Chapter 14
Minimizing Ruin Probabilities by Reinsurance and Investment: A Markovian Decision Approach

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14.1 Introduction

Consider a classical Cramér-Lundberg model

$$X_t = x - ct - \sum_{i=1}^{N_t} Y_i, \tag{14.1}$$

where the claim number process $\{N_t\}$ is a Poisson process with intensity λ, and the claim payment $\{Y_t\}$ is a sequence of independent and identically distributed (i.i.d.) positive random variables independent of $\{N_t\}$ and with support on the positive half line. Let c be the constant premium rate (income) paid by the insurer. We assume the *safety loading* condition $c > \lambda E[Y]$.

One of the key quantities in the classical risk model is the *ruin probability*, as a function of the initial reserve x,

$$\psi(x) = \Pr\{X_t < 0 \ \text{ for some } \ t > 0\}.$$

In general, it is very difficult to derive explicit and closed expressions for the ruin probability. The pioneering works on approximations to the ruin probability were achieved by Cramér and Lundberg as early as the 1930s under Cramér-Lundberg condition. This condition is to assume that there exists a constant $\kappa > 0$ called *adjustment coefficient*, satisfying the following Lundberg equation:

$$\int_0^{\infty} \mathrm{e}^{\kappa y} \bar{F}(y) \mathrm{d}y = \frac{c}{\lambda},$$

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with $F(y) = \Pr\{Y \le y\}$. Under this condition, the Cramér-Lundberg asymptotic formula states that if

$$\int_0^\infty y e^{\kappa y} dP(y) < \infty,$$

where $P(y) = \frac{1}{E[Y]} \int_0^y \bar{F}(s) ds$ is the equilibrium distribution of F, then

$$\psi(x) \le e^{-\kappa x}, x \ge 0. \qquad (14.2)$$

The insurer has now the possibility to reinsure the claims. In the case of proportional reinsurance Schmidli [22] showed that there exists an optimal strategy that can be derived from the corresponding Hamilton-Jacobi-Bellman equation. Hipp and Vogt [15] derived by similar methods the same result for excess of loss reinsurance. In Schäl [21], the control problem of controlling ruin by investment in a financial market is studied. The insurance business is described by the usual Cramér-Lundberg-type model, and the risk driver of the financial market is a compound Poisson process. Conditions for investment to be profitable are derived by means of discrete-time dynamic programming. Moreover, Lundberg bounds are established for the controlled model. Diasparra and Romera [3, 4] consider a discrete-time process driven by proportional reinsurance and an interest rate process which behaves as a Markov chain. To reduce the risk of ruin, the insurer may reinsure a part or even all of the reserve. Recursive and integral equations for the ruin probabilities are given, and generalized Lundberg inequalities for the ruin probabilities are derived.

We consider a discrete-time insurance risk/reserve process which can be controlled by reinsurance and investment in the financial market, and we study the ruin probability problem in the finite-horizon case. Although controlling a risk/reserve process is a very active area of research (see [2, 6, 16, 24, 26], and references therein), obtaining explicit optimal solutions minimizing the ruin probability is in general a difficult task even for the classical Cramér-Lundberg risk process. Thus, an alternative method commonly used in ruin theory is to derive inequalities for ruin probabilities. The inequalities can be used to obtain upper bounds for the ruin probabilities [8, 23, 27], and this is the approach followed in this chapter. The basis of this approach is the well-known fact that in the classical Cramér-Lundberg model, if the claim sizes have finite exponential moments, then the ruin probability decays exponentially as the initial surplus increases (see for instance the book by Asmussen [1]). For the heavy-tailed claims' case, it is also shown to decay with a rate depending on the distribution of the claim size (see, e.g., [7]). Paulsen [18] reviews general processes for the ruin problem when the insurance company invests in a risky asset. Xiong and Yang [25] give conditions for the ruin probability to be equal to 1 for any initial endowment and without any assumption on the distribution of the claim size as long as it is not identically zero.

Control problems for risk/reserve processes are commonly formulated in continuous time. Schäl [20] introduces a formulation of the problem where events (arrivals of claims and asset price changes) occur at discrete points in time that may be deterministic or random, but their total number is fixed. Diasparra and Romera [3] consider a similar formulation in discrete time. Having a fixed total number of events implies that in the case of random time points the horizon is random as well.

