T + k_0 h$, and we conclude that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$. Hence, the previously defined value $\Delta(t_0,\varepsilon)$ satisfies Definition 5.2.

Necessity: In this part of the proof we make use of functional (5.8). In the proof of Theorem 5.4 it was demonstrated that the functional is positive definite and does not increase along the solutions of system (5.1). This means that the functional satisfies the first condition of the theorem.

We address the second condition of the theorem and choose the value $\mu(t_0)$ as follows:

$$\mu(t_0) = \Delta(H,t_0) > 0.$$

Now for any initial function $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h < \mu(t_0)$, we know that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$. This means that for any $\varepsilon_1 > 0$ there exists $t_1 \ge t_0$ such that

$$\|x(t,t_0,\varphi)\| < \frac{1}{1+d\rho}\varepsilon_1, \quad t \ge t_1.$$

According to Eq. (5.8), we have

$$\begin{aligned} v(t,x_t(t_0,\varphi)) &= \sup_{s \ge t} \|x(s,t_0,\varphi) - Dx(s-h,t_0,\varphi)\| \\ &\le \frac{1}{1+d\rho}\varepsilon_1 + \frac{d\rho}{1+d\rho}\varepsilon_1 = \varepsilon_1, \quad t \ge t_1 + h. \end{aligned}$$

The preceding observation proves that $v(t,x_t(t_0,\varphi))$ tends to zero as $t - t_0 \to \infty$. □

The following statement gives sufficient conditions for the asymptotic stability of the trivial solution of system (5.1).

Theorem 5.7. *The trivial solution of system (5.1) is asymptotically stable if there exist a positive-definite functional $v(t,\varphi)$ and a positive-definite function $w(x)$ such that along the solutions of the system the functional $v(t,\varphi)$ is differentiable and its time derivative satisfies the inequality*

$$\frac{\mathrm{d}v(t,x_t)}{\mathrm{d}t} \le -w(x(t) - Dx(t-h)).$$

Proof. Since the matrix D is Schur stable, there exists $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\left\|D^k\right\| \leq d\rho^k$ holds for $k \geq 0$.

Observe first that the theorem conditions imply that of Theorem 5.4; therefore, the trivial solution of system (5.1) is stable, i.e., for any $t_0 \geq 0$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, t_0) > 0$, which satisfies Definition 5.1. Let us set

$$\Delta(\varepsilon, t_0) = \delta(\frac{\varepsilon}{1+d\rho}, t_0) > 0.$$

Given $t_0 \geq 0$ and an initial function $\varphi \in PC^1\left([-h,0], R^n\right)$ such that $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, we have that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon}{1+d\rho}, \quad t \geq t_0,$$

and

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon, \quad t \geq t_0 + h. \tag{5.10}$$

First we demonstrate that

$$x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi) \to 0, \text{ as } t - t_0 \to \infty. \tag{5.11}$$

Assume by contradiction that this is not the case; then there exists $\alpha > 0$ and a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k - t_0 \to \infty$, as $k \to \infty$ such that

$$\|x(t_k,t_0,\varphi) - Dx(t_k-h,t_0,\varphi)\| \geq \alpha, \quad k \geq 1.$$

Without loss of generality we may assume that $t_{k+1} - t_k \geq h$ for $k \geq 0$. It follows from system (5.1) that

$$x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi) = [x(t_k,t_0,\varphi) - Dx(t_k-h,t_0,\varphi)] + \int_{t_k}^{t} f(s, x_s(t_0,\varphi))\mathrm{d}s, \quad t \geq t_k,$$

and since $\|x(t_k,t_0,\varphi) - Dx(t_k-h,t_0,\varphi)\| \geq \alpha$, then

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \geq \alpha - M(\varepsilon)(t-t_k), \quad t \geq t_k$$

(see condition *(i)* of Theorem 5.1). Hence, for any $k \geq 1$ the following inequality holds:

