Chapter 7
Nonmonotone Hazard Quantile Functions

Abstract The existence of nonmonotonic hazard rates was recognized from the study of human mortality three centuries ago. Among such hazard rates, ones with bathtub or upside-down bathtub shape have received considerable attention during the last five decades. Several models have been suggested to represent lifetimes possessing bathtub-shaped hazard rates. In this chapter, we review the existing results and also discuss some new models based on quantile functions. We discuss separately bathtub-shaped distributions with two parameters, three parameters, and then more flexible families. Among the two-parameter models, the Topp-Leone distribution, exponential power, lognormal, inverse Gaussian, Birnbaum and Saunders distributions, Dhillon's model, beta, Haupt-Schäbe models, loglogistic, Avinadev and Raz model, inverse Weibull, Chen's model and a flexible Weibull extension are presented along with their quantile functions. The quadratic failure rate distribution, truncated normal, cubic exponential family, Hjorth model, generalized Weibull model of Mudholkar and Kollia, exponentiated Weibull, Marshall-Olkin family, generalized exponential, modified Weibull extension, modified Weibull, generalized power Weibull, logistic exponential, generalized linear failure rate distribution, generalized exponential power, upper truncated Weibull, geometric-exponential, Weibull-Poisson and transformed model are some of the distributions considered under three-parameter versions. Distributions with more than three parameters introduced by Murthy et al., Jiang et al., Xie and Lai, Phani, Agarwal and Kalla, Kalla, Gupta and Lvin, and Carrasco et al. are presented as more flexible families. We also introduce general methods that enable the construction of distributions with nonmonotone hazard functions. In the case of many of the models so far specified, the hazard quantile functions and their analysis are also presented to facilitate a quantile-based study. Finally, the properties of total time on test transforms and Parzen's score function are utilized to develop some new methods of deriving quantile functions that have bathtub hazard quantile functions.

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7.1 Introduction

The recognition of the existence of nonmonotonic hazard rates dates back to three centuries in the study of human mortality when researchers found that the force of mortality (alternative name for hazard rate) first decreases, then remains more or less constant and then increases. Since then, the problem of modelling such curves through different distributions has been taken up in many disciplines such as reliability, survival analysis, demography and actuarial science. Among nonmonotonic hazard rates, those with bathtub shape or upside-down bathtub shape have received much attention during the last 5 decades. There is an extensive literature on finding appropriate models for representing them and also on methods of analysing their behaviour, in several practical problems. Earlier in Sect. 4.3, we have introduced the notions of bathtub (BT) and upside-down bathtub (UBT) hazard rates and the corresponding hazard quantile functions. Recall that a random variable X with differentiable $h(x)(H(u))$ possesses a BT hazard rate (hazard quantile function) if and only if $h'(x)(H'(u)) < 0$ for $x(u)$ in $(0,x_0)((0,u_0))$, $h'(x_0) = 0$ $(H'(u_0) = 0)$ and $h'(x)(H'(u)) > 0$ for $x(u)$ in $(x_0,\infty)((u_0,1))$. In the UBT case, $H'(u) > 0$ for u in $(0,u_0)$, $H'(u_0) = 0$ and $H'(u_0) < u_0$ in $(u_0,1)$. Thus, BT distributions are characterized by a hazard rate (hazard quantile function) that is first decreasing and then increasing with a unique change point. A more general definition that considers $H(u)$ as a constant in an interval (see Definition 4.2) is also available, but this extended definition will not be considered in the sequel. The three phases of a BT hazard rate represent an 'infant mortality' period in which $H(u)$ decreases, a 'useful period' in which $H(u)$ is approximately constant, and a 'wear out' stage in which the hazard function increases leading to the ultimate failure of a unit. To avoid infant mortality in large proportions, 'burn-in' procedures are often employed to enhance the reliability of products. On the other hand, replacement policies aim at improving the reliability of units by eliminating those with short lives before the wear out process is at an advanced stage. We make use of the sign of the derivative of $H(u)$ $(h(x))$ to ascertain the nonmonotonicity. In case when the survival function is not tractable, giving complicated expressions for $H(u)$ or $h(x)$, Theorems 4.1 and 4.2 will be employed, as demonstrated in Example 4.4. Several methods of construction of models with BT or UBT have been proposed in the literature. However, in the following discussion, we will distinguish the models by the number of parameters they involve. Quantile functions and corresponding hazard quantile functions are presented whenever the proposed distributions have such functions in tractable forms. A review of bathtub-shaped distributions is given in Rajarshi and Rajarshi [500] and Lai et al. [369],

7.2 Two-Parameter BT and UBT Hazard Functions

The Topp-Leone [566] distribution with density function

$$f(x) = \frac{2\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}\left(1-\frac{x}{\theta}\right)\left(2-\frac{x}{\theta}\right)^{\alpha-1}, \quad 0 \le x \le \theta,\ 0 < \alpha < 1,$$

has its survival function as

$$\overline{F}(x) = 1-\left(\frac{x}{\theta}\right)^{\alpha}\left(2-\frac{x}{\theta}\right)^{\alpha}, \quad 0 \le x \le \theta. \tag{7.1}$$

Thus, the hazard rate turns out to be

$$h(x) = \frac{2\alpha}{\theta}\frac{(\frac{x}{\theta})^{\alpha-1}(1-\frac{x}{\theta})(2-\frac{x}{\theta})^{\alpha-1}}{1-(\frac{x}{\theta})^{\alpha}(2-\frac{x}{\theta})^{\alpha}}.$$

By differentiating $h(x)$, it can be seen that $h(x)$ has a bathtub shape with change point x_0 for every α, where x_0 satisfies the equation

$$\left(\frac{x_0}{\theta}\right)^{\alpha}+\frac{2\alpha(\theta-x_0)}{2\theta-x_0}-1=0.$$

The distribution in (7.1) admits a convenient quantile function. Applying the transformation $Y = 1-\frac{X}{\theta}$ to (7.1), we have

$$F_Y(x) = (1-x^2)^{\alpha}$$

and so

$$Q_Y(u) = (1-u^{\frac{1}{\alpha}})^{\frac{1}{2}}.$$

Retransforming this expression to X, we readily obtain

$$\begin{aligned} Q_X(u) &= \theta-\theta Q_Y(u) \\ &= \theta\left\{1-(1-u^{\frac{1}{\alpha}})^{\frac{1}{2}}\right\}. \end{aligned}$$

The corresponding hazard quantile function is

$$H(u) = \frac{1}{(1-u)q(u)} = \frac{2\alpha}{\theta}\frac{(1-u^{\frac{1}{\alpha}})^{\frac{1}{2}}}{u^{\frac{1}{\alpha}-1}(1-u)}.$$

Smith and Bain [544] introduced the exponential power model with survival function

$$\overline{F}(x) = \exp[-e^{(\lambda x)^{\alpha}}+1], \quad 0 < x < \infty. \tag{7.2}$$

Notice that the hazard rate is

$$h(x) = \lambda^{\alpha}\alpha x^{\alpha-1}e^{(\lambda x)^{\alpha}}, \tag{7.3}$$

which is strictly convex in $(0,\infty)$ satisfying $\int_0^\infty h(x)dx = \infty$. Sometimes, the choice of such a function is adopted as a method of deriving a BT distribution. A feature of the function (7.3) is that $h(x) \to \infty$ when $x \to 0$ or ∞. The hazard function is BT for $\alpha < 1$ with a change point at $x_0 = \frac{(1-\alpha)}{(\lambda\alpha)^{\frac{1}{\alpha}}}$. For further detailed study of the distribution including the estimation of parameters and applications to other disciplines, one may refer to Dhillon [175], Paranjpe and Rajarshi [481], Leemis [379] and Chen [141]. A closed-form expression is available for the quantile function of (7.2) as

$$Q(u) = \frac{1}{\lambda}[\log\{1-\log(1-u)\}]^{\frac{1}{\alpha}},$$

which can be used to simulate observations from the distribution from the uniform (0,1) random numbers. Observing that the quantile density function is

$$q(u) = \frac{[\log(1-\log(1-u))]^{\frac{1}{\alpha}-1}}{\lambda\alpha(1-u)(1-\log(1-u))},$$

it becomes clear that

$$H(u) = \frac{\lambda\alpha(1-\log(1-u))}{[\log(1-\log(1-u))]^{\frac{1}{\alpha}-1}}.$$

Two standard distributions possessing nonmonotone hazard rates that were considered reliability analysis are the lognormal and the inverse Gaussian. The lognormal distribution has its density function as

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left[-\frac{(\log x-\mu)^2}{2\sigma^2}\right], \quad x \geq 0,\ -\infty < \mu < \infty,\ \sigma > 0, \tag{7.4}$$

and survival function as

$$\overline{F}(x) = 1-\Phi\left(\frac{\log x - x}{\sigma}\right) = 1-\Phi[\log(\alpha x)^{\frac{1}{\sigma}}], \quad \text{with } \alpha = e^{-\mu},$$

where Φ is the distribution function of the standard normal distribution. The hazard rate is

$$h(x) = \frac{1}{\sqrt{2\pi}\sigma x}\frac{\exp[-(\log\alpha x)^2/2\sigma^2]}{1-\Phi(\log(\alpha x)/\sigma)}.$$

A detailed study of the hazard rate has been carried out by Sweet [556]. The book by Crow and Shimizu [159] details all methods and applications of lognormal distribution. Moreover, from Marshall and Olkin [412] and Johnson et al. [303], we note the following properties:

1. $h(x) = \sigma^{-1} h_N(t) \exp[-\sigma t - \mu]$, where $t = \sigma^{-1}(\log x - \mu)$ and $h_N(x)$ is the hazard rate of the normal distribution;
2. For all real θ,

$$\lim_{x \to 0} x^{\theta} h(x) = 0, \quad \lim_{x \to \infty} h(x) = 0;$$

3. $h(x)$ is unimodal with mode at $\exp(\sigma x^* + \mu)$, where x^* is the unique solution of the equation $h_N(x) = x + \sigma$. This solution is less than $\exp[1 + \mu - \sigma^2]$, but greater than $\exp[\mu - \sigma^2]$. As $\sigma \to \infty$, $x^* \to \exp[\mu - \sigma^2]$ and so for large σ, we have

$$\max h(x) \doteq \frac{\exp(\mu - \frac{\sigma^2}{2})}{\sigma\sqrt{2\pi}};$$

as $\sigma \to 0$, $x^* \to \exp[\mu - \sigma^2 + 1]$ and so for small σ, we have

$$\max h(x) \doteq \{\sigma^2 \exp(\mu - \sigma^2 + 1)\}^{-1}.$$

The quantile function corresponding to (7.4) is

$$Q(u) = \exp[\mu + \sigma \Phi^{-1}(u)]$$

and so $H(u)$ does not have a nice algebraic form for manipulations.

Inverse Gaussian distribution, discussed in detail by Chhikara and Folks [146] and Seshadri [528] as a lifetime model, has its density function as

$$f(x) = \frac{\theta\mu}{(2\pi^3 x^3)^{\frac{1}{2}}} \exp\left\{-\frac{(\theta x - \mu)^2}{2\theta x}\right\}, \quad x, \theta, \mu > 0. \tag{7.5}$$

Its survival function is

$$\overline{F}(x) = \frac{1}{2}\left[\overline{G}\left(\frac{(\theta x - \mu)^2}{\theta x}\right) - e^{2\mu}\overline{G}\left(\frac{(\theta x + \mu)^2}{\theta x}\right)\right],$$

where

$$\overline{G}(y) = \int_y^{\infty} (2\pi x)^{-\frac{1}{2}} e^{-\frac{x}{2}} dx$$

is the survival function of a chi-square variable with one degree of freedom. Needless to say, the hazard rate function is of a complicated form to study its behaviour explicitly. The hazard rate is UBT with change point x_0 that is the solution of the equation

$$h(x) = \frac{3}{2x} + \frac{\theta}{2} - \frac{\mu^2}{2\theta x^2}.$$

For various applications in reliability and lifetime data analysis, we refer the readers to Padgett and Tsai [479] and Bhattacharya and Fries [99], and similarly to Hougaard [283] in survival analysis and Feaganes and Suchindran [195] as a distribution of frailty.

