# Chapter 8
# Stochastic Orders in Reliability

**Abstract** Stochastic orders enable global comparison of two distributions in terms of their characteristics. Specifically, for a given characteristic $A$, stochastic order says that the distribution of $X$ has lesser (greater) $A$ than the distribution of $Y$. For example, one may use hazard rate or mean residual life for such a comparison. In this chapter, we discuss various stochastic orders useful in reliability modelling and analysis.

The stochastic order treated here are the usual stochastic order, hazard rate order, mean residual life order, harmonic mean residual life order, renewal and harmonic renewal mean residual life orders, variance residual life order, percentile residual life order, reversed hazard rate order, mean inactivity time order, variance inactivity time order, the total time on test transform order, the convex transform (IHR) order, star (IHRA) order, DMRL order, superadditive (NBU) order, NBUE order, NBUHR and NBUHRA orders and MTTF order. The interpretation of ageing concepts, preservation properties with reference to convolution, mixing and coherent structures are also discussed in relation to each of these orders. Implications among the different orders are also presented. Examples of the stochastic orders and counter examples where certain implications do not hold are also provided. Some special models used in reliability like proportional hazard and reverse hazard models, mean residual life models and weighted distributions have been discussed in earlier chapters. Some applications of these stochastic models are reviewed as well.

## 8.1 Introduction

There are many situations in practice wherein we need to compare the characteristics of two distributions. In certain cases, descriptive measures like mean and variance have been used for this purpose. Since these measures are summary measures of the data, they become less informative and so cannot capture all the essential features inherent in the data. An alternative approach to assess the relative behaviour of the properties of distributions is provided by stochastic orders which provide a global

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comparison by taking into account different features of the underlying models. Specifically, for a given characteristic $A$, a stochastic order says that the distribution $F_X$ of a random variable $X$ has lesser (greater) $A$ than the distribution $F_Y$ of $Y$ and we express it as $F_X \leq_A F_Y$ ($F_X \geq_A F_Y$), or equivalently in terms of the random variables $X \leq_A Y$ ($X \geq_A Y$). For example, in the context of reliability theory, if two manufacturers produce devices for the same purpose, the natural interest is to know which is more reliable. The reliability functions of the two devices then become natural objects for comparison and the characteristic in question may be their mean lives. But, when both devices were working for a specified time, the characteristic in question may change to the mean residual life and the comparison confirms which one of the two has more remaining life on an average. In all cases of comparison, the characteristic of comparison should have an appropriate measure $\omega(A)$, which should satisfy $\omega_X(A) \leq \omega_Y(A)$. Marshall and Olkin [412] point out that Mann and Whitney [409] used this approach initially and Birnbaum [101] subsequently to study peakedness. There is a phenomenal growth in the study of stochastic orders in recent years in such diverse fields as reliability theory, queueing theory, survival studies, biology, economics, insurance, operations research, actuarial science and management. In this chapter, we take up such stochastic orders and present results relevant to reliability analysis using quantile functions. Details of other orderings, proofs of results using the distribution function approach and so on are well documented; see, e.g., Szekli [557] and Shaked and Shantikumar [531].

Some notation need to be introduced first for the developments in subsequent discussions. Let $\Omega$ be a nonempty set. A binary relation $\leq$ on this set is called a preorder if

(i) $x \leq x, x \in \Omega$ (reflexivity),
(ii) $x \leq y, y \leq z \Rightarrow x \leq z$ (transitivity).

If, in addition, we also have

(iii) $x \leq y, y \leq x \Rightarrow x = y$ (anti-symmetry),
then $\leq$ is called a partial order. The term stochastic order considered here include both preorders and partial orders.

Let $F$ and $G$ be distribution functions of random variables $X$ and $Y$, respectively. Then, the function

$$\psi_{F,G}(x) = G^{-1}(F(x)), \tag{8.1}$$

for all real $x$, is called the relative inverse function of $F$ and $G$. If $F$ is continuous and supported by an interval of reals, then $\psi(X)$ and $Y$ are identically distributed. If $U$ is uniformly distributed over $[0,1]$, then $\psi_{F_U,G}(U)$ has the same distribution as $Y$. On the other hand, if $Y$ is exponential, $\psi_{\text{Exp},F}(Y)$ has the same distribution as $X$ for $X \geq 0$. These are easy to verify from the definition of the $\psi$ function. Further, if $F$ and $G$ are strictly increasing with derivatives $f$ and $g$, then

$$\frac{d}{dx}\psi(x) = \frac{f(x)}{g(G^{-1}F(x))} \tag{8.2}$$

and

$$\frac{d}{dx}FG^{-1}(x) = \frac{f(G^{-1}(x))}{gG^{-1}(x)}. \tag{8.3}$$

If $G$ is continuous with interval support, then

$$\psi^{-1}_{F,G}(x) = \psi_{G,F}(x). \tag{8.4}$$

## 8.2 Usual Stochastic Order

The usual stochastic order is basic in the sense that it compares the distribution functions of two random variables.

**Definition 8.1.** Let $X$ and $Y$ be random variables with quantile functions $Q_X(u)$ and $Q_Y(u)$, respectively. We say that $X$ is smaller than $Y$ in the usual stochastic order, denoted by $X \le_{\text{st}} Y$, if and only if

$$Q_X(u) \le Q_Y(u) \text{ for all } u \text{ in } (0,1).$$

The $\le_{\text{st}}$ ordering is usually employed to compare the distributions of two random variable $X$ and $Y$ or to compare the distribution of $X$ at two chosen parameter values.

*Example 8.1.* Let $X$ follow Pareto II distribution with

$$Q_X(u) = (1-u)^{-\frac{1}{c}} - 1, \quad c > 0,$$

and $Y$ follow the beta distribution with

$$Q_Y(u) = 1 - (1-u)^{\frac{1}{c}}, \quad c > 0.$$

Then,

$$\begin{aligned} Q_Y(u) - Q_X(u) &= 1 - (1-u)^{\frac{1}{c}} - \frac{1-(1-u)^{\frac{1}{c}}}{(1-u)^{\frac{1}{c}}} \\ &= -(1-u)^{-\frac{1}{c}}\left\{1-(1-u)^{\frac{1}{c}}\right\}^2 \\ &\le 0 \quad \text{for all } u. \end{aligned}$$

Thus, $X \ge_{\text{st}} Y$.

*Example 8.2.* Assume that $X_\lambda$ has exponential distribution with

$$Q(u) = -\frac{1}{\lambda}\log(1-u)$$

for $\lambda > 0$. It is easy to verify that for $\lambda_1 < \lambda_2$, $X_{\lambda_1} \leq_{\text{st}} X_{\lambda_2}$.

There are several equivalent forms of Definition 8.1 that are useful in establishing stochastic ordering results. We list them in the following theorem.

**Theorem 8.1.** *The following conditions are equivalent:*

*(i)* $X \leq_{st} Y$*;*
*(ii)* $\overline{F}_X(x) \leq \overline{F}_Y(x)$ *or* $F_X(x) \geq F_Y(x)$ *for all x;*
*(iii)* $E\phi(X) \leq E\phi(Y)$ *for all increasing functions* $\phi$ *for which the expectations exist. As a consequence, it is apparent that if* $\phi(x) = x^r$*, then*

$$X \leq_{st} Y \Rightarrow \begin{cases} E(X^r) \leq E(Y^r), & r \geq 0 \\ E(X^r) \geq E(Y^r), & r \leq 0 \end{cases}$$

*which connects the moments of the two distributions. Another function of interest is* $\phi(x) = e^{tx}$*, with which we have a comparison of moment generating functions as*

$$X \leq_{st} Y \Rightarrow \begin{cases} E(e^{tX}) \leq E(e^{tY}), & t \geq 0 \\ E(e^{tX}) \geq E(e^{tY}), & t \leq 0. \end{cases}$$

*Proof of the main result is available in Szekli [557]. If* $\phi$ *is strictly increasing and* $X \leq_{st} Y$*, then X and Y are identically distributed if* $E\phi(X) = E\phi(Y)$*;*
*(iv)* $\phi(X) \leq_{st} \phi(Y)$ *for all increasing functions* $\phi$*;*
*(v)* $Q_Y^{-1}(Q_X(u)) \leq u$*;*
*(vi)* $\phi(X,Y) \leq_{st} \phi(Y,X)$ *for all* $\phi(x,y)$*, where* $\phi(x,y)$ *is increasing in x and decreasing in y and X and Y are independent.*

One important advantage of studying stochastic orders is that many of the ageing concepts discussed earlier in Chap. 4 can be expressed in terms of some ordering. This in turn assists us in deriving many new properties and bounds based on the properties of the orderings, which are otherwise not explicit. We now present some theorems defining the IHR (DHR), NBU (NWU), NBUE, NBUC, RNBU, DMRL and RNBRU classes discussed in Chap. 4.

**Theorem 8.2.** *The lifetime variable X is IHR (DHR) if and only if* $X_t \leq_{st} (\geq) X_{t'}$ *whenever* $t < t'$*, where* $X_t = (X - t | X > t)$ *is the residual life.*

*Proof.* The quantile function of the residual life at $t$ is given by (1.4) as

$$Q_1(u) = Q(u_0 + (1-u_0)u) - Q(u_0),$$

where $u_0 = F(t)$ and $Q(\cdot)$ is the quantile function of $X$. Similarly, for $X_{t'}$, we have

$$Q_2(u) = Q(u_1 + (1-u_1)u) - Q(u_1),$$

with $u_1 = F(t') > u_0$. Now assume that $X_t \leq_{\text{st}} X_{t'}$. Then, by Definition 8.1, we have

$$\begin{aligned}
&Q(u_0+(1-u_0)u) - Q(u_0) \leq Q(u_1+(1-u_1)u) - Q(u_1)\\
&\quad \Leftrightarrow Q(u_1) - Q(u_0) \leq Q(u_1+(1-u_1)u) - Q(u_0+(1-u_0)u)\\
&\quad \Leftrightarrow \frac{Q(u_1)-Q(u_0)}{(1-u)(u_1-u_0)} \leq \frac{Q(u_1+(1-u_1)u) - Q(u_0+(1-u_0)u)}{(u_1+(1-u_1)u) - (u_0+(1-u_0)u)}\\
&\quad \Rightarrow \frac{1}{1-u}q(u_0) \leq q(u_0+(1-u_0)u) \qquad (8.5)\\
&\quad \Rightarrow \frac{1}{(1-u_0)q(u_0)} \leq \frac{1}{(1-u_0-(1-u_0)u)q(u_0+(1-u_0)u)}\\
&\quad \Rightarrow H(u_0) \leq H(u_0+(1-u_0)u) \text{ for every } u_0 \text{ in } (0,1).\\
&\quad \Rightarrow X \text{ is IHR.}
\end{aligned}$$

Conversely, when $X$ is IHR, we can retrace the above steps up to (8.5). However, (8.5) is equivalent to

$$\frac{d}{du_0}\left\{\frac{1}{1-u}Q(u_0) - \frac{1}{1-u}Q(u_0+(1-u_0))u\right\} \leq 0$$

which means that

$$Q(u_0) - Q(u_0+(1-u_0)u)$$

is a decreasing function of $u_0$. Hence,

$$Q(u_0) - Q(u_0+(1-u_0)u) \geq Q(u_1) - Q(u_1+(1-u_1)u)$$

for $u_1 > u_0$ or $Q_1(u) \leq Q_2(u)$ as we wished to prove. The proof of the DHR case is obtained by simply reversing the above inequalities.

**Theorem 8.3.** *A lifetime $X$ is NBU (NWU) if and only if $X \geq_{st} (\leq_{st}) X_t$.*

The result is a straightforward application of Definition 4.22.

**Theorem 8.4.** *If $X$ is a lifetime random variable with $E(X) < \infty$, then $X$ is NBUE (NWUE) if and only if $X \geq_{st} (\leq_{st}) Z$, where $Z$ is the equilibrium random variable with survival function* (4.7)*.*

*Proof.* Assume that $X \geq_{\text{st}} Z$. Then, from (4.9), we have

$$Q_X(u) \geq Q_Z(u) = \mu Q_X(T_X^{-1}(u)),$$

where $T_X(x) = \int_0^u (1-p)q(p)dp$ and $\mu = E(X)$. This gives

$$X \geq_{\text{st}} Z \Leftrightarrow Q_X(T_X(u)) \geq Q_X(\mu u) \Leftrightarrow \int_0^u (1-p)q(p)dp \geq \mu u$$
$$\Leftrightarrow \mu - \int_u^1 (1-p)q(p)dp \geq \mu u$$
$$\Leftrightarrow \frac{1}{1-\mu}\int_u^1 (1-p)q(p)dp \leq \mu \Leftrightarrow X \text{ is NBUE.}$$

from Definition 4.33.

**Theorem 8.5 (Nair and Sankaran [446]).**

*(a)* $X \geq_{st} Z_t$ *for all* $t \geq 0 \Leftrightarrow X$ *is NBUC, where* $Z_t = Z - t|(Z > t)$ *is the residual life of* $Z$*;*
*(b)* $Z \geq_{st} X_t \Leftrightarrow X$ *is RNBU;*
*(c)* $X_t \geq_{st} Z_t \Leftrightarrow X$ *is DMRL;*
*(d)* $Z \geq_{st} Z_t \Leftrightarrow X$ *is RNBRU.*

As with ageing criteria, it is customary to study the preservation properties of stochastic orders. With regard to the usual stochastic order, the following properties hold:

1. Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two sets of independent random variables. For every increasing function $\phi$, we have

$$\phi(X_1, X_2, \ldots, X_n) \leq_{\text{st}} \phi(Y_1, Y_2, \ldots, Y_n)$$

whenever $X_i \leq_{\text{st}} Y_i$. Hence, if $X_i \leq_{\text{st}} Y_i$, then

$$\sum_{i=1}^n X_i \leq_{\text{st}} \sum_{i=1}^n Y_i.$$

Thus the usual stochastic order preserves convolution property or is closed under the formation of additional lifelengths.
2. The ordering $\leq_{\text{st}}$ is preserved under convergence in distribution. That is, if $(X_n)$ and $(Y_n)$ are sequences such that $X_n \to X$ and $Y_n \to Y$ as $n \to \infty$ in distribution and if $X_n \leq_{\text{st}} Y_n$, $n = 1, 2, \ldots$, then $X \leq_{\text{st}} Y$.
3. Under the formulation of mixture distributions, $\leq_{\text{st}}$ is closed. This means that if $X, Y$ and $\Theta$ are random variables satisfying

$$[X|\Theta = \theta] \leq_{\text{st}} [Y|\Theta = \theta]$$

for all $\theta \in \Theta$, then $X \leq_{st} Y$.

