# Chapter 1
# General Theory

This chapter serves as a brief introduction to the theory of the retarded type time-delay system. It starts with a discussion of such basic notions as solutions, initial conditions, and the state of a time-delay system. Then some results on the existence and uniqueness of an initial value problem are presented. Continuity properties of the solutions are discussed as well. The main part of the chapter is devoted to stability analysis. Here we define concepts of stability, asymptotic stability, and exponential stability of the trivial solution of a time-delay system. Classical stability results, obtained using the Lyapunov–Krasovskii approach, are given in the form of necessary and sufficient conditions. A short section with historical comments concludes the chapter.

## 1.1 Preliminaries

We begin with a class of retarded type time-delay systems of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = g(t, x(t), x(t-h)), \tag{1.1}$$

where $x \in R^n$ and the time delay $h > 0$. Let the vector-valued function $g(t,x,y)$ be defined for $t \geq 0$, $x \in R^n$, and $y \in R^n$. We assume that this function is continuous in the variables.

### *1.1.1 Initial Value Problem*

It is well known that a particular solution of a delay-free system, $\dot{x} = G(t,x)$, is defined by its initial conditions, which include an initial time instant $t_0$ and an initial

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state $x_0 \in R^n$. This is not the case when dealing with a solution of system (1.1). Here the knowledge of $t_0$ and $x_0$ is not sufficient even to define the value of the time derivative of $x(t)$ at the initial time instant $t_0$. To define a solution of system (1.1), one needs to select an initial time instant $t_0 \geq 0$ and an initial function $\varphi : [-h,0] \rightarrow R^n$. The initial value problem for system (1.1) is formulated as follows. Given an initial time instant $t_0 \geq 0$ and an initial function $\varphi$, find a solution of the system that satisfies the condition

$$x(t_0+\theta) = \varphi(\theta), \quad \theta \in [-h,0]. \tag{1.2}$$

The initial function $\varphi$ belongs to a certain functional space. It may be the space of continuous functions, $C([-h,0],R^n)$, the space of piecewise continuous functions, $PC([-h,0],R^n)$, or some other functional space. The choice of the space is dictated by a specific problem under investigation. In our case we assume that initial functions belong to the space $PC([-h,0],R^n)$. Recall that the function $\varphi$ belongs to the space if it admits at most a finite number of discontinuity points and for each continuity interval $(\alpha,\beta) \in [-h,0]$ the function has a finite right-hand-side limit at $\theta = \alpha$, $\varphi(\alpha+0) = \lim_{\varepsilon \to 0} \varphi(\alpha+|\varepsilon|)$, and a finite left-hand-side limit at $\theta = \beta$, $\varphi(\beta-0) = \lim_{\varepsilon \to 0} \varphi(\beta-|\varepsilon|)$.

The Euclidean norm is used for vectors and the corresponding induced norm for matrices. The space $PC([-h,0],R^n)$ is supplied with the standard uniform norm [24, 65, 66],

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|.$$

On the one hand, the fact that initial functions belong to a functional space gives rise to the interpretation of time-delay systems as a particular class of infinite-dimensional systems. On the other hand, the trajectories of a time-delay system lie in $R^{n+1}$; therefore, to some extent such systems can also be treated as systems in the finite-dimensional space.

### *1.1.2 Solutions*

In this section we discuss the existence issue for the initial value problem (1.1)–(1.2). The approach presented here is known as the "step-by-step" method [3].

First, we consider a system on the segment $[t_0, t_0+h]$. Here $t-h \in [t_0-h, t_0]$, and $x(t-h)$ is defined by Eq. (1.2), $x(t-h) = \varphi(t-t_0-h)$, and the system takes the form of the following auxiliary system of ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G^{(1)}(t,x) = g(t,x,\varphi(t-t_0-h)), \qquad t \in [t_0, t_0+h].$$

We are looking for a solution of the system that satisfies the condition $x(t_0) = \varphi(0)$.

If such a solution $\widetilde{x}(t)$ can be defined on the whole segment $[t_0, t_0+h]$, then we address the next segment $[t_0+h, t_0+2h]$. Here $t-h \in [t_0, t_0+h]$, and the delay state $x(t-h)$ was already defined in the previous step, $x(t-h) = \widetilde{x}(t-h)$. Thus, on this segment system (1.1) is a delay-free system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G^{(2)}(t,x) = g(t,x,\widetilde{x}(t-h)), \qquad t \in [t_0+h, t_0+2h],$$

and we are looking for a solution of the initial value problem $x(t_0+h) = \widetilde{x}(t_0+h)$.

Applying the step-by-step method, we reduce the computation of a solution of the initial value problem (1.1)–(1.2) to a series of standard initial value problems for a set of auxiliary systems of ordinary differential equations.

### *1.1.3 State Concept*

In the theory of dynamic systems the concept of a system state occupies center stage. In general, we can say that the state of a system at a given time instant $t_1 \geq t_0$ should include the minimal information that allows one to continue the dynamic for $t \geq t_1$. If we adopt this point of view, then the state should be defined in the same manner as it was for the initial value problem.

The definition of the initial conditions and the step-by-step method of construction of the system solutions presented previously demonstrate that we need to know $x(t_1+\theta)$, for $\theta \in [-h,0]$, in order to continue a solution for $t \geq t_1$. Therefore, along a given solution of system (1.1) the state of the system at a time instant $t \geq t_0$ is defined as the restriction of the solution on the segment $[t-h,t]$. We use the following notation for the system state

$$x_t : \theta \to x(t+\theta), \quad \theta \in [-h,0].$$

In the case where the initial condition $(t_0, \varphi)$ should be indicated explicitly we use the notations $x(t,t_0,\varphi)$ and $x_t(t_0,\varphi)$. For time-invariant systems we usually assume that $t_0 = 0$ and omit the argument $t_0$ in these notations.

## 1.2 Existence and Uniqueness Issues

The dynamic of a time-delay system may depend not only on a delay state, $x(t-h)$, as happens in system (1.1), but on the complete state, $x_t$, of the system. An example of such a situation is given by the system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_{-h}^{0} g(t,x(t+\theta))\mathrm{d}\theta.$$

Here the right-hand side of the system depends on the values of $x(t+\theta)$, $\theta \in [-h,0]$. This means that the right-hand side is no longer a function but a functional that is defined on a particular functional space. It is clear that for such systems the step-by-step method is no longer applicable. Thus, we must look for an alternative procedure to compute solutions. Here we present such a procedure, but first we introduce a definition.

**Definition 1.1 ([45]).** Given a functional

$$F : PC([-h,0], R^n) \to R^n,$$

we say that the functional is continuous at a point $\varphi_0 \in PC([-h,0], R^n)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $\varphi \in PC([-h,0], R^n)$ the inequality $\|\varphi - \varphi_0\|_h < \delta$ implies that

$$\|F(\varphi) - F(\varphi_0)\| < \varepsilon.$$

Functional $F$ is said to be continuous on a set $\Phi \subset PC([-h,0], R^n)$ if it is continuous at each point of the set.

