1

Numbers and Sets

1.1 Numbers

Sets and Number Systems

All of discrete mathematics—and, in fact, all of mathematics—rests on the foundations of set theory and numbers. We start this chapter by reminding you of some basic definitions and notations and some further properties of numbers and sets.

A *set* is any collection of objects. The objects in the collection are called the members or elements of the set. If x is a member of a set S, we write $x \in S$, and $x \notin S$ means that x is *not* a member of S. One way of defining a set is to list all the elements, usually between braces; thus if S is the set consisting of the numbers 0, 1, and 3, we could write $S = \{0, 1, 3\}$.

Another method is to use the membership law of the set: for example, since the numbers 0, 1, and 3 are precisely the numbers that satisfy the equation $x^3 - 4x^2 + 3x = 0$, we could write the set S as

$$S = \left\{x : x^3 - 4x^2 + 3x = 0\right\}$$

(which we read as "the set of all x such that $x^3 - 4x^2 + 3x = 0$"). Often we use a vertical line instead of the colon in this expression, as in

$$S = \left\{x \mid x^3 - 4x^2 + 3x = 0\right\}.$$

This form is sometimes called *set-builder notation*.

Sample Problem 1.1. *Write three different expressions for the set with elements* 1 *and* −1.

Solution. Three possibilities are $\{1, -1\}$, $\{x : x^2 = 1\}$, and "the set of square roots of 1". There are others.

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
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Your Turn. Write three different expressions for the set with elements 1, 2, and 3.

We define several sets of numbers, denoted $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, that we shall use later. The set $\mathbb{N}$ of all positive integers or natural numbers is the first number system we encounter; the natural numbers are used to count things. If one includes zero, to account for the possibility of there being nothing to count, and negatives for subtraction, the result is the set $\mathbb{Z}$ of integers. It is useful to have a notation for the set of all *non-negative* integers—that is, the members of $\mathbb{N}$, together with 0. We shall write $\mathbb{Z}^*$ for that set, but the reader should note that a minority of authors, particularly in Computer Science, include 0 as a member of $\mathbb{N}$.

We sometimes write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The use of a string of dots (an *ellipsis*) is not precise, but is understood to mean that the set continues without end. Such sets are called *infinite* (as opposed to *finite* sets like $\{0, 1, 3\}$). We also write $\{1, 2, \ldots, 20\}$ to mean the set of all positive integers from 1 to 20. A more precise notation for this set, borrowed from the computer language *Pascal*, is (1..20).

The set $\mathbb{Q}$ of rational numbers consists of all fractions, the ratios $\frac{p}{q}$, where p and q are integers and q is not 0. In other words,

$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\right\}.$$

In this notation, p is called the *numerator* of the fraction, and q is the *denominator*. An alternative definition is that $\mathbb{Q}$ is the set of all numbers with a repeating or terminating decimal expansion. Examples are

$$\begin{aligned}\frac{1}{2} &= 0.5,\\ \frac{-12}{5} &= -2.4,\\ \frac{3}{7} &= 0.428571428571\ldots.\end{aligned}$$

In the last example, the digit string 428571 repeats forever, and we usually indicate this by writing

$$\frac{3}{7} = 0.\overline{428571}.$$

The denominator q cannot be zero. In fact, division by zero is never possible. This is not a made-up rule, but rather it follows from the definition of division. When we write $x = \frac{p}{q}$, we mean "x is the number that, when multiplied by q, gives p". What would $x = 2/0$ mean? There is no number that, when multiplied by 0, gives 2. Similarly, $x = 0/0$ would be meaningless. In this case, there *are* suitable numbers x, in fact every number will give 0 when multiplied by 0, but we wanted a uniquely defined answer.

Different decimal expansions do not always mean different numbers. The exception is an infinite string of 9s. These can be rounded up: $0.\overline{9} = 1$, $0.7\overline{9} = 0.8$, and so on. Proving this requires some understanding of the theory of limits, but the following discussion illustrates the ideas of the proof.

Suppose x is the number represented by $0.\overline{9}$. Then $x = 0.\overline{9}$, and multiplying both sides by 10, $10x = 9.\overline{9}$. If we subtract x from both sides, we get $9x = 10x - x = 10x - 0.\overline{9}$. So $9x = 9.\overline{9} - 0.\overline{9} = 9$ and $x = 1$.

Integers are rational numbers, and in fact they are the rational numbers with numerator 1. For example, $5 = 5/1$.

Each rational number has infinitely many representations as a ratio:

$$1/2 = 2/4 = 3/6 = \cdots .$$

The final number system we shall use is the set $\mathbb{R}$ of real numbers, consisting of all numbers that are decimal expansions, all numbers that represent lengths. If n is any natural number other than a perfect square (one of $1, 4, 9, 16, \ldots$), then $\sqrt{n}$ is not rational. Another important number that is not rational is the ratio π of the circumference of a circle to its diameter.

Remember that every natural number is an integer; every integer is a rational number; every rational number is a real number. Do not fall into the common error of thinking that "rational number" excludes the integers, and so on. There are special words that exclude: for example, real numbers that are not rational are called *irrational numbers*.

This is not the end of number systems. For example, the set $\mathbb{C}$ of *complex numbers* is derived from the real numbers by including square roots of negative numbers, plus all the sums of these square roots with real numbers. However, we do not encounter them in this book.

Intervals

The set of all real numbers between two bounds is called an *interval*. For example, the set of all real numbers greater than -2 but smaller than 2, which we would write as

$$S = \{x : x \in \mathbb{R}, -2 < x < 2\},$$

is the interval from -2 to 2, exclusive. These intervals occur so often that there is a special notation for them. The above example is written $(-2, 2)$, where the parenthesis (round bracket) indicates that the bound is not included. If the bound is included, a square bracket is used. So

$$\begin{aligned}
(-2, 2) &= \{x : x \in \mathbb{R}, -2 < x < 2\},\\
[-2, 2) &= \{x : x \in \mathbb{R}, -2 \le x < 2\},\\
(-2, 2] &= \{x : x \in \mathbb{R}, -2 < x \le 2\},\\
[-2, 2] &= \{x : x \in \mathbb{R}, -2 \le x \le 2\}.
\end{aligned}$$

The set of all real numbers greater than 1 is denoted $(1, \infty)$, and the set of all real numbers less than or equal to 3 is $(-\infty, 3]$. Read the symbol ∞ as *infinity*.

Sample Problem 1.2. *Write expressions for the following intervals:*

$$\{x : x \in \mathbb{R}, 3 < x \leq 4\}, \qquad \{x : x \in \mathbb{R}, 0 \leq x\},$$
$$\{x : x \in \mathbb{R}, x < 2\}, \qquad \{x : x \in \mathbb{R}, x \geq 2\}.$$

Solution. $(3, 4]$, $[0, \infty)$, $(-\infty, 2)$, $[2, \infty)$.

Your Turn. Write expressions for the following intervals:

$$\{x : x \in \mathbb{R}, 0 < x\}, \qquad \{x : x \in \mathbb{R}, 0 < x < 1\},$$
$$\{x : x \in \mathbb{R}, 1 < x \leq 4\}, \qquad \{x : x \in \mathbb{R}, x < 3\}.$$

Factors

When x and y are integers, we use the phrase "x divides y", and write $x \mid y$, to mean "$\frac{y}{x}$ is an integer", or alternatively "there is an integer z such that $y = x \cdot z$". We say x is a *divisor* of y. If $x \mid y$ is false, one can write $x \nmid y$. Thus 2 divides 6 (because $\frac{6}{2}$ equals the integer 3), -2 divides 6 (because $\frac{6}{-2} = -3$), 2 divides -6 (because $\frac{-6}{2} = -3$). Putting it the other way, the reasons are respectively, $6 = 2 \cdot 3$, $6 = (-2) \cdot (-3)$, and $-6 = 2 \cdot (-3)$. In symbols, we would write $2 \mid 6$ and $-2 \mid 6$. As 4 does not divide 6, we write $4 \nmid 6$.

Some students are confused by the case $y = 0$, but according to our definition x divides 0 for any non-zero integer x. We define the *factors* of a positive integer x to be the positive divisors of x (in our terminology, negative divisors are not called factors so -2 is not a factor of 6).

If x divides both y and z, then we call x a *common divisor* of y and z. Among the common divisors of y and z there is naturally a greatest one, called (not surprisingly) *the greatest common divisor* of y and z, and denoted (y, z), or sometimes $gcd(y, z)$. If $(y, z) = 1$, y and z are called *coprime* or *relatively prime*. For example, $(4, 10) = 2$, so 4 and 10 are not coprime; $(4, 9) = 1$, so 4 and 9 are coprime.

A *prime number* is a positive integer x other than 1 whose only factors are 1 and x. The first few primes are 2, 3, 5, 7, 11, and 13. The number 1 is excluded, and is called a *unit*.

Sample Problem 1.3. *What are the factors of* 12, 22, *and* 17*? What is* $(12, 22)$*?*

Solution.

12 has factors 1, 2, 3, 4, 6, and 12.

22 has factors 1, 2, 11, and 22.

17 has factors 1 and 17.

The common factors of 12 and 22 are 1, 2, so $(12, 22) = 2$.

Your Turn. What are the factors of 24, 15, and 13? What is (24,15)?

Suppose x is any positive integer. If x is not a prime we can find positive integers y and z, neither equal to 1, such that $x = yz$. We could then break down y and z, if possible, until finally we obtain

$$x = x_1, x_2, \ldots, x_k$$

where the x_i are all primes. So every positive integer is the product of prime factors. In fact, it can be shown that such a decomposition is unique (up to the order of the factors): for example, if

$$2^a 3^b 5^c = 2^x 3^y 7^z,$$

it must be true that $a = x$, $b = y$, and c and z are both zero.

Exponents

If x is a positive integer, we know that b^x is the product of x copies of b; in this expression b is ellipsis called the *base* and x the *exponent*. It is easy to deduce such properties as

$$\begin{aligned} b^x b^y &= b^{x+y}, \\ \left(b^x\right)^y &= b^{xy}, \\ (ab)^x &= a^x b^x. \end{aligned}$$

Negative exponents are handled by defining $b^{-x} = \frac{1}{b^x}$, and also $b^0 = 1$ whenever b is non-zero. The rule $b^x b^y = b^{x+y}$ leads us to define $b^{\frac{1}{x}}$ to be the xth root of b. (When x is even, we take the positive root for positive b and say $b^{\frac{1}{x}}$ is not defined for negative b).

Sample Problem 1.4. *Express the following in the simplest form, as decimal numbers if possible:*

$$(612)^0, \quad \left(x^3\right)^5, \quad \left(x^4\right)^0, \quad (-10)^{-4}, \quad \frac{1}{10^{-2}}.$$

Solution. $(612)^0 = 1$ ($b^0 = 1$ for non-zero b); $(x^3)^5 = x^{3\cdot5} = x^{15}$; $(x^4)^0 = 1$ (again, $b^0 = 1$ for any b, or you could also argue that $(x^4)^0 = x^{4\cdot0} = x^0 = 1$); $(-10)^{-4} = \frac{1}{(-10)^4} = \frac{1}{(-1)^4\times10^4} = \frac{1}{10^4} = 0.0001$; $\frac{1}{10^{-2}} = 10^2 = 100$.

Your Turn. Do the same to

$$1^{15}, \quad 0^5, \quad \left(x^{-1}\right)^0, \quad \frac{x^6}{x^3}, \quad (-2)^{-1}, \quad \frac{1}{5^{-2}}.$$

Sample Problem 1.5. *Express in the simplest possible form, with positive exponents:*

$$\frac{1}{x^{-4}}, \quad \frac{u^{-2}}{v^{-3}}, \quad \left(3a^2\right)\left(5a^{-3}\right).$$

Solution.

$$\frac{1}{x^{-4}} = \left(x^{-4}\right)^{-1} = x^{(-4)\cdot(-1)} = x^4;$$
$$\frac{u^{-2}}{v^{-3}} = \left(u^{-2}\right)\left(v^{-3}\right)^{-1} = u^{-2}v^3 = \frac{v^3}{u^{-2}};$$
$$\left(3a^2\right)\left(5a^{-3}\right) = 3\cdot 5\cdot a^2\cdot a^{-3} = 15a^{-1} = \frac{15}{a}.$$

Your Turn. Express in the simplest possible form, with positive exponents:

$$\frac{t^{-2}}{t^{-3}}, \quad y^{5-2}, \quad \left(4x^{-2}\right)\left(3x^4\right).$$

Absolute Value, Floor, and Ceiling

The *absolute value* or *modulus* of the number x, which is written $|x|$, is the positive number equal to either x or $-x$. For example, $|5.3| = 5.3$, $|-7.2| = 7.2$.

