8

Financial Mathematics

8.1 Simple Interest

In this chapter, we study the mathematics of finance: interest, investments and loans. There are some very deep mathematical ideas related to finance, and a full study is well beyond our scope. For example, many topics require the use of calculus. In our brief overview, we can only touch on a few elementary topics. Even for this, a calculator will be essential.

In all our financial calculations, we use the idea of percentages. Recall that "per cent" means "out of 100". Therefore, $R\%$ is just another way of saying $\frac{R}{100}$.

How Interest Works

The idea of simple interest is well-known. Suppose you put \$100 in a bank account, and at the end of one year the bank pays you back your \$100 plus \$6. This is called *interest* on your investment, and the rate is 6%.

This interest rate is always stated in terms of the annual interest. Suppose you put your \$100 in the bank for three months, and receive \$1.50 in interest. This is 1.5% of your original investment, but the rate is still quoted as 6%, the equivalent rate if the money had been kept for a year.

Sometimes you reinvest your money; in the first example, you would have \$106 in your account at the beginning of year 2. If this process is carried out automatically by the bank, it is called *compounding*. We shall discuss compounding and compound interest in the next section. *Simple interest* is just the case where compounding does not occur. The obvious model is where you take out the interest and spend it as, for example, when somebody retires and lives on the interest from their savings.

The arithmetic of borrowing money (loans, mortgages) is similar to that for investing. Typically, you do not wait until the end of a loan period to pay back a loan.

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
DOI 10.1007/978-0-8176-8319-1_8,

The usual practice is to pay equal amounts each month (or each week or ...). For this reason, most loans involve compound interest. However, some loans use simple interest. We shall give examples at the end of this section.

The original amount you borrow is called the *principal*, or *present value* of an investment or loan. Suppose you draw simple interest on a principal $\$P$ for n years at $R\$$ interest. The total amount you would receive is $\$A$, where

$$A = P\left(1 + n\frac{R}{100}\right) = P(1 + nr).$$

(The fraction $r = R/100$ is sometimes more convenient to use.) The total interest is $\$I$, where

$$I = Pn\frac{R}{100} = Pnr.$$

When dealing in periods shorter than a year, it is common to calculate as though the year consisted of 12 months, each of 30 days. This 360 day "year" is called a "standard" year. The regular 365 day year is called an "exact" year. Simple interest may be calculated for part of a year—n need not be an integer—and in that case, standard years are used. In the case of an investment, simple interest is usually paid at the end of each year, but sometimes at the end of the loan period; for a loan, both interest and principal are typically paid at the end of the whole loan period. (Compound interest is normally used in cases where parts are paid throughout the loan period.)

Sample Problem 8.1. *You borrow* \$1600 *at* 12% *simple interest for four months. How much must you pay at the end of the period? How much would you pay if the loan were for two years?*

Solution. We use the formula $A = P(1+nR/100)$ with $P = 1600$ and $R = 12$. In the first case, n is $\frac{1}{3}$ (four months is one-third of a year), so

$$A = 1600 \times (1 + 0.12/3) = 1600 \times (1.04) = 1664$$

and you repay \$1664. In the second case, n is 2, so

$$A = 1600 \times (1 + 0.12 \times 2) = 1600 \times (1.24) = 1984$$

and you repay \$1984.

Your Turn. You borrow \$1200 at 10% simple interest for three months. How much must you pay at the end of the period? How much would you pay if the loan were for three years?

Using the "Simple Interest" Formula

The simple interest formula can be inverted. Given the final amount, interest rate and term, you can calculate the principal:

$$P = A\left(1 + n\frac{R}{100}\right).$$

Similarly, you can calculate the interest rate from the other data.

Sample Problem 8.2. *You borrow* \$1800 *at simple interest for six months. At the end of the period you owe* \$1863, *including the principal. What was the interest rate?*

Solution. Again we use $A = P(1 + nR/100)$. We know $A = 1863$, $P = 1800$, and $n = \frac{1}{2}$. So

$$1863 = 1800 \cdot \left(1 + \frac{R}{200}\right) = 1800 + 9 \cdot R,$$

and therefore $9 \cdot R = 63$, $R = 7$. So the interest rate was 7%.

Your Turn. You borrow \$1400 at simple interest for four months. At he end of the period you owe \$1428, including the principal. What was the interest rate?

Sample Problem 8.3. *You need to borrow some money for three months. Your lender offers a rate of* 12%. *At the end of the period you repay* \$824. *How much was the principal?*

Solution. Using the formula with $R = 12$ and $n = \frac{1}{4}$, we get

$$824 = P(1 + 12/400), \qquad A = \frac{824}{1.03} = 800.$$

The principal was \$800.

Sample Problem 8.4. *You borrow* \$1400 *at simple interest for three years. At the end of that period, your interest is* \$210. *What was the interest rate?*

Solution. In this case $P = 1400$, $I = 210$, and $n = 3$. So

$$210 = 1400 \cdot 3 \cdot \frac{R}{100} = 42R; \qquad R = \frac{210}{42} = 5.$$

The rate was 5%.

Your Turn. You borrow \$2200 at simple interest for two years. At the end of that period, you owe a total of \$2233. What was the interest rate?

Add-on Loans

One example of a loan calculated with simple interest is the *add-on loan*. In an add-on loan, you add the whole amount of simple interest to the principal and pay off that total. If the principal is $\$P$, the interest is $R\%$, and the period is n years, then the total to be paid back is $\$P(1 + \frac{nR}{100})$.

Very often these loans are for short periods, and in those cases the interest is high.

Sample Problem 8.5. *What is your monthly payment on an add-on loan if you borrow* \$12000 *over five years at* 8% *per year?*

Solution. Simple interest is \$960 per year, so the total (simple) interest is for five years is \$4800. Therefore, the total to be paid is \$16800. There are 60 monthly payments, so your monthly payment will be \$16800/60, which is \$280.

Your Turn. What is your monthly payment on an add-on loan if you borrow \$500 over three months at 36% per year?