In this chapter, we follow an approach inspired by Edoli and Runggaldier [5] who claim that a more natural way to formulate the problem in case of random time points is to consider a given fixed time horizon so that also the number of event times becomes random, and this makes the problem nonstandard. Accordingly, it is reasonable to assume that also the control decisions (level of reinsurance and amount invested) correspond to these random time points. Notice that this formulation can be seen equivalently in discrete or continuous time. The stochastic elements that affect the evolution of the risk/reserve process are thus the timing and size of the claims, as well as the dynamics of the prices of the assets in which the insurer is investing. This evolution is controlled by the sequential choice of the reinsurance and investment levels. Claims occur at random points in time and also their sizes are random, while asset price evolutions are usually modeled as continuous-time processes. On small time scales, prices actually change at discrete random time points and vary by tick size. In the proposed model, we let also asset prices change only at discrete random time points with their sizes being random as well. This will allow us to consider the timing of the events, namely, the arrivals of claims and the changes of the asset prices, to be triggered by a same continuous-time semi-Markov process (SMP), that is, a stochastic process where the embedded jump chain (the discrete process registering what values the process takes) is a Markov chain, and where the holding times (time between jumps) are random variables, whose distribution function may depend on the two states between which the move is made. Since between event times the situation for the insurer does not change, we shall consider controls only at event times.

Under this setting, we construct a methodology to achieve an optimal solution that minimizes the bounds of the ruin probability previously derived. Admissible strategies ranging in a compact set are in fact reinsurance levels and investment positions. From a general perspective, and due to the Markovian feature of the risk process, it seems quite natural to look at the minimization of the ruin probability as a Markov decision problem (MDP) for which suitable MDP techniques may hold. Although this is not a standard approach in actuarial risk models, we present in this chapter a connection between our optimization problem and the use of MDP techniques. Many of the most relevant contributions in the literature related to MDP techniques have been developed by Onesimo Hernández-Lerma and his coauthors, and some of them have inspired the optimization part of this chapter [9–14].

The rest of the chapter is organized as follows: Section 14.2 describes the elements of the considered model and introduces the formulation of the risk process. Section 14.3 presents the previous recursive relations on ruin probabilities needed to derived our main contributions on the exponential bounds for the ruin probabilities. In Sect. 14.4, the derivation of the reinsurance and investment policy that minimizes an exponential bound is described in connection with MDP techniques, namely, policy improvement and value iteration.

14.2 The Risk Process

We start this section by fixing the elements of the model studied in this chapter. According to the model proposed in Romera and Runggaldier [19], we consider a finite time horizon $T > 0$. More precisely, to model the timing of the events (arrival of claims and asset price changes), inspired by Schäl [21], we introduce the process $\{K_t\}_{t>0}$ for $t \leq T$, a continuous-time SMP on $\{0, 1\}$, where $K_t = 0$ holds for the arrival of a claim, and $K_t = 1$ for a change in the asset price. The embedded Markov chain, that is, the jump chain associated to the SMP $\{K_t\}_{t>0}$, evolves according to a transition probability matrix $P = \left\|p_{ij}\right\|_{i,j\in\{0,1\}}$ that is supposed to be given, and the holding times (time between jumps) are random variables whose probability distribution function may depend on the two states between which the move is made.

Let T_n be the random time of the $n-th$ event, $n \geq 1$, and let the counting process N_t denote the number of events having occurred up to time t defined as follows:

$$N_t = \sum_{j=1}^{\infty} 1_{\{T_j \leq t\}} \tag{14.3}$$

and so

$$T_n = \min\{t \geq 0 | N_t = n\}. \tag{14.4}$$

We introduce now the dynamics of the controlled risk process X_t for $t \in [0,T]$ with T a given fixed horizon. For this purpose, let Y_n be the $n-th$ $(n \geq 1)$ claim payment represented by a sequence of i.i.d. random variables with common probability distribution function (p.d.f.) $F(y)$. Let Z_n be the random variable denoting the time between the occurrence of the $n-1st$ and nth $(n \geq 1)$ jumps of the SMP $\{K_t\}_{t>0}$. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with p.d.f. $G(.)$. From this, we may consider that the transition probabilities of the SMP $\{K_t\}_{t>0}$ are