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \geq \frac{\alpha}{2}, \quad t \in [t_k, t_k + \tau],$$

where

$$\tau = \min\left\{h, \frac{\alpha}{2M(\varepsilon)}\right\}.$$

Because the function $w(x)$ is positive definite, we have that

$$\beta = \min_{\frac{\alpha}{2}\leq\|x\|\leq\varepsilon} w(x) > 0.$$

The second condition of the theorem implies that

$$\begin{aligned} v(t, x_t(t_0, \varphi)) &\leq v(t_0, \varphi) - \int_{t_0}^{t} w(x(s, t_0, \varphi) - Dx(s-h, t_0, \varphi))\mathrm{d}s \\ &\leq v(t_0, \varphi) - \tau\beta N(t), \end{aligned}$$

where $N(t)$ denotes the number of segments $[t_k, t_k + \tau]$ that belong to $[t_0, t]$. Since $N(t) \to \infty$ as $t - t_0 \to \infty$, we have that $v(t, x_t(t_0, \varphi))$ becomes negative for sufficiently large t, which contradicts the positive definiteness of the functional. The contradiction proves Eq. (5.11). This means that

$$x(t, t_0, \varphi) = Dx(t-h, t_0, \varphi) + \xi(t), \quad t \geq t_0,$$

and $\xi(t) \to 0$ as $t - t_0 \to \infty$. Given a positive value $\varepsilon_1 < \varepsilon$, there exists $t_1 > t_0$ such that

$$\|\xi(t)\| < \frac{1-\rho}{2d}\varepsilon_1, \quad t \geq t_1.$$

Let us define k_0 such that $d\rho^k\varepsilon < \frac{1}{2}\varepsilon_1$ for $k \geq k_0$. Now for any $t \geq t_1 + k_0 h$ we have

$$x(t, t_0, \varphi) = \sum_{j=0}^{k_0-1} D^j \xi(t - jh) + D^{k_0} x(t - k_0 h, t_0, \varphi)$$

and

$$\begin{aligned} \|x(t, t_0, \varphi)\| &\leq \sum_{j=0}^{k_0-1} \left\|D^j\right\| \|\xi(t - jh)\| + \left\|D^{k_0}\right\| \|x(t - k_0 h, t_0, \varphi)\| \\ &< \frac{d}{1-\rho}\left(\frac{1-\rho}{2d}\varepsilon_1\right) + d\rho^{k_0}\varepsilon < \varepsilon_1, \end{aligned}$$

and we arrive at the conclusion that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. This means that the previously defined positive value $\Delta(\varepsilon, t_0)$ satisfies Definition 5.2, and the trivial solution of system (5.1) is asymptotically stable. □

Now we provide a criterion of the uniform asymptotic stability of the trivial solution of system (5.1).

Theorem 5.8. *The trivial solution of system (5.1) is uniformly asymptotically stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions hold.*

1. *The functional is continuous in φ at the point 0_h, uniformly for $t \geq 0$.*
2. *There exists a positive value μ_1 such that $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t-t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$.*

Proof. Sufficiency: Comparing this theorem with Theorems 5.5 and 5.6 we conclude that the trivial solution of system (5.1) is uniformly stable and asymptotically stable. Therefore, for a given $\varepsilon > 0$ there exists

$$\Delta(\varepsilon) = \min\left\{\frac{1}{2}\delta(\varepsilon), \mu_1\right\} > 0$$

such that the following properties hold.

1. Given $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \Delta(\varepsilon)$, we have that $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$.
2. $x(t,t_0,\varphi) \to 0$, as $t-t_0 \to \infty$.

Now we define the positive value

$$H_1 = \Delta(H).$$

The functional $v(t,\varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that for $t \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H$, the following inequality holds:

$$v_1(\varphi(0) - D\varphi(-h)) \leq v(t,\varphi).$$

For a given $\varepsilon_1 > 0(\varepsilon_1 < \varepsilon)$ we set

$$\varepsilon_2 = \frac{1-\rho}{2d}\varepsilon_1 > 0$$

and define the positive value

$$\alpha = \min_{\varepsilon_2 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that for any $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$, the following inequality holds:

$$v(t,x_t(t_0,\varphi)) < \alpha,\ t-t_0 \geq T.$$

This implies that

$$v_1(x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)) < \alpha, \quad t - t_0 \geq T,$$

and we conclude that

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_2, \quad t - t_0 \geq T,$$

for any $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. And we again arrive at equality (5.9). Applying the arguments used in the proof of the sufficiency part of Theorem 5.6 we obtain the inequality

$$\|x(t,t_0,\varphi)\| \leq \frac{1}{2}\varepsilon_1 + d\rho^k H, \quad t - t_0 \geq T.$$

Since $\rho^k \to 0$ as $k \to \infty$, then, starting from some k_0, the following inequality holds:

$$d\rho^k H < \frac{1}{2}\varepsilon_1, \quad k \geq k_0.$$

This means that $\|x(t,t_0,\varphi)\| < \varepsilon_1$for $t - t_0 \geq \max\{T, k_0 h\}$, and we conclude that $x(t,t_0,\varphi) \to 0$ as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. Therefore, the previously defined values $\Delta(\varepsilon)$ and H_1 satisfy Definition 5.2. This concludes the proof of the sufficiency part of the theorem.