Discounted Loans

If the interest is $r\%$, then in a *discounted loan* you subtract the interest from the amount borrowed. Suppose your loan says you will borrow $\$P$ at an interest rate of $R\%$, and the period is n years. Instead of $\$P$ you receive $\$P(1 - Rn/100)$. For example, if your loan has principal \$12000 over 5 years at 8%, you only receive

$$\$12000 \times (1 - 0.4) = \$7200.$$

At the end of the period you repay the original principal, \$12000 in the example.

These loans are sometimes used by auto sales companies, for lease agreements with an option to buy.

Sample Problem 8.6. *You need to pay* \$12000. *What will be your payments for a five-year discounted loan at* 8% *per year?*

Solution. If you need \$12000 then your principal will be $\$A$ where

$$A \times (1 - 0.4) = 12000$$

so $\$A = \$(12000/0.6) = 20000$. Your monthly payments total \$20000 over 60 months, so they equal \$333.33 per month, to the nearest cent. (In actual fact, you would probably pay \$333.34 per month, with a last payment of \$332.94, or maybe \$334 per month, with the last payment adjusted down.)

Observe the difference between the payments in Sample Problems 8.5 and 8.6. This is not an isolated example. An add-on loan is always better than a discounted loan at the same (non-zero) interest rate.

To see this, suppose you need \$100. If you borrow \$100 for n years at $R\%$ interest, using an add-on loan, you eventually pay $\$100(1 + \frac{nR}{100}) = \$(100 + nR)$. In order to obtain \$100 using a discounted loan at $R\%$, your "principal" is $\$P$, where $P(1 - nR/100) = 100$.

Suppose the discounted loan were as good a deal as the add-on. Then $P \leq 100 + nR$. Then

$$\begin{aligned} 100 = P(1 - nR/100) &\leq (100 + nR)(1 - nR/100) \\ &= (100 + nR)(100 - nR)/100 \end{aligned}$$

from which

$$10000 \leq (100 + nR)(100 - nR) = 10000 - n^2R^2.$$

This would mean $n^2R^2 \leq 0$. This is never true.

In any case, for a discounted loan or an add-on loan to be worthwhile, the interest rate must be low. They are better for short-term loans.

Exercises 8.1 A

1. You borrow \$5000 at 7% simple interest. What is the total you must pay if the loan is for a period of:

(i) One year; (ii) Three years; (iii) Five years?

2. You borrow \$4800 at 4% simple interest. How much is the interest if the loan is for a period of:

(i) Two years; (ii) Five years; (iii) Eight years?

3. You borrow \$12000 at 4% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Three months; (iii) Five months?

4. You borrow \$2500 at 6% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Two months; (iii) Six months?

5. You borrow \$500 at simple interest for two years. At the end of the loan, you owe \$80 in interest. What was the interest rate?

6. You borrow \$1200 at simple interest for six months. At the end of the loan, you must repay a total of \$1260. What was the interest rate?

7. You borrow some money at 7% simple interest. (Assume the interest is calculated on a standard (360-day) year.) How much did you borrow, if:

(i) Your total repayment at the end of two years is \$2508;

(ii) Your total repayment at the end of eight months is \$1570;

(iii) The interest you owe at the end of six months is \$42?

8. You borrowed \$1000 from a loan company at simple interest. After three months, your total debt was \$1060. What interest did they charge?

9. You buy a treasury bill for \$980. After three months, you sell it back to the Government for \$1000. What was the interest rate?

10. What is your monthly payment on an add-on loan if you borrow \$3000 over three years at 12% per year?

11. What is your monthly payment on an add-on loan if you borrow \$1200 over six months at 36% per year?

12. You need to pay \$24000. How much must you borrow for a five-year discounted loan at 6%? What will be your monthly payments?

13. You need to pay \$16000. How much must you borrow for a three-year discounted loan at 9%? What will be your monthly payments?

Exercises 8.1 B

1. You borrow \$4000 at 8% simple interest. What is the total you must pay if the loan is for a period of:

(i) One year; (ii) Three years; (iii) Four years?

2. You borrow \$8000 at 3% simple interest. What is the total interest if the loan is for a period of:

(i) Two years; (ii) Three years; (iii) Six years?

3. You borrow \$2500 at 5% simple interest. What is the total you must pay if the loan is for a period of:

(i) Three years; (ii) Four years; (iii) Seven years?

4. You borrow \$3000 at 5% simple interest. What is the total interest if the loan is for a period of:

(i) Two years; (ii) Four years; (iii) Five years?

5. You borrow \$2000 at 3% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the total you must pay if the loan is for a period of:

(i) One month; (ii) Four months; (iii) Six months?

6. You borrow \$4800 at 5% simple interest. Assume the interest is calculated on a standard (360-day) year. What is the interest if the loan is for a period of:

(i) One month; (ii) Three months; (iii) Six months?

7. You borrow \$400 at simple interest for three years. At the end of the loan, you owe \$120 in interest. What was the interest rate?

8. You borrow \$600 at simple interest for two years. At the end of the loan, you must repay \$690 in total. What was the interest rate?

9. You borrow \$1000 at simple interest for six months. At the end of the loan, you owe \$60 in interest. What was the interest rate?

10. You borrow \$5400 at simple interest for nine months. At the end of the loan, you must repay a total of \$5562. What was the interest rate?

11. You borrow some money at 8% simple interest. How much did you borrow, if:

(i) The interest you owe at the end of four years is \$1344;

(ii) Your total repayment at the end of eight months is \$13272;

(iii) The interest you owe at the end of six months is \$44?

12. You borrow some money at 6% simple interest, calculated on a standard (360-day) year. How much did you borrow, if:

(i) Your total repayment at the end of two years and four months is \$3648;

(ii) Your total repayment at the end of eight months is \$2912;

(iii) The interest you owe at the end of six months is \$198?

13. A loan of \$30000 is repaid after six years; the total repayment is \$48000. What was the interest rate?

14. A company borrows \$17300 at 6.8% simple interest to cover short-term costs, and pays interest of \$882.30. How long was the period of the loan?

15. What is your monthly payment on an add-on loan if you borrow \$500 over four months at 30% per year?

16. What is your monthly payment on an add-on loan if you borrow \$1200 over five years at 6% per year?

17. You need to pay \$15000. How much must you borrow for a five-year discounted loan at 5%? What will be your monthly payments (to the nearest cent)?