Now we consider a functional

$$f : [0,\infty) \times PC([-h,0], R^n) \longrightarrow R^n.$$

The functional defines the time-delay system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x_t). \tag{1.3}$$

**Theorem 1.1.** *Given a time-delay system (1.3), where the functional*

$$f : [0,\infty) \times PC([-h,0], R^n) \longrightarrow R^n$$

*satisfies the following conditions:*

*(i) For any $H > 0$ there exists $M(H) > 0$ such that*

$$\|f(t,\varphi)\| \le M(H), \quad (t,\varphi) \in [0,\infty) \times PC([-h,0], R^n), \text{ and } \|\varphi\|_h \le H;$$

*(ii) The functional $f(t,\varphi)$ is continuous on the set $[0,\infty) \times PC([-h,0], R^n)$ with respect to both arguments;*

*(iii) The functional $f(t,\varphi)$ satisfies the Lipschitz condition with respect to the second argument, i.e., for any $H > 0$ there exists a Lipschitz constant $L(H) > 0$ such that the inequality*

$$\left\| f(t,\varphi^{(1)}) - f(t,\varphi^{(2)}) \right\| \le L(H) \left\| \varphi^{(1)} - \varphi^{(2)} \right\|_h$$

*holds for* $t \geq 0$, $\varphi^{(k)} \in PC^1([-h,0],R^n)$, *and* $\left\|\varphi^{(k)}\right\|_h \leq H$, $k = 1,2$.
*Then, for a given* $t_0 \geq 0$ *and an initial function* $\varphi \in PC([-h,0],R^n)$ *there exists* $\tau > 0$ *such that the system admits a unique solution* $x(t)$ *of the initial value problem (1.2), and the solution is defined on the segment* $[t_0 - h, t_0 + \tau]$.

*Proof.* Given $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, let us select $H > 0$ such that the inequality

$$H > H_0 = \|\varphi\|_h$$

holds. Now we can define the corresponding values $M = M(H)$ and $L = L(H)$.

Let us select $\tau > 0$ such that

$$\tau L < 1, \text{ and } \tau M < H - H_0,$$

and let us define a function $u : [t_0 - h, t_0 + \tau] \to R^n$ such that

$$u(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-h,0],$$

and the function is continuous on $[t_0, t_0 + \tau]$. Assume additionally that the following inequality holds:

$$\|u(t) - \varphi(0)\| \leq (t - t_0)M, \quad t \in [t_0, t_0 + \tau].$$

It follows from the definition that

$$\|u(t)\| \leq \|\varphi(0)\| + (t - t_0)M \leq \|\varphi\|_h + \tau M < H, \quad t \in [t_0, t_0 + \tau].$$

We denote by $U$ the set of all such functions. On the set $U$ we define the operator $\mathcal{A}$ that acts on the functions of the set as follows:

$$\mathcal{A}(u)(t) = \begin{cases} \varphi(t - t_0), & t \in [t_0 - h, t_0], \\ \varphi(0) + \int\limits_{t_0}^{t} f(s, u_s)\mathrm{d}s, & t \in [t_0, t_0 + \tau], \end{cases}$$

where $u_s : \theta \to u(s + \theta)$, $\quad \theta \in [-h,0]$. It is a matter of simple calculation to check that the theorem conditions (i) and (ii) guarantee that the transformed function $\mathcal{A}(u)$ belongs to the same set $U$:

$$u \in U \Rightarrow \mathcal{A}(u) \in U.$$

Let $x(t, t_0, \varphi)$ be a solution of the initial value problem (1.3)–(1.2); then

$$x(t,t_0,\varphi) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h,t_0], \\ \varphi(0) + \int\limits_{t_0}^{t} f(s,x_s(t_0,\varphi))\mathrm{d}s, & t \in [t_0,t_0+\tau], \end{cases}$$

and we conclude that this solution defines a fixed point of the operator $\mathcal{A}$.

Observe that

$$\mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) = \begin{cases} 0, & t \in [t_0-h,t_0], \\ \int\limits_{t_0}^{t} \left[ f(s,u_s^{(1)}) - f(s,u_s^{(2)}) \right] \mathrm{d}s, & t \in [t_0,t_0+\tau]. \end{cases}$$

Hence for $t \in [t_0-h,t_0]$

$$\left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| = 0,$$

and for $t \in [t_0,t_0+\tau]$

$$\begin{aligned} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| &\le \left\| \int\limits_{t_0}^{t} [f(s,u_s^{(1)}) - f(s,u_s^{(2)})]\mathrm{d}s \right\| \\ &\le \int\limits_{t_0}^{t_0+\tau} \left\| f(s,u_s^{(1)}) - f(s,u_s^{(2)}) \right\| \mathrm{d}s. \end{aligned}$$

The Lipschitz condition (iii) implies that for $t \in [t_0,t_0+\tau]$

$$\begin{aligned} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| &\le \int\limits_{t_0}^{t_0+\tau} L \left\| u_s^{(1)} - u_s^{(2)} \right\|_h \mathrm{d}s \\ &\le \tau L \sup_{s\in[t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|. \end{aligned}$$

Since the preceding inequality holds for all $t \in [t_0-h,t_0+\tau]$, we conclude that

$$\sup_{s\in[t_0-h,t_0+\tau]} \left\| \mathcal{A}(u^{(1)})(t) - \mathcal{A}(u^{(2)})(t) \right\| \le \tau L \sup_{s\in[t_0-h,t_0+\tau]} \left\| u^{(1)}(s) - u^{(2)}(s) \right\|.$$

Now, as $\tau L < 1$, the operator $\mathcal{A}$ satisfies the conditions of the contraction mapping theorem [45], and there exists a unique function $u^{(*)} \in U$ such that

$$u^{(*)}(t) = \mathcal{A}(u^{(*)})(t) = \begin{cases} \varphi(t-t_0), & t \in [t_0-h, t_0], \\ \varphi(0) + \int\limits_{t_0}^{t} f(s, u_s^{(*)})\mathrm{d}s, & t \in [t_0, t_0+\tau]. \end{cases}$$

The functional $f(t,\varphi)$ is continuous, so differentiating the preceding equality,

$$\frac{\mathrm{d}u^{(*)}(t)}{\mathrm{d}t} = f(t, u_t^{(*)}), \quad t \in [t_0, t_0+\tau],$$

we arrive at the conclusion that $u^{(*)}(t)$ is the unique solution of the initial value problem (1.3)–(1.2). □

*Remark 1.1.* We can take the new initial time instant, $t_1 = t_0 + \tau$, and define the new initial function

$$\varphi^{(1)}(\theta) = u^{(*)}(t_1+\theta), \quad \theta \in [-h, 0].$$

Then the procedure can be repeated, and we extend the solution to the next segment $[t_1, t_1+\widetilde{\tau}]$. This extension process can be continued as long as the solution remains bounded.

For each solution there exists a maximal interval $[t_0, t_0+T)$ on which the solution is defined. Here we present conditions under which any solution of system (1.3) is defined on $[t_0, \infty)$.