$$P\{K_{T_{n+1}} = j, Z_{n+1} \leq s | K_{T_n} = i\} = p_{ij} G(s).$$

Notice that for a full SMP model, the distribution function $G(.)$ depends also on i and j. While the results derived below go through in the same way also for a $G_{ij}(.)$depending on i,j, we restrict ourselves to independent $G(.)$. A specific form of SMP arises, for example, as follows: let N_t^1 and N_t^2 be independent Poisson processes with intensities λ^1 and λ^2, respectively. We may think of N_t^1 as counting the number of claims and N_t^2 that of price changes and we have that $N_t = N_t^1 + N_t^2$ is again a Poisson process with intensity $\lambda = \lambda^1 + \lambda^2$. It then follows easily that

$$\begin{cases} p_{ij} &= \frac{\lambda^j}{\lambda} := p_j\,, \quad \forall i, \\ G(s) &= \left[1 - \mathrm{e}^{-\lambda s}\right]. \end{cases}$$

The risk process is controlled by reinsurance and investment. In general, this means that we may choose adaptively at the event times T_{N_t} (they correspond to the jump times of N_t) the retention level (or proportionality factor or risk exposure) b_{N_t} of a reinsurance contract as well as the amount δ_{N_t} to be invested in the risky asset, namely, in S_{N_t} with S_t denoting discounted prices. For the values b that the various b_{N_t} may take, we assume that $b \in (b_{\min}, 1] \subset (0,1]$, where $b_{\min}$ will be introduced below, and for the values of δ for the various δ_{N_t}, we assume $\delta \in [\underline{\delta}, \bar{\delta}]$ with $\underline{\delta} \leq 0$ and $\overline{\delta} > 0$ exogenously given. Notice that this condition allows also for negative values of δ meaning that, see also [22], short selling of stocks is allowed. On the other hand, with an exogenously given upper bound $\bar{\delta}$, it might occasionally happen that $\delta_{N_t} > X_{N_t}$, implying a temporary debt of the agent beyond his/her current wealth in order to invest optimally in the financial market. By choosing a policy that minimizes the ruin probability, this debt is however only instantaneous and with high probability leads to a positive wealth already at the next event time.

Assume that prices change only according to

$$\frac{S_{N_t+1} - S_{N_t}}{S_{N_t}} = \left(\mathrm{e}^{W_{N_t+1}} - 1\right) K_{T_{N_t+1}}, \tag{14.5}$$

where W_n is a sequence of i.i.d. random variables taking values in $[\underline{w}, \bar{w}]$ with $\underline{w} < 0 < \bar{w}$, where one may also have $\underline{w} = -\infty$, $\bar{w} = +\infty$ and with p.d.f. $H(w)$. For simplicity and without loss of generality, we consider only one asset to invest in. An immediate generalization would be to allow for investment also in the money market account.

Let c be the premium rate (income) paid by the customer to the company, fixed in the contract. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b_{N_t} chosen at the various event times T_{N_t}, we denote by $C(b_{N_t})$ the net income rate of the insurer at time $t \in [0,T]$. For $b \in (b_{\min}, 1]$, we let $h(b,Y)$ represent the part of the generic claim Y paid by the insurer, and in what follows, we take the function $h(b,Y)$ to be of the form $h(b,Y) = b \cdot Y$ (proportional reinsurance). We shall call *policy* a sequence $\pi = (b_n, \delta_n)$ of *control actions*. Control actions over a single period will be denoted by $\phi_n = (b_n, \delta_n)$. According to the expected value principle with safety loading θ of the reinsurer, for a given starting time $t < T$, the function $C(b)$ can be chosen as follows:

$$C(b) := c - (1+\theta)\frac{E\{Y_1 - h(b,Y_1)\}}{E\{Z_1 \wedge (T-t)\}}, \; 0 < t < T, \tag{14.6}$$

We use Z_1 and Y_1 in the above formula since, by our i.i.d. assumption, the various Z_n and Y_n are all independent copies of Z_1 and Y_1. Notice also that, in order to keep formula (14.6) simple and possibly similar to standard usage, in the denominator of the right-hand side, we have considered the random time Z_1 between to successive events, while more correctly, we should have taken the random time between two successive claims, which is larger. For this, we can however play with the safety loading factor. As explained in Romera and Runggaldier [19], we can now define