The *floor* $\lfloor x\rfloor$ of x is the largest integer not greater than x. If x is an integer, $\lfloor x\rfloor = x$. Some other examples are $\lfloor 6.1\rfloor = 6$, $\lfloor -6.1\rfloor = -7$. It is easy to deduce the following properties:

(1) $\lfloor x\rfloor = n$ if and only if n is an integer and $n \le x < n+1$.

(2) If x is non-negative, then $\lfloor x\rfloor$ equals the integer part of x. If x is negative and non-integral then $\lfloor x\rfloor$ is 1 less than the integer part of x.

(3) If n and k are integers, then k divides n if and only if $\frac{n}{k} = \lfloor \frac{n}{k}\rfloor$.

The *ceiling* $\lceil x\rceil$ is defined analogously as the smallest integer not less than x.

Sample Problem 1.6. *What are* $\lfloor 6.3\rfloor$, $\lceil 7.2\rceil$, $\lceil -2.4\rceil$, $|3.4|$, $|-3.4|$*?*

Solution. 6, 8, −2, 3.4, 3.4.

Your Turn. What are $\lfloor 2.6\rfloor$, $\lfloor -1.7\rfloor$, $\lceil 4.8\rceil$, $\lceil -4\rceil$, $|3.1|$, $|-4,4|$?

Exercises 1.1 A

1. Is the given statement true or false?

(i) $3 \in \{2, 3, 4, 6\}$; (ii) $4 \notin \{2, 3, 4, 6\}$;
(iii) $5 \in \{2, 3, 4, 6\}$; (iv) $\{3, 2\} = \{2, 3\}$;
(v) $\{1, 2\} \in \{1, 2, 3\}$; (vi) $\{1, 2\} = \{1, 2, 3\}$;
(vii) $2 \in \mathbb{Z}$; (viii) $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$;
(ix) $\mathbb{R} \notin \mathbb{Z}$; (x) $\{4\} \in \{2, 4, 6\}$.

2. In each case, write the list of all members of the set.
(i) $\{x : x$ is an even positive integer less than $12\}$;
(ii) $\{x : x$ is a color on the American flag$\}$;
(iii) $\{x : x$ is a day of the weekend$\}$.

3. Which of the following are true?
(i) All natural numbers are integers.
(ii) All integers are natural numbers.

4. Give an example of a real number that is not a rational number.

5. For each of the following numbers, to which of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ does it belong?
(i) 1.308; (ii) $1.\overline{3}$; (iii) -7;
(iv) $\sqrt{3}$; (v) 10508; (vi) $1 - \sqrt{7}$.

6. Give three different expressions for the set with members $1, -1$, and 0.

7. In each case, list all the factors of the number:
(i) 36; (ii) 48; (iii) 50;
(iv) 81; (v) 72; (vi) 61.

8. Decompose the following numbers into primes:
(i) 1044; (ii) 268;
(iii) 256; (iv) 333.

9. In each case, decompose the two numbers into primes, and then compute their greatest common divisor:
(i) 88 and 132; (ii) 256 and 224;
(iii) 1080 and 855; (iv) 168 and 231.

10. Simplify the following expressions, writing the answers using positive exponents only:
(i) $t^3 t^{-3}$; (ii) $\left(2x^2 y^{-3}\right)^2$;
(iii) $\dfrac{(xy)^3}{(xy)^2}$; (iv) $\dfrac{5^6 2^4}{10^3}$;

(v) $\left(4b^2\right)^2(2b)^{-3}$;

(vi) $(x+y)^{-3}$;

(vii) $\left(\dfrac{a^{-4}}{a^{-3}}\right)^{-2}$;

(viii) $\dfrac{4p^2q^2}{8p^2q}$.

11. Evaluate the following:

(i) $\lfloor 117 \rfloor$;

(ii) $\lceil 44.3 \rceil$;

(iii) $\lfloor 28.4 \rfloor$;

(iv) $\lceil -107.7 \rceil$;

(v) $|11.4|$;

(vi) $\lceil \sqrt{3} \rceil$;

(vii) $|-27|$;

(viii) $\lfloor -73 \rfloor$;

(ix) $\lfloor \sqrt{2} \rfloor$;

(x) $\lfloor -7.3 \rfloor$.

Exercises 1.1 B

1. Is the given statement true or false?

(i) $5 \in \{1, 3, 4, 7\}$;

(ii) $6 \notin \{1, 3, 4, 7\}$;

(iii) $4 \in \{1, 3, 4, 7\}$;

(iv) $\{1, 3\} = \{1, 2, 3\}$;

(v) $3 \in \{1, 2, 3, 4, 5, 6\}$;

(vi) $\mathbb{R} \subseteq \mathbb{Z}$;

(vii) $4 \subseteq \{1, 3, 4, 5, 7\}$;

(viii) $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1, 0\}$;

(ix) $\{1, 3, 2\} = \{1, 2, 3\}$;

(x) $\{2, 3, 4\} = \{1, 2, 3\}$.

2. In each case, write the list of all members of the set:

(i) $\{x : x$ is a month whose name starts with J$\}$;

(ii) $\{x : x$ is an odd integer between -6 and $6\}$;

(iii) $\{x : x$ is a letter in the word “Mississippi”$\}$.

3. Which of the following are true?

(i) All rational numbers are real.

(ii) All real numbers are rational.

(iii) No irrational numbers are rational.

(iv) No irrational numbers are real.

4. For each of the following numbers, to which of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ does it belong?

(i) -2.13;

(ii) $\sqrt{5}$;

(iii) $\sqrt{4}$;

(iv) 2π;

(v) $1.8\overline{34}$;

(vi) $2+\sqrt{2}$;

(vii) 0;

(viii) $\sqrt{2}+\sqrt{2}$.

5. In each case, list all the factors of the number:

(i) 63; (ii) 24; (iii) 40; (iv) 28;

(v) 29; (vi) 66; (vii) 112; (viii) 70.

6. Give three different expressions for the set with members 0, 1, and 2.

7. Decompose the following numbers into primes:

(i) 96; (ii) 1029; (iii) 38 and 56; (iv) 112 and 54;

(v) 187; (vi) 588; (vii) 175; (viii) 127.

8. In each case, decompose the two numbers into primes, and then compute their greatest common divisor:

(i) 240 and 84; (ii) 162 and 45;

(iii) 231 and 275; (iv) 444 and 629;

(v) 95 and 125; (vi) 462 and 252.

9. Simplify the following expressions, writing the answers using positive exponents only:

(i) x^4x^{-4}; (ii) $\dfrac{2^6 3^2}{6^3}$;

(iii) $(2xy)^{-2}$; (iv) $5y^2z^{-3}$;

(v) $(2x)^2y^3(xy)^{-2}$; (vi) $(x-y)^{-2}$;

(vii) $\left(\dfrac{x-3}{x-2}\right)^4$; (viii) $\dfrac{2x^2y}{4xy^2}$;

(ix) $6x^2(3x)^{-2}$; (x) $\dfrac{(2y)^2x^6}{x^4}$;

(xi) $(3xy)^{-2}z$; (xii) $\dfrac{3x^2}{4x^3}$.

10. Evaluate the following:

(i) $\lfloor 8.1 \rfloor$; (ii) $\lfloor -4.6 \rfloor$;

(iii) $\lceil 308 \rceil$; (iv) $\lceil 77.7 \rceil$;

(v) $\lfloor \sqrt{5} \rfloor$; (vi) $|-7.6|$;

(vii) $\lceil 21.5 \rceil$; (viii) $\lceil -7.3 \rceil$;

(ix) $|1.73|$; (x) $|-5.2|$;

(xi) $\lceil -105 \rceil$; (xii) $\lfloor \sqrt{2} \rfloor$;

(xiii) $\lfloor 2\pi \rfloor$; (xiv) $\lceil |-9.6| \rceil$;

(xv) $\lfloor \lfloor 7.5 \rfloor \rfloor$; (xvi) $\lceil \lfloor 7.5 \rfloor \rceil$.

1.2 Equations and Inequalities

Equations

An *equation* is any expression of equality. Sometimes an equation simply expresses a relation between numbers, such as $5-3=2$. But more often we do not write down the values of all the numbers concerned. For example, we might write $x-3=2$.

In the expression $x-3=2$, the quantity x is called a *variable* and can represent any real number. Then $x-3=2$ is *true* when $x=5$, and we say it is *false* otherwise. We say $x=5$ *satisfies* the equation, and we say 5 is the *solution* of the equation.

Sometimes equations can have more than one solution. For example, both $x=1$ and $x=2$ satisfy the equation $x^2-3x=-2$, as you can easily check. We can also show that there are no other solutions. The set of all solutions is called, naturally enough, the *solution set* of the equation. The solution set of $x^2-3x=-2$ is $\{1, 2\}$. Two equations with the same solution set are called *equivalent*.

If a problem asks you to solve an equation, you should find all solutions, or in other words find the solution set.

Equations are not restricted to one variable. For example, the equation $x-y=2$ contains the two variables x and y, and we usually think of it as expressing the relationship between x and y. This equation has an infinitude of solutions. One example is $x=3$, $y=1$, which can also be written $(x, y)=(3, 1)$. We might also use this equation to express one variable in terms of the other, as in $y=x-2$.

Some equations are satisfied by every number; one such equation is $(x-1)^2 = x^2-2x+1$. Such an equation is called an *identity*. There are also equations with no solutions, and these are called *inconsistent* equations or *contradictions*.

Sample Problem 1.7. *Which of the following are solutions of* $2x+3y=1$*?* (i) $x=2, y=3$*;* (ii) $x=-1, y=1$*;* (iii) $x=5, y=-3$.

Solution. If we replace x by 2 and y by 3, the equation becomes $4+9=1$, which is false. The other two suggested solutions lead to $-2+3=1$ and $10-9=1$, respectively, and both of these are true. So (ii) and (iii) are solutions, but (i) is not.

Your Turn. Which of the following are solutions of $3x-2y=4$?
(i) $x=2, y=2$; (ii) $x=4, y=4$; (iii) $x=3, y=1$.

When an equation is first presented to you, you know nothing about the variables. When you solve an equation, you find out more about the variables. Assuming the equation to be true, you know that the variables belong to the solution set. For this reason, the variables are often called *unknowns*.

Linear Equations with One Variable

The *degree* of an equation is the highest power of a variable that occurs in it. For example, $x - y = 2$ is of degree 1, or *first degree*, while $x^2 - 3x = -2$ is of degree 2. The simplest equations are those of the first degree. These equations are called *linear* because linear equations in two variables can be represented by straight lines, as we shall see in Section 5.1.

We shall now discuss the solution of linear equations containing one variable, such as $3x + 4 = 0$, $2x = 3$, and so on. These equations can be written in the form

$$ax = b,$$

which we shall call *standard form*. The symbols a and b may represent any numbers provided $a \neq 0$; they are *constants*, not variables, and in any particular equation their values will be known.

To solve a linear equation with one variable, we first reduce it to standard form. If any term on the right-hand side is a multiple of x, add its negative to both sides. For example, given $x + 2 = 1 - 2x$, add $2x$, the negative of $-2x$, to both sides. The $-2x$ cancels, leaving $x + 2 + 2x = 1$. If any constant is on the left, add its negative to both sides. In the example, the new equation is $x + 2x = 1 - 2$. So you must move all terms involving x to the left-hand side of the equation, and all other terms to the right-hand side; but when you move a term from one side to the other, you must change its sign, from positive to negative, or from negative to positive. Finally, add all terms on each side of the new equation.

Once the equation is in standard form, its solution is immediate. As a is non-zero, we can divide both sides by it, obtaining

$$x = \frac{b}{a},$$

and this is the only possible solution.

Sample Problem 1.8. *Solve*

$$3x - 4 = x + 2.$$

Solution. First we move all x terms to the left-hand side. The term x becomes $-x$ on the left, thus we have:

$$3x - 4 - x = 2.$$

Then move the -4. It becomes 4:

$$3x - x = 4 + 2.$$

Next, gather terms, obtaining

$$2x = 6.$$

Finally, divide by 2:

$$x = 3.$$

Your Turn. Solve $3x + 5 = x + 7$.