4. A further extension of Property 1 above for random convolution is possible. If $X_i$'s and $Y_i$'s are non-negative, $M$ is a non-negative integer valued random variable independent of the $X_i$'s and $N$ is non-negative integer valued random variable and independent of the $Y_i$'s, then

$$X_i \leq_{\text{st}} Y_i \Rightarrow \sum_{i=1}^{M} X_i \leq_{\text{st}} \sum_{i=1}^{N} Y_i$$

provided $M \leq_{\text{st}} N$.

5. The ordering $X \leq_{\text{st}} Y$ is closed under shifting and scaling meaning that

$$X \leq_{\text{st}} Y \Rightarrow CX \leq_{\text{st}} CY$$

and

$$X \leq_{\text{st}} Y \Rightarrow X + a \leq_{\text{st}} Y + a.$$

More properties of the $\leq_{\text{st}}$ ordering will appear in connection with other orderings discussed later. Further properties of $\leq_{\text{st}}$ can be found in Muller and Stoyan [432], Scarsini and Shaked [521], Barlow and Proschan [68] and Ma [406].

## 8.3 Hazard Rate Order

In hazard rate ordering, we compare two distributions by means of the relative magnitude of their hazard rates. The idea behind this comparison is that when the hazard rate becomes larger, the variable becomes stochastically smaller.

**Definition 8.2.** If $X$ and $Y$ are lifetime random variables with absolutely continuous distribution functions, we say that $X$ is smaller than $Y$ in hazard rate order, denoted by $X \leq_{\text{hr}} Y$, if

$$H_X(u) \geq H_Y^*(u),$$

where $H_X(u) = h_X(Q_X(u))$ and $H_Y^*(u) = h_Y(Q_X(u))$ and $h(\cdot)$ denotes the hazard rate function.

*Example 8.3.* The hazard quantile function of the Pareto II distribution (Table 2.4) is

$$H_X(u) = \frac{c(1-u)^{\frac{1}{c}}}{\alpha}$$

and the hazard rate function of the beta distribution with $R = 1$ is $h_Y(x) = \frac{c}{1-x}$. Hence,

$$H_Y^*(u) = h_Y(Q_X(u)) = h_Y((1-u)^{-\frac{1}{c}} - 1)$$
$$= \frac{c}{2-(1-u)^{-\frac{1}{c}}}.$$

It is easy to check that for $0 < u < 1$, $H_X(u) < H_Y^*(u)$ and so $X \geq_{\text{hr}} Y$.

Some equivalent conditions that ensure hazard rate order are presented in the following theorem.

**Theorem 8.6.** *X is less than Y in hazard rate order if and only if*

*(i)* $u^{-1}F_Y(Q_X(1-u))$ *is decreasing in u;*
*(ii)* $u^{-1}[1-F_X(Q_Y(1-u))]$ *is decreasing in u;*
*(iii)* $\frac{\overline{F}_Y(x)}{\overline{F}_X(x)}$ *is increasing in x;*
*(iv)* $\overline{F}_X(x)\overline{F}_Y(y) \geq \overline{F}_X(y)\overline{F}_Y(x)$ *for all* $x \leq y$*;*
*(v)* $\frac{\overline{F}_X(x+y)}{\overline{F}_X(x)} \leq \frac{\overline{F}_Y(x+y)}{\overline{F}_Y(x)}$ *for all* $x, y \geq 0$*;*
*(vi)* $(X|X > x) \leq_{st} (Y|Y > x)$.

*Proof.* (i) From (8.3), we have

$$\frac{\overline{F}_Y(Q_X(1-u))}{u} \text{ is decreasing in } u \Leftrightarrow uf_Y(Q_X(1-u))q_X(1-u) - \overline{F}_Y(Q_X(1-u)) \leq 0$$
$$\Leftrightarrow \frac{f_Y(Q_X(1-u))}{\overline{F}_y(Q_X(1-u))} \leq \frac{1}{uq_X(1-u)}$$
$$\Leftrightarrow h_Y(Q_X(1-u)) \leq H_X(1-u)$$
$$\Leftrightarrow H_Y^*(1-u) \leq H_X(1-u) \text{ for all } 0 < u < 1$$
$$\Leftrightarrow X \leq_{\text{hr}} Y.$$

The proof of (ii) is exactly similar. Result (iii) is obtained from (i) by setting $u = \overline{F}(x)$ and noting that since $u = \overline{F}(x)$ when $u$ is decreasing $x$ is increasing. Notice that (iv) is a consequence of (iii) while (v) is equivalent to (iv) and (vi) to (v).

When different stochastic orders are studied, the implications, if any, between them is also an important aspect. The relationship between $\leq_{\text{st}}$ and $\leq_{\text{hr}}$, e.g., is explained in the following theorem.

**Theorem 8.7.**

$$X \leq_{hr} Y \Rightarrow X \leq_{st} Y,$$

*but not conversely.*

*Proof.*

$$X \leq_{\rm hr} Y \Leftrightarrow \frac{\overline{F}_X(x+y)}{\overline{F}_X(x)} \leq \frac{\overline{F}_Y(x+y)}{\overline{F}_Y(x)}, \text{ for all } x \leq y$$

$$\Rightarrow \overline{F}_X(y) \leq \overline{F}_Y(y) \text{ for all } y > 0, \text{ when } x \to 0.$$

$$\Rightarrow X \leq_{\rm st} Y.$$

To prove that the converse need not be true, let $X$ be distributed as exponential with $Q_X(u) = -\log(1-u)$ and $Y$ follow distribution with survival function

$$\overline{F}_Y = e^{-x} + e^{-2x} - e^{-3x}, \quad x > 0.$$

Since $\overline{F}_X(x) = e^{-x}$, it is easy to verify that $\overline{F}_X(x) \leq \overline{F}_Y(x)$ and so $X \leq_{\rm st} Y$. On the other hand,

$$Q(1-u) = \overline{F}^{-1}(u) = -\log u$$

and so

$$u^{-1}F_Y(Q(1-u)) = \frac{F_Y(-\log u)}{u} = 1 + u - u^2.$$

The last expression is increasing for $u$ in $(0, \frac{1}{2}]$ and decreasing for $u$ in $[\frac{1}{2}, 1)$. The hazard rates are therefore not ordered by (i) of Theorem 8.6. Hazard ordering allows definition of certain ageing classes encountered previously in Chap. 4 as the following theorems illustrate.

**Theorem 8.8.** *The random variable $X$ is IHR (DHR) if and only if any one of the following conditions hold:*

*(i)* $(X-t|X>t) \geq_{hr} (\leq_{hr})(X-s|X>s)$ *for all* $t \leq s$;
*(ii)* $X \geq_{hr} (X-t|X>t)$ *for all* $t \geq 0$;
*(iii)* $X+t \leq_{hr} X+s$, $t \leq s$.

The proof of the theorem rests on the fact that $(X-t|X>t)$ has its hazard rate as $h(x+t)$.

**Theorem 8.9.** *If $E(X) < \infty$, then:*

*(a)* *$X$ is DMRL* $\Leftrightarrow X \geq_{hr} Z$;
*(b)* *$X$ is IMRL* $\Leftrightarrow X \leq_{hr} Z$.

*Proof.* (a) We see that

$$X \geq_{\rm hr} Z \Leftrightarrow H_X(u) \leq H_Z(u) = \frac{1}{M_X(u)}$$

$$\Leftrightarrow H_X(u)M_X(u) \leq 1$$

$$\Leftrightarrow 1 - (1-u)H_X(u)M'_X(u) \leq 1$$

$$\Leftrightarrow M'_X(u) \leq 0 \Leftrightarrow X \text{ is DMRL}.$$

The proof of (b) is obtained by reversing the inequalities in the above argument.

**Theorem 8.10.** *If $E(X) < \infty$, then:*

*(a)* $Z \geq_{hr} (Z - t|Z > t) \Leftrightarrow X$ *is DMRL;*
*(b)* $Z_{t_1} \geq_{hr} Z_{t_2}$ *for* $0 < t_1 < t_2 \Leftrightarrow X$ *is DMRL.*

*Proof.* By Part (ii) of Theorem 8.8, we see that

$$Z \geq_{\text{hr}} (Z - t|Z > t) \Leftrightarrow Z \text{ is IHR } \Leftrightarrow X \text{ is DMRL.}$$

From proving (b), we use Part (i) of Theorem 8.8 and the same argument as for (a).

Some preservation properties useful in reliability analysis concerning the hazard rate ordering are as follows:

1. For every increasing function $\phi(x)$, $\phi(X) \leq_{\text{hr}} \phi(Y)$, whenever $X \leq_{\text{hr}} Y$;
2. In general, convolution is not preserved under hazard rate ordering. However, if $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are both independent collections such that $X_i \leq_{\text{hr}} Y_i$, $i = 1, 2, \ldots, n$, and $X_i$ and $Y_i$ are IHR for all $i$, then
$$\sum_{i=1}^{n} X_i \leq_{\text{hr}} \sum_{i=1}^{n} Y_i.$$
3. If $X_1, X_2, \ldots, X_n$ is a sequence of independent IHR lifetime variables and $M$ and $N$ are discrete positive integer valued random variables such that $M \leq_{\text{hr}} N$ and are independent of the $X_i$'s, then
$$\sum_{i=1}^{M} X_i \leq_{\text{hr}} \sum_{i=1}^{N} X_i.$$
Thus, the ordering '$\leq_{\text{hr}}$' is only conditionally closed under the formation of random convolutions.
4. If $X, Y$ and $\Theta$ are random variables such that $X|(\Theta = \theta) \leq_{\text{hr}} Y|(\Theta = \theta')$ for all $\theta$ and $\theta'$ in the support of $\Theta$, then $X \leq_{\text{hr}} Y$ (Lehmann and Rojo [383]).
5. For $0 < a \leq 1$ and $X$ is IHR, $aX \leq_{\text{hr}} X$ (Kochar [346]).
6. If $X_1, X_2, \ldots, X_n$ are independent, then:
   (a) $X_{k:n} \leq_{\text{hr}} X_{k+1:n}$ (Boland et al. [114, 115]);
   (b) $X_{1:1} \geq_{\text{hr}} X_{1:2} \geq_{\text{hr}} \cdots \geq_{\text{hr}} X_{1:n}$;
   (c) $X_{k:n-1} \geq_{\text{hr}} X_{k:n}$, $k = 1, 2, \ldots, n-1$.

   The results in (b) and (c) are due to Korwar [352] in connection with $k$-out-of-$n$ system. Proofs of the above properties along with some more general results are given in Sect. 1.B of Shaked and Shantikumar [531].
7. If the hazard rate $h(x)$ of $X$ is such that $xh(x)$ is increasing, then $Y = aX$, $a \geq 1$, satisfies $X \leq_{\text{hr}} Y$.

## 8.4 Mean Residual Life Order

Let $X$ be a non-negative random variable representing the lifetime of a device with $E(X) = \mu < \infty$. Then, the comparison of the mean residual lives of $X$ and $Y$ by their magnitudes provides a stochastic ordering of the distributions of $X$ and $Y$. Assume also that $E(Y) < \infty$.

**Definition 8.3.** $X$ is said to be smaller than $Y$ in mean residual quantile function order if

$$M_X(u) \leq M_Y^*(u),$$

written as $X \leq_{\text{mrl}} Y$, where

$$M_X(u) = m_X(Q_X(u)) \quad \text{and} \quad M_Y^*(u) = m_Y(Q_X(u)).$$

*Example 8.4.* Let $X$ and $Y$ have distributions with quantile functions

$$Q_X(u) = 1 - (1-u)^{\frac{1}{c}}, \quad c > 0,$$

and

$$Q_Y(u) = 1 - (1-u)^{-\frac{1}{c}} - 1, \quad c > 0,$$

respectively. Then,

$$\overline{F}_Y(x) = (1+x)^{-c}, \quad x > 0.$$

We have

$$M_X(u) = \frac{1}{1-u}\int_u^1 (1-p)q(p)dp = \frac{(1-u)^{\frac{1}{c}}}{c+1},$$
$$M_Y(x) = \frac{1+x}{c-1},$$

and

$$M_Y^*(u) = m_Y(Q_X(u)) = \frac{2-(1-u)^{\frac{1}{c}}}{c-1}, \quad c > 1,$$
$$M_X(u) - M_Y^*(u) = 2c(1-u)^{\frac{1}{c}} - 2(c+1)$$
$$= 2c\left\{(1-u)^{\frac{1}{c}} - \frac{c+1}{c}\right\} < 0.$$

Hence, $X \leq_{\text{mrl}} Y$.

There are several equivalent conditions for the validity of $X \leq_{\text{mrl}} Y$ as presented in the following theorem.

**Theorem 8.11.** *$X \leq_{mrl} Y$ if and only if any of the following conditions hold:*

*(a)* $m_X(x) \leq m_Y(x)$ *for all* $x > 0$;
*(b)* $\frac{\int_x^\infty \overline{F}_Y(t)dt}{\int_x^\infty \overline{F}_X(t)dt}$ *is an increasing function of x, or equivalently*

$$\frac{1}{\overline{F}_Y(Q_X(u))}\int_{Q_X(u)}^{\infty} \overline{F}_Y(x)dx \geq \frac{1}{1-u}\int_u^1 (1-p)q_X(p)dp;$$

*(c)* $\frac{P_X(u)}{P_Y^*(x)}$ *is an increasing function of u, when* $P_X(u)$ *is the partial mean*

$$P_X(u) = \int_u^1 (1-p)q(p)dp$$

*defined in* (6.47) *and*

$$P_Y^*(u) = P_Y(Q_X(u)) = \int_{Q_X(u)}^{\infty} \overline{F}_Y(t)dt.$$

Notice that (a) is the definition of the mean residual life order in the distribution function approach. Differentiating (b) and noting that the derivative is non-negative, we get (a). Setting $x = Q(u)$ in (b), we obtain (c) which is equivalent to (b).