**Theorem 1.2.** *Let system (1.3) satisfy the conditions of Theorem 1.1. Assume additionally that $f(t,\varphi)$ satisfies the inequality*

$$\|f(t,\varphi)\| \le \eta(\|\varphi\|_h), \quad t \ge 0,\ \varphi \in PC([-h,0], R^n),$$

*where the function $\eta(r)$, $r \in [0,\infty)$, is continuous, nondecreasing, and such that for any $r_0 \ge 0$ the following condition holds:*

$$\lim_{R\to\infty} \int\limits_{r_0}^{R} \frac{\mathrm{d}r}{\eta(r)} = \infty.$$

*Then any solution $x(t, t_0, \varphi)$ of the system is defined on $[t_0, \infty)$.*

*Proof.* Given $t_0 \ge 0$ and $\varphi \in PC([-h,0], R^n)$, there exists a maximal interval $[t_0, t_0+T)$ on which the corresponding solution $x(t, t_0, \varphi)$ is defined. For the sake of simplicity we denote $x(t, t_0, \varphi)$ by $x(t)$.

Assume by contradiction that $T < \infty$. Then there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \in [t_0, t_0+T)$,

$$\lim_{k\to\infty} t_k = t_0 + T,$$

and

$$\lim_{k\to\infty} \|x(t_k)\| \to \infty;$$

otherwise, by Remark 1.1, the solution can be defined on a wider segment $[t_0, t_0 + T + \tau]$, where $\tau > 0$.

The solution satisfies the equality

$$x(t) = \varphi(0) + \int_{t_0}^{t} f(s, x_s)\mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

It follows from the preceding equality and the theorem conditions that

$$\|x_t\|_h \le \|\varphi\|_h + \int_{t_0}^{t} \eta(\|x_s\|_h)\mathrm{d}s, \quad t \in [t_0, t_0 + T).$$

Denote the right-hand side of the last inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \eta(\|x_t\|_h) \le \eta(v(t)), \quad t \in [t_0, t_0 + T).$$

This implies that

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} \le t_k - t_0, \quad k = 1, 2, 3, \ldots.$$

On the one hand, as

$$\int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \int_{r_0}^{r_k} \frac{\mathrm{d}\xi}{\eta(\xi)},$$

where $r_0 = v(t_0) = \|\varphi\|_h \ge 0$, and

$$r_k = v(t_k) \ge \left\|x_{t_k}\right\|_h \ge \|x(t_k)\| \to \infty, \text{ as } k \to \infty,$$

then

$$\lim_{k\to\infty} \int_{t_0}^{t_k} \frac{\mathrm{d}v(s)}{\eta(v(s))} = \infty.$$

On the other hand,

$$\lim_{k\to\infty} (t_k - t_0) = T.$$

Therefore, $T = \infty$, and we arrive at the contradiction with our assumption that $T < \infty$. This ends the proof of the statement. □

## 1.3 Continuity Properties

In this section we analyze the continuity properties of the solutions of system (1.3) with respect to the initial conditions and with respect to the system perturbations. These continuity properties are a direct consequence of the following theorem.

**Theorem 1.3.** *Assume that $f(t,\varphi)$ satisfies the conditions of Theorem 1.1. Let $x(t,t_0,\varphi)$ be a solution of system (1.3) such that*

$$x(t_0+\theta)=\varphi(\theta),\ \theta\in[-h,0].$$

*Given the perturbed system*

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}=f(t,y_t)+g(t,y_t),$$

*where the functional $g(t,\varphi)$ is continuous on the set $[0,\infty)\times PC([-h,0],R^n)$, satisfies the Lipschitz condition with respect to the second argument, and*

$$\|g(t,\varphi)\|\le m,\quad t\ge 0,\ \varphi\in PC([-h,0],R^n),$$

*let $y(t,t_0,\psi)$ be a solution of the perturbed system with the initial condition*

$$y(t_0+\theta)=\psi(\theta),\ \theta\in[-h,0].$$

*If the solutions are defined for $t\in[t_0-h,t_0+T]$, and if $H$ is such that*

$$\|x(t,t_0,\varphi)\|\le H,\quad \|y(t,t_0,\psi)\|\le H,\quad t\in[t_0-h,t_0+T],$$

*then the inequality*

$$\begin{aligned}\|x(t,t_0,\varphi)-y(t,t_0,\psi)\| &\le \|x_t(t_0,\varphi)-y_t(t_0,\psi)\|_h\\ &\le\left(\|\psi-\varphi\|_h+\frac{m}{L(H)}\right)\mathrm{e}^{L(H)(t-t_0)}\end{aligned}$$

*holds for $t\in[t_0,t_0+T]$.*

*Proof.* For the sake of simplicity we will use the following shorthand notations for the solutions $x(t)=x(t,t_0,\varphi)$ and $y(t)=y(t,t_0,\psi)$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-y(t)]=f(t,x_t)-f(t,y_t)-g(t,y_t),\quad t\in[t_0,t_0+T].$$

Integrating the preceding equality we obtain

$$x(t) - y(t) = \varphi(0) - \psi(0) + \int_{t_0}^{t} [f(s,x_s) - f(s,y_s) - g(s,y_s)]\mathrm{d}s, \quad t \in [t_0, t_0 + T].$$

The last equality implies that for $t \in [t_0, t_0 + T]$ the following inequalities hold:

$$\begin{aligned}\|x(t) - y(t)\| &\le \|\varphi(0) - \psi(0)\| + \int_{t_0}^{t} \|f(s,x_s) - f(s,y_s) - g(s,y_s)\|\,\mathrm{d}s \\ &\le \|\varphi(0) - \psi(0)\| + m\,(t - t_0) + L(H)\int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s.\end{aligned}$$

Since $\|\varphi(0) - \psi(0)\| \le \|\psi - \varphi\|_h$, we have

$$\begin{aligned}\|x(t) - y(t)\| \le\ &\|\varphi - \psi\|_h + m\,(t - t_0) \\ &+ L(H)\int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s, \quad t \in [t_0, t_0 + T].\end{aligned}$$

Using similar arguments we can conclude that for $t_1 \in [t - h, t]$, the inequality

$$\|x(t_1) - y(t_1)\| \le \|\varphi - \psi\|_h + m\,(t - t_0) + L(H)\int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s$$

holds, which implies

$$\sup_{t_1 \in [t-h,t]} \|x(t_1) - y(t_1)\| \le \|\varphi - \psi\|_h + m\,(t - t_0) + L(H)\int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s.$$

So we have

$$\begin{aligned}\|x_t - y_t\|_h \le\ &\|\varphi - \psi\|_h + m\,(t - t_0) \\ &+ L(H)\int_{t_0}^{t} \|x_s - y_s\|_h\,\mathrm{d}s, \quad t \in [t_0, t_0 + T].\end{aligned}$$

Denote the right-hand side of the preceding inequality by $v(t)$; then

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = m + L(H)\,\|x_t - y_t\|_h \le m + L(H)v(t), \quad t \in [t_0, t_0 + T].$$

Integrating this inequality we arrive at the desired one:

$$\begin{aligned}\|x(t,t_0,\varphi)-y(t,t_0,\psi)\| &\le \|x_t(t_0,\varphi)-y_t(t_0,\psi)\|_h\\ &\le \|\psi-\varphi\|_h\,\mathrm{e}^{L(H)(t-t_0)}+\frac{m}{L(H)}(\mathrm{e}^{L(H)(t-t_0)}-1)\\ &\le \left(\|\psi-\varphi\|_h+\frac{m}{L(H)}\right)\mathrm{e}^{L(H)(t-t_0)},\quad t\in[t_0,t_0+T].\end{aligned}$$

□

**Corollary 1.1.** *Let $g(t,\varphi)\equiv 0$; then $m=0$, and both $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ are solutions of system (1.3). Assume that these solutions are defined for $t\in[t_0,t_0+T]$. For any $\varepsilon>0$ there exists $\delta>0$ such that if $\|\psi-\varphi\|_h<\delta$, then the following inequality holds:*

$$\|x(t,t_0,\varphi)-x(t,t_0,\psi)\|<\varepsilon,\qquad t\in[t_0,t_0+T].$$

*In other words, $x(t,t_0,\varphi)$ depends continuously on $\varphi$.*

*Proof.* The statement follows directly from Theorem 1.3 if we set $\delta=\varepsilon\mathrm{e}^{-L(H)T}$.

