Chapter 6
L-Moments of Residual Life and Partial Moments

Abstract The residual life distribution and various descriptive measures derived from it form the basis of modelling, characterization and ageing concepts in reliability theory. Of these, the moment-based descriptive measures such as mean, variance and coefficient of variation of residual life and their quantile forms were all discussed earlier in Chaps. 2 and 4. The role of *L*-moments as alternatives to conventional moments in all forms of statistical analysis was also highlighted in Chap. 1. *L*-moments generally outperform the usual moments in providing smaller sampling variance, robustness against outliers and easier characterization of distributional characteristics, especially in the case of models with explicit quantile functions but no tractable distribution functions. For this reason, we discuss in this chapter the properties of the first two *L*-moments of residual life. After introducing the definitions, several identities that connect *L*-moments of residual life with the hazard quantile function, and mean and variance of residual quantile function, are derived. A comparison between the second *L*-moment and variance of residual life points out the situations in which the former is better. Expressions for the *L*-moments of residual life of quantile function models of Chap. 3 are derived and their behaviour is discussed in relation to the mean residual quantile function. Characterization of lifetime models based on the functional form of the second *L*-moment as well as in terms of its relationship with the hazard and mean residual quantile functions are also presented. The upper and lower partial moments have been found to be of use in reliability analysis, economics, insurance and risk theory. Quantile-based definitions of these moments and their relationships with various reliability functions are presented in this chapter. Many of the results on *L*-moments of residual life have potential applications in economics. For example, income distributions can be characterized by means of some simple properties of concepts like income gap ratio, truncated Gini index and poverty measures. Quantile forms of all these measures are defined and their usefulness in establishing characterizations are explored.

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6.1 Introduction

The notion of residual life, based on the information that a unit has functioned satisfactorily for a specified period of time, has been fundamental in reliability theory and practice. As seen already in Chaps. 2 and 4, the residual life distribution and various descriptive measures derived from it form the basis for the definition of various ageing concepts. Of these measures, the moment-based descriptive measures such as mean, variance and coefficient of variation of residual life are used commonly in modelling lifetime data, characterizing life distributions, defining classes of life distributions, and in evolving strategies for maintenance and repair of equipments. The Lorenz curve and Bonferroni curve used in measuring income inequality in economics and the Leimkuhler curve in informatics are all characterized by the mean residual life and variance residual life along with other reliability functions; see Chap. 5 for details. Upper and lower partial moments of X are closely related to the moments of residual life. If X has finite moment of order r, the rth upper partial moment (also called the stop-loss moment) about x is defined as

$$p_r(x) = E[(X-x)^+]^r = \int_x^\infty (t-x)^r dF(t),$$

where $(X-x)^+ = \max(X-x,0)$. The quantity $(X-x)^+$ is interpreted as a residual life in the context of lifelength studies (Lin [401]) and the moments $p_r(x)$ are used in actuarial studies in the analysis of risks (Denuit [170]). In the assessment of income tax, x can be taken as the tax exemption level, so that $(X-x)^+$ then becomes the taxable income. Obviously, from the expression

$$m(x) = \bar{F}(x) p_1(x),$$

various identities connecting $p_1(x)$ and the different reliability functions follow. For characterizations of distributions using $p_r(x)$ for $r=1$ and in the general case, we refer to Chong [147], Nair [438], Lin [401], Sunoj [554] and Abraham et al. [14]. If we consider

$$(X-x)^- = \begin{cases} x-X & \text{if } X \le x \\ 0 & \text{if } X > x \end{cases},$$

we have similarly the lower partial moments as $E[(X-x)^-]^r$. Sunoj and Maya [555] have discussed characterizations of distributions and various applications of lower partial moments in the context of risk analysis and income analysis for the poor.

The use of L-moments as an alternative to the conventional moments, for all purposes in which the latter is prescribed, is well known. Our discussions and the references earlier in Chap. 1 do emphasize this aspect. L-moments generally outperform the usual moments in providing smaller sampling variance, robustness against outliers and easier characterization of distributional characteristics, especially for models with explicit quantile functions but no tractable distribution functions. All these considerations apply to lifetime data analysis as well as in the discussion of properties of residual life distributions. Heavy-tailed distributions occur as models of reliability data in which case the usual sample moments lack efficiency. Nair and Vineshkumar [452] pointed out the usefulness of L-moments of residual life in reliability analysis and then studied their properties in comparison with the mean and variance of residual life.

6.2 Definition and Properties of *L*-Moments of Residual Life

Recall from Sect. 1.6 that the L-moment of order r is given by

$$\begin{aligned} L_r &= \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r}),\ r=1,2,\ldots \\ &= \sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}^2\int_0^\infty x(F(x))^{r-k-1}(1-F(x))^k f(x)dx. \end{aligned} \tag{6.1}$$

The truncated variable $X(t)=X|(X>t)$ has its survival function as

$$\bar{F}_{(t)}(x)=\frac{\bar{F}(x)}{\bar{F}(t)},\quad x>t,$$

so that the L-moment of $X(t)$ is given by

$$L_r(t)=\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}^2\int_t^\infty x\left(\frac{\bar{F}(t)-\bar{F}(x)}{\bar{F}(t)}\right)^{r-k-1}\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^k\frac{f(x)}{\bar{F}(t)}dx. \tag{6.2}$$

In particular, setting $r=1$ in (6.2), we obtain

$$L_1(t)=\frac{1}{\bar{F}(t)}\int_t^\infty xf(x)dx=E[X|(X>t)]$$

which is the conditional mean function studied in Chap. 3. Further, $r=2$ in (6.2) leads to

$$\begin{aligned} L_2(t) &= \sum_{k=0}^{1}(-1)^k\binom{1}{k}^2\int_t^\infty x\left(\frac{\bar{F}(t)-\bar{F}(x)}{\bar{F}(t)}\right)^{1-k}\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^k\frac{f(x)}{\bar{F}(t)}dx \\ &= \frac{1}{\bar{F}^2(t)}\int_t^\infty x[\bar{F}(t)-2\bar{F}(x)]f(x)dx \end{aligned}$$

$$= \frac{1}{\bar{F}(t)} \int_t^{\infty} x f(x) dx - \frac{2}{\bar{F}^2(t)} \int_t^{\infty} x \bar{F}(x) f(x) dx$$

$$= L_1(t) - t - \frac{1}{\bar{F}^2(t)} \int_t^{\infty} \bar{F}^2(x) dx$$

$$= m(t) - \frac{1}{\bar{F}^2(t)} \int_t^{\infty} \bar{F}^2(x) dx, \tag{6.3}$$

where $m(t)$ is the usual mean residual life function. It thus follows that $L_2(t) \le m(t)$, where the equality sign does not hold for any non-degenerate distribution. Thus, $L_2(t)$ is strictly less than the mean residual life function for all non-degenerate distributions. Differentiating (6.2) and simplifying the resulting expression, we get

$$L_2'(t) = h(t)(2L_2(t) - m(t)). \tag{6.4}$$

Now, setting $F(x) = p$ and $F(t) = u$ in (6.2), we get

$$l_r(u) = L_r(Q(u))$$
$$= \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}^2 \int_u^1 \left(\frac{p-u}{1-u}\right)^{r-k-1} \left(\frac{1-p}{1-u}\right)^k \frac{Q(p)}{1-u} dp.$$

In particular, we have

$$l_1(u) = (1-u)^{-1} \int_u^1 Q(p) dp \tag{6.5}$$

and

$$l_2(u) = (1-u)^{-2} \int_u^1 (2p-u-1) Q(p) dp. \tag{6.6}$$

The properties of $l_1(u)$, equivalent to $E[X|(X > t)]$, have been studied rather extensively and so we concentrate here more on $l_2(u)$. However, notice that $l_1(u)$ uniquely determines the distribution through the formula

$$Q(u) = l_1(u) - (1-u) l_1'(u) \tag{6.7}$$

which is evident from (6.5). From (6.6), as $u \to 0$, we have

$$l_2(0) = \int_0^1 (2p-1) Q(p) dp = \int_0^1 p(1-p) q(p) dp = 2\Delta,$$

where Δ is the mean difference of X as defined in (1.12).

Theorem 6.1. *The functions $l_1(u)$, $l_2(u)$ and $M(u)$ determine each other and $Q(u)$ uniquely.*

Proof. We begin with $M(u)$, and the identity

$$\begin{aligned} M(u) &= l_1(u) - Q(u) \\ &= l_1(u) - \{l_1(u) - (1-u)l_1'(u)\} \\ &= (1-u)l_1'(u). \end{aligned} \tag{6.8}$$

Differentiating (6.6), we have

$$\begin{aligned} (1-u)^2 l_2'(u) - 2(1-u)l_2(u) &= -2uQ(u) + (u+1)Q(u) - \int_u^1 Q(p)dp \\ &= Q(u) - uQ(u) - \int_u^1 Q(p)dp \\ &= Q(u) - uQ(u) - (1-u)(M(u)+Q(u)) \\ &= -(1-u)M(u), \end{aligned}$$

or equivalently

$$M(u) = 2l_2(u) - (1-u)l_2'(u). \tag{6.9}$$

Finally, from (2.38), we have

$$Q(u) = \mu - M(u) + \int_0^u \frac{M(p)}{1-p}dp, \tag{6.10}$$

and thus $M(u)$ determines $Q(u)$, and $l_1(u)$ and $l_2(u)$ determine $M(u)$. In the case of $l_1(u)$, we have

$$\begin{aligned} l_1(u) &= \int_0^u \frac{M(p)}{1-p}dp &(6.11) \\ &= \int_0^u \frac{2l_2(p) - (1-u)l_2'(p)}{1-p}dp. &(6.12) \end{aligned}$$

Equations (6.11) and (6.12) express $l_1(u)$ in terms of $M(u)$, while $l_2(u)$ and (6.7) recover $Q(u)$ from $l_1(u)$. We also have

$$\begin{aligned} l_2(u) &= (1-u)^{-2}\int_u^1 (1-p)M(p)dp \\ &= (1-u)^{-2}\int_u^1 (1-p)^2 l_1'(p)dp \end{aligned}$$

determining $l_2(u)$ from $M(u)$ and $l_1(u)$. Given $l_2(u)$, $M(u)$ is derived from (6.9) and so $Q(u)$ from (6.10). This completes the proof of the theorem.

Gini's mean difference of $X(t)$ is

$$G(t) = 2\int_t^\infty F_{(t)}(x)\bar{F}_{(t)}(x)dx.$$

In terms of quantile functions, we have

$$\Delta(u) = G(Q(u)) = 2\int_u^1 \frac{(1-p)(p-u)}{(1-u)^2} q(p)dp. \tag{6.13}$$

Integrating the RHS of (6.13) by parts, we obtain

$$\Delta(u) = 2l_2(u).$$

Thus, the second L-moment of the conditional distribution of $X|(X > t)$ is half the mean difference of $X|(X > t)$. Since the mean difference is location invariant, the second L-moment of $X(t)$ is the same as that of $X_t = X - t|(X > t)$, and so we refer to $l_2(u)$ as the second L-moment of residual life. Mean difference is a measure of dispersion and so $l_2(u)$ will be treated as a measure of variation in the residual life. Thus, $l_2(u)$ can be viewed as an alternative to the variance residual life in future discussions.

In addition to the mean residual quantile function, other quantile-based reliability functions are also connected with $l_2(u)$. Some typical examples are worked out below. The others can be obtained by exploiting various identities presented earlier in Chap. 2. Invoking (2.36), we have

$$l_2(u) = (1-u)^{-2}\int_u^1 \left(\int_p^1 H^{-1}(s)ds\right) dp.$$

Similarly, from (2.46), we have

$$\begin{aligned} V(u) &= \frac{1}{1-u}\int_u^1 M^2(p)dp \\ &= \frac{1}{1-u}\int_u^1 \left\{2l_2(p) - (1-u)l_2'(p)\right\}^2 dp. \end{aligned}$$

Using the relation (2.36) once again, the total time on test transform satisfies

$$l_2(u) = \frac{1}{(1-u)^2}\int_u^1 (\mu - T(p))dp.$$

Table 6.1 Second *L*-moment of residual life for some distributions

Distribution	$l_2(u)$
Exponential	$(2\lambda)^{-1}$
Pareto II	$\frac{\alpha c}{(c-1)(2c-1)}(1-u)^{-\frac{1}{c}}$
Rescaled beta	$\frac{Rc}{(c+1)(2c+1)}(1-u)^{-\frac{1}{c}}$
Half-logistic	$\frac{2\sigma}{(1-u)^2}\left\{1-u-(1+u)\log\left(\frac{2}{1+u}\right)\right\}$
Power	$\frac{\alpha}{(1+\beta)(1-u)^2}\left\{\beta+(\beta+1-u)u^{\frac{1}{\beta}}\right\}$
Exponential geometric	$\frac{1-p}{p(1-u)^2}\left\{\frac{1-pu}{p}\log\left(\frac{1-pu}{1-p}\right)-(1-u)\right\}$

Example 6.1. The linear hazard quantile distribution is specified by

$$q(u) = [(1-u)(a+bu)]^{-1}$$

and so

$$\begin{aligned} l_2(u) &= \frac{1}{(1-u)^2}\int_u^1 (1-p)(p-u)q(p)dp \\ &= \frac{1}{(1-u)^2}\int_u^1 \frac{p-u}{a+bp}dp \\ &= \frac{1}{b(1-u)^2}\left\{1-u+\frac{a+bu}{b}\log\left(\frac{a+bu}{a+b}\right)\right\}. \end{aligned}$$

The expressions of $l_2(u)$ of some life distributions are presented in Table 6.1.

Since both variance residual life quantile function and $l_2(u)$ are measures of variability, it is appropriate to compare the two. The functional form of $l_2(u)$ characterizes the life distribution and hence it can be used to identify the distribution. Although $V(u)$ also characterizes the distribution, unlike $l_2(u)$, there is no simple expression that relates $Q(u)$ in terms of $V(u)$ or between $\bar{F}(x)$ and $\sigma^2(x)$. See the corresponding discussion in Sect. 2.1.3. Yitzhaki [596] has pointed out that the mean difference is a better measure than variance in deriving properties of distributions which are non-normal. Nair and Vineshkumar [452] have provided empirical evidence that supports this observation. They simulated random samples from the exponential population with varying parameter values. Using $V(u) = \lambda^{-2}$ and $l_2(u) = (2\lambda)^{-1}$, the parameter λ was estimated by equating the sample and population values. They then noted that $l_2(u)$ gave a better approximation to the model as well as estimates with less bias in at least 75 % of the samples.

Another important advantage of the L-moments is that, if the mean exists, all higher-order L-moments exist, which may not be the case with the usual moments. The data on annual flood discharge rates of Floyd river at James, Iowa, considered by Mudholkar and Hutson [423], was reanalysed using the power-Pareto law which gave a good fit at the parameter values $\hat{c} = 3{,}495.2$, $\hat{\lambda}_1 = 0.6226$ and $\hat{\lambda}_2 = 0.5946$. Note that, since $\lambda_2 > 0.5$, the function $V(u)$ does not exist for the distribution, while $l_2(u)$ can be used for further analysis.

The variance residual quantile function has an important role in analysing the ageing aspects. Some additional life distributions were identified based on their monotone behaviour, such as DVRL and IVRL classes (Chap. 4). An important implication observed earlier was that decreasing (increasing) mean residual quantile function implied decreasing (increasing) $V(u)$. By comparison, $V(u)$ and $l_2(u)$ may not show the same type of monotonicity. Even when $V(u)$ increases for larger u, $l_2(u)$ may show a decreasing trend. For example, for the distribution

$$Q(u) = 4u^3 - 3u^4, \quad 0 \le u \le 1,$$

after performing some algebra, we obtain

$$V(u) = \frac{1}{175}\left\{22 - 6u - 34u^2 - 62u^3 + 50u^4 + 78u^5 + 106u^6 + 9u^7 - 38u^8\right\}.$$

Thus,

$$\frac{dV(u)}{du} = \frac{1}{175}\left\{-6 - 68u - 186u^2 + 200u^3 + 390u^4 + 636u^6 + 63u^6 - 304u^7\right\},$$

from which we find that $V(u)$ decreases in $(0, u_0)$ and then increases in $(u_0, 1)$ with the change point u_0 which is approximately 0.554449. At the same time, we have

$$l_2(u) = \frac{(1-u^2)^2}{5}$$

and

$$\frac{dl_2(u)}{du} = -\frac{4}{5}u(1-u^2) < 0$$

showing that $l_2(u)$ is decreasing for all u in $(0,1)$. Neither the implications between the mean residual quantile function $M(u)$ and $V(u)$ hold good when $V(u)$ is replaced by $l_2(u)$. This is well established in the following illustrations that involve some quantile function models discussed earlier in Chap. 3.

Example 6.2. The generalized Tukey-lambda distribution of Freimer et al. [203] with

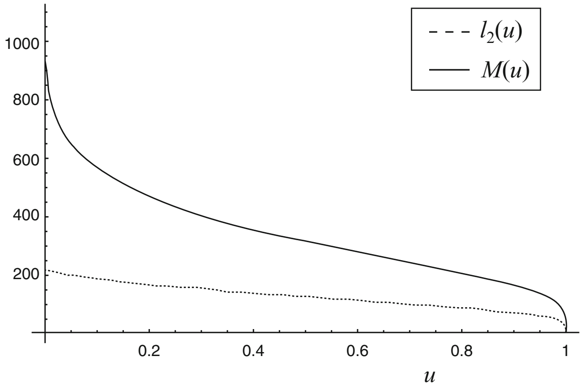

Fig. 6.1 Plot of $M(u)$ and $l_2(u)$ for the data on lifetimes of aluminum coupons

$$Q(u) = \lambda_1 + \lambda_2^{-1}\left\{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right\}$$

has

$$M(u) = \frac{(1-u)^{\lambda_4}}{\lambda_2(\lambda_4+1)} - \frac{u^{\lambda_3}}{\lambda_2\lambda_3} + \frac{1-u^{\lambda_3+1}}{\lambda_2(1+\lambda_3)(1-u)}$$

and

$$l_2(u) = \frac{1-u}{\lambda_2\lambda_4} - \frac{2(1-u^{\lambda_3+2})}{\lambda_2\lambda_3(\lambda_3+1)(\lambda_3+2)(1-u)^2} + \frac{1-u^{\lambda_4}}{\lambda_2(1+\lambda_4)(2+\lambda_4)} + \frac{(1-u)(1+u^{\lambda_3+1})}{\lambda_2\lambda_3(\lambda_3+1)(1-u)^2}.$$

The distribution provides satisfactory fit to the aluminum coupon data discussed earlier (first 100 observations) with parameter values

$$\hat{\lambda}_1 = 1382.18, \quad \hat{\lambda}_2 = 0.0033, \quad \hat{\lambda}_3 = 0.2706 \text{ and } \hat{\lambda}_4 = 0.2211.$$

The graphs of $M(u)$ and $l_2(u)$ given in Fig. 6.1 show that both are decreasing functions of u.

Example 6.3. Govindarajulu [224] fitted the distribution

$$Q(u) = ((\beta+1)u^{\beta} - \beta u^{\beta+1})$$

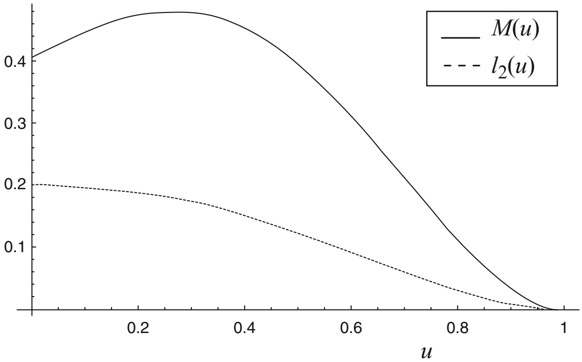

Fig. 6.2 Plot of $M(u)$ and $l_2(u)$ for the data on failure times of a set of refrigerators

to the data on failure times of a set of refrigerators, with the estimate of β being $\hat{\beta} = 2.94$. The mean residual quantile function and the second L-moment function for the distribution are

$$M(u) = \frac{2-(\beta+1)(\beta+2)u^{\beta}+2\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}}{(\beta+2)(1-u)}$$

and

$$l_2(u) = \frac{2\beta-2(\beta+3)u+(\beta+2)(\beta+3)u^{\beta+1}-2\beta(\beta+3)u^{\beta+2}+\beta(\beta+1)u^{\beta+3}}{(\beta+2)(\beta+3)(1-u)^2}.$$

In the case of the data mentioned above, $M(u)$ initially increases and then decreases with approximate change point $u = 0.2673$, but $l_2(u)$ decreases for all u, as displayed in Fig. 6.2.

Example 6.4. Consider the power-Pareto distribution with

$$Q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}, \quad C, \lambda_1, \lambda_2 > 0,$$

for which

$$M(u) = c(1-u)^{-1}\{B_{1-u}(\lambda_1+1, 1-\lambda_2) - u^{\lambda_1}(1-u)^{1-\lambda_2}\}$$

and

$$l_2(u) = c(1-u)^{-2}\{2B_{1-u}(\lambda_1+2, 1-\lambda_2) - (u+1)B_{1-u}(\lambda_1+1, 1-\lambda_2)\},$$

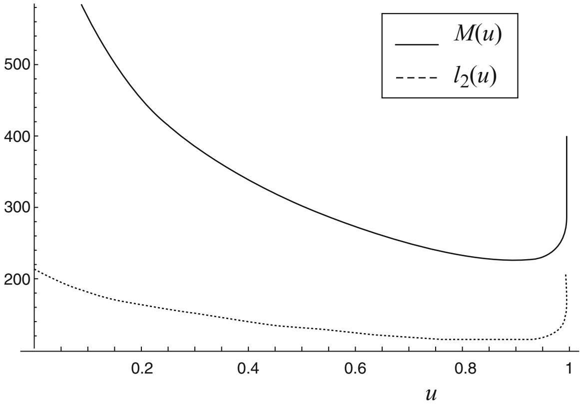

Fig. 6.3 Plot of $M(u)$ and $l_2(u)$ for the data on failure times of electric carts

where

$$B_u(p,q) = \int_0^u t^{p-1}(1-t)^{q-1}dt$$

is the incomplete beta integral. Applying the model to the times of failure of 20 electric carts reported in Zimmer et al. [604], the fit by the method of L-moments with

$$\hat{\lambda}_1 = 0.234612, \quad \hat{\lambda}_2 = 0.09669912, \quad \hat{C} = 1530.53,$$

is observed to be satisfactory. Both $M(u)$ and $l_2(u)$ are seen to possess the similar behaviour, decreasing first and then increasing, as displayed in Fig. 6.3.

Arising from the mean and variance of residual life, the coefficient of variation of residual life is also of importance in reliability. We refer to Sects. 2.1.3 and 2.5 for pertinent definitions and other details. Just as the coefficient of variation of residual life uniquely determines a distribution, it is possible to show that the L-coefficient of variation $c(u) = \frac{l_2(u)}{l_1(u)}$ also possesses a similar property. Nair and Vineshkumar [452] have shown that if $C(u)$ is differentiable, from the definitions of $l_2(u)$ and $l_1(u)$, we can write

$$\int_0^1 (2p-u-1)Q(p)dp = (1-u)c(u)\int_u^1 Q(p)dp.$$

Differentiating this expression and simplifying, we obtain

$$\frac{Q(u)}{\int_u^1 Q(p)dp} = \frac{(1-u)c'(u)-c(u)+1}{(1-u)(1+c(u))}.$$

Upon integrating, we get

$$-\log \int_u^1 Q(p)dp = \int \frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))} du$$

so that

$$Q(u) = g(u) \exp\left\{-\int g(u)du\right\}, \tag{6.14}$$

where

$$g(u) = \frac{(1-u)c'(u) - c(u) + 1}{(1-u)(1+c(u))}.$$

Equation (6.14) retrieves $Q(u)$ from $c(u)$ only up to a change of scale. We illustrate this result in the next theorem.

Theorem 6.2. *X has L-coefficient of variation of the form*

$$c(u) = \frac{1-u}{3(1+u)} \tag{6.15}$$

if and only if it has uniform distribution.

Proof. When X has uniform distribution over (α, β), $0 < \alpha < \beta$, we have $Q(u) = u(\beta - \alpha)$,

$$l_1(u) = \frac{(\beta - \alpha)}{2}(1+u)$$

and

$$l_2(u) = \frac{(\beta - \alpha)}{6}(1-u)$$

giving (6.15). Conversely, applying (6.14) with $c(u)$ as in (6.15), we get

$$g(u) = \frac{2u}{(1-u)(1+u)}.$$

Upon substituting this in (6.14), we obtain

$$Q(u) = 2u$$

which is uniform (with a change of scale).

6.3 *L*-Moments of Reversed Residual Life

On lines similar to those in the preceding section, we can look at the L-moments of ${}_tX = X|(X \leq t)$ whose distribution function is $\frac{F(x)}{F(t)}$, $0 < x \leq t$. Using (6.1), the rth L-moment of ${}_tX$ has the expression

$$B_r(t) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}^2 \int_0^t x \left(\frac{F(x)}{F(t)}\right)^{r-k-1} \left(1 - \frac{F(x)}{F(t)}\right)^k \frac{f(x)}{F(t)} dx.$$

In particular, we have

$$B_1(t) = \int_0^t \frac{xf(x)}{F(t)} dx = E[X|X \leq x], \tag{6.16}$$

$$B_2(t) = \frac{1}{F^2(t)} \int_0^t (2F(x) - F(t))xf(x)dx. \tag{6.17}$$

Setting $u = F(t)$ and $p = F(x)$, we have

$$\theta_1(u) = B_1(Q(u)) = u^{-1} \int_0^u Q(p)dp \tag{6.18}$$

and

$$\theta_2(u) = B_2(Q(u)) = u^{-2} \int_0^u (2p - u)Q(p)dp. \tag{6.19}$$

By differentiating (6.16) and using the definitions of the reversed hazard rate in (2.22) and the reversed mean residual life in (2.24), we get

$$\lambda(t) = \frac{B_1'(t)}{t - B_1(t)}$$

and

$$r(t) = t - B_1(t)$$

so that

$$\lambda(t) = B_1'(t)r(t).$$

Also, by differentiating (6.17) and simplifying, we get

$$B_2'(t) = \lambda(t)[r(t) - 2B_2(t)].$$

Table 6.2 Expressions of $\theta_2(u)$ for some quantile models

Distribution	$\theta_2(u)$
Power	$\frac{\alpha\beta}{(\beta+1)(2\beta+1)}u^{\frac{1}{\beta}}$
Govindarajulu	$\frac{\sigma\beta u^{\beta}}{(\beta+2)(\beta+3)}\{\beta+3-(\beta+1)u\}$
Generalized lambda	$\frac{1}{u^2}\Big[\frac{\lambda_3 u^{\lambda_3+2}}{\lambda_2(\lambda_3+1)} + \frac{1}{\lambda_2(\lambda_4+1)}\Big(u\{(1-u)^{\lambda_3+1}-1\}+\frac{2}{\lambda_4+2}\{(1-u)^{\lambda_4+2}-1\}\Big)\Big]$
Power Pareto	$Cu^{-2}\{2B_u(\lambda_1+2,1-\lambda_2)-uB(\lambda_1+1,1-\lambda_2)\}$

Likewise, we have the following relationships connecting $\theta_1(u)$ and $\theta_2(u)$ with the reliability functions

$$\begin{aligned} R(u) &= Q(u)-\theta_1(u) \\ &= \theta_1(u)+u\theta_1'(u)-\theta_1(u)=u\theta_1'(u), \\ \theta_2(u) &= \frac{1}{u^2}\int_0^u pR(p)dp, \\ Q(u) &= R(u)+\int_0^u p^{-1}R(p)dp, \end{aligned} \tag{6.20}$$

and

$$R(u)=u\theta_2'(u)+2\theta_2(u).$$

As in Sect. 6.2, each of $Q(u)$, $\theta_1(u)$, $R(u)$ and $\theta_2(u)$ determine others uniquely. We further have

$$D(u)=u^{-1}\int_0^u p\{\theta_2'(p)+2\theta_2(p)\}^2 dp.$$

Examples of $\theta_2(u)$ for some quantile models are presented in Table 6.2.

The L-coefficient of variation of ${}_tX$, i.e., $\theta(u)=\frac{\theta_2(u)}{\theta_1(u)}$, determines the distribution of X up to a change of scale through the formula

$$Q(u)=\frac{u\theta'(u)+\theta(u)+1}{u(1-\theta(u))}\exp\left\{\int\frac{u\theta'(u)+\theta(u)+1}{u(1-\theta(u))}du\right\}.$$

Theorem 6.3. *X follows the power distribution if and only if $\theta(u)$ is a constant.*

Proof. For the power distribution with

$$Q(u) = \alpha u^{\frac{1}{\beta}}, \quad 0 \le u \le 1,\ \beta \ne 0, \beta > 0,$$

we have

$$\theta_1(u) = \frac{\alpha\beta}{1+\beta} u^{\frac{1}{\beta}} \quad \text{and } \theta_2(u) = \frac{\alpha\beta}{(\beta+1)(2\beta+1)} u^{\frac{1}{\beta}}$$

so that

$$\theta(u) = \frac{1}{2\beta+1},$$

a constant. Conversely when $\theta(u) = c$, a constant, the expression given above for $Q(u)$ in terms of $\theta(u)$ yields

$$Q(u) = \frac{c+1}{u(1-c)} \exp\left\{ \int \frac{c+1}{u(1-c)} du \right\}$$
$$= \frac{c+1}{1-c} u^{\frac{2c}{1-c}}, \quad c \ne 1,$$

which corresponds to a power distribution. Hence, the theorem.

The relevance of this characterization in economics is explained later in Sect. 6.5.

6.4 Characterizations

Like other reliability functions, the second L-moments $l_2(u)$ and $\theta_2(u)$ also characterize life distributions through special relationships. We now present several such results. The first result is the characterization of the generalized Pareto distribution by simple relationships between $l_2(u)$, $M(u)$ and $l_1(u)$.

Theorem 6.4. *Let X be a continuous non-negative random variable with $E(X) < \infty$. Then, X follows the generalized Pareto distribution with*

$$Q(u) = \frac{b}{a}\left\{ (1-u)^{-\frac{a}{a+1}} - 1 \right\}, \quad a > -1, b > 0, \tag{6.21}$$

if and only if the following conditions are satisfied:

(i) $l_2(u) = CM(u)$, $0 < C < 1$;
(ii) $l_2(u) = a_1 l_1(u) + a_2$, $a_1 > -1$, $a_2 > 0$;
(iii) $l_1(u) = AM(u) + B$.

Proof. First, we calculate $l_1(u)$, $l_2(u)$ and $M(u)$, using (6.21), to be

$$l_1(u) = ba^{-1}(a+1)\left\{(1-u)^{-\frac{a}{a+1}} - 1\right\},$$
$$l_2(u) = b(a+1)(a+2)^{-1}(1-u)^{-\frac{a}{a+1}},$$
$$M(u) = b(1-u)^{-\frac{a}{a+1}},$$

so that

$$l_2(u) = \frac{a+1}{a+2}M(u),$$
$$l_2(u) = \frac{a}{a+2}l_1(u) + \frac{b(a+1)}{a+2},$$
$$l_1(u) = \frac{a}{a+1}M(u) - \frac{ba}{a+1}.$$

Thus, the conditions (i), (ii) and (iii) are satisfied for the generalized Pareto distribution. Conversely, condition (i) is equivalent to

$$C(1-u)^2 M(u) = \int_0^1 (1-p)M(p)dp$$

or

$$\frac{(1-u)M(u)}{\int_u^1 (1-p)M(p)dp} = \frac{1}{C(1-u)}.$$

Upon solving the last equation, we get

$$M(u) = K(1-u)^{\frac{1-2C}{C}}, \tag{6.22}$$

where K is found to be $K = M(0) = \mu$. Since $0 < C < 1$, we can write it as $C = \frac{a+1}{a+2}$ for $a > -1$ and obtain (6.21). To prove the sufficiency of (ii), we note that it implies

$$(1-u)^{-2}\int_u^1 (2p-u-1)Q(p)dp = (1-u)^{-1}a_1\int_u^1 Q(p)dp + a_2.$$

Differentiating the above equation twice, we get

$$Q'(u) - \frac{2a_1 Q(u)}{(1+a_1)(1-u)} = \frac{2a_2}{(1+a_1)(1-u)}. \tag{6.23}$$

Noticing that (6.23) is a first-order linear differential equation with integrating factor $(1-u)^{\frac{2a_1}{1+a_1}}$, we have its unique solution as

$$(1-u)^{\frac{2a_1}{1+a_1}}Q(u) = K - \frac{a_2}{a_1}(1-u)^{\frac{2a_1}{1+a_1}}.$$

Evaluating K at $u = 0$, we obtain $K = \frac{a_2}{a_1}$, and thus

$$Q(u) = \frac{a_2}{a_1}\left\{(1-u)^{-\frac{2a_1}{1+a_1}} - 1\right\}$$

which corresponds to a generalized Pareto $\big($reduces to (6.21) when $a_1 = \frac{a}{a+2}$, $a_2 = \frac{b}{a+2}\big)$. The result in (iii) is a consequence of (i) and (ii), and this completes the proof of the theorem.

Remark 6.1. Conditions (i), (ii) and (iii) show that each of $l_1(u)$, $l_2(u)$ and $M(u)$ is a linear function of the other.

Remark 6.2. The generalized Pareto law reduces to the exponential distribution as $a \to 0$, rescaled beta for $-1 < a < 0$, and Pareto II for $a > 0$. Thus, the exponential (rescaled beta; Pareto II) is characterized by $l_2(u) = \frac{1}{2}M(u)$ $\big(< \frac{1}{2}M(u);\ > \frac{1}{2}M(u)\big)$ corresponding to the values $C = \frac{1}{2}$ $\big(< \frac{1}{2},\ > \frac{1}{2}\big)$ in result (i).

Remark 6.3. It is seen from direct calculations that

$$V(u) = \frac{1+a}{1-a}b^2(1-u)^{-\frac{2a}{a+1}} = Kl_2^2(u), \quad \text{with } K = \frac{(a+2)^2}{1-a^2}.$$

But, Nair and Vineshkumar [452] have shown that this is not a characteristic property of the generalized Pareto.

Life distributions characterized by simple forms of various reliability functions have been of interest in reliability theory. They are quite useful in modelling lifetime data. The linear and quadratic hazard rate distributions belong to this category. One may refer to Bain [45, 46], Sen and Bhattacharya [525] and Gore et al. [223] for details. A second example is the generalized Pareto distribution which is uniquely determined by a linear mean residual life function (also by a reciprocal linear hazard rate function). It has been seen that $L_2(t) = c$ $(l_2(u) = c)$, where c is a constant characterizing the exponential law. In the same manner, let us consider the linearity

$$L_2(t) = A + Bt,$$

or equivalently

$$l_2(u) = A + BQ(u) \tag{6.24}$$

and identify the corresponding life distribution. Using (6.9), we then have

$$M(u) = 2A + 2BQ(u) - B(1-u)q(u.)$$

To express the RHS also in terms of $M(u)$, we make use of (2.38) and (2.39) to arrive at

$$M(u) = 2A + 2B\left\{\int_0^u \frac{M(p)dp}{1-p} - M(u) + \mu\right\} - B[M(u) - (1-u)M'(u)].$$

Differentiating with respect to u and simplifying the resulting expression, we obtain the homogeneous linear differential equation of order two with variable coefficients

$$B(1-u)^2M''(u) - (4B+1)(1-u)M'(u) + 2BM(u) = 0. \tag{6.25}$$

To solve (6.25), we set $M(u) = (1-u)^m$ to get the auxiliary equation

$$Bm(m-1) + (4B+1)m + 2B = 0,$$

or the quadratic equation (in m)

$$Bm^2 + (3B+1)m + 2B = 0. \tag{6.26}$$

Suppose (6.26) has two distinct roots m_1 and m_2. Then, the general solution of (6.25) is of the form

$$M(u) = C_1(1-u)^{m_1} + C_2(1-u)^{m_2}. \tag{6.27}$$

As u tends to zero, we get

$$\mu = C_1 + C_2. \tag{6.28}$$

Thus, from (6.27), the distribution satisfying (2.26) is recovered as

$$\begin{aligned} Q(u) &= \int_0^u \frac{M(p)}{1-p}dp - M(u) + \mu \\ &= \mu + C_1\left\{\frac{1}{m_1} - \frac{1}{m_1+1}(1-u)^{m_1}\right\} + C_2\left(\frac{1}{m_2} - \frac{1}{m_2+1}\right)(1-u)^{m_2}. \end{aligned}$$

Upon substituting for μ from (6.28), we obtain the final expression

$$Q(u) = C_1\frac{1+m_1}{m_1}\{1-(1-u)^{m_1}\} + C_2\frac{1+m_2}{m_2}\{1-(1-u)^{m_2}\}. \tag{6.29}$$

When the roots are equal, we must have

$$(3B+1)^2 - 8B^2 = 0,$$

or equivalently

$$B^2 + 6B + 1 = 0. \tag{6.30}$$

Also, in this situation, we have

$$m_1 = -\frac{(1+3B)}{2B}.$$

Since the product of the roots is 2, $m_1 = \pm\sqrt{2}$, and therefore from (6.30), we have

$$B = -\frac{1}{-3-2\sqrt{2}} = -3+2\sqrt{2}$$

or

$$B = -\frac{1}{-3+2\sqrt{2}} = -3-2\sqrt{2}.$$

Both values satisfy (6.30). We then use the method of variation of parameters to extract the solution of (6.25). Assume that the solution in (6.27), of the form

$$M(u) = C_1 M_1(u) + C_2 M_2(u),$$

where $M_i(u) = (1-u)^{m_i}$, $i = 1,2$, is such that $M_2(u) = yM_1(u)$ is a solution with y being some function of u. Then,

$$M_2'(u) = yM_1'(u) + y'M_1(u)$$

and

$$M_2''(u) = yM_1''(u) + 2y'M_1'(u) + M_1(u)y''.$$

Substituting these in (6.25), we get

$$\begin{aligned}&[B(1-u^2)M_1''(u) - 4(B+1)(1-u)M_1'(u) + 2BM_1(u)]y \\ &\qquad + (1-u)^2 M_1(y)y'' - (4B+1)(1-u)y'M_1(u) + 2(1-u)^2 y'M_1'(u) = 0.\end{aligned}$$

Since $M_1(u)$ is a particular solution of (6.25), the first term vanishes and so we get

$$(1-u)^2 M_1(u)y'' - (4B+1)(1-u)y'M_1(u) + 2(1-u)B^2 y'M_1'(u) = 0. \qquad (6.31)$$

The transformation

$$M_1(u) = (1-u)^{-\frac{3B+1}{2B}}$$

in (6.31) shows that

$$(1-u)y'' - By' = 0,$$

which has its solution as

$$y' = (1-u)^{-B}$$

or

$$y = \frac{(1-u)^{1-B}}{B-1}.$$

Thus, we have

$$M_2(u) = \frac{(1-u)^{1-B}}{B-1} M_1(u)$$

which gives the second solution corresponding to $m_1 = m_2$ as

$$\begin{aligned} M(u) &= C_1 M_1(u) + C_2 \frac{(1-u)^{1-B}}{B-1} M_1(u) \\ &= \left\{ C_1 + \frac{C_2}{B-1}(1-u)^{1-B} \right\} (1-u)^{m_1}. \end{aligned} \tag{6.32}$$

The quantile function corresponding to (6.32) is calculated from (2.38) as

$$Q(u) = C_1 \frac{m_1+1}{m_1}\{1-(1-u)^{m_1}\} + \frac{c_2(m_1-B+2)}{(B-1)(m_1-B+1)}\{1-(1-u)^{m_1-B+1}\}. \tag{6.33}$$

To complete the required characterization, it remains to be shown that the identity in (6.24) holds for the quantile functions in (6.28) and (6.33). By direct calculation from (6.28), we see that

$$\begin{aligned} l_2(u) &= \frac{c_1(1-u)^{m_1}}{m_1+2} + \frac{c_2(1-u)^{m_2}}{m_2+2} \\ &= \frac{c_1(1-u)^{m_1}}{m_1+2} + \frac{c_2 m_1 (1-u)^{\frac{2}{m_1}}}{2(1+m_1)}, \end{aligned}$$

where we have used the fact that $m_1 m_2 = 2$. Then, (6.24) holds with

$$\begin{aligned} A &= \frac{C_1}{2+m_1} + \frac{C_2 m_1}{2(1+m_1)}, \\ B &= -\frac{m_1}{(1+m_1)(2+m_2)}. \end{aligned}$$

In the second case, $A = C_1 \frac{m_1+1}{m_1} + \frac{C_2(m_1-B+2)}{(B-1)(m_1-B+1)}$, where m_1 and B have the values determined earlier.

Thus, we have established the following theorem.

Theorem 6.5. *A continuous non-negative random variable X with finite mean satisfies*

$$l_2(u) = A + BQ(u) \quad (L_2(t) = A + Bt)$$

if and only if its distribution is specified by the quantile functions (6.29) *or* (6.33).

Remark 6.4. The conditions on the parameters are determined such that $Q(u)$ is a quantile function. Notice also that the quantile functions in this case cannot be inverted into analytically tractable distribution functions.

Remark 6.5. The generalized Pareto distribution arises as a particular solution when $C_2 = 0$ and $m_1 = -\frac{a}{a+1}$.

The next result is based on a simple relationship between $l_2(u)$ and the hazard quantile function $H(u)$.

Theorem 6.6. *A continuous non-negative random variable with finite mean satisfies*

$$l_2(u) = K[H(u)]^{-1}, \quad K > 0, \tag{6.34}$$

for all $0 < u < 1$*, if and only if*

$$Q(u) = C_1 \frac{1+m_1}{m_1}\{1-(1-u)^{m_1}\} + C_2 \frac{1+m_2}{m_2}\{1-(1-u)^{m_2}\}, \tag{6.35}$$

where m_1 *and* m_2 *are the roots of the quadratic equation*

$$Km^2 + 3Km + (2K-1) = 0.$$

Proof. The condition (6.34) is equivalent to

$$\frac{1}{(1-u)^2}\int_u^1 (1-p)M(p)dp = K\{M(u) - (1-u)M'(u)\},$$

or

$$\int_u^1 (1-p)M(p)dp = K(1-u)^2 M(u) - K(1-u)^3 M'(u).$$

Differentiating and simplifying the expression, we get

$$K(1-u)^2 M''(u) - 4K(1-u)M'(u) + (1-2K)M(u) = 0. \tag{6.36}$$

Now by setting $M(u) = (1-u)^m$ and proceeding as in the previous theorem, we have the auxiliary equation

$$Km^2 + 3Km + 2K - 1 = 0.$$

Let m_1 and m_2 be the roots of this quadratic equation. Then, a general solution to (6.36) is

$$M(u) = C_1(1-u)^{m_1} + C_2(1-u)^{m_2},$$

where $m_1 + m_2 = -3$. Then,

$$Q(u) = \frac{C_1(1+m_1)}{m_1}\{1-(1-u)^{m_1}\} + C_2\frac{1+m_2}{m_2}\{1-(1-u)^{m_2}\}.$$

If the roots are the same, the condition for this is

$$K^2 + 4K = 0.$$

However, the roots $K = 0$ and $K = -4$ are both inadmissible. Hence, (6.35) represents the unique distribution satisfying (6.34).

Now, for the distribution (6.35), we have

$$l_2(u) = \frac{C_1(1-u)^{m_1}}{m_1+2} + \frac{C_2(1-u)^{m_2}}{m_2+2}$$

and

$$q(u) = C_1(1+m_1)(1-u)^{m_1-1} + C_2(1+m_2)(1-u)^{m_2-1}.$$

Hence,

$$\begin{aligned} H(u) &= \{(1-u)q(u)\}^{-1} \\ &= \{C_1(1+m_1)(1-u)^{m_1} + C_2(1+m_2)(1-u)^{m_2}\}^{-1}. \end{aligned}$$

Since $(1+m_1)(2+m_1) = (1+m_2)(2+m_2)$ by virtue of $m_1 + m_2 = -3$, we have

$$l_2(u) = K(H(u))^{-1} \quad \text{with } K = (1+m_1)(2+m_2).$$

The proof of the theorem is thus completed.

There exist similar results for the reversed hazard quantile functions. Since the proof proceeds along the same lines, we just briefly outline the proofs.

Theorem 6.7. *If X is a continuous non-negative random variable with finite mean, then*

$$\theta_2(u)\Lambda(u) = C, \tag{6.37}$$

a positive constant, if and only if

$$Q(u) = \frac{1+m_1}{m_1} C_1 u^{m_1} + \frac{1+m_2}{m_2} C_2 u^{m_2}, \tag{6.38}$$

where C_1 and C_2 are the roots of the quadratic equation

$$Cm^2 + 3Cm + 2C - 1 = 0.$$

Proof. Condition (6.37) is same as

$$\frac{1}{u^2} \int_0^u pR(p)dp = C\{R(u) + R'(u)\}$$

leading to

$$Cu^2 R''(u) + 4CuR'(u) + (2C-1)R(u) = 0.$$

Assuming $R(u) = u^m$, the auxiliary equation becomes

$$Cm(m-1) + 4m + (2C-1) = 0,$$

and so

$$R(u) = C_1 u^{m_1} + C_2 u^{m_2}$$

which gives $Q(u)$ in (6.38) on applying (2.51). The condition for equal roots is $C = -4$ or 0, which are both inadmissible.

Conversely, when (6.38) holds, we have

$$\theta_2(u) = \frac{C_1 u^{m_1}}{m_1 + 2} + \frac{C_2 u^{m_2}}{m_2 + 2}$$
$$= C \wedge (u),$$

where $C^{-1} = (1+m_1)(m_1+2) = (1+m_2)(m_2+2)$, on using $m_1 + m_2 = -3$.

Theorem 6.8. *If X is a non-negative random variable with finite mean, the identity*

$$\theta_2(u) = CR(u) \tag{6.39}$$

holds if and only if X has power distribution

$$Q(u) = \alpha u^{1/\theta}, \quad \textit{i.e., } F(x) = \left(\frac{x}{\alpha}\right)^{\beta}, \ 0 \le x \le \alpha. \tag{6.40}$$

Proof. For the power distribution, we have

$$\theta_2(u) = \frac{\alpha\beta}{(\beta+1)(2\beta+1)} u^{\frac{1}{\beta}}$$

and

$$R(u) = \frac{\alpha}{\beta+1} u^{\frac{1}{\beta}}$$

so that (6.39) is satisfied with $C = \frac{\beta}{(2\beta+1)}$. Conversely, (6.39) means that

$$\frac{1}{u^2}\int_0^u pR(p)dp = cR(u),$$

or equivalently

$$CuR'(u) = (1-2c)R(u).$$

The last equation yields the solution as

$$R(u) = Ku^{\frac{1-2c}{c}} \quad \text{and} \quad Q(u) = \frac{K(1-c)}{1-2c} u^{\frac{1-2c}{c}}$$

which is of the from (6.40) with $C = \frac{\beta}{1+2\beta}$ and $\alpha = \frac{K(1-C)}{1-2C}$.

Theorem 6.9. *If X is a non-negative random variable with finite mean, the identity*

$$\theta_2(u) = AQ(u) \tag{6.41}$$

holds if and only if X has a distribution with

$$Q(u) = C_1\left(\frac{1+m_1}{m_1}\right)u^{m_1} + C_2\left(\frac{1+m_2}{m_2}\right)u^{m_2}, \tag{6.42}$$

where m_1 and m_2 are the distinct roots of

$$Am^2 + (3A-1)m + 2A = 0. \tag{6.43}$$

If (6.43) has equal roots, then

$$Q(u) = \left\{C_1\left(\frac{1+m_1}{m_1}\right) + C_2\frac{1+m_1}{m_1}\log u - \frac{C_2}{m_1^2}\right\}u^{m_1} \tag{6.44}$$

with $m_1 = \frac{3\sqrt{2}-4}{3-2\sqrt{2}}$ and $A = 3-2\sqrt{2}$.

Proof. Let us assume the identity in (6.41). Then, from (2.51) and (6.20), we have

$$\frac{1}{u^2}\int_0^u pR(p)dp = AR(u) + A\int_0^u \frac{R(p)}{p}dp.$$

Differentiation of this equation yields

$$Au^2R''(u) + (4A-1)uR'(u) + 2AR(u) = 0.$$

Substitution of $R(u) = u^m$ gives the auxiliary equation

$$Am^2 + (3A-1)m + 2A = 0. \tag{6.45}$$

When the roots of the quadratic equation in (6.45) are distinct, we get

$$R(u) = C_1u^{m_1} + C_2u^{m_2} \tag{6.46}$$

and then for (2.51)

$$Q(u) = C_1\frac{1+m_1}{m_1}u^{m_1} + C_2\frac{1+m_2}{m_2}u^{m_2},$$

where m_1 and m_2 are such that

$$m_1m_2 = 2 \quad \text{and} \quad m_1 + m_2 = \frac{1-3A}{A}. \tag{6.47}$$

Using (6.46), we have

$$\theta_2(u) = \frac{1}{u_2}\int_0^u pR(p)dp = \frac{C_1u^{m_1}}{m_1+2} + \frac{C_2u^{m_2}}{m_2+2}.$$

One can verify that

$$\theta_2(u) = AQ(u)$$

with

$$A = \frac{m_1}{(1+m_1)(m_1+2)} = \frac{m_2}{(1+m_2)(m_2+2)},$$

where the last equality holds since $m_1m_2 = 2$. When the roots of (6.45) are equal, say m_1, we see that

$$A^2 - 6A + 1 = 0$$

holds whenever $A = 3 \pm 2\sqrt{2}$ both of which are admissible values. Taking

$$R(u) = C_1 u^{m_1} + C_2 u^{m_2} = C_1 R_1(u) + C_2 R_2(u)$$

from (6.46) and setting $R_2(u) = yR_1(u)$, we get, by the method of variation of parameters,

$$Au^2 R_1(u) y'' + 2Au^2 R_1'(u) y' + u(4A-1) R_1(u) y' = 0$$

when $R_1(u) = u^{\frac{1-3A}{2A}}$,

$$Au^2 y'' + Auy' = 0$$

or

$$uy'' + y' = 0.$$

The solution is $y = \log u$, and so the quantile function simplifies to

$$Q(u) = C_1 \frac{1+m_1}{m_1} u^{m_1} + C_2 \frac{1+m_1}{m_1} u^{m_1} \log u - \frac{C_2 u^{m_1}}{m_1^2},$$

as in (6.44). Notice that $Q(u)$ becomes a quantile function only when $m_1 > 0$. In this case, $m_1 = \frac{3\sqrt{2}-4}{3-2\sqrt{2}}$ and

$$\begin{aligned}\theta_2(u) &= \left\{ \frac{C_1}{m_1+2} + \frac{C_2}{m_1+2} \log u - \frac{C_2}{(m_1+2)^2} \right\} u^{m_1} \\ &= 2(3-2\sqrt{2}) Q(u).\end{aligned}$$

This completes the proof of the theorem.

6.5 Ageing Properties

When conceived as a reliability function, the L-moment $l_2(u)$ can also be employed in distinguishing life distributions based on its monotone behaviour. Since $l_2(u)$ is twice the mean difference residual quantile function, we have the following definitions.

Definition 6.1. A lifetime random variable X is said to have increasing (decreasing) mean difference residual quantile function, IMDR (DMDR), according to whether $l_2(u)$ is an increasing (decreasing) function of u.

Example 6.5. From the expressions in Table 6.1, the Pareto II distribution has increasing mean difference residual quantile function, while the rescaled beta has decreasing mean difference residual quantile function.

The mean difference residual quantile function is known to be

$$\Delta(u) = 2l_2(u),$$

and accordingly, from (6.9), we have

$$\Delta'(u) = \frac{2}{1-u}(\Delta(u) - M(u)).$$

Thus, a necessary and sufficient condition that $\Delta(u)$ is increasing (decreasing) is $\Delta(u) \geq (\leq) M(u)$. It is evident that the graph of $\Delta(u)$ lies above (below) that of $M(u)$ when the former is increasing (decreasing). Also, when $\Delta(u)$ crosses $M(u)$ at some point u_0 from below (above), then it is a change point of $\Delta(u)$ that indicates $\Delta(u)$ is increasing (decreasing) first and then decreasing (increasing). Since $\Delta(u)$ is directly related to $M(u)$, it is also clear that

$$X \text{ is DMRL (IMRL) } \Leftrightarrow 3\Delta'(u) \leq (\geq)(1-u)\Delta''(u).$$

The comparison of the implications of monotonicities of $V(u)$, $\Delta(u)$ and $M(u)$ have all been addressed earlier in Sect. 6.2.

6.6 Partial Moments

The partial moments, whose definitions were given earlier in Sect. 6.1, can also be viewed as reliability functions. Since the first two moments are of interest to the concepts discussed earlier, we recall their definitions as

$$p_1(x) = E[(X-x)^+] = \int_x^\infty (t-x)f(t)dt \tag{6.48}$$

and

$$p_2(x) = E[(X-x)^{+2}] = \int_x^\infty (t-x)^2 f(t)dt. \tag{6.49}$$

Gupta and Gupta [231] have discussed the general properties of the rth partial moment. They proved that the rth moment determines the underlying distribution for any positive real r. Also, when r is a positive integer, there exists a recurrence relation between two consecutive partial moments. Earlier, Chong [147] characterized the exponential distribution by the property

$$E(X-t-s)^+E(X) = E(X-t)^+E(X-s)^+.$$

The expression for the survival function in terms of $p_r(x)$ is

$$\bar{F}(x) = \frac{(-1)^r}{r!} \frac{d^r p_r(x)}{dx^r};$$

see Navarro et al. [465] and Sunoj [554]. Sunoj [554] also obtained the partial moments of the length-biased distribution, equilibrium distribution and characterizations thereof. Gupta [236] extended this result to show that the kth order equilibrium distribution has survival function

$$S_K(x) = \frac{E[(X-x)^+]^k}{E(X^k)}.$$

Lin [401] and Abraham et al. [14] characterized the exponential, beta and Lomax distributions by relationships between the partial moments. The quantile forms of (6.48) and (6.49) are

$$\begin{aligned} P_1(u) = p_1(Q(u)) &= \int_u^1 (Q(p)-Q(u))dp \qquad (6.50) \\ &= \int_u^1 (1-p)q(p)dp \\ &= (1-u)M(u) \end{aligned}$$

and

$$P_2(u) = p_2(Q(u)) = \int_u^1 [Q(p)-Q(u)]^2 dp.$$

Accordingly, the variance of $(X-x)^+$ has the form

$$V_+(u) = \int_u^1 [Q(p)-Q(u)]^2 dp - P_1^2(u). \qquad (6.51)$$

We then have

$$P_1'(u) = -(1-u)q(u) \qquad (6.52)$$

and

$$\begin{aligned} V_+(u) &= \int_u^1 Q^2(p)dp - 2Q(u)\int_u^1 Q(p)dp + (1-u)Q^2(u) - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp - 2Q(u)[P_1(u)+(1-u)Q(u)] + (1-u)Q^2(u) - P_1^2(u) \\ &= \int_u^1 Q^2(p)dp - [P_1(u)+Q(u)]^2 + uQ^2(u). \end{aligned}$$

Differentiating the above expression, we get

$$V'_+(u) = -2[P_1(u) + Q(u)][P'_1(u) + q(u)] + 2uQ(u)q(u).$$

Eliminating $Q(u)$ and $q(u)$ by using (6.50) and (6.52), we obtain

$$V'_+(u) = \frac{2uP_1(u)P'_1(u)}{1-u}. \tag{6.53}$$

Equation (6.53) shows that both $P_1(u)$ and $V_+(u)$ determine each other as

$$V_+(u) = -\int_u^1 \frac{2pP_1(p)P'_1(p)}{1-p}dp$$

and

$$P_1^2(u) = -\int_u^1 \frac{(1-p)}{p}V'_+(p)dp.$$

Thus, for all practical purposes, it is enough if the first partial moment (stop loss transform) is available. The relationships that the partial moments have with the reliability functions is immediate from the above discussions. We notice that

$$H(u) = -\frac{1}{P'_1(u)}, \tag{6.54}$$

$$M(u) = (1-u)^{-1}P_1(u),$$

$$V(u) = \frac{1}{(1-u)}\int_u^1 (1-p)^{-2}P_1(p)dp,$$

$$T(u) = \mu - P_1(u,) \tag{6.55}$$

and

$$(1-u)P_1(u) = 2l_2(u) - (1-u)l'_2(u).$$

The ageing properties of X can also be characterized in terms of $P_1(u)$. These can be expressed with the use of Theorems in Sect. 5.4 and (6.55). Some examples are

(i) X is IHR (DHR) if and only if $P_1(u)$ is convex (concave). This result follows from Theorem 5.2 and (6.54);
(ii) A necessary and sufficient condition that X is DMTTF (IMTTF) is that $\frac{\mu - P_1(u)}{\mu u}$ is decreasing (increasing), which simplifies to

$$P_1(u) - uP'_1(u) < \mu.$$

The other ageing properties result from Theorems 5.4–5.6.

Table 6.3 Stop-loss transforms for some distributions

Distribution	$P_1(u)$
Exponential	$\lambda^{-1}(1-u)$
Pareto II	$\frac{\alpha}{C-1}(1-u)^{-\frac{1}{c}+1}$
Rescaled beta	$\frac{C}{R+1}(1-u)^{\frac{1}{c}+1}$
Half logistic	$2\sigma\log\frac{2}{1+u}$
Exponential geometric	$\frac{1-p}{\lambda p}\log\frac{1-pu}{1-p}$
Power	$\frac{\alpha}{1+\beta}\{\beta-(1+\beta)u^{\frac{1}{\beta}}+u^{\frac{1}{\beta}+1}\}$
Linear hazard quantile	$\frac{1}{b}\log\frac{a+b}{a+bu}$
Generalized lambda	$\frac{1}{\lambda_2}\left\{\frac{\lambda_4}{1+\lambda_4}(1-u)^{\lambda_4+1}+\frac{(1-u^{\lambda_3+1})}{1+\lambda_3}-(1-u)u^{\lambda_3}\right\}$
Generalized Tukey lambda	$(1-u)\left\{\frac{(1-u)^{\lambda_4}}{\lambda_2(\lambda_4+1)}+\frac{1-u^{\lambda_3+1}}{\lambda_2\lambda_3(1+\lambda_3)(1-u)}-\frac{u^{\lambda_3}}{\lambda_2\lambda_3}\right\}$
van Staden and Loots	$\lambda_2\left[\frac{(1-\lambda_3)}{\lambda_4}\left\{\frac{1-u^{\lambda_4+1}}{\lambda_4+1}-(1-u)u^{\lambda_4}\right\}+\frac{\lambda_3}{\lambda_4+1}(1-u)^{\lambda_4+1}\right]$
Generalized Weibull	$\frac{\sigma^\alpha}{\lambda^\alpha}(1-u)B_{(1-u)^\lambda}(\frac{1}{\lambda}+1,\alpha)$
Power-Pareto	$c(1-u)\{\lambda_1 B_{1-u}(2-\lambda_2,\lambda_1)+\lambda_2 B_{1-u}(1-\lambda_2,\lambda_1)\}$
Govindarajulu	$\frac{\sigma}{\beta+2}\{2-(\beta+1)(\beta+2)u^\beta+2\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}\}$

The stop-loss transforms of several distributions are presented in Table 6.3.

The lower partial moments of order r in the case of a non-negative random variable is defined as

$$p_r^*(t)=E[(X-t)^-]^r,$$

where

$$(X-t)^-=\begin{cases}t-X, & X\le t\\ 0, & X\ge t\end{cases}.$$

The first two moments, in terms of quantile functions, become

$$\begin{aligned}P_1^*(u)=p_1^*(Q(u))&=\int_0^u[Q(u)-Q(p)]dp\\&=\int_0^u pq(p)dp,\end{aligned}\tag{6.56}$$

$$P_2^*(u)=p_2^*(Q(u)=\int_0^u[Q(u)-Q(p)]^2dp.\tag{6.57}$$

From (6.56) and (6.57), the variance of $(X-t)^{-}$ is obtained as

$$v_{-}(u)=\int_0^u [Q(u)-Q(p)]^2-[P_1^*(u)]^2 dp.$$

Using now the relations

$$\int_0^u Q(p)dp = P_1^*(u)-uQ(u),$$
$$\frac{dP_1^*(u)}{du}=uq(u),$$

we can eliminate $Q(u)$ and $q(u)$ from

$$v'_{-}(u)=2q(u)P_1^*(u)-2uQ(u)q(u)-2uQ(u)q(u)$$

to arrive at the identity

$$v'_{-}(u)=\frac{2(1-u)}{u}P_1^*(u)\frac{dP_1^*(u)}{du}. \tag{6.58}$$

Thus, $P_1^*(u)$ determines $v_{-}(u)$ uniquely as

$$v_{-}(u)=\int_0^u \frac{2(1-p)}{p}P_1^*(p)\frac{dP_1^*}{dp}dp,$$

and conversely

$$[P_1^*(u)]^2=\int_0^u \frac{pv'_{-}(p)}{1-p}dp.$$

From the reliability theory perspective, the partial mean is useful in defining the reversed quantile functions. The basic relationships are as follows:

$$\Lambda(u)=\frac{dP_1^*(u)}{du},$$
$$R(u)=u^{-1}P_1^*(u),$$
$$D(u)=\frac{1}{u}\int_0^u R^2(p)dp=\frac{1}{u}\int_0^u\left\{\frac{P^*(p)}{p}\right\}^2 dp.$$

6.7 Some Applications

The *L*-moments and the two kinds of partial moments discussed so far are known in some other disciplines than reliability for their applications. We now give a brief account of the important ones, partly because the models discussed have relevance in reliability theory as well. One major application is related to income analysis in economics. Let X denote the non-negative continuous random variable representing incomes of individuals in a population. Income is often conceived as an indicator to differentiate between the strata of the society, notably the poor and the affluent, with generally more attention to the former. A poverty line $X = t$ is set such that those having income below t is considered poor. Then, $\alpha = F(t)$ represents the proportion of poor in the population, and their income has the distribution

$$_tF(x) = \begin{cases} \frac{F(x)}{F(t)}, & x \le t \\ 1, & x > t \end{cases}.$$

The extent to which poverty exists among the poor is measured by the income gap ratio defined as

$$\begin{aligned} G(t) &= 1 - E\left[\frac{X}{t}|(X \le t)\right] \\ &= 1 - \frac{1}{t}E[X|(X \le t)] = 1 - \frac{B_1(t)}{t}. \end{aligned} \tag{6.59}$$

In terms of quantile functions, we have

$$g(u) = G(Q(u)) = 1 - \frac{\theta_1(u)}{Q(u)}. \tag{6.60}$$

Traditionally, the income gap ratio is computed from the income distribution; but, the reverse process is also valid. Nair et al. [440] have shown that there exists a one-to-one relationship between income gap ratio and the income distribution and the latter can be retrieved from the former as explained in the following theorems. Empirically, it is possible to draw some ideas about the approximate form of $G(t)$ from the data.

Theorem 6.10. *If X has a finite mean and income gap ratio $G(t)$, then the distribution of X is*

$$F(x) = \exp\left\{-\int_x^\infty \frac{1 - G(t) - tG'(t)}{tG(t)} dt\right\}, \quad x > 0.$$

Remark 6.6. Using the above theorem, it follows that the only continuous distribution for which $G(t) =$ a constant is the power distribution.

Remark 6.7. The analogue of Theorem 6.10 is

$$Q(u) = \frac{\mu}{u(1-g(u))} \exp\left\{-\int_u^1 \frac{dp}{p(1-g(p))}\right\}.$$

A popular measure for the income inequality in a population is the Gini index defined as

$$I = 1 - \frac{2}{\mu}\int_0^\infty x\bar{F}(x)f(x)dx.$$

In the case of the poor (below the poverty line or $X \le t$), the index has the form

$$I(t) = 1 - \frac{2}{E[X|(X \le t)]}\int_0^t x\left(1 - \frac{F(x)}{F(t)}\right)\frac{f(x)}{F(t)}dx. \tag{6.61}$$

Using the transformation $x = Q(u)$, we have the quantile version as

$$i(u) = I(Q(u)) = 1 - \frac{2}{\theta_1(u)}\int_0^u Q(p)\left(\frac{u-p}{u^2}\right)dp. \tag{6.62}$$

From (6.18) and (6.19), we then have

$$\int_0^u Q(p)dp = u\theta_1(u) \tag{6.63}$$

and

$$\int_0^u pQ(p)dp = \frac{u^2}{2}(\theta_1(u) + \theta_2(u)). \tag{6.64}$$

Eliminating the integral on the right-hand side of (6.62) with the use of (6.63) and (6.64), we obtain

$$i(u) = \frac{\theta_2(u)}{\theta_1(u)},$$

which is the L-coefficient of variation $\theta(u)$ considered earlier in Sect. 6.3. By virtue of Theorem 6.10, we conclude that $i(u) =$ a constant if and only if X has power distribution. Theorem 6.10 leaves scope for characterizing income distributions by the form of their truncated Gini index. A further example is that the form

$$i(u) = \frac{(\beta+3) - (\beta+1)u}{(\beta+2) - \beta u}$$

determines the Govindarajulu distribution.

The income gap ratio and truncated Gini index play a crucial role in defining index of poverty. For example, Sen [524] suggested the index

$$s(t) = F(t)[G(t) + (1 - G(t))I(t)]$$

for a measure of poverty. This turns out to be equivalent to

$$\begin{aligned} S(u) &= u[g(u) + (1 - g(u))i(u)] \\ &= u\left\{1 - \frac{\theta_1(u)}{\theta(u)} + \frac{\theta_2(u)}{\theta_1(u)}\left(\frac{\theta_1(u)}{Q(u)}\right)\right\}. \end{aligned}$$

Since $Q(u) = u\theta_1'(u) + \theta_1(u)$, we have on simplification

$$S(u) = u\left[\frac{u\theta_1'(u) + \theta_2(u)}{u\theta_1'(u) + \theta_1(u)}\right]. \tag{6.65}$$

Instead of distribution functions as models of income, Tarsitano [562] used the generalized lambda distribution and Haritha et al. [260] employed the generalized Tukey lambda distribution. Since both these distributions do not have closed-form expressions for their distribution functions, the expressions in (6.57), (6.62) and (6.65) become important.

Theorem 6.11. *Let X be a non-negative random variable with finite mean. Then, $S(u) = cu$ if and only if X has power distribution of the form*

$$Q(u) = \alpha u^{\frac{1}{\beta}}, \quad \alpha, \beta > 0,\ 0 \le u \le 1.$$

Proof. In the case of the power distribution, we have

$$\begin{aligned} \theta_1(u) &= \frac{\alpha\beta}{\beta+1}u^{\frac{1}{\beta}}, \\ \theta_2(u) &= \frac{\alpha\beta}{(\beta+1)(1+2\beta)}u^{\frac{1}{\beta}}, \end{aligned}$$

and so from (6.65), we obtain

$$S(u) = \frac{1+3\beta}{(1+\beta)(1+2\beta)}u$$

which proves the 'if' part. Conversely, when $S(u) = cu$, (6.1) provides

$$c[u\theta_1'(u) + \theta_1(u)] = u\theta_1'(u) + u\theta_2(u)$$

or equivalently

$$cu^2Q(u) = u^2Q(u) - 2u\int_0^u Q(p)dp + 2\int_0^u pQ(p)dp$$

upon using the expressions of $\theta_1(u)$ and $\theta_2(u)$. Differentiating and simplifying the resulting expression, we get

$$(c-1)u^2Q''(u) + 4(c-1)uQ'(u) + 2cQ(u) = 0. \tag{6.66}$$

Now, by setting $Q(u) = u^m$, the auxiliary equation for the solution of (6.66) is

$$(c-1)m^2 + 3(c-1)m + 2c = 0,$$

which has its roots as

$$m = -\frac{3(c-1) \pm \sqrt{a(c-1)^2 - 8c(c-1)}}{2(c-1)},$$

that simplify to

$$m = -\frac{3}{2} \pm \frac{1}{2}\sqrt{\frac{c-9}{c-1}}.$$

Hence, the solution of (6.66) becomes

$$Q(u) = C_1 u^{-\frac{3}{2}+\frac{1}{2}(\frac{c-9}{c-1})^{\frac{1}{2}}} + C_2 u^{-\frac{3}{2}-\frac{1}{2}(\frac{c-9}{c-1})^{\frac{1}{2}}}.$$

Since $Q(u)$ has to be increasing for all u, $C_2 = 0$ and so

$$Q(u) = \alpha u^{\frac{1}{\beta}},$$

which corresponds to the power distribution with $\beta = \frac{1}{2}[(\frac{c-9}{c-1})^{\frac{1}{2}} - 3]$ and $\alpha = C_1$. This completes the proof of the theorem.

The lower partial moments have an important role in the measurement of risk associated with management, industrial and insurance strategies. Sunoj and Maya [555] discussed their role in stochastic modelling that includes characterization of distributions, weighted and equilibrium models. In $p_r^*(t)$ defined earlier, t is a target, that separates gains and losses and the main interest is in 'downside risk' measured by $p_1^*(t)$. Portfolio theory is concerned about maximizing the return for a given risk, where X stands for the random return and t the target return. In this context, lower partial moments provides summary measures of downside risk. The second moment $p_2^*(t)$ is called target semi-variance which fits investors' risk preference better than the traditional variance. Some references in this connection are Bawa [81], Fishburn [199], Harlow [261], Brogan and Stidham [121], Willmot et al. [582] and Hesselager et al. [270].