$b_{\min} := \min\{b \in (0,1] \mid c \geq C(b) \geq c^*\}$, where $c^* \geq 0$ denotes the minimal value of the premium considered by the insurer. We have to consider the following technical restrictions:

Assumption 14.2.1. *Let*

(i) The random variables $(Z_n, Y_n, W_n)_{n\geq 1}$ *are, conditionally on* K_t*, mutually independent.*
(ii) $E\left\{e^{rY_1}\right\} < +\infty$ *for* $r \in (0, \bar{r})$ *with* $\bar{r} \in (0, \infty)$.
(iii) $c - (1+\theta)\frac{E\{Y_1\}}{E\{Z_1 \wedge T\}} \geq 0$.

Notice that, since the support of Y_1 is the positive half line, we have $\lim_{r\uparrow\bar{r}}\{E\{e^{rY_1}\}\} = \infty$ ($\bar{r}$ may be equal to $+\infty$, e.g., if the support of Y_1 is bounded).

In the given setting, we obtain for the insurance risk process (surplus) X the following one-step transition dynamics between the generic random times T_n and T_{n+1} when at T_n a control action $\phi = (b, \delta)$ is taken for a certain $b \in (b_{\min}, 1] \subset (0,1]$, and $\delta \in [\underline{\delta}, \bar{\delta}]$,

$$X_{T_{n+1}} = X_{T_n} + C(b)Z_{n+1} - (1 - K_{T_{n+1}})h(b, Y_{n+1}) + K_{T_{n+1}}\delta(e^{W_{n+1}} - 1). \tag{14.7}$$

Definition 14.2.1. Letting $U := [b_{\min}, 1] \times [\underline{\delta}, \overline{\delta}]$, we shall say that a control action $\phi = (b, \delta)$ is *admissible* if $(b, \delta) \in U$. Notice that U is compact.

We want now to express the one-step dynamics in (14.7) when starting from a generic time instant $t < T$ with a capital x. For this purpose, note that if, for a given $t < T$ one has $N_t = n$, the time T_{N_t} is the random time of the $n-th$ event and $T_n \leq t \leq T_{n+1}$. Since, when standing at time t, we observe the time that has elapsed since the last event in T_{N_t}, it is not restrictive to assume that $t = T_{N_t}$ [see the comment below after (14.8)]. Furthermore, since Z_n, Y_n, W_n are i.i.d., in the one-step random dynamics for the risk process X_t, we may replace the generic $(Z_{n+1}, Y_{n+1}, W_{n+1})$ by (Z_1, Y_1, W_1). We may thus write

$$X_{N_t+1} = x + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1) \tag{14.8}$$

for $0 < t < T$, $T > 0$ and with $X_t = x \geq 0$ (recall that we assumed $t = T_{N_t}$). Notice that, if we had $t \neq T_{N_t}$ and therefore $t > T_{N_t}$, the second term on the right in (14.8) would become $C(b)[Z_1 - (t - T_{N_t})]$, and (14.8) could then be rewritten as

$$X_{N_t+1} = [x - C(b)(t - T_{N_t})] + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(e^{W_1} - 1)$$

with the quantity $[x - C(b)(t - T_{N_t})]$, which is known at time t, replacing x. This is the sense in which above, we mentioned that it is not restrictive to assume that $t = T_{N_t}$. In what follows, we shall work with the risk process X_t, (or X_{N_t}) as defined by (14.8). For convenience, we shall denote by (b_n, δ_n) the values of $\phi = (b, \delta)$ at $t = T_{N_t}$. Accordingly, we shall also write (b_{N_t}, δ_{N_t}) for $\left(b_{T_{N_t}}, \delta_{T_{N_t}}\right)$.

Following [24], we shall also introduce an absorbing (cemetery) state κ, such that if $X_{N_t} < 0$ or $X_{N_t} = \kappa$, then $X_{N_t+1} = \kappa$, $\forall t \leq T$. The state space is then $\mathbb{R} \cup \{\kappa\}$.

