Chapter 17
A Direct Approach to the Solution of Optimal Multiple-Stopping Problems*

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17.1 Introduction

With the deregulation of the energy markets in the United States, options to purchase electricity for a preset price have become an important risk-management tool; many of these options allow the holder the opportunity to exercise it each day during the contract period. In the world of water usage, rather than negotiate permanent sales of water rights, owners negotiate contracts in which the other party may divert a certain amount of water for other usage (such as from agricultural to urban), and these contracts often allow more than one diversion. Some employee compensation packages include stock options with the possibility of a number of reloads before expiration. A common feature of these various contracts is the opportunity for a decision-maker to act a finite number of times and receive some reward for each action. Rather than tie our presentation to a particular application, we examine a general formulation.

This chapter considers a broad class of optimal multiple-stopping problems, a natural extension to optimal (single-) stopping problems. Though the extension seems natural, there are nevertheless significant challenges to determining the value and optimal stopping policies. Our objective is to demonstrate a tractable method of

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*It is with great pleasure that we contribute a paper to this Festschrift in honor of Onésimo Hernández-Lerma's 65th birthday. He has made many contributions to the stochastic control of Markov processes literature; our interests have many intersections with his work. This contribution honors his career at this important milestone and is dedicated to him.

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D. Hernández-Hernández and A. Minjárez-Sosa (eds.), *Optimization, Control, and Applications of Stochastic Systems*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-8337-5_17,

solution for models in which the distribution of the process is known. To establish the problem, we assume X is a solution of the stochastic differential equation

$$\mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t), \qquad X(0) = x \tag{17.1}$$

in which W is a standard Brownian motion process and the drift and diffusion coefficients are such that X takes values in an interval $(x_\ell, x_r) \subset \mathbb{R}$. The decision-maker may select up to N times (with N fixed) at which to receive a reward. However, after each decision time, a lag of at least $\delta > 0$ units of time (the refraction period) must pass before the next decision to receive a reward is made; this time lag increases the complexity of the problem. We assume the time horizon is $T = \infty$; that is, there is no imposed limit on the time by which decisions must be made. Let $\{\tau_n : n = 1, \ldots, N\}$ denote the decision times. Throughout this chapter, the subscript will denote the number of remaining decisions, so τ_1 is the last decision and τ_N is the first. Note that for each $i = 1, \ldots, N-1$, $\tau_{i+1} < \tau_{i+1} + \delta \leq \tau_i$ on the set where τ_{i+1} is finite. For $i = 1, \ldots, N$, let $R_i : (x_\ell, x_r) \to \mathbb{R}$ denote the payoff function for the ith last decision. Letting $\alpha > 0$ denote the discount rate, the objective is to maximize the expected payoff

$$\sum_{i=1}^{N} \mathbb{E}\left[\mathrm{e}^{-\alpha\tau_i} I_{\{\tau_i < \infty\}} R_i(X(\tau_i))\right] \tag{17.2}$$

over all decision times $\tau_1, \ldots, \tau_N$ satisfying the refraction period condition.

As indicated in the first paragraph, recent interest in multiple-stopping problems has developed due to deregulation and new types of options, though multiple-stopping problems have previously been studied in sequential analysis (see, e.g., Haggstrom [7]). Villinski [14] discusses contracts involving multiple decisions for water rights from an economic point of view and describes a dynamic programming formulation for the valuation of these contracts. From a more mathematical point of view, Thompson [13] examines a discrete-time binomial tree model for the evolution of the process and concentrates on developing a Monte Carlo method to value a path-dependent contingent claim. Zeghal and Mnif [15] consider the valuation of swing options for Lévy models using Snell envelopes and illustrates this approach using Monte Carlo techniques on a put option having a maturity time of 1. Carmona and Touzi [4] analyze the valuation of a perpetual put swing option with infinitely many exercises in a continuous-time Black-Scholes market. The paper independently develops a theoretical foundation to the solution using Snell envelopes and obtains exercise rules by discrete approximation. The paper by Carmona and Dayanik [3] examines the same type of problem for a more general one-dimensional diffusion model having a more general reward function and determines the solution using a generalized convex function approach. Dai and Kwok [5] examine the pricing of reload and shout options in which the refraction period models the time until the employee is vested. The paper uses a Black-Scholes model having continuous dividend rate, approaches the solution using a variational inequality which is then approximately solved using a binomial tree model and dynamic programming. Interestingly, the authors relate the reload option to a lookback feature of the stock price process. Aleksandrov and Hambly [1] use a dual approach to analyze multiple exercise options under constraints, though the formulation allows multiple exercises at the same time (no refraction period). The authors solve the problem by considering the marginal value of one additional exercise time. Kobylanski and Quenez [12] discuss the general theory of multiple-stopping time problems using Snell envelopes.

This paper seeks a numerically tractable approach to the solution of multiple-stopping problems. It considers the same model and general reward as Carmona and Dayanik [3] though it approaches the analysis of the problem using a quite different method. As in several of the aforementioned papers, the multiple-stopping problem is reduced to an iterated sequence of N single-stopping problems through a conditioning argument. This paper then utilizes the results in Helmes and Stockbridge [8] to characterize the value of each single-stopping problem in two ways. This characterization enables the value function for each single-stopping problem to be determined in closed form for many payoff functions. We also employ the argument in Helmes and Stockbridge [9] in which we first obtain an upper bound on the value and then identify a stopping rule which achieves the bound. The problem formulation in terms of stochastic processes is given in Sect. 17.2 along with the reduction to the sequence of single-stopping problems. Section 17.3 then summarizes the approaches to determining the value function of Helmes and Stockbridge [8, 9]. The tractability of this method is then illustrated in Sect. 17.4.

The current paper is similar to Helmes and Stockbridge [10] in that both papers consider a finite number of decision times at which a reward is earned and analyze a sequence of single-stopping problems by solving nonlinear optimization problems. The significant difference is the requirement in this paper that successive decisions to stop must wait at least the length of the refraction period. The time lag increases the complexity of the analysis in a nontrivial way.

17.2 Problem Formulation

We begin with a precise formulation of the class of multiple-stopping problems examined in this chapter. We assume the coefficients μ and σ of (17.1) are continuous and are such that X takes values in some interval $(x_\ell, x_r) \subseteq \mathbb{R}$. The process X has generator A given by $Af(x) = (1/2)\sigma^2(x)f''(x) + \mu(x)f'(x))$ operating on $f \in C^2(x_\ell, x_r)$ (see [2, II.9, p. 17] for sufficient conditions).Further assume X is a weak solution of (17.1) while $X(t) \in (x_\ell, x_r)$ (see Ethier and Kurtz [6, Sect. 5.3, p. 291] for details) and that the solution to (17.1) is unique in distribution. This existence and uniqueness imply that the martingale problem for A is well posed and hence that X is a strong Markov process (see [6, Theorem 4.4.2, p. 184]). We denote the filtration for the weak solution by $\{\mathscr{F}_t\}$. Throughout this chapter we assume $x_\ell < x < x_r$. We emphasize that x will always represent the initial position for the multiple-stopping problem in this chapter.