The classes of life distributions induced by $\leq_{\text{mrl}}$ are presented in the following theorem.

**Theorem 8.12.** *(a) X is DMRL if and only if any one of the following properties hold:*

*(i)* $X_t \geq_{mrl} X_{t'}$ *for* $t' \geq t$;
*(ii)* $X \geq_{mrl} X_t$;
*(iii)* $X + t \leq_{mrl} X + t'$.

*(b) X is DRMRL if and only if any one of the following properties hold:*

*(i)* $X \geq_{mrl} Z$;
*(ii)* $X_t \geq_{mrl} Z_t$;
*(iii)* $Z \leq_{mrl} Z_t$.

Part (a) follows readily from the fact that the mean residual life of $X_t$ is $m(x+t)$ and the definition of $\leq_{\text{mrl}}$. To prove (b), recall Definition 4.17. $X$ is said to DRMRL if and only if $e_X(u) \leq M_X(u)$, where (4.24)

$$e(u) = \frac{\int_u^1 [Q(p) - Q(u)](1-p)q(p)dp}{\int_u^1 (1-p)q(p)}.$$

The mean residual functions of $X$, $Z$, $X_t$ and $Z_t$ are, respectively, $m(x)$, $e(x)$, $m(x+t)$ and $m^*(x+t)$ (4.23). Hence, (i) implies

$$X \geq_{\text{mrl}} Z \Leftrightarrow m(x) \geq e(x)$$
$$\Leftrightarrow M_X(u) \geq e_X(u)$$
$$\Leftrightarrow X \text{ is DRMRL.}$$

Other properties follow similarly.

Regarding the closure properties enjoyed by $\leq_{\text{mrl}}$, some of the important ones are as follows:

1. For every increasing convex function $\phi(x)$, $X \leq_{\text{mrl}} Y$ implies $\phi(X) \leq_{\text{mrl}} \phi(Y)$.
2. The mean residual life order is closed with respect to the formation of mixtures under certain conditions only. If $X|(\Theta = \theta) \leq Y|(\Theta = \theta')$ for all $\theta$, $\theta'$ in the support of $\Theta$, then $X \leq_{\text{mrl}} Y$ (Nanda et al. [460]).
3. $(X_i, Y_i)$, $i = 1, 2, \ldots, n$, are independent pairs of IHR random variables such that $X_i \leq_{\text{mrl}} Y_i$ for all $i$, then (Pellerey [490])
$$\sum_{i=1}^{n} X_i \leq_{\text{mrl}} \sum_{i=1}^{n} Y_i.$$
4. For a sequence $\{X_n\}$, $n = 1, 2, \ldots$, of independent and identically distributed IHR random variables,
$$\sum_{i=1}^{M} X_i \leq_{\text{mrl}} \sum_{i=1}^{N} X_i,$$
where $M$ and $N$ are positive integer valued random variables such that $M \leq_{\text{mrl}} N$ (Pellerey [490]).
5. If $X$ is DMRL and $0 < a \leq 1$, then $aX \leq_{\text{mrl}} X$.
6. Let $X_1, X_2, \ldots X_n$ be independent. If $X_i \leq_{\text{mrl}} X_n$, for $i = 1, 2, \ldots, n-1$, then $X_{n-1:n-1} \leq_{\text{mrl}} X_{n:n}$.
7. Let $U$ be a random variable with mixture distribution function $\alpha F_X(x) + (1-\alpha)F_Y(x)$, $0 < \alpha < 1$. If $X \leq_{\text{mrl}} Y$, then $X \leq_{\text{mrl}} U \leq_{\text{mrl}} Y$.

The hazard quantile function and the mean residual quantile function are closely related and determine each other. Moreover, the IHR class of life distributions is a subclass of the DMRL class. We now examine how the orderings based on the hazard quantile and mean residual quantile functions imply each other.

**Theorem 8.13.** *If $X \leq_{hr} Y$, then $X \leq_{mrl} Y$, but the converse need not be true.*

*Proof.* We have

$$\begin{aligned}
X \leq_{\text{hr}} Y &\Rightarrow H_X(u) \geq H_Y^*(u) \\
&\Rightarrow \int_u^1 \frac{dp}{H_X(p)} \leq \int_u^1 \frac{dp}{H_{Y^*}(p)} \\
&\Rightarrow \int_u^1 (1-p)q_X(p)dp \leq \int_{Q(u)}^{\infty} \frac{\overline{F}_Y(t)dt}{f_Y(t)} \\
&\Rightarrow M_X(u) \leq M_Y^*(u) \\
&\Rightarrow X \leq_{\text{mrl}} Y.
\end{aligned}$$

To prove the second part, let $X$ have standard exponential distribution with

$$Q_X(u) = -\log(1-u)$$

so that $E(X) = 1$, and $Y$ be Weibull with

$$Q_Y(u) = \sigma(-\log(1-u))^{\frac{1}{\lambda}}.$$

The parameters of $Y$ be chosen such that $\lambda > 1$ and $E(Y) < 1$. Since $\lambda > 1$, $Y$ is IHR and hence NBUE. This means that

$$M_Y(u) \leq 1 = E(X) = M_X(u) \text{ for all } 0 < u < 1.$$

Thus, $Y \leq_{\text{mrl}} X$. On the other hand, $H_X(u) = 1$ and

$$h_Y(x) = \frac{\lambda}{\sigma^{\lambda}} x^{\lambda-1}.$$

This gives

$$H_Y^*(u) = \frac{\lambda}{\sigma^{\lambda}}(-\log(1-u))^{\lambda-1}$$

or

$$H_X(u) - H_Y^*(u) = 1 - \frac{\lambda}{\sigma^{\lambda}}(-\log(1-u))^{\lambda-1}.$$

We can see that $X$ and $Y$ are not ordered in hazard rate since

$$H_X(u) \leq H_Y^*(u) \text{ for } u \text{ in } \left(0, 1-\exp\left(\frac{\sigma^{\lambda}}{\lambda}\right)^{\frac{1}{\lambda-1}}\right)$$

and

$$H_X(u) \geq H_Y^*(u) \text{ in } \left(1-\exp\left(\frac{\sigma^{\lambda}}{\lambda}\right)^{\frac{1}{\lambda-1}}, 1\right).$$

The above result leads us to seek conditions under which the $\leq_{\text{hr}}$ ordering can be generated from the $\leq_{\text{mrl}}$ ordering.

**Theorem 8.14 (Belzunce et al. [86]).**

1. *$X \leq_{hr} Y \Rightarrow \min(X,Z) \leq_{mrl} \min(Y,Z)$ for any non-negative random variable $Z$ independent of $X$ and $Y$;*
2. *$X \leq_{hr} Y \Rightarrow 1-e^{-sX} \leq_{mrl} 1-e^{-sY}$, $s>0$.*

A result that is helpful in establishing the mrl ordering is stated in the following theorem.

**Theorem 8.15.** *If $X$ and $Y$ have finite means,*

$$X \leq_{mrl} Y \Leftrightarrow Z_X \leq_{hr} Z_Y,$$

*where $Z_X$ and $Z_Y$ denote the equilibrium random variables corresponding to $X$ and $Y$, respectively.*

This result is immediate from the fact that the hazard quantile function of $Z_X(Z_Y)$ is the reciprocal of the mean residual quantile function of $X(Y)$. A comparison between the usual stochastic order and the mrl order is even more interesting. Although the mean residual life function determines the distribution uniquely, there is no implication between $\leq_{\text{st}}$ and $\leq_{\text{mrl}}$. This is seen from the following examples furnished by Gupta and Kirmani [241]. Upon choosing

$$F_X(x) = \begin{cases} e^{-x}, & 0 \leq x < 1, \\ e^{-x^{\frac{1}{2}}}, & x \geq 1, \end{cases}$$

and

$$F_Y(x) = e^{-x^{\frac{1}{2}}}, \quad x > 0,$$

we see that $F_Y(x) \leq F_X(x)$ or $X \geq_{\text{hr}} Y$. At the same time, $m_X(x)$ and $m_Y(x)$ are not ordered. Secondly, in the counter example in Theorem 8.13, $\overline{F}_X(x) - \overline{F}_Y(x)$ can have both negative and positive signs ruling out either $X \leq_{\text{hr}} Y$ or $X \geq_{\text{hr}} Y$. But, $X \geq_{\text{mrl}} Y$. With additional assumptions on $X$ and $Y$, implications between the two orders can be established as provided in the following theorem.

**Theorem 8.16 (Gupta and Kirmani [241]).**

1. *If $\frac{M_X(u)}{M_Y^*(u)}$ is increasing in $u$, then*

$$X \leq_{mrl} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y;$$

2. *If $\frac{M_X(u)}{M_Y^*(u)} \geq \frac{E(X)}{E(Y)}$, then*

$$X \leq_{mrl} Y \Rightarrow X \leq_{st} Y.$$

*We have conditions under which the mrl order ensures stochastic equality of $X$ and $Y$. If $X \geq_{mrl} Y$, $E(Y) > 0$, $E(X) = E(Y)$ and $V(X) = V(Y)$, then $X$ and $Y$ have the same distribution.*

For some additional results on mrl ordering, one may refer to Alzaid [35], Ahmed [28], Joag-Dev et al. [298], Fagiouli and Pellerey [190, 192], Hu et al. [288], Zhao and Balakrishnan [602] and Nanda et al. [459].

Another stochastic order that involves the mean residual life is the harmonic mean residual life order defined as follows.

**Definition 8.4.** $X$ is said to be smaller than $Y$ in harmonically mean residual life order, denoted by $X \leq_{\text{hmrl}} Y$, if and only if

$$\left\{\frac{1}{x}\int_0^x \frac{dt}{m_X(t)}\right\}^{-1} \leq \left\{\frac{1}{x}\int_0^x \frac{dt}{m_Y(t)}\right\}^{-1},$$

or equivalently

$$\int_0^u \frac{q_X(p)dp}{M_X(p)} \geq \int_0^u \frac{q_X(p)dp}{M_Y(Q_X(p))}.$$

*Example 8.5.* Let $X$ be distributed as Pareto I with $\overline{F}_X(x) = (\frac{x}{\sigma})^{-\alpha_1}$. Then, we have

$$Q_X(u) = \sigma(1-u)^{-\frac{1}{\alpha_1}},$$

$$M_X(u) = \sigma\frac{(1-u)^{-\frac{1}{\alpha_1}}}{\alpha_1 - 1},$$

$$\int_0^u \frac{q_X(p)dp}{M_X(p)} = \frac{\alpha_1 - 1}{\alpha_1}(-\log(1-u)).$$

Assume that $Y$ has Pareto distribution with

$$\overline{F}_Y(x) = \left(\frac{x}{\sigma}\right)^{-\alpha_2},$$

$$M_Y(Q_X(u)) = \sigma\frac{(1-u)^{-\frac{1}{\alpha_1}}}{\alpha_2 - 1},$$

$$\int_0^u \frac{q_X(p)dp}{M_Y^*(p)} = \frac{\alpha_2 - 1}{\alpha_1}(-\log(1-u)).$$

Hence, $X \leq_{\text{hmrl}} Y$ if and only if $\alpha_1 \geq \alpha_2$.

Some equivalent conditions for $X \leq_{\text{hmrl}} Y$ are as follows:

(i)
$$\frac{\int_x^\infty \overline{F}_X(t)dt}{E(X)} \leq \frac{\int_x^\infty \overline{G}(t)dt}{E(Y)} \left(\frac{\int_u^1 (1-p)q(p)dp}{E(X)} \leq \frac{\int_{Q_X(u)}^\infty \overline{G}(t)dt}{E(Y)}\right);$$

(ii) $\frac{E\phi(X)}{E(X)} \leq \frac{E\phi(Y)}{E(Y)}$ for all increasing convex functions $\phi(x)$;

(iii) $\frac{P_X(u)}{E(X)} \leq \frac{P_Y^*(u)}{E(Y)}$, where $P_Y^*(u)$ is as in Part (c) of Theorem 8.11.

As a further consequence of the hmrl order, we also have

$$X \leq_{\text{hmrl}} Y \Rightarrow E(X) \leq E(Y)$$

and in addition if $Y$ is NWUE (Kirmani [328, 329]), then

$$V(X) \leq V(Y);$$

(iv) $Z_X \leq_{\text{st}} Z_Y$.

The preservation properties enjoyed by the hmrl order are summarized in the following theorem. Here, all the variables involved $X, Y, X_i$ and $Y_i$ are non-negative. For proofs and other details, we refer the reader to Pellerey [490] and Nanda et al. [460].

**Theorem 8.17.** *(a) $(X_i, Y_i)$, $i = 1, 2, \ldots, n$, are independent pairs of random variables such that $X_i \leq_{hmrl} Y_i$ for all i. If $X_i, Y_i$ are all NBUE, then*

$$\sum_{i=1}^{n} X_i \leq_{hmrl} \sum_{i=1}^{n} Y_i;$$

*(b) $(X_n)$ and $(Y_n)$ are sequences of NBUE independent and identically distributed random variables satisfying $X_n \leq_{hmrl} Y_n$, $n = 1, 2, \ldots$. If M and N are positive integer-valued random variables independent of the sequences $\{X_n\}$ and $\{Y_n\}$ such that $M \leq_{hmrl} N$, then*

$$\sum_{i=1}^{M} X_i \leq_{hmrl} \sum_{j=1}^{N} Y_j;$$

*(c) Let X, Y and $\Theta$ be random variables with $X|(\Theta = \theta) \leq_{hmrl} Y|(\Theta = \theta')$ for all $\theta$ and $\theta'$ in the support of $\Theta$. Then, $X \leq_{hmrl} Y$;*

*(d) If $X, Y$ and $\Theta$ are random variables such that $X|(\Theta = \theta) \leq_{hmrl} Y|(\Theta = \theta)$ for all $\theta$ in the support of $\Theta$ along with the additional condition*

$$E(Y|\Theta = \theta) = kE(X|\Theta = \theta),$$

*where k is independent of $\theta$, then $X \leq_{hmrl} Y$;*

*(e) If $E(X), E(Y) > 0$ and $E(X) \leq E(Y)$, then $X =_{hmrl} Y$ if and only if $X =_{st} UY$, where U is a Bernoulli variable independent of Y;*

*(f) If U has mixture distribution*

$$F_U(x) = \alpha F_X(x) + (1-\alpha) F_Y(x), \quad 0 < \alpha < 1,$$

*then*

$$X \leq_{hmrl} Y \Rightarrow X \leq_{hmrl} U \leq_{hmrl} Y.$$

The DMRL class and NBUE class of life distributions can be characterized by the hmrl order as given in the following theorem.