Sample Problem 1.9. *Solve*

$$\frac{x}{2} - 4 = -\frac{5x}{3}.$$

Solution. First move $-\frac{5x}{3}$ to the left:

$$\frac{x}{2} - 4 + \frac{5x}{3} = 0.$$

Then move the -4. It becomes 4:

$$\frac{x}{2} + \frac{5x}{3} = 4.$$

Next, gather terms. To simplify this equation, we multiply both sides by the common denominator 6:

$$3x + 10x = 24.$$

Finally, divide by 13:

$$x = \frac{24}{13}.$$

Your Turn. Solve

$$\frac{3x}{2} - 5 = \frac{2x}{3}.$$

Equations in General

In solving linear equations with one variable, our procedure is to change the given equation into a simpler equation that is equivalent to the original. This principle can be applied to any equation. When we add terms to both sides of an equation, or equivalently move terms to the other side of the equation, we are actually using the following general property of equality:

If a, b and c are any quantities, then:

$$\text{If } a = b, \quad \text{then } a + c = b + c;$$
$$\text{If } a = b, \quad \text{then } a - c = b - c.$$

The process of multiplying by a common denominator, and the process of dividing by the coefficient of x, use:

If a, b and c are any quantities, then:

$$\text{If } a = b, \quad \text{then } c \times a = c \times b;$$
$$\text{If } a = b, \quad \text{then } \frac{a}{c} = \frac{b}{c}, \text{ provided } c \neq 0.$$

These ideas can be used in general.

Sample Problem 1.10. *Use the equation* $4x^2 + 3y = y + 6$ *to express* y *in terms of* x.

Solution. Even though x is a variable, we move it to the right-hand side because we want to find an expression for y. So,

$$3y = y + 6 - 4x^2.$$

All y terms belong on the left:

$$3y - y = 6 - 4x^2.$$

Now we simplify:

$$2y = 6 - 4x^2,$$
$$y = 3 - 2x^2.$$

Your Turn. Use the equation $x^2 - 2y = 2y - 8$ to express y in terms of x.

Inequalities

An *inequality* is similar to an equation, with one of the signs $<$, $>$, $\leq$, or $\geq$ replacing the equality. Solving inequalities is similar to solving equations. Again there will be a solution set, and inequalities with the same solution set are called equivalent. The solution sets of inequalities usually involve intervals. A *linear* inequality is one with only constants and first powers of variables.

The most important difference between solving inequalities and solving equations is the effect of multiplying both sides by a constant. If $a < b$, and c is a *positive* constant, then $c \times a < c \times b$, but if d is a *negative* constant, then $d \times a > d \times b$. If the sign is $\leq$, then multiplication by a positive constant leaves the sign unchanged, while multiplication by a negative constant changes $\leq$ to $\geq$.

Sample Problem 1.11. *Solve the following inequality for* x*:*

$$4x - 3 \leq 2x + 1.$$

Solution. We first rewrite the inequality as:

$$4x - 2x \leq 3 + 1$$

or

$$2x \leq 4.$$

Dividing both sides by 2, we obtain $x \leq 2$, or solution set $(-\infty, 2)$.

Your Turn. Solve for x: $2x + 3 \leq x + 6$.

Sample Problem 1.12. *Solve the following inequality for x:*

$$3x - 2 > 5x + 4.$$

Solution. We first rewrite the inequality as:

$$3x - 5x > 2 + 4$$

or

$$-2x > 6.$$

Dividing both sides by -2, we obtain $x < -3$. Observe the change in the inequality sign.

Your Turn. Solve for x: $2x + 3 \leq 4x + 5$.

Sample Problem 1.13. *Use the following inequality to express y in terms of x:*

$$3x - 4y > 2(x + y) - 3.$$

Solution. We first rewrite the inequality with all y terms on the left and all other terms on the right:

$$-4y - 2y > 2x - 3x - 3$$

or

$$-6y > -x - 3.$$

Dividing both sides by -6, we obtain $y < \frac{1}{6}x + \frac{1}{2}$.

Exercises 1.2 A

1. In each case, are the indicated values a solution to the equation?

(i) Equation $3x + y = 4$; solution $x = 3$, $y = -5$;

(ii) Equation $3x + y = 4$; solution $x = 4$, $y = -7$;

(iii) Equation $2x + 3y = 8$; solution $x = 3$, $y = 1$;
(iv) Equation $2x - 5y = -1$; solution $x = 2$, $y = 1$;
(v) Equation $x^2 + y^2 = 4$; solution $x = 2$, $y = 0$;
(vi) Equation $x^2 + y^2 = 1$; solution $x = 1$, $y = 1$.

2. Solve these equations for x:

(i) $2x - 7 = 5$;
(ii) $x + 3 = 3 - x$;
(iii) $3x - 5 = x - 3$;
(iv) $3(x - 1) = 2(x + 1)$;
(v) $14 = 4 - 5x$;
(vi) $3(x - 1) + 2x = 7$;
(vii) $\frac{1}{2}x - 1 = 2$;
(viii) $\frac{3}{2}x + 1 = 5 - \frac{4}{3}x$;
(ix) $2x + 2 = -1 - 4x$;
(x) $3 - 3x = 4 - 2x$;
(xi) $4x - 8 = 7x - 14$;
(xii) $2(3 + x) = 4x - 7$.

3. Use the following equations to express y in terms of x:

(i) $4x - 6y = 9$;
(ii) $2x + 2y = 6$;
(iii) $3x + y = 4y + 6$;
(iv) $x^2 - 2y = y - 3$.

4. Solve these inequalities for x:

(i) $2x - 5 < 7x - 2$;
(ii) $3x + 1 > 5 - x$;
(iii) $2x + 4 \geq 4x$;
(iv) $x - 4 \geq 2 - 4x$;
(v) $3 - x \geq x + 5$;
(vi) $3x - 3 < 2 + x$;
(vii) $5 - 3x \leq x + 3$;
(viii) $2x + 3 > 3 - x$;
(ix) $2 - 2x > 3 + x$;
(x) $4(1 - x) < 2(x + 4)$.

5. Use the following inequalities to express y in terms of x:

(i) $4x + 2y \leq 4$;
(ii) $x + y \leq 3 - x$;
(iii) $x + 3y > 2y + 5x$;
(iv) $3x - 2y < 2x + 3$;
(v) $x + 3y + 2 \geq 3x - y - 6$;
(vi) $3 - 2x > 5 - 2y$.

Exercises 1.2 B

1. In each case, are the indicated values a solution to the equation?

(i) Equation $3x + y = 4$; solution $x = 1$, $y = 1$;
(ii) Equation $3x + y = 4$; solution $x = 2$, $y = 2$;
(iii) Equation $4x - 2y = 3$; solution $x = 2$, $y = 2.5$;
(iv) Equation $4x - 2y = 6$; solution $x = 3$, $y = 3$;
(v) Equation $2x + 3y = 5$; solution $x = 1$, $y = -1$;
(vi) Equation $x^2 + 2x = y$; solution $x = 3$, $y = -5$;
(vii) Equation $x^2 + y^2 = 3$; solution $x = 2$, $y = 1$;
(viii) Equation $x^2 = y^2$; solution $x = 1$, $y = -1$.

2. Solve these equations for x:

(i) $2x - 4 = 2$; (ii) $6x - 3 = 9$;

(iii) $8 - 6x = 2$; (iv) $2x + 7 = 5$;

(v) $6x - 2 = 4x - 4$; (vi) $3 + x = 2x + 1$;

(vii) $7x - 3 = 2 - 4x$; (viii) $2 - 6x = 3x - 7$.

3. Solve these equations for x:

(i) $5x - 4 = -4$; (ii) $2x - 3 = 5$;

(iii) $4 - 2x = 3$; (iv) $3x + 7 = 6$;

(v) $2x - 4 = 3x - 3$; (vi) $3 - x = 3x + 1$;

(vii) $5x - 3 = 7 - 2x$; (viii) $2 - 3x = 3x + 7$;

(ix) $3x + 7 = -1 - 5x$; (x) $3 - 3x = 2 - 4x$;

(xi) $3x - 2 = 7x - 14$; (xii) $2(2 + x) = 3x - 7$;

(xiii) $2(x + 3) = 1 - 3x$; (xiv) $5 - x = 2(1 - 2x)$;

(xv) $2x - 5 = 7x - 17$; (xvi) $3(x - 1) = 4x - 2$.

4. Use the following equations to express y in terms of x:

(i) $3x + 2y = x + 4$; (ii) $3 - 2x - 2y = x + y$;

(iii) $2(x + y) = 5 - (x - y)$; (iv) $3 + 3x = 7 + 2y$;

(v) $2x - \frac{1}{3} = y - \frac{1}{2}$; (vi) $3x - y = x - 3y$.

5. Solve these inequalities for x:

(i) $3 - 4x \le 2$; (ii) $3x - 5 < 5x - 3$;

(iii) $5x + 2 > 8 - x$; (iv) $4x + 6 \le 2x$;

(v) $x + 10 \ge 12 - 3x$; (vi) $2 - x \le 2x + 5$;

(vii) $3x - 3 < 2 + 2x$; (viii) $3(x + 1) \le 3(1 - 2x)$.

6. Solve these inequalities for x:

(i) $3 - 2x \le 1$; (ii) $5x - 5 < 7x - 3$;

(iii) $4x + 2 > 7 - x$; (iv) $2x + 6 \le 4x$;

(v) $x + 8 \ge 12 - 5x$; (vi) $3 - x \le 3x + 1$;

(vii) $7x - 3 < 2 + 4x$; (viii) $2(x + 1) \le 3(1 - x)$;

(ix) $4 - 7x > 2 + x$; (x) $3(1 - 2x) < 2(x + 3)$;

(xi) $2(x - 3) \ge 1 + x$; (xii) $3x + 2 \le 3(3 - x)$;

(xiii) $1 - 5x > 2(3 - 4x)$; (xiv) $3(1 + x) < 5(x - 3)$.

7. Use the following inequalities to express y in terms of x:

(i) $2x + 2y \le 2$;
(ii) $6x - 3y + 12 < 0$;
(iii) $2x + 2y \le 1 - 2x$;
(iv) $x - 3y \le 3 - 2x$;
(v) $4x + 3y > 2y + 5x + 3$;
(vi) $2x - 5y < x + y - 3$;
(vii) $4x + 3y - 2 \ge 2(x + y)$;
(viii) $6 + 3x > x + 2y$;
(ix) $5(x + 1) \ge 3x + 2y + 1$;
(x) $3(1 - x) < 2(x - y)$.

8. Use the following inequalities to express x in terms of y:

(i) $2x + y \le 3$;
(ii) $4x - 2y + 2 < 0$;
(iii) $2x + 2y \le 2 - 4x$;
(iv) $x - y \le 2x - 3$;
(v) $2x + 3y > y + 6x + 4$;
(vi) $6x - 4y < x + y - 3$;
(vii) $3x + y - 2 \ge 4(x - y)$;
(viii) $2 + 3y > 2x + 2y$;
(ix) $5(x + 1) \ge 2x + 3y + 2$;
(x) $4(1 - x) < 2(x + y)$.

1.3 Sums

Sigma Notation

Suppose you want to write the sum of the first 16 positive integers. You could write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16$$

but, instead of this clumsy form, it is more usual to write $1 + 2 + \cdots + 16$, assuming that the reader will take the three dots to mean "continue in this fashion until you reach the last number shown" (and, more importantly, hoping it is clear that each number in the sum is obtained by adding 1 to the preceding number).

There is a standard mathematical notation for long sums, which uses the Greek capital letter sigma, or $\sum$. We write the above sum as

$$\sum_{i=1}^{16} i$$

which means we take the sum of all the values $i = 1, i = 2, \ldots$, up to $i = 16$. In the same way,

$$\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2;$$

the notation, called *sigma notation*, means "first evaluate the expression after the $\sum$ (that is, i^2) when $i = 1$, then when $i = 2, \ldots$, then when $i = 6$, and add the results". More generally, suppose a_1, a_2, a_3, and a_4 are any four numbers. (This use of a *subscript* on a letter, like the 1, 2, ... on a, is common in mathematics—otherwise we would run out of symbols!) Then

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4.$$

The definition of a set does not allow for ordering of its elements, or for repetition of its elements. For example, $\{1, 2, 3\}$, $\{1, 3, 2\}$, and $\{1, 2, 3, 1\}$ all represent the same set (which could be written $\{x \mid x \in \mathbb{N} \text{ and } x \leq 3\}$, or $\{x \in \mathbb{N} \mid x \leq 3\}$). However, ordering is often useful, so we define a *sequence* to be an ordered set. Sequences can be denoted by parentheses; (1, 3, 2) is the sequence with first element 1, second element 3 and third element 2, and is different from $(1, 2, 3)$. Sequences can contain repetitions, and $(1, 2, 1, 3)$ is quite different from $(1, 2, 3)$; the two occurrences of object 1 are distinguished by the fact that they lie in different positions in the ordering. Formally, a *sequence* (a_i) of length n is a collection of n objects $\{a_1, a_2, \ldots, a_n\}$, or $\{a_i : 1 \leq i \leq n\}$ together with an *ordering*: a_1 is first, a_2 is second, and so on. Entry a_i, where i is any one of the positive integers $1, 2, \ldots, n$, is called the ith *member of the sequence*. Two members can have the same value, but are counted as different if they are in different positions. For example, $(1, 2, 1, 3)$ is a sequence of length 4, but if we ignore the ordering then its elements form the set $\{1, 2, 3\}$ of order 3, called the *underlying set* of the sequence.

