Chapter 9
Optimal Execution of Derivatives: A Taylor Expansion Approach

Gerardo Hernandez-del-Valle and Yuemeng Sun

9.1 Introduction

The problem of optimal execution is a very general problem in which a trader who wishes to buy or sell a *large* position K of a given asset S—for instance, wheat, shares, derivatives, etc.—is confronted with the dilemma of executing slowly or as quick as possible. In the first case, he/she would be exposed to volatility, and in the second, to the laws of offer and demand. Thus, the trader must hedge between the *market impact* (due to his trade) and the *volatility* (due to the market).

The main *aim* of this chapter is to study and characterize the so-called Markowitz-optimal open-loop execution trajectory of contingent claims.

The problem of minimizing expected overall liquidity costs has been analyzed using different market models by [1, 6, 8], and [2], just to mention a few. However, some of these approaches miss the volatility risk associated with time delay. Instead, [3, 4] suggested studying and solving a mean-variance optimization for sales revenues in the class of deterministic strategies. Further on, [5] allowed for intertemporal updating and proved that this can *strictly* improve the mean-variance performance. Nevertheless, in [9], the authors study the original problem of expected utility maximization with CARA utility functions. Their main result states that for CARA investors there is surprisingly no added utility from allowing for intertemporal updating of strategies. Finally, we mention that the Hamilton-Jacobi-Bellman approach has also recently been studied in [7].

G. Hernandez-del-Valle (✉)
Statistics Department, Columbia University, New York, NY 10027, USA
e-mail: gerardo@stat.columbia.edu

Y. Sun
Cornell University, Theca, NY 14853, USA
e-mail: ys273@cornell.edu

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The chapter is organized as follows: in Sect. 9.2, we state the optimal execution contingent claim problem. Next, in Sect. 9.3, we provide its closed form solution. In Sect. 9.4, a numerical example is studied, and finally we conclude in Sect. 9.5 with some final remarks and comments.

9.2 The Problem

The Model. A trader wishes to execute $K = k_0 + \cdots + k_n$ units of a *contingent claim* C with underlying S by time T. The quantity to optimize is given by the so-called execution shortfall, defined as

$$Y = \sum_{j=0}^{n} k_j C_j - KC_0,$$

and the problem is then to find $k_0, \ldots, k_n$ such that attain the minimum

$$\min_{k_0,\ldots,k_n} \left(\mathbf{E}[Y] + \lambda \mathbf{V}[Y]\right),$$

for some $\lambda > 0$. Assuming the derivative C is smooth in terms of its underlying S, it follows from the Taylor series expansion that

$$C_j = f(S_0) + f'(S_0)(\tilde{S}_j - S_0) + \frac{1}{2} f''(S_0)(\tilde{S}_j - S_0)^2 + R_3,$$

where $\tilde{S}$ is the effective price and R_3 is the remainder which is $o((\tilde{S}_j - S_0)^3)$. Hence,

$$\begin{aligned}
\sum_{j=0}^{n} k_j C_j &= \sum_{j=0}^{n} k_j f(S_0) + \sum_{j=0}^{n} f'(S_0) k_j (\tilde{S}_j - S_0) + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3 \\
&= KC_0 + f'(S_0) \left(\sum_{j=0}^{n} k_j \tilde{S}_j - KS_0 \right) s + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3.
\end{aligned}$$

That is,

$$\begin{aligned}
Y &= \sum_{j=0}^{n} k_j C_j - KC_0 \\
&= f'(S_0) \left(\sum_{j=0}^{n} k_j \tilde{S}_j - KS_0 \right) + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j (\tilde{S}_j - S_0)^2 + \sum_{j=0}^{n} k_j R_3. \qquad (9.1)
\end{aligned}$$

Note that if we use only the first-order approximation, then our optimization problem has already been solved and corresponds to [4] trading trajectory.

9.3 Second-Order Taylor Approximation

In this section, we extend [4] market impact model for the case of a contingent claim. We provide our main result which is the closed form objective function by adapting a second order-Taylor approximation.

9.3.1 *Effective Price Process*

Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. Gaussian random variables with mean zero and variance 1, and let the execution times be equally spaced, that is, $\tau := T/n$. Then, the price and "effective" processes are respectively defined as

$$S_j = S_{j-1} - \tau g\left(\frac{k_j}{\tau}\right) + \sigma\tau^{1/2}\xi_j,$$

$$\tilde{S}_j = S_j - h\left(\frac{k_j}{\tau}\right),$$

and the permanent and temporary market impact will be modeled, for simplicity, as

$$g\left(\frac{k_j}{\tau}\right) = \alpha\frac{k_j}{\tau}, \qquad h\left(\frac{k_j}{\tau}\right) = \beta\frac{k_j}{\tau},$$

for some constant α and β. Hence, letting

$$x_j := K - \sum_{m=0}^{j} k_m \quad \text{and}$$

$$W_j := \sum_{m=1}^{j} \xi_m, \quad \text{i.e. } W_j \sim \mathbf{N}(0, j), \operatorname{Cov}(W_j, W_i) = \min(i, j),$$

it follows that

$$\tilde{S}_j - S_0 = \sigma\tau^{1/2}W_j - \alpha(K - x_j) - \frac{\beta}{\tau}k_j. \tag{9.2}$$

9.3.2 *Second-Order Approximation*

From (9.1) and (9.2), the second-order approximation of the execution shortfall Y is given by

$$Y \approx f'(S_0)\sum_{j=0}^{n} k_j\left(\sigma\tau^{1/2}W_j - \alpha(K-x_j) - \frac{\beta}{\tau}k_j\right)$$
$$+\frac{1}{2}f''(S_0)\sum_{j=0}^{n} k_j\left(\sigma\tau^{1/2}W_j - \alpha(K-x_j) - \frac{\beta}{\tau}k_j\right)^2. \quad (9.3)$$

Next, expanding the squared term, we get

$$\left(\sigma\tau^{1/2}W_j - \alpha(K-x_j) - \frac{\beta}{\tau}k_j\right)^2 = \sigma^2\tau W_j^2 + \alpha^2(K-x_j)^2 + \frac{\beta^2}{\tau^2}k_j^2 - 2\frac{\beta\sigma}{\tau^{1/2}}k_jW_j$$
$$-2\alpha\sigma\tau^{1/2}(K-x_j)W_j + 2\frac{\alpha\beta}{\tau}k_j(K-x_j),$$

Thus the expected value of Y is approximately

$$\mathbf{E}[Y] = f'(S_0)\sum_{j=0}^{n} k_j\left(-\alpha(K-x_j) - \frac{\beta}{\tau}k_j\right)$$
$$+\frac{1}{2}f''(S_0)\sum_{j=0}^{n} k_j\left[\sigma^2\tau j + \alpha^2(K-x_j)^2 + \frac{\beta^2}{\tau^2}k_j^2 + 2\frac{\alpha\beta}{\tau}k_j(K-x_j)\right], \quad (9.4)$$

to compute the variance $\mathbf{V}$ of Y we rearrange (9.3) as

$$Y \approx \sum_{j=0}^{n} \nu_j k_j W_j + \sum_{j=0}^{n} \eta_j k_j W_j^2 + D,$$

where D are all the deterministic terms and

$$\nu_j := f'(S_0)\sigma\tau^{1/2} - f''(S_0)\left[\alpha\sigma\tau^{1/2}(K-x_j) + \frac{\beta\sigma}{\tau^{1/2}}k_j\right]$$
$$\eta_j := \frac{1}{2}f''(S_0)\sigma^2\tau.$$

It follows that the variance of Y is

$$\mathbf{V}[Y] = \mathbf{V}\left[\sum_{j=0}^{n} \nu_j k_j W_j\right] + \mathbf{V}\left[\sum_{j=0}^{n} \eta_j k_j W_j^2\right] + 2\,\mathrm{Cov}\left(\sum_{j=0}^{n} \nu_j k_j W_j, \sum_{j=0}^{n} \eta_j k_j W_j^2\right)$$
$$= \sum_{j=0}^{n} \nu_j^2 k_j^2 j + 2\sum_{0\le i<j\le n} \nu_i k_i \nu_j k_j i + \sum_{j=0}^{n} \eta_j^2 k_j^2 \cdot 2j^2 + 2\sum_{0\le i<j\le n} \eta_i k_i \eta_j k_j \cdot 2i^2 \quad (9.5)$$

and the last term equals zero.

9.3.3 *Optimal Trading Schedule for the Second-Order Approximation*

To find the optimal trading schedule for the second-order approximation of Y, we need find the sequence of $k_0, \ldots, k_n$ such that

$$\mathbf{E}[Y] + \lambda \mathbf{V}[Y]$$

is minimized for a given λ and where $\mathbf{E}[Y]$ and $\mathbf{V}[Y]$ are as in (9.4) and (9.5), respectively. After some simplification,

$$\begin{aligned}\mathbf{E}[Y] + \lambda \mathbf{V}[Y] = f'(S_0) \sum_{j=0}^{n} k_j \left[\alpha(x_j - K) - \frac{\beta}{\tau} k_j\right] \\ + \frac{1}{2} f''(S_0) \sum_{j=0}^{n} k_j \left[\sigma^2 \tau j + \alpha^2 (K - x_j)^2 + \frac{\beta^2}{\tau^2} k_j^2 + \frac{2\alpha\beta}{\tau}(K - x_j) k_j\right] \\ + \lambda \sum_{j=0}^{n} j k_j \left[v_j^2 k_j + 2 j k_j \eta_j^2 + 2 v_j \sum_{m=j+1}^{n} v_m k_m + 4 j \eta_j \sum_{m=j+1}^{n} \eta_m k_m\right].\end{aligned}$$

9.4 Numerical Solution

For Y as in (9.3), the optimization problem we aim to solve is

$$\min_{k_0, k_1, \ldots, k_n} \left(\mathbf{E}[Y] + \lambda \mathbf{V}[Y]\right)$$

subject to

$$\sum_{j=0}^{n} k_j = K.$$

We solve the problem using *fmincon* in the **Matlab**.

Example 9.4.1. For this example let

$$n = 2; \quad K = 1000; \quad \alpha = 0.1; \quad \beta = 0.5; \quad \lambda = 0.4; \quad \tau = 1;$$
$$\delta = f'(S_0) = 0.5; \quad \gamma = f'(S_0) = 0.2; \quad \sigma = 0.5,$$

the optimal trading strategy is

$$k_0 = 333.3348, \quad k_1 = 333.3336, \quad k_2 = 333.3316,$$

and the optimal objective function is 5.5736×10^8.

Remark 9.4.1. The trading trajectory has a downward trend. Intuitively, and on contrast to executing a large size at a single transaction, our result suggests to split the overall position in almost even trades. The linear assumption that we made on the temporary and the permanent impacts seems to explain the almost equal execution quantities.

9.5 Concluding Remarks

In this work, we study the Markowitz-optimal execution trajectory of contingent claims. In order to do so, we use a second-order Taylor approximation with respect to the contingent claim C evaluated at the initial value of the underlying S. We obtain the closed form objective function given a risk averse criterion. Our approach allows us to obtain the explicit numerical solution and we provide an example.

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