18. You need to pay \$12000. How much must you borrow for a four-year discounted loan at 9%? What will be your monthly payments?

8.2 Compound Interest

In this section, students will find it very useful if their calculator enables them to calculate powers of numbers: for example, if it has a key (possibly one marked x^y) that enables the user to input two numbers and automatically calculate the result of raising the first number to the power of the second number. (Only positive whole number powers will occur.)

Compounding

Say, you have $100 and every year you double your money.

One year from now you have $200.

2 years from now you have $400.

3 years from now you have $800.

So in 1 year you gain 100%; in three years you gain 700%—much more than $3 \times 100\%$. This process is called *compounding*. Tt also happens for interest less than 100%.

Sample Problem 8.7. *Suppose you put* $1000 *in the bank for five years at* 10% *interest paid annually. If you take your interest out of the bank at the end of each year, how much do you have at the end of five years? If you allow it to compound, how much do you have at the end of five years?*

Solution. If you take your interest out of the bank at the end of each year, you get $100 each year. After five years you have a total of $1500, a profit of $500.

If you put your interest back in the bank at 10%:

- After year 1 you get $100, so you have a total of $1100 in the bank.
- After year 2 you get $110 (10% of $1100), so you have a total of $1210 in the bank.
- After year 3 you get $121, for a total of $1331.
- After year 4 get $133.10, for a total of $1464.10.
- After year 5 get $146.41, for a total of $1610.51.

So after five years you have $1610.51, a profit of $610.

Let's look at this in general. Say you put $\$P$ in the bank at $R\%$ for N years, and reinvest all the interest. You end up with

$$P\left(1+\frac{R}{100}\right)^N.$$

This process is called *geometric growth.*

On the other hand, simple interest is the same as "we'll take the interest out each year". After N years at $R\%$ you would finish with

$$P\left(1+\frac{RN}{100}\right).$$

This is called *arithmetic growth.*

Sample Problem 8.8. *Suppose you invest* \$1200 *at* 10% *interest for three years with interest paid each year. How much interest is earned in total, if you take the interest out each year? How much if you reinvest the interest each year?*

Solution. We use the two formulas. For arithmetic growth, you end with

$$1200\left(1+\frac{10\cdot 3}{100}\right)=1200\cdot 1.3=1560$$

and the interest is $\$(1560-1200)=\$360$. Under geometric growth, the amount received after three years is

$$1200\left(1+\frac{10}{100}\right)^3=1200\cdot 1.1^3=1200\cdot 1.331=1597.2$$

and the interest is \$397.20.

Your Turn. What are the results for the above problem if your period of investment is four years?

Interest Periods

Very often a bank pays interest more frequently than once a year. Say you invest at R% per annum for one year but interest is paid four times per year. The bank pays $\frac{R}{4}\%$ every 3 months; this is called the *interest period*, or *term*. (Unfortunately, *term* is also used to denote the life of the loan.) If you invest $\$A$, your capital after a year is

$$\$P\left(1+\frac{R}{400}\right)^4,$$

just as if the interest rate was divided by 4 and the number of years was multiplied by 4.

(Notice that we used the simplifying assumption that three months is a quarter of a year. In fact, some quarters can be 92 days, others 91 or 90, but banks seldom take this into account. Similarly, when interest is calculated every month, it is usual to assume that each month is one-twelfth of a year.)

Sample Problem 8.9. *Say your bank pays $R = 8\%$ annual interest, and interest is paid four times per year. What interest rate, compounded annually, would give you the same return?*

Solution. $(1 + \frac{R}{400})^4 = (1.02)^4 = 1.0824\ldots$ so the effect is the same as compounding annually with $8.24\ldots\%$ interest rate.

Say interest is added t times per year. The result after one year is the same as if the interest rate were $(R/t)\%$ and the investment had been held for t years: after one year,

$$P\left(1 + \frac{R}{t \cdot 100}\right)^t,$$

so after N years,

$$P\left(1 + \frac{R}{t \cdot 100}\right)^{tN}.$$

The calculations are exactly the same when you borrow money as they are when you invest money.

Sample Problem 8.10. *You borrow* \$50000 *at* 10% *annual interest, compounded every three months, for ten years. Assuming you make no payments until the end of the period, how much will you owe* (*to the nearest dollar*)*?*

Solution. We have $P = 50000$, $R = 10$, $t = 4$, and $N = 10$. So

$$P\left(1 + \frac{R}{t \cdot 100}\right)^{tN}$$

becomes

$$50000 \times \left(1 + \frac{10}{400}\right)^{40} = 50000 \cdot (1.025)^{40} = 50000 \cdot 2.68506$$

which comes to $134253.19\ldots$ and you owe \$134253.

Your Turn. You borrow \$40000 at 12% annual interest, compounded every three months, for 15 years. Assuming you make no payments until the end of the period, how much will you owe?

In Sample Problem 8.9, 8% is called the *nominal rate* of interest. The nominal rate doesn't tell you how often compounding takes place. 8.24... % per annum is the *effective rate.* (These terms are established in the Truth in Savings Act.)

Note: when discussing a loan, the nominal rate per annum is called the *annual percentage rate* or APR.

When the period is a year ("per annum") the effective rate is called the *annual percentage yield* (APY) or *effective annual rate* (EAR). To avoid confusion, we shall refer to the APY, both for investments and loans.

Banks and other lenders like to tell you the APR, but what you really want to know is the APY. To calculate the APY, one works out how much would be owed on \$100 at the end of a year, if no payments were made. This is not a real-world calculation: credit card companies and mortgage-holders normally require some minimum payment, or a penalty is charged. In calculating the APY, act as though all penalties are waived.

Sample Problem 8.11. *Your credit card company charges an APR of* 18%. *Payments are required monthly, and interest is charged each month. What is the corresponding APY?*

Solution. The amount owing from a \$100 loan at the end of one year is

$$\begin{aligned} A &= 100\left(1+\frac{18}{12\cdot 100}\right)^{12} \\ &= 100\cdot 1.015^{12} \\ &= 119.56, \end{aligned}$$

so the APY is 19.56%.

Your Turn. Suppose your credit card company charges an APR of 12%, under the above conditions. What is the corresponding APY?