If (a_i) is a sequence of length n or longer, then $\sum_{i=1}^{n} a_i$ is defined by the rules

$$\sum_{i=1}^{1} a_i = a_1,$$

$$\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n.$$

The use of a_i to mean a "general" or "typical" member of $\{a_1, a_2, \ldots, a_n\}$ is very common. When a set of numbers $\{a_1, a_2, \ldots, a_n\}$ is being discussed, we say a property is true "for all a_i" when we mean it is true for each member of the set.

Usually, the sigma notation is used with a formula involving i for the term following $\sum$, as in the following examples. Notice that the range need not start at 1; we can write $\sum_{i=j}^{n}$ when j and n are any integers, provided $j < n$.

Sample Problem 1.14. *Write out the following as sums and evaluate them:*

$$\sum_{i=1}^{4} i^2; \quad \sum_{i=3}^{6} i(i+1).$$

Solution.

$$\begin{aligned}\sum_{i=1}^{4} i^2 &= 1^2 + 2^2 + 3^2 + 4^2 \\ &= 1 + 4 + 9 + 16 \\ &= 30; \\ \sum_{i=3}^{6} i(i+1) &= 3\cdot 4 + 4\cdot 5 + 5\cdot 6 + 6\cdot 7 \\ &= 12 + 20 + 30 + 42 \\ &= 104.\end{aligned}$$

Your Turn. Write out the following as sums and evaluate them:

$$\sum_{i=3}^{5} i(i-1); \quad \sum_{i=2}^{6} i.$$

Sample Problem 1.15. *Write the following in sigma notation:*

$$2 + 6 + 10 + 14; \quad 1 + 16 + 81.$$

Solution.

$$\sum_{i=1}^{4} (4i-2); \quad \sum_{i=1}^{3} i^4.$$

Your Turn. Write the following in sigma notation:

$$1 + 3 + 5 + 7 + 9; \quad 8 + 27 + 64 + 125.$$

Some properties of sums

It is easy to see that the following properties of sums are true:

(1) If c is any given number, then

$$\sum_{i=1}^{n} c = nc.$$

(2) If c is any given number and (a_i) is any sequence of length n, then

$$\sum_{i=1}^{n} (ca_i) = c\cdot\left(\sum_{i=1}^{n} a_i\right).$$

(3) If (a_i) and (b_i) are any two sequences, both of length n, then

$$\sum_{i=1}^{n}(a_i + b_i) = \left(\sum_{i=1}^{n} a_i\right) + \left(\sum_{i=1}^{n} b_i\right).$$

(4) If (a_i) is any sequence of length n, $1 \leq j < n$, then

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{j} a_i + \sum_{i=j+1}^{n} a_i.$$

The following is a standard result on sums.

Theorem 1. *The sum of the first n positive integers is* $\frac{1}{2}n(n+1)$*, or*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof. Let us write s for the answer. Then $s = \sum_{i=1}^{n} i$. We shall define two very simple sequences of length n. Write $a_i = i$ and $b_i = n + 1 - i$. Then (a_i) is the sequence $(1, 2, \ldots, n)$ and (b_i) is $(n, n-1, \ldots, 1)$. They both have the same elements, although they are written in different order, so they have the same sum,

$$s = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

So

$$\begin{aligned} 2s &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\ &= \sum_{i=1}^{n} (a_i + b_i) \\ &= \sum_{i=1}^{n} \big(i + (n + 1 - i)\big) \\ &= \sum_{i=1}^{n} (n + 1) \\ &= n \cdot (n + 1). \end{aligned}$$

Therefore, dividing by 2, we get $s = n(n+1)/2$. □

Since adding 0 does not change a sum, this result could also be stated as

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

That form is sometimes more useful.

Sample Problem 1.16. *Find the sum of the numbers from* 11 *to* 30 *inclusive.*

Solution. $\sum_{i=11}^{30} i$ is required. Now:

$$\sum_{i=1}^{30} i = \sum_{i=1}^{10} i + \sum_{i=11}^{30} i,$$

and, using the theorem and substituting, we have:

$$\frac{30 \cdot 31}{2} = \frac{10 \cdot 11}{2} + \sum_{i=11}^{30} i,$$
$$465 = 55 + \sum_{i=11}^{30} i,$$

so $\sum_{i=11}^{30} i = 465 - 55 = 410$.

Sample Problem 1.17. *Find* $2 + 6 + 10 + 14 + 18 + 22 + \cdots + 122$.

Solution. This is the sum of the terms $2 + 4i$, where i goes from 0 to 30.

$$\begin{aligned}\sum_{i=0}^{30} (2 + 4i) &= \left(\sum_{i=0}^{30} 2\right) + \left(\sum_{i=0}^{30} 4i\right) \\ &= \left(\sum_{i=0}^{30} 2\right) + 4 \cdot \left(\sum_{i=0}^{30} (i)\right) \\ &= 31 \cdot 2 + 4 \cdot \frac{30 \cdot 31}{2} \\ &= 62 + 1860 \\ &= 1922.\end{aligned}$$

Your Turn. Find $3 + 7 + \cdots + 43$.

Two other standard results, that will be proven in the exercises, are:

Theorem 2.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Theorem 3.

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1} \quad \textit{unless } x = 1.$$

Exercises 1.3 A

1. Write each of the following as sums and evaluate them:

(i) $\sum_{i=1}^{6} i^2 + 1$;

(ii) $\sum_{i=1}^{3} \frac{1}{10^i}$;

(iii) $\sum_{i=3}^{9} i(i-3)$;

(iv) $\sum_{i=2}^{5} \frac{1}{i}$;

(v) $\sum_{i=2}^{4} (-2)^i$;

(vi) $\sum_{i=1}^{4} 1 + (-1)^i$;

(vii) $\sum_{i=2}^{5} \frac{1}{i-1}$;

(viii) $\sum_{i=1}^{6} 1 + 3i$.

2. Write each of the following sums in sigma notation:

(i) $1 + 4 + 7 + 10 + 13$;

(ii) $2 + 6 + 10$;

(iii) $1 - 4 + 7 - 10 + 13$;

(iv) $0 - 1 + 4 - 9 + 16$;

(v) $3 + 7 + 3 + 7 + 3 + 7$;

(vi) $0 + 3 + 8 + 15$.

3. Suppose $a_i = i^2$, so that $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and so on.

(i) Say $b_i = i^3 - (i-1)^3$. Prove that $\sum_{i=1}^{n} b_i = n^3$.

(ii) Prove that $b_i = 3i^2 - 3i + 1$, and therefore

$$\sum_{i=1}^{n} b_i = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + n.$$

(iii) Use this to prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. Suppose $\sum_{i=1}^{n} a_i = A$ and $\sum_{i=1}^{n} b_i = B$. Evaluate $\sum_{i=1}^{n} c_i$ in each of the following cases:

(i) $c_i = 2a_i + 1$;

(ii) $c_i = 3a_i - b_i$;

(iii) $c_i = a_i + b_i + 2$;

(iv) $c_i = a_i + b_i + (-1)^n$.

5. Use the standard results of Theorems 1, 2, and 3 and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=7}^{12} (i-3)$; (ii) $\sum_{i=3}^{8} (5i^2-4)$;

(iii) $\sum_{i=2}^{6} 2^i$; (iv) $\sum_{i=1}^{n} (i^2+1)$;

(v) $\sum_{i=1}^{n} 3^i$; (vi) $\sum_{i=1}^{n} (6i^2-1)$;

(vii) $\sum_{i=1}^{n} i(i-2)$; (viii) $\sum_{i=1}^{n} \frac{1}{2^i}$.

6. Prove that

$$\sum_{i=1}^{n} 2n - 1 = n^2$$

(in other words, the sum of the first n odd positive whole numbers is n^2).

Exercises 1.3 B

1. Write the following as sums and evaluate them:

(i) $\sum_{i=1}^{4} i^2$; (ii) $\sum_{i=1}^{5} 1 + (-1)^i$;

(iii) $\sum_{i=3}^{6} 1 - i$; (iv) $\sum_{i=6}^{15} 4$;

(v) $\sum_{i=3}^{6} i(i-1)$; (vi) $\sum_{i=2}^{4} \frac{1}{i^2}$.

2. Write the following as sums and evaluate them:

(i) $\sum_{i=1}^{3} i^3$; (ii) $\sum_{i=1}^{6} (-1)^i$;

(iii) $\sum_{i=2}^{4} \sqrt{i}$; (iv) $\sum_{i=1}^{5} 1 + \frac{i}{2}$;

(v) $\sum_{i=3}^{6} i(i-2)$; (vi) $\sum_{i=1}^{5} (-1)^{i+1}i^2$;

(vii) $\sum_{i=12}^{15} i + \frac{1}{i}$; (viii) $\sum_{i=2}^{5} i + (-1)^i$;

(ix) $\sum_{i=2}^{5} i^2 - 3$; (x) $\sum_{i=2}^{5} i^2 - 4i$.

3. Write each of the following sums in sigma notation:

(i) $1 + 4 + 9 + 16 + 25$;

(ii) $2 + 5 + 8 + 11$;

(iii) $-1 + 2 - 3 + 4$;

(iv) $6 + 5 + 4 + 3 + 2 + 1$;

(v) $1 + 3 + 9 + 27 + 81$;

(vi) $4 - 6 + 8 - 10 + 12 - 14$;

(vii) $2 + 4 + 10 + 28 + 82$;

(viii) $4 + 8 + 12 + 16 + 20$;

(ix) $-1 - 2 - 3 - 4 - 5 - 6$;

(x) $11 + 17 + 23$.

4. Suppose $a_1 = 1, a_2 = x, a_3 = x^2$, and, in general, $a_i = x^{i-1}$.

(i) Prove that $\sum_{i=2}^{n+1} a_i = x\sum_{i=1}^{n} a_i$;

(ii) Use this to show that $\sum_{i=1}^{n} a_i = 1 + x\sum_{i=1}^{n} a_i - x^n$;

(iii) Use part (ii) to find the value of $\sum_{i=1}^{n} a_i$ when $x \neq 1$;

(iv) Why did we have to require "$x \neq 1$" in the preceding part?

5. Suppose $\sum_{i=1}^{n} a_i = A$ and $\sum_{i=1}^{n} b_i = B$. Evaluate $\sum_{i=1}^{n} c_i$ in each of the following cases:

(i) $c_i = 5a_i$; (ii) $c_i = 2a_i - b_i$;

(iii) $c_i = a_i + 2b_i$; (iv) $c_i = a_i + b_i - 1$;

(v) $c_i = 2a_i + b_i$; (vi) $c_i = 2 - b_i$.

6. Use that standard results of Theorems 1, 2, and 3 in the text and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=1}^{6}(2i-5)$;

(ii) $\sum_{i=1}^{10}\left(2i^2\right)$;

(iii) $\sum_{i=1}^{20}\left(i^2-3i\right)$;

(iv) $\sum_{i=0}^{5}\left(\frac{1}{2}\right)^i$;

(v) $\sum_{i=1}^{12} i^2-5i-1$;

(vi) $\sum_{i=1}^{n} 2i+1$.

7. Use that standard results of Theorems 1, 2, and 3 and the four properties of sums to evaluate the following expressions:

(i) $\sum_{i=5}^{20}(2i+1)$;

(ii) $\sum_{i=1}^{n}\left(i^2-i\right)$;

(iii) $\sum_{i=4}^{10}\left(i^2+1\right)$;

(iv) $\sum_{i=1}^{n}(1+i)^2$;

(v) $\sum_{i=1}^{9} 2^{-i}$;

(vi) $\sum_{i=0}^{n} 2^{-i}$.

1.4 Elements of Set Theory

More About Sets

We defined the notation

$$s \in S$$

to mean "s belongs to S" or "s is an element of S". If S and T are two sets, we shall write $T \subseteq S$ to mean that every member of T is also a member of S. In other words, "If s is any element of T then s is a member of S", or

$$s \in T \quad \Rightarrow \quad s \in S,$$

where $\Rightarrow$ is shorthand for *implies*. When $T \subseteq S$ we say T is a *subset* of S. Sets S and T are equal, $S = T$, if and only if $S \subseteq T$ and $T \subseteq S$ are both true. If necessary, we can represent the situation where T is a subset of S but S is not equal to T, that is, there is at least one member of S that is not a member of T, by writing $S \subset T$, and we call T a *proper* subset of S.

Suppose $R \subseteq S$ and $S \subseteq T$ are both true. Any member of R will also be a member of S, which means it is a member of T. So $R \subseteq T$. This sort of rule is called a *transitive law*.