A distribution that is related to the inverse Gaussian, but derived independently as a lifetime model based on shocks that arrive at regular intervals of time causing random damages, was derived by Birnbaum and Saunders [105, 106]. It models fatigue life of metals subject to periodic stress. The distribution has density function

$$f(x) = \frac{\lambda}{2\alpha\sqrt{2\pi}} \frac{1}{\sqrt{\lambda x}} \left(1 + \frac{1}{\lambda x}\right) \exp\left\{-\frac{1}{2\alpha^2}\left(\lambda x - 2 + \frac{1}{\lambda x}\right)\right\}. \tag{7.6}$$

Desmond [174] pointed out that (7.6) can be written as a mixture in equal proportions of an inverse Gaussian and a reciprocal inverse Gaussian. The distribution function is given by

$$F(x) = \Phi(\alpha^{-1} g(\lambda x)), \tag{7.7}$$

where $\lambda, \alpha > 0$ and Φ is the standard normal distribution function and

$$g(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}.$$

Note the resemblance between (7.7) and the distribution of the lognormal law in which case $g(x) = \log x$. Various properties and inferential procedures of the distribution have been discussed by Chang and Tang [136,137], Johnson et al. [302], Dupuis and Mills [182], Rieck [507], Ng et al. [470, 471], Owen [478], Leomonte et al. [384], Balakrishnan and Zhu [62] and Xie and Wei [591]. Recently, Kundu et al. [359] expressed the hazard rate function of the Birnbaum-Saunders distribution in (7.6) as

$$h(x, \alpha) = \frac{\frac{1}{\sqrt{2\pi}} g'(x) \exp\left\{-\frac{1}{2\alpha^2} g^2(x)\right\}}{\Phi(e^{-\frac{g(x)}{\alpha}})}$$

by taking $\lambda = 1$, without loss of generality, since λ is a scale parameter. They then showed that $h(x)$ is UBT for all $x > 0$ and for all α and λ. The change point x_0 is the solution of the equation

$$\Phi\left(-\frac{1}{\alpha} g(x)\right) \{-(g'(x))^2 g(x) + \alpha^2 g''(x)\} + \alpha \Phi\left(-\frac{1}{\alpha} g(x)\right) (g'(x))^2 = 0,$$

which has to be solved by numerical methods. An approximation has been given as

$$x_0 = (-0.4604 + 1.8417\alpha)^{-2}, \quad \alpha > 0.25;$$

see also Bebbington et al. [84]. A comparison of the hazard rates of (7.7) and the lognormal has been made by Nelson [469].

Some useful generalizations of the Birnbaum-Saunders distribution have been developed in order to provide more flexible models in terms of the range of skewness as well as varying shapes of the hazard function. For example, with the choice of the function $g(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$, instead of basing the distribution in (7.7) on a normal distribution, one could base it on general family of elliptically contoured distributions or scale-mixture distributions; see, for example, Diaz-Garcia and Leiva [176], Leiva et al. [386] and Balakrishnan et al. [54]. Properties of such models and their reliability characteristics have also been studied; for instance, Azevedo et al. [42] recently discussed the shape and change points of the hazard function of the BS-t (Birnbaum-Saunders model based on t-distribution) model.

Dhillon [175] introduced a two-parameter survival function

$$\overline{F}(x) = \exp\left[-\{\log(\lambda x+1)\}^{\beta+1}\right], \quad x \geq 0,\ \beta \geq 0,\ \lambda > 0, \tag{7.8}$$

and density function

$$f(x) = \frac{\lambda(\beta+1)}{\lambda x+1}\{\log(\lambda x+1)\}^{\beta}\exp\left[-\{\log(\lambda x+1)\}^{\beta+1}\right].$$

The corresponding hazard rate is

$$h(x) = \frac{(\beta+1)\lambda\{\log(\lambda x+1)\}^{\beta}}{\lambda x+1}.$$

It can be seen that $h(x)$ is UBT with change point $x_0 = \lambda^{-1}(e^{\beta}-1)$. We see that (7.8) is also expressible as

$$Q(u) = \frac{1}{\lambda}[e^{-\{\log(1-u)\}^{\frac{1}{\beta+1}}} - 1].$$

Its hazard quantile function is

$$H(u) = \frac{(\beta+1)\lambda\exp\left[\{\log(1-u)\}^{\frac{1}{\beta+1}}\right]}{\{\log(1-u)\}^{\frac{1}{\beta+1}-1}},$$

which becomes UBT with change point

$$u_0 = 1 - e^{-\beta(\beta+1)}.$$

Mukherjee and Islam [430] and Lai and Mukherjee [367] considered the power distribution with

$$F(x) = \left(\frac{x}{\alpha}\right)^{\beta}, \ 0 \le x \le \alpha, \ \beta < 1, \tag{7.9}$$

and hazard rate

$$h(x) = \frac{\beta x^{\beta-1}}{\alpha^{\beta} - x^{\beta}}$$

which has a BT shape with change point $x_0 = \alpha(1-\beta)^{\frac{1}{\beta}}$. The quantile function of this distributions and its properties has been discussed several times in the preceding chapters. The distribution in (7.9) forms a special case of the beta distribution with density function

$$f(x) = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}, \quad 0 \le x \le 1, \ p,q > 0, \tag{7.10}$$

when $q = 1$ and then rescaled to the interval $(0, \theta)$. Pham and Turkkan [494] have considered standby systems with component lives distributed as beta and Ganter [209] used it in the context of accelerated test of electronic assemblies. However, a detailed analysis of the hazard rate and mean residual life has been carried out much later by Gupta and Gupta [232] and Ghitany [212]. The hazard rate of the beta model is

$$h(x) = \frac{x^{p-1}(1-x)}{B(p,q) - B_x(p,q)},$$

where

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1} dt$$

is the incomplete beta integral. Ghitany [212] has shown that Glaser's result mentioned earlier in Sect. 4.3 is valid only when the upper end of the support is ∞ and $f(\infty) = 0$, and that it fails to determine the shape of the hazard rate when the support of a distribution is $(0,b)$ with $b < \infty$. He then modified Glaser's result as follows.

Theorem 7.1. *Let X be a continuous random variable on $(0,b)$, $b < \infty$, with twice differentiable density $f(x)$. Define $\eta(x) = -\frac{f'(x)}{f(x)}$. Then:*

(a) If $\eta(x)$ is decreasing and $f(b) = 0$, then $h(x)$ is decreasing;
(b) If $\eta(x)$ is increasing, then $h(x)$ is increasing;
(c) If $\eta(x)$ is BT and $f(0) = 0$ ($f(0) = \infty$), then $h(x)$ is increasing (BT);

(d) If $\eta(x)$ is UBT, $f(0) = 0$ ($f(0) = \infty$), and $f(b) = 0$, then $h(x)$ is UBT (decreasing);
(e) If $\eta(x) \leq 0$ and $f(b) > 0$, then $h(x)$ is decreasing;
(f) If $\eta(x)$ is decreasing and $f(0) = f(b) = \infty$, then $h(x)$ is BT.

In Theorem 7.1, the monotonicities involved are strict. Using the above results, it has been shown that the hazard rate of the beta distribution is BT (increasing) if $p < 1$ ($p \geq 1$). Also, the mean residual life is UBT (decreasing) if $p < 1$ ($p \geq 1$). Notice that the adaptation of Glaser's result in Theorem 4.1 also requires corresponding changes in the cases discussed in Theorem 7.1. For more details on beta distribution and its applications, one may refer to the volume by Gupta and Nadarajah [230].

Haupt and Schäbe [265] proposed the distribution with

$$F(x) = \begin{cases} 1, & x \geq x_0 \\ -\beta + \sqrt{\beta^2 + \frac{(1+2\beta)x}{x_0}}, & 0 \leq x \leq x_0 \end{cases}. \tag{7.11}$$

In this model, β is a shape parameter and it varies over $(-\frac{1}{2}, \infty)$ and x_0 is a scale parameter. The corresponding hazard rate is

$$h(x) = \frac{1+2\beta}{2x_0\left(\beta^2 + \frac{(1+2\beta)x}{x_0}\right)^{\frac{1}{2}}\left\{1+\beta-\left(\beta^2+\frac{(1+2\beta)x}{x_0}\right)^{\frac{1}{2}}\right\}}$$

which is BT for $\frac{1}{3} < \beta < 1$ and decreasing for $\beta \geq 1$ and $\beta \leq -\frac{1}{3}$. Construction of lifetime distributions with bathtub-shaped hazard rates from DHR distributions was proposed by Schäbe [522]. For $0 < \theta < \infty$, let us define

$$G(x) = \frac{F(x)}{F(\theta)}, \quad x \leq \theta.$$

Then, $G(x)$ has BT hazard rate if

$$h'(x)[\overline{F}(x) - \overline{F}(\theta)] + h^2(x)\overline{F}(\theta)$$

has one and only one zero in the interval $(0, \theta)$ and changes its sign from $-$ to $+$. An illustration of this result has been given with the Pareto II distribution.

Paranjpe and Rajarshi [481] suggested the survival function

$$\overline{F}(x) = \exp[-\exp\{\exp(\beta x^\alpha) - 1\}], \quad \beta > 0,\ \alpha < 1, \tag{7.12}$$

to model BT hazard rates. The hazard rate function has the form

$$h(x) = \beta\alpha x^{\alpha-1}\exp(\beta x^\alpha)\exp\{\exp(\beta x^\alpha) - 1\}.$$

The quantile function of (7.12) becomes

$$Q(u) = \frac{1}{\beta}\left[\log\{1+\log(-\log(1-u))\}\right]^{\frac{1}{\alpha}}$$

with hazard quantile function of the form

$$H(u) = -\frac{\beta\alpha\log(1-u)\{1+\log(-\log(1-u))\}}{[\log\{1+\log(-\log(1-u))\}]^{\frac{1}{\alpha}-1}}.$$

Another distribution of interest proposed by Lai et al. [366] has hazard rate

$$h(x) = x^{a-1}(1-x)^{b-1}\{a-(a+b)x\}, \quad 0<x<1, a>0, b<1. \tag{7.13}$$

In both (7.12) and (7.13), $h(x)$ tends to infinity at both end points of the support thus supporting the BT shape.

A method of constructing BT-shaped hazard rates is given in Haupt and Schäbe [266]. Let $G(u)$ be a twice differentiable function satisfying the following conditions:

(a) $G(0)=0$, $G(1)=1$, $0 \le G(u) \le 1$;
(b) the solution $F(x)$ of the differential equation

$$\frac{\theta G(F(x))dF(x)}{\bar{F}(x)} = dx, \quad \theta = T(1) > 0,$$

where $T(u)$ is the TTT;
(c) the scaled TTT $\phi(u)$ (see Chap. 5 for pertinent details) has one inflexion point u_0 such that $0<u_0<1$ and $\phi(u)$ is convex on $[0,u_0]$ and concave on $[u_0,1]$.

They then illustrated this method for

$$G(u) = -\frac{1}{3}\alpha u^3 + \frac{1}{2}(\alpha-\alpha\beta)u^2 + \alpha\beta u$$

to arrive at the model in (7.11), discussed earlier by Haupt and Schäbe [266]. It appears that (a) and (b) are redundant, since (c) alone can produce a BT curve (see Theorem 5.2).

The loglogistic distribution with density function

$$f(x) = \frac{\alpha\rho^{\alpha}x^{\alpha-1}}{(1+\rho^{\alpha}x^{\alpha})^2} \quad x>0,\ \alpha,\rho>0,$$

has its survival function as

$$S(x) = (1+\rho^{\alpha}x^{\alpha})^{-1}, \tag{7.14}$$

and thus the hazard rate as

$$h(x) = \frac{\alpha\rho^{\alpha}x^{\alpha-1}}{1+\rho^{\alpha}x^{\alpha}}$$

which is UBT for $\alpha > 1$ with change point $x_0 = \rho^{-1}(\alpha-1)^{\frac{1}{\alpha}}$. It is easy to convert (7.14) into a quantile function in the simple form

$$Q(u) = \frac{1}{\rho}\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}$$

giving the hazard quantile function

$$H(u) = \frac{\rho\alpha(1-u)^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}-1}}.$$

A direct differentiation of $H(u)$ shows that it is UBT with change point at $u_0 = \frac{\alpha-1}{\alpha}$. For a detailed discussion of the model in reliability analysis, see Bennet [88] and Gupta et al. [237]. One may also refer to Balakrishnan et al. [55] and Balakrishnan and Saleh [58] for some inferential methods for this model based on censored lifetime data.

Employing what is referred to as the logWeibull time displacement transformation,

$$y = \log(1+\rho x),$$

to the Weibull survival function $\overline{G}(y) = \exp(-y^{\alpha})$, Avinadav and Raz [41] obtained the distribution with survival function

$$\overline{F}(x) = \exp[-\{\log(1+\rho x)\}^{\alpha}]. \tag{7.15}$$

The corresponding density function is

$$f(x) = \frac{\alpha\rho}{(1+\rho x)}\{\log(1+\rho x)\}^{\alpha-1}\exp[-\{\log(1+\rho x)\}^{\alpha}],$$

and so

$$h(x) = \frac{\alpha\rho\{\log(1+\rho x)\}^{\alpha-1}}{1+\rho x}.$$

For $\alpha > 1$, $h(x)$ has upside-down bathtub shape with maximum value at $x_0 = \frac{e^{\alpha-1}-1}{\rho}$. A quantile analysis of the distribution can be made with

$$Q(u) = \frac{1}{\rho}\left[\exp\left[\{-\log(1-u)\}^{\frac{1}{\alpha}}\right]-1\right]$$

and

$$H(u) = \frac{\rho\alpha}{e^{\{-\log(1-u)\}^{\frac{1}{\alpha}}}\{-\log(1-u)\}^{\frac{1}{\alpha}-1}}.$$

An interesting feature of the distribution is that it is closer to the loglogistic distribution until a certain point of time and then becomes closer to the Weibull law.

Remark 7.1. Upon comparing the survival function in (7.15) with that of the Dhillon model in (7.8), we immediately observe that the above distribution is identical to the two-parameter Dhillon model in (7.8) with $\lambda = \rho$ and $\beta = \alpha - 1$.