Exercises 8.2 A

1. You borrow \$1000 at 12% interest, and repay it after one year. What is the total payment if the interest is compounded

(i) Every three months;

(ii) Every six months;

(iii) Once a year?

2. You borrow \$100 at 6% interest for four years. What is the total interest if compounding takes place

(i) Every month;

(ii) Every three months;

(iii) Once a year?

3. You invest \$3250 at 2% interest compounded annually. What is its value after five years?

4. You borrow \$625 at 8% interest compounded quarterly, and repay it after 12 years. How much interest must you pay?

5. You invest \$200 for one year at interest rate 12%, compounded monthly. What is the value of your investment at the end of the year?

6. You invest \$2000 for three years at interest rate 6%, compounded every six months. What is the value of your investment at the end of the period?

7. You invest a sum for twelve years at interest rate 12%, compounded quarterly. At the end of the period your investment is worth \$10000. How much did you invest initially?

8. You wish to deposit a sum at 6% interest, compounded every six months, in order to pay \$10000 due in five years. How much must you deposit?

9. You deposit \$4000 at 7% interest, compounded monthly. How many years will it take until your investment exceeds \$9000?

10. You invest your money at 12% compound interest paid quarterly. When will it double in value?

11. What is the APY on a loan:

(i) At 5% APR, compounded monthly;

(ii) At 6% APR, compounded quarterly;

(iii) At 3% APR, compounded monthly?

12. What is the APY corresponding to compound interest at an annual rate of 12%, compounded:

(i) Annually;

(ii) Quarterly;

(iii) Monthly;

(iv) Daily (assume a 360-day year);

(v) Daily (assume a 365-day year)?

Round your answers to one-hundredth of one percent.

Exercises 8.2 B

1. You borrow \$2500 at 8% interest, and repay it after one year. What is the total payment if the interest is compounded

(i) Every three months;

(ii) Every six months;

(iii) Once a year?

2. You borrow \$5000 at 5% interest, compounded monthly. What is the total payment if the period is

(i) Two years;

(ii) Four years?

3. You invest \$2575 at 4% interest compounded quarterly, for two years. How much interest do you receive?

4. You borrow \$460 at 6% interest compounded quarterly, and repay it after seven years. How much must you repay, in total?

5. You invest \$80000 for 25 years at interest rate 6%, compounded annually. What is the value of your investment at the end of the period?

6. You invest \$1000 for two months at interest rate 24%, compounded each month. What is the value of your investment at the end of the period?

7. You invest a sum for fifteen years at interest rate 6%, compounded every two months. At the end of the period your investment is worth \$18000. How much did you invest initially?

8. You make a deposit at 6% interest, compounded every six months, in order to pay a promissory note for \$5000 that falls due in ten years. How much should you deposit?

9. You invest at 6% interest, compounded annually. How many years must you wait to double your money?

10. You invest your money at 10% compound interest paid quarterly. When will it double in value?

11. What is the APY on a loan:

(i) At 7% APR, compounded monthly;

(ii) At 6% APR, compounded monthly;

(iii) At 8% APR, compounded quarterly?

12. Which investment would give a better return: one at 5.8% APR, compounded monthly, or one at 5.95% APR, compounded annually?

8.3 Regular Deposits

As we pointed out in Sect. 8.1, you do not usually wait until the end of a loan period to pay back a loan. The usual practice is to pay equal amounts each month (or each week or ...). Another situation where equal deposits are made is the periodic savings account, such as a Christmas club or retirement account, where a fixed amount is deposited into savings each period.

Regular Savings

Consider a periodic savings account. Suppose you deposit $\$D$ each month. The interest each month is $M\%$; write $m = M/100$. Assume the account is empty to start, and you pay in for n months. (Often $n = 11$ or 12 because people use these accounts to save for vacations or Christmas shopping.)

The calculations to find the amount at the end of the nth month might start:

Principal, start of month 1	$\$D$
Interest earned in month 1	$\$mD$
Principal, end of month 1	$\$(1+m)D$
Add to principle	$\$D$
Principal, start of month 2	$\$D + \$(1+m)D$ $= \$(2+m)D$
Interest earned in month 2	$\$m(2+m)D$
Principal, end of month 2	$\$(2+m)D + \$m(2+m)D$ $= \$(1+m)(2+m)D$
Add to principle	$\$D$
Principal, start of month 3	$\$D + \$(1+m)(2+m)D,$

and so on.

This soon becomes complicated. An easier way is to calculate the effect of putting each new payment in a new bank account. The total in all the accounts at the end of n months will be the required amount.

Payment 1 draws interest for n months, so the amount in that account at the end is $\$D(1+m)^n$; payment 2 draws interest for $n-1$ months, so the amount in that account at the end is $\$D(1+m)^{n-1}$. The total of accounts 1 and 2 is

$$\begin{aligned}&\$D(1+m)^n + \$D(1+m)^{n-1}\\ &\quad = \$D\big[(1+m)^n + (1+m)^{n-1}\big].\end{aligned}$$

If we proceed in this way, the total after n months (n accounts) is

$$\$D\big[(1+m)^n + (1+m)^{n-1} + \cdots + (1+m)\big].$$

If $m = 0$ then the total is simply $\$nD$. We assume $m \neq 0$ and evaluate

$$(1+m)^n + (1+m)^{n-1} + \cdots + (1+m).$$

Write

$$X = (1+m)^n + a^{n-1} + \cdots + (1+m)^2 + (1+m).$$

Then

$$(1+m)X = (1+m)^{n+1} + (1+m)^n + \cdots + (1+m)^3 + (1+m)^2.$$

Subtracting, $mX = (1+m)^{n+1} - (1+m)$ so $X = \frac{(1+m)^{n+1}}{m}$ (this is where we need to assume $m \neq 0$).

So the amount in the account after n months is

$$\begin{aligned} &\frac{\$D \cdot [(1+m)^{n+1} - (1+m)]}{(1+m) - 1} \\ &= \frac{\$D \cdot [(1+m)^{n+1} - (1+m)]}{m}. \end{aligned}$$

We call this amount the *accumulation*.