It is important not to confuse the two symbols $\in$ and $\subseteq$, or their meanings:

Sample Problem 1.18. *Suppose* $S = \{0, 1\}$. *Which of the following are true:* $0 \in S$, $\{0\} \in S$, $0 \subset S$, $\{0\} \subset S$, $0 \subseteq S$, $\{0\} \subseteq S$, $S \in S$, $S \subset S$, $S \subseteq S$?

Solution. 0 is a member of S, but $\{0\}$ and S are not, so $0 \in S$ is true but $\{0\} \in S$, and $S \in S$ are false. As 0 is a member of S, $\{0\} \subset S$ and $\{0\} \subseteq S$ are true. But 0 is not a *set* of elements of S, so $0 \subset S$ and $0 \subseteq S$ are false. Finally, $S \subseteq S$ is true, but $S \subset S$ would imply $S \neq S$, so it is false.

Among the standard number sets, many subset relationships exist. Every natural number is an integer, every integer is a rational number, and every rational number is a real number, so $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$. We could write all these relationships down in one expression:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

In fact, we know that no two of these sets are equal, so we could write

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Some special sets of numbers are given special names: we have already used $\mathbb{Z}^*$ to denote the set of all *non-negative* integers, and similarly we could write $\mathbb{R}^*$ for the set of all non-negative real numbers. The notation $^+$ is often used for the set of all *positive* members of a number set; in particular, $\mathbb{Z}^+$ is another name for $\mathbb{N}$.

An important concept is the *empty* set, or *null* set, which has no elements. This set, denoted by $\emptyset$, is unique and is a subset of every other set.

In all the discussions of sets in this book, we shall assume (usually without bothering to mention the fact) that all the sets we are dealing with are subsets of some given universal set, U. The set U may be chosen to be as large as necessary in any problem we deal with; in many of our discussions we can choose $U = \mathbb{Z}$ or $U = \mathbb{R}$. The universal set can often be chosen to be a finite set.

The Basic Operations on Sets

Given sets S and T, we define three operations: the *union* of S and T is the set

$$S \cup T = \big\{x : x \in S \text{ or } x \in T \text{ (or both)}\big\};$$

the *intersection* of S and T is the set

$$S \cap T = \{x : x \in S \text{ and } x \in T\};$$

the *relative complement* of T with respect to S (or alternatively the *set-theoretic difference* or *relative difference* between S and T) is the set

$$S \backslash T = \{x : x \in S \text{ and } x \notin T\}.$$

In particular, the relative complement $U\backslash T$ with respect to the universal set U is denoted by $\overline{T}$ and called the *complement* of T.

We may write $R\backslash S = R \cap \overline{S}$, since each of these sets consists of the elements belonging to R but not to S. Hence we see that $R \subseteq S$ if and only if $R\backslash S = \emptyset$. If S is any set, $S\backslash S$ will equal $\emptyset$.

There is also a special relationship between the other operations and subsets. If S is any subset of T, then $S \cap T = S$ and $S \cup T = T$.

Sample Problem 1.19. *If $\mathbb{E}$ is the set of all even integers and Π is the set of all prime numbers, what are $\mathbb{E} \cup \Pi$, $\mathbb{E} \cap \Pi$, $\mathbb{E}\backslash\Pi$, and $\mathbb{Z} \cup \mathbb{N}$, $\mathbb{Z}^*\backslash\Pi$?*

Solution. $\mathbb{E} \cup \Pi$ contains all primes and all even numbers. (2 is the only number we have described twice.) $\mathbb{E} \cap \Pi$ contains only the common element 2, the only even prime. $\mathbb{E}\backslash\Pi$ contains all the even numbers *except* 2. All natural numbers are integers, so $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{Z} \cup \mathbb{N} = \mathbb{Z}$. $\mathbb{Z}^*\backslash\Pi$ contains all the positive numbers that are not primes, and zero. In symbols,

$$\begin{aligned}
\mathbb{E} \cup \Pi &= \{\ldots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, 8, 10, 11, \ldots\},\\
\mathbb{E} \cap \Pi &= \{2\},\\
\mathbb{E}\backslash\Pi &= \{\ldots, -8, -6, -4, -2, 0, 4, 6, 8, \ldots\},\\
\mathbb{Z} \cup \mathbb{N} &= \mathbb{Z},\\
\mathbb{Z}^*\backslash\Pi &= \{0, 1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \ldots\}.
\end{aligned}$$

Your Turn. What are $\mathbb{Z}\backslash\mathbb{Z}^*$, $\mathbb{Z} \cap \mathbb{N}$, $(\mathbb{N}\backslash\mathbb{E}) \cup \Pi$?

If two sets, S and T, have no common element, so that $S \cap T = \emptyset$, then we say that S and T are *disjoint*. Observe that $S\backslash T$ and T must be disjoint sets; in particular, T and $\overline{T}$ are disjoint.

Sample Problem 1.20. *In each case, are the sets S and T disjoint? If not, what is their intersection?*

(i) *S is the set of perfect squares, $T = \mathbb{R}\backslash\mathbb{R}^*$.*

(ii) *S is the set of perfect squares, $T = \mathbb{R}\backslash\mathbb{R}^+$.*

(iii) *S is the set of all multiples of 5, T is the set of all multiples of 7.*

Solution.

(i) The sets are disjoint.

(ii) They are not disjoint, because 0 is a perfect square ($0 = 0^2$); $S \cap T = \{0\}$.

(iii) They are not disjoint. $S \cap T$ is the set of all multiples of 35.

Your Turn. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) $S = \Pi$, T is the set of even integers.

(ii) $S = \{1, 3, 5, 7, 9\}$, $T = \{2, 4, 6, 8, 10\}$.

Properties of the Operations

We now investigate some of the easier properties of the operations $\cup$, $\cap$, and $\setminus$; for the more difficult problems, we shall introduce some techniques in the next section.

Union and intersection both satisfy *idempotence laws*: for any set S,

$$S \cup S = S \cap S = S.$$

Both operations satisfy *commutative laws*; in other words,

$$S \cup T = T \cup S$$

and

$$S \cap T = T \cap S,$$

for any sets S and T. Similarly, the *associative laws*

$$R \cup (S \cup T) = (R \cup S) \cup T$$

and

$$R \cap (S \cap T) = (R \cap S) \cap T$$

are always satisfied. The associative law means that we can omit brackets in a string of unions (or a string of intersections); expressions like $(A \cup B) \cup (C \cup D)$, $((A \cup B) \cup C) \cup D$, and $(A \cup (B \cup C)) \cup D$ are all equal, and we usually omit all the parentheses and simply write $A \cup B \cup C \cup D$. But be careful not to mix operations. $(A \cup B) \cap C$ and $A \cup (B \cap C)$ are quite different. Combining these two laws, we see that any string of unions can be rewritten in any order: for example,

$$(D \cup B) \cup (C \cup A) = \big(C \cup \big(B \cup (A \cup D)\big)\big) = (A \cup B \cup C \cup D).$$

Sample Problem 1.21. *Prove that* $(A \cup B) \cap C = A \cup (B \cap C)$ *is not always true.*

Solution. To prove that a general rule is not true, it suffices to find just one case in which it is false. As an example we take the case $A = \mathbb{R}$, $B = \mathbb{Z}$, $C = \{0\}$. Then $(A \cup B) \cap C = \{0\}$, while $A \cup (B \cap C) = \mathbb{R}$.

Cartesian Product

We define the *Cartesian product* (or *cross product*) $S \times T$ of sets S and T to be the set of all ordered pairs (s, t) where $s \in S$ and $t \in T$:

$$S \times T = \{(s, t) : s \in S, t \in T\}.$$

There is no requirement that S and T should be disjoint; in fact, it is often useful to consider $S \times S$.

The number of elements of $S \times T$ is $|S| \cdot |T|$.

Sample Problem 1.22. *Suppose* $S = \{0, 1\}$ *and* $T = \{1, 2\}$. *What is* $S \times T$*?*

Solution. $S \times T = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$, the set of all four of the possible ordered pairs.

Your Turn. What is $S \times T$ if $S = \{1, 2\}$ and $T = \{1, 4, 5\}$?

The sets $(R \times S) \times T$ and $R \times (S \times T)$ are not equal; one consists of an ordered pair whose *first* element is itself an ordered pair, and the other of pairs in which the *second* is an ordered pair. So there is no associative law, and no natural meaning for $R \times S \times T$. On the other hand, it is sometimes natural to talk about ordered *triples* of elements, so we define

$$R \times S \times T = \{(r, s, t) : r \in R, s \in S, t \in T\}.$$

This notation can be extended to ordered collections of any length.

Exercises 1.4 A

1. In each case, list all elements of the set described:
- (i) The set of all odd numbers between 2 and 10;
- (ii) The set of all days in a week;
- (iii) The set of all positive multiples of 3 between 10 and 20;
- (iv) The set of all prime numbers smaller than 10;
- (v) The set of all even prime numbers;
- (vi) The set of all positive factors of 36.

2. Write brief descriptions of the following sets. (There will often be many correct answers.)

(i) $\{3, 6, 9\}$;
(ii) $\{3, 6, 9, \ldots\}$;
(iii) $\{1, -10\}$;
(iv) $\{1, 4, 9\}$;
(v) $\{1, 3\}$.

3. Suppose

$$A = \{a, b, c, d, e\}, \qquad B = \{a, c, e, g, i\}, \qquad C = \{c, f, i, e, o\}.$$

Write down the elements of the following sets:

(i) $A \cup B$;
(ii) $A \cap C$;
(iii) $A \backslash B$;
(iv) $A \cup (B \backslash C)$.

4. Suppose

$$A = \{1, 2, 4, 5, 6, 7\}, \qquad B = \{1, 3, 5, 7, 9\}, \qquad C = \{2, 4, 6, 7, 8, 9\}.$$

Write down the elements of the following sets:

(i) $A \cup B \cup C$;
(ii) $A \cup (B \cap C)$;
(iii) $A \backslash C$;
(iv) $A \cap (B \backslash C)$.

5. Suppose

$$A = \{1, 3, 5, 6, 7\}, \qquad B = \{1, 2, 3, 4, 5\}, \qquad C = \{5, 6, 7, 8\}.$$

Write down the elements of the following sets:

(i) $A \cap B$;
(ii) $A \cap C$;
(iii) $A \backslash (B \cap C)$;
(iv) $A \cup B \cup C$;
(v) $A \backslash (B \cup C)$;
(vi) $(A \cup B) \backslash C$.

6. Consider the sets

$$\begin{aligned} S_1 &= \Pi, \\ S_2 &= \{2, 4, 6, 8\}, \\ S_3 &= \{2, 3, 5, 7\}, \\ S_4 &= \{x : x \text{ is a prime divisor of } 30\}, \\ S_5 &= \{x \in \mathbb{N} : x \text{ is a power of } 2\}. \end{aligned}$$

(i) For which i and j, if any, is $S_i \subseteq S_j$? For which i and j, if any, is $S_i = S_j$?
(ii) Write down the elements of $S_i \cap S_j$ for every case where $i < j$.

7. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) S is the set of all multiples of 5; T is the set of all perfect squares.

(ii) S is the set of all students in your class; T is the set of all students in your college.

8. In each case, list all elements of $S \times T$:

(i) $S = \{a, b, c\}$, $T = \{a, c, f\}$;

(ii) $S = \{x \mid x^2 = 4\}$, $T = \{2, 3, 4, 5\}$;

(iii) $S = \{x \mid x$ is a prime less than $4\}$, $T = \{4, 5\}$.

9. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2\}$, $S = \{3, 4\}$, $T = \{5, 6\}$;

(ii) $R = \{x, y\}$, $S = \{z\}$, $T = \{1, 2\}$.

Exercises 1.4 B

1. In each case, list all elements of the set described:

(i) The set of all integers between two and ten inclusive;

(ii) The set of all months in a year;

(iii) The set of odd positive multiples of 5, less than 50;

(iv) The set of all positive factors of 24;

(v) The set of all factors of 24;

(vi) The set of perfect squares smaller than 20;

(vii) The set of all vowels in English;

(viii) The set of all even integers between two and ten inclusive;

(ix) The set of all seasons in a year;

(x) The set of positive multiples of 5, less than 30.

2. Write brief descriptions of the following sets. (There will often be many correct answers.)

(i) $\{2, 4, 6, 8\}$;

(ii) $\{2, 4, 6, 8, \ldots\}$;

(iii) $\{2, 3, 5\}$;

(iv) $\{-1, -2, -3, \ldots\}$;

(v) $\{0, 1\}$.

