Chapter 4
Systems with Distributed Delay

In this chapter a linear retarded type system with distributed delays is studied. First, we introduce quadratic functionals and Lyapunov matrices for the system. Then we present the existence and uniqueness conditions for the matrices and provide some numerical schemes for the computation of the matrices. In the last part of the chapter functionals of the complete type are introduced, and some applications of the functionals are discussed.

4.1 System Description

We start with the following retarded type time-delay system:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{4.1}$$

Here A_0 and A_1 are given real $n \times n$ matrices, delay $h > 0$, and $G(\theta)$ is a continuous matrix defined for $\theta \in [-h, 0]$.

4.2 Quadratic Functionals

Given a symmetric matrix W, we are looking for a quadratic functional

$$v_0 : PC([-h,0], R^n) \rightarrow R$$

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such that along the solutions of system (4.1) the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0. \tag{4.2}$$

Definition 4.1. The matrix $U(\tau)$ is said to be a Lyapunov matrix of system (4.1) associated with a symmetric matrix W if it satisfies the following properties:

1. Dynamic property:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) = U(\tau)A_0 + U(\tau-h)A_1 + \int_{-h}^{0} U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau \geq 0; \tag{4.3}$$

2. Symmetry property:

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0; \tag{4.4}$$

3. Algebraic property:

$$\begin{aligned} -W &= U(0)A_0 + U(-h)A_1 + \int_{-h}^{0} U(\theta)G(\theta)\mathrm{d}\theta + A_0^T U(0) \\ &\quad + A_1^T U(h) + \int_{-h}^{0} G^T(\theta)U(-\theta)\mathrm{d}\theta. \end{aligned} \tag{4.5}$$

Remark 4.1. The algebraic property can also be written as

$$U'(+0) - U'(-0) = -W. \tag{4.6}$$

For a given matrix $U(\tau)$ we define on $PC([-h,0],R^n)$ a functional of the form

$$\begin{aligned} v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \\ &\quad + \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1 \\ &\quad + 2\varphi^T(0)\int_{-h}^{0} \left(\int_{-h}^{\theta} U(\xi-\theta)G(\xi)\mathrm{d}\xi \right) \varphi(\theta)\mathrm{d}\theta \end{aligned}$$

$$+2\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_2}U(h+\theta_1-\theta_2+\xi)G(\xi)\mathrm{d}\xi\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

$$+\int_{-h}^{0}\varphi^T(\theta_1)\left\{\int_{-h}^{0}\left[\int_{-h}^{\theta_1}G^T(\xi_1)\left(\int_{-h}^{\theta_2}U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\right.$$

$$\left.\times\,\varphi(\theta_2)\mathrm{d}\theta_2\right\}\mathrm{d}\theta_1. \qquad (4.7)$$

We can now prove the theorem.

Theorem 4.1. *Let $U(\tau)$ be a Lyapunov matrix of system (4.1) associated with W. Then the time derivative of functional (4.7) along the solutions of the system satisfies equality (4.2).*

Proof. Let $x(t)$, $t\geq 0$, be a solution of system (4.1); then

$$v_0(x_t)=x^T(t)U(0)x(t)+2x^T(t)\int_{-h}^{0}U(-h-\theta)A_1x(t+\theta)\mathrm{d}\theta$$

$$+\int_{-h}^{0}x^T(t+\theta_1)A_1^T\left(\int_{-h}^{0}U(\theta_1-\theta_2)A_1x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

$$+2x^T(t)\int_{-h}^{0}\left[\int_{-h}^{\theta}U(\xi-\theta)G(\xi)\mathrm{d}\xi\right]x(t+\theta)\mathrm{d}\theta$$

$$+2\int_{-h}^{0}x^T(t+\theta_1)A_1^T\left[\int_{-h}^{0}\left(\int_{-h}^{\theta_2}U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\right.$$

$$\left.\times\,x(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$

$$+\int_{-h}^{0}x^T(t+\theta_1)\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_1}G^T(\xi_1)\left(\int_{-h}^{\theta_2}U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\right.$$

$$\left.\times\,x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1.$$

At the first stage we compute the time derivative of each term of the functional.

For the first term, $R_0(t) = x^T(t)U(0)x(t)$, the time derivative is computed as

$$\frac{\mathrm{d}}{\mathrm{d}t}R_0(t) = \underline{2x^T(t)U(0)A_0x(t)} + \underline{2x^T(t)U(0)A_1x(t-h)} \\ \underline{+2x^T(t)U(0)\int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta}.$$

The time derivative of the term

$$R_1(t) = 2x^T(t)\int_{-h}^{0} U(-h-\theta)A_1x(t+\theta)\mathrm{d}\theta \\ = 2x^T(t)\int_{t-h}^{t} [U(h+s-t)]^T A_1x(s)\mathrm{d}s$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) = \underbrace{2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^T\int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s} \\ +\underline{2x^T(t)U(-h)A_1x(t)} - \underline{2x^T(t)U(0)A_1x(t-h)} \\ \underbrace{\underbrace{-2x^T(t)\int_{t-h}^{t}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\Big|_{\tau=h+s-t}\right]^T A_1x(s)\mathrm{d}s}}.$$

For the term

$$R_2(t) = \int_{-h}^{0} x^T(t+\theta_1)A_1^T\left(\int_{-h}^{0} U(\theta_1-\theta_2)A_1x(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\ = \int_{t-h}^{t} x^T(s_1)A_1^T\left(\int_{t-h}^{t} U(s_1-s_2)A_1x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) = x^T(t)A_1^T\int_{t-h}^{t} U(t-s)A_1x(s)\mathrm{d}s \\ -x^T(t-h)A_1^T\int_{t-h}^{t} U(t-h-s)A_1x(s)\mathrm{d}s$$

$$+\left(\int_{t-h}^{t} x^T(s)A_1^T U(s-t)\mathrm{d}s\right) A_1 x(t)$$

$$-\left(\int_{t-h}^{t} x^T(s)A_1^T U(s-t+h)\mathrm{d}s\right) A_1 x(t-h)$$

$$= \underbrace{\underbrace{2x^T(t)\int_{t-h}^{t} [U(s-t)A_1]^T A_1 x(s)\mathrm{d}s}}$$

$$\underbrace{-2x^T(t-h)A_1^T\int_{t-h}^{t} U(t-s-h)A_1 x(s)\mathrm{d}s}.$$

Now we consider the term

$$R_3(t) = 2x^T(t)\int_{-h}^{0}\left[\int_{-h}^{\theta} U(\xi-\theta)G(\xi)\mathrm{d}\xi\right] x(t+\theta)\mathrm{d}\theta$$

$$= 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{s-t} U^T(-\xi+s-t)G(\xi)\mathrm{d}\xi\right] x(s)\mathrm{d}s.$$

Its time derivative is given as

$$\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) = \underline{\underline{2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right]^T\int_{t-h}^{t}\left[\int_{-h}^{s-t} U(\xi-s+t)G(\xi)\mathrm{d}\xi\right] x(s)\mathrm{d}s}}$$

$$\underline{+\,2x^T(t)\left[\int_{-h}^{0} U(\xi)G(\xi)\mathrm{d}\xi\right] x(t)} \underline{-\,2x^T(t)U(0)\int_{t-h}^{t} G(s-t)x(s)\mathrm{d}s}$$

$$-2x^T(t)\int_{t-h}^{t}\left(\int_{-h}^{s-t}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\Big|_{\tau=-\xi+s-t}\right]^T G(\xi)\mathrm{d}\xi\right) x(s)\mathrm{d}s.$$

The time derivative of the next term

$$R_4(t) = 2\int_{-h}^{0} x^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} U(h+\theta_1-\theta_2+\xi)G(\xi)\mathrm{d}\xi \right) x(t+\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$
$$= 2\int_{t-h}^{t} x^T(s_1)A_1^T \left[\int_{t-h}^{t} \left(\int_{-h}^{s_2-t} U(h+s_1-s_2+\xi)G(\xi)\mathrm{d}\xi \right) x(s_2)\mathrm{d}s_2 \right] \mathrm{d}s_1$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}R_4(t) = 2x^T(t)\int_{t-h}^{t} \left(\int_{-h}^{s-t} A_1^T U(h+t-s+\xi)G(\xi)\mathrm{d}\xi \right) x(s)\mathrm{d}s$$
$$\underline{\underline{-2\left[A_1x(t-h)\right]^T \int_{t-h}^{t} \left(\int_{-h}^{s-t} U(t-s+\xi)G(\xi)\mathrm{d}\xi \right) x(s)\mathrm{d}s}}$$
$$\underbrace{\underbrace{+2\left[\int_{t-h}^{t} x^T(s) \left(\int_{-h}^{0} A_1^T U(h+s-t+\xi)G(\xi)\mathrm{d}\xi \right) \mathrm{d}s \right] x(t)}}$$
$$\underbrace{-2\int_{t-h}^{t}\int_{t-h}^{t} x^T(s_1)A_1^T U(h+s_1-t)G(s_2-t)x(s_2)\mathrm{d}s_1\mathrm{d}s_2}.$$