14.3 Ruin Probabilities

We present first the general expression of the ruin probability corresponding to the risk model (14.8). Thus, using the policy π, given the initial surplus x at time t and initial event $k \in \{0,1\}$ for the Markov chain K_t at time t, the ruin probability is given by

$$\psi^{\pi}(t,x;k) := P^{\pi}\left(\bigcup_{s=N_t+1}^{N_T} \{X_s < 0 \,|X_{N_t} = x, K_t = k\}\right). \qquad (14.9)$$

Note that the finite-horizon character of the considered model imposes a specific definition for the ruin probabilities. In order to obtain recursive relations for the ruin probability, we specify some notation and introduce the basic definitions concerning the finite-horizon ruin probabilities when one or n intra-event periods are considered.

Given a policy π, namely. a predictable process pair $\pi_t := (b_t, \delta_t)$ with (b_t, δ_t) in U [of which in the definitions below we need just to consider the generic individual control action $\phi = (b, \delta)$], we introduce the following functions:

$$\begin{aligned} u^{\pi}(y,z,w,k) &: = (1-k)by - C(b)z - k\delta(\mathrm{e}^{w}-1), \\ \tau^{\pi}(y,w,k,x) &: = \frac{(1-k)by - k\delta(\mathrm{e}^{w}-1) - x}{C(b)}, \end{aligned} \qquad (14.10)$$

so that $u^{\pi}(y,z,w,k) < x \iff z > \tau^{\pi}(y,w,k,x)$.

The ruin probability over one intra-event period (namely, the period between to successive event times) when using the control action $\phi = (b, \delta)$ is, for a given initial surplus x at time $t \in (0,T)$ and initial event $K_{T_{N_t}} = k \in \{0,1\}$,

$$\psi_1^{\pi}(t,x;k) := \sum_{h=0}^{1} p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^{\infty} G(\tau^{\pi}(y,w,h,x) \wedge (T-t))\mathrm{d}F(y)\mathrm{d}H(w). \qquad (14.11)$$

We want to obtain a recursive relation for

$$\begin{aligned} \psi_n^{\pi}(t,x;k) &: = P^{\pi}\left\{\bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\} \,|\, X_{N_t} = x, K_{T_{N_t}} = k\right\} \\ &: = P^{\pi}_{x,k}\left\{\bigcup_{k=N_t+1}^{(N_t+n)\wedge N_T} \{X_k < 0\}\right\}, \end{aligned} \qquad (14.12)$$

namely, for the ruin probability when at most n events are considered in the interval $[t,T]$ and a policy π is adopted.

In Romera and Runggaldier [19], the following recursive relation is obtained:

Proposition 14.3.1. *For an initial surplus x at a given time $t \in [0,T]$, as well as an initial event $K_{T_{N_t}} = k$ and a given policy π, one has*

$$\begin{aligned}
&\psi_n^\pi(t,x,k) \\
&\quad = P\{N_T - N_t > 0\} \textstyle\sum_{h=0}^1 p_{k,h} \displaystyle\int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) \mathrm{d}F(y) \mathrm{d}H(w) \\
&\qquad + P\{N_T - N_t > 1\} \textstyle\sum_{h=0}^1 p_{k,h} \\
&\qquad \cdot \displaystyle\int_{\underline{w}}^{\bar{w}} \int_0^\infty \int_{\tau^\pi(y,w,h,x)}^{T-t} \psi_{n-1}^\pi(t+z, x - u^\pi(y,z,w,h), h) \mathrm{d}G(z) \mathrm{d}F(y) \mathrm{d}H(w)
\end{aligned} \tag{14.13}$$

from which it immediately also follows that

$$\psi_1^\pi(t,x,k) = P\{N_T - N_t = 1\} \sum_{h=0}^1 p_{k,h} \int_{\underline{w}}^{\bar{w}} \int_0^\infty G(\tau^\pi(y,w,h,x) \wedge (T-t)) \mathrm{d}F(y) \mathrm{d}H(w) \tag{14.14}$$

since in the case of at most one jump, one has that $P\{N_T - N_t > 0\} = P\{N_T - N_t = 1\}$ and $P\{N_T - N_t > 1\} = 0$.

14.4 Minimizing the Bounds

In the following, we base ourselves on results in Diasparra and Romera [3,4] that are here extended to the general setup of this chapter to obtain the exponential bounds and then to minimize them.

To stress the fact that the process X defined in (14.7) corresponds to the choice of a specific policy π, in what follows, we shall use the notation X^π.