Applying transformation $X = \frac{\beta^2}{Y}$ when Y is a two-parameter Weibull distribution with survival function

$$\overline{G}(y) = \exp\left\{-\left(\frac{y}{\beta}\right)^{\alpha}\right\}, \quad y > 0; \alpha > 0, \beta > 0,$$

we obtain the inverse Weibull law with distribution function

$$F(x) = \exp\left\{-\left(\frac{\beta}{x}\right)^{\alpha}\right\}, \quad \alpha, \beta > 0; x > 0, \tag{7.16}$$

and density function

$$f(x) = \alpha\beta^{\alpha}x^{-\alpha-1}\exp\left\{-\left(\frac{\beta}{x}\right)^{\alpha}\right\}.$$

Applications of (7.16) in lifetime modelling has been discussed by Erto [188] and Jiang et al. [297]. In this case, we have

$$h(x) = \frac{\alpha\beta^{\alpha}x^{-\alpha-1}\exp[-(\frac{\beta}{x})^{\alpha}]}{1+\exp[-(\frac{\beta}{x})^{\alpha}]}$$

which is UBT shaped with change point x_0 as the solution of the equation

$$\frac{(\frac{\beta}{x})^{\alpha}}{1-e^{-(\frac{\beta}{x})^{\alpha}}} = \frac{\alpha+1}{\alpha}.$$

The quantile function has a simple form

$$Q(u) = \beta(-\log u)^{-\frac{1}{\alpha}}$$

and

$$H(u) = \frac{\alpha u(-\log u)^{\frac{1}{\alpha}+1}}{\beta(1-u)}.$$

Differentiation of $H(u)$ yields the change point as u_0 satisfying the equation

$$\log u_0 + \frac{\alpha+1}{\alpha}(1-u_0) = 0.$$

Cooray [156] and de Gusmao et al. [168] have discussed a generalization of the inverse Weibull distribution.

Chen [142] modified the exponential power distribution in (7.2) by setting $\lambda = 1$ and introducing a new parameter by taking the survival function as

$$\overline{F}(x) = \exp[-\lambda(e^{x^{\alpha}} - 1)]. \tag{7.17}$$

The hazard rate function has the modified form

$$h(x) = \lambda \alpha x^{\alpha-1} e^{x^{\alpha}}.$$

Since the parameter λ does not alter the monotonic behaviour of $h(x)$, we have its shape identified to that of (7.3). However, the form (7.17) becomes amenable to developing a three-parameter model as discussed later (see also Tang et al. [560]). Bebbington et al. [83] proposed a flexible Weibull extension by the model

$$\overline{F}(x) = \exp\left[e^{-\alpha x} - \frac{\beta}{x}\right], \quad x > 0; \alpha, \beta > 0, \tag{7.18}$$

with its hazard rate given by

$$h(x) = \left(\alpha + \frac{\beta}{x^2}\right)\exp\left(\alpha x - \frac{\beta}{x}\right).$$

In this case, $\lim_{x\to 0} h(x) = 0$ and so a pure bathtub curve is not envisaged. When $\alpha\beta < \frac{27}{64}$, the hazard rate is strictly increasing in $(0, x_0)$, strictly decreasing in (x_0, x_1), and strictly increasing on (x_1, ∞), where

$$x_0 = \frac{1}{2}\left[-\frac{4\beta}{3\alpha} + A + B\right]^{\frac{1}{2}} - \frac{1}{2}\left[\frac{8\beta}{3\alpha} - A - B + \frac{4\beta}{\alpha^2(-\frac{4\beta}{3\alpha} + A + B)^{\frac{1}{2}}}\right]^{\frac{1}{2}},$$

$$x_1 = \frac{1}{2}\left[-\frac{4\beta}{3\alpha} + A + B\right]^{\frac{1}{2}} + \frac{1}{2}\left[\frac{8\beta}{3\alpha} - A - B + \frac{4\beta}{\alpha^2(-\frac{4\beta}{3\alpha} + A + B)^{\frac{1}{2}}}\right]^{\frac{1}{2}},$$

with

$$A = \frac{2^{\frac{11}{3}}\beta^2}{3[27\alpha^2\beta^2 - 32\alpha^3\beta^3 + 3\sqrt{3}(27\alpha^4\beta^4 - 64\alpha^5\beta^5)^{\frac{1}{2}}]^{\frac{1}{3}}}$$

and

$$B = 2^{\frac{1}{3}}[27\alpha^2\beta^2 - 32\alpha^3\beta^3 + 3\sqrt{3}(27\alpha^4\beta^4 - 64\alpha^5\beta^5)^{\frac{1}{2}}]^{\frac{1}{3}}.$$

7.3 Three-Parameter BT and UBT Models

A majority of models described in the context of nonmonotonic hazard functions contain three parameters, some of them being extensions of two-parameter versions discussed in the preceding section. Some others are postulated in terms of hazard rates, rather than distribution functions. The quadratic hazard rate

$$h(x) = a + bx + cx^2, \quad a \geq 0, -2(ac)^{\frac{1}{2}} \leq b < 0, c > 0,$$

generating the survival function

$$\overline{F}(x) = \exp\left[-\left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)\right], \quad x > 0, \tag{7.19}$$

is one such model discussed at some length in Bain [45, 46] and Gore et al. [223]. The parameters of the model are estimated by the method of maximum likelihood or by regression of the empirical hazard rate on a quadratic polynomial. Hazard rates of the form

$$h(x) = \exp(a_0 + a_1 x + a_2 x^2)$$

were studied by Lewis and Shedler [385, 387] using simulations of homogeneous Poisson process. The truncated normal distribution as a failure time model, with only one failure mechanism, has been studied by Bosch [118]. Generalizing this, Glaser [220], Cobb [152] and Cobb et al. [153] have studied the distribution with density function

$$f(x) = C\exp[-\alpha x - \beta x^2 + r\log x], \quad x > 0,$$

with α real, $\beta > 0$, $\gamma > -1$ or $\alpha > 0$, $\beta = 0$ and $\gamma > -1$ (giving also extended gamma densities) which gives a BT hazard rate for $\gamma < 0$. They also discussed the cubic exponential family with density function

$$f(x) = C\exp[-\alpha x - \beta x^2 - \gamma x^3]$$

with $C < \alpha$ resulting in BT hazard rate.

The lifetime model introduced by Hjorth [272] is an interesting one as it has some physical interpretations. Relying upon the practical interest in mechanical units that are subject to wear, a distribution with minimal number of parameters and with enough flexibility lead to the study of a distribution with survival function

$$\overline{F}(x) = \frac{\exp\left(-\alpha\frac{x^2}{2}\right)}{(1+\beta x)^{\frac{\theta}{\beta}}} \quad x \geq 0,\ \alpha,\beta,\theta > 0,\ \alpha+\theta > 0, \tag{7.20}$$

and density function

$$f(x) = \frac{(1+\beta x)\alpha x + \theta}{(1+\beta x)^{\frac{\theta}{\beta}+1}} e^{-\frac{\alpha x^2}{2}}.$$

So, the hazard rate is given by

$$h(x) = \alpha x + \frac{\theta}{1+\beta x}.$$

As special cases, we have the Rayleigh distribution ($\theta = 0$), exponential ($\alpha = \beta = 0$), decreasing hazard ($\alpha = 0$), increasing hazard ($\alpha \geq \theta\beta$), and the bathtub curve ($0 < \alpha < \theta\beta$). Hjorth [272] has given two physical interpretations for the model in (7.20). Assuming that every produced or maintained unit has linear hazard rate

$$h^*(x) = u + \alpha x,$$

where α is the same for all units, but u is the realization from the gamma distribution with density

$$g(u) = \frac{u^{a-1}e^{-\frac{u}{\beta}}}{\beta^a \Gamma a},$$

we have

$$\overline{F}(x) = \frac{1}{(1+\beta x)^a}\exp\left(-\frac{\alpha x^2}{2}\right)$$

which is of the same form as (7.20). Alternatively, if failures are classified as type A and type B caused by $h_A(x) = \alpha x$ or $h_B(x) = \frac{\theta}{(1+\beta x)}$, then (7.20) is the distribution of $\min(X_A, X_B)$, where X_A and X_B are independent lifetimes with hazard rates h_A

and h_B, respectively. Maximum likelihood method can be used to estimate the parameters. We can also see that (7.20) belongs to the class of additive hazard models discussed in Nair and Sankaran [446].

Let X be a lifetime random variable with hazard rate $h(x)$ and Y be a non-negative random variable representing changes in the conditions so that h has an additive effect on X through the relationship

$$h(x|y) = a(y) + h(x) \tag{7.21}$$

for some positive function $a(y)$. If X^* is the random variable corresponding to X satisfying the relationship in (7.21), then the survival function of X^* is

$$S^*(x) = S(x)S_E(x),$$

where

$$S_E(x) = \int_0^\infty e^{-xa(y)} g(y)dy$$

and $g(y)$ is the density function of Y. Equivalently, we arrive at the additive hazard model

$$h^*(x) = h(x) + h_E(x),$$

where $h^*(\cdot)$ and $h_E(\cdot)$ are the hazard rates of $S^*(x)$ and $S_E(x)$, respectively. Now, when

$$g(y) = [\Gamma(\alpha)]^{-1} c\lambda^{C\alpha} y^{C\alpha-1} \exp[-(\lambda y)^c],$$

and $a(y) = y^c$ for $c > 0$, we get

$$S_E(x) = \lambda^{c\alpha}(x+\lambda^c)^{-\alpha}.$$

Then, the additive model takes on the form

$$h^*(x) = h(x) + \alpha(x+\lambda^c)^{-1}.$$

It is easy to see that the Hjorth model arises from a particular choice of the hazard rate function $h^*(x) = \alpha x$.

Mudholkar and Kollia [426] and Mudholkar et al. [428] introduced a generalization of the Weibull distribution with its survival function as

$$\overline{F}(x) = \left\{1 - \lambda\left(\frac{x}{\alpha}\right)^\beta\right\}^{\frac{1}{\lambda}}, \quad \alpha, \beta > 0, \tag{7.22}$$

where X has support $(0,\infty)$ for $\lambda \le 0$ and $\left(0, \frac{\alpha}{\lambda^{\frac{1}{\beta}}}\right)$ for $\lambda > 0$. It is easy to see that, as $\lambda \to 0$,

$$\overline{F}(x) = \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right],$$

which is the standard two-parameter Weibull model. The hazard rate corresponding to (7.22) is

$$h(x) = \frac{\beta(\frac{x}{\alpha})^{\beta-1}}{\alpha\left\{1-\lambda(\frac{x}{\alpha})^{\beta}\right\}},$$

which is BT for $\beta < 1$, $\lambda > 0$; UBT for $\beta > 1$, $\lambda < 0$; IHR for $\beta \ge 1$, $\lambda > 0$; and DHR for $\beta \le 1$, $\lambda \le 0$. For the estimation of parameters, they discussed the maximum likelihood method. Corresponding to (7.22), the quantile function is

$$Q(u) = \begin{cases} \alpha\left\{\frac{1-(1-u)^{\lambda}}{\lambda}\right\}^{\frac{1}{\beta}}, & \lambda \ne 0 \\ \alpha\{-\log(1-u)\}^{\frac{1}{\beta}}, & \lambda = 0 \end{cases},$$

which has been discussed in detail in Chap. 3. Another modification to the Weibull model is the exponentiated Weibull distribution with distribution function

$$F(x) = \left\{1-\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\right\}^{\theta}, \quad \alpha,\beta,\theta > 0,\ x \ge 0, \tag{7.23}$$

density function

$$f(x) = \frac{\beta\alpha\theta}{\alpha^{\beta}} e^{-(\frac{x}{\alpha})^{\beta}} \left\{1-\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right)\right\}^{\theta-1},$$

and hazard function

$$h(x) = \frac{\beta\theta e^{-(\frac{x}{\alpha})^{\beta}}\left\{1-\exp(-(\frac{x}{\alpha})^{\beta})\right\}^{\theta-1}}{1-\left\{1-\exp(-(\frac{x}{\alpha})^{\beta})\right\}^{\theta}}.$$

The nature of $h(x)$ within the parameter space, other properties, and estimation of parameters have all been discussed by Mudholkar et al. [428], Mudholkar and Hutson [423], Jiang and Murthy [296], Nassar and Eissa [463, 464], Singh et al. [543] and Shanmukhapriya and Lakshmi [534]. It is seen that $h(x)$ is BT for $\beta > 1$, $\beta\theta < 1$; UBT for $\beta < 1$, $\beta\theta > 1$; IHR for $\beta \ge 1$, $\beta\theta \ge 1$ and DHR for $\beta \le 1$; and $\beta\theta \le 1$. Reverting to quantile function, (7.23) becomes

$$Q(u) = \alpha \left\{ -\log(1 - u^{\frac{1}{\theta}}) \right\}^{\frac{1}{\beta}}.$$

The hazard quantile function is then

$$H(u) = \frac{\beta\theta(1 - u^{\frac{1}{\theta}})}{\alpha(1-u)u^{\frac{1}{\theta}-1} \left\{ -\log(1 - u^{\frac{1}{\theta}}) \right\}^{\frac{1}{\beta}-1}}. \tag{7.24}$$

Differentiating (7.24), we find that the change points in (7.24) are the solutions of

$$\left\{ 1 - u - \theta(1 - u^{\frac{1}{\theta}}) \right\} \log(1 - u^{\frac{1}{\theta}}) = \frac{1-\beta}{\beta}(1-u)u^{\frac{1}{\theta}}.$$

Marshall and Olkin [411] devised a new method of introducing more flexibility to a given distribution $G(x)$ by adding a new parameter. Their scheme is to construct distribution $\overline{F}(x)$ from $\overline{G}(x)$ through the formula

$$\overline{F}(x) = \frac{\theta \overline{G}(x)}{1-(1-\theta)\overline{G}(x)}, \quad \theta > 0. \tag{7.25}$$

Assuming $\overline{G}(x)$ to be a two-parameter Weibull, $\overline{G}(x) = \exp\left\{ -(\frac{x}{\alpha})^\beta \right\}$, for example, we find from (7.25) that

$$\overline{F}(x) = \frac{\theta \exp\left\{ -(\frac{x}{\alpha})^\beta \right\}}{1-(1-\theta)\exp\left\{ -(\frac{x}{\alpha})^\beta \right\}}$$

yielding the density function

$$\begin{aligned} f(x) &= \frac{\theta g(x)}{[1-(1-\theta)\overline{G}(x)]^2} \\ &= \frac{\theta\beta}{\alpha} \frac{(\frac{x}{\alpha})^{\beta-1} \exp\left\{ -(\frac{x}{\alpha})^\beta \right\}}{\left\{ 1-(1-\theta)\exp(-(\frac{x}{\alpha})^\beta) \right\}^2} \end{aligned}$$

and hazard rate

$$h(x) = \frac{(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}}{1-(1-\theta)\exp\left\{ -(\frac{x}{\alpha}\right\}^\beta)}.$$

Teiling and Xie [565] have carried out a failure time data analysis by using (7.25).