□

**Corollary 1.2.** *Let $\psi(\theta)=\varphi(\theta)$, $\theta\in[-h,0]$; this means that the solutions $x(t,t_0,\varphi)$ and $y(t,t_0,\psi)$ have the same initial conditions. Assume that these solutions are defined for $t\in[t_0,t_0+T]$. For any $\varepsilon>0$ there exists $\delta>0$ such that if $m<\delta$, then*

$$\|x(t,t_0,\varphi)-y(t,t_0,\varphi)\|<\varepsilon,\qquad t\in[t_0,t_0+T].$$

*This means that $x(t,t_0,\varphi)$ depends continuously on the right-hand side of system (1.3).*

*Proof.* The statement follows directly from Theorem 1.3 if we set $\delta=L(H)\mathrm{e}^{-L(H)T}\varepsilon$.

□

## 1.4 Stability Concepts

In this section we introduce some stability concepts for system (1.3). Let the system satisfy the conditions of Theorem 1.1. Assume additionally that the system admits a trivial solution, i.e., $f(t,0_h)\equiv 0$, for $t\ge 0$. Here $0_h$ stands for the trivial function, $0_h:\theta\to 0\in R^n$, $\theta\in[-h,0]$.

**Definition 1.2 ([46]).** The trivial solution of system (1.3) is said to be stable if for any $\varepsilon>0$ and $t_0\ge 0$ there exists $\delta(\varepsilon,t_0)>0$ such that for every initial function $\varphi\in PC([-h,0],R^n)$, $\|\varphi\|_h<\delta(\varepsilon,t_0)$, the following inequality holds:

$$\|x(t,t_0,\varphi)\|<\varepsilon,\quad t\ge t_0.$$

If $\delta(\varepsilon, t_0)$ can be chosen independently of $t_0$, then the trivial solution is said to be uniformly stable.

*Remark 1.2.* The value $\delta(\varepsilon, t_0)$ is always smaller than or equal to $\varepsilon$.

*Proof.* Assume that for some $\varepsilon > 0$ and $t_0 \geq 0$ we have $\delta(\varepsilon, t_0) > \varepsilon$; then there is $\varphi \in PC([-h,0], R^n)$ such that $\|\varphi\|_h < \delta(\varepsilon, t_0)$, and $\|\varphi(0)\| > \varepsilon$. On the one hand, the corresponding solution $x(t, t_0, \varphi)$ should satisfy the inequality

$$\|x(t, t_0, \varphi)\| < \varepsilon, \quad t \geq t_0,$$

and, in particular, $\|x(t_0, t_0, \varphi)\| < \varepsilon$. On the other hand, $x(t_0, t_0, \varphi) = \varphi(0)$, so $\|x(t_0, t_0, \varphi)\| = \|\varphi(0)\| > \varepsilon$. This contradiction proves the remark. □

**Definition 1.3.** The trivial solution of system (1.3) is said to be asymptotically stable if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\Delta(\varepsilon, t_0) > 0$ such that for every initial function $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, the following conditions hold.

1. $\|x(t, t_0, \varphi)\| < \varepsilon$, for $t \geq t_0$.
2. $x(t, t_0, \varphi) \to 0$ as $t - t_0 \longrightarrow \infty$.
   If $\Delta(\varepsilon, t_0)$ can be chosen independently of $t_0$ and there exists $H_1 > 0$ such that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \longrightarrow \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h \leq H_1$, then the trivial solution is said to be uniformly asymptotically stable.

**Definition 1.4.** The trivial solution of system (1.3) is said to be exponentially stable if there exist $\Delta_0 > 0$, $\sigma > 0$, and $\gamma \geq 1$ such that for every $t_0 \geq 0$ and any initial function $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h < \Delta_0$, the following inequality holds:

$$\|x(t, t_0, \varphi)\| \leq \gamma \|\varphi\|_h \mathrm{e}^{-\sigma(t-t_0)}, \quad t \geq t_0.$$

## 1.5 Lyapunov–Krasovskii Approach

First we show why the direct application of the classical Lyapunov approach does not work for time-delay systems. To this end, we consider a scalar linear equation of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t-h), \quad t \geq 0,$$

where $a, b$ are real constants. Since the equation is linear, it seems natural to apply the positive-definite Lyapunov function $v(x) = x^2$. The time derivative of the function along the solutions of the equation is

$$\frac{\mathrm{d}v(x(t))}{\mathrm{d}t} = 2x(t)\left[ax(t) + bx(t-h)\right] = 2ax^2(t) + 2bx(t)x(t-h).$$

For the case $b = 0$ the equation is delay free, and the time derivative is negative definite when $a < 0$. According to the Lyapunov stability theory, this implies the asymptotic stability of the equation.

The situation becomes different when $b \neq 0$. In this case the time derivative includes two terms and, despite the fact that the first term remains negative definite for $a < 0$, we are not able to state the same about the time derivative because nothing certain can be said about the sign and the value of the second term, $2bx(t)x(t-h)$. Therefore, some modifications of the Lyapunov approach should be made if we would like to apply it to a stability analysis of time-delay systems.

Such modifications have been proposed in two distinct ways.

1. The first one is due to N. N. Krasovskii, who proposed to replace classical Lyapunov functions that depend on the instant state, $x(t)$, of a system by functionals that depend on the true state, $x_t$. This modification is now known as the Lyapunov–Krasovskii approach [46–48].
2. The other modification was proposed by Razumikhin [61,62]. It uses the classical Lyapunov functions but adds an additional condition that allows one to compare the values of $x(t)$ and $x(t-h)$ and provides negativity conditions for the time derivative of the functions along the solutions of the system.

In this book we do not treat the Razumikhin approach but concentrate on the Lyapunov–Krasovskii one. We start with the definition of positive-definite functions.

**Definition 1.5.** A function $v_1(x)$ is said to be positive definite if there exists $H > 0$ such that the function is continuous on the set $\{x \mid \|x\| \leq H\}$ and satisfies the following conditions:

1. $v_1(0) = 0$;
2. $v_1(x) > 0$ for $0 < \|x\| \leq H$.

Now we extend the positive-definiteness concept to the case of functionals.