**Theorem 8.18.** *(i) $X$ is DMRL $\Leftrightarrow X_t \geq_{hmrl} X_{t'}$, $t' \geq t \geq 0$;*
*(ii) $X$ is NBUE $\Leftrightarrow X \leq_{hmrl} Y$, where $Y$ is independent of $X$ and $E(Y) > 0$;*
*(iii) $X$ is NBUE $\Leftrightarrow X + Y_1 \leq_{hmrl} X + Y_2$, where $Y_1$ and $Y_2$, are independent of $X$, $E(Y_i) < \infty$, $i = 1, 2$, and $Y_1 \leq_{hmrl} Y_2$.*

The results in Parts (ii) and (iii) are due to Lefevre and Utev [381].

Finally, we study the relationships the hmrl order have with some other orders. First of all, by the increasing nature of harmonic averages, we have

$$X \leq_{\text{mrl}} Y \Rightarrow X \leq_{\text{hmrl}} Y.$$

Even otherwise, in terms of quantile functions,

$$\begin{aligned} X \leq_{\text{mrl}} Y &\Rightarrow M_X(u) \leq M_Y^*(u), \text{ where } M_Y^*(u) = M_Y(Q_X(u)). \\ &\Rightarrow \frac{q_X(u)}{M_X(u)} \geq \frac{q_X(u)}{M_{Y^*}(u)} \\ &\Rightarrow \int_0^u \frac{q_X(p)dp}{M_X(p)} \geq \int_0^u \frac{q_X(p)dp}{M_Y^*(p)dp} \\ &\Leftrightarrow X \leq_{\text{hmrl}} Y. \end{aligned}$$

The converse need not be true and so the $\leq_{\text{hmrl}}$ order is weaker than the $\leq_{\text{mrl}}$ order. Moreover, neither the usual stochastic order nor the hmrl order imply the other (see Deshpande et al. [173]).

## 8.5 Renewal and Harmonic Renewal Mean Residual Life Orders

Recall the definition of the renewal mean residual life function (4.23)

$$m^*(x) = \frac{\int_x^\infty (t-x)\overline{F}(t)dt}{\int_x^\infty \overline{F}(t)dt}, \tag{8.6}$$

which is an alternative to the traditional mean residual life function, as it facilitates all the functions and calculations enjoyed by the latter. The quantile-based definition is

$$
\begin{aligned}
e(u) = m^*(Q(u)) &= \frac{\int_u^1 [Q(p)-Q(u)](1-p)q(p)dp}{\int_u^1 (1-p)q(p)dp} \\
&= \left\{ \int_u^1 (1-p)q(p)dp \right\}^{-1} \int_u^1 \int_p^1 (1-t)q(t)q(p)dtdp.
\end{aligned}
\tag{8.7}
$$

In this section, we discuss the properties of a stochastic order based on the $e(u)$ in (8.7), and these results are taken from Nair and Sankaran [446].

**Definition 8.5.** The random variable $X$ is said to be less (greater) than $Y$ in renewal mean residual life order, denoted by $X \leq_{\text{rmrl}} Y$, if and only if

$$m_X^*(x) \leq (\geq) m_Y^*(x) \text{ for all } x \geq 0,$$

or equivalently

$$e_X(u) \leq (\geq) e_Y^*(u) \text{ for all } 0 < u < 1,$$

where $e_Y^*(u) = m_Y^*(Q_X(u))$ and $e_X(u) = m_X^*(Q_X(u))$.

*Example 8.6.* Let $X$ be distributed with quantile function

$$Q_X(u) = 1-(1-u)^{\frac{1}{3}}$$

and $Y$ have its quantile function as

$$Q_Y(u) = (1-u)^{-\frac{1}{12}} - 1.$$

Then, from (8.7), we have

$$e_X(u) = \frac{(1-u)^{\frac{1}{3}}}{5}.$$

Again, $m_Y^*(x) = \frac{2+x}{10}$ so that

$$e_Y^*(u) = \frac{3-(1-u)^{\frac{1}{3}}}{10}.$$

It is easy to see that $e_X(u) \leq e_Y^*(u)$ for all $u$, and so $X \leq_{\text{rmrl}} Y$.

Some other conditions that characterize the rmrl order are as follows:

(a) $\frac{\int_x^\infty \int_u^\infty \overline{F}_X(t)dtdu}{\int_x^\infty \int_u^\infty \overline{F}_Y(t)dtdu}$ is increasing in $x$ over $\{x | \int_x^\infty \overline{F}_Y(t)dt > 0\}$;

(b) $(\int_x^\infty \overline{F}_Y(t)dt)(\int_x^\infty \int_u^\infty \overline{F}_X(t)dtdu) \leq (\int_x^\infty \overline{F}_X(t)dt)(\int_x^\infty \int_u^\infty \overline{F}_Y(t)dtdu)$;

(c) $\frac{\int_x^\infty E(X-t)^+ dt}{\int_x^\infty E(Y-t)^+ dt}$ is decreasing.

By the methods used earlier, the results in (a), (b) and (c) above can also be expressed in terms of quantile functions.

One issue of primary interest is the relationship between the usual mrl order and the rmrl order, which is described in the following theorem.

**Theorem 8.19.** *If $X \leq_{mrl} Y$, then $X \leq_{rmrl} Y$. But, the converse is not true.*

*Proof.* For simplicity, we write $Q_X(u) = Q(u)$ throughout the proof. We have

$$\begin{aligned}
X \leq_{\text{mrl}} Y &\Rightarrow \frac{1}{1-u}\int_u^1 (1-p)q(p)dp \leq \frac{1}{\overline{G}(Q(u))}\int_u^1 \overline{G}(Q(p))q(p)dp \\
&\Rightarrow \frac{1-u}{\int_u^1 (1-p)q(p)dp} \geq \frac{\overline{G}Q(u)}{\int_u^1 \overline{G}(Q(p))q(p)dp} \\
&\Rightarrow \frac{d}{du}\log\int_u^1 (1-p)q(p)dp \leq \frac{d}{du}\log\int_u^1 \overline{G}(Q(p))q(p)dp \\
&\Rightarrow \int_p^u \left(\frac{d}{dt}\log\int_t^1 (1-p)q(p)dpdt\right) \leq \int_p^u \left(\frac{d}{dt}\log\int_t^1 \overline{G}(Q(p))q(p)dp\right) \\
&\Rightarrow \frac{\int_u^1 (1-p)q(p)dp}{\int_p^1 (1-t)q(t)dt} \leq \frac{\int_u^1 \overline{G}(Q(p))q(p)dp}{\int_p^1 \overline{G}(Q(t))q(t)dt} \\
&\Rightarrow \frac{\int_p^1\int_u^1 (1-t)q(t)q(u)du}{\int_p^1 (1-t)q(t)dt} \leq \frac{\int_p^1\int_u^1 \overline{G}(Q(t))q(t)q(u)du}{\int_p^1 \overline{G}Q(t)q(t)dt} \\
&\Rightarrow e_X(p) \leq e_Y^*(p) \Leftrightarrow X \leq_{\text{rmrl}} Y.
\end{aligned}$$

To prove the latter part of the theorem, we reconsider Example 8.5 wherein we had established that for the random variables $X$ and $Y$ described therein, $X \leq_{rmrl} Y$. In this case, we also have

$$M_X(u) = \frac{(1-u)^{\frac{1}{3}}}{4}$$

and

$$m_Y(x) = \frac{2+x}{11}$$

giving

$$M_Y^*(u) = \frac{3-(1-u)^{\frac{1}{3}}}{11}.$$

Thus,

$$M_X(u) - M_Y^*(u) = \frac{3}{44}\left\{5(1-u)^{\frac{1}{3}} - 4\right\}$$

which is decreasing in $(0, \frac{61}{125})$ and increasing in $(\frac{61}{125}, 1)$. Hence, $X$ and $Y$ are not ordered in mrl.

*Remark.* One could see that the $\leq_{\text{rmrl}}$ order is strictly weaker than the $\leq_{\text{mrl}}$ order and consequently generates a larger class of life distributions.

As was done in the mrl order, we consider conditions under which the two orders become equivalent in the following theorem.

**Theorem 8.20.** *If $\frac{e_X(u)}{e_Y^*(u)}$, is an increasing function of u, then*

$$X \leq_{mrl} Y \Leftrightarrow X \leq_{rmrl} Y.$$

*Proof.* Since $\frac{e_X(u)}{e_Y^*(u)}$ is an increasing function of $u$, we have

$$\frac{e_X'(u)}{e_X(u)} \geq \frac{e_Y^{*'}(u)}{e_Y^*(u)}. \tag{8.8}$$

From (4.25), we have

$$M_X(u) = \frac{e_X(u)q_X(u)}{q_X(u) + e_X'(u)}. \tag{8.9}$$

But, by definition, we have

$$\begin{aligned} e_Y^*(u) &= m_Y^*(Q_X(u)) \\ &= \frac{\int_{Q_X(u)}^{\infty}(t-x)\overline{F}_Y(t)dt}{\int_{Q_X(u)}^{\infty}\overline{F}_Y(t)dt} \\ &= \frac{\int_u^1\int_p^1\overline{F}_Y(Q_X(t))q_X(t)dt}{\int_u^1\overline{F}_Y(Q_X(p))q_X(p)dp}. \end{aligned}$$

Differentiating and simplifying, we obtain

$$M_Y^*(u) = \frac{e_Y^*(u)q_X(u)}{q_X(u) + e_X'(u)} \tag{8.10}$$

From (8.8), (8.9) and (8.10), whenever $X \leq_{\text{rmrl}} Y$, we must have

$$\begin{aligned} \frac{1}{M_X(u)} &= \frac{1}{e_X(u)} + \frac{e_X'(u)}{e_X(u)} \\ &= \frac{1}{e_Y^*(u)} + \frac{e_Y^{*'}(u)}{e_Y^*(u)} = \frac{1}{M_Y^*(u)} \end{aligned}$$

and so

$$M_X(u) \le M_Y^*(u) \Leftrightarrow X \le_{\text{mrl}} Y.$$

The reverse inequality $X \le_{\text{mrl}} Y \Rightarrow X \le_{\text{rmrl}} Y$ has already been established in Theorem 8.8 and this completes the proof.

The procedure of taking harmonic averages and then comparing life distributions based on them is also possible with renewal mean residual life functions as described below.

**Definition 8.6.** $X$ is said to be smaller than $Y$ in harmonic renewal mean residual life, denoted by $X \le_{\text{hrmrl}} Y$, if and only if

$$\frac{1}{x}\int_0^x \frac{dt}{m_X^*(t)} \le \frac{1}{x}\int_0^x \frac{dt}{m_Y^*(t)}.$$

An equivalent definition is

$$\int_0^u \frac{q_X(p)dp}{e_X(p)} \ge \int_0^u \frac{q_X(p)dp}{e_Y^*(p)}. \tag{8.11}$$

It can be shown that (8.11) is equivalent to

$$\frac{E[(X-x)^+]^2}{E(X^2)} \le \frac{E[(Y-x)^+]^2}{E(Y^2)}.$$

The following properties hold for the $\le_{\text{hrmrl}}$ ordering:

(i) If $\frac{e_X(u)}{e_Y^*(u)}$, is increasing in $u$, then $X \le_{\text{hrmrl}} Y \Leftrightarrow X \le_{\text{hrmrl}} Y$;
(ii) In general,

$$X \le_{\text{hmrl}} Y \Rightarrow X \le_{\text{hrmrl}} Y;$$

(iii) $X \le_{\text{rmrl}} Y \Rightarrow X \le_{hrmrl} Y$.

The preservation properties and other implications of the rmrl and hrmrl orders have not yet been studied in detail.

## 8.6 Variance Residual Life Order

Earlier in Sect. 4.3, we have defined the variance residual life of $X$ as

$$\sigma^2(x) = \frac{2}{\overline{F}(x)}\int_x^\infty \int_u^\infty \overline{F}(t)dtdu - m^2(x),$$

or in terms of quantile function as

$$V(u) = \sigma^2 Q(u) = (1-u)^{-1} \int_u^1 M^2(p)dp, \tag{8.12}$$

where $M(u)$ is the mean residual quantile function.

**Definition 8.7.** We say that $X$ is smaller than $Y$ in variance residual life, denoted by $X \leq_{\text{vrl}} Y$, if and only if any of the following equivalent conditions hold:

(i) $\sigma_X^2(x) \leq \sigma_Y^2(x)$ for all $x > 0$;
(ii) $V_X(u) \leq V_Y^*(u)$ for all $0 < u < 1$, where $V_Y^*(u) = \sigma_Y^2(Q_X(u))$.

For the definition in (i) and properties of the vrl ordering, one may refer to Singh [541].

Connection of the $\leq_{\text{vrl}}$ ordering with the $\leq_{\text{mrl}}$ ordering is presented in the next theorem.

**Theorem 8.21.** *If $X \leq_{mrl} Y$, then $X \leq_{vrl} Y$.*

*Proof.* The result easily follows from the fact

$$X \leq_{\text{mrl}} Y \Rightarrow M_X(u) \leq M_Y^*(u)$$

and (8.12).

If $\overline{F}_1$ and $\overline{F}_2$ are survival functions of the equilibrium random variables of $X$ and $Y$, respectively, Fagiouli and Pellery [192] defined

$$X \leq_{\text{vrl}} Y \text{ if } \frac{\int_x^\infty \overline{F}_1(t)dt}{\int_x^\infty \overline{F}_2(t)dt}$$

is nonincreasing in $x \geq 0$. There has not been much investigation on the preservation properties and other aspects of the vrl order.

## 8.7 Percentile Residual Life Order

The percentile life ordering was introduced by Joe and Proschan [301] in the context of testing the hypothesis of the equality of two distributions. Earlier, we have defined the $\alpha$th percentile residual life function for any $0 < \alpha < 1$ as

$$p_\alpha(x) = F^{-1}(1-(1-\alpha)\overline{F}(x)) - x$$

or

$$p_\alpha(u) = p_\alpha(Q(u)) = Q[1-(1-\alpha)(1-u)] - Q(u).$$

Franco-Pereira et al. [202] have discussed some properties of the percentile order.