A key additional assumption on the coefficients is required, which we separate out for later reference.

Condition 17.2.1. *The eigenvalue problem $Af(\cdot) = \alpha f(\cdot)$ has both a positive, strictly decreasing solution ϕ and a nonnegative, strictly increasing solution ψ.*

The conditions assumed in this paper are sufficient to imply Condition 17.2.1 (see Borodin and Salminen [2, II.10, p. 18,19]). The functions ϕ and ψ depend on the discount factor α; since we assume the discount factor is fixed, we omit this dependence from the notation.

Before proceeding further, we briefly digress to consider the boundary points. We restrict the models to those for which x_ℓ is either an entrance-not-exit boundary point or a natural boundary point [2, II.10, p. 14–19]. The analysis also applies when x_ℓ is an exit boundary point, but the expressions are slightly more complicated, so we have chosen this restriction on the type of boundary point for clarity of presentation. When x_ℓ is either an entrance or natural boundary, X will almost surely never reach x_ℓ in finite time.The distinction between entrance and natural boundaries is that the process will immediately enter the interval (x_ℓ, x_r) when $x = x_\ell$ is an entrance point (we assume $x > x_\ell$), after which it will never return to the boundary, and thus x_ℓ is in the state space of the process. This behavior does not happen with a natural boundary point so such an x_ℓ will not be in the state space of X. We place the same restrictions on the model for x_r. In the event either $x_\ell = -\infty$ or $x_r = \infty$, we require these to be natural boundary points with the implication that the process X will not "explode to ∞ or $-\infty$" in finite time.

The importance of the type of boundary points for this chapter is the properties that $\psi(x_\ell) \geq 0$ and $\phi(x_\ell+) = \infty$ [2, pp. 14–19]. When $x_\ell = -\infty$, the natural boundary point assumption implies $\phi(-\infty) = \infty$ and $\psi(-\infty) = 0$. Symmetric properties hold for x_r with the roles of ϕ and ψ reversed.

The reward earned by the decision-maker is the sum of the expected discounted payoffs at each decision time given in (17.2). Denote the optimal value by $V^{(N)}(x)$, in which the superscript indicates the number of decisions. We assume that for each $i = 1, \ldots, N$, the reward function $R_i : (x_\ell, x_r) \mapsto \mathbb{R}$ is upper-semicontinuous, is positive for some $y \in (x_\ell, x_r)$, and satisfies

$$\lim_{y \searrow x_\ell} \frac{R_i(y)}{\phi(y)} = 0, \quad \text{and} \lim_{y \nearrow x_r} \frac{R_i(y)}{\psi(y)} = 0. \tag{17.3}$$

We further assume that $\tau_1, \ldots, \tau_N$ are $\{\mathscr{F}_t\}$-stopping times satisfying $0 \leq \tau_N$, and for each $i = 1, \ldots, N-1$, on the set $\{\tau_{i+1} < \infty\}$ the stopping times satisfy $\tau_{i+1} < \tau_{i+1} + \delta \leq \tau_i$. Let $\mathscr{A}_N$ denote the set of these N-tuples of stopping times. Since the multiple-stopping problem will be reduced to a sequence of single-stopping problems, it will be beneficial to denote the set of nonnegative (single-) stopping times by $\mathscr{A}_1$, in which the subscript denotes that the set consists of stopping times and not N-tuples of stopping times.

We now present the key conditioning argument which reduces (17.2) to a sequence of single-stopping problems. The argument uses the strong Markov property so it is helpful to designate the expectation relative to the initial position of the process X using a subscript. It is necessary to develop some additional notation.

Set $\tilde{V}_0^{(1)} \equiv 0$ and define the "modified" payoff function $\tilde{R}_1 = R_1 = R_1 + \tilde{V}_0^{(1)}$ for the reward received upon making the final decision. Define the corresponding value function $V_1^{(1)}$ by

$$V_1^{(1)}(y) = \sup_{\tau \in \mathscr{A}_1} \mathbb{E}_y[\mathrm{e}^{-\alpha\tau} I_{\{\tau<\infty\}} \tilde{R}_1(X(\tau))], \qquad y \in (x_\ell, x_r).$$

Proceeding recursively, for $i = 2, \ldots, N$ and $y \in (x_\ell, x_r)$, define $\tilde{V}_i^{(1)}(y) = \mathbb{E}_y[\mathrm{e}^{-\alpha\delta} V_i^{(1)}(X(\delta))]$, the modified payoff function $\tilde{R}_i = R_i + \tilde{V}_{i-1}^{(1)}$ and

$$V_i^{(1)}(y) = \sup_{\tau \in \mathscr{A}_1} \mathbb{E}_y[\mathrm{e}^{-\alpha\tau} I_{\{\tau<\infty\}} \tilde{R}_i(X(\tau))]. \tag{17.4}$$

Theorem 17.2.1. *The value of optimal multiple-stopping problem of maximizing* (17.2) *over decision times* $(\tau_1, \ldots, \tau_N) \in \mathscr{A}_N$ *at which to stop the process* X *satisfying* (17.1) *is obtained through recursion by solving the* N *single-stopping problems; that is,* $V^{(N)}(x) = V_N^{(1)}(x)$.