It is of interest to know the quantile version of (7.25) for studying the properties of the new distribution further. Setting $x = Q(u)$, where Q is the quantile function of G, after writing

$$F(x) = \frac{1-\overline{G}(x)}{1-(1-\theta)\overline{G}(x)},$$

$$F(Q(u)) = \frac{u}{1-(1-\theta)(1-u)} = \frac{u}{u+\theta-\theta u},$$

we have

$$Q(u) = Q_1\left(\frac{u}{u+\theta-\theta u}\right)$$

with $Q_1(u)$ being the quantile function corresponding to $F(x)$. Equivalently, we have

$$Q_1(u) = Q\left(\frac{u\theta}{1-(1-\theta)u}\right). \tag{7.26}$$

Applying (7.26) in the case of the Weibull distribution for which

$$Q(u) = \alpha\{-\log(1-u)\}^{\frac{1}{\beta}},$$

we obtain

$$Q_1(u) = \alpha\left\{-\log\left(\frac{1-u}{1-u+u\theta}\right)\right\}^{\frac{1}{\beta}}.$$

The hazard quantile of $Q_1(u)$ is

$$H(u) = \frac{\beta(1-u+u\theta)}{\alpha\theta\left\{-\log\left(\frac{1-u}{1-u+u\theta}\right)\right\}^{\frac{1}{\beta}-1}}.$$

The sign of $H'(u)$ depends on the expression

$$(\theta-1)\log\frac{1-u+u\theta}{1-u} - \frac{\theta(1-\beta)}{\beta(1-u)}$$

and $F(x)$ is IHR for $\theta > 1$, $\beta > 1$, and DHR for $0 < \theta < 1$ and $0 < \beta < 1$. Change points of $H(u)$ are the solutions of the equation

$$(1-u)\log\frac{1-u+u\theta}{1-u} = \frac{\theta(1-\beta)}{\beta(\theta-1)}.$$

A special case of the exponentiated Weibull distribution is the exponentiated exponential distribution considered in Gupta et al. [239] and Gupta and Kundu [250, 253]. We look at the general form with survival function

$$F(x) = \left\{1 - \exp\left(-\frac{x-\mu}{\sigma}\right)\right\}^{\alpha}, \quad x > \mu;\ \alpha, \sigma > 0. \tag{7.27}$$

We find the expression for the hazard rate as

$$h(x) = \frac{\alpha}{\sigma} \frac{\left\{1 - \exp(-\frac{x-\mu}{\sigma})\right\}^{\alpha-1} \exp(-\frac{x-\mu}{\sigma})}{1 - \left[1 - \left\{1 - \exp(-\frac{x-\mu}{\sigma})\right\}^{\alpha}\right]}.$$

It could be seen that $h(x) = \frac{1}{\sigma}$ for $\alpha = 1$, increases from 0 to $\frac{1}{\sigma}$ for $\alpha > 1$, and decreases from ∞ to $\frac{1}{\sigma}$ for $\alpha < 1$. Quantile analysis of (7.27) is straightforward with

$$Q(u) = \mu - \sigma \log(1 - u^{\frac{1}{\alpha}})$$

and

$$H(u) = \frac{\sigma}{\alpha} \frac{u^{\frac{1}{\alpha}-1}}{1 - u^{\frac{1}{\alpha}}}.$$

A comparative study of (7.27) with the gamma, Weibull and lognormal distributions has been carried out by Gupta and Kundu [251], Kundu et al. [358] and Gupta and Kundu [252].

The two-parameter Chen's [142] model in (7.17) has been generalized by Xie et al. [595] to provide a new distribution with survival function

$$\overline{F}(x) = \exp\left[-\lambda\alpha\left\{\exp\left(\frac{x}{\alpha}\right)^{\beta} - 1\right\}\right], \quad x \geq 0;\ \alpha, \beta, \lambda > 0, \tag{7.28}$$

called the modified Weibull extension. From the corresponding density function

$$f(x) = \exp\left[-\lambda\alpha\left\{\exp\left(\frac{x}{\alpha}\right)^{\beta} - 1\right\}\right] \lambda\beta e^{(\frac{x}{\alpha})^{\beta}} \left(\frac{x}{\alpha}\right)^{\beta-1},$$

we obtain the hazard function as

$$h(x) = \lambda\beta e^{(\frac{x}{\alpha})^{\beta}} \left(\frac{x}{\alpha}\right)^{\beta-1}.$$

The name Weibull extension comes from the fact that (7.28) reduces to the Weibull distribution when $\lambda \to \alpha$ in such a way that $\alpha^{\beta-1}\lambda^{-1}$ remains constant. Two special cases are the exponential power distribution in (7.2) when $\lambda = 1$ and the Chen's model when $\alpha = 1$. When $\beta \geq 1$, $h(x)$ is IHR and when $0 < \alpha < 1$, $h(x) \to \infty$ as $x \to 0$ or ∞. In this case, we have BT shape with change point $x_0 = \alpha\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{\beta}}$. The change point increases as β decreases from 1 to 0. The quantile function of (7.28) takes on the expression

$$Q(u) = \alpha \left[\log\left\{1 - \frac{\log(1-u)}{\lambda\alpha}\right\}\right]^{\frac{1}{\beta}}$$

and therefrom we get

$$H(u) = \frac{\beta(\lambda\alpha - \log(1-u))}{\alpha\left[\log\left\{1 - \frac{\log(1-u)}{\lambda\alpha}\right\}\right]^{\frac{1}{\beta}-1}}.$$

Yet another extension of the Weibull law is due to Lai et al. [370], called the modified Weibull distribution. It has its density function as

$$f(x) = \beta(\alpha + \lambda x)x^{\alpha-1}\exp\{\lambda x - \beta x^{\alpha} e^{\lambda x}\} \tag{7.29}$$

and survival function as

$$\overline{F}(x) = \exp(-\beta x^{\alpha} e^{\lambda x}).$$

As $\lambda \to 0$, we have the usual Weibull distribution. Note that (7.29) has a hazard rate of the form

$$h(x) = \beta(\alpha + \lambda x)x^{\alpha-1}e^{\lambda x},$$

so that the shape of $h(x)$ is independent of β and λ. For $\alpha \geq 1$, the distribution is IHR, and for $0 < \alpha < 1$, we have BT shape with change point $x_0 = \frac{\alpha^{\frac{1}{2}} - \alpha}{\lambda}$. There is no simple closed-form expression for the quantile function. Nikulin and Haghighi [472] (see also Dimitrakopoulas et al. [178]) proposed a generalized power Weibull distribution with survival function

$$\overline{F}(x) = \exp\left[1 - \left\{1 + \left(\frac{x}{\beta}\right)^{\alpha}\right\}^{\theta}\right], \quad x \geq 0,\, \alpha, \beta > 0,\, \theta > 0, \tag{7.30}$$

which is a general family consisting of Weibull ($\theta = 1$) and exponential ($\theta = 1$, $\alpha = 1$) distributions as particular members.

The transformation $Y = 1 + (\frac{X}{\beta})^{\alpha}$ gives the Weibull distribution with parameters 1 and θ in $(1, \infty)$. Similarly, transforming by $\log\left\{1 + (\frac{x}{\alpha})^{\beta}\right\}$ and $\left[\log\left\{1 + (\frac{x}{\alpha})^{\beta}\right\}\right]^{\frac{1}{\beta}}$, respectively, we obtain the modified extreme value distribution and the power exponential distribution of Smith and Bain [544] in (7.2). From the density function

$$f(x) = \frac{\theta\alpha}{\beta^\alpha} \exp\left[1 - \left\{1 + \left(\frac{x}{\beta}\right)^\alpha\right\}^\theta\right] \left\{1 + \left(\frac{x}{\beta}\right)^\alpha\right\}^{\theta-1} x^{\alpha-1},$$

we obtain the hazard function as

$$h(x) = \frac{\theta\alpha}{\beta^\alpha} \left\{1 + \left(\frac{x}{\beta}\right)^\alpha\right\}^{\theta-1} x^{\alpha-1}.$$

The above expression yields flexible hazard rate shapes, like IHR if either $\alpha > 1$ and $\alpha > \frac{1}{\theta}$ or $\alpha = 1$ and $\theta > 1$, DHR if either $0 < \alpha < 1$ and $\alpha < \frac{1}{\theta}$ or $\alpha\theta = 1$ and $0 < \alpha < 1$, and UBT whenever $\frac{1}{\theta} > \alpha > 1$. For a quantile-based analysis, we can use

$$Q(u) = \beta\left[\{1 - \log(1-u)\}^{\frac{1}{\theta}} - 1\right]^{\frac{1}{\alpha}}$$

and

$$H(u) = \frac{\alpha\theta}{\beta}\left[\{1 - \log(1-u)\}^{\frac{1}{\theta}} - 1\right]^{1-\frac{1}{\alpha}} \{1 - \log(1-u)\}^{1-\frac{1}{\theta}}.$$

Differentiating $H(u)$ and setting $H'(u) = 0$, we find the change point u_0 as

$$u_0 = 1 - \exp\left\{1 - \left(\frac{\alpha\theta - \alpha}{1 - \alpha\theta}\right)^\theta\right\}.$$

Lan and Leemis [372] presented the logistic exponential distribution as a model for lifetimes with flexible hazard rate shapes. Their two-parametric version has its survival function as

$$\overline{F}(x) = \left\{1 + (e^{\lambda x} - 1)^k\right\}^{-1}, \quad x \geq 0. \tag{7.31}$$

Clearly, the distribution reduces to the exponential case when $k = 1$ having constant hazard rate. In general, the hazard function is

$$h(x) = \frac{\lambda k e^{\lambda x}(e^{\lambda x} - 1)^{k-1}}{1 + (e^{\lambda x} - 1)^k}.$$

For $0 < k < 1$, $h(x)$ is BT, while for $k > 1$, it is UBT. The change point in both cases is

$$x_0 = \lambda^{-1} \log(x_k + 1),$$

where x_k is the positive root of the equation

$$kx - x^k = 1 - k.$$

It can be shown that the quantile function of (7.31) is

$$Q(u) = \frac{1}{\lambda} \log \left\{ 1 + \left(\frac{u}{1-u} \right)^{\frac{1}{k}} \right\}$$

and

$$H(u) = k\lambda \frac{u^{\frac{1}{k}} + (1-u)^{\frac{1}{k}}}{u^{\frac{1}{k}} - 1}.$$

The change point of $H(u)$ is the solution of the equation

$$ku^{\frac{1}{k}} + (1-u)^{\frac{1}{k}-1}(k - ku - 1) = 0.$$

Introducing yet another parameter into (7.31), we have the more general model with survival function

$$\overline{F}(x) = \frac{1 + (e^{\lambda\theta} - 1)^k}{1 + \{e^{\lambda(x+\theta)} - 1\}^k}, \quad \theta \geq 0,\ k > 0, \lambda > 0, \tag{7.32}$$

and corresponding hazard function

$$h(x) = \frac{C\lambda k \{e^{\lambda(x+\theta)} - 1\}^{k-1} e^{\lambda(x+\theta)}}{1 + \{e^{\lambda(x+\theta)} - 1\}^k},$$

where

$$c = 1 + (e^{\lambda\theta} - 1)^k.$$

By proceeding along the same lines as in the reduced model, we see that $\overline{F}(x)$ has highly flexible hazard rate being exponential for $k = 1$; BT for $0 < k < 1$, $\lambda\theta < \log(x_k + 1)$ with minimum at $\frac{\log(1+x_k)}{\lambda} - \theta$; IHR for $0 < k < 1$ and $\lambda\theta > \log(x_k + 1)$; and UBT for $k > 1$ and $\lambda\theta > \log(1 + x_k)$. The quantile function, with a slightly more complicated from than the two-parameter version, given by

$$Q(u) = \frac{1}{\lambda_2} \log \left\{ 1 + \left(\frac{c+u-1}{1-u} \right)^{\frac{1}{k}} - \theta \right\}$$

can be employed to find $H(u)$ and its change points as done before.