And, finally, the time derivative of the last term,

$$R_5(t) = \int_{-h}^{0} x^T(t+\theta_1) \left(\int_{-h}^{0} \left[\int_{-h}^{\theta_1} G^T(\xi_1) \left(\int_{-h}^{\theta_2} U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right.$$
$$\left. \times\ x(t+\theta_2)\mathrm{d}\theta_2 \right) \mathrm{d}\theta_1$$
$$= \int_{t-h}^{t} x^T(s_1) \left(\int_{t-h}^{t} \left[\int_{-h}^{s_1-t} G^T(\xi_1) \left(\int_{-h}^{s_2-t} U(s_1-s_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right.$$
$$\left. \times\ x(s_2)\mathrm{d}s_2 \right) \mathrm{d}s_1,$$

can be computed as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_5(t) &= x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{0} G^T(\xi_1)\left(\int_{-h}^{s-t} U(t-s-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]x(s)\mathrm{d}s\\
&\quad+\left(\int_{t-h}^{t} x^T(s)\left[\int_{-h}^{s-t} G^T(\xi_1)\left(\int_{-h}^{0} U(s_1-t-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]\mathrm{d}s\right)x(t)\\
&\quad-\int_{t-h}^{t} x^T(s_1)\left(\int_{t-h}^{t}\left[\int_{-h}^{s_2-t} G^T(s_1-t)U(-s_2+t+\xi)G(\xi)\mathrm{d}\xi\right]x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1\\
&\quad-\int_{t-h}^{t} x^T(s_1)\left(\int_{t-h}^{t}\left[\int_{-h}^{s_1-t} G^T(\xi)U(s_1-\xi-t)G(s_2-t)\mathrm{d}\xi\right]x(s_2)\mathrm{d}s_2\right)\mathrm{d}s_1\\
&= 2x^T(t)\int_{t-h}^{t}\left[\int_{-h}^{0} G^T(\xi_1)\left(\int_{-h}^{s-t} U(t-s-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1\right]x(s)\mathrm{d}s\\
&\quad-\underline{\underline{2\left[\int_{t-h}^{t} G(s_1-t)x(s_1)\mathrm{d}s_1\right]^T\left[\int_{t-h}^{t}\left(\int_{-h}^{s_2-t} U(-s_2+t+\xi)G(\xi)\mathrm{d}\xi\right)x(s_2)\mathrm{d}s_2\right]}}.
\end{aligned}$$

At the next stage we collect terms in the computed time derivatives. We start with the terms that are underlined by a single straight line. Their sum is

$$\begin{aligned}
S_1(t) &= 2x^T(t)U(0)A_0x(t)+2x^T(t)U(-h)A_1x(t)+2x^T(t)\left[\int_{-h}^{0} U(\xi)G(\xi)\mathrm{d}\xi\right]x(t)\\
&= x^T(t)\Bigg(U(0)A_0+U(-h)A_1+\int_{-h}^{0} U(\xi)G(\xi)\mathrm{d}\xi\\
&\qquad\qquad +A_0^TU(0)+A_1^TU^T(-h)+\int_{-h}^{0} G^T(\xi)U^T(\xi)\mathrm{d}\xi\Bigg)x(t)\\
&= -x^T(t)Wx(t).
\end{aligned}$$

Now we collect the terms underlined by a single curved line. Their sum is

$$S_2(t) = 2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - A_1x(t-h) - \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T \times \int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s$$

$$= 2x^T(t)A_0^T\int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s. \tag{4.8}$$

The sum of the terms underlined by a double curved line is equal to

$$S_3(t) = 2x^T(t)\int_{t-h}^{t}\left[-\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) + U(\tau-h)A_1 + \int_{-h}^{0} U(\tau+\xi)G(\xi)\mathrm{d}\xi\bigg|_{\tau=h+s-t}\right]^T A_1x(s)\mathrm{d}s$$

$$= -2x^T(t)A_0^T\int_{t-h}^{t} U(t-s-h)A_1x(s)\mathrm{d}s,$$

and it is cancelled by (4.8). The sum of the terms underlined by a double straight line is equal to

$$S_4(t) = 2\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t} - A_1x(t-h) - \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right]^T \times \int_{t-h}^{t}\left(\int_{-h}^{s-t} U(\xi-s+t)G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s$$

$$= 2x^T(t)A_0^T\int_{t-h}^{t}\left(\int_{-h}^{s-t} U(\xi-s+t)G(\xi)\mathrm{d}\xi\right)x(s)\mathrm{d}s. \tag{4.9}$$

Finally, the sum of the nonunderlined terms is

$$S_5(t) = 2x^T(t) \int_{t-h}^{t} \left[\int_{-h}^{s-t} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) + U(\tau - h) A_1 \right.\right.$$

$$\left.\left. + \int_{-h}^{0} U(\tau + \xi_2) G(\xi_2) \mathrm{d}\xi_2 \Bigg|_{\tau = -\xi + s - t} \right)^T G(\xi) \mathrm{d}\xi \right] x(s) \mathrm{d}s$$

$$= -2x^T(t) A_0^T \int_{t-h}^{t} \left(\int_{-h}^{s-t} U(t - s + \xi) G(\xi) \mathrm{d}\xi \right) x(s) \mathrm{d}s,$$

and it is cancelled by (4.9).

Summarizing our computations we arrive at the conclusion that the time derivative of the functional $v_0(\varphi)$ along the solutions of system (4.1) satisfies equality (4.2). □

4.3 Lyapunov Matrices: Existence Issue

In this section we study the existence issue for the Lyapunov matrices of system (4.1).

The characteristic function of the system is of the form

$$f(s) = \det \left(sI - A_0 - \mathrm{e}^{-sh} A_1 - \int_{-h}^{0} \mathrm{e}^{s\theta} G(\theta) \mathrm{d}\theta \right). \tag{4.10}$$

We define the matrix

$$H(s) = \left(sI - A_0 - \mathrm{e}^{-sh} A_1 - \int_{-h}^{0} \mathrm{e}^{s\theta} G(\theta) \mathrm{d}\theta \right)^{-1}.$$

The poles of $H(s)$ form the spectrum,

$$\Lambda = \{ s \mid f(s) = 0 \},$$

of the system. If system (4.1) satisfies the Lyapunov condition, then the spectrum can be divided into two parts; the first one, $\Lambda^{(+)}$, includes eigenvalues with positive real part, whereas the second one, $\Lambda^{(-)}$, includes eigenvalues with negative real part.

Theorem 4.2 ([26]). *Let system (4.1) satisfy the Lyapunov condition; then for any symmetric matrix W matrix*

$$\widetilde{U}(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(s)WH(-s)\mathrm{e}^{-\tau s}, s_0\right\}$$
$$+\sum_{s_0\in\Lambda^{(+)}} \text{Res}\left\{H^T(-s)WH(s)\mathrm{e}^{\tau s}, s_0\right\} \tag{4.11}$$

is a Lyapunov matrix of the system associated with W.

Proof. System (4.1) satisfies the Lyapunov condition, so neither the matrix $H(s)$ nor the matrix $H(-s)$ has a pole on the imaginary axis of the complex plane. Let ξ be a real number; then for sufficiently large $|\xi|$ the matrix $H^T(i\xi)WH(-i\xi)\mathrm{e}^{-i\tau\xi}$ is of the order $|\xi|^{-2}$. This means that the improper integral on the right-hand side of (4.11) is well defined for all real τ.

Part 1: The proof of symmetry property (4.4) coincides with that of Theorem 3.5.

Part 2: We address now the algebraic property. To check (4.5), we compute the following matrix:

$$\mathcal{O} = \widetilde{U}(0)A_0 + \widetilde{U}(-h)A_1 + \int_{-h}^{0}\widetilde{U}(\theta)G(\theta)\mathrm{d}\theta + A_0^T\widetilde{U}(0)$$
$$+A_1^T\widetilde{U}(h) + \int_{-h}^{0} G^T(\theta)\widetilde{U}(-\theta)\mathrm{d}\theta$$
$$= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left(H^T(\xi)WH(-\xi)\left[A_0 + \mathrm{e}^{\xi h}A_1 + \int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\right.$$
$$\left. + \left[A_0 + \mathrm{e}^{-\xi h}A_1 + \int_{-h}^{0}\mathrm{e}^{\xi\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(\xi)WH(-\xi)\right)\mathrm{d}\xi$$
$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(s)WH(-s)\left[A_0 + \mathrm{e}^{sh}A_1 + \int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}$$
$$+\sum_{s_0\in\Lambda^{(+)}}\text{Res}\left\{H^T(-s)WH(s)\left[A_0 + \mathrm{e}^{-sh}A_1 + \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right], s_0\right\}$$

$$+\sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{\left[A_0+\mathrm{e}^{-sh}A_1+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(s)WH(-s),s_0\right\}$$
$$+\sum_{s_0\in\Lambda^{(+)}} \mathrm{Res}\left\{\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]^T H^T(-s)WH(s),s_0\right\}.$$

It is a matter of simple calculation to verify the identities

$$H(s)\left[A_0+\mathrm{e}^{-sh}+\int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]=sH(s)-I$$

and

$$H(-s)\left[A_0+\mathrm{e}^{sh}A_1+\int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]=-sH(-s)-I.$$

Additionally,

$$\frac{\mathrm{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}WH(-\xi)\mathrm{d}\xi=\langle\lambda=-\xi\rangle=\frac{\mathrm{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}WH(\lambda)\mathrm{d}\lambda.$$

Now, the matrix $\mathcal{O}$ can be written as

$$\begin{aligned}\mathcal{O}=&-\frac{\mathrm{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W+WH(\xi)\right]\mathrm{d}\xi\\&-\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)W,s_0\right\}-\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)W,s_0\right\}\\&-\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(-s)W,s_0\right\}-\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(-s)W,s_0\right\}.\end{aligned}$$

Since the Lyapunov condition implies that no poles of the matrix $H(-s)$ lie in the set $\Lambda^{(+)}$, the last two sums on the right-hand side of the preceding equality disappear and

$$\mathcal{O}=-\frac{\mathrm{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty}\left[H^T(\xi)W+WH(\xi)\right]\mathrm{d}\xi-\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)W+WH(s),s_0\right\}.$$

The remainder of the proof of this part is identical to that of Theorem 3.5.