Necessity: The uniform asymptotic stability of the trivial solution of system (5.1) implies that functional (5.8) satisfies the first condition of the theorem. Set

$$\mu_1 = \frac{1}{2}\Delta(H),$$

where $\Delta(\varepsilon)$ is from Definition 5.2. Now, given $\varepsilon_1 > 0$, then for any $t_0 \geq 0$ and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$, there exists $T > 0$ such that

$$\|x(t,t_0,\varphi)\| < \frac{\varepsilon_1}{1 + d\rho}, \quad t - t_0 \geq T,$$

and

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| < \varepsilon_1, \quad t - t_0 \geq T + h.$$

This means that functional (5.8) satisfies the inequality

$$v(t, x_t(t_0,\varphi)) \leq \varepsilon_1, \quad t - t_0 \geq T + h,$$

for any $t_0 \geq 0$, and $\varphi \in PC^1([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$. In other words, under the conditions of the theorem, the value $v(t, x_t(t_0,\varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h \leq \mu_1$. This concludes the proof of the necessity part. □

Theorem 5.9. *The trivial solution of system (5.1) is exponentially stable if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions are satisfied.*

1. *There are two positive constants α_1, α_2 for which the inequalities*

$$\alpha_1 \|\varphi(0) - D\varphi(-h)\|^2 \leq v(t,\varphi) \leq \alpha_2 \|\varphi\|_h^2$$

   hold for $t \geq 0$, and $\varphi \in PC^1([-h,0], R^n)$, with $\|\varphi\|_h \leq H$.
2. *The functional is differentiable along the solutions of the system, and there exists a positive constant σ_1 such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t,x_t) + 2\sigma_1 v(t,x_t) \leq 0.$$

Proof. Because the matrix D is Schur stable, there exist $d \geq 1$ and $\rho \in (0,1)$ such that the inequality $\|D^k\| \leq d\rho^k$ holds for $k \geq 0$. There exists $\sigma_2 > 0$ such that $\rho = \mathrm{e}^{-\sigma_2 h}$.

If we define the positive-definite function $v_1(x) = \alpha_1 \|x\|^2$ and the positive-definite functional $v_2(\varphi) = \alpha_2 \|\varphi\|_h^2$, then it becomes evident that the functional $v(t,\varphi)$ satisfies the conditions of Theorem 5.5. Therefore, the trivial solution of system (5.1) is uniformly stable. This means that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality $\|\varphi\|_h < \delta(\varepsilon)$ implies $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. Let us set

$$\Delta_0 = \Delta(H).$$

We will demonstrate that this value satisfies Definition 5.3. To this end, we assume that $t_0 \geq 0$ and $\varphi \in PC^1([-h,0], R^n)$, $\|\varphi\|_h < \Delta_0$. The corresponding solution $x(t,t_0,\varphi)$ is such that

$$\|x(t,t_0,\varphi)\| < H, \quad t \geq t_0.$$

The second condition of the theorem implies

$$v(t,x_t(t_0,\varphi)) \leq v(t_0,\varphi)\mathrm{e}^{-2\sigma_1(t-t_0)}, \quad t \geq t_0.$$

Applying the first condition we obtain the inequalities

$$\begin{aligned}\alpha_1 \|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\|^2 &\leq v(t_0,\varphi)\mathrm{e}^{-2\sigma_1(t-t_0)} \\ &\leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma_1(t-t_0)}, \quad t \geq t_0.\end{aligned}$$

And, finally, we arrive at the exponential estimate

$$\|x(t,t_0,\varphi) - Dx(t-h,t_0,\varphi)\| \leq \gamma_1 \|\varphi\|_h \mathrm{e}^{-\sigma_1(t-t_0)}, \quad t \geq t_0,$$

where

$$\gamma_1 = \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

This means that

$$x(t,t_0,\varphi) = Dx(t-h,t_0,\varphi) + \eta(t), \quad t \geq t_0, \tag{5.12}$$

where

$$\|\eta(t)\| \leq \gamma_1 \|\varphi\|_h \mathrm{e}^{-\sigma_1(t-t_0)}, \quad t \geq t_0.$$