Given a policy $\pi_t = (b_t, \delta_t)$and defining for $t \in [0,T]$, the random variable

$$V_t^\pi := C(b)(Z_1 \wedge (T-t)) - \mathbf{1}_{\{Z_1 \le T-t\}} \left[(1 - K_{T_{N_t}+1}) b Y_1 + K_{T_{N_t}+1} \delta \left(\mathrm{e}^{W_1} - 1 \right) \right], \tag{14.15}$$

where $b = b_t$ and $\delta = \delta_t$ let, for $r \in (0, \bar{r})$ and $k \in \{0,1\}$,

$$l_r^\pi(t,k) := E_{t,k}\{\mathrm{e}^{-rV_t^\pi}\} - 1, \tag{14.16}$$

where, for reasons that should become clear below, we distinguish the dependence of l^π on r from that on (t,k).

Remark 14.4.1. Notice that, by its definition, $l_r^\pi(t,k)$ is, for given π and $r \in (0,\bar{r})$, a bounded function of $(t,k) \in [0,T] \times \{0,1\}$.Given its continuity in r, it is uniformly bounded on any compact subset of $(0,\bar{r})$, for example, on $[\eta, \bar{r} - \eta]$ for $\eta \in (0,\bar{r})$. Having fixed $\eta > 0$,denote this bound by L, that is,

$$\sup_{(t,k) \in [0,T] \times \{0,1\},\, r \in [\eta, \bar{r} - \eta]} |l_r^\pi(t,k)| \le L. \tag{14.17}$$

Definition 14.4.2. We shall call a policy π *admissible* and denote their set by A if at each $t \in [0,T]$, the corresponding control action $(b_t, \delta_t) \in U$, and for any $t \in [0,T]$ and $k \in \{0,1\}$, $E_{t,k}\{V_t^\pi\} > 0 \; \forall \pi \in A$.

Notice that A is nonempty since (see Assumption 14.2.1, (iii) it contains at least the stationary policy $(b_{N_t}, \delta_{N_t}) \equiv (b_{\min}, 0)$.

According to Romera and Runggaldier [19], we obtain the following result:

Proposition 14.4.2. *For each (t,k) and each $\pi \in A$, we have that:*

(a) As a function of $r \in (0,\bar{r})$, $l_r^\pi(t,k)$ is convex with a negative slope at $r = 0$.
(b) The equation $l_r^\pi(t,k) = 0$ has a unique positive root in $(0,\bar{r})$ that we simply denote by R^π so that the defining relation for R^π is

$$l_{R^\pi}^\pi(t,k) = 0. \tag{14.18}$$

Notice that R^π actually depends also on (t,k), but for simplicity of notation, we denote it just by R^π.

In view of the main result of this section, namely, Theorem 14.4.1 below, we first obtain [19]:

Lemma 14.4.1. *Given a surplus $x > 0$ at a given initial time $t \in [0,T]$ and an initial event $k \in \{0,1\}$, we have*

$$\psi_1^\pi(t,x,k) \le \mathrm{e}^{-R^\pi x} \tag{14.19}$$

for each $\pi \in A$, where R^π is the unique positive root of (14.18) that depends on t and k but is independent of x.

Lemma 14.4.2. *For given (t,x,k), we have*

$$\psi_n^\pi(t,x,k) \le \gamma_n \mathrm{e}^{-R^\pi x} \tag{14.20}$$

for all $n \in N, \pi \in A$, where R^π is the unique positive solution with respect to r of $l_r^\pi(t,k) = 0$ (see (14.18)), and γ_n is defined recursively b

$$\begin{aligned} \gamma_1 &= 1, \\ \gamma_n &= \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\}. \end{aligned} \tag{14.21}$$

Remark 14.4.2. Due to the defining relations (14.21), it follows immediately that $\gamma_n \le 1$ for all $n \in N$. In fact, using forward induction, we see that the inequality is true for $n = 1$, and assuming it true for $n - 1$, we have

$$\gamma_n = \gamma_{n-1} P\{N_T - N_t > 1\} + P\{N_T - N_t = 1\} \le P\{N_T - N_t > 0\} \le 1. \tag{14.22}$$

We come now to our main result in this section, namely, Theorem 14.4.1 whose proof follows immediately from Lemma 14.4.2 noticing that, see Remark 14.4.2, one has $\gamma_n \le 1$.