Part 3: Let us address property (4.3). For a given $\tau > 0$ we compute the matrix

$$
\begin{aligned}
F(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{U}(\tau) - \widetilde{U}(\tau)A_0 - \widetilde{U}(\tau - h)A_1 - \int_{-h}^{0}\widetilde{U}(\tau+\theta)G(\theta)\mathrm{d}\theta \\
&= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\left[-\xi I - A_0 - \mathrm{e}^{\xi h}A_1 - \int_{-h}^{0}\mathrm{e}^{-\xi\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau\xi}\mathrm{d}\xi \\
&\quad + \sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)WH(-s)\left[-sI - A_0 - \mathrm{e}^{sh}A_1 - \int_{-h}^{0}\mathrm{e}^{-s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{-\tau s}, s_0\right\} \\
&\quad + \sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(-s)WH(s)\left[sI - A_0 - \mathrm{e}^{-sh}A_1 - \int_{-h}^{0}\mathrm{e}^{s\theta}G(\theta)\mathrm{d}\theta\right]\mathrm{e}^{\tau s}, s_0\right\} \\
&= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)W\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)W\mathrm{e}^{-\tau s}, s_0\right\} \\
&\quad + \sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(-s)W\mathrm{e}^{\tau s}, s_0\right\}.
\end{aligned}
$$

Since the matrix $H(-s)$ has no poles in the set $\Lambda^{(+)}$, the sum

$$
\sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(-s)W\mathrm{e}^{\tau s}, s_0\right\} = 0_{n\times n},
$$

and we obtain

$$
F(\tau) = \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)W\mathrm{e}^{-\tau\xi}\mathrm{d}\xi + \sum_{s_0\in\Lambda^{(+)}}\mathrm{Res}\left\{H^T(s)W\mathrm{e}^{-\tau s}, s_0\right\}.
$$

The remainder of the proof of this part repeats that of Theorem 3.5. □

Corollary 4.1. *If system (4.1) is exponentially stable, then the Lyapunov matrix associated with a symmetric matrix W can be written as*

$$
\begin{aligned}
U(\tau) &= \frac{\text{V.P.}}{2\pi i}\int_{-i\infty}^{i\infty} H^T(\xi)WH(-\xi)\mathrm{e}^{-\tau\xi}\mathrm{d}\xi \\
&= \int_{0}^{\infty} K(t)WK(t+\tau)\mathrm{d}\tau.
\end{aligned}
$$

Here $K(t)$ is the fundamental matrix of the system.

4.4 Lyapunov Matrices: Uniqueness Issue

Here we study the uniqueness issue for Lyapunov matrices.

Lemma 4.1. *Given an integral-differential system of the form*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}z(\tau) = Az(\tau) + \int_0^{\tau} B(s,\tau)z(s)\mathrm{d}s \quad \tau \geq 0, \tag{4.12}$$

where A is a constant matrix and $B(s,\tau)$ is a continuous bivariate matrix, the only solution of the system that satisfies the condition $z(0) = 0$ is the trivial one.

Proof. Given $H > 0$, let us consider the system on the segment $[0,H]$. Compute the values

$$a = \|A\|, \quad b = \max_{(s,\tau)\in[0,H]^2} \|B(s,\tau)\|.$$

Integrating Eq. (4.12) from 0 to τ we obtain

$$z(\tau) = z(0) + A\int_0^{\tau} z(\xi)\mathrm{d}\xi + \int_0^{\tau}\left(\int_0^{\xi} B(s,\xi)z(s)\mathrm{d}s\right)\mathrm{d}\xi.$$

Thus,

$$\begin{aligned}\|z(\tau)\| &\leq \|z(0)\| + a\int_0^{\tau}\|z(\xi)\|\mathrm{d}\xi + b\int_0^{\tau}\left(\int_0^{\xi}\|z(s)\|\mathrm{d}s\right)\mathrm{d}\xi \\ &= \|z(0)\| + a\int_0^{\tau}\|z(\xi)\|\mathrm{d}\xi + b\int_0^{\tau}(\tau - s)\|z(s)\|\mathrm{d}s \\ &\leq \|z(0)\| + (a+bH)\int_0^{\tau}\|z(s)\|\mathrm{d}s.\end{aligned}$$

Now, by the Gronwall lemma,

$$\|z(\tau)\| \leq \mathrm{e}^{(a+bH)\tau}\|z(0)\|, \quad \tau \in [0,H].$$

In our case $z(0) = 0$, and we arrive at the conclusion that

$$z(\tau) = 0, \quad \tau \in [0,H].$$

□

Theorem 4.3. *Let system (4.1) satisfy the Lyapunov condition. Then for any symmetric matrix W there exists a unique Lyapunov matrix associated with W.*

Proof. Part 1: The fact that under the theorem condition matrix (4.11) satisfies Definition 4.1 was demonstrated in Theorem 4.2. Assume that for a given symmetric matrix W there exist two Lyapunov matrices, $U^{(1)}(\tau)$ and $U^{(2)}(\tau)$. Each of the matrices defines the corresponding functional, $v_0^{(j)}(\varphi)$, $j = 1,2$, of the form (4.7). The functionals satisfy the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0^{(j)}(x_t) = -x^T(t) W x(t), \quad j = 1,2,$$

along the solutions of system (4.1). The difference, $\Delta v(x_t) = v_0^{(2)}(x_t) - v_0^{(1)}(x_t)$, is such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta v(x_t) = 0, \quad t \geq 0,$$

and we obtain that for any $\varphi \in PC([-h,0],R^n)$ the identity

$$\Delta v(x_t(\varphi)) = \Delta v(\varphi), \quad t \geq 0, \tag{4.13}$$

holds along the solution $x(t,\varphi)$ of the system. In the case where system (4.1) is exponentially stable, $x_t(\varphi) \to 0_h$ as $t \to \infty$, and we arrive at the conclusion that

$$\Delta v(\varphi) = 0, \quad \varphi \in PC([-h,0],R^n). \tag{4.14}$$

If system (4.1) is not exponentially stable, then by the Lyapunov condition it has no eigenvalues on the imaginary axis of the complex plane, and there is a finite number of the eigenvalues in the open right half-plane of the complex plane. Let $\chi > 0$ be an upper bound for the real part of the eigenvalues in the right half-plane. Only a finite number of the system eigenvalues, $s_1, s_2, \ldots, s_N$, lies in the vertical stripe

$$Z = \{\ s \mid -\chi \leq \mathrm{Re}(s) \leq \chi\ \}$$

of the complex plane. Every solution $x(t,\varphi)$ of the system can be presented as the sum

$$x(t,\varphi) = x^{(1)}(t) + x^{(2)}(t),$$

where $x^{(1)}(t)$ corresponds to the part of the system spectrum that lies in Z and $x^{(2)}(t)$ corresponds to the rest of the spectrum, which lies to the left of the vertical line $\mathrm{Re}(s) = -\chi$.

The first term, $x^{(1)}(t)$, is a finite sum of the form

$$x^{(1)}(t) = \sum_{\ell=1}^{N} \mathrm{e}^{s_\ell t} p^{(\ell)}(t),$$

where $p^{(\ell)}(t)$ is a polynomial with vector coefficients of degree less than the multiplicity of s_ℓ as a zero of the system characteristic function (4.10), $\ell = 1, 2, \dots, N$.

The second term, $x^{(2)}(t)$, admits an upper estimate of the form

$$\left\| x^{(2)}(t) \right\| \leq c\mathrm{e}^{-(\chi+\varepsilon)t}, \quad t \geq 0. \tag{4.15}$$

Here c is a positive constant and ε is a small positive number.

The functional $\Delta v(x_t(\varphi))$ can be decomposed as follows:

$$\Delta v\left(x_t(\varphi)\right) = \Delta v\left(x_t^{(1)}\right) + 2\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) + \Delta v\left(x_t^{(2)}\right),$$

where

$$\begin{aligned}
\Delta z\left(x_t^{(1)}, x_t^{(2)}\right) &= \left[x^{(1)}(t)\right]^T \Delta U(0) x^{(2)}(t) \\
&+ \left[x^{(1)}(t)\right]^T \int_{-h}^{0} \Delta U(-h-\theta) A_1 x^{(2)}(t+\theta)\mathrm{d}\theta \\
&+ \left[x^{(2)}(t)\right]^T \int_{-h}^{0} \Delta U(-h-\theta) A_1 x^{(1)}(t+\theta)\mathrm{d}\theta \\
&+ \left[x^{(1)}(t)\right]^T \int_{-h}^{0} \left[\int_{-h}^{\theta} \Delta U(\xi-\theta) G(\xi)\mathrm{d}\xi\right] x^{(2)}(t+\theta)\mathrm{d}\theta \\
&+ \left[x^{(2)}(t)\right]^T \int_{-h}^{0} \left[\int_{-h}^{\theta} \Delta U(\xi-\theta) G(\xi)\mathrm{d}\xi\right] x^{(1)}(t+\theta)\mathrm{d}\theta \\
&+ \int_{-h}^{0} \left[x^{(1)}(t+\theta_1)\right]^T A_1^T \left(\int_{-h}^{0} \Delta U(\theta_1-\theta_2) A_1 x^{(2)}(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1 \\
&+ \int_{-h}^{0} \left[x^{(1)}(t+\theta_1)\right]^T A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2) G(\xi_2)\mathrm{d}\xi_2\right)\right. \\
&\left.\times x^{(2)}(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\
&+ \int_{-h}^{0} \left[x^{(2)}(t+\theta_1)\right]^T A_1^T \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2) G(\xi_2)\mathrm{d}\xi_2\right)\right.
\end{aligned}$$

$$\times x^{(1)}(t+\theta_2)\mathrm{d}\theta_2\Bigg]\mathrm{d}\theta_1$$

$$+\int_{-h}^{0}\left[x^{(1)}(t+\theta_1)\right]^T\left\{\int_{-h}^{0}\left(\int_{-h}^{\theta_1}G^T(\xi_1)\right.\right.$$

$$\left.\left.\times\left[\int_{-h}^{\theta_2}\Delta U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1\right)x^{(2)}(t+\theta_2)\mathrm{d}\theta_2\right\}\mathrm{d}\theta_1.$$