**Definition 1.6.** Functional $v(t,\varphi)$ is said to be positive definite if there exists $H > 0$ such that the following conditions are satisfied.

1. The functional $v(t,\varphi)$ is defined for $t \geq 0$ and any $\varphi \in PC([-h,0],R^n)$ with $\|\varphi\|_h \leq H$.
2. $v(t,0_h) = 0$, $t \geq 0$.
3. There exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t,\varphi), \quad t \geq 0, \text{ and } \varphi \in PC([-h,0],R^n), \; \|\varphi\|_h \leq H.$$

4. For any given $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in $\varphi$ at the point $0_h$, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|\varphi\|_h < \delta$ implies

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) < \varepsilon.$$

We are now ready to present some basic statements of the Lyapunov–Krasovskii approach.

**Theorem 1.4.** *The trivial solution of system (1.3) is stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that along the solutions of the system the value of the functional $v(t,x_t)$ as a function of $t$ does not increase.*

*Proof. Sufficiency*: The positive definiteness of the functional $v(t,\varphi)$ implies that there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \le v(t,\varphi), \quad t \ge 0, \text{ and } \varphi \in PC([-h,0],R^n), \quad \|\varphi\|_h \le H.$$

For a given $\varepsilon > 0$ $(\varepsilon < H)$ we define the positive value

$$\lambda(\varepsilon) = \min_{\|x\|=\varepsilon} v_1(x). \tag{1.4}$$

Since for a given $t_0 \ge 0$ the functional $v(t_0,\varphi)$ is continuous in $\varphi$ at the point $0_h$, there exists $\delta > 0$ such that $v(t_0,\varphi) < \lambda(\varepsilon)$ for any $\varphi \in PC([-h,0],R^n)$ with $\|\varphi\|_h < \delta$.

It is clear that $\delta \le \varepsilon$; otherwise we could present an initial function $\varphi \in PC([-h,0],R^n)$ such that $\|\varphi\|_h < \delta$ and $\|\varphi(0)\| = \varepsilon$. On the one hand, for this initial function we have $v_1(\varphi(0)) \ge \lambda(\varepsilon)$. On the other hand, $v_1(\varphi(0)) \le v(t_0,\varphi) < \lambda(\varepsilon)$. The contradiction proves the inequality $\delta \le \varepsilon$.

Now let $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta$. Then the theorem condition implies that

$$v_1(x(t,t_0,\varphi)) \le v(t,x_t(t_0,\varphi)) \le v(t_0,\varphi) < \lambda(\varepsilon), \quad t \ge t_0. \tag{1.5}$$

Assume by contradiction that there exists a time instant $t_1 \ge t_0$ for which $\|x(t_1,t_0,\varphi)\| \ge \varepsilon$. Since for $t \ge t_0$ the function $\|x(t,t_0,\varphi)\|$ is continuous in $t$, and since $\|x(t_0,t_0,\varphi)\| = \|\varphi(0)\| \le \|\varphi\|_h < \delta \le \varepsilon$, there exists $t^* \in [t_0,t_1]$ such that $\|x(t^*,t_0,\varphi)\| = \varepsilon$. So, on the one hand, by Eq. (1.4), we know that

$$v_1(x(t^*,t_0,\varphi)) \ge \lambda(\varepsilon).$$

On the other hand, inequality (1.5) implies the inequality

$$v_1(x(t^*,t_0,\varphi)) < \lambda(\varepsilon),$$

which contradicts the previous one. The contradiction proves that our assumption is wrong, and the following inequality holds:

$$\|x(t,t_0,\varphi)\| < \varepsilon, \quad t \ge t_0.$$

This means that $\delta$ satisfies Definition 1.2, and therefore the trivial solution of system (1.3) is stable.

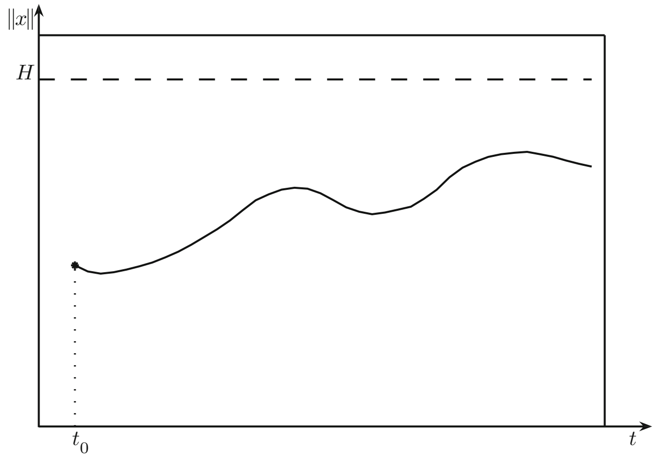


**Fig. 1.1** Value of $\|x(t,t_0,\varphi\|$, the first case

*Necessity*: Now, the trivial solution of system (1.3) is stable, and we must prove that there exists a functional $v(t,\varphi)$ that satisfies the theorem condition.

*Construction of the functional*: Since the trivial solution of system (1.3) is stable, for $\varepsilon = H$ there exists $\delta(H,t_0) > 0$ such that the inequality $\|\varphi\|_h < \delta(H,t_0)$ implies that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. We define the functional $v(t,\varphi)$ as follows:

$$v(t_0,\varphi) = \begin{cases} \sup\limits_{t\geq t_0} \|x(t,t_0,\varphi)\|, \text{ if } \|x(t,t_0,\varphi)\| < H, \text{ for } t \geq t_0, \\ H, \text{ if there exists } T \geq t_0 \text{ such that } \|x(T,t_0,\varphi)\| = H. \end{cases} \tag{1.6}$$

These two possibilities are illustrated in Figs. 1.1 and 1.2, respectively.

We check first that the functional $v(t,\varphi)$ is positive definite. To this end, we must verify that it satisfies the conditions of Definition 1.6.

*Condition 1*: The value $v(t_0,\varphi)$ is defined for all $t_0 \geq 0$, and every initial function $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.

*Condition 2*: For the trivial initial function, $\varphi = 0_h$, the corresponding solution is trivial, $x(t,t_0,0_h) = 0$, for $t \geq t_0$. Thus $v(t_0,0_h) = 0$, $t_0 \geq 0$.