Proof. Consider a single generic term of the form

$$\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_i} I_{\{\tau_i<\infty\}} g(X(\tau_i))\right],$$

in which g is some measurable function such that the integrand is integrable and τ_i is one of the stopping times in an N-tuple $(\tau_1, \ldots, \tau_N) \in \mathscr{A}_N$ in which $i \in \{1, \ldots, N-1\}$. On the set $\{\tau_{i+1} < \infty\}$, notice that $\tau_i \geq \tau_{i+1} + \delta$ so we can define $\tilde{\tau}_i = \tau_i - \tau_{i+1} - \delta$ and have $\tilde{\tau}_i \in \mathscr{A}_1$, where the stopping times are relative to the filtration $\{\mathscr{G}_t\} = \left\{\mathscr{F}_{\tau_{i+1}+t}\right\}$. Using the strong Markov property of X in the third equality below yields

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_i} I_{\{\tau_i<\infty\}} g(X(\tau_i))\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_i} I_{\{\tau_i<\infty\}} g(X(\tau_i)) \middle| \mathscr{F}_{\tau_{i+1}+\delta}\right]\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-\alpha(\tau_{i+1}+\delta)} I_{\{\tau_{i+1}<\infty\}} \mathbb{E}_x\left[\mathrm{e}^{-\alpha\tilde{\tau}_i} I_{\{\tilde{\tau}_i<\infty\}} g(X(\tau_{i+1}+\delta+\tilde{\tau}_i)) \middle| \mathscr{F}_{\tau_{i+1}+\delta}\right]\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-\alpha(\tau_{i+1}+\delta)} I_{\{\tau_{i+1}<\infty\}} \mathbb{E}_{X(\tau_{i+1}+\delta)}\left[\mathrm{e}^{-\alpha\tilde{\tau}_i} I_{\{\tilde{\tau}_i<\infty\}} g(X(\tilde{\tau}_i))\right]\right].
\end{aligned}$$

The key to the tractability of the problem lies in a second conditioning argument. Observe that for any integrable random variable Y,

$$\mathbb{E}_x\left[\mathbb{E}_{X(\tau_{i+1}+\delta)}[Y]\right] = \mathbb{E}_x\left[\mathbb{E}_x[\mathbb{E}_{X(\tau_{i+1}+\delta)}[Y] | \mathscr{F}_{\tau_{i+1}}]\right] = \mathbb{E}_x\left[\mathbb{E}_{X(\tau_{i+1})}[\mathbb{E}_{X(\delta)}[Y]]\right]$$

and thus

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_i}I_{\{\tau_i<\infty\}}g(X(\tau_i))\right]\\&\quad=\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_{i+1}}I_{\{\tau_{i+1}<\infty\}}\mathbb{E}_{X(\tau_{i+1})}\left[\mathrm{e}^{-\alpha\delta}\mathbb{E}_{X(\delta)}\left[\mathrm{e}^{-\alpha\tilde{\tau}_i}I_{\{\tilde{\tau}_i<\infty\}}g(X(\tilde{\tau}_i))\right]\right]\right].\end{aligned}$$

Now specify $i=1$ and $g=R_1$. For $i=1,\dots,N$, define the set $\mathscr{A}_{N,i}=\{\tau_i:(\tau_1,\dots,\tau_N)\in\mathscr{A}_N\}$. Taking the supremum over $\tau_1\in\mathscr{A}_{N,1}$ of the left-hand side and then over $\tau_2\in\mathscr{A}_{N,2}$ and $\tilde{\tau}_1\in\mathscr{A}_1$ on the right-hand side produces one inequality, whereas taking the suprema in the opposite order yields the opposite inequality and hence

$$\begin{aligned}&\sup_{\tau_1\in\mathscr{A}_{N,1}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_1}I_{\{\tau_1<\infty\}}R_1(X(\tau_1))\right]\\&\quad=\sup_{\tau_2\in\mathscr{A}_{N,2}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_2}I_{\{\tau_2<\infty\}}\mathbb{E}_{X(\tau_2)}\left[\mathrm{e}^{-\alpha\delta}V_1^{(1)}(X(\delta))\right]\right]\\&\quad=\sup_{\tau_2\in\mathscr{A}_{N,2}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_2}I_{\{\tau_2<\infty\}}\tilde{V}_1^{(1)}(X(\tau_2))\right].\end{aligned}\tag{17.5}$$

An important reduction occurs when we consider the two successive terms of (17.2) involving τ_1 and τ_2. Observe

$$\begin{aligned}&\sup_{\substack{\tau_1\in\mathscr{A}_{N,1},\\ \tau_2\in\mathscr{A}_{N,2}}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_2}I_{\{\tau_2<\infty\}}R_2(X(\tau_2))+\mathrm{e}^{-\alpha\tau_1}I_{\{\tau_1<\infty\}}R_1(X(\tau_1))\right]\\&\quad=\sup_{\tau_2\in\mathscr{A}_{N,2}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_2}I_{\{\tau_2<\infty\}}\left(R_2(X(\tau_2))+\tilde{V}_1(X(\tau_2))\right)\right]\\&\quad=\sup_{\tau_2\in\mathscr{A}_{N,2}}\mathbb{E}_x\left[\mathrm{e}^{-\alpha\tau_2}I_{\{\tau_2<\infty\}}\tilde{R}_2(X(\tau_2))\right]\end{aligned}$$

in which we recall $\tilde{R}_2(y)=R_2(y)+\tilde{V}_1(y)$. Using induction, we obtain

$$\begin{aligned}V^{(N)}(x)&=\sup_{(\tau_1,\dots,\tau_N)\in\mathscr{A}_N}\sum_{i=1}^{N}\mathbb{E}_x[\mathrm{e}^{-\alpha\tau_i}I_{\{\tau_i<\infty\}}R_i(X(\tau_i))]\\&=\sup_{\tau_N\in\mathscr{A}_{N,N}}\mathbb{E}_x[\mathrm{e}^{-\alpha\tau_N}I_{\{\tau_N<\infty\}}\tilde{R}_N(X(\tau_N))]\\&=V_N^{(1)}(x).\end{aligned}$$

□

The implications of Theorem 17.2.1 is that the N-stopping problem can be solved using an iteration of three steps. First, obtain the value V_i for the successor stopping time as a function of the initial position y; that is, determine the successor value function. Next, find the expected discounted value (discounted by the refraction time δ) of this function evaluated at the new position $X(\delta)$ of the process. Finally, add this function to the predecessor (more decisions to make) payoff function R_{i+1} to form a new payoff function for the predecessor stopping problem, leading again to a single-stopping problem.

Thus, the main tasks to solve the multiple-stopping problem are to determine the sequence of single-step value functions and to utilize the distribution of $X(\delta)$, parametrized by an arbitrary initial position $y \in (x_\ell, x_r)$.

17.3 Solution Approaches for Single-Stopping Problems

The single-stopping problem seeks to maximize

$$J(\tau;x) := \mathbb{E}_x[\mathrm{e}^{-\alpha\tau} I_{\{\tau<\infty\}} R(X(\tau))] \tag{17.6}$$

over the set of all $\{\mathscr{F}_t\}$-stopping times τ in which X is a weak solution of (17.1). Let $\mathscr{A}$ denote this set of stopping times and define $V(x) = \sup_{\tau\in\mathscr{A}} J(\tau;x)$. This section briefly states the line of reasoning in Helmes and Stockbridge [9] and then recalls the results in Helmes and Stockbridge [8]. The first method of solution identifies an upper bound on the value with the goal of identifying a stopping time that achieves this value. The second approach involves maximizing the expected reward over all two-point stopping rules, whereas the final technique utilizes duality theory. We wish to emphasize that the optimal stopping problem is solved for a single initial value x, rather than seeking the value function, though the structure of the values is such that the value function can often be determined.