**Definition 8.8.** We say that $X$ is smaller than $Y$ in the $\alpha$-percentile residual life, denoted by $X \leq_{\text{prl}-\alpha} Y$, if and only if

$$p_{\alpha,X}(x) \leq p_{\alpha,Y}(x) \quad (P_{\alpha,X}(u) \leq P^*_{\alpha,Y}(u))$$

for all $x$ (for all $u$) and $P^*_{\alpha,Y}(u) = p_{\alpha,Y}(Q(u))$.

One specific aspect about the prl order is that, unlike other orderings we have discussed, it is indexed by $\alpha$ which can take any value in $(0,1)$. Moreover, the percentile residual life function $P_\alpha(u)$, for a given $\alpha$, does not determine the distribution uniquely. If $X \leq_{\text{prl}-\alpha} Y$, then the upper end point of the support of $X$ cannot exceed that of $Y$, but it is not necessary that a corresponding result hold for the left end point of the supports of the random variables.

*Example 8.7.* Consider the distribution (Pareto) with quantile function

$$\begin{aligned} Q(u) &= (1-u)^{-\frac{1}{\alpha}}, \quad 0 < u < 1 \\ P_\alpha(u) &= [1 - \{1-(1-\alpha)(1-u)\}]^{-\frac{1}{\alpha}} - (1-u)^{-\frac{1}{\alpha}} \\ &= (1-u)^{-\frac{1}{\alpha}}[(1-u)^{-\frac{1}{\alpha}} - 1]. \end{aligned}$$

Let $X$ and $Y$ be random variables with the above distribution with parameters $\alpha_1$ and $\alpha_2$, respectively. Then, we find

$$P_{\alpha,X}(u) - P^*_{\alpha,Y}(u) = (1-u)^{-\frac{1}{\alpha_1}} \left\{ (1-\alpha)^{-\frac{1}{\alpha_1}} - (1-\alpha)^{-\frac{1}{\alpha_2}} \right\},$$

and so

$$X \leq_{\text{prl}-\alpha} Y \text{ for } \alpha_2 \leq \alpha_1.$$

Two useful characterizations of the $\leq_{\text{prl}}$ order, one in terms of quantile functions and the other in terms of distribution functions, are presented in the following theorem both of which are direct consequences of the definition.

**Theorem 8.22.** *$X \leq_{prl-\alpha} Y$ and only if*

*(i)* $Q_X(\alpha + (1-\alpha)u) \leq Q_Y(\alpha + (1-\alpha)Q_Y^{-1}(Q_X(u)))$,
*(ii)* $\frac{\overline{F}_Y(Q_X(u))}{u} \leq \frac{\overline{F}_Y(Q_X(1-\alpha)(u))}{(1-\alpha)u}$ *for all* $0 < u < 1$.

The following relationships exist between the prl order and some other orders we have discussed:

(a) $X \leq_{\text{hr}} Y \Leftrightarrow X \leq_{\text{prl}-\alpha} Y$ for all $\alpha$ in $(0,1)$;
(b) For a specific $\alpha$, $X \leq_{\text{hr}} Y \Rightarrow X \leq_{\text{prl}-\alpha} Y$. So, the result in (a) is not practically useful;
(c) Percentile life orders do not preserve expectations and as such $\leq_{\text{prl}-\alpha}$ neither implies the usual stochastic order, mean residual life order, and hmrl order, for any $\alpha$. Further, stochastic order does not imply prl order, or mrl or hmrl orders;
(d) If, for $0 < \beta < 1$, $X \leq_{\text{prl}-\alpha} Y$ for every $\alpha$ in $(0,\beta)$, then $X \leq_{\text{hr}} Y$. Naturally, if $X \leq_{\text{prl}-\alpha} Y$ for all $\alpha$ in $(0,\beta)$, then $X \leq_{\text{prl}-\alpha} Y$ for all $\alpha$.

Some interesting preservation properties, established by Franco-Pereira et al. [202], are as follows:

1. For an increasing function $\phi(\cdot)$, we have

$$X \leq_{\text{prl}-\alpha} Y \Leftrightarrow \phi(X) \leq_{\text{prl}-\alpha} \phi(Y);$$

2. Let $(X_n)$, $(Y_n)$, $n = 1,2,\ldots$, be two sequences of random variables such that $X_n \to X$ and $Y_n \to Y$ in distribution as $n \to \infty$. If $X$ and $Y$ have continuous distributions with interval support, then for any $\alpha$, if $X_n \leq_{\text{prl}-\alpha} Y_n$ holds, $n = 1,2,\ldots$, then $X \leq_{\text{prl}-\alpha} Y$;
3. Let $X_\theta$, $\theta \in \Theta$, and $Y_\theta$, $\theta \in \Theta$, be two families of random variables with continuous distributions. If

$$F_W(x) = \int_\Theta F_X(x|\theta)dH(\theta)$$

and

$$F_Z(x) = \int_\Theta F_Y(x|\theta)dH(\theta),$$

where $H$ is some distribution function on $\Theta$ and $U$ is a random variable such that

$$X_\theta \leq_{\text{prl}-\alpha} U \leq_{\text{prl}-\alpha} Y_\theta \quad \text{for all } \theta \in \Theta,$$

then

$$W \leq_{\text{prl}-\alpha} Z.$$

In particular, if $W$ has the mixture distribution function

$$F_W = pF_X + (1-p)F_Y$$

for some $0 \leq p \leq 1$, then

$$X \leq_{\text{prl}-\alpha} Y \Rightarrow X \leq_{\text{prl}-\alpha} W \leq_{\text{prl}-\alpha} Y;$$

4. The prl-$\alpha$ order is not closed under the formation of parallel or series systems. However, if $X_i$, $Y_i$, $i = 1,2,\ldots,n$, are independent and identically distributed random variables with continuous distributions, satisfying $X_1 \leq_{\text{prl}-\alpha} Y_1$, then

$$\min(X_1, X_2, \ldots, X_n) \leq_{\text{prl}-\beta} (Y_1, Y_2, \ldots, Y_n),$$

where $\beta = 1 - (1-\alpha)^n$.

## 8.8 Stochastic Order by Functions in Reversed Time

Earlier in Sect. 2.4, we have defined and given examples of reliability functions in reversed time like the reversed hazard quantile function and the reversed mean residual quantile function. These functions have also been used in Sect. 4.5 to introduce various ageing classes. It is therefore possible to order life distributions on the basis of their magnitudes, and this is the focus of the present section.

### *8.8.1 Reversed Hazard Rate Order*

Let $X$ and $Y$ be two absolutely continuous random variables with reversed hazard rates

$$\lambda_X(x) = \frac{f_X(x)}{F_X(x)} \text{ and } \lambda_Y(x) = \frac{f_Y(x)}{F_Y(x)},$$

respectively.

**Definition 8.9.** $X$ is said to be smaller than $Y$ in reversed hazard rate order, denoted by $X \leq_{\text{rh}} Y$, if and only if

$$\lambda_X(x) \leq \lambda_Y(x) \text{ for all } x > 0,$$

or equivalently

$$\Lambda_X(u) \leq \Lambda_Y^*(u) \text{ for all } 0 < u < 1,$$

where $\Lambda_Y^*(u) = \lambda_Y(Q_X(u))$ (see (2.50)).

Some other conditions that characterize the $\leq_{\text{rh}}$ order are presented in the following theorem.

**Theorem 8.23.** *$X \le_{rh} Y$ if and only if*

*(a)* $\frac{Q_Y^{-1}(Q_X(u))}{u} \le \frac{Q_Y^{-1}Q_X(v)}{v}$ *for all* $0 < u \le v < 1$*;*
*(b)* $\frac{F_Y(x)}{F_X(x)}$ *increases in x;*
*(c)* $X|(X \le x) \le_{st} Y|(Y \le x)$ *for all* $x > 0$.

Nanda and Shaked [461] have proved a basic relationship between the $\le_{\rm hr}$ order and the $\le_{\rm rh}$ order as presented in the following theorem, and it simplifies the proofs of many results.

**Theorem 8.24.** *For two continuous random variables X and Y,*

$$X \le_{hr} Y \Rightarrow \phi(X) \ge_{rh} \phi(Y)$$

*for any continuous function $\phi$ which is strictly decreasing on $(a_1, b_2)$, where $a_1$ is the lower end of the support of X and $b_2$ is the upper end of the support of Y. Furthermore,*

$$X \le_{rh} Y \Rightarrow \phi(X) \le_{rh} \phi(Y)$$

*when $\phi$ is strictly increasing.*

Various properties of the $\le_{\rm rh}$ order have been studied by many authors including Kebir [321], Shaked and Wang [533], Kijima [325], Block et al. [111], Hu and He [285], Nanda and Shaked [461], Gupta and Nanda [254], Yu [597], Zang and Li [599] and Brito et al. [120]. There exists a relationship between the $\le_{st}$ and the $\le_{\rm rh}$ orders which is stated in the following theorem.

**Theorem 8.25.** *If $X \le_{rh} Y$, then $X \le_{st} Y$.*

*Proof.* We observe that

$$\begin{aligned} X \le_{\rm rh} Y &\Rightarrow \lambda_X(u) \le \lambda_Y(Q_X(u)) \Rightarrow \frac{1}{uq(u)} \le \frac{1}{F_Y(Q_X(u))q_Y(Q_X(u))} \\ &\Rightarrow -\log u \le -\log F_Y(Q_X(u)) \Rightarrow \frac{1}{u} \le \frac{1}{F_Y(Q_X(u))} \\ &\Rightarrow Q_X(u) \le Q_Y(u) \Rightarrow X \le_{\rm st} Y, \end{aligned}$$

as required.

The preservation properties enjoyed by the $\le_{\rm rh}$ order are as follows:

(i) *Convolution property* Let $(X_i, Y_i)$, $i = 1, 2, \ldots, n$, be $n$ pairs of random variables such that $X_i \le_{\rm rh} Y_i$ for all $i$. If all $X_i, Y_i$ have decreasing reversed hazard rates, then

$$\sum_{i=1}^{n} X_i \le_{\rm rh} \sum_{i=1}^{n} Y_i;$$

(ii) *Mixture function* If $X|(\Theta = \theta) \le_{\rm rh} Y|(\Theta = \theta')$ for all $\theta, \theta'$ in the support of $\Theta$, then $X \le_{\rm rh} Y$;

(iii) *Order statistics*

(a) If $X_i$ are independent, $i = 1,2,\ldots,n$, then

$$X_{k:n} \leq_{\text{rh}} X_{k+1:n}, \quad k = 1,2,\ldots,n-1;$$

(b) If $X_n \leq_{\text{rh}} X_i$ for $i = 1,2,\ldots,n-1$, then

$$X_{k-1:n-1} \leq_{\text{rh}} X_{k:n}, \quad k = 2,3,\ldots,n;$$

(c) Let $X_i, Y_i$ be pairs of independent absolutely continuous random variables with $X_i \leq_{\text{rh}} Y_i$, $i = 1,2,\ldots,n$. If the $X_i$'s and $Y_i$'s are also identically distributed, then

$$X_{k:n} \leq_{\text{rh}} Y_{k:n}, \quad k = 1,2,\ldots m.$$

Under slightly different conditions, without the assumption of identical distributions for $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$, if $X_i \leq_{\text{rh}} Y_j$ for all $i, j$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, the result that

$$X_{i:n} \leq_{\text{rh}} Y_{j:m}$$

holds for $i - j \geq \max(0, m-n)$.

### 8.8.2 Other Orders in Reversed Time

The reversed mean residual life function and the corresponding reversed mean residual quantile function have been defined earlier as

$$r(x) = E[x - X | X \leq x] = \frac{1}{F(x)} \int_0^x F(t) dt$$

and

$$R(u) = r(Q(u)) = \frac{1}{u} \int_0^u p q(p) dp.$$

Nanda et al. [459] introduced an ordering of reversed mean residual life, and their definition and the equivalent version in terms of quantile function are presented in the following theorem.

**Definition 8.10.** The random variable $X$ is said to be smaller than the random variable $Y$ in reversed mean residual life, denoted by $X \leq_{\text{MIT}} Y$, if and only if

$$r_X(x) \geq r_Y(x) \text{ for all } x,$$

or equivalently

$$R_X(u) \geq R_Y^*(u) \text{ for all } 0 < u < 1,$$

where $R_Y^*(u) = r_Y(Q_X(u))$.

Sometimes, the reversed mean residual life is also called the mean inactivity time and so the corresponding ordering is called the mean inactivity time order or simply the MIT order. The relationship of the MIT order to some other orders has been discussed in the literature; see, e.g., Nanda et al. [462], Kayid and Ahmad [319] and Ahmed et al. [24]. It has been shown that, for $0 < t_1 < t_2$, $X$ is DRHR if and only if

(i) $X_{(t_1)} \leq_{\text{st}} X_{(t_2)}$, $X_{(t)} = t - X|(X \leq t)$ is the inactivity time;
(ii) $X_{(t_1)} \leq_{\text{hr}} X_{(t_2)}$;
(iii) for all positive integers $m$ and $n$,

$$F^{m+n}(x) \geq F^m\left(\frac{n}{m}x\right) F^n\left(\frac{m}{n}x\right).$$

Further,

$$X \leq_{\text{rh}} Y \Rightarrow X \leq_{\text{MIT}} Y,$$

but the converse need not be true.

Ahmed and Kayid [23] have shown that if $\frac{r_X(x)}{r_Y(x)}$ is an increasing function of $x$, then the $\leq_{\text{rh}}$ order and the $\leq_{\text{MIT}}$ order are equivalent. Li and Xu [393] have made a comparison of the residual $X_t$ and the inactivity time $X_{(t)}$ of series and parallel systems. Instead of considering the life at a specified time $t$, Li and Zuo [395] discussed the residual life at a random time $Y$ through the random residual life of the form

$$X_Y = (X - Y)|(X > Y)$$

and the inactivity at the random time of the form

$$X_{(Y)} = (Y - X)|(X \leq Y).$$

Notice that the distribution function of $X_Y$ then becomes

$$\begin{aligned} P(X_Y \leq x) &= P(X - Y \leq x | X > Y) \\ &= \frac{\int_0^\infty [F_X(y+x) - F_X(y)] dF_Y(y)}{\int_0^\infty F_Y(y) dF(y)}. \end{aligned}$$

They then established that $X$ has increasing mean inactivity time if and only if $X \leq_{\text{MIT}} X + Y$ for any $Y$ independent of $X$. Moreover, if $\phi$ is a strictly increasing concave function with $\phi(0) = 0$, then

$$X \leq_{\rm MIT} Y \Rightarrow \phi(X) \leq_{\rm MIT} \phi(Y).$$

Ortega [474] has some additional results concerning the $\leq_{\rm rh}$ and $\leq_{\rm MIT}$ orders presented in the following theorem.