Theorem 14.4.1. *Given an initial surplus $x > 0$ at a given time $t \in [0,T]$, we have, for all $n \in N$ and any initial event $k \in \{0,1\}$ and for all $\pi \in A$,*

$$\psi_n^\pi(t,x,k) \le \mathrm{e}^{-R^\pi x}$$

with R^π that may depend on (t,k) in $[0,T] \times \{0,1\}$.

14.4.1 Minimizing the Bounds by a Policy Improvement Procedure

As mentioned previously, it is in general a difficult task to obtain an explicit solution to the given reinsurance-investment problem in order to minimize the ruin probability and this even for a classical risk process. We shall thus choose the reinsurance level and the investment in order to minimize the bounds that we have derived. By Theorem 14.4.1, this amounts to choosing a strategy $\pi \in A$ such that, for each pair (t,k), the value of R^π is as large as possible. This strategy will thus depend in general also on t and k but not on the level x of wealth. By Proposition 14.4.2, this R^π is, for each $\pi \in A$, the unique positive solution of the equation $l_r^\pi(t,k)) = 0$, where $l_r^\pi(t,k)$ is, as a function of $r \in [0,\bar{r}]$ (and for every fixed (t,k)), convex with a negative slope at $r = 0$. To obtain, for a given (t,k), the largest value of R^π, it thus suffices to choose $\pi \in A$ that minimizes $l_r^\pi(t,k)$ just at $r = R^\pi$. This, in fact, appeals also to intuition since, according to the definition in (14.16), minimizing $l_r^\pi(t,k)$ amounts to penalizing negative values of $X_t^\pi = x + V_t^\pi$, thereby minimizing the possibility of ruin.

Concerning the minimization of $l_r^\pi(t,k)$ at $r = R^\pi$, notice that decisions concerning the control actions $\phi = (b,\delta)$ have to be made only at the event times T_n. The minimization of $l_r^\pi(t,k)$ with respect to $\pi \in A$ has thus to be performed only for pairs (t,k) corresponding to event times, namely, those of the form (T_n, K_{T_n}), thus leading to a policy π with individual control actions $\phi_{T_n} = (b_{T_n}, \delta_{T_n})$.

Our problem to determine an investment and insurance policy to minimize the bounds on the ruin probability may thus be solved by solving the following subproblems:

1. For a given policy, $\bar{\pi} \in A$ determine $l_r^{\bar{\pi}}(t,k)$ for pairs (t,k) of the form (T_n, K_{T_n}).
2. Determine $R^{\bar{\pi}}(T_n, K_{T_n})$ that is solution with respect to r of $l_r^{\bar{\pi}}(T_n, K_{T_n}) = 0$.
3. Improve the policy by minimizing $l_{R^{\bar{\pi}}}^\pi(T_n, K_{T_n})$ with respect to $\pi \in A$.

This leads to a policy improvement-type approach, more precisely, one can proceed as follows:

- Start from a given policy π^0 (e.g., the one requiring minimal reinsurance and no investment in the financial market).
- Determine R^{π^0} corresponding to π^0 for the various (T_n, K_{T_n}).

- For $r = R^{\pi^0}$, determine π^1 that minimizes $l^{\pi}_{R^{\pi^0}}(T_n, K_{T_n})$.
- Repeat the procedure until a stopping criterion is met (notice that by the above procedure $R^{\pi^n} > R^{\pi^{n-1}}$).

One crucial step in this procedure is determining the function $l_r^{\pi}(t,k)$ corresponding to a given $\pi \in A$, and this will be discussed in the next section.

14.4.2 Computing the Value Function in the Policy Improvement Procedure

Recall again that the decisions have to be made only at the event times over a given finite horizon, and consequently, the function $l_r^{\pi}(t,k)$ has to be computed only for pairs of the form (T_n, K_{T_n}). The number of these events is however random and may be arbitrarily large; furthermore, the timing of these events is random as well. On the other hand, notice that if we can represent the function $l_r^{\pi}(t,k)$ to be minimized as the fixed point of a contraction operator involving expectations of functions of a Markov process, then the computation can be performed iteratively as in value iteration algorithms.