Sample Problem 8.12. *At the beginning of each month you put* \$100 *into an account that pays* 6% *annual interest. How much have you accumulated at the end of the year?*

Solution. 6% annual interest is 0.5% per month. So $m = 0.005$, $n = 12$, $D = 100$, and you get

$$\begin{aligned} &\$100(1.005^{13} - 1.005)/0.005 \\ &= \$100(1.066986 - 1.005) \cdot 200 \\ &= \$20000(0.061986) \\ &= \$1239.72. \end{aligned}$$

Your Turn. In the above example, suppose you started saving in April, so that you only made nine payments. How much will you accumulate?

Some investment funds are set up so that you make your payment at the *end* of the payment period, rather than the beginning. In these cases, it is usual to add the last payment to the accumulation, even though it accrues no interest. In that case, the accumulation is

$$\frac{\$D \cdot [(1+m)^n - 1]}{m}.$$

For example, in a Christmas club, you might make your first payment on January 31st and withdraw the money early in December. There are ten payments. If the annual interest is again 6%, your accumulation is

$$\frac{\$D \cdot [(1.005)^{10} - 1]}{0.005}.$$

Compound Interest Loans

When you borrow money at compound interest and make regular repayments, interest is normally calculated on the amount you owe. Usually the interest is calculated at the time your payment is due; when you buy a house on a mortgage and make monthly payments, the interest is compounded monthly. There is a penalty for late payments, in addition to the interest on the missed payment. Usually there is no reward for paying early in the month. We shall refer to this arrangement as a *standard compound interest loan*, or simply a *compound interest loan*.

To clarify this, suppose your house payment of \$2000 is due on the first of the month, and on June 1st this year your total indebtedness is \$100000. For simplicity, say your annual interest rate is 12%, so the interest for one month is 1%. On July 1st, interest of \$1000 (1% of \$100000) is added to your debt. Then payments are credited. Assuming you made the standard \$2000 payment, this is subtracted from your debt, which becomes (\$100000 + \$1000 − \$2000) = \$99000. When August 1st comes around, the new interest will be \$990 (1% of \$99000). It does not matter when you made the payment, provided it is on or before July 1st; even if you paid on June 2nd, it is applied on July 1st. If you made a payment greater than \$2000 during June, the total would be subtracted from your debt on July 1st.

If you miss a payment, or pay less than \$2000, some penalty is exacted. The arrangements differ from loan to loan. Some lenders charge a higher interest rate on the amount in arrears; some charge a fee; many do both.

Payments

The amount of payment required to pay off a loan can be calculated from the data about the loan.

The calculation proceeds as follows. Say you borrow $\$P$ at $M\%$ monthly interest (compounded monthly), and pay it back at the end of Y years. The arithmetic is the same as if you put $\$D$ into a savings account each month at $M\%$ interest compounded monthly, and at the end of Y years you have exactly enough money to pay off your loan. This amount is $\$P(1+m)^n$, where $n = 12Y$ and $m = M/100$. Then the required monthly payment is $\$D$.

The only difference between this and the example of accumulated savings is that loan repayments usually start at the *end* of the first month, so for Y years the total number of months for which your money accumulates is $12Y - 1$, not $12Y$, and you gain nothing during the first month. If we continue to write n for the number of months in the life of the loan, we use $n-1$ in place of n in the accumulation formula. This actually simplifies the formula: we want to sum

$$\$D\big[(1+m)^{n-1} + (1+m)^{n-2} + \cdots + 1\big].$$

The required payment $\$D$ for a loan is calculated from

$$P(1+m)^n = \frac{D \cdot [(1+m)^n - 1]}{m}.$$

Sample Problem 8.13. *You take out a compound interest loan of* \$100000 *at* 6% *annual interest to pay off your house. The period is* 30 *years. What payment is required each month?*

Solution. Suppose the monthly payment is $\$D$. The interest rate is $0.5\% = 0.005$ per month. There are 360 months in thirty years. So $P(1+m)^n$ is

$$\$100000 \times (1.005)^{360}$$

or \$602257.52. In this example

$$\begin{aligned}\frac{D \times [(1+m)n - 1]}{m} &= \frac{\$D \times [(1.005)^{360} - 1)]}{0.005} \\ &= \$D \times 5.0225752 \times 200 = \$D \times 1004.515\end{aligned}$$

so $\$D \times 1004.515 = 602257.52$ and $D = 599.55$. You would pay \$599.55 per month.

Your Turn. To buy your car, you borrow \$12000 over five years at 8% interest, compounded monthly. To pay it off you pay $\$D$ per month. What is D?

It is interesting to observe the differences that follow from small changes in a long-term loan. Suppose we change the annual rate of interest in the preceding example from 6% to 8%, leaving the principal and period unchanged. The interest rate is $0.00666\ldots$ or $1/150$ per month. So the accumulation after 30 years at 8% is

$$\$100000 \cdot (151/150)^{360},$$

or \$1093572.96. In the "regular savings" model, depositing $\$D$ per month, your accumulated savings would be

$$\frac{\$D \cdot [(151/150)^{360} - 1)]}{1/150} = \$D \cdot 9.9357296 \cdot 150 = \$D \cdot 1490.36.$$

So your monthly payment is $D = 733.76$.

On the other hand, changing the period of the loan makes less difference than you might think. Reducing the period by 20%, to 24 years, adds less than 10% to your monthly payment:

Sample Problem 8.14. *You take out a compound interest loan of* \$100000 *at* 6% *annual interest to pay off your house. The period is* 24 *years. What payment is required each month?*

Solution. As in Sample Problem 8.14, the interest rate is $0.5\% = 0.005$ per month. There are 288 monthly payments. So in this case $P(1+m)^n$ is

$$\$100000 \times (1.005)^{288}$$

or \$420557.89, and

$$\frac{D \times [(1+m)^n - 1]}{m} = \frac{\$D \times [(1.005)^{288} - 1)]}{0.005}$$
$$= \$D \times 3.2055789 \times 200 = \$D \times 641.116.$$

So $D \times 641.116 = 420557.89$. You would pay \$655.98 per month.

Your Turn. Repeat the preceding calculation when the interest rate is 8%.

Here is another way of interpreting the above calculation: suppose you contracted to buy your \$100000 house at 6% annual interest over 30 years. Your bank requires a monthly payment of \$600 (actually \$599.55 per month, but the bank rounds up slightly; the last payment would be reduced a little). If you decided to pay an extra \$56 each month, you would finish paying for your house six years ahead of schedule.