*Condition 3*: The function $v_1(x) = \|x\|$ is positive definite. Given $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$, in the case where $\|x(t,t_0,\varphi)\| < H$, for $t \geq t_0$, we have

$$v_1(\varphi(0)) = \|\varphi(0)\| \leq \sup_{t\geq t_0} \|x(t,t_0,\varphi)\| = v(t_0,\varphi).$$

In the other case, where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| = H$, we have

$$v_1(\varphi(0)) = \|\varphi(0)\| \leq H = v(t_0,\varphi).$$

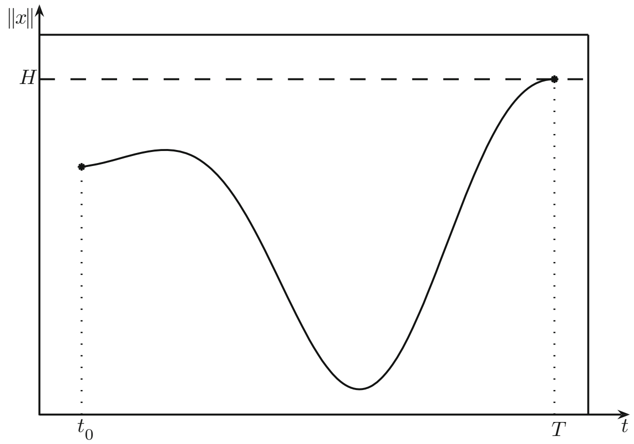


**Fig. 1.2** Value of $\|x(t,t_0,\varphi\|$, the second case

*Condition 4*: Given $t_0 \geq 0$, the stability of the trivial solution means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\varphi\|_h < \delta$ implies $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. In other words, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\varphi\|_h < \delta$ implies

$$v(t_0,\varphi) = |v(t_0,\varphi) - v(t_0,0_h)| \leq \varepsilon.$$

This observation makes it clear that for a fixed $t_0 \geq 0$ the functional $v(t_0,\varphi)$ is continuous in $\varphi$ at the point $0_h$.

Now we check that functional (1.6) satisfies the theorem condition. First, we consider the case where $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$. In this case, given two time instants, $t_1$ and $t_2$, such that $t_2 > t_1 \geq t_0$, we compare the values

$$v(t_1,x_{t_1}(t_0,\varphi)) = \sup_{t \geq t_1} \|x(t,t_0,\varphi)\|$$

and

$$v(t_2,x_{t_2}(t_0,\varphi)) = \sup_{t \geq t_2} \|x(t,t_0,\varphi)\|.$$

Since for the second value the range of the supremum is smaller than that for the first value, we conclude that

$$v(t_2,x_{t_2}(t_0,\varphi)) \leq v(t_1,x_{t_1}(t_0,\varphi)).$$

This means that the functional $v(t,x_t(t_0,\varphi))$ does not increase along the solution. In the second case, where there exists $T \geq t_0$ such that $\|x(T,t_0,\varphi)\| = H$, we have the equality

$$v(t_2,x_{t_2}(t_0,\varphi)) = v(t_1,x_{t_1}(t_0,\varphi)) = H,$$

and, once again, the functional does not increase along the solution of system (1.3). □

*Remark 1.3.* The functional $v(t,\varphi)$, defined in the proof of the necessity part of Theorem 1.4, is of academic interest only. Obviously, we cannot use such functionals in applications. The computation of practically useful Lyapunov functionals is not a simple task.

**Theorem 1.5.** *The trivial solution of system (1.3) is uniformly stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions are satisfied.*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, does not increase.*
2. *The functional is continuous in $\varphi$ at the point $0_h$, uniformly for $t \geq 0$.*

*Proof. Sufficiency*: In the proof of the sufficiency part of Theorem 1.4 the value $\delta = \delta(\varepsilon,t_0)$ was chosen such that for any $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta$, the value of the functional for a given $t_0 \geq 0$ satisfies the inequality $v(t_0,\varphi) < \lambda(\varepsilon)$. Since now the functional is continuous in $\varphi$ at the point $0_h$, uniformly for $t \geq 0$, the value $\delta$ can be chosen independently of $t_0$.

*Necessity*: The uniform stability of the trivial solution of system (1.3) implies that $\delta$ can be chosen independently of $t_0$, $\delta = \delta(\varepsilon)$. It was demonstrated in the proof of Theorem 1.4 that functional (1.6) is positive definite and does not increase along the solutions of system (1.3). We show that this functional satisfies the second condition of the theorem. For any $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h < \delta(\varepsilon)$, and any $t_0 \geq 0$ we have that $\|x(t,t_0,\varphi)\| < \varepsilon$ for $t \geq t_0$. This means that

$$|v(t_0,\varphi) - v(t_0,0_h)| = v(t_0,\varphi) \leq \varepsilon.$$

In other words, functional (1.6) is continuous in $\varphi$ at the point $0_h$, uniformly for $t \geq 0$. □

*Remark 1.4.* The second condition of Theorem 1.5 is satisfied when $v(t,\varphi)$ admits an upper estimate of the form

$$v(t,\varphi) \leq v_2(\varphi), \quad t \geq 0,\ \varphi \in PC([-h,0],R^n),\ \|\varphi\|_h \leq H,$$

with a positive-definite functional $v_2(\varphi)$.

**Theorem 1.6.** *The trivial solution of system (1.3) is asymptotically stable if and only if there exists a positive-definite functional $v(t,\varphi)$ such that the following conditions hold.*

1. *The value of the functional along the solutions of the system, $v(t,x_t)$, does not increase.*
2. *For any $t_0 \geq 0$ there exists a positive value $\mu(t_0)$ such that if $\varphi \in PC([-h,0],R^n)$ and $\|\varphi\|_h < \mu(t_0)$, then $v(t,x_t(t_0,\varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$.*

*Proof. Sufficiency*: The first condition of the theorem implies the stability of the trivial solution of system (1.3) (Theorem 1.4). Thus, for any $\varepsilon > 0$ $(\varepsilon < H)$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0) > 0$ such that if $\|\varphi\|_h < \delta(\varepsilon, t_0)$, then $\|x(t, t_0, \varphi)\| < \varepsilon$ for $t \geq t_0$. Let us define the value

$$\Delta(\varepsilon, t_0) = \min\{\delta(\varepsilon, t_0), \mu(t_0)\}.$$

For any given initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon, t_0)$, the following inequality holds:

$$\|x(t, t_0, \varphi)\| < \varepsilon, \quad t \geq t_0.$$

We will demonstrate that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. The functional $v(t, \varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t, \varphi), \text{ for } t \geq 0, \text{ and } \varphi \in PC([-h, 0], R^n), \|\varphi\|_h \leq H.$$

The function $v_1(x)$ is continuous, so for any given $\varepsilon_1 > 0$ $(\varepsilon_1 < \varepsilon)$ we may define the positive value

$$\alpha = \min_{\varepsilon_1 \leq \|x\| \leq \varepsilon} v_1(x).$$

By the second condition of the theorem, there exists $T > 0$ such that $v(t, x_t(t_0, \varphi)) < \alpha$ for $t \geq t_0 + T$. This implies the inequality

$$v_1(x(t, t_0, \varphi)) < \alpha, \quad t \geq t_0 + T,$$

and we conclude that

$$\|x(t, t_0, \varphi)\| < \varepsilon_1, \quad t \geq t_0 + T.$$

This means that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$, and we must accept that the previously defined value $\Delta(t_0, \varepsilon)$ satisfies Definition 1.3.

*Necessity*: In this part of the proof we make use of functional (1.6). In the proof of Theorem 1.4 it was demonstrated that the functional is positive definite and does not increase along the solutions of system (1.3). This means that the functional satisfies the first condition of the theorem.