3. Write at least two different brief descriptions of the set $\{2, 3, 5, 7\}$.

4. Consider the sets

$$A_1 = \{1, 2, 3, 4\},$$
$$A_2 = \{1, 2, 3, 4, 5, 6, 7, 8\},$$
$$A_3 = \{3, 4, 5, 8, 9\},$$

$$A_4 = \{4, 3, 2, 1\},$$
$$A_5 = \{2, 4, 6, 8\}.$$

(i) For which i and j, if any, is $A_i \subseteq A_j$? For which i and j, if any, is $A_i = A_j$?

(ii) Write down the elements of $A_i \cap A_j$ for every case where $i < j$.

5. Consider the sets

$$A_1 = \{1, 2, 4\},$$
$$A_2 = \{1, 2, 3, 4, 5\},$$
$$A_3 = \{1, 3, 4, 6\},$$
$$A_4 = \{4, 2, 1\}.$$

(i) For which i and j, if any, is $A_i \subseteq A_j$? For which i and j, if any, is $A_i = A_j$?

(ii) Write down the elements of $A_i \cap A_j$ for every case where $i < j$.

6. Suppose $A = \{2, 3, 5, 6, 8, 9\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{5, 6, 7, 8, 9\}$. Write down the elements of

(i) $A \cap B$; (ii) $A \cup C$;

(iii) $A \backslash (B \cap C)$; (iv) $(A \cup B) \backslash C$.

7. Suppose $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 5, 7, 8\}$, $C = \{2, 4, 6, 7, 8, 9\}$. Write down the elements of

(i) $(A \cap B) \cup C$; (ii) $A \cup (B \cap C)$;

(iii) $A \backslash C$; (iv) $A \cap B \cap C$.

8. Suppose $A = \Pi$, $B = \{2, 4, 6, 7, 9\}$, $C = \{1, 2, 3, 4, 5, 6\}$. Write down the elements of

(i) $A \cap B$; (ii) $C \backslash A$;

(iii) $A \cap (B \backslash C)$; (iv) $A \cap (B \cup C)$.

9. Suppose $A = \mathbb{Z}^+$, $B = \{-4, -2, 1, 3, 5, 7\}$, $C = \{x \mid x^2 = 1\}$. Write down the elements of

(i) $(A \cap B) \cup C$; (ii) $A \cap B \cap C$;

(iii) $C \backslash A$; (iv) $A \cap (B \backslash C)$.

10. Suppose $A = \{1, 2, 3, 5, 7\}$, $B = \{1, 2, 3, 4, 5, 6, 8\}$, $C = \{2, 4, 5, 6, 7\}$. Write down

(i) $A \cap B$; (ii) $A \cap C$;

(iii) $A \cup B$; (iv) $A \cup C$;

(v) $A \backslash (B \cap C)$; (vi) $A \backslash (B \cup C)$;

(vii) $(A \cup B)\backslash C$; (viii) $A \cap B \cap C$.

11. Consider the sets

$$S_1 = \{2, 5\},$$
$$S_2 = \{1, 2, 4\},$$
$$S_3 = \{1, 2, 4, 5, 10\},$$
$$S_4 = \{x \in \mathbb{N} : x \text{ is a divisor of } 20\},$$
$$S_5 = \{x \in \mathbb{N} : x \text{ is a power of 2 and a divisor of } 20\}.$$

(i) For which i and j, if any, is $S_i \subseteq S_j$? For which i and j, if any, is $S_i = S_j$?

(ii) Write down the elements of $S_i \cap S_j$ for every case where $i < j$.

12. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) $S = \{1, 2, 3, 4, 5\}$, $T = \{6, 7, 8, 9, 10\}$;

(ii) $S = \{1, 2, 3, 4, 5\}$, $T = \{5, 6, 7, 8, 9\}$;

(iii) $S = \{1, 3, 5, 7, 9\}$, $T = \{2, 4, 6, 8, 10\}$;

(iv) $S = \{1, 2, 3, 4, 5\}$; $T = \{2, 4, 6, 8, 10\}$.

13. In each case, are the sets S and T disjoint? If not, what is their intersection?

(i) S is the set of squares $1, 4, 9, \ldots$ and T is the set of cubes $1, 8, 27, \ldots$ of positive integers.

(ii) S is the set of perfect squares; $T = \mathbb{R}\backslash\mathbb{R}^+$.

(iii) S is the set of perfect squares $1, 4, 9, \ldots$; T is the set Π of primes.

14. In each case, list all elements of $S \times T$:

(i) $S = \{1, 3\}$, $T = \{1, 5\}$;

(ii) $S = \{0, 1, 2\}$, $T = \{2, 3, 4\}$;

(iii) $S = \{1, 3, 5, 7\}$, $T = \{1, 2, 3\}$;

(iv) $S = \{1, 3, 4\}$, $T = \{\text{red}, \text{blue}\}$;

(v) $S = \{x \mid x^2 = 1\}$, $T = \{y \mid y^2 = 4\}$;

(vi) $S = \{x \mid x \in \Pi, x < 10\}$, $T = \{1, 2\}$;

(vii) $S = \{1, 2\}$, $T = \{3, 4\}$;

(viii) $S = \{1, 2\}$, $T = \{2, 3\}$;

(ix) $S = \{1, 7\}$, $T = \{1, 2, 3\}$;

(x) $S = \{x \mid x^2 = 1\}$, $T = \{y \mid y^2 = 1\}$.

15. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2\}$, $S = \{1, 3\}$, $T = \{2, 3\}$;

(ii) $R = \{12, 13, 14\}$, $S = \{1\}$, $T = \{1, 2, 3\}$.

16. In each case, list all elements of $R \times S \times T$:

(i) $R = \{1, 2, 3\}$, $S = \{1, 3\}$, $T = \{2\}$;

(ii) $R = \{2, 3\}$, $S = \{1, 3\}$, $T = \{1, 2\}$.

17. In each case, find the sets $A \cap (B \cup C)$ and $(A \cap B) \cup C$:

(i) $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$, $C = \{5, 6, 7, 8\}$;

(ii) $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 4, 6\}$, $C = \{2, 3, 7, 8\}$.

18. (i) If $S = \emptyset$, $T \neq \emptyset$, what is $S \times T$?

(ii) If $S \times T = T \times S$, what can you say about S and T?

19. (i) If $A \subseteq S$, $B \subseteq T$, show that $A \times B \subseteq S \times T$.

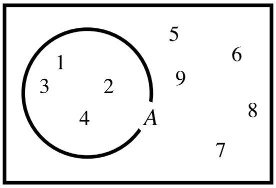
(ii) Find an example of sets A, B, S, T, such that $A \times B \subseteq S \times T$ and $B \subseteq T$, but $A \not\subseteq S$.

1.5 Venn Diagrams

Venn Diagrams

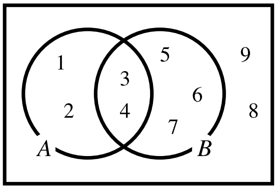
It is common to illustrate sets and operations on sets by diagrams. A set A is represented by a circle, and it is assumed that the elements of A correspond to the points (or some of the points) inside the circle. The universal set is usually shown as a rectangle enclosing all the other sets; if it is not needed, the universal set is sometimes omitted. Such an illustration is called a *Venn diagram*.[1]

For example, suppose the universal set consists of the nine integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then the diagram will be enclosed in a rectangle, and the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 will be represented by points in the rectangle. If A is the set $\{1, 2, 3, 4\}$, then a circle will be drawn inside the rectangle, with 1, 2, 3, and 4 inside it and the other numbers outside:

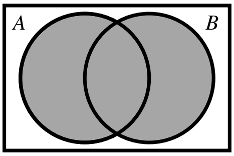


[1] Similar diagram methods were first used by Leibniz and Euler. George Venn used the diagrams extensively. Some textbooks use the name "Euler diagram" for the case where not all possible regions are represented, but we think it more appropriate simply to say "Venn diagram" because Venn formalized and unified diagram methods. For some reason, Leibniz has escaped credit.

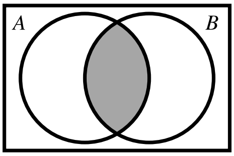
If $B = \{3, 4, 5, 6, 7\}$ then A and B can be shown on the same diagram as follows, with the common elements of A and B in the area common to the two circles:



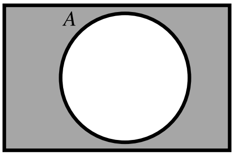
This same method can be used to represent more general sets. The following figures show Venn diagrams representing $A \cup B$, $A \cap B$, $\overline{A}$, and $A \backslash B$; in each case, the set represented is shown by the shaded area. The universal set is shown in each figure. Of course, we do not know the individual members of the sets, but they could be represented by points in the diagram if they were known.



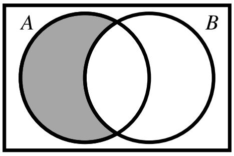
$A \cup B$



$A \cap B$



$\overline{A}$



$A \backslash B$

Two sets are equal if and only if they have the same Venn diagram. In order to illustrate this, consider the set equation

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T),$$

where R, S, and T can be any sets. The Venn diagram for $R \cup (S \cap T)$ is constructed in Figure 1.1, and that for $(R \cup S) \cap (R \cup T)$ is constructed in Figure 1.2. The two are obviously identical.

This rule is called a *distributive law*, and it should remind you of the distributive law

$$a(b + c) = ab + ac$$

for numbers. There are in fact two distributive laws for sets; the second is:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

To prove $A \subseteq B$, it is sufficient to show that all the shaded areas in the diagram for A are also shaded in B. We illustrate this idea with another set law.

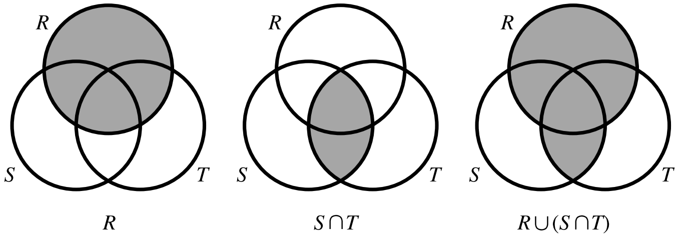


Fig. 1.1. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

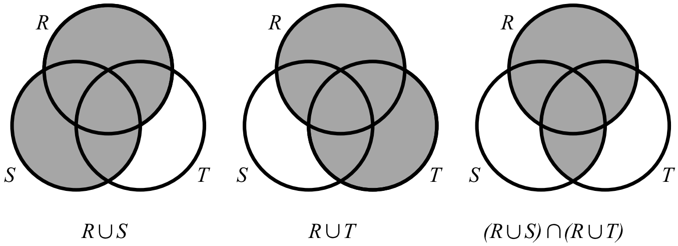
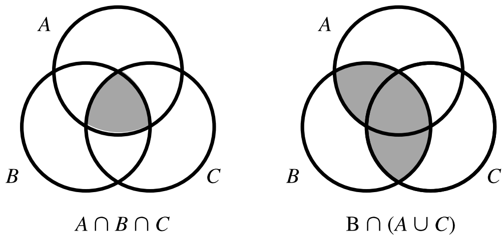


Fig. 1.2. $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

Sample Problem 1.23. *Use Venn diagrams to prove that*

$$A \cap B \cap C \subseteq B \cap (A \cup C).$$

Solution.



Your Turn. Use Venn diagrams to prove that

$$A \cap C \subseteq (\overline{A} \cap B) \cup C.$$

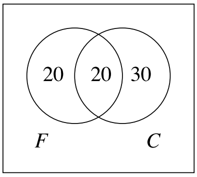
Sometimes we draw a Venn diagram so as to represent some properties of sets. For example, if A and B are disjoint sets, the diagram can be drawn with A and B shown as disjoint circles. If $A \subseteq B$, the circle for A is entirely inside the circle for B.

Counting the Elements

Venn diagrams can also be used to calculate the numbers of elements in sets. We write the numbers of elements in the areas of the diagram, instead of the elements themselves.

Sample Problem 1.24. *There are* 40 *students in Dr. Brown's Finite Mathematics course and* 50 *in his Calculus section. If these are his only classes, and if* 20 *of the students are taking both subjects, how many students does he have altogether? Represent the data in a Venn diagram.*

Solution. We use the notation F for the set of students in the finite class and C for Calculus. There are four areas in the Venn diagram: the set of students in both classes (the center area, $F \cap C$) has 20 members; the set of students in Finite Mathematics only (the left-hand enclosed area) has 20 members—subtract 20 from 40; the area corresponding to Calculus-only students, has $50 - 20 = 30$ members; and the outside area has no members (only Dr. Brown's students are being considered). So Dr. Brown has 70 students; the diagram is

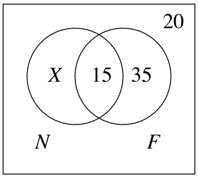


Your Turn. A survey shows that 12 newspaper readers buy the morning edition and seven buy the evening edition; three of these also bought the morning paper. Represent the data in a Venn diagram. How many readers were surveyed?