**Theorem 8.26.** *When X and Y are absolutely continuous random variables,*

$$X \leq_{rh} Y \Leftrightarrow \exp[sX] \leq_{MIT} \exp(sY) \textit{ for all } s > 0.$$

It may be noted that Theorem 8.26 characterizes the $\leq_{\rm rh}$ order in terms of the $\leq_{\rm MIT}$ order. Conversely, the reverse characterization is apparent from

$$X \leq_{\rm MIT} Y \Leftrightarrow \log X^{\frac{1}{S}} \leq_{\rm rh} \log Y^{\frac{1}{S}} \text{ for all } s > 0.$$

The MIT order is also related to the mrl order as

$$X \leq_{\rm MIT} Y \Rightarrow \phi(X) \geq_{\rm mrl} \phi(Y)$$

for any strictly decreasing convex function $\phi : [0,\infty) \to [0,\infty)$.

The following preservation properties of order statistics and convolutions hold in this case.

**Theorem 8.27.** *(i) Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ be two sets of independent and identically distributed random variable with support $[0,\infty)$. Then,*

$$X_1 \leq_{MIT} Y_1 \Rightarrow X_{k:n} \leq_{rh} Y_{l:m}, \quad k \geq l \textit{ and } n-k \leq m-l;$$

*(ii) If $X_n \leq_{MIT} X_i$, $i = 1,2,\ldots,n-1$, then*

$$X_{k+1:n} \leq_{rh} X_{k:n-1}, \quad k = 1,2,\ldots,m-1;$$

*also, when $X_1, X_2, \ldots, X_n$ are independent absolutely continuous random variables with $X_i \leq_{MIT} Y_j$ for all $i, j$, then:*

*(a) $X_{l:n} \leq_{rh} Y_{l:n}$, $l = 1,2,\ldots,n$;*
*(b) $X_{k:n} \leq_{rh} Y_{l:n}$, $k \geq l$, $n \leq m$.*

**Theorem 8.28.** *Let $X = \sum_{i=1}^{N} X_i$ and $Y = \sum_{i=1}^{M} Y_i$, where $(X_i, Y_i)$ are independent pairs of random variables such that $X_i$ has decreasing reversed hazard rate, $Y_i$ also has decreasing reversed hazard rate, and $X_i \geq_{MIT} Y_i$, $i = 1,2,\ldots$, and $N \geq_{rh} M$, then $X \geq_{MIT} Y$.*

Another function in reversed time for which stochastic orders can be defined is the reversed variance residual life (variance of inactivity time, VIT) given by

$$v(x) = E\left[(x-X)^2|X \le x\right] - r^2(x)$$
$$= \frac{2}{F(x)}\int_0^x \int_0^y F(t)dtdy - r^2(x),$$

or equivalently in quantile form as

$$D(u) = \frac{1}{u}\int_0^u R^2(p)dp$$

(see (2.53)). Mahdy [408] has then defined the following stochastic order.

**Definition 8.11.** We say that $X$ is smaller than $Y$ in variance inactivity time order, denoted by $X \le_{\text{VIT}} Y$, if and only if

$$\frac{\int_0^x \int_0^t F_X(y)dydt}{F_X(x)} \ge \frac{\int_0^x \int_0^t F_Y(y)dydt}{F_Y(x)}$$

for all $x \ge 0$. In other words,

$$\frac{1}{u}\int_0^u R_X^2(p)dp \ge \frac{1}{F_Y(Q_X(u))}\int_0^u R_X^{*2}(p)dp$$

for all $u$ in $(0,1)$, where $R_Y^*(p) = v_Y(Q_X(p))$

Some properties of the $\le_{\text{VIT}}$ order are as follows:

1. A necessary and sufficient condition for $X \le_{\text{VIT}} Y$ is that

$$\frac{\int_0^x \int_0^t F_X(y)dydt}{\int_0^x \int_0^t F_Y(y)dydt}$$

is an increasing function of $x$;
2. $X$ has increasing VIT $\Leftrightarrow X \le_{\text{VIT}} X+Y$, where $Y$ is independent of $X$;
3. If $\phi$ is strictly increasing and concave with $\phi(0) = 0$, then

$$X \le_{\text{VIT}} \Rightarrow \phi(X) \le_{\text{VIT}} \phi(Y);$$

4. If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent copies of $X$ and $Y$, respectively, then

$$\max_{1\le i\le n} X_i \le_{\text{VIT}} \max_{1\le i\le n} Y_i \Rightarrow X \le_{\text{VIT}} Y.$$

## 8.9 Total Time on Test Transform Order

Recall from (5.6) that the total time on test transform (TTT) of $X$ is defined as

$$T(u) = \int_0^u (1-p)q(p)dp.$$

The role of this function in characterizing life distributions, ageing properties and in various other applications have been described earlier in Chap. 5. Here, $T(u)$ represents the quantile function of a random variable, say $X_T$, in the support of $[0,\mu]$, where $\mu = E(X)$. In this section, we define and study some properties of an order obtained through the comparison of the TTT's of two random variables; for further details, one may refer to Kochar et al. [349] and Li and Shaked [392].

**Definition 8.12.** A random variable $X$ is said to be smaller than another random variable $Y$ in total time on test transform order, denoted by $X \leq_{\text{TTT}} Y$, if

$$T_X(u) \leq T_Y(u)$$

for all $u \in (0,1)$.

*Example 8.8.* Let $X$ be exponential with mean $\frac{1}{4}$, i.e.,

$$Q_X(u) = -4\log(1-u),$$

and $Y$ be uniform with

$$Q_Y(u) = u.$$

Then, we have $T_X(u) = \frac{u}{4}$ and $T_Y(u) = \frac{u(2-u)}{4}$ so that

$$T_X(u) - T_Y(u) = \frac{4}{u}(u-1) < 0 \text{ for all } 0 < u < 1.$$

Hence, $X \leq_{\text{TTT}} Y$.

Some interesting relationships possessed by the $\leq_{\text{TTT}}$ order are presented in the following theorem.

**Theorem 8.29.** *(i)* $X \leq_{st} Y \Rightarrow X \leq_{TTT} Y$*;*
*(ii)* $X \leq_{TTT} Y \Rightarrow aX \leq_{TTT} aY$, $a > 0$*;*
*(iii)* $X_T \leq_{st} Y_T \Leftrightarrow X \leq_{TTT} Y$*, where* $X_T$ *denotes the random variable with quantile function* $T(u)$*;*
*(iv)* $X \leq_{TTT} Y \Rightarrow X_T \leq_{TTT} Y_T$*;*
*(v)* $X \leq_{st} Y \Rightarrow X_T \leq_{st} Y_T$.

*Proof.* (i) We note that

$$T(u) = \int_0^u (1-p)q(p)dp$$
$$= (1-u)Q(u) + \int_0^u Q(p)dp.$$

Now,

$$\begin{aligned} X \le_{\rm st} Y &\Rightarrow Q_X(u) \le Q_Y(u) \\ &\Rightarrow (1-u)Q_X(u) + \int_0^u Q_X(p)dp \le (1-u)Q_Y(u) + \int_0^u Q_Y(p)dp \\ &\Rightarrow T_X(u) \le T_Y(u) \Rightarrow X \le_{\rm TTT} Y. \end{aligned}$$

Part (ii) follows from the fact that $Q_{aX}(a) = aQ_X(u)$ and (iii) is obvious from the definitions of the stochastic and TTT orders. To prove Part (iv), we note that the transform of $X_T$ is

$$T_{X_T}(u) = \int_0^u (1-u)t_X(u),$$

where $t_X(u) = T_X'(u)$, the quantile density function of $X_T$. The last equation, using integration by parts, becomes

$$T_{X_T}(u) = (1-u)T_X(u) + \int_0^u T_X(p)dp.$$

The proof of Part (iv) is then similar to that of (i). Part (v) is a direct consequence of Parts (iii) and (i).

**Theorem 8.30.** *If $X$ and $Y$ have zero as the common left end point of their supports, then for an increasing concave function $\phi$ with $\phi(0) = 0$,*

$$X \le_{TTT} Y \Rightarrow \phi(X) \le_{TTT} \phi(Y).$$

**Theorem 8.31 (Li and Zuo [395]).** *Let $\{X_n\}$, $\{Y_n\}$, $n = 1,2,\ldots$, be two sequences of independent and identically distributed random variables and $N$ be a positive integer valued random variable independent of the $X$'s and $Y$'s. If $X_1 \le_{TTT} Y_1$, then*

$$\min_{1 \le i \le N} X_i \le_{TTT} \min_{1 \le i \le N} Y_i.$$

Extensions of the above results are possible if we consider total time on test transform of order $n$ $(\text{TTT}-n)$ introduced earlier in (5.26). Recall that TTT$-n$ is defined as

$$T_n(u) = \int_0^u (1-p)t_{n-1}(p)dp, \quad n = 1,2,\ldots,$$

with $T_0(u) = Q(u)$ and $t_n(u) = \frac{dT_n(u)}{du}$, provided $\mu_{n-1} = \int_0^1 T_{n-1}(u)du < \infty$.

**Definition 8.13.** $X$ is said to be smaller than $Y$ in TTT of order $n$, written as $X \leq_{\mathrm{TTT}-n} Y$, if and only if $T_{n+1,X} \leq T_{n+1,Y}$ for all $u$ in $(0,1)$. Denote by $X_n$ and $Y_n$ the random variables with quantile functions $T_{n,X}(u)$ and $T_{n,Y}(u)$, respectively.

As in the case of the first order transforms $T(u)$, we have the following relationships:

(i) $X \leq_{\mathrm{TTT}-n} Y \Leftrightarrow X_{n+1} \leq_{\mathrm{st}} Y_{n+1}$;
(ii) $X \leq_{\mathrm{TTT}} Y \Rightarrow X \leq_{\mathrm{TTT}-n} Y$.

If $(X_1, X_2, \ldots X_n)$ and $(Y_1, Y_2, \ldots Y_n)$ are independent copies of $X$ and $Y$ that are identically distributed and $X \leq_{\mathrm{TTT}-n} Y$, then $\min(X_1, X_2, \ldots, X_n) \leq_{\mathrm{TTT}-n} \min(Y_1, Y_2, \ldots, Y_n)$. For further results and other aspects of TTT$-n$ order, we refer the reader to Nair et al. [447].

## 8.10 Stochastic Orders Based on Ageing Criteria

So far, our attention has focussed on partial orders that compare life distributions on the basis of reliability concepts. In view of the predominant role ageing criteria have in modelling and in the analysis of reliability data, it will be natural to consider similar comparisons that spell out which of the two given distributions is more positively ageing than the other in terms of concepts like IHR, IHRA, NBU, etc. This idea has resulted in some partial orders that are discussed in this section.

We begin with the convex transform order defined by Barlow and Proschan [68].

**Definition 8.14.** Let $X$ and $Y$ have continuous distributions with $F_X(0) = F_Y(0) = 0$, and $F_Y(x)$ be strictly increasing on an interval support. Then, we say that $X$ is less than $Y$ in convex transform order, denoted by $X \leq_c Y$, if $F_Y^{-1}(F_X(x))$ is a convex function in $x$ on the support of $X$, assumed to be an interval.

Notice that according to (8.1), $\psi_{F_X,F_Y}(x) = F_Y^{-1}(F(x))$ is the relative inverse function of $F_X$ and $F_Y$, and it enjoys the properties of $\psi$ mentioned earlier in Sect. 8.1. An immediate consequence of Definition 8.14 is that if $Y$ is exponential, then

$$\psi_{F_X,F_Y}(x) = F_Y^{-1} F_X(x) = -\frac{1}{\lambda} \log(1 - F(x))$$

is convex, which means that

$$\psi'(x) = \frac{1}{\lambda} \frac{f(x)}{\overline{F}(x)} = \frac{1}{\lambda} h(x)$$

is increasing, or $X$ is IHR. It is easy to see that the converse also holds. Thus, we have an equivalent condition for $X$ to be IHR in terms of $\leq_c$ as follows.

**Theorem 8.32.** *$X$ is IHR if and only if $X \le_c Y$, where $Y$ is exponential.*

In the above result, $Y$ can have any scale parameter. In general, in terms of distribution function,

$$F_X <_c F_Y \Leftrightarrow F_X(\alpha x) <_c F_Y(\beta x)$$

for all $\alpha, \beta > 0$, and so $<_c$ is unaffected by scaling. Kochar and Wiens [350] have developed an ordering based on IHR from the above facts.

**Definition 8.15.** We say that $X$ is more IHR than $Y$ if $X \le_c Y$. Making use of (8.3) and (8.2) and assuming that $X$ and $Y$ have densities, we find

$$\begin{aligned} \frac{d}{dx} F_Y^{-1} F_X(x) &= \frac{f_X(F_X^{-1}(x))}{f_Y(F_Y^{-1}(x))} \\ &= \frac{f_X(Q_X(u))}{f_Y(Q_Y(u))} = \frac{q_Y(u)}{q_X(u)}. \end{aligned}$$

Hence, $X \le_c Y$ if and only if $\frac{q_Y(u)}{q_X(u)}$ is increasing in $u$ in $[0,1]$.

**Theorem 8.33.**

$$X \le_c Y \Leftrightarrow X_T \le_c Y_T.$$

*Proof.* From the above discussion, we have seen that $X \le_c Y$ if and only if the ratio of the quantile density functions $\frac{q_Y(u)}{q_X(u)}$ of $X$ and $Y$ is increasing in $u$. The quantile density functions of $X_T$ and $Y_T$ are

$$t_{X_T}(u) = (1-u) q_X(u)$$

and

$$t_{Y_T}(u) = (1-u) q_Y(u).$$

Since $\frac{q_Y}{q_X}$ is increasing by hypothesis, $\frac{t_{Y_T}}{t_{X_T}}$ is also increasing by virtue of the fact that $\frac{t_{Y_T}}{t_{X_T}} = \frac{q_Y}{q_X}$. Hence, $X_T \le_c Y_T$, as required.