We address the second condition of the theorem and choose the value $\mu(t_0)$ as follows:

$$\mu(t_0) = \Delta(H, t_0) > 0.$$

Now, for any initial function $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \mu(t_0)$, we know that $x(t, t_0, \varphi) \to 0$ as $t - t_0 \to \infty$. This means that for any $\varepsilon_1 > 0$ there exists $T > 0$ such that $\|x(t, t_0, \varphi)\| < \varepsilon_1$ for $t \geq t_0 + T$. According to Eq. (1.6), we have

$$v(t, x_t(t_0, \varphi)) = \sup_{s \geq t} \|x(s, t_0, \varphi)\| \leq \varepsilon_1, \text{ for } t \geq t_0 + T.$$

The preceding observation means that $v(t, x_t(t_0, \varphi))$ tends to zero as $t \to \infty$. □

**Theorem 1.7.** *The trivial solution of system (1.3) is uniformly asymptotically stable if and only if there exists a positive-definite functional $v(t, \varphi)$ such that the following conditions hold.*

1. *The value of the functional along the solutions of the system, $v(t, x_t)$, does not increase.*
2. *The functional is continuous in $\varphi$ at the point $0_h$, uniformly for $t \geq 0$.*
3. *There exists a positive value $\mu_1$ such that $v(t, x_t(t_0, \varphi))$ decreases monotonically to zero as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h \leq \mu_1$.*

*Proof. Sufficiency*: Comparing this theorem with Theorem 1.5 we conclude that the trivial solution of system (1.3) is uniformly stable. Therefore, for a given $\varepsilon > 0$ the value

$$\Delta(\varepsilon) = \min\{\mu_1, \delta(\varepsilon)\} > 0$$

is such that the following properties hold:

1. Given $t_0 \geq 0$ and $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h < \Delta(\varepsilon)$, then $\|x(t, t_0, \varphi)\| < \varepsilon$ for $t \geq t_0$.
2. $v(t, x_t(t_0, \varphi)) \to 0$ as $t - t_0 \to \infty$.

   Now we define

$$H_1 = \frac{1}{2}\Delta(H) > 0.$$

The functional $v(t, \varphi)$ is positive definite, so there exists a positive-definite function $v_1(x)$ such that

$$v_1(\varphi(0)) \leq v(t, \varphi), \text{ for } t \geq 0, \text{ and } \varphi \in PC([-h, 0], R^n), \ \|\varphi\|_h \leq H.$$

The function $v_1(x)$ is continuous; therefore, for any $\varepsilon_1 > 0$ $(\varepsilon_1 < H)$ we may define the positive value

$$\alpha = \min_{\varepsilon_1 \leq \|x\| \leq H} v_1(x).$$

By the third condition of the theorem there exists $T > 0$ such that for any $t_0 \geq 0$ and $\varphi \in PC([-h, 0], R^n)$, with $\|\varphi\|_h \leq H_1$, the following inequality holds:

$$v(t, x_t(t_0, \varphi)) < \alpha, \ t - t_0 \geq T.$$

This implies that

$$v_1(x(t, t_0, \varphi)) < \alpha, \quad t - t_0 \geq T,$$

and we conclude that

$$\|x(t, t_0, \varphi)\| < \varepsilon_1, \quad t - t_0 \geq T,$$

for any $t_0 \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H_1$. Therefore, the previously defined values $\Delta(\varepsilon)$ and $H_1$ satisfy Definition 1.3. This ends the proof of the sufficiency part of the theorem.

*Necessity*: The uniform asymptotic stability of the trivial solution of system (1.3) implies that functional (1.6) satisfies the first two conditions of the theorem. Let us set

$$\mu_1 = \frac{1}{2}\Delta(H),$$

where $\Delta(\varepsilon)$ is from Definition 1.3. Now, given $\varepsilon_1 > 0$, for any $t_0 \geq 0$ and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$, there exists $T > 0$ such that

$$\|x(t,t_0,\varphi)\| < \varepsilon_1, \quad t - t_0 \geq T.$$

This means that functional (1.6) satisfies the inequality

$$v(t,x_t(t_0,\varphi)) = \sup_{s\geq 0} \|x(s,t_0,\varphi)\| \leq \varepsilon_1, \quad t - t_0 \geq T,$$

i.e., the functional decreases monotonically to zero as $t - t_0 \to \infty$, uniformly with respect to $t_0 \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq \mu_1$. This ends the proof of the necessity part. □

The following statement provides sufficient conditions of the uniform asymptotic stability of the trivial solution of system (1.3).

**Theorem 1.8 ([46]).** *The trivial solution of system (1.3) is uniformly asymptotically stable if there exist two positive-definite functionals, $v(t,\varphi)$ and $v_2(\varphi)$, and a positive-definite function $w(x)$ such that the following two conditions hold.*

1. *$v(t,\varphi) \leq v_2(\varphi)$, for $t \geq 0$, and $\varphi \in PC([-h,0],R^n)$, with $\|\varphi\|_h \leq H$.*
2. *The value of the functional along the solutions of the system is differentiable by $t$, and its time derivative satisfies the inequality*

$$\frac{\mathrm{d}v(t,x_t)}{\mathrm{d}t} \leq -w(x(t)).$$

*Proof.* Observe that the first condition of the theorem implies that the functional $v(t,\varphi)$ is continuous in $\varphi$ at the point $0_h$, uniformly for $t \geq 0$ (Corollary 1.4). This means that the second condition of Theorem 1.7 is satisfied. The first condition of Theorem 1.7 follows directly from the second condition of this theorem.

Now we show that the third condition of Theorem 1.7 is also satisfied. It is evident that the theorem conditions guarantee that the trivial solution is uniformly stable, i.e., for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ that satisfies the definition of the uniform stability. The functional $v_2(\varphi)$ is positive definite, so there exists a positive value $\eta$ such that the following inequality holds:

$$v_2(\varphi) < H, \quad \varphi \in PC([-h,0], R^n), \text{ with } \|\varphi\|_h \leq \eta.$$

Let us set

$$\mu_1 = \min\left\{\frac{1}{2}\delta(H), \eta\right\}.$$

We are going to demonstrate that for any given $\alpha > 0$ there exists $T > 0$ such that if $t - t_0 \geq T$, then the inequality

$$v(t, x_t(t_0, \varphi)) < \alpha$$

holds for any $t_0 \geq 0$ and $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h \leq \mu_1$. Since the functional $v_2(\varphi)$ is positive definite, there exists $\beta > 0$ such that the inequality $\|\varphi\|_h < \beta$ implies $v_2(\varphi) < \alpha$. The function $w(x)$ is positive definite, and we can define a positive constant $\gamma$ as follows:

$$\gamma = \min_{\frac{\beta}{2} \leq \|x\| \leq H} w(x).$$

For any function $\varphi \in PC([-h,0], R^n)$, with $\|\varphi\|_h \leq H$, we have

$$\|f(t,\varphi)\| \leq M(H), \text{ for } t \geq 0.$$

Now we set

$$\tau = \min\left\{h, \frac{\beta}{2M(H)}\right\}$$

and select an entire number $N$ satisfying the inequality

$$H - \gamma\tau N < 0.$$

Finally, we define a positive value $T$ as follows:

$$T = 2hN.$$

Given an initial instant $t_0 \geq 0$ and function $\varphi \in PC([-h,0], R^n)$ such that $\|\varphi\|_h \leq \mu_1$, we will demonstrate that $v(t, x_t(t_0,\varphi)) < \alpha$ for $t - t_0 \geq T$. First we observe that the second condition of the theorem implies that $v(t, x_t(t_0,\varphi))$ is a decreasing function of $t$, so it is enough to check that $v\left(t_0 + T, x_{t_0+T}(t_0,\varphi)\right) < \alpha$. Assume by contradiction that this is not the case, and $v\left(t_0 + T, x_{t_0+T}(t_0,\varphi)\right) \geq \alpha$. This means that

$$\alpha \leq v(t, x_t(t_0,\varphi)) \leq v_2(x_t(t_0,\varphi))$$

for $t \in [t_0, t_0 + T]$. The inequality $\alpha \leq v_2(x_t(t_0,\varphi))$ implies that $\|x_t(t_0,\varphi)\|_h \geq \beta$ for $t \in [t_0, t_0 + T]$, i.e., in each segment $[t - h, t] \subset [t_0, t_0 + T]$ there exists a point $t^* \in [t-h, t]$ such that $\|x(t^*, t_0, \varphi)\| \geq \beta$. These arguments demonstrate that we can define an increasing sequence, $\{t_j\}_{j=1}^N$, such that at the points of the sequence $\left\|x(t_j,t_0,\varphi)\right\| \geq \beta$. Without any loss of generality we assume that any two consecutive points of the sequence satisfy the inequalities $h < t_{j+1} - t_j < 2h$.

According to the choice of the initial function $\varphi$, we know that $\|x(t,t_0,\varphi)\| < H$ for $t \geq t_0$, and at the points of the sequence the following inequality holds:

$$\beta \leq \left\|x(t_j,t_0,\varphi)\right\|, \quad j = 1,2,\dots,N.$$

Now observe that

$$x(t,t_0,\varphi) = x(t_j,t_0,\varphi) + \int_{t_j}^{t} f(s,x_s(t_0,\varphi))\mathrm{d}s, \quad t \geq t_j,$$

and, since $\|f(s,x_s(t_0,\varphi))\| \leq M(H)$, for $t \geq 0$ we have

$$\begin{aligned}\left\|x(t,t_0,\varphi) - x(t_j,t_0,\varphi)\right\| &\leq \int_{t_j}^{t} \|f(s,x_s(t_0,\varphi))\|\,\mathrm{d}s \\ &\leq \tau M(H), \text{ for } t \in [t_j,t_j+\tau].\end{aligned}$$

According to our choice of $\tau$, we conclude that for $t \in [t_j,t_j+\tau]$

$$\left\|x(t,t_0,\varphi) - x(t_j,t_0,\varphi)\right\| \leq \frac{\beta}{2}.$$

As $\left\|x(t_j,t_0,\varphi)\right\| \geq \beta$, the inequality

$$\|x(t,t_0,\varphi)\| \geq \frac{\beta}{2}, \quad t \in [t_j,t_j+\tau],$$

holds for $j = 1,2,\dots,N$. It is evident that

$$w(x(t,t_0,\varphi)) \geq \gamma,\ t \in [t_j,t_j+\tau], \quad j = 1,2,\dots,N,$$

and the second condition of the theorem implies that

$$\begin{aligned}v(t_0+T,x_{t_0+T}(t_0,\varphi)) &\leq v(t_0,\varphi) - \int_{t_0}^{t_0+T} w(x(s,t_0,\varphi)\mathrm{d}s \\ &\leq H - \gamma\tau N < 0.\end{aligned}$$

This means that $v(t_0+T,x_{t_0+T}(t_0,\varphi))$ is negative, which contradicts the positive definiteness of the functional $v(t,\varphi)$. The contradiction proves that

$$v(t, x_t(t_0, \varphi)) < \alpha, \text{ for } t - t_0 \geq T.$$

Now to end the proof, it is enough to refer to Theorem 1.7. □

**Theorem 1.9.** *The trivial solution of system (1.3) is exponentially stable if there exists a positive-definite functional $v(t, \varphi)$ such that the following conditions hold.*

1. *There are two positive constants $\alpha_1, \alpha_2$ for which*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(t, \varphi) \leq \alpha_2 \|\varphi\|_h^2, \text{ for } t \geq 0,$$

   *for $t \geq 0$, and $\varphi \in PC([-h, 0], R^n)$ with $\|\varphi\|_h \leq H$.*
2. *The functional is differentiable along the solutions of the system, and there exists a positive constant $\sigma$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t, x_t) + 2\sigma v(t, x_t) \leq 0.$$

*Proof.* Let us define the positive-definite function $v_1(x) = \alpha_1 \|x\|^2$ and the positive-definite functional $v_2(\varphi) = \alpha_1 \|\varphi\|_h^2$. It is evident that the functional $v(t, \varphi)$ satisfies the conditions of Theorem 1.5. Therefore, the trivial solution of system (1.3) is uniformly stable, and for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality $\|\varphi\|_h < \delta(\varepsilon)$ implies $\|x(t, t_0, \varphi)\| < \varepsilon$ for $t \geq t_0$. We will show that the value $\Delta_0 = \delta(H)$ satisfies Definition 1.4. To this end, assume that $t_0 \geq 0$ and $\varphi \in PC([-h, 0], R^n)$, $\|\varphi\|_h < \Delta_0$. The corresponding solution $x(t, t_0, \varphi)$ is such that

$$\|x(t, t_0, \varphi)\| < H, \text{ for } t \geq t_0.$$

The second condition of the theorem implies the inequality

$$v(t, x_t(t_0, \varphi)) \leq v(t_0, \varphi) \mathrm{e}^{-2\sigma(t - t_0)}, \quad t \geq t_0.$$

Applying the first condition of the theorem we obtain that

$$\alpha_1 \|x(t, t_0, \varphi)\|^2 \leq v(t_0, \varphi) \mathrm{e}^{-2\sigma(t - t_0)} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma(t - t_0)}, \quad t \geq t_0.$$

The preceding inequalities provide the desired exponential estimate

$$\|x(t, t_0, \varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|\varphi\|_h \mathrm{e}^{-\sigma(t - t_0)}, \quad t \geq t_0.$$

□

## 1.6 Notes and References

The origins of the time-delay systems go back to such giants as L. Euler, J. L. Lagrange, and P. Laplace. A systematic development of the theory of functional differential equations began in the twentieth century with Volterra [69, 70], Myshkis [57], Krasovskii [46], Bellman and Cooke [3], Halanay [19], and Hale [21], to mention just the principal contributors.

The restriction of a solution, $x_t : \theta \to x(t+\theta)$, $\theta \in [-h, 0]$, as the true state of a time-delay system was introduced by Krasovskii [48]. This allowed him to develop the stability theory of time-delay systems to the same level as that of ordinary differential equations [46].

In the exposition of the basic existence and continuity results we follow the excellent book by Halanay [19]; see also [3, 6, 10, 11, 20, 23, 49].

The foundations of the Lyapunov second approach for time-delay systems, which is now known as the Lyapunov–Krasovskii approach, were developed by Krasovskii [46–48]; see also [44,58]. The form of presentation of the stability results in Sect. 1.5 was inspired by Zubov [72].