On the one hand, since $x^{(1)}(t)$ and $x^{(2)}(t)$ are solutions of system (4.1), $\Delta v\left(x_t^{(1)}\right)$ and $\Delta v\left(x_t^{(2)}\right)$ maintain constant values, and we conclude that $\Delta z\left(x_t^{(1)},x_t^{(2)}\right)$ is also constant. On the other hand, the choice of χ and inequality (4.15) guarantee that

$$\Delta v\left(x_t^{(2)}\right)\to 0,\ \text{and}\ \Delta z\left(x_t^{(1)},x_t^{(2)}\right)\to 0,\quad \text{as}\ t\to\infty.$$

This means that

$$\Delta v\left(x_t^{(2)}\right)=0,\ \text{and}\ \Delta z\left(x_t^{(1)},x_t^{(2)}\right)=0,\quad t\geq 0.$$

The first summand, $\Delta v\left(x_t^{(1)}\right)$, can be written as follows:

$$\Delta v\left(x_t^{(1)}\right)=\sum_{\ell=1}^{N}\sum_{r=1}^{N}\mathrm{e}^{(s_\ell+s_r)t}\alpha_{\ell r}(t),$$

where the functions $\alpha_{\ell r}(t)$, $\ell,r=1,2,\dots,N$, are of the form

$$\alpha_{\ell r}(t)=\left[p^{(\ell)}(t)\right]^T\Delta U(0)p^{(r)}(t)+2\left[p^{(\ell)}(t)\right]^T\int_{-h}^{0}\Delta U(-h-\theta)A_1\mathrm{e}^{s_r\theta}p^{(r)}(t+\theta)\mathrm{d}\theta$$

$$+2\left[p^{(\ell)}(t)\right]^T\int_{-h}^{0}\left[\int_{-h}^{\theta}\Delta U(\xi-\theta)G(\xi)\mathrm{d}\xi\right]\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta)\mathrm{d}\theta$$

$$+\int_{-h}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^TA_1^T\left(\int_{-h}^{0}\Delta U(\theta_1-\theta_2)A_1\mathrm{e}^{s_r\theta_2}p^{(r)}(t+\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$

$$+2\int_{-h}^{0}\left[\mathrm{e}^{s_\ell\theta_1}p^{(\ell)}(t+\theta_1)\right]^TA_1^T$$

$$\times \left[\int_{-h}^{0} \left(\int_{-h}^{\theta_2} \Delta U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right) \mathrm{e}^{s_r\theta_2} p^{(r)}(t+\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1$$

$$+ \int_{-h}^{0} \left[\mathrm{e}^{s_\ell \theta_1} p^{(\ell)}(t+\theta_1) \right]^T$$

$$\times \left\{ \int_{-h}^{0} \left(\int_{-h}^{\theta_1} G^T(\xi_1) \left[\int_{-h}^{\theta_2} \Delta U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \right] \mathrm{d}\xi_1 \right) \right.$$

$$\left. \times\, \mathrm{e}^{s_r\theta_2} p^{(r)}(t+\theta_2)\mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1.$$

A careful inspection of $\alpha_{\ell r}(t)$ reveals that it is a polynomial in t of degree less than the sum of the multiplicities of s_ℓ and s_r as zeros of the characteristic function (4.10). This means that identity (4.13) takes the form

$$\sum_{\ell=1}^{N} \sum_{r=1}^{N} \mathrm{e}^{(s_\ell+s_r)t} \alpha_{\ell r}(t) = \mathrm{e}^{0t} \Delta v(\varphi), \quad t \geq 0.$$

Part 2: According to the Lyapunov condition, no one of the sums $(s_\ell + s_r)$, $\ell, r \in \{1,2,\ldots,N\}$, is equal to zero. Therefore, by Lemma 3.8, we conclude from the last identity that equality (4.14) holds for any initial function $\varphi \in PC([-h,0],R^n)$.

Part 3: Equality (4.14) can be written as follows:

$$\begin{aligned} 0 &= \varphi^T(0)\Delta U(0)\varphi(0) \\ &\quad +2\varphi^T(0) \int_{-h}^{0} \left[\Delta U(-h-\theta)A_1 + \int_{-h}^{\theta} \Delta U(\xi-\theta)G(\xi)\mathrm{d}\xi \right] \varphi(\theta)\mathrm{d}\theta \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\theta_1) \left[A_1^T \Delta U(\theta_1-\theta_2)A_1 \right. \\ &\quad + 2\int_{-h}^{\theta_2} A_1^T \Delta U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2 \\ &\quad \left. + \int_{-h}^{\theta_1}\int_{-h}^{\theta_2} G^T(\xi_1)\Delta U(\theta_1-\theta_2-\xi_1+\xi_2)G(\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1 \right] \varphi(\theta_2)\mathrm{d}\theta_2\mathrm{d}\theta_1. \end{aligned} \tag{4.16}$$

For a given vector $\gamma \in R^n$ we define the initial function

$$\varphi^{(1)}(\theta) = \begin{cases} \gamma, & \text{for } \theta = 0 \\ 0, & \text{for } \theta \in [-h, 0) \end{cases}.$$

For this function equality (4.16) takes the form

$$\gamma^T \Delta U(0) \gamma = 0.$$

Since the last equality holds for any vector γ and the matrix $\Delta U(0)$ is symmetric, we conclude that

$$\Delta U(0) = 0_{n\times n}. \tag{4.17}$$

Now, given vectors $\gamma \in R^n$ and $\mu \in R^n$, let us select $\tau \in (0, h]$ and $\varepsilon > 0$ such that $-\tau + \varepsilon < 0$. Then we define the following initial function:

$$\varphi^{(2)}(\theta) = \begin{cases} \gamma, & \text{for } \theta = 0, \\ \mu, & \text{for } \theta \in [-\tau, -\tau + \varepsilon], \\ 0, & \text{for all other points of segment } [-h, 0]. \end{cases}$$

For this initial function equality (4.16) takes the form

$$0 = 2\varepsilon \gamma^T \left[\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi \right] \mu + o(\varepsilon),$$

where

$$\lim_{\varepsilon \to +0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Since γ and μ are arbitrary vectors and $\varepsilon > 0$ may be arbitrarily small, we conclude that the equality

$$\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi = 0_{n\times n}$$

holds for $\tau \in (0, h]$. By continuity arguments, we obtain

$$\Delta U(\tau - h) A_1 + \int_{-h}^{-\tau} \Delta U(\tau + \xi) G(\xi) \mathrm{d}\xi = 0_{n\times n}, \quad \tau \in [0, h]. \tag{4.18}$$

The matrix $\Delta U(\tau)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \Delta U(\tau - h)A_1 + \int_{-h}^{0} \Delta U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau \in [0,h].$$

Condition (4.18) makes it possible to present the preceding equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \int_{-\tau}^{0} \Delta U(\tau+\theta)G(\theta)\mathrm{d}\theta, \quad \tau \in [0,h]$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \int_{0}^{\tau} \Delta U(s)G(s-\tau)\mathrm{d}s, \quad \tau \in [0,h].$$

We are looking for a solution of this equation that satisfies condition (4.17). By Lemma 4.1, the solution is trivial, and

$$\Delta U(\tau) = U^{(2)}(\tau) - U^{(1)}(\tau) = 0_{n\times n}, \quad \tau \in [0,h]. \qquad \square$$

4.5 Lyapunov Matrices: Computational Issue

In this section we present some approaches to the computation of Lyapunov matrices for system (4.1). The main difficulty that appears in the computation of the matrices as solutions of delay equation (4.3) is the luck of the corresponding initial conditions. To some extent, symmetry condition (4.4) compensates this deficiency, but the computation problem remains complicated.