For this purpose, recalling that Z_n are i.i.d. random variables with probability distribution function $G(.)$ and that, for given $\pi \in A$ and $r \in [\eta, \bar{r} - \eta]$,the functions $l_r^{\pi}(t,k)$ are bounded by some L (see Remark 14.4.1), we start with the following:

Definition 14.4.3. For given $\pi \in A$, define T^{π} as the operator acting on bounded functions $v(t,k)$ of (t,k) in the following way:

$$\begin{aligned} &T^{\pi}(v(t,k)) \\ &= \mathbf{1}_{\{t \le T\}} E^{\pi}_{t,k} \left\{ \mathbf{1}_{\{t+Z_1 \le T\}} v(t+Z_1, K_{t+Z_1}) + \mathbf{1}_{\{t \le T \le t+Z_1\}} \left[\mathrm{e}^{-rC(b)(T-t)} - 1 \right] \right\} \\ &= \sum_{h=0}^{1} p_{k,h} \left\{ \int_0^{T-t} v(t+z,h) \mathrm{d}G(z) + \bar{G}(T-t) \left[\mathrm{e}^{-rC(b)(T-t)} - 1 \right] \right\} \end{aligned}$$

with $\bar{G}(z) = 1 - G(z)$ and where, given $\pi_t = (b_t, \delta_t)$, the value of b is $b = b_t$.

The following lemma is now rather straightforward:

Lemma 14.4.3. *For a given $\pi \in A$ and any value of the parameter $r \in [\eta, \bar{r} - \eta]$, the function $l_r^{\pi}(.)$ is a fixed point of T^{π}, that is,*

$$l_r^{\pi}(t,k) = T^{\pi}(l_r^{\pi})(t,k). \tag{14.23}$$

On the basis of the above definitions and results, we may now consider the following recursive relations:

$$\begin{aligned} l_r^{\pi,0}(T_n,k) &= \bar{G}(T - T_n)[\mathrm{e}^{-rC(b_n)(T-T_n)} - 1], \\ l_r^{\pi,m}(T_n,k) &= T^{\pi}(l_r^{\pi,m-1})(T_n,k) \;\; for\; m = 1,2,.. \end{aligned} \tag{14.24}$$

that we may view as a *value iteration* algorithm. Since between any event time T_n and the terminal time T there may be any number of events occurring, to obtain $l_r^{\pi}(.)$, the recursions in (14.24) would have to be iterated an infinite number of times. If however the mappings T^{π} are contracting in the sense that

$$\|T^{\pi}(v_1) - T^{\pi}(v_2)\| \leq \gamma \|v_1 - v_2\| \tag{14.25}$$

for bounded functions $v_1(.)$ and $v_2(.)$ and with $\gamma < 1$, then $l_r^{\pi,m}(T_n,k)$ approximates $l_r^{\pi}(T_n,k)$ arbitrarily well in the sup-norm, provided m is sufficiently large.

The above assumption can be seen to be satisfied in various cases of practical interest [19].

14.4.2.1 Reduction of Dimensionality and Particular Cases

For the policy improvement and value iteration-type procedure in the previous section, the "Markovian state" was seen to be the tuple (T_n, K_{T_n}), which makes the problem two dimensional. It is shown in Romera and Runggaldier [19] that in the particular case when the inter-event time and the claim size distributions are (negative) exponential, a case that has been most discussed in the literature under different settings, then the state space can be further reduced to only the time variable t (the sequence of event times T_n is then in fact a Markov process by itself), and so, the optimal policy becomes dependent only on the event time. This particular case can also be shown [19] to be a concrete example where the mappings T^{π} are contracting as assumed in (14.25). Always for this particular case, it can furthermore be shown (see again [19]) that the fixed point l_r^{π} of the mapping T^{π} which, as discussed above, depends here only on t, can be computed as a semianalytic solution involving a Volterra integral equation.

We conclude this section by pointing out that, although by our procedure, we minimize only an upper bound on the ruin probability, the optimal upper bound can also be used as a benchmark with respect to which other standard policies may be evaluated.

Finally, as explained in Romera and Runggaldier [19], our procedure allows also to obtain some qualitative insight into the impact that investment in the financial market may have on the ruin probability. It turns in fact out that, in line with some of the findings in the more recent literature (see e.g., [17]), the choice of investing also in the financial market has little impact on the ruin probability unless, as we do here, one allows also for reinsurance.

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