There is a preservation property for the order statistics as well as described below.

**Theorem 8.34.** *Let $\{X_n\}$, $\{Y_n\}$ be two sequences of independent and identically distributed random variables and $N$ be a positive integer valued random variable independent of the $X_i$'s and $Y_i$'s. If $X_1 \le_c Y_1$, then*

$$\min_{1 \le i \le N} X_i \le_c \min_{1 \le i \le N} Y_i \text{ and } \max_{1 \le i \le N} X_i \le_c \max_{1 \le i \le N} Y_i.$$

A weaker order than the convex transform order is the star order defined as follows.

**Definition 8.16.** We say that $X$ is smaller than $Y$ in star order, written as $X \leq_* Y$, if and only if $F_Y^{-1}(F_X(x))$ is star-shaped in $x$.

By definition of star-shaped functions, it means that, for $X \leq_* Y$, we should have $\frac{1}{x}F_Y^{-1}(F_X(x))$ increasing in $x \geq 0$. Now,

$$\begin{aligned} xq_Y(F_X(x))f_X(x) - Q_Y(F_X(x)) \geq 0 & \\ \Rightarrow q_Y(u)\frac{Q_X(u)}{q_X(u)} - Q_Y(u) \geq 0 & \\ \Rightarrow Q_X(u)q_Y(u) - Q_Y(u)q_X(u) \geq 0 & \\ \Rightarrow \frac{Q_Y(u)}{Q_X(u)} \text{ is increasing in } u. & \end{aligned}$$

Since $X \leq_c Y$ implies $\frac{q_Y}{q_X}$ is increasing, it follows that

$$X \leq_c Y \Rightarrow X \leq_* Y.$$

The converse need not be true. Bartoszewicz and Skolimowska [78] have shown that

(a) if $X \leq_* Y$, $\log Q_Y$ is convex and $\log Q_X$ is concave, then $X \leq_c Y$;
(b) if $F_X$ and $F_Y$ are absolutely continuous and $X \leq_* Y$, $xf_X(x)$ is increasing and $xg_X(x)$ is decreasing, then $X \leq_c Y$.

Assume that $Y$ is exponential with scale parameter $\lambda$. Then,

$$X \leq_* Y \Rightarrow -\frac{1}{\lambda}\frac{\log(1-u)}{Q(u)}$$

is increasing. Hence, by Definition 4.9, $X$ is IHRA. Thus, the star ordering can be used to define increasing hazard quantile distributions, giving an ordering of IHRA distributions as follows.

**Definition 8.17.** $X$ is said to be more IHRA than $Y$ if and only if $X \leq_* Y$.

The star ordering enjoys properties similar to the convex transform ordering, and they are:

(i) $X \leq_* Y \Rightarrow X_T \leq_* Y_T$;
(ii) Theorem 8.34 holds when $\leq_c$ is replaced by $\leq_*$;
(iii) $X \leq_* Y \Rightarrow X^p \leq_* Y^p$ for any $p \neq 0$.

Ordering life distributions by the NBU property requires the superadditive property which is defined as follows.

**Definition 8.18.** We say that $X$ is more NBU than $Y$ if $F_Y^{-1}(F_X(x))$ is superadditive in $x$, i.e., if

$$F_Y^{-1}F_X(x+y) \geq F_Y^{-1}(F_X(x)) + F_Y^{-1}(F_X(y)) \text{ for all } x,y \geq 0. \tag{8.13}$$

This is denoted by $X \leq_{\text{su}} Y$.

To justify the above definition, we note that when $Y$ is exponential, (8.13) becomes

$$-\frac{1}{\lambda}\log(1-F_X(x+y)) \geq -\frac{1}{\lambda}\log(1-F_X(x)) - \frac{1}{\lambda}\log(1-F_Y(x)),$$

or

$$\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y).$$

Hence, $X$ is NBU by (4.26). Thus, we have the following theorem.

**Theorem 8.35.** *When $Y$ is exponential, $X \leq_{su} Y \Leftrightarrow X$ is NBU.*

Some other properties of the $\leq_{\text{su}}$ order are:

(a) $X \leq_* Y \Rightarrow X \leq_{\text{su}} Y$;
(b) Theorem 8.34 holds when $\leq_c$ is replaced by $\leq_{\text{su}}$.

A more general result holds for order statistics that involves all three orders discussed in this section in the context of $k$-out-of-$n$ systems as stated in the following theorem.

**Theorem 8.36.** *If $(X_i, Y_i)$, $i = 1,2,\ldots,n$, are independent pairs of random variables with the property $X_i \leq_c (\leq_*, \leq_{su}) Y_i$ for all $i$, and $X_i$'s and $Y_i$'s are identically distributed, then*

$$X_{k:n} \leq_c (\leq_*, \leq_{su}) Y_{k:n}, \quad k = 1,2,\ldots,n.$$

The orderings with respect to other ageing criteria discussed below are due to Kochar and Weins [350] and Kochar [347].

**Definition 8.19.** We say that $X$ is more decreasing mean residual life than $Y$, denoted by $X <_{\text{DMRL}} Y$, if

$$\frac{M_X(u)}{M_Y(u)} \text{ is nonincreasing in } u.$$

Since the reciprocal of the hazard quantile function of $Z$ is the mean residual quantile function of $X$, an equivalent condition for $X \leq_{\text{DMRL}} Y$ is that

$$\frac{H_{Z,X}(u)}{H_{Z,Y}(u)} \text{ is non-decreasing in } u,$$

where $H_{Z,X}$ is the hazard quantile function of the equilibrium distribution of $X$. Observe that the definition

$$M_X(u) = m_X(Q_X(u)) = \frac{1}{1-u}\int_u^1 (1-p)q_X(p)dp$$

is the mean residual quantile of $X$, and similarly

$$M_Y(u) = m_Y(Q_Y(u)) = \frac{1}{1-u}\int_u^1 (1-p)q_Y(p)dp.$$

**Theorem 8.37.** *If $Y$ is exponential, then*

$$X \leq_{DMRL} Y \Leftrightarrow X \text{ is DMRL.}$$

The proof is immediate upon substituting $M_Y(u) = \frac{1}{\lambda}$ in Definition 8.19.

**Theorem 8.38.**

$$X \leq_{DMRL} Y \Leftrightarrow \frac{\mu_Y - T_Y(u)}{\mu_X - T_X(u)} \text{ is increasing in } u.$$

*Proof.* We have

$$X \leq_{\text{DMRL}} Y \Leftrightarrow \frac{M_X(u)}{M_Y(u)} \text{ is increasing}$$

$$\Leftrightarrow \frac{\int_u^1 (1-p)q_X(p)dp}{\int_u^1 (1-p)q_Y(p)dp} \text{ is increasing}$$

The proof is completed simply by noting that $\int_u^1 (1-p)q_X(p)dp = \mu - T(u)$.

**Theorem 8.39.**

$$X \leq_c Y \Rightarrow X \leq_{DMRL} Y.$$

*In other words, the IHR order implies the DMRL order.*

**Definition 8.20.** $X$ is said to be smaller than $Y$ in NBUE order ($X$ is more NBUE than $Y$) if and only if

$$\frac{M_X(u)}{M_Y(u)} \leq \frac{\mu_X}{\mu_Y} \text{ for all } u \text{ in } [0,1],$$

and we denote it by $X \leq_{\text{NBUE}} Y$.

Two equivalent conditions for the $\leq_{NBUE}$ order are:

(a) $\frac{H_{Z,X}(u)}{H_{Z,Y}(u)} \geq \frac{\mu_Y}{\mu_X}$;
(b) $\frac{T_X(u)}{T_Y(u)} \geq \frac{\mu_X}{\mu_Y}$.

**Theorem 8.40.** *Let $Y$ be an exponential random variable. Then,*

$$X \leq_{NBUE} Y \Leftrightarrow X \text{ is NBUE.}$$

*Proof.* Since $M_Y(u) = \mu_Y = \frac{1}{\lambda}$, the definition of $\leq_{\text{NBUE}}$ gives the desired result.

**Theorem 8.41.** *If $X$ and $Y$ have supports of the form $[0,a)$, then:*

*(i)* $X \leq_{DMRL} Y \Rightarrow X \leq_{NBUE} Y$;
*(ii)* $X \leq_{*} Y \Rightarrow X \leq_{NBUE} Y$.

The proof of Part (i) is straightforward from the definitions of the two orderings. To prove Part (ii), we note that

$$\begin{aligned} X \leq_{*} Y &\Rightarrow X_T \leq_{*} Y_T \\ &\Rightarrow \frac{T_Y(u)}{T_X(u)} \text{ is increasing in } u \\ &\Rightarrow \frac{T_Y(u)}{T_X(u)} \leq \frac{T_Y(1)}{T_X(1)} = \frac{\mu_Y}{\mu_X} \\ &\Rightarrow X \leq_{\text{NBUE}} Y. \end{aligned}$$

The characterization of the class of distributions for which $X \leq_{\text{su}} Y$ implies $X \leq_{\text{NBUE}} Y$ remains open.

**Definition 8.21.** We say that $F$ is more NBUHR (new better than used in hazard rate) if $\frac{d}{dx}\psi_{F_X,F_Y}(x) \geq \psi'(0)$, and is denoted by $X \leq_{\text{NBUHR}} Y$.

From this definition, we see that

$$\frac{d}{dx}\psi(x) = \frac{d}{dx}F_Y^{-1}F(x) = \frac{H_X(u)}{H_Y(u)}$$

from the discussion following Definition 8.15. Hence,

$$X \leq_{\text{NBUHR}} Y \Leftrightarrow \frac{H_X(u)}{H_X(0)} > \frac{H_Y(u)}{H_Y(0)},$$

using which we obtain the interpretation in the following theorem.

**Theorem 8.42.** *If $Y$ is exponential, then $X \leq_{NBUHR} Y \Leftrightarrow X$ is NBUHR.*

*Proof.* We observe that

$$
\begin{aligned}
X \leq_{\text{NBUHR}} Y &\Leftrightarrow \frac{d}{dx}\psi(x) \geq \psi'(0) \\
&\Leftrightarrow \frac{H_X(u)}{\lambda} \geq \frac{H_X(0)}{\lambda} \\
&\Leftrightarrow X \text{ is NBUHR}
\end{aligned}
$$

by Definition 4.6.

A similar definition for the NBUHRA order can be provided as follows.

**Definition 8.22.** $X$ is more NBUHRA (new better than used in hazard rate average than $Y$), denoted by $X \leq_{\text{NBUHRA}} Y$, if and only if

$$\psi(x) \geq x\psi'(0).$$

We then have

$$X \leq_{\text{NBUHRA}} Y \Rightarrow X \text{ is NBUHRA}$$

and

$$X \leq_{\text{NBU}} Y \Rightarrow X \leq_{\text{NBUHRA}} Y \Rightarrow X \leq_{\text{NBUHRA}} Y.$$

## 8.11 MTTF Order

Earlier in Sect. 4.2, we have defined the mean time to failure (MTTF) in an age replacement model as (see (4.19)).

$$M(T) = \frac{1}{F(T)} \int_0^T \overline{F}(t)dt.$$

Another formulation of MTTF is

$$\mu(u) = M(Q(u)) = \frac{1}{u} \int_0^u (1-p)q(p)dp.$$

Now, a comparison of life distributions by the magnitude of MTTF is possible by considering an appropriate stochastic order.

**Definition 8.23.** A lifetime random variable $X$ is smaller than another lifetime random variable $Y$ in MTTF order, denoted by $X \leq_{\text{MTTF}} Y$, if and only if $\mu_X(u) \leq \mu_Y^*(u)$ for all $u$ in $(0,1)$ (or equivalently, $M_X(T) \leq M_Y(T)$ for all $T > 0$), where $\mu_Y^*(u) = M_Y(Q_X(u))$.

First, we discuss the relationship of the MTTF order with other stochastic orders discussed earlier.

**Theorem 8.43.** *If $X \le_{st} Y$, then $X \le_{MTTF} Y$, but the converse is not always true.*

The proof of this result and a counter example are given in Asha and Nair [39]. Resulting from Theorem 8.43, we have the following chain of implications:

$$\begin{array}{c} X \le_{\mathrm{hr}} Y \Rightarrow X \le_{\mathrm{st}} Y \Rightarrow X \le_{\mathrm{MTTF}} Y \\ \Uparrow \\ X \le_{\mathrm{rh}} Y. \end{array}$$

Two other basic reliability orders are $\le_{\mathrm{mrl}}$ and $\le_{\mathrm{MIT}}$, comparing the mean residual life and the mean inactivity time. As already seen, the hr order implies the mrl order and the hr order also implies the MTTF order. Hence, the point of interest is to know whether there exist any implications between the $\le_{\mathrm{mrl}}$ and the $\le_{\mathrm{MTTF}}$ orders. By taking

$$f_Y(x) = \frac{1}{2}\exp\left(-\frac{x}{2}\right)$$

and

$$f_X(x) = xe^{-x}, \quad x > 0,$$

we see that $X \ge_{\mathrm{MTTF}} Y$, but $X \le_{\mathrm{mrl}} Y$.

Conditions under which the $\le_{\mathrm{st}}$ and the $\le_{\mathrm{mrl}}$ orders have implications with the $\le_{\mathrm{MTTF}}$ order are of interest. These are presented in the next theorem. The conditions can be stated in terms of quantiles by setting $x = Q(u)$ as usual.