We have calculated D from A (answering the question "What payment must I make?"). We can also calculate A from D ("Given the maximum payment I can make, how much can I afford?").

Sample Problem 8.15. *You want to buy a car. You can get an* 8% *loan over five years. You can pay* \$200 *per month. How much can you afford to pay for the car?*

Solution. $m = 8/12\% = 1/150$, $D = 200$, $n = 60$. So

$$mP(1+m)^n = D \cdot \big[(1+m)^n - 1\big]$$

becomes

$$P(151/150)^{60}/150 = 200\big[(151/150)^{60} - 1\big].$$

So $P \cdot 1.49646 = 150 \cdot 200 \cdot 0.49646$, implying $P = 9952.69$, and you can afford about \$9950.

Your Turn. In the above example, suppose you negotiated a loan at 7.5%. How much could you then afford?

One can use this method to compare compound interest loans with other sorts of loans. For example, an add-on loan of \$1000 at 5% interest for a four-year period requires monthly payments of \$25. If you took out a \$1000 loan for four years at

6% compound interest, and found your monthly payment to be \$25, your principal was \$$P$, where

$$0.005 \cdot P \cdot 1.005^{48} = 25 \cdot \left[1.005^{48} - 1\right].$$

Then

$$\begin{aligned} P &= \frac{25 \cdot [1.005^{48} - 1]}{0.005 \cdot 1.005^{48}} \\ &= \frac{6.76223}{0.0063525} \\ &= 1064.50. \end{aligned}$$

So the APY for the four-year 5% add-on loan is greater than 6%.

Equity

Suppose you have finished 3 years' payment on a 5-year loan of \$9952 at 8% annual interest for a car. As we saw in Sample Problem 8.14, your payments were \$200 per month.

Think about the remaining 24 months. Your situation is as though you had just taken a loan of \$$A$ at 8% per annum, where

$$A(151/150)^{24}/150 = 200\left[(151/150)^{24} - 1\right],$$

so $1.17288A = 150 \cdot 200 \cdot 0.17288$, $A = 4421.94$.

We say your *equity in the loan* is \$9952 − \$4421.94, about \$5530.

You might think that, after making payments for three-fifths of the payment period, you would own 60% of your car. However, your equity is a little less than that amount: around 55.5%.

The difference is *much* greater on longer-term loans. For example, suppose you take out a 30-year house loan for \$100000 at 8% per annum, with equal monthly payments of \$733.76 (as we calculated above). After three-quarters of the term—270 of the 360 payments have been made—it is as if you had just taken a loan at 8% per annum with principal \$$A$, where

$$A(151/150)^{90}/150 = 733.76\left[(151/150)^{24} - 1\right],$$

that is,

$$A \cdot 0.012123295 = 600.578$$

so $A = 49539.20$ and your equity is \$50460.80.

After three-quarters of your payments, you own about half of your house.

Exercises 8.3 A

1. You invest \$200 every quarter for 20 years in an annuity that pays 5% interest compounded quarterly. What is the final value of the annuity?

2. You invest \$4000 every year for five years in an annuity that pays 10.5% interest compounded annually. What is the final value of the annuity?

3. Credit card interest is 18% interest compounded monthly. How much must be paid each month to eliminate a debt of \$1000 in one year?

4. \$200 is invested per month in an fund that pays 9% interest compounded monthly. The first payment is made at the end of the first month. What is the value of the annuity after:

(i) 1 year; (ii) 3 years; (iii) 5 years; (iv) 8 years?

5. A house mortgage is set at 9%, compounded monthly. If the house costs \$200000, what is the monthly payment if the term of the mortgage is

(i) 15 years; (ii) 30 years?

6. You take out a compound interest loan of \$120000 at 6.5% annual interest to pay off your house. The period is 30 years. What payment is required each month?

7. You want to buy a car for \$8000. The dealer offers you a 5% add-on loan for four years, with monthly payments. You can borrow \$8000 for four years from your credit union at 7.5% interest. What would be your monthly payment in each case? Which is the better deal?

8. You take out a compound interest loan of \$100000 at 6% annual interest to pay off your house. The period is 30 years. (We saw earlier that your monthly payment is \$599.55.) What is your equity after:

(i) 15 years; (ii) 20 years; (iii) 22.5 years?

Exercises 8.3 B

1. You invest \$160 every quarter for four years in an annuity that pays 4% interest compounded quarterly. What is the final value of the annuity?

2. You invest \$2000 every six months for 10 years in an annuity that pays 8% interest compounded twice yearly. What is the final value of the annuity?

3. You need to have \$100000 in ten years, so you set aside a fixed sum every three months in a savings account. How much should you set aside each quarter if the interest is:

(i) 6%; (ii) 8%; (iii) 10%?

4. You take out a compound interest loan of \$200000 at 6% annual interest to pay off your house. The period is 30 years. What payment is required each month?

5. You wish to accumulate \$200000 over a 30 year period by making monthly payments into a fund that pays 9% annual interest. how much must you pay each month?

6. You buy a car costing \$16995 (after down-payment) and are offered a loan at 1.6% annual interest over 60 months. What payment is required each month?

7. You take out a compound interest loan of \$100000 at 6% annual interest to pay off your house. The period is 24 years. As we saw, your monthly payment is \$655.98. What is your equity after:

(i) 15 years; (ii) 20 years?

8.4 Exponential Growth

In this section, we examine another aspect of compounding, in which either the compounding period is very short or the number of periods is very great. Again you may wish to have a calculator that enables you to raise numbers to integer powers while studying this section. It is also useful to have the constant e available on your calculator.

Continuous Compounding

In some cases, compounding takes place after a very short interest. For example, in some bank accounts, interest is calculated every day. Over one year, there are 365 compounding periods. We saw in Exercise 8.2A.12 that the interest rate differs very little between the final interest if the standard year is used instead of the exact year.

In another case, even though the period is longer, there are still a large number of compounding periods. For example, suppose a company invests some of its funds and loses track of the investment for 100 years. It is found that there is little difference whether compounding took place monthly or quarterly.