4.5.1 A Particular Case

In what follows we show that in the case of a polynomial matrix

$$G(\theta) = \sum_{j=1}^{m} \theta^{j-1}B_j, \tag{4.19}$$

where $B_1,\dots,B_m$ are constant $n\times n$ matrices, a Lyapunov matrix $U(\tau)$ may be computed as a solution of an auxiliary delay-free system of linear ordinary differential matrix equations. To this end, we first define the matrices

$$Z(\tau)=U(\tau),\quad V(\tau)=U(\tau-h),\quad \tau\in[0,h],$$

and the set of $2m$ auxiliary matrices

$$X_j(\tau)=\int_{-h}^{0}\theta^{j-1}U(\tau+\theta)\mathrm{d}\theta,\quad Y_j(\tau)=\int_{-h}^{0}\theta^{j-1}U(\tau-\theta-h)\mathrm{d}\theta,\quad j=1,\dots,m.$$

Then Eq. (4.3) can be written as

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}=Z(\tau)A_0+V(\tau)A_1+\sum_{j=1}^{m}X_j(\tau)B_j.$$

Now we compute the first derivative of the matrix $V(\tau)$:

$$\begin{aligned}\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau}&=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[U(h-\tau)\right]^T\\&=-\left[U(h-\tau)A_0+U(-\tau)A_1+\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta B_j\right]^T.\end{aligned}$$

Observe that

$$U(h-\tau)=V^T(\tau),\quad U(-\tau)=Z^T(\tau)$$

and

$$\int_{-h}^{0}\theta^{j-1}U(h-\tau+\theta)\mathrm{d}\theta=\left[\int_{-h}^{0}\theta^{j-1}U(\tau-\theta-h)\mathrm{d}\theta\right]^T=Y_j^T(\tau),\quad j=1,2,\dots,m,$$

hence

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau}=-A_0^TV(\tau)-A_1^TZ(\tau)-\sum_{j=1}^{m}B_j^TY_j(\tau).$$

The first derivatives of the matrices $X_1(\tau)$ and $Y_1(\tau)$ are

$$\frac{\mathrm{d}X_1(\tau)}{\mathrm{d}\tau}=U(\tau)-U(\tau-h)=Z(\tau)-V(\tau),$$

$$\frac{\mathrm{d}Y_1(\tau)}{\mathrm{d}\tau}=-V(\tau)+V(\tau+h)=-V(\tau)+Z(\tau).$$

Now, for $j = 2, \dots, m$,

$$\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau} = -(-h)^{j-1}U(\tau-h) - (j-1)\int_{-h}^{0} \theta^{j-2}U(\tau+\theta)\mathrm{d}\theta$$
$$= -(-h)^{j-1}V(\tau) - (j-1)X_{j-1}(\tau)$$

and

$$\frac{\mathrm{d}Y_j(\tau)}{\mathrm{d}\tau} = (-h)^{j-1}U(\tau) + (j-1)\int_{-h}^{0} \theta^{j-2}U(\tau-\theta-h)\mathrm{d}\theta$$
$$= (-h)^{j-1}Z(\tau) + (j-1)Y_{j-1}(\tau).$$

As a result, we arrive at the conclusion that the set of matrices

$$\{Z(\tau), V(\tau), X_1(\tau), \dots, X_m(\tau), Y_1(\tau), \dots, Y_m(\tau)\}$$

satisfies the following delay-free system of $2(m+1)$ ordinary differential matrix equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}Z = ZA_0 + VA_1 + \sum\limits_{j=1}^{m} X_j B_j, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}V = -A_1^T Z - A_0^T V - \sum\limits_{j=1}^{m} B_j^T Y_j, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}X_1 = Z - V, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y_1 = Z - V, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}X_j = -(-h)^{j-1}V - (j-1)X_{j-1}, \quad j = 2, \dots, m, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y_j = (-h)^{j-1}Z + (j-1)Y_{j-1}, \quad j = 2, \dots, m. \end{cases} \tag{4.20}$$

Lemma 4.2. *The spectrum of system (4.20) is symmetrical with respect to the origin of the complex plane.*

Proof. A complex number s_0 is an eigenvalue of system (4.20) if and only if there exists a nontrivial set of $n \times n$ matrices

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \dots, X_m^{(0)}, Y_1^{(0)}, \dots, Y_m^{(0)}\right\}$$

satisfying the following system of matrix equations:

$$\begin{cases} s_0 Z^{(0)} = Z^{(0)} A_0 + V^{(0)} A_1 + \sum_{j=1}^{m} X_j^{(0)} B_j, \\ s_0 V^{(0)} = -A_1^T Z^{(0)} - A_0^T V^{(0)} - \sum_{j=1}^{m} B_j^T Y_j^{(0)}, \\ s_0 X_1^{(0)} = Z^{(0)} - V^{(0)}, \\ s_0 Y_1^{(0)} = Z^{(0)} - V^{(0)}, \\ s_0 X_j^{(0)} = -(-h)^{j-1} V^{(0)} - (j-1) X_{j-1}^{(0)}, \quad j = 2, \dots, m, \\ s_0 Y_j^{(0)} = (-h)^{j-1} Z^{(0)} + (j-1) Y_{j-1}^{(0)}, \quad j = 2, \dots, m. \end{cases} \tag{4.21}$$

It is easy to check that the matrices

$$\widetilde{Z}^{(0)} = \left(V^{(0)}\right)^T, \quad \widetilde{V}^{(0)} = \left(Z^{(0)}\right)^T, \quad \widetilde{X}_j^{(0)} = \left(Y_j^{(0)}\right)^T,$$
$$\widetilde{Y}_j^{(0)} = \left(X_j^{(0)}\right)^T, \quad j = 1, \dots, m,$$

satisfy the system

$$\begin{cases} -s_0 \widetilde{Z}^{(0)} = \widetilde{Z}^{(0)} A_0 + \widetilde{V}^{(0)} A_1 + \sum_{j=1}^{m} \widetilde{X}_j^{(0)} B_j, \\ -s_0 \widetilde{V}^{(0)} = -A_1^T \widetilde{Z}^{(0)} - A_0^T \widetilde{V}^{(0)} - \sum_{j=1}^{m} B_j^T \widetilde{Y}_j^{(0)}, \\ -s_0 \widetilde{X}_1^{(0)} = \widetilde{Z}^{(0)} - \widetilde{V}^{(0)}, \\ -s_0 \widetilde{Y}_1^{(0)} = \widetilde{Z}^{(0)} - \widetilde{V}^{(0)}, \\ -s_0 \widetilde{X}_j^{(0)} = -(-h)^{j-1} \widetilde{V}^{(0)} - (j-1) \widetilde{X}_{j-1}^{(0)}, \quad j = 2, \dots, m, \\ -s_0 \widetilde{Y}_j^{(0)} = (-h)^{j-1} \widetilde{Z}^{(0)} + (j-1) \widetilde{Y}_{j-1}^{(0)}, \quad j = 2, \dots, m. \end{cases}$$

This means that $-s_0$ belongs to the spectrum of system (4.20). □

The solution of system (4.20) defined by the matrix $U(\tau)$ satisfies also the following set of boundary value conditions:

$$Z(0) = V^T(h),$$

$$X_j(0) = \int_{-h}^{0} \theta^{j-1} U(\theta) \mathrm{d}\theta = \left[\int_{-h}^{0} \theta^{j-1} U(h-\theta-h) \mathrm{d}\theta \right]^T$$
$$= Y_j^T(h), \quad j = 1, \ldots, m,$$

$$Y_j(0) = \int_{-h}^{0} \theta^{j-1} U(-\theta-h) \mathrm{d}\theta = \left[\int_{-h}^{0} \theta^{j-1} U(h+\theta) \mathrm{d}\theta \right]^T$$
$$= X_j^T(h), \quad j = 1, \ldots, m,$$

as well as the algebraic condition

$$Z(0)A_0 + V(0)A_1 + \sum_{j=1}^{m} X_j(0)B_j + A_0^T V(h) + A_1^T Z(h) + \sum_{j=1}^{m} B_j^T Y_j(h) = -W.$$

We finally arrive at the following statement.

Theorem 4.4. *Given a time-delay system (4.1), where matrix $G(\theta)$ is of the form (4.19), let $U(\tau)$ be a Lyapunov matrix of the system associated with the matrix W. Then the set of matrices*

$$\{Z(\tau), V(\tau), X_1(\tau), \ldots, X_m(\tau), Y_1(\tau), \ldots, Y_m(\tau)\}$$

is a solution of system (4.20) that satisfies the boundary value conditions

$$\begin{cases} Z(0) = V^T(h), \\ X_j(0) = Y_j^T(h), \text{ and } Y_j(0) = X_j^T(h), \quad j = 1, \ldots, m, \\ Z(0)A_0 + V(0)A_1 + \sum_{j=1}^{m} X_j(0)B_j + A_0^T V(h) + A_1^T Z(h) \\ \quad + \sum_{j=1}^{m} B_j^T Y_j(h) = -W. \end{cases} \tag{4.22}$$

There exist some relations between the auxiliary matrices that are described in the following lemma.

Lemma 4.3. *The auxiliary matrices $X_j(\tau)$ and $Y_j(\tau)$, $j = 1, \ldots, m$, satisfy the relations*

$$X_j(\tau) = (-1)^{j-1} \sum_{k=0}^{j-1} \frac{(j-1)!}{k!(j-k-1)!} h^k Y_{j-k}(\tau), \quad j = 1, \ldots, m,$$

and

$$Y_j(\tau) = (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k X_{j-k}(\tau), \quad j=1,\dots,m.$$

Proof. The first set of relations can be easily obtained as follows:

$$\begin{aligned}
X_j(\tau) &= \int_{-h}^{0}\theta^{j-1}U(\tau+\theta+h-h)\mathrm{d}\theta \\
&= \langle \xi = -\theta - h\rangle = \int_{-h}^{0}(-h-\xi)^{j-1}U(\tau-\xi-h)\mathrm{d}\xi \\
&= (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k\int_{-h}^{0}\xi^{j-k-1}U(\tau-\xi-h)\mathrm{d}\xi \\
&= (-1)^{j-1}\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}h^k Y_{j-k}(\tau).
\end{aligned}$$

The second set of relations can be obtained in a similar way. □

Lemma 4.3 provides a reduction of system (4.20). We have the sum

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{j=1}^{m}(-1)^{j-1}B_j^T\left(\sum_{k=0}^{j-1}\frac{(j-1)!}{k!\,(j-k-1)!}h^k X_{j-k}(\tau)\right).$$