By exponentiating the linear failure rate model, Sarhan and Kundu [520] arrived at the generalized linear failure rate distribution with survival function

$$\overline{F}(x) = \left[1 - \exp\left\{-\left(ax + \frac{b}{2}x^2\right)\right\}\right]^\theta, \quad x \geq 0. \tag{7.33}$$

It contains as special cases the linear failure rate model, the generalized exponential distribution in (7.27) and generalized Rayleigh distribution discussed by Kundu and Raqab [361]. The hazard rate becomes

$$h(x) = \frac{\theta(a+bx)\left[1 - \exp\left\{-\left(ax + \frac{b}{2}x^2\right)\right\}\right]^{\theta-1} \exp\left\{-(ax + \frac{bx^2}{2})\right\}}{1 - \left[1 - \exp\left\{-(ax + \frac{bx^2}{2})\right\}\right]^\theta}.$$

Analysing $h(x)$, it is seen that the hazard rate is constant or increasing when $\theta = 1$, increasing when $\theta > 1$, either decreasing ($b = 0$) or bathtub ($b > 0$) when $\theta < 1$. The same approach is made by Barreto-Souza and Cribari-Neto [71] to extend the exponential Poisson distribution of Kus [364] given by

$$F(x) = \frac{1 - \exp(-\lambda + \lambda e^{-\beta x})}{1 - e^{-\lambda}}, \quad x, \lambda, \beta > 0,$$

to the general form

$$F(x) = \left\{\frac{1 - \exp(-\lambda + \lambda e^{-\beta x})}{1 - e^{-\lambda}}\right\}^\theta, \quad x, \theta > 0. \tag{7.34}$$

The hazard rate of (7.34) is given by

$$h(x) = \frac{\theta\lambda\beta(1 - e^{-\lambda+\lambda e^{-\beta x}})^{\theta-1} e^{-\lambda-\beta x+\lambda e^{-\beta x}}}{(1 - e^{-\lambda})^\theta - \{1 - \exp(-\lambda + \lambda e^{-\beta x})\}^\theta}$$

which can be IHR, DHR or UBT. A closed-form quantile function for (7.34) is given by

$$Q(u) = \frac{1}{\beta} \log\left[-\frac{1}{\lambda} \log\left\{1 - (1 - e^{-\lambda})u^{\frac{1}{\theta}}\right\}\right].$$

When the baseline distribution in (7.33) or (7.34) is changed to the exponential power distribution in (7.2), we have the model proposed by Barriga et al. [72], for which the survival function is

$$\overline{F}(x) = 1 - \left[1 - \exp\left\{1 - \exp\left(\frac{x}{\alpha}\right)^\beta\right\}\right]^\theta, \quad x > 0. \tag{7.35}$$

The corresponding hazard rate function

$$h(x) = \frac{\beta\theta x^{\beta-1}\exp\left\{1+(\frac{x}{\alpha})^{\beta}-\exp(\frac{x}{\alpha})^{\beta}\right\}\exp\left[1-\exp\left\{1-\exp(\frac{x}{\alpha})^{\beta}\right\}\right]^{\theta-1}}{\alpha^{\beta}[1-\{1-\exp(1-\exp(\frac{x}{\alpha})^{\beta})\}]^{\theta}}$$

has the following properties:

(i) $h(0)=0$ for $\beta>1$ and $h(0)=\frac{1}{\alpha}$ for $\beta=1$, $\theta=1$;
(ii) $h(x)$ is increasing for $\beta>1$, $\theta\leq 1$;
(iii) $h(x)$ is decreasing for $\beta\theta\leq 1$, $\theta>1$;
(iv) $h(x)$ is UBT for $\beta<1$, $\beta\theta>1$;
(v) $h(x)$ is BT for $\theta\leq 1$ or $\beta>1$ and $\beta\theta<1$.

The parameter estimation is carried out by the maximum likelihood method. Distribution (7.35) is specified by the quantile function

$$Q(u) = \alpha\left[\log\left\{1-\log(1-u)^{\frac{1}{\theta}}\right\}\right]^{\frac{1}{\beta}}.$$

Zhang and Xie [600] considered the upper truncated Weibull distribution given by

$$G(x) = \frac{F(x)-F(a)}{F(T)-F(a)}, \quad a\leq x<T,$$

with

$$F(x) = 1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}$$

yielding a hazard rate

$$h(x) = \frac{(\frac{\beta}{\eta})(\frac{x}{\eta})^{\beta-1}\exp\left\{-\left(\frac{x}{\eta}\right)^{\beta}\right\}}{F(T)-F(x)}, \quad a\leq x<T,$$

which is increasing for $\beta\geq 1$ and BT for $\beta<1$.

Silva et al. [539] introduced the generalized geometric exponential distribution with distribution function

$$F(x) = \left(\frac{1-e^{-px}}{1-pe^{-\beta x}}\right)^{\theta}, \quad x>0,\ 0<p<1,\ \theta,\beta>0. \tag{7.36}$$

When $\theta>0$ is an integer, (7.36) is the distribution of $X=\max_{1\leq i\leq\alpha}Y_i$, where $Y_1,Y_2,\ldots,Y_{\alpha}$ is a random sample from the exponential geometric distribution. Note that

$$h(x) = \frac{\theta\beta(1-p)e^{-\beta x}(1-e^{-\beta x})^{\theta-1}}{(1-pe^{-\beta x})\{(1-pe^{-\beta x})^{\alpha}-(1-e^{-\beta x})^{\alpha}\}}$$

which has the following properties:

(a) decreasing for p and α in $(0,1)$;
(b) increasing for p in $(0, \frac{\alpha-1}{\alpha+1})$ and α in $(1,\infty)$;
(c) UBT for p in $(\frac{\alpha-1}{\alpha+1}, 1)$ and α in $(1,\infty)$.

Let $Y_1, Y_2, \ldots, Y_N$ be a random sample from a Weibull distribution with density function

$$f(y) = \beta\alpha^{\beta}y^{\beta-1}e^{-(\alpha y)^{\beta}}, \quad y, \alpha, \beta > 0,$$

where N is a zero-truncated Poisson random variable with probability mass function

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{\Gamma(n+1)(1-e^{-\lambda})}, \quad n = 1, 2, \ldots.$$

Assuming that Y_i and N are independent, the distribution of $X = \min(Y_1, Y_2, \ldots, Y_N)$ is called a Weibull-Poisson distribution by Hemmati et al. [267]. It has density function

$$f(x) = \frac{\lambda\beta\alpha}{1-e^{-\lambda}}(\alpha x)^{\beta-1}\exp\left\{-\lambda-(\alpha x)^{\beta}+\lambda e^{-(\alpha x)^{\beta}}\right\}$$

and survival function

$$\overline{F}(x) = \left\{1-\exp(\lambda e^{-(\alpha x)^{\beta}})\right\}(1-e^{-\lambda})^{-1}.$$

The corresponding hazard rate is

$$h(x) = \frac{\lambda\beta\alpha}{1-e^{-\lambda}}\frac{(\alpha x)^{\beta-1}(1-e^{\lambda})\exp\left\{-\lambda-(\alpha x)^{\beta}+\lambda e^{-(\alpha x)^{\beta}}\right\}}{\left\{1-\exp(\lambda e^{-(\alpha x)^{\beta}})\right\}}$$

which can be either increasing, decreasing or modified bathtub shaped.

A lifetime model for bathtub failure rate data by transforming them to the Weibull was considered in Mudholkar et al. [422]. Consider the data in the form of pairs of independent and identically distributed random variables (X_i, δ_i), where $X_i = \min(T_i, C_i)$ and $\delta_i = 1$ when $T_i \leq C_i$ (uncensored case) and $\delta_i = 0$ if $T_i > C_i$ (censored case), with T_i as the lifetime and C_i as the censoring time. Mudholkar et al. [422] assumed that there exists a transformation

$$y = g(x,\theta) = \frac{x}{1-\theta x}$$

that transforms the data to the Weibull form $\overline{F}(y) = e^{-(\frac{y}{\alpha})^\beta}$. The range of the transformation is $(0,\alpha)$; $g(x,\theta)$ should be invertible and θ can be zero in which case the original data is retained. We have $x = \frac{y}{y+\theta}$, and so

$$\overline{F}(x) = P\Big(\frac{Y}{1+\theta Y} > x\Big) = \exp\Big\{-\Big(\frac{1}{\alpha}\frac{x}{1+\theta x}\Big)^\beta\Big\}, \quad 0 < x < \frac{1}{\theta}.$$

The corresponding hazard rate

$$h(x) = \frac{\beta}{\alpha^\beta}\frac{x^{\beta-1}}{(1-\theta x)^{\beta+1}}$$

has

$$h'(x) = (1-\theta x)^\beta x^{\beta-2}(\beta - 1 + \theta x)$$

which reveals that $h(x)$ is increasing for $\beta > 1$ and BT for $0 < \beta < 1$. The distribution has its quantile function as

$$Q(u) = \frac{\alpha\{-\log(1-u)\}^{\frac{1}{\beta}}}{1+\theta\alpha\{-\log(1-u)\}^{\frac{1}{\beta}}}.$$

7.4 More Flexible Hazard Rate Functions

When we examine the models in the last two sections chronologically, it is seen that model parsimony was an important concern in earlier works with many two-parametric models. With improvement in computational technology, the number of parameters and complexities in estimating them became less problematic. Consequently, models with more than one shape parameter that provide more richness in the shapes of the hazard rate began to appear. In this section, we deal with such models that have at least four parameters.

Models with hazard rates as a sum were proposed by many. These include

$$h(x) = \frac{\theta}{1+\beta x} + \lambda\alpha x^{\alpha-1}, \quad \theta,\beta,\lambda,\alpha > 0, \tag{7.37}$$

suggested by Murthy et al. [435]. Note that the first term is the hazard rate of a Pareto II distribution, while the second is that of the Weibull. Here, $h(0) = \theta$ and $h(\infty) = \infty$ suggesting a bathtub shape. The Hjorth model in (7.20) is a special case of this family with survival function

$$\overline{F}(x) = \frac{e^{-\lambda x^{\alpha}}}{(1+\beta x)^{\frac{\theta}{\beta}}}.$$

Jaising et al. [291] extended the hazard rate in (7.37) to the form

$$h(x) = \lambda + \frac{\theta}{x+\beta} + \lambda x^{\delta}, \tag{7.38}$$

while Canfield and Borgman [126] considered the representation

$$h(x) = \alpha_1 \beta_1 x^{\beta_1 - 1} + \alpha_2 \beta_2 x^{\beta_2 - 1} + \alpha_3 \tag{7.39}$$

with $\beta_2 > 2$ and $\beta_1 < 1$. Notice that (7.38) is a construction of the hazard rates of exponential Pareto II and Weibull, while (7.39) considers the sum of an exponential and two Weibull hazard rates.

In the case of (7.39), the change point is given by

$$x_0 = \left(\frac{\alpha_1 \beta_1 (1-\beta_1)}{\alpha_2 \beta_2}\right)^{\frac{1}{\beta_2 - \beta_1}}, \quad \beta_1 < 1.$$

Similar representation of $h(x)$ with only two Weibull hazard rates as components was also discussed by Xie and Lai [594], Jiang and Murthy [293] and Usagaonkar and Maniappan [570]. Instead of two Weibull distributions, if we take the hazard rates of Burr distributions with survival functions

$$\overline{F}_i(x) = \frac{1}{(1+(\frac{x}{a_i})^{c_i})^{k_i}}, \quad i = 1,2,$$

we get the hazard rate

$$h(x) = \frac{k_1 c_1 x^{c_1 - 1}}{a_1^{c_1}\{1+(\frac{x}{a_1})^{c_1}\}} + \frac{k_2 c_2 x^{c_2 - 1}}{a_2^{c_2}\{1+(\frac{x}{a_2})^{c_2}\}}, \tag{7.40}$$

for $k_1, k_2, a_1, a_2 \geq 0$, $0 < c_1 < 1$ and $c_2 > 2$. A bathtub model arising from the above hazard rate has been discussed by Wang [577].

Phani [495] considered a new distribution with survival function

$$\overline{F}(x) = \exp\left\{-\lambda \frac{(x-\alpha)^{\theta_1}}{(\beta - x)^{\theta_2}}\right\}, \quad \lambda > 0,\ \beta_1, \beta_2 > 0,\ 0 \leq \theta_1 \leq x \leq \theta_2, \tag{7.41}$$

and corresponding hazard rate as

$$h(x) = \frac{\lambda (x-\alpha)^{\theta_1 - 1} (\beta - x)^{\theta_2 - 1} \{\theta_1 (\beta - x) + \theta_2 (x - \alpha)\}}{(\beta - x)^{2\theta_2}}.$$

The sign of $h(x)$ is determined by a quadratic function in x, and so provides a BT shape. In the reduced case of $\theta_1 = \theta_2$, the condition for BT is $0 < \theta_1 < 1$. Subsequently, Moore and Lai [420] proposed the version

$$h(x) = c(x+p)^{a-1}(q-x)^{b-1}, \quad 0 < a < 1,\, b < -1,\, c > 0,\, p \geq 0,\, 0 \leq x < q,$$

which also gives a BT form since $h(0) = cp^{a-1}q^{b-1}$ and $h(x) \to \infty$. The change point is $x_0 = (a+b-2)^{-1}\{(a-1)q-(b-1)p\}$.