For our calculations it will be convenient to work in terms of the fraction $r = R/100$ rather than the percentage R. If your bank applies interest to your account n times per year, the interest is $R\%$ and your initial investment ("capital") was $\$A$, then your ending capital after N years is

$$\$A(1 + r/n)^{nN}.$$

Consider the example of 100% interest, the case $r = 1$. The following table shows the values of $(1 + 1/n)^n$ and $(1 + 1/n)^n - 1$ for several values of n. The right-hand number is the APY corresponding to an APR of 100% with compounding n times annually.

n	$(1+1/n)^n$	
1	2.0000000	1.0000000
2	2.2500000	1.2500000
5	2.4883200	1.4883200
10	2.5937424	1.5937424
100	2.6915880	1.6915880
1000	2.7169239	1.7169239
10000	2.7181459	1.7181459
100000	2.7182682	1.7182682

Eventually the right-hand number gets very close to a fixed value, usually denoted e, about $2.7182818\ldots$

A similar calculation can be made for other values of r. For large n, $(1 + r/n)^n$ gets very close to e^r. For example, if $r = 0.1$, then $e^r = 0.10517\ldots$

We define *continuous compounding* to be a process where, after N years, the ending capital is

$$\$Pe^{rN}.$$

The calculations above show that continuous compounding is a very good approximation to daily interest, and many lenders use it instead of daily interest because it is easy to calculate.

Sample Problem 8.16. *You borrow* \$50000 *at* 10% *annual interest, compounded continuously, for ten years. Assuming you make no payments until the end of the period, how much will you owe* (*to the nearest dollar*)*?*

Solution. We have $P = 50000$, $R = 10$ and $N = 10$. So $r = 0.1$ and

$$Pe^{rN} = 50000 \cdot e = 135914.09\ldots$$

and you owe \$135914. Compare this with the result of Sample Problem 8.10, where compounding was quarterly and the answer was \$134253. The difference is not great.

Your Turn. You borrow \$40000 at 12% annual interest, compounded continuously, for 15 years. Assuming you make no payments until the end of the period, how much will you owe?

One advantage of continuous compounding is that one can easily calculate the interest when principal and interest are known. For example, suppose you invest \$100 for two years and receive \$21 interest. Your money grew from \$100 to \$121 with continuous compounding, so the rate r satisfied $100e^{2r} = 121$, so $e^{2r} = 1.21$ and $e^r = \sqrt{1.21} = 1.1$. From this, r will equal the *natural logarithm*, or logarithm to base e, of 1.1, which equals $0.0953\ldots$, so the interest rate was 9.53%. (The natural logarithm function, written *ln*, is available on most scientific calculators.)

However, you don't even need to calculate the value of r in order to find the interest; all you need is e^r. In this example, after n years, your capital will be $\$100 \cdot (1.1)^n$ and your interest will be $\$100 \cdot [(1.1)^n - 1]$. Of course, you can easily calculate the APY — in the example, your money would grow from \$100 to \$110 in the first year, so the APY is 10%. If you don't want to deal with natural logarithms, there are tables to calculate the value of R (and r) from the APY.

Sample Problem 8.17. *You borrow* \$50000 *at continuous compounding for ten years. At the end of the period, you owe* \$100000*. What was the approximate APY?*

Solution. Suppose the rate is r. Then $50000e^{10r} = 100000$ so $e^{10r} = 2$. Therefore, e^r equals the tenth root of 2, which equals 1.0718 approximately. The APY is approximately 7.18%.

Your Turn. You borrow \$40000 for seven years. At the end of the period, you owe \$80000. What was the approximate APY?

Observe that the figures of \$50000 and \$100000 were not an essential part of the above Sample Problem. The main point was the *ratio*: the debt at the end was double the principal. The answer would be the same for any value of P, provided the final amount owed was twice P.

If k is any constant, the function $f(x) = k^x$ is called an *exponential function*, and the situation where a quantity changes over x units of time from A to Ak^x is called *exponential growth*; x is the *exponent*. So continuous compounding is an example of exponential growth.

Inflation and the Consumer Price Index

There is a tendency for the purchasing power of money to decrease over time; this is called *inflation*. Sometimes the rate of inflation will change rapidly, but often it stays roughly constant for several years. Inflation follows the same model as continuous compounding.

For example, let's suppose inflation rate is 5% from 2007 to 2012. If something costs \$100 in January 2007, then five years later, in January 2012, we expect it to cost $\$100 \cdot (1.05)^5 = \$127.63$. Of course, individual items do not increase at a uniform rate, but this would be a useful approximate guide to the cost of living.

As inflation is not constant, governments often calculate tables to show the purchasing power of today's dollar in earlier years. In the United States, the Consumer Price Index (CPI) is calculated each month by finding the cost of a standard set of items (food, housing, vehicles, and so on). There are in fact several CPIs constructed. We shall always refer to the CPI-U, an index that reflects the cost of living in urban areas (about 80% of America). There is also a CPI-W, for wage-earners, and there are other indices. Tables of the CPI-U are available online at http://stats/bls/gov/cpi/#tables.

The total CPI is divided by the average for 1982–1984 (the *base period*), and multiplied by 100. For example, the CPI for February 2006 was 198.7, so a collection of goods that cost \$198.70 in February 2006 would have cost an average of about \$100 in the base period. The average for a year is also published; the average for 1988 was 118.3; the figures for May and June were 118.0 and 18.5, respectively.

This can be used to compare two different years. The CPI for June 2002 was 179.9. The ratio $\frac{179.9}{118.5} = 1.518\ldots$ provides a comparison between June 2002 and June 1988 prices: if something cost \$1000 in 1988, our best guess is that it would cost about \$1518 in 2002. These figures are approximate because the prices of different items do not increase at the same rate. However, it is reasonable to say "the cost of living was about 50% higher in 2002 than in 1988." A speaker in 2002 might say: "A dollar today is worth about two-thirds of what it was worth in 1988."