If we define the matrix

$$B(\xi) = \sum_{j=1}^{m}(-\xi)^{j-1}B_j^T,$$

then we obtain the sum

$$\sum_{j=1}^{m}B_j^T Y_j(\tau) = \sum_{k=1}^{m}\left[\frac{1}{(k-1)!}B^{(k-1)}(h)\right]X_k(\tau),$$

where

$$B^{(k-1)}(h) = \left.\frac{\mathrm{d}^{k-1}B(\xi)}{\mathrm{d}\xi^{k-1}}\right|_{\xi=h}, \quad k=1,2,\dots,m.$$

The second equation of system (4.20) takes the form

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau} = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{k=1}^{m}\left[\frac{1}{(k-1)!}B^{(k-1)}(h)\right]X_k(\tau).$$

Therefore, system (4.20) is reduced to the following system of $(m+2)$ matrix equations:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}Z = ZA_0 + VA_1 + \sum_{j=1}^{m} X_j B_j, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}V = -A_1^T Z - A_0^T V - \sum_{j=1}^{m} \left[\frac{1}{(k-1)!} B^{(k-1)}(h) \right] X_k(\tau), \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_1 = Z - V, \\ \\ \frac{\mathrm{d}}{\mathrm{d}\tau}X_j = -(-h)^{j-1}V - (j-1)X_{j-1}, \quad j = 2, \dots, m. \end{cases} \tag{4.23}$$

In a similar way, the set of boundary value conditions (4.22) is reduced to the next one:

$$\begin{cases} Z(0) = V^T(h), \\ X_k(0) = (-1)^{k-1} \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-j-1)!} h^j X_{k-j}^T(h), \quad k = 1, \dots, m, \\ Z(0)A_0 + A_0^T Z(0) + V(0)A_1 + A_1^T V^T(0) + \sum_{j=1}^{m} \left[X_j(0)B_j + B_j^T X_j^T(0) \right] = -W. \end{cases}$$

In the following statement we show that the spectrum of system (4.1) and that of system (4.20) are connected.

Theorem 4.5. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.19), let s_0 be an eigenvalue of the time-delay system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of delay-free system (4.20).*

Proof. The characteristic matrix of system (4.1) is

$$G(s) = sI - A_0 - \mathrm{e}^{-hs}A_1 - \sum_{k=1}^{m} f^{(k-1)}(s)B_k,$$

where

$$f^{(0)}(s) = \frac{1-\mathrm{e}^{-hs}}{s}, \text{ and } f^{(k-1)}(s) = \frac{\mathrm{d}^{k-1} f(s)}{\mathrm{d}s^{k-1}}, \quad ,k = 2, \dots, m.$$

Because s_0 and $-s_0$ are eigenvalues of the system, there exist nonzero vectors γ and μ such that

$$\gamma^T G(s_0) = 0, \quad G^T(-s_0)\mu = 0. \tag{4.24}$$

A complex number s_0 belongs to the spectrum of system (4.20) if and only if there exists a nontrivial set of $n \times n$ matrices

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \dots, X_m^{(0)}, Y_1^{(0)}, \dots, Y_m^{(0)}\right\}$$

that satisfies (4.21). Multiplying the first equality in (4.24) by μ from the left and the second equality by $-\mathrm{e}^{-hs_0}\gamma^T$ from the right we obtain

$$s_0\mu\gamma^T - \mu\gamma^T A_0 - \mathrm{e}^{-hs_0}\mu\gamma^T A_1 - \sum_{k=1}^{m} f^{(k-1)}(s_0)\mu\gamma^T B_k = 0_{n\times n}$$

and

$$s_0\mathrm{e}^{-hs_0}\mu\gamma^T + A_0^T\mathrm{e}^{-hs_0}\mu\gamma^T + A_1^T\mu\gamma^T + \sum_{k=1}^{m}\mathrm{e}^{-hs_0} f^{(k-1)}(-s_0)B_k^T\mu\gamma^T = 0_{n\times n}.$$

If we introduce the nontrivial matrices

$$Z^{(0)} = \mu\gamma^T,\ V^{(0)} = \mathrm{e}^{-hs_0}\mu\gamma^T,$$

$$X_j^{(0)} = f^{(j-1)}(s_0)\mu\gamma^T,\ Y_j^{(0)} = \mathrm{e}^{-hs_0} f^{(j-1)}(-s_0)\mu\gamma^T, \quad j = 1, \dots, m,$$

then the preceding equalities take the form

$$s_0 Z^{(0)} - Z^{(0)}A_0 - V^{(0)}A_1 - \sum_{k=1}^{m} X_k^{(0)}B_k = 0_{n\times n},$$

$$s_0 V^{(0)} + A_0^T V^{(0)} + A_1^T Z^{(0)} + \sum_{k=1}^{m} B_k^T Y_k^{(0)} = 0_{n\times n}.$$

In other words, the matrices satisfy the first two equations of system (4.21). To verify that these matrices satisfy the remaining $2(m+1)$ matrix equations in (4.21), we multiply the identity

$$sf^{(0)}(s) = 1 - \mathrm{e}^{-hs}$$

by the matrix $\mu\gamma^T$ and set $s = s_0$; then we obtain the equality

$$s_0 X_1^{(0)} = Z^{(0)} - V^{(0)}.$$

Now we compute the derivatives

$$\begin{aligned}\frac{\mathrm{d}^{j-1}}{\mathrm{d}s^{j-1}}\left[sf^{(0)}(s)\right] &= sf^{(j-1)}(s) + (j-1)f^{(j-2)}(s)\\ &= -(-h)^{j-1}\mathrm{e}^{-hs}, \quad j = 2, \dots, m-1.\end{aligned}$$

This means that the following identities hold:

$$sf^{(j-1)}(s) = -(-h)^{j-1}\mathrm{e}^{-hs} - (j-1)f^{(j-2)}(s), \quad j = 2, \ldots, m.$$

If we multiply these identities by the matrix $\mu\gamma^T$ and set $s = s_0$, then we obtain the desired set of matrix equalities

$$s_0 X_j^{(0)} = -(-h)^{j-1}V^{(0)} - (j-1)X_{j-1}^{(0)}, \quad j = 2, \ldots, m.$$

In a similar way one can verify the remaining equalities in (4.21).

It is evident that the set of matrices introduced previously,

$$\left\{Z^{(0)}, V^{(0)}, X_1^{(0)}, \ldots, X_m^{(0)}, Y_1^{(0)}, \ldots, Y_m^{(0)}\right\},$$

is not trivial. Therefore, the complex value s_0 belongs to the spectrum of system (4.20). The same is true for $-s_0$. □

Remark 4.2. The statement remains valid if we replace in Theorem 4.5 system (4.20) by the reduced system (4.23).

4.5.2 A Special Case

Now we consider the case where the matrix $G(\theta)$ is of the form

$$G(\theta) = \sum_{j=1}^{m} \eta_j(\theta) B_j, \tag{4.25}$$

where $B_1, \ldots, B_m$ are given $n \times n$ matrices and the scalar functions $\eta_1(\theta), \ldots, \eta_m(\theta)$ are such that

$$\frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = \sum_{k=1}^{m} \alpha_{jk}\eta_k(\theta), \quad j = 1, \ldots, m.$$

Remark 4.3. In the previous subsection we had $\eta_j(\theta) = \theta^{j-1}$, $j = 1, \ldots, m$. These functions satisfy the equations

$$\frac{\mathrm{d}\eta_1(\theta)}{\mathrm{d}\theta} = 0, \quad \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} = (j-1)\,\eta_{j-1}(\theta), \quad j = 2, \ldots, m.$$

The time-delay matrix equation for $U(\tau)$ is now of the form

$$\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = U(\tau)A_0 + U(\tau - h)A_1 + \sum_{j=1}^{m} \int_{-h}^{0} \eta_j(\theta) U(\tau + \theta) B_j \mathrm{d}\theta, \quad \tau \geq 0. \tag{4.26}$$

Let us define for $\tau \in [0,h]$ the matrices $Z(\tau) = U(\tau)$, $V(\tau) = U(\tau - h)$, and

$$X_j(\tau) = \int_{-h}^{0} \eta_j(\theta) U(\tau+\theta) \mathrm{d}\theta, \quad Y_j(\tau) = \int_{-h}^{0} \eta_j(\theta) U(\tau-\theta-h) \mathrm{d}\theta, \quad j = 1, \ldots, m.$$

Then Eq. (4.26) has the form

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau} = Z(\tau)A_0 + V(\tau)A_1 + \sum_{j=1}^{m} X_j(\tau)B_j, \quad \tau \in [0,h],$$

and

$$\frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau} = -A_1^T Z(\tau) - A_0^T V(\tau) - \sum_{j=1}^{m} B_j^T Y_j(\tau).$$

Now

$$\begin{aligned}
\frac{\mathrm{d}X_j(\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{-h}^{0} \eta_j(\theta) U(\tau+\theta) \mathrm{d}\theta \right) \\
&= \eta_j(0)U(\tau) - \eta_j(-h)U(\tau-h) - \int_{-h}^{0} \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} U(\tau+\theta) \mathrm{d}\theta \\
&= \eta_j(0)Z(\tau) - \eta_j(-h)V(\tau) - \sum_{k=1}^{m} \alpha_{jk} X_k(\tau), \quad j = 1, \ldots, m,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}Y_j(\tau)}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_{-h}^{0} \eta_j(\theta) U(\tau-\theta-h) \mathrm{d}\theta \right) \\
&= -\eta_j(0)U(\tau-h) + \eta_j(-h)U(\tau) + \int_{-h}^{0} \frac{\mathrm{d}\eta_j(\theta)}{\mathrm{d}\theta} U(\tau-\theta-h) \mathrm{d}\theta \\
&= \eta_j(-h)Z(\tau) - \eta_j(0)V(\tau) + \sum_{k=1}^{m} \alpha_{jk} Y_k(\tau), \quad j = 0, 1, \ldots, m.
\end{aligned}$$

We arrive at the following system of delay-free matrix equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}Z = ZA_0 + VA_1 + \sum\limits_{j=1}^{m} X_j B_j, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}V = -A_0^T V - A_1^T Z - \sum\limits_{j=1}^{m} B_j^T Y_j, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}X_j = \eta_j(0)Z - \eta_j(-h)V - \sum\limits_{k=1}^{m} \alpha_{jk} X_k, \quad j = 1,\dots,m, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}Y_j = \eta_j(-h)Z - \eta_j(0)V + \sum\limits_{k=1}^{m} \alpha_{jk} Y_k, \quad j = 1,\dots,m. \end{cases} \tag{4.27}$$

Because the auxiliary matrices $Z(\tau)$, $V(\tau)$, $X_j(\tau)$, $Y_j(\tau)$, $j = 1,\dots,m$, satisfy the boundary value conditions (4.22), the following result holds.