Mixtures of distributions form an important aspect in the consideration of bathtub-shaped models. Many authors like Glaser [221], Kunitz and Pamme [362], Pamme and Kunitz [480] and Gupta and Warren [249] have focused on this formulation. If $f_1(x)$ and $f_2(x)$ are density functions with hazard rates $h_1(x)$ and $h_2(x)$, respectively, the two-component mixture

$$f(x) = \alpha f_1(x) + (1-\alpha)f_2(x), \quad 0 < \alpha < 1,$$

has its hazard rate as

$$h(x) = \frac{\alpha f_1(x) + (1-\alpha)f_2(x)}{\alpha \overline{F}_1(x) + (1-\alpha)\overline{F}_2(x)};$$

see the discussion in Sect. 4.2. Assuming

$$f_i(x) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\frac{t}{\beta_i}}, \quad i = 1,2$$

Gupta and Warren [249] showed that $h(x)$ is BT in the cases (i) $\beta_1 = \beta_2 = \beta$ and $\alpha_1 > 1$, $\alpha_2 > 1$; (ii) $\alpha_2 = 1$, $\alpha_1 > 2$; (iii) $\alpha_1 > 1$, $\alpha_2 > 1$. With $\alpha_1 - \alpha_2 > 1$ and $(\alpha_1 - \alpha_2 - 1)^2 - 4(\alpha_2 - 1) > 0$, $h(x)$ can be UBT.

The properties of mixtures of Weibull distributions have been studied by Jiang and Murthy [295] and Wondamagegnehu [584]. Assuming $\overline{F}_1(x) = \exp(-\lambda_1 x^\alpha)$ and $\overline{F}_2(x) = \exp(-\lambda_2 x^\alpha)$, the mixture hazard rate given by

$$h(x) = \frac{pe^{-\lambda_1 x^\alpha}\lambda_1 \alpha x^{\alpha-1} + (1-p)e^{-\lambda_2 x^\alpha}\lambda_2 \alpha x^{\alpha-1}}{pe^{-\lambda_1 x^\alpha} + (1-p)e^{-\lambda_2 x^\alpha}}$$

will be modified BT shaped when $0 < p < \theta$ and IHR when $\theta \leq p < 1$, where $\alpha > 1$ and

$$\theta = \frac{\alpha(\beta-1)+A}{2\alpha(\beta-1)\exp\left[\frac{(\alpha-1)(\beta+1)+A}{\alpha(\beta-1)}\right]},$$

$$A = (\alpha^2(\beta-1)^2 + 4(\alpha-1)^2\beta)^{\frac{1}{2}},$$

$$\beta = \frac{\lambda_2}{\lambda_1}.$$

While the mixtures can have IHR, DHR, UBT, modified bathtub or roller-coaster type, they cannot be BT.

Navarro and Hernandez [466] have discussed the nature of the failure rates of truncated normal mixtures. They considered the truncated normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma\Phi(\frac{\mu}{\sigma})}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big), \quad x>0, \tag{7.42}$$

where $\Phi(x)$ is the standard normal distribution function, and formed the mixture

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad 0<p<1, \tag{7.43}$$

where $f_i(x)$ is distributed as (7.42) with parameters (μ_0,σ_0) and (μ_1,σ_1). When $\sigma_1=\sigma_0$ and $\delta = \frac{\sigma_0^2}{(\mu_0-\mu_1)^2}$, they proved that if

(i) $\delta > \frac{1}{4}$, $f(x)$ is IHR;
(ii) $\delta \le \frac{1}{4}$, $w(0) \le \frac{1}{2}$ and $w(0)(1-w(0)) < \delta$, $f(x)$ is IHR or BT;
(iii) $\delta \le \frac{1}{4}$, $w(0) \le \frac{1}{2}$ and $w(0)(1-w(0)) \ge \delta$, $f(x)$ is IHR or BT;
(iv) $\delta \le \frac{1}{4}$, $w(0) < \frac{1}{2}$ and $w(0)(1-w(0)) \le \delta$, $f(x)$ is IHR, BT or modified BT,

where

$$w(t) = \left\{1 + \frac{1-p}{p}\frac{f_1(x)}{f_2(x)}\right\}^{-1}.$$

Further, the change points of $\eta(x) = -\frac{f'(x)}{f(x)}$ in (7.43) is found from the equation

$$w(x)(1-w(x)) = \delta.$$

In the general case when the variances are unequal, let us assume $\sigma_1 > \sigma_0$,

$$x_0 = \frac{\sigma_1^2\mu_0 - \sigma_0^2\mu_1}{\sigma_1^2 - \sigma_0^2} > 0,$$

$$\theta(x) = \frac{\frac{w(x)}{\sigma_1^2} - \frac{(1-w(x))}{\sigma_0^2}}{w(x)(1-w(x))} - \Big(\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_2}{\sigma_0^2}\Big)^2,$$

and

$$y_1 = w(x_1).$$

Then, the following results hold:

1. If $w(x_0) \ge y_1$, then f is IHR;

2. If $w(x_0) < y_1$ and
   (a) $w(0) < y_1$, $\theta(0) \geq 0$ and $\theta(x_1) \geq 0$, then f is IHR;
   (b) $w(0) < y_1$, $\theta(0) < 0$ and $\theta(x_1) \geq 0$, then f is IHR or BT;
   (c) $w(0) < y_1$, $\theta(0) \geq 0$ and $\theta(x_1) < 0$, then f is IHR, BT or modified BT (MBT);
   (d) $w(0) < y_1$, $\theta(0) < 0$ and $\theta(x_1) < 0$, then f is IHR, BT, MBT or BBT (BT in $(0,x_0)$ and BT in (x_0,∞));
   (e) $w(0) \geq y_1$, $\theta(0) > 0$ and $\theta(x_1) \geq 0$ and $\theta(x_2) \geq 0$, then f is IHR;
   (f) $w(0) \geq y_1$, $\theta(0) \leq 0$ and $\theta(x_2) \geq 0$, then f is IHR or BT;
   (g) $w(0) \geq y_1$, $\theta(0) > 0$ and $\theta(x_i) \geq 0$ for $i = 1$ or 2, then f is IHR, BT or MBT;
   (h) $w(0) \geq y_1$, $\theta(0) \leq 0$ and $\theta(x_2) < 0$, then f is IHR, BT, MBT or BBT;
   (i) $w(0) \geq y_1$, $\theta(0) > 0$ and $\theta(x_1) < 0$, $\theta(x_2) < 0$, then f is IHR, BT, MBT, BBT or IBT.

Sultan et al. [553] considered a mixture of inverse Weibull distributions with survival function

$$\overline{F}(x) = p\left\{1 - \exp(-\alpha_1 x)^{-\beta_1}\right\} + (1-p)\left\{1 - \exp(-\alpha_2 x)^{-\beta_2}\right\}$$

and the corresponding hazard rate

$$h(x) = \frac{p\beta_1\alpha_1^{-\beta_1}x^{-(\beta_1+1)}e^{-(\alpha_1 x)^{-\beta_1}} + (1-p)\beta_2\alpha_2^{-\beta_2}x^{-(\beta_2+1)}e^{-(\alpha_2 x)^{-\beta_1}}}{p\{1 - e^{-(\alpha_1 x)^{-\beta_1}}\} + (1-p)\{1 - e^{-(\alpha_2 x)^{-\beta_2}}\}}$$

which can be unimodal and bimodal.

From the above illustrations, one might have noticed that the analysis of hazard rates of mixtures is quite complicated. Also, the shape of the hazard rates changes with the mixing proportion p and the component distributions. In most cases, the quantile functions are not invertible into explicit forms and so have to be evaluated numerically. There are several discussions on the shapes of hazard rates in the general case as well as for mixtures of distributions with specified components. For more details, one may refer to Gurland and Sethuraman [255], Lynch [405], AL-Hussaini and Sultan [29], Shaked and Spizzichino [532], Block et al. [109, 110, 112], Wondamagegnehu et al. [585], Bebbington et al. [82], Sultan et al. [553] and Ahmed et al. [27].

Agarwal and Kalla [20] studied a generalized gamma model of the form

$$f(x) = \frac{x^m e^{-\delta x}(n+x)^\lambda}{\delta^{\lambda-m}\Gamma_\lambda(m+1, n\delta)}, \quad x > 0,\ \lambda, \delta, n, m > 0,$$

where

$$\Gamma_\lambda(m,n) = \int_0^\infty \frac{e^{-t}t^{m-1}}{(t+n)^\lambda}dt$$

which was further extended by Kalla et al. [309] to the model

$$f(x) = \frac{\beta x^{m+\beta-1} e^{-\delta x^{\beta}} (n+x^{\beta})^{\lambda}}{\delta^{\lambda-\frac{m}{\beta}} \Gamma_{\lambda}(\frac{m}{\beta}+1, n\delta)}. \tag{7.44}$$

The distribution in (7.44) includes the Stacy distribution when $\lambda = 1$ and appropriate reparametrization that gives

$$f(x) = Cx^{\alpha\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\alpha}\right\}. \tag{7.45}$$

The gamma distribution and Weibull distribution are particular cases of (7.45) when $\alpha = 1$ and $\beta = 1$, respectively. Glaser [220] and McDonald and Richards [414, 415] have discussed the shape of the hazard rate and conditions on the parameters that produce IHR, DHR, BT and UBT curves. The most general form in (7.44) has been analysed by Gupta and Lvin [248].

In order to accommodate early failures, Muraleedharan and Lathika [433] proposed mixing a Weibull distribution with a singular distribution at $x = \delta$, where δ is small and specified in advance. Thus, their model has the representation

$$F(x) = (1-\alpha)F_1(x) + \alpha F_2(x),$$

where F_1 is the singular component and F_2 is Weibull. Mitra and Basu [418] considered the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a homogeneous Poisson process:

$$\overline{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \overline{P}_K, \quad 1 = \overline{P}_0 \geq \overline{P}_1 \geq \overline{P}_2 \geq \cdots .$$

They derived conditions under which $\overline{H}(t)$ has a BT hazard rate in terms of certain properties of $\overline{P}_K$.

Mitra and Basu [419] have presented some general properties of BT distributions. Their main results resemble the properties of ageing concepts described earlier in Chap. 4. Suppose F has a BT hazard rate with a change point x_0. Then:

1. $\overline{F}(x) < \overline{G}(x)$, where G is exponential with mean $[h(x_0)]^{-1}$;
2. $\mu_r' \leq \frac{\Gamma(r+1)}{[h(x_0)]^k}$ with equality sign holding true when X is exponential;
3. BT-shaped hazard rate distributions are not preserved under convolution or mixing. They are also not closed under the formation of parallel systems. However, if each component in a series system has a BT hazard rate with change point x_0, then the system also has a BT hazard rate with x_0 as one of the change points.

The modified Weibull distribution was further generalized by Carrasco et al. [129] to a density function of the form

$$f(x) = \frac{\alpha\beta x^{r-1}(r+\lambda t)\exp(\lambda x - \alpha x^r e^{\lambda x})}{\{1-\exp(-\alpha x^r e^{\lambda x})\}^{1-\beta}}. \tag{7.46}$$

Correspondingly, the hazard rate function is

$$h(x) = \frac{\alpha\beta x^{r-1}(r+\lambda x)\exp(\lambda x - \alpha x^r e^{\lambda x})\{1-\exp(\alpha x^r e^{\lambda x})\}}{1-\{1-\exp(-\alpha x^r e^{\lambda x})\}^{\beta}}.$$

Special cases of the distribution are Weibull ($\lambda = 0$, $\beta = 1$), type I extreme value ($r = 0, \beta = 1$), exponentiated Weibull ($\lambda = 0$), exponentiated exponential ($\lambda = 0$, $r = 1$), generalized Rayleigh ($r = 2$, $\lambda = 0$), and modified Weibull ($\beta = 1$). We see that $h(x)$ is increasing for $r \geq 1$, $0 < \beta < 1$, decreasing for $0 < r < 1$, $\beta > 1$, unimodal for $0 < r < 1$, $\beta \to \infty$, and BT for $\lambda = 0$, $r > 1$, $r\beta < 1$.

In the past three sections, we have reviewed only models of a representative nature. Further models, inference procedures, and applications to data analysis can all be seen from the papers cited in the text and the references therein. More references and details are available in Lai and Xie [368], Lai et al. [369], Bebbington et al. [85], Nadarajah [437] and Silva et al. [538].

7.5 Some General Methods of Construction

In this section, we present some general methods that lead to the construction of a model with BT-shaped hazard function.

- *Using Glaser's theorem*

  Let X a non-negative random with positive density function $f(x)$ that is twice differentiable. Define $\eta(x) = -\frac{f'(x)}{f(x)}$ and $g(x) = [h(x)]^{-1}$. If there exists a point x_0 such that $\eta'(x) < 0$ for $x < x_0$, $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for $x > x_0$, and further there exists a y_0 such that

  $$g'(y_0) = \int_{y_0}^{\infty} \frac{f(y)}{f(y_0)}[\eta(y_0) - \eta(y)]dy.$$

  Then, the corresponding distribution has BT-shaped hazard rate. Verification of the BT nature of several distributions discussed earlier like (7.4), (7.5), (7.45) and mixtures of gamma is in fact accomplished in this manner.
- *From convex functions*

  A BT hazard rate distribution emerges from a strictly convex positive function on $(0, \alpha)$ satisfying the condition $\int h(x)dx = \infty$. Also, a strictly increasing function of BT hazard rate is also a BT hazard rate. Models (7.12) and (7.19) are examples of this form.

- *Series systems (Addition of hazard rates)*

  The hazard rate of a series system with n independent components is the sum of the hazard rates of the components. By choosing some hazard rates to be IHR and the rest to be DHR, one may arrive at a BT hazard rate. Models (7.19), (7.20), (7.38), (7.39) and (7.40) all belong to this category. Generalizing this idea, lifetime distributions with hazard functions of the form

  $$h(x) = A_1 h_1(x) + A_2 h_2(x), \quad x > 0,$$

  were investigated by Shaked [529]. In the above formulation, A_1 and A_2 are independent of $h_1(x)$ and $h_2(x)$, while both $h_1(x)$ and $h_2(x)$ may be assumed to be of known forms. Shaked [529] chose $h_1(x) = 1$ and $h_2(x) = \sin x$, for example, in modelling hazard rate influenced by periodic fluctuations of temperature. Gaver and Acar [210] discussed models with hazard rates of the form

  $$h(x) = h_1(x) + \lambda + h_2(x),$$

  where $h_1(x)$ is positive and decreasing and tends to zero as $x \to \infty$, and $h_2(x)$ is increasing. One can see several hazard functions of the above two forms in our earlier discussions. Closely related to these are distributions with polynomial form for $h(x)$.
- *Stochastic hazard rates*

  Rajarshi and Rajarshi [500] identified a stochastic hazard rate as

  $$h^*(x) = u + h_1(x),$$

  where u is the realization of a continuous positive random variable U, and $h_1(x)$ need not be a hazard rate, and $h^*(x)$ is the hazard rate of X given $U = u$. It is obvious that the above representation is a special case of the additive hazard rate model of Nair and Sankaran [446] discussed in connection with the Hjorth [272] model. The BT-shaped hazard functions obtained earlier as the sum of hazard rate models of Murthy et al. [435], Shaked [529] and Davis and Feldstein [167] also belong to this category. The ageing properties and stochastic order relations connecting the random variable X^* (corresponding to $h^*(x)$) and the baseline variable X have been studied in Nair and Sankaran [446]).
- *Mixtures*

  Mixtures of two distributions, with one of the components being IHR and the other being DHR, may yield a BT-shaped hazard rate model. See the mixture distributions discussed in Sect. 7.4 for illustration.
- *Introduction of additional parameters*

  Introducing additional parameters that influence the shape of a baseline distribution has become a standard practice to generate new models with BT hazard rates. One simple method is exponentiation, i.e., to consider $[F(x)]^\theta$, $\theta > 0$, where $F(x)$ is a life distribution. The exponentiated Weibull, generalized exponential, and generalized linear failure rate distributions are all examples of this kind.

Another method is to use the Marshall-Olkin [411] method. A given survival function $\overline{G}(x)$ is modified into the form

$$\overline{F}(x) = \frac{\theta \overline{G}(x)}{1-(1-\theta)\overline{G}(x)}.$$

Several such models are discussed, along with their hazard rates, by Marshall and Olkin [411]. See also various generalizations arising from the Weibull distributions in Sects. 7.4 and 7.5.

- *Upside-down mean residual life models*

  Like the hazard rate, the mean residual life function can also have BT and UBT shapes. The following theorem, from Ghai and Mi [211], is of interest in the pursuit of BT or UBT hazard rates.

Theorem 7.2. *Let x_0 be the unique change point of a UBT (BT) mean residual life function $m(x)$. Suppose there exists a $t_0 \in [x_0, \infty)$ such that $m(x)$ is concave (convex) in $[0, t_0]$ and convex (concave) in $[t_0, \infty)$. If $m'(x)$ is convex (concave) in $[x_0, t_0)$, then either of the following is true:*

(a) $h(x)$ exhibits a BT (UBT) that has two change points $x_1 < x_2$, where $x_0 \leq x_1 < x_2 \leq t_0$;
(b) $h(x)$ exhibits a BT (UBT) that has a unique change point x^*, where $x_0 \leq x^* \leq t_0$.

Hence, a known mean residual life satisfying Theorem 7.2 can generate a BT or UBT hazard rate. Other methods of obtaining BT shapes for the hazard rate function will be discussed in the following section.

7.6 Quantile Function Models

So far, we have discussed in this chapter models based on distribution functions that possess nonmonotone hazard rates. Since many of the models have tractable quantile functions, a quantile-based analysis is possible in all such cases. While analysing the standard quantile functions in Chap. 3, the nonmonotonicity of their hazard quantile functions was witnessed to make use of them in data analysis. The primary objective of the present section is to enrich the domain of applications by finding some more new quantile functions.

7.6.1 Bathtub Hazard Quantile Functions Using Total Time on Test Transforms

Recall from Chap. 5 that the total time on test transform (TTT) of order n of a non-negative random variable with quantile functions $Q(u)$ is defined as

$$T_n(u) = \int_0^u (1-p)t_{n-1}(p)dp, \quad n = 1,2,3,\dots, \tag{7.47}$$

with $T_0(u) = Q(u)$ and $t_n(u) = T_n'(n)$, provided $\mu_{n-1} = \int_0^1 T_{n-1}(p)dp < \infty$. Since $T_n(u)$ is also a quantile function, let us denote by X_n the corresponding random variable. Let $\mu_n = E(X_n)$ and $H_n(u)$ be the hazard quantile function of X_n. We then have

$$t_n(u) = (1-u)t_{n-1}(u) = [H_{n-1}(u)]^{-1}$$

and

$$t_n(u) = (1-u)^n t_0(u) = (1-u)^n q(u) = \frac{(1-u)^{n-1}}{H(u)}.$$

Finally,

$$H(u) = (1-u)^n H_n(u), \quad n = 0,1,2,3,\dots \tag{7.48}$$

Definition (7.47) applies to negative integers as well; for example, $Q(u)$ can be thought of as the transform of $T_{-1}(u)$. In that case, the hazard quantile function $H_{-n}(u)$ corresponds to

$$H_{-n}(u) = (1-u)^n H(u), \quad n = 0,1,2,3,\dots. \tag{7.49}$$

Equation (7.48) reveals that, in successive transforms, the hazard quantile function increases when n is positive and decreases when n is negative. The following results (Nair et al. [448]) are useful in this connection.

Theorem 7.3. *1. The random variable X has BT hazard quantile function if there exists a u_0 for which $Q(u) \geq L(u)$ in $[0,u_0]$ and $Q(u) \leq L(u)$ in $[u_0,1]$, where $L(u)$ is the quantile function of the Pareto II distribution with parameters $(k, \frac{1}{n})$. Then, u_0 will be the change point;*

2. The random variable X_n has UBT hazard quantile function if there exists a u_0 for which $T_n(u) \leq B(u)$ in $[0,y_0]$ and $T_n(u) \geq B(u)$ in $[u_0,1]$, where $B(u)$ is the quantile function of the rescaled beta distribution with parameters $(\frac{k}{n+1}, \frac{1}{n+1})$. Then, we have u_0 as the change point.

From (7.48) and (7.49), we see that for DHR (IHR) distributions the hazard quantile function of X_n has a tendency to increase (decrease). In effect, we look at the successive transforms where a change point occurs in the corresponding hazard quantile function to construct a model with BT- or UBT-shaped hazard quantile function. This technique will be used to develop new quantile functions with the above property from some standard distributions.

- *Weibull distribution*

It has been seen in the previous sections that many of the models with nonmonotone hazard rates were generated by either generalizing or modifying the Weibull distribution. In the same spirit, the present example also considers the Weibull distribution with survival function

$$\overline{F}(x) = \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}, \quad x > 0; \alpha, \beta > 0,$$

and mean $\mu = \alpha\Gamma(1+\frac{1}{\beta})$, as the baseline model. Using the quantile function

$$Q(u) = \alpha\{-\log(1-u)\}^{\frac{1}{\beta}},$$

we have

$$H(u) = \beta\alpha^{-1}\{-\log(1-u)\}^{1-\frac{1}{\beta}}$$

and from (7.48),

$$H_n(u) = \beta\alpha^{-1}(1-u)^{-n}\{-\log(1-u)\}^{1-\frac{1}{\beta}}$$

and

$$H_n'(u) = \beta\alpha^{-1}\{-\log(1-u)\}^{-\frac{1}{\beta}}(1-u)^{-n+1}\left\{1-\frac{1}{\beta}-n\log(1-u)\right\}. \quad (7.50)$$

Since $H_n(u)$ has the tendency to increase with n, the only possibility to get a BT hazard quantile function is to consider DHR distributions. Accordingly, we take the DHR Weibull distribution with $\beta \leq 1$. Equation (7.50) reveals that $H_n(u)$ is concave in $[u_0, 1]$ and convex on $[0, u_0]$, where $u_0 = 1-\exp(\frac{\beta-1}{n\beta})$, $\beta \leq 1$. Hence, X_n has BT distribution for $n \geq 1$. As seen from the expression for u_0, the change point u_0 also increases with n so that X_n becomes IHR for a larger range, along with increasing n.

Take the case when $n = 1$. We have the random variable X_1 in the support of $(0, \mu)$ with quantile function $T_1(u)$ and hazard quantile function as

$$H_1(u) = \beta\alpha^{-1}(1-u)^{-1}\{-\log(1-u)\}^{1-\frac{1}{\beta}}.$$

The quantile density function is

$$t_1(u) = \alpha\beta^{-1}\{-\log(1-u)\}^{1-\frac{1}{\beta}}, \quad 0 \leq u \leq 1, \quad (7.51)$$

which is bathtub-shaped hazard quantile function with change point $u_0 = 1 - \exp(\frac{\beta-1}{\beta})$. We can find the distributional characteristics of X_1 from (7.50). Quantile function corresponding to (7.50) is expressed in terms of the incomplete gamma function as

$$T_1(u) = \frac{\alpha}{\beta} \Gamma_{-\log(1-u)}\left(\frac{1}{\beta}\right), \quad 0 < \beta < \infty,$$

where

$$\Gamma_x(p) = \int_0^x e^{-t} t^{p-1} dt.$$

The first four L-moments of the distribution are as follows:

$$L_1 = E(X) = \frac{\alpha \Gamma\left(\frac{1}{\beta}\right)}{\beta 2^{\frac{1}{\beta}}},$$

$$L_2 = \frac{\alpha}{\beta}\left(2^{-\frac{1}{\beta}} - 3^{-\frac{1}{\beta}}\right)\Gamma\left(\frac{1}{\beta}\right),$$

$$L_3 = \frac{\alpha}{\beta}\left\{2^{-\frac{1}{\beta}} - 3(3^{-\frac{1}{\beta}}) + 2(4^{-\frac{1}{\beta}})\right\}\Gamma\left(\frac{1}{\beta}\right),$$

$$L_4 = \frac{\alpha}{\beta}\left\{2^{-\frac{1}{\beta}} - 6(3^{-\frac{1}{\beta}}) + 10(4^{-\frac{1}{\beta}}) - 5(5^{-\frac{1}{\beta}})\right\}\Gamma\left(\frac{1}{\beta}\right).$$

Thus, the L-skewness has the simple expression

$$\begin{aligned}\tau_3 &= \frac{2^{-\theta} - 3^{1-\theta} + 4^{\frac{1}{2}-\theta}}{2^{-\theta} - 3^{-\theta}} \\ &= 1 - \frac{2(3^{-\theta} - 4^{-\theta})}{2^{-\theta} - 3^{-\theta}} \\ &= 1 - \frac{2\left\{1 - (\frac{3}{4})^{\theta}\right\}}{(\frac{3}{2})^{\theta} - 1}, \quad \text{with } \theta = \beta^{-1}.\end{aligned}$$

As $\theta \to \infty$ or $\beta \to 0$, we see that τ_3 tends to 1, and as $\theta \to 0$, we have τ_3 appropriately -0.53. Hence, the distribution covers skewness in the range $(-0.53, 1)$. On the other hand, the L-kurtosis is

$$\tau_4 = 1 - \frac{5 - (\frac{3}{4})^{\theta} + 5(\frac{3}{5})^{\theta}}{(\frac{3}{2})^{\theta} - 1}$$

which tends to 1 as $\beta \to 0$. The parameters of the distribution allows easy estimation by equating the first two L-moments of the sample with those of the population. Thus, (7.50) gives a two-parameter life distribution with BT-shaped hazard quantile function.

7.6.2 Models Using Properties of Score Function

The results discussed here are mainly based on the work of Nair et al. [448]. Recall from Sect. 4.3 that the definition of the score function is

$$J(u) = \frac{q'(u)}{q^2(u)},$$

where $q(u)$ is the quantile density function of the lifetime X. We see that

$$J(u) = -\frac{d}{du}\frac{1}{q(u)} = -\frac{d}{du}(1-u)H(u),$$

or equivalently

$$(1-u)H'(u) = H(u) - J(u).$$

Thus, X is $I(D)$ according as $H(u) \geq J(u)$ for all u. Further, if $H(u)$ is nonmonotonic, the change points of $H(u)$ are zeros of $H(u) - J(u)$. Geometrically, for increasing (decreasing) $H(u)$, the $H(u)$ curve lies above (below) that of $J(u)$ and for BT (UBT) hazard quantile function $H(u)$ crosses $J(u)$ from below (above). An interesting property of $J(u)$ is that there exists some simple relationships between $J(u)$ and $H(u)$ that characterize many life distributions.

Theorem 7.4. *The random variable X is distributed as generalized Pareto with*

$$Q(u) = ba^{-1}\left\{(1-u)^{-\frac{a}{a+1}} - 1\right\}, \quad a > -1, b > 0, \tag{7.52}$$

if and only if

$$J(u) = cH(u) \tag{7.53}$$

for a positive constant c.

Proof. Assuming (7.50), we find $J(u)$ and $H(u)$ as

$$J(u) = \frac{2a+1}{b}(1-u)^{\frac{a}{a+1}}$$

and

$$H(u) = \frac{a+1}{b}(1-u)^{\frac{a}{a+1}}.$$

This readily verifies (7.53) with $c = \frac{2a+1}{a+1}$. Conversely, if (7.53) applies to a random variable X, then

$$\frac{H'(u)}{H(u)} = \frac{1-c}{1-u}$$

and

$$H(u) = K(1-u)^{c-1},$$
$$q(u) = K(1-u)^{c-1},$$

which is the quantile density function of the generalized Pareto with $c = \frac{2a+1}{a+1}$. Hence, the theorem.

Remark 7.2. When $c = 1$, we have the exponential distribution and $c > (<)1$ leads to the Pareto II (rescaled beta) model. It is apparent that by generalizing the identity in (7.53), we can obtain more flexible models. This fact is illustrated in the following theorems.

Theorem 7.5. *The relationship*

$$J(u) = AH(u) + B$$

is satisfied for all u and real constants A and B if and only if the distribution of X is given by

$$Q(u) = \begin{cases} \log\left\{\left(1+\frac{B}{1-A}\right)^{\frac{1}{B}}\left(c+\frac{B}{1-A}(1-u)^{1-A}\right)^{-\frac{1}{B}}\right\}, & c \le 1,\ A \ne 1 \\ \frac{1}{B}\log\left\{\frac{c}{c+B\log(1-u)}\right\}, & A = 1,\ c > 0. \end{cases} \tag{7.54}$$

Theorems 7.4 and 7.5 do not provide models with nonmonotone hazard quantile functions. The distribution in Theorem 7.5 contains known models like the exponential, Pareto, rescaled beta, half-logistic and Gompertz as special cases. In general, $H(u)$ is increasing for (7.54) when $A < 1$, $0 < c \le 1$ or $A > 1$, $C < 0$ and decreasing when $A > 1$, $C < 0$ or $A > 1$, $0 < c \le 1$. Some other properties of the distribution have been studied by Nair et al. [448].

Returning to the construction of bathtub-shaped $H(u)$, we have the following characterization that generates a distribution with BT-shaped hazard quantile function.

Theorem 7.6. *If the functions $J(u)$ and $H(u)$ are such that*

$$J(u) = \left(A + \frac{\alpha}{u}\right)H(u) \tag{7.55}$$

for all u, then it is necessary and sufficient that the distribution is specified by the quantile density function

$$q(u) = Ku^{\alpha}(1-u)^{-(A+\alpha)}, \tag{7.56}$$

where α, A and K are real constants.

Proof. Equation (7.55) is equivalent to

$$\frac{q'(u)}{q^2(u)} = \frac{(A+\frac{\alpha}{u})}{(1-u)q(u)},$$

or

$$\frac{d\log q(u)}{du} = \frac{A+\frac{\alpha}{u}}{(1-u)}.$$

Integrating the above equation, we obtain (7.56). Conversely, if the distribution is of the form (7.56), then by direct calculations, we have

$$H(u) = K^{-1}u^{-\alpha}(1-u)^{A+\alpha-1}$$

and

$$J(u) = K^{-1}u^{-\alpha-1}(1-u)^{A+\alpha-1}(uA+\alpha),$$

thus verifying (7.55).

The family of distributions in (7.55) includes several well-known distributions as special cases. Of these are

- the exponential $(\alpha = 0, A = 1)$ with constant $H(u)$;
- Pareto II $(\alpha = 0, A < 1)$ with decreasing $H(u)$;
- rescaled beta $(\alpha = 0, A > 1)$ with increasing $H(u)$;
- loglogistic $(A = 2, \alpha = \lambda - 1)$, specified by

$$\overline{F}(x) = \frac{x^{\frac{1}{\lambda}}}{\alpha^{\frac{1}{\lambda}} + x^{\frac{1}{\lambda}}}, \quad x > 0,\ \lambda, \alpha > 0.$$

  The reliability aspects of this distribution have been studied by Gupta et al. [237]. Since

$$J(u) = \frac{2u+\alpha-1}{u}H(u)$$

  in this case, X is UBT with change point at $u_0 = 1-\lambda$;
- Govindarajulu's distribution with

$$Q(u) = \theta + \sigma\left\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\right\}$$

  on setting $\alpha = \beta - 1$, $A = -\beta$ and $K = \sigma\beta(\beta+1)$. See Chap. 3 and Nair et al. [448] for a detailed discussion on the properties and reliability implications. For $A > -1$, $H(u)$ is increasing while for $A < -1$, $H(u)$ has bathtub shape with

Table 7.1 Observed and expected frequencies for the gastric carcinoma data

Class intervals	0–111.8	111.8–197	197–289.4	289.4–401	401–550.3	550.3–782	782–1265.5	> 1265.5
Observed frequencies	13	10	9	15	14	10	14	10
Expected frequencies	12	12	12	12	12	12	12	11

change point $u = \frac{A+1}{A-1}$. The hazard quantile can be differentiated to study its shape for various values of the parameters. We have

$$H'(u) = K^{-1}u^{-\alpha-1}(1-u)^{A+\alpha-2}\{-\alpha+u(1-A)\}.$$

Thus, $H(u)$ is increasing for $\alpha < 0, A < 1$, and decreasing for $\alpha < 0, A > 1$ for all u giving the IHR and DHR cases. The BT and UBT cases also hold, respectively, when $\alpha > 0, A < 1$ and $\alpha \not< 0, A > 1$. Accordingly, the model can cover all the cases.

Example 7.1. The use of the model was tested against the data on survival times in days from a clinical trial on gastric carcinoma on 90 patients, as given by Kleinbaum [342], by considering the survival times alone in a single set. In order to estimate the parameters of the model, the 25th, 50th and 75th percentiles of the sample and the population are matched. This procedure results in the estimates

$$\hat{\alpha} = -0.3128,\ \hat{A} = 1.7693 \text{ and } \hat{K} = 296.267.$$

We then calculated the observed and expected frequencies for various classes and these are reported in Table 7.1. The χ^2 value of 3.14 does not reject the model in (7.56) for the data at 5 % level of significance.

Some distributional aspects of (7.56) will also be interest in further analysis. The first four L-moments, for example, are as follows:

$$L_1 = KB(\alpha+1, 2-A-\alpha), \quad \text{with } A+\alpha < 2,$$
$$L_2 = KB(\alpha+2, 2-A-\alpha),$$
$$L_3 = K\{B(\alpha+3, 2-A-\alpha) - B(\alpha+2, 3-A-\alpha)\},$$
$$L_4 = K\{B(\alpha+2, 2-A-\alpha) - 5B(\alpha+3, 3-A-\alpha)\}.$$

Hence, as a location measure, the mean is

$$\mu = KB(\alpha+1, 2-A-\alpha)$$

and as a dispersion measure, the mean difference is

$$\Delta = 2KB(\alpha+1, 2-A-\alpha).$$

The L-skewness is

$$\tau_3 = \frac{L_3}{L_2} = \frac{(\alpha+2)-(2-A-\alpha)(4-A)}{(4-A)},$$

and the L-Kurtosis is

$$\tau_4 = 1 - \frac{5(\alpha+2)(2-A-\alpha)}{(5-A)(4-A)}.$$

Theorem 7.7. *The relationship*

$$J(u) = [A + M\{\log(1-u)\}^{-1}]H(u) \tag{7.57}$$

is satisfied for all u and real A and M if and only if

$$q(u) = K(1-u)^{-A}\{-\log(1-u)\}^{-M}. \tag{7.58}$$

Proof. Rewriting (7.57) as

$$\frac{q'(u)}{q^2(u)} = \left[A + M\{\log(1-u)\}^{-1}\right]\frac{1}{(1-u)q(u)},$$

we have

$$\frac{q'(u)}{q(u)} = \frac{A}{1-u} + \frac{M}{(1-u)q(u)}.$$

Integrating, we obtain (7.58). Conversely, logarithmic differentiation of (7.58) leads to (7.57). Hence, the theorem.

We can write the quantile function in terms of special function as

$$Q(u) = K(1-A)^{M-1}I(1-M, \log(1-u)^{A+1}),$$

where

$$I(a,x) = \int_0^x e^{-t}t^{a-1}dt$$

is the incomplete gamma function. The density function of X can be written in terms of the survival function as

$$f(x) = C[\overline{F}(x)]^A\{1-\log\overline{F}(x)\}^M, \quad x > 0.$$

Some special cases of (7.58) are

- The Weibull distribution with shape parameter λ and scale parameter $\sigma = K\lambda$, and in particular, exponential and Rayleigh distributions when $\lambda = 1$ and 2, respectively;
- Pareto II ($A > 1$, $M = 0$), rescaled beta ($A < 1$, $M = 0$), and uniform ($A = 0$, $M = 0$).

Thus, (7.58) is a generalized Weibull model belonging to the category of several such models discussed in the preceding sections. The hazard quantile function has the form

$$H(u) = K^{-1}(1-u)^{A-1}\{-\log(1-u)\}^{M}.$$

Upon taking the derivative, we get

$$H'(u) = K^{-1}(1-u)^{A-2}\{-\log(1-u)\}^{M-1}\{M+(A-1)\log(1-u)\}. \tag{7.59}$$

Equation (7.59) shows that $H(u)$ is capable of taking on different shapes. In fact,

$$X \text{ is IHR when } A \le 1,\, M > 0;\, A < 1, M = 0;$$
$$X \text{ is DHR when } A \le 1,\, M < 0;\, A > 1, M = 0;$$
$$X \text{ is BT when } A < 1,\, M < 0;$$
$$X \text{ is UBT when } A > 1,\, M > 0;$$
$$X \text{ is exponential when } A = 1,\, M = 0.$$

We now look at some distributional properties of this family. First, we see that the members of the family are either unimodal or monotonic with modal value at $u_0 = 1-\exp(\frac{M}{A})$. The summary measures can be described in terms of the quantiles or. We have the first four L-moments as follows:

$$L_1 = \frac{K\Gamma(1-M)}{(2-A)^{1-M}}, \quad M < 1,\, A < 2,$$

$$L_2 = \left\{1 - \left(\frac{2-A}{3-A}\right)^{1-M}\right\} L_1,$$

$$L_3 = \left\{1 - 3\left(\frac{2-A}{3-A}\right)^{1-M} + 2\left(\frac{1-A}{4-A}\right)^{1-M}\right\} L_1,$$

$$L_4 = \left\{1 - 6\left(\frac{2-A}{3-A}\right)^{1-M} + 10\left(\frac{2-A}{4-A}\right)^{1-M} - 5\left(\frac{2-A}{3-A}\right)^{1-M}\right\} L_1.$$

The mean, mean difference, L-skewness and L-kurtosis are all readily obtained from the above expressions.

Table 7.2 Observed and expected frequencies of the failure time data

Class intervals	0–3.184	3.184–13.5	13.5–29.48	29.48–48.5	48.5–67.47	67.47–83.25	>83.25
Observed freq.	9	4	6	6	9	6	8
Expected freq.	6	6	6	6	6	6	14

For an empirical validation of the model, the data on the failure times of 50 devices given in Lai and Xie [368, p. 353] is considered. Matching the 25th, 50th and 75th percentiles of the sample with the corresponding percentiles of the population, the estimates of the model parameters are found to be

$$\hat{A} = -1.8224, \quad \hat{M} = -1.2576 \quad \text{and} \quad \hat{K} = 875.927.$$

A χ^2 value of 4.509 is found from the observed and expected frequencies presented in Table 7.2, which does not lead to rejection of the model.

The methods suggested in this section, using the total time on test transform as well as the relationship between $J(u)$ and $H(u)$, are quite general in nature. The above examples illustrate how we can work with them. It will, of course, be of interest to develop more flexible families of distributions that generalize the existing distributions and present varying shapes and characteristics to become practically useful!