**Theorem 8.44.** *(a) If $\frac{\int_0^x F_X(t)dt}{\int_0^x F_Y(t)dt}$ is decreasing, then $X \ge_{MTTF} Y \Rightarrow X \ge_{st} Y$;*
*(b) If $\frac{m_X(x)}{m_Y(x)}$ is decreasing, then $X \ge_{mrl} Y \Rightarrow X \ge_{MTTF} Y$.*

A similar result holds for the MIT order as well. It has been mentioned earlier that if $\frac{r_X(x)}{r_Y(x)}$ is an increasing function of $x$, then the $\le_{\mathrm{rh}}$ and the $\le_{\mathrm{MIT}}$ orders are equivalent. Accordingly, when $\frac{r_X(x)}{r_Y(x)}$ is decreasing,

$$X \ge_{\mathrm{MIT}} Y \Rightarrow X \ge_{MTTF} Y.$$

Further, if $X \ge_{\mathrm{st}} Y$, then $X \ge_{\mathrm{MTTF}} Y \Rightarrow X \ge_{\mathrm{hmrl}} Y$. Returning to decreasing mean time to failure as an ageing concept (see Sect. 4.3), we have a stochastic order comparison based on DMTTF as follows.

**Definition 8.24.** $X$ has more DMTTF than $Y$ if $\frac{\mu_X(u)}{\mu_Y(u)}$ is decreasing in $u$ for all $0 \le u \le 1$, and we denote it by $X \le_{\mathrm{DMTTF}} Y$.

Suppose $Y$ is exponential. Then, $\mu_Y(u) = \frac{1}{\lambda}$ and so in this particular case, we have

$$X \geq_{\text{DMTTF}} Y \Leftrightarrow X \text{ is DMTTF.}$$

Two other properties of this ordering are as follows:

1. $X \geq_{\text{DMRL}} Y \Rightarrow X \leq_{\text{DMTTF}} Y$;
2. $X \leq_{\text{NBUE}} Y \Leftrightarrow \frac{\mu_X(u)}{\mu_x} \geq \frac{\mu_Y(u)}{\mu_Y}$.

## 8.12 Some Applications

When $X$ represents a continuous lifetime with distribution function $F(x)$, the proportional reversed hazard model is represented by a non-negative absolutely continuous random variable $U$ whose distribution function is

$$F_U(x) = [F_X(x)]^{\theta},$$

where $\theta$ is a positive real number (see Example 1.3). When $F(x)$ is strictly increasing, $F_X(x) = u$ gives the quantile function of $U$ as

$$Q_U(\theta) = Q_X(u^{\frac{1}{\theta}}).$$

For this model, the reversed hazard rates of $U$ and $X$ are proportional, i.e., $\lambda_U(x) = \theta\lambda_X(x)$ or $\Lambda_U^*(u) = \theta\Lambda_X(u)$, where

$$\Lambda_U^*(u) = \lambda_U(Q_X(u)).$$

Gupta et al. [239] and Di Crecenzo [177] have studied the order relationship between $X$ and $U$ and also between two random variable $X$ and $Y$ and their proportional reversed hazard models $U$ and $V$. Let

$$\mathscr{H}(x) = -\log F_X(x) = \int_x^{\infty} \lambda(t)dt$$

be the cumulative reversed hazard rate of $X$.

**Theorem 8.45.** *Let $[\mathscr{H}(x)]^{-1}$ be star-shaped (antistarshaped). Then:*

*(i) If $\theta < 1$, $\theta X \leq_{st} U(\theta X) \geq_{st} U$;*
*(ii) If $\theta > 1$, $\theta X \geq_{st} U(\theta X) \leq_{st} U$.*

**Theorem 8.46.** *(i) $X \leq_{st} Y \Leftrightarrow U \leq_{st} V$;*
*(ii) $X \leq_{rh} Y \Leftrightarrow U \leq_{rh} V$;*
*(iii) $X \leq_{hr} Y$ and $\theta > 1 \Leftrightarrow U \leq_{hr} V$.*

Gupta and Nanda [254] have considered $X_i$, $i = 1,2$, with distribution functions $F_i(x)$ and $U_i$ as proportional reversed hazards models of $X_i$ with distribution functions $[F_i(x)]^{\theta_i}$, $i = 1,2$.

**Theorem 8.47.** *$\theta_1 \geq \theta_2$ and $X_1 \geq_{rh} X_2 \Rightarrow Y_1 \geq_{rh} Y_2$.*

In particular, if

$$S_i(x) = 1 - e^{-(\frac{x}{\sigma_i})^{\lambda}},$$

then $X_1 \geq_{\text{rh}} X_2$ if and only if $\sigma_1 \geq \sigma_2$ $(> 0)$, irrespective of the value of $\lambda$. Similarly, for the exponentiated Weibull distribution with

$$F_i(x) = [1 - e^{-(\frac{x}{\sigma_i})^{\alpha}}]^{\theta},$$

$X_1 \geq_{\text{rh}} X_2$ if and only if $\sigma_1 \geq \sigma_2$. If $X_1, X_2, \ldots$ are independent and identically distributed random variables and $N$ is geometric with $P(N = n) = p(1-p)^{n-1}$, $n = 1,2,\ldots$, independent of the $X_i$'s, then the sum

$$S_N = X_1 + \cdots + X_N$$

is said to be a geometric compound. It is easy to see that $S_N$ belongs to the random convolution discussed earlier. Hu and Lin [284] have given several characterizations of the exponential distribution using stochastic orders, some of which are presented in the following theorem.

**Theorem 8.48.** *1. If $F$, the common distribution function of the $X_i$'s, is NWU and $pS_N \leq_{st} T\min(X_1,\ldots,X_T)$, then $F$ is exponential, where $T$ is an integer valued random variable. If $F$ is NBU and $T\min(X_1 \ldots X_T) \leq_{st} pS_N$, then $F$ is exponential;*

*2. If $pS_N \leq_{st} X_1$, then $F$ is exponential;*

*3. In the renewal process $(S_n)_{n=1}^{\infty}$, $S_n = \sum_{k=1}^{n} X_k$ and $r(t) = S_{N(t)+1} - t$ is the residual life at time $t$, if $F$ is NBU and $pS_N \leq_{st} r(t)$, then $F$ is exponential.*

Nanda et al. [458] have discussed stochastic orderings in terms of the proportional mean residual life model. Let $X$ be a non-negative random variable with absolutely continuous distribution function and finite mean and $V$ be another non-negative random variable with the same properties. Then, we say that $V$ is the proportional mean residual life model (PMRLM) of $X$ if

$$m_V(x) = cm_X(x),$$

where $m_X(x)$ is as usual the mean residual life function. An equivalent condition is

$$M_V^*(u) = cM_X(u),$$

where $M_V^*(u) = m_v(Q_X(u))$. For this model, we have the following properties:

(i) $X \leq_{\text{hr}} (\geq) V$ if $c > (< 1)$;
(ii) Let $X \leq_{\text{st}} Y$. If either (a) $c < 1$ and

$$\frac{m_Y(x)}{\mu_Y} \geq \frac{m_X(x)}{\mu_X},$$

or (b) $c > 1$ and

$$\frac{m_Y(x)}{\mu_Y} \leq \frac{m_X(x)}{\mu_X},$$

then $V_X \leq_{\text{st}} V_Y$, where $V_X(V_Y)$ is the PMRLM corresponding to $X(Y)$;
(iii) $X \leq_{\text{hr}} (\geq_{\text{hr}})Y$ and $c < 1 \Rightarrow V_X \leq_{\text{hr}} (\geq_{\text{hr}})V_Y$;
(iv) $X \leq_{\text{mrl}} (\geq_{\text{mrl}})Y \Leftrightarrow V_X \leq_{\text{mrl}} (\geq_{\text{mrl}})V_Y$;
(v) $X \leq_{\text{hmrl}} (\geq_{\text{hmrl}})Y \Leftrightarrow V_X \leq_{\text{hmrl}} (\geq_{\text{hmrl}})V_Y$.

The preservation of stochastic orders among weighted distributions has been discussed in Misra et al. [417]. Let $X_1$ and $Y_1$ be weighted versions of $X$ and $Y$ defined as

$$F_{X_1}(x) = \frac{\int_0^x w_1(t) f_X(t) dt}{EW_1(X)}$$

and

$$F_{Y_1}(x) = \frac{\int_0^x w_2(t) f_Y(t) dt}{EW_2(Y)}.$$

We then have the following results.

**Theorem 8.49.** *(i) If $X \leq_{st} Y$, $w_1(\cdot)$ is decreasing and $w_2(\cdot)$ is increasing, then $X_1 \leq_{st} Y_1$;*
*(ii) If $X$ and $Y$ have a common support, $X \leq_{hr} Y$ and $w(x) = w_1(x) = w_2(x)$ is increasing, then $X_1 \leq_{hr} Y_1$;*
*(iii) If in (ii) $w(\cdot)$ is decreasing and $X \leq_{rh} Y$, then $X_1 \leq_{rh} Y_1$;*
*(iv) Let $X \leq_{hr} Y$ ($X \leq_{rh} Y$), $w_2(x)$ is increasing ($w_1(x)$ is decreasing) and $\frac{w_2(x_1)}{w_1(x_1)}$ is increasing on the intersection of the supports, then $X_1 \leq_{hr} Y_1$ ($X_1 \leq_{rh} Y_1$) provided that $l_1 \leq l_2$, $u_1 \leq u_2$, where $(l_1, u_1)$ and $(l_2, u_2)$ are the supports of $X_1$ and $Y_1$, respectively.*

Yu [597] has discussed stochastic comparisons between exponential family of distributions and their mixtures with respect to various stochastic orders. Members of this family have been frequently used in reliability analysis and for this reason we present some results relevant in this regard. The exponential family is expressed by the probability density function

$$f(x,\theta) = a(x)e^{b(\theta)x}h(\theta),$$

where the support is $(0,\infty)$. Let

$$g(x) = \int f(x;t)d\mu(t)$$

be the mixture of $f(x,\theta)$. Then we have the order relations, between $X$ and $Y$, the random variables corresponding to $f(x;\theta)$ and $g(x)$, as follows:

(a) $X \leq_{\text{st}} Y$ ($X \leq_{\text{hr}} Y$) if and only if $\int h(t)d\mu(t) \leq h(\theta)$;
(b) $X \leq_{\text{rh}} Y$ if and only if

$$b(\theta) \leq \frac{\int b(t)h(t)d\mu(t)}{\int h(t)d\mu(t)}.$$

Let $X = \sum_{i=1}^{\infty} \beta_i X_i$, where $X_i$ is gamma $(\alpha_i, 1)$ independently and $\beta_i > 0$. The order relations between $X$ and $Y$ which is gamma $(\sum_{i=1}^{n} \alpha_i, \beta)$ have been discussed by many authors. When $X_i$'s are independent exponential with different scale parameters (i.e., when $\alpha_i = 1$), Boland et al. [114] have established that

$$\beta \leq \frac{n}{\sum_{i=1}^{n} \beta_i^{-1}} \Rightarrow X \leq_{\text{rh}} Y$$

and Bon and Paltanea [117] have extended this result to

$$Y \leq_{\text{st}} X \Leftrightarrow Y \leq_{\text{hr}} X \Leftrightarrow \beta \leq \left(\prod_{i=1}^{n} \beta_i\right)^{\frac{1}{n}}.$$

Yu [597] has further established that

$$Y \leq_{\text{st}} X (Y \leq_{\text{hr}} X) \text{ if and only if } \beta \leq \left(\prod_{i=1}^{n} \beta_i^{\alpha_i}\right)^{\sum_i^n \frac{1}{\alpha_i}},$$

$$Y \leq_{\text{rh}} X \text{ if and only if } \beta \leq \frac{\sum_1^n \alpha_i}{\sum_i^n \frac{\alpha_i}{\beta_i}}.$$

These results are useful in developing bounds for the hazard rate of $X$ through simpler hazard rate of $Y$.

If $X$ and $Y$ are lifetime variables with cumulative hazard functions $\mathscr{H}_X(x)$ and $\mathscr{H}_Y(x)$, Sengupta and Deshpande [526] have defined $X$ to be ageing faster than $Y$ if and only if $\mathscr{H}_1 \mathscr{H}_2^{-1}$ is superadditive, i.e.,

$$\mathscr{H}_1 \mathscr{H}_2^{-1}(x+y) \geq \mathscr{H}_1 \mathscr{H}_2^{-1}(x) + \mathscr{H}_1 \mathscr{H}_2^{-1}(y).$$

Abraham and Nair [13] have proposed a relative ageing factor

$$B(x,y) = \frac{\mathscr{H}^{-1}(\mathscr{H}(x) + \mathscr{H}(y)) - x}{y}$$

between a new component and an old component that survived up to time $x$. They then defined an order $X \leq_{B:\text{NBU}} Y$ by the relation $B_X(x,y) \leq B_Y(x,y)$ for all $x, y > 0$. They provided the result that

$$B_X(x,y) \leq B_Y(x,y) \Leftrightarrow X \text{ is NBU},$$

where the NBU part arises from the fact that $Y$ is exponential. The relative ageing defined by the superadditive order now becomes

$$X \leq_{\text{su}} Y \Leftrightarrow X \leq_{B:\text{NBU}} Y.$$

Thus, an ageing criterion is prescribed in terms of $B(x,y)$ to assess the concept of '$X$ ageing faster than $Y$'.

If $X$ is a random variable with survival function $\overline{F}(x)$ and $Z$ has survival function $\overline{F}_2(x) = [\overline{F}(x)]^{\theta}$, $\theta > 0$, then $F_Z(x)$ is called the proportional hazards model corresponding to $X$. The terminology is evident from the fact that $h_Z(x) = \theta h_X(x)$. There are other interpretations also for $Z$. If $\theta < 1$, $Z$ represents the lifetime of a component in which the original lifetime of the component $X$ is subjected imperfect repair procedure, where $\theta$ is the probability of a minimal repair. If $\theta = n$, obviously we have $(\overline{F}(x))^n$ as the survival function of a series system consisting of $n$ independent and identical components whose lifetimes are distributed as $X$. Franco-Pereira et al. [202] have shown that if $X$ and $Y$ are continuous random variables on interval supports, the $\alpha$-percentile life order satisfies

$$X \leq_{\text{prl}-\alpha} Y \Rightarrow Z_X \leq_{\text{prl}-\beta} Z_Y,$$

where $\beta = 1 - (1-\alpha)^{\theta}$ and $Z_X$ ($Z_Y$) is the proportional hazards model corresponding to $X(Y)$.

Extensions of some of the stochastic orders discussed above as well as a variety of applications of all these stochastic orders can be found in Kayid et al. [320], Aboukalam and Kayid [11], Li and Shaked [388], Boland et al. [115], Navarro and Lai [467], Zhang and Li [599], Hu and Wei [286], and Da et al. [164] and the references contained therein.