Theorem 4.6. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.25), let $U(\tau)$ be a Lyapunov matrix of the delay system associated with the matrix W. Then the matrices $Z(\tau)$, $V(\tau)$, $X_j(\tau)$, $Y_j(\tau)$, $j = 1,\dots,m$, define a solution of the auxiliary boundary value problem (4.27), (4.22).*

For the special case the statement of Theorem 4.5 remains true.

Theorem 4.7. *Given a time-delay system (4.1), where the matrix $G(\theta)$ is of the form (4.25), let s_0 be an eigenvalue of the time-delay system such that $-s_0$ is also an eigenvalue of the system. Then s_0 belongs to the spectrum of system (4.27).*

Sometimes it is possible to perform a reduction of delay-free system (4.27). This happens when the functions $\eta_j(\theta)$, $j = 1,\dots,m$, satisfy the conditions

$$\eta_j(-\theta - h) = \sum_{k=1}^{m} \gamma_{jk}\eta_k(\theta), \quad \theta \in [-h, 0], \quad j = 1,\dots,m.$$

In this case

$$\begin{aligned} Y_j(\tau) &= \int_{-h}^{0} \eta_j(\theta)U(\tau - \theta - h)\mathrm{d}\theta = \langle \xi = -\theta - h \rangle \\ &= \int_{-h}^{0} \eta_j(-\xi - h)U(\tau + \xi)\mathrm{d}\xi = \sum_{k=1}^{m} \gamma_{jk} \int_{-h}^{0} \eta_k(\xi)U(\tau + \xi)\mathrm{d}\xi \\ &= \sum_{k=1}^{m} \gamma_{jk} X_k(\tau), \quad j = 1,\dots,m, \end{aligned}$$

and one can exclude the matrices $Y_j(\tau)$ of system (4.27), as well as those of boundary value conditions (4.22).

4.5.3 Numerical Scheme

In this section we propose a numerical scheme to approximate Lyapunov matrices.

Given a symmetric matrix W, we are looking for an approximate initial condition for the Lyapunov matrix associated with W of the form

$$\Phi(\theta) = \sum_{j=0}^{m} \theta^j \Phi_j, \quad \theta \in [-h, 0],$$

where Φ_j, $j = 0, 1, \ldots, m$, are $n \times n$ constant matrices. We address symmetry property (4.4). According to this property,

$$\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} = (-1)^k \left[\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0} \right]^T, \quad k \geq 0. \tag{4.28}$$

Here $\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0}$ and $\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0}$ stand for the right-hand side and the left-hand side derivatives of $U(\tau)$ of the order k at $\tau = 0$, respectively. It follows from (4.3) that

$$\left.\frac{\mathrm{d}^{k+1} U(\tau)}{\mathrm{d}\tau^{k+1}}\right|_{\tau=+0} = \left(\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} \right) A_0 + \left(\left.\frac{\mathrm{d}^k U(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-h+0} \right) A_1$$
$$+ \int_{-h}^{0} \frac{\mathrm{d}^k U(\theta)}{\mathrm{d}\theta^k} G(\theta) \mathrm{d}\theta, \quad k \geq 0.$$

If we replace $U(\theta)$ in the preceding equality by $\Phi(\theta)$, then we obtain that

$$\left.\frac{\mathrm{d}^{k+1} \widehat{U}(\tau)}{\mathrm{d}\tau^{k+1}}\right|_{\tau=+0} = \left(\left.\frac{\mathrm{d}^k \widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} \right) A_0 + \sum_{j=k}^{m} j(j-1)\ldots(j-k-1)\Phi_j$$
$$\times \left[(-h)^{j-k} A_1 + \int_{-h}^{0} \theta^{j-k} G(\theta) \mathrm{d}\theta \right], \quad k \geq 0.$$

For $k = 0$ we have

$$\left.\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} = \Phi_0 A_0 + \sum_{j=0}^{m} (-h)^j \Phi_j A_1 + \sum_{j=0}^{m} \Phi_j \int_{-h}^{0} \theta^j G(\theta) \mathrm{d}\theta$$
$$= \sum_{j=0}^{m} \Phi_j L_j^{(1)},$$

where

$$L_0^{(1)} = A_0 + A_1 + \int_{-h}^{0} G(\theta)\mathrm{d}\theta, \quad L_j^{(1)} = (-h)^j A_1 + \int_{-h}^{0} \theta^j G(\theta)\mathrm{d}\theta, \quad j = 1,2,\ldots,m.$$

For $k = 1$

$$\left.\frac{\mathrm{d}^2\widehat{U}(\tau)}{\mathrm{d}\tau^2}\right|_{\tau=+0} = \left(\left.\frac{\mathrm{d}\widehat{U}(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0}\right) A_0 + \sum_{j=1}^{m} j\Phi_j \left[(-h)^{j-1}A_1 + \int_{-h}^{0} \theta^{j-1} G(\theta)\mathrm{d}\theta\right]$$
$$= \sum_{j=0}^{m} \Phi_j L_j^{(2)},$$

where

$$L_0^{(2)} = L_0^{(1)} A_0, \quad L_j^{(2)} = L_j^{(1)} A_0 + jL_{j-1}^{(1)}, \quad j = 1,2,\ldots,m.$$

On the one hand, repeating this process we obtain the following expressions for the right-hand-side derivatives:

$$\left.\frac{\mathrm{d}^k\widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=+0} = \sum_{j=0}^{m} \Phi_j L_j^{(k)}, \quad k = 1,2,\ldots,m.$$

Here

$$L_j^{(k)} = \begin{cases} L_j^{(k-1)} A_0, & j = 0,1,\ldots,k-2, \\ L_j^{(k-1)} A_0 + j(j-1)\ldots(j-k+2)L_{j-k+1}^{(1)}, & j = k-1,k,\ldots,m. \end{cases}$$

On the other hand, the left-hand-side derivatives at $\tau = 0$ are of the form

$$\left.\frac{\mathrm{d}^k\widehat{U}(\tau)}{\mathrm{d}\tau^k}\right|_{\tau=-0} = k!\Phi_k, \quad k = 1,2,\ldots,m.$$

Substituting these expressions into (4.28) we obtain a system of $(m+1)$ matrix equations for $(m+1)$ matrices Φ_j, $j = 0,1,\ldots,m$:

$$\begin{cases} (-1)^k k!\Phi_k^T = \sum_{j=0}^{m} \Phi_j L_j^{(k)}, & k = 1,2,\ldots,m, \\ \sum_{j=0}^{m} \Phi_j L_j^{(1)} + \Phi_1^T = -W. \end{cases} \tag{4.29}$$

The last equation of this system is property (4.6), written in terms of the matrices.

If system (4.29) admits a solution, Φ_j, $j=0,1,\ldots,m$, then we arrive at the matrix

$$\Phi(\theta)=\sum_{j=0}^{m}\theta_j^j\Phi_j,\quad \theta\in[-h,0].$$

The desired approximation of the Lyapunov matrix associated with W is now of the form

$$\widehat{U}(\tau)=[\Phi(-\tau)]^T,\quad \tau\in[0,h].$$

4.6 Complete Type Functionals

Here we define a new class of quadratic functionals. But first we prove the statement.

Theorem 4.8. *Define for the given symmetric matrices W_0, W_1, and W_2 the functional*

$$\begin{aligned} w(\varphi) &= \varphi^T(0)W_0\varphi(0)+\varphi^T(-h)W_1\varphi(-h) \\ &\quad +\int_{-h}^{0}\varphi^T(\theta)W_2\varphi(\theta)\mathrm{d}\theta,\quad \varphi\in PC([-h,0],R^n). \end{aligned} \tag{4.30}$$

Let there exist a Lyapunov matrix $U(\tau)$ associated with matrix

$$W=W_0+W_1+hW_2.$$

This Lyapunov matrix defines the functional $v_0(\varphi)$; see (4.7). The time derivative of the functional

$$v(\varphi)=v_0(\varphi)+\int_{-h}^{0}\varphi^T(\theta)\left[W_1+(h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta,\quad \varphi\in PC([-h,0],R^n), \tag{4.31}$$

along the solutions of system (4.1) is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x_t)=-w(x_t),\quad t\geq 0.$$

Proof. The proof is similar to that of Theorem 3.4. □

Definition 4.2. We say that functional (4.31) is of the complete type if the matrices W_0, W_1, and W_2 are positive definite.

Lemma 4.4. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, the complete type functional (4.31) admits a lower bound of the form*

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n),$$

where β_1 and β_2 are positive constants.

Proof. We define an auxiliary functional of the form

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_1 \|\varphi(0)\|^2 - \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where β_1 and β_2 are assumed to be positive constants. The time derivative of the functional along the solution of system (4.1) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 x^T(t)\left[A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)\mathrm{d}\theta\right] \\ &+ \beta_2 x^T(t)x(t) - \beta_2 x^T(t-h)x(t-h), \quad t \geq 0.\end{aligned}$$

The functional $\widetilde{w}(\varphi)$ admits a lower estimation of the form

$$\widetilde{w}(\varphi) \geq [\varphi^T(0), \varphi^T(-h)]R_1(\beta_1,\beta_2)\begin{bmatrix}\varphi(0)\\ \varphi(-h)\end{bmatrix} + \int_{-h}^{0} \varphi^T(\theta)R_2(\theta,\beta_1)\varphi(\theta)\mathrm{d}\theta,$$

where

$$R_1(\beta_1,\beta_2) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \beta_1 \begin{pmatrix} A_0 + A_0^T - hI & A_1 \\ A_1^T & 0_{n\times n} \end{pmatrix} + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}$$

and

$$R_2(\theta,\beta_1) = W_2 - \beta_1 G^T(\theta)G(\theta).$$

The matrices W_0, W_1, and W_2 are positive definite, so there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that the following inequalities hold

$$R_1(\beta_1,\beta_2) \geq 0, \quad R_2(\theta,\beta_1) \geq 0, \quad \theta \in [-h,0].$$

For these β_1 β_2 we have

$$\widetilde{w}(\varphi) \geq 0, \quad \varphi \in PC([-h,0],R^n).$$

Therefore,

$$\widetilde{v}(\varphi) = \int_0^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0, \quad \varphi \in PC([-h,0],R^n),$$

and we arrive at the conclusion that

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \qquad \square$$

Corollary 4.2. *If we assume $\beta_2 = 0$ and set $\beta_1 = \alpha_1$, then there exists $\alpha_1 > 0$ such that the following inequalities hold:*

$$R_1(\alpha_1,0) \geq 0, \quad R_2(\theta,\alpha_1) \geq 0, \quad \theta \in [-h,0].$$

Therefore, the complete type functional $v(\varphi)$ admits a lower bound of the form

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \tag{4.32}$$

Lemma 4.5. *Let system (4.1) satisfy the Lyapunov condition. Given the symmetric matrices W_0, W_1, and W_2, there exist positive constants δ_1 and δ_2 such that functional (4.31) admits an upper bound of the form*

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n). \tag{4.33}$$

Proof. The Lyapunov condition implies that there exists a Lyapunov matrix $U(\tau)$ associated with matrix $W = W_0 + W_1 + hW_2$. We define the following constants:

$$\nu = \max_{\tau\in[0,h]} \|U(\tau)\|, \quad a = \|A_1\|, \quad g = \max_{\theta\in[-h,0]} \|G(\theta)\|.$$

Now we estimate the summands that constitute functional (4.31). The sum of the first two terms admits the upper bound

$$R_1+R_2 = \varphi^T(0)U(0)\varphi(0)+2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta$$
$$\leq \nu(1+ha)\|\varphi(0)\|^2+\nu a\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.$$

The sum of the next two terms can be estimated as follows:

$$R_3+R_4 = 2\varphi^T(0)\int_{-h}^{0}\left(\int_{-h}^{\theta}U(\xi-\theta)G(\xi)\mathrm{d}\xi\right)\varphi(\theta)\mathrm{d}\theta$$
$$+\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0}U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$
$$\leq 2\nu g\|\varphi(0)\|\int_{-h}^{0}(h+\theta)\|\varphi(\theta)\|\mathrm{d}\theta+\nu a^2\left(\int_{-h}^{0}\|\varphi(\theta)\|\mathrm{d}\theta\right)^2$$
$$\leq \nu gh\|\varphi(0)\|^2+\nu h\left(\frac{gh}{3}+a^2\right)\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.$$

The fifth term admits the estimation

$$R_5 = 2\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left(\int_{-h}^{0}\left[\int_{-h}^{\theta_2}U(h+\theta_1-\theta_2+\xi_2)G(\xi_2)\mathrm{d}\xi_2\right]\varphi(\theta_2)\mathrm{d}\theta_2\right)\mathrm{d}\theta_1$$
$$\leq 2\nu ag\left(\int_{-h}^{0}\|\varphi(\theta_1)\|\mathrm{d}\theta_1\right)\left(\int_{-h}^{0}(h+\theta_2)\|\varphi(\theta_2)\|\mathrm{d}\theta_2\right)$$
$$\leq 2\nu ag\left(\sqrt{h\int_{-h}^{0}\|\varphi(\theta_1)\|^2\mathrm{d}\theta_1}\right)\left(\sqrt{\frac{h^3}{3}\int_{-h}^{0}\|\varphi(\theta_2)\|^2\mathrm{d}\theta_2}\right)$$
$$\leq \frac{2}{\sqrt{3}}\nu h^2ag\int_{-h}^{0}\|\varphi(\theta)\|^2\mathrm{d}\theta.$$

The next term can be estimated as follows:

$$R_6 = \int_{-h}^{0} \varphi^T(\theta_1) \left\{ \int_{-h}^{0} \left[\int_{-h}^{\theta_1} G^T(\xi_1) \left(\int_{-h}^{\theta_2} U(\theta_1 - \theta_2 - \xi_1 + \xi_2) G(\xi_2) \mathrm{d}\xi_2 \right) \mathrm{d}\xi_1 \right] \right.$$
$$\left. \times\ \varphi(\theta_2)\mathrm{d}\theta_2 \right\} \mathrm{d}\theta_1$$
$$\leq \nu g^2 \left(\int_{-h}^{0} (h+\theta_1) \|\varphi(\theta_1)\| \mathrm{d}\theta_1 \right) \left(\int_{-h}^{0} (h+\theta_2) \|\varphi(\theta_2)\| \mathrm{d}\theta_2 \right)$$
$$\leq \frac{1}{3} \nu h^3 g^2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

And, finally,

$$R_7 = \int_{-h}^{0} \varphi^T(\theta) \left[W_1 + (h+\theta) W_2 \right] \varphi(\theta) \mathrm{d}\theta$$
$$\leq (\|W_1\| + h\|W_2\|) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta.$$

If we collect the estimations, then inequality (4.33) holds for

$$\delta_1 = \nu (1 + ha + hg),$$
$$\delta_2 = \nu a(1 + ha) + \frac{1}{3} \nu g h^2 \left(1 + 2\sqrt{3} a + hg \right) + \|W_1\| + h\|W_2\|. \qquad \square$$

Corollary 4.3. *If we assume that* $\alpha_2 = \delta_1 + h\delta_2$*, then functional (4.31) admits an upper bound of the form*

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h, 0], R^n). \tag{4.34}$$

4.7 Exponential Estimates

Lemma 4.6. *Given the positive-definite matrices* W_0*,* W_1*, and* W_2*, functional (4.30) admits the following exponential estimate:*

$$\lambda_{\min}(W_0)\left\|\varphi(0)\right\|^2+\lambda_{\min}(W_2)\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2\mathrm{d}\theta\leq w(\varphi),\quad \varphi\in PC([-h,0],R^n).$$

Proof. The proof follows directly from (4.30). □

Lemma 4.7. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, there exists $\sigma>0$ such that the complete type functional (4.31) satisfies the inequality*

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t}+2\sigma v(x_t)\leq 0,\quad t\geq 0, \tag{4.35}$$

along the solutions of the system.

Proof. On the one hand, by Lemma 4.5, there exist positive constants δ_1 and δ_2 such that

$$v(\varphi)\leq\delta_1\left\|\varphi(0)\right\|^2+\delta_2\int_{-h}^{0}\left\|\varphi(\theta)\right\|^2\mathrm{d}\theta.$$

On the other hand, Lemma 4.6 provides the estimate

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t}=-w(x_t)\leq-\lambda_{\min}(W_0)\left\|x(t)\right\|^2-\lambda_{\min}(W_2)\int_{-h}^{0}\left\|x(t+\theta)\right\|^2\mathrm{d}\theta.$$

Therefore, any $\sigma>0$ that satisfies the inequalities

$$2\sigma\delta_1\leq\lambda_{\min}(W_0)\text{ and }2\sigma\delta_2\leq\lambda_{\min}(W_2)$$

also satisfies (4.35). □

Theorem 4.9. *Let system (4.1) be exponentially stable. Given the positive-definite matrices W_0, W_1, and W_2, the inequality*

$$\left\|x(t,\varphi)\right\|\leq\sqrt{\frac{\alpha_2}{\alpha_1}}\left\|\varphi\right\|_h\mathrm{e}^{-\sigma t},\quad t\geq 0,$$

holds for any solution of the system. Here α_1 and α_2 are as defined in Corollaries 4.2 and 4.3, respectively, and $\sigma>0$ is as computed in Lemma 4.7.

Proof. Let $\sigma>0$ satisfy Lemma 4.7. Then, integrating inequality (4.35), we obtain that

$$v(x_t(\varphi))\leq v(\varphi)\mathrm{e}^{-2\sigma t},\quad t\geq 0.$$

Now inequalities (4.32) and (4.34) imply that

$$\alpha_1 \|x(t,\varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

The desired exponential estimate is a direct consequence of the preceding inequalities. □