This method can be used to find the number of elements in the set represented by any of the areas or unions of areas, provided there is enough information.

Sample Problem 1.25. 100 *people were surveyed to find out whether newspaper advertisements or flyers were more efficacious in advertising supermarket specials.* 20 *of them said they pay no attention to either medium.* 50 *said they read the flyers, and* 15 *of those said they also check the newspapers. How many use the newspaper ads, in total?*

Solution. We do not know how many people read the newspaper advertisements but not the flyers. Suppose there are X of them. Then we get the Venn diagram

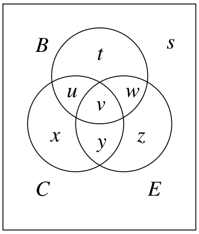


In order for the total to add to 100, $X = 30$, so the total who use the newspaper ads is $X + 15 = 45$.

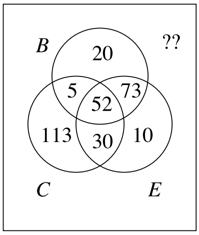
These methods can be applied to three or more sets.

Sample Problem 1.26. 500 *people were asked about their morning vitamin intake. It was found that* 150 *take vitamin B,* 200 *take vitamin C,* 165 *take vitamin E,* 57 *take both B and C,* 125 *take both B and E,* 82 *take both C and E, and* 52 *take all three vitamins. How many take both B and E but do not take C? How many take none of these vitamins?*

Solution. We start with the diagram



From the data, $v = 52$, and $u + v = 57$, so $u = 5$. Similarly, $w = 73$ and $y = 30$. Now $t + u + v + w = 150$, so $t = 20$. The other sizes are calculated similarly, and we get the diagram



The cell corresponding to "B and E but not C" has 73 elements. There are 303 elements in total, so there are 197 people who take none.

Your Turn. In a survey of 150 moviegoers, it was found that 80 like horror movies, 75 like police procedurals, and 60 like romances. In total, 35 like both horror and procedurals, 25 like horror and romance, and 30 like procedurals and romance. 15 like all three types of movie. Represent these data in a Venn diagram. How many like horror and police procedural movies, but do not like romances? How many like romances only? How many like none of these three types?

Exercises 1.5 A

1. Represent the following sets in a Venn diagram:

(i) $R \cup S \cup T$; (ii) $R \cup (S \cap \overline{T})$;

(iii) $(R \cap S)\backslash(S \cap T)$; (iv) $(R\backslash S) \cap T$.

2. Use Venn diagrams to prove the commutative and associative laws for $\cup$:

(i) $(R \cup S) = (S \cup R)$;

(ii) $((R \cup S) \cup T) = (R \cup (S \cup T))$.

3. Prove the following rules using Venn diagrams:

(i) $\overline{S \cap T} = \overline{S} \cup \overline{T}$;

(ii) $(S \cap T) \subseteq S$.

4. 1865 voters were surveyed about a new highway. Of them, 805 were in favor financing it with a new tax, 627 favored tolls on the highway, while 438 said they would vote in favor of either measure.

(i) How many favor taxes but not tolls?

(ii) How many favor tolls but not taxes?

(iii) How many would vote against either form of funding?

5. Some shoppers buy bread, butter, and coffee. Ten buy bread, 15 buy butter, and 14 buy coffee. The number buying both bread and butter, both bread and coffee, and both butter and coffee are four, five, and eight, respectively. Three buy all three products. How many shoppers were there (assuming all bought at least one of the three)?

6. 18 people are interviewed. Of these, seven dislike the Republican party, ten dislike the Democrats, and 11 dislike the Green party. Moreover, five dislike both Democrats and Republicans, five dislike both Republicans and Greens, while six dislike both Democrats and Greens. And four dislike all three types of politicians, on principle. How many *like* all three?

7. In a survey at Dave's Bagel Barn, 75 customers were asked about their preferences. It was found that 35 like poppy seed bagels, 33 like onion bagels, and 26 like chocolate bagels. There were 18 who like both poppy seed and onion, and nine who like both poppy seed and chocolate, while 12 enjoyed only chocolate from the three being discussed.

(i) How many customers enjoy onion and chocolate but do not like poppy seed?

(ii) How many liked onion but neither of the others?

(iii) How many did not like any of those flavors?

(iv) Can you tell how many liked all three?

8. Elizabeth's Chocolate Company sells milk chocolates, dark chocolates, and white chocolates. On one day, 164 customers bought chocolates. 120 of them purchased a selection that included milk chocolates, 114 included dark chocolates in their order, and 84 included white chocolates. There were 80 who bought both dark and milk and 64 who bought both white and milk, while 40 bought all three varieties. Draw a Venn diagram to represent these data.

(i) How many bought both dark and white chocolate but no milk chocolate?

(ii) How many bought a milk and white assortment that included no dark chocolate?

(iii) How many bought white chocolate only?

9. A shopping mall contains K-Mart, Wal-Mart, and Target stores. In a survey of 100 shoppers, it is found that

(a) 47 shopped at Wal-Mart;

(b) 61 shopped at K-Mart;

(c) 52 shopped at Target;

(d) 32 shopped at both Wal-Mart and K-Mart;

(e) 35 shopped at both K-Mart and Target;

(f) 22 shopped at both Wal-Mart and Target;

(g) 12 shopped at all three stores.

(i) How many of those surveyed shopped only at Wal-Mart?

(ii) How many shopped at Wal-Mart and K-Mart, but not at Target?

(iii) How many shopped at exactly one of the stores?

(iv) How many shopped at none of the three stores?

Exercises 1.5 B

1. Represent the following sets in a Venn diagram.

(i) $\overline{R \cup S \cup T}$; (ii) $(R \cap S) \cap T$;

(iii) $R \cap (S \backslash T)$; (iv) $(R \cup T) \backslash (S \cap T)$.

2. Prove: $R = (\overline{\overline{R} \cup S}) \cup (R \cap S)$.

3. Find a simpler expression for $S \cup (\overline{(\overline{R} \cup S) \cap R})$.

4. Use Venn diagrams to prove the commutative and associative laws for $\cap$:

(i) $(R \cap S) = (S \cap R)$;

(ii) $((R \cap S) \cap T) = (R \cap (S \cap T))$.

5. Use Venn diagrams to prove the distributive law

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

6. Prove the following rules using Venn diagrams:

(i) $S \cap \overline{S} = \emptyset$;

(ii) $\overline{S \cup T} = \overline{S} \cap \overline{T}$;

(iii) $S \subseteq (S \cup T)$.

7. For any sets R and S, prove $R \cap (R \cup S) = R$.

8. Among 1000 personal computer users it was found that 375 have a scanner and 450 have a DVD player attached to their computer. Moreover, 150 had both devices.

(i) How many had either a scanner or a DVD player?

(ii) How many had neither device?

9. A survey was carried out. It was found that ten of the people surveyed were drinkers, while five were smokers; only two both drank and smoked, but seven did neither. How many people were interviewed?

10. 50 people were asked about their book purchases. 20 said they had bought at least one fiction book in the last week, 30 had bought at least one non-fiction book, and ten had bought no books. Assuming that any book can be classified as either fiction or non-fiction, how many of the people interviewed had bought both fiction and non-fiction during the week?

11. 60 people were asked about news magazines. It was found that 32 read *Newsweek* regularly, 25 read *Time*, and 20 read *U.S. News and World Report*. Nine read both *Time* and *U.S. News and World Report*, 11 read both *Newsweek* and *Time*, and eight read both *Newsweek* and *U.S. News and World Report*. Eight of the people do not read any of the magazines.

(i) How many read all three magazines?

(ii) Represent the data in a Venn diagram.

(iii) How many people read exactly one of the three magazines?

12. Of 100 high school students, 45 were currently enrolled in Mathematics courses, 38 were enrolled in Science, and 21 were enrolled in Geography. There were 18 in both Math and Science and nine in both Math and Geography, while 12 were taking Geography but neither of the other subjects. Exactly five students were taking all three subjects, and there were 23 not currently enrolled in any of the three subjects. Represent these figures in a Venn diagram. How many students were taking precisely one of the three subjects? How many were taking precisely two?

13. Of 100 personal computer users surveyed, 27 use Dell, 35 use Gateway, and 35 use Hewlett-Packard. Ten of them use both Dell and Gateway, eight use both Dell and Hewlett-Packard, and 12 use both Gateway and Hewlett-Packard. Four use all three.

(i) How many use exactly one of these brands?

(ii) How many only use other brands?

14. Freshmen at New College are required to attend at least one orientation lecture. Three lectures are held. Of the 450 freshmen, 136 attended the morning lecture, 185 the afternoon lecture, and 127 the evening lecture. There were 20 who attended both morning and afternoon, 20 who attended both afternoon and evening, and five who attended both morning and evening. A total of 41 students attended more than one lecture.

(i) How many students attended all three lectures?

(ii) How many students failed to attend any lecture?

15. Researchers at IBM were surveyed about their qualifications in computer science. It was found that 214 have a Bachelor's degree, 123 have a Master's degree, and 99 have a Ph.D. It was further found that 57 have both Bachelor's and Master's degrees, 74 have both Master's and Ph.D., and 22 have both Bachelor's and Ph.D. degrees.

(i) Show that at least eight of the researchers have all three degrees.

(ii) Suppose that all of the researchers with both a Bachelor's degree and a Ph.D. also hold a Master's degree in the field. How many have all three degrees? How many came into computer science from another field: that is, they have either a Master's or a Ph.D., but no Bachelor's degree?

16. A survey of students found that:

(a) 62 were enrolled in Calculus;

(b) 71 were enrolled in Algebra;

(c) 67 were enrolled in Discrete Mathematics;

(d) 37 were enrolled in both Calculus and Algebra;

(e) 32 were enrolled in both Calculus and Discrete Mathematics;

(f) 40 were enrolled in both Algebra and Discrete Mathematics;

(g) 12 were enrolled in all three subjects;

(h) 44 were enrolled in none of the three.

(i) How many were enrolled in both Calculus and Algebra but not in Discrete Mathematics?

(ii) How many were enrolled in exactly one of the three subjects?

(iii) How many students were surveyed?

17. A pet store surveyed 120 of its customers. It was found that 55 of them owned dogs, 50 owned cats, and 40 owned goldfish. Further, 20 own both dogs and cats, 15 own both dogs and goldfish, and 12 own both cats and goldfish. Draw a Venn diagram to represent these data.

(i) How many customers own dogs and cats but do not own goldfish?

(ii) How many own none of these three kinds of pets?

18. Prove, using Venn diagrams, that $(R\backslash S)\backslash T = R\backslash(S\backslash T)$ does not hold for *all* choices of sets R, S and T. In other words, show that there are some choices for R, S and T such that the equation is not true.

1.6 Averages

Central Tendency

Suppose you measure the noon temperature in your home town on ten summer days. The Fahrenheit temperature readings, arranged in ascending order, might be

$$F = \{76, 79, 80, 81, 83, 83, 84, 84, 84, 86\}.$$

If somebody asked you what is a typical noon summer temperature, there are three logical answers.

First, you might add the ten readings and divide by 10, giving the answer 82°. This is called the *mean* or *average* temperature.

Second, you might say 84° because that is the most common answer. It came up three times. This is called the *mode* of F.

Finally, the *median* or half-way point is the reading at the midpoint of the list (after it has been put in ascending or descending order). As F has ten elements, the midpoint lies between the fifth and sixth readings. Each equals 83, so the median is 83°.

All three measures are attempts to answer the question, "what is the center of the collection of temperatures?" For this reason the mean, mode, and median are all called measures of *central tendency*.

Sample Problem 1.27. *What are the mean, mode, and median of the following collections?*

$$S = \{13, 13, 11, 14, 15\},$$
$$T = \{8, 13, 12, 15, 10, 26\}.$$

Solution. S has sum 66, so the mean is $66/5 = 13.2$. The mode is 13, the only reading to appear twice. To find the median, first write the readings in ascending order: 11, 13, 13, 14, 15. The "center" reading is 13, so the median is 13.

T has sum 84, so its mean is 14. Its median is 12.5 (after the readings are reordered as 8, 10, 12, 13, 15, 26, the two "center" values are 12 and 13; when they are unequal, take the average of the two). There is no mode because no reading occurs more frequently than any other.

Your Turn. What are the means, modes, and medians of the following collections?

$$U = \{7, 9, 2, 3, 6, 7, 15\},$$
$$V = \{14, 22, 24, 17, 21, 19\}.$$

Notice the use of the word "collection" rather than "set" because often the data or observations will not form sets; there will be repetitions. Also observe from the examples that neither the mean nor the median need be members of the collection. In fact, all the collections we looked at were integers, but the mean and median were not always integers.