For a given $t \geq t_0$ we define an integer number k such that $t \in [t_0 + (k-1)h, t_0 + kh)$. After $k-1$ iterations of equality (5.12) we obtain

$$x(t,t_0,\varphi) = \sum_{j=0}^{k-1} D^j \eta(t-jh) + D^k x(t-kh,t_0,\varphi).$$

The last equality implies that

$$\begin{aligned}
\|x(t,t_0,\varphi)\| &\leq \sum_{j=0}^{k-1} \left\|D^j\right\| \|\eta(t-jh)\| + \left\|D^k\right\| \|\varphi\|_h \\
&\leq \sum_{j=0}^{k-1} \left(d\mathrm{e}^{-\sigma_2 jh}\right) \left(\gamma_1 \|\varphi\|_h \mathrm{e}^{-\sigma_1(t-jh-t_0)}\right) + d\mathrm{e}^{-\sigma_2 kh} \|\varphi\|_h \\
&\leq \gamma_1 d \left(\sum_{j=0}^{k-1} \mathrm{e}^{-\sigma_2 jh} \mathrm{e}^{-\sigma_1(t-jh-t_0)}\right) \|\varphi\|_h + d\mathrm{e}^{-\sigma_2 kh} \|\varphi\|_h .
\end{aligned}$$

If we set $\sigma_0 = \min\{\sigma_1, \sigma_2\}$, then

$$\|x(t,t_0,\varphi)\| \leq d\left[\gamma_1 k\mathrm{e}^{-\sigma_0(t-t_0)} + \mathrm{e}^{-\sigma_0 kh}\right] \|\varphi\|_h .$$

It follows from the definition of k that $(k-1)h \leq t - t_0 < kh$, hence

$$\begin{aligned}
\|x(t,t_0,\varphi)\| &\leq d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right]\mathrm{e}^{-\sigma_0(t-t_0)} \|\varphi\|_h \\
&= \left(d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right]\mathrm{e}^{-\mu(t-t_0)}\right)\mathrm{e}^{-(\sigma_0-\mu)(t-t_0)} \|\varphi\|_h ,
\end{aligned}$$

where $\mu \in (0,\sigma)$. Observe that the function

$$d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right]\mathrm{e}^{-\mu(t-t_0)} \to 0, \text{ as } t - t_0 \to \infty,$$

i.e., the function is bounded,

$$d\left[\gamma_1\left(\frac{t-t_0}{h}+1\right)+1\right]\mathrm{e}^{-\mu(t-t_0)}\leq\gamma,\quad t\geq t_0,$$

and we arrive at the exponential estimate for the solutions of system (5.1):

$$\|x(t,t_0,\varphi)\|\leq\gamma\mathrm{e}^{-\sigma(t-t_0)}\|\varphi\|_h,\quad t\geq t_0,$$

where $\gamma\geq 1$ and $\sigma=\sigma_0-\mu>0$. □

5.6 Notes and References

There are several forms in which to present neutral type time-delay systems. In this book we use the one proposed in the fundamental monograph [23]. This form assumes that solutions may have discontinuity points, but the difference $x(t)-Dx(t-h)$ remains continuous for $t\geq t_0$ (Assumption 5.1). The main reason to restrict our study to the case of systems with this simple difference operator is that a highly complicated stability analysis of more general classes of difference operators would be required. To the best of our knowledge, even in the case of several delays, stability conditions often become extremely sensitive to small variations in delays. An exhaustive stability study of more general classes of difference operators can be found in [23].

In the exposition of the existence and uniqueness theorem in Sect. 5.2 we follow an excellent source [19]. For the continuity properties of the solutions see [3, 19, 23].

A comprehensive treatise on the Lyapunov–Krasovskii approach to the stability analysis of neutral type time-delay systems is given in [44]. Our method of presenting basic stability results in Sect. 5.5 was inspired by [19, 72].

A list of contributions with sufficient stability results, mainly presented in the form of special linear matrix inequalities, can be found in [64]; see also [58] and references therein.