17.3.1 Linear Programming Imbedding

A common imbedding of the stochastic problem underlies these methods. We briefly describe the derivation of the linear program and then, in the next sections, utilize this in two related ways. Applying Itô's formula to $\mathrm{e}^{-\alpha t} f(X(t))$ for $f \in C_c^2(x_\ell, x_r)$ yields

$$\mathrm{e}^{-\alpha t} f(X(t)) = f(x) + \int_0^t \mathrm{e}^{-\alpha s}[Af - \alpha f](X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\alpha s} f'(X(s))\,\mathrm{d}W(s).$$

For any $\tau \in \mathscr{A}$, the optional sampling theorem indicates that

$$\mathrm{e}^{-\alpha(t\wedge\tau)} f(X(t\wedge\tau)) - f(x) - \int_0^{t\wedge\tau} \mathrm{e}^{-\alpha s}[Af - \alpha f](X(s))\,\mathrm{d}s$$

is a mean 0 martingale, so taking expectations then letting $t \to \infty$ establishes Dynkin's formula

$$\mathbb{E}[e^{-\alpha\tau} I_{\{\tau<\infty\}} f(X(\tau))] - \mathbb{E}\left[\int_0^\tau e^{-\alpha s}[Af - \alpha f](X(s))\,ds\right] = f(x). \tag{17.7}$$

Defining ν_τ to be the discounted (stopping) distribution of $X(\tau)$ and μ_0 to be the expected, discounted occupation measure of X over the interval $[0,\tau]$, (17.7) can be written as $\int f\,d\nu_\tau - \int[Af - \alpha f]\,d\mu_0 = f(x)$ and the single-stopping objective function (17.6) becomes $\int R\,d\nu_\tau$. The optimal stopping problem is therefore imbedded in the infinite-dimensional linear program

$$\begin{cases} \text{Maximize} & \int R\,d\nu_\tau \\ \text{Subject to} & \int f\,d\nu_\tau - \int [Af - \alpha f]\,d\mu_0 = f(x), \quad \forall f \in C_c^2(x_\ell, x_r). \end{cases} \tag{17.8}$$

We note that the variables in this linear program are the measures ν_τ and μ_0 and that ν_τ arises from the stopping time τ so is the decision variable.

17.3.2 Achieving an Upper Bound

A first auxiliary linear program is obtained by limiting the constraints to a single test function. One implication is that the feasible set of measures may be larger and hence the value of the auxiliary problem gives an upper bound for (17.8). We may take $f = \psi$ in (17.8) (see [9] for details justifying the use of ψ as a test function since $\psi \notin C_c^2(x_\ell, x_r)$). The benefit of this choice is that $A\psi - \alpha\psi \equiv 0$ so the integral with respect to the occupation measure μ_0 drops from the constraints. Notice the constraint can be written as

$$\int \psi/\psi(x)\,d\nu_\tau = 1,$$

so the integrand forms the density for a probability measure $\tilde{\nu}_\tau$ on (x_ℓ, x_r).

Proposition 17.3.1. *Assume X is a weak solution of* (17.1) *and Condition 17.2.1 is satisfied. Let R satisfy the conditions in Sect. 17.2. Then*

$$V(x) \le \sup_{y\in(x_\ell,x_r)} \frac{R(y)}{\psi(y)} \cdot \psi(x).$$

In addition, if $\lim_{y\searrow x_\ell} R(y)/\psi(y) = 0$, then there exists a maximizer y^* and τ_{y^*} is an optimal stopping rule when $x \le y^*$.

Proof. Examining the objective function, we have

$$\int R(y)\,\nu_\tau(dy) = \int [R(y)\psi(x)/\psi(y)]\,\tilde{\nu}_\tau(dy) \le \sup_{y\in(x_\ell,x_r)} (R(y)/\psi(y)) \cdot \psi(x).$$

The conditions on R imply the existence of a maximizer y^* of $R(y)/\psi(y)$. It is well known (see [2]) that $\mathbb{E}_x[\mathrm{e}^{-\alpha\tau_{y^*}}] = \psi(x)/\psi(y^*)$ when $x \le y^*$, so this stopping rule achieves the upper bound. □

It will be helpful to note that when R is differentiable, an interior optimizer for the function $R(y)/\psi(y)$ occurs where $\psi(y)R'(y) - \psi'(y)R(y) = 0$. This necessary optimality condition implies the elasticities of the function ψ, and the payoff function R must be the same at an optimizing level.

17.3.3 Maximization Over Two-Point Hitting Rules

The previous approach is sufficient when the structure of the problem is such that stopping to the right of the initial position is optimal. A second auxiliary problem provides a general solution and is also obtained from (17.8), this time by limiting the test functions to the pair ϕ and ψ.

Consider points a and b such that $x_\ell < a \le x \le b < x_r$ but $a < b$. Define $\tau_a = \inf\{t \ge 0 : X(t) = a\}$ and τ_b similarly. Define $\tau_{a,b} = \tau_a \wedge \tau_b$. The payoff associated with the decision rule $\tau_{a,b}$ is

$$\begin{aligned} J(\tau_{a,b};x) &= R(a)\cdot\tfrac{\phi(x)\,\psi(b)-\phi(b)\,\psi(x)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)} + R(b)\cdot\tfrac{\phi(a)\,\psi(x)-\phi(x)\,\psi(a)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)} \\ &= \tfrac{R(a)\psi(b)-R(b)\psi(a)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)}\cdot\phi(x) + \tfrac{R(b)\phi(a)-R(a)\phi(b)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)}\cdot\psi(x). \end{aligned} \tag{17.9}$$

Several observations are helpful. First, the fractional terms in the first expressions of (17.9) are the masses of $\nu_{\tau_{a,b}}$. Next, when $x = a$, the expression for $J(\tau_{a,b};x) = R(a)$ and similarly for $x = b$. This agrees with one's intuition that stopping occurs immediately resulting in a non-discounted payoff. Also when holding $b > x$ fixed and letting $a \to x_\ell$, the fractional terms in the first expression converge to 0 and $\psi(x)/\psi(b) = E[\mathrm{e}^{-\alpha\tau_b}]$, respectively, and hence $J(\tau_{a,b};x) \to J(\tau_b;x)$. Similarly, when $b \to x_r$ with a fixed, $J(\tau_{a,b};x) \to J(\tau_a;x)$. Finally, by examining the second expression of (17.9), one observes that, as a function of x, the value of $J(\tau_{a,b};x)$ is continuous on (a,b).