The Mean

The *mean* m_X of an n-element collection $X = \{x_1, x_2, \ldots, x_n\}$ is defined by the formula

$$m_X = \frac{\sum_{i=1}^{n} x_i}{n}.$$

If only one set is involved, the subscript X is often omitted.

Suppose a new collection $Y = \{y_1, y_2, \ldots, y_n\}$ is defined by the rule

$$y_i = a + x_i.$$

Then

$$m_Y = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{\sum_{i=1}^{n} a + x_i}{n} = \frac{na + \sum_{i=1}^{n} x_i}{n} = a + m_X.$$

So we have the rule:

> If the members of Y are obtained by adding a to each member of X, then the mean of Y is obtained by adding a to the mean of X.

The number a can be positive or negative. This rule is useful in practice, as the following example shows.

Sample Problem 1.28. *Find the mean of*

$$X = \{1051, 1053, 1054, 1055, 1059\}.$$

Solution. We define Y by the rule "$y_i = x_i - 1050$". Then $Y = \{1, 3, 4, 5, 9\}$; $m_Y = 22/5 = 4.4$, so $m_X = 4.4 + 1050 = 1054.4$.

Your Turn. Use this method to find the mean of

$$\{833, 834, 836, 837, 841, 841\}.$$

When the collection of data is large, there will sometimes be several copies of the same value. If there are f members all equal to the same value x, then f is called the *frequency* of x. For example, the collection F of temperatures that we discussed at the beginning of this section might be described by saying

> 76, 79, 80, 81, and 86 each have frequency 1; 83 has frequency 2; 84 has frequency 3.

If we wish, we can also say that 77, 78, 82, and 85 (and all values outside the range) have frequency 0.

If the collection X consists of elements $\{x_1, x_2, \ldots, x_k\}$, where x_1 has frequency f_1, x_2 has frequency f_2, and so on, then its mean is

$$m_X = \frac{\sum_{i=1}^{k} x_i f_i}{\sum_{i=1}^{k} f_i}.$$

Sample Problem 1.29. *The scores in a quiz are integers ranging from* 0 *to* 5. *The frequencies of the scores are:*

x	0	1	2	3	4	5
f	2	0	10	14	4	1

What is the mean score?

Solution. The sum of the terms $x_i f_i$ is

$$0 \times 2 + 1 \times 0 + 2 \times 10 + 3 \times 14 + 4 \times 4 + 5 \times 1 = 83$$

while the sum of the frequencies is

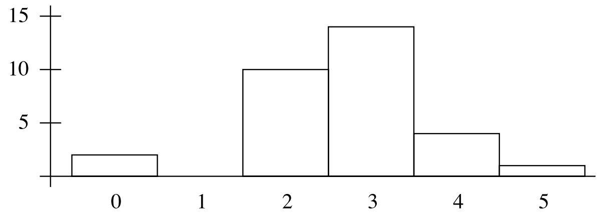
$$2 + 0 + 10 + 14 + 4 + 1 = 31$$

so the mean is $83/31 = 2.6774\ldots$ or about 2.68. In practice we would say the mean is "about 2.7", or even "between 2 and 3", because fractional scores were not given.

Your Turn. Find the mean of the following data:

x	2	3	4	5	6	7	8
f	1	3	7	11	12	4	1

Instead of a table, data like this is often represented graphically in a *histogram*, a diagram with a column for each possible value, where the height of the column is proportional to the frequency of that value. Here is the histogram for the data of Sample Problem 1.29.



Measures of Variation

Even when you have decided which measure of central tendency to use, and calculate it, you do not know all about your collection of readings. For example, consider two towns. Placidville has similar weather on most summer days; the temperatures on 11 consecutive days were

$$P = \{78, 79, 80, 81, 82, 83, 83, 84, 84, 84, 84\}.$$

In Wildtown the weather is far more extreme, with sudden cold and hot changes; over eleven days, the recorded temperatures were

$$W = \{69, 71, 75, 81, 82, 83, 84, 84, 89, 91, 93\}.$$

In both cities, the mean, mode and median were 82°, 84°, and 83°, respectively. However, there is far more temperature variation in Wildtown than in Placidville.

Several measures of *variation*, or *spread*, are used. We shall define the *standard deviation*.

The Standard Deviation

If a set of observations has mean m, the deviation of an observation x from m is $x - m$. At first it seems that the average of the deviations would be a good measure of variation. However, the mean deviation is always zero! The negative and positive contributions cancel each other out.

Better measures can be found by averaging the absolute values of the deviations. Given an n-element collection $X = \{x_1, x_2, \ldots, x_n\}$ with mean m_X, we consider the values

$$|x_1 - m_X|, \quad |x_2 - m_X|, \quad \ldots, \quad |x_m - m_X|.$$

For technical reasons, the most reliable measure is found by averaging the squares of the deviations and taking the square root of this average. So we define the *standard deviation* s_X of the set X to be

$$s_X = \sqrt{\frac{\sum_{i=1}^{n}(x_i - m_X)^2}{n}}.$$

The quantity s_X^2, which must be calculated before the square root is taken, is the *variance* of the set X.

Sample Problem 1.30. *Calculate the standard deviation of the daily temperatures in Wildtown and Placidville.*

Solution. The mean for Wildtown is 82, so the variance is

$$\begin{aligned}&\big[(69-82)^2 + (71-82)^2 + (75-82)^2 + (81-82)^2 + (82-82)^2\\ &\quad + (83-82)^2 + (84-82)^2 + (84-82)^2 + (89-82)^2 + (91-82)^2\\ &\quad + (93-82)^2\big]/11\\ &= [169 + 121 + 49 + 1 + 0 + 1 + 4 + 4 + 49 + 81 + 121]/11\\ &= 599/11 = 54.4545\ldots\end{aligned}$$

and the standard deviation is about 7.2. The mean for Placidville is again 82, so its variance is

$$\begin{aligned}&\big[(78-82)^2 + (79-82)^2 + (80-82)^2 + (81-82)^2 + (82-82)^2\\ &\quad + (83-82)^2 + (83-82)^2 + (84-82)^2 + (84-82)^2 + (84-82)^2\\ &\quad + (84-82)^2\big]/11\\ &= [16 + 9 + 4 + 1 + 0 + 1 + 1 + 4 + 4 + 4 + 4]/11\\ &= 48/11 = 4.3636\ldots\end{aligned}$$

and the standard deviation is about 2.1.

The standard deviation of a set of observations will be unchanged if any constant is added to, or subtracted from, each observation. For example, in discussing Placidville, we could just as easily have subtracted 80 from each reading, and found the standard deviation of $\{-2, -1, 0, 1, 2, 3, 3, 4, 4, 4, 4\}$.

If the collection X consists of elements $\{x_1, x_2, \ldots, x_k\}$, where x_1 has frequency f_1, x_2 has frequency f_2, and so on, then its standard deviation is

$$s_X = \frac{\sum_{i=1}^{k} x_i - m_X{}^2 f_i}{\sum_{i=1}^{k} f_i}.$$

A Formula for Standard Deviation

The formula for variance can be rewritten as

$$\begin{aligned} ns_X^2 &= \sum_{i=1}^{n}(x_i - m_X)^2 \\ &= \sum_{i=1}^{n}\left(x_i^2 - 2x_i m_X + m_X^2\right) \\ &= \left(\sum_{i=1}^{n} x_i^2\right) - 2\left(\sum_{i=1}^{n} x_i m_X\right) + \left(\sum_{i=1}^{n} m_X^2\right) \\ &= \left(\sum_{i=1}^{n} x_i^2\right) - 2m_X\left(\sum_{i=1}^{n} x_i\right) + nm_X^2. \end{aligned}$$

Now $\sum_{i=1}^{n} x_i = nm_X$ (this is just the equation for m_X, multiplied on both sides by n), so

$$2m_X\left(\sum_{i=1}^{n} x_i\right) = 2nm_X^2,$$

and

$$ns_X^2 = \left(\sum_{i=1}^{n} x_i^2\right) - 2nm_X^2 + nm_X^2 = \left(\sum_{i=1}^{n} x_i^2\right) - nm_X^2,$$

giving the formula

$$s_X = \sqrt{\frac{(\sum_{i=1}^{n} x_i^2) - nm_X^2}{n}}.$$

This formula is often more convenient for calculations, as the following (very small) example shows.

Sample Problem 1.31. *Find the standard deviation of the data set* $\{1, 1, 1, 2, 3, 3, 5\}$.

Solution. The data add to 16, so the mean is $\frac{16}{7} = 2.\overline{285714}$. Using the original standard deviation formula, each calculation must include several decimal places in order to ensure accuracy. However, using the new formula, we find

$$
\begin{aligned}
7s_X^2 &= \left[1^2 + 1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 5^2\right] - 7\left(\frac{16}{7}\right)^2 \\
&= [1 + 1 + 1 + 4 + 9 + 9 + 25] - \frac{256}{7} \\
&= 50 - 36.5714 = 13.4286
\end{aligned}
$$

so $s_X^2 = 1.91837$, and $s_X = 1.39$ (correct to two decimal places).

Your Turn. Find the standard deviation of the data $\{1, 1, 2, 2, 3, 4, 4\}$.

Exercises 1.6 A

1. Find the mean and median of the following sets of data:
(i) $\{17, 19, 23, 30\}$;
(ii) $\{8, 13, 20, 12, 29\}$;
(iii) $\{14, 15, 23, 22, 24, 19\}$.

2. Find the mean, mode, and median of the following sets of data:
(i) $\{3, 16, 10, 1, 4, 2, 6, 2\}$;
(ii) $\{-3, 5, 8, 1, -4, 6, 6, 6\}$;
(iii) $\{2, 10, 10, 1, 1, 2, 7, 9, 11, 12\}$.

3. Find the mean of the data $\{2, 3, 5, 2, 7, 9, 7\}$, and list the deviations of the data from the mean. Verify that the mean deviation is zero.

4. For each set of data in Exercise 2, what is the standard deviation?

5. Find the mean and standard deviation of the following data. Under each score is listed its frequency

x	1	2	3	4	5	6
f	5	2	4	4	7	4

.

6. Draw a histogram representing the data in the preceding exercise.

Exercises 1.6 B

1. Find the mean and median of the following collections of data:
(i) $\{15, 17, 28, 21\}$;
(ii) $\{14, 13, 18, 12, 24\}$;

(iii) $\{17, 25, 13, 22, 14, 29\}$;

(iv) $\{12, 13, 19, 20, 16\}$;

(v) $\{11, 20, 15, 21, 14, 28, 17\}$.

2. Find the mean, mode, and median of the following collections of data:

(i) $\{5, 3, 4, 7, 7\}$;

(ii) $\{8, 11, 5, 4, 9\}$;

(iii) $\{8, 12, 10, 4, 4, 7, 6, 11\}$;

(iv) $\{-3, -5, 6, 4, -4, 6, 6, 8\}$;

(v) $\{9, 10, 7, 9, 1, 2, 7, 9, 11, 7\}$;

(vi) $\{73, 77, 72, 70, 77, 83, 86, 84\}$;

(vii) $\{43, 46, 50, 43, 47, 53, 46, 44\}$;

(viii) $\{1081, 1083, 1088, 1086, 1087\}$;

(ix) $\{944, 942, 940, 947, 949\}$.

3. Find the mean of the data $\{6, 3, 4, 2, 7, 3, 3\}$, and list the deviations of the data from the mean. Verify that the mean deviation is zero.

4. For each set of data in Exercise 2, what is the standard deviation?

5. Find the mean, mode, median, and standard deviation of the following collections of data:

(i) $\{2, 4, 8, 3, 8\}$;

(ii) $\{7, 11, 2, 4, 6, 6\}$;

(iii) $\{-2, -2, 3, 1, 4, 2, 4, -2\}$;

(iv) $\{-3, -2, 1, 1, -2, 3, 5, 5\}$;

(v) $\{2, -2, 5, -3, 3, 7\}$;

(vi) $\{27, 25, 28, 30, 30\}$;

(vii) $\{-5, -6, 4, 14, 17, -7, 6, 4, 3, -5\}$;

(viii) $\{21, 23, 23, 25, 23\}$.

6. In the following data, under each score is listed its frequency. In each case:

(a) Draw a histogram representing the data;

(b) Find the mean and standard deviation:

(i)

x	1	2	3	4	5	7	9
f	2	1	3	2	4	5	3

;

(ii)

x	27	28	30	31	33	35
f	1	2	3	1	2	1

;

(iii)

x	1	2	3	4	5	6	7	8	9
f	3	3	2	3	5	4	3	2	3

;

(iv)

x	17	18	19	20	21	22	23	24
f	2	3	3	2	4	8	4	3

.