Suppose the cost of a major item at time A is $\$X_A$. Suppose the CPI at time A is C_A, and at time B it is C_B. Then your estimate of the cost at time B is

$$\frac{C_B}{C_A} X_A.$$

Sample Problem 8.18. *A house cost* \$150000 *in June* 1988. *What would you expect a similar house to cost in February* 2006*?*

Solution. If the house cost \$150000 in mid-1988, it is equivalent to a house that cost $\frac{198.7}{118.5} \cdot \$150000 = \$251518.99$ in February 2006. Your realistic answer might be "about \$250,000."

Your Turn. A house costs \$225000 in February 2006. What would a similar house have cost in June 2002?

Animal Populations

Another example of exponential growth is the growth of an animal population. Given two animals (male and female), we know how frequently they will reproduce on average, and how many offspring will be produced. These numbers are not precise, but with large numbers the errors average out. If the animals reproduce an average of three offspring per year, and on average two die per year, the end result is as if the number of animals grows by 50% annually.

Of course, the animals do not all reproduce at the same time. The process is more like continuous compounding. In the example, the appropriate model is continuous compounding with an APY of 50%.

This model is more accurate with shorter breeding periods. When studying microscopic creatures, that reproduce within hours, reasonable predictions can be made of the population growth over periods of shorter than a day. For insects, a few days is often long enough for an accurate model. With humans, we need decades or even centuries. The "continuous compounding" model of a human population is used only for predicting the population movement in large cities, states, or whole countries.

Sample Problem 8.19. *A fish population doubles every year. At present it is* 10000*. Approximately when will it reach* 100000*? When will it reach* 1000000*?*

Solution. After n years, the total population is $10000 \cdot 2^n$, so the questions are, *"when is* $2^n = 10$*?"* and *"when is* $2^n = 100$*?"*

Now $2^3 = 8$, $2^4 = 16$, $2^6 = 64$, $2^7 = 128$, so the answers are

$$\begin{aligned} 100000 &: \quad \text{during the 4th year;} \\ 1000000 &: \quad \text{during the 7th year.} \end{aligned}$$

Radioactive Decay

Radioactive decay works like continuous compounding in reverse. A radioactive material will dissipate with time, its molecules breaking down into molecules of other substances. If you have a certain amount present at a given time, it is found that the proportion that is lost depends only on the type of material and the time elapsed.

If you start with 100 grams, then at the end of one day there will remain $100k$ grams, where k is a constant, between 0 and 1, depending only on the material. After n days, the amount remaining is $100k^n$ grams. This is exactly the same formula as continuous compounding, with $e^r = k$.

Of course, if e^r is to be smaller than 1, r must be negative. So radioactive decay is an example of exponential growth with a negative component.

The *half-life* of an element is the time it takes for the amount of it present to halve. For example, if you have 500 grams with half-life 1 year, there will be 250 grams after one year, 125 after two years, and so on. After n half-lives amount A decays to $A/2^n$.

Sample Problem 8.20. *An artificial element has a half-life of one hour. You have* 450 *grams. Approximately how long will it take until only* 50 *grams is left?*

Solution. You want $450/2^n = 50$.

$$\begin{aligned} n = 3 &: 450/23 = 450/8 = 56.25, \\ n = 4 &: 450/24 = 450/16 = 28.125, \end{aligned}$$

so the approximate answer is: a little over 3 hours.

Your Turn. An artificial element has a half-life of one day. You start with 240 grams. How much will be left after four days?

Exercises 8.4 A

1. \$1000 is invested at 10% annual interest, compounded continuously. What is the value of the investment after:

(i) Three months; (ii) Two years; (iii) Ten years?

2. You invest \$10000 with continuous compounding. After two years, your investment is worth \$11200. What is its value after

(i) Four years; (ii) Ten years?

3. A house was bought for \$120000 in March, 1991, when the CPI was 135.0. What is its expected value in February 2006?

4. In January 2000 you have the choice of investing in houses for five years, or investing in a five-year certificate of deposit that pays 3% compounded annually. The CPI for January 2000 was 168.8, and for January 2005 it was 190.7. Assuming the housing market shows the same growth as the CPI, which is the better investment? Why?

5. The 2000 census shows the population of Cook County, IL as 5376741. If the population grows at the rate of 1.2% per year, what is the expected population in:

(i) 2010; (ii) 2050?

6. A fish hatchery has 2550 fish at the beginning of 1998. Each year the population grows by 50% at the end of the year, 30% of the fish are sold. How many fish are there at the beginning of 1999? How many at the beginning of 2002 ?

7. A certain insect doubles in population every week. There are 3400 in a colony on Monday March 1st, and the numbers are checked every Monday. When do the numbers exceed 100000?

8. There are 1280 grams of an isotope present at noon on Monday. If the half-life is 12 hours, how much is left at noon on Thursday?

Exercises 8.4 B

1. \$1000 is invested at 6% annual interest, compounded continuously. What is the value of the investment after:

(i) One year; (ii) Two years; (iii) Four years?

2. \$1000 is invested, and after a year the value of the investment is \$1050. Assuming continuous compounding, what is the value after:

(i) Two years; (ii) Four years; (iii) Six years?

3. You invest \$20000 with continuous compounding. After two years, your investment is worth \$23000. What is its value after:

(i) Four years; (ii) Six years?

4. In January, 1995, you have the choice of investing in houses for five years, or investing in a five-year certificate of deposit that pays 3% compounded annually. The CPI for January 1995 was 150.3, and for January 2000 was 168.8. Assuming the housing market shows the same growth as the CPI, which is the better investment? Why?

5. A house was bought for \$86000 in May, 1989, when the CPI was 123.8. What is its expected value in February 2006?

6. You bought your house for \$167000 in January, 1999, when the CPI was 164.3. In December, 2005, when the CPI was 196.4, you were offered \$195000. Is this a good deal? Why?

7. The 2000 census shows the population of Cook County, Illinois as 59612. What is the expected population in 2050:

(i) If the population grows at the rate of 1.2% per year;

(ii) If the population grows at the rate of 1.5% per year?

8. A colony of birds had 4400 members in April 2001. In April, 2003, there are 5234 birds. What is the annual growth rate?

9. A certain class of bacteria double in population in a day if unchecked. How many days will it take for a colony of 10000 to exceed 1000000?

10. A radioactive material has a half-life of 11 hours. It is safe to move quantities of 125 grams or less. If you start with 1 kilogram, how long will it be until the material is safe to move?