Proposition 17.3.2. *Assume X is a weak solution of* (17.1) *and Condition 17.2.1 is satisfied. Let R satisfy the conditions in Sect. 17.2. Then*

$$V(x) = \sup_{a\le x\le b} J(\tau_{a,b};x).$$

Moreover, there exist $a^*, b^* \in [x_\ell, x_r]$ such that $J(\tau_{a^*,b^*};x) = V(x)$; that is, τ_{a^*,b^*} is an optimal stopping rule.

When $a* = x_\ell$, the two-point hitting rule is actually a one-point hitting rule at b^* and hence τ_{b^*} is an optimal stopping time. Similar comments apply when

$b^* = x_r$. We observe that it will never occur that both $a^* = x_\ell$ and $b^* = x_r$ since, by assumption on the model, the process will never hit either x_ℓ or x_r so the "stopping" time $\tau_{x_\ell,x_r} = \infty$ a.s. and the value is 0, but a positive value can be obtained by choosing to stop at a point where R is strictly positive.

17.3.4 Minimization of α-Harmonic Functions

As indicated previously, to establish the optimality of a two-point hitting rule in [8], the stochastic problem is imbedded in an infinite-dimensional linear program, and an upper bound is obtained by restricting the constraints (and increasing the feasible set). A dual linear program to this auxiliary linear program is also derived for which it is easy to prove a weak duality result between the values of the linear programs, with more involved arguments establishing strong duality [8]. As a result, the optimal value can be obtained by solving the following two-dimensional linear program:

$$\begin{cases} \text{Minimize} & c_1\phi(x) + c_2\psi(x) \\ & c_1\phi(y) + c_2\psi(y) \geq R(y),\ \forall y \in (x_l, x_r), \\ \text{Subject to} & c_1, c_2 \text{ unrestricted.} \end{cases} \tag{17.10}$$

We note that this problem involves minimizing a linear combination of the functions ϕ and ψ of Condition 17.2.1 evaluated at the initial position of the process. To be feasible, this linear combination is required to majorize the payoff function R.

A further observation will be helpful. As in Proposition 17.3.2, take a^* and b^* to be maximizers. Section 4.3 of [8] proves that when the payoff function R is continuously differentiable in a neighborhood of a^* and in a neighborhood of b^*, then these are points which satisfy the principle of smooth pasting; namely,

$$\begin{cases} c_1\phi(a) + c_2\psi(a) = R(a) \\ c_1\phi'(a) + c_2\psi'(a) = R'(a) \end{cases} \quad \text{and} \quad \begin{cases} c_1\phi(b) + c_2\psi(b) = R(b) \\ c_1\phi'(b) + c_2\psi'(b) = R'(b). \end{cases} \tag{17.11}$$

To obtain this result, one analyzes the maximization over two-point stopping rules and shows how to optimally select c_1 and c_2. Notice there are four equations in the four variables a, b, c_1, and c_2.

It will be helpful to consider a particular case of smooth pasting more extensively. Consider the situation in which it is optimal to stop immediately at the initial time; this means that the smooth pasting conditions must be satisfied when $a = x$. In this case, the coefficients c_1 and c_2 are easily determined to be

$$c_1 = \frac{\psi'(x)R(x) - \psi(x)R'(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \ \text{and}\ c_2 = \frac{\phi(x)R'(x) - \phi'(x)R(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)}. \tag{17.12}$$

Notice, in particular, that c_2 is always positive when R is positive and increasing and similarly that c_1 is always positive when R is positive and decreasing. Considering further the case that R is positive and increasing, observe that the denominator of c_1 in (17.12) is always positive, so c_1 is positive when $\psi'(x)R(x) - \psi(x)R'(x) > 0$, and equals 0 when the same elasticity condition as in Sect. 17.3.2 is satisfied. Moreover, comparing the numerator of c_1 in (17.12) with the numerator of $(R/\psi)'$, we see that c_1 will be positive whenever R/ψ is strictly decreasing. A similar comment holds for c_2 when R is positive and decreasing by analyzing $(R/\phi)'$.

Finally, recall that the optimal stopping problem is solved for a single initial value x, rather than seeking the value function. But the structure of this approach typically determines the value for initial positions in regions, and hence the value function can be typically obtained through a limited number of optimizations. In fact, to determine the value function, it is often easiest to use different methods for x in different regions.

17.4 Drifted Brownian Motion

The process X satisfies $\mathrm{d}X(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)$; that is, $X(t) = x + \mu t + \sigma W(t)$, in which $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ and the process takes values in $\mathbb{R}$. It is easily verified that $\phi(y) = \mathrm{e}^{\gamma_1 y}$ and $\psi(y) = \mathrm{e}^{\gamma_2 y}$, where $\gamma_1 = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\alpha}{\sigma^2}}$ and $\gamma_2 = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\alpha}{\sigma^2}}$. We note that $\gamma_1 < 0 < \gamma_2$ and that these values are in fact the roots of the quadratic equation $(\sigma^2/2)y^2 + \mu y - \alpha = 0$.

We consider a triple-stopping problem so assume $N = 3$ and we take $R_i(y) = y^+$ for $i = 1, 2, 3$; recall throughout the paper, the subscript denotes the number of decisions that remain to be made. Proceeding in a recursive manner with the final stopping decision, we must first determine the value function $V_1^{(1)}(x)$.

Consider first the minimization approach to determining the value of this last stopping problem. To be feasible, the α-harmonic function $c_1\phi + c_2\psi$ must lie above the payoff function $R_1(y) = y^+$. For $y \neq 0$, R_1 is differentiable, and hence we can apply the smooth pasting argument. Since both ϕ and ψ are strictly positive functions, $R_1(x) = 0$ for $x < 0$, and it is not possible to find a linear combination $c_1\phi + c_2\psi$ which has $c_1\phi(x) + c_2\psi(x) = 0$ and majorizes R_1. Thus, the optimal value is not 0, a fact that also follows directly from the observation that using a stopping rule of τ_{y_0}, where $R_1(y_0) > 0$, yields a strictly positive value.

We next investigate whether it is possible to have a feasible α-harmonic function that equals the payoff function at x when $x > 0$. The coefficients c_1 and c_2 must satisfy

$$\begin{cases} \mathrm{e}^{\gamma_1 x} c_1 + \mathrm{e}^{\gamma_2 x} c_2 = x, \\ \gamma_1 \mathrm{e}^{\gamma_1 x} c_1 + \gamma_2 \mathrm{e}^{\gamma_2 x} c_2 = 1. \end{cases}$$

The solution to this linear system is

$$c_1 = \frac{\gamma_2 x - 1}{(\gamma_2 - \gamma_1)\mathrm{e}^{\gamma_1 x}} \qquad \text{and} \qquad c_2 = \frac{1 - \gamma_1 x}{(\gamma_2 - \gamma_1)\mathrm{e}^{\gamma_2 x}}\,. \tag{17.13}$$

Since $R_1 \geq 0$ and $\phi(y) = \mathrm{e}^{\gamma_1 y}$ and $\psi(y) = \mathrm{e}^{\gamma_2 y}$ are both positive functions, to be feasible the coefficients c_1 and c_2 must both be nonnegative. The coefficient c_2 is always positive since $\gamma_1 < 0$ and $x > 0$. The coefficient c_1, however, is only nonnegative when the initial value satisfies $x \geq 1/\gamma_2$. Thus, for x in this range, the value of the single-optimal stopping problem is $V_1^{(1)}(x) = x$, and an optimal stopping rule is to stop immediately, $\tilde{\tau}_1^* = \tau_x$.

Now consider an initial position x with $x < 1/\gamma_2$. First we note that the function $c_1^*\phi(y) + c_2^*\psi(y) := (\gamma_2 e)^{-1}\mathrm{e}^{\gamma_2 y}$, which is obtained using the coefficients (17.13) with $x = 1/\gamma_2$, is feasible for the minimization problem. Now consider $c_1, c_2 > 0$ such that $c_1\phi(x) + c_2\psi(x) < c_2^*\psi(x)$. Simple algebra demonstrates that $c_1\phi(x)/\psi(x) < c_2^* - c_2$ and hence $c_2^* - c_2 > 0$. Moreover, the inequality can be rearranged to show $c_1 < (c_2^* - c_2)\psi(x)/\phi(x)$, and thus evaluating the new α-harmonic function at $1/\gamma_2$, we have

$$\begin{aligned}
c_1\phi(1/\gamma_2) + c_2\psi(1/\gamma_2) &< (c_2^* - c_2)\frac{\psi(x)\phi(1/\gamma_2)}{\phi(x)} + c_2\psi(1/\gamma_2) \\
&= (c_2^* - c_2)\left[\tfrac{\phi(1/\gamma_2)}{\phi(x)}\,\psi(x) - \psi(1/\gamma_2)\right] + c_2^*\psi(1/\gamma_2) \\
&< R_1(1/\gamma_2);
\end{aligned}$$

the final inequality follows from the facts that ϕ is strictly decreasing and ψ is strictly increasing along with the observation that $c_2^*\psi(1/\gamma_2) = R_1(1/\gamma_2)$. From this, we see that no linear combination with positive coefficients and $c_1\phi(x) + c_2\psi(x) < c_2^*\psi(x)$ is feasible for the minimization problem.

The above argument utilizes the minimization approach in both regions. Consider now the upper bound method of Sect. 17.3.2. Maximizing $h(y) := y^+/\psi(y)$ immediately results in a unique maximizer at $y_1^* = 1/\gamma_2$ and a corresponding upper bound of $(\gamma_2 e)^{-1}\mathrm{e}^{\gamma_2 x}$. As noted in Proposition 17.3.1, this upper bound is achieved by the stopping rule $\tau_{(1/\gamma_2)}$ when $x \leq 1/\gamma_2$.

The value function is therefore

$$V_1^{(1)}(x) = \begin{cases} (1/\gamma_2)\mathrm{e}^{\gamma_2 x - 1}, & \text{for } x \leq 1/\gamma_2, \\ x, & \text{for } x \geq 1/\gamma_2. \end{cases}$$

This value function is displayed in Fig. 17.1along with the payoff function $\tilde{R}_1(y) = y^+$ (dotted). We also display the ratio $\tilde{R}_1/\psi$; notice one is able to observe the ratio achieves its maximum at the location of the maximizer $1/\gamma_2 \approx 20.5$ and the function is strictly decreasing above this maximizer, confirming graphically that the coefficient c_1 of (17.12) will be positive and that it is optimal to stop immediately.

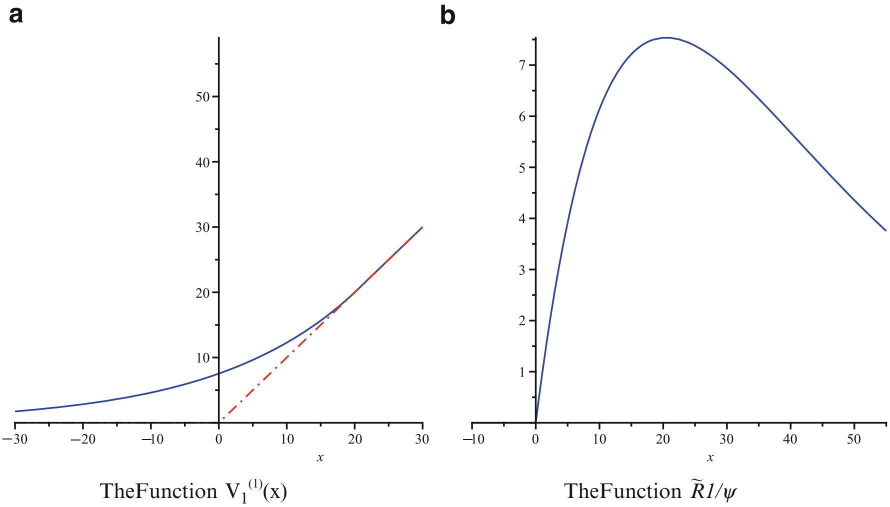

Fig. 17.1 The value function and ratio for final stopping problem; $\mu = \sigma = \delta = 1$, $\alpha = 0.05$

The next stopping decision is the point at which one must take into account the refraction period. Since we have $V_1^{(1)}$ in explicit form, we can determine the function $\tilde{V}_1^{(1)}$. Notice that, with initial position y, $X(\delta)$ is $N(y+\mu\delta, \sigma^2\delta)$-distributed. Let Φ denote the standard normal distribution function and set $\overline{\Phi} = 1 - \Phi$. Now, recalling that $(\sigma^2/2)\gamma_2^2 + \mu\gamma_2 - \alpha = 0$,

$$\begin{aligned}
\tilde{V}_1^{(1)}(y) &= \mathbb{E}_y[\mathrm{e}^{-\alpha\delta} V_3^{(1)}(X(\delta))] \\
&= \int_{-\infty}^{1/\gamma_2} (1/\gamma_2)\mathrm{e}^{\gamma_2 z - 1 - \alpha\delta} \cdot (2\pi\sigma^2\delta)^{-1/2} \mathrm{e}^{-(z-y-\mu\delta)^2/(2\sigma^2\delta)}\,\mathrm{d}z \\
&\quad + \int_{1/\gamma_2}^{\infty} z\mathrm{e}^{-\alpha\delta} \cdot (2\pi\sigma^2\delta)^{-1/2} \mathrm{e}^{-(z-y-\mu\delta)^2/(2\sigma^2\delta)}\,\mathrm{d}z \\
&= (1/\gamma_2)\mathrm{e}^{\gamma_2 y - 1}\, \Phi\left(\frac{-y-\mu\delta-\gamma_2\sigma^2\delta+\gamma_2^{-1}}{\sigma\sqrt{\delta}}\right) \\
&\quad + \mathrm{e}^{-\alpha\delta}\left(\sigma\sqrt{\delta/(2\pi)}\right)\, \mathrm{e}^{-(y+\mu\delta-\gamma_2^{-1})^2/(2\sigma^2\delta)} \\
&\quad + \mathrm{e}^{-\alpha\delta}(y+\mu\delta)\, \overline{\Phi}\left(\frac{-y-\mu\delta+\gamma_2^{-1}}{\sigma\sqrt{\delta}}\right).
\end{aligned}$$

It is easy to show that $\tilde{V}_1^{(1)}(y) \to 0$ as $y \to -\infty$ and that $\tilde{V}_1^{(1)}$ is asymptotic to the line $z = \mathrm{e}^{-\alpha\delta}(y+\mu\delta)$ as y goes to ∞.

Now recall $\tilde{R}_2(y) = R_2(y) + \tilde{V}_1^{(1)}(y)$, and the second last decision time is chosen to satisfy

$$J_2(\tau_2^*; y) = \sup_{\tau \in \mathscr{A}_1} \mathbb{E}_y \left[\mathrm{e}^{-\alpha\tau} I_{\{\tau<\infty\}} \tilde{R}_2(X(\tau)) \right].$$

The value of the modified payoff function $\tilde{R}_2(y)$ is asymptotic to $(1/(\gamma_2 e))\mathrm{e}^{\gamma_2 y}$ as $y \to -\infty$ (and hence converges to 0) and is asymptotically linear as $y \to \infty$ with asymptote $z = (1 + \mathrm{e}^{-\alpha\delta})y + \mu\delta\mathrm{e}^{-\alpha\delta}$. Examining the value $\tilde{R}_2(1/\gamma_2)$, we have

$$\frac{\tilde{R}_2(1/\gamma_2)}{\psi(1/\gamma_2)} = (1/(\gamma_2 e)) + \mathrm{e}^{-1}\tilde{V}_1^{(1)}(1/\gamma_2) > 1/(\gamma_2 e),$$

which implies the existence of some $y_2^* \in (x_\ell, x_r)$ at which $\tilde{R}_2(y)/\psi(y)$ achieves its maximum. Observe that since y_2^* is an interior maximizer, $\psi(y_2^*)\tilde{R}_2'(y_2^*) - \psi'(y_2^*)\tilde{R}_2(y_2^*) = 0$. Using the upper bound approach of Sect. 17.3.2 therefore implies that for $x \le y_2^*$, $V_2^{(1)}(x) = \mathrm{e}^{\gamma_2(x-y_2^*)}\tilde{R}_2(y_2^*)$ and an optimal stopping rule is given by $\tau_{y_2^*}$.

We believe that when $x > y_2^*$, an optimal value is obtained by stopping immediately. One way to verify this claim would be to show the existence of feasible c_1 and c_2 such that the smooth pasting conditions (17.11) are satisfied with $x = a$. Recalling the values c_1 and c_2 in (17.12), feasibility requires that $\psi'(x)\tilde{R}_2(x) - \psi(x)\tilde{R}_2'(x) > 0$ for $x > y_2^*$, and since $\psi'(x) = \gamma_2\psi(x)$, we must examine the function

$$\gamma_2\tilde{R}_2(y) - \tilde{R}_2'(y) = -\mathrm{e}^{-\alpha\delta}\Bigg[(1-\gamma_2 y)\,\mathrm{e}^{\alpha\delta} + (1-\gamma_2 y - \gamma_2\alpha\delta)\Phi\left(\frac{y+\mu\delta-\gamma_2^{-1}}{\sigma\sqrt{\delta}}\right) - \gamma_2\sigma\sqrt{\delta/(2\pi)}\,\mathrm{e}^{-(y+\mu\delta-\gamma_2^{-1})^2/(2\sigma^2\delta)}\Bigg]. \tag{17.14}$$

Note that $\gamma_2\tilde{R}(y_2^*) - \tilde{R}_2'(y_2^*) = 0$ since y_2^* is an interior maximizer. At this point, the dependence of the expression (17.14) on y is such that a general proof is not clear, so numerical tractability becomes advantageous. Figure 17.2 displays the function $\gamma_2\tilde{R}_2 - \tilde{R}_2'$ for a particular choice of parameters. Notice, in particular, for $y > y_2^*$ the values are positive and hence the value of c_1 is also positive resulting in a feasible solution to the minimization problem which has value $\tilde{R}_2(x)$. The value function $V_2^{(1)}$ is displayed in Fig. 17.2 as well for this choice of parameters.

Summarizing, the optimal value for the second last single-stopping problem is

$$V_2^{(1)}(x) = \begin{cases} \tilde{R}_2(y_2^*)\mathrm{e}^{\gamma_2(x-y_2^*)}, & x \le y_2^*, \\ \tilde{R}_2(x), & x > y_2^*. \end{cases} \tag{17.15}$$

At this point, it is clear that determining closed-form expressions for the maximizer and the value function is not possible. However, some progress can be made theoretically, and one may also continue to employ numerical and graphical techniques for particular parameters. The analysis of the third single-stopping

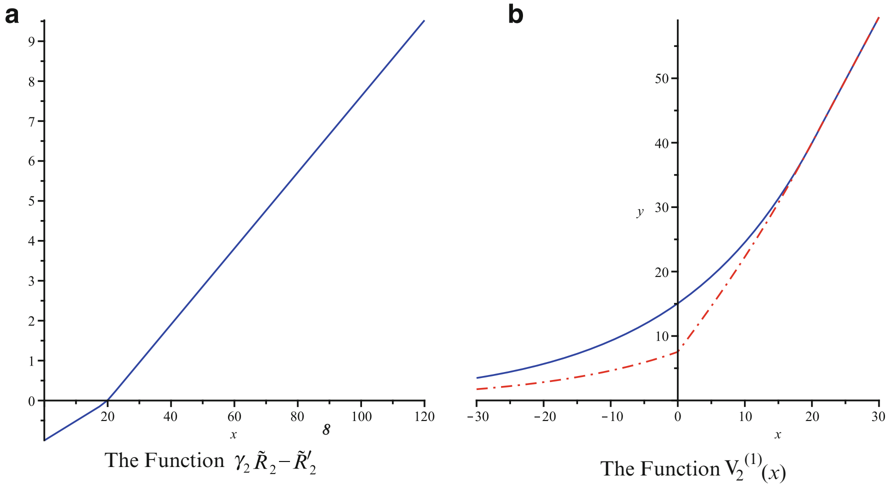

Fig. 17.2 $\gamma_2 \tilde{R}_2 - \tilde{R}_2'$ and $V_1^{(1)}$ for the second stopping problem; $\mu = \sigma = \delta = 1$, $\alpha = 0.05$

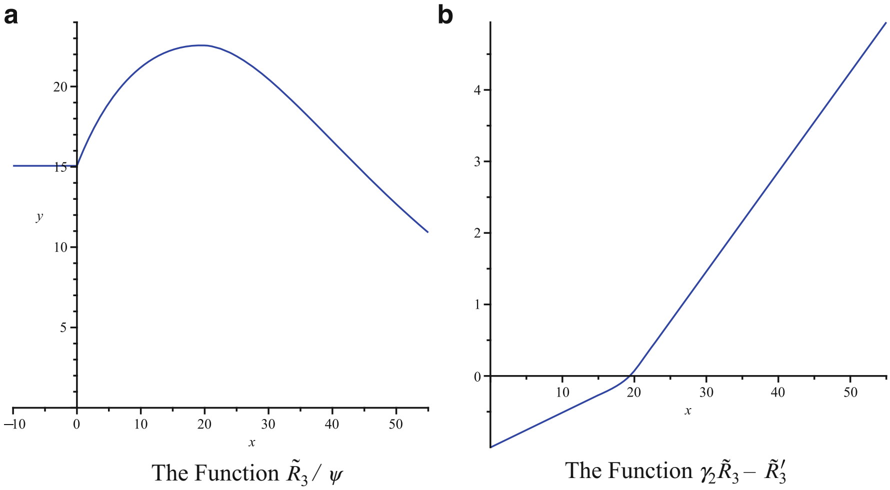

Fig. 17.3 Checking for optimality; $\mu = 1$, $\sigma = 1$, $\alpha = 0.05$

problem follows along the same line as for the second. In particular, one may show that $\tilde{R}_3$ is asymptotic to $\tilde{R}_2(y_2^*)e^{\gamma_2(y-y_2^*)}$ as $y \to -\infty$ and has a linear asymptote as $y \to \infty$. Moreover, $\tilde{R}_3(y_2^*)/\psi(y_2^*) > \tilde{R}_2(y_2^*)e^{-\gamma_2 y_2^*}$ which implies the existence of some finite y_3^* at which $\tilde{R}_3/\psi$ achieves its maximum. Therefore, the upper bound approach establishes that the value function is $\tilde{R}_3(y_3^*)e^{\gamma_2(y-y_3^*)}$ for $y \le y_3^*$. Figure 17.3 displays graphs of the ratio $\tilde{R}_3/\psi$ and the function $\gamma_2\tilde{R}_3 - \tilde{R}_3'$ to graphically verify that the form of the value function is the same as (17.15).

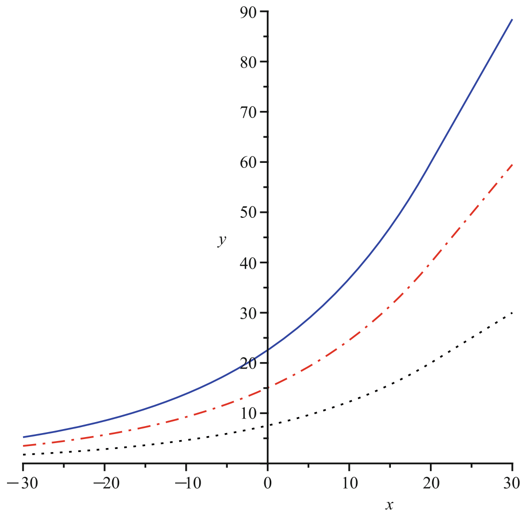

Fig. 17.4 Comparison of value functions: $V_1^{(1)}$ (*dotted*); $V_2^{(1)}$ (*dashed*); $V_3^{(1)}$ (*solid*); $\mu = 1$, $\sigma = 1$, $\alpha = 0.05$

All three value functions are displayed in Fig. 17.4 for comparison purposes. In particular, one can notice the increase in the value functions as the number of available decisions increases. Finally, we identify optimal decision times for the original triple-stopping problem:

$$\begin{aligned}
\tau_3^* &= \inf\{t \geq 0 : X(t) \in [y_3^*, \infty)\}, \\
\tau_2^* &= \inf\{t \geq \tau_3^* + \delta : X(t) \in [y_2^*, \infty)\}, \\
\tau_1^* &= \inf\{t \geq \tau_2^* + \delta : X(t) \in [y_1^*, \infty)\},
\end{aligned}$$

where the critical values of the stopping locations are $y_3^* = 19.346$, $y_3^* = 19.888$, and $y_3^* = 20.488$, when $\mu = \sigma = \delta = 1$ and $\alpha = 0.05$.

17.5 Concluding Remarks

This chapter demonstrates that multiple-stopping problems of one-dimensional diffusions in the presence of refraction periods reduce to a sequence of single-stopping problems in which the reward for an earlier action must include the optimal payoff for the subsequent action. The presence of the refraction period introduces the need to evaluate the expectation of the value function for a later action according to the distribution of the process at a time dependent on the length of the refraction period. This becomes numerically tractable when this distribution is known. Three solution approaches to the single-stopping problems are briefly discussed based on an imbedding of the original stochastic problem in an infinite-dimensional linear program; a similar linear programming approach to stochastic control of discrete-time processes has been studied by O. Hernańdez-Lerma (e.g., [11]). Tractability of these type of problems is illustrated in detail for a particular example.

Acknowledgements The research of Richard H. Stockbridge was supported in part by the U.S. National Security Agency under Grant Agreement Number H98230-09-1-0002. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein. The research of Chao Zhu was supported in part by a grant from the UWM Research Growth Initiative and under NSF grant